Wolfram|Alpha Step-by-Step Solution

Wolfram|Alpha Input:

STEP 1

Solve $\frac{d^2 y(t)}{dt^2} + 4\, y(t) = \cos(a\, t)$, such that $y(0) = \frac{1}{2}$ and $y'(0) = 0$:

STEP 2

The general solution will be the sum of the complementary solution and particular solution.

Find the complementary solution by solving $\frac{d^2 y(t)}{dt^2} + 4\, y(t) = 0$:

STEP 3

Assume a solution will be proportional to $e^{\lambda t}$ for some constant λ.

Substitute $y(t) = e^{\lambda t}$ into the differential equation:

$$\frac{d^2}{dt^2}\left(e^{\lambda t}\right) + 4\, e^{\lambda t} = 0$$

STEP 4

Substitute $\frac{d^2}{dt^2}\left(e^{\lambda t}\right) = \lambda^2\, e^{\lambda t}$:

$$\lambda^2\, e^{\lambda t} + 4\, e^{\lambda t} = 0$$

STEP 5

Factor out $e^{\lambda t}$:

$$\left(\lambda^2 + 4\right) e^{\lambda t} = 0$$

STEP 6

Since $e^{\lambda t} \neq 0$ for any finite λ, the zeros must come from the polynomial:

$$\lambda^2 + 4 = 0$$

STEP 7

Solve for λ:

$\lambda = 2\, i$ or $\lambda = -2\, i$

STEP 8

The roots $\lambda = \pm 2\,i$ give $y_1(t) = c_1\,e^{2\,i\,t}$, $y_2(t) = c_2\,e^{-2\,i\,t}$ as solutions, where c_1 and c_2 are arbitrary constants.
The general solution is the sum of the above solutions:
$y(t) = y_1(t) + y_2(t) = c_1\,e^{2\,i\,t} + c_2\,e^{-2\,i\,t}$

STEP 9

Apply Euler's identity $e^{\alpha + i\,\beta} = e^{\alpha}\cos(\beta) + i\,e^{\alpha}\sin(\beta)$:
$y(t) = c_1\,(\cos(2\,t) + i\sin(2\,t)) + c_2\,(\cos(2\,t) - i\sin(2\,t))$

STEP 10

Regroup terms:
$y(t) = (c_1 + c_2)\cos(2\,t) + i\,(c_1 - c_2)\sin(2\,t)$

STEP 11

Redefine $c_1 + c_2$ as c_1 and $i\,(c_1 - c_2)$ as c_2, since these are arbitrary constants:
$y(t) = c_1\cos(2\,t) + c_2\sin(2\,t)$

STEP 12

Determine the particular solution to $\frac{d^2 y(t)}{dt^2} + 4\,y(t) = \cos(a\,t)$ by variation of parameters:
List the basis solutions in $y_c(t)$:
$y_{b_1}(t) = \cos(2\,t)$ and $y_{b_2}(t) = \sin(2\,t)$

STEP 13

Compute the Wronskian of $y_{b_1}(t)$ and $y_{b_2}(t)$:

$$\mathcal{W}(t) = \begin{vmatrix} \cos(2\,t) & \sin(2\,t) \\ \frac{d}{dt}(\cos(2\,t)) & \frac{d}{dt}(\sin(2\,t)) \end{vmatrix} = \begin{vmatrix} \cos(2\,t) & \sin(2\,t) \\ -2\sin(2\,t) & 2\cos(2\,t) \end{vmatrix} = 2$$

STEP 14

Let $f(t) = \cos(a\,t)$:
Let $v_1(t) = -\int \frac{f(t)\,y_{b_2}(t)}{\mathcal{W}(t)}\,dt$ and $v_2(t) = \int \frac{f(t)\,y_{b_1}(t)}{\mathcal{W}(t)}\,dt$:
The particular solution will be given by:
$y_p(t) = v_1(t)\,y_{b_1}(t) + v_2(t)\,y_{b_2}(t)$

STEP 15

Compute $v_1(t)$:

$$v_1(t) = -\int \frac{1}{2} \cos(a\,t) \sin(2\,t)\,dt = -\frac{\cos(t\,(a-2))}{4\,(a-2)} + \frac{\cos(t\,(a+2))}{4\,(a+2)}$$

STEP 16

Compute $v_2(t)$:

$$v_2(t) = \int \frac{1}{2} \cos(2\,t) \cos(a\,t)\,dt = \frac{1}{2}\left(\frac{\sin(t\,(a-2))}{2\,(a-2)} + \frac{\sin(t\,(a+2))}{2\,(a+2)}\right)$$

STEP 17

The particular solution is thus:

$$y_p(t) = v_1(t)\, y_{b_1}(t) + v_2(t)\, y_{b_2}(t) =$$
$$\cos(2\,t)\left(-\frac{\cos((a-2)\,t)}{4\,(a-2)} + \frac{\cos((a+2)\,t)}{4\,(a+2)}\right) + \frac{1}{2}\left(\frac{\sin((a-2)\,t)}{2\,(a-2)} + \frac{\sin((a+2)\,t)}{2\,(a+2)}\right)\sin(2\,t)$$

STEP 18

Simplify:

$$y_p(t) = -\frac{\cos(a\,t)}{a^2-4}$$

STEP 19

The general solution is given by:

$$y(t) = y_c(t) + y_p(t) = c_1 \cos(2\,t) + c_2 \sin(2\,t) - \frac{\cos(a\,t)}{a^2-4}$$

STEP 20

Solve for the unknown constants using the initial conditions:

Compute $\frac{dy(t)}{dt}$:

$$\frac{dy(t)}{dt} = \frac{d}{dt}\left(-\frac{\cos(a\,t)}{a^2-4} + c_1 \cos(2\,t) + c_2 \sin(2\,t)\right)$$
$$= \frac{a \sin(a\,t)}{a^2-4} - 2\,c_1 \sin(2\,t) + 2\,c_2 \cos(2\,t)$$

STEP 21

Substitute $y(0) = \frac{1}{2}$ into $y(t) = -\frac{\cos(a\,t)}{a^2-4} + \cos(2\,t)\,c_1 + \sin(2\,t)\,c_2$:

$$-\frac{1}{a^2-4} + c_1 = \frac{1}{2}$$

STEP 22

Substitute $y'(0) = 0$ into $\frac{dy(t)}{dt} = \frac{a\sin(a\,t)}{a^2-4} - 2\sin(2\,t)\,c_1 + 2\cos(2\,t)\,c_2$:

$2\,c_2 = 0$

STEP 23

Solve the system:

$c_1 = \frac{a^2-2}{2(a^2-4)}$

$c_2 = 0$

STEP 24

Substitute $c_1 = \frac{a^2-2}{2(a^2-4)}$ and $c_2 = 0$ into $y(t) = -\frac{\cos(a\,t)}{a^2-4} + \cos(2\,t)\,c_1 + \sin(2\,t)\,c_2$:

Answer:

$$y(t) = \frac{-2\cos(a\,t) + (a^2-2)\cos(2\,t)}{2(a^2-4)}$$