y'' + 4y = cos(at), y(0) = 1/2, y'(0) =0

Input:

$$\left\{y''(t) + 4\,y(t) = \cos(a\,t),\ y(0) = \frac{1}{2},\ y'(0) = 0\right\}$$

ODE classification:

second-order linear ordinary differential equation

Alternate forms:

$$\left\{y''(t) = \cos(a\,t) - 4\,y(t),\ y(0) = \frac{1}{2},\ y'(0) = 0\right\}$$

$$\{\cos(a\,t) = y''(t) + 4\,y(t),\ 2\,y(0) = 1,\ y'(0) = 0\}$$

$$\left\{y''(t) + 4\,y(t) = \frac{1}{2}\,e^{-i\,a\,t} + \frac{1}{2}\,e^{i\,a\,t},\ y(0) = \frac{1}{2},\ y'(0) = 0\right\}$$

Differential equation solution: Step-by-step solution

$$y(t) = \frac{(a^2 - 2)\cos(2\,t) - 2\cos(a\,t)}{2\,(a^2 - 4)}$$

Wolfram|Alpha Step-by-Step Solution

Differential equation solutions: Solve with undetermined coefficients

Wolfram|Alpha Step-by-Step Solution

Wolfram|Alpha Input:

STEP 1

Solve $\frac{d^2 y(t)}{dt^2} + 4\,y(t) = \cos(a\,t)$, such that $y(0) = \frac{1}{2}$ and $y'(0) = 0$:

STEP 2

The general solution will be the sum of the complementary solution and particular solution.

Find the complementary solution by solving $\frac{d^2 y(t)}{dt^2} + 4\, y(t) = 0$:

STEP 3

Assume a solution will be proportional to $e^{\lambda t}$ for some constant λ.

Substitute $y(t) = e^{\lambda t}$ into the differential equation:

$$\frac{d^2}{dt^2}\left(e^{\lambda t}\right) + 4\, e^{\lambda t} = 0$$

STEP 4

Substitute $\frac{d^2}{dt^2}\left(e^{\lambda t}\right) = \lambda^2 e^{\lambda t}$:

$$\lambda^2 e^{\lambda t} + 4\, e^{\lambda t} = 0$$

STEP 5

Factor out $e^{\lambda t}$:

$$\left(\lambda^2 + 4\right) e^{\lambda t} = 0$$

STEP 6

Since $e^{\lambda t} \neq 0$ for any finite λ, the zeros must come from the polynomial:

Solve for λ:

$\lambda = 2\, i$ or $\lambda = -2\, i$

STEP 8

The roots $\lambda = \pm 2\, i$ give $y_1(t) = c_1\, e^{2\, i\, t}$, $y_2(t) = c_2\, e^{-2\, i\, t}$ as solutions, where c_1 and c_2 are arbitrary constants.

The general solution is the sum of the above solutions:

$$y(t) = y_1(t) + y_2(t) = c_1\, e^{2\, i\, t} + c_2\, e^{-2\, i\, t}$$

STEP 9

Apply Euler's identity $e^{\alpha + i\beta} = e^{\alpha} \cos(\beta) + i e^{\alpha} \sin(\beta)$:
$y(t) = c_1 (\cos(2t) + i \sin(2t)) + c_2 (\cos(2t) - i \sin(2t))$

STEP 10

Regroup terms:
$y(t) = (c_1 + c_2) \cos(2t) + i (c_1 - c_2) \sin(2t)$

STEP 11

Redefine $c_1 + c_2$ as c_1 and $i(c_1 - c_2)$ as c_2, since these are arbitrary constants:
$y(t) = c_1 \cos(2t) + c_2 \sin(2t)$

STEP 12

Determine the particular solution to $\frac{d^2 y(t)}{dt^2} + 4 y(t) = \cos(a t)$ by the method of undetermined coefficients:
The particular solution to $\frac{d^2 y(t)}{dt^2} + 4 y(t) = \cos(a t)$ is of the form:
$y_p(t) = b_1 \cos(a t) + b_2 \sin(a t)$

STEP 13

Solve for the unknown constants b_1 and b_2:
Compute $\frac{d^2 y_p(t)}{dt^2}$:

$$\frac{d^2 y_p(t)}{dt^2} = \frac{d^2}{dt^2}(b_1 \cos(a t) + b_2 \sin(a t))$$
$$= -a^2 \cos(a t) b_1 - a^2 \sin(a t) b_2$$

STEP 14

Substitute the particular solution $y_p(t)$ into the differential equation:
$\frac{d^2 y_p(t)}{dt^2} + 4 y_p(t) = \cos(a t)$
$-a^2 b_1 \cos(a t) - a^2 b_2 \sin(a t) + 4 (b_1 \cos(a t) + b_2 \sin(a t)) = \cos(a t)$

STEP 15

Simplify:
$(4 b_1 - a^2 b_1) \cos(a t) + (4 b_2 - a^2 b_2) \sin(a t) = \cos(a t)$

STEP 16

Equate the coefficients of $\cos(a\, t)$ on both sides of the equation:

$$4\, b_1 - a^2\, b_1 = 1$$

STEP 17

Equate the coefficients of $\sin(a\, t)$ on both sides of the equation:

$$4\, b_2 - a^2\, b_2 = 0$$

STEP 18

Solve the system:

$$b_1 = \frac{1}{-a^2 + 4}$$

$$b_2 = 0$$

STEP 19

Substitute b_1 and b_2 into $y_p(t) = \cos(a\, t)\, b_1 + \sin(a\, t)\, b_2$:

$$y_p(t) = \frac{\cos(a\, t)}{-a^2 + 4}$$

STEP 20

The general solution is:

$$y(t) = y_c(t) + y_p(t) = \frac{\cos(a\, t)}{-a^2 + 4} + c_1 \cos(2\, t) + c_2 \sin(2\, t)$$

STEP 21

Solve for the unknown constants using the initial conditions:

Compute $\frac{d y(t)}{dt}$:

$$\begin{aligned}\frac{d y(t)}{dt} &= \frac{d}{dt}\left(\frac{\cos(a\, t)}{-a^2 + 4} + c_1 \cos(2\, t) + c_2 \sin(2\, t)\right) \\ &= -\frac{a \sin(a\, t)}{-a^2 + 4} - 2\, c_1 \sin(2\, t) + 2\, c_2 \cos(2\, t)\end{aligned}$$

STEP 22

Substitute $y(0) = \frac{1}{2}$ into $y(t) = \frac{\cos(a\, t)}{-a^2 + 4} + \cos(2\, t)\, c_1 + \sin(2\, t)\, c_2$:

$$\frac{1}{-a^2 + 4} + c_1 = \frac{1}{2}$$

STEP 23

Substitute $y'(0) = 0$ into $\frac{dy(t)}{dt} = -\frac{a \sin(a\,t)}{-a^2+4} - 2\sin(2\,t)c_1 + 2\cos(2\,t)c_2$:

$2\,c_2 = 0$

STEP 24

Solve the system:

$c_1 = \frac{a^2-2}{2(a^2-4)}$

$c_2 = 0$

STEP 25

Substitute $c_1 = \frac{a^2-2}{2(a^2-4)}$ and $c_2 = 0$ into $y(t) = \frac{\cos(a\,t)}{-a^2+4} + \cos(2\,t)c_1 + \sin(2\,t)c_2$:

Answer:

$$y(t) = \frac{-2\cos(a\,t) + (a^2-2)\cos(2\,t)}{2(a^2-4)}$$

Related Queries:

- y''(t) + sin(y(t))=0
- x''(t) + 3 x(t) = 2 cos(4t), x(0) = 1, x'(0) = 2
- pendulum formula
- double pendulum
- (d^2/dy^2 c(x, y)) + (d^2/dy^2 c(x, y)) = 0