

CÁLCULO

Octava edición

SOLUCIONARIO

LARSON
HOSTETLER
EDWARDS

PART I

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CHAPTER P

Preparation for Calculus

Section P.1 Graphs and Models

Solutions to Odd-Numbered Exercises

1. $y = -\frac{1}{2}x + 2$

x -intercept: $(4, 0)$

y -intercept: $(0, 2)$

Matches graph (b)

3. $y = 4 - x^2$

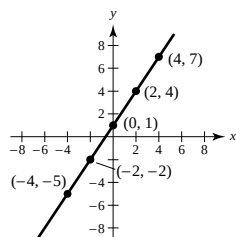
x -intercepts: $(2, 0), (-2, 0)$

y -intercept: $(0, 4)$

Matches graph (a)

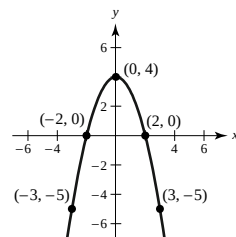
5. $y = \frac{3}{2}x + 1$

x	-4	-2	0	2	4
y	-5	-2	1	4	7



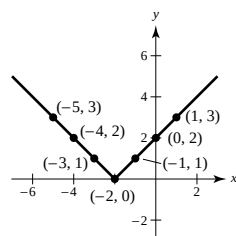
7. $y = 4 - x^2$

x	-3	-2	0	2	3
y	-5	0	4	0	-5



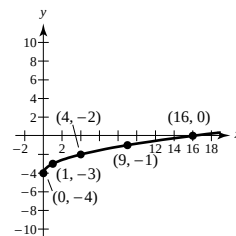
9. $y = |x + 2|$

x	-5	-4	-3	-2	-1	0	1
y	3	2	1	0	1	2	3



11. $y = \sqrt{x} - 4$

x	0	1	4	9	16
y	-4	-3	-2	-1	0



13.
$$\begin{array}{l} \text{Xmin} = -3 \\ \text{Xmax} = 5 \\ \text{Xscl} = 1 \\ \text{Ymin} = -3 \\ \text{Ymax} = 5 \\ \text{Yscl} = 1 \end{array}$$

Note that $y = 4$ when $x = 0$.

17. $y = x^2 + x - 2$

y-intercept: $y = 0^2 + 0 - 2$

$y = -2; (0, -2)$

x-intercepts: $0 = x^2 + x - 2$

$0 = (x + 2)(x - 1)$

$x = -2, 1; (-2, 0), (1, 0)$

21. $y = \frac{3(2 - \sqrt{x})}{x}$

y-intercept: None. x cannot equal 0.

x-intercepts: $0 = \frac{3(2 - \sqrt{x})}{x}$

$0 = 2 - \sqrt{x}$

$x = 4; (4, 0)$

25. Symmetric with respect to the y-axis since

$y = (-x)^2 - 2 = x^2 - 2.$

29. Symmetric with respect to the origin since

$(-x)(-y) = xy = 4.$

33. Symmetric with respect to the origin since

$-y = \frac{-x}{(-x)^2 + 1}$

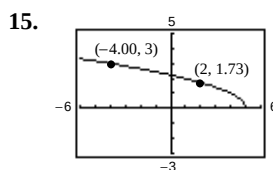
$y = \frac{x}{x^2 + 1}.$

37. $y = -3x + 2$

Intercepts:

$(\frac{2}{3}, 0), (0, 2)$

Symmetry: none



(a) $(2, y) = (2, 1.73)$ ($y = \sqrt{5 - 2} = \sqrt{3} \approx 1.73$)

(b) $(x, 3) = (-4, 3)$ ($3 = \sqrt{5 - (-4)}$)

19. $y = x^2\sqrt{25 - x^2}$

y-intercept: $y = 0^2\sqrt{25 - 0^2}$

$y = 0; (0, 0)$

x-intercepts: $0 = x^2\sqrt{25 - x^2}$

$0 = x^2\sqrt{(5 - x)(5 + x)}$

$x = 0, \pm 5; (0, 0); (\pm 5, 0)$

23. $x^2y - x^2 + 4y = 0$

y-intercept:

$0^2(y) - 0^2 + 4y = 0$

$y = 0; (0, 0)$

x-intercept:

$x^2(0) - x^2 + 4(0) = 0$

$x = 0; (0, 0)$

27. Symmetric with respect to the x-axis since

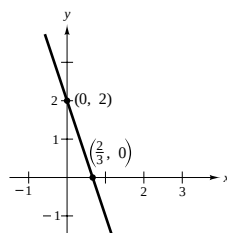
$(-y)^2 = y^2 = x^3 - 4x.$

31. $y = 4 - \sqrt{x + 3}$

No symmetry with respect to either axis or the origin.

35. $y = |x^3 + x|$ is symmetric with respect to the y-axis

since $y = |(-x)^3 + (-x)| = |-(x^3 + x)| = |x^3 + x|.$

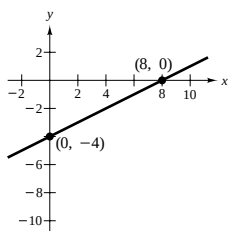


39. $y = \frac{x}{2} - 4$

Intercepts:

$(8, 0), (0, -4)$

Symmetry: none

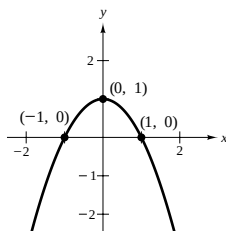


41. $y = 1 - x^2$

Intercepts:

$(1, 0), (-1, 0), (0, 1)$

Symmetry: y-axis

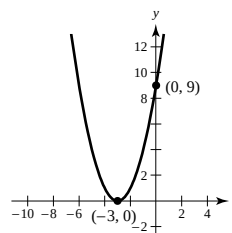


43. $y = (x + 3)^2$

Intercepts:

$(-3, 0), (0, 9)$

Symmetry: none

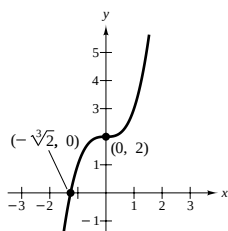


45. $y = x^3 + 2$

Intercepts:

$(-\sqrt[3]{2}, 0), (0, 2)$

Symmetry: none

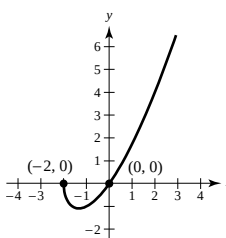


47. $y = x\sqrt{x+2}$

Intercepts:

$(0, 0), (-2, 0)$

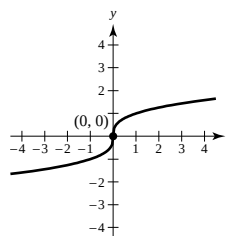
Symmetry: none

Domain: $x \geq -2$ 

49. $x = y^3$

Intercepts: $(0, 0)$

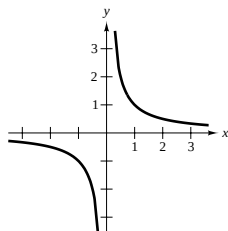
Symmetry: origin



51. $y = \frac{1}{x}$

Intercepts: none

Symmetry: origin

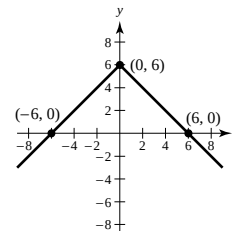


53. $y = 6 - |x|$

Intercepts:

$(0, 6), (-6, 0), (6, 0)$

Symmetry: y-axis



55. $y^2 - x = 9$

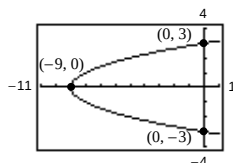
$y^2 = x + 9$

$y = \pm\sqrt{x+9}$

Intercepts:

$(0, 3), (0, -3), (-9, 0)$

Symmetry: x-axis



57. $x + 3y^2 = 6$

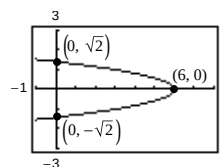
$3y^2 = 6 - x$

$y = \pm\sqrt{2 - \frac{x}{3}}$

Intercepts:

$(6, 0), (0, \sqrt{2}), (0, -\sqrt{2})$

Symmetry: x-axis



59. $y = (x + 2)(x - 4)(x - 6)$ (other answers possible)

63. $x + y = 2 \Rightarrow y = 2 - x$

$$2x - y = 1 \Rightarrow y = 2x - 1$$

$$2 - x = 2x - 1$$

$$3 = 3x$$

$$1 = x$$

The corresponding y -value is $y = 1$.

Point of intersection: $(1, 1)$

67. $x^2 + y = 6 \Rightarrow y = 6 - x^2$

$$x + y = 4 \Rightarrow y = 4 - x$$

$$6 - x^2 = 4 - x$$

$$0 = x^2 - x - 2$$

$$0 = (x - 2)(x + 1)$$

$$x = 2, -1$$

The corresponding y -values are $y = 2$ (for $x = 2$) and $y = 5$ (for $x = -1$).

Points of intersection: $(2, 2), (-1, 5)$

71. $y = x^3$

$$y = x$$

$$x^3 = x$$

$$x^3 - x = 0$$

$$x(x + 1)(x - 1) = 0$$

$$x = 0, x = -1, \text{ or } x = 1$$

The corresponding y -values are $y = 0, y = -1$, and $y = 1$.

Points of intersection: $(0, 0), (-1, -1), (1, 1)$

61. Some possible equations:

$$y = x$$

$$y = x^3$$

$$y = 3x^3 - x$$

$$y = \sqrt[3]{x}$$

65. $x + y = 7 \Rightarrow y = 7 - x$

$$3x - 2y = 11 \Rightarrow y = \frac{3x - 11}{2}$$

$$7 - x = \frac{3x - 11}{2}$$

$$14 - 2x = 3x - 11$$

$$-5x = -25$$

$$x = 5$$

The corresponding y -value is $y = 2$.

Point of intersection: $(5, 2)$

69. $x^2 + y^2 = 5 \Rightarrow y^2 = 5 - x^2$

$$x - y = 1 \Rightarrow y = x - 1$$

$$5 - x^2 = (x - 1)^2$$

$$5 - x^2 = x^2 - 2x + 1$$

$$0 = 2x^2 - 2x - 4 = 2(x + 1)(x - 2)$$

$$x = -1 \text{ or } x = 2$$

The corresponding y -values are $y = -2$ and $y = 1$.

Points of intersection: $(-1, -2), (2, 1)$

73. $y = x^3 - 2x^2 + x - 1$

$$y = -x^2 + 3x - 1$$

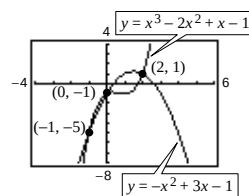
$$x^3 - 2x^2 + x - 1 = -x^2 + 3x - 1$$

$$x^3 - x^2 - 2x = 0$$

$$x(x - 2)(x + 1) = 0$$

$$x = -1, 0, 2$$

$(-1, -5), (0, -1), (2, 1)$



75. $5.5\sqrt{x} + 10,000 = 3.29x$

$$(5.5\sqrt{x})^2 = (3.29x - 10,000)^2$$

$$30.25x = 10.8241x^2 - 65,800x + 100,000,000$$

$$0 = 10.8241x^2 - 65,830.25x + 100,000,000 \quad \text{Use the Quadratic Formula.}$$

$$x \approx 3133 \text{ units}$$

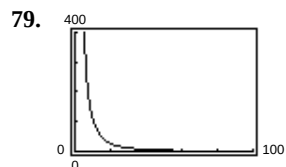
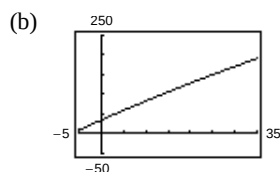
The other root, $x \approx 2949$, does not satisfy the equation $R = C$.

This problem can also be solved by using a graphing utility and finding the intersection of the graphs of C and R .

77. (a) Using a graphing utility, you obtain

$$y = -0.0153t^2 + 4.9971t + 34.9405$$

(c) For the year 2004, $t = 34$ and
 $y \approx 187.2$ CPI.



If the diameter is doubled, the resistance is changed by approximately a factor of $(1/4)$. For instance, $y(20) \approx 26.555$ and $y(40) \approx 6.36125$.

81. False; x -axis symmetry means that if $(1, -2)$ is on the graph, then $(1, 2)$ is also on the graph.

83. True; the x -intercepts are

$$\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, 0 \right).$$

85. Distance to the origin $= K \times$ Distance to $(2, 0)$

$$\sqrt{x^2 + y^2} = K\sqrt{(x - 2)^2 + y^2}, K \neq 1$$

$$x^2 + y^2 = K^2(x^2 - 4x + 4 + y^2)$$

$$(1 - K^2)x^2 + (1 - K^2)y^2 + 4K^2x - 4K^2 = 0$$

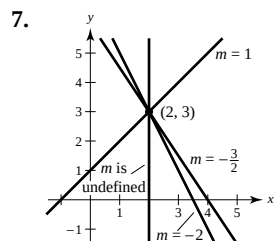
Note: This is the equation of a circle!

Section P.2 Linear Models and Rates of Change

1. $m = 1$

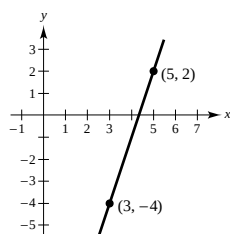
3. $m = 0$

5. $m = -12$



9.
$$m = \frac{2 - (-4)}{5 - 3}$$

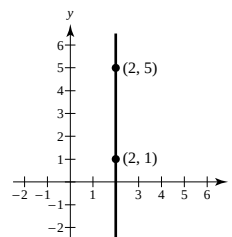
$$= \frac{6}{2} = 3$$



11.
$$m = \frac{5 - 1}{2 - 2}$$

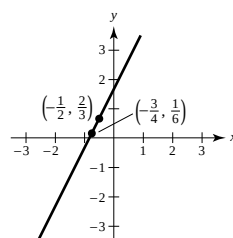
$$= \frac{4}{0}$$

undefined



13.
$$m = \frac{2/3 - 1/6}{-1/2 - (-3/4)}$$

$$= \frac{1/2}{1/4} = 2$$



15. Since the slope is 0, the line is horizontal and its equation is $y = 1$. Therefore, three additional points are (0, 1), (1, 1), and (3, 1).

17. The equation of this line is

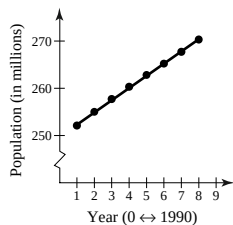
$$y - 7 = -3(x - 1)$$

$$y = -3x + 10.$$

Therefore, three additional points are (0, 10), (2, 4), and (3, 1).

19. Given a line L , you can use any two distinct points to calculate its slope. Since a line is straight, the ratio of the change in y -values to the change in x -values will always be the same. See Section P.2 Exercise 93 for a proof.

21. (a)



(b) The slopes of the line segments are

$$\frac{255.0 - 252.1}{2 - 1} = 2.9$$

$$\frac{257.7 - 255.0}{3 - 2} = 2.7$$

$$\frac{260.3 - 257.7}{4 - 3} = 2.6$$

$$\frac{262.8 - 260.3}{5 - 4} = 2.5$$

$$\frac{265.2 - 262.8}{6 - 5} = 2.4$$

$$\frac{267.7 - 265.2}{7 - 6} = 2.5$$

$$\frac{270.3 - 267.7}{8 - 7} = 2.6$$

The population increased most rapidly from 1991 to 1992.

 $(m = 2.9)$

23. $x + 5y = 20$

$$y = -\frac{1}{5}x + 4$$

Therefore, the slope is $m = -\frac{1}{5}$ and the y -intercept is $(0, 4)$.

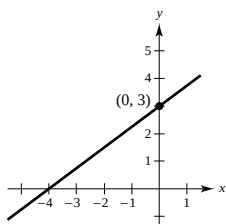
25. $x = 4$

The line is vertical. Therefore, the slope is undefined and there is no y -intercept.

27. $y = \frac{3}{4}x + 3$

$$4y = 3x + 12$$

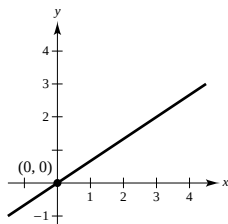
$$0 = 3x - 4y + 12$$



29. $y = \frac{2}{3}x$

$$3y = 2x$$

$$2x - 3y = 0$$

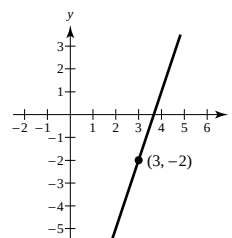


31. $y + 2 = 3(x - 3)$

$$y + 2 = 3x - 9$$

$$y = 3x - 11$$

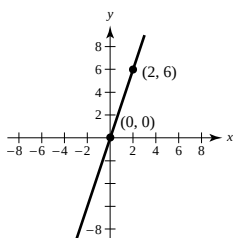
$$y - 3x + 11 = 0$$



33. $m = \frac{6 - 0}{2 - 0} = 3$

$$y - 0 = 3(x - 0)$$

$$y = 3x$$

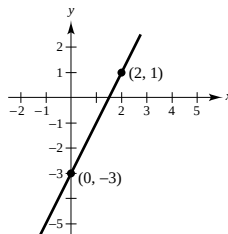


35. $m = \frac{1 - (-3)}{2 - 0} = 2$

$$y - 1 = 2(x - 2)$$

$$y - 1 = 2x - 4$$

$$0 = 2x - y - 3$$

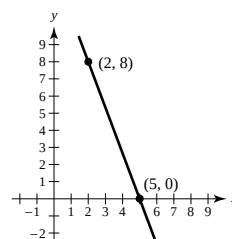


37. $m = \frac{8 - 0}{2 - 5} = -\frac{8}{3}$

$$y - 0 = -\frac{8}{3}(x - 5)$$

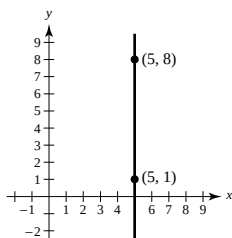
$$y = -\frac{8}{3}x + \frac{40}{3}$$

$$3y + 8x - 40 = 0$$



$$39. m = \frac{8-1}{5-5} \text{ Undefined.}$$

Vertical line $x = 5$

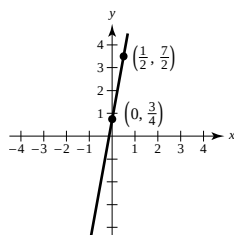


$$41. m = \frac{7/2 - 3/4}{1/2 - 0} = \frac{11/4}{1/2} = \frac{11}{2}$$

$$y - \frac{3}{4} = \frac{11}{2}(x - 0)$$

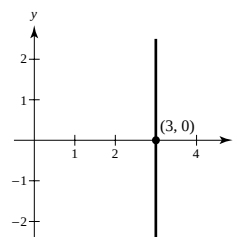
$$y = \frac{11}{2}x + \frac{3}{4}$$

$$22x - 4y + 3 = 0$$



$$43. x = 3$$

$$x - 3 = 0$$



$$45. \frac{x}{2} + \frac{y}{3} = 1$$

$$3x + 2y - 6 = 0$$

$$47. \frac{x}{a} + \frac{y}{a} = 1$$

$$\frac{1}{a} + \frac{2}{a} = 1$$

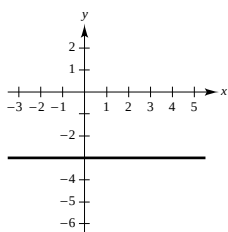
$$\frac{3}{a} = 1$$

$$a = 3 \Rightarrow x + y = 3$$

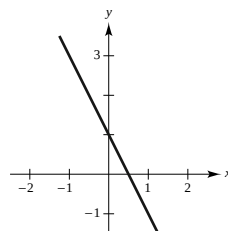
$$x + y - 3 = 0$$

$$49. y = -3$$

$$y + 3 = 0$$



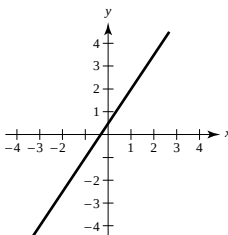
$$51. y = -2x + 1$$



$$53. y - 2 = \frac{3}{2}(x - 1)$$

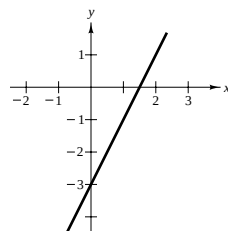
$$y = \frac{3}{2}x + \frac{1}{2}$$

$$2y - 3x - 1 = 0$$

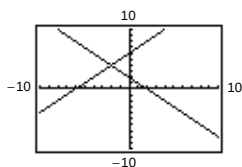


$$55. 2x - y - 3 = 0$$

$$y = 2x - 3$$

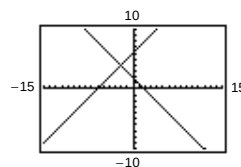


57.



The lines do not appear perpendicular.

The lines are perpendicular because their slopes 1 and -1 are negative reciprocals of each other. You must use a square setting in order for perpendicular lines to appear perpendicular.



The lines appear perpendicular.

59. $4x - 2y = 3$

$$y = 2x - \frac{3}{2}$$

$$m = 2$$

(a) $y - 1 = 2(x - 2)$

$$y - 1 = 2x - 4$$

$$2x - y - 3 = 0$$

(b) $y - 1 = -\frac{1}{2}(x - 2)$

$$2y - 2 = -x + 2$$

$$x + 2y - 4 = 0$$

61. $5x - 3y = 0$

$$y = \frac{5}{3}x$$

$$m = \frac{5}{3}$$

(a) $y - \frac{7}{8} = \frac{5}{3}(x - \frac{3}{4})$

$$24y - 21 = 40x - 30$$

$$24y - 40x + 9 = 0$$

(b) $y - \frac{7}{8} = -\frac{3}{5}(x - \frac{3}{4})$

$$40y - 35 = -24x + 18$$

$$40y + 24x - 53 = 0$$

63. (a) $x = 2 \Rightarrow x - 2 = 0$

(b) $y = 5 \Rightarrow y - 5 = 0$

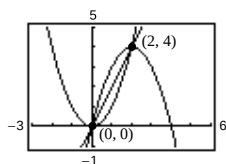
65. The slope is 125. Hence, $V = 125(t - 1) + 2540$

$$= 125t + 2415$$

67. The slope is -2000 . Hence, $V = -2000(t - 1) + 20,400$

$$= -2000t + 22,400$$

69.



You can use the graphing utility to determine that the points of intersection are $(0, 0)$ and $(2, 4)$. Analytically,

$$x^2 = 4x - x^2$$

$$2x^2 - 4x = 0$$

$$2x(x - 2) = 0$$

$$x = 0 \Rightarrow y = 0 \Rightarrow (0, 0)$$

$$x = 2 \Rightarrow y = 4 \Rightarrow (2, 4).$$

The slope of the line joining $(0, 0)$ and $(2, 4)$ is $m = (4 - 0)/(2 - 0) = 2$. Hence, an equation of the line is

$$y - 0 = 2(x - 0)$$

$$y = 2x.$$

$$71. m_1 = \frac{1 - 0}{-2 - (-1)} = -1$$

$$m_2 = \frac{-2 - 0}{2 - (-1)} = -\frac{2}{3}$$

$$m_1 \neq m_2$$

The points are not collinear.

73. Equations of perpendicular bisectors:

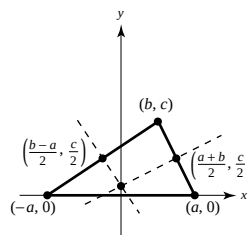
$$y - \frac{c}{2} = \frac{a - b}{c} \left(x - \frac{a + b}{2} \right)$$

$$y - \frac{c}{2} = \frac{a + b}{-c} \left(x - \frac{b - a}{2} \right)$$

Letting $x = 0$ in either equation gives the point of intersection:

$$\left(0, \frac{-a^2 + b^2 + c^2}{2c} \right).$$

This point lies on the third perpendicular bisector, $x = 0$.



75. Equations of altitudes:

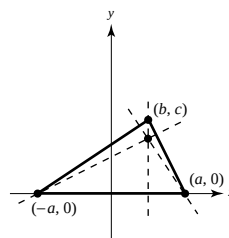
$$y = \frac{a - b}{c}(x + a)$$

$$x = b$$

$$y = -\frac{a + b}{c}(x - a)$$

Solving simultaneously, the point of intersection is

$$\left(b, \frac{a^2 - b^2}{c} \right).$$



77. Find the equation of the line through the points (0, 32) and (100, 212).

$$m = \frac{180}{100} = \frac{9}{5}$$

$$F - 32 = \frac{9}{5}(C - 0)$$

$$F = \frac{9}{5}C + 32$$

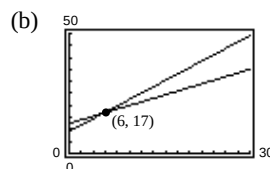
$$5F - 9C - 160 = 0$$

For $F = 72^\circ$, $C \approx 22.2^\circ$.

79. (a) $W_1 = 0.75x + 12.50$

$$W_2 = 1.30x + 9.20$$

(c) Both jobs pay \$17 per hour if 6 units are produced. For someone who can produce more than 6 units per hour, the second offer would pay more. For a worker who produces less than 6 units per hour, the first offer pays more.



Using a graphing utility, the point of intersection is approximately (6, 17). Analytically,

$$0.75x + 12.50 = 1.30x + 9.20$$

$$3.3 = 0.55x \Rightarrow x = 6$$

$$y = 0.75(6) + 12.50 = 17.$$

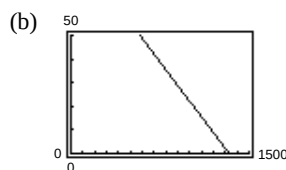
81. (a) Two points are (50, 580) and (47, 625). The slope is

$$m = \frac{625 - 580}{47 - 50} = -15.$$

$$p - 580 = -15(x - 50)$$

$$p = -15x + 750 + 580 = -15x + 1330$$

$$\text{or } x = \frac{1}{15}(1330 - p)$$



$$\text{If } p = 655, x = \frac{1}{15}(1330 - 655) = 45 \text{ units.}$$

$$(c) \text{ If } p = 595, x = \frac{1}{15}(1330 - 595) = 49 \text{ units.}$$

$$83. 4x + 3y - 10 = 0 \Rightarrow d = \frac{|4(0) + 3(0) - 10|}{\sqrt{4^2 + 3^2}} = \frac{10}{5} = 2$$

$$85. x - y - 2 = 0 \Rightarrow d = \frac{|1(-2) + (-1)(1) - 2|}{\sqrt{1^2 + 1^2}} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

87. A point on the line
- $x + y = 1$
- is (0, 1). The distance from the point (0, 1) to
- $x + y - 5 = 0$
- is

$$d = \frac{|1(0) + 1(1) - 5|}{\sqrt{1^2 + 1^2}} = \frac{|1 - 5|}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

89. If
- $A = 0$
- , then
- $By + C = 0$
- is the horizontal line
- $y = -C/B$
- . The distance to
- (x_1, y_1)
- is

$$d = \left| y_1 - \left(\frac{-C}{B} \right) \right| = \frac{|By_1 + C|}{|B|} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

If $B = 0$, then $Ax + C = 0$ is the vertical line $x = -C/A$. The distance to (x_1, y_1) is

$$d = \left| x_1 - \left(\frac{-C}{A} \right) \right| = \frac{|Ax_1 + C|}{|A|} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

(Note that A and B cannot both be zero.)

The slope of the line $Ax + By + C = 0$ is $-A/B$. The equation of the line through (x_1, y_1) perpendicular to $Ax + By + C = 0$ is:

$$y - y_1 = \frac{B}{A}(x - x_1)$$

$$Ay - Ay_1 = Bx - Bx_1$$

$$Bx_1 - Ay_1 = Bx - Ay$$

The point of intersection of these two lines is:

$$Ax + By = -C \quad \Rightarrow \quad A^2x + AB y = -AC \quad (1)$$

$$Bx - Ay = Bx_1 - Ay_1 \Rightarrow \frac{B^2x - AB y}{A^2 + B^2} = \frac{B^2x_1 - AB y_1}{A^2 + B^2} \quad (2)$$

$$(A^2 + B^2)x = -AC + B^2x_1 - AB y_1 \quad (\text{By adding equations (1) and (2)})$$

$$x = \frac{-AC + B^2x_1 - AB y_1}{A^2 + B^2}$$

$$Ax + By = -C \quad \Rightarrow \quad ABx + B^2y = -BC \quad (3)$$

$$Bx - Ay = Bx_1 - Ay_1 \Rightarrow \frac{-ABx + A^2y}{A^2 + B^2} = \frac{-ABx_1 + A^2y_1}{A^2 + B^2} \quad (4)$$

$$(A^2 + B^2)y = -BC - ABx_1 + A^2y_1 \quad (\text{By adding equations (3) and (4)})$$

$$y = \frac{-BC - ABx_1 + A^2y_1}{A^2 + B^2}$$

89. —CONTINUED—

$$\left(\frac{-AC + B^2x_1 - ABy_1}{A^2 + B^2}, \frac{-BC - ABx_1 + A^2y_1}{A^2 + B^2} \right) \text{ point of intersection}$$

The distance between (x_1, y_1) and this point gives us the distance between (x_1, y_1) and the line $Ax + By + C = 0$.

$$\begin{aligned} d &= \sqrt{\left[\frac{-AC + B^2x_1 - ABy_1}{A^2 + B^2} - x_1 \right]^2 + \left[\frac{-BC - ABx_1 + A^2y_1}{A^2 + B^2} - y_1 \right]^2} \\ &= \sqrt{\left[\frac{-AC - AB^2y_1 - A^2x_1}{A^2 + B^2} \right]^2 + \left[\frac{-BC - ABx_1 - B^2y_1}{A^2 + B^2} \right]^2} \\ &= \sqrt{\left[\frac{-A(C + By_1 + Ax_1)}{A^2 + B^2} \right]^2 + \left[\frac{-B(C + Ax_1 + By_1)}{A^2 + B^2} \right]^2} \\ &= \sqrt{\frac{(A^2 + B^2)(C + Ax_1 + By_1)^2}{(A^2 + B^2)^2}} \\ &= \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \end{aligned}$$

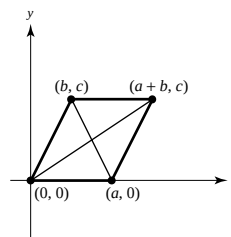
91. For simplicity, let the vertices of the rhombus be $(0, 0)$, $(a, 0)$, (b, c) , and $(a + b, c)$, as shown in the figure. The slopes of the diagonals are then

$$m_1 = \frac{c}{a + b} \text{ and } m_2 = \frac{c}{b - a}.$$

Since the sides of the Rhombus are equal, $a^2 = b^2 + c^2$, and we have

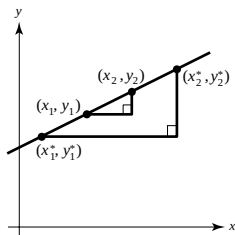
$$m_1 m_2 = \frac{c}{a + b} \cdot \frac{c}{b - a} = \frac{c^2}{b^2 - a^2} = \frac{c^2}{-c^2} = -1.$$

Therefore, the diagonals are perpendicular.



93. Consider the figure below in which the four points are collinear. Since the triangles are similar, the result immediately follows.

$$\frac{y_2^* - y_1^*}{x_2^* - x_1^*} = \frac{y_2 - y_1}{x_2 - x_1}$$



95. True.

$$ax + by = c_1 \Rightarrow y = -\frac{a}{b}x + \frac{c_1}{b} \Rightarrow m_1 = -\frac{a}{b}$$

$$bx - ay = c_2 \Rightarrow y = \frac{b}{a}x - \frac{c_2}{a} \Rightarrow m_2 = \frac{b}{a}$$

$$m_2 = -\frac{1}{m_1}$$

Section P.3 Functions and Their Graphs

1. (a) $f(0) = 2(0) - 3 = -3$

(b) $f(-3) = 2(-3) - 3 = -9$

(c) $f(b) = 2b - 3$

(d) $f(x - 1) = 2(x - 1) - 3 = 2x - 5$

3. (a) $g(0) = 3 - 0^2 = 3$

(b) $g(\sqrt{3}) = 3 - (\sqrt{3})^2 = 3 - 3 = 0$

(c) $g(-2) = 3 - (-2)^2 = 3 - 4 = -1$

(d) $g(t - 1) = 3 - (t - 1)^2 = -t^2 + 2t + 2$

5. (a) $f(0) = \cos(2(0)) = \cos 0 = 1$

(b) $f\left(-\frac{\pi}{4}\right) = \cos\left(2\left(-\frac{\pi}{4}\right)\right) = \cos\left(-\frac{\pi}{2}\right) = 0$

(c) $f\left(\frac{\pi}{3}\right) = \cos\left(2\left(\frac{\pi}{3}\right)\right) = \cos\frac{2\pi}{3} = -\frac{1}{2}$

7.
$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{(x + \Delta x)^3 - x^3}{\Delta x} = \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2, \Delta x \neq 0$$

9.
$$\begin{aligned} \frac{f(x) - f(2)}{x - 2} &= \frac{(1/\sqrt{x-1}) - 1}{x - 2} \\ &= \frac{1 - \sqrt{x-1}}{(x-2)\sqrt{x-1}} \cdot \frac{1 + \sqrt{x-1}}{1 + \sqrt{x-1}} = \frac{2 - x}{(x-2)\sqrt{x-1}(1 + \sqrt{x-1})} = \frac{-1}{\sqrt{x-1}(1 + \sqrt{x-1})}, x \neq 2 \end{aligned}$$

11. $h(x) = -\sqrt{x+3}$

Domain: $x + 3 \geq 0 \Rightarrow [-3, \infty)$

Range: $(-\infty, 0]$

13. $f(t) = \sec \frac{\pi t}{4}$

$$\frac{\pi t}{4} \neq \frac{(2k+1)\pi}{2} \Rightarrow t \neq 4k + 2$$

Domain: all $t \neq 4k + 2, k$ an integer

Range: $(-\infty, -1], [1, \infty)$

15. $f(x) = \frac{1}{x}$

Domain: $(-\infty, 0), (0, \infty)$

Range: $(-\infty, 0), (0, \infty)$

17. $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$

(a) $f(-1) = 2(-1) + 1 = -1$

(b) $f(0) = 2(0) + 2 = 2$

(c) $f(2) = 2(2) + 2 = 6$

(d) $f(t^2 + 1) = 2(t^2 + 1) = 2t^2 + 4$

(Note: $t^2 + 1 \geq 0$ for all t)

Domain: $(-\infty, \infty)$

Range: $(-\infty, 1), [2, \infty)$

19. $f(x) = \begin{cases} |x| + 1, & x < 1 \\ -x + 1, & x \geq 1 \end{cases}$

(a) $f(-3) = |-3| + 1 = 4$

(b) $f(1) = -1 + 1 = 0$

(c) $f(3) = -3 + 1 = -2$

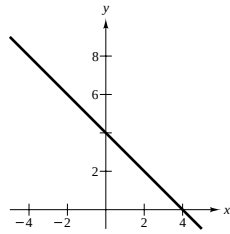
(d) $f(b^2 + 1) = -(b^2 + 1) + 1 = -b^2$

Domain: $(-\infty, \infty)$

Range: $(-\infty, 0] \cup [1, \infty)$

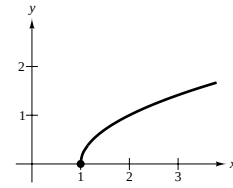
21. $f(x) = 4 - x$

 Domain: $(-\infty, \infty)$

 Range: $(-\infty, \infty)$


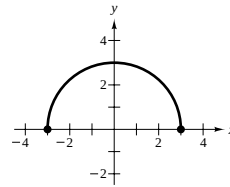
23. $h(x) = \sqrt{x - 1}$

 Domain: $[1, \infty)$

 Range: $[0, \infty)$


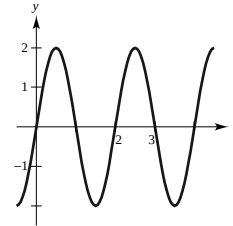
25. $f(x) = \sqrt{9 - x^2}$

 Domain: $[-3, 3]$

 Range: $[0, 3]$


27. $g(t) = 2 \sin \pi t$

 Domain: $(-\infty, \infty)$

 Range: $[-2, 2]$


29. $x - y^2 = 0 \Rightarrow y = \pm \sqrt{x}$

y is not a function of x . Some vertical lines intersect the graph twice.

33. $x^2 + y^2 = 4 \Rightarrow y = \pm \sqrt{4 - x^2}$

y is not a function of x since there are two values of y for some x .

31. y is a function of x . Vertical lines intersect the graph at most once.

35. $y^2 = x^2 - 1 \Rightarrow y = \pm \sqrt{x^2 - 1}$

y is not a function of x since there are two values of y for some x .

37. $f(x) = |x| + |x - 2|$

If $x < 0$, then $f(x) = -x - (x - 2) = -2x + 2 = 2(1 - x)$.

If $0 \leq x < 2$, then $f(x) = x - (x - 2) = 2$.

If $x \geq 2$, then $f(x) = x + (x - 2) = 2x - 2 = 2(x - 1)$.

Thus,

$$f(x) = \begin{cases} 2(1 - x), & x < 0 \\ 2, & 0 \leq x < 2 \\ 2(x - 1), & x \geq 2. \end{cases}$$

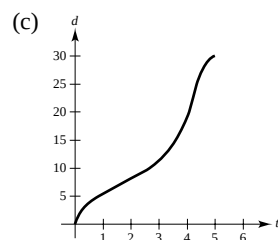
39. The function is $g(x) = cx^2$. Since $(1, -2)$ satisfies the equation, $c = -2$. Thus, $g(x) = -2x^2$.

41. The function is $r(x) = c/x$, since it must be undefined at $x = 0$. Since $(1, 32)$ satisfies the equation, $c = 32$. Thus, $r(x) = 32/x$.

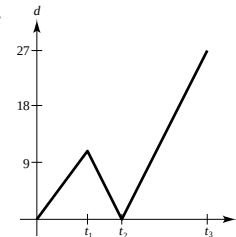
43. (a) For each time t , there corresponds a depth d .

(b) Domain: $0 \leq t \leq 5$

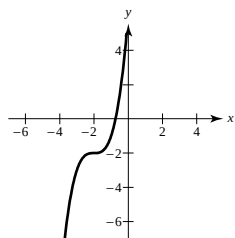
Range: $0 \leq d \leq 30$



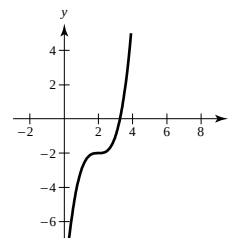
45.



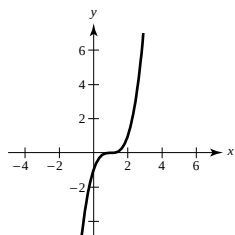
47. (a) The graph is shifted 3 units to the left.



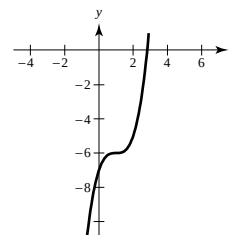
- (b) The graph is shifted 1 unit to the right.



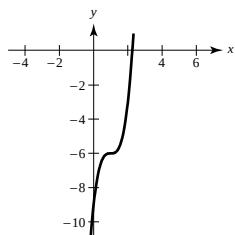
- (c) The graph is shifted 2 units upward.



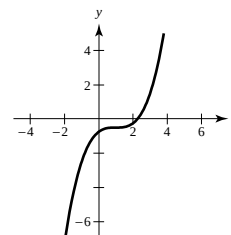
- (d) The graph is shifted 4 units downward.



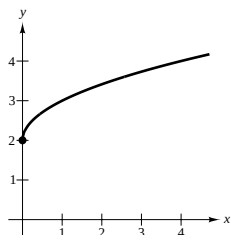
- (e) The graph is stretched vertically by a factor of 3.



- (f) The graph is stretched vertically by a factor of $\frac{1}{4}$.

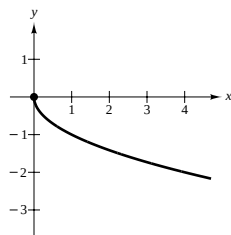


49. (a) $y = \sqrt{x} + 2$



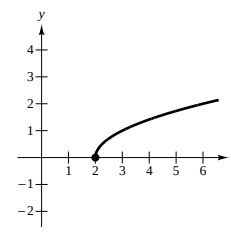
Vertical shift 2 units upward

- (b) $y = -\sqrt{x}$



Reflection about the x-axis

- (c) $y = \sqrt{x - 2}$



Horizontal shift 2 units to the right

51. (a) $T(4) = 16^\circ$, $T(15) \approx 23^\circ$

(b) If $H(t) = T(t - 1)$, then the program would turn on (and off) one hour later.

(c) If $H(t) = T(t) - 1$, then the overall temperature would be reduced 1 degree.

53. $f(x) = x^2$, $g(x) = \sqrt{x}$

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x, \quad x \geq 0$$

Domain: $[0, \infty)$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = |x|$$

Domain: $(-\infty, \infty)$

No. Their domains are different. $(f \circ g) = (g \circ f)$ for $x \geq 0$.

55. $f(x) = \frac{3}{x}$, $g(x) = x^2 - 1$

$$(f \circ g)(x) = f(g(x)) = f(x^2 - 1) = \frac{3}{x^2 - 1}$$

Domain: all $x \neq \pm 1$

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{3}{x}\right) = \left(\frac{3}{x}\right)^2 - 1 = \frac{9}{x^2} - 1 = \frac{9 - x^2}{x^2}$$

Domain: all $x \neq 0$

No, $f \circ g \neq g \circ f$.

57. $(A \circ r)(t) = A(r(t)) = A(0.6t) = \pi(0.6t)^2 = 0.36\pi t^2$

$(A \circ r)(t)$ represents the area of the circle at time t .

61. $f(-x) = (-x) \cos(-x) = -x \cos x = -f(x)$

Odd

63. (a) If f is even, then $(\frac{3}{2}, 4)$ is on the graph.

(b) If f is odd, then $(\frac{3}{2}, -4)$ is on the graph.

65.
$$\begin{aligned} f(-x) &= a_{2n+1}(-x)^{2n+1} + \cdots + a_3(-x)^3 + a_1(-x) \\ &= -[a_{2n+1}x^{2n+1} + \cdots + a_3x^3 + a_1x] \\ &= -f(x) \end{aligned}$$

Odd

67. Let $F(x) = f(x)g(x)$ where f and g are even. Then

$$F(-x) = f(-x)g(-x) = f(x)g(x) = F(x).$$

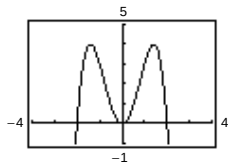
Thus, $F(x)$ is even. Let $F(x) = f(x)g(x)$ where f and g are odd. Then

$$F(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = F(x).$$

Thus, $F(x)$ is even.

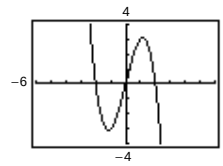
69. $f(x) = x^2 + 1$ and $g(x) = x^4$ are even.

$$f(x)g(x) = (x^2 + 1)(x^4) = x^6 + x^4 \text{ is even.}$$



$f(x) = x^3 - x$ is odd and $g(x) = x^2$ is even.

$$f(x)g(x) = (x^3 - x)(x^2) = x^5 - x^3 \text{ is odd.}$$



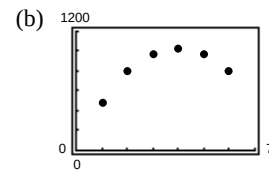
71. (a)

x	length and width	volume V
1	$24 - 2(1)$	484
2	$24 - 2(2)$	800
3	$24 - 2(3)$	972
4	$24 - 2(4)$	1024
5	$24 - 2(5)$	980
6	$24 - 2(6)$	864

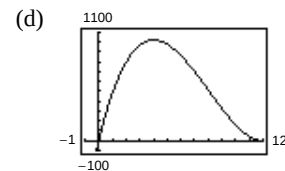
The maximum volume appears to be 1024 cm^3 .

(c) $V = x(24 - 2x)^2 = 4x(12 - x)^2$

Domain: $0 < x < 12$



Yes, V is a function of x .



Maximum volume is $V = 1024 \text{ cm}^3$ for box having dimensions $4 \times 16 \times 16 \text{ cm}$.

73. False; let $f(x) = x^2$.

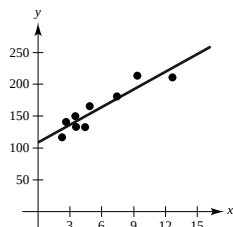
Then $f(-3) = f(3) = 9$, but $-3 \neq 3$.

75. True, the function is even.

Section P.4 Fitting Models to Data

1. Quadratic function

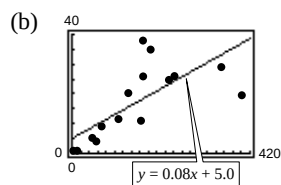
5. (a), (b)



Yes. The cancer mortality increases linearly with increased exposure to the carcinogenic substance.

(c) If $x = 3$, then $y \approx 136$.

9. (a) Let x = per capita energy usage (in millions of Btu)
 y = per capita gross national product (in thousands)
 $y = 0.0764x + 4.9985 \approx 0.08x + 5.0$
 $r = 0.7052$



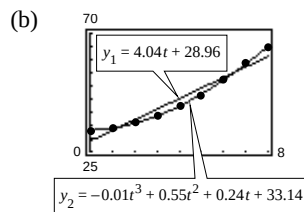
(c) Denmark, Japan, and Canada

(d) Deleting the data for the three countries above,

$$y = 0.0959x + 1.0539$$

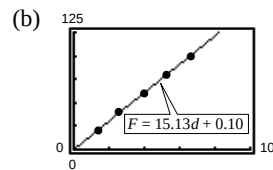
($r = 0.9202$ is much closer to 1.)

13. (a) $y_1 = 4.0367t + 28.9644$
 $y_2 = -0.0099t^3 + 0.5488t^2 + 0.2399t + 33.1414$



(c) The cubic model is better.

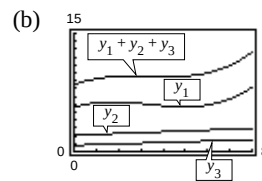
3. Linear function

7. (a) $d = 0.066F$ or $F = 15.1d + 0.1$ 

The model fits well.

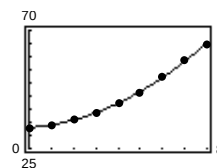
(c) If $F = 55$, then $d \approx 0.066(55) = 3.63$ cm.

11. (a) $y_1 = 0.0343t^3 - 0.3451t^2 + 0.8837t + 5.6061$
 $y_2 = 0.1095t + 2.0667$
 $y_3 = 0.0917t + 0.7917$



For 2002, $t = 12$ and $y_1 + y_2 + y_3 \approx 31.06$ cents/mile

(d) $y_3 = 0.4297t^2 + 0.5994t + 32.9745$

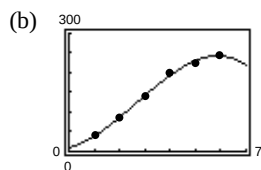


(e) The slope represents the average increase per year in the number of people (in millions) in HMOs.

(f) For 2000, $t = 10$, and $y_1 \approx 69.3$ million. (linear)

$y_2 \approx 80.5$ million (cubic)

15. (a) $y = -1.81x^3 + 14.58x^2 + 16.39x + 10$



(c) If $x = 4.5$, $y \approx 214$ horsepower.

17. (a) Yes, y is a function of t . At each time t , there is one and only one displacement y .

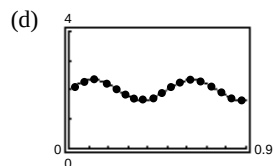
(b) The amplitude is approximately

$$(2.35 - 1.65)/2 = 0.35.$$

The period is approximately

$$2(0.375 - 0.125) = 0.5.$$

(c) One model is $y = 0.35 \sin(4\pi t) + 2$.



19. Answers will vary.

Review Exercises for Chapter P

1. $y = 2x - 3$

$$x = 0 \Rightarrow y = 2(0) - 3 = -3 \Rightarrow (0, -3) \quad y\text{-intercept}$$

$$y = 0 \Rightarrow 0 = 2x - 3 \Rightarrow x = \frac{3}{2} \Rightarrow (\frac{3}{2}, 0) \quad x\text{-intercept}$$

3. $y = \frac{x-1}{x-2}$

$$x = 0 \Rightarrow y = \frac{0-1}{0-2} = \frac{1}{2} \Rightarrow (0, \frac{1}{2}) \quad y\text{-intercept}$$

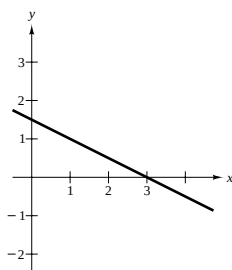
$$y = 0 \Rightarrow 0 = \frac{x-1}{x-2} \Rightarrow x = 1 \Rightarrow (1, 0) \quad x\text{-intercept}$$

5. Symmetric with respect to y -axis since

$$(-x)^2y - (-x)^2 + 4y = 0$$

$$x^2y - x^2 + 4y = 0.$$

7. $y = -\frac{1}{2}x + \frac{3}{2}$



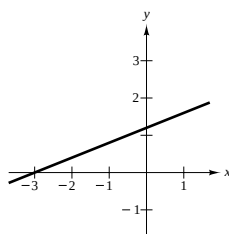
9. $-\frac{1}{3}x + \frac{5}{6}y = 1$

$$-\frac{2}{5}x + y = \frac{6}{5}$$

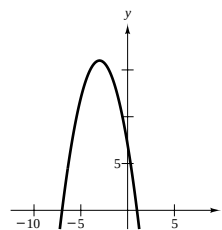
$$y = \frac{2}{5}x + \frac{6}{5}$$

Slope: $\frac{2}{5}$

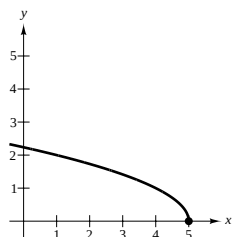
y -intercept: $\frac{6}{5}$



11. $y = 7 - 6x - x^2$



13. $y = \sqrt{5 - x}$

Domain: $(-\infty, 5]$ 

15. $y = 4x^2 - 25$

Xmin = -5
Xmax = 5
Xscl = 1
Ymin = -30
Ymax = 10
Yscl = 5

17. $3x - 4y = 8$

$$\frac{4x + 4y = 20}{7x} = 28$$

$$7x = 28$$

$$x = 4$$

$$y = 1$$

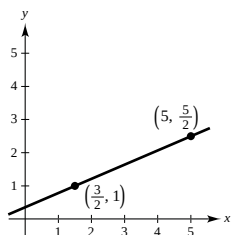
Point: (4, 1)

19. You need factors $(x + 2)$ and $(x - 2)$. Multiply by x to obtain origin symmetry

$$y = x(x + 2)(x - 2).$$

$$= x^3 - 4x.$$

21.



$$\text{Slope} = \frac{(5/2) - 1}{5 - (3/2)} = \frac{3/2}{7/2} = \frac{3}{7}$$

23. $\frac{1 - t}{1 - 0} = \frac{1 - 5}{1 - (-2)}$

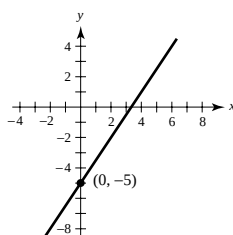
$$1 - t = -\frac{4}{3}$$

$$t = \frac{7}{3}$$

25. $y - (-5) = \frac{3}{2}(x - 0)$

$$y = \frac{3}{2}x - 5$$

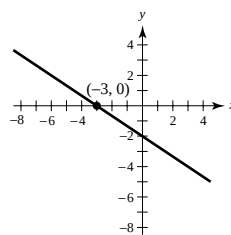
$$2y - 3x + 10 = 0$$



27. $y - 0 = -\frac{2}{3}(x - (-3))$

$$y = -\frac{2}{3}x - 2$$

$$3y + 2x + 6 = 0$$



29. (a) $y - 4 = \frac{7}{16}(x + 2)$

$$16y - 64 = 7x + 14$$

$$0 = 7x - 16y + 78$$

(c) $m = \frac{4 - 0}{-2 - 0} = -2$

$$y = -2x$$

$$2x + y = 0$$

(b) Slope of line is $\frac{5}{3}$.

$$y - 4 = \frac{5}{3}(x + 2)$$

$$3y - 12 = 5x + 10$$

$$0 = 5x - 3y + 22$$

(d) $x = -2$

$$x + 2 = 0$$

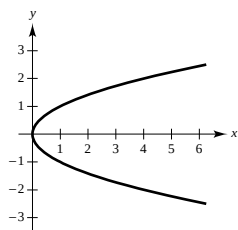
31. The slope is -850 . $V = -850t + 12,500$.

$$V(3) = -850(3) + 12,500 = \$9950$$

33. $x - y^2 = 0$

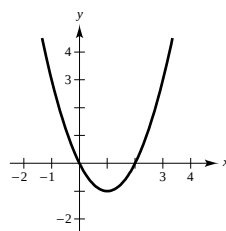
$$y = \pm \sqrt{x}$$

Not a function of x since there are two values of y for some x .



35. $y = x^2 - 2x$

Function of x since there is one value of y for each x .



37. $f(x) = \frac{1}{x}$

(a) $f(0)$ does not exist.

(b)
$$\frac{f(1 + \Delta x) - f(1)}{\Delta x} = \frac{\frac{1}{1 + \Delta x} - \frac{1}{1}}{\Delta x} = \frac{1 - 1 - \Delta x}{(1 + \Delta x)\Delta x}$$

$$= \frac{-1}{1 + \Delta x}, \Delta x \neq -1, 0$$

39. (a) Domain: $36 - x^2 \geq 0 \Rightarrow -6 \leq x \leq 6$ or $[-6, 6]$

Range: $[0, 6]$

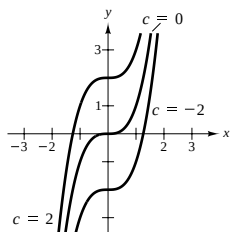
(b) Domain: all $x \neq 5$ or $(-\infty, 5), (5, \infty)$

Range: all $y \neq 0$ or $(-\infty, 0), (0, \infty)$

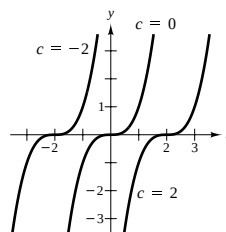
(c) Domain: all x or $(-\infty, \infty)$

Range: all y or $(-\infty, \infty)$

41. (a) $f(x) = x^3 + c, c = -2, 0, 2$

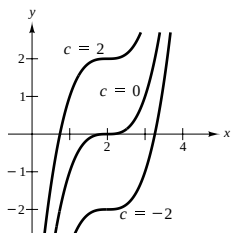


(b) $f(x) = (x - c)^3, c = -2, 0, 2$

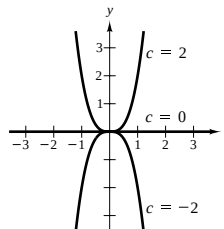


41. —CONTINUED—

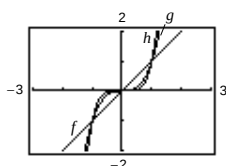
(c) $f(x) = (x - 2)^3 + c$, $c = -2, 0, 2$



(d) $f(x) = cx^3$, $c = -2, 0, 2$



43. (a) Odd powers: $f(x) = x$, $g(x) = x^3$, $h(x) = x^5$

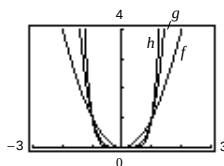


The graphs of f , g , and h all rise to the right and fall to the left. As the degree increases, the graph rises and falls more steeply. All three graphs pass through the points $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

(b) $y = x^7$ will look like $h(x) = x^5$, but rise and fall even more steeply.

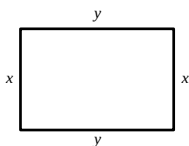
$y = x^8$ will look like $h(x) = x^6$, but rise even more steeply.

Even powers: $f(x) = x^2$, $g(x) = x^4$, $h(x) = x^6$



The graphs of f , g , and h all rise to the left and to the right. As the degree increases, the graph rises more steeply. All three graphs pass through the points $(0, 0)$, $(1, 1)$, and $(-1, 1)$.

45. (a)

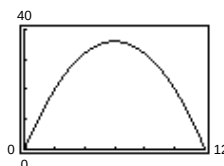


$2x + 2y = 24$

$y = 12 - x$

$A = xy = x(12 - x) = 12x - x^2$

(b) Domain: $0 < x < 12$



(c) Maximum area is $A = 36$. In general, the maximum area is attained when the rectangle is a square. In this case, $x = 6$.

47. (a) 3 (cubic), negative leading coefficient

(b) 4 (quartic), positive leading coefficient

(c) 2 (quadratic), negative leading coefficient

(d) 5, positive leading coefficient

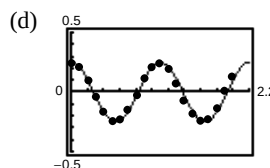
49. (a) Yes, y is a function of t . At each time t , there is one and only one displacement y .

(b) The amplitude is approximately

$(0.25 - (-0.25))/2 = 0.25$.

The period is approximately 1.1.

(c) One model is $y = \frac{1}{4} \cos\left(\frac{2\pi}{1.1}t\right) \approx \frac{1}{4} \cos(5.7t)$



PART II

CHAPTER P Preparation for Calculus

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CHAPTER P

Preparation for Calculus

Section P.1 Graphs and Models

Solutions to Even-Numbered Exercises

2. $y = \sqrt{9 - x^2}$

x -intercepts: $(-3, 0), (3, 0)$

y -intercept: $(0, 3)$

Matches graph (d)

4. $y = x^3 - x$

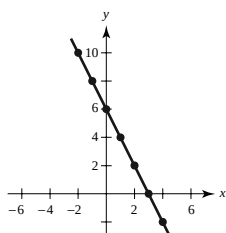
x -intercepts: $(0, 0), (-1, 0), (1, 0)$

y -intercept: $(0, 0)$

Matches graph (c)

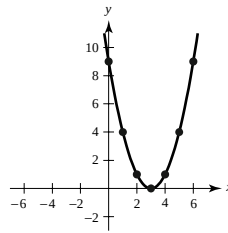
6. $y = 6 - 2x$

x	-2	-1	0	1	2	3	4
y	10	8	6	4	2	0	-2



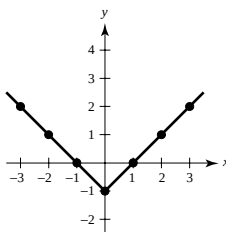
8. $y = (x - 3)^2$

x	0	1	2	3	4	5	6
y	9	4	1	0	1	4	9



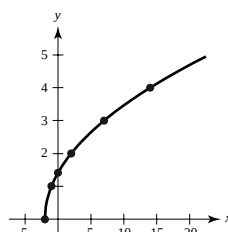
10. $y = |x| - 1$

x	-3	-2	-1	0	1	2	3
y	2	1	0	-1	0	1	2



12. $y = \sqrt{x + 2}$

x	-2	-1	0	2	7	14
y	0	1	$\sqrt{2}$	2	3	4



14.
$$\begin{array}{l} \text{Xmin} = -30 \\ \text{Xmax} = 30 \\ \text{Xscl} = 5 \\ \text{Ymin} = -10 \\ \text{Ymax} = 40 \\ \text{Yscl} = 5 \end{array}$$

Note that $y = 10$ when $x = 0$ or $x = 10$.

18. $y^2 = x^3 - 4x$

y -intercept: $y^2 = 0^3 - 4(0)$

$y = 0; (0, 0)$

x -intercepts: $0 = x^3 - 4x$

$0 = x(x - 2)(x + 2)$

$x = 0, \pm 2; (0, 0), (\pm 2, 0)$

22. $y = \frac{x^2 + 3x}{(3x + 1)^2}$

y -intercept: $y = \frac{0^2 + 3(0)}{[3(0) + 1]^2}$

$y = 0; (0, 0)$

x -intercepts: $0 = \frac{x^2 + 3x}{(3x + 1)^2}$

$0 = \frac{x(x + 3)}{(3x + 1)^2}$

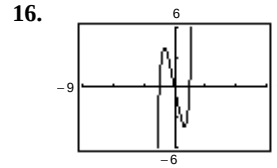
$x = 0, -3; (0, 0), (-3, 0)$

26. $y = x^2 - x$

No symmetry with respect to either axis or the origin.

30. Symmetric with respect to the x -axis since

$x(-y)^2 = xy^2 = -10.$



(a) $(-0.5, y) = (-0.5, 2.47)$

(b) $(x, -4) = (-1.65, -4)$ and $(x, -4) = (1, -4)$

20. $y = (x - 1)\sqrt{x^2 + 1}$

y -intercept: $y = (0 - 1)\sqrt{0^2 + 1}$

$y = -1; (0, -1)$

x -intercepts: $0 = (x - 1)\sqrt{x^2 + 1}$

$x = 1; (1, 0)$

24. $y = 2x - \sqrt{x^2 + 1}$

y -intercept: $y = 2(0) - \sqrt{0^2 + 1}$

$y = -1; (0, -1)$

x -intercepts: $0 = 2x - \sqrt{x^2 + 1}$

$2x = \sqrt{x^2 + 1}$

$4x^2 = x^2 + 1$

$3x^2 = 1$

$x^2 = \frac{1}{3}$

$x = \pm \frac{\sqrt{3}}{3}$

$x = \frac{\sqrt{3}}{3}; \left(\frac{\sqrt{3}}{3}, 0\right)$

Note: $x = -\sqrt{3}/3$ is an extraneous solution.

28. Symmetric with respect to the origin since

$(-y) = (-x)^3 + (-x)$

$-y = -x^3 - x$

$y = x^3 + x.$

32. Symmetric with respect to the origin since

$(-x)(-y) - \sqrt{4 - (-x)^2} = 0$

$xy - \sqrt{4 - x^2} = 0.$

34. $y = \frac{x^2}{x^2 + 1}$ is symmetric with respect to the y -axis

since $y = \frac{(-x)^2}{(-x)^2 + 1} = \frac{x^2}{x^2 + 1}$.

36. $|y| - x = 3$ is symmetric with respect to the x -axis

since $|-y| - x = 3$

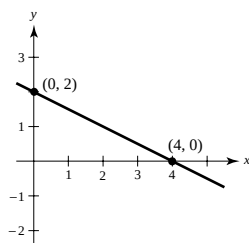
$|y| - x = 3$.

38. $y = -\frac{x}{2} + 2$

Intercepts:

$(4, 0), (0, 2)$

Symmetry: none

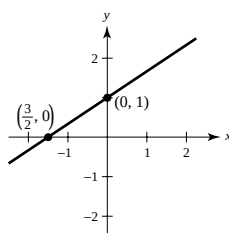


40. $y = \frac{2}{3}x + 1$

Intercepts:

$(0, 1), (-\frac{3}{2}, 0)$

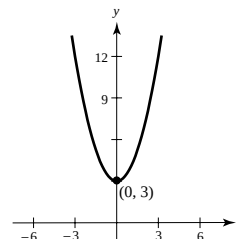
Symmetry: none



42. $y = x^2 + 3$

Intercept: $(0, 3)$

Symmetry: y -axis

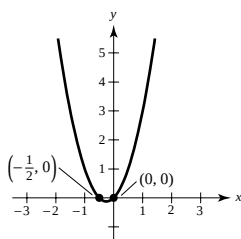


44. $y = 2x^2 + x = x(2x + 1)$

Intercepts:

$(0, 0), (-\frac{1}{2}, 0)$

Symmetry: none

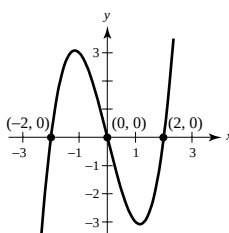


46. $y = x^3 - 4x$

Intercepts:

$(0, 0), (2, 0), (-2, 0)$

Symmetry: origin



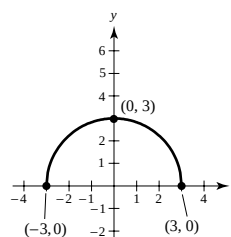
48. $y = \sqrt{9 - x^2}$

Intercepts:

$(-3, 0), (3, 0), (0, 3)$

Symmetry: y -axis

Domain: $[-3, 3]$

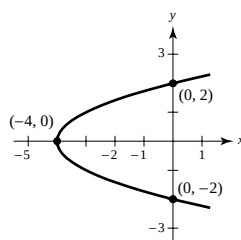


50. $x = y^2 - 4$

Intercepts:

$(0, 2), (0, -2), (-4, 0)$

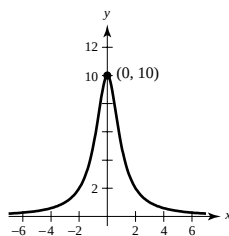
Symmetry: x -axis



52. $y = \frac{10}{x^2 + 1}$

Intercepts: $(0, 10)$

Symmetry: y -axis

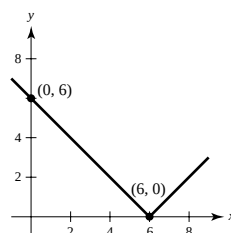


54. $y = |6 - x|$

Intercepts:

$(0, 6), (6, 0)$

Symmetry: none



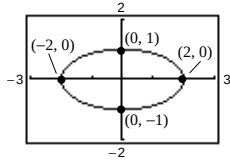
$$56. x^2 + 4y^2 = 4 \Rightarrow y = \pm \frac{\sqrt{4 - x^2}}{2}$$

Intercepts:

$$(-2, 0), (2, 0), (0, -1), (0, 1)$$

Symmetry: origin and both axes

Domain: $[-2, 2]$



$$60. y = (x + \frac{5}{2})(x - 2)(x - \frac{3}{2}) \text{ (other answers possible)}$$

$$64. 2x - 3y = 13 \Rightarrow y = \frac{2x - 13}{3}$$

$$5x + 3y = 1 \Rightarrow y = \frac{1 - 5x}{3}$$

$$\frac{2x - 13}{3} = \frac{1 - 5x}{3}$$

$$2x - 13 = 1 - 5x$$

$$7x = 14$$

$$x = 2$$

The corresponding y-value is $y = -3$.

Point of intersection: $(2, -3)$

$$68. x = 3 - y^2 \Rightarrow y^2 = 3 - x$$

$$y = x - 1$$

$$3 - x = (x - 1)^2$$

$$3 - x = x^2 - 2x + 1$$

$$0 = x^2 - x - 2 = (x + 1)(x - 2)$$

$$x = -1 \text{ or } x = 2$$

The corresponding y-values are $y = -2$ and $y = 1$.

Points of intersection: $(-1, -2), (2, 1)$

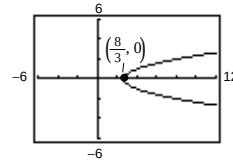
$$58. 3x - 4y^2 = 8$$

$$4y^2 = 3x - 8$$

$$y = \pm \sqrt{\frac{3}{4}x - 2}$$

Intercept: $(\frac{8}{3}, 0)$

Symmetry: x-axis



62. Some possible equations:

$$x = y^2$$

$$x = |y|$$

$$x = y^4 + 1$$

$$x^2 + y^2 = 25$$

$$66. 5x - 6y = 9 \Rightarrow y = \frac{5x - 9}{6}$$

$$-7x + 3y = -18 \Rightarrow y = \frac{7x - 18}{3}$$

$$\frac{5x - 9}{6} = \frac{7x - 18}{3}$$

$$5x - 9 = 14x - 36$$

$$27 = 9x$$

$$x = 3$$

The corresponding y-value is $y = 1$.

Point of intersection: $(3, 1)$

$$70. x^2 + y^2 = 25 \Rightarrow y^2 = 25 - x^2$$

$$2x + y = 10 \Rightarrow y = 10 - 2x$$

$$25 - x^2 = (10 - 2x)^2$$

$$25 - x^2 = 100 - 40x + 4x^2$$

$$0 = 5x^2 - 40x + 75 = 5(x - 3)(x - 5)$$

$$x = 3 \text{ or } x = 5$$

The corresponding y-values are $y = 4$ and $y = 0$.

Points of intersection: $(3, 4), (5, 0)$

72. $y = x^3 - 4x$

$y = -(x + 2)$

$x^3 - 4x = -(x + 2)$

$x^3 - 3x + 2 = 0$

$(x - 1)^2(x + 2) = 0$

$x = 1 \quad \text{or} \quad x = -2$

The corresponding y -values are $y = -3$ and $y = 0$.

Points of intersection: $(1, -3), (-2, 0)$

74. $y = x^4 - 2x^2 + 1$

$y = 1 - x^2$

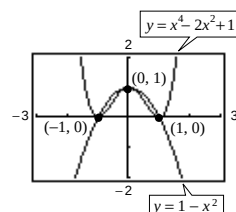
$1 - x^2 = x^4 - 2x^2 + 1$

$0 = x^4 - x^2$

$0 = x^2(x + 1)(x - 1)$

$x = -1, 0, 1$

$(-1, 0), (0, 1), (1, 0)$



76. $y = kx + 5$ matches (b).

Use $(1, 7)$: $7 = k(1) + 5 \Rightarrow k = 2$, thus, $y = 2x + 5$.

$y = x^2 + k$ matches (d).

Use $(1, -9)$: $-9 = (1)^2 + k \Rightarrow k = -10$, thus, $y = x^2 - 10$.

$y = kx^{3/2}$ matches (a).

Use $(1, 3)$: $3 = k(1)^{3/2} \Rightarrow k = 3$, thus, $y = 3x^{3/2}$.

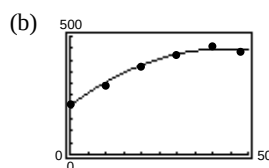
$xy = k$ matches (c).

Use $(1, 36)$: $(1)(36) = k \Rightarrow k = 36$, thus, $xy = 36$.

78. (a) Using a graphing utility, you obtain

$y = -0.1283t^2 + 11.0988t + 207.1116$

(c) For the year 2004, $t = 54$ and
 $y \approx 432.3$ acres per farm.



80. (a) If (x, y) is on the graph, then so is $(-x, y)$ by y -axis symmetry. Since $(-x, y)$ is on the graph, then so is $(-x, -y)$ by x -axis symmetry. Hence, the graph is symmetric with respect to the origin. The converse is not true. For example, $y = x^3$ has origin symmetry but is not symmetric with respect to either the x -axis or the y -axis.

(b) Assume that the graph has x -axis and origin symmetry. If (x, y) is on the graph, so is $(x, -y)$ by x -axis symmetry. Since $(x, -y)$ is on the graph, then so is $(-x, -(-y)) = (-x, y)$ by origin symmetry. Therefore, the graph is symmetric with respect to the y -axis. The argument is similar for y -axis and origin symmetry.

82. True

84. True; the x -intercept is

$$\left(-\frac{b}{2a}, 0\right).$$

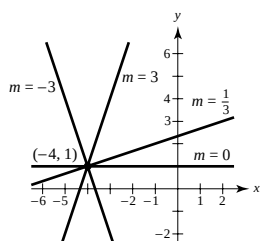
Section P.2 Linear Models and Rates of Change

2. $m = 2$

4. $m = -1$

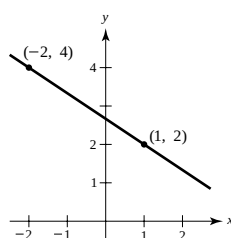
6. $m = \frac{40}{3}$

8.

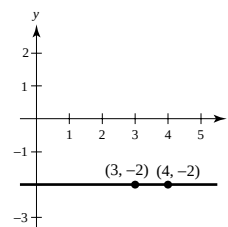


10.
$$m = \frac{4 - 2}{-2 - 1}$$

$$= -\frac{2}{3}$$

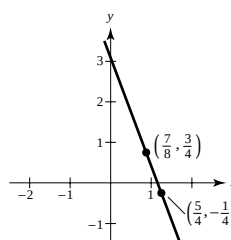


12.
$$m = \frac{-2 - (-2)}{4 - 1} = 0$$



14.
$$m = \frac{(3/4) - (-1/4)}{(7/8) - (5/4)}$$

$$= \frac{1}{-3/8} = -\frac{8}{3}$$



16. Since the slope is undefined, the line is vertical and its equation is $x = -3$. Therefore, three additional points are $(-3, 2)$, $(-3, 3)$, and $(-3, 5)$.

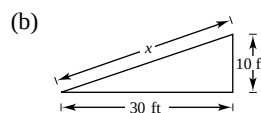
18. The equation of this line is

$$y + 2 = 2(x + 2)$$

$$y = 2x + 2.$$

Therefore, three additional points are $(-3, -4)$, $(-1, 0)$, and $(0, 2)$.

20. (a) $\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{1}{3}$



By the Pythagorean Theorem,

$$x^2 = 30^2 + 10^2 = 1000$$

$$x \approx 31.623 \text{ feet.}$$

22. (a) $m = 400$ indicates that the revenues increase by 400 in one day.

(b) $m = 100$ indicates that the revenues increase by 100 in one day.

(c) $m = 0$ indicates that the revenues do not change from one day to the next.

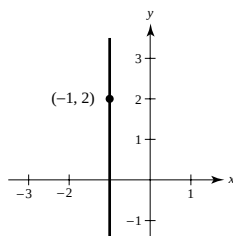
24. $6x - 5y = 15$

$$y = \frac{6}{5}x - 3$$

Therefore, the slope is $m = \frac{6}{5}$ and the y-intercept is $(0, -3)$.

28. $x = -1$

$$x + 1 = 0$$

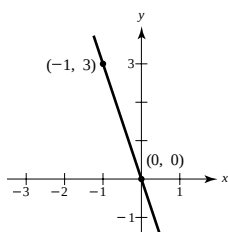


34. $m = \frac{3 - 0}{-1 - 0} = -3$

$$y - 0 = -3(x - 0)$$

$$y = -3x$$

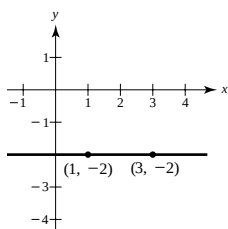
$$3x + y = 0$$



40. $m = 0$

$$y = -2$$

$$y + 2 = 0$$

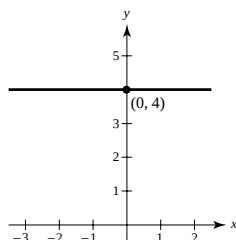


26. $y = -1$

The line is horizontal. Therefore, the slope is $m = 0$ and the y-intercept is $(0, -1)$.

30. $y = 4$

$$y - 4 = 0$$

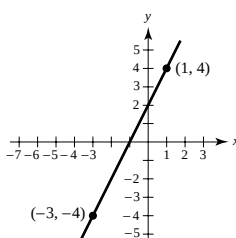


36. $m = \frac{4 - (-4)}{1 - (-3)} = \frac{8}{4} = 2$

$$y - 4 = 2(x - 1)$$

$$y - 4 = 2x - 2$$

$$0 = 2x - y + 2$$



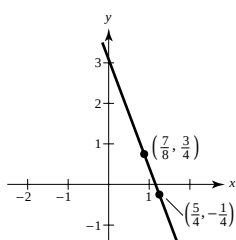
42. $m = \frac{(3/4) - (-1/4)}{(7/8) - (5/4)}$

$$= \frac{1}{-3/8} = -\frac{8}{3}$$

$$y + \frac{1}{4} = -\frac{8}{3}\left(x - \frac{5}{4}\right)$$

$$12y + 3 = -32x + 40$$

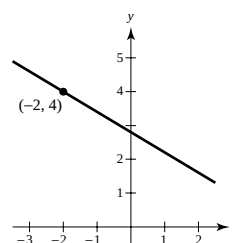
$$32x + 12y - 37 = 0$$



32. $y - 4 = -\frac{3}{5}(x + 2)$

$$5y - 20 = -3x - 6$$

$$3x + 5y - 14 = 0$$

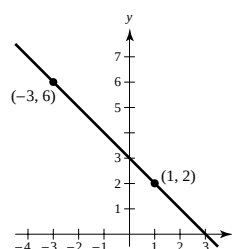


38. $m = \frac{6 - 2}{-3 - 1} = \frac{4}{-4} = -1$

$$y - 2 = -1(x - 1)$$

$$y - 2 = -x + 1$$

$$x + y - 3 = 0$$

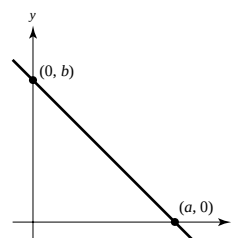


44. $m = -\frac{b}{a}$

$$y = -\frac{b}{a}x + b$$

$$\frac{b}{a}x + y = b$$

$$\frac{x}{a} + \frac{y}{b} = 1$$



46. $\frac{x}{-2/3} + \frac{y}{-2} = 1$

$$\frac{-3x}{2} - \frac{y}{2} = 1$$

$$3x + y = -2$$

$$3x + y + 2 = 0$$

48. $\frac{x}{a} + \frac{y}{a} = 1$

$$\frac{-3}{a} + \frac{4}{a} = 1$$

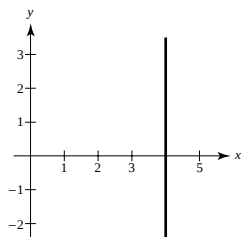
$$\frac{1}{a} = 1$$

$$a = 1 \Rightarrow x + y = 1$$

$$x + y - 1 = 0$$

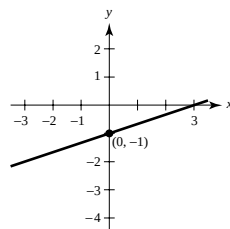
50. $x = 4$

$$x - 4 = 0$$



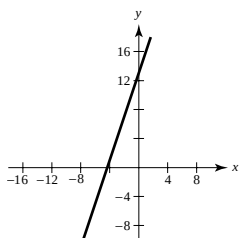
52. $y = \frac{1}{3}x - 1$

$$3y - x + 3 = 0$$



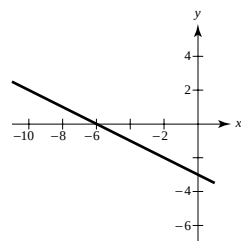
54. $y - 1 = 3(x + 4)$

$$y = 3x + 13$$

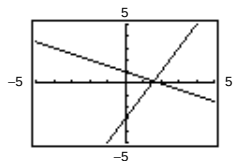


56. $x + 2y + 6 = 0$

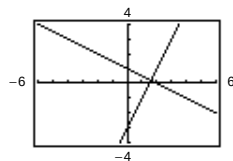
$$y = -\frac{1}{2}x - 3$$



58.



The lines do not appear perpendicular.



The lines appear perpendicular.

The lines are perpendicular because their slopes 2 and $-\frac{1}{2}$ are negative reciprocals of each other. You must use a square setting in order for perpendicular lines to appear perpendicular.

60. $x + y = 7$

$$y = -x + 7$$

$$m = -1$$

(a) $y - 2 = -1(x + 3)$

$$y - 2 = -x - 3$$

$$x + y + 1 = 0$$

(b) $y - 2 = 1(x + 3)$

$$y - 2 = x + 3$$

$$x - y + 5 = 0$$

62. $3x + 4y = 7$

$$y = -\frac{3}{4}x + \frac{7}{4}$$

$$m = -\frac{3}{4}$$

(a) $y - 4 = -\frac{3}{4}(x + 6)$

$$4y - 16 = -3x - 18$$

$$3x + 4y + 2 = 0$$

(b) $y - 4 = \frac{4}{3}(x + 6)$

$$3y - 12 = 4x + 24$$

$$4x - 3y + 36 = 0$$

64. (a) $y = 0$

(b) $x = -1 \Rightarrow x + 1 = 0$

66. The slope is 4.50.

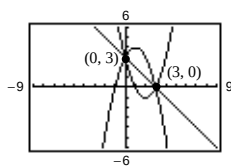
$$\text{Hence, } V = 4.5(t - 1) + 156$$

$$= 4.5t + 151.5$$

68. The slope is -5600 . Hence, $V = -5600(t - 1) + 245,000$

$$= -5600t + 250,600$$

70.



You can use the graphing utility to determine that the points of intersection are $(0, 3)$ and $(3, 0)$. Analytically,

$$x^2 - 4x + 3 = -x^2 + 2x + 3$$

$$2x^2 - 6x = 0$$

$$2x(x - 3) = 0$$

$$x = 0 \Rightarrow y = 3 \Rightarrow (0, 3)$$

$$x = 3 \Rightarrow y = 0 \Rightarrow (3, 0).$$

The slope of the line joining $(0, 3)$ and $(3, 0)$ is $m = (0 - 3)/(3 - 0) = -1$. Hence, an equation of the line is

$$y - 3 = -1(x - 0)$$

$$y = -x + 3.$$

72. $m_1 = \frac{-6 - 4}{7 - 0} = -\frac{10}{7}$

$$m_2 = \frac{11 - 4}{-5 - 0} = -\frac{7}{5}$$

$$m_1 \neq m_2$$

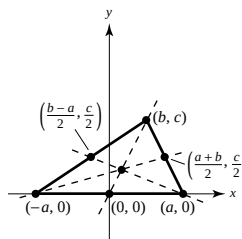
The points are not collinear.

74. Equations of medians:

$$y = \frac{c}{b}x$$

$$y = \frac{c}{3a + b}(x + a)$$

$$y = \frac{c}{-3a + b}(x - a)$$



Solving simultaneously, the point of intersection is

$$\left(\frac{b}{3}, \frac{c}{3}\right).$$

76. The slope of the line segment from $\left(\frac{b}{3}, \frac{c}{3}\right)$ to $\left(b, \frac{a^2 - b^2}{c}\right)$ is:

$$m_1 = \frac{[(a^2 - b^2)/c] - (c/3)}{b - (b/3)} = \frac{(3a^2 - 3b^2 - c^2)/(3c)}{(2b)/3} = \frac{3a^2 - 3b^2 - c^2}{2bc}$$

The slope of the line segment from $\left(\frac{b}{3}, \frac{c}{3}\right)$ to $\left(0, \frac{-a^2 + b^2 + c^2}{2c}\right)$ is:

$$m_2 = \frac{[(-a^2 + b^2 + c^2)/(2c)] - (c/3)}{0 - (b/3)} = \frac{(-3a^2 + 3b^2 + 3c^2 - 2c^2)/(6c)}{-b/3} = \frac{3a^2 - 3b^2 - c^2}{2bc}$$

$$m_1 = m_2$$

Therefore, the points are collinear.

78. $C = 0.34x + 150$. If $x = 137$, $C = 0.34(137) + 150 = \$196.58$

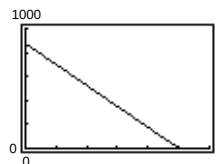
80. (a) Depreciation per year:

$$\frac{875}{5} = \$175$$

$$y = 875 - 175x$$

where $0 \leq x \leq 5$.

$$(b) y = 875 - 175(2) = \$525$$

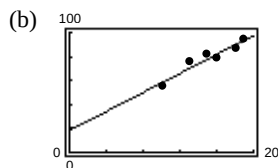


$$(c) 200 = 875 - 175x$$

$$175x = 675$$

$$x \approx 3.86 \text{ years}$$

82. (a) $y = 18.91 + 3.97x$ (x = quiz score, y = test score)



$$(c) \text{ If } x = 17, y = 18.91 + 3.97(17) = 86.4.$$

(d) The slope shows the average increase in exam score for each unit increase in quiz score.

(e) The points would shift vertically upward 4 units. The new regression line would have a y-intercept 4 greater than before: $y = 22.91 + 3.97x$.

$$84. 4x + 3y - 10 = 0 \Rightarrow d = \frac{|4(2) + 3(3) - 10|}{\sqrt{4^2 + 3^2}} = \frac{7}{5}$$

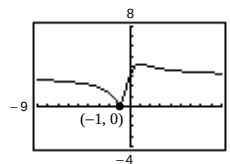
$$86. x + 1 = 0 \Rightarrow d = \frac{|1(6) + (0)(2) + 1|}{\sqrt{1^2 + 0^2}} = 7$$

88. A point on the line $3x - 4y = 1$ is $(-1, -1)$. The distance from the point $(-1, -1)$ to $3x - 4y - 10 = 0$ is

$$d = \frac{|-3 + 4 - 10|}{5} = \frac{9}{5}.$$

90. $y = mx + 4 \Rightarrow mx + (-1)y + 4 = 0$

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|m(3) + (-1)(1) + 4|}{\sqrt{m^2 + (-1)^2}} = \frac{|3m + 3|}{\sqrt{m^2 + 1}}$$



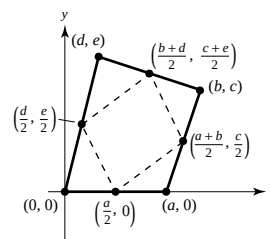
The distance is 0 when $m = -1$. In this case, the line $y = -x + 4$ contains the point $(3, 1)$.

92. For simplicity, let the vertices of the quadrilateral be $(0, 0)$, $(a, 0)$, (b, c) , and (d, e) , as shown in the figure. The midpoints of the sides are

$$\left(\frac{a}{2}, 0\right), \left(\frac{a+b}{2}, \frac{c}{2}\right), \left(\frac{b+d}{2}, \frac{c+e}{2}\right), \text{ and } \left(\frac{d}{2}, \frac{e}{2}\right).$$

The slope of the opposite sides are equal:

$$\begin{aligned} \frac{\frac{c}{2} - 0}{\frac{a+b}{2} - \frac{a}{2}} &= \frac{\frac{c+e}{2} - \frac{e}{2}}{\frac{b+d}{2} - \frac{d}{2}} = \frac{c}{b} \\ 0 - \frac{e}{2} &= \frac{\frac{c}{2} - \frac{c+e}{2}}{\frac{a+b}{2} - \frac{b+d}{2}} = -\frac{e}{a-d} \end{aligned}$$



Therefore, the figure is a parallelogram.

94. If $m_1 = -1/m_2$, then $m_1 m_2 = -1$. Let L_3 be a line with slope m_3 that is perpendicular to L_1 . Then $m_1 m_3 = -1$. Hence, $m_2 = m_3 \Rightarrow L_2$ and L_3 are parallel. Therefore, L_2 and L_1 are also perpendicular.

96. False; if m_1 is positive, then $m_2 = -1/m_1$ is negative.

Section P.3 Functions and Their Graphs

2. (a) $f(-2) = \sqrt{-2+3} = \sqrt{1} = 1$

(b) $f(6) = \sqrt{6+3} = \sqrt{9} = 3$

(c) $f(c) = \sqrt{c+3}$

(d) $f(x + \Delta x) = \sqrt{x + \Delta x + 3}$

4. (a) $g(4) = 4^2(4-4) = 0$

(b) $g\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2\left(\frac{3}{2}-4\right) = \frac{9}{4}\left(-\frac{5}{2}\right) = -\frac{45}{8}$

(c) $g(c) = c^2(c-4) = c^3 - 4c^2$

(d) $g(t+4) = (t+4)^2(t+4-4)$
 $= (t+4)^2 t = t^3 + 8t^2 + 16t$

6. (a) $f(\pi) = \sin \pi = 0$

(c) $f\left(\frac{2\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

(b) $f\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) = \frac{-\sqrt{2}}{2}$

8. $\frac{f(x) - f(1)}{x - 1} = \frac{3x - 1 - (3 - 1)}{x - 1} = \frac{3(x - 1)}{x - 1} = 3, x \neq 1$

10. $\frac{f(x) - f(1)}{x - 1} = \frac{x^3 - x - 0}{x - 1} = \frac{x(x+1)(x-1)}{x - 1} = x(x+1), x \neq 1$

12. $g(x) = x^2 - 5$

 Domain: $(-\infty, \infty)$

 Range: $[-5, \infty)$

16. $g(x) = \frac{2}{x-1}$

 Domain: $(-\infty, 1), (1, \infty)$

 Range: $(-\infty, 0), (0, \infty)$

18. $f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ 2x^2 + 2, & x > 1 \end{cases}$

(a) $f(-2) = (-2)^2 + 2 = 6$

(b) $f(0) = 0^2 + 2 = 2$

(c) $f(1) = 1^2 + 2 = 3$

(d) $f(s^2 + 2) = 2(s^2 + 2)^2 = 2s^4 + 8s^2 + 10$

 (Note: $s^2 + 2 > 1$ for all s)

 Domain: $(-\infty, \infty)$

 Range: $[2, \infty)$

14. $h(t) = \cot t$

 Domain: all $t \neq k\pi$, k an integer

 Range: $(-\infty, \infty)$

20. $f(x) = \begin{cases} \sqrt{x+4}, & x \leq 5 \\ (x-5)^2, & x > 5 \end{cases}$

(a) $f(-3) = \sqrt{-3+4} = \sqrt{1} = 1$

(b) $f(0) = \sqrt{0+4} = 2$

(c) $f(5) = \sqrt{5+4} = 3$

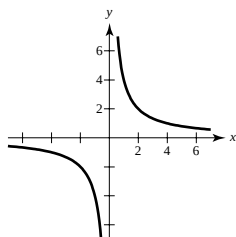
(d) $f(10) = (10-5)^2 = 25$

 Domain: $[-4, \infty)$

 Range: $[0, \infty)$

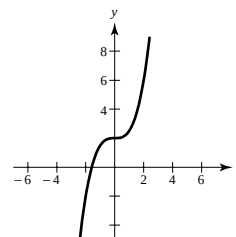
22. $g(x) = \frac{4}{x}$

 Domain: $(-\infty, 0), (0, \infty)$

 Range: $(-\infty, 0), (0, \infty)$


24. $f(x) = \frac{1}{2}x^3 + 2$

 Domain: $(-\infty, \infty)$

 Range: $(-\infty, \infty)$


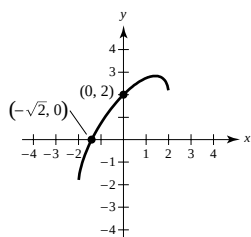
26. $f(x) = x + \sqrt{4-x^2}$

 Domain: $[-2, 2]$

Range:

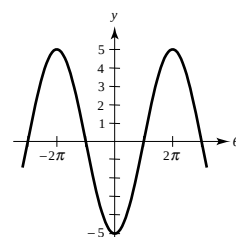
$$[-2, 2\sqrt{2}] \approx [-2, 2.83]$$

 y-intercept: $(0, 2)$

 x-intercept: $(-\sqrt{2}, 0)$


28. $h(\theta) = -5 \cos \frac{\theta}{2}$

 Domain: $(-\infty, \infty)$

 Range: $[-5, 5]$


30. $\sqrt{x^2 - 4} - y = 0 \Rightarrow y = \sqrt{x^2 - 4}$

y is a function of x . Vertical lines intersect the graph at most once.

32. $x^2 + y^2 = 4$

$$y = \pm \sqrt{4 - x^2}$$

y is not a function of x . Some vertical lines intersect the graph twice.

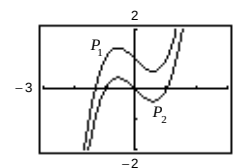
34. $x^2 + y = 4 \Rightarrow y = 4 - x^2$

y is a function of x since there is one value of y for each x .

36. $x^2y - x^2 + 4y = 0 \Rightarrow y = \frac{x^2}{x^2 + 4}$

y is a function of x since there is one value of y for each x .

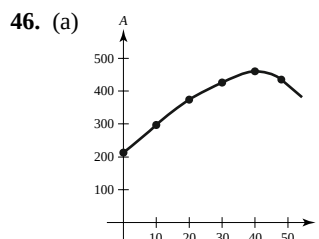
38. $p_1(x) = x^3 - x + 1$ has one zero. $p_2(x) = x^3 - x$ has three zeros. Every cubic polynomial has at least one zero. Given $p(x) = Ax^3 + Bx^2 + Cx + D$, we have $p \rightarrow -\infty$ as $x \rightarrow -\infty$ and $p \rightarrow \infty$ as $x \rightarrow \infty$ if $A > 0$. Furthermore, $p \rightarrow \infty$ as $x \rightarrow -\infty$ and $p \rightarrow -\infty$ as $x \rightarrow \infty$ if $A < 0$. Since the graph has no breaks, the graph must cross the x -axis at least one time.



40. The function is $f(x) = cx$. Since $(1, 1/4)$ satisfies the equation, $c = 1/4$. Thus, $f(x) = (1/4)x$.

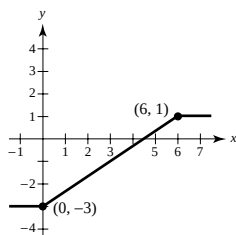
42. The function is $h(x) = c\sqrt{|x|}$. Since $(1, 3)$ satisfies the equation, $c = 3$. Thus, $h(x) = 3\sqrt{|x|}$.

44. The student travels $\frac{2-0}{4-0} = \frac{1}{2}$ mi/min during the first 4 minutes. The student is stationary for the following 2 minutes. Finally, the student travels $\frac{6-2}{10-6} = 1$ mi/min during the final 4 minutes.

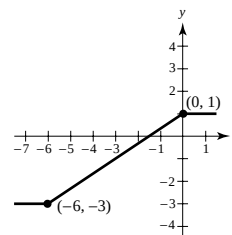


(b) $A(15) \approx 345$ acres/farm

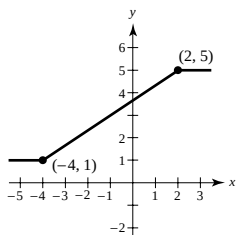
48. (a) $g(x) = f(x - 4)$
 $g(6) = f(2) = 1$
 $g(0) = f(-4) = -3$
 Shift f right 4 units



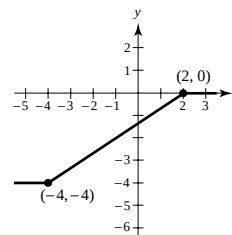
- (b) $g(x) = f(x + 2)$
 Shift f left 2 units



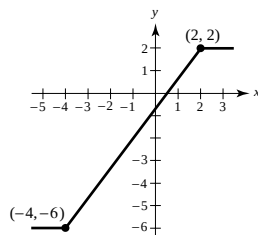
- (c) $g(x) = f(x) + 4$
 Vertical shift upwards
 4 units



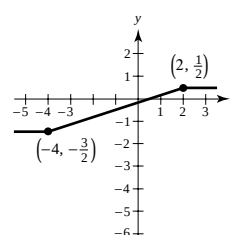
- (d) $g(x) = f(x) - 1$
 Vertical shift down 1 unit



- (e) $g(x) = 2f(x)$
 $g(2) = 2f(2) = 2$
 $g(-4) = 2f(-4) = -6$



- (f) $g(x) = \frac{1}{2}f(x)$
 $g(2) = \frac{1}{2}f(2) = \frac{1}{2}$
 $g(-4) = \frac{1}{2}f(-4) = -\frac{3}{2}$



50. (a) $h(x) = \sin(x + (\pi/2)) + 1$ is a horizontal shift $\pi/2$ units to the left, followed by a vertical shift 1 unit upwards.
 (b) $h(x) = -\sin(x - 1)$ is a horizontal shift 1 unit to the right followed by a reflection about the x -axis.

52. (a) $f(g(1)) = f(0) = 0$

(b) $g(f(1)) = g(1) = 0$

(c) $g(f(0)) = g(0) = -1$

(d) $f(g(-4)) = f(15) = \sqrt{15}$

(e) $f(g(x)) = f(x^2 - 1) = \sqrt{x^2 - 1}$

(f) $g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 - 1 = x - 1 \quad (x \geq 0)$

54. $f(x) = x^2 - 1, g(x) = \cos x$

$(f \circ g)(x) = f(g(x)) = f(\cos x) = \cos^2 x - 1$

Domain: $(-\infty, \infty)$

$(g \circ f)(x) = g(x^2 - 1) = \cos(x^2 - 1)$

Domain: $(-\infty, \infty)$

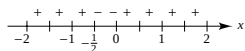
No, $f \circ g \neq g \circ f$.

56. $(f \circ g)(x) = f(\sqrt{x+2}) = \frac{1}{\sqrt{x+2}}$

Domain: $(-2, \infty)$

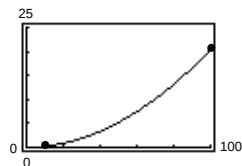
$(g \circ f)(x) = g\left(\frac{1}{x}\right) = \sqrt{\frac{1}{x} + 2} = \sqrt{\frac{1+2x}{x}}$

You can find the domain of $g \circ f$ by determining the intervals where $(1 + 2x)$ and x are both positive, or both negative.



Domain: $(-\infty, -\frac{1}{2}], (0, \infty)$

58. (a)



$$\begin{aligned} \text{(b) } H(1.6x) &= 0.002(1.6x)^2 + 0.005(1.6x) - 0.029 \\ &= 0.00512x^2 + 0.008x - 0.029 \end{aligned}$$

60. $f(-x) = \sqrt[3]{-x} = -\sqrt[3]{x} = -f(x)$

Odd

62. $f(-x) = \sin^2(-x) = \sin(-x) \sin(-x) = (-\sin x)(-\sin x) = \sin^2 x$

Even

64. (a) If f is even, then $(-4, 9)$ is on the graph.

(b) If f is odd, then $(-4, -9)$ is on the graph.

$$\begin{aligned} \text{66. } f(-x) &= a_{2n}(-x)^{2n} + a_{2n-2}(-x)^{2n-2} + \cdots + a_2(-x)^2 + a_0 \\ &= a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdots + a_2x^2 + a_0 \\ &= f(x) \end{aligned}$$

Even

68. Let $F(x) = f(x)g(x)$ where f is even and g is odd. Then

$F(-x) = f(-x)g(-x) = f(x)[-g(x)] = -f(x)g(x) = -F(x).$

Thus, $F(x)$ is odd.

70. (a) Let $F(x) = f(x) \pm g(x)$ where f and g are even. Then, $F(-x) = f(-x) \pm g(-x) = f(x) \pm g(x) = F(x)$. Thus, $F(x)$ is even.
- (b) Let $F(x) = f(x) \pm g(x)$ where f and g are odd. Then, $F(-x) = f(-x) \pm g(-x) = -f(x) \mp g(x) = -F(x)$. Thus, $F(x)$ is odd.
- (c) Let $F(x) = f(x) \pm g(x)$ where f is odd and g is even. Then, $F(-x) = f(-x) \pm g(-x) = -f(x) \pm g(x)$. Thus, $F(x)$ is neither odd nor even.

72. By equating slopes, $\frac{y-2}{0-3} = \frac{0-2}{x-3}$

$$y-2 = \frac{6}{x-3}$$

$$y = \frac{6}{x-3} + 2 = \frac{2x}{x-3},$$

$$L = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(\frac{2x}{x-3}\right)^2}.$$

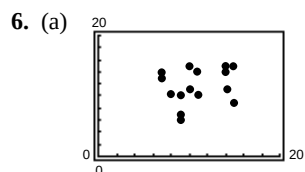
74. True

76. False; let $f(x) = x^2$. Then $f(3x) = (3x)^2 = 9x^2$ and $3f(x) = 3x^2$. Thus, $3f(x) \neq f(3x)$.

Section P.4 Fitting Models to Data

2. Trigonometric function

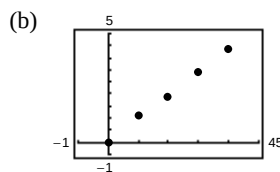
4. No relationship



No, the relationship does not appear to be linear.

- (b) Quiz scores are dependent on several variables such as study time, class attendance, etc. These variables may change from one quiz to the next.

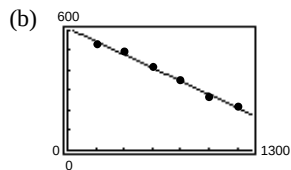
8. (a) $s = 9.7t + 0.4$



The model fits well.

- (c) If $t = 2.5$, $s = 24.65$ meters/second.

10. (a) Linear model: $H = -0.3323t + 612.9333$

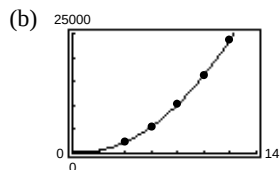


The fit is very good.

- (c) When $t = 500$,

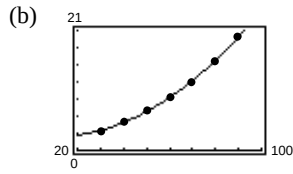
$$H = -0.3323(500) + 612.9333 \approx 446.78.$$

12. (a) $S = 180.89x^2 - 205.79x + 272$



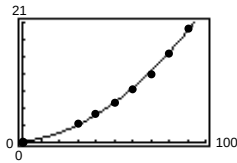
- (c) When $x = 2$, $S \approx 583.98$ pounds.

14. (a) $t = 0.00271s^2 - 0.0529s + 2.671$



(c) The curve levels off for $s < 20$.

(d) $t = 0.0025s^2 + 0.0346s + 0.183$

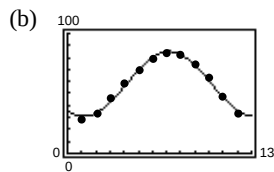


The model is better for low speeds.

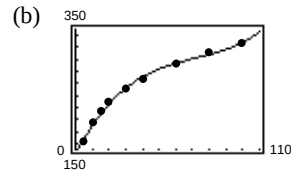
18. (a) $H(t) = 84.4 + 4.28 \sin\left(\frac{\pi t}{6} + 3.86\right)$

One model is

$$C(t) = 58 + 27 \sin\left(\frac{\pi t}{6} + 4.1\right).$$

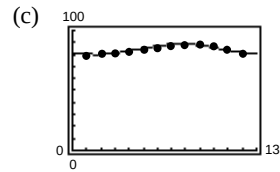


16. (a) $T = 2.9856 \times 10^{-4} p^3 - 0.0641 p^2 + 5.2826 p + 143.1$



(c) For $T = 300^\circ\text{F}$, $p \approx 68.29$ pounds per square inch.

(d) The model is based on data up to 100 pounds per square inch.



(d) The average in Honolulu is 84.4.

The average in Chicago is 58.

(e) The period is 12 months (1 year).

(f) Chicago has greater variability ($27 > 4.28$).

20. Answers will vary.

Review Exercises for Chapter P

2. $y = (x - 1)(x - 3)$

$x = 0 \Rightarrow y = (0 - 1)(0 - 3) = 3 \Rightarrow (0, 3)$ y-intercept

$y = 0 \Rightarrow 0 = (x - 1)(x - 3) \Rightarrow x = 1, 3 \Rightarrow (1, 0), (3, 0)$ x-intercepts

4. $xy = 4$

$x = 0$ and $y = 0$ are both impossible. No intercepts.

6. Symmetric with respect to y-axis since

$$y = (-x)^4 - (-x)^2 + 3$$

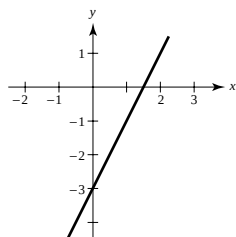
$$y = x^4 - x^2 + 3.$$

8. $4x - 2y = 6$

$$y = 2x - 3$$

Slope: 2

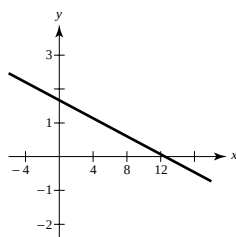
y-intercept: -3



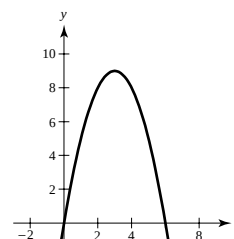
10. $0.02x + 0.15y = 0.25$

$$2x + 15y = 25$$

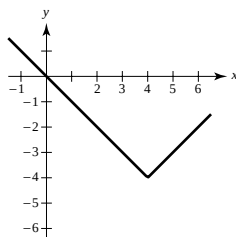
$$y = -\frac{2}{15}x + \frac{5}{3}$$

Slope: $-\frac{2}{15}$ y-intercept: $\frac{5}{3}$ 

12. $y = x(6 - x)$



14. $y = |x - 4| - 4$



16. $y = 8\sqrt[3]{x - 6}$

Xmin = -40
 Xmax = 40
 Xscl = 10
 Ymin = -40
 Ymax = 40
 Yscl = 10

18. $y = x + 1$

$$(x + 1) - x^2 = 7$$

$$0 = x^2 - x + 6$$

No real solution

No points of intersection

The graphs of $y = x + 1$ and $y = x^2 + 7$ do not intersect.

20. $y = kx^3$

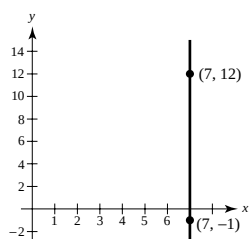
(a) $4 = k(1)^3 \Rightarrow k = 4$ and $y = 4x^3$

(c) $0 = k(0)^3 \Rightarrow$ any k will do!

(b) $1 = k(-2)^3 \Rightarrow k = -\frac{1}{8}$ and $y = -\frac{1}{8}x^3$

(d) $-1 = k(-1)^3 \Rightarrow k = 1 \Rightarrow y = x^3$

22.



The line is vertical and has no slope.

24. $\frac{3 - (-1)}{-3 - t} = \frac{3 - 6}{-3 - 8}$

$$\frac{4}{-3 - t} = \frac{-3}{-11}$$

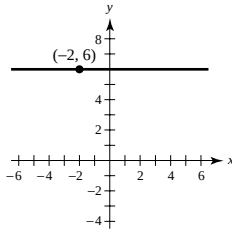
$$-44 = 9 + 3t$$

$$-53 = 3t$$

$$t = -\frac{53}{3}$$

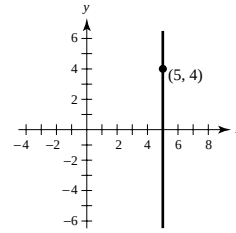
26. $y - 6 = 0(x - (-2))$

$y = 6$ Horizontal line



28. m is undefined. Line is vertical.

$x = 5$



30. (a) $y - 3 = -\frac{2}{3}(x - 1)$

$3y - 9 = -2x + 2$

$2x + 3y - 11 = 0$

(b) Slope of perpendicular line is 1.

$y - 3 = 1(x - 1)$

$y = x + 2$

$0 = x - y + 2$

(c) $m = \frac{4 - 3}{2 - 1} = 1$

$y - 3 = 1(x - 1)$

$y = x + 2$

$0 = x - y + 2$

(d) $y = 3$

$y - 3 = 0$

32. (a) $C = 9.25t + 13.50t + 36,500$

$= 22.75t + 36,500$

(b) $R = 30t$

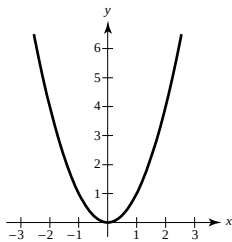
(c) $30t = 22.75t + 36,500$

$7.25t = 36,000$

$t \approx 5034.48$ hours to break even.

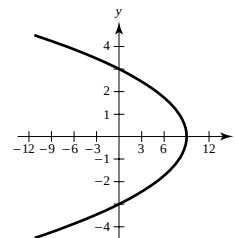
34. $x^2 - y = 0$

Function of x since there is one value for y for each x .



36. $x = 9 - y^2$

Not a function of x since there are two values of y for some x .



38. (a) $f(-4) = (-4)^2 + 2 = 18$ (because $-4 < 0$)

(b) $f(0) = |0 - 2| = 2$

(c) $f(1) = |1 - 2| = 1$

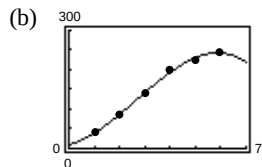
40. $f(x) = 1 - x^2$ and $g(x) = 2x + 1$

(a) $f(x) - g(x) = (1 - x^2) - (2x + 1) = -x^2 - 2x$

(b) $f(x)g(x) = (1 - x^2)(2x + 1) = -2x^3 - x^2 + 2x + 1$

(c) $g(f(x)) = g(1 - x^2) = 2(1 - x^2) + 1 = 3 - 2x^2$

15. (a) $y = -1.81x^3 + 14.58x^2 + 16.39x + 10$



(c) If $x = 4.5$, $y \approx 214$ horsepower.

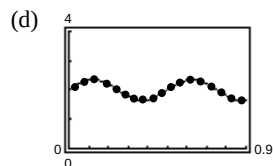
17. (a) Yes, y is a function of t . At each time t , there is one and only one displacement y .

(b) The amplitude is approximately

$$(2.35 - 1.65)/2 = 0.35.$$

The period is approximately

$$2(0.375 - 0.125) = 0.5.$$

(c) One model is $y = 0.35 \sin(4\pi t) + 2$.

19. Answers will vary.

Review Exercises for Chapter P

1. $y = 2x - 3$

$x = 0 \Rightarrow y = 2(0) - 3 = -3 \Rightarrow (0, -3) \quad y\text{-intercept}$

$y = 0 \Rightarrow 0 = 2x - 3 \Rightarrow x = \frac{3}{2} \Rightarrow (\frac{3}{2}, 0) \quad x\text{-intercept}$

3. $y = \frac{x-1}{x-2}$

$x = 0 \Rightarrow y = \frac{0-1}{0-2} = \frac{1}{2} \Rightarrow (0, \frac{1}{2}) \quad y\text{-intercept}$

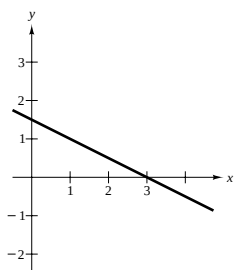
$y = 0 \Rightarrow 0 = \frac{x-1}{x-2} \Rightarrow x = 1 \Rightarrow (1, 0) \quad x\text{-intercept}$

5. Symmetric with respect to y -axis since

$$(-x)^2y - (-x)^2 + 4y = 0$$

$$x^2y - x^2 + 4y = 0.$$

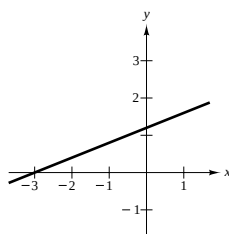
7. $y = -\frac{1}{2}x + \frac{3}{2}$



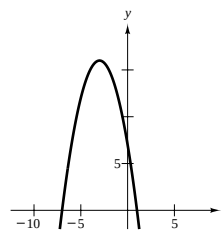
9. $-\frac{1}{3}x + \frac{5}{6}y = 1$

$$-\frac{2}{5}x + y = \frac{6}{5}$$

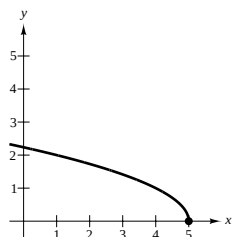
$$y = \frac{2}{5}x + \frac{6}{5}$$

Slope: $\frac{2}{5}$ y -intercept: $\frac{6}{5}$ 

11. $y = 7 - 6x - x^2$



13. $y = \sqrt{5 - x}$

Domain: $(-\infty, 5]$ 

15. $y = 4x^2 - 25$

Xmin = -5
Xmax = 5
Xscl = 1
Ymin = -30
Ymax = 10
Yscl = 5

17. $3x - 4y = 8$

$$\frac{4x + 4y = 20}{7x} = 28$$

$$7x = 28$$

$$x = 4$$

$$y = 1$$

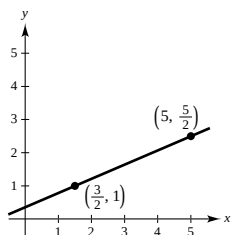
Point: (4, 1)

19. You need factors $(x + 2)$ and $(x - 2)$. Multiply by x to obtain origin symmetry

$$y = x(x + 2)(x - 2).$$

$$= x^3 - 4x.$$

21.



$$\text{Slope} = \frac{(5/2) - 1}{5 - (3/2)} = \frac{3/2}{7/2} = \frac{3}{7}$$

23. $\frac{1 - t}{1 - 0} = \frac{1 - 5}{1 - (-2)}$

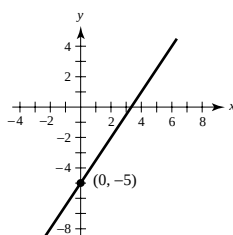
$$1 - t = -\frac{4}{3}$$

$$t = \frac{7}{3}$$

25. $y - (-5) = \frac{3}{2}(x - 0)$

$$y = \frac{3}{2}x - 5$$

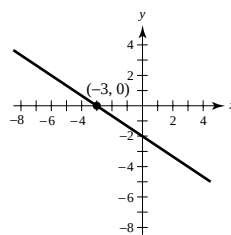
$$2y - 3x + 10 = 0$$



27. $y - 0 = -\frac{2}{3}(x - (-3))$

$$y = -\frac{2}{3}x - 2$$

$$3y + 2x + 6 = 0$$



29. (a) $y - 4 = \frac{7}{16}(x + 2)$

$$16y - 64 = 7x + 14$$

$$0 = 7x - 16y + 78$$

(c) $m = \frac{4 - 0}{-2 - 0} = -2$

$$y = -2x$$

$$2x + y = 0$$

(b) Slope of line is $\frac{5}{3}$.

$$y - 4 = \frac{5}{3}(x + 2)$$

$$3y - 12 = 5x + 10$$

$$0 = 5x - 3y + 22$$

(d) $x = -2$

$$x + 2 = 0$$

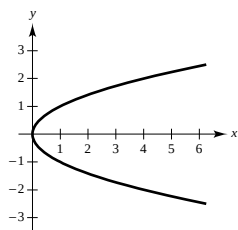
31. The slope is -850 . $V = -850t + 12,500$.

$$V(3) = -850(3) + 12,500 = \$9950$$

33. $x - y^2 = 0$

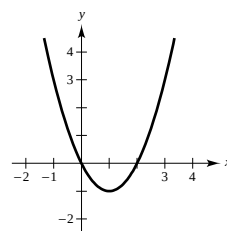
$$y = \pm \sqrt{x}$$

Not a function of x since there are two values of y for some x .



35. $y = x^2 - 2x$

Function of x since there is one value of y for each x .



37. $f(x) = \frac{1}{x}$

(a) $f(0)$ does not exist.

(b)
$$\frac{f(1 + \Delta x) - f(1)}{\Delta x} = \frac{\frac{1}{1 + \Delta x} - \frac{1}{1}}{\Delta x} = \frac{1 - 1 - \Delta x}{(1 + \Delta x)\Delta x}$$

$$= \frac{-1}{1 + \Delta x}, \Delta x \neq -1, 0$$

39. (a) Domain: $36 - x^2 \geq 0 \Rightarrow -6 \leq x \leq 6$ or $[-6, 6]$

Range: $[0, 6]$

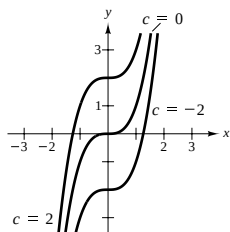
(b) Domain: all $x \neq 5$ or $(-\infty, 5), (5, \infty)$

Range: all $y \neq 0$ or $(-\infty, 0), (0, \infty)$

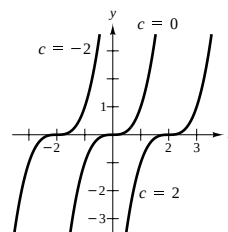
(c) Domain: all x or $(-\infty, \infty)$

Range: all y or $(-\infty, \infty)$

41. (a) $f(x) = x^3 + c, c = -2, 0, 2$



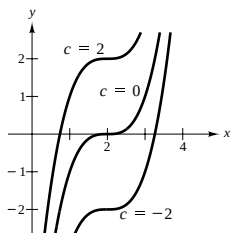
(b) $f(x) = (x - c)^3, c = -2, 0, 2$



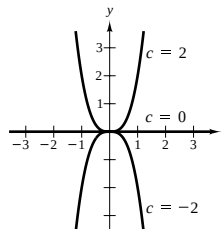
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41. —CONTINUED—

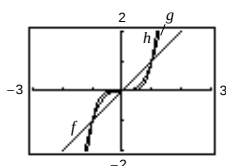
(c) $f(x) = (x - 2)^3 + c$, $c = -2, 0, 2$



(d) $f(x) = cx^3$, $c = -2, 0, 2$



43. (a) Odd powers: $f(x) = x$, $g(x) = x^3$, $h(x) = x^5$

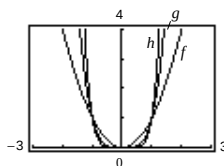


The graphs of f , g , and h all rise to the right and fall to the left. As the degree increases, the graph rises and falls more steeply. All three graphs pass through the points $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

(b) $y = x^7$ will look like $h(x) = x^5$, but rise and fall even more steeply.

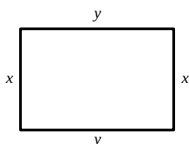
$y = x^8$ will look like $h(x) = x^6$, but rise even more steeply.

Even powers: $f(x) = x^2$, $g(x) = x^4$, $h(x) = x^6$



The graphs of f , g , and h all rise to the left and to the right. As the degree increases, the graph rises more steeply. All three graphs pass through the points $(0, 0)$, $(1, 1)$, and $(-1, 1)$.

45. (a)

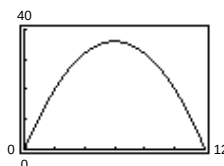


$2x + 2y = 24$

$y = 12 - x$

$A = xy = x(12 - x) = 12x - x^2$

(b) Domain: $0 < x < 12$



(c) Maximum area is $A = 36$. In general, the maximum area is attained when the rectangle is a square. In this case, $x = 6$.

47. (a) 3 (cubic), negative leading coefficient

(b) 4 (quartic), positive leading coefficient

(c) 2 (quadratic), negative leading coefficient

(d) 5, positive leading coefficient

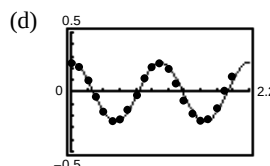
49. (a) Yes, y is a function of t . At each time t , there is one and only one displacement y .

(b) The amplitude is approximately

$(0.25 - (-0.25))/2 = 0.25$.

The period is approximately 1.1.

(c) One model is $y = \frac{1}{4} \cos\left(\frac{2\pi}{1.1}t\right) \approx \frac{1}{4} \cos(5.7t)$



Problem Solving for Chapter P

1. (a) $x^2 - 6x + y^2 - 8y = 0$
 $(x^2 - 6x + 9) + (y^2 - 8y + 16) = 9 + 16$
 $(x - 3)^2 + (y - 4)^2 = 25$

Center: (3, 4) Radius: 5

(c) Slope of line from (6, 0) to (3, 4) is $\frac{4 - 0}{3 - 6} = -\frac{4}{3}$.

Slope of tangent line is $\frac{3}{4}$. Hence,

$y - 0 = \frac{3}{4}(x - 6) \Rightarrow y = \frac{3}{4}x - \frac{9}{2}$ Tangent line

(b) Slope of line from (0, 0) to (3, 4) is $\frac{4}{3}$ Slope of tangent line is $-\frac{3}{4}$ Hence,

$y - 0 = -\frac{3}{4}(x - 0) \Rightarrow y = -\frac{3}{4}x$ Tangent line

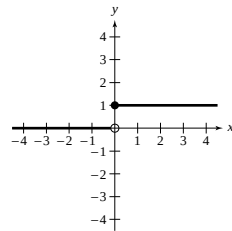
(d) $-\frac{3}{4}x = \frac{3}{4}x - \frac{9}{2}$

$\frac{3}{2}x = \frac{9}{2}$

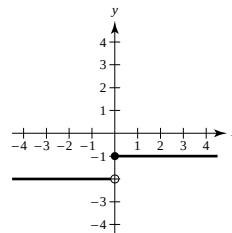
$x = 3$

Intersection: $\left(3, -\frac{9}{4}\right)$

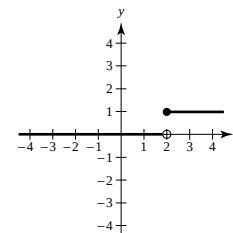
3. $H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$



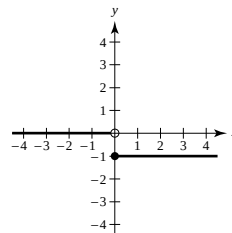
(a) $H(x) - 2$



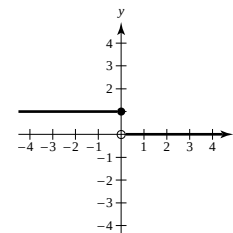
(b) $H(x - 2)$



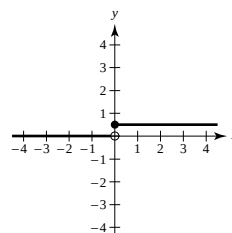
(c) $-H(x)$



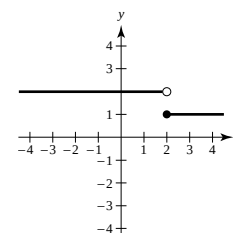
(d) $H(-x)$



(e) $\frac{1}{2}H(x)$



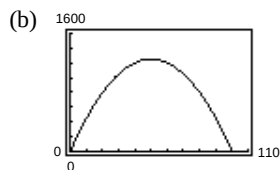
(f) $-H(x - 2) + 2$



5. (a) $x + 2y = 100 \Rightarrow y = \frac{100 - x}{2}$

$$A(x) = xy = x\left(\frac{100 - x}{2}\right) = -\frac{x^2}{2} + 50x$$

Domain: $0 < x < 100$



Maximum of 1250 m^2 at $x = 50 \text{ m}$, $y = 25 \text{ m}$.

$$\begin{aligned} \text{(c) } A(x) &= -\frac{1}{2}(x^2 - 100x) \\ &= -\frac{1}{2}(x^2 - 100x + 2500) + 1250 \\ &= -\frac{1}{2}(x - 50)^2 + 1250 \end{aligned}$$

$A(50) = 1250 \text{ m}^2$ is the maximum. $x = 50 \text{ m}$, $y = 25 \text{ m}$.

9. (a) Slope $= \frac{9 - 4}{3 - 2} = 5$. Slope of tangent line is less than 5.

(b) Slope $= \frac{4 - 1}{2 - 1} = 3$. Slope of tangent line is greater than 3.

(c) Slope $= \frac{4.41 - 4}{2.1 - 2} = 4.1$. Slope of tangent line is less than 4.1.

$$\begin{aligned} \text{(d) Slope} &= \frac{f(2 + h) - f(2)}{(2 + h) - 2} \\ &= \frac{(2 + h)^2 - 4}{h} \\ &= \frac{4h + h^2}{h} \\ &= 4 + h, h \neq 0 \end{aligned}$$

(e) Letting h get closer and closer to 0, the slope approaches 4. Hence, the slope at $(2, 4)$ is 4.

11. (a) At $x = 1$ and $x = -3$ the sounds are equal.

(b) $\frac{I}{\sqrt{x^2 + y^2}} = \frac{2I}{\sqrt{(x - 3)^2 + y^2}}$

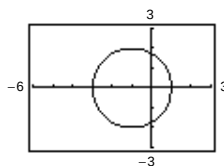
$$(x - 3)^2 + y^2 = 4(x^2 + y^2)$$

$$3x^2 + 3y^2 + 6x = 9$$

$$x^2 + 2x + y^2 = 3$$

$$(x + 1)^2 + y^2 = 4$$

Circle of radius 2 centered at $(-1, 0)$



7. The length of the trip in the water is $\sqrt{2^2 + x^2}$, and the length of the trip over land is $\sqrt{1 + (3 - x)^2}$. Hence, the total time is

$$T = \frac{\sqrt{4 + x^2}}{2} + \frac{\sqrt{1 + (3 - x)^2}}{4} \text{ hours.}$$

13.

$$d_1 d_2 = 1$$

$$[(x+1)^2 + y^2][(x-1)^2 + y^2] = 1$$

$$(x+1)^2(x-1)^2 + y^2[(x+1)^2 + (x-1)^2] + y^4 = 1$$

$$(x^2 - 1)^2 + y^2[2x^2 + 2] + y^4 = 1$$

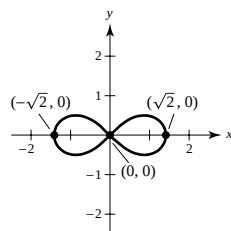
$$x^4 - 2x^2 + 1 + 2x^2y^2 + 2y^2 + y^4 = 1$$

$$(x^4 + 2x^2y^2 + y^4) - 2x^2 + 2y^2 = 0$$

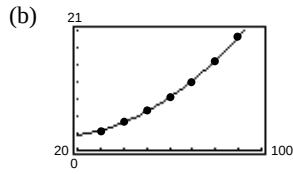
$$(x^2 + y^2)^2 = 2(x^2 - y^2)$$

Let $y = 0$. Then $x^4 = 2x^2 \Rightarrow x = 0$ or $x^2 = 2$.

Thus, $(0, 0)$, $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$ are on the curve.

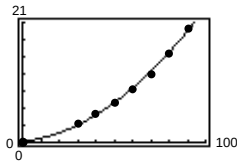


14. (a) $t = 0.00271s^2 - 0.0529s + 2.671$



(c) The curve levels off for $s < 20$.

(d) $t = 0.002s^2 + 0.0346s + 0.183$

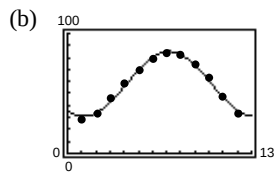


The model is better for low speeds.

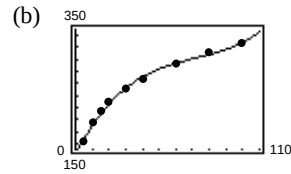
18. (a) $H(t) = 84.4 + 4.28 \sin\left(\frac{\pi t}{6} + 3.86\right)$

One model is

$$C(t) = 58 + 27 \sin\left(\frac{\pi t}{6} + 4.1\right).$$

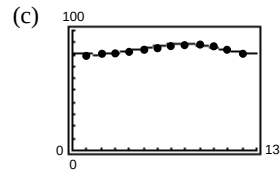


16. (a) $T = 2.9856 \times 10^{-4} p^3 - 0.0641 p^2 + 5.2826 p + 143.1$



(c) For $T = 300^\circ\text{F}$, $p \approx 68.29$ pounds per square inch.

(d) The model is based on data up to 100 pounds per square inch.



(d) The average in Honolulu is 84.4.

The average in Chicago is 58.

(e) The period is 12 months (1 year).

(f) Chicago has greater variability ($27 > 4.28$).

20. Answers will vary.

Review Exercises for Chapter P

2. $y = (x - 1)(x - 3)$

$x = 0 \Rightarrow y = (0 - 1)(0 - 3) = 3 \Rightarrow (0, 3)$ y-intercept

$y = 0 \Rightarrow 0 = (x - 1)(x - 3) \Rightarrow x = 1, 3 \Rightarrow (1, 0), (3, 0)$ x-intercepts

4. $xy = 4$

$x = 0$ and $y = 0$ are both impossible. No intercepts.

6. Symmetric with respect to y-axis since

$$y = (-x)^4 - (-x)^2 + 3$$

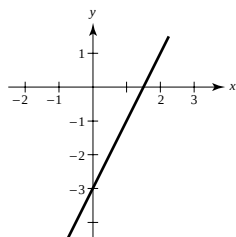
$$y = x^4 - x^2 + 3.$$

8. $4x - 2y = 6$

$$y = 2x - 3$$

Slope: 2

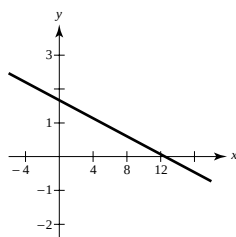
y-intercept: -3



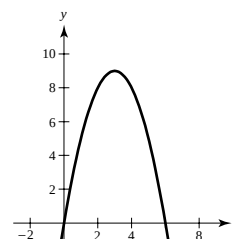
10. $0.02x + 0.15y = 0.25$

$$2x + 15y = 25$$

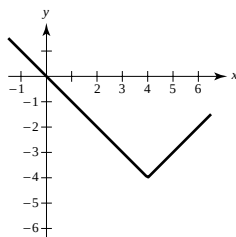
$$y = -\frac{2}{15}x + \frac{5}{3}$$

Slope: $-\frac{2}{15}$ y-intercept: $\frac{5}{3}$ 

12. $y = x(6 - x)$



14. $y = |x - 4| - 4$



16. $y = 8\sqrt[3]{x - 6}$

Xmin = -40
 Xmax = 40
 Xscl = 10
 Ymin = -40
 Ymax = 40
 Yscl = 10

18. $y = x + 1$

$$(x + 1) - x^2 = 7$$

$$0 = x^2 - x + 6$$

No real solution

No points of intersection

The graphs of $y = x + 1$ and $y = x^2 + 7$ do not intersect.

20. $y = kx^3$

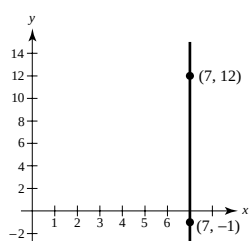
(a) $4 = k(1)^3 \Rightarrow k = 4$ and $y = 4x^3$

(c) $0 = k(0)^3 \Rightarrow$ any k will do!

(b) $1 = k(-2)^3 \Rightarrow k = -\frac{1}{8}$ and $y = -\frac{1}{8}x^3$

(d) $-1 = k(-1)^3 \Rightarrow k = 1 \Rightarrow y = x^3$

22.



The line is vertical and has no slope.

24. $\frac{3 - (-1)}{-3 - t} = \frac{3 - 6}{-3 - 8}$

$$\frac{4}{-3 - t} = \frac{-3}{-11}$$

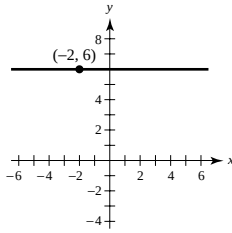
$$-44 = 9 + 3t$$

$$-53 = 3t$$

$$t = -\frac{53}{3}$$

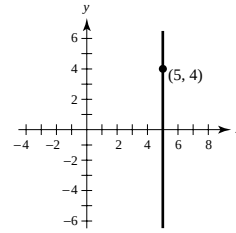
26. $y - 6 = 0(x - (-2))$

$y = 6$ Horizontal line



28. m is undefined. Line is vertical.

$x = 5$



30. (a) $y - 3 = -\frac{2}{3}(x - 1)$

$3y - 9 = -2x + 2$

$2x + 3y - 11 = 0$

(b) Slope of perpendicular line is 1.

$y - 3 = 1(x - 1)$

$y = x + 2$

$0 = x - y + 2$

(c) $m = \frac{4 - 3}{2 - 1} = 1$

$y - 3 = 1(x - 1)$

$y = x + 2$

$0 = x - y + 2$

(d) $y = 3$

$y - 3 = 0$

32. (a) $C = 9.25t + 13.50t + 36,500$

$= 22.75t + 36,500$

(b) $R = 30t$

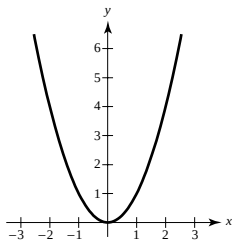
(c) $30t = 22.75t + 36,500$

$7.25t = 36,000$

$t \approx 5034.48$ hours to break even.

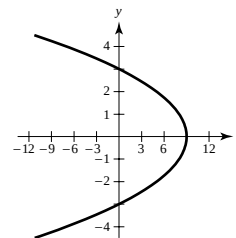
34. $x^2 - y = 0$

Function of x since there is one value for y for each x .



36. $x = 9 - y^2$

Not a function of x since there are two values of y for some x .



38. (a) $f(-4) = (-4)^2 + 2 = 18$ (because $-4 < 0$)

(b) $f(0) = |0 - 2| = 2$

(c) $f(1) = |1 - 2| = 1$

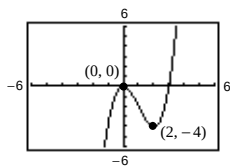
40. $f(x) = 1 - x^2$ and $g(x) = 2x + 1$

(a) $f(x) - g(x) = (1 - x^2) - (2x + 1) = -x^2 - 2x$

(b) $f(x)g(x) = (1 - x^2)(2x + 1) = -2x^3 - x^2 + 2x + 1$

(c) $g(f(x)) = g(1 - x^2) = 2(1 - x^2) + 1 = 3 - 2x^2$

42. $f(x) = x^3 - 3x^2$



- (a) The graph of g is obtained from f by a vertical shift down 1 unit, followed by a reflection in the x -axis:

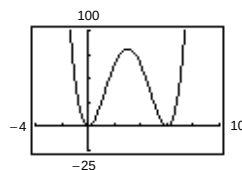
$$\begin{aligned} g(x) &= -[f(x) - 1] \\ &= -x^3 + 3x^2 + 1 \end{aligned}$$

- (b) The graph of g is obtained from f by a vertical shift upwards of 1 and a horizontal shift of 2 to the right.

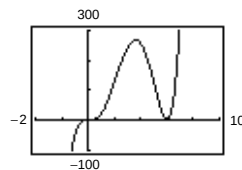
$$\begin{aligned} g(x) &= f(x - 2) + 1 \\ &= (x - 2)^3 - 3(x - 2)^2 + 1 \end{aligned}$$

46. For company (a) the profit rose rapidly for the first year, and then leveled off. For the second company (b), the profit dropped, and then rose again later.

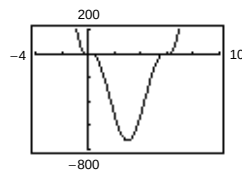
44. (a) $f(x) = x^2(x - 6)^2$



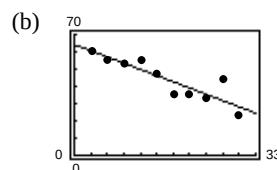
(b) $g(x) = x^3(x - 6)^2$



(c) $h(x) = x^3(x - 6)^3$



48. (a) $y = -1.204x + 64.2667$



- (c) The data point (27, 44) is probably an error. Without this point, the new model is

$$y = -1.4344x + 66.4387.$$

Problem Solving for Chapter P

2. Let $y = mx + 1$ be a tangent line to the circle from the point (0, 1). Then

$$x^2 + (y + 1)^2 = 1$$

$$x^2 + (mx + 1 + 1)^2 = 1$$

$$(m^2 + 1)x^2 + 4mx + 3 = 0$$

Setting the discriminant $b^2 - 4ac$ equal to zero,

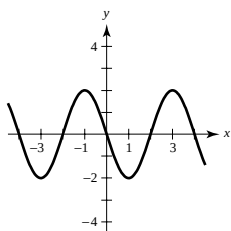
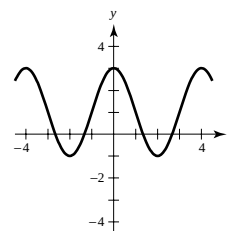
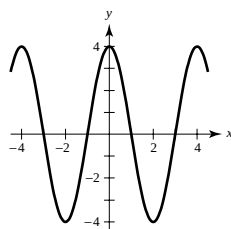
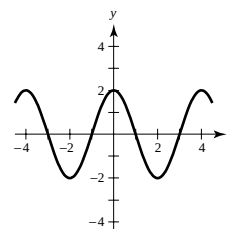
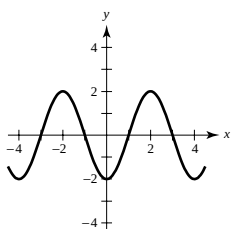
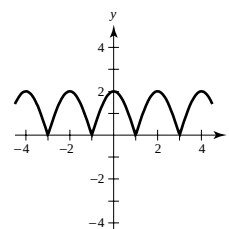
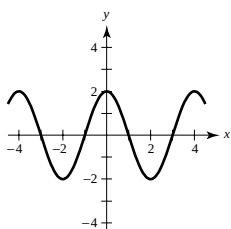
$$16m^2 - 4(m^2 + 1)(3) = 0$$

$$16m^2 - 12m^2 = 12$$

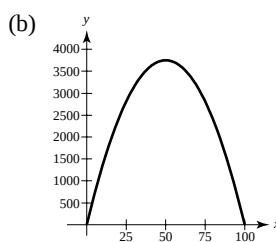
$$4m^2 = 12$$

$$m = \pm\sqrt{3}$$

Tangent lines: $y = \sqrt{3}x + 1$ and $y = -\sqrt{3}x + 1$.

4. (a) $f(x + 1)$

 (b) $f(x) + 1$

 (c) $2f(x)$

 (d) $f(-x)$

 (e) $-f(x)$

 (f) $|f(x)|$

 (g) $f(|x|)$

 6. (a) $4y + 3x = 300 \Rightarrow y = \frac{300 - 3x}{4}$

$$A(x) = x(2y) = x\left(\frac{300 - 3x}{2}\right) = \frac{-3x^2 + 300x}{2}$$

 Domain: $0 < x < 100$

 Maximum of 3750 ft² at $x = 50$ ft, $y = 37.5$ ft.

 (c) $A(x) = -\frac{3}{2}(x^2 - 100x)$

$$= -\frac{3}{2}(x^2 - 100x + 2500) + 3750$$

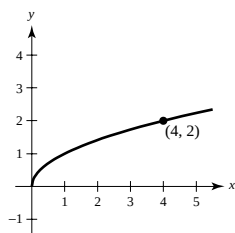
$$= -\frac{3}{2}(x - 50)^2 + 3750$$

$A(50) = 3750$ square feet is the maximum area,
where $x = 50$ ft and $y = 37.5$ ft.

8. Let d be the distance from the starting point to the beach.

$$\begin{aligned}
 \text{Average velocity} &= \frac{\text{distance}}{\text{time}} \\
 &= \frac{2d}{\frac{d}{120} + \frac{d}{60}} \\
 &= \frac{2}{\frac{1}{120} + \frac{1}{60}} \\
 &= 80 \text{ km/hr}
 \end{aligned}$$

10.



(a) Slope $= \frac{3-2}{9-4} = \frac{1}{5}$. Slope of tangent line is greater than $\frac{1}{5}$.

(b) Slope $= \frac{2-1}{4-1} = \frac{1}{3}$. Slope of tangent line is less than $\frac{1}{3}$.

(c) Slope $= \frac{2.1-2}{4.1-4} = \frac{10}{41}$. Slope of tangent line is greater than $\frac{10}{41}$.

(d) Slope $= \frac{f(4+h) - f(4)}{(4+h) - 4}$
 $= \frac{\sqrt{4+h} - 2}{h}$

(e) $\frac{\sqrt{4+h} - 2}{h} = \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2}$
 $= \frac{(4+h) - 4}{h(\sqrt{4+h} + 2)}$
 $= \frac{1}{\sqrt{4+h} + 2}, h \neq 0$

As h gets closer to 0, the slope gets closer to $\frac{1}{4}$. The slope is $\frac{1}{4}$ at the point $(4, 2)$.

$$12. (a) \quad \frac{I}{\sqrt{x^2 + y^2}} = \frac{kI}{\sqrt{(x-4)^2 + y^2}}$$

$$(x-4)^2 + y^2 = k^2(x^2 + y^2)$$

$$(k^2 - 1)x^2 + (k^2 - 1)y^2 + 8x = 16$$

If $k = 1$, then $x = 2$ is a vertical line. So, assume $k^2 - 1 \neq 0$. Then

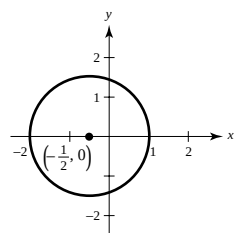
$$x^2 + y^2 + \frac{8x}{k^2 - 1} = \frac{16}{k^2 - 1}$$

$$\left(x + \frac{4}{k^2 - 1}\right)^2 + y^2 = \frac{16}{k^2 - 1} + \frac{16}{(k^2 - 1)^2}$$

$$\left(x + \frac{4}{k^2 - 1}\right)^2 + y^2 = \left(\frac{4k}{k^2 - 1}\right)^2, \text{ Circle}$$

$$(b) \text{ If } k = 3, \left(x + \frac{1}{2}\right)^2 + y^2 = \left(\frac{3}{2}\right)^2$$

(c) For large k , the center of the circle is near $(0, 0)$, and the radius becomes smaller.



$$14. f(x) = y = \frac{1}{1-x}$$

(a) Domain: all $x \neq 1$

Range: all $y \neq 0$

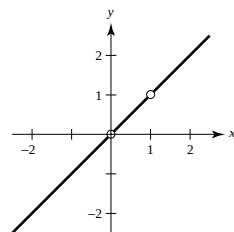
$$(b) f(f(x)) = f\left(\frac{1}{1-x}\right) = \frac{1}{1 - \left(\frac{1}{1-x}\right)} = \frac{1}{\frac{1-x-1}{1-x}} = \frac{1-x}{-x} = \frac{x-1}{x}$$

Domain: all $x \neq 0, 1$

$$(c) f(f(f(x))) = f\left(\frac{x-1}{x}\right) = \frac{1}{1 - \left(\frac{x-1}{x}\right)} = \frac{1}{\frac{1}{x}} = x$$

Domain: all $x \neq 0, 1$

(d) The graph is not a line. It has holes at $(0, 0)$ and $(1, 1)$.



CHAPTER 1

Limits and Their Properties

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CHAPTER 1

Limits and Their Properties

Section 1.1 A Preview of Calculus

Solutions to Odd-Numbered Exercises

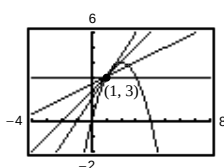
1. Precalculus: $(20 \text{ ft/sec})(15 \text{ seconds}) = 300 \text{ feet}$

3. Calculus required: slope of tangent line at $x = 2$ is rate of change, and equals about 0.16.

5. Precalculus: Area $= \frac{1}{2}bh = \frac{1}{2}(5)(3) = \frac{15}{2}$ sq. units

7. Precalculus: Volume $= (2)(4)(3) = 24$ cubic units

9. (a)



(b) The graphs of y_2 are approximations to the tangent line to y_1 at $x = 1$.

(c) The slope is approximately 2. For a better approximation make the list numbers smaller:
 $\{0.2, 0.1, 0.01, 0.001\}$

11. (a) $D_1 = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{16+16} \approx 5.66$

(b) $D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{3} - \frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - 1\right)^2}$
 $\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$

(c) Increase the number of line segments.

Section 1.2 Finding Limits Graphically and Numerically

1.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.3448	0.3344	0.3334	0.3332	0.3322	0.3226

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2-x-2} \approx 0.3333 \quad (\text{Actual limit is } \frac{1}{3}.)$$

3.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.2911	0.2889	0.2887	0.2887	0.2884	0.2863

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+3} - \sqrt{3}}{x} \approx 0.2887 \quad (\text{Actual limit is } 1/(2\sqrt{3}).)$$

5.

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	-0.0641	-0.0627	-0.0625	-0.0625	-0.0623	-0.0610

$$\lim_{x \rightarrow 3} \frac{[1/(x+1)] - (1/4)}{x-3} \approx -0.0625 \quad (\text{Actual limit is } -\frac{1}{16}.)$$

7.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	1.0000	1.0000	0.99998	0.9983

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \approx 1.0000 \quad (\text{Actual limit is } 1.) \quad (\text{Make sure you use radian mode.})$$

9. $\lim_{x \rightarrow 3} (4 - x) = 1$

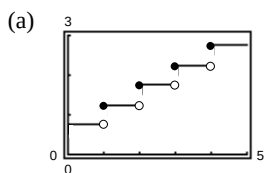
11. $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (4 - x) = 2$

13. $\lim_{x \rightarrow 5} \frac{|x-5|}{x-5}$ does not exist. For values of x to the left of 5, $|x-5|/(x-5)$ equals -1 , whereas for values of x to the right of 5, $|x-5|/(x-5)$ equals 1 .

15. $\lim_{x \rightarrow \pi/2} \tan x$ does not exist since the function increases and decreases without bound as x approaches $\pi/2$.

17. $\lim_{x \rightarrow 0} \cos(1/x)$ does not exist since the function oscillates between -1 and 1 as x approaches 0 .

19. $C(t) = 0.75 - 0.50\lfloor -(t-1) \rfloor$



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	1.75	2.25	2.25	2.25	2.25	2.25	2.25

$$\lim_{t \rightarrow 3.5} C(t) = 2.25$$

(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	1.25	1.75	1.75	1.75	2.25	2.25	2.25

$\lim_{t \rightarrow 3} C(t)$ does not exist. The values of C jump from 1.75 to 2.25 at $t = 3$.

21. You need to find δ such that $0 < |x-1| < \delta$ implies

$$|f(x) - 1| = \left| \frac{1}{x} - 1 \right| < 0.1. \quad \text{That is,}$$

$$-0.1 < \frac{1}{x} - 1 < 0.1$$

$$1 - 0.1 < \frac{1}{x} < 1 + 0.1$$

$$\frac{9}{10} < \frac{1}{x} < \frac{11}{10}$$

$$\frac{10}{9} > x > \frac{10}{11}$$

$$\frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1$$

$$\frac{1}{9} > x - 1 > -\frac{1}{11}.$$

So take $\delta = \frac{1}{11}$. Then $0 < |x-1| < \delta$ implies

$$-\frac{1}{11} < x - 1 < \frac{1}{11}$$

$$-\frac{1}{11} < x - 1 < \frac{1}{9}.$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 1| = \left| \frac{1}{x} - 1 \right| < \epsilon < 0.1.$$

$$23. \lim_{x \rightarrow 2} (3x + 2) = 8 = L$$

$$|(3x + 2) - 8| < 0.01$$

$$|3x - 6| < 0.01$$

$$3|x - 2| < 0.01$$

$$0 < |x - 2| < \frac{0.01}{3} \approx 0.0033 = \delta$$

Hence, if $0 < |x - 2| < \delta = \frac{0.01}{3}$, you have

$$3|x - 2| < 0.01$$

$$|3x - 6| < 0.01$$

$$|(3x + 2) - 8| < 0.01$$

$$|f(x) - L| < 0.01$$

$$27. \lim_{x \rightarrow 2} (x + 3) = 5$$

Given $\epsilon > 0$:

$$|(x + 3) - 5| < \epsilon$$

$$|x - 2| < \epsilon = \delta$$

Hence, let $\delta = \epsilon$.

Hence, if $0 < |x - 2| < \delta = \epsilon$, you have

$$|x - 2| < \epsilon$$

$$|(x + 3) - 5| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$31. \lim_{x \rightarrow 6} 3 = 3$$

Given $\epsilon > 0$:

$$|3 - 3| < \epsilon$$

$$0 < \epsilon$$

Hence, any $\delta > 0$ will work.

Hence, for any $\delta > 0$, you have

$$|3 - 3| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$25. \lim_{x \rightarrow 2} (x^2 - 3) = 1 = L$$

$$|(x^2 - 3) - 1| < 0.01$$

$$|x^2 - 4| < 0.01$$

$$|(x + 2)(x - 2)| < 0.01$$

$$|x + 2| |x - 2| < 0.01$$

$$|x - 2| < \frac{0.01}{|x + 2|}$$

If we assume $1 < x < 3$, then $\delta = 0.01/5 = 0.002$.

Hence, if $0 < |x - 2| < \delta = 0.002$, you have

$$|x - 2| < 0.002 = \frac{1}{5}(0.01) < \frac{1}{|x + 2|}(0.01)$$

$$|x + 2| |x - 2| < 0.01$$

$$|x^2 - 4| < 0.01$$

$$|(x^2 - 3) - 1| < 0.01$$

$$|f(x) - L| < 0.01$$

$$29. \lim_{x \rightarrow -4} \left(\frac{1}{2}x - 1\right) = \frac{1}{2}(-4) - 1 = -3$$

Given $\epsilon > 0$:

$$\left|\left(\frac{1}{2}x - 1\right) - (-3)\right| < \epsilon$$

$$\left|\frac{1}{2}x + 2\right| < \epsilon$$

$$\frac{1}{2}|x - (-4)| < \epsilon$$

$$|x - (-4)| < 2\epsilon$$

Hence, let $\delta = 2\epsilon$.

Hence, if $0 < |x - (-4)| < \delta = 2\epsilon$, you have

$$|x - (-4)| < 2\epsilon$$

$$\left|\frac{1}{2}x + 2\right| < \epsilon$$

$$\left|\left(\frac{1}{2}x - 1\right) + 3\right| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$33. \lim_{x \rightarrow 0} \sqrt[3]{x} = 0$$

Given $\epsilon > 0$: $|\sqrt[3]{x} - 0| < \epsilon$

$$|\sqrt[3]{x}| < \epsilon$$

$$|x| < \epsilon^3 = \delta$$

Hence, let $\delta = \epsilon^3$.

Hence for $0 < |x - 0| < \delta = \epsilon^3$, you have

$$|x| < \epsilon^3$$

$$|\sqrt[3]{x}| < \epsilon$$

$$|\sqrt[3]{x} - 0| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$35. \lim_{x \rightarrow -2} |x - 2| = |(-2) - 2| = 4$$

Given $\epsilon > 0$:

$$||x - 2| - 4| < \epsilon$$

$$|-(x - 2) - 4| < \epsilon \quad (x - 2 < 0)$$

$$|-x - 2| = |x + 2| = |x - (-2)| < \epsilon$$

Hence, $\delta = \epsilon$.

Hence for $0 < |x - (-2)| < \delta = \epsilon$, you have

$$|x + 2| < \epsilon$$

$$|-(x + 2)| < \epsilon$$

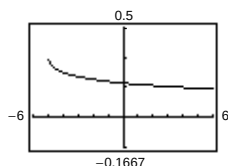
$$|-(x - 2) - 4| < \epsilon$$

$$||x - 2| - 4| < \epsilon \quad (\text{because } x - 2 < 0)$$

$$|f(x) - L| < \epsilon$$

$$39. f(x) = \frac{\sqrt{x+5} - 3}{x - 4}$$

$$\lim_{x \rightarrow 4} f(x) = \frac{1}{6}$$



The domain is $[-5, 4) \cup (4, \infty)$.

The graphing utility does not show the hole at $(4, \frac{1}{6})$.

$$37. \lim_{x \rightarrow 1} (x^2 + 1) = 2$$

Given $\epsilon > 0$:

$$|(x^2 + 1) - 2| < \epsilon$$

$$|x^2 - 1| < \epsilon$$

$$|(x + 1)(x - 1)| < \epsilon$$

$$|x - 1| < \frac{\epsilon}{|x + 1|}$$

If we assume $0 < x < 2$, then $\delta = \epsilon/3$.

Hence for $0 < |x - 1| < \delta = \frac{\epsilon}{3}$, you have

$$|x - 1| < \frac{1}{3}\epsilon < \frac{1}{|x + 1|}\epsilon$$

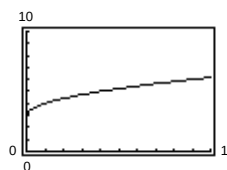
$$|x^2 - 1| < \epsilon$$

$$|(x^2 + 1) - 2| < \epsilon$$

$$|f(x) - 2| < \epsilon$$

$$41. f(x) = \frac{x - 9}{\sqrt{x} - 3}$$

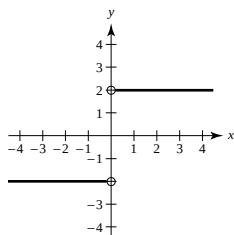
$$\lim_{x \rightarrow 9} f(x) = 6$$



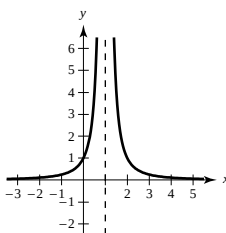
The domain is all $x \geq 0$ except $x = 9$. The graphing utility does not show the hole at $(9, 6)$.

43. $\lim_{x \rightarrow 8} f(x) = 25$ means that the values of f approach 25 as x gets closer and closer to 8.

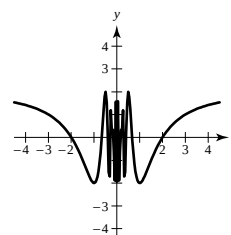
45. (i) The values of f approach different numbers as x approaches c from different sides of c :



(ii) The values of f increase without bound as x approaches c :

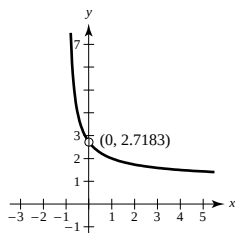


(iii) The values of f oscillate between two fixed numbers as x approaches c :



$$47. f(x) = (1 + x)^{1/x}$$

$$\lim_{x \rightarrow 0} (1 + x)^{1/x} = e \approx 2.71828$$



x	$f(x)$	x	$f(x)$
-0.1	2.867972	0.1	2.593742
-0.01	2.731999	0.01	2.704814
-0.001	2.719642	0.001	2.716942
-0.0001	2.718418	0.0001	2.718146
-0.00001	2.718295	0.00001	2.718268
-0.000001	2.718283	0.000001	2.718280

49. False; $f(x) = (\sin x)/x$ is undefined when $x = 0$.
From Exercise 7, we have

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

51. False; let

$$f(x) = \begin{cases} x^2 - 4x, & x \neq 4 \\ 10, & x = 4 \end{cases}$$

$$f(4) = 10$$

$$\lim_{x \rightarrow 4} f(x) = \lim_{x \rightarrow 4} (x^2 - 4x) = 0 \neq 10$$

53. Answers will vary.

55. If $\lim_{x \rightarrow c} f(x) = L_1$ and $\lim_{x \rightarrow c} f(x) = L_2$, then for every $\epsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \epsilon$ and $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \epsilon$. Let δ equal the smaller of δ_1 and δ_2 . Then for $|x - c| < \delta$, we have
- $$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \epsilon + \epsilon.$$

Therefore, $|L_1 - L_2| < 2\epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

57. $\lim_{x \rightarrow c} [f(x) - L] = 0$ means that for every $\epsilon > 0$ there exists $\delta > 0$ such that if

$$0 < |x - c| < \delta,$$

then

$$|(f(x) - L) - 0| < \epsilon.$$

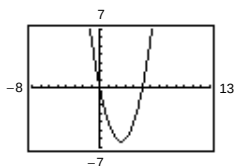
This means the same as $|f(x) - L| < \epsilon$ when

$$0 < |x - c| < \delta.$$

Thus, $\lim_{x \rightarrow c} f(x) = L$.

Section 1.3 Evaluating Limits Analytically

1.

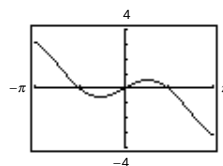


$$h(x) = x^2 - 5x$$

(a) $\lim_{x \rightarrow 5} h(x) = 0$

(b) $\lim_{x \rightarrow -1} h(x) = 6$

3.



$$f(x) = x \cos x$$

(a) $\lim_{x \rightarrow 0} f(x) = 0$

(b) $\lim_{x \rightarrow \pi/3} f(x) \approx 0.524$
 $\left(= \frac{\pi}{6} \right)$

5. $\lim_{x \rightarrow 2} x^4 = 2^4 = 16$

7. $\lim_{x \rightarrow 0} (2x - 1) = 2(0) - 1 = -1$

9. $\lim_{x \rightarrow -3} (x^2 + 3x) = (-3)^2 + 3(-3) = 9 - 9 = 0$

11. $\lim_{x \rightarrow -3} (2x^2 + 4x + 1) = 2(-3)^2 + 4(-3) + 1 = 18 - 12 + 1 = 7$

13. $\lim_{x \rightarrow 2} \frac{1}{x} = \frac{1}{2}$

15. $\lim_{x \rightarrow 1} \frac{x-3}{x^2+4} = \frac{1-3}{1^2+4} = \frac{-2}{5} = -\frac{2}{5}$

17. $\lim_{x \rightarrow 7} \frac{5x}{\sqrt{x+2}} = \frac{5(7)}{\sqrt{7+2}} = \frac{35}{\sqrt{9}} = \frac{35}{3}$

19. $\lim_{x \rightarrow 3} \sqrt{x+1} = \sqrt{3+1} = 2$

$$21. \lim_{x \rightarrow -4} (x + 3)^2 = (-4 + 3)^2 = 1$$

$$25. \begin{aligned} \text{(a)} \quad & \lim_{x \rightarrow 1} f(x) = 4 - 1 = 3 \\ \text{(b)} \quad & \lim_{x \rightarrow 3} g(x) = \sqrt{3 + 1} = 2 \\ \text{(c)} \quad & \lim_{x \rightarrow 1} g(f(x)) = g(3) = 2 \end{aligned}$$

$$29. \lim_{x \rightarrow 2} \cos \frac{\pi x}{3} = \cos \frac{\pi 2}{3} = -\frac{1}{2}$$

$$33. \lim_{x \rightarrow 5\pi/6} \sin x = \sin \frac{5\pi}{6} = \frac{1}{2}$$

$$37. \begin{aligned} \text{(a)} \quad & \lim_{x \rightarrow c} [5g(x)] = 5 \lim_{x \rightarrow c} g(x) = 5(3) = 15 \\ \text{(b)} \quad & \lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) = 2 + 3 = 5 \\ \text{(c)} \quad & \lim_{x \rightarrow c} [f(x)g(x)] = \left[\lim_{x \rightarrow c} f(x) \right] \left[\lim_{x \rightarrow c} g(x) \right] = (2)(3) = 6 \\ \text{(d)} \quad & \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} = \frac{2}{3} \end{aligned}$$

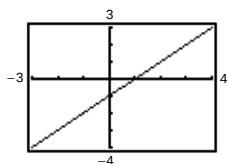
$$41. f(x) = -2x + 1 \text{ and } g(x) = \frac{-2x^2 + x}{x} \text{ agree except at } x = 0.$$

$$\text{(a)} \quad \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} f(x) = 1$$

$$\text{(b)} \quad \lim_{x \rightarrow -1} g(x) = \lim_{x \rightarrow -1} f(x) = 3$$

$$45. f(x) = \frac{x^2 - 1}{x + 1} \text{ and } g(x) = x - 1 \text{ agree except at } x = -1.$$

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} g(x) = -2$$



$$49. \begin{aligned} \lim_{x \rightarrow 5} \frac{x - 5}{x^2 - 25} &= \lim_{x \rightarrow 5} \frac{x - 5}{(x + 5)(x - 5)} \\ &= \lim_{x \rightarrow 5} \frac{1}{x + 5} = \frac{1}{10} \end{aligned}$$

$$23. \text{(a)} \quad \lim_{x \rightarrow 1} f(x) = 5 - 1 = 4$$

$$\text{(b)} \quad \lim_{x \rightarrow 4} g(x) = 4^3 = 64$$

$$\text{(c)} \quad \lim_{x \rightarrow 1} g(f(x)) = g(f(1)) = g(4) = 64$$

$$27. \lim_{x \rightarrow \pi/2} \sin x = \sin \frac{\pi}{2} = 1$$

$$31. \lim_{x \rightarrow 0} \sec 2x = \sec 0 = 1$$

$$35. \lim_{x \rightarrow 3} \tan\left(\frac{\pi x}{4}\right) = \tan \frac{3\pi}{4} = -1$$

$$39. \text{(a)} \quad \lim_{x \rightarrow c} [f(x)]^3 = \left[\lim_{x \rightarrow c} f(x) \right]^3 = (4)^3 = 64$$

$$\text{(b)} \quad \lim_{x \rightarrow c} \sqrt{f(x)} = \sqrt{\lim_{x \rightarrow c} f(x)} = \sqrt{4} = 2$$

$$\text{(c)} \quad \lim_{x \rightarrow c} [3f(x)] = 3 \lim_{x \rightarrow c} f(x) = 3(4) = 12$$

$$\text{(d)} \quad \lim_{x \rightarrow c} [f(x)]^{3/2} = \left[\lim_{x \rightarrow c} f(x) \right]^{3/2} = (4)^{3/2} = 8$$

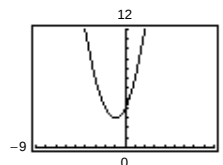
$$43. f(x) = x(x + 1) \text{ and } g(x) = \frac{x^3 - x}{x - 1} \text{ agree except at } x = 1.$$

$$\text{(a)} \quad \lim_{x \rightarrow 1} g(x) = \lim_{x \rightarrow 1} f(x) = 2$$

$$\text{(b)} \quad \lim_{x \rightarrow -1} g(x) = \lim_{x \rightarrow -1} f(x) = 0$$

$$47. f(x) = \frac{x^3 - 8}{x - 2} \text{ and } g(x) = x^2 + 2x + 4 \text{ agree except at } x = 2.$$

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} g(x) = 12$$



$$51. \begin{aligned} \lim_{x \rightarrow -3} \frac{x^2 + x - 6}{x^2 - 9} &= \lim_{x \rightarrow -3} \frac{(x + 3)(x - 2)}{(x + 3)(x - 3)} \\ &= \lim_{x \rightarrow -3} \frac{x - 2}{x - 3} = \frac{-5}{-6} = \frac{5}{6} \end{aligned}$$

$$\begin{aligned}
 53. \lim_{x \rightarrow 0} \frac{\sqrt{x+5} - \sqrt{5}}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{x+5} - \sqrt{5}}{x} \cdot \frac{\sqrt{x+5} + \sqrt{5}}{\sqrt{x+5} + \sqrt{5}} \\
 &= \lim_{x \rightarrow 0} \frac{(x+5) - 5}{x(\sqrt{x+5} + \sqrt{5})} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+5} + \sqrt{5}} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}
 \end{aligned}$$

$$\begin{aligned}
 55. \lim_{x \rightarrow 4} \frac{\sqrt{x+5} - 3}{x-4} &= \lim_{x \rightarrow 4} \frac{\sqrt{x+5} - 3}{x-4} \cdot \frac{\sqrt{x+5} + 3}{\sqrt{x+5} + 3} \\
 &= \lim_{x \rightarrow 4} \frac{(x+5) - 9}{(x-4)(\sqrt{x+5} + 3)} = \lim_{x \rightarrow 4} \frac{1}{\sqrt{x+5} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}
 \end{aligned}$$

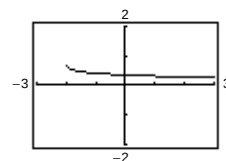
$$57. \lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = \lim_{x \rightarrow 0} \frac{\frac{2 - (2+x)}{2(2+x)}}{x} = \lim_{x \rightarrow 0} \frac{-1}{2(2+x)} = -\frac{1}{4}$$

$$59. \lim_{\Delta x \rightarrow 0} \frac{2(x + \Delta x) - 2x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2x + 2\Delta x - 2x}{\Delta x} = \lim_{\Delta x \rightarrow 0} 2 = 2$$

$$\begin{aligned}
 61. \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - x^2 + 2x - 1}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x - 2) = 2x - 2
 \end{aligned}$$

$$63. \lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} \approx 0.354$$

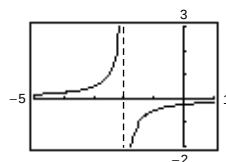
x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.345	?	0.354	0.353	0.349



$$\begin{aligned}
 \text{Analytically, } \lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} \cdot \frac{\sqrt{x+2} + \sqrt{2}}{\sqrt{x+2} + \sqrt{2}} \\
 &= \lim_{x \rightarrow 0} \frac{x + 2 - 2}{x(\sqrt{x+2} + \sqrt{2})} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354
 \end{aligned}$$

$$65. \lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = -\frac{1}{4}$$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.263	-0.251	-0.250	?	-0.250	-0.249	-0.238



$$\text{Analytically, } \lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = \lim_{x \rightarrow 0} \frac{2 - (2+x)}{2(2+x)} \cdot \frac{1}{x} = \lim_{x \rightarrow 0} \frac{-x}{2(2+x)} \cdot \frac{1}{x} = \lim_{x \rightarrow 0} \frac{-1}{2(2+x)} = -\frac{1}{4}$$

$$67. \lim_{x \rightarrow 0} \frac{\sin x}{5x} = \lim_{x \rightarrow 0} \left[\left(\frac{\sin x}{x} \right) \left(\frac{1}{5} \right) \right] = (1) \left(\frac{1}{5} \right) = \frac{1}{5}$$

$$69. \lim_{x \rightarrow 0} \frac{\sin x(1 - \cos x)}{2x^2} = \lim_{x \rightarrow 0} \left[\frac{1}{2} \cdot \frac{\sin x}{x} \cdot \frac{1 - \cos x}{x} \right] \\ = \frac{1}{2}(1)(0) = 0$$

$$71. \lim_{x \rightarrow 0} \frac{\sin^2 x}{x} = \lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \sin x \right] = (1) \sin 0 = 0$$

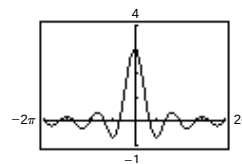
$$73. \lim_{h \rightarrow 0} \frac{(1 - \cos h)^2}{h} = \lim_{h \rightarrow 0} \left[\frac{1 - \cos h}{h} (1 - \cos h) \right] \\ = (0)(0) = 0$$

$$75. \lim_{x \rightarrow \pi/2} \frac{\cos x}{\cot x} = \lim_{x \rightarrow \pi/2} \sin x = 1$$

$$77. \lim_{t \rightarrow 0} \frac{\sin 3t}{2t} = \lim_{t \rightarrow 0} \left(\frac{\sin 3t}{3t} \right) \left(\frac{3}{2} \right) = (1) \left(\frac{3}{2} \right) = \frac{3}{2}$$

$$79. f(t) = \frac{\sin 3t}{t}$$

t	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(t)$	2.96	2.9996	3	?	3	2.9996	2.96

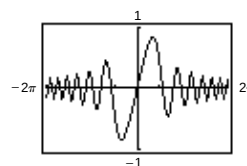


The limit appear to equal 3.

$$\text{Analytically, } \lim_{t \rightarrow 0} \frac{\sin 3t}{t} = \lim_{t \rightarrow 0} 3 \left(\frac{\sin 3t}{3t} \right) = 3(1) = 3.$$

$$81. f(x) = \frac{\sin x^2}{x}$$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.099998	-0.01	-0.001	?	0.001	0.01	0.099998



$$\text{Analytically, } \lim_{x \rightarrow 0} \frac{\sin x^2}{x} = \lim_{x \rightarrow 0} x \left(\frac{\sin x^2}{x^2} \right) = 0(1) = 0.$$

$$83. \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{2(x+h) + 3 - (2x+3)}{h} = \lim_{h \rightarrow 0} \frac{2x+2h+3-2x-3}{h} = \lim_{h \rightarrow 0} \frac{2h}{h} = 2$$

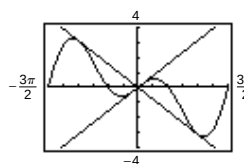
$$85. \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{4}{x+h} - \frac{4}{x}}{h} = \lim_{h \rightarrow 0} \frac{4x - 4(x+h)}{(x+h)xh} = \lim_{h \rightarrow 0} \frac{-4}{(x+h)x} = \frac{-4}{x^2}$$

$$87. \lim_{x \rightarrow 0} (4 - x^2) \leq \lim_{x \rightarrow 0} f(x) \leq \lim_{x \rightarrow 0} (4 + x^2)$$

$$4 \leq \lim_{x \rightarrow 0} f(x) \leq 4$$

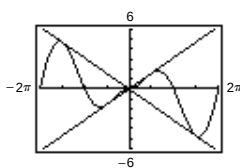
$$\text{Therefore, } \lim_{x \rightarrow 0} f(x) = 4.$$

$$89. f(x) = x \cos x$$



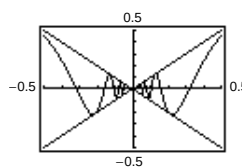
$$\lim_{x \rightarrow 0} (x \cos x) = 0$$

91. $f(x) = |x| \sin x$



$$\lim_{x \rightarrow 0} |x| \sin x = 0$$

93. $f(x) = x \sin \frac{1}{x}$



$$\lim_{x \rightarrow 0} \left(x \sin \frac{1}{x} \right) = 0$$

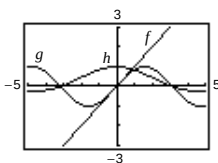
95. We say that two functions f and g agree at all but one point (on an open interval) if $f(x) = g(x)$ for all x in the interval except for $x = c$, where c is in the interval.

97. An indeterminate form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as $0/0$. That is,

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$$

$$\text{for which } \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$$

99. $f(x) = x$, $g(x) = \sin x$, $h(x) = \frac{\sin x}{x}$



When you are “close to” 0 the magnitude of f is approximately equal to the magnitude of g .
Thus, $|g|/|f| \approx 1$ when x is “close to” 0.

101. $s(t) = -16t^2 + 1000$

$$\lim_{t \rightarrow 5} \frac{s(5) - s(t)}{5 - t} = \lim_{t \rightarrow 5} \frac{600 - (-16t^2 + 1000)}{5 - t} = \lim_{t \rightarrow 5} \frac{16(t + 5)(t - 5)}{-(t - 5)} = \lim_{t \rightarrow 5} -16(t + 5) = -160 \text{ ft/sec.}$$

Speed = 160 ft/sec

103. $s(t) = -4.9t^2 + 150$

$$\begin{aligned} \lim_{t \rightarrow 3} \frac{s(3) - s(t)}{3 - t} &= \lim_{t \rightarrow 3} \frac{-4.9(3^2) + 150 - (-4.9t^2 + 150)}{3 - t} = \lim_{t \rightarrow 3} \frac{-4.9(9 - t^2)}{3 - t} \\ &= \lim_{x \rightarrow 3} \frac{-4.9(3 - t)(3 + t)}{3 - t} = \lim_{x \rightarrow 3} -4.9(3 + t) = -29.4 \text{ m/sec} \end{aligned}$$

105. Let $f(x) = 1/x$ and $g(x) = -1/x$. $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 0} g(x)$ do not exist.

$$\lim_{x \rightarrow 0} [f(x) + g(x)] = \lim_{x \rightarrow 0} \left[\frac{1}{x} + \left(-\frac{1}{x} \right) \right] = \lim_{x \rightarrow 0} [0] = 0$$

107. Given $f(x) = b$, show that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - b| < \epsilon$ whenever $|x - c| < \delta$. Since $|f(x) - b| = |b - b| = 0 < \epsilon$ for any $\epsilon > 0$, then any value of $\delta > 0$ will work.

109. If $b = 0$, then the property is true because both sides are equal to 0. If $b \neq 0$, let $\epsilon > 0$ be given. Since $\lim_{x \rightarrow c} f(x) = L$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon/|b|$ whenever $0 < |x - c| < \delta$. Hence, wherever $0 < |x - c| < \delta$,

we have

$$|b||f(x) - L| < \epsilon \quad \text{or} \quad |bf(x) - bL| < \epsilon$$

which implies that $\lim_{x \rightarrow c} [bf(x)] = bL$.

111. $-M|f(x)| \leq f(x)g(x) \leq M|f(x)|$

$$\lim_{x \rightarrow c} (-M|f(x)|) \leq \lim_{x \rightarrow c} f(x)g(x) \leq \lim_{x \rightarrow c} (M|f(x)|)$$

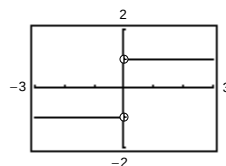
$$-M(0) \leq \lim_{x \rightarrow c} f(x)g(x) \leq M(0)$$

$$0 \leq \lim_{x \rightarrow c} f(x)g(x) \leq 0$$

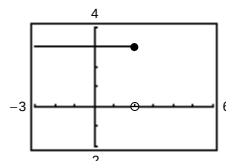
Therefore, $\lim_{x \rightarrow c} f(x)g(x) = 0$.

115. True.

113. False. As x approaches 0 from the left, $\frac{|x|}{x} = -1$.



117. False. The limit does not exist.



119. Let

$$f(x) = \begin{cases} 4, & \text{if } x \geq 0 \\ -4, & \text{if } x < 0 \end{cases}$$

$$\lim_{x \rightarrow 0} |f(x)| = \lim_{x \rightarrow 0} 4 = 4.$$

$\lim_{x \rightarrow 0} f(x)$ does not exist since for $x < 0$, $f(x) = -4$ and for $x \geq 0$, $f(x) = 4$.

121. $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$$

$\lim_{x \rightarrow 0} f(x)$ does not exist.

No matter how “close to” 0 x is, there are still an infinite number of rational and irrational numbers so that $\lim_{x \rightarrow 0} f(x)$ does not exist.

$$\lim_{x \rightarrow 0} g(x) = 0.$$

When x is “close to” 0, both parts of the function are “close to” 0.

123. (a) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \cdot \frac{1 + \cos x}{1 + \cos x}$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x^2(1 + \cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} \cdot \frac{1}{1 + \cos x}$$

$$= (1) \left(\frac{1}{2} \right) = \frac{1}{2}$$

(b) Thus, $\frac{1 - \cos x}{x^2} \approx \frac{1}{2} \Rightarrow 1 - \cos x \approx \frac{1}{2}x^2$

$$\Rightarrow \cos x \approx 1 - \frac{1}{2}x^2 \text{ for } x \approx 0.$$

(c) $\cos(0.1) \approx 1 - \frac{1}{2}(0.1)^2 = 0.995$

(d) $\cos(0.1) \approx 0.9950$, which agrees with part (c).

Section 1.4 Continuity and One-Sided Limits

1. (a) $\lim_{x \rightarrow 3^+} f(x) = 1$

(b) $\lim_{x \rightarrow 3^-} f(x) = 1$

(c) $\lim_{x \rightarrow 3} f(x) = 1$

The function is continuous at $x = 3$.

3. (a) $\lim_{x \rightarrow 3^+} f(x) = 0$

(b) $\lim_{x \rightarrow 3^-} f(x) = 0$

(c) $\lim_{x \rightarrow 3} f(x) = 0$

The function is NOT continuous at $x = 3$.

5. (a) $\lim_{x \rightarrow 4^+} f(x) = 2$

(b) $\lim_{x \rightarrow 4^-} f(x) = -2$

(c) $\lim_{x \rightarrow 4} f(x)$ does not exist

The function is NOT continuous at $x = 4$.

7. $\lim_{x \rightarrow 5^+} \frac{x-5}{x^2-25} = \lim_{x \rightarrow 5^+} \frac{1}{x+5} = \frac{1}{10}$

9. $\lim_{x \rightarrow -3^-} \frac{x}{\sqrt{x^2-9}}$ does not exist because $\frac{x}{\sqrt{x^2-9}}$ grows without bound as $x \rightarrow -3^-$.

11. $\lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1$.

13.
$$\begin{aligned} \lim_{\Delta x \rightarrow 0^-} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} &= \lim_{\Delta x \rightarrow 0^-} \frac{x - (x+\Delta x)}{x(x+\Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} \frac{-\Delta x}{x(x+\Delta x)} \cdot \frac{1}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0^-} \frac{-1}{x(x+\Delta x)} \\ &= \frac{-1}{x(x+0)} = -\frac{1}{x^2} \end{aligned}$$

15. $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} \frac{x+2}{2} = \frac{5}{2}$

17. $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x+1) = 2$

$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^3+1) = 2$

$\lim_{x \rightarrow 1} f(x) = 2$

19. $\lim_{x \rightarrow \pi} \cot x$ does not exist since

$\lim_{x \rightarrow \pi^+} \cot x$ and $\lim_{x \rightarrow \pi^-} \cot x$ do not exist.

21. $\lim_{x \rightarrow 4^-} (3\llbracket x \rrbracket - 5) = 3(3) - 5 = 4$

$(\llbracket x \rrbracket = 3 \text{ for } 3 < x < 4)$

23. $\lim_{x \rightarrow 3} (2 - \llbracket -x \rrbracket)$ does not exist

because

$\lim_{x \rightarrow 3^-} (2 - \llbracket -x \rrbracket) = 2 - (-3) = 5$

and

$\lim_{x \rightarrow 3^+} (2 - \llbracket -x \rrbracket) = 2 - (-4) = 6$.

25. $f(x) = \frac{1}{x^2-4}$

has discontinuities at $x = -2$ and $x = 2$ since $f(-2)$ and $f(2)$ are not defined.

27. $f(x) = \frac{\llbracket x \rrbracket}{2} + x$

has discontinuities at each integer k since $\lim_{x \rightarrow k^-} f(x) \neq \lim_{x \rightarrow k^+} f(x)$.

29. $g(x) = \sqrt{25-x^2}$ is continuous on $[-5, 5]$.

31. $\lim_{x \rightarrow 0^-} f(x) = 3 = \lim_{x \rightarrow 0^+} f(x)$.
 f is continuous on $[-1, 4]$.

33. $f(x) = x^2 - 2x + 1$ is continuous for all real x .

35. $f(x) = 3x - \cos x$ is continuous for all real x .

37. $f(x) = \frac{x}{x^2 - x}$ is not continuous at $x = 0, 1$. Since

$\frac{x}{x^2 - x} = \frac{1}{x - 1}$ for $x \neq 0, x = 0$ is a removable discontinuity, whereas $x = 1$ is a nonremovable discontinuity.

39. $f(x) = \frac{x}{x^2 + 1}$ is continuous for all real x .

41. $f(x) = \frac{x + 2}{(x + 2)(x - 5)}$

has a nonremovable discontinuity at $x = 5$ since $\lim_{x \rightarrow 5} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ since

$$\lim_{x \rightarrow -2} f(x) = \lim_{x \rightarrow -2} \frac{1}{x - 5} = -\frac{1}{7}.$$

43. $f(x) = \frac{|x + 2|}{x + 2}$ has a nonremovable discontinuity at $x = -2$ since $\lim_{x \rightarrow -2} f(x)$ does not exist.

45. $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1$

2. $\left. \begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} x = 1 \\ \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} x^2 = 1 \end{aligned} \right\} \lim_{x \rightarrow 1} f(x) = 1$

3. $f(1) = \lim_{x \rightarrow 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x .

47. $f(x) = \begin{cases} \frac{x}{2} + 1, & x \leq 2 \\ 3 - x, & x > 2 \end{cases}$ has a **possible** discontinuity at $x = 2$.

1. $f(2) = \frac{2}{2} + 1 = 2$

2. $\left. \begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} \left(\frac{x}{2} + 1 \right) = 2 \\ \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (3 - x) = 1 \end{aligned} \right\} \lim_{x \rightarrow 2} f(x) \text{ does not exist.}$

Therefore, f has a nonremovable discontinuity at $x = 2$.

49. $f(x) = \begin{cases} \tan \frac{\pi x}{4}, & |x| < 1 \\ x, & |x| \geq 1 \end{cases} = \begin{cases} \tan \frac{\pi x}{4}, & -1 < x < 1 \\ x, & x \leq -1 \text{ or } x \geq 1 \end{cases}$ has **possible** discontinuities at $x = -1, x = 1$.

1. $f(-1) = -1$ $f(1) = 1$

2. $\lim_{x \rightarrow -1} f(x) = -1$ $\lim_{x \rightarrow 1} f(x) = 1$

3. $f(-1) = \lim_{x \rightarrow -1} f(x)$ $f(1) = \lim_{x \rightarrow 1} f(x)$

f is continuous at $x = \pm 1$, therefore, f is continuous for all real x .

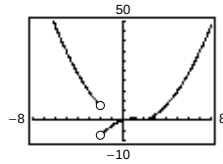
51. $f(x) = \csc 2x$ has nonremovable discontinuities at integer multiples of $\pi/2$.

53. $f(x) = \llbracket x - 1 \rrbracket$ has nonremovable discontinuities at each integer k .

55. $\lim_{x \rightarrow 0^+} f(x) = 0$

$\lim_{x \rightarrow 0^-} f(x) = 0$

f is not continuous at $x = -2$.



57. $f(2) = 8$

Find a so that $\lim_{x \rightarrow 2^+} ax^2 = 8 \Rightarrow a = \frac{8}{2^2} = 2$.

59. Find a and b such that $\lim_{x \rightarrow -1^+} (ax + b) = -a + b = 2$ and $\lim_{x \rightarrow 3^-} (ax + b) = 3a + b = -2$.

$$a - b = -2$$

$$\begin{array}{r} (+) 3a + b = -2 \\ \hline 4a = -4 \end{array}$$

$$4a = -4$$

$$a = -1$$

$$b = 2 + (-1) = 1$$

$$f(x) = \begin{cases} 2, & x \leq -1 \\ -x + 1, & -1 < x < 3 \\ -2, & x \geq 3 \end{cases}$$

61. $f(g(x)) = (x - 1)^2$

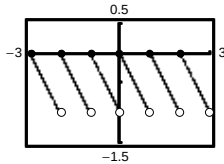
Continuous for all real x .

63. $f(g(x)) = \frac{1}{(x^2 + 5) - 6} = \frac{1}{x^2 - 1}$

Nonremovable discontinuities at $x = \pm 1$

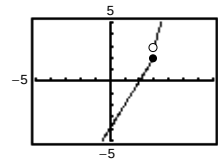
65. $y = \llbracket x \rrbracket - x$

Nonremovable discontinuity at each integer



67. $f(x) = \begin{cases} 2x - 4, & x \leq 3 \\ x^2 - 2x, & x > 3 \end{cases}$

Nonremovable discontinuity at $x = 3$



69. $f(x) = \frac{x}{x^2 + 1}$

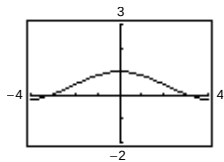
Continuous on $(-\infty, \infty)$

71. $f(x) = \sec \frac{\pi x}{4}$

Continuous on:

$\dots, (-6, -2), (-2, 2), (2, 6), (6, 10), \dots$

73. $f(x) = \frac{\sin x}{x}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Since $f(0)$ is not defined, we know that f has a discontinuity at $x = 0$. This discontinuity is removable so it does not show up on the graph.

75. $f(x) = \frac{1}{16}x^4 - x^3 + 3$ is continuous on $[1, 2]$.

$f(1) = \frac{33}{16}$ and $f(2) = -4$. By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 1 and 2.

- 77.
- $f(x) = x^2 - 2 - \cos x$
- is continuous on
- $[0, \pi]$
- .

$f(0) = -3$ and $f(\pi) = \pi^2 - 1 > 0$. By the Intermediate Value Theorem, $f(c) = 0$ for the least one value of c between 0 and π .

- 81.
- $g(t) = 2 \cos t - 3t$

g is continuous on $[0, 1]$.

$$g(0) = 2 > 0 \text{ and } g(1) \approx -1.9 < 0.$$

By the Intermediate Value Theorem, $g(t) = 0$ for at least one value c between 0 and 1. Using a graphing utility, we find that $t \approx 0.5636$.

- 85.
- $f(x) = x^3 - x^2 + x - 2$

f is continuous on $[0, 3]$.

$$f(0) = -2 \text{ and } f(3) = 19$$

$$-2 < 4 < 19$$

The Intermediate Value Theorem applies.

$$x^3 - x^2 + x - 2 = 4$$

$$x^3 - x^2 + x - 6 = 0$$

$$(x - 2)(x^2 + x + 3) = 0$$

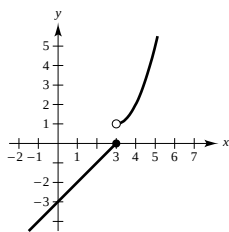
$$x = 2$$

$(x^2 + x + 3)$ has no real solution.)

$$c = 2$$

Thus, $f(2) = 4$.

- 89.



The function is not continuous at $x = 3$ because

$$\lim_{x \rightarrow 3^+} f(x) = 1 \neq 0 = \lim_{x \rightarrow 3^-} f(x).$$

- 79.
- $f(x) = x^3 + x - 1$

$f(x)$ is continuous on $[0, 1]$.

$$f(0) = -1 \text{ and } f(1) = 1$$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1. Using a graphing utility, we find that $x \approx 0.6823$.

- 83.
- $f(x) = x^2 + x - 1$

f is continuous on $[0, 5]$.

$$f(0) = -1 \text{ and } f(5) = 29$$

$$-1 < 11 < 29$$

The Intermediate Value Theorem applies.

$$x^2 + x - 1 = 11$$

$$x^2 + x - 12 = 0$$

$$(x + 4)(x - 3) = 0$$

$$x = -4 \text{ or } x = 3$$

$$c = 3 \text{ (} x = -4 \text{ is not in the interval.)}$$

Thus, $f(3) = 11$.

87. (a) The limit does not exist at
- $x = c$
- .

(b) The function is not defined at $x = c$.

(c) The limit exists at $x = c$, but it is not equal to the value of the function at $x = c$.

(d) The limit does not exist at $x = c$.

91. The functions agree for integer values of
- x
- :

$$\left. \begin{aligned} g(x) &= 3 - \lfloor -x \rfloor = 3 - (-x) = 3 + x \\ f(x) &= 3 + \lfloor x \rfloor = 3 + x \end{aligned} \right\} \text{ for } x \text{ an integer}$$

However, for non-integer values of x , the functions differ by 1.

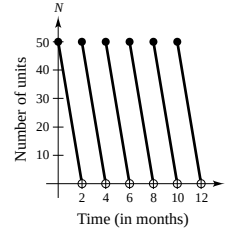
$$f(x) = 3 + \lfloor x \rfloor = g(x) - 1 = 2 - \lfloor -x \rfloor.$$

For example, $f(\frac{1}{2}) = 3 + 0 = 3$, $g(\frac{1}{2}) = 3 - (-1) = 4$.

93. $N(t) = 25\left(2\left\lceil\frac{t+2}{2}\right\rceil - t\right)$

t	0	1	1.8	2	3	3.8
$N(t)$	50	25	5	50	25	5

Discontinuous at every positive even integer. The company replenishes its inventory every two months.



95. Let $V = \frac{4}{3}\pi r^3$ be the volume of a sphere of radius r .

$$V(1) = \frac{4}{3}\pi \approx 4.19$$

$$V(5) = \frac{4}{3}\pi(5^3) \approx 523.6$$

Since $4.19 < 275 < 523.6$, the Intermediate Value Theorem implies that there is at least one value r between 1 and 5 such that $V(r) = 275$. (In fact, $r \approx 4.0341$.)

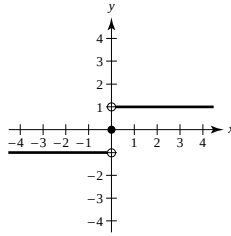
97. Let c be any real number. Then $\lim_{x \rightarrow c} f(x)$ does not exist since there are both rational and irrational numbers arbitrarily close to c . Therefore, f is not continuous at c .

99. $\text{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$

(a) $\lim_{x \rightarrow 0^-} \text{sgn}(x) = -1$

(b) $\lim_{x \rightarrow 0^+} \text{sgn}(x) = 1$

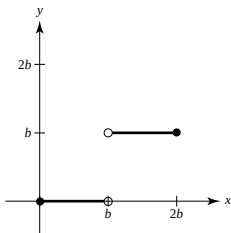
(c) $\lim_{x \rightarrow 0} \text{sgn}(x)$ does not exist.



101. True; if $f(x) = g(x)$, $x \neq c$, then $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x)$ and at least one of these limits (if they exist) does not equal the corresponding function at $x = c$.

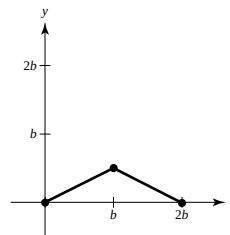
103. False; $f(1)$ is not defined and $\lim_{x \rightarrow 1} f(x)$ does not exist.

105. (a) $f(x) = \begin{cases} 0 & 0 \leq x < b \\ b & b < x \leq 2b \end{cases}$



NOT continuous at $x = b$.

(b) $g(x) = \begin{cases} \frac{x}{2} & 0 \leq x \leq b \\ b - \frac{x}{2} & b < x \leq 2b \end{cases}$



Continuous on $[0, 2b]$.

107. $f(x) = \frac{\sqrt{x+c^2}-c}{x}, c > 0$

Domain: $x + c^2 \geq 0 \Rightarrow x \geq -c^2$ and $x \neq 0, [-c^2, 0) \cup (0, \infty)$

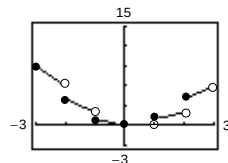
$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{x+c^2}-c}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{x+c^2}-c}{x} \cdot \frac{\sqrt{x+c^2}+c}{\sqrt{x+c^2}+c} \\ &= \lim_{x \rightarrow 0} \frac{(x+c^2)-c^2}{x[\sqrt{x+c^2}+c]} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+c^2}+c} = \frac{1}{2c}\end{aligned}$$

Define $f(0) = 1/(2c)$ to make f continuous at $x = 0$.

109. $h(x) = x\llbracket x \rrbracket$

h has nonremovable discontinuities at

$$x = \pm 1, \pm 2, \pm 3, \dots$$



Section 1.5 Infinite Limits

1. $\lim_{x \rightarrow -2^+} 2 \left| \frac{x}{x^2 - 4} \right| = \infty$

3. $\lim_{x \rightarrow -2^+} \tan \frac{\pi x}{4} = -\infty$

$\lim_{x \rightarrow -2^-} 2 \left| \frac{x}{x^2 - 4} \right| = \infty$

$\lim_{x \rightarrow -2^-} \tan \frac{\pi x}{4} = \infty$

5. $f(x) = \frac{1}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	0.308	1.639	16.64	166.6	-166.7	-16.69	-1.695	-0.364

$\lim_{x \rightarrow -3^-} f(x) = \infty$

$\lim_{x \rightarrow -3^+} f(x) = -\infty$

7. $f(x) = \frac{x^2}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	3.769	15.75	150.8	1501	-1499	-149.3	-14.25	-2.273

$\lim_{x \rightarrow -3^-} f(x) = \infty$

$\lim_{x \rightarrow -3^+} f(x) = -\infty$

$$9. \lim_{x \rightarrow 0^+} \frac{1}{x^2} = \infty = \lim_{x \rightarrow 0^-} \frac{1}{x^2}$$

Therefore, $x = 0$ is a vertical asymptote.

$$13. \lim_{x \rightarrow -2^-} \frac{x^2}{x^2 - 4} = \infty \text{ and } \lim_{x \rightarrow -2^+} \frac{x^2}{x^2 - 4} = -\infty$$

Therefore, $x = -2$ is a vertical asymptote.

$$\lim_{x \rightarrow 2^-} \frac{x^2}{x^2 - 4} = -\infty \text{ and } \lim_{x \rightarrow 2^+} \frac{x^2}{x^2 - 4} = \infty$$

Therefore, $x = 2$ is a vertical asymptote.

$$17. f(x) = \tan 2x = \frac{\sin 2x}{\cos 2x} \text{ has vertical asymptotes at}$$

$$x = \frac{(2n+1)\pi}{4} = \frac{\pi}{4} + \frac{n\pi}{2}, n \text{ any integer.}$$

$$21. \lim_{x \rightarrow -2^+} \frac{x}{(x+2)(x-1)} = \infty$$

$$\lim_{x \rightarrow -2^-} \frac{x}{(x+2)(x-1)} = -\infty$$

Therefore, $x = -2$ is a vertical asymptote.

$$\lim_{x \rightarrow 1^+} \frac{x}{(x+2)(x-1)} = \infty$$

$$\lim_{x \rightarrow 1^-} \frac{x}{(x+2)(x-1)} = -\infty$$

Therefore, $x = 1$ is a vertical asymptote.

$$25. f(x) = \frac{(x-5)(x+3)}{(x-5)(x^2+1)} = \frac{x+3}{x^2+1}, x \neq 5$$

No vertical asymptotes. The graph has a hole at $x = 5$.

$$11. \lim_{x \rightarrow 2^+} \frac{x^2 - 2}{(x-2)(x+1)} = \infty$$

$$\lim_{x \rightarrow 2^-} \frac{x^2 - 2}{(x-2)(x+1)} = -\infty$$

Therefore, $x = 2$ is a vertical asymptote.

$$\lim_{x \rightarrow -1^+} \frac{x^2 - 2}{(x-2)(x+1)} = \infty$$

$$\lim_{x \rightarrow -1^-} \frac{x^2 - 2}{(x-2)(x+1)} = -\infty$$

Therefore, $x = -1$ is a vertical asymptote.

15. No vertical asymptote since the denominator is never zero.

$$19. \lim_{t \rightarrow 0^+} \left(1 - \frac{4}{t^2}\right) = -\infty = \lim_{t \rightarrow 0^-} \left(1 - \frac{4}{t^2}\right)$$

Therefore, $t = 0$ is a vertical asymptote.

$$23. f(x) = \frac{x^3 + 1}{x + 1} = \frac{(x+1)(x^2 - x + 1)}{x + 1}$$

has no vertical asymptote since

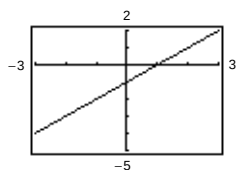
$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} (x^2 - x + 1) = 3$$

$$27. s(t) = \frac{t}{\sin t} \text{ has vertical asymptotes at } t = n\pi, n$$

a nonzero integer. There is no vertical asymptote at $t = 0$ since

$$\lim_{t \rightarrow 0} \frac{t}{\sin t} = 1.$$

$$29. \lim_{x \rightarrow -1} \frac{x^2 - 1}{x + 1} = \lim_{x \rightarrow -1} (x - 1) = -2$$



Removable discontinuity at $x = -1$

$$33. \lim_{x \rightarrow 2^+} \frac{x - 3}{x - 2} = -\infty$$

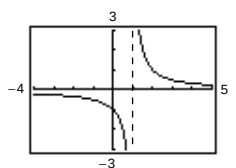
$$37. \lim_{x \rightarrow -3} \frac{x^2 + 2x - 3}{x^2 + x - 6} = \lim_{x \rightarrow -3} \frac{x - 1}{x - 2} = \frac{4}{5}$$

$$41. \lim_{x \rightarrow 0^-} \left(1 + \frac{1}{x}\right) = -\infty$$

$$45. \lim_{x \rightarrow \pi} \frac{\sqrt{x}}{\csc x} = \lim_{x \rightarrow \pi} (\sqrt{x} \sin x) = 0$$

$$49. f(x) = \frac{x^2 + x + 1}{x^3 - 1}$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{1}{x - 1} = \infty$$

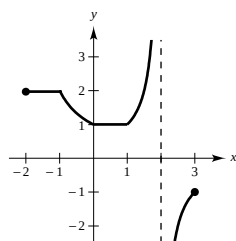


53. A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an infinite limit. ∞ is not a number. Rather, the symbol

$$\lim_{x \rightarrow c} f(x) = \infty$$

says how the limit fails to exist.

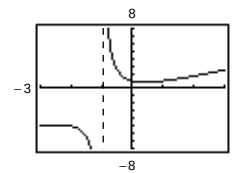
57.



$$31. \lim_{x \rightarrow -1^+} \frac{x^2 + 1}{x + 1} = \infty$$

$$\lim_{x \rightarrow -1^-} \frac{x^2 + 1}{x + 1} = -\infty$$

Vertical asymptote at $x = -1$



$$35. \lim_{x \rightarrow 3^+} \frac{x^2}{(x - 3)(x + 3)} = \infty$$

$$39. \lim_{x \rightarrow 1} \frac{x^2 - x}{(x^2 + 1)(x - 1)} = \lim_{x \rightarrow 1} \frac{x}{x^2 + 1} = \frac{1}{2}$$

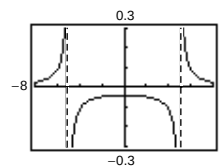
$$43. \lim_{x \rightarrow 0^+} \frac{2}{\sin x} = \infty$$

$$47. \lim_{x \rightarrow (1/2)^-} x \sec(\pi x) = \infty \text{ and } \lim_{x \rightarrow (1/2)^+} x \sec(\pi x) = -\infty.$$

Therefore, $\lim_{x \rightarrow (1/2)} x \sec(\pi x)$ does not exist.

$$51. f(x) = \frac{1}{x^2 - 25}$$

$$\lim_{x \rightarrow 5^-} f(x) = -\infty$$



$$55. \text{ One answer is } f(x) = \frac{x - 3}{(x - 6)(x + 2)} = \frac{x - 3}{x^2 - 4x - 12}.$$

$$59. S = \frac{k}{1 - r}, 0 < |r| < 1. \text{ Assume } k \neq 0.$$

$$\lim_{r \rightarrow 1} S = \lim_{r \rightarrow 1} \frac{k}{1 - r} = \infty \quad (\text{or } -\infty \text{ if } k < 0)$$

61. $C = \frac{528x}{100 - x}, 0 \leq x < 100$

(a) $C(25) = \$176$ million

(b) $C(50) = \$528$ million

(c) $C(75) = \$1584$ million

(d) $\lim_{x \rightarrow 100^-} \frac{528}{100 - x} = \infty$ Thus, it is not possible.

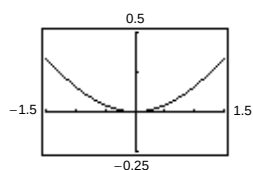
63. (a) $r = \frac{2(7)}{\sqrt{625 - 49}} = \frac{7}{12}$ ft/sec

(b) $r = \frac{2(15)}{\sqrt{625 - 225}} = \frac{3}{2}$ ft/sec

(c) $\lim_{x \rightarrow 25^-} \frac{2x}{\sqrt{625 - x^2}} = \infty$

65. (a)

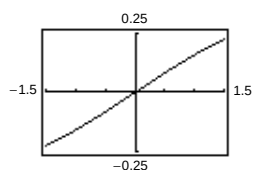
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0411	0.0067	0.0017	≈ 0	≈ 0	≈ 0



$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x} = 0$$

(b)

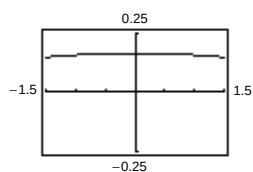
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0823	0.0333	0.0167	0.0017	≈ 0	≈ 0



$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x^2} = 0$$

(c)

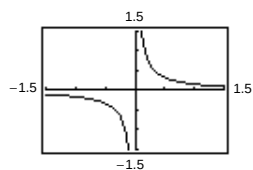
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.1646	0.1663	0.1666	0.1667	0.1667	0.1667



$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x^3} = 0.1167 \text{ (1/6)}$$

(d)

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.3292	0.8317	1.6658	16.67	166.7	1667.0



$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x^4} = \infty$$

For $n \geq 3$, $\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x^n} = \infty$.

67. (a) Because the circumference of the motor is half that of the saw arbor, the saw makes $1700/2 = 850$ revolutions per minute.
- (c) $2(20 \cot \phi) + 2(10 \cot \phi)$: straight sections.
The angle subtended in each circle is

$$2\pi - \left(2\left(\frac{\pi}{2} - \phi\right)\right) = \pi + 2\phi.$$

Thus, the length of the belt around the pulleys is

$$20(\pi + 2\phi) + 10(\pi + 2\phi) = 30(\pi + 2\phi).$$

$$\text{Total length} = 60 \cot \phi + 30(\pi + 2\phi)$$

$$\text{Domain: } \left(0, \frac{\pi}{2}\right)$$

69. False; for instance, let

$$f(x) = \frac{x^2 - 1}{x - 1} \text{ or}$$

$$g(x) = \frac{x}{x^2 + 1}.$$

73. Given $\lim_{x \rightarrow c} f(x) = \infty$ and $\lim_{x \rightarrow c} g(x) = L$:

(2) Product:

If $L > 0$, then for $\epsilon = L/2 > 0$ there exists $\delta_1 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_1$. Thus, $L/2 < g(x) < 3L/2$. Since $\lim_{x \rightarrow c} f(x) = \infty$ then for $M > 0$, there exists $\delta_2 > 0$ such that $f(x) > M(2/L)$ whenever $|x - c| < \delta_2$. Let δ be the smaller of δ_1 and δ_2 . Then for $0 < |x - c| < \delta$, we have $f(x)g(x) > M(2/L)(L/2) = M$. Therefore $\lim_{x \rightarrow c} f(x)g(x) = \infty$. The proof is similar for $L < 0$.

(3) Quotient: Let $\epsilon > 0$ be given.

There exists $\delta_1 > 0$ such that $f(x) > 3L/2\epsilon$ whenever $0 < |x - c| < \delta_1$ and there exists $\delta_2 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_2$. This inequality gives us $L/2 < g(x) < 3L/2$. Let δ be the smaller of δ_1 and δ_2 . Then for $0 < |x - c| < \delta$, we have

$$\left| \frac{g(x)}{f(x)} \right| < \frac{3L/2}{3L/2\epsilon} = \epsilon.$$

$$\text{Therefore, } \lim_{x \rightarrow c} \frac{g(x)}{f(x)} = 0.$$

75. Given $\lim_{x \rightarrow c} \frac{1}{f(x)} = 0$.

Suppose $\lim_{x \rightarrow c} f(x)$ exists and equals L . Then,

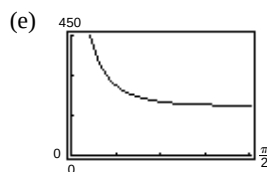
$$\lim_{x \rightarrow c} \frac{1}{f(x)} = \frac{\lim_{x \rightarrow c} 1}{\lim_{x \rightarrow c} f(x)} = \frac{1}{L} = 0.$$

This is not possible. Thus, $\lim_{x \rightarrow c} f(x)$ does not exist.

- (b) The direction of rotation is reversed.

(d)

ϕ	0.3	0.6	0.9	1.2	1.5
L	306.2	217.9	195.9	189.6	188.5



(f) $\lim_{\phi \rightarrow (\pi/2)^-} L = 60\pi \approx 188.5$

(All the belts are around pulleys.)

(g) $\lim_{\phi \rightarrow 0^+} L = \infty$

71. False; let

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 3, & x = 0. \end{cases}$$

The graph of f has a vertical asymptote at $x = 0$, but $f(0) = 3$.

CHAPTER 1

Limits and Their Properties

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CHAPTER 1

Limits and Their Properties

Section 1.1 A Preview of Calculus

Solutions to Even-Numbered Exercises

2. Calculus: velocity is not constant

$$\text{Distance} \approx (20 \text{ ft/sec})(15 \text{ seconds}) = 300 \text{ feet}$$

4. Precalculus: rate of change = slope = 0.08

6. Precalculus: Area = $\pi(\sqrt{2})^2$
= 2π

8. Precalculus: Volume = $\pi(3)^2 6 = 54\pi$

10. (a) Area $\approx 5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} \approx 10.417$

$$\text{Area} \approx \frac{1}{2} \left(5 + \frac{5}{1.5} + \frac{5}{2} + \frac{5}{2.5} + \frac{5}{3} + \frac{5}{3.5} + \frac{5}{4} + \frac{5}{4.5} \right) \approx 9.145$$

(b) You could improve the approximation by using more rectangles.

Section 1.2 Finding Limits Graphically and Numerically

2.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2-4} \approx 0.25 \quad (\text{Actual limit is } \frac{1}{4}.)$$

4.

x	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9
$f(x)$	-0.2485	-0.2498	-0.2500	-0.2500	-0.2502	-0.2516

$$\lim_{x \rightarrow -3} \frac{\sqrt{1-x}-2}{x+3} \approx -0.25 \quad (\text{Actual limit is } -\frac{1}{4}.)$$

6.

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$	0.0408	0.0401	0.0400	0.0400	0.0399	0.0392

$$\lim_{x \rightarrow 4} \frac{[x/(x+1)] - (4/5)}{x-4} \approx 0.04 \quad (\text{Actual limit is } \frac{1}{25}.)$$

8.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.0500	0.0050	0.0005	-0.0005	-0.0050	-0.0500

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} \approx 0.0000 \quad (\text{Actual limit is } 0.) \quad (\text{Make sure you use radian mode.})$$

10. $\lim_{x \rightarrow 1} (x^2 + 2) = 3$

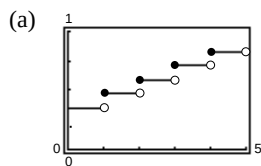
12. $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (x^2 + 2) = 3$

14. $\lim_{x \rightarrow 3} \frac{1}{x - 3}$ does not exist since the function increases and decreases without bound as x approaches 3.

16. $\lim_{x \rightarrow 0} \sec x = 1$

18. $\lim_{x \rightarrow 1} \sin(\pi x) = 0$

20. $C(t) = 0.35 - 0.12\lfloor -(t - 1) \rfloor$



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
$C(t)$	0.59	0.71	0.71	0.71	0.71	0.71	0.71

$$\lim_{t \rightarrow 3.5} C(t) = 0.71$$

(c)

t	3	2.5	2.9	3	3.1	3.5	4
$C(t)$	0.47	0.59	0.59	0.59	0.71	0.71	0.71

$\lim_{t \rightarrow 3.5} C(t)$ does not exist. The values of C jump from 0.59 to 0.71 at $t = 3$.

22. You need to find δ such that $0 < |x - 2| < \delta$ implies $|f(x) - 3| = |x^2 - 1 - 3| = |x^2 - 4| < 0.2$. That is,

$$\begin{aligned} -0.2 &< x^2 - 4 < 0.2 \\ 4 - 0.2 &< x^2 < 4 + 0.2 \\ 3.8 &< x^2 < 4.2 \\ \sqrt{3.8} &< x < \sqrt{4.2} \\ \sqrt{3.8} - 2 &< x - 2 < \sqrt{4.2} - 2 \end{aligned}$$

So take $\delta = \sqrt{4.2} - 2 \approx 0.0494$.

Then $0 < |x - 2| < \delta$ implies

$$\begin{aligned} -(\sqrt{4.2} - 2) &< x - 2 < \sqrt{4.2} - 2 \\ \sqrt{3.8} - 2 &< x - 2 < \sqrt{4.2} - 2. \end{aligned}$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 3| = |x^2 - 4| < \epsilon = 0.2.$$

24. $\lim_{x \rightarrow 4} \left(4 - \frac{x}{2}\right) = 2$

$$\left| \left(4 - \frac{x}{2}\right) - 2 \right| < 0.01$$

$$\left| 2 - \frac{x}{2} \right| < 0.01$$

$$\left| -\frac{1}{2}(x - 4) \right| < 0.01$$

$$0 < |x - 4| < 0.02 = \delta$$

Hence, if $0 < |x - 4| < \delta = 0.02$, you have

$$\left| -\frac{1}{2}(x - 4) \right| < 0.01$$

$$\left| 2 - \frac{x}{2} \right| < 0.01$$

$$\left| \left(4 - \frac{x}{2}\right) - 2 \right| < 0.01$$

$$|f(x) - L| < 0.01$$

26. $\lim_{x \rightarrow 5} (x^2 + 4) = 29$

$$|(x^2 + 4) - 29| < 0.01$$

$$|x^2 - 25| < 0.01$$

$$|(x + 5)(x - 5)| < 0.01$$

$$|x - 5| < \frac{0.01}{|x + 5|}$$

If we assume $4 < x < 6$, then $\delta = 0.01/11 \approx 0.0009$.

Hence, if $0 < |x - 5| < \delta = \frac{0.01}{11}$, you have

$$|x - 5| < \frac{0.01}{11} < \frac{1}{|x + 5|}(0.01)$$

$$|x - 5||x + 5| < 0.01$$

$$|x^2 - 25| < 0.01$$

$$|(x^2 + 4) - 29| < 0.01$$

$$|f(x) - L| < 0.01$$

30. $\lim_{x \rightarrow 1} (\frac{2}{3}x + 9) = \frac{2}{3}(1) + 9 = \frac{29}{3}$

Given $\epsilon > 0$:

$$\left| \left(\frac{2}{3}x + 9 \right) - \frac{29}{3} \right| < \epsilon$$

$$\left| \frac{2}{3}x - \frac{2}{3} \right| < \epsilon$$

$$\frac{2}{3}|x - 1| < \epsilon$$

$$|x - 1| < \frac{3}{2}\epsilon$$

Hence, let $\delta = (3/2)\epsilon$.

Hence, if $0 < |x - 1| < \delta = \frac{3}{2}\epsilon$, you have

$$|x - 1| < \frac{3}{2}\epsilon$$

$$\left| \frac{2}{3}x - \frac{2}{3} \right| < \epsilon$$

$$\left| \left(\frac{2}{3}x + 9 \right) - \frac{29}{3} \right| < \epsilon$$

$$|f(x) - L| < \epsilon$$

34. $\lim_{x \rightarrow 4} \sqrt{x} = \sqrt{4} = 2$

Given $\epsilon > 0$:

$$|\sqrt{x} - 2| < \epsilon$$

$$|\sqrt{x} - 2| |\sqrt{x} + 2| < \epsilon |\sqrt{x} + 2|$$

$$|x - 4| < \epsilon |\sqrt{x} + 2|$$

Assuming $1 < x < 9$, you can choose $\delta = 3\epsilon$. Then,

$$\begin{aligned} 0 < |x - 4| < \delta = 3\epsilon &\Rightarrow |x - 4| < \epsilon |\sqrt{x} + 2| \\ &\Rightarrow |\sqrt{x} - 2| < \epsilon. \end{aligned}$$

28. $\lim_{x \rightarrow -3} (2x + 5) = -1$

Given $\epsilon > 0$:

$$|(2x + 5) - (-1)| < \epsilon$$

$$|2x + 6| < \epsilon$$

$$2|x + 3| < \epsilon$$

$$|x + 3| < \frac{\epsilon}{2} = \delta$$

Hence, let $\delta = \epsilon/2$.

Hence, if $0 < |x + 3| < \delta = \frac{\epsilon}{2}$, you have

$$|x + 3| < \frac{\epsilon}{2}$$

$$|2x + 6| < \epsilon$$

$$|(2x + 5) - (-1)| < \epsilon$$

$$|f(x) - L| < \epsilon$$

32. $\lim_{x \rightarrow 2} (-1) = -1$

Given $\epsilon > 0$: $|-1 - (-1)| < \epsilon$

$$0 < \epsilon$$

Hence, any $\delta > 0$ will work.

Hence, for any $\delta > 0$, you have

$$|(-1) - (-1)| < \epsilon$$

$$|f(x) - L| < \epsilon$$

36. $\lim_{x \rightarrow 3} |x - 3| = 0$

Given $\epsilon > 0$:

$$|(x - 3) - 0| < \epsilon$$

$$|x - 3| < \epsilon = \delta$$

Hence, let $\delta = \epsilon$.

Hence for $0 < |x - 3| < \delta = \epsilon$, you have

$$|x - 3| < \epsilon$$

$$||x - 3| - 0| < \epsilon$$

$$|f(x) - L| < \epsilon$$

38. $\lim_{x \rightarrow -3} (x^2 + 3x) = 0$

Given $\epsilon > 0$:

$$|(x^2 + 3x) - 0| < \epsilon$$

$$|x(x + 3)| < \epsilon$$

$$|x + 3| < \frac{\epsilon}{|x|}$$

If we assume $-4 < x < -2$, then $\delta = \epsilon/4$.

Hence for $0 < |x - (-3)| < \delta = \frac{\epsilon}{4}$, you have

$$|x + 3| < \frac{1}{4}\epsilon < \frac{1}{|x|}\epsilon$$

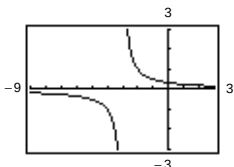
$$|x(x + 3)| < \epsilon$$

$$|x^2 + 3x - 0| < \epsilon$$

$$|f(x) - L| < \epsilon$$

42. $f(x) = \frac{x-3}{x^2-9}$

$$\lim_{x \rightarrow 3} f(x) = \frac{1}{6}$$



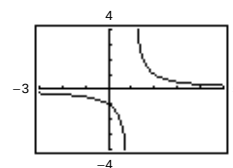
The domain is all $x \neq \pm 3$. The graphing utility does not show the hole at $(3, \frac{1}{6})$.

46. Let $p(x)$ be the atmospheric pressure in a plane at altitude x (in feet).

$$\lim_{x \rightarrow 0^+} p(x) = 14.7 \text{ lb/in}^2$$

40. $f(x) = \frac{x-3}{x^2-4x+3}$

$$\lim_{x \rightarrow 3} f(x) = \frac{1}{2}$$



The domain is all $x \neq 1, 3$. The graphing utility does not show the hole at $(3, \frac{1}{2})$.

44. (a) No. The fact that $f(2) = 4$ has no bearing on the existence of the limit of $f(x)$ as x approaches 2.

(b) No. The fact that $\lim_{x \rightarrow 2} f(x) = 4$ has no bearing on the value of f at 2.

50. True

52. False; let

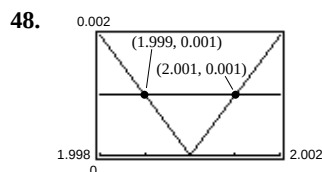
$$f(x) = \begin{cases} x^2 - 4x, & x \neq 4 \\ 10, & x = 4 \end{cases}$$

$$\lim_{x \rightarrow 4} f(x) = \lim_{x \rightarrow 4} (x^2 - 4x) = 0 \text{ and } f(4) = 10 \neq 0$$

54. $\lim_{x \rightarrow 4} \frac{x^2 - x - 12}{x - 4} = 7$

n	$4 + [0.1]^n$	$f(4 + [0.1]^n)$
1	4.1	7.1
2	4.01	7.01
3	4.001	7.001
4	4.0001	7.0001

n	$4 - [0.1]^n$	$f(4 - [0.1]^n)$
1	3.9	6.9
2	3.99	6.99
3	3.999	6.999
4	3.9999	6.9999



Using the zoom and trace feature, $\delta = 0.001$. That is, for

$$0 < |x - 2| < 0.001, \left| \frac{x^2 - 4}{x - 2} - 4 \right| < 0.001.$$

56. $f(x) = mx + b$, $m \neq 0$. Let $\epsilon > 0$ be given. Take $\delta = \frac{\epsilon}{|m|}$.

If $0 < |x - c| < \delta = \frac{\epsilon}{|m|}$, then

$$|m||x - c| < \epsilon$$

$$|mx - mc| < \epsilon$$

$$|(mx + b) - (mc + b)| < \epsilon$$

which shows that $\lim_{x \rightarrow c} (mx + b) = mc + b$.

58. $\lim_{x \rightarrow c} g(x) = L$, $L > 0$. Let $\epsilon = \frac{1}{2}L$. There exists $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|g(x) - L| < \epsilon = \frac{1}{2}L$.

That is,

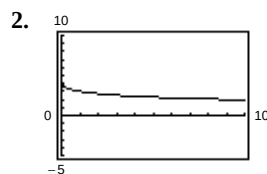
$$-\frac{1}{2}L < g(x) - L < \frac{1}{2}L$$

$$\frac{1}{2}L < g(x) < \frac{3}{2}L$$

Hence for x in the interval $(c - \delta, c + \delta)$, $x \neq c$,

$$g(x) > \frac{1}{2}L > 0.$$

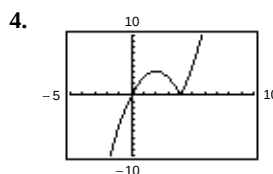
Section 1.3 Evaluating Limits Analytically



(a) $\lim_{x \rightarrow 4} g(x) = 2.4$

(b) $\lim_{x \rightarrow 0} g(x) = 4$

$$g(x) = \frac{12(\sqrt{x} - 3)}{x - 9}$$



(a) $\lim_{t \rightarrow 4} f(t) = 0$

(b) $\lim_{t \rightarrow -1} f(t) = -5$

$$f(t) = t|t - 4|$$

6. $\lim_{x \rightarrow -2} x^3 = (-2)^3 = -8$

8. $\lim_{x \rightarrow -3} (3x + 2) = 3(-3) + 2 = -7$

10. $\lim_{x \rightarrow 1} (-x^2 + 1) = -(1)^2 + 1 = 0$

12. $\lim_{x \rightarrow 1} (3x^3 - 2x^2 + 4) = 3(1)^3 - 2(1)^2 + 4 = 5$

14. $\lim_{x \rightarrow -3} \frac{2}{x + 2} = \frac{2}{-3 + 2} = -2$

16. $\lim_{x \rightarrow 3} \frac{2x - 3}{x + 5} = \frac{2(3) - 3}{3 + 5} = \frac{3}{8}$

18. $\lim_{x \rightarrow 3} \frac{\sqrt{x + 1}}{x - 4} = \frac{\sqrt{3 + 1}}{3 - 4} = -2$

20. $\lim_{x \rightarrow 4} \sqrt[3]{x + 4} = \sqrt[3]{4 + 4} = 2$

22. $\lim_{x \rightarrow 0} (2x - 1)^3 = [2(0) - 1]^3 = -1$

24. (a) $\lim_{x \rightarrow -3} f(x) = (-3) + 7 = 4$

(b) $\lim_{x \rightarrow 4} g(x) = 4^2 = 16$

(c) $\lim_{x \rightarrow -3} g(f(x)) = g(4) = 16$

26. (a) $\lim_{x \rightarrow 4} f(x) = 2(4^2) - 3(4) + 1 = 21$

(b) $\lim_{x \rightarrow 21} g(x) = \sqrt[3]{21 + 6} = 3$

(c) $\lim_{x \rightarrow 4} g(f(x)) = g(21) = 3$

28. $\lim_{x \rightarrow \pi} \tan x = \tan \pi = 0$

30. $\lim_{x \rightarrow 1} \sin \frac{\pi x}{2} = \sin \frac{\pi}{2} = 1$

32. $\lim_{x \rightarrow \pi} \cos 3x = \cos 3\pi = -1$

34. $\lim_{x \rightarrow 5\pi/3} \cos x = \cos \frac{5\pi}{3} = \frac{1}{2}$

36. $\lim_{x \rightarrow 7} \sec\left(\frac{\pi x}{6}\right) = \sec \frac{7\pi}{6} = \frac{-2\sqrt{3}}{3}$

$$38. (a) \lim_{x \rightarrow c} [4f(x)] = 4 \lim_{x \rightarrow c} f(x) = 4\left(\frac{3}{2}\right) = 6$$

$$(b) \lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) = \frac{3}{2} + \frac{1}{2} = 2$$

$$(c) \lim_{x \rightarrow c} [f(x)g(x)] = \left[\lim_{x \rightarrow c} f(x)\right]\left[\lim_{x \rightarrow c} g(x)\right] = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = \frac{3}{4}$$

$$(d) \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} = \frac{3/2}{1/2} = 3$$

$$40. (a) \lim_{x \rightarrow c} \sqrt[3]{f(x)} = \sqrt[3]{\lim_{x \rightarrow c} f(x)} = \sqrt[3]{27} = 3$$

$$(b) \lim_{x \rightarrow c} \frac{f(x)}{18} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} 18} = \frac{27}{18} = \frac{3}{2}$$

$$(c) \lim_{x \rightarrow c} [f(x)]^2 = \left[\lim_{x \rightarrow c} f(x)\right]^2 = (27)^2 = 729$$

$$(d) \lim_{x \rightarrow c} [f(x)]^{2/3} = \left[\lim_{x \rightarrow c} f(x)\right]^{2/3} = (27)^{2/3} = 9$$

$$42. f(x) = x - 3 \text{ and } h(x) = \frac{x^2 - 3x}{x} \text{ agree except at } x = 0.$$

$$(a) \lim_{x \rightarrow -2} h(x) = \lim_{x \rightarrow -2} f(x) = -5$$

$$(b) \lim_{x \rightarrow 0} h(x) = \lim_{x \rightarrow 0} f(x) = -3$$

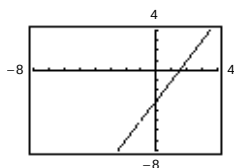
$$44. g(x) = \frac{1}{x-1} \text{ and } f(x) = \frac{x}{x^2 - x} \text{ agree except at } x = 0.$$

$$(a) \lim_{x \rightarrow 1} f(x) \text{ does not exist.}$$

$$(b) \lim_{x \rightarrow 0} f(x) = -1$$

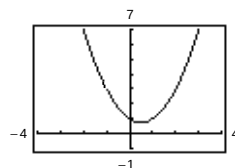
$$46. f(x) = \frac{2x^2 - x - 3}{x + 1} \text{ and } g(x) = 2x - 3 \text{ agree except at } x = -1.$$

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} g(x) = -5$$



$$48. f(x) = \frac{x^3 + 1}{x + 1} \text{ and } g(x) = x^2 - x + 1 \text{ agree except at } x = -1.$$

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} g(x) = 3$$



$$50. \lim_{x \rightarrow 2} \frac{2 - x}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{-(x - 2)}{(x - 2)(x + 2)} = \lim_{x \rightarrow 2} \frac{-1}{x + 2} = -\frac{1}{4}$$

$$52. \lim_{x \rightarrow 4} \frac{x^2 - 5x + 4}{x^2 - 2x - 8} = \lim_{x \rightarrow 4} \frac{(x - 4)(x - 1)}{(x - 4)(x + 2)} = \lim_{x \rightarrow 4} \frac{(x - 1)}{(x + 2)} = \frac{3}{6} = \frac{1}{2}$$

$$54. \lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} \cdot \frac{\sqrt{2+x} + \sqrt{2}}{\sqrt{2+x} + \sqrt{2}} = \lim_{x \rightarrow 0} \frac{2+x-2}{(\sqrt{2+x} + \sqrt{2})x} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{2+x} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

$$56. \lim_{x \rightarrow 3} \frac{\sqrt{x+1} - 2}{x-3} = \lim_{x \rightarrow 3} \frac{\sqrt{x+1} - 2}{x-3} \cdot \frac{\sqrt{x+1} + 2}{\sqrt{x+1} + 2} = \lim_{x \rightarrow 3} \frac{x-3}{(x-3)(\sqrt{x+1} + 2)} = \lim_{x \rightarrow 3} \frac{1}{\sqrt{x+1} + 2} = \frac{1}{4}$$

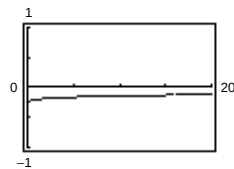
$$58. \lim_{x \rightarrow 0} \frac{\frac{1}{x+4} - \frac{1}{4}}{x} = \lim_{x \rightarrow 0} \frac{\frac{4 - (x+4)}{4(x+4)}}{x} = \lim_{x \rightarrow 0} \frac{-1}{4(x+4)} = -\frac{1}{16}$$

$$60. \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x$$

$$\begin{aligned}
 62. \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(3x^2 + 3x\Delta x + (\Delta x)^2)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2
 \end{aligned}$$

$$64. f(x) = \frac{4 - \sqrt{x}}{x - 16}$$

x	15.9	15.99	15.999	16	16.001	16.01	16.1
$f(x)$	-.1252	-.125	-.125	?	-.125	-.125	-.1248

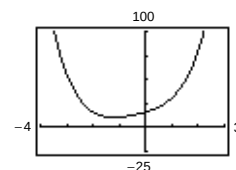


It appears that the limit is -0.125 .

$$\begin{aligned}
 \text{Analytically, } \lim_{x \rightarrow 16} \frac{4 - \sqrt{x}}{x - 16} &= \lim_{x \rightarrow 16} \frac{(4 - \sqrt{x})}{(\sqrt{x} + 4)(\sqrt{x} - 4)} \\
 &= \lim_{x \rightarrow 16} \frac{-1}{\sqrt{x} + 4} = -\frac{1}{8}.
 \end{aligned}$$

$$66. \lim_{x \rightarrow 2} \frac{x^5 - 32}{x - 2} = 80$$

x	1.9	1.99	1.999	1.9999	2.0	2.0001	2.001	2.01	2.1
$f(x)$	72.39	79.20	79.92	79.99	?	80.01	80.08	80.80	88.41



$$\begin{aligned}
 \text{Analytically, } \lim_{x \rightarrow 2} \frac{x^5 - 32}{x - 2} &= \lim_{x \rightarrow 2} \frac{(x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16)}{x - 2} \\
 &= \lim_{x \rightarrow 2} (x^4 + 2x^3 + 4x^2 + 8x + 16) = 80.
 \end{aligned}$$

(Hint: Use long division to factor $x^5 - 32$.)

$$68. \lim_{x \rightarrow 0} \frac{3(1 - \cos x)}{x} = \lim_{x \rightarrow 0} \left[3 \left(\frac{1 - \cos x}{x} \right) \right] = (3)(0) = 0$$

$$70. \lim_{\theta \rightarrow 0} \frac{\cos \theta \tan \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

$$\begin{aligned}
 72. \lim_{x \rightarrow 0} \frac{\tan^2 x}{x} &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x \cos^2 x} = \lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \cdot \frac{\sin x}{\cos^2 x} \right] \\
 &= (1)(0) = 0
 \end{aligned}$$

$$74. \lim_{\phi \rightarrow \pi} \phi \sec \phi = \pi(-1) = -\pi$$

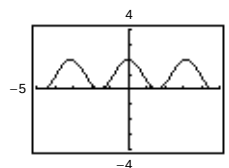
$$\begin{aligned}
 76. \lim_{x \rightarrow \pi/4} \frac{1 - \tan x}{\sin x - \cos x} &= \lim_{x \rightarrow \pi/4} \frac{\cos x - \sin x}{\sin x \cos x - \cos^2 x} \\
 &= \lim_{x \rightarrow \pi/4} \frac{-(\sin x - \cos x)}{\cos x(\sin x - \cos x)} \\
 &= \lim_{x \rightarrow \pi/4} \frac{-1}{\cos x} \\
 &= \lim_{x \rightarrow \pi/4} (-\sec x) \\
 &= -\sqrt{2}
 \end{aligned}$$

$$78. \lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x} = \lim_{x \rightarrow 0} \left[2 \left(\frac{\sin 2x}{2x} \right) \left(\frac{1}{3} \right) \left(\frac{3x}{\sin 3x} \right) \right] = 2(1) \left(\frac{1}{3} \right) (1) = \frac{2}{3}$$

80. $f(h) = (1 + \cos 2h)$

h	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(h)$	1.98	1.9998	2	?	2	1.9998	1.98

Analytically, $\lim_{h \rightarrow 0} (1 + \cos 2h) = 1 + \cos(0) = 1 + 1 = 2$.

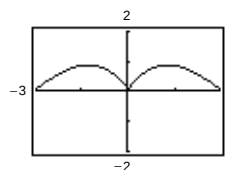


The limit appear to equal 2.

82. $f(x) = \frac{\sin x}{\sqrt[3]{x}}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.215	0.0464	0.01	?	0.01	0.0464	0.215

Analytically, $\lim_{x \rightarrow 0} \frac{\sin x}{\sqrt[3]{x}} = \lim_{x \rightarrow 0} \sqrt[3]{x^2} \left(\frac{\sin x}{x} \right) = (0)(1) = 0$.



The limit appear to equal 0.

$$\begin{aligned}
 84. \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
 &= \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
 \end{aligned}$$

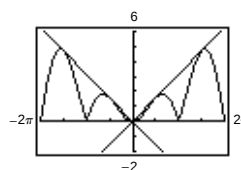
$$\begin{aligned}
 86. \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - 4(x+h) - (x^2 - 4x)}{h} = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - 4x - 4h - x^2 + 4x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(2x + h - 4)}{h} = \lim_{h \rightarrow 0} (2x + h - 4) = 2x - 4
 \end{aligned}$$

88. $\lim_{x \rightarrow a} [b - |x - a|] \leq \lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} [b + |x - a|]$

$$b \leq \lim_{x \rightarrow a} f(x) \leq b$$

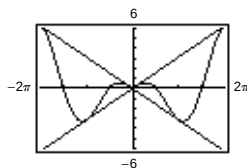
Therefore, $\lim_{x \rightarrow a} f(x) = b$.

90. $f(x) = |x \sin x|$



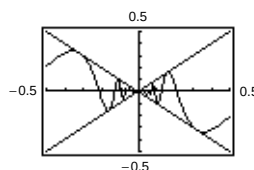
$$\lim_{x \rightarrow 0} |x \sin x| = 0$$

92. $f(x) = |x| \cos x$



$$\lim_{x \rightarrow 0} |x| \cos x = 0$$

94. $h(x) = x \cos \frac{1}{x}$

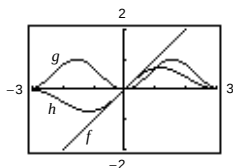


$$\lim_{x \rightarrow 0} \left(x \cos \frac{1}{x} \right) = 0$$

96. $f(x) = \frac{x^2 - 1}{x - 1}$ and $g(x) = x + 1$ agree at all points except $x = 1$.

98. If a function f is squeezed between two functions h and g , $h(x) \leq f(x) \leq g(x)$, and h and g have the same limit L as $x \rightarrow c$, then $\lim_{x \rightarrow c} f(x)$ exists and equals L .

100. $f(x) = x$, $g(x) = \sin^2 x$, $h(x) = \frac{\sin^2 x}{x}$



When you are “close to” 0 the magnitude of g is “smaller” than the magnitude of f and the magnitude of g is approaching zero “faster” than the magnitude of f . Thus, $|g|/|f| \approx 0$ when x is “close to” 0

102. $s(t) = -16t^2 + 1000 = 0$ when $t = \sqrt{\frac{1000}{16}} = \frac{5\sqrt{10}}{2}$ seconds

$$\begin{aligned} \lim_{t \rightarrow 5\sqrt{10}/2} \frac{s\left(\frac{5\sqrt{10}}{2}\right) - s(t)}{\frac{5\sqrt{10}}{2} - t} &= \lim_{t \rightarrow 5\sqrt{10}/2} \frac{0 - (-16t^2 + 1000)}{\frac{5\sqrt{10}}{2} - t} \\ &= \lim_{t \rightarrow 5\sqrt{10}/2} \frac{16\left(t^2 - \frac{125}{2}\right)}{\frac{5\sqrt{10}}{2} - t} = \lim_{t \rightarrow 5\sqrt{10}/2} \frac{16\left(t + \frac{5\sqrt{10}}{2}\right)\left(t - \frac{5\sqrt{10}}{2}\right)}{-\left(t - \frac{5\sqrt{10}}{2}\right)} \\ &= \lim_{t \rightarrow 5\sqrt{10}/2} -16\left(t + \frac{5\sqrt{10}}{2}\right) = -80\sqrt{10} \text{ ft/sec} \approx -253 \text{ ft/sec} \end{aligned}$$

104. $-4.9t^2 + 150 = 0$ when $t = \sqrt{\frac{150}{4.9}} = \sqrt{\frac{1500}{49}} \approx 5.53$ seconds.

The velocity at time $t = a$ is

$$\begin{aligned} \lim_{t \rightarrow a} \frac{s(a) - s(t)}{a - t} &= \lim_{t \rightarrow a} \frac{(-4.9a^2 + 150) - (-4.9t^2 + 150)}{a - t} = \lim_{t \rightarrow a} \frac{-4.9(a - t)(a + t)}{a - t} \\ &= \lim_{t \rightarrow a} -4.9(a + t) = -2a(4.9) = -9.8a \text{ m/sec.} \end{aligned}$$

Hence, if $a = \sqrt{1500/49}$, the velocity is $-9.8\sqrt{1500/49} \approx -54.2$ m/sec.

106. Suppose, on the contrary, that $\lim_{x \rightarrow c} g(x)$ exists. Then, since $\lim_{x \rightarrow c} f(x)$ exists, so would $\lim_{x \rightarrow c} [f(x) + g(x)]$, which is a contradiction. Hence, $\lim_{x \rightarrow c} g(x)$ does not exist.

108. Given $f(x) = x^n$, n is a positive integer, then

$$\begin{aligned} \lim_{x \rightarrow c} x^n &= \lim_{x \rightarrow c} (x x^{n-1}) = \left[\lim_{x \rightarrow c} x \right] \left[\lim_{x \rightarrow c} x^{n-1} \right] \\ &= c \left[\lim_{x \rightarrow c} (x x^{n-2}) \right] = c \left[\lim_{x \rightarrow c} x \right] \left[\lim_{x \rightarrow c} x^{n-2} \right] \\ &= c(c) \lim_{x \rightarrow c} (x x^{n-3}) = \dots = c^n. \end{aligned}$$

110. Given $\lim_{x \rightarrow c} f(x) = 0$:

For every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - 0| < \epsilon$ whenever $0 < |x - c| < \delta$.

Now $|f(x) - 0| = |f(x)| = ||f(x)| - 0| < \epsilon$ for $|x - c| < \delta$. Therefore, $\lim_{x \rightarrow c} |f(x)| = 0$.

112. (a) If $\lim_{x \rightarrow c} |f(x)| = 0$, then $\lim_{x \rightarrow c} [-|f(x)|] = 0$.

$$-|f(x)| \leq f(x) \leq |f(x)|$$

$$\lim_{x \rightarrow c} [-|f(x)|] \leq \lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} |f(x)|$$

$$0 \leq \lim_{x \rightarrow c} f(x) \leq 0$$

Therefore, $\lim_{x \rightarrow c} f(x) = 0$.

(b) Given $\lim_{x \rightarrow c} f(x) = L$:

For every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$.

Since $||f(x)| - |L|| \leq |f(x) - L| < \epsilon$ for $|x - c| < \delta$, then $\lim_{x \rightarrow c} |f(x)| = |L|$.

114. True. $\lim_{x \rightarrow 0} x^3 = 0^3 = 0$

116. False. Let $f(x) = \begin{cases} x & x \neq 1 \\ 3 & x = 1 \end{cases}$, $c = 1$

Then $\lim_{x \rightarrow 1} f(x) = 1$ but $f(1) \neq 1$.

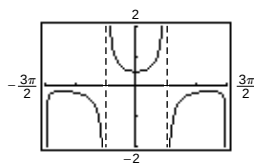
118. False. Let $f(x) = \frac{1}{2}x^2$ and $g(x) = x^2$. Then $f(x) < g(x)$ for all $x \neq 0$. But $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} g(x) = 0$.

$$\begin{aligned} 120. \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x} \\ &= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x(1 + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x} \\ &= \left[\lim_{x \rightarrow 0} \frac{\sin x}{x} \right] \left[\lim_{x \rightarrow 0} \frac{\sin x}{1 + \cos x} \right] \\ &= (1)(0) = 0 \end{aligned}$$

122. $f(x) = \frac{\sec x - 1}{x^2}$

(a) The domain of f is all $x \neq 0, \pi/2 + n\pi$.

(b)



The domain is not obvious. The hole at $x = 0$ is not apparent.

(c) $\lim_{x \rightarrow 0} f(x) = \frac{1}{2}$

$$\begin{aligned} (d) \frac{\sec x - 1}{x^2} &= \frac{\sec x - 1}{x^2} \cdot \frac{\sec x + 1}{\sec x + 1} = \frac{\sec^2 x - 1}{x^2(\sec x + 1)} \\ &= \frac{\tan^2 x}{x^2(\sec x + 1)} = \frac{1}{\cos^2 x} \left(\frac{\sin^2 x}{x^2} \right) \frac{1}{\sec x + 1} \end{aligned}$$

$$\begin{aligned} \text{Hence, } \lim_{x \rightarrow 0} \frac{\sec x - 1}{x^2} &= \lim_{x \rightarrow 0} \frac{1}{\cos^2 x} \left(\frac{\sin^2 x}{x^2} \right) \frac{1}{\sec x + 1} \\ &= 1(1) \left(\frac{1}{2} \right) = \frac{1}{2}. \end{aligned}$$

124. The calculator was set in degree mode, instead of radian mode.

Section 1.4 Continuity and One-Sided Limits

2. (a) $\lim_{x \rightarrow -2^+} f(x) = -2$

(b) $\lim_{x \rightarrow -2^-} f(x) = -2$

(c) $\lim_{x \rightarrow -2} f(x) = -2$

The function is continuous at $x = -2$.

4. (a) $\lim_{x \rightarrow -2^+} f(x) = 2$

(b) $\lim_{x \rightarrow -2^-} f(x) = 2$

(c) $\lim_{x \rightarrow -2} f(x) = 2$

The function is NOT continuous at $x = -2$.

6. (a) $\lim_{x \rightarrow -1^+} f(x) = 0$

(b) $\lim_{x \rightarrow -1^-} f(x) = 2$

(c) $\lim_{x \rightarrow -1} f(x)$ does not exist.

The function is NOT continuous at $x = -1$.

8. $\lim_{x \rightarrow 2^+} \frac{2-x}{x^2-4} = \lim_{x \rightarrow 2^+} -\frac{1}{x+2} = -\frac{1}{4}$

$$\begin{aligned} 10. \lim_{x \rightarrow 4^-} \frac{\sqrt{x}-2}{x-4} &= \lim_{x \rightarrow 4^-} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} \\ &= \lim_{x \rightarrow 4^-} \frac{x-4}{(x-4)(\sqrt{x}+2)} \\ &= \lim_{x \rightarrow 4^-} \frac{1}{\sqrt{x}+2} = \frac{1}{4} \end{aligned}$$

12. $\lim_{x \rightarrow 2^+} \frac{|x-2|}{x-2} = \lim_{x \rightarrow 2^+} \frac{x-2}{x-2} = 1$

$$\begin{aligned} 14. \lim_{\Delta x \rightarrow 0^+} \frac{(x+\Delta x)^2 + (x+\Delta x) - (x^2+x)}{\Delta x} &= \lim_{\Delta x \rightarrow 0^+} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 + x + \Delta x - x^2 - x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0^+} \frac{2x(\Delta x) + (\Delta x)^2 + \Delta x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0^+} (2x + \Delta x + 1) \\ &= 2x + 0 + 1 = 2x + 1 \end{aligned}$$

$$\begin{aligned} 16. \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (-x^2 + 4x - 2) = 2 \\ \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} (x^2 - 4x + 6) = 2 \\ \lim_{x \rightarrow 2} f(x) &= 2 \end{aligned}$$

18. $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (1-x) = 0$

$$\begin{aligned} 20. \lim_{x \rightarrow \pi/2} \sec x &\text{ does not exist since} \\ \lim_{x \rightarrow (\pi/2)^+} \sec x &\text{ and } \lim_{x \rightarrow (\pi/2)^-} \sec x \text{ do not exist.} \end{aligned}$$

22. $\lim_{x \rightarrow 2^+} (2x - \lfloor x \rfloor) = 2(2) - 2 = 2$

24. $\lim_{x \rightarrow 1} \left(1 - \left\lfloor -\frac{x}{2} \right\rfloor \right) = 1 - (-1) = 2$

26. $f(x) = \frac{x^2-1}{x+1}$

has a discontinuity at $x = -1$ since $f(-1)$ is not defined.

$$28. f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x-1, & x > 1 \end{cases} \text{ has discontinuity at } x = 1 \text{ since } f(1) = 2 \neq \lim_{x \rightarrow 1} f(x) = 1.$$

30. $f(t) = 3 - \sqrt{9-t^2}$ is continuous on $[-3, 3]$.

32. $g(2)$ is not defined. g is continuous on $[-1, 2)$.

34. $f(x) = \frac{1}{x^2 - 1}$ is continuous for all real x .

36. $f(x) = \cos \frac{\pi x}{2}$ is continuous for all real x .

38. $f(x) = \frac{x}{x^2 - 1}$ has nonremovable discontinuities at $x = 1$ and $x = -1$ since $\lim_{x \rightarrow 1} f(x)$ and $\lim_{x \rightarrow -1} f(x)$ do not exist.

40. $f(x) = \frac{x - 3}{x^2 - 9}$ has a nonremovable discontinuity at $x = -3$ since $\lim_{x \rightarrow -3} f(x)$ does not exist, and has a removable discontinuity at $x = 3$ since

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{1}{x + 3} = \frac{1}{6}.$$

42. $f(x) = \frac{x - 1}{(x + 2)(x - 1)}$

has a nonremovable discontinuity at $x = -2$ since

$\lim_{x \rightarrow -2} f(x)$ does not exist, and has a removable discontinuity at $x = 1$ since

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{1}{x + 2} = \frac{1}{3}.$$

44. $f(x) = \frac{|x - 3|}{x - 3}$

has a nonremovable discontinuity at $x = 3$ since $\lim_{x \rightarrow 3} f(x)$ does not exist.

46. $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \geq 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1^2 = 1$

2. $\left. \begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} (-2x + 3) = 1 \\ \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} x^2 = 1 \end{aligned} \right\} \lim_{x \rightarrow 1} f(x) = 1$

3. $f(1) = \lim_{x \rightarrow 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x .

48. $f(x) = \begin{cases} -2x, & x \leq 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$ has a **possible** discontinuity at $x = 2$.

1. $f(2) = -2(2) = -4$

2. $\left. \begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} (-2x) = -4 \\ \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (x^2 - 4x + 1) = -3 \end{aligned} \right\} \lim_{x \rightarrow 2} f(x) \text{ does not exist.}$

Therefore, f has a nonremovable discontinuity at $x = 2$.

50. $f(x) = \begin{cases} \csc \frac{\pi x}{6}, & |x - 3| \leq 2 \\ 2, & |x - 3| > 2 \end{cases} = \begin{cases} \csc \frac{\pi x}{6}, & 1 \leq x \leq 5 \\ 2, & x < 1 \text{ or } x > 5 \end{cases}$ has **possible** discontinuities at $x = 1$, $x = 5$.

1. $f(1) = \csc \frac{\pi}{6} = 2$ $f(5) = \csc \frac{5\pi}{6} = 2$

2. $\lim_{x \rightarrow 1} f(x) = 2$ $\lim_{x \rightarrow 5} f(x) = 2$

3. $f(1) = \lim_{x \rightarrow 1} f(x)$ $f(5) = \lim_{x \rightarrow 5} f(x)$

f is continuous at $x = 1$ and $x = 5$, therefore, f is continuous for all real x .

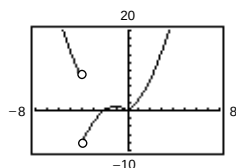
52. $f(x) = \tan \frac{\pi x}{2}$ has nonremovable discontinuities at each $2k + 1$, k is an integer.

54. $f(x) = 3 - \llbracket x \rrbracket$ has nonremovable discontinuities at each integer k .

56. $\lim_{x \rightarrow 0^+} f(x) = 0$

$\lim_{x \rightarrow 0^-} f(x) = 0$

f is not continuous at $x = -4$



58. $\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0^-} \frac{4 \sin x}{x} = 4$

$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} (a - 2x) = a$

Let $a = 4$.

60. $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} \frac{x^2 - a^2}{x - a}$

$= \lim_{x \rightarrow a} (x + a) = 2a$

Find a such that $2a = 8 \Rightarrow a = 4$.

62. $f(g(x)) = \frac{1}{\sqrt{x-1}}$

Nonremovable discontinuity at $x = 1$. Continuous for all $x > 1$.

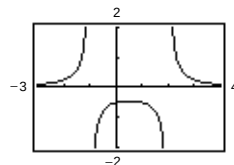
Because $f \circ g$ is not defined for $x < 1$, it is better to say that $f \circ g$ is discontinuous from the right at $x = 1$.

64. $f(g(x)) = \sin x^2$

Continuous for all real x

66. $h(x) = \frac{1}{(x+1)(x-2)}$

Nonremovable discontinuity at $x = -1$ and $x = 2$.



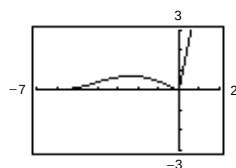
68. $f(x) = \begin{cases} \frac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \geq 0 \end{cases}$

$f(0) = 5(0) = 0$

$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{(\cos x - 1)}{x} = 0$

$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (5x) = 0$

Therefore, $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$ and f is continuous on the entire real line. ($x = 0$ was the only possible discontinuity.)



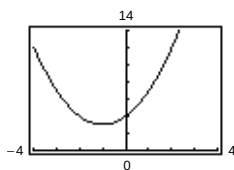
70. $f(x) = x\sqrt{x+3}$

Continuous on $[-3, \infty]$

72. $f(x) = \frac{x+1}{\sqrt{x}}$

Continuous on $(0, \infty)$

74. $f(x) = \frac{x^3 - 8}{x - 2}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Since $f(2)$ is not defined, we know that f has a discontinuity at $x = 2$. This discontinuity is removable so it does not show up on the graph.

78. $f(x) = \frac{-4}{x} + \tan \frac{\pi x}{8}$ is continuous on $[1, 3]$.

$$f(1) = -4 + \tan \frac{\pi}{8} < 0 \text{ and } f(3) = -\frac{4}{3} + \tan \frac{3\pi}{8} > 0.$$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 1 and 3.

82. $h(\theta) = 1 + \theta - 3 \tan \theta$

h is continuous on $[0, 1]$.

$$h(0) = 1 > 0 \text{ and } h(1) \approx -2.67 < 0.$$

By the Intermediate Value Theorem, $h(\theta) = 0$ for at least one value θ between 0 and 1. Using a graphing utility, we find that $\theta \approx 0.4503$.

86. $f(x) = \frac{x^2 + x}{x - 1}$

f is continuous on $[\frac{5}{2}, 4]$. The nonremovable discontinuity, $x = 1$, lies outside the interval.

$$f\left(\frac{5}{2}\right) = \frac{35}{6} \text{ and } f(4) = \frac{20}{3}$$

$$\frac{35}{6} < 6 < \frac{20}{3}$$

76. $f(x) = x^3 + 3x - 2$ is continuous on $[0, 1]$.

$$f(0) = -2 \text{ and } f(1) = 2$$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1.

80. $f(x) = x^3 + 3x - 2$

$f(x)$ is continuous on $[0, 1]$.

$$f(0) = -2 \text{ and } f(1) = 2$$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1. Using a graphing utility, we find that $x \approx 0.5961$.

84. $f(x) = x^2 - 6x + 8$

f is continuous on $[0, 3]$.

$$f(0) = 8 \text{ and } f(3) = -1$$

$$-1 < 0 < 8$$

The Intermediate Value Theorem applies.

$$x^2 - 6x + 8 = 0$$

$$(x - 2)(x - 4) = 0$$

$$x = 2 \text{ or } x = 4$$

$$c = 2 \text{ (} x = 4 \text{ is not in the interval.)}$$

Thus, $f(2) = 0$.

The Intermediate Value Theorem applies.

$$\frac{x^2 + x}{x - 1} = 6$$

$$x^2 + x = 6x - 6$$

$$x^2 - 5x + 6 = 0$$

$$(x - 2)(x - 3) = 0$$

$$x = 2 \text{ or } x = 3$$

$$c = 3 \text{ (} x = 2 \text{ is not in the interval.)}$$

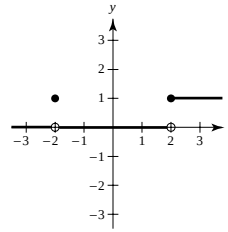
Thus, $f(3) = 6$.

88. A discontinuity at $x = c$ is removable if you can define (or redefine) the function at $x = c$ in such a way that the new function is continuous at $x = c$. Answers will vary.

$$(a) f(x) = \frac{|x - 2|}{x - 2}$$

$$(b) f(x) = \frac{\sin(x + 2)}{x + 2}$$

$$(c) f(x) = \begin{cases} 1, & \text{if } x \geq 2 \\ 0, & \text{if } -2 < x < 2 \\ 1, & \text{if } x = -2 \\ 0, & \text{if } x < -2 \end{cases}$$



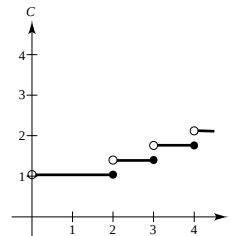
90. If f and g are continuous for all real x , then so is $f + g$ (Theorem 1.11, part 2). However, f/g might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then f and g are continuous for all real x , but f/g is not continuous at $x = \pm 1$.

$$92. C = \begin{cases} 1.04, & 0 < t \leq 2 \\ 1.04 + 0.36\lfloor t - 1 \rfloor, & t > 2, t \text{ is not an integer} \\ 1.04 + 0.36(t - 2), & t > 2, t \text{ is an integer} \end{cases}$$

Nonremovable discontinuity at each integer greater than 2.

You can also write C as

$$C = \begin{cases} 1.04, & 0 < t \leq 2 \\ 1.04 - 0.36\lfloor 2 - t \rfloor, & t > 2 \end{cases}$$



94. Let $s(t)$ be the position function for the run up to the campsite. $s(0) = 0$ ($t = 0$ corresponds to 8:00 A.M., $s(20) = k$ (distance to campsite)). Let $r(t)$ be the position function for the run back down the mountain: $r(0) = k$, $r(10) = 0$. Let $f(t) = s(t) - r(t)$.

When $t = 0$ (8:00 A.M.), $f(0) = s(0) - r(0) = 0 - k < 0$.

When $t = 10$ (8:10 A.M.), $f(10) = s(10) - r(10) > 0$.

Since $f(0) < 0$ and $f(10) > 0$, then there must be a value t in the interval $[0, 10]$ such that $f(t) = 0$. If $f(t) = 0$, then $s(t) - r(t) = 0$, which gives us $s(t) = r(t)$. Therefore, at some time t , where $0 \leq t \leq 10$, the position functions for the run up and the run down are equal.

96. Suppose there exists x_1 in $[a, b]$ such that $f(x_1) > 0$ and there exists x_2 in $[a, b]$ such that $f(x_2) < 0$. Then by the Intermediate Value Theorem, $f(x)$ must equal zero for some value of x in $[x_1, x_2]$ (or $[x_2, x_1]$ if $x_2 < x_1$). Thus, f would have a zero in $[a, b]$, which is a contradiction. Therefore, $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

98. If $x = 0$, then $f(0) = 0$ and $\lim_{x \rightarrow 0} f(x) = 0$. Hence, f is continuous at $x = 0$.

If $x \neq 0$, then $\lim_{t \rightarrow x} f(t) = 0$ for x rational, whereas

$\lim_{t \rightarrow x} f(t) = \lim_{t \rightarrow x} kt = kx \neq 0$ for x irrational. Hence, f is not continuous for all $x \neq 0$.

100. True

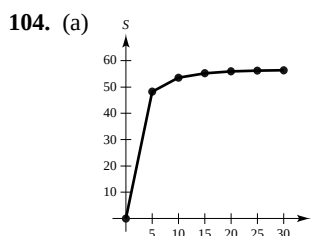
1. $f(c) = L$ is defined.

2. $\lim_{x \rightarrow c} f(x) = L$ exists.

3. $f(c) = \lim_{x \rightarrow c} f(x)$

All of the conditions for continuity are met.

102. False; a rational function can be written as $P(x)/Q(x)$ where P and Q are polynomials of degree m and n , respectively. It can have, at most, n discontinuities.



- (b) There appears to be a limiting speed and a possible cause is air resistance.

106. Let y be a real number. If $y = 0$, then $x = 0$. If $y > 0$, then let $0 < x_0 < \pi/2$ such that $M = \tan x_0 > y$ (this is possible since the tangent function increases without bound on $[0, \pi/2)$). By the Intermediate Value Theorem, $f(x) = \tan x$ is continuous on $[0, x_0]$ and $0 < y < M$, which implies that there exists x between 0 and x_0 such that $\tan x = y$. The argument is similar if $y < 0$.

108. 1. $f(c)$ is defined.

2. $\lim_{x \rightarrow c} f(x) = \lim_{\Delta x \rightarrow 0} f(c + \Delta x) = f(c)$ exists.
[Let $x = c + \Delta x$. As $x \rightarrow c$, $\Delta x \rightarrow 0$]

3. $\lim_{x \rightarrow c} f(x) = f(c)$.

Therefore, f is continuous at $x = c$.

110. Define $f(x) = f_2(x) - f_1(x)$. Since f_1 and f_2 are continuous on $[a, b]$, so is f .

$$f(a) = f_2(a) - f_1(a) > 0 \quad \text{and} \quad f(b) = f_2(b) - f_1(b) < 0.$$

By the Intermediate Value Theorem, there exists c in $[a, b]$ such that $f(c) = 0$.

$$f(c) = f_2(c) - f_1(c) = 0 \Rightarrow f_1(c) = f_2(c)$$

Section 1.5 Infinite Limits

2. $\lim_{x \rightarrow -2^+} \frac{1}{x+2} = \infty$

$$\lim_{x \rightarrow -2^-} \frac{1}{x+2} = -\infty$$

4. $\lim_{x \rightarrow -2^+} \sec \frac{\pi x}{4} = \infty$

$$\lim_{x \rightarrow -2^-} \sec \frac{\pi x}{4} = -\infty$$

6. $f(x) = \frac{x}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	-1.077	-5.082	-50.08	-500.1	499.9	49.92	4.915	0.9091

$$\lim_{x \rightarrow -3^-} f(x) = -\infty$$

$$\lim_{x \rightarrow -3^+} f(x) = \infty$$

8. $f(x) = \sec \frac{\pi x}{6}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	-3.864	-19.11	-191.0	-1910	1910	191.0	19.11	3.864

$$\lim_{x \rightarrow -3^-} f(x) = -\infty$$

$$\lim_{x \rightarrow -3^+} f(x) = \infty$$

10. $\lim_{x \rightarrow 2^+} \frac{4}{(x-2)^3} = \infty$

$$\lim_{x \rightarrow 2^-} \frac{4}{(x-2)^3} = -\infty$$

Therefore, $x = 2$ is a vertical asymptote.

12. $\lim_{x \rightarrow 0^-} \frac{2+x}{x^2(1-x)} = \lim_{x \rightarrow 0^+} \frac{2+x}{x^2(1-x)} = \infty$

Therefore, $x = 0$ is a vertical asymptote.

$$\lim_{x \rightarrow 1^-} \frac{2+x}{x^2(1-x)} = \infty$$

$$\lim_{x \rightarrow 1^+} \frac{2+x}{x^2(1-x)} = -\infty$$

Therefore, $x = 1$ is a vertical asymptote.

14. No vertical asymptote since the denominator is never zero.

16. $\lim_{s \rightarrow -5^-} h(s) = -\infty$ and $\lim_{s \rightarrow -5^+} h(s) = \infty$.

Therefore, $s = -5$ is a vertical asymptote.

$$\lim_{s \rightarrow 5^-} h(s) = -\infty \text{ and } \lim_{s \rightarrow 5^+} h(s) = \infty.$$

Therefore, $s = 5$ is a vertical asymptote.

18. $f(x) = \sec \pi x = \frac{1}{\cos \pi x}$ has vertical asymptotes at

$$x = \frac{2n+1}{2}, n \text{ any integer.}$$

20. $g(x) = \frac{(1/2)x^3 - x^2 - 4x}{3x^2 - 6x - 24} = \frac{1}{6} \frac{x(x^2 - 2x - 8)}{x^2 - 2x - 8}$
 $= \frac{1}{6}x,$

$$x \neq -2, 4$$

No vertical asymptotes. The graph has holes at $x = -2$ and $x = 4$.

22. $f(x) = \frac{4(x^2 + x - 6)}{x(x^3 - 2x^2 - 9x + 18)} = \frac{4(x+3)(x-2)}{x(x-2)(x^2-9)} = \frac{4}{x(x-3)}, x \neq -3, 2$

Vertical asymptotes at $x = 0$ and $x = 3$. The graph has holes at $x = -3$ and $x = 2$.

24. $h(x) = \frac{x^2 - 4}{x^3 + 2x^2 + x + 2} = \frac{(x+2)(x-2)}{(x+2)(x^2+1)}$

has no vertical asymptote since

$$\lim_{x \rightarrow -2} h(x) = \lim_{x \rightarrow -2} \frac{x-2}{x^2+1} = -\frac{4}{5}.$$

26. $h(t) = \frac{t(t-2)}{(t-2)(t+2)(t^2+4)} = \frac{t}{(t+2)(t^2+4)}, t \neq 2$

Vertical asymptote at $t = -2$. The graph has a hole at $t = 2$.

28. $g(\theta) = \frac{\tan \theta}{\theta} = \frac{\sin \theta}{\theta \cos \theta}$ has vertical asymptotes at

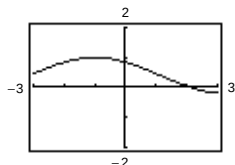
$$\theta = \frac{(2n+1)\pi}{2} = \frac{\pi}{2} + n\pi, n \text{ any integer.}$$

There is no vertical asymptote at $\theta = 0$ since

$$\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1.$$

32. $\lim_{x \rightarrow -1} \frac{\sin(x+1)}{x+1} = 1$

Removable discontinuity at $x = -1$



36. $\lim_{x \rightarrow 4} \frac{x^2}{x^2 + 16} = \frac{1}{2}$

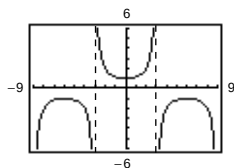
40. $\lim_{x \rightarrow 3} \frac{x-2}{x^2} = \frac{1}{9}$

44. $\lim_{x \rightarrow (\pi/2)^+} \frac{-2}{\cos x} = \infty$

48. $\lim_{x \rightarrow (1/2)^-} x^2 \tan \pi x = \infty$ and $\lim_{x \rightarrow (1/2)^+} x^2 \tan \pi x = -\infty$.
Therefore, $\lim_{x \rightarrow (1/2)} x^2 \tan \pi x$ does not exist.

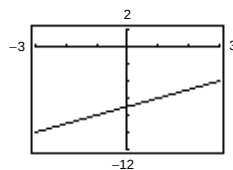
52. $f(x) = \sec \frac{\pi x}{6}$

$$\lim_{x \rightarrow 3^+} f(x) = -\infty$$



56. No. For example, $f(x) = \frac{1}{x^2 + 1}$ has no vertical asymptote.

30. $\lim_{x \rightarrow -1} \frac{x^2 - 6x - 7}{x + 1} = \lim_{x \rightarrow -1} (x - 7) = -8$



Removable discontinuity at $x = -1$

34. $\lim_{x \rightarrow 1^+} \frac{2+x}{1-x} = -\infty$

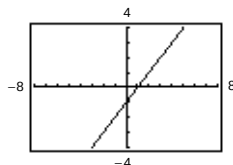
38. $\lim_{x \rightarrow -(1/2)^+} \frac{6x^2 + x - 1}{4x^2 - 4x - 3} = \lim_{x \rightarrow -(1/2)^+} \frac{3x - 1}{2x - 3} = \frac{5}{8}$

42. $\lim_{x \rightarrow 0^-} \left(x^2 - \frac{1}{x} \right) = \infty$

46. $\lim_{x \rightarrow 0} \frac{(x+2)}{\cot x} = \lim_{x \rightarrow 0} [(x+2)\tan x] = 0$

50. $f(x) = \frac{x^3 - 1}{x^2 + x + 1}$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} (x - 1) = 0$$



54. The line $x = c$ is a vertical asymptote if the graph of f approaches $\pm\infty$ as x approaches c .

58. $P = \frac{k}{V}$

$$\lim_{V \rightarrow 0^+} \frac{k}{V} = k(\infty) = \infty \quad (\text{In this case we know that } k > 0.)$$

$$60. (a) r = 50\pi \sec^2 \frac{\pi}{6} = \frac{200\pi}{3} \text{ ft/sec}$$

$$(b) r = 50\pi \sec^2 \frac{\pi}{3} = 200\pi \text{ ft/sec}$$

$$(c) \lim_{\theta \rightarrow (\pi/2)^-} [50\pi \sec^2 \theta] = \infty$$

$$64. (a) \text{Average speed} = \frac{\text{Total distance}}{\text{Total time}}$$

$$50 = \frac{2d}{(d/x) + (d/y)}$$

$$50 = \frac{2xy}{y + x}$$

$$50y + 50x = 2xy$$

$$50x = 2xy - 50y$$

$$50x = 2y(x - 25)$$

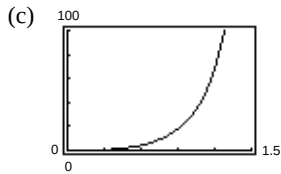
$$\frac{25x}{x - 25} = y$$

Domain: $x > 25$

$$66. (a) A = \frac{1}{2}bh - \frac{1}{2}r^2\theta = \frac{1}{2}(10)(10 \tan \theta) - \frac{1}{2}(10)^2\theta$$

$$= 50 \tan \theta - 50\theta$$

Domain: $\left(0, \frac{\pi}{2}\right)$



68. False; for instance, let

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

The graph of f has a hole at $(1, 2)$, not a vertical asymptote.

$$72. \text{ Let } f(x) = \frac{1}{x^2} \text{ and } g(x) = \frac{1}{x^4}, \text{ and } c = 0.$$

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty \text{ and } \lim_{x \rightarrow 0} \frac{1}{x^4} = \infty, \text{ but}$$

$$\lim_{x \rightarrow 0} \left(\frac{1}{x^2} - \frac{1}{x^4} \right) = \lim_{x \rightarrow 0} \left(\frac{x^2 - 1}{x^4} \right) = -\infty \neq 0.$$

$$62. m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}$$

$$\lim_{v \rightarrow c} m = \lim_{v \rightarrow c} \frac{m_0}{\sqrt{1 - (v^2/c^2)}} = \infty$$

(b)

x	30	40	50	60
y	150	66.667	50	42.857

$$(c) \lim_{x \rightarrow 25^+} \frac{25x}{x - 25} = \infty$$

As x gets close to 25 mph, y becomes larger and larger.

(b)

ϕ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$	0.47	4.21	18.0	68.6	630.1

$$(d) \lim_{\theta \rightarrow \pi/2^-} A = \infty$$

70. True

$$74. \text{ Given } \lim_{x \rightarrow c} f(x) = \infty, \text{ let } g(x) = 1. \text{ then } \lim_{x \rightarrow c} \frac{g(x)}{f(x)} = 0$$

by Theorem 1.15.

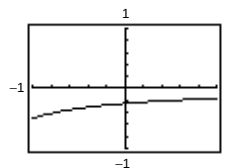
Review Exercises for Chapter 1

1. Calculus required. Using a graphing utility, you can estimate the length to be 8.3.
Or, the length is slightly longer than the distance between the two points, 8.25.

3.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-0.26	-0.25	-0.250	-0.2499	-0.249	-0.24

$$\lim_{x \rightarrow 0} f(x) \approx -0.25$$



5. $h(x) = \frac{x^2 - 2x}{x}$

(a) $\lim_{x \rightarrow 0} h(x) = -2$

(b) $\lim_{x \rightarrow -1} h(x) = -3$

7. $\lim_{x \rightarrow 1} (3 - x) = 3 - 1 = 2$

Let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. Then for

$$0 < |x - 1| < \delta = \epsilon, \text{ you have}$$

$$|x - 1| < \epsilon$$

$$|1 - x| < \epsilon$$

$$|(3 - x) - 2| < \epsilon$$

$$|f(x) - L| < \epsilon$$

9. $\lim_{x \rightarrow 2} (x^2 - 3) = 1$

Let $\epsilon > 0$ be given. We need $|x^2 - 3 - 1| < \epsilon \Rightarrow |x^2 - 4| = |(x - 2)(x + 2)| < \epsilon \Rightarrow |x - 2| < \frac{1}{|x + 2|} \epsilon$.

Assuming, $1 < x < 3$, you can choose $\delta = \epsilon/5$. Hence, for $0 < |x - 2| < \delta = \epsilon/5$ you have

$$|x - 2| < \frac{\epsilon}{5} < \frac{1}{|x + 2|} \epsilon$$

$$|x - 2||x + 2| < \epsilon$$

$$|x^2 - 4| < \epsilon$$

$$|(x^2 - 3) - 1| < \epsilon$$

$$|f(x) - L| < \epsilon$$

11. $\lim_{t \rightarrow 4} \sqrt{t + 2} = \sqrt{4 + 2} = \sqrt{6} \approx 2.45$

13. $\lim_{t \rightarrow -2} \frac{t + 2}{t^2 - 4} = \lim_{t \rightarrow -2} \frac{1}{t - 2} = -\frac{1}{4}$

15. $\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \lim_{x \rightarrow 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$

17. $\lim_{x \rightarrow 0} \frac{[1/(x + 1)] - 1}{x} = \lim_{x \rightarrow 0} \frac{1 - (x + 1)}{x(x + 1)} = \lim_{x \rightarrow 0} \frac{-1}{x + 1} = -1$

19. $\lim_{x \rightarrow -5} \frac{x^3 + 125}{x + 5} = \lim_{x \rightarrow -5} \frac{(x + 5)(x^2 - 5x + 25)}{x + 5} = \lim_{x \rightarrow -5} (x^2 - 5x + 25) = 75$

21. $\lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x} = \lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \right) \left(\frac{1 - \cos x}{x} \right) = (1)(0) = 0$

$$\begin{aligned}
 23. \lim_{\Delta x \rightarrow 0} \frac{\sin[(\pi/6) + \Delta x] - (1/2)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{\sin(\pi/6) \cos \Delta x + \cos(\pi/6) \sin \Delta x - (1/2)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{1}{2} \cdot \frac{(\cos \Delta x - 1)}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{\sqrt{3}}{2} \cdot \frac{\sin \Delta x}{\Delta x} \\
 &= 0 + \frac{\sqrt{3}}{2}(1) = \frac{\sqrt{3}}{2}
 \end{aligned}$$

$$25. \lim_{x \rightarrow c} [f(x) \cdot g(x)] = \left(-\frac{3}{4}\right)\left(\frac{2}{3}\right) = -\frac{1}{2}$$

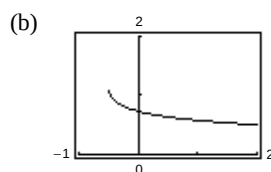
$$27. f(x) = \frac{\sqrt{2x+1} - \sqrt{3}}{x-1}$$

(a)

x	1.1	1.01	1.001	1.0001
$f(x)$	0.5680	0.5764	0.5773	0.5773

$$\lim_{x \rightarrow 1^+} \frac{\sqrt{2x+1} - \sqrt{3}}{x-1} \approx 0.577 \quad (\text{Actual limit is } \sqrt{3}/3)$$

$$\begin{aligned}
 (c) \lim_{x \rightarrow 1^+} \frac{\sqrt{2x+1} - \sqrt{3}}{x-1} &= \lim_{x \rightarrow 1^+} \frac{\sqrt{2x+1} - \sqrt{3}}{x-1} \cdot \frac{\sqrt{2x+1} + \sqrt{3}}{\sqrt{2x+1} + \sqrt{3}} \\
 &= \lim_{x \rightarrow 1^+} \frac{(2x+1) - 3}{(x-1)(\sqrt{2x+1} + \sqrt{3})} \\
 &= \lim_{x \rightarrow 1^+} \frac{2}{\sqrt{2x+1} + \sqrt{3}} \\
 &= \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}
 \end{aligned}$$



$$\begin{aligned}
 29. \lim_{t \rightarrow a} \frac{s(a) - s(t)}{a - t} &= \lim_{t \rightarrow 4} \frac{(-4.9(4)^2 + 200) - (-4.9t^2 + 200)}{4 - t} \\
 &= \lim_{t \rightarrow 4} \frac{4.9(t-4)(t+4)}{4 - t} \\
 &= \lim_{t \rightarrow 4} -4.9(t+4) = -39.2 \text{ m/sec}
 \end{aligned}$$

$$31. \lim_{x \rightarrow 3^-} \frac{|x-3|}{x-3} = \lim_{x \rightarrow 3^-} \frac{-(x-3)}{x-3} = -1$$

$$33. \lim_{x \rightarrow 2} f(x) = 0$$

$$\begin{aligned}
 35. \lim_{t \rightarrow 1} h(t) \text{ does not exist because } \lim_{t \rightarrow 1^-} h(t) &= 1 + 1 = 2 \text{ and} \\
 \lim_{t \rightarrow 1^+} h(t) &= \frac{1}{2}(1 + 1) = 1.
 \end{aligned}$$

$$37. f(x) = \llbracket x + 3 \rrbracket$$

$$\lim_{x \rightarrow k^+} \llbracket x + 3 \rrbracket = k + 3 \text{ where } k \text{ is an integer.}$$

$$\lim_{x \rightarrow k^-} \llbracket x + 3 \rrbracket = k + 2 \text{ where } k \text{ is an integer.}$$

Nonremovable discontinuity at each integer k

Continuous on $(k, k+1)$ for all integers k

$$39. f(x) = \frac{3x^2 - x - 2}{x-1} = \frac{(3x+2)(x-1)}{x-1}$$

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (3x+2) = 5$$

Removable discontinuity at $x = 1$

Continuous on $(-\infty, 1) \cup (1, \infty)$

$$41. f(x) = \frac{1}{(x-2)^2}$$

$$\lim_{x \rightarrow 2} \frac{1}{(x-2)^2} = \infty$$

Nonremovable discontinuity at $x = 2$

Continuous on $(-\infty, 2) \cup (2, \infty)$

$$43. f(x) = \frac{3}{x+1}$$

$$\lim_{x \rightarrow 1^-} f(x) = -\infty$$

$$\lim_{x \rightarrow 1^+} f(x) = \infty$$

Nonremovable discontinuity at $x = -1$

Continuous on $(-\infty, -1) \cup (-1, \infty)$

45. $f(x) = \csc \frac{\pi x}{2}$

Nonremovable discontinuities at each even integer.
Continuous on

$$(2k, 2k + 2)$$

for all integers k .

49. f is continuous on $[1, 2]$. $f(1) = -1 < 0$ and $f(2) = 13 > 0$. Therefore by the Intermediate Value Theorem, there is at least one value c in $(1, 2)$ such that $2c^3 - 3 = 0$.

53. $g(x) = 1 + \frac{2}{x}$

Vertical asymptote at $x = 0$

57. $\lim_{x \rightarrow -2^-} \frac{2x^2 + x + 1}{x + 2} = -\infty$

61. $\lim_{x \rightarrow 1^-} \frac{x^2 + 2x + 1}{x - 1} = -\infty$

65. $\lim_{x \rightarrow 0^+} \frac{\sin 4x}{5x} = \lim_{x \rightarrow 0^+} \left[\frac{4}{5} \left(\frac{\sin 4x}{4x} \right) \right] = \frac{4}{5}$

69. $C = \frac{80,000p}{100 - p}, 0 \leq 0 < 100$

(a) $C(15) \approx \$14,117.65$ (b) $C(50) = \$80,000$

(c) $C(90) = \$720,000$ (d) $\lim_{p \rightarrow 100^-} \frac{80,000p}{100 - p} = \infty$

47. $f(2) = 5$

Find c so that $\lim_{x \rightarrow 2^+} (cx + 6) = 5$.

$$c(2) + 6 = 5$$

$$2c = -1$$

$$c = -\frac{1}{2}$$

51. $f(x) = \frac{x^2 - 4}{|x - 2|} = (x + 2) \left[\frac{x - 2}{|x - 2|} \right]$

(a) $\lim_{x \rightarrow 2^-} f(x) = -4$

(b) $\lim_{x \rightarrow 2^+} f(x) = 4$

(c) $\lim_{x \rightarrow 2} f(x)$ does not exist.

55. $f(x) = \frac{8}{(x - 10)^2}$

Vertical asymptote at $x = 10$

59. $\lim_{x \rightarrow -1^+} \frac{x + 1}{x^3 + 1} = \lim_{x \rightarrow -1^+} \frac{1}{x^2 - x + 1} = \frac{1}{3}$

63. $\lim_{x \rightarrow 0^+} \left(x - \frac{1}{x^3} \right) = -\infty$

67. $\lim_{x \rightarrow 0^+} \frac{\csc 2x}{x} = \lim_{x \rightarrow 0^+} \frac{1}{x \sin 2x} = \infty$

Problem Solving for Chapter 1

1. (a) Perimeter $\triangle PAO = \sqrt{x^2 + (y - 1)^2} + \sqrt{x^2 + y^2} + 1$
 $= \sqrt{x^2 + (x^2 - 1)^2} + \sqrt{x^2 + x^4} + 1$

Perimeter $\triangle PBO = \sqrt{(x - 1)^2 + y^2} + \sqrt{x^2 + y^2} + 1$
 $= \sqrt{(x - 1)^2 + x^4} + \sqrt{x^2 + x^4} + 1$

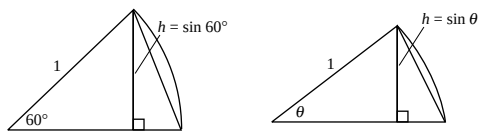
(c) $\lim_{x \rightarrow 0^+} r(x) = \frac{1 + 0 + 1}{1 + 0 + 1} = \frac{2}{2} = 1$

(b) $r(x) = \frac{\sqrt{x^2 + (x^2 - 1)^2} + \sqrt{x^2 + x^4} + 1}{\sqrt{(x - 1)^2 + x^4} + \sqrt{x^2 + x^4} + 1}$

x	4	2	1	0.1	0.01
Perimeter $\triangle PAO$	33.02	9.08	3.41	2.10	2.01
Perimeter $\triangle PBO$	33.77	9.60	3.41	2.00	2.00
$r(x)$	0.98	0.95	1	1.05	1.005

3. (a) There are 6 triangles, each with a central angle of $60^\circ = \pi/3$. Hence,

$$\begin{aligned}\text{Area hexagon} &= 6 \left[\frac{1}{2}bh \right] = 6 \left[\frac{1}{2}(1) \sin \frac{\pi}{3} \right] \\ &= \frac{3\sqrt{3}}{2} \approx 2.598.\end{aligned}$$



$$\text{Error: } \pi - \frac{3\sqrt{3}}{2} \approx 0.5435.$$

- (b) There are n triangles, each with central angle of $\theta = 2\pi/n$. Hence,

$$An = n \left[\frac{1}{2}bh \right] = n \left[\frac{1}{2}(1) \sin \frac{2\pi}{n} \right] = \frac{n \sin(2\pi/n)}{2}.$$

(c)

n	6	12	24	48	96
An	2.598	3	3.106	3.133	3.139

- (d) As n gets larger and larger, $2\pi/n$ approaches 0.

Letting $x = 2\pi/n$,

$$An = \frac{\sin(2\pi/n)}{2/n} = \frac{\sin(2\pi/n)}{(2\pi/n)} \pi = \frac{\sin x}{x} \pi$$

which approaches $(1)\pi = \pi$.

5. (a) Slope = $-\frac{12}{5}$

(b) Slope of tangent line is $\frac{5}{12}$.

$$y + 12 = \frac{5}{12}(x - 5)$$

$$y = \frac{5}{12}x - \frac{169}{12} \quad \text{Tangent line}$$

(c) $Q = (x, y) = (x, \sqrt{169 - x^2})$

$$m_x = \frac{-\sqrt{169 - x^2} + 12}{x - 5}$$

(d) $\lim_{x \rightarrow 5} m_x = \lim_{x \rightarrow 5} \frac{12 - \sqrt{169 - x^2}}{x - 5} \cdot \frac{12 + \sqrt{169 - x^2}}{12 + \sqrt{169 - x^2}}$

$$= \lim_{x \rightarrow 5} \frac{144 - (169 - x^2)}{(x - 5)(12 + \sqrt{169 - x^2})}$$

$$= \lim_{x \rightarrow 5} \frac{x^2 - 25}{(x - 5)(12 + \sqrt{169 - x^2})}$$

$$= \lim_{x \rightarrow 5} \frac{(x + 5)}{12 + \sqrt{169 - x^2}}$$

$$= \frac{10}{12 + 12} = \frac{5}{12}$$

This is the same slope as part (b).

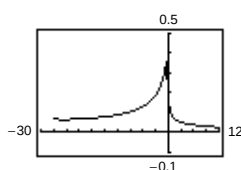
7. (a) $3 + x^{1/3} \geq 0$

$$x^{1/3} \geq -3$$

$$x \geq -27$$

Domain: $x \geq -27, x \neq 1$

(b)



(c) $\lim_{x \rightarrow -27^+} f(x) = \frac{\sqrt{3 + (-27)^{1/3}} - 2}{-27 - 1}$

$$= \frac{-2}{-28} = \frac{1}{14} \approx 0.0714$$

(d) $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{\sqrt{3 + x^{1/3}} - 2}{x - 1} \cdot \frac{\sqrt{3 + x^{1/3}} + 2}{\sqrt{3 + x^{1/3}} + 2}$

$$= \lim_{x \rightarrow 1} \frac{3 + x^{1/3} - 4}{(x - 1)(\sqrt{3 + x^{1/3}} + 2)}$$

$$= \lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{(x^{1/3} - 1)(x^{2/3} + x^{1/3} + 1)(\sqrt{3 + x^{1/3}} + 2)}$$

$$= \lim_{x \rightarrow 1} \frac{1}{(x^{2/3} + x^{1/3} + 1)(\sqrt{3 + x^{1/3}} + 2)}$$

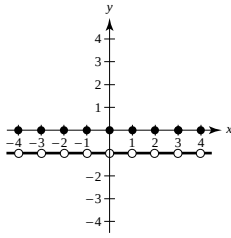
$$= \frac{1}{(1 + 1 + 1)(2 + 2)} = \frac{1}{12}$$

9. (a) $\lim_{x \rightarrow 2} f(x) = 3$: g_1, g_4

(b) f continuous at 2: g_1

(c) $\lim_{x \rightarrow 2^-} f(x) = 3$: g_1, g_3, g_4

11.



$$(a) \quad f(1) = \llbracket 1 \rrbracket + \llbracket -1 \rrbracket = 1 + (-1) = 0$$

$$f(0) = 0$$

$$f\left(\frac{1}{2}\right) = 0 + (-1) = -1$$

$$f(-2.7) = -3 + 2 = -1$$

$$(b) \quad \lim_{x \rightarrow 1^-} f(x) = -1$$

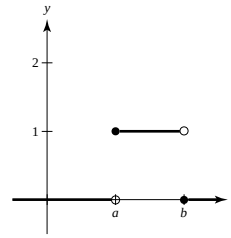
$$\lim_{x \rightarrow 1^+} f(x) = -1$$

$$\lim_{x \rightarrow 1/2} f(x) = -1$$

(c) f is continuous for all real numbers except

$$x = 0, \pm 1, \pm 2, \pm 3, \dots$$

13. (a)



$$(b) \quad (i) \quad \lim_{x \rightarrow a^+} P_{a,b}(x) = 1$$

$$(ii) \quad \lim_{x \rightarrow a^-} P_{a,b}(x) = 0$$

$$(iii) \quad \lim_{x \rightarrow b^+} P_{a,b}(x) = 0$$

$$(iv) \quad \lim_{x \rightarrow b^-} P_{a,b}(x) = 1$$

(c) $P_{a,b}$ is continuous for all positive real numbers except $x = a, b$.

(d) The area under the graph of u , and above the x -axis, is 1.

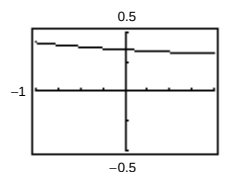
Review Exercises for Chapter 1

2. Precalculus. $L = \sqrt{(9-1)^2 + (3-1)^2} \approx 8.25$

4.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.354	0.354	0.353	0.349

$$\lim_{x \rightarrow 0} f(x) \approx 0.2$$



6. $g(x) = \frac{3x}{x-2}$

(a) $\lim_{x \rightarrow 2} g(x)$ does not exist.

(b) $\lim_{x \rightarrow 0} g(x) = 0$

8. $\lim_{x \rightarrow 9} \sqrt{x} = \sqrt{9} = 3$.

Let $\epsilon > 0$ be given. We need

$$|\sqrt{x} - 3| < \epsilon \Rightarrow |\sqrt{x} + 3| |\sqrt{x} - 3| < \epsilon |\sqrt{x} + 3|$$

$$|x - 9| < \epsilon |\sqrt{x} + 3|$$

Assuming $4 < x < 16$, you can choose $\delta = 5\epsilon$.

Hence, for $0 < |x - 9| < \delta = 5\epsilon$, you have

$$|x - 9| < 5\epsilon < |\sqrt{x} + 3| \epsilon$$

$$|\sqrt{x} - 3| < \epsilon$$

$$|f(x) - L| < \epsilon$$

10. $\lim_{x \rightarrow 5} 9 = 9$. Let $\epsilon > 0$ be given. δ can be any positive number. Hence, for $0 < |x - 5| < \delta$, you have

$$|9 - 9| < \epsilon$$

$$|f(x) - L| < \epsilon$$

12. $\lim_{y \rightarrow 4} 3|y - 1| = 3|4 - 1| = 9$

14. $\lim_{t \rightarrow 3} \frac{t^2 - 9}{t - 3} = \lim_{t \rightarrow 3} (t + 3) = 6$

16. $\lim_{x \rightarrow 0} \frac{\sqrt{4+x} - 2}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{4+x} - 2}{x} \cdot \frac{\sqrt{4+x} + 2}{\sqrt{4+x} + 2}$

$$= \lim_{x \rightarrow 0} \frac{1}{\sqrt{4+x} + 2} = \frac{1}{4}$$

18. $\lim_{s \rightarrow 0} \frac{(1/\sqrt{1+s}) - 1}{s} = \lim_{s \rightarrow 0} \left[\frac{(1/\sqrt{1+s}) - 1}{s} \cdot \frac{(1/\sqrt{1+s}) + 1}{(1/\sqrt{1+s}) + 1} \right]$

$$= \lim_{s \rightarrow 0} \frac{[1/(1+s)] - 1}{s[(1/\sqrt{1+s}) + 1]} = \lim_{s \rightarrow 0} \frac{-1}{(1+s)[(1/\sqrt{1+s}) + 1]} = -\frac{1}{2}$$

20. $\lim_{x \rightarrow -2} \frac{x^2 - 4}{x^3 + 8} = \lim_{x \rightarrow -2} \frac{(x+2)(x-2)}{(x+2)(x^2 - 2x + 4)}$

$$= \lim_{x \rightarrow -2} \frac{x-2}{x^2 - 2x + 4}$$

$$= -\frac{4}{12} = -\frac{1}{3}$$

22. $\lim_{x \rightarrow (\pi/4)} \frac{4x}{\tan x} = \frac{4(\pi/4)}{1} = \pi$

$$\begin{aligned}
 24. \lim_{\Delta x \rightarrow 0} \frac{\cos(\pi + \Delta x) + 1}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{\cos \pi \cos \Delta x - \sin \pi \sin \Delta x + 1}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \left[-\frac{(\cos \Delta x - 1)}{\Delta x} \right] - \lim_{\Delta x \rightarrow 0} \left[\sin \pi \frac{\sin \Delta x}{\Delta x} \right] \\
 &= -0 - (0)(1) = 0
 \end{aligned}$$

$$26. \lim_{x \rightarrow c} [f(x) + 2g(x)] = -\frac{3}{4} + 2\left(\frac{2}{3}\right) = \frac{7}{12}$$

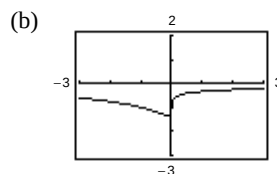
$$28. f(x) = \frac{1 - \sqrt[3]{x}}{x - 1}$$

(a)

x	1.1	1.01	1.001	1.0001
$f(x)$	-0.3228	-0.3322	-0.3332	-0.3333

$$\lim_{x \rightarrow 1^+} \frac{1 - \sqrt[3]{x}}{x - 1} \approx -0.333 \quad (\text{Actual limit is } -\frac{1}{3}.)$$

$$\begin{aligned}
 (c) \lim_{x \rightarrow 1^+} \frac{1 - \sqrt[3]{x}}{x - 1} &= \lim_{x \rightarrow 1^+} \frac{1 - \sqrt[3]{x}}{x - 1} \cdot \frac{1 + \sqrt[3]{x} + (\sqrt[3]{x})^2}{1 + \sqrt[3]{x} + (\sqrt[3]{x})^2} \\
 &= \lim_{x \rightarrow 1^+} \frac{1 - x}{(x - 1)[1 + \sqrt[3]{x} + (\sqrt[3]{x})^2]} \\
 &= \lim_{x \rightarrow 1^+} \frac{-1}{1 + \sqrt[3]{x} + (\sqrt[3]{x})^2} \\
 &= -\frac{1}{3}
 \end{aligned}$$



$$30. s(t) = 0 \Rightarrow -4.9t^2 + 200 = 0 \Rightarrow t^2 \approx 40.816 \Rightarrow t \approx 6.39 \text{ sec}$$

When $t = 6.39$, the velocity is approximately

$$\begin{aligned}
 \lim_{t \rightarrow a} \frac{s(a) - s(t)}{a - t} &= \lim_{t \rightarrow a} -4.9(a + t) \\
 &= \lim_{t \rightarrow 6.39} -4.9(6.39 + 6.39) = -62.6 \text{ m/sec.}
 \end{aligned}$$

$$32. \lim_{x \rightarrow 4} \llbracket x - 1 \rrbracket \text{ does not exist. The graph jumps from 2 to 3 at } x = 4.$$

$$34. \lim_{x \rightarrow 1^+} g(x) = 1 + 1 = 2.$$

$$36. \lim_{s \rightarrow -2} f(s) = 2$$

$$38. f(x) = \begin{cases} \frac{3x^2 - x - 2}{x - 1}, & x \neq 1 \\ 0, & x = 1 \end{cases}$$

$$\begin{aligned}
 \lim_{x \rightarrow 1} f(x) &= \lim_{x \rightarrow 1} \frac{3x^2 - x - 2}{x - 1} \\
 &= \lim_{x \rightarrow 1} (3x + 2) = 5 \neq 0
 \end{aligned}$$

Removable discontinuity at $x = 1$
Continuous on $(-\infty, 1) \cup (1, \infty)$

$$40. f(x) = \begin{cases} 5 - x, & x \leq 2 \\ 2x - 3, & x > 2 \end{cases}$$

$$\lim_{x \rightarrow 2^-} (5 - x) = 3$$

$$\lim_{x \rightarrow 2^+} (2x - 3) = 1$$

Nonremovable discontinuity at $x = 2$

Continuous on $(-\infty, 2) \cup (2, \infty)$

$$44. f(x) = \frac{x + 1}{2x + 2}$$

$$\lim_{x \rightarrow -1} \frac{x + 1}{2(x + 1)} = \frac{1}{2}$$

Removable discontinuity at $x = -1$

Continuous on $(-\infty, -1) \cup (-1, \infty)$

$$48. \lim_{x \rightarrow 1^+} (x + 1) = 2$$

$$\lim_{x \rightarrow 3^-} (x + 1) = 4$$

Find b and c so that $\lim_{x \rightarrow 1^-} (x^2 + bx + c) = 2$ and $\lim_{x \rightarrow 3^+} (x^2 + bx + c) = 4$.

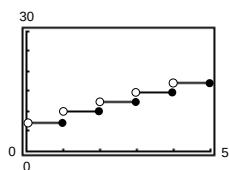
Consequently we get $1 + b + c = 2$ and $9 + 3b + c = 4$.

Solving simultaneously, $b = -3$ and $c = 4$.

$$50. C = 9.80 + 2.50[-\lfloor -x \rfloor - 1], \quad x > 0$$

$$= 9.80 - 2.50[\lfloor -x \rfloor + 1]$$

C has a nonremovable discontinuity at each integer.



$$54. h(x) = \frac{4x}{4 - x^2}$$

Vertical asymptotes at $x = 2$ and $x = -2$

$$58. \lim_{x \rightarrow (1/2)^+} \frac{x}{2x - 1} = \infty$$

$$62. \lim_{x \rightarrow -1^+} \frac{x^2 - 2x + 1}{x + 1} = \infty$$

$$66. \lim_{x \rightarrow 0^+} \frac{\sec x}{x} = \infty$$

$$42. f(x) = \sqrt{\frac{x+1}{x}} = \sqrt{1 + \frac{1}{x}}$$

$$\lim_{x \rightarrow 0^+} \sqrt{1 + \frac{1}{x}} = \infty$$

Domain: $(-\infty, -1], (0, \infty)$

Nonremovable discontinuity at $x = 0$

Continuous on $(-\infty, -1] \cup (0, \infty)$

$$46. f(x) = \tan 2x$$

Nonremovable discontinuities when

$$x = \frac{(2n + 1)\pi}{4}$$

Continuous on

$$\left(\frac{(2n - 1)\pi}{4}, \frac{(2n + 1)\pi}{4} \right)$$

for all integers n .

$$52. f(x) = \sqrt{(x - 1)x}$$

(a) Domain: $(-\infty, 0] \cup [1, \infty)$

$$(b) \lim_{x \rightarrow 0^-} f(x) = 0$$

$$(c) \lim_{x \rightarrow 1^+} f(x) = 0$$

$$56. f(x) = \csc \pi x$$

Vertical asymptote at every integer k

$$60. \lim_{x \rightarrow -1^-} \frac{x + 1}{x^4 - 1} = \lim_{x \rightarrow -1^-} \frac{1}{(x^2 + 1)(x - 1)} = -\frac{1}{4}$$

$$64. \lim_{x \rightarrow 2^-} \frac{1}{\sqrt[3]{x^2 - 4}} = -\infty$$

$$68. \lim_{x \rightarrow 0^-} \frac{\cos^2 x}{x} = -\infty$$

70. $f(x) = \frac{\tan 2x}{x}$

(a)

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	2.0271	2.0003	2.0000	2.0000	2.0003	2.0271

$$\lim_{x \rightarrow 0} \frac{\tan 2x}{x} = 2$$

(b) Yes, define

$$f(x) = \begin{cases} \frac{\tan 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

Now $f(x)$ is continuous at $x = 0$.

Problem Solving for Chapter 1

2. (a) Area $\triangle PAO = \frac{1}{2}bh = \frac{1}{2}(1)(x) = \frac{x}{2}$

$$\text{Area } \triangle PBO = \frac{1}{2}bh = \frac{1}{2}(1)(y) = \frac{y}{2} = \frac{x^2}{2}$$

(b) $a(x) = \frac{\text{Area } \triangle PBO}{\text{Area } \triangle PAO} = \frac{x^2/2}{x/2} = x$

x	4	2	1	0.1	0.01
Area $\triangle PAO$	2	1	1/2	1/20	1/200
Area $\triangle PBO$	8	2	1/2	1/200	1/20,000
$a(x)$	4	2	1	1/10	1/100

(c) $\lim_{x \rightarrow 0^+} a(x) = \lim_{x \rightarrow 0^+} x = 0$

6.
$$\frac{\sqrt{a+bx} - \sqrt{3}}{x} = \frac{\sqrt{a+bx} - \sqrt{3}}{x} \cdot \frac{\sqrt{a+bx} + \sqrt{3}}{\sqrt{a+bx} + \sqrt{3}}$$

$$= \frac{(a+bx) - 3}{x(\sqrt{a+bx} + \sqrt{3})}$$

Letting $a = 3$ simplifies the numerator.

Thus,

$$\lim_{x \rightarrow 0} \frac{\sqrt{3+bx} - \sqrt{3}}{x} = \lim_{x \rightarrow 0} \frac{bx}{x(\sqrt{3+bx} + \sqrt{3})}$$

$$= \lim_{x \rightarrow 0} \frac{b}{\sqrt{3+bx} + \sqrt{3}}$$

Setting $\frac{b}{\sqrt{3} + \sqrt{3}} = \sqrt{3}$, you obtain $b = 6$.

Thus, $a = 3$ and $b = 6$.

4. (a) Slope $= \frac{4-0}{3-0} = \frac{4}{3}$

(b) Slope $= -\frac{3}{4}$ Tangent line: $y - 4 = -\frac{3}{4}(x - 3)$

$$y = -\frac{3}{4}x + \frac{25}{4}$$

(c) Let $Q = (x, y) = (x, \sqrt{25-x^2})$

$$m_x = \frac{\sqrt{25-x^2} - 4}{x - 3}$$

(d) $\lim_{x \rightarrow 3} m_x = \lim_{x \rightarrow 3} \frac{\sqrt{25-x^2} - 4}{x - 3} \cdot \frac{\sqrt{25-x^2} + 4}{\sqrt{25-x^2} + 4}$

$$= \lim_{x \rightarrow 3} \frac{25 - x^2 - 16}{(x - 3)(\sqrt{25-x^2} + 4)}$$

$$= \lim_{x \rightarrow 3} \frac{(3-x)(3+x)}{(x-3)(\sqrt{25-x^2} + 4)}$$

$$= \lim_{x \rightarrow 3} \frac{-(3+x)}{\sqrt{25-x^2} + 4} = \frac{-6}{4+4} = -\frac{3}{4}$$

This is the slope of the tangent line at P .

8. $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (a^2 - 2) = a^2 - 2$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{ax}{\tan x} = a \left(\text{because } \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right)$$

Thus,

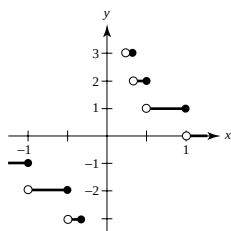
$$a^2 - 2 = a$$

$$a^2 - a - 2 = 0$$

$$(a-2)(a+1) = 0$$

$$a = -1, 2$$

10.



$$\begin{aligned} \text{(a)} \quad f\left(\frac{1}{4}\right) &= \llbracket 4 \rrbracket = 4 \\ f(3) &= \llbracket \frac{1}{3} \rrbracket = 0 \\ f(1) &= \llbracket 1 \rrbracket = 1 \end{aligned}$$

$$\text{(b)} \quad \lim_{x \rightarrow 1^-} f(x) = 1$$

$$\lim_{x \rightarrow 1^+} f(x) = 0$$

$$\lim_{x \rightarrow 0^-} f(x) = -\infty$$

$$\lim_{x \rightarrow 0^+} f(x) = \infty$$

(c) f is continuous for all real numbers except

$$x = 0, \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \dots$$

$$12. \text{ (a)} \quad v^2 = \frac{192,000}{r} + v_0^2 - 48$$

$$\frac{192,000}{r} = v^2 - v_0^2 + 48$$

$$r = \frac{192,000}{v - v_0^2 + 48}$$

$$\lim_{v \rightarrow 0} r = \frac{192,000}{48 - v_0^2}$$

$$\text{Let } v_0 = \sqrt{48} = 4\sqrt{3} \text{ feet/sec.}$$

$$\text{(b)} \quad v^2 = \frac{1920}{r} + v_0^2 - 2.17$$

$$\frac{1920}{r} = v^2 - v_0^2 + 2.17$$

$$r = \frac{1920}{v^2 - v_0^2 + 2.17}$$

$$\lim_{v \rightarrow 0} r = \frac{1920}{2.17 - v_0^2}$$

$$\text{Let } v_0 = \sqrt{2.17} \text{ mi/sec } (\approx 1.47 \text{ mi/sec}).$$

$$\text{(c)} \quad r = \frac{10,600}{v^2 - v_0^2 + 6.99}$$

$$\lim_{v \rightarrow 0} r = \frac{10,600}{6.99 - v_0^2}$$

$$\text{Let } v_0 = \sqrt{6.99} \approx 2.64 \text{ mi/sec.}$$

Since this is smaller than the escape velocity for earth, the mass is less.

14. Let $a \neq 0$ and let $\epsilon > 0$ be given. There exists $\delta_1 > 0$ such that if $0 < |x - 0| < \delta$, then $|f(x) - L| < \epsilon$. Let $\delta = \delta_1/|a|$. Then for $0 < |x - 0| < \delta = \delta_1/|a|$, you have

$$|x| < \frac{\delta_1}{|a|}$$

$$|ax| < \delta_1$$

$$|f(ax) - L| < \epsilon.$$

$$\text{As a counterexample, let } f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}.$$

$$\text{Then } \lim_{x \rightarrow 0} f(x) = 1 = L,$$

$$\text{but } \lim_{x \rightarrow 0} f(ax) = \lim_{x \rightarrow 0} f(0) = 2.$$

CHAPTER 2

Differentiation

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CHAPTER 2

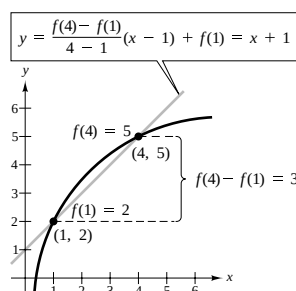
Differentiation

Section 2.1 The Derivative and the Tangent Line Problem

Solutions to Odd-Numbered Exercises

1. (a) $m = 0$
(b) $m = -3$

3. (a), (b)



$$\begin{aligned} \text{(c) } y &= \frac{f(4) - f(1)}{4 - 1}(x - 1) + f(1) \\ &= \frac{3}{3}(x - 1) + 2 \\ &= 1(x - 1) + 2 \\ &= x + 1 \end{aligned}$$

5. $f(x) = 3 - 2x$ is a line. Slope $= -2$

$$\begin{aligned} 7. \text{ Slope at } (1, -3) &= \lim_{\Delta x \rightarrow 0} \frac{g(1 + \Delta x) - g(1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(1 + \Delta x)^2 - 4 - (-3)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1 + 2(\Delta x) + (\Delta x)^2 - 1}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} [2 + 2(\Delta x)] = 2 \end{aligned}$$

$$\begin{aligned} 9. \text{ Slope at } (0, 0) &= \lim_{\Delta t \rightarrow 0} \frac{f(0 + \Delta t) - f(0)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{3(\Delta t) - (\Delta t)^2 - 0}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} (3 - \Delta t) = 3 \end{aligned}$$

$$\begin{aligned} 11. f(x) &= 3 \\ f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{3 - 3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} 0 = 0 \end{aligned}$$

13. $f(x) = -5x$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-5(x + \Delta x) - (-5x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} -5 = -5 \end{aligned}$$

15. $h(s) = 3 + \frac{2}{3}s$

$$\begin{aligned} h'(s) &= \lim_{\Delta s \rightarrow 0} \frac{h(s + \Delta s) - h(s)}{\Delta s} \\ &= \lim_{\Delta s \rightarrow 0} \frac{3 + \frac{2}{3}(s + \Delta s) - \left(3 + \frac{2}{3}s\right)}{\Delta s} \\ &= \lim_{\Delta s \rightarrow 0} \frac{\frac{2}{3}\Delta s}{\Delta s} = \frac{2}{3} \end{aligned}$$

17. $f(x) = 2x^2 + x - 1$

$$\begin{aligned}f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \frac{[2(x + \Delta x)^2 + (x + \Delta x) - 1] - [2x^2 + x - 1]}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \frac{(2x^2 + 4x\Delta x + 2(\Delta x)^2 + x + \Delta x - 1) - (2x^2 + x - 1)}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \frac{4x\Delta x + 2(\Delta x)^2 + \Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} (4x + 2\Delta x + 1) = 4x + 1\end{aligned}$$

19. $f(x) = x^3 - 12x$

$$\begin{aligned}f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^3 - 12(x + \Delta x)] - [x^3 - 12x]}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12x - 12\Delta x - x^3 + 12x}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12\Delta x}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + (\Delta x)^2 - 12) = 3x^2 - 12\end{aligned}$$

21. $f(x) = \frac{1}{x - 1}$

$$\begin{aligned}f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x + \Delta x - 1} - \frac{1}{x - 1}}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \frac{(x - 1) - (x + \Delta x - 1)}{\Delta x(x + \Delta x - 1)(x - 1)} \\&= \lim_{\Delta x \rightarrow 0} \frac{-\Delta x}{\Delta x(x + \Delta x - 1)(x - 1)} \\&= \lim_{\Delta x \rightarrow 0} \frac{-1}{(x + \Delta x - 1)(x - 1)} \\&= -\frac{1}{(x - 1)^2}\end{aligned}$$

23. $f(x) = \sqrt{x + 1}$

$$\begin{aligned}f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x + 1} - \sqrt{x + 1}}{\Delta x} \cdot \left(\frac{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}}{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}} \right) \\&= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x + 1) - (x + 1)}{\Delta x[\sqrt{x + \Delta x + 1} + \sqrt{x + 1}]} \\&= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}} \\&= \frac{1}{\sqrt{x + 1} + \sqrt{x + 1}} = \frac{1}{2\sqrt{x + 1}}\end{aligned}$$

25. (a) $f(x) = x^2 + 1$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 + 1] - [x^2 + 1]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x
 \end{aligned}$$

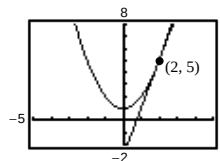
At $(2, 5)$, the slope of the tangent line is
 $m = 2(2) = 4$. The equation of the tangent line is

$$y - 5 = 4(x - 2)$$

$$y - 5 = 4x - 8$$

$$y = 4x - 3.$$

(b)



27. (a) $f(x) = x^3$

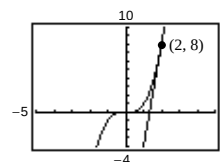
$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2
 \end{aligned}$$

At $(2, 8)$, the slope of the tangent is $m = 3(2)^2 = 12$.
 The equation of the tangent line is

$$y - 8 = 12(x - 2)$$

$$y = 12x - 16.$$

(b)



29. (a) $f(x) = \sqrt{x}$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
 \end{aligned}$$

At $(1, 1)$, the slope of the tangent line is

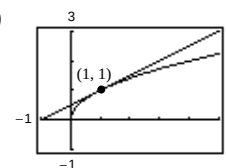
$$m = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

The equation of the tangent line is

$$y - 1 = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x + \frac{1}{2}.$$

(b)



31. (a) $f(x) = \frac{4}{x}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) + \frac{4}{x + \Delta x} - \left(x + \frac{4}{x}\right)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x(x + \Delta x)(x + \Delta x) + 4x - x^2(x + \Delta x) - 4(x + \Delta x)}{x(\Delta x)(x + \Delta x)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 2x^2(\Delta x) + x(\Delta x)^2 - x^3 - x^2(\Delta x) - 4(\Delta x)}{x(\Delta x)(x + \Delta x)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2(\Delta x) + x(\Delta x)^2 - 4(\Delta x)}{x(\Delta x)(x + \Delta x)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + x(\Delta x) - 4}{x(x + \Delta x)} \\ &= \frac{x^2 - 4}{x^2} = 1 - \frac{4}{x^2} \end{aligned}$$

At (4, 5), the slope of the tangent line is

$$m = 1 - \frac{4}{16} = \frac{3}{4}$$

The equation of the tangent line is

$$y - 5 = \frac{3}{4}(x - 4)$$

$$y = \frac{3}{4}x + 2$$

33. From Exercise 27 we know that $f'(x) = 3x^2$. Since the slope of the given line is 3, we have

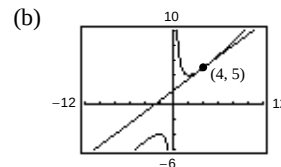
$$3x^2 = 3$$

$$x = \pm 1.$$

Therefore, at the points (1, 1) and (-1, -1) the tangent lines are parallel to $3x - y + 1 = 0$. These lines have equations

$$y - 1 = 3(x - 1) \quad \text{and} \quad y + 1 = 3(x + 1)$$

$$y = 3x - 2 \quad \quad \quad y = 3x + 2.$$



35. Using the limit definition of derivative,

$$f'(x) = \frac{-1}{2x\sqrt{x}}.$$

Since the slope of the given line is $-\frac{1}{2}$, we have

$$-\frac{1}{2x\sqrt{x}} = -\frac{1}{2}$$

$$x = 1.$$

Therefore, at the point (1, 1) the tangent line is parallel to $x + 2y - 6 = 0$. The equation of this line is

$$y - 1 = -\frac{1}{2}(x - 1)$$

$$y - 1 = -\frac{1}{2}x + \frac{1}{2}$$

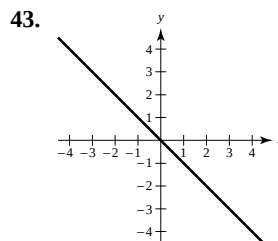
$$y = -\frac{1}{2}x + \frac{3}{2}.$$

37. $g(5) = 2$ because the tangent line passes through (5, 2)

$$g'(5) = \frac{2 - 0}{5 - 9} = \frac{2}{-4} = -\frac{1}{2}$$

39. $f(x) = x \Rightarrow f'(x) = 1$ (b)

41. $f(x) = \sqrt{x} \Rightarrow f'(x)$ matches (a)
(decreasing slope as $x \rightarrow \infty$)



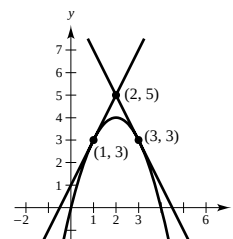
Answers will vary.

Sample answer: $y = -x$

45. (a) If $f'(c) = 3$ and f is odd, then $f'(-c) = f'(c) = 3$
(b) If $f'(c) = 3$ and f is even, then $f'(-c) = -f'(c) = -3$

47. Let (x_0, y_0) be a point of tangency on the graph of f . By the limit definition for the derivative, $f'(x) = 4 - 2x$. The slope of the line through $(2, 5)$ and (x_0, y_0) equals the derivative of f at x_0 :

$$\begin{aligned}\frac{5 - y_0}{2 - x_0} &= 4 - 2x_0 \\ 5 - y_0 &= (2 - x_0)(4 - 2x_0) \\ 5 - (4x_0 - x_0^2) &= 8 - 8x_0 + 2x_0^2 \\ 0 &= x_0^2 - 4x_0 + 3 \\ 0 &= (x_0 - 1)(x_0 - 3) \Rightarrow x_0 = 1, 3\end{aligned}$$



Therefore, the points of tangency are $(1, 3)$ and $(3, 3)$, and the corresponding slopes are 2 and -2 . The equations of the tangent lines are

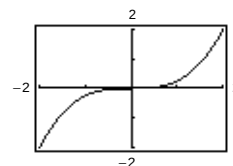
$$\begin{aligned}y - 5 &= 2(x - 2) & y - 5 &= -2(x - 2) \\ y &= 2x + 1 & y &= -2x + 9\end{aligned}$$

49. (a) $g'(0) = -3$
(b) $g'(3) = 0$
(c) Because $g'(1) = -\frac{8}{3}$, g is decreasing (falling) at $x = 1$.
(d) Because $g'(-4) = \frac{7}{5}$, g is increasing (rising) at $x = -4$.
(e) Because $g'(4)$ and $g'(6)$ are both positive, $g(6)$ is greater than $g(4)$, and $g(6) - g(4) > 0$.
(f) No, it is not possible. All you can say is that g is decreasing (falling) at $x = 2$.

51. $f(x) = \frac{1}{4}x^3$

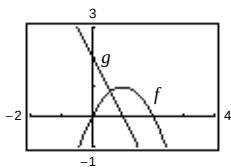
By the limit definition of the derivative we have $f'(x) = \frac{3}{4}x^2$.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	-2	$-\frac{27}{32}$	$-\frac{1}{4}$	$-\frac{1}{32}$	0	$\frac{1}{32}$	$\frac{1}{4}$	$\frac{27}{32}$	2
$f'(x)$	3	$\frac{27}{16}$	$\frac{3}{4}$	$\frac{3}{16}$	0	$\frac{3}{16}$	$\frac{3}{4}$	$\frac{27}{16}$	3



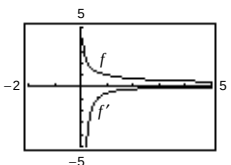
$$53. g(x) = \frac{f(x + 0.01) - f(x)}{0.01}$$

$$= (2(x + 0.01) - (x + 0.01)^2 - 2x + x^2) + 100$$



The graph of $g(x)$ is approximately the graph of $f'(x)$.

$$57. f(x) = \frac{1}{\sqrt{x}} \text{ and } f'(x) = \frac{-1}{2x^{3/2}}.$$



As $x \rightarrow \infty$, f is nearly horizontal and thus $f' \approx 0$.

$$59. f(x) = 4 - (x - 3)^2$$

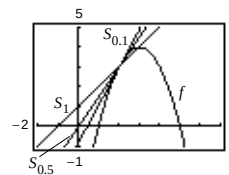
$$S_{\Delta x}(x) = \frac{f(2 + \Delta x) - f(2)}{\Delta x}(x - 2) + f(2)$$

$$= \frac{4 - (2 + \Delta x - 3)^2 - 3}{\Delta x}(x - 2) + 3 = \frac{1 - (\Delta x - 1)^2}{\Delta x}(x - 2) + 3 = (-\Delta x + 2)(x - 2) + 3$$

$$(a) \Delta x = 1: S_{\Delta x} = (x - 2) + 3 = x + 1$$

$$\Delta x = 0.5: S_{\Delta x} = \left(\frac{3}{2}\right)(x - 2) + 3 = \frac{3}{2}x$$

$$\Delta x = 0.1: S_{\Delta x} = \left(\frac{19}{10}\right)(x - 2) + 3 = \frac{19}{10}x - \frac{4}{5}$$



(b) As $\Delta x \rightarrow 0$, the line approaches the tangent line to f at $(2, 3)$.

$$61. f(x) = x^2 - 1, c = 2$$

$$f'(2) = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2} \frac{(x^2 - 1) - 3}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 2) = 4$$

$$63. f(x) = x^3 + 2x^2 + 1, c = -2$$

$$f'(-2) = \lim_{x \rightarrow -2} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \rightarrow -2} \frac{x^2(x + 2)}{x + 2} = \lim_{x \rightarrow -2} \frac{(x^3 + 2x^2 + 1) - 1}{x + 2} = \lim_{x \rightarrow -2} x^2 = 4$$

$$65. g(x) = \sqrt{|x|}, c = 0$$

$$g'(0) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\sqrt{|x|}}{x}. \text{ Does not exist.}$$

$$\text{As } x \rightarrow 0^-, \frac{\sqrt{|x|}}{x} = \frac{-1}{\sqrt{x}} \rightarrow -\infty$$

$$\text{As } x \rightarrow 0^+, \frac{\sqrt{|x|}}{x} = \frac{1}{\sqrt{x}} \rightarrow \infty$$

$$67. f(x) = (x - 6)^{2/3}, c = 6$$

$$f'(6) = \lim_{x \rightarrow 6} \frac{f(x) - f(6)}{x - 6}$$

$$= \lim_{x \rightarrow 6} \frac{(x - 6)^{2/3} - 0}{x - 6}$$

$$= \lim_{x \rightarrow 6} \frac{1}{(x - 6)^{1/3}}$$

Does not exist.

69. $h(x) = (x + 5)$, $c = -5$

$$\begin{aligned} h'(-5) &= \lim_{x \rightarrow -5} \frac{h(x) - h(-5)}{x - (-5)} \\ &= \lim_{x \rightarrow -5} \frac{|x + 5| - 0}{x + 5} \\ &= \lim_{x \rightarrow -5} \frac{|x + 5|}{x + 5} \end{aligned}$$

Does not exist.

73. $f(x)$ is differentiable everywhere except at $x = -1$.
(Discontinuity)

77. $f(x)$ is differentiable on the interval $(1, \infty)$.
(At $x = 1$ the tangent line is vertical)

81. $f(x) = |x - 1|$

The derivative from the left is

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{|x - 1| - 0}{x - 1} = -1.$$

The derivative from the right is

$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{|x - 1| - 0}{x - 1} = 1.$$

The one-sided limits are not equal. Therefore, f is not differentiable at $x = 1$.

71. $f(x)$ is differentiable everywhere except at $x = -3$.
(Sharp turn in the graph.)

75. $f(x)$ is differentiable everywhere except at $x = 3$.
(Sharp turn in the graph)

79. $f(x)$ is differentiable everywhere except at $x = 0$.
(Discontinuity)

83. $f(x) = \begin{cases} (x - 1)^3, & x \leq 1 \\ (x - 1)^2, & x > 1 \end{cases}$

The derivative from the left is

$$\begin{aligned} \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1^-} \frac{(x - 1)^3 - 0}{x - 1} \\ &= \lim_{x \rightarrow 1^-} (x - 1)^2 = 0. \end{aligned}$$

The derivative from the right is

$$\begin{aligned} \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1^+} \frac{(x - 1)^2 - 0}{x - 1} \\ &= \lim_{x \rightarrow 1^+} (x - 1) = 0. \end{aligned}$$

These one-sided limits are equal. Therefore, f is differentiable at $x = 1$. ($f'(1) = 0$)

85. Note that f is continuous at $x = 2$. $f(x) = \begin{cases} x^2 + 1, & x \leq 2 \\ 4x - 3, & x > 2 \end{cases}$

The derivative from the left is $\lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{(x^2 + 1) - 5}{x - 2} = \lim_{x \rightarrow 2^-} (x + 2) = 4$.

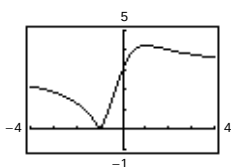
The derivative from the right is $\lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^+} \frac{(4x - 3) - 5}{x - 2} = \lim_{x \rightarrow 2^+} 4 = 4$.

The one-sided limits are equal. Therefore, f is differentiable at $x = 2$. ($f'(2) = 4$)

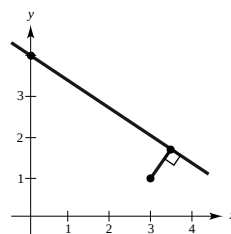
87. (a) The distance from $(3, 1)$ to the line $mx - y + 4 = 0$ is

$$\begin{aligned} d &= \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \\ &= \frac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}} = \frac{|3m + 3|}{\sqrt{m^2 + 1}} \end{aligned}$$

(b)



The function d is not differentiable at $m = -1$. This corresponds to the line $y = -x + 4$, which passes through the point $(3, 1)$.



89. False. the slope is $\lim_{\Delta x \rightarrow 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x}$.

91. False. If the derivative from the left of a point does not equal the derivative from the right of a point, then the derivative does not exist at that point. For example, if $f(x) = |x|$, then the derivative from the left at $x = 0$ is -1 and the derivative from the right at $x = 0$ is 1 . At $x = 0$, the derivative does not exist.

$$93. f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Using the Squeeze Theorem, we have $-|x| \leq x \sin(1/x) \leq |x|$, $x \neq 0$. Thus, $\lim_{x \rightarrow 0} x \sin(1/x) = 0 = f(0)$ and f is continuous at $x = 0$. Using the alternative form of the derivative we have

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x \sin(1/x) - 0}{x - 0} = \lim_{x \rightarrow 0} \left(\sin \frac{1}{x} \right).$$

Since this limit does not exist (it oscillates between -1 and 1), the function is not differentiable at $x = 0$.

$$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Using the Squeeze Theorem again we have $-x^2 \leq x^2 \sin(1/x) \leq x^2$, $x \neq 0$. Thus, $\lim_{x \rightarrow 0} x^2 \sin(1/x) = 0 = f(0)$ and f is continuous at $x = 0$. Using the alternative form of the derivative again we have

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0.$$

Therefore, g is differentiable at $x = 0$, $g'(0) = 0$.

Section 2.2 Basic Differentiation Rules and Rates of Change

$$\begin{aligned} 1. \quad (a) \quad y &= x^{1/2} \\ y' &= \frac{1}{2}x^{-1/2} \\ y'(1) &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} (b) \quad y &= x^{3/2} \\ y' &= \frac{3}{2}x^{1/2} \\ y'(1) &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} (c) \quad y &= x^2 \\ y' &= 2x \\ y'(1) &= 2 \end{aligned}$$

$$\begin{aligned} (d) \quad y &= x^3 \\ y' &= 3x^2 \\ y'(1) &= 3 \end{aligned}$$

$$\begin{aligned} 3. \quad y &= 8 \\ y' &= 0 \end{aligned}$$

$$\begin{aligned} 5. \quad y &= x^6 \\ y' &= 6x^5 \end{aligned}$$

$$\begin{aligned} 7. \quad y &= \frac{1}{x^7} = x^{-7} \\ y' &= -7x^{-8} = -\frac{7}{x^8} \end{aligned}$$

$$\begin{aligned} 9. \quad y &= \sqrt[5]{x} = x^{1/5} \\ y' &= \frac{1}{5}x^{-4/5} = \frac{1}{5x^{4/5}} \end{aligned}$$

$$\begin{aligned} 11. \quad f(x) &= x + 1 \\ f'(x) &= 1 \end{aligned}$$

$$\begin{aligned} 13. \quad f(t) &= -2t^2 + 3t - 6 \\ f'(t) &= -4t + 3 \end{aligned}$$

$$\begin{aligned} 15. \quad g(x) &= x^2 + 4x^3 \\ g'(x) &= 2x + 12x^2 \end{aligned}$$

$$\begin{aligned} 17. \quad s(t) &= t^3 - 2t + 4 \\ s'(t) &= 3t^2 - 2 \end{aligned}$$

$$\begin{aligned} 19. \quad y &= \frac{\pi}{2} \sin \theta - \cos \theta \\ y' &= \frac{\pi}{2} \cos \theta + \sin \theta \end{aligned}$$

$$\begin{aligned} 21. \quad y &= x^2 - \frac{1}{2} \cos x \\ y' &= 2x + \frac{1}{2} \sin x \end{aligned}$$

$$\begin{aligned} 23. \quad y &= \frac{1}{x} - 3 \sin x \\ y' &= -\frac{1}{x^2} - 3 \cos x \end{aligned}$$

<u>Function</u>	<u>Rewrite</u>	<u>Derivative</u>	<u>Simplify</u>
25. $y = \frac{5}{2x^2}$	$y = \frac{5}{2}x^{-2}$	$y' = -5x^{-3}$	$y' = \frac{-5}{x^3}$
27. $y = \frac{3}{(2x)^3}$	$y = \frac{3}{8}x^{-3}$	$y' = \frac{-9}{8}x^{-4}$	$y' = \frac{-9}{8x^4}$
29. $y = \frac{\sqrt{x}}{x}$	$y = x^{-1/2}$	$y' = -\frac{1}{2}x^{-3/2}$	$y' = -\frac{1}{2x^{3/2}}$
-
31. $f(x) = \frac{3}{x^2} = 3x^{-2}, (1, 3)$

$f'(x) = -6x^{-3} = \frac{-6}{x^3}$

$f'(1) = -6$

33. $f(x) = -\frac{1}{2} + \frac{7}{5}x^3, \left(0, -\frac{1}{2}\right)$

$f'(x) = \frac{21}{5}x^2$

$f'(0) = 0$

35. $y = (2x + 1)^2, (0, 1)$

$= 4x^2 + 4x + 1$

$y' = 8x + 4$

$y'(0) = 4$
-
37. $f(\theta) = 4 \sin \theta - \theta, (0, 0)$

$f'(\theta) = 4 \cos \theta - 1$

$f'(0) = 4(1) - 1 = 3$

39. $f(x) = x^2 + 5 - 3x^{-2}$

$f'(x) = 2x + 6x^{-3} = 2x + \frac{6}{x^3}$

41. $g(t) = t^2 - \frac{4}{t^3} = t^2 - 4t^{-3}$

$g'(t) = 2t + 12t^{-4} = 2t + \frac{12}{t^4}$
-
43. $f(x) = \frac{x^3 - 3x^2 + 4}{x^2} = x - 3 + 4x^{-2}$

$f'(x) = 1 - \frac{8}{x^3} = \frac{x^3 - 8}{x^3}$

45. $y = x(x^2 + 1) = x^3 + x$

$y' = 3x^2 + 1$
-
47. $f(x) = \sqrt{x} - 6\sqrt[3]{x} = x^{1/2} - 6x^{1/3}$

$f'(x) = \frac{1}{2}x^{-1/2} + 2x^{-2/3} = \frac{1}{2\sqrt{x}} + \frac{2}{x^{2/3}}$

49. $h(s) = s^{4/5} - s^{2/3}$

$h'(s) = \frac{4}{5}s^{-4/5} - \frac{2}{3}s^{-1/3} = \frac{4}{5s^{4/5}} - \frac{2}{3s^{1/3}}$
-
51. $f(x) = 6\sqrt{x} + 5 \cos x = 6x^{1/2} + 5 \cos x$

$f'(x) = 3x^{-1/2} - 5 \sin x = \frac{3}{\sqrt{x}} - 5 \sin x$

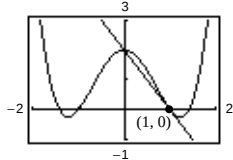
53. (a) $y = x^4 - 3x^2 + 2$

$y' = 4x^3 - 6x$

At $(1, 0)$: $y' = 4(1)^3 - 6(1) = -2$.

Tangent line: $y - 0 = -2(x - 1)$

$2x + y - 2 = 0$
-
- (b)



55. (a) $f(x) = \frac{2}{\sqrt[4]{x^3}} = 2x^{-3/4}$

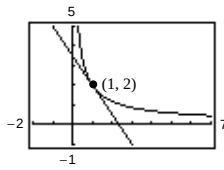
$f'(x) = \frac{-3}{2}x^{-7/4} = \frac{-3}{2x^{7/4}}$

At $(1, 2)$, $f'(1) = \frac{-3}{2}$

Tangent line: $y - 2 = -\frac{3}{2}(x - 1)$

$y = -\frac{3}{2}x + \frac{7}{2}$

$3x + 2y - 7 = 0$
-
- (b)



57. $y = x^4 - 8x^2 + 2$

$$y' = 4x^3 - 16x$$

$$= 4x(x^2 - 4)$$

$$= 4x(x - 2)(x + 2)$$

$$y' = 0 \Rightarrow x = 0, \pm 2$$

Horizontal tangents: $(0, 2)$, $(2, -14)$, $(-2, -14)$

61. $y = x + \sin x$, $0 \leq x < 2\pi$

$$y' = 1 + \cos x = 0$$

$$\cos x = -1 \Rightarrow x = \pi$$

At $x = \pi$, $y = \pi$.Horizontal tangent: (π, π)

65. $\frac{k}{x} = -\frac{3}{4}x + 3$ Equate functions

$$-\frac{k}{x^2} = -\frac{3}{4}$$
 Equate derivatives

$$\text{Hence, } k = \frac{3}{4}x^2 \text{ and } \frac{\frac{3}{4}x^2}{x} = -\frac{3}{4}x + 3 \Rightarrow \frac{3}{4}x = -\frac{3}{4}x + 3 \Rightarrow \frac{3}{2}x = 3 \Rightarrow x = 2 \Rightarrow k = 3.$$

67. (a) The slope appears to be steepest between A and B.

(b) The average rate of change between A and B is **greater** than the instantaneous rate of change at B.

59. $y = \frac{1}{x^2} = x^{-2}$

$$y' = -2x^{-3} = \frac{-2}{x^3} \text{ cannot equal zero.}$$

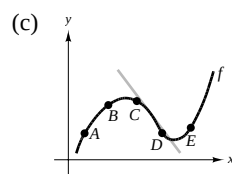
Therefore, there are no horizontal tangents.

63. $x^2 - kx = 4x - 9$ Equate functions

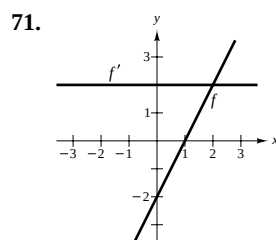
$$2x - k = 4$$
 Equate derivatives

Hence, $k = 2x - 4$ and

$$x^2 - (2x - 4)x = 4x - 9 \Rightarrow -x^2 = -9 \Rightarrow x = \pm 3.$$

For $x = 3$, $k = 2$ and for $x = -3$, $k = -10$.

69. $g(x) = f(x) + 6 \Rightarrow g'(x) = f'(x)$

If f is linear then its derivative is a constant function.

$$f(x) = ax + b$$

$$f'(x) = a$$

73. Let (x_1, y_1) and (x_2, y_2) be the points of tangency on $y = x^2$ and $y = -x^2 + 6x - 5$, respectively. The derivatives of these functions are

$$y' = 2x \Rightarrow m = 2x_1 \quad \text{and} \quad y' = -2x + 6 \Rightarrow m = -2x_2 + 6.$$

$$m = 2x_1 = -2x_2 + 6$$

$$x_1 = -x_2 + 3$$

Since $y_1 = x_1^2$ and $y_2 = -x_2^2 + 6x_2 - 5$,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-x_2^2 + 6x_2 - 5) - (-x_1^2)}{x_2 - x_1} = -2x_2 + 6.$$

$$\frac{(-x_2^2 + 6x_2 - 5) - (-x_2 + 3)^2}{x_2 - (-x_2 + 3)} = -2x_2 + 6$$

$$(-x_2^2 + 6x_2 - 5) - (x_2^2 - 6x_2 + 9) = (-2x_2 + 6)(2x_2 - 3)$$

$$-2x_2^2 + 12x_2 - 14 = -4x_2^2 + 18x_2 - 18$$

$$2x_2^2 - 6x_2 + 4 = 0$$

$$2(x_2 - 2)(x_2 - 1) = 0$$

$$x_2 = 1 \text{ or } 2$$

$$x_2 = 1 \Rightarrow y_2 = 0, x_1 = 2 \text{ and } y_1 = 4$$

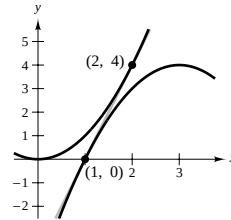
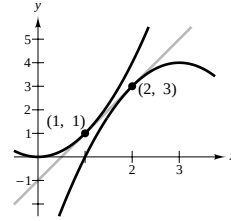
Thus, the tangent line through $(1, 0)$ and $(2, 4)$ is

$$y - 0 = \left(\frac{4 - 0}{2 - 1}\right)(x - 1) \Rightarrow y = 4x - 4.$$

$$x_2 = 2 \Rightarrow y_2 = 3, x_1 = 1 \text{ and } y_1 = 1$$

Thus, the tangent line through $(2, 3)$ and $(1, 1)$ is

$$y - 1 = \left(\frac{3 - 1}{2 - 1}\right)(x - 1) \Rightarrow y = 2x - 1.$$



75. $f(x) = \sqrt{x}, (-4, 0)$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{1}{2\sqrt{x}} = \frac{0 - y}{-4 - x}$$

$$4 + x = 2\sqrt{xy}$$

$$4 + x = 2\sqrt{x}\sqrt{y}$$

$$4 + x = 2x$$

$$x = 4, y = 2$$

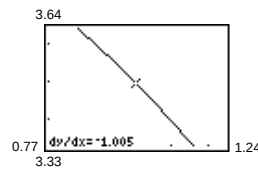
The point $(4, 2)$ is on the graph of f .

Tangent line: $y - 2 = \frac{0 - 2}{-4 - 4}(x - 4)$

$$4y - 8 = x - 4$$

$$0 = x - 4y + 4$$

77. $f'(1) = -1$

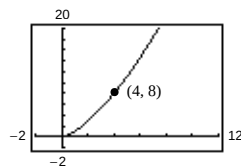


79. (a) One possible secant is between (3.9, 7.7019) and (4, 8):

$$y - 8 = \frac{8 - 7.7019}{4 - 3.9}(x - 4)$$

$$y - 8 = 2.981(x - 4)$$

$$y = S(x) = 2.981x - 3.924$$

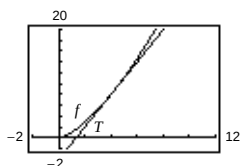


$$(b) f'(x) = \frac{3}{2}x^{1/2} \Rightarrow f'(4) = \frac{3}{2}(2) = 3$$

$$T(x) = 3(x - 4) + 8 = 3x - 4$$

$S(x)$ is an approximation of the tangent line $T(x)$.

- (c) As you move further away from (4, 8), the accuracy of the approximation T gets worse.



(d)

Δx	-3	-2	-1	-0.5	-0.1	0	0.1	0.5	1	2	3
$f(4 + \Delta x)$	1	2.828	5.196	6.548	7.702	8	8.302	9.546	11.180	14.697	18.520
$T(4 + \Delta x)$	-1	2	5	6.5	7.7	8	8.3	9.5	11	14	17

81. False. Let $f(x) = x^2$ and $g(x) = x^2 + 4$. Then $f'(x) = g'(x) = 2x$, but $f(x) \neq g(x)$.

83. False. If $y = \pi^2$, then $dy/dx = 0$. (π^2 is a constant.)

85. True. If $g(x) = 3f(x)$, then $g'(x) = 3f'(x)$.

87. $f(t) = 2t + 7, [1, 2]$

$$f'(t) = 2$$

Instantaneous rate of change is the constant 2.

Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{[2(2) + 7] - [2(1) + 7]}{1} = 2$$

(These are the same because f is a line of slope 2.)

89. $f(x) = -\frac{1}{x}, [1, 2]$

$$f'(x) = \frac{1}{x^2}$$

Instantaneous rate of change:

$$(1, -1) \Rightarrow f'(1) = 1$$

$$\left(2, -\frac{1}{2}\right) \Rightarrow f'(2) = \frac{1}{4}$$

Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{(-1/2) - (-1)}{1} = \frac{1}{2}$$

91. (a) $s(t) = -16t^2 + 1362$

$$v(t) = -32t$$

(b) $\frac{s(2) - s(1)}{2 - 1} = 1298 - 1346 = -48 \text{ ft/sec}$

(c) $v(t) = s'(t) = -32t$

When $t = 1$: $v(1) = -32 \text{ ft/sec}$.

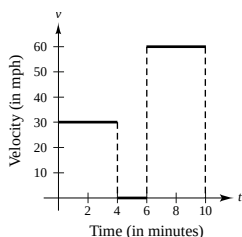
When $t = 2$: $v(2) = -64 \text{ ft/sec}$.

(d) $-16t^2 + 1362 = 0$

$$t^2 = \frac{1362}{16} \Rightarrow t = \frac{\sqrt{1362}}{4} \approx 9.226 \text{ sec}$$

(e) $v\left(\frac{\sqrt{1362}}{4}\right) = -32\left(\frac{\sqrt{1362}}{4}\right) = -8\sqrt{1362} \approx -295.242 \text{ ft/sec}$

95.



(The velocity has been converted to miles per hour)

99. (a) Using a graphing utility, you obtain

$$R = 0.167v - 0.02.$$

(c) $T = R + B = 0.00586v^2 + 0.1431v + 0.44$

(e) $\frac{dT}{dv} = 0.01172v + 0.1431$

For $v = 40$, $T'(40) \approx 0.612$.

For $v = 80$, $T'(80) \approx 1.081$.

For $v = 100$, $T'(100) \approx 1.315$.

101. $A = s^2, \frac{dA}{ds} = 2s$

When $s = 4 \text{ m}$,

$$\frac{dA}{ds} = 8 \text{ square meters per meter change in } s.$$

93. $s(t) = -4.9t^2 + v_0t + s_0$

$$= -4.9t^2 + 120t$$

$$v(t) = -9.8t + 120$$

$$v(5) = -9.8(5) + 120 = 71 \text{ m/sec}$$

$$v(10) = -9.8(10) + 120 = 22 \text{ m/sec}$$

97. $v = 40 \text{ mph} = \frac{2}{3} \text{ mi/min}$

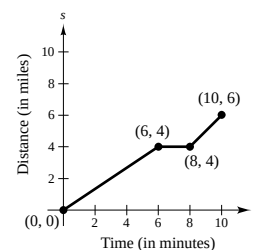
$$\left(\frac{2}{3} \text{ mi/min}\right)(6 \text{ min}) = 4 \text{ mi}$$

$$v = 0 \text{ mph} = 0 \text{ mi/min}$$

$$(0 \text{ mi/min})(2 \text{ min}) = 0 \text{ mi}$$

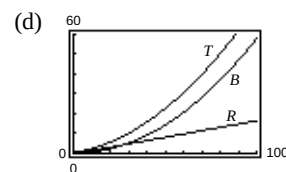
$$v = 60 \text{ mph} = 1 \text{ mi/min}$$

$$(1 \text{ mi/min})(2 \text{ min}) = 2 \text{ mi}$$



(b) Using a graphing utility, you obtain

$$B = 0.00586v^2 - 0.0239v + 0.46.$$



(f) For increasing speeds, the total stopping distance increases.

103. $C = \frac{1,008,000}{Q} + 6.3Q$

$$\frac{dC}{dQ} = -\frac{1,008,000}{Q^2} + 6.3$$

$$C(351) - C(350) \approx 5083.095 - 5085 \approx -\$1.91$$

When $Q = 350$, $\frac{dC}{dQ} \approx -\$1.93$.

 105. (a) $f'(1.47)$ is the rate of change of the amount of gasoline sold when the price is \$1.47 per gallon.

 (b) $f'(1.47)$ is usually negative. As prices go up, sales go down.

107. $y = ax^2 + bx + c$

Since the parabola passes through (0, 1) and (1, 0), we have

$$(0, 1): 1 = a(0)^2 + b(0) + c \Rightarrow c = 1$$

$$(1, 0): 0 = a(1)^2 + b(1) + 1 \Rightarrow b = -a - 1.$$

Thus, $y = ax^2 + (-a - 1)x + 1$. From the tangent line $y = x - 1$, we know that the derivative is 1 at the point (1, 0).

$$y' = 2ax + (-a - 1)$$

$$1 = 2a(1) + (-a - 1)$$

$$1 = a - 1$$

$$a = 2$$

$$b = -a - 1 = -3$$

Therefore, $y = 2x^2 - 3x + 1$.

109. $y = x^3 - 9x$

$$y' = 3x^2 - 9$$

Tangent lines through (1, -9):

$$y + 9 = (3x^2 - 9)(x - 1)$$

$$(x^3 - 9x) + 9 = 3x^3 - 3x^2 - 9x + 9$$

$$0 = 2x^3 - 3x^2 = x^2(2x - 3)$$

$$x = 0 \text{ or } x = \frac{3}{2}$$

The points of tangency are (0, 0) and $(\frac{3}{2}, -\frac{81}{8})$. At (0, 0) the slope is $y'(0) = -9$. At $(\frac{3}{2}, -\frac{81}{8})$ the slope is $y'(\frac{3}{2}) = -\frac{9}{4}$.

Tangent lines:

$$y - 0 = -9(x - 0) \quad \text{and} \quad y + \frac{81}{8} = -\frac{9}{4}(x - \frac{3}{2})$$

$$y = -9x \quad y = -\frac{9}{4}x - \frac{27}{4}$$

$$9x + y = 0 \quad 9x + 4y + 27 = 0$$

111. $f(x) = \begin{cases} ax^3, & x \leq 2 \\ x^2 + b, & x > 2 \end{cases}$

f must be continuous at $x = 2$ to be differentiable at $x = 2$.

$$\left. \begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} ax^3 = 8a \\ \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (x^2 + b) = 4 + b \end{aligned} \right\} \begin{aligned} 8a &= 4 + b \\ 8a - 4 &= b \end{aligned}$$

$$f'(x) = \begin{cases} 3ax^2, & x < 2 \\ 2x, & x > 2 \end{cases}$$

For f to be differentiable at $x = 2$, the left derivative must equal the right derivative.

$$3a(2)^2 = 2(2)$$

$$12a = 4$$

$$a = \frac{1}{3}$$

$$b = 8a - 4 = -\frac{4}{3}$$

113. Let $f(x) = \cos x$.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\cos x(\cos \Delta x - 1)}{\Delta x} - \lim_{\Delta x \rightarrow 0} \sin x \left(\frac{\sin \Delta x}{\Delta x} \right) \\ &= 0 - \sin x(1) = -\sin x \end{aligned}$$

Section 2.3 The Product and Quotient Rules and Higher-Order Derivatives

1. $g(x) = (x^2 + 1)(x^2 - 2x)$

$$\begin{aligned} g'(x) &= (x^2 + 1)(2x - 2) + (x^2 - 2x)(2x) \\ &= 2x^3 - 2x^2 + 2x - 2 + 2x^3 - 4x^2 \\ &= 4x^3 - 6x^2 + 2x - 2 \end{aligned}$$

5. $f(x) = x^3 \cos x$

$$\begin{aligned} f'(x) &= x^3(-\sin x) + \cos x(3x^2) \\ &= 3x^2 \cos x - x^3 \sin x \end{aligned}$$

9. $h(x) = \frac{\sqrt[3]{x}}{x^3 + 1} = \frac{x^{1/3}}{x^3 + 1}$

$$\begin{aligned} h'(x) &= \frac{(x^3 + 1) \frac{1}{3} x^{-2/3} - x^{1/3}(3x^2)}{(x^3 + 1)^2} \\ &= \frac{(x^3 + 1) - x(9x^2)}{3x^{2/3}(x^3 + 1)^2} \\ &= \frac{1 - 8x^3}{3x^{2/3}(x^3 + 1)^2} \end{aligned}$$

13. $f(x) = (x^3 - 3x)(2x^2 + 3x + 5)$

$$\begin{aligned} f'(x) &= (x^3 - 3x)(4x + 3) + (2x^2 + 3x + 5)(3x^2 - 3) \\ &= 10x^4 + 12x^3 - 3x^2 - 18x - 15 \\ f'(0) &= -15 \end{aligned}$$

17. $f(x) = x \cos x$

$$f'(x) = (x)(-\sin x) + (\cos x)(1) = \cos x - x \sin x$$

$$f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\pi}{4} \left(\frac{\sqrt{2}}{2} \right) = \frac{\sqrt{2}}{8} (4 - \pi)$$

3. $h(t) = \sqrt[3]{t}(t^2 + 4) = t^{1/3}(t^2 + 4)$

$$\begin{aligned} h'(t) &= t^{1/3}(2t) + (t^2 + 4) \frac{1}{3} t^{-2/3} \\ &= 2t^{4/3} + \frac{t^2 + 4}{3t^{2/3}} \\ &= \frac{7t^2 + 4}{3t^{2/3}} \end{aligned}$$

7. $f(x) = \frac{x}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

11. $g(x) = \frac{\sin x}{x^2}$

$$g'(x) = \frac{x^2(\cos x) - \sin x(2x)}{(x^2)^2} = \frac{x \cos x - 2 \sin x}{x^3}$$

15. $f(x) = \frac{x^2 - 4}{x - 3}$

$$\begin{aligned} f'(x) &= \frac{(x - 3)(2x) - (x^2 - 4)(1)}{(x - 3)^2} = \frac{2x^2 - 6x - x^2 + 4}{(x - 3)^2} \\ &= \frac{x^2 - 6x + 4}{(x - 3)^2} \end{aligned}$$

$$f'(1) = \frac{1 - 6 + 4}{(1 - 3)^2} = -\frac{1}{4}$$

<u>Function</u>	<u>Rewrite</u>	<u>Derivative</u>	<u>Simplify</u>
19. $y = \frac{x^2 + 2x}{3}$	$y = \frac{1}{3}x^2 + \frac{2}{3}x$	$y' = \frac{2}{3}x + \frac{2}{3}$	$y' = \frac{2x + 2}{3}$
21. $y = \frac{7}{3x^3}$	$y = \frac{7}{3}x^{-3}$	$y' = -7x^{-4}$	$y' = -\frac{7}{x^4}$
23. $y = \frac{4x^{3/2}}{x}$	$y = 4\sqrt{x}, x > 0$	$y' = 2x^{-1/2}$	$y' = \frac{2}{\sqrt{x}}$
25. $f(x) = \frac{3 - 2x - x^2}{x^2 - 1}$			
	$f'(x) = \frac{(x^2 - 1)(-2 - 2x) - (3 - 2x - x^2)(2x)}{(x^2 - 1)^2}$		
	$= \frac{2x^2 - 4x + 2}{(x^2 - 1)^2} = \frac{2(x - 1)^2}{(x^2 - 1)^2}$		
	$= \frac{2}{(x + 1)^2}, x \neq 1$		
27. $f(x) = x\left(1 - \frac{4}{x + 3}\right) = x - \frac{4x}{x + 3}$			
	$f'(x) = 1 - \frac{(x + 3)4 - 4x(1)}{(x + 3)^2} = \frac{(x^2 + 6x + 9) - 12}{(x + 3)^2}$		
	$= \frac{x^2 + 6x - 3}{(x + 3)^2}$		
29. $f(x) = \frac{2x + 5}{\sqrt{x}} = 2x^{1/2} + 5x^{-1/2}$			
	$f'(x) = x^{-1/2} - \frac{5}{2}x^{-3/2} = x^{-3/2}\left[x - \frac{5}{2}\right]$		
	$= \frac{2x - 5}{2x\sqrt{x}} = \frac{2x - 5}{2x^{3/2}}$		
31. $h(s) = (s^3 - 2)^2 = s^6 - 4s^3 + 4$			
	$h'(s) = 6s^5 - 12s^2 = 6s^2(s^3 - 2)$		
33. $f(x) = \frac{2 - \frac{1}{x}}{x - 3} = \frac{2x - 1}{x(x - 3)} = \frac{2x - 1}{x^2 - 3x}$			
	$f'(x) = \frac{(x^2 - 3x)2 - (2x - 1)(2x - 3)}{(x^2 - 3x)^2} = \frac{2x^2 - 6x - 4x^2 + 8x - 3}{(x^2 - 3x)^2}$		
	$= \frac{-2x^2 + 2x - 3}{(x^2 - 3x)^2} = -\frac{2x^2 - 2x + 3}{x^2(x - 3)^2}$		
35. $f(x) = (3x^3 + 4x)(x - 5)(x + 1)$			
	$f'(x) = (9x^2 + 4)(x - 5)(x + 1) + (3x^3 + 4x)(1)(x + 1) + (3x^3 + 4x)(x - 5)(1)$		
	$= (9x^2 + 4)(x^2 - 4x - 5) + 3x^4 + 3x^3 + 4x^2 + 4x + 3x^4 - 15x^3 + 4x^2 - 20x$		
	$= 9x^4 - 36x^3 - 41x^2 - 16x - 20 + 6x^4 - 12x^3 + 8x^2 - 16x$		
	$= 15x^4 - 48x^3 - 33x^2 - 32x - 20$		
37. $f(x) = \frac{x^2 + c^2}{x^2 - c^2}$			
	$f'(x) = \frac{(x^2 - c^2)(2x) - (x^2 + c^2)(2x)}{(x^2 - c^2)^2}$		
	$= \frac{-4xc^2}{(x^2 - c^2)^2}$		
39. $f(x) = t^2 \sin t$			
	$f'(t) = t^2 \cos t + 2t \sin t$		
	$= t(t \cos t + 2 \sin t)$		

$$41. f(t) = \frac{\cos t}{t}$$

$$f'(t) = \frac{-t \sin t - \cos t}{t^2} = -\frac{t \sin t + \cos t}{t^2}$$

$$45. g(t) = \sqrt[4]{t} + 8 \sec t = t^{1/4} + 8 \sec t$$

$$g'(t) = \frac{1}{4}t^{-3/4} + 8 \sec t \tan t = \frac{1}{4t^{3/4}} + 8 \sec t \tan t$$

$$49. y = -\csc x - \sin x$$

$$y' = \csc x \cot x - \cos x$$

$$= \frac{\cos x}{\sin^2 x} - \cos x$$

$$= \cos x(\csc^2 x - 1)$$

$$= \cos x \cot^2 x$$

$$53. y = 2x \sin x + x^2 \cos x$$

$$y' = 2x \cos x + 2 \sin x + x^2(-\sin x) + 2x \cos x$$

$$= 4x \cos x + 2 \sin x - x^2 \sin x$$

$$57. g(\theta) = \frac{\theta}{1 - \sin \theta}$$

$$g'(\theta) = \frac{1 - \sin \theta + \theta \cos \theta}{(\sin \theta - 1)^2} \quad (\text{form of answer may vary})$$

$$59. y = \frac{1 + \csc x}{1 - \csc x}$$

$$y' = \frac{(1 - \csc x)(-\csc x \cot x) - (1 + \csc x)(\csc x \cot x)}{(1 - \csc x)^2} = \frac{-2 \csc x \cot x}{(1 - \csc x)^2}$$

$$y'\left(\frac{\pi}{6}\right) = \frac{-2(2)(\sqrt{3})}{(1 - 2)^2} = -4\sqrt{3}$$

$$61. h(t) = \frac{\sec t}{t}$$

$$h'(t) = \frac{t(\sec t \tan t) - (\sec t)(1)}{t^2}$$

$$= \frac{\sec t(t \tan t - 1)}{t^2}$$

$$h'(\pi) = \frac{\sec \pi(\pi \tan \pi - 1)}{\pi^2} = \frac{1}{\pi^2}$$

$$43. f(x) = -x + \tan x$$

$$f'(x) = -1 + \sec^2 x = \tan^2 x$$

$$47. y = \frac{3(1 - \sin x)}{2 \cos x} = \frac{3}{2}(\sec x - \tan x)$$

$$y' = \frac{3}{2}(\sec x \tan x - \sec^2 x) = \frac{3}{2}\sec x(\tan x - \sec x)$$

$$= \frac{3}{2}(\sec x \tan x - \tan^2 x - 1)$$

$$51. f(x) = x^2 \tan x$$

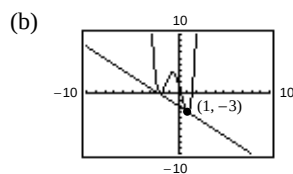
$$f'(x) = x^2 \sec^2 x + 2x \tan x$$

$$= x(x \sec^2 x + 2 \tan x)$$

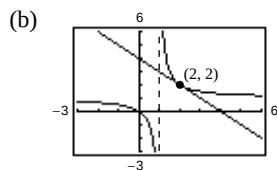
$$55. g(x) = \left(\frac{x+1}{x+2}\right)(2x-5)$$

$$g'(x) = \frac{2x^2 + 8x - 1}{(x+2)^2} \quad (\text{form of answer may vary})$$

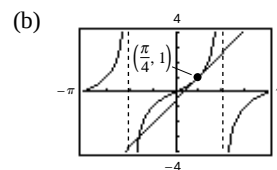
63. (a) $f(x) = (x^3 - 3x + 1)(x + 2)$, $(1, -3)$
 $f'(x) = (x^3 - 3x + 1)(1) + (x + 2)(3x^2 - 3)$
 $= 4x^3 + 6x^2 - 6x - 5$
 $f'(1) = -1 = \text{slope at } (1, -3).$
Tangent line: $y + 3 = -1(x - 1) \Rightarrow y = -x - 2$



65. (a) $f(x) = \frac{x}{x-1}$, $(2, 2)$
 $f'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$
 $f'(2) = \frac{-1}{(2-1)^2} = -1 = \text{slope at } (2, 2).$
Tangent line: $y - 2 = -1(x - 2) \Rightarrow y = -x + 4$



67. (a) $f(x) = \tan x$, $\left(\frac{\pi}{4}, 1\right)$
 $f'(x) = \sec^2 x$
 $f'\left(\frac{\pi}{4}\right) = 2 = \text{slope at } \left(\frac{\pi}{4}, 1\right).$
Tangent line:



$$y - 1 = 2\left(x - \frac{\pi}{4}\right)$$

$$y - 1 = 2x - \frac{\pi}{2}$$

$$4x - 2y - \pi + 2 = 0$$

69. $f(x) = \frac{x^2}{x-1}$
 $f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2}$
 $= \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$
 $f'(x) = 0$ when $x = 0$ or $x = 2$.

Horizontal tangents are at $(0, 0)$ and $(2, 4)$.

71. $f'(x) = \frac{(x+2)3 - 3x(1)}{(x+2)^2} = \frac{6}{(x+2)^2}$
 $g'(x) = \frac{(x+2)5 - (5x+4)(1)}{(x+2)^2} = \frac{6}{(x+2)^2}$
 $g(x) = \frac{5x+4}{(x+2)} = \frac{3x}{(x+2)} + \frac{2x+4}{(x+2)} = f(x) + 2$
 f and g differ by a constant.

73. $f(x) = x^n \sin x$
 $f'(x) = x^n \cos x + nx^{n-1} \sin x$
 $= x^{n-1}(x \cos x + n \sin x)$
When $n = 1$: $f'(x) = x \cos x + \sin x$.
When $n = 2$: $f'(x) = x(x \cos x + 2 \sin x)$.
When $n = 3$: $f'(x) = x^2(x \cos x + 3 \sin x)$.
When $n = 4$: $f'(x) = x^3(x \cos x + 4 \sin x)$.
For general n , $f'(x) = x^{n-1}(x \cos x + n \sin x)$.

75. Area = $A(t) = (2t + 1)\sqrt{t} = 2t^{3/2} + t^{1/2}$
 $A'(t) = 2\left(\frac{3}{2}t^{1/2}\right) + \frac{1}{2}t^{-1/2}$
 $= 3t^{1/2} + \frac{1}{2}t^{-1/2}$
 $= \frac{6t + 1}{2\sqrt{t}} \text{ cm}^2/\text{sec}$

$$77. C = 100\left(\frac{200}{x^2} + \frac{x}{x+30}\right), \quad 1 \leq x$$

$$\frac{dC}{dx} = 100\left(-\frac{400}{x^3} + \frac{30}{(x+30)^2}\right)$$

$$(a) \text{ When } x = 10: \frac{dC}{dx} = -\$38.13.$$

$$(b) \text{ When } x = 15: \frac{dC}{dx} = -\$10.37.$$

$$(c) \text{ When } x = 20: \frac{dC}{dx} = -\$3.80.$$

As the order size increases, the cost per item decreases.

$$79. P(t) = 500\left[1 + \frac{4t}{50 + t^2}\right]$$

$$P'(t) = 500\left[\frac{(50 + t^2)(4) - (4t)(2t)}{(50 + t^2)^2}\right]$$

$$= 500\left[\frac{200 - 4t^2}{(50 + t^2)^2}\right]$$

$$= 2000\left[\frac{50 - t^2}{(50 + t^2)^2}\right]$$

$$P'(2) \approx 31.55 \text{ bacteria per hour}$$

$$81. (a) \quad \sec x = \frac{1}{\cos x}$$

$$\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{(\cos x)(0) - (1)(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos x \cos x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

$$(b) \quad \csc x = \frac{1}{\sin x}$$

$$\frac{d}{dx}[\csc x] = \frac{d}{dx}\left[\frac{1}{\sin x}\right] = \frac{(\sin x)(0) - (1)(\cos x)}{(\sin x)^2} = -\frac{\cos x}{\sin x \sin x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

$$(c) \quad \cot x = \frac{\cos x}{\sin x}$$

$$\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{\sin x(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

$$83. f(x) = 4x^{3/2}$$

$$f'(x) = 6x^{1/2}$$

$$f''(x) = 3x^{-1/2} = \frac{3}{\sqrt{x}}$$

$$85. f(x) = \frac{x}{x-1}$$

$$f'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

$$f''(x) = \frac{2}{(x-1)^3}$$

$$87. f(x) = 3 \sin x$$

$$f'(x) = 3 \cos x$$

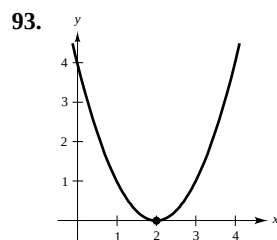
$$f''(x) = -3 \sin x$$

$$89. f'(x) = x^2$$

$$f''(x) = 2x$$

$$91. f'''(x) = 2\sqrt{x}$$

$$f^{(4)}(x) = \frac{1}{2}(2)x^{-1/2} = \frac{1}{\sqrt{x}}$$



$$f(2) = 0$$

One such function is $f(x) = (x-2)^2$.

$$95. f(x) = 2g(x) + h(x)$$

$$f'(x) = 2g'(x) + h'(x)$$

$$f'(2) = 2g'(2) + h'(2)$$

$$= 2(-2) + 4$$

$$= 0$$

$$97. f(x) = \frac{g(x)}{h(x)}$$

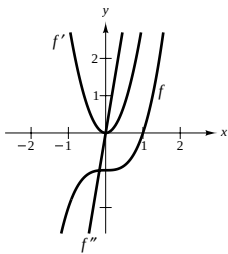
$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$

$$f'(2) = \frac{h(2)g'(2) - g(2)h'(2)}{[h(2)]^2}$$

$$= \frac{(-1)(-2) - (3)(4)}{(-1)^2}$$

$$= -10$$

99.



It appears that f is cubic; so f' would be quadratic and f'' would be linear.

$$101. v(t) = 36 - t^2, \quad 0 \leq t \leq 6$$

$$a(t) = -2t$$

$$v(3) = 27 \text{ m/sec}$$

$$a(3) = -6 \text{ m/sec}$$

The speed of the object is decreasing.

$$103. v(t) = \frac{100t}{2t + 15}$$

$$a(t) = \frac{(2t + 15)(100) - (100t)(2)}{(2t + 15)^2}$$

$$= \frac{1500}{(2t + 15)^2}$$

$$(a) \quad a(5) = \frac{1500}{[2(5) + 15]^2} = 2.4 \text{ ft/sec}^2$$

$$(b) \quad a(10) = \frac{1500}{[2(10) + 15]^2} \approx 1.2 \text{ ft/sec}^2$$

$$(c) \quad a(20) = \frac{1500}{[2(20) + 15]^2} \approx 0.5 \text{ ft/sec}^2$$

$$105. f(x) = g(x)h(x)$$

$$(a) \quad f'(x) = g(x)h'(x) + h(x)g'(x)$$

$$f''(x) = g(x)h''(x) + g'(x)h'(x) + h(x)g''(x) + h'(x)g'(x)$$

$$= g(x)h''(x) + 2g'(x)h'(x) + h(x)g''(x)$$

$$f'''(x) = g(x)h'''(x) + g'(x)h''(x) + 2g''(x)h'(x) + 2g'(x)h''(x) + h(x)g'''(x) + h'(x)g''(x)$$

$$= g(x)h'''(x) + 3g'(x)h''(x) + 3g''(x)h'(x) + g'''(x)h(x)$$

$$f^{(4)}(x) = g(x)h^{(4)}(x) + g'(x)h'''(x) + 3g''(x)h''(x) + 3g'''(x)h'(x) + 3g''(x)h''(x) + 3g'''(x)h'(x)$$

$$+ g^{(4)}(x)h(x) + g^{(4)}(x)h(x)$$

$$= g(x)h^{(4)}(x) + 4g'(x)h'''(x) + 6g''(x)h''(x) + 4g'''(x)h'(x) + g^{(4)}(x)h(x)$$

$$(b) \quad f^{(n)}(x) = g(x)h^{(n)}(x) + \frac{n(n-1)(n-2) \cdots (2)(1)}{1[(n-1)(n-2) \cdots (2)(1)]} g'(x)h^{(n-1)}(x) + \frac{n(n-1)(n-2) \cdots (2)(1)}{(2)(1)[(n-2)(n-3) \cdots (2)(1)]} g''(x)h^{(n-2)}(x)$$

$$+ \frac{n(n-1)(n-2) \cdots (2)(1)}{(3)(2)(1)[(n-3)(n-4) \cdots (2)(1)]} g'''(x)h^{(n-3)}(x) + \cdots$$

$$+ \frac{n(n-1)(n-2) \cdots (2)(1)}{[(n-1)(n-2) \cdots (2)(1)](1)} g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)$$

$$= g(x)h^{(n)}(x) + \frac{n!}{1!(n-1)!} g'(x)h^{(n-1)}(x) + \frac{n!}{2!(n-2)!} g''(x)h^{(n-2)}(x) + \cdots$$

$$+ \frac{n!}{(n-1)!1!} g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)$$

Note: $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ (read “ n factorial.”)

$$107. f(x) = \cos x \quad f\left(\frac{\pi}{3}\right) = \cos \frac{\pi}{3} = \frac{1}{2}$$

$$f'(x) = -\sin x \quad f'\left(\frac{\pi}{3}\right) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

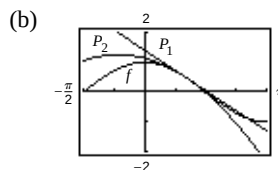
$$f''(x) = -\cos x \quad f''\left(\frac{\pi}{3}\right) = -\cos \frac{\pi}{3} = -\frac{1}{2}$$

$$(a) P_1(x) = f'(a)(x - a) + f(a) = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}$$

$$P_2(x) = \frac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a)$$

$$= -\frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}$$

(c) P_2 is a better approximation.



(d) The accuracy worsens as you move farther away from $x = a = (\pi/3)$.

109. False. If $y = f(x)g(x)$, then

$$\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x).$$

111. True

$$\begin{aligned} h'(c) &= f(c)g'(c) + g(c)f'(c) \\ &= f(c)(0) + g(c)(0) \\ &= 0 \end{aligned}$$

113. True

$$115. f(x) = x|x| = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ -2x, & \text{if } x < 0 \end{cases} = 2|x|$$

$$f''(x) = \begin{cases} 2, & \text{if } x > 0 \\ -2, & \text{if } x < 0 \end{cases}$$

$f''(0)$ does not exist since the left and right derivatives are not equal.

Section 2.4 The Chain Rule

$$y = f(g(x))$$

$$u = g(x)$$

$$y = f(u)$$

$$1. y = (6x - 5)^4$$

$$u = 6x - 5$$

$$y = u^4$$

$$3. y = \sqrt{x^2 - 1}$$

$$u = x^2 - 1$$

$$y = \sqrt{u}$$

$$5. y = \csc^3 x$$

$$u = \csc x$$

$$y = u^3$$

$$7. y = (2x - 7)^3$$

$$y' = 3(2x - 7)^2(2) = 6(2x - 7)^2$$

$$9. g(x) = 3(4 - 9x)^4$$

$$g'(x) = 12(4 - 9x)^3(-9) = -108(4 - 9x)^3$$

$$11. f(x) = (9 - x^2)^{2/3}$$

$$f'(x) = \frac{2}{3}(9 - x^2)^{-1/3}(-2x) = -\frac{4x}{3(9 - x^2)^{1/3}}$$

$$13. f(t) = (1 - t)^{1/2}$$

$$f'(t) = \frac{1}{2}(1 - t)^{-1/2}(-1) = -\frac{1}{2\sqrt{1 - t}}$$

15. $y = (9x^2 + 4)^{1/3}$

$$y' = \frac{1}{3}(9x^2 + 4)^{-2/3}(18x) = \frac{6x}{(9x^2 + 4)^{2/3}}$$

19. $y = (x - 2)^{-1}$

$$y' = -1(2 - x)^{-2}(1) = \frac{-1}{(x - 2)^2}$$

23. $y = (x + 2)^{-1/2}$

$$\frac{dy}{dx} = -\frac{1}{2}(x + 2)^{-3/2} = -\frac{1}{2(x + 2)^{3/2}}$$

27. $y = x\sqrt{1 - x^2} = x(1 - x^2)^{1/2}$

$$\begin{aligned} y' &= x \left[\frac{1}{2}(1 - x^2)^{-1/2}(-2x) \right] + (1 - x^2)^{1/2}(1) \\ &= -x^2(1 - x^2)^{-1/2} + (1 - x^2)^{1/2} \\ &= (1 - x^2)^{-1/2}[-x^2 + (1 - x^2)] \\ &= \frac{1 - 2x^2}{\sqrt{1 - x^2}} \end{aligned}$$

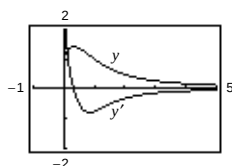
31. $g(x) = \left(\frac{x + 5}{x^2 + 2} \right)^2$

$$\begin{aligned} g'(x) &= 2 \left(\frac{x + 5}{x^2 + 2} \right) \left(\frac{(x^2 + 2) - (x + 5)(2x)}{(x^2 + 2)^2} \right) \\ &= \frac{2(x + 5)(2 - 10x - x^2)}{(x^2 + 2)^3} \end{aligned}$$

35. $y = \frac{\sqrt{x} + 1}{x^2 + 1}$

$$y' = \frac{1 - 3x^2 - 4x^{3/2}}{2\sqrt{x}(x^2 + 1)^2}$$

The zero of y' corresponds to the point on the graph of y where the tangent line is horizontal.



17. $y = 2(4 - x^2)^{1/4}$

$$\begin{aligned} y' &= 2 \left(\frac{1}{4} \right) (4 - x^2)^{-3/4} (-2x) \\ &= \frac{-x}{\sqrt[4]{(4 - x^2)^3}} \end{aligned}$$

21. $f(t) = (t - 3)^{-2}$

$$f'(t) = -2(t - 3)^{-3} = \frac{-2}{(t - 3)^3}$$

25. $f(x) = x^2(x - 2)^4$

$$\begin{aligned} f'(x) &= x^2[4(x - 2)^3(1)] + (x - 2)^4(2x) \\ &= 2x(x - 2)^3[2x + (x - 2)] \\ &= 2x(x - 2)^3(3x - 2) \end{aligned}$$

29. $y = \frac{x}{\sqrt{x^2 + 1}} = x(x^2 + 1)^{-1/2}$

$$\begin{aligned} y' &= x \left[-\frac{1}{2}(x^2 + 1)^{-3/2}(2x) \right] + (x^2 + 1)^{-1/2}(1) \\ &= -x^2(x^2 + 1)^{-3/2} + (x^2 + 1)^{-1/2} \\ &= (x^2 + 1)^{-3/2}[-x^2 + (x^2 + 1)] \\ &= \frac{1}{(x^2 + 1)^{3/2}} \end{aligned}$$

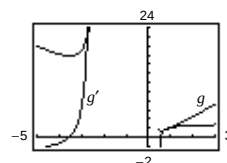
33. $f(v) = \left(\frac{1 - 2v}{1 + v} \right)^3$

$$\begin{aligned} f'(v) &= 3 \left(\frac{1 - 2v}{1 + v} \right)^2 \left(\frac{(1 + v)(-2) - (1 - 2v)}{(1 + v)^2} \right) \\ &= \frac{-9(1 - 2v)^2}{(1 + v)^4} \end{aligned}$$

37. $g(t) = \frac{3t^2}{\sqrt{t^2 + 2t - 1}}$

$$g'(t) = \frac{3t(t^2 + 3t - 2)}{(t^2 + 2t - 1)^{3/2}}$$

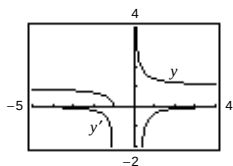
The zeros of g' correspond to the points on the graph of g where the tangent lines are horizontal.



$$39. y = \sqrt{\frac{x+1}{x}}$$

$$y' = -\frac{\sqrt{(x+1)/x}}{2x(x+1)}$$

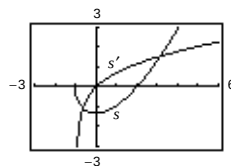
y' has no zeros.



$$41. s(t) = \frac{-2(2-t)\sqrt{1+t}}{3}$$

$$s'(t) = \frac{t}{\sqrt{1+t}}$$

The zero of $s'(t)$ corresponds to the point on the graph of $s(t)$ where the tangent line is horizontal.

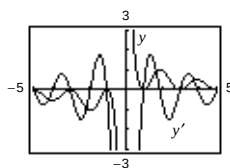


$$43. y = \frac{\cos \pi x + 1}{x}$$

$$\frac{dy}{dx} = \frac{-\pi x \sin \pi x - \cos \pi x - 1}{x^2}$$

$$= -\frac{\pi x \sin \pi x + \cos \pi x + 1}{x^2}$$

The zeros of y' correspond to the points on the graph of y where the tangent lines are horizontal.



$$45. (a) y = \sin x$$

$$y' = \cos x$$

$$y'(0) = 1$$

$$1 \text{ cycle in } [0, 2\pi]$$

$$(b) y = \sin 2x$$

$$y' = 2 \cos 2x$$

$$y'(0) = 2$$

$$2 \text{ cycles in } [0, 2\pi]$$

The slope of $\sin ax$ at the origin is a .

$$47. y = \cos 3x$$

$$\frac{dy}{dx} = -3 \sin 3x$$

$$49. g(x) = 3 \tan 4x$$

$$g'(x) = 12 \sec^2 4x$$

$$51. y = \sin(\pi x)^2 = \sin(\pi^2 x^2)$$

$$y' = \cos(\pi x^2)[2\pi^2 x] = 2\pi^2 x \cos(\pi^2 x^2)$$

$$53. h(x) = \sin 2x \cos 2x$$

$$h'(x) = \sin 2x(-2 \sin 2x) + \cos 2x(2 \cos 2x)$$

$$= 2 \cos^2 2x - 2 \sin^2 2x$$

$$= 2 \cos 4x.$$

Alternate solution: $h(x) = \frac{1}{2} \sin 4x$

$$h'(x) = \frac{1}{2} \cos 4x(4) = 2 \cos 4x$$

$$55. f(x) = \frac{\cot x}{\sin x} = \frac{\cos x}{\sin^2 x}$$

$$f'(x) = \frac{\sin^2 x(-\sin x) - \cos x(2 \sin x \cos x)}{\sin^4 x}$$

$$= \frac{-\sin^2 x - 2 \cos^2 x}{\sin^3 x} = \frac{-1 - \cos^2 x}{\sin^3 x}$$

57. $y = 4 \sec^2 x$

$$y' = 8 \sec x \cdot \sec x \tan x = 8 \sec^2 x \tan x$$

61. $f(x) = 3 \sec^2(\pi t - 1)$

$$\begin{aligned} f'(t) &= 6 \sec(\pi t - 1) \sec(\pi t - 1) \tan(\pi t - 1)(\pi) \\ &= 6\pi \sec^2(\pi t - 1) \tan(\pi t - 1) = \frac{6\pi \sin(\pi t - 1)}{\cos^3(\pi t - 1)} \end{aligned}$$

65. $y = \sin(\cos x)$

$$\begin{aligned} \frac{dy}{dx} &= \cos(\cos x) \cdot (-\sin x) \\ &= -\sin x \cos(\cos x) \end{aligned}$$

69. $f(x) = \frac{3}{x^3 - 4} = 3(x^3 - 4)^{-1}, \left(-1, -\frac{3}{5}\right)$

$$f'(x) = -3(x^3 - 4)^{-2}(3x^2) = -\frac{9x^2}{(x^3 - 4)^2}$$

$$f'(-1) = -\frac{9}{25}$$

73. $y = 37 - \sec^3(2x), (0, 36)$

$$y' = -3 \sec^2(2x)[2 \sec(2x) \tan(2x)]$$

$$= -6 \sec^3(2x) \tan(2x)$$

$$y'(0) = 0$$

75. (a) $f(x) = \sqrt{3x^2 - 2}, (3, 5)$

$$f'(x) = \frac{1}{2}(3x^2 - 2)^{-1/2}(6x)$$

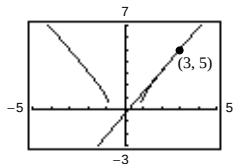
$$= \frac{3x}{\sqrt{3x^2 - 2}}$$

$$f'(3) = \frac{9}{5}$$

Tangent line:

$$y - 5 = \frac{9}{5}(x - 3) \Rightarrow 9x - 5y - 2 = 0$$

(b)



59. $f(\theta) = \frac{1}{4} \sin^2 2\theta = \frac{1}{4}(\sin 2\theta)^2$

$$\begin{aligned} f'(\theta) &= 2\left(\frac{1}{4}\right)(\sin 2\theta)(\cos 2\theta)(2) \\ &= \sin 2\theta \cos 2\theta = \frac{1}{2} \sin 4\theta \end{aligned}$$

63. $y = \sqrt{x} + \frac{1}{4} \sin(2x)^2$

$$= \sqrt{x} + \frac{1}{4} \sin(4x^2)$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} + \frac{1}{4} \cos(4x^2)(8x)$$

$$= \frac{1}{2\sqrt{x}} + 2x \cos(2x)^2$$

67. $s(t) = (t^2 + 2t + 8)^{1/2}, (2, 4)$

$$s'(t) = \frac{1}{2}(t^2 + 2t + 8)^{-1/2}(2t + 2)$$

$$= \frac{t + 1}{\sqrt{t^2 + 2t + 8}}$$

$$s'(2) = \frac{3}{4}$$

71. $f(t) = \frac{3t + 2}{t - 1}, (0, -2)$

$$f'(t) = \frac{(t - 1)(3) - (3t + 2)(1)}{(t - 1)^2} = \frac{-5}{(t - 1)^2}$$

$$f'(0) = -5$$

77. (a) $f(x) = \sin 2x, (\pi, 0)$

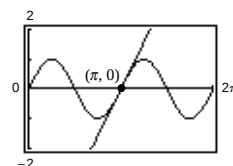
$$f'(x) = 2 \cos 2x$$

$$f'(\pi) = 2$$

Tangent line:

$$y = 2(x - \pi) \Rightarrow 2x - y - 2\pi = 0$$

(b)



79. $f(x) = 2(x^2 - 1)^3$

$$f'(x) = 6(x^2 - 1)^2(2x)$$

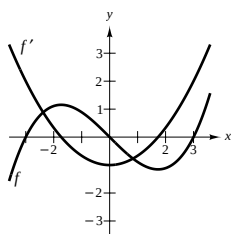
$$= 12x(x^4 - 2x^2 + 1)$$

$$= 12x^5 - 24x^3 + 12x$$

$$f''(x) = 60x^4 - 72x^2 + 12$$

$$= 12(5x^2 - 1)(x^2 - 1)$$

83.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

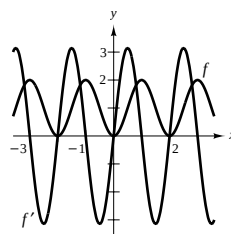
81. $f(x) = \sin x^2$

$$f'(x) = 2x \cos x^2$$

$$f''(x) = 2x[2x(-\sin x^2)] + 2 \cos x^2$$

$$= 2[\cos x^2 - 2x^2 \sin x^2]$$

85.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

87. $g(x) = f(3x)$

$$g'(x) = f'(3x)(3) \Rightarrow g'(x) = 3f'(3x)$$

89. (a) $f(x) = g(x)h(x)$

$$f'(x) = g(x)h'(x) + g'(x)h(x)$$

$$f'(5) = (-3)(-2) + (6)(3) = 24$$

(c) $f(x) = \frac{g(x)}{h(x)}$

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$

$$f'(5) = \frac{(3)(6) - (-3)(-2)}{(3)^2} = \frac{12}{9} = \frac{4}{3}$$

91. (a) $f = 132,400(331 - v)^{-1}$

$$f' = (-1)(132,400)(331 - v)^{-2}(-1)$$

$$= \frac{132,400}{(331 - v)^2}$$

When $v = 30$, $f' \approx 1.461$.

(b) $f(x) = g(h(x))$

$$f'(x) = g'(h(x))h'(x)$$

$$f'(5) = g'(3)(-2) = -2g'(3)$$

Need $g'(3)$ to find $f'(5)$.

(d) $f(x) = [g(x)]^3$

$$f'(x) = 3[g(x)]^2g'(x)$$

$$f'(5) = 3(-3)^2(6) = 162$$

(b) $f = 132,400(331 + v)^{-1}$

$$f' = (-1)(132,400)(331 + v)^{-2}(1)$$

$$= \frac{-132,400}{(331 + v)^2}$$

When $v = 30$, $f' \approx -1.016$.

93. $\theta = 0.2 \cos 8t$

The maximum angular displacement is $\theta = 0.2$ (since $-1 \leq \cos 8t \leq 1$).

$$\frac{d\theta}{dt} = 0.2[-8 \sin 8t] = -1.6 \sin 8t$$

When $t = 3$, $d\theta/dt = -1.6 \sin 24 \approx 1.4489$ radians per second.

95. $S = C(R^2 - r^2)$

$$\frac{dS}{dt} = C\left(2R \frac{dR}{dt} - 2r \frac{dr}{dt}\right)$$

Since r is constant, we have $dr/dt = 0$ and

$$\frac{dS}{dt} = (1.76 \times 10^5)(2)(1.2 \times 10^{-2})(10^{-5})$$

$$= 4.224 \times 10^{-2} = 0.04224.$$

97. (a) $x = -1.6372t^3 + 19.3120t^2 - 0.5082t - 0.6161$

(b) $C = 60x + 1350$

$$= 60(-1.6372t^3 + 19.3120t^2 - 0.5082t - 0.6161) + 1350$$

$$\frac{dC}{dt} = 60(-4.9116t^2 + 38.624t - 0.5082)$$

$$= -294.696t^2 + 2317.44t - 30.492$$

The function $\frac{dC}{dt}$ is quadratic, not linear. The cost function levels off at the end of the day, perhaps due to fatigue.

99. $f(x) = \sin \beta x$

(a) $f'(x) = \beta \cos \beta x$

$$f''(x) = -\beta^2 \sin \beta x$$

$$f'''(x) = -\beta^3 \cos \beta x$$

$$f^{(4)}(x) = \beta^4 \sin \beta x$$

(b) $f''(x) + \beta^2 f(x) = -\beta^2 \sin \beta x + \beta^2 (\sin \beta x) = 0$

(c) $f^{(2k)}(x) = (-1)^k \beta^{2k} \sin \beta x$

$$f^{(2k-1)}(x) = (-1)^{k+1} \beta^{2k-1} \cos \beta x$$

101. (a) $r'(x) = f'(g(x))g'(x)$

$$r'(1) = f'(g(1))g'(1)$$

$$\text{Note that } g(1) = 4 \text{ and } f'(4) = \frac{5-0}{6-2} = \frac{5}{4}$$

$$\text{Also, } g'(1) = 0. \text{ Thus, } r'(1) = 0$$

(b) $s'(x) = g'(f(x))f'(x)$

$$s'(4) = g'(f(4))f'(4)$$

$$\text{Note that } f(4) = \frac{5}{2}, g'\left(\frac{5}{2}\right) = \frac{6-4}{6-2} = \frac{1}{2} \text{ and}$$

$$f'(4) = \frac{5}{4}.$$

$$\text{Thus, } s'(4) = \frac{1}{2}\left(\frac{5}{4}\right) = \frac{5}{8}.$$

103. $g = \sqrt{x(x+n)}$

$$= \sqrt{x^2 + nx}$$

$$\frac{dg}{dx} = \frac{1}{2}(x^2 + nx)^{-1/2}(2x + n)$$

$$= \frac{2x + n}{2\sqrt{x^2 + nx}}$$

$$= \frac{(2x + n)/2}{\sqrt{x(x+n)}}$$

$$= \frac{[x + (x + n)]/2}{\sqrt{x(x+n)}}$$

$$= \frac{a}{g}$$

105. $g(x) = |2x - 3|$

$$g'(x) = 2\left(\frac{2x - 3}{|2x - 3|}\right), \quad x \neq \frac{3}{2}$$

107. $h(x) = |x|\cos x$

$$h'(x) = -|x|\sin x + \frac{x}{|x|}\cos x, \quad x \neq 0$$

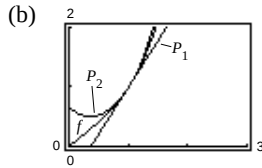
109. (a) $f(x) = \tan \frac{\pi x}{4}$ $f(1) = 1$

$f'(x) = \frac{\pi}{4} \sec^2 \frac{\pi x}{4}$ $f'(1) = \frac{\pi}{4}(2) = \frac{\pi}{2}$

$f''(x) = \frac{\pi}{2} \sec^2 \frac{\pi x}{4} \cdot \tan \frac{\pi x}{4} \left(\frac{\pi}{4} \right)$ $f''(1) = \frac{\pi}{8}(2)(1) = \frac{\pi}{4}$

$P_1(x) = f'(1)(x - 1) + f(1) = \frac{\pi}{2}(x - 1) + 1.$

$P_2(x) = \frac{1}{2} \left(\frac{\pi}{4} \right) (x - 1)^2 + f'(1)(x - 1) + f(1) = \frac{\pi}{8}(x - 1)^2 + \frac{\pi}{2}(x - 1) + 1$



(c) P_2 is a better approximation than P_1

(d) The accuracy worsens as you move away from $x = c = 1$.

111. False. If $y = (1 - x)^{1/2}$, then $y' = \frac{1}{2}(1 - x)^{-1/2}(-1)$.

113. True

Section 2.5 Implicit Differentiation

1. $x^2 + y^2 = 36$

$2x + 2yy' = 0$

$y' = \frac{-x}{y}$

3. $x^{1/2} + y^{1/2} = 9$

$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$

$y' = -\frac{x^{-1/2}}{y^{-1/2}} = -\sqrt{\frac{y}{x}}$

5. $x^3 - xy + y^2 = 4$

$3x^2 - xy' - y + 2yy' = 0$

$(2y - x)y' = y - 3x^2$

$y' = \frac{y - 3x^2}{2y - x}$

7. $x^3y^3 - y - x = 0$

$3x^3y^2y' + 3x^2y^3 - y' - 1 = 0$

$(3x^3y^2 - 1)y' = 1 - 3x^2y^3$

$y' = \frac{1 - 3x^2y^3}{3x^3y^2 - 1}$

9. $x^3 - 3x^2 + 2xy^2 = 12$

$3x^2 - 3x^2y' - 6xy + 4xyy' + 2y^2 = 0$

$(4xy - 3x^2)y' = 6xy - 3x^2 - 2y^2$

$y' = \frac{6xy - 3x^2 - 2y^2}{4xy - 3x^2}$

11. $\sin x + 2\cos 2y = 1$

$\cos x - 4(\sin 2y)y' = 0$

$y' = \frac{\cos x}{4 \sin 2y}$

13. $\sin x = x(1 + \tan y)$

$$\cos x = x(\sec^2 y)y' + (1 + \tan y)(1)$$

$$y' = \frac{\cos x - \tan y - 1}{x \sec^2 y}$$

17. (a) $x^2 + y^2 = 16$

$$y^2 = 16 - x^2$$

$$y = \pm \sqrt{16 - x^2}$$

(c) Explicitly:

$$\begin{aligned} \frac{dy}{dx} &= \pm \frac{1}{2}(16 - x^2)^{-1/2}(-2x) \\ &= \frac{\mp x}{\sqrt{16 - x^2}} = \frac{-x}{\pm \sqrt{16 - x^2}} = \frac{-x}{y} \end{aligned}$$

19. (a) $16y^2 = 144 - 9x^2$

$$y^2 = \frac{1}{16}(144 - 9x^2) = \frac{9}{16}(16 - x^2)$$

$$y = \pm \frac{3}{4}\sqrt{16 - x^2}$$

(c) Explicitly:

$$\begin{aligned} \frac{dy}{dx} &= \pm \frac{3}{8}(16 - x^2)^{-1/2}(-2x) \\ &= \mp \frac{3x}{4\sqrt{16 - x^2}} = \frac{-3x}{4(4/3)y} = \frac{-9x}{16y} \end{aligned}$$

21. $xy = 4$

$$xy' + y(1) = 0$$

$$xy' = -y$$

$$y' = \frac{-y}{x}$$

$$\text{At } (-4, -1): y' = -\frac{1}{4}$$

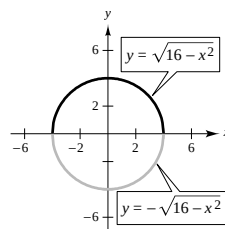
15. $y = \sin(xy)$

$$y' = [xy' + y] \cos(xy)$$

$$y' - x \cos(xy)y' = y \cos(xy)$$

$$y' = \frac{y \cos(xy)}{1 - x \cos(xy)}$$

(b)

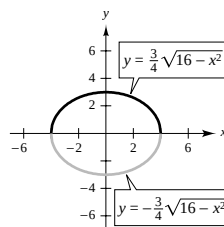


(d) Implicitly:

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

(b)



(d) Implicitly:

$$18x + 32yy' = 0$$

$$y' = \frac{-9x}{16y}$$

23. $y^2 = \frac{x^2 - 4}{x^2 + 4}$

$$2yy' = \frac{(x^2 + 4)(2x) - (x^2 - 4)(2x)}{(x^2 + 4)^2}$$

$$2yy' = \frac{16x}{(x^2 + 4)^2}$$

$$y' = \frac{8x}{y(x^2 + 4)^2}$$

At $(2, 0)$, y' is undefined.

25. $x^{2/3} + y^{2/3} = 5$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = \frac{-x^{-1/3}}{y^{-1/3}} = -\sqrt[3]{\frac{y}{x}}$$

At (8, 1): $y' = -\frac{1}{2}$.

29. $(x^2 + 4)y = 8$

$$(x^2 + 4)y' + y(2x) = 0$$

$$\begin{aligned} y' &= \frac{-2xy}{x^2 + 4} \\ &= \frac{-2x[8/(x^2 + 4)]}{x^2 + 4} \\ &= \frac{-16x}{(x^2 + 4)^2} \end{aligned}$$

At (2, 1): $y' = \frac{-32}{64} = -\frac{1}{2}$

(Or, you could just solve for y : $y = \frac{8}{x^2 + 4}$)

33. $\tan y = x$

$$y' \sec^2 y = 1$$

$$y' = \frac{1}{\sec^2 y} = \cos^2 y, -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$y' = \frac{1}{1 + x^2}$$

37. $x^2 - y^2 = 16$

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$

$$x - yy' = 0$$

$$1 - yy'' - (y')^2 = 0$$

$$1 - yy'' - \left(\frac{x}{y}\right)^2 = 0$$

$$y^2 - y^3y'' = x^2$$

$$y'' = \frac{y^2 - x^2}{y^3} = \frac{-16}{y^3}$$

27. $\tan(x + y) = x$

$$(1 + y') \sec^2(x + y) = 1$$

$$\begin{aligned} y' &= \frac{1 - \sec^2(x + y)}{\sec^2(x + y)} \\ &= \frac{-\tan^2(x + y)}{\tan^2(x + y) + 1} = -\sin^2(x + y) \\ &= -\frac{x^2}{x^2 + 1} \end{aligned}$$

At (0, 0): $y' = 0$.

31. $(x^2 + y^2)^2 = 4x^2y$

$$2(x^2 + y^2)(2x + 2yy') = 4x^2y' + y(8x)$$

$$4x^3 + 4x^2yy' + 4xy^2 + 4y^3y' = 4x^2y' + 8xy$$

$$4x^2yy' + 4y^3y' - 4x^2y' = 8xy - 4x^3 - 4xy^2$$

$$4y'(x^2y + y^3 - x^2) = 4(2xy - x^3 - xy^2)$$

$$y' = \frac{2xy - x^3 - xy^2}{x^2y + y^3 - x^2}$$

At (1, 1): $y' = 0$.

35. $x^2 + y^2 = 36$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y}$$

$$y'' = \frac{y(-1) + xy'}{y^2} = \frac{-y + x\left(\frac{-x}{y}\right)}{y^2} = \frac{-y^2 - x^2}{y^3} = \frac{-36}{y^3}$$

39. $y^2 = x^3$

$$2yy' = 3x^2$$

$$y' = \frac{3x^2}{2y} = \frac{3x^2}{2y} \cdot \frac{xy}{xy} = \frac{3y}{2x} \cdot \frac{x^3}{y^2} = \frac{3y}{2x}$$

$$y'' = \frac{2x(3y') - 3y(2)}{4x^2}$$

$$= \frac{2x[3 \cdot (3y/2x)] - 6y}{4x^2}$$

$$= \frac{3y}{4x^2} = \frac{3x}{4y}$$

41. $\sqrt{x} + \sqrt{y} = 4$

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$$

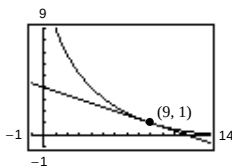
$$y' = \frac{-\sqrt{y}}{\sqrt{x}}$$

$$\text{At } (9, 1), y' = -\frac{1}{3}$$

$$\text{Tangent line: } y - 1 = -\frac{1}{3}(x - 9)$$

$$y = -\frac{1}{3}x + 4$$

$$x + 3y - 12 = 0$$



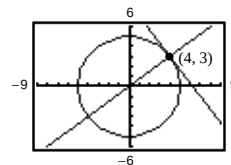
43. $x^2 + y^2 = 25$

$$y' = \frac{-x}{y}$$

At (4, 3):

$$\text{Tangent line: } y - 3 = \frac{-4}{3}(x - 4) \Rightarrow 4x + 3y - 25 = 0$$

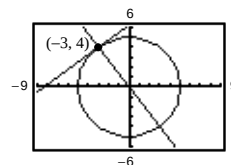
$$\text{Normal line: } y - 3 = \frac{3}{4}(x - 4) \Rightarrow 3x - 4y = 0.$$



At (-3, 4):

$$\text{Tangent line: } y - 4 = \frac{3}{4}(x + 3) \Rightarrow 3x - 4y + 25 = 0$$

$$\text{Normal line: } y - 4 = \frac{-4}{3}(x + 3) \Rightarrow 4x + 3y = 0.$$



45. $x^2 + y^2 = r^2$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y} = \text{slope of tangent line}$$

$$\frac{y}{x} = \text{slope of normal line}$$

Let (x_0, y_0) be a point on the circle. If $x_0 = 0$, then the tangent line is horizontal, the normal line is vertical and, hence, passes through the origin. If $x_0 \neq 0$, then the equation of the normal line is

$$y - y_0 = \frac{y_0}{x_0}(x - x_0)$$

$$y = \frac{y_0}{x_0}x$$

which passes through the origin.

47. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

$$50x + 32yy' + 200 - 160y' = 0$$

$$y' = \frac{200 + 50x}{160 - 32y}$$

Horizontal tangents occur when $x = -4$:

$$25(16) + 16y^2 + 200(-4) - 160y + 400 = 0$$

$$y(y - 10) = 0 \Rightarrow y = 0, 10$$

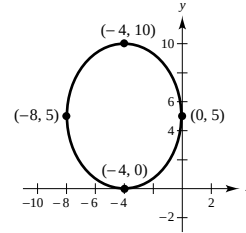
Horizontal tangents: $(-4, 0), (-4, 10)$.

Vertical tangents occur when $y = 5$:

$$25x^2 + 400 + 200x - 800 + 400 = 0$$

$$25x(x + 8) = 0 \Rightarrow x = 0, -8$$

Vertical tangents: $(0, 5), (-8, 5)$.



49. Find the points of intersection by letting $y^2 = 4x$ in the equation $2x^2 + y^2 = 6$.

$$2x^2 + 4x = 6 \quad \text{and} \quad (x + 3)(x - 1) = 0$$

The curves intersect at $(1, \pm 2)$.

Ellipse:

$$4x + 2yy' = 0$$

$$y' = -\frac{2x}{y}$$

Parabola:

$$2yy' = 4$$

$$y' = \frac{2}{y}$$

At $(1, 2)$, the slopes are:

$$y' = -1$$

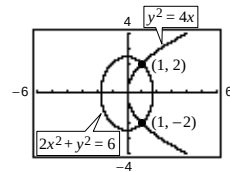
$$y' = 1.$$

At $(1, -2)$, the slopes are:

$$y' = 1$$

$$y' = -1.$$

Tangents are perpendicular.



51. $y = -x$ and $x = \sin y$

Point of intersection: $(0, 0)$

$$y = -x:$$

$$y' = -1$$

$$x = \sin y:$$

$$1 = y' \cos y$$

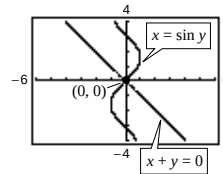
$$y' = \sec y$$

At $(0, 0)$, the slopes are:

$$y' = -1$$

$$y' = 1.$$

Tangents are perpendicular.

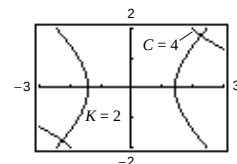
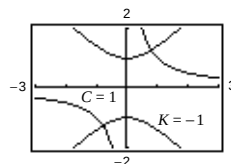


53. $xy = C$ $x^2 - y^2 = K$

$$xy' + y = 0 \quad 2x - 2yy' = 0$$

$$y' = -\frac{y}{x}$$

$$y' = \frac{x}{y}$$



At any point of intersection (x, y) the product of the slopes is $(-y/x)(x/y) = -1$. The curves are orthogonal.

55. $2y^2 - 3x^4 = 0$

(a) $4yy' - 12x^3 = 0$

$$4yy' = 12x^3$$

$$y' = \frac{12x^3}{4y} = \frac{3x^3}{y}$$

(b) $4y \frac{dy}{dt} - 12x^3 \frac{dx}{dt} = 0$

$$y \frac{dy}{dt} = 3x^3 \frac{dx}{dt}$$

57. $\cos \pi y - 3 \sin \pi x = 1$

(a) $-\pi \sin(\pi y)y' - 3\pi \cos \pi x = 0$

$$y' = \frac{-3 \cos \pi x}{\sin \pi y}$$

(b) $-\pi \sin(\pi y) \frac{dy}{dt} - 3\pi \cos(\pi x) \frac{dx}{dt} = 0$

$$-\sin(\pi y) \frac{dy}{dt} = 3 \cos(\pi x) \frac{dx}{dt}$$

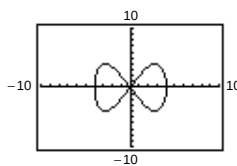
59. A function is in explicit form if y is written as a function of x : $y = f(x)$. For example, $y = x^3$. An implicit equation is not in the form $y = f(x)$. For example, $x^2 + y^2 = 5$.

61. (a) $x^4 = 4(4x^2 - y^2)$

$$4y^2 = 16x^2 - x^4$$

$$y^2 = 4x^2 - \frac{1}{4}x^4$$

$$y = \pm \sqrt{4x^2 - \frac{1}{4}x^4}$$



(b) $y = 3 \Rightarrow 9 = 4x^2 - \frac{1}{4}x^4$

$$36 = 16x^2 - x^4$$

$$x^4 - 16x^2 + 36 = 0$$

$$x^2 = \frac{16 \pm \sqrt{256 - 144}}{2} = 8 \pm \sqrt{28}$$

Note that $x^2 = 8 \pm \sqrt{28} = 8 \pm 2\sqrt{7} = (1 \pm \sqrt{7})^2$.

Hence, there are four values of x :

$$-1 - \sqrt{7}, 1 - \sqrt{7}, -1 + \sqrt{7}, 1 + \sqrt{7}$$

To find the slope, $2yy' = 8x - x^3 \Rightarrow y' = \frac{x(8 - x^2)}{2(3)}$.

For $x = -1 - \sqrt{7}$, $y' = \frac{1}{3}(\sqrt{7} + 7)$, and the line is

$$y_1 = \frac{1}{3}(\sqrt{7} + 7)(x + 1 + \sqrt{7}) + 3 = \frac{1}{3}[(\sqrt{7} + 7)x + 8\sqrt{7} + 23].$$

For $x = 1 - \sqrt{7}$, $y' = \frac{1}{3}(\sqrt{7} - 7)$, and the line is

$$y_2 = \frac{1}{3}(\sqrt{7} - 7)(x - 1 + \sqrt{7}) + 3 = \frac{1}{3}[(\sqrt{7} - 7)x + 23 - 8\sqrt{7}].$$

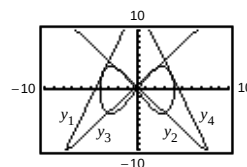
For $x = -1 + \sqrt{7}$, $y' = -\frac{1}{3}(\sqrt{7} - 7)$, and the line is

$$y_3 = -\frac{1}{3}(\sqrt{7} - 7)(x + 1 - \sqrt{7}) + 3 = -\frac{1}{3}[(\sqrt{7} - 7)x - (23 - 8\sqrt{7})].$$

For $x = 1 + \sqrt{7}$, $y' = -\frac{1}{3}(\sqrt{7} + 7)$, and the line is

$$y_4 = -\frac{1}{3}(\sqrt{7} + 7)(x - 1 - \sqrt{7}) + 3 = -\frac{1}{3}[(\sqrt{7} + 7)x - (8\sqrt{7} + 23)].$$

—CONTINUED—



61. —CONTINUED—

(c) Equating y_3 and y_4 ,

$$\begin{aligned}
 -\frac{1}{3}(\sqrt{7} - 7)(x + 1 - \sqrt{7}) + 3 &= -\frac{1}{3}(\sqrt{7} + 7)(x - 1 - \sqrt{7}) + 3 \\
 (\sqrt{7} - 7)(x + 1 - \sqrt{7}) &= (\sqrt{7} + 7)(x - 1 - \sqrt{7}) \\
 \sqrt{7}x + \sqrt{7} - 7 - 7x - 7 + 7\sqrt{7} &= \sqrt{7}x - \sqrt{7} - 7 + 7x - 7 - 7\sqrt{7} \\
 16\sqrt{7} &= 14x \\
 x &= \frac{8\sqrt{7}}{7}
 \end{aligned}$$

If $x = \frac{8\sqrt{7}}{7}$, then $y = 5$ and the lines intersect at $\left(\frac{8\sqrt{7}}{7}, 5\right)$.

63. Let $f(x) = x^n = x^{p/q}$, where p and q are nonzero integers and $q > 0$. First consider the case where $p = 1$. The derivative of $f(x) = x^{1/q}$ is given by

$$\frac{d}{dx}[x^{1/q}] = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x}$$

where $t = x + \Delta x$. Observe that

$$\begin{aligned}
 \frac{f(t) - f(x)}{t - x} &= \frac{t^{1/q} - x^{1/q}}{t - x} = \frac{t^{1/q} - x^{1/q}}{(t^{1/q})^q - (x^{1/q})^q} \\
 &= \frac{t^{1/q} - x^{1/q}}{(t^{1/q} - x^{1/q})(t^{1-(1/q)} + t^{1-(2/q)}x^{1/q} + \cdots + t^{1/q}x^{1-(2/q)} + x^{1-(1/q)})} \\
 &= \frac{1}{t^{1-(1/q)} + t^{1-(2/q)}x^{1/q} + \cdots + t^{1/q}x^{1-(2/q)} + x^{1-(1/q)}}.
 \end{aligned}$$

As $t \rightarrow x$, the denominator approaches $qx^{1-(1/q)}$. That is,

$$\frac{d}{dx}[x^{1/q}] = \frac{1}{qx^{1-(1/q)}} = \frac{1}{q}x^{(1/q)-1}.$$

Now consider $f(x) = x^{p/q} = (x^p)^{1/q}$. From the Chain Rule,

$$f'(x) = \frac{1}{q}(x^p)^{(1/q)-1} \frac{d}{dx}[x^p] = \frac{1}{q}(x^p)^{(1/q)-1} px^{p-1} = \frac{p}{q}x^{[(p/q)-p]+(p-1)} = \frac{p}{q}x^{(p/q)-1} = nx^{n-1} \left(n = \frac{p}{q}\right).$$

Section 2.6 Related Rates

1. $y = \sqrt{x}$

$$\frac{dy}{dt} = \left(\frac{1}{2\sqrt{x}}\right) \frac{dx}{dt}$$

$$\frac{dx}{dt} = 2\sqrt{x} \frac{dy}{dt}$$

(a) When $x = 4$ and $dx/dt = 3$,

$$\frac{dy}{dt} = \frac{1}{2\sqrt{4}}(3) = \frac{3}{4}.$$

(b) When $x = 25$ and $dy/dt = 2$,

$$\frac{dx}{dt} = 2\sqrt{25}(2) = 20.$$

3. $xy = 4$

$$x \frac{dy}{dt} + y \frac{dx}{dt} = 0$$

$$\frac{dy}{dt} = \left(-\frac{y}{x}\right) \frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(-\frac{x}{y}\right) \frac{dy}{dt}$$

(a) When $x = 8$, $y = 1/2$, and $dx/dt = 10$,

$$\frac{dy}{dt} = -\frac{1/2}{8}(10) = -\frac{5}{8}.$$

(b) When $x = 1$, $y = 4$, and $dy/dt = -6$,

$$\frac{dx}{dt} = -\frac{1}{4}(-6) = \frac{3}{2}.$$

5. $y = x^2 + 1$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = 2x \frac{dx}{dt}$$

(a) When $x = -1$,

$$\frac{dy}{dt} = 2(-1)(2) = -4 \text{ cm/sec.}$$

(b) When $x = 0$,

$$\frac{dy}{dt} = 2(0)(2) = 0 \text{ cm/sec.}$$

(c) When $x = 1$,

$$\frac{dy}{dt} = 2(1)(2) = 4 \text{ cm/sec.}$$

9. (a) $\frac{dx}{dt}$ negative $\Rightarrow \frac{dy}{dt}$ positive

(b) $\frac{dy}{dt}$ positive $\Rightarrow \frac{dx}{dt}$ negative

13. $D = \sqrt{x^2 + y^2} = \sqrt{x^2 + (x^2 + 1)^2} = \sqrt{x^4 + 3x^2 + 1}$

$$\frac{dx}{dt} = 2$$

$$\frac{dD}{dt} = \frac{1}{2}(x^4 + 3x^2 + 1)^{-1/2}(4x^3 + 6x) \frac{dx}{dt} = \frac{2x^3 + 3x}{\sqrt{x^4 + 3x^2 + 1}} \frac{dx}{dt} = \frac{4x^3 + 6x}{\sqrt{x^4 + 3x^2 + 1}}$$

15. $A = \pi r^2$

$$\frac{dr}{dt} = 3$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

(a) When $r = 6$,

$$\frac{dA}{dt} = 2\pi(6)(3) = 36\pi \text{ cm}^2/\text{min.}$$

(b) When $r = 24$,

$$\frac{dA}{dt} = 2\pi(24)(3) = 144\pi \text{ cm}^2/\text{min.}$$

7. $y = \tan x$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \sec^2 x \frac{dx}{dt}$$

(a) When $x = -\pi/3$,

$$\frac{dy}{dt} = (2)^2(2) = 8 \text{ cm/sec.}$$

(b) When $x = -\pi/4$,

$$\frac{dy}{dt} = (\sqrt{2})^2(2) = 4 \text{ cm/sec.}$$

(c) When $x = 0$,

$$\frac{dy}{dt} = (1)^2(2) = 2 \text{ cm/sec.}$$

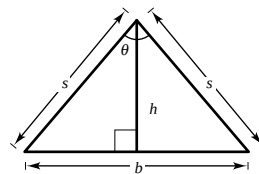
11. Yes, y changes at a constant rate: $\frac{dy}{dt} = a \cdot \frac{dx}{dt}$.

No, the rate $\frac{dy}{dt}$ is a multiple of $\frac{dx}{dt}$.

17. (a) $\sin \frac{\theta}{2} = \frac{(1/2)b}{s} \Rightarrow b = 2s \sin \frac{\theta}{2}$

$$\cos \frac{\theta}{2} = \frac{h}{s} \Rightarrow h = s \cos \frac{\theta}{2}$$

$$\begin{aligned} A &= \frac{1}{2}bh = \frac{1}{2}\left(2s \sin \frac{\theta}{2}\right)\left(s \cos \frac{\theta}{2}\right) \\ &= \frac{s^2}{2}\left(2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right) = \frac{s^2}{2} \sin \theta \end{aligned}$$



(b) $\frac{dA}{dt} = \frac{s^2}{2} \cos \theta \frac{d\theta}{dt}$ where $\frac{d\theta}{dt} = \frac{1}{2} \text{ rad/min.}$

$$\text{When } \theta = \frac{\pi}{6}, \frac{dA}{dt} = \frac{s^2}{2} \left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{2}\right) = \frac{\sqrt{3}s^2}{8}$$

$$\text{When } \theta = \frac{\pi}{3}, \frac{dA}{dt} = \frac{s^2}{2} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) = \frac{s^2}{8}$$

(c) If $d\theta/dt$ is constant, dA/dt is proportional to $\cos \theta$.

19. $V = \frac{4}{3}\pi r^3, \frac{dV}{dt} = 800$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \left(\frac{dV}{dt} \right) = \frac{1}{4\pi r^2} (800)$$

(a) When $r = 30$, $\frac{dr}{dt} = \frac{1}{4\pi(30)^2} (800) = \frac{2}{9\pi}$ cm/min.

(b) When $r = 60$, $\frac{dr}{dt} = \frac{1}{4\pi(60)^2} (800) = \frac{1}{18\pi}$ cm/min.

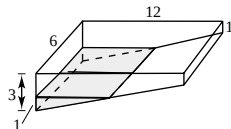
23. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{9}{4}h^2 \right) h$ [since $2r = 3h$]
 $= \frac{3\pi}{4}h^3$

$$\frac{dV}{dt} = 10$$

$$\frac{dV}{dt} = \frac{9\pi}{4}h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{4(dV/dt)}{9\pi h^2}$$

When $h = 15$, $\frac{dh}{dt} = \frac{4(10)}{9\pi(15)^2} = \frac{8}{405\pi}$ ft/min.

25.



(a) Total volume of pool $= \frac{1}{2}(2)(12)(6) + (1)(6)(12) = 144 \text{ m}^3$

Volume of 1m. of water $= \frac{1}{2}(1)(6)(6) = 18 \text{ m}^3$

(see similar triangle diagram)

% pool filled $= \frac{18}{144}(100\%) = 12.5\%$

(b) Since for $0 \leq h \leq 2$, $b = 6h$, you have

$$V = \frac{1}{2}bh(6) = 3bh = 3(6h)h = 18h^2$$

$$\frac{dV}{dt} = 36h \frac{dh}{dt} = \frac{1}{4} \Rightarrow \frac{dh}{dt} = \frac{1}{144h} = \frac{1}{144(1)} = \frac{1}{144} \text{ m/min.}$$

21. $s = 6x^2$

$$\frac{dx}{dt} = 3$$

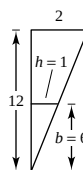
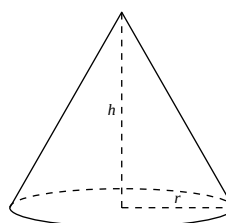
$$\frac{ds}{dt} = 12x \frac{dx}{dt}$$

(a) When $x = 1$,

$$\frac{ds}{dt} = 12(1)(3) = 36 \text{ cm}^2/\text{sec.}$$

(b) When $x = 10$,

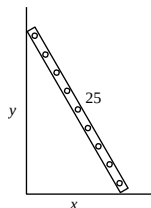
$$\frac{ds}{dt} = 12(10)(3) = 360 \text{ cm}^2/\text{sec.}$$



27. $x^2 + y^2 = 25^2$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-x}{y} \cdot \frac{dx}{dt} = \frac{-2x}{y} \text{ since } \frac{dx}{dt} = 2.$$



(a) When $x = 7, y = \sqrt{576} = 24, \frac{dy}{dt} = \frac{-2(7)}{24} = \frac{-7}{12}$ ft/sec.

When $x = 15, y = \sqrt{400} = 20, \frac{dy}{dt} = \frac{-2(15)}{20} = \frac{-3}{2}$ ft/sec.

When $x = 24, y = 7, \frac{dy}{dt} = \frac{-2(24)}{7} = \frac{-48}{7}$ ft/sec.

(b) $A = \frac{1}{2}xy$

$$\frac{dA}{dt} = \frac{1}{2} \left(x \frac{dy}{dt} + y \frac{dx}{dt} \right)$$

From part (a) we have $x = 7, y = 24, \frac{dx}{dt} = 2,$

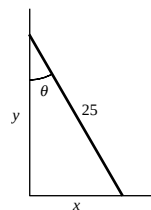
and $\frac{dy}{dt} = -\frac{7}{12}.$

$$\begin{aligned} \text{Thus, } \frac{dA}{dt} &= \frac{1}{2} \left[7 \left(-\frac{7}{12} \right) + 24(2) \right] \\ &= \frac{527}{24} \approx 21.96 \text{ ft}^2/\text{sec}. \end{aligned}$$

(c) $\tan \theta = \frac{x}{y}$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{y} \cdot \frac{dx}{dt} - \frac{x}{y^2} \cdot \frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \cos^2 \theta \left[\frac{1}{y} \cdot \frac{dx}{dt} - \frac{x}{y^2} \cdot \frac{dy}{dt} \right]$$



Using $x = 7, y = 24, \frac{dx}{dt} = 2, \frac{dy}{dt} = -\frac{7}{12}$ and $\cos \theta = \frac{24}{25}$, we have $\frac{d\theta}{dt} = \left(\frac{24}{25} \right)^2 \left[\frac{1}{24}(2) - \frac{7}{(24)^2} \left(-\frac{7}{12} \right) \right] = \frac{1}{12}$ rad/sec.

29. When $y = 6, x = \sqrt{12^2 - 6^2} = 6\sqrt{3}$, and

$$s = \sqrt{x^2 + (12 - y)^2}$$

$$= \sqrt{108 + 36} = 12.$$

$$x^2 + (12 - y)^2 = s^2$$

$$2x \frac{dx}{dt} + 2(12 - y)(-1) \frac{dy}{dt} = 2s \frac{ds}{dt}$$

$$x \frac{dx}{dt} + (y - 12) \frac{dy}{dt} = s \frac{ds}{dt}$$

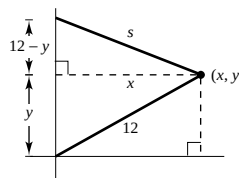
Also, $x^2 + y^2 = 12^2$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt}.$$

Thus, $x \frac{dx}{dt} + (y - 12) \left(\frac{-x}{y} \frac{dx}{dt} \right) = s \frac{ds}{dt}$

$$\frac{dx}{dt} \left[x - x + \frac{12x}{y} \right] = s \frac{ds}{dt} \Rightarrow \frac{dx}{dt} = \frac{sy}{12x} \cdot \frac{ds}{dt} = \frac{(12)(6)}{(12)(6\sqrt{3})} (-0.2) = \frac{-1}{5\sqrt{3}} = \frac{-\sqrt{3}}{15} \text{ m/sec (horizontal)}$$

$$\frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt} = \frac{-6\sqrt{3}}{6} \cdot \frac{(-\sqrt{3})}{15} = \frac{1}{5} \text{ m/sec (vertical).}$$



31. (a) $s^2 = x^2 + y^2$

$$\frac{dx}{dt} = -450$$

$$\frac{dy}{dt} = -600$$

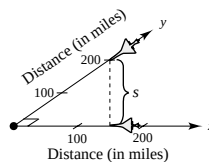
$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{x(dx/dt) + y(dy/dt)}{s}$$

When $x = 150$ and $y = 200$, $s = 250$ and

$$\frac{ds}{dt} = \frac{150(-450) + 200(-600)}{250} = -750 \text{ mph.}$$

(b) $t = \frac{250}{750} = \frac{1}{3} \text{ hr} = 20 \text{ min}$



33. $s^2 = 90^2 + x^2$

$$x = 30$$

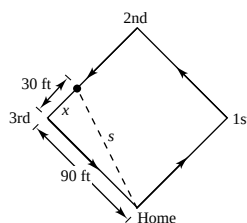
$$\frac{dx}{dt} = -28$$

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} \Rightarrow \frac{ds}{dt} = \frac{x}{s} \cdot \frac{dx}{dt}$$

When $x = 30$,

$$s = \sqrt{90^2 + 30^2} = 30\sqrt{10}$$

$$\frac{ds}{dt} = \frac{30}{30\sqrt{10}}(-28) = \frac{-28}{\sqrt{10}} \approx -8.85 \text{ ft/sec.}$$



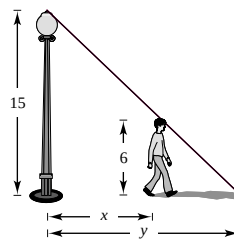
35. (a) $\frac{15}{6} = \frac{y}{y-x} \Rightarrow 15y - 15x = 6y$

$$y = \frac{5}{3}x$$

$$\frac{dx}{dt} = 5$$

$$\frac{dy}{dt} = \frac{5}{3} \cdot \frac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3} \text{ ft/sec}$$

(b) $\frac{d(y-x)}{dt} = \frac{dy}{dt} - \frac{dx}{dt} = \frac{25}{3} - 5 = \frac{10}{3} \text{ ft/sec}$



37. $x(t) = \frac{1}{2} \sin \frac{\pi t}{6}, x^2 + y^2 = 1$

(a) Period: $\frac{2\pi}{\pi/6} = 12$ seconds

(b) When $x = \frac{1}{2}, y = \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$ m.

Lowest point: $\left(0, \frac{\sqrt{3}}{2}\right)$

(c) When $x = \frac{1}{4}, y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$ and $t = 1$

$$\frac{dx}{dt} = \frac{1}{2} \left(\frac{\pi}{6}\right) \cos \frac{\pi t}{6} = \frac{\pi}{12} \cos \frac{\pi t}{6}$$

$$x^2 + y^2 = 1$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt}$$

Thus,

$$\begin{aligned} \frac{dy}{dt} &= -\frac{1/4}{\sqrt{15}/4} \cdot \frac{\pi}{12} \cos\left(\frac{\pi}{6}\right) \\ &= \frac{-\pi}{\sqrt{15}} \left(\frac{1}{12}\right) \frac{\sqrt{3}}{2} = \frac{-\pi}{24} \frac{1}{\sqrt{5}} = \frac{-\sqrt{5}\pi}{120} \end{aligned}$$

$$\text{Speed} = \left| \frac{-\sqrt{5}\pi}{120} \right| = \frac{\sqrt{5}\pi}{120} \text{ m/sec}$$

41. $pV^{1.3} = k$

$$1.3 pV^{0.3} \frac{dV}{dt} + V^{1.3} \frac{dp}{dt} = 0$$

$$V^{0.3} \left(1.3p \frac{dV}{dt} + V \frac{dp}{dt} \right) = 0$$

$$1.3p \frac{dV}{dt} = -V \frac{dp}{dt}$$

43. $\tan \theta = \frac{y}{30}$

$$\frac{dy}{dt} = 3 \text{ m/sec.}$$

$$\sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{30} \frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{30} \cos^2 \theta \cdot \frac{dy}{dt}$$

When $y = 30$, $\theta = \pi/4$ and $\cos \theta = \sqrt{2}/2$. Thus,

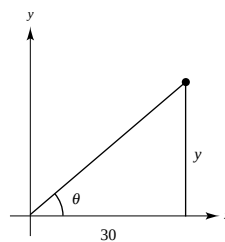
$$\frac{d\theta}{dt} = \frac{1}{30} \left(\frac{1}{2}\right) (3) = \frac{1}{20} \text{ rad/sec.}$$

39. Since the evaporation rate is proportional to the surface area, $dV/dt = k(4\pi r^2)$. However, since $V = (4/3)\pi r^3$, we have

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

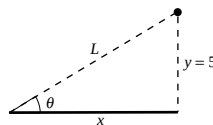
Therefore,

$$k(4\pi r^2) = 4\pi r^2 \frac{dr}{dt} \Rightarrow k = \frac{dr}{dt}.$$



45. $\tan \theta = \frac{y}{x}, y = 5$

$$\frac{dx}{dt} = -600 \text{ mi/hr}$$



$$(\sec^2 \theta) \frac{d\theta}{dt} = -\frac{5}{x^2} \cdot \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \cos^2 \theta \left(-\frac{5}{x^2} \right) \frac{dx}{dt} = \frac{x^2}{L^2} \left(-\frac{5}{x^2} \right) \frac{dx}{dt}$$

$$= \left(-\frac{5^2}{L^2} \right) \left(\frac{1}{5} \right) \frac{dx}{dt} = (-\sin^2 \theta) \left(\frac{1}{5} \right) (-600) = 120 \sin^2 \theta$$

(a) When $\theta = 30^\circ$, $\frac{d\theta}{dt} = \frac{120}{4} = 30 \text{ rad/hr} = \frac{1}{2} \text{ rad/min}$.

(b) When $\theta = 60^\circ$, $\frac{d\theta}{dt} = 120 \left(\frac{3}{4} \right) = 90 \text{ rad/hr} = \frac{3}{2} \text{ rad/min}$.

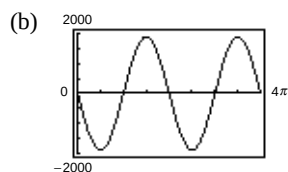
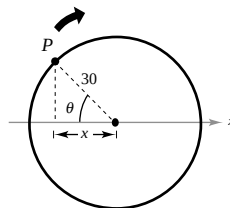
(c) When $\theta = 75^\circ$, $\frac{d\theta}{dt} = 120 \sin^2 75^\circ \approx 111.96 \text{ rad/hr} \approx 1.87 \text{ rad/min}$.

47. $\frac{d\theta}{dt} = (10 \text{ rev/sec})(2\pi \text{ rad/rev}) = 20\pi \text{ rad/sec}$

(a) $\cos \theta = \frac{x}{30}$

$$-\sin \theta \frac{d\theta}{dt} = \frac{1}{30} \frac{dx}{dt}$$

$$\frac{dx}{dt} = -30 \sin \theta \frac{d\theta}{dt} = -30 \sin \theta (20\pi) = -600\pi \sin \theta$$



(c) $|dx/dt| = |-600\pi \sin \theta|$ is greatest when $\sin \theta = 1 \Rightarrow \theta = (\pi/2) + n\pi$ (or $90^\circ + n \cdot 180^\circ$)

$|dx/dt|$ is least when $\theta = n\pi$ (or $n \cdot 180^\circ$).

(d) For $\theta = 30^\circ$, $\frac{dx}{dt} = -600\pi \sin(30^\circ) = -600\pi \frac{1}{2} = -300\pi \text{ cm/sec}$.

For $\theta = 60^\circ$, $\frac{dx}{dt} = -600\pi \sin(60^\circ) = -600\pi \frac{\sqrt{3}}{2} = -300\sqrt{3} \pi \text{ cm/sec}$

49. $\tan \theta = \frac{x}{50} \Rightarrow x = 50 \tan \theta$

$$\frac{dx}{dt} = 50 \sec^2 \theta \frac{d\theta}{dt}$$

$$2 = 50 \sec^2 \theta \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{25} \cos^2 \theta, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$

51. $x^2 + y^2 = 25$; acceleration of the top of the ladder $= \frac{d^2y}{dt^2}$

First derivative: $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$

$$x \frac{dx}{dt} + y \frac{dy}{dt} = 0$$

Second derivative: $x \frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{dx}{dt} + y \frac{d^2y}{dt^2} + \frac{dy}{dt} \cdot \frac{dy}{dt} = 0$

$$\frac{d^2y}{dt^2} = \left(\frac{1}{y}\right) \left[-x \frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 \right]$$

When $x = 7$, $y = 24$, $\frac{dy}{dt} = -\frac{7}{12}$, and $\frac{dx}{dt} = 2$ (see Exercise 27). Since $\frac{dx}{dt}$ is constant, $\frac{d^2x}{dt^2} = 0$.

$$\frac{d^2y}{dt^2} = \frac{1}{24} \left[-7(0) - (2)^2 - \left(-\frac{7}{12}\right)^2 \right] = \frac{1}{24} \left[-4 - \frac{49}{144} \right] = \frac{1}{24} \left[-\frac{625}{144} \right] \approx -0.1808 \text{ ft/sec}^2$$

53. (a) Using a graphing utility, you obtain $m(s) = -0.881s^2 + 29.10s - 206.2$

(b) $\frac{dm}{dt} = \frac{dm}{ds} \frac{ds}{dt} = (-1.762s + 29.10) \frac{ds}{dt}$

(c) If $t = s$ (1995), then $s = 15.5$ and $\frac{ds}{dt} = 1.2$.

Thus, $\frac{dm}{dt} = (-1.762(15.5) + 29.10)(1.2) \approx 2.15$ million.

Review Exercises for Chapter 2

1. $f(x) = x^2 - 2x + 3$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 - 2(x + \Delta x) + 3] - [x^2 - 2x + 3]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + 3) - (x^2 - 2x + 3)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2x(\Delta x) + (\Delta x)^2 - 2(\Delta x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x - 2) = 2x - 2 \end{aligned}$$

3. $f(x) = \sqrt{x} + 1$

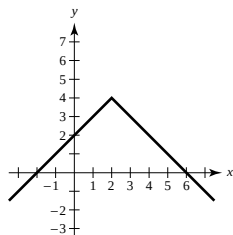
5. f is differentiable for all $x \neq -1$.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(\sqrt{x + \Delta x} + 1) - (\sqrt{x} + 1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

7. $f(x) = 4 - |x - 2|$

(a) Continuous at $x = 2$.

(b) Not differentiable at $x = 2$ because of the sharp turn in the graph.

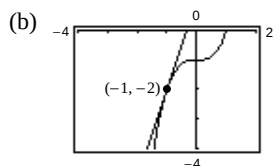


11. (a) Using the limit definition, $f'(x) = 3x^2$.

At $x = -1$, $f'(-1) = 3$. The tangent line is

$$y - (-2) = 3(x - (-1))$$

$$y = 3x + 1$$

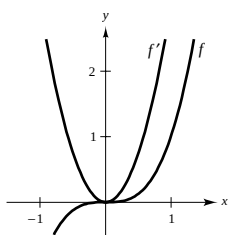


9. Using the limit definition, you obtain $g'(x) = \frac{4}{3}x - \frac{1}{6}$.

At $x = -1$, $g'(-1) = -\frac{4}{3} - \frac{1}{6} = -\frac{3}{2}$

$$\begin{aligned} 13. \quad g'(2) &= \lim_{x \rightarrow 2} \frac{g(x) - g(2)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{x^2(x - 1) - 4}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{x^3 - x^2 - 4}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{(x - 2)(x^2 + x + 2)}{x - 2} \\ &= \lim_{x \rightarrow 2} (x^2 + x + 2) = 8 \end{aligned}$$

15.



17. $y = 25$

$$y' = 0$$

19. $f(x) = x^8$

$$f'(x) = 8x^7$$

21. $h(t) = 3t^4$

$$h'(t) = 12t^3$$

23. $f(x) = x^3 - 3x^2$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

25. $h(x) = 6\sqrt{x} + 3\sqrt[3]{x} = 6x^{1/2} + 3x^{1/3}$

$$h'(x) = 3x^{-1/2} + x^{-2/3} = \frac{3}{\sqrt{x}} + \frac{1}{\sqrt[3]{x^2}}$$

27. $g(t) = \frac{2}{3}t^{-2}$

$$g'(t) = \frac{-4}{3}t^{-3} = -\frac{4}{3t^3}$$

29. $f(\theta) = 2\theta - 3\sin \theta$

$$f'(\theta) = 2 - 3\cos \theta$$

31. $f(\theta) = 3\cos \theta - \frac{\sin \theta}{4}$

$$f'(\theta) = -3\sin \theta - \frac{\cos \theta}{4}$$

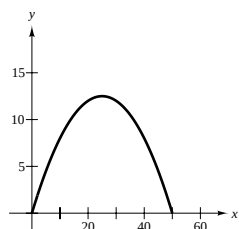
33. $F = 200\sqrt{T}$

$$F'(t) = \frac{100}{\sqrt{T}}$$

(a) When $T = 4$, $F'(4) = 50$ vibrations/sec/lb.

(b) When $T = 9$, $F'(9) = 33\frac{1}{3}$ vibrations/sec/lb.

37. (a)



Total horizontal distance: 50

(b) $0 = x - 0.02x^2$

$$0 = x\left(1 - \frac{x}{50}\right) \text{ implies } x = 50.$$

39. $x(t) = t^2 - 3t + 2 = (t - 2)(t - 1)$

(a) $v(t) = x'(t) = 2t - 3$

$$a(t) = v'(t) = 2$$

(c) $v(t) = 0$ for $t = \frac{3}{2}$.

$$x = \left(\frac{3}{2} - 2\right)\left(\frac{3}{2} - 1\right) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$$

41. $f(x) = (3x^2 + 7)(x^2 - 2x + 3)$

$$\begin{aligned} f'(x) &= (3x^2 + 7)(2x - 2) + (x^2 - 2x + 3)(6x) \\ &= 2(6x^3 - 9x^2 + 16x - 7) \end{aligned}$$

45. $f(x) = 2x - x^{-2}$

$$\begin{aligned} f'(x) &= 2 + 2x^{-3} = 2\left(1 + \frac{1}{x^3}\right) \\ &= \frac{2(x^3 + 1)}{x^3} \end{aligned}$$

49. $f(x) = (4 - 3x^2)^{-1}$

$$f'(x) = -(4 - 3x^2)^{-2}(-6x) = \frac{6x}{(4 - 3x^2)^2}$$

53. $y = 3x^2 \sec x$

$$y' = 3x^2 \sec x \tan x + 6x \sec x$$

35. $s(t) = -16t^2 + s_0$

$$s(9.2) = -16(9.2)^2 + s_0 = 0$$

$$s_0 = 1354.24$$

The building is approximately 1354 feet high (or 415 m).

(c) Ball reaches maximum height when $x = 25$.

(d) $y = x - 0.02x^2$

$$y' = 1 - 0.04x$$

$$y'(0) = 1$$

$$y'(10) = 0.6$$

$$y'(25) = 0$$

$$y'(30) = -0.2$$

$$y'(50) = -1$$

(e) $y'(25) = 0$

(b) $v(t) < 0$ for $t < \frac{3}{2}$.

(d) $x(t) = 0$ for $t = 1, 2$.

$$|v(1)| = |2(1) - 3| = 1$$

$$|v(2)| = |2(2) - 3| = 1$$

The speed is 1 when the position is 0.

43. $h(x) = \sqrt{x} \sin x = x^{1/2} \sin x$

$$h'(x) = \frac{1}{2\sqrt{x}} \sin x + \sqrt{x} \cos x$$

47. $f(x) = \frac{x^2 + x - 1}{x^2 - 1}$

$$\begin{aligned} f'(x) &= \frac{(x^2 - 1)(2x + 1) - (x^2 + x - 1)(2x)}{(x^2 - 1)^2} \\ &= \frac{-(x^2 + 1)}{(x^2 - 1)^2} \end{aligned}$$

51. $y = \frac{x^2}{\cos x}$

$$y' = \frac{\cos x (2x) - x^2(-\sin x)}{\cos^2 x} = \frac{2x \cos x + x^2 \sin x}{\cos^2 x}$$

55. $y = -x \tan x$

$$y' = -x \sec^2 x - \tan x$$

57. $y = x \cos x - \sin x$

$$y' = -x \sin x + \cos x - \cos x = -x \sin x$$

61. $f(\theta) = 3 \tan \theta$

$$f'(\theta) = 3 \sec^2 \theta$$

$$f''(\theta) = 6 \sec \theta (\sec \theta \tan \theta) = 6 \sec^2 \theta \tan \theta$$

65. $f(x) = (1 - x^3)^{1/2}$

$$f'(x) = \frac{1}{2}(1 - x^3)^{-1/2}(-3x^2)$$

$$= -\frac{3x^2}{2\sqrt{1 - x^3}}$$

69. $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$

$$f'(s) = (s^2 - 1)^{5/2}(3s^2) + (s^3 + 5)\left(\frac{5}{2}\right)(s^2 - 1)^{3/2}(2s)$$

$$= s(s^2 - 1)^{3/2}[3s(s^2 - 1) + 5(s^3 + 5)]$$

$$= s(s^2 - 1)^{3/2}(8s^3 - 3s + 25)$$

73. $y = \frac{1}{2} \csc 2x$

$$y' = \frac{1}{2}(-\csc 2x \cot 2x)(2)$$

$$= -\csc 2x \cot 2x$$

77. $y = \frac{2}{3} \sin^{3/2} x - \frac{2}{7} \sin^{7/2} x$

$$y' = \sin^{1/2} x \cos x - \sin^{5/2} x \cos x$$

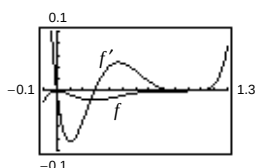
$$= (\cos x) \sqrt{\sin x}(1 - \sin^2 x)$$

$$= (\cos^3 x) \sqrt{\sin x}$$

81. $f(t) = t^2(t - 1)^5$

$$f'(t) = t(t - 1)^4(7t - 2)$$

The zeros of f' correspond to the points on the graph of f where the tangent line is horizontal.



59. $g(t) = t^3 - 3t + 2$

$$g'(t) = 3t^2 - 3$$

$$g''(t) = 6t$$

63. $y = 2 \sin x + 3 \cos x$

$$y' = 2 \cos x - 3 \sin x$$

$$y'' = -2 \sin x - 3 \cos x$$

$$y'' + y = -(2 \sin x + 3 \cos x) + (2 \sin x + 3 \cos x) = 0$$

67. $h(x) = \left(\frac{x-3}{x^2+1}\right)^2$

$$h'(x) = 2\left(\frac{x-3}{x^2+1}\right)\left(\frac{(x^2+1)(1) - (x-3)(2x)}{(x^2+1)^2}\right)$$

$$= \frac{2(x-3)(-x^2+6x+1)}{(x^2+1)^3}$$

71. $y = 3 \cos(3x + 1)$

$$y' = -9 \sin(3x + 1)$$

75. $y = \frac{x}{2} - \frac{\sin 2x}{4}$

$$y' = \frac{1}{2} - \frac{1}{4} \cos 2x(2)$$

$$= \frac{1}{2}(1 - \cos 2x) = \sin^2 x$$

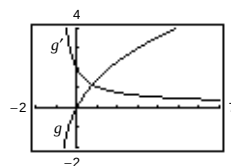
79. $y = \frac{\sin \pi x}{x+2}$

$$y' = \frac{(x+2)\pi \cos \pi x - \sin \pi x}{(x+2)^2}$$

83. $g(x) = 2x(x+1)^{-1/2}$

$$g'(x) = \frac{x+2}{(x+1)^{3/2}}$$

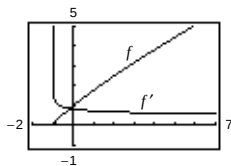
g' does not equal zero for any value of x in the domain. The graph of g has no horizontal tangent lines.



85. $f(t) = (t + 1)^{1/2}(t + 1)^{1/3} = (t + 1)^{5/6}$

$$f'(t) = \frac{5}{6(t + 1)^{1/6}}$$

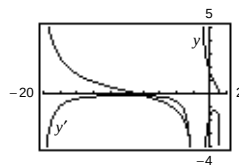
f' does not equal zero for any x in the domain. The graph of f has no horizontal tangent lines.



87. $y = \tan \sqrt{1 - x}$

$$y' = -\frac{\sec^2 \sqrt{1 - x}}{2\sqrt{1 - x}}$$

y' does not equal zero for any x in the domain. The graph has no horizontal tangent lines.



89. $y = 2x^2 + \sin 2x$

$$y' = 4x + 2 \cos 2x$$

$$y'' = 4 - 4 \sin 2x$$

91. $f(x) = \cot x$

$$f'(x) = -\csc^2 x$$

$$\begin{aligned} f'' &= -2 \csc x (-\csc x \cdot \cot x) \\ &= 2 \csc^2 x \cot x \end{aligned}$$

93. $f(t) = \frac{t}{(1 - t)^2}$

$$f'(t) = \frac{t + 1}{(1 - t)^3}$$

$$f''(t) = \frac{2(t + 2)}{(1 - t)^4}$$

95. $g(\theta) = \tan 3\theta - \sin(\theta - 1)$

$$g'(\theta) = 3 \sec^2 3\theta - \cos(\theta - 1)$$

$$g''(\theta) = 18 \sec^2 3\theta \tan 3\theta + \sin(\theta - 1)$$

97. $T = 700(t^2 + 4t + 10)^{-1}$

$$T' = \frac{-1400(t + 2)}{(t^2 + 4t + 10)^2}$$

(a) When $t = 1$,

$$T' = \frac{-1400(1 + 2)}{(1 + 4 + 10)^2} \approx -18.667 \text{ deg/hr.}$$

(b) When $t = 3$,

$$T' = \frac{-1400(3 + 2)}{(9 + 12 + 10)^2} \approx -7.284 \text{ deg/hr.}$$

(c) When $t = 5$,

$$T' = \frac{-1400(5 + 2)}{(25 + 30 + 10)^2} \approx -3.240 \text{ deg/hr.}$$

(d) When $t = 10$,

$$T' = \frac{-1400(10 + 2)}{(100 + 40 + 10)^2} \approx -0.747 \text{ deg/hr.}$$

99. $x^2 + 3xy + y^3 = 10$

$$2x + 3xy' + 3y + 3y^2y' = 0$$

$$3(x + y^2)y' = -(2x + 3y)$$

$$y' = \frac{-(2x + 3y)}{3(x + y^2)}$$

101. $y\sqrt{x} - x\sqrt{y} = 16$

$$y\left(\frac{1}{2}x^{-1/2}\right) + x^{1/2}y' - x\left(\frac{1}{2}y^{-1/2}y'\right) - y^{1/2} = 0$$

$$\left(\sqrt{x} - \frac{x}{2\sqrt{y}}\right)y' = \sqrt{y} - \frac{y}{2\sqrt{x}}$$

$$\frac{2\sqrt{xy} - x}{2\sqrt{y}}y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}}$$

$$y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}} \cdot \frac{2\sqrt{y}}{2\sqrt{xy} - x} = \frac{2y\sqrt{x} - y\sqrt{y}}{2x\sqrt{y} - x\sqrt{x}}$$

103. $x \sin y = y \cos x$

$$(x \cos y)y' + \sin y = -y \sin x + y' \cos x$$

$$y'(x \cos y - \cos x) = -y \sin x - \sin y$$

$$y' = \frac{y \sin x + \sin y}{\cos x - x \cos y}$$

105. $x^2 + y^2 = 20$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

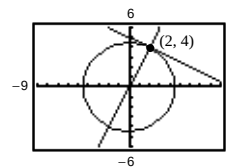
At $(2, 4)$: $y' = -\frac{1}{2}$

Tangent line: $y - 4 = -\frac{1}{2}(x - 2)$

$$x + 2y - 10 = 0$$

Normal line: $y - 4 = 2(x - 2)$

$$2x - y = 0$$



107. $y = \sqrt{x}$

$$\frac{dy}{dt} = 2 \text{ units/sec}$$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{x}} \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 2\sqrt{x} \frac{dy}{dt} = 4\sqrt{x}$$

(a) When $x = \frac{1}{2}$, $\frac{dx}{dt} = 2\sqrt{2}$ units/sec.

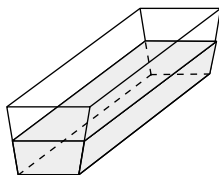
(b) When $x = 1$, $\frac{dx}{dt} = 4$ units/sec.

(c) When $x = 4$, $\frac{dx}{dt} = 8$ units/sec.

109. $\frac{s}{h} = \frac{1/2}{2}$

$$s = \frac{1}{4}h$$

$$\frac{dV}{dt} = 1$$



Width of water at depth h :

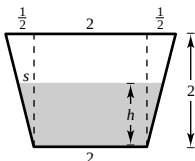
$$w = 2 + 2s = 2 + 2\left(\frac{1}{4}h\right) = \frac{4 + h}{2}$$

$$V = \frac{5}{2}\left(2 + \frac{4 + h}{2}\right)h = \frac{5}{4}(8 + h)h$$

$$\frac{dV}{dt} = \frac{5}{2}(4 + h)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2(dV/dt)}{5(4 + h)}$$

When $h = 1$, $\frac{dh}{dt} = \frac{2}{25}$ m/min.



111. $s(t) = 60 - 4.9t^2$

$$s'(t) = -9.8t$$

$$s = 35 = 60 - 4.9t^2$$

$$4.9t^2 = 25$$

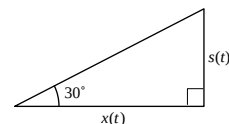
$$t = \frac{5}{\sqrt{4.9}}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{s(t)}{x(t)}$$

$$x(t) = \sqrt{3}s(t)$$

$$\frac{dx}{dt} = \sqrt{3} \frac{ds}{dt} = \sqrt{3}(-9.8) \frac{5}{\sqrt{4.9}}$$

$$\approx -38.34 \text{ m/sec}$$



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CHAPTER 2

Differentiation

Section 2.1 The Derivative and the Tangent Line Problem

Solutions to Even-Numbered Exercises

2. (a) $m = \frac{1}{4}$

(b) $m = 1$

4. (a) $\frac{f(4) - f(1)}{4 - 1} = \frac{5 - 2}{3} = 1$

$$\frac{f(4) - f(3)}{4 - 3} \approx \frac{5 - 4.75}{1} = 0.25$$

Thus, $\frac{f(4) - f(1)}{4 - 1} > \frac{f(4) - f(3)}{4 - 3}$

(b) The slope of the tangent line at $(1, 2)$ equals $f'(1)$. This slope is steeper than the slope of the line through $(1, 2)$ and $(4, 5)$. Thus,

$$\frac{f(4) - f(1)}{4 - 1} < f'(1).$$

6. $g(x) = \frac{3}{2}x + 1$ is a line. Slope = $\frac{3}{2}$

8. Slope at $(2, 1) = \lim_{\Delta x \rightarrow 0} \frac{g(2 + \Delta x) - g(2)}{\Delta x}$

$$= \lim_{\Delta x \rightarrow 0} \frac{5 - (2 + \Delta x)^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{5 - 4 - 4(\Delta x) - (\Delta x)^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} (-4 - \Delta x) = -4$$

10. Slope at $(-2, 7) = \lim_{\Delta t \rightarrow 0} \frac{h(-2 + \Delta t) - h(-2)}{\Delta t}$

$$= \lim_{\Delta t \rightarrow 0} \frac{(-2 + \Delta t)^2 + 3 - 7}{\Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} \frac{4 - 4(\Delta t) + (\Delta t)^2 - 4}{\Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} (-4 + \Delta t) = -4$$

12. $g(x) = -5$

$$g'(x) = \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-5 - (-5)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = 0$$

14. $f(x) = 3x + 2$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{[3(x + \Delta x) + 2] - [3x + 2]}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{3\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} 3 = 3$$

16. $f(x) = 9 - \frac{1}{2}x$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{[9 - (1/2)(x + \Delta x)] - [9 - (1/2)x]}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \left(-\frac{1}{2}\right) = -\frac{1}{2}$$

18. $f(x) = 1 - x^2$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[1 - (x + \Delta x)^2] - [1 - x^2]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1 - x^2 - 2x\Delta x - (\Delta x)^2 - 1 + x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} (-2x - \Delta x) = -2x \end{aligned}$$

20. $f(x) = x^3 + x^2$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^3 + (x + \Delta x)^2] - [x^3 + x^2]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + x^2 + 2x\Delta x + (\Delta x)^2 - x^3 - x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + (\Delta x)^2 + 2x + (\Delta x)) = 3x^2 + 2x \end{aligned}$$

22. $f(x) = \frac{1}{x^2}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{(x + \Delta x)^2} - \frac{1}{x^2}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 - (x + \Delta x)^2}{\Delta x(x + \Delta x)^2x^2} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x(x + \Delta x)^2x^2} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2x - \Delta x}{(x + \Delta x)^2x^2} \\ &= \frac{-2x}{x^4} \\ &= -\frac{2}{x^3} \end{aligned}$$

24. $f(x) = \frac{4}{\sqrt{x}}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{4}{\sqrt{x + \Delta x}} - \frac{4}{\sqrt{x}}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{4\sqrt{x} - 4\sqrt{x + \Delta x}}{\Delta x\sqrt{x}\sqrt{x + \Delta x}} \cdot \left(\frac{\sqrt{x} + \sqrt{x + \Delta x}}{\sqrt{x} + \sqrt{x + \Delta x}} \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{4x - 4(x + \Delta x)}{\Delta x\sqrt{x}\sqrt{x + \Delta x}(\sqrt{x} + \sqrt{x + \Delta x})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-4}{\sqrt{x}\sqrt{x + \Delta x}(\sqrt{x} + \sqrt{x + \Delta x})} \\ &= \frac{-4}{\sqrt{x}\sqrt{x}(\sqrt{x} + \sqrt{x})} = \frac{-2}{x\sqrt{x}} \end{aligned}$$

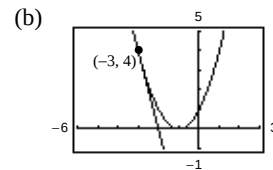
26. (a) $f(x) = x^2 + 2x + 1$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 + 2(x + \Delta x) + 1] - [x^2 + 2x + 1]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2 + 2\Delta x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x + 2) = 2x + 2
 \end{aligned}$$

At $(-3, 4)$, the slope of the tangent line is $m = 2(-3) + 2 = -4$.

The equation of the tangent line is

$$\begin{aligned}
 y - 4 &= -4(x + 3) \\
 y &= -4x - 8.
 \end{aligned}$$



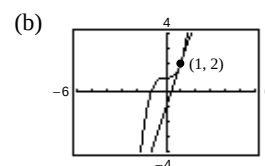
28. (a) $f(x) = x^3 + 1$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^3 + 1] - (x^3 + 1)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 + 1 - x^3 - 1}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} [3x^2 + 3x(\Delta x) + (\Delta x)^2] = 3x^2
 \end{aligned}$$

At $(1, 2)$, the slope of the tangent line is $m = 3(1)^2 = 3$.

The equation of the tangent line is

$$\begin{aligned}
 y - 2 &= 3(x - 1) \\
 y &= 3x - 1.
 \end{aligned}$$



30. (a) $f(x) = \sqrt{x - 1}$

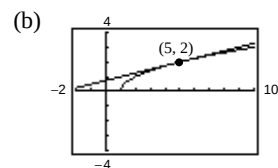
$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x - 1} - \sqrt{x - 1}}{\Delta x} \cdot \left(\frac{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}}{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}} \right) \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x - 1) - (x - 1)}{\Delta x(\sqrt{x + \Delta x - 1} + \sqrt{x - 1})} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}} = \frac{1}{2\sqrt{x - 1}}
 \end{aligned}$$

At $(5, 2)$, the slope of the tangent line is

$$m = \frac{1}{2\sqrt{5 - 1}} = \frac{1}{4}$$

The equation of the tangent line is

$$\begin{aligned}
 y - 2 &= \frac{1}{4}(x - 5) \\
 y &= \frac{1}{4}x + \frac{3}{4}
 \end{aligned}$$



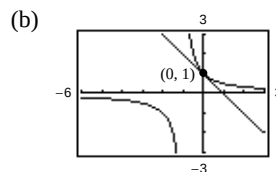
32. (a) $f(x) = \frac{1}{x+1}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x + \Delta x + 1} - \frac{1}{x + 1}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + 1) - (x + \Delta x + 1)}{\Delta x(x + \Delta x + 1)(x + 1)} \\ &= \lim_{\Delta x \rightarrow 0} -\frac{1}{(x + \Delta x + 1)(x + 1)} \\ &= -\frac{1}{(x + 1)^2} \end{aligned}$$

At $(0, 1)$, the slope of the tangent line is

$$m = \frac{-1}{(0 + 1)^2} = -1.$$

The equation of the tangent line is $y = -x + 1$.



34. Using the limit definition of derivative, $f'(x) = 3x^2$. Since the slope of the given line is 3, we have

$$3x^2 = 3$$

$$x^2 = 1 \Rightarrow x = \pm 1.$$

Therefore, at the points $(1, 3)$ and $(-1, 1)$ the tangent lines are parallel to $3x - y - 4 = 0$. These lines have equations

$$y - 3 = 3(x - 1) \text{ and } y - 1 = 3(x + 1)$$

$$y = 3x \quad y = 3x + 4$$

36. Using the limit definition of derivative, $f'(x) = \frac{-1}{2(x-1)^{3/2}}$.

Since the slope of the given line is $-\frac{1}{2}$, we have

$$\frac{-1}{2(x-1)^{3/2}} = -\frac{1}{2}$$

$$1 = (x-1)^{3/2}$$

$$1 = x - 1 \Rightarrow x = 2$$

At the point $(2, 1)$, the tangent line is parallel to $x + 2y + 7 = 0$. The equation of the tangent line is

$$y - 1 = -\frac{1}{2}(x - 2)$$

$$y = -\frac{1}{2}x + 2$$

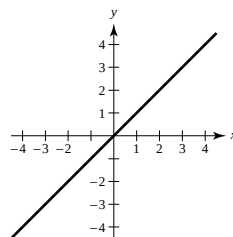
38. $h(-1) = 4$ because the tangent line passes through $(-1, 4)$

$$h'(-1) = \frac{6 - 4}{3 - (-1)} = \frac{2}{4} = \frac{1}{2}$$

40. $f(x) = x^2 \Rightarrow f'(x) = 2x$ (d)

42. f' does not exist at $x = 0$. Matches (c)

44.



Answers will vary.

Sample answer: $y = x$

46. (a) Yes. $\lim_{\Delta x \rightarrow 0} \frac{f(x + 2\Delta x) - f(x)}{2\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$

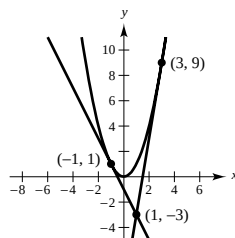
(b) No. The numerator does not approach zero.

(c) Yes.
$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x) - f(x - \Delta x) + f(x)}{2\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{f(x + \Delta x) - f(x)}{2\Delta x} + \frac{f(x - \Delta x) - f(x)}{2(-\Delta x)} \right] \\ &= \frac{1}{2}f'(x) + \frac{1}{2}f'(x) = f'(x) \end{aligned}$$

(d) Yes. $\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$

48. Let (x_0, y_0) be a point of tangency on the graph of f . By the limit definition for the derivative, $f'(x) = 2x$. The slope of the line through $(1, -3)$ and (x_0, y_0) equals the derivative of f at x_0 :

$$\begin{aligned} \frac{-3 - y_0}{1 - x_0} &= 2x_0 \\ -3 - y_0 &= (1 - x_0)2x_0 \\ -3 - x_0^2 &= 2x_0 - 2x_0^2 \\ x_0^2 - 2x_0 - 3 &= 0 \\ (x_0 - 3)(x_0 + 1) &= 0 \Rightarrow x_0 = 3, -1 \end{aligned}$$



Therefore, the points of tangency are $(3, 9)$ and $(-1, 1)$, and the corresponding slopes are 6 and -2 . The equations of the tangent lines are

$$\begin{aligned} y + 3 &= 6(x - 1) & y + 3 &= -2(x - 1) \\ y &= 6x - 9 & y &= -2x - 1 \end{aligned}$$

50. (a) $f(x) = x^2$

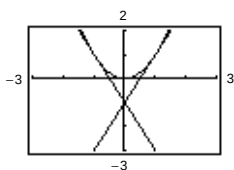
$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x \end{aligned}$$

At $x = -1$, $f'(-1) = -2$ and the tangent line is

$$y - 1 = -2(x + 1) \quad \text{or} \quad y = -2x - 1.$$

At $x = 0$, $f'(0) = 0$ and the tangent line is $y = 0$.

At $x = 1$, $f'(1) = 2$ and the tangent line is $y = 2x - 1$.



For this function, the slopes of the tangent lines are always distinct for different values of x .

(b)
$$\begin{aligned} g'(x) &= \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(3x^2 + 3x(\Delta x) + (\Delta x)^2)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x(\Delta x) + (\Delta x)^2) = 3x^2 \end{aligned}$$

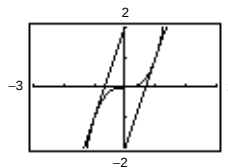
At $x = -1$, $g'(-1) = 3$ and the tangent line is

$$y + 1 = 3(x + 1) \quad \text{or} \quad y = 3x + 2.$$

At $x = 0$, $g'(0) = 0$ and the tangent line is $y = 0$.

At $x = 1$, $g'(1) = 3$ and the tangent line is

$$y - 1 = 3(x - 1) \quad \text{or} \quad y = 3x - 2.$$

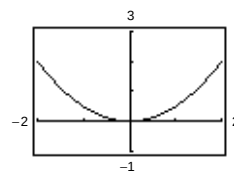


For this function, the slopes of the tangent lines are sometimes the same.

52. $f(x) = \frac{1}{2}x^2$

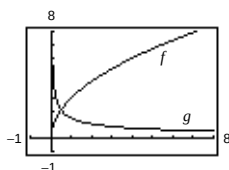
By the limit definition of the derivative we have $f'(x) = x$.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	2	1.125	0.5	0.125	0	0.125	0.5	1.125	2
$f'(x)$	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2



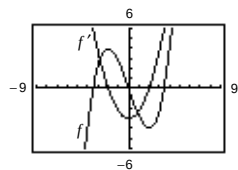
54.
$$g(x) = \frac{f(x + 0.01) - f(x)}{0.01}$$

$$= (3\sqrt{x + 0.01} - 3\sqrt{x})100$$



The graph of $g(x)$ is approximately the graph of $f'(x)$.

58. $f(x) = \frac{x^3}{4} - 3x$ and $f'(x) = \frac{3}{4}x^2 - 3$



60. $f(x) = x + \frac{1}{x}$

$$S_{\Delta x}(x) = \frac{f(2 + \Delta x) - f(2)}{\Delta x}(x - 2) + f(2) = \frac{(2 + \Delta x) + \frac{1}{2 + \Delta x} - \frac{5}{2}}{\Delta x}(x - 2) + \frac{5}{2}$$

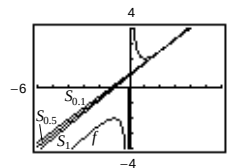
$$= \frac{2(2 + \Delta x)^2 + 2 - 5(2 + \Delta x)}{2(2 + \Delta x)\Delta x}(x - 2) + \frac{5}{2} = \frac{(2\Delta x + 3)}{2(2 + \Delta x)}(x - 2) + \frac{5}{2}$$

(a) $\Delta x = 1$: $S_{\Delta x} = \frac{5}{6}(x - 2) + \frac{5}{2} = \frac{5}{6}x + \frac{5}{6}$

$\Delta x = 0.5$: $S_{\Delta x} = \frac{4}{5}(x - 2) + \frac{5}{2} = \frac{4}{5}x + \frac{9}{10}$

$\Delta x = 0.1$: $S_{\Delta x} = \frac{16}{21}(x - 2) + \frac{5}{2} = \frac{16}{21}x + \frac{41}{42}$

(b) As $\Delta x \rightarrow 0$, the line approaches the tangent line to f at $(2, \frac{5}{2})$.



62. $g(x) = x(x - 1) = x^2 - x$, $c = 1$

$$g'(1) = \lim_{x \rightarrow 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x^2 - x - 0}{x - 1} = \lim_{x \rightarrow 1} \frac{x(x - 1)}{x - 1}$$

$$= \lim_{x \rightarrow 1} x = 1$$

64. $f(x) = x^3 + 2x, c = 1$

$$f'(1) = \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x^3 + 2x - 3}{x - 1} = \lim_{x \rightarrow 1} \frac{(x - 1)(x^2 + x + 3)}{x - 1} = \lim_{x \rightarrow 1} (x^2 + x + 3) = 5$$

66. $f(x) = \frac{1}{x}, c = 3$

$$f'(3) = \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \rightarrow 3} \frac{(1/x) - (1/3)}{x - 3} = \lim_{x \rightarrow 3} \frac{3 - x}{3x} \cdot \frac{1}{x - 3} = \lim_{x \rightarrow 3} \left(-\frac{1}{3x} \right) = -\frac{1}{9}$$

68. $g(x) = (x + 3)^{1/3}, c = -3$

$$g'(-3) = \lim_{x \rightarrow -3} \frac{g(x) - g(-3)}{x - (-3)} = \lim_{x \rightarrow -3} \frac{(x + 3)^{1/3} - 0}{x + 3} = \lim_{x \rightarrow -3} \frac{1}{(x + 3)^{2/3}}$$

Does not exist.

70. $f(x) = |x - 4|, c = 4$

$$f'(4) = \lim_{x \rightarrow 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \rightarrow 4} \frac{|x - 4| - 0}{x - 4} = \lim_{x \rightarrow 4} \frac{|x - 4|}{x - 4}$$

Does not exist.

72. $f(x)$ is differentiable everywhere except at $x = \pm 3$. (Sharp turns in the graph.)

74. $f(x)$ is differentiable everywhere except at $x = 1$. (Discontinuity)

76. $f(x)$ is differentiable everywhere except at $x = 0$. (Sharp turn in the graph)

78. $f(x)$ is differentiable everywhere except at $x = \pm 2$. (Discontinuities)

80. $f(x)$ is differentiable everywhere except at $x = 1$. (Discontinuity)

82. $f(x) = \sqrt{1 - x^2}$

The derivative from the left does not exist because

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{\sqrt{1 - x^2} - 0}{x - 1} = \lim_{x \rightarrow 1^-} \frac{\sqrt{1 - x^2}}{x - 1} \cdot \frac{\sqrt{1 - x^2}}{\sqrt{1 - x^2}} = \lim_{x \rightarrow 1^-} -\frac{1 + x}{\sqrt{1 - x^2}} = -\infty. \text{ (Vertical tangent)}$$

The limit from the right does not exist since f is undefined for $x > 1$. Therefore, f is not differentiable at $x = 1$.

84. $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

The derivative from the left is

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{x - 1}{x - 1} = \lim_{x \rightarrow 1^-} 1 = 1.$$

The derivative from the right is

$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1^+} (x + 1) = 2.$$

These one-sided limits are not equal. Therefore, f is not differentiable at $x = 1$.

86. Note that f is continuous at $x = 2$. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \geq 2 \end{cases}$

The derivative from the left is

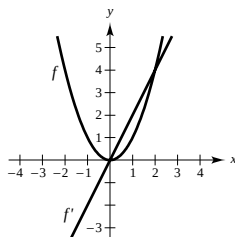
$$\lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{(\frac{1}{2}x + 1) - 2}{x - 2} = \lim_{x \rightarrow 2^-} \frac{\frac{1}{2}(x - 2)}{x - 2} = \frac{1}{2}.$$

The derivative from the right is

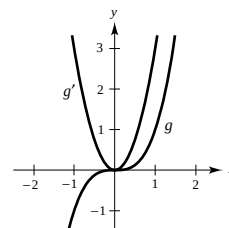
$$\begin{aligned} \lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} &= \lim_{x \rightarrow 2^+} \frac{\sqrt{2x} - 2}{x - 2} \cdot \frac{\sqrt{2x} + 2}{\sqrt{2x} + 2} \\ &= \lim_{x \rightarrow 2^+} \frac{2x - 4}{(x - 2)(\sqrt{2x} + 2)} = \lim_{x \rightarrow 2^+} \frac{2(x - 2)}{(x - 2)(\sqrt{2x} + 2)} = \lim_{x \rightarrow 2^+} \frac{2}{\sqrt{2x} + 2} = \frac{1}{2}. \end{aligned}$$

The one-sided limits are equal. Therefore, f is differentiable at $x = 2$. ($f'(2) = \frac{1}{2}$)

88. (a) $f(x) = x^2$ and $f'(x) = 2x$



- (b) $g(x) = x^3$ and $g'(x) = 3x^2$



(c) The derivative is a polynomial of degree 1 less than the original function. If $h(x) = x^n$, then $h'(x) = nx^{n-1}$.

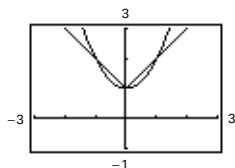
$$\begin{aligned} \text{(d) If } f(x) = x^4, \text{ then } f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^4 - x^4}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^4 + 4x^3(\Delta x) + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4 - x^4}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3) = 4x^3 \end{aligned}$$

Hence, if $f(x) = x^4$, then $f'(x) = 4x^3$ which is consistent with the conjecture. However, this is not a proof, since you must verify the conjecture for all integer values of n , $n \geq 2$.

90. False. $y = |x - 2|$ is continuous at $x = 2$, but is not differentiable at $x = 2$. (Sharp turn in the graph)

92. True—see Theorem 2.1

- 94.



As you zoom in, the graph of $y_1 = x^2 + 1$ appears to be locally the graph of a horizontal line, whereas the graph of $y_2 = |x| + 1$ always has a sharp corner at $(0, 1)$. y_2 is not differentiable at $(0, 1)$.

Section 2.2 Basic Differentiation Rules and Rates of Change

2. (a) $y = x^{-1/2}$
 $y' = -\frac{1}{2}x^{-3/2}$
 $y'(1) = -\frac{1}{2}$
- (b) $y = x^{-1}$
 $y' = -x^{-2}$
 $y'(1) = -1$
- (c) $y = x^{-3/2}$
 $y' = -\frac{3}{2}x^{-5/2}$
 $y'(1) = -\frac{3}{2}$
- (d) $y = x^{-2}$
 $y' = -2x^{-3}$
 $y'(1) = -2$
4. $f(x) = -2$
 $f'(x) = 0$
6. $y = x^8$
 $y' = 8x^7$
8. $y = \frac{1}{x^8} = x^{-8}$
 $y' = 8x^{-9} = \frac{-8}{x^9}$
10. $y = \sqrt[4]{x} = x^{1/4}$
 $y' = \frac{1}{4}x^{-3/4} = \frac{1}{4x^{3/4}}$
12. $g(x) = 3x - 1$
 $g'(x) = 3$
14. $y = t^2 + 2t - 3$
 $y' = 2t + 2$
16. $y = 8 - x^3$
 $y' = -3x^2$
18. $f(x) = 2x^3 - x^2 + 3x$
 $f'(x) = 6x^2 - 2x + 3$
20. $g(t) = \pi \cos t$
 $g'(t) = -\pi \sin t$
22. $y = 5 + \sin x$
 $y' = \cos x$
24. $y = \frac{5}{(2x)^3} + 2 \cos x = \frac{5}{8}x^{-3} + 2 \cos x$
 $y' = \frac{5}{8}(-3)x^{-4} - 2 \sin x = \frac{-15}{8x^4} - 2 \sin x$

<u>Function</u>	<u>Rewrite</u>	<u>Derivative</u>	<u>Simplify</u>
26. $y = \frac{2}{3x^2}$	$y = \frac{2}{3}x^{-2}$	$y' = -\frac{4}{3}x^{-3}$	$y' = -\frac{4}{3x^3}$
28. $y = \frac{\pi}{(3x)^2}$	$y = \frac{\pi}{9}x^{-2}$	$y' = -\frac{2\pi}{9}x^{-3}$	$y' = -\frac{2\pi}{9x^3}$
30. $y = \frac{4}{x^{-3}}$	$y = 4x^3$	$y' = 12x^2$	$y' = 12x^2$
32. $f(t) = 3 - \frac{3}{5t}, \left(\frac{3}{5}, 2\right)$ $f'(t) = \frac{3}{5t^2}$ $f'\left(\frac{3}{5}\right) = \frac{5}{3}$	34. $y = 3x^3 - 6, (2, 18)$ $y' = 9x^2$ $y'(2) = 36$	36. $f(x) = 3(5 - x)^2, (5, 0)$ $= 3x^2 - 30x + 75$ $f'(x) = 6x - 30$ $f'(5) = 0$	
38. $g(t) = 2 + 3 \cos t, (\pi, -1)$ $g'(t) = -3 \sin t$ $g'(\pi) = 0$	40. $f(x) = x^2 - 3x - 3x^{-2}$ $f'(x) = 2x - 3 + 6x^{-3}$ $= 2x - 3 + \frac{6}{x^3}$	42. $f(x) = x + x^{-2}$ $f'(x) = 1 - 2x^{-3}$ $= 1 - \frac{2}{x^3}$	
44. $h(x) = \frac{2x^2 - 3x + 1}{x} = 2x - 3 + x^{-1}$ $h'(x) = 2 - \frac{1}{x^2} = \frac{2x^2 - 1}{x^2}$	46. $y = 3x(6x - 5x^2) = 18x^2 - 15x^3$ $y' = 36x - 45x^2$		
48. $f(x) = \sqrt[3]{x} + \sqrt[5]{x} = x^{1/3} + x^{1/5}$ $f'(x) = \frac{1}{3}x^{-2/3} + \frac{1}{5}x^{-4/5} = \frac{1}{3x^{2/3}} + \frac{1}{5x^{4/5}}$	50. $f(t) = t^{2/3} - t^{1/3} + 4$ $f'(t) = \frac{2}{3}t^{-1/3} - \frac{1}{3}t^{-2/3} = \frac{2}{3t^{1/3}} - \frac{1}{3t^{2/3}}$		

$$52. f(x) = \frac{2}{\sqrt[3]{x}} + 3 \cos x = 2x^{-1/3} + 3 \cos x$$

$$f'(x) = \frac{-2}{3}x^{-4/3} - 3 \sin x = \frac{-2}{3x^{4/3}} - 3 \sin x$$

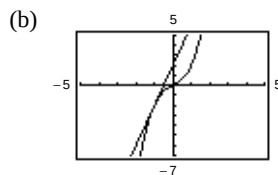
$$54. (a) y = x^3 + x$$

$$y' = 3x^2 + 1$$

$$\text{At } (-1, -2): y' = 3(-1)^2 + 1 = 4.$$

$$\text{Tangent line: } y + 2 = 4(x + 1)$$

$$4x - y + 2 = 0$$



$$56. (a) y = (x^2 + 2x)(x + 1)$$

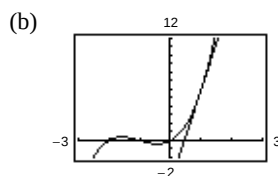
$$= x^3 + 3x^2 + 2x$$

$$y' = 3x^2 + 6x + 2$$

$$\text{At } (1, 6): y' = 3(1)^2 + 6(1) + 2 = 11.$$

$$\text{Tangent line: } y - 6 = 11(x - 1)$$

$$0 = 11x - y - 5$$



$$58. y = x^3 + x$$

$$y' = 3x^2 + 1 > 0 \text{ for all } x.$$

Therefore, there are no horizontal tangents.

$$60. y = x^2 + 1$$

$$y' = 2x = 0 \Rightarrow x = 0$$

$$\text{At } x = 0, y = 1.$$

Horizontal tangent: $(0, 1)$

$$62. y = \sqrt{3}x + 2 \cos x, 0 \leq x < 2\pi$$

$$y' = \sqrt{3} - 2 \sin x = 0$$

$$\sin x = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

$$\text{At } x = \frac{\pi}{3}, y = \frac{\sqrt{3}\pi + 3}{3}.$$

$$\text{At } x = \frac{2\pi}{3}, y = \frac{2\sqrt{3}\pi - 3}{3}.$$

$$\text{Horizontal tangents: } \left(\frac{\pi}{3}, \frac{\sqrt{3}\pi + 3}{3}\right), \left(\frac{2\pi}{3}, \frac{2\sqrt{3}\pi - 3}{3}\right)$$

$$64. k - x^2 = -4x + 7 \quad \text{Equate functions}$$

$$-2x = -4 \quad \text{Equate derivatives}$$

$$\text{Hence, } x = 2 \text{ and } k - 4 = -8 + 7 \Rightarrow k = 3$$

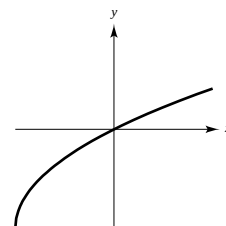
$$66. k\sqrt{x} = x + 4 \quad \text{Equate functions}$$

$$\frac{k}{2\sqrt{x}} = 1 \quad \text{Equate derivatives}$$

$$\text{Hence, } k = 2\sqrt{x} \text{ and}$$

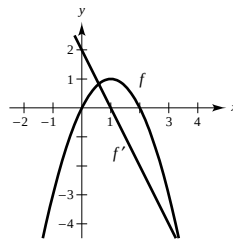
$$(2\sqrt{x})\sqrt{x} = x + 4 \Rightarrow 2x = x + 4 \Rightarrow x = 4 \Rightarrow k = 4$$

68. The graph of a function f such that $f' > 0$ for all x and the rate of change the function is decreasing (i.e. $f'' < 0$) would, in general, look like the graph at the right.



70. $g(x) = -5f(x) \Rightarrow g'(x) = -5f'(x)$

72.



If f is quadratic, then its derivative is a linear function.

$$f(x) = ax^2 + bx + c$$

$$f'(x) = 2ax + b$$

74. m_1 is the slope of the line tangent to $y = x$. m_2 is the slope of the line tangent to $y = 1/x$. Since

$$y = x \Rightarrow y' = 1 \Rightarrow m_1 = 1 \text{ and } y = \frac{1}{x} \Rightarrow y' = \frac{-1}{x^2} \Rightarrow m_2 = \frac{-1}{x^2}.$$

The points of intersection of $y = x$ and $y = 1/x$ are

$$x = \frac{1}{x} \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$$

At $x = \pm 1$, $m_2 = -1$. Since $m_2 = -1/m_1$, these tangent lines are perpendicular at the points intersection.

76. $f(x) = \frac{2}{x}, (5, 0)$

$$f'(x) = -\frac{2}{x^2}$$

$$-\frac{2}{x^2} = \frac{0 - y}{5 - x}$$

$$-10 + 2x = -x^2y$$

$$-10 + 2x = -x^2\left(\frac{2}{x}\right)$$

$$-10 + 2x = -2x$$

$$4x = 10$$

$$x = \frac{5}{2}, y = \frac{4}{5}$$

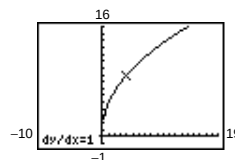
The point $\left(\frac{5}{2}, \frac{4}{5}\right)$ is on the graph of f . The slope of the tangent line is $f'\left(\frac{5}{2}\right) = -\frac{8}{25}$.

Tangent line: $y - \frac{4}{5} = -\frac{8}{25}\left(x - \frac{5}{2}\right)$

$$25y - 20 = -8x + 20$$

$$8x + 25y - 40 = 0$$

78. $f'(4) = 1$

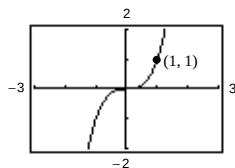


80. (a) Nearby point: (1.0073138, 1.0221024)

$$\text{Secant line: } y - 1 = \frac{1.0221024 - 1}{1.0073138 - 1}(x - 1)$$

$$y = 3.022(x - 1) + 1$$

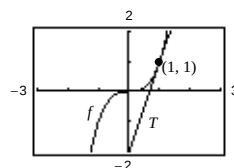
(Answers will vary.)



(b) $f'(x) = 3x^2$

$$T(x) = 3(x - 1) + 1 = 3x - 2$$

- (c) The accuracy worsens as you move away from (1, 1).



(d)

Δx	-3	-2	-1	-0.5	-0.1	0	0.1	0.5	1	2	3
$f(x)$	-8	-1	0	0.125	0.729	1	1.331	3.375	8	27	64
$T(x)$	-8	-5	-2	-0.5	0.7	1	1.3	2.5	4	7	10

The accuracy decreases more rapidly than in Exercise 59 because $y = x^3$ is less “linear” than $y = x^{3/2}$.

82. True. If $f(x) = g(x) + c$, then $f'(x) = g'(x) + 0 = g'(x)$.

84. True. If $y = x/\pi = (1/\pi) \cdot x$, then $dy/dx = (1/\pi)(1) = 1/\pi$.

86. False. If $f(x) = \frac{1}{x^n} = x^{-n}$, then $f'(x) = -nx^{-n-1} = \frac{-n}{x^{n+1}}$

88. $f(t) = t^2 - 3, [2, 2.1]$

$$f'(t) = 2t$$

Instantaneous rate of change:

$$(2, 1) \Rightarrow f'(2) = 2(2) = 4$$

$$(2.1, 1.41) \Rightarrow f'(2.1) = 4.2$$

Average rate of change:

$$\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{1.41 - 1}{0.1} = 4.1$$

90. $f(x) = \sin x, \left[0, \frac{\pi}{6}\right]$

$$f'(x) = \cos x$$

Instantaneous rate of change:

$$(0, 0) \Rightarrow f'(0) = 1$$

$$\left(\frac{\pi}{6}, \frac{1}{2}\right) \Rightarrow f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \approx 0.866$$

Average rate of change:

$$\frac{f(\pi/6) - f(0)}{(\pi/6) - 0} = \frac{(1/2) - 0}{(\pi/6) - 0} = \frac{3}{\pi} \approx 0.955$$

92. $s(t) = -16t^2 - 22t + 220$

$$v(t) = -32t - 22$$

$$v(3) = -118 \text{ ft/sec}$$

$$s(t) = -16t^2 - 22t + 220$$

$$= 112 \text{ (height after falling 108 ft)}$$

$$-16t^2 - 22t + 108 = 0$$

$$-2(t - 2)(8t + 27) = 0$$

$$t = 2$$

$$v(2) = -32(2) - 22$$

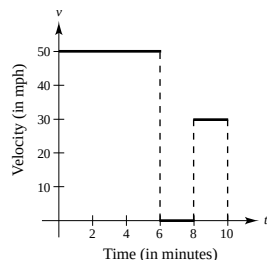
$$= -86 \text{ ft/sec}$$

94. $s(t) = -4.9t^2 + v_0t + s_0$

$$= -4.9t^2 + s_0 = 0 \text{ when } t = 6.8.$$

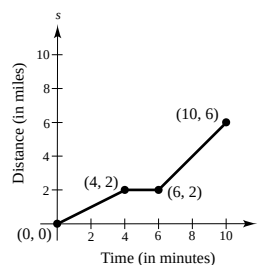
$$s_0 = 4.9t^2 = 4.9(6.8)^2 = 226.6 \text{ m}$$

96.



(The velocity has been converted to miles per hour)

98. This graph corresponds with Exercise 75.



100. $s(t) = -\frac{1}{2}at^2 + c$ and $s'(t) = -at$.

$$\begin{aligned} \text{Average velocity: } \frac{s(t_0 + \Delta t) - s(t_0 - \Delta t)}{(t_0 + \Delta t) - (t_0 - \Delta t)} &= \frac{[-(1/2)a(t_0 + \Delta t)^2 + c] - [-(1/2)a(t_0 - \Delta t)^2 + c]}{2\Delta t} \\ &= \frac{-(1/2)a(t_0^2 + 2t_0\Delta t + (\Delta t)^2) + (1/2)a(t_0^2 - 2t_0\Delta t + (\Delta t)^2)}{2\Delta t} \\ &= \frac{-2at_0\Delta t}{2\Delta t} \\ &= -at_0 \\ &= s'(t_0) \quad \text{Instantaneous velocity at } t = t_0 \end{aligned}$$

102. $V = s^3, \frac{dV}{ds} = 3s^2$

When $s = 4$ cm, $\frac{dV}{ds} = 48 \text{ cm}^2$.

104. $C = (\text{gallons of fuel used})(\text{cost per gallon})$

$$= \left(\frac{15,000}{x}\right)(1.25) = \frac{18,750}{x}$$

$$\frac{dC}{dx} = -\frac{18,750}{x^2}$$

x	10	15	20	25	30	35	40
C	1875	1250	537.5	750	625	535.71	468.75
$\frac{dC}{dx}$	-187.5	-83.333	-46.875	-30	-20.833	-15.306	-11.719

The driver who gets 15 miles per gallon would benefit more from a 1 mile per gallon increase in fuel efficiency. The rate of change is larger when $x = 15$.

106. $\frac{dT}{dt} = K(T - T_a)$

108. $y = \frac{1}{x}, x > 0$

$$y' = -\frac{1}{x^2}$$

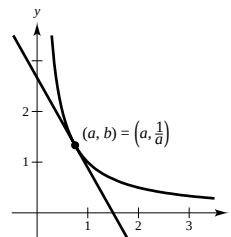
At (a, b) , the equation of the tangent line is

$$y - \frac{1}{a} = -\frac{1}{a^2}(x - a) \quad \text{or} \quad y = -\frac{x}{a^2} + \frac{2}{a}.$$

The x -intercept is $(2a, 0)$.

The y -intercept is $(0, \frac{2}{a})$.

The area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(2a)\left(\frac{2}{a}\right) = 2$.



110. $y = x^2$

$$y' = 2x$$

(a) Tangent lines through $(0, a)$:

$$y - a = 2x(x - 0)$$

$$x^2 - a = 2x^2$$

$$-a = x^2$$

$$\pm \sqrt{-a} = x$$

The points of tangency are $(\pm \sqrt{-a}, -a)$. At $(\sqrt{-a}, -a)$ the slope is $y'(\sqrt{-a}) = 2\sqrt{-a}$. At $(-\sqrt{-a}, -a)$ the slope is $y'(-\sqrt{-a}) = -2\sqrt{-a}$.

Tangent lines: $y + a = 2\sqrt{-a}(x - \sqrt{-a})$ and $y + a = -2\sqrt{-a}(x + \sqrt{-a})$

$$y = 2\sqrt{-a}x + a$$

$$y = -2\sqrt{-a}x + a$$

Restriction: a must be negative.

(b) Tangent lines through $(a, 0)$:

$$y - 0 = 2x(x - a)$$

$$x^2 = 2x^2 - 2ax$$

$$0 = x^2 - 2ax = x(x - 2a)$$

The points of tangency are $(0, 0)$ and $(2a, 4a^2)$. At $(0, 0)$ the slope is $y'(0) = 0$. At $(2a, 4a^2)$ the slope is $y'(2a) = 4a$.

Tangent lines: $y - 0 = 0(x - 0)$ and $y - 4a^2 = 4a(x - 2a)$

$$y = 0$$

$$y = 4ax - 4a^2$$

Restriction: None, a can be any real number.

112. $f_1(x) = |\sin x|$ is differentiable for all $x \neq n\pi$, n an integer.

$$f_2(x) = \sin|x| \text{ is differentiable for all } x \neq 0.$$

You can verify this by graphing f_1 and f_2 and observing the locations of the sharp turns.

Section 2.3 The Product and Quotient Rules and Higher-Order Derivatives

2. $f(x) = (6x + 5)(x^3 - 2)$

$$\begin{aligned} f'(x) &= (6x + 5)(3x^2) + (x^3 - 2)(6) \\ &= 18x^3 + 15x^2 + 6x^3 - 12 \\ &= 24x^3 + 15x^2 - 12 \end{aligned}$$

6. $g(x) = \sqrt{x} \sin x$

$$g'(x) = \sqrt{x} \cos x + \sin x \left(\frac{1}{2\sqrt{x}} \right) = \sqrt{x} \cos x + \frac{1}{2\sqrt{x}} \sin x$$

10. $h(s) = \frac{s}{\sqrt{s} - 1}$

$$\begin{aligned} h'(s) &= \frac{(\sqrt{s} - 1)(1) - s\left(\frac{1}{2}s^{-1/2}\right)}{(\sqrt{s} - 1)^2} \\ &= \frac{\sqrt{s} - 1 - \frac{1}{2}\sqrt{s}}{(\sqrt{s} - 1)^2} = \frac{\sqrt{s} - 2}{2(\sqrt{s} - 1)^2} \end{aligned}$$

14. $f(x) = (x^2 - 2x + 1)(x^3 - 1)$

$$\begin{aligned} f'(x) &= (x^2 - 2x + 1)(3x^2) + (x^3 - 1)(2x - 2) \\ &= 3x^2(x - 1)^2 + 2(x - 1)^2(x^2 + x + 1) \\ &= (x - 1)^2(5x^2 + 2x + 2) \\ f'(1) &= 0 \end{aligned}$$

18. $f(x) = \frac{\sin x}{x}$

$$\begin{aligned} f'(x) &= \frac{(x)(\cos x) - (\sin x)(1)}{x^2} \\ &= \frac{x \cos x - \sin x}{x^2} \\ f'\left(\frac{\pi}{6}\right) &= \frac{(\pi/6)(\sqrt{3}/2) - (1/2)}{\pi^2/36} \\ &= \frac{3\sqrt{3}\pi - 18}{\pi^2} \\ &= \frac{3(\sqrt{3}\pi - 6)}{\pi^2} \end{aligned}$$

4. $g(s) = \sqrt{s}(4 - s^2) = s^{1/2}(4 - s^2)$

$$\begin{aligned} g'(s) &= s^{1/2}(-2s) + (4 - s^2)\frac{1}{2}s^{-1/2} = -2s^{3/2} + \frac{4 - s^2}{2s^{1/2}} \\ &= \frac{4 - 5s^2}{2s^{1/2}} \end{aligned}$$

8. $g(t) = \frac{t^2 + 2}{2t - 7}$

$$g'(t) = \frac{(2t - 7)(2t) - (t^2 + 2)(2)}{(2t - 7)^2} = \frac{2t^2 - 14t - 4}{(2t - 7)^2}$$

12. $f(t) = \frac{\cos t}{t^3}$

$$f'(t) = \frac{t^3(-\sin t) - \cos t(3t^2)}{(t^3)^2} = -\frac{t \sin t + 3 \cos t}{t^4}$$

16. $f(x) = \frac{x + 1}{x - 1}$

$$\begin{aligned} f'(x) &= \frac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2} \\ &= \frac{x - 1 - x - 1}{(x - 1)^2} \\ &= -\frac{2}{(x - 1)^2} \\ f'(2) &= -\frac{2}{(2 - 1)^2} = -2 \end{aligned}$$

Function

Rewrite

Derivative

Simplify

20. $y = \frac{5x^2 - 3}{4}$

$y = \frac{5}{4}x^2 - \frac{3}{4}$

$y' = \frac{10}{4}x$

$y' = \frac{5x}{2}$

<u>Function</u>	<u>Rewrite</u>	<u>Derivative</u>	<u>Simplify</u>
22. $y = \frac{4}{5x^2}$	$y = \frac{4}{5}x^{-2}$	$y' = -\frac{8}{5}x^{-3}$	$y' = -\frac{8}{5x^3}$
24. $y = \frac{3x^2 - 5}{7}$	$y = \frac{3}{7}x^2 - \frac{5}{7}$	$y' = \frac{6x}{7}$	$y' = \frac{6}{7}x$
26. $f(x) = \frac{x^3 + 3x + 2}{x^2 - 1}$ $f'(x) = \frac{(x^2 - 1)(3x^2 + 3) - (x^3 + 3x + 2)(2x)}{(x^2 - 1)^2}$ $= \frac{x^4 - 6x^2 - 4x - 3}{(x^2 - 1)^2}$		28. $f(x) = x^4 \left[1 - \frac{2}{x+1} \right] = x^4 \left[\frac{x-1}{x+1} \right]$ $f'(x) = x^4 \left[\frac{(x+1) - (x-1)}{(x+1)^2} \right] + \left[\frac{x-1}{x+1} \right] (4x^3)$ $= 2x^3 \left[\frac{2x^2 + x - 2}{(x+1)^2} \right]$	
30. $f(x) = \sqrt[3]{x}(\sqrt{x} + 3) = x^{1/3}(x^{1/2} + 3)$ $f'(x) = x^{1/3} \left(\frac{1}{2}x^{-1/2} \right) + (x^{1/2} + 3) \left(\frac{1}{3}x^{-2/3} \right)$ $= \frac{5}{6}x^{-1/6} + x^{-2/3}$ $= \frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$		32. $h(x) = (x^2 - 1)^2 = x^4 - 2x^2 + 1$ $h'(x) = 4x^3 - 4x = 4x(x^2 - 1)$	
Alternate solution:			
$f(x) = \sqrt[3]{x}(\sqrt{x} + 3)$ $= x^{5/6} + 3x^{1/3}$ $f'(x) = \frac{5}{6}x^{-1/6} + x^{-2/3}$ $= \frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$			
34. $g(x) = x^2 \left(\frac{2}{x} - \frac{1}{x+1} \right) = 2x - \frac{x^2}{x+1}$ $g'(x) = 2 - \frac{(x+1)2x - x^2(1)}{(x+1)^2} = \frac{2(x^2 + 2x + 1) - x^2 - 2x}{(x+1)^2} = \frac{x^2 + 2x + 2}{(x+1)^2}$			
36. $f(x) = (x^2 - x)(x^2 + 1)(x^2 + x + 1)$ $f'(x) = (2x - 1)(x^2 + 1)(x^2 + x + 1) + (x^2 - x)(2x)(x^2 + x + 1) + (x^2 - x)(x^2 + 1)(2x + 1)$ $= (2x - 1)(x^4 + x^3 + 2x^2 + x + 1) + (x^2 - x)(2x^3 + 2x^2 + 2x) + (x^2 - x)(2x^3 + x^2 + 2x + 1)$ $= 2x^5 + x^4 + 3x^3 + x - 1 + 2x^5 - 2x^2 + 2x^5 - x^4 + x^3 - x^2 - x$ $= 6x^5 + 4x^3 - 3x^2 - 1$			
38. $f(x) = \frac{c^2 - x^2}{c^2 + x^2}$ $f'(x) = \frac{(c^2 + x^2)(-2x) - (c^2 - x^2)(2x)}{(c^2 + x^2)^2}$ $= \frac{-4xc^2}{(c^2 + x^2)^2}$		40. $f(\theta) = (\theta + 1) \cos \theta$ $f'(\theta) = (\theta + 1)(-\sin \theta) + (\cos \theta)(1)$ $= \cos \theta - (\theta + 1) \sin \theta$	

$$42. f(x) = \frac{\sin x}{x}$$

$$f'(x) = \frac{x \cos x - \sin x}{x^2}$$

$$46. h(s) = \frac{1}{s} - 10 \csc s$$

$$h'(s) = -\frac{1}{s^2} + 10 \csc s \cot s$$

$$50. y = x \sin x + \cos x$$

$$y' = x \cos x + \sin x - \sin x = x \cos x$$

$$54. h(\theta) = 5\theta \sec \theta + \theta \tan \theta$$

$$h'(\theta) = 5\theta \sec \theta \tan \theta + 5 \sec \theta + \theta \sec^2 \theta + \tan \theta$$

$$58. f(\theta) = \frac{\sin \theta}{1 - \cos \theta}$$

$$f'(\theta) = \frac{1}{\cos \theta - 1} = \frac{\cos \theta - 1}{(1 - \cos \theta)^2}$$

(form of answer may vary)

$$62. f(x) = \sin x(\sin x + \cos x)$$

$$\begin{aligned} f'(x) &= \sin x(\cos x - \sin x) + (\sin x + \cos x)\cos x \\ &= \sin x \cos x - \sin^2 x + \sin x \cos x + \cos^2 x \\ &= \sin 2x + \cos 2x \end{aligned}$$

$$f'\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{2} + \cos \frac{\pi}{2} = 1$$

$$64. (a) f(x) = (x-1)(x^2-2), (0, 2)$$

$$f'(x) = (x-1)(2x) + (x^2-2)(1) = 3x^2 - 2x - 2$$

$$f'(0) = -2 = \text{slope at } (0, 2).$$

$$\text{Tangent line: } y - 2 = -2x \Rightarrow y = -2x + 2$$

$$66. (a) f(x) = \frac{x-1}{x+1}, \left(2, \frac{1}{3}\right)$$

$$f'(x) = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$$f'(2) = \frac{2}{9} = \text{slope at } \left(2, \frac{1}{3}\right).$$

$$\text{Tangent line: } y - \frac{1}{3} = \frac{2}{9}(x-2) \Rightarrow y = \frac{2}{9}x - \frac{1}{9}$$

$$44. y = x + \cot x$$

$$y' = 1 - \csc^2 x = -\cot^2 x$$

$$48. y = \frac{\sec x}{x}$$

$$y' = \frac{x \sec x \tan x - \sec x}{x^2}$$

$$= \frac{\sec x(x \tan x - 1)}{x^2}$$

$$52. f(x) = \sin x \cos x$$

$$f'(x) = \sin x(-\sin x) + \cos x(\cos x)$$

$$= \cos 2x$$

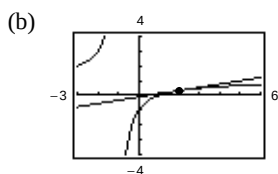
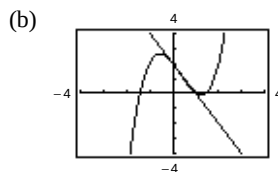
$$56. f(x) = \left(\frac{x^2 - x - 3}{x^2 + 1}\right)(x^2 + x + 1)$$

$$f'(x) = 2 \frac{x^5 + 2x^3 + 2x^2 - 2}{(x^2 + 1)^2} \quad (\text{form of answer may vary})$$

$$60. f(x) = \tan x \cot x = 1$$

$$f'(x) = 0$$

$$f'(1) = 0$$



68. (a) $f(x) = \sec x, \left(\frac{\pi}{3}, 2\right)$

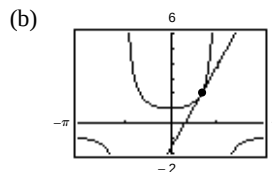
$$f'(x) = \sec x \tan x$$

$$f'\left(\frac{\pi}{3}\right) = 2\sqrt{3} = \text{slope at } \left(\frac{\pi}{3}, 2\right).$$

Tangent line:

$$y - 2 = 2\sqrt{3}\left(x - \frac{\pi}{3}\right)$$

$$6\sqrt{3}x - 3y + 6 - 2\sqrt{3}\pi = 0$$



70. $f(x) = \frac{x^2}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(2x) - (x^2)(2x)}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}$$

$$f'(x) = 0 \text{ when } x = 0.$$

Horizontal tangent is at $(0, 0)$.

72. $f'(x) = \frac{x(\cos x - 3) - (\sin x - 3x)(1)}{x^2} = \frac{x \cos x - \sin x}{x^2}$

$$g'(x) = \frac{x(\cos x + 2) - (\sin x + 2x)(1)}{x^2} = \frac{x \cos x - \sin x}{x^2}$$

$$g(x) = \frac{\sin x + 2x}{x} = \frac{\sin x - 3x + 5x}{x} = f(x) + 5$$

f and g differ by a constant.

74. $f(x) = \frac{\cos x}{x^n} = x^{-n} \cos x$

$$f'(x) = -x^{-n} \sin x - nx^{-n-1} \cos x$$

$$= -x^{-n-1}(x \sin x + n \cos x)$$

$$= -\frac{x \sin x + n \cos x}{x^{n+1}}$$

When $n = 1$: $f'(x) = -\frac{x \sin x + \cos x}{x^2}$.

When $n = 2$: $f'(x) = -\frac{x \sin x + 2 \cos x}{x^3}$.

When $n = 3$: $f'(x) = -\frac{x \sin x + 3 \cos x}{x^4}$.

When $n = 4$: $f'(x) = -\frac{x \sin x + 4 \cos x}{x^5}$.

For general n , $f'(x) = -\frac{x \sin x + n \cos x}{x^{n+1}}$.

76. $V = \pi r^2 h = \pi(t + 2)\left(\frac{1}{2}\sqrt{t}\right)$

$$= \frac{1}{2}(t^{3/2} + 2t^{1/2})\pi$$

$$V'(t) = \frac{1}{2}\left(\frac{3}{2}t^{1/2} + t^{-1/2}\right)\pi = \frac{3t + 2}{4t^{1/2}}\pi \text{ cubic inches/sec}$$

78. $P = \frac{k}{V}$

$$\frac{dP}{dV} = -\frac{k}{V^2}$$

80. $f(x) = \sec x$

$$g(x) = \csc x, [0, 2\pi)$$

$$f'(x) = g'(x)$$

$$\sec x \tan x = -\csc x \cot x \Rightarrow \frac{\sec x \tan x}{\csc x \cot x} = -1 \Rightarrow \frac{\frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}}{\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}} = -1 \Rightarrow$$

$$\frac{\sin^3 x}{\cos^3 x} = -1 \Rightarrow \tan^3 x = -1 \Rightarrow \tan x = -1$$

$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

82. (a) $n(t) = -9.6643t^2 + 90.7414t + 77.5029$

$$v(t) = -276.4643t^2 + 2987.6929t + 1809.9714$$

(b) $A = \frac{v(t)}{n(t)} \approx \frac{-276.46t^2 + 2987.69t + 1809.97}{-9.66t^2 + 90.74t + 77.50}$

A represents the average retail value (in millions of dollars) per 1000 motor homes.

(c) $A'(t) \approx \frac{40.46(x^2 - 2.09x + 17.83)}{(x^2 - 9.39x - 8.02)^2}$

86. $f(x) = \frac{x^2 + 2x - 1}{x} = x + 2 - \frac{1}{x}$

$$f'(x) = 1 + \frac{1}{x^2}$$

$$f''(x) = -\frac{2}{x^3}$$

90. $f''(x) = 2 - 2x^{-1}$

$$f'''(x) = 2x^{-2} = \frac{2}{x^2}$$

92. $f^{(4)}(x) = 2x + 1$

$$f^{(5)}(x) = 2$$

$$f^{(6)}(x) = 0$$

84. $f(x) = x + \frac{32}{x^2}$

$$f'(x) = 1 - \frac{64}{x^3}$$

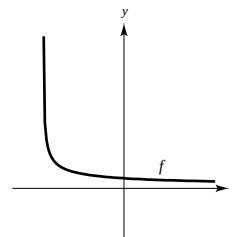
$$f''(x) = \frac{192}{x^4}$$

88. $f(x) = \sec x$

$$f'(x) = \sec x \tan x$$

$$\begin{aligned} f''(x) &= \sec x(\sec^2 x) + \tan x(\sec x \tan x) \\ &= \sec x(\sec^2 x + \tan^2 x) \end{aligned}$$

94. The graph of a differentiable function f such that $f' > 0$ and $f'' < 0$ for all real numbers x would in general look like the graph below.



96. $f(x) = 4 - h(x)$

$$f'(x) = -h'(x)$$

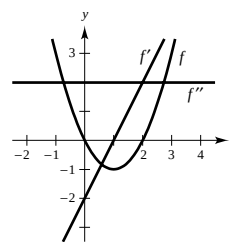
$$f'(2) = -h'(2) = -4$$

98. $f(x) = g(x)h(x)$

$$f'(x) = g(x)h'(x) + h(x)g'(x)$$

$$\begin{aligned} f'(2) &= g(2)h'(2) + h(2)g'(2) \\ &= (3)(4) + (-1)(-2) \\ &= 14 \end{aligned}$$

100.



It appears that f is quadratic; so f' would be linear and f'' would be constant.

102. $s(t) = -8.25t^2 + 66t$

$$v(t) = -16.50t + 66$$

$$a(t) = -16.50$$

$t(\text{sec})$	0	1	2	3	4
$s(t)$ (ft)	0	57.75	99	123.75	132
$v(t) = s'(t)$ (ft/sec)	66	49.5	33	16.5	0
$a(t) = v'(t)$ (ft/sec ²)	-16.5	-16.5	-16.5	-16.5	-16.5

Average velocity on:

$$[0, 1] \text{ is } \frac{57.75 - 0}{1 - 0} = 57.75.$$

$$[1, 2] \text{ is } \frac{99 - 57.75}{2 - 1} = 41.25.$$

$$[2, 3] \text{ is } \frac{123.75 - 99}{3 - 2} = 24.75.$$

$$[3, 4] \text{ is } \frac{132 - 123.75}{4 - 3} = 8.25.$$

104. (a) $f(x) = x^n$

$$f^n(x) = n(n-1)(n-2) \cdots (2)(1) = n!$$

(b) $f(x) = \frac{1}{x}$

$$f^{(n)}(x) = \frac{(-1)^n(n)(n-1)(n-2) \cdots (2)(1)}{x^{n+1}}$$

$$= \frac{(-1)^n n!}{x^{n+1}}$$

Note: $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ (read “ n factorial.”)

106. $[xf(x)]' = xf'(x) + f(x)$

$$[xf(x)]'' = xf''(x) + f'(x) + f'(x) = xf''(x) + 2f'(x)$$

$$[xf(x)]''' = xf'''(x) + f''(x) + 2f''(x) = xf'''(x) + 3f''(x)$$

$$\text{In general, } [xf(x)]^{(n)} = xf^{(n)}(x) + nf^{(n-1)}(x).$$

108. $f(x) = \sin x \quad f\left(\frac{\pi}{2}\right) = 1$

$$f'(x) = \cos x \quad f'\left(\frac{\pi}{2}\right) = 0$$

$$f''(x) = -\sin x \quad f''\left(\frac{\pi}{2}\right) = -1$$

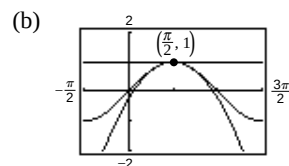
(a) $P_1(x) = f'(a)(x-a) + f(a) = 0\left(x - \frac{\pi}{2}\right) + 1 = 1$

$$P_2(x) = \frac{1}{2}f''(a)(x-a)^2 + f'(a)(x-a) + f(a) = \frac{1}{2}(-1)\left(x - \frac{\pi}{2}\right)^2 + 1$$

$$= 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2$$

(c) P_2 is a better approximation than P_1 .

(d) The accuracy worsens as you move farther away from $x = a = \frac{\pi}{2}$.



110. True. y is a fourth-degree polynomial.

$$\frac{d^n y}{dx^n} = 0 \text{ when } n > 4.$$

112. True

114. True. If $v(t) = c$ then

$$a(t) = v'(t) = 0.$$

116. (a) $(fg' - f'g)' = fg'' + f'g' - f'g' - f''g$

$$= fg'' - f''g \quad \text{True}$$

(b) $(fg)'' = (fg' + f'g)'$

$$= fg'' + f'g' + f'g' + f''g$$

$$= fg'' + 2f'g' + f''g$$

$$\neq fg'' + f''g \quad \text{False}$$

Section 2.4 The Chain Rule

$$y = f(g(x))$$

$$u = g(x)$$

$$y = f(u)$$

$$2. y = \frac{1}{\sqrt{x+1}}$$

$$u = x + 1$$

$$y = u^{-1/2}$$

$$4. y = 3 \tan(\pi x^2)$$

$$u = \pi x^2$$

$$y = 3 \tan u$$

$$6. y = \cos \frac{3x}{2}$$

$$u = \frac{3x}{2}$$

$$y = \cos u$$

$$8. y = (2x^3 + 1)^2$$

$$y' = 2(2x^3 + 1)(6x^2) = 12x^2(2x^3 + 1)$$

$$10. y = 3(4 - x^2)^5$$

$$y' = 15(4 - x^2)(-2x) = -30x(4 - x^2)$$

$$12. f(t) = (9t + 2)^{2/3}$$

$$f'(t) = \frac{2}{3}(9t + 2)^{-1/3}(9) = \frac{6}{\sqrt[3]{9t + 2}}$$

$$14. g(x) = \sqrt{5 - 3x} = (5 - 3x)^{1/2}$$

$$g'(x) = \frac{1}{2}(5 - 3x)^{-1/2}(-3) = \frac{-3}{2\sqrt{5 - 3x}}$$

$$16. g(x) = \sqrt{x^2 - 2x + 1} = \sqrt{(x - 1)^2} = |x - 1|$$

$$g'(x) = \begin{cases} 1, & x > 1 \\ -1, & x < 1 \end{cases}$$

$$18. f(x) = -3(2 - 9x)^{1/4}$$

$$f'(x) = -\frac{3}{4}(2 - 9x)^{-3/4}(-9) = \frac{27}{4(2 - 9x)^{3/4}}$$

$$20. s(t) = (t^2 + 3t - 1)^{-1}$$

$$\begin{aligned} s'(t) &= -1(t^2 + 3t - 1)^{-2}(2t + 3) \\ &= \frac{-(2t + 3)}{(t^2 + 3t - 1)^2} \end{aligned}$$

$$22. y = -5(t + 3)^{-3}$$

$$y' = 15(t + 3)^{-4} = \frac{15}{(t + 3)^4}$$

$$24. g(t) = (t^2 - 2)^{-1/2}$$

$$g'(t) = -\frac{1}{2}(t^2 - 2)^{-3/2}(2t) = -\frac{t}{(t^2 - 2)^{3/2}}$$

$$26. f(x) = x(3x - 9)^3$$

$$\begin{aligned} f'(x) &= x[3(3x - 9)^2(3)] + (3x - 9)^3(1) \\ &= (3x - 9)^2[9x + 3x - 9] \\ &= 27(x - 3)^2(4x - 3) \end{aligned}$$

$$28. y = \frac{1}{2}x^2\sqrt{16 - x^2}$$

$$\begin{aligned} y' &= \frac{1}{2}x^2\left(\frac{1}{2}(16 - x^2)^{-1/2}(-2x)\right) + x(16 - x^2)^{1/2} \\ &= \frac{-x^3}{2\sqrt{16 - x^2}} + x\sqrt{16 - x^2} \\ &= \frac{-x(3x^2 - 32)}{2\sqrt{16 - x^2}} \end{aligned}$$

$$30. y = \frac{x}{\sqrt{x^4 + 4}}$$

$$\begin{aligned} y' &= \frac{(x^4 + 4)^{1/2}(1) - x\frac{1}{2}(x^4 + 4)^{-1/2}(4x^3)}{x^4 + 4} \\ &= \frac{x^4 + 4 - 2x^4}{(x^4 + 4)^{3/2}} = \frac{4 - x^4}{(x^4 + 4)^{3/2}} \end{aligned}$$

$$32. h(t) = \left(\frac{t^2}{t^3 + 2}\right)^2$$

$$\begin{aligned} h'(t) &= 2\left(\frac{t^2}{t^3 + 2}\right)\left(\frac{(t^3 + 2)(2t) - t^2(3t^2)}{(t^3 + 2)^2}\right) \\ &= \frac{2t^2(4t - t^4)}{(t^3 + 2)^3} = \frac{2t^3(4 - t^3)}{(t^3 + 2)^3} \end{aligned}$$

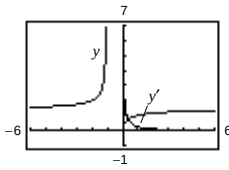
$$34. g(x) = \left(\frac{3x^2 - 2}{2x + 3} \right)^3$$

$$\begin{aligned} g'(x) &= 3 \left(\frac{3x^2 - 2}{2x + 3} \right)^2 \left(\frac{(2x + 3)(6x) - (3x^2 - 2)(2)}{(2x + 3)^2} \right) \\ &= \frac{3(3x^2 - 2)^2(6x^2 + 18x + 4)}{(2x + 3)^4} = \frac{6(3x^2 - 2)^2(3x^2 + 9x + 2)}{(2x + 3)^4} \end{aligned}$$

$$36. y = \sqrt{\frac{2x}{x+1}}$$

$$y' = \frac{1}{\sqrt{2x}(x+1)^{3/2}}$$

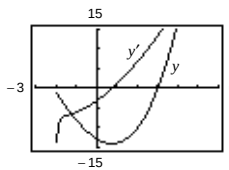
y' has no zeros.



$$40. y = (t^2 - 9)\sqrt{t+2}$$

$$y' = \frac{5t^2 + 8t - 9}{2\sqrt{t+2}}$$

The zero of y' corresponds to the point on the graph of y where the tangent line is horizontal.



$$44. y = x^2 \tan \frac{1}{x}$$

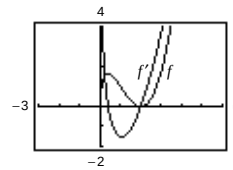
$$\frac{dy}{dx} = 2x \tan \frac{1}{x} - \sec^2 \frac{1}{x}$$

The zeros of y' correspond to the points on the graph of y where the tangent lines are horizontal.

$$38. f(x) = \sqrt{x}(2-x)^2$$

$$f'(x) = \frac{(x-2)(5x-2)}{2\sqrt{x}}$$

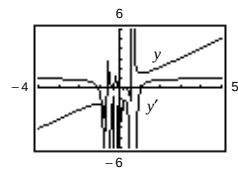
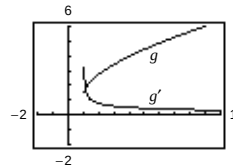
The zeros of f' correspond to the points on the graph of f where the tangent lines are horizontal.



$$42. g(x) = \sqrt{x-1} + \sqrt{x+1}$$

$$g'(x) = \frac{1}{2\sqrt{x-1}} + \frac{1}{2\sqrt{x+1}}$$

g' has no zeros.



$$46. (a) y = \sin 3x$$

$$y' = 3 \cos 3x$$

$$y'(0) = 3$$

3 cycles in $[0, 2\pi]$

$$(b) y = \sin\left(\frac{x}{2}\right)$$

$$y' = \left(\frac{1}{2}\right) \cos\left(\frac{x}{2}\right)$$

$$y'(0) = \frac{1}{2}$$

Half cycle in $[0, 2\pi]$

The slope of $\sin ax$ at the origin is a .

48. $y = \sin \pi x$

$$\frac{dy}{dx} = \pi \cos \pi x$$

52. $y = \cos(1 - 2x)^2 = \cos((1 - 2x)^2)$

$$y' = -\sin(1 - 2x)^2(2(1 - 2x)(-2)) = 4(1 - 2x) \sin(1 - 2x)^2$$

54. $g(\theta) = \sec\left(\frac{1}{2}\theta\right) \tan\left(\frac{1}{2}\theta\right)$

$$\begin{aligned} g'(\theta) &= \sec\left(\frac{1}{2}\theta\right) \sec^2\left(\frac{1}{2}\theta\right) \frac{1}{2} + \tan\left(\frac{1}{2}\theta\right) \sec\left(\frac{1}{2}\theta\right) \tan\left(\frac{1}{2}\theta\right) \frac{1}{2} \\ &= \frac{1}{2} \sec\left(\frac{1}{2}\theta\right) \left[\sec^2\left(\frac{1}{2}\theta\right) + \tan^2\left(\frac{1}{2}\theta\right) \right] \end{aligned}$$

56. $g(v) = \frac{\cos v}{\csc v} = \cos v \cdot \sin v$

$$g'(v) = \cos v(\cos v) + \sin v(-\sin v) = \cos^2 v - \sin^2 v = \cos 2v$$

58. $y = 2 \tan^3 x$

$$y' = 6 \tan^2 x \cdot \sec^2 x$$

50. $h(x) = \sec(x^2)$

$$h'(x) = 2x \sec(x^2) \tan(x^2)$$

60. $g(t) = 5 \cos^2 \pi t = 5(\cos \pi t)^2$

$$\begin{aligned} g'(t) &= 10 \cos \pi t (-\sin \pi t)(\pi) \\ &= -10\pi(\sin \pi t)(\cos \pi t) = -5\pi \sin 2\pi t \end{aligned}$$

62. $h(t) = 2 \cot^2(\pi t + 2)$

$$\begin{aligned} h'(t) &= 4 \cot(\pi t + 2)(-\csc^2(\pi t + 2)(\pi)) \\ &= -4\pi \cot(\pi t + 2) \csc^2(\pi t + 2) \end{aligned}$$

64. $y = 3x - 5 \cos(\pi x)^2$

$$= 3x - 5 \cos(\pi^2 x^2)$$

$$\begin{aligned} \frac{dy}{dx} &= 3 + 5 \sin(\pi^2 x^2)(2\pi^2 x) \\ &= 3 + 10\pi^2 x \sin(\pi x)^2 \end{aligned}$$

66. $y = \sin x^{1/3} + (\sin x)^{1/3}$

$$\begin{aligned} y' &= \cos x^{1/3} \left(\frac{1}{3} x^{-2/3} \right) + \frac{1}{3} (\sin x)^{-2/3} \cos x \\ &= \frac{1}{3} \left[\frac{\cos x^{1/3}}{x^{2/3}} + \frac{\cos x}{(\sin x)^{2/3}} \right] \end{aligned}$$

68. $y = (3x^3 + 4x)^{1/5}, (2, 2)$

$$\begin{aligned} y' &= \frac{1}{5} (3x^3 + 4x)^{-4/5} (9x^2 + 4) \\ &= \frac{9x^2 + 4}{5(3x^3 + 4x)^{4/5}} \end{aligned}$$

$$y'(2) = \frac{1}{2}$$

70. $f(x) = \frac{1}{(x^2 - 3x)^2} = (x^2 - 3x)^{-2}, \left(4, \frac{1}{16}\right)$

$$f'(x) = -2(x^2 - 3x)^{-3}(2x - 3) = \frac{-2(2x - 3)}{(x^2 - 3x)^3}$$

$$f'(4) = -\frac{5}{32}$$

72. $f(x) = \frac{x+1}{2x-3}, (2, 3)$

$$f'(x) = \frac{(2x-3)(1) - (x+1)(2)}{(2x-3)^2} = \frac{-5}{(2x-3)^2}$$

$$f'(2) = -5$$

74. $y = \frac{1}{x} + \sqrt{\cos x}, \left(\frac{\pi}{2}, \frac{2}{\pi}\right)$

$$y' = -\frac{1}{x^2} - \frac{\sin x}{2\sqrt{\cos x}}$$

$$y'(\pi/2) \text{ is undefined.}$$

76. (a) $f(x) = \frac{1}{3}x\sqrt{x^2 + 5}$, $(2, 2)$

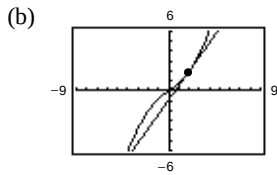
$$f'(x) = \frac{1}{3}x \left[\frac{1}{2}(x^2 + 5)^{-1/2}(2x) \right] + \frac{1}{3}(x^2 + 5)^{1/2}$$

$$= \frac{x^2}{3\sqrt{x^2 + 5}} + \frac{1}{3}\sqrt{x^2 + 5}$$

$$f'(2) = \frac{4}{3(3)} + \frac{1}{3}(3) = \frac{13}{9}$$

Tangent line:

$$y - 2 = \frac{13}{9}(x - 2) \Rightarrow 13x - 9y - 8 = 0$$



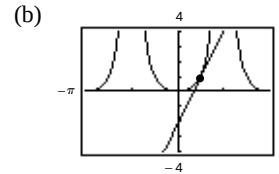
78. (a) $f(x) = \tan^2 x$, $\left(\frac{\pi}{4}, 1\right)$

$$f'(x) = 2 \tan x \sec^2 x$$

$$f'\left(\frac{\pi}{4}\right) = 2(1)(2) = 4$$

Tangent line:

$$y - 1 = 4\left(x - \frac{\pi}{4}\right) \Rightarrow 4x - y + (1 - \pi) = 0$$



80. $f(x) = (x - 2)^{-1}$

$$f'(x) = -(x - 2)^{-2} = \frac{-1}{(x - 2)^2}$$

$$f''(x) = 2(x - 2)^{-3} = \frac{2}{(x - 2)^3}$$

82. $f(x) = \sec^2 \pi x$

$$f'(x) = 2 \sec \pi x (\pi \sec \pi x \tan \pi x)$$

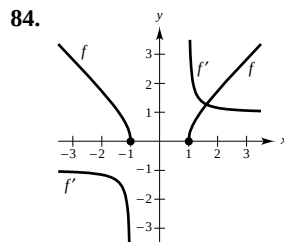
$$= 2\pi \sec^2 \pi x \tan \pi x$$

$$f''(x) = 2\pi \sec^2 \pi x (\sec^2 \pi x)(\pi) + 2\pi \tan \pi x (2\pi \sec^2 \pi x \tan \pi x)$$

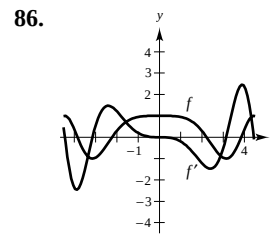
$$= 2\pi^2 \sec^4 \pi x + 4\pi^2 \sec^2 \pi x \tan^2 \pi x$$

$$= 2\pi^2 \sec^2 \pi x (\sec^2 \pi x + 2 \tan^2 \pi x)$$

$$= 2\pi^2 \sec^2 \pi x (3 \sec^2 \pi x - 2)$$



f is decreasing on $(-\infty, -1)$ so f' must be negative there. f is increasing on $(1, \infty)$ so f' must be positive there.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

88. $g(x) = f(x^2)$

$$g'(x) = f'(x^2)(2x) \Rightarrow g'(x) = 2x f'(x^2)$$

90. (a) $g(x) = \sin^2 x + \cos^2 x = 1 \Rightarrow g'(x) = 0$

$$g'(x) = 2 \sin x \cos x + 2 \cos x(-\sin x) = 0$$

(b) $\tan^2 x + 1 = \sec^2 x$

$$g(x) + 1 = f(x)$$

Taking derivatives of both sides,

$$g'(x) = f'(x).$$

Equivalently, $f'(x) = 2 \sec x \cdot \sec x \cdot \tan x$ and

$g'(x) = 2 \tan x \cdot \sec^2 x$, which are the same.

94. $y = A \cos \omega t$

(a) Amplitude: $A = \frac{3.5}{2} = 1.75$

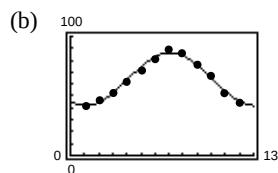
$$y = 1.75 \cos \omega t$$

Period: $10 \Rightarrow \omega = \frac{2\pi}{10} = \frac{\pi}{5}$

$$y = 1.75 \cos \frac{\pi t}{5}$$

96. (a) Using a graphing utility, or by trial and error, you obtain a model of the form

$$T(t) = 64.18 - 22.15 \sin\left(\frac{\pi t}{6} + 1\right)$$



92. $y = \frac{1}{3} \cos 12t - \frac{1}{4} \sin 12t$

$$v = y' = \frac{1}{3}[-12 \sin 12t] - \frac{1}{4}[12 \cos 12t]$$

$$= -4 \sin 12t - 3 \cos 12t$$

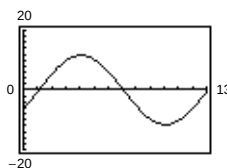
When $t = \pi/8$, $y = 0.25$ feet and $v = 4$ feet per second.

(b) $v = y' = 1.75 \left[-\frac{\pi}{5} \sin \frac{\pi t}{5} \right]$

$$= -0.35 \pi \sin \frac{\pi t}{5}$$

(c) $T'(t) = -22.15 \cos\left(\frac{\pi t}{6} + 1\right) \left(\frac{\pi}{6}\right)$

$$= -11.60 \cos\left(\frac{\pi t}{6} + 1\right)$$



(d) The temperature changes most rapidly when $t \approx 4.1$ (April) and $t \approx 10.1$ (October). The temperature changes most slowly ($T'(t) = 0$) when $t \approx 1.1$ (January) and $t \approx 7.1$ (July).

98. (a) $g(x) = f(x) - 2 \Rightarrow g'(x) = f'(x)$

(b) $h(x) = 2f(x) \Rightarrow h'(x) = 2f'(x)$

(c) $r(x) = f(-3x) \Rightarrow r'(x) = f'(-3x)(-3) = -3f'(-3x)$

Hence, you need to know $f'(-3x)$.

$$r'(0) = -3f'(0) = (-3)\left(-\frac{1}{3}\right) = 1$$

$$r'(-1) = -3f'(3) = (-3)(-4) = 12$$

(d) $s(x) = f(x + 2) \Rightarrow s'(x) = f'(x + 2)$

Hence, you need to know $f'(x + 2)$.

$$s'(-2) = f'(0) = -\frac{1}{3}, \text{ etc.}$$

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$h'(x)$	8	$\frac{4}{3}$	$-\frac{2}{3}$	-2	-4	-8
$r'(x)$		12	1			
$s'(x)$	$-\frac{1}{3}$	-1	-2	-4		

100. $f(x + p) = f(x)$ for all x .

(a) Yes, $f'(x + p) = f'(x)$, which shows that f' is periodic as well.

(b) Yes, let $g(x) = f(2x)$, so $g'(x) = 2f'(2x)$. Since f' is periodic, so is g' .

102. If $f(-x) = -f(x)$, then

$$\frac{d}{dx}[f(-x)] = \frac{d}{dx}[-f(x)]$$

$$f'(-x)(-1) = -f'(x)$$

$$f'(-x) = f'(x).$$

Thus, $f'(x)$ is even.

104. $|u| = \sqrt{u^2}$

$$\frac{d}{dx}[|u|] = \frac{d}{dx}[\sqrt{u^2}] = \frac{1}{2}(u^2)^{-1/2}(2uu') = \frac{uu'}{\sqrt{u^2}} = u' \frac{u}{|u|}, u \neq 0$$

106. $f(x) = |x^2 - 4|$

$$f'(x) = 2x \left(\frac{x^2 - 4}{|x^2 - 4|} \right), x \neq \pm 2$$

110. (a) $f(x) = \sec(2x)$

$$f'(x) = 2(\sec 2x)(\tan 2x)$$

$$\begin{aligned} f''(x) &= 2[2(\sec 2x)(\tan 2x)] \tan 2x + 2(\sec 2x)(\sec^2 2x)(2) \\ &= 4[(\sec 2x)(\tan^2 2x) + \sec^3 2x] \end{aligned}$$

$$f\left(\frac{\pi}{6}\right) = \sec\left(\frac{\pi}{3}\right) = 2$$

$$f'\left(\frac{\pi}{6}\right) = 2 \sec\left(\frac{\pi}{3}\right) \tan\left(\frac{\pi}{3}\right) = 4\sqrt{3}$$

$$f''\left(\frac{\pi}{6}\right) = 4[2(3) + 2^3] = 56$$

$$P_1(x) = 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

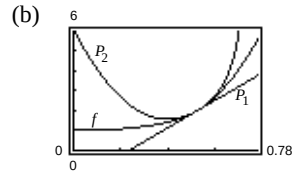
$$P_2(x) = \frac{1}{2}(56)\left(x - \frac{\pi}{6}\right)^2 + 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

$$= 28\left(x - \frac{\pi}{6}\right)^2 + 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

(c) P_2 is a better approximation than P_1 .

108. $f(x) = |\sin x|$

$$f'(x) = \cos x \left(\frac{\sin x}{|\sin x|} \right), x \neq k\pi$$



(d) The accuracy worsens as you move away from $x = \pi/6$.

112. False. If $f(x) = \sin^2 2x$, then $f'(x) = 2(\sin 2x)(2 \cos 2x)$.

114. False. First apply the Product Rule.

Section 2.5 Implicit Differentiations

2. $x^2 - y^2 = 16$

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$

6. $x^2y + y^2x = -2$

$$x^2y' + 2xy + y^2 + 2yxy' = 0$$

$$(x^2 + 2xy)y' = -(y^2 + 2xy)$$

$$y' = \frac{-y(y + 2x)}{x(x + 2y)}$$

10. $2 \sin x \cos y = 1$

$$2[\sin x(-\sin y)y' + \cos y(\cos x)] = 0$$

$$y' = \frac{\cos x \cos y}{\sin x \sin y}$$

$$= \cot x \cot y$$

14. $\cot y = x - y$

$$(-\csc^2 y)y' = 1 - y'$$

$$y' = \frac{1}{1 - \csc^2 y}$$

$$= \frac{1}{-\cot^2 y} = -\tan^2 y$$

18. (a) $(x^2 - 4x + 4) + (y^2 + 6y + 9) = -9 + 4 + 9$

$$(x - 2)^2 + (y + 3)^2 = 4 \text{ (Circle)}$$

$$(y + 3)^2 = 4 - (x - 2)^2$$

$$y = -3 \pm \sqrt{4 - (x - 2)^2}$$

(c) Explicitly:

$$\frac{dy}{dx} = \pm \frac{1}{2} [4 - (x - 2)^2]^{-1/2} (-2)(x - 2)$$

$$= \frac{\mp(x - 2)}{(\sqrt{4 - (x - 2)^2})^2}$$

$$= \frac{-(x - 2)}{\pm \sqrt{4 - (x - 2)^2}}$$

$$= \frac{-(x - 2)}{-3 \pm \sqrt{4 - (x - 2)^2} + 3}$$

$$= \frac{-(x - 2)}{y + 3}$$

4. $x^3 + y^3 = 8$

$$3x^2 + 3y^2y' = 0$$

$$y' = -\frac{x^2}{y^2}$$

8. $(xy)^{1/2} - x + 2y = 0$

$$\frac{1}{2}(xy)^{-1/2}(xy' + y) - 1 + 2y' = 0$$

$$\frac{x}{2\sqrt{xy}}y' + \frac{y}{2\sqrt{xy}} - 1 + 2y' = 0$$

$$xy' + y - 2\sqrt{xy} + 4\sqrt{xy}y' = 0$$

$$y' = \frac{2\sqrt{xy} - y}{4\sqrt{xy} + x}$$

12. $(\sin \pi x + \cos \pi y)^2 = 2$

$$2(\sin \pi x + \cos \pi y)[\pi \cos \pi x - \pi(\sin \pi y)y'] = 0$$

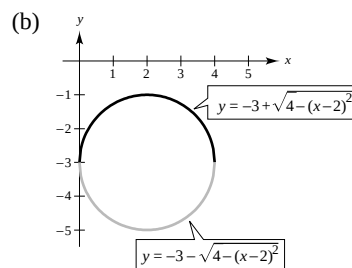
$$\pi \cos \pi x - \pi(\sin \pi y)y' = 0$$

$$y' = \frac{\cos \pi x}{\sin \pi y}$$

16. $x = \sec \frac{1}{y}$

$$1 = -\frac{y'}{y^2} \sec \frac{1}{y} \tan \frac{1}{y}$$

$$y' = \frac{-y^2}{\sec(1/y) \tan(1/y)} = -y^2 \cos\left(\frac{1}{y}\right) \cot\left(\frac{1}{y}\right)$$



(d) Implicitly:

$$2x + 2yy' - 4 + 6y' = 0$$

$$(2y + 6)y' = -2(x - 2)$$

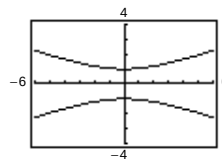
$$y' = \frac{-(x - 2)}{y + 3}$$

20. (a) $9y^2 = x^2 + 9$

$$y^2 = \frac{x^2}{9} + 1 = \frac{x^2 + 9}{9}$$

$$y = \frac{\pm\sqrt{x^2 + 9}}{3}$$

(b)



(c) Explicitly: $\frac{dy}{dx} = \frac{\pm\frac{1}{2}(x^2 + 9)^{-1/2}(2x)}{3} = \frac{\pm x}{3\sqrt{x^2 + 9}} = \frac{\pm x}{3(\pm 3y)} = \frac{x}{9y}$

(d) Implicitly: $9y^2 - x^2 = 9$

$$18yy' - 2x = 0$$

$$18yy' = 2x$$

$$y' = \frac{2x}{18y} = \frac{x}{9y}$$

22. $x^2 - y^3 = 0$

$$2x - 3y^2y' = 0$$

$$y' = \frac{2x}{3y^2}$$

At (1, 1): $y' = \frac{2}{3}$.

24.

$$(x + y)^3 = x^3 + y^3$$

$$x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3$$

$$3x^2y + 3xy^2 = 0$$

$$x^2y + xy^2 = 0$$

$$x^2y' + 2xy + 2xyy' + y^2 = 0$$

$$(x^2 + 2xy)y' = -(y^2 + 2xy)$$

$$y' = -\frac{y(y + 2x)}{x(x + 2y)}$$

At (-1, 1): $y' = -1$.

26. $x^3 + y^3 = 4xy + 1$

$$3x^2 + 3y^2y' = 4xy' + 4y$$

$$(3y^2 - 4x)y' = 4y - 3x^2$$

$$y' = \frac{4y - 3x^2}{(3y^2 - 4x)}$$

At (2, 1), $y' = \frac{4 - 12}{3 - 8} = \frac{8}{5}$

28.

$$x \cos y = 1$$

$$x[-y' \sin y] + \cos y = 0$$

$$y' = \frac{\cos y}{x \sin y}$$

$$= \frac{1}{x} \cot y = \frac{\cot y}{x}$$

At $\left(2, \frac{\pi}{3}\right)$: $y' = \frac{1}{2\sqrt{3}}$.

30. $(4 - x)y^2 = x^3$

$$(4 - x)(2yy') + y^2(-1) = 3x^2$$

$$y' = \frac{3x^2 + y^2}{2y(4 - x)}$$

At (2, 2): $y' = 2$.

32.

$$x^3 + y^3 - 6xy = 0$$

$$3x^2 + 3y^2y' - 6xy' - 6y = 0$$

$$y'(3y^2 - 6x) = 6y - 3x^2$$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$$

At $\left(\frac{4}{3}, \frac{8}{3}\right)$: $y' = \frac{(16/3) - (16/9)}{(64/9) - (8/3)} = \frac{32}{40} = \frac{4}{5}$.

34. $\cos y = x$

$$-\sin y \cdot y' = 1$$

$$y' = \frac{-1}{\sin y}, 0 < y < \pi$$

$$\sin^2 y + \cos^2 y = 1$$

$$\sin^2 y = 1 - \cos^2 y$$

$$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$$

$$y' = \frac{-1}{\sqrt{1 - x^2}}, -1 < x < 1$$

36. $x^2 y^2 - 2x = 3$

$$2x^2 y y' + 2xy^2 - 2 = 0$$

$$x^2 y y' + xy^2 - 1 = 0$$

$$y' = \frac{1 - xy^2}{x^2 y}$$

$$2xyy' + x^2(y')^2 + x^2yy'' + 2xyy' + y^2 = 0$$

$$4xyy' + x^2(y')^2 + x^2yy'' + y^2 = 0$$

$$\frac{4 - 4xy^2}{x} + \frac{(1 - xy^2)^2}{x^2 y^2} + x^2 yy'' + y^2 = 0$$

$$4xy^2 - 4x^2 y^4 + 1 - 2xy^2 + x^2 y^4 + x^4 y^3 y'' + x^2 y^4 = 0$$

$$x^4 y^3 y'' = 2x^2 y^4 - 2xy^2 - 1$$

$$y'' = \frac{2x^2 y^4 - 2xy^2 - 1}{x^4 y^3}$$

38. $1 - xy = x - y$

$$y - xy = x - 1$$

$$y = \frac{x - 1}{1 - x} = -1$$

$$y' = 0$$

$$y'' = 0$$

40. $y^2 = 4x$

$$2yy' = 4$$

$$y' = \frac{2}{y}$$

$$y'' = -2y^{-2} y' = \left[\frac{-2}{y^2} \right] \cdot \frac{2}{y} = \frac{-4}{y^3}$$

42. $y^2 = \frac{x - 1}{x^2 + 1}$

$$2yy' = \frac{(x^2 + 1)(1) - (x - 1)(2x)}{(x^2 + 1)^2}$$

$$= \frac{x^2 + 1 - 2x^2 + 2x}{(x^2 + 1)^2}$$

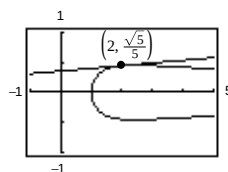
$$y' = \frac{1 + 2x - x^2}{2y(x^2 + 1)^2}$$

At $\left(2, \frac{\sqrt{5}}{5}\right)$: $y' = \frac{1 + 4 - 4}{\left[(2\sqrt{5})/5\right](4 + 1)^2} = \frac{1}{10\sqrt{5}}$.

Tangent line: $y - \frac{\sqrt{5}}{5} = \frac{1}{10\sqrt{5}}(x - 2)$

$$10\sqrt{5}y - 10 = x - 2$$

$$x - 10\sqrt{5}y + 8 = 0$$



44. $x^2 + y^2 = 9$

$$y' = \frac{-x}{y}$$

At $(0, 3)$:

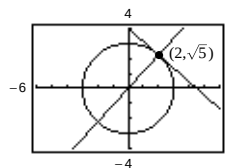
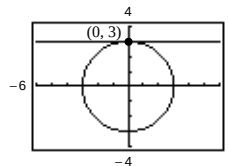
Tangent line: $y = 3$

Normal line: $x = 0$.

At $(2, \sqrt{5})$:

$$\text{Tangent line: } y - \sqrt{5} = \frac{-2}{\sqrt{5}}(x - 2) \Rightarrow 2x + \sqrt{5}y - 9 = 0$$

$$\text{Normal line: } y - \sqrt{5} = \frac{\sqrt{5}}{2}(x - 2) \Rightarrow \sqrt{5}x - 2y = 0.$$



46. $y^2 = 4x$

$$2yy' = 4$$

$$y' = \frac{2}{y} = 1 \text{ at } (1, 2)$$

Equation of normal at $(1, 2)$ is $y - 2 = -1(x - 1)$, $y = 3 - x$. The centers of the circles must be on the normal and at a distance of 4 units from $(1, 2)$. Therefore,

$$(x - 1)^2 + [(3 - x) - 2]^2 = 16$$

$$2(x - 1)^2 = 16$$

$$x = 1 \pm 2\sqrt{2}.$$

Centers of the circles: $(1 + 2\sqrt{2}, 2 - 2\sqrt{2})$ and $(1 - 2\sqrt{2}, 2 + 2\sqrt{2})$

$$\text{Equations: } (x - 1 - 2\sqrt{2})^2 + (y - 2 + 2\sqrt{2})^2 = 16$$

$$(x - 1 + 2\sqrt{2})^2 + (y - 2 - 2\sqrt{2})^2 = 16$$

48. $4x^2 + y^2 - 8x + 4y + 4 = 0$

$$8x + 2yy' - 8 + 4y' = 0$$

$$y' = \frac{8 - 8x}{2y + 4} = \frac{4 - 4x}{y + 2}$$

Horizontal tangents occur when $x = 1$:

$$4(1)^2 + y^2 - 8(1) + 4y + 4 = 0$$

$$y^2 + 4y = y(y + 4) = 0 \Rightarrow y = 0, -4$$

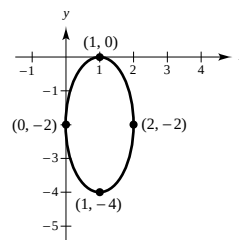
Horizontal tangents: $(1, 0), (1, -4)$.

Vertical tangents occur when $y = -2$:

$$4x^2 + (-2)^2 - 8x + 4(-2) + 4 = 0$$

$$4x^2 - 8x = 4x(x - 2) = 0 \Rightarrow x = 0, 2$$

Vertical tangents: $(0, -2), (2, -2)$.



50. Find the points of intersection by letting $y^2 = x^3$ in the equation $2x^2 + 3y^2 = 5$.

$$2x^2 + 3x^3 = 5 \quad \text{and} \quad 3x^3 + 2x^2 - 5 = 0$$

Intersect when $x = 1$.

Points of intersection: $(1, \pm 1)$

$$\begin{array}{ll} y^2 = x^3: & 2x^2 + 3y^2 = 5: \\ 2yy' = 3x^2 & 4x + 6yy' = 0 \\ y' = \frac{3x^2}{2y} & y' = -\frac{2x}{3y} \end{array}$$

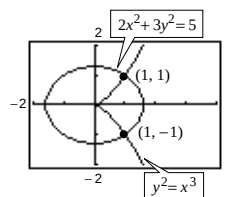
At $(1, 1)$, the slopes are:

$$y' = \frac{3}{2} \qquad y' = -\frac{2}{3}$$

At $(1, -1)$, the slopes are:

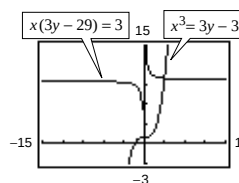
$$y' = -\frac{3}{2} \qquad y' = \frac{2}{3}$$

Tangents are perpendicular.



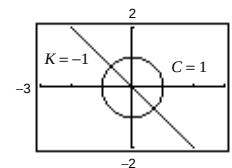
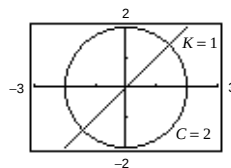
52. Rewriting each equation and differentiating,

$$\begin{array}{ll} x^3 = 3(y - 1) & x(3y - 29) = 3 \\ y = \frac{x^3}{3} + 1 & y = \frac{1}{3}\left(\frac{3}{x} + 29\right) \\ y' = x^2 & y' = -\frac{1}{x^2} \end{array}$$



For each value of x , the derivatives are negative reciprocals of each other. Thus, the tangent lines are orthogonal at both points of intersection.

54. $x^2 + y^2 = C^2 \quad y = Kx$
 $2x + 2yy' = 0 \quad y' = K$
 $y' = -\frac{x}{y}$



At the point of intersection (x, y) the product of the slopes is $(-x/y)(K) = (-x/Kx)(K) = -1$. The curves are orthogonal.

56. $x^2 - 3xy^2 + y^3 = 10$

(a) $2x - 3y^2 - 6xyy' + 3y^2y' = 0$

$$(-6xy + 3y^2)y' = 3y^2 - 2x$$

$$y' = \frac{3y^2 - 2x}{3y^2 - 6xy}$$

(b) $2x \frac{dx}{dt} - 3y^2 \frac{dx}{dt} - 6xy \frac{dy}{dt} + 3y^2 \frac{dy}{dt} = 0$

$$(2x - 3y^2) \frac{dx}{dt} = (6xy - 3y^2) \frac{dy}{dt}$$

58. (a) $4 \sin x \cos y = 1$

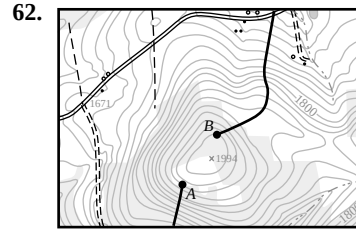
$$4 \sin x(-\sin y)y' + 4 \cos x \cos y = 0$$

$$y' = \frac{\cos x \cos y}{\sin x \sin y}$$

(b) $4 \sin x(-\sin y) \frac{dy}{dt} + 4 \cos x \frac{dx}{dt} \cos y = 0$

$$\cos x \cos y \frac{dx}{dt} = \sin x \sin y \frac{dy}{dt}$$

60. Given an implicit equation, first differentiate both sides with respect to x . Collect all terms involving y' on the left, and all other terms to the right. Factor out y' on the left side. Finally, divide both sides by the left-hand factor that does not contain y' .



Use starting point B.

64. $\sqrt{x} + \sqrt{y} = \sqrt{c}$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Tangent line at (x_0, y_0) :

$$y - y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$$

$$x\text{-intercept: } (x_0 + \sqrt{x_0}\sqrt{y_0}, 0)$$

$$y\text{-intercept: } (0, y_0 + \sqrt{x_0}\sqrt{y_0})$$

Sum of intercepts:

$$(x_0 + \sqrt{x_0}\sqrt{y_0}) + (y_0 + \sqrt{x_0}\sqrt{y_0}) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = (\sqrt{x_0} + \sqrt{y_0})^2 = (\sqrt{c})^2 = c.$$

Section 2.6 Related Rates

2. $y = 2(x^2 - 3x)$

$$\frac{dy}{dt} = (4x - 6) \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{1}{4x - 6} \frac{dy}{dt}$$

(a) When $x = 3$ and $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = [4(3) - 6](2) = 12$

(b) When $x = 1$ and $\frac{dy}{dt} = 5$, $\frac{dx}{dt} = \frac{1}{4(1) - 6}(5) = -\frac{5}{2}$

4. $x^2 + y^2 = 25$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \left(-\frac{x}{y}\right) \frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(-\frac{y}{x}\right) \frac{dy}{dt}$$

(a) When $x = 3$, $y = 4$, and $dx/dt = 8$,

$$\frac{dy}{dt} = -\frac{3}{4}(8) = -6$$

(b) When $x = 4$, $y = 3$, and $dy/dt = -2$,

$$\frac{dx}{dt} = -\frac{3}{4}(-2) = \frac{3}{2}.$$

$$6. \quad y = \frac{1}{1+x^2}$$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \left[\frac{-2x}{(1+x^2)^2} \right] \frac{dx}{dt}$$

(a) When $x = -2$,

$$\frac{dy}{dt} = \frac{-2(-2)(2)}{25} = \frac{8}{25} \text{ cm/sec.}$$

(b) When $x = 0$,

$$\frac{dy}{dt} = 0 \text{ cm/sec.}$$

(c) When $x = 2$,

$$\frac{dy}{dt} = \frac{-2(2)(2)}{25} = \frac{-8}{25} \text{ cm/sec.}$$

$$10. \quad (a) \quad \frac{dx}{dt} \text{ negative} \Rightarrow \frac{dy}{dt} \text{ negative}$$

$$(b) \quad \frac{dy}{dt} \text{ positive} \Rightarrow \frac{dx}{dt} \text{ positive}$$

$$14. \quad D = \sqrt{x^2 + y^2} = \sqrt{x^2 + \sin^2 x}$$

$$\frac{dx}{dt} = 2$$

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + \sin^2 x)^{-1/2}(2x + 2 \sin x \cos x) \frac{dx}{dt} = \frac{x + \sin x \cos x}{\sqrt{x^2 + \sin^2 x}} \frac{dx}{dt} = \frac{2 + 2 \sin x \cos x}{\sqrt{x^2 + \sin^2 x}}$$

$$16. \quad A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

If dr/dt is constant, dA/dt is not constant.

$$\frac{dA}{dt} \text{ depends on } r \text{ and } \frac{dr}{dt}.$$

$$20. \quad V = x^3$$

$$\frac{dx}{dt} = 3$$

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

(a) When $x = 1$,

$$\frac{dV}{dt} = 3(1)^2(3) = 9 \text{ cm}^3/\text{sec.}$$

$$8. \quad y = \sin x$$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \cos x \frac{dx}{dt}$$

(a) When $x = \pi/6$,

$$\frac{dy}{dt} = \left(\cos \frac{\pi}{6} \right) (2) = \sqrt{3} \text{ cm/sec.}$$

(b) When $x = \pi/4$,

$$\frac{dy}{dt} = \left(\cos \frac{\pi}{4} \right) (2) = \sqrt{2} \text{ cm/sec.}$$

(c) When $x = \pi/3$,

$$\frac{dy}{dt} = \left(\cos \frac{\pi}{3} \right) (2) = 1 \text{ cm/sec.}$$

12. Answers will vary. See page 145.

$$18. \quad V = \frac{4}{3}\pi r^3$$

$$\frac{dr}{dt} = 2$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$(a) \quad \text{When } r = 6, \frac{dV}{dt} = 4\pi(6)^2(2) = 288\pi \text{ in}^3/\text{min.}$$

$$\text{When } r = 24, \frac{dV}{dt} = 4\pi(24)^2(2) = 4608\pi \text{ in}^3/\text{min.}$$

(b) If dr/dt is constant, dV/dt is proportional to r^2 .

(b) When $x = 10$,

$$\frac{dV}{dt} = 3(10)^2(3) = 900 \text{ cm}^3/\text{sec.}$$

$$22. V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(3r) = \pi r^3$$

$$\frac{dr}{dt} = 2$$

$$\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}$$

(a) When $r = 6$,

$$\frac{dV}{dt} = 3\pi(6)^2(2) = 216\pi \text{ in}^3/\text{min}.$$

(b) When $r = 24$,

$$\frac{dV}{dt} = 3\pi(24)^2(2) = 3456\pi \text{ in}^3/\text{min}.$$

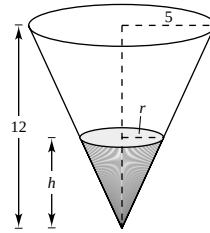
$$24. V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{25}{144} h^3 = \frac{25\pi}{3(144)} h^3$$

(By similar triangles, $\frac{r}{5} = \frac{h}{12} \Rightarrow r = \frac{5}{12}h$.)

$$\frac{dV}{dt} = 10$$

$$\frac{dV}{dt} = \frac{25\pi}{144} h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \left(\frac{144}{25\pi h^2} \right) \frac{dV}{dt}$$

$$\text{When } h = 8, \frac{dh}{dt} = \frac{144}{25\pi(64)}(10) = \frac{9}{10\pi} \text{ ft/min}.$$

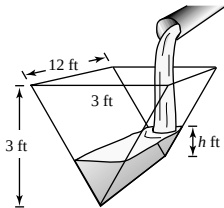


$$26. V = \frac{1}{2}bh(12) = 6bh = 6h^2 \text{ (since } b = h\text{)}$$

$$(a) \frac{dV}{dt} = 12h \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{12h} \frac{dV}{dt}$$

$$\text{When } h = 1 \text{ and } \frac{dV}{dt} = 2, \frac{dh}{dt} = \frac{1}{12(1)}(2) = \frac{1}{6} \text{ ft/min}$$

$$(b) \text{ If } \frac{dh}{dt} = \frac{3}{8} \text{ and } h = 2, \text{ then } \frac{dV}{dt} = 12(2)\left(\frac{3}{8}\right) = 9 \text{ ft}^3/\text{min}.$$



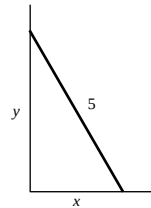
$$28. x^2 + y^2 = 25$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dx}{dt} = -\frac{y}{x} \cdot \frac{dy}{dt} = -\frac{0.15y}{x} \text{ since } \frac{dy}{dt} = 0.15.$$

When $x = 2.5$,

$$y = \sqrt{18.75}, \frac{dx}{dt} = -\frac{\sqrt{18.75}}{2.5} 0.15 \approx -0.26 \text{ m/sec}$$



30. Let L be the length of the rope.

$$(a) L^2 = 144 + x^2$$

$$2L \frac{dL}{dt} = 2x \frac{dx}{dt}$$

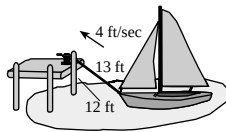
$$\frac{dx}{dt} = \frac{L}{x} \cdot \frac{dL}{dt} = -\frac{4L}{x} \text{ since } \frac{dL}{dt} = -4 \text{ ft/sec}.$$

When $L = 13$,

$$x = \sqrt{L^2 - 144} = \sqrt{169 - 144} = 5$$

$$\frac{dx}{dt} = -\frac{4(13)}{5} = -\frac{52}{5} = -10.4 \text{ ft/sec}.$$

Speed of the boat increases as it approaches the dock.



$$(b) \text{ If } \frac{dx}{dt} = -4, \text{ and } L = 13,$$

$$\frac{dL}{dt} = \frac{x}{L} \frac{dx}{dt}$$

$$= \frac{5}{13}(-4)$$

$$= -\frac{20}{13} \text{ ft/sec}$$

As $L \rightarrow 0$, $\frac{dL}{dt}$ increases.

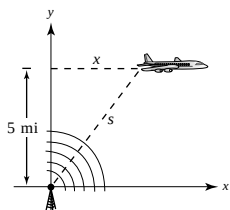
32. $x^2 + y^2 = s^2$

$$2x \frac{dx}{dt} + 0 = 2s \frac{ds}{dt} \quad \left(\text{since } \frac{dy}{dt} = 0 \right)$$

$$\frac{dx}{dt} = \frac{s}{x} \frac{ds}{dt}$$

When $s = 10$, $x = \sqrt{100 - 25} = \sqrt{75} = 5\sqrt{3}$

$$\frac{dx}{dt} = \frac{10}{5\sqrt{3}}(-240) = \frac{-480}{\sqrt{3}} = -160\sqrt{3} \approx -277.13 \text{ mph.}$$



36. (a) $\frac{20}{6} = \frac{y}{y-x}$

$$20y - 20x = 6y$$

$$14y = 20x$$

$$y = \frac{10}{7}x$$

$$\frac{dx}{dt} = -5$$

$$\frac{dy}{dt} = \frac{10}{7} \frac{dx}{dt} = \frac{10}{7}(-5) = \frac{-50}{7} \text{ ft/sec}$$

(b) $\frac{d(y-x)}{dt} = \frac{dy}{dt} - \frac{dx}{dt} = \frac{-50}{7} - (-5) = \frac{-50}{7} + \frac{35}{7} = \frac{-15}{7} \text{ ft/sec}$

38. $x(t) = \frac{3}{5} \sin \pi t$, $x^2 + y^2 = 1$

(a) Period: $\frac{2\pi}{\pi} = 2$ seconds

(b) When $x = \frac{3}{5}$, $y = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5} \text{ m.}$

Lowest point: $\left(0, \frac{4}{5}\right)$

34. $s^2 = 90^2 + x^2$

$$x = 60$$

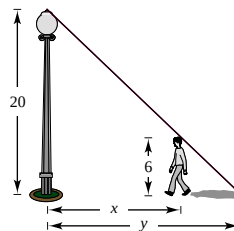
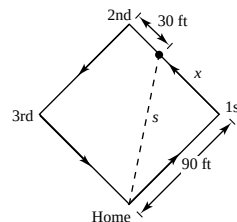
$$\frac{dx}{dt} = 28$$

$$\frac{ds}{dt} = \frac{x}{s} \cdot \frac{dx}{dt}$$

When $x = 60$,

$$s = \sqrt{90^2 + 60^2} = 30\sqrt{13}$$

$$\frac{ds}{dt} = \frac{60}{30\sqrt{13}}(28) = \frac{56}{\sqrt{13}} \approx 15.53 \text{ ft/sec.}$$



(c) When $x = \frac{3}{10}$, $y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$ and

$$\frac{3}{10} = \frac{3}{5} \sin \pi t \Rightarrow \sin \pi t = \frac{1}{2} \Rightarrow t = \frac{1}{6}$$

$$\frac{dx}{dt} = \frac{3}{5} \pi \cos \pi t$$

$$x^2 + y^2 = 1$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt}$$

Thus, $\frac{dy}{dt} = \frac{-3/10}{\sqrt{15}/4} \cdot \frac{3}{5} \pi \cos\left(\frac{\pi}{6}\right)$

$$= \frac{-9\pi}{25\sqrt{5}} = \frac{-9\sqrt{5}\pi}{125}$$

Speed = $\left| \frac{-9\sqrt{5}\pi}{125} \right| \approx 0.5058 \text{ m/sec}$

$$40. \quad \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$\frac{dR_1}{dt} = 1$$

$$\frac{dR_2}{dt} = 1.5$$

$$\frac{1}{R^2} \cdot \frac{dR}{dt} = \frac{1}{R_1^2} \cdot \frac{dR_1}{dt} + \frac{1}{R_2^2} \cdot \frac{dR_2}{dt}$$

When $R_1 = 50$ and $R_2 = 75$,

$$R = 30$$

$$\frac{dR}{dt} = (30)^2 \left[\frac{1}{(50)^2} (1) + \frac{1}{(75)^2} (1.5) \right]$$

$$= 0.6 \text{ ohms/sec.}$$

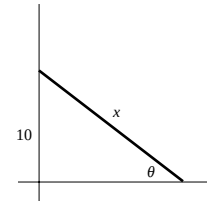
$$44. \quad \sin \theta = \frac{10}{x}$$

$$\frac{dx}{dt} = (-1) \text{ ft/sec}$$

$$\cos \theta \left(\frac{d\theta}{dt} \right) = \frac{-10}{x^2} \cdot \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{-10}{x^2} \frac{dx}{dt} (\sec \theta)$$

$$= \frac{-10}{25^2} (-1) \frac{25}{\sqrt{25^2 - 10^2}} = \frac{10}{25} \frac{1}{5\sqrt{21}} = \frac{2}{25\sqrt{21}} = \frac{2\sqrt{21}}{525} \approx 0.017 \text{ rad/sec}$$



$$46. \quad \tan \theta = \frac{x}{50}$$

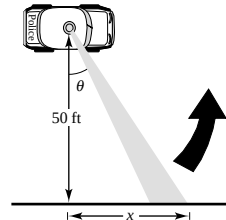
$$\frac{d\theta}{dt} = 30(2\pi) = 60\pi \text{ rad/min} = \pi \text{ rad/sec}$$

$$\sec^2 \theta \left(\frac{d\theta}{dt} \right) = \frac{1}{50} \left(\frac{dx}{dt} \right)$$

$$\frac{dx}{dt} = 50 \sec^2 \theta \left(\frac{d\theta}{dt} \right)$$

(a) When $\theta = 30^\circ$, $\frac{dx}{dt} = \frac{200\pi}{3} \text{ ft/sec.}$

(c) When $\theta = 70^\circ$, $\frac{dx}{dt} \approx 427.43\pi \text{ ft/sec.}$

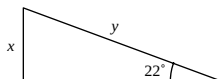


(b) When $\theta = 60^\circ$, $\frac{dx}{dt} = 200\pi \text{ ft/sec.}$

$$48. \quad \sin 22^\circ = \frac{x}{y}$$

$$0 = -\frac{x}{y^2} \cdot \frac{dy}{dt} + \frac{1}{y} \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x}{y} \cdot \frac{dy}{dt} = (\sin 22^\circ)(240) \approx 89.9056 \text{ mi/hr}$$



$$42. \quad rg \tan \theta = v^2$$

$$32r \tan \theta = v^2, r \text{ is a constant.}$$

$$32r \sec^2 \theta \frac{d\theta}{dt} = 2v \frac{dv}{dt}$$

$$\frac{dv}{dt} = \frac{16r}{v} \sec^2 \theta \frac{d\theta}{dt}$$

$$\text{Likewise, } \frac{d\theta}{dt} = \frac{v}{16r} \cos^2 \theta \frac{dv}{dt}.$$

50. (a) $dy/dt = 3(dx/dt)$ means that y changes three times as fast as x changes.

(b) y changes slowly when $x \approx 0$ or $x \approx L$. y changes more rapidly when x is near the middle of the interval.

52. $L^2 = 144 + x^2$; acceleration of the boat $= \frac{d^2x}{dt^2}$.

First derivative: $2L \frac{dL}{dt} = 2x \frac{dx}{dt}$

$$L \frac{dL}{dt} = x \frac{dx}{dt}$$

Second derivative: $L \frac{d^2L}{dt^2} + \frac{dL}{dt} \cdot \frac{dL}{dt} = x \frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{dx}{dt}$

$$\frac{d^2x}{dt^2} = \left(\frac{1}{x}\right) \left[L \frac{d^2L}{dt^2} + \left(\frac{dL}{dt}\right)^2 - \left(\frac{dx}{dt}\right)^2 \right]$$

When $L = 13$, $x = 5$, $\frac{dx}{dt} = -10.4$, and $\frac{dL}{dt} = -4$ (see Exercise 30). Since $\frac{dL}{dt}$ is constant, $\frac{d^2L}{dt^2} = 0$.

$$\begin{aligned} \frac{d^2x}{dt^2} &= \frac{1}{5} [13(0) + (-4)^2 - (-10.4)^2] \\ &= \frac{1}{5} [16 - 108.16] = \frac{1}{5} [-92.16] = -18.432 \text{ ft/sec}^2 \end{aligned}$$

54. $y(t) = -4.9t^2 + 20$

$$\frac{dy}{dt} = -9.8t$$

$$y(1) = -4.9 + 20 = 15.1$$

$$y'(1) = -9.8$$

By similar triangles, $\frac{20}{x} = \frac{y}{x - 12}$

$$20x - 240 = xy.$$

When $y = 15.1$, $20x - 240 = x(15.1)$

$$(20 - 15.1)x = 240$$

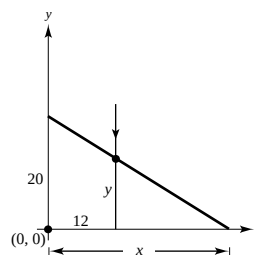
$$x = \frac{240}{4.9}.$$

$$20x - 240 = xy$$

$$20 \frac{dx}{dt} = x \frac{dy}{dt} + y \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x}{20 - y} \frac{dy}{dt}$$

At $t = 1$, $\frac{dx}{dt} = \frac{240/4.9}{20 - 15.1} (-9.8) \approx -97.96 \text{ m/sec}.$



51. $x^2 + y^2 = 25$; acceleration of the top of the ladder $= \frac{d^2y}{dt^2}$

First derivative: $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$

$$x \frac{dx}{dt} + y \frac{dy}{dt} = 0$$

Second derivative: $x \frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{dx}{dt} + y \frac{d^2y}{dt^2} + \frac{dy}{dt} \cdot \frac{dy}{dt} = 0$

$$\frac{d^2y}{dt^2} = \left(\frac{1}{y}\right) \left[-x \frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 \right]$$

When $x = 7$, $y = 24$, $\frac{dy}{dt} = -\frac{7}{12}$, and $\frac{dx}{dt} = 2$ (see Exercise 27). Since $\frac{dx}{dt}$ is constant, $\frac{d^2x}{dt^2} = 0$.

$$\frac{d^2y}{dt^2} = \frac{1}{24} \left[-7(0) - (2)^2 - \left(-\frac{7}{12}\right)^2 \right] = \frac{1}{24} \left[-4 - \frac{49}{144} \right] = \frac{1}{24} \left[-\frac{625}{144} \right] \approx -0.1808 \text{ ft/sec}^2$$

53. (a) Using a graphing utility, you obtain $m(s) = -0.881s^2 + 29.10s - 206.2$

(b) $\frac{dm}{dt} = \frac{dm}{ds} \frac{ds}{dt} = (-1.762s + 29.10) \frac{ds}{dt}$

(c) If $t = s$ (1995), then $s = 15.5$ and $\frac{ds}{dt} = 1.2$.

Thus, $\frac{dm}{dt} = (-1.762(15.5) + 29.10)(1.2) \approx 2.15$ million.

Review Exercises for Chapter 2

1. $f(x) = x^2 - 2x + 3$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 - 2(x + \Delta x) + 3] - [x^2 - 2x + 3]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + 3) - (x^2 - 2x + 3)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2x(\Delta x) + (\Delta x)^2 - 2(\Delta x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x - 2) = 2x - 2 \end{aligned}$$

3. $f(x) = \sqrt{x} + 1$

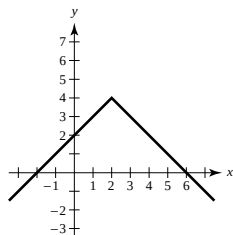
5. f is differentiable for all $x \neq -1$.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(\sqrt{x + \Delta x} + 1) - (\sqrt{x} + 1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

7. $f(x) = 4 - |x - 2|$

(a) Continuous at $x = 2$.

(b) Not differentiable at $x = 2$ because of the sharp turn in the graph.

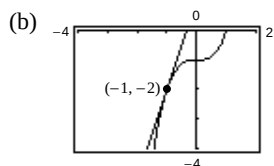


11. (a) Using the limit definition, $f'(x) = 3x^2$.

At $x = -1$, $f'(-1) = 3$. The tangent line is

$$y - (-2) = 3(x - (-1))$$

$$y = 3x + 1$$

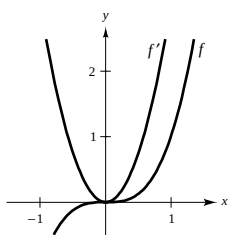


9. Using the limit definition, you obtain $g'(x) = \frac{4}{3}x - \frac{1}{6}$.

$$\text{At } x = -1, g'(-1) = -\frac{4}{3} - \frac{1}{6} = -\frac{3}{2}$$

$$\begin{aligned} 13. \quad g'(2) &= \lim_{x \rightarrow 2} \frac{g(x) - g(2)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{x^2(x - 1) - 4}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{x^3 - x^2 - 4}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{(x - 2)(x^2 + x + 2)}{x - 2} \\ &= \lim_{x \rightarrow 2} (x^2 + x + 2) = 8 \end{aligned}$$

15.



17. $y = 25$

$$y' = 0$$

19. $f(x) = x^8$

$$f'(x) = 8x^7$$

21. $h(t) = 3t^4$

$$h'(t) = 12t^3$$

23. $f(x) = x^3 - 3x^2$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

25. $h(x) = 6\sqrt{x} + 3\sqrt[3]{x} = 6x^{1/2} + 3x^{1/3}$

$$h'(x) = 3x^{-1/2} + x^{-2/3} = \frac{3}{\sqrt{x}} + \frac{1}{\sqrt[3]{x^2}}$$

27. $g(t) = \frac{2}{3}t^{-2}$

$$g'(t) = \frac{-4}{3}t^{-3} = -\frac{4}{3t^3}$$

29. $f(\theta) = 2\theta - 3\sin \theta$

$$f'(\theta) = 2 - 3\cos \theta$$

31. $f(\theta) = 3\cos \theta - \frac{\sin \theta}{4}$

$$f'(\theta) = -3\sin \theta - \frac{\cos \theta}{4}$$

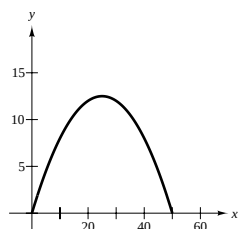
33. $F = 200\sqrt{T}$

$$F'(t) = \frac{100}{\sqrt{T}}$$

(a) When $T = 4$, $F'(4) = 50$ vibrations/sec/lb.

(b) When $T = 9$, $F'(9) = 33\frac{1}{3}$ vibrations/sec/lb.

37. (a)



Total horizontal distance: 50

(b) $0 = x - 0.02x^2$

$$0 = x\left(1 - \frac{x}{50}\right) \text{ implies } x = 50.$$

39. $x(t) = t^2 - 3t + 2 = (t - 2)(t - 1)$

(a) $v(t) = x'(t) = 2t - 3$

$$a(t) = v'(t) = 2$$

(c) $v(t) = 0$ for $t = \frac{3}{2}$.

$$x = \left(\frac{3}{2} - 2\right)\left(\frac{3}{2} - 1\right) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$$

41. $f(x) = (3x^2 + 7)(x^2 - 2x + 3)$

$$\begin{aligned} f'(x) &= (3x^2 + 7)(2x - 2) + (x^2 - 2x + 3)(6x) \\ &= 2(6x^3 - 9x^2 + 16x - 7) \end{aligned}$$

45. $f(x) = 2x - x^{-2}$

$$\begin{aligned} f'(x) &= 2 + 2x^{-3} = 2\left(1 + \frac{1}{x^3}\right) \\ &= \frac{2(x^3 + 1)}{x^3} \end{aligned}$$

49. $f(x) = (4 - 3x^2)^{-1}$

$$f'(x) = -(4 - 3x^2)^{-2}(-6x) = \frac{6x}{(4 - 3x^2)^2}$$

53. $y = 3x^2 \sec x$

$$y' = 3x^2 \sec x \tan x + 6x \sec x$$

35. $s(t) = -16t^2 + s_0$

$$s(9.2) = -16(9.2)^2 + s_0 = 0$$

$$s_0 = 1354.24$$

The building is approximately 1354 feet high (or 415 m).

(c) Ball reaches maximum height when $x = 25$.

(d) $y = x - 0.02x^2$

$$y' = 1 - 0.04x$$

$$y'(0) = 1$$

$$y'(10) = 0.6$$

$$y'(25) = 0$$

$$y'(30) = -0.2$$

$$y'(50) = -1$$

(e) $y'(25) = 0$

(b) $v(t) < 0$ for $t < \frac{3}{2}$.

(d) $x(t) = 0$ for $t = 1, 2$.

$$|v(1)| = |2(1) - 3| = 1$$

$$|v(2)| = |2(2) - 3| = 1$$

The speed is 1 when the position is 0.

43. $h(x) = \sqrt{x} \sin x = x^{1/2} \sin x$

$$h'(x) = \frac{1}{2\sqrt{x}} \sin x + \sqrt{x} \cos x$$

47. $f(x) = \frac{x^2 + x - 1}{x^2 - 1}$

$$\begin{aligned} f'(x) &= \frac{(x^2 - 1)(2x + 1) - (x^2 + x - 1)(2x)}{(x^2 - 1)^2} \\ &= \frac{-(x^2 + 1)}{(x^2 - 1)^2} \end{aligned}$$

51. $y = \frac{x^2}{\cos x}$

$$y' = \frac{\cos x (2x) - x^2(-\sin x)}{\cos^2 x} = \frac{2x \cos x + x^2 \sin x}{\cos^2 x}$$

55. $y = -x \tan x$

$$y' = -x \sec^2 x - \tan x$$

57. $y = x \cos x - \sin x$

$$y' = -x \sin x + \cos x - \cos x = -x \sin x$$

61. $f(\theta) = 3 \tan \theta$

$$f'(\theta) = 3 \sec^2 \theta$$

$$f''(\theta) = 6 \sec \theta (\sec \theta \tan \theta) = 6 \sec^2 \theta \tan \theta$$

65. $f(x) = (1 - x^3)^{1/2}$

$$f'(x) = \frac{1}{2}(1 - x^3)^{-1/2}(-3x^2)$$

$$= -\frac{3x^2}{2\sqrt{1 - x^3}}$$

69. $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$

$$f'(s) = (s^2 - 1)^{5/2}(3s^2) + (s^3 + 5)\left(\frac{5}{2}\right)(s^2 - 1)^{3/2}(2s)$$

$$= s(s^2 - 1)^{3/2}[3s(s^2 - 1) + 5(s^3 + 5)]$$

$$= s(s^2 - 1)^{3/2}(8s^3 - 3s + 25)$$

73. $y = \frac{1}{2} \csc 2x$

$$y' = \frac{1}{2}(-\csc 2x \cot 2x)(2)$$

$$= -\csc 2x \cot 2x$$

77. $y = \frac{2}{3} \sin^{3/2} x - \frac{2}{7} \sin^{7/2} x$

$$y' = \sin^{1/2} x \cos x - \sin^{5/2} x \cos x$$

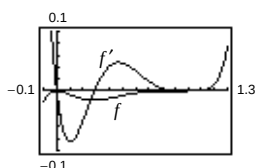
$$= (\cos x) \sqrt{\sin x}(1 - \sin^2 x)$$

$$= (\cos^3 x) \sqrt{\sin x}$$

81. $f(t) = t^2(t - 1)^5$

$$f'(t) = t(t - 1)^4(7t - 2)$$

The zeros of f' correspond to the points on the graph of f where the tangent line is horizontal.



59. $g(t) = t^3 - 3t + 2$

$$g'(t) = 3t^2 - 3$$

$$g''(t) = 6t$$

63. $y = 2 \sin x + 3 \cos x$

$$y' = 2 \cos x - 3 \sin x$$

$$y'' = -2 \sin x - 3 \cos x$$

$$y'' + y = -(2 \sin x + 3 \cos x) + (2 \sin x + 3 \cos x) = 0$$

67. $h(x) = \left(\frac{x-3}{x^2+1}\right)^2$

$$h'(x) = 2\left(\frac{x-3}{x^2+1}\right)\left(\frac{(x^2+1)(1) - (x-3)(2x)}{(x^2+1)^2}\right)$$

$$= \frac{2(x-3)(-x^2+6x+1)}{(x^2+1)^3}$$

71. $y = 3 \cos(3x + 1)$

$$y' = -9 \sin(3x + 1)$$

75. $y = \frac{x}{2} - \frac{\sin 2x}{4}$

$$y' = \frac{1}{2} - \frac{1}{4} \cos 2x(2)$$

$$= \frac{1}{2}(1 - \cos 2x) = \sin^2 x$$

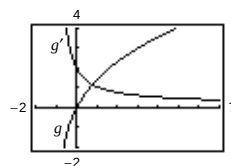
79. $y = \frac{\sin \pi x}{x+2}$

$$y' = \frac{(x+2)\pi \cos \pi x - \sin \pi x}{(x+2)^2}$$

83. $g(x) = 2x(x+1)^{-1/2}$

$$g'(x) = \frac{x+2}{(x+1)^{3/2}}$$

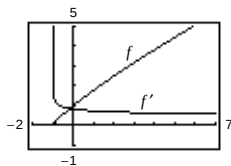
g' does not equal zero for any value of x in the domain. The graph of g has no horizontal tangent lines.



85. $f(t) = (t + 1)^{1/2}(t + 1)^{1/3} = (t + 1)^{5/6}$

$$f'(t) = \frac{5}{6(t + 1)^{1/6}}$$

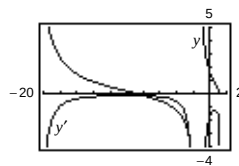
f' does not equal zero for any x in the domain. The graph of f has no horizontal tangent lines.



87. $y = \tan \sqrt{1 - x}$

$$y' = -\frac{\sec^2 \sqrt{1 - x}}{2\sqrt{1 - x}}$$

y' does not equal zero for any x in the domain. The graph has no horizontal tangent lines.



89. $y = 2x^2 + \sin 2x$

$$y' = 4x + 2 \cos 2x$$

$$y'' = 4 - 4 \sin 2x$$

91. $f(x) = \cot x$

$$f'(x) = -\csc^2 x$$

$$\begin{aligned} f'' &= -2 \csc x (-\csc x \cdot \cot x) \\ &= 2 \csc^2 x \cot x \end{aligned}$$

93. $f(t) = \frac{t}{(1 - t)^2}$

$$f'(t) = \frac{t + 1}{(1 - t)^3}$$

$$f''(t) = \frac{2(t + 2)}{(1 - t)^4}$$

95. $g(\theta) = \tan 3\theta - \sin(\theta - 1)$

$$g'(\theta) = 3 \sec^2 3\theta - \cos(\theta - 1)$$

$$g''(\theta) = 18 \sec^2 3\theta \tan 3\theta + \sin(\theta - 1)$$

97. $T = 700(t^2 + 4t + 10)^{-1}$

$$T' = \frac{-1400(t + 2)}{(t^2 + 4t + 10)^2}$$

(a) When $t = 1$,

$$T' = \frac{-1400(1 + 2)}{(1 + 4 + 10)^2} \approx -18.667 \text{ deg/hr.}$$

(b) When $t = 3$,

$$T' = \frac{-1400(3 + 2)}{(9 + 12 + 10)^2} \approx -7.284 \text{ deg/hr.}$$

(c) When $t = 5$,

$$T' = \frac{-1400(5 + 2)}{(25 + 30 + 10)^2} \approx -3.240 \text{ deg/hr.}$$

(d) When $t = 10$,

$$T' = \frac{-1400(10 + 2)}{(100 + 40 + 10)^2} \approx -0.747 \text{ deg/hr.}$$

99. $x^2 + 3xy + y^3 = 10$

$$2x + 3xy' + 3y + 3y^2y' = 0$$

$$3(x + y^2)y' = -(2x + 3y)$$

$$y' = \frac{-(2x + 3y)}{3(x + y^2)}$$

101. $y\sqrt{x} - x\sqrt{y} = 16$

$$y\left(\frac{1}{2}x^{-1/2}\right) + x^{1/2}y' - x\left(\frac{1}{2}y^{-1/2}y'\right) - y^{1/2} = 0$$

$$\left(\sqrt{x} - \frac{x}{2\sqrt{y}}\right)y' = \sqrt{y} - \frac{y}{2\sqrt{x}}$$

$$\frac{2\sqrt{xy} - x}{2\sqrt{y}}y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}}$$

$$y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}} \cdot \frac{2\sqrt{y}}{2\sqrt{xy} - x} = \frac{2y\sqrt{x} - y\sqrt{y}}{2x\sqrt{y} - x\sqrt{x}}$$

103. $x \sin y = y \cos x$
 $(x \cos y)y' + \sin y = -y \sin x + y' \cos x$
 $y'(x \cos y - \cos x) = -y \sin x - \sin y$
 $y' = \frac{y \sin x + \sin y}{\cos x - x \cos y}$

105. $x^2 + y^2 = 20$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

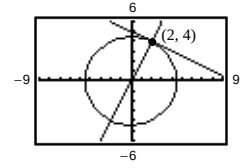
At $(2, 4)$: $y' = -\frac{1}{2}$

Tangent line: $y - 4 = -\frac{1}{2}(x - 2)$

$$x + 2y - 10 = 0$$

Normal line: $y - 4 = 2(x - 2)$

$$2x - y = 0$$



107. $y = \sqrt{x}$

$$\frac{dy}{dt} = 2 \text{ units/sec}$$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{x}} \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 2\sqrt{x} \frac{dy}{dt} = 4\sqrt{x}$$

(a) When $x = \frac{1}{2}$, $\frac{dx}{dt} = 2\sqrt{2}$ units/sec.

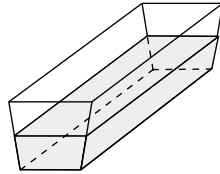
(b) When $x = 1$, $\frac{dx}{dt} = 4$ units/sec.

(c) When $x = 4$, $\frac{dx}{dt} = 8$ units/sec.

109. $\frac{s}{h} = \frac{1/2}{2}$

$$s = \frac{1}{4}h$$

$$\frac{dV}{dt} = 1$$



Width of water at depth h :

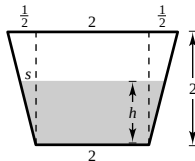
$$w = 2 + 2s = 2 + 2\left(\frac{1}{4}h\right) = \frac{4 + h}{2}$$

$$V = \frac{5}{2}\left(2 + \frac{4 + h}{2}\right)h = \frac{5}{4}(8 + h)h$$

$$\frac{dV}{dt} = \frac{5}{2}(4 + h)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2(dV/dt)}{5(4 + h)}$$

When $h = 1$, $\frac{dh}{dt} = \frac{2}{25}$ m/min.



111. $s(t) = 60 - 4.9t^2$

$$s'(t) = -9.8t$$

$$s = 35 = 60 - 4.9t^2$$

$$4.9t^2 = 25$$

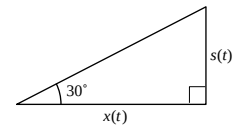
$$t = \frac{5}{\sqrt{4.9}}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{s(t)}{x(t)}$$

$$x(t) = \sqrt{3}s(t)$$

$$\frac{dx}{dt} = \sqrt{3} \frac{ds}{dt} = \sqrt{3}(-9.8) \frac{5}{\sqrt{4.9}}$$

$$\approx -38.34 \text{ m/sec}$$



Problem Solving for Chapter 2

1. (a) $x^2 + (y - r)^2 = r^2$ Circle

$$x^2 = y \quad \text{Parabola}$$

Substituting,

$$(y - r)^2 = r^2 - y$$

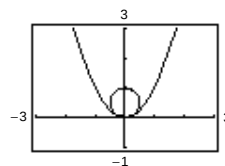
$$y^2 - 2ry + r^2 = r^2 - y$$

$$y^2 - 2ry + y = 0$$

$$y(y - 2r + 1) = 0$$

Since you want only one solution, let $1 - 2r = 0 \Rightarrow r = \frac{1}{2}$

$$\text{Graph } y = x^2 \text{ and } x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$



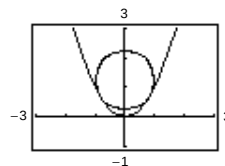
- (b) Let (x, y) be a point of tangency: $x^2 + (y - b)^2 = 1 \Rightarrow 2x + 2(y - b)y' = 0 \Rightarrow y' = \frac{x}{b - y}$ (circle).

$y = x^2 \Rightarrow y' = 2x$ (parabola). Equating,

$$2x = \frac{x}{b - y}$$

$$2(b - y) = 1$$

$$b - y = \frac{1}{2} \Rightarrow b = y + \frac{1}{2}$$



Also, $x^2 + (y - b)^2 = 1$ and $y = x^2$ imply

$$y + (y - b)^2 = 1 \Rightarrow y + \left[y - \left(y + \frac{1}{2}\right)\right]^2 = 1 \Rightarrow y - \frac{1}{2} = 1 \Rightarrow y = \frac{3}{4} \text{ and } b = \frac{5}{4}.$$

$$\text{Center: } \left(0, \frac{5}{4}\right)$$

$$\text{Graph } y = x^2 \text{ and } x^2 + \left(y - \frac{5}{4}\right)^2 = 1$$

3. (a) $f(x) = \cos x$ $P_1(x) = a_0 + a_1x$
 $f(0) = 1$ $P_1(0) = a_0 \Rightarrow a_0 = 1$
 $f'(0) = 0$ $P'_1(0) = a_1 \Rightarrow a_1 = 0$
 $P_1(x) = 1$

- (b) $f(x) = \cos x$ $P_2(x) = a_0 + a_1x + a_2x^2$
 $f(0) = 1$ $P_2(0) = a_0 \Rightarrow a_0 = 1$
 $f'(0) = 0$ $P'_2(0) = a_1 \Rightarrow a_1 = 0$
 $f''(0) = -1$ $P''_2(0) = 2a_2 \Rightarrow a_2 = -\frac{1}{2}$
 $P_2(x) = 1 - \frac{1}{2}x^2$

(c)

x	-1.0	-0.1	-0.001	0	0.001	0.1	1.0
$\cos x$	0.5403	0.9950	≈ 1	1	≈ 1	0.9950	0.5403
$P_2(x)$	0.5	0.9950	≈ 1	1	≈ 1	0.9950	0.5

$P_2(x)$ is a good approximation of $f(x) = \cos x$ when x is near 0.

- (d) $f(x) = \sin x$ $P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$
 $f(0) = 0$ $P_3(0) = a_0 \Rightarrow a_0 = 0$
 $f'(0) = 1$ $P'_3(0) = a_1 \Rightarrow a_1 = 1$
 $f''(0) = 0$ $P''_3(0) = 2a_2 \Rightarrow a_2 = 0$
 $f'''(0) = -1$ $P'''_3(0) = 6a_3 \Rightarrow a_3 = -\frac{1}{6}$
 $P_3(x) = x - \frac{1}{6}x^3$

5. Let $p(x) = Ax^3 + Bx^2 + Cx + D$

$$p'(x) = 3Ax^2 + 2Bx + C$$

At (1, 1): $A + B + C + D = 1$ Equation 1

$$3A + 2B + C = 14$$
 Equation 2

At (-1, -3): $-A + B - C + D = -3$ Equation 3

$$3A - 2B + C = -2$$
 Equation 4

Adding Equations 1 and 3: $2B + 2D = -2$

Subtracting Equations 1 and 3: $2A + 2C = 4$

Adding Equations 2 and 4: $6A + 2C = 12$

Subtracting Equations 2 and 4: $4B = 16$

Hence, $B = 4$ and $D = \frac{1}{2}(-2 - 2B) = -5$

Subtracting $2A + 2C = 4$ and $6A + 2C = 12$, you obtain $4A = 8 \Rightarrow A = 2$. Finally, $C = \frac{1}{2}(4 - 2A) = 0$

Thus, $p(x) = 2x^3 + 4x^2 - 5$.

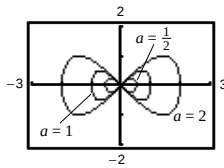
7. (a) $x^4 = a^2x^2 - a^2y^2$

$$a^2y^2 = a^2x^2 - x^4$$

$$y = \frac{\pm \sqrt{a^2x^2 - x^4}}{a}$$

Graph: $y_1 = \frac{\sqrt{a^2x^2 - x^4}}{a}$ and $y_2 = -\frac{\sqrt{a^2x^2 - x^4}}{a}$

(b)



$(\pm a, 0)$ are the x-intercepts, along with $(0, 0)$.

(c) Differentiating implicitly,

$$4x^3 = 2a^2x - 2a^2yy'$$

$$y' = \frac{2a^2x - 4x^3}{2a^2y} = \frac{x(a^2 - 2x^2)}{a^2y} = 0 \Rightarrow 2x^2 = a^2 \Rightarrow x = \frac{\pm a}{\sqrt{2}}$$

$$\left(\frac{a^2}{2}\right)^2 = a^2\left(\frac{a^2}{2}\right) - a^2y^2$$

$$\frac{a^4}{4} = \frac{a^4}{2} - a^2y^2$$

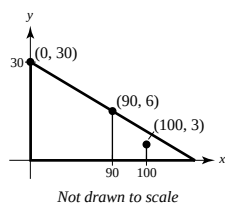
$$a^2y^2 = \frac{a^4}{4}$$

$$y^2 = \frac{a^2}{4}$$

$$y = \pm \frac{a}{2}$$

Four points: $\left(\frac{a}{\sqrt{2}}, \frac{a}{2}\right), \left(\frac{a}{\sqrt{2}}, -\frac{a}{2}\right), \left(-\frac{a}{\sqrt{2}}, \frac{a}{2}\right), \left(-\frac{a}{\sqrt{2}}, -\frac{a}{2}\right)$

9. (a)

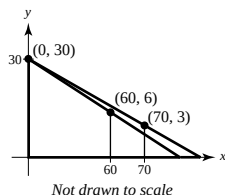


Line determined by (0, 30) and (90, 6):

$$y - 30 = \frac{30 - 6}{0 - 90}(x - 0) = -\frac{24}{90}x = -\frac{4}{15}x \Rightarrow y = -\frac{4}{15}x + 30$$

When $x = 100$, $y = -\frac{4}{15}(100) + 30 = \frac{10}{3} > 3 \Rightarrow$ Shadow determined by man.

(b)



Line determined by (0, 30) and (60, 6):

$$y - 30 = \frac{30 - 6}{0 - 60}(x - 0) = -\frac{2}{5}x \Rightarrow y = -\frac{2}{5}x + 30$$

When $x = 70$, $y = -\frac{2}{5}(70) + 30 = 2 < 3 \Rightarrow$ Shadow determined by child.

(c) Need (0, 30), (d, 6), (d + 10, 3) collinear.

$$\frac{30 - 6}{0 - d} = \frac{6 - 3}{d - (d + 10)} \Rightarrow \frac{24}{-d} = \frac{3}{-10} \Rightarrow d = 80 \text{ feet}$$

(d) Let y be the length of the street light to the tip of the shadow. We know that $\frac{dx}{dt} = -5$.For $x > 80$, the shadow is determined by the man.

$$\frac{y}{30} = \frac{y - x}{6} \Rightarrow y = \frac{5}{4}x \text{ and } \frac{dy}{dt} = \frac{5}{4} \frac{dx}{dt} = \frac{-25}{4}.$$

For $x < 80$, the shadow is determined by the child.

$$\frac{y}{30} = \frac{y - x - 10}{3} \Rightarrow y = \frac{10}{9}x + \frac{100}{9} \text{ and } \frac{dy}{dt} = \frac{10}{9} \frac{dx}{dt} = \frac{-50}{9}.$$

Therefore,

$$\frac{dy}{dt} = \begin{cases} -\frac{25}{4} & x > 80 \\ -\frac{50}{9} & 0 < x < 80 \end{cases}$$

 $\frac{dy}{dt}$ is not continuous at $x = 80$.

$$\begin{aligned} 11. L'(x) &= \lim_{\Delta x \rightarrow 0} \frac{L(x + \Delta x) - L(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{L(x) + L(\Delta x) - L(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{L(\Delta x)}{\Delta x} \end{aligned}$$

$$\text{Also, } L'(0) = \lim_{\Delta x \rightarrow 0} \frac{L(\Delta x) - L(0)}{\Delta x}$$

But, $L(0) = 0$ because $L(0) = L(0 + 0) = L(0) + L(0) \Rightarrow L(0) = 0$.Thus, $L'(x) = L'(0)$, for all x .The graph of L is a line through the origin of slope $L'(0)$.

13. (a)

z (degrees)	0.1	0.01	0.0001
$\frac{\sin z}{z}$	0.0174524	0.0174533	0.0174533

(b) $\lim_{z \rightarrow 0} \frac{\sin z}{z} \approx 0.0174533$

In fact, $\lim_{z \rightarrow 0} \frac{\sin z}{z} = \frac{\pi}{180}$

(c) $\frac{d}{dz}(\sin z) = \lim_{\Delta z \rightarrow 0} \frac{\sin(z + \Delta z) - \sin z}{\Delta z}$

$$= \lim_{\Delta z \rightarrow 0} \frac{\sin z \cdot \cos \Delta z + \sin \Delta z \cdot \cos z - \sin z}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \left[\sin z \left(\frac{\cos \Delta z - 1}{\Delta z} \right) \right] + \lim_{\Delta z \rightarrow 0} \left[\cos z \left(\frac{\sin \Delta z}{\Delta z} \right) \right]$$

$$= \sin z(0) + \cos z \left(\frac{\pi}{180} \right) = \frac{\pi}{180} \cos z$$

(d) $S(90) = \sin\left(\frac{\pi}{180} 90\right) = \sin \frac{\pi}{2} = 1$; $C(180) = \cos\left(\frac{\pi}{180} 180\right) = -1$

$$\frac{d}{dz}S(z) = \frac{d}{dz}\sin(cz) = c \cdot \cos(cz) = \frac{\pi}{180}C(z)$$

(e) The formulas for the derivatives are more complicated in degrees.

15. $j(t) = a'(t)$

(a) $j(t)$ is the rate of change of the acceleration.

(b) From Exercise 102 in Section 2.3,

$$s(t) = -8.25t^2 + 66t$$

$$v(t) = -16.5t + 66$$

$$a(t) = -16.5$$

$$a'(t) = j(t) = 0$$

Review Exercises for Chapter 2

$$2. f(x) = \frac{x+1}{x-1}$$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{x + \Delta x + 1}{x + \Delta x - 1} - \frac{x + 1}{x - 1}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x + 1)(x - 1) - (x + \Delta x - 1)(x + 1)}{\Delta x(x + \Delta x - 1)(x - 1)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x^2 + x\Delta x + x - x - \Delta x - 1) - (x^2 + x\Delta x - x + x + \Delta x - 1)}{\Delta x(x + \Delta x - 1)(x - 1)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2\Delta x}{\Delta x(x + \Delta x - 1)(x - 1)} = \lim_{\Delta x \rightarrow 0} \frac{-2}{(x + \Delta x - 1)(x - 1)} = \frac{-2}{(x - 1)^2} \end{aligned}$$

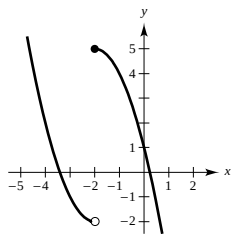
$$4. f(x) = \frac{2}{x}$$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{2}{x + \Delta x} - \frac{2}{x}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2x - (2x + 2\Delta x)}{\Delta x(x + \Delta x)x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2\Delta x}{\Delta x(x + \Delta x)x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2}{(x + \Delta x)x} = \frac{-2}{x^2} \end{aligned}$$

$$8. f(x) = \begin{cases} x^2 + 4x + 2, & \text{if } x < -2 \\ 1 - 4x - x^2, & \text{if } x \geq -2 \end{cases}$$

(a) Nonremovable discontinuity at $x = -2$.

(b) Not differentiable at $x = -2$ because the function is discontinuous there.



$$12. (a) \text{ Using the limit definition, } f'(x) = \frac{-2}{(x+1)^2}.$$

At $x = 0$, $f'(0) = -2$. The tangent line is

$$y - 2 = -2(x - 0)$$

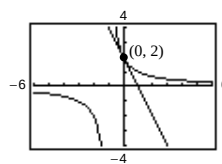
$$y = -2x + 2$$

6. f is differentiable for all $x \neq -3$.

10. Using the limit definition, you obtain $h'(x) = \frac{3}{8} - 4x$.

$$\text{At } x = -2, h'(-2) = \frac{3}{8} - 4(-2) = \frac{67}{8}.$$

(b)



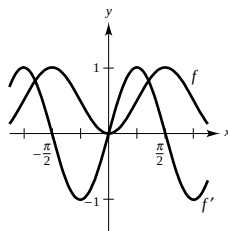
$$14. f'(2) = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \rightarrow 2} \frac{\frac{1}{x+1} - \frac{1}{3}}{x - 2}$$

$$= \lim_{x \rightarrow 2} \frac{3 - x - 1}{(x - 2)(x + 1)3}$$

$$= \lim_{x \rightarrow 2} \frac{-1}{(x + 1)3} = \frac{-1}{9}$$

16.



$$18. y = -12$$

$$y' = 0$$

$$20. g(x) = x^{12}$$

$$g'(x) = 12x^{11}$$

$$22. f(t) = -8t^5$$

$$f'(t) = -40t^4$$

$$24. g(s) = 4s^4 - 5s^2$$

$$g'(s) = 16s^3 - 10s$$

$$26. f(x) = x^{1/2} - x^{-1/2}$$

$$f'(x) = \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{-3/2} = \frac{x+1}{2x^{3/2}}$$

$$28. h(x) = \frac{2}{9}x^{-2}$$

$$h'(x) = \frac{-4}{9}x^{-3} = \frac{-4}{9x^3}$$

$$30. g(\alpha) = 4 \cos \alpha + 6$$

$$g'(\alpha) = -4 \sin \alpha$$

$$32. g(\alpha) = \frac{5 \sin \alpha}{3} - 2\alpha$$

$$g'(\alpha) = \frac{5 \cos \alpha}{3} - 2$$

$$34. s = -16t^2 + s_0$$

First ball:

$$-16t^2 + 100 = 0$$

$$t = \sqrt{\frac{100}{16}} = \frac{10}{4} = 2.5 \text{ seconds to hit ground}$$

Second ball:

$$-16t^2 + 75 = 0$$

$$t^2 = \sqrt{\frac{75}{16}} = \frac{5\sqrt{3}}{4} \approx 2.165 \text{ seconds to hit ground}$$

Since the second ball was released one second after the first ball, the first ball will hit the ground first. The second ball will hit the ground $3.165 - 2.5 = 0.665$ second later.

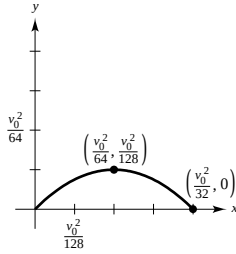
$$36. s(t) = -16t^2 + 14,400 = 0$$

$$16t^2 = 14,400$$

$$t = 30 \text{ sec}$$

Since $600 \text{ mph} = \frac{1}{6} \text{ mi/sec}$, in 30 seconds the bomb will move horizontally $(\frac{1}{6})(30) = 5$ miles.

38.



$$(a) \ y = x - \frac{32}{v_0^2}x^2 = x\left(1 - \frac{32}{v_0^2}x\right)$$

$$= 0 \text{ if } x = 0 \text{ or } x = \frac{v_0^2}{32}.$$

Projectile strikes the ground when $x = v_0^2/32$.

Projectile reaches its maximum height at $x = v_0^2/64$.
(one-half the distance)

$$(c) \ y = x - \frac{32}{v_0^2}x^2 = x\left(1 - \frac{32}{v_0^2}x\right) = 0$$

when $x = 0$ and $x = v_0^2/32$. Therefore, the range is $x = v_0^2/32$. When the initial velocity is doubled the range is

$$x = \frac{(2v_0)^2}{32} = \frac{4v_0^2}{32}$$

or four times the initial range. From part (a), the maximum height occurs when $x = v_0^2/64$. The maximum height is

$$y\left(\frac{v_0^2}{64}\right) = \frac{v_0^2}{64} - \frac{32}{v_0^2}\left(\frac{v_0^2}{64}\right)^2 = \frac{v_0^2}{64} - \frac{v_0^2}{128} = \frac{v_0^2}{128}.$$

If the initial velocity is doubled, the maximum height is

$$y\left[\frac{(2v_0)^2}{64}\right] = \frac{(2v_0)^2}{128} = 4\left(\frac{v_0^2}{128}\right)$$

or four times the original maximum height.

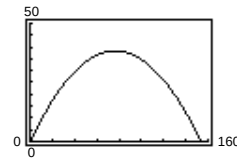
$$(b) \ y' = 1 - \frac{64}{v_0^2}x$$

$$\text{When } x = \frac{v_0^2}{64}, y' = 1 - \frac{64}{v_0^2}\left(\frac{v_0^2}{64}\right) = 0.$$

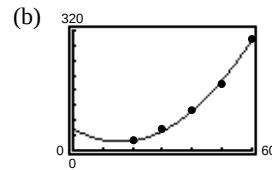
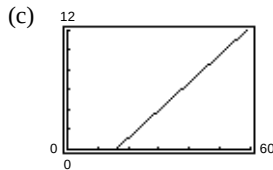
$$(d) \ v_0 = 70 \text{ ft/sec}$$

$$\text{Range: } x = \frac{v_0^2}{32} = \frac{(70)^2}{32} = 153.125 \text{ ft}$$

$$\text{Maximum height: } y = \frac{v_0^2}{128} = \frac{(70)^2}{128} \approx 38.28 \text{ ft}$$



$$40. (a) \ y = 0.14x^2 - 4.43x + 58.4$$



$$(d) \ \text{If } x = 65, y \approx 362 \text{ feet.}$$

(e) As the speed increases, the stopping distance increases at an increasing rate.

$$42. g(x) = (x^3 - 3x)(x + 2)$$

$$\begin{aligned} g'(x) &= (x^3 - 3x)(1) + (x + 2)(3x^2 - 3) \\ &= x^3 - 3x + 3x^3 + 6x^2 - 3x - 6 \\ &= 4x^3 + 6x^2 - 6x - 6 \end{aligned}$$

$$46. f(x) = \frac{x+1}{x-1}$$

$$\begin{aligned} f'(x) &= \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} \\ &= \frac{-2}{(x-1)^2} \end{aligned}$$

$$50. f(x) = 9(3x^2 - 2x)^{-1}$$

$$f'(x) = -9(3x^2 - 2x)^{-2}(6x - 2) = \frac{18(1 - 3x)}{(3x^2 - 2x)^2}$$

$$54. y = 2x - x^2 \tan x$$

$$y' = 2 - x^2 \sec^2 x - 2x \tan x$$

$$58. v(t) = 36 - t^2, 0 \leq t \leq 6$$

$$a(t) = v'(t) = -2t$$

$$v(4) = 36 - 16 = 20 \text{ m/sec}$$

$$a(4) = -8 \text{ m/sec}$$

$$62. h(t) = 4 \sin t - 5 \cos t$$

$$h'(t) = 4 \cos t + 5 \sin t$$

$$h''(t) = -4 \sin t + 5 \cos t$$

$$66. f(x) = (x^2 - 1)^{1/3}$$

$$\begin{aligned} f'(x) &= \frac{1}{3}(x^2 - 1)^{-2/3}(2x) \\ &= \frac{2x}{3(x^2 - 1)^{2/3}} \end{aligned}$$

$$44. f(t) = t^3 \cos t$$

$$\begin{aligned} f'(t) &= t^3(-\sin t) + \cos t(3t^2) \\ &= -t^3 \sin t + 3t^2 \cos t \end{aligned}$$

$$48. f(x) = \frac{6x - 5}{x^2 + 1}$$

$$\begin{aligned} f'(x) &= \frac{(x^2 + 1)(6) - (6x - 5)(2x)}{(x^2 + 1)^2} \\ &= \frac{2(3 + 5x - 3x^2)}{(x^2 + 1)^2} \end{aligned}$$

$$52. y = \frac{\sin x}{x^2}$$

$$y' = \frac{(x^2) \cos x - (\sin x)(2x)}{x^4} = \frac{x \cos x - 2 \sin x}{x^3}$$

$$56. y = \frac{1 + \sin x}{1 - \sin x}$$

$$\begin{aligned} y' &= \frac{(1 - \sin x) \cos x - (1 + \sin x)(-\cos x)}{(1 - \sin x)^2} \\ &= \frac{2 \cos x}{(1 - \sin x)^2} \end{aligned}$$

$$60. f(x) = 12x^{1/4}$$

$$f'(x) = 3x^{-3/4}$$

$$f''(x) = \frac{-9}{4}x^{-7/4} = \frac{-9}{4x^{7/4}}$$

$$64. y = \frac{(10 - \cos x)}{x}$$

$$xy + \cos x = 10$$

$$xy' + y - \sin x = 0$$

$$xy' = \sin x - y$$

$$xy' + y = (\sin x - y) + y = \sin x$$

$$68. f(x) = \left(x^2 + \frac{1}{x}\right)^5$$

$$f'(x) = 5\left(x^2 + \frac{1}{x}\right)^4\left(2x - \frac{1}{x^2}\right)$$

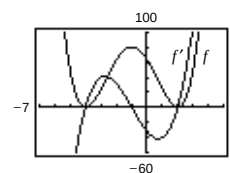
$$\begin{aligned}
 70. \quad h(\theta) &= \frac{\theta}{(1-\theta)^3} \\
 h'(\theta) &= \frac{(1-\theta)^3 - \theta[3(1-\theta)^2(-1)]}{(1-\theta)^6} \\
 &= \frac{(1-\theta)^2(1-\theta+3\theta)}{(1-\theta)^6} = \frac{2\theta+1}{(1-\theta)^4}
 \end{aligned}$$

$$\begin{aligned}
 74. \quad y &= \csc 3x + \cot 3x \\
 y' &= -3 \csc 3x \cot 3x - 3 \csc^2 3x \\
 &= -3 \csc 3x(\cot 3x + \csc 3x)
 \end{aligned}$$

$$\begin{aligned}
 78. \quad f(x) &= \frac{3x}{\sqrt{x^2+1}} \\
 f'(x) &= \frac{3(x^2+1)^{1/2} - 3x \frac{1}{2}(x^2+1)^{-1/2}(2x)}{x^2+1} \\
 &= \frac{3(x^2+1) - 3x^2}{(x^2+1)^{3/2}} \\
 &= \frac{3}{(x^2+1)^{3/2}}
 \end{aligned}$$

$$\begin{aligned}
 82. \quad f(x) &= [(x-2)(x+4)]^2 = (x^2+2x-8)^2 \\
 f'(x) &= 4(x^3+3x^2-6x-8) \\
 &= 4(x-2)(x+1)(x+4)
 \end{aligned}$$

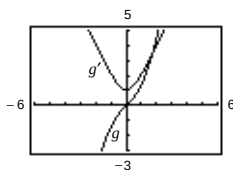
The zeros of f' correspond to the points on the graph of f where the tangent line is horizontal.



$$84. \quad g(x) = x(x^2+1)^{1/2}$$

$$g'(x) = \frac{2x^2+1}{\sqrt{x^2+1}}$$

g' does not equal zero for any value of x . The graph of g has no horizontal tangent lines.



$$\begin{aligned}
 72. \quad y &= 1 - \cos 2x + 2 \cos^2 x \\
 y' &= 2 \sin 2x - 4 \cos x \sin x \\
 &= 2[2 \sin x \cos x] - 4 \sin x \cos x \\
 &= 0
 \end{aligned}$$

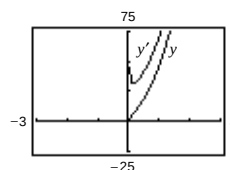
$$\begin{aligned}
 76. \quad y &= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} \\
 y' &= \sec^6 x(\sec x \tan x) - \sec^4 x(\sec x \tan x) \\
 &= \sec^5 x \tan x(\sec^2 x - 1) \\
 &= \sec^5 x \tan^3 x
 \end{aligned}$$

$$\begin{aligned}
 80. \quad y &= \frac{\cos(x-1)}{x-1} \\
 y' &= \frac{-(x-1) \sin(x-1) - \cos(x-1)(1)}{(x-1)^2} \\
 &= -\frac{1}{(x-1)^2}[(x-1) \sin(x-1) + \cos(x-1)]
 \end{aligned}$$

$$86. \quad y = \sqrt{3x}(x+2)^3$$

$$y' = \frac{3(x+2)^2(7x+2)}{2\sqrt{3x}}$$

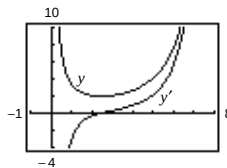
y' does not equal zero for any x in the domain. The graph has no horizontal tangent lines.



88. $y = 2 \csc^3(\sqrt{x})$

$$y' = -\frac{3}{\sqrt{x}} \csc^3 \sqrt{x} \cot \sqrt{x}$$

The zero of y' corresponds to the point on the graph of y where the tangent line is horizontal.



90. $y = x^{-1} + \tan x$

$$y' = -x^{-2} + \sec^2 x$$

$$y'' = 2x^{-3} + 2 \sec x (\sec x \tan x)$$

$$= \frac{2}{x^3} + 2 \sec^2 x \tan x$$

92. $y = \sin^2 x$

$$y' = 2 \sin x \cos x = \sin 2x$$

$$y'' = 2 \cos 2x$$

94. $g(x) = \frac{6x - 5}{x^2 + 1}$

$$g'(x) = \frac{2(-3x^2 + 5x + 3)}{(x^2 + 1)^2}$$

$$g''(x) = \frac{2(6x^3 - 15x^2 - 18x + 5)}{(x^2 + 1)^3}$$

96. $h(x) = x\sqrt{x^2 - 1}$

$$h'(x) = \frac{2x^2 - 1}{\sqrt{x^2 - 1}}$$

$$h''(x) = \frac{x(2x^2 - 3)}{(x^2 - 1)^{3/2}}$$

98. $v = \sqrt{2gh} = \sqrt{2(32)h} = 8\sqrt{h}$

$$\frac{dv}{dh} = \frac{4}{\sqrt{h}}$$

(a) When $h = 9$, $\frac{dv}{dh} = \frac{4}{3}$ ft/sec.

(b) When $h = 4$, $\frac{dv}{dh} = 2$ ft/sec.

100. $x^2 + 9y^2 - 4x + 3y = 0$

$$2x + 18yy' - 4 + 3y' = 0$$

$$3(6y + 1)y' = 4 - 2x$$

$$y' = \frac{4 - 2x}{3(6y + 1)}$$

102.

$$y^2 = x^3 - x^2y + xy - y^2$$

$$0 = x^3 - x^2y + xy - 2y^2$$

$$0 = 3x^2 - x^2y' - 2xy + xy' + y - 4yy'$$

$$(x^2 - x + 4y)y' = 3x^2 - 2xy + y$$

$$y' = \frac{3x^2 - 2xy + y}{x^2 - x + 4y}$$

104. $\cos(x + y) = x$

$$-(1 + y') \sin(x + y) = 1$$

$$-y' \sin(x + y) = 1 + \sin(x + y)$$

$$y' = -\frac{1 + \sin(x + y)}{\sin(x + y)}$$

$$= -\csc(x + 1) - 1$$

106. $x^2 - y^2 = 16$

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$

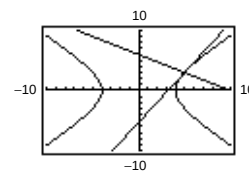
At (5, 3): $y' = \frac{5}{3}$

Tangent line: $y - 3 = \frac{5}{3}(x - 5)$

$$5x - 3y - 16 = 0$$

Normal line: $y - 3 = -\frac{3}{5}(x - 5)$

$$3x + 5y - 30 = 0$$



108. Surface area = $A = 6x^2$, x length of edge.

$$\frac{dx}{dt} = 5$$

$$\frac{da}{dt} = 12x \frac{dx}{dt} = 12(4.5)(5) = 270 \text{ cm}^2/\text{sec}$$

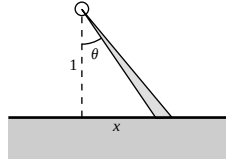
110. $\tan \theta = x$

$$\frac{d\theta}{dt} = 3(2\pi) \text{ rad/min}$$

$$\sec^2 \theta \left(\frac{d\theta}{dt} \right) = \frac{dx}{dt}$$

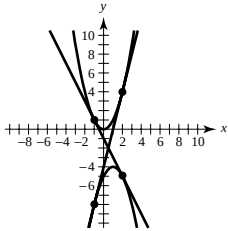
$$\frac{dx}{dt} = (\tan^2 \theta + 1)(6\pi) = 6\pi(x^2 + 1)$$

When $x = \frac{1}{2}$, $\frac{dx}{dt} = 6\pi \left(\frac{1}{4} + 1 \right) = \frac{15\pi}{2} \text{ km/min} = 450\pi \text{ km/hr.}$



Problem Solving for Chapter 2

2.



Let (a, a^2) and $(b, -b^2 + 2b - 5)$ be the points of tangency.

For $y = x^2$, $y' = 2x$ and for $y = -x^2 + 2x - 5$, $y' = -2x + 2$.

Thus, $2a = -2b + 2 \Rightarrow a + b = 1$, or $a = 1 - b$. Furthermore, the slope of the common tangent line is

$$\frac{a^2 - (-b^2 + 2b - 5)}{a - b} = \frac{(1 - b)^2 + b^2 - 2b + 5}{(1 - b) - b} = -2b + 2$$

$$\Rightarrow \frac{1 - 2b + b^2 + b^2 - 2b + 5}{1 - 2b} = -2b + 2$$

$$\Rightarrow 2b^2 - 4b + 6 = 4b^2 - 6b + 2$$

$$\Rightarrow 2b^2 - 2b - 4 = 0$$

$$\Rightarrow b^2 - b - 2 = 0$$

$$\Rightarrow (b - 2)(b + 1) = 0$$

$$b = 2, -1$$

For $b = 2$, $a = 1 - b = -1$ and the points of tangency are $(-1, 1)$, $(2, -5)$. The tangent line has slope -2 :

$$y - 1 = -2(x + 1) \Rightarrow y = -2x - 1$$

For $b = -1$, $a = 1 - b = 2$ and the points of tangency are $(2, 4)$ and $(-1, -8)$. The tangent line has slope 4 :

$$y - 4 = 4(x - 2) \Rightarrow y = 4x - 4.$$

4. (a) $y = x^2$, $y' = 2x$. Slope = 4 at $(2, 4)$.

Tangent line: $y - 4 = 4(x - 2)$

$$y = 4x - 4$$

(b) Slope of normal line: $-\frac{1}{4}$.

Normal line: $y - 4 = -\frac{1}{4}(x - 2)$

$$y = -\frac{1}{4}x + \frac{9}{2}$$

$$y = -\frac{1}{4}x + \frac{9}{2} = x^2 \Rightarrow 4x^2 + x - 18 = 0 \Rightarrow (4x + 9)(x - 2) = 0$$

$$x = 2, -\frac{9}{4}. \text{ Second intersection point: } \left(-\frac{9}{4}, \frac{81}{16}\right)$$

(c) Tangent line: $y = 0$

Normal line: $x = 0$

—CONTINUED—

4. —CONTINUED—

(d) Let (a, a^2) , $a \neq 0$, be a point on the parabola $y = x^2$. Tangent line at (a, a^2) is $y = 2a(x - a) + a^2$. Normal line at (a, a^2) is $y = -\frac{1}{2a}(x - a) + a^2$. To find points of intersection, solve

$$x^2 = -\frac{1}{2a}(x - a) + a^2$$

$$x^2 + \frac{1}{2a}x = a^2 + \frac{1}{2}$$

$$x^2 + \frac{1}{2a}x + \frac{1}{16a^2} = a^2 + \frac{1}{2} + \frac{1}{16a^2}$$

$$\left(x + \frac{1}{4a}\right)^2 = \left(a + \frac{1}{4a}\right)^2$$

$$x + \frac{1}{4a} = \pm \left(a + \frac{1}{4a}\right)$$

$$x + \frac{1}{4a} = a + \frac{1}{4a} \Rightarrow x = a \text{ (Point of tangency)}$$

$$x + \frac{1}{4a} = -\left(a + \frac{1}{4a}\right) \Rightarrow x = -a - \frac{1}{2a} = -\frac{2a^2 + 1}{2a}$$

The normal line intersects a second time at $x = -\frac{2a^2 + 1}{2a}$.

6. $f(x) = a + b \cos cx$

$$f'(x) = -bc \sin cx$$

At $(0, 1)$: $a + b = 1$ Equation 1

At $\left(\frac{\pi}{4}, \frac{3}{2}\right)$: $a + b \cos\left(\frac{c\pi}{4}\right) = \frac{3}{2}$ Equation 2

$$-bc \sin\left(\frac{c\pi}{4}\right) = 1 \quad \text{Equation 3}$$

From Equation 1, $a = 1 - b$. Equation 2 becomes $(1 - b) + b \cos\left(\frac{c\pi}{4}\right) = \frac{3}{2} \Rightarrow -b + b \cos\frac{c\pi}{4} = \frac{1}{2}$

From Equation 3, $b = \frac{-1}{c \sin\left(\frac{c\pi}{4}\right)}$. Thus $\frac{1}{c \sin\left(\frac{c\pi}{4}\right)} + \frac{-1}{c \sin\left(\frac{c\pi}{4}\right)} \cos\left(\frac{c\pi}{4}\right) = \frac{1}{2}$

$$1 - \cos\left(\frac{c\pi}{4}\right) = \frac{1}{2} c \sin\left(\frac{c\pi}{4}\right)$$

Graphing the equation $g(c) = \frac{1}{2} c \sin\left(\frac{c\pi}{4}\right) + \cos\left(\frac{c\pi}{4}\right) - 1$, you see that many values of c will work.

One answer: $c = 2$, $b = -\frac{1}{2}$, $a = \frac{3}{2} \Rightarrow f(x) = \frac{3}{2} - \frac{1}{2} \cos 2x$

8. (a) $b^2y^2 = x^3(a - x)$; $a, b > 0$

$$y^2 = \frac{x^3(a - x)}{b^2}$$

$$\text{Graph } y_1 = \frac{\sqrt{x^3(a - x)}}{b} \text{ and } y_2 = -\frac{\sqrt{x^3(a - x)}}{b}$$

- (c) Differentiating implicitly.

$$2b^2yy' = 3x^2(a - x) - x^3 = 3ax^2 - 4x^3$$

$$y' = \frac{(3ax^2 - 4x^3)}{2b^2y} = 0$$

$$\Rightarrow 3ax^2 = 4x^3$$

$$3a = 4x$$

$$x = \frac{3a}{4}$$

$$b^2y^2 = \left(\frac{3a}{4}\right)^3 \left(a - \frac{3a}{4}\right) = \frac{27a^3}{64} \left(\frac{1}{4}a\right)$$

$$y^2 = \frac{27a^4}{256b^2} \Rightarrow y = \pm \frac{3\sqrt{3}a^2}{16b}$$

$$\text{Two points: } \left(\frac{3a}{4}, \frac{3\sqrt{3}a^2}{16b}\right), \left(\frac{3a}{4}, -\frac{3\sqrt{3}a^2}{16b}\right)$$

- (b) a determines the x -intercept on the right: $(a, 0)$.

b affects the height.

10. (a) $y = x^{1/3} \Rightarrow \frac{dy}{dt} = \frac{1}{3}x^{-2/3} \frac{dx}{dt}$

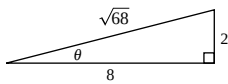
$$1 = \frac{1}{3}(8)^{-2/3} \frac{dx}{dt}$$

$$\frac{dx}{dt} = 12 \text{ cm/sec}$$

- (b) $D = \sqrt{x^2 + y^2} \Rightarrow \frac{dD}{dt} = \frac{1}{2}(x^2 + y^2) \left(2x \frac{dx}{dt} + 2y \frac{dy}{dt}\right)$

$$\begin{aligned} &= \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{\sqrt{x^2 + y^2}} \\ &= \frac{8(12) + 2(1)}{\sqrt{64 + 4}} = \frac{98}{\sqrt{68}} = \frac{49}{\sqrt{17}} \text{ cm/sec.} \end{aligned}$$

- (c) $\tan \theta = \frac{y}{x} \Rightarrow \sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{x \frac{dy}{dt} - y \frac{dx}{dt}}{x^2}$



$$\text{From the triangle, } \sec \theta = \frac{\sqrt{68}}{8}. \text{ Hence } \frac{d\theta}{dt} = \frac{8(1) - 2(12)}{64 \left(\frac{68}{64}\right)} = \frac{-16}{68} = \frac{-4}{17} \text{ rad/sec}$$

$$\begin{aligned} 12. \quad E'(x) &= \lim_{\Delta x \rightarrow 0} \frac{E(x + \Delta x) - E(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{E(x)E(\Delta x) - E(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} E(x) \left(\frac{E(\Delta x) - 1}{\Delta x} \right) \\ &= E(x) \lim_{\Delta x \rightarrow 0} \frac{E(\Delta x) - 1}{\Delta x} \end{aligned}$$

$$\text{But, } E'(0) = \lim_{\Delta x \rightarrow 0} \frac{E(\Delta x) - E(0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{E(\Delta x) - 1}{\Delta x} = 1.$$

Thus, $E'(x) = E(x)E'(0) = E(x)$ exists for all x .

For example: $E(x) = e^x$.

$$14. \quad (a) \quad v(t) = -\frac{27}{5}t + 27 \text{ ft/sec}$$

$$a(t) = -\frac{27}{5} \text{ ft/sec}^2$$

$$(b) \quad v(t) = -\frac{27}{5}t + 27 = 0 \Rightarrow \frac{27}{5}t = 27 \Rightarrow t = 5 \text{ seconds}$$

$$S(5) = -\frac{27}{10}(5)^2 + 27(5) + 6 = 73.5 \text{ feet}$$

(c) The acceleration due to gravity on Earth is greater in magnitude than that on the moon.

CHAPTER 3

Applications of Differentiation

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CHAPTER 3

Applications of Differentiation

Section 3.1 Extrema on an Interval

Solutions to Odd-Numbered Exercises

$$1. f(x) = \frac{x^2}{x^2 + 4}$$

$$f'(x) = \frac{(x^2 + 4)(2x) - (x^2)(2x)}{(x^2 + 4)^2} = \frac{8x}{(x^2 + 4)^2}$$

$$f'(0) = 0$$

$$5. f(x) = (x + 2)^{2/3}$$

$$f'(x) = \frac{2}{3}(x + 2)^{-1/3}$$

$f'(-2)$ is undefined.

$$9. \text{Critical numbers: } x = 1, 2, 3$$

$x = 1, 3$: absolute maximum

$x = 2$: absolute minimum

$$13. g(t) = t\sqrt{4-t}, t < 3$$

$$g'(t) = t\left[\frac{1}{2}(4-t)^{-1/2}(-1)\right] + (4-t)^{1/2}$$

$$= \frac{1}{2}(4-t)^{-1/2}[-t + 2(4-t)]$$

$$= \frac{8-3t}{2\sqrt{4-t}}$$

Critical number is $t = \frac{8}{3}$.

$$17. f(x) = 2(3-x), [-1, 2]$$

$$f'(x) = -2 \Rightarrow \text{No critical numbers}$$

Left endpoint: $(-1, 8)$ Maximum

Right endpoint: $(2, 2)$ Minimum

$$3. f(x) = x + \frac{27}{2x^2} = x + \frac{27}{x}x^{-2}$$

$$f'(x) = 1 - 27x^{-3} = 1 - \frac{27}{x^3}$$

$$f'(3) = 1 - \frac{27}{3^3} = 1 - 1 = 0$$

$$7. \text{Critical numbers: } x = 2$$

$x = 2$: absolute maximum

$$11. f(x) = x^2(x-3) = x^3 - 3x^2$$

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

Critical numbers: $x = 0, x = 2$

$$15. h(x) = \sin^2 x + \cos x, 0 < x < 2\pi$$

$$h'(x) = 2 \sin x \cos x - \sin x = \sin x(2 \cos x - 1)$$

On $(0, 2\pi)$, critical numbers: $x = \frac{\pi}{3}, x = \pi, x = \frac{5\pi}{3}$

$$19. f(x) = -x^2 + 3x, [0, 3]$$

$$f'(x) = -2x + 3$$

Left endpoint: $(0, 0)$ Minimum

Critical number: $\left(\frac{3}{2}, \frac{9}{4}\right)$ Maximum

Right endpoint: $(3, 0)$ Minimum

21. $f(x) = x^3 - \frac{3}{2}x^2, [-1, 2]$

$$f'(x) = 3x^2 - 3x = 3x(x - 1)$$

Left endpoint: $\left(-1, -\frac{5}{2}\right)$ Minimum

Right endpoint: $(2, 2)$ Maximum

Critical number: $(0, 0)$

Critical number: $\left(1, -\frac{1}{2}\right)$

25. $g(t) = \frac{t^2}{t^2 + 3}, [-1, 1]$

$$g'(t) = \frac{6t}{(t^2 + 3)^2}$$

Left endpoint: $\left(-1, \frac{1}{4}\right)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $\left(1, \frac{1}{4}\right)$ Maximum

29. $f(x) = \cos \pi x, \left[0, \frac{1}{6}\right]$

$$f'(x) = -\pi \sin \pi x$$

Left endpoint: $(0, 1)$ Maximum

Right endpoint: $\left(\frac{1}{6}, \frac{\sqrt{3}}{2}\right)$ Minimum

33. (a) Minimum: $(0, -3)$

Maximum: $(2, 1)$

(b) Minimum: $(0, -3)$

(c) Maximum: $(2, 1)$

(d) No extrema

23. $f(x) = 3x^{2/3} - 2x, [-1, 1]$

$$f'(x) = 2x^{-1/3} - 2 = \frac{2(1 - \sqrt[3]{x})}{\sqrt[3]{x}}$$

Left endpoint: $(-1, 5)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $(1, 1)$

27. $h(s) = \frac{1}{s - 2}, [0, 1]$

$$h'(s) = \frac{-1}{(s - 2)^2}$$

Left endpoint: $\left(0, -\frac{1}{2}\right)$ Maximum

Right endpoint: $(1, -1)$ Minimum

31. $y = \frac{4}{x} + \tan \frac{\pi x}{8}, [1, 2]$

$$y' = \frac{-4}{x^2} + \frac{\pi}{8} \sec^2 \frac{\pi x}{8} = 0$$

$$\frac{\pi}{8} \sec^2 \frac{\pi x}{8} = \frac{4}{x^2}$$

On the interval $[1, 2]$, this equation has no solutions. Thus, there are no critical numbers.

Left endpoint: $(1, \sqrt{2} + 3) \approx (1, 4.4142)$ Maximum

Right endpoint: $(2, 3)$ Minimum

35. $f(x) = x^2 - 2x$

(a) Minimum: $(1, -1)$

Maximum: $(-1, 3)$

(b) Maximum: $(3, 3)$

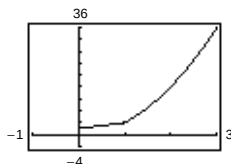
(c) Minimum: $(1, -1)$

(d) Minimum: $(1, -1)$

$$37. f(x) = \begin{cases} 2x + 2, & 0 \leq x \leq 1 \\ 4x^2, & 1 < x \leq 3 \end{cases}$$

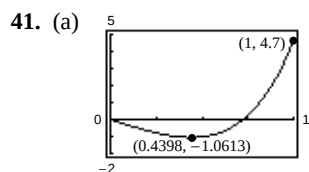
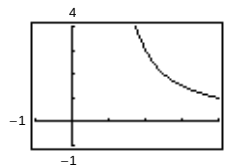
Left endpoint: (0, 2) Minimum

Right endpoint: (3, 36) Maximum



$$39. f(x) = \frac{3}{x-1}, (1, 4]$$

Right endpoint: (4, 1) Minimum



Maximum: (1, 4.7) (endpoint)

Minimum: (0.4398, -1.0613)

(b) $f(x) = 3.2x^5 + 5x^3 - 3.5x, [0, 1]$

$$f'(x) = 16x^4 + 15x^2 - 3.5$$

$$16x^4 + 15x^2 - 3.5 = 0$$

$$x^2 = \frac{-15 \pm \sqrt{(15)^2 - 4(16)(-3.5)}}{2(16)}$$

$$= \frac{-15 \pm \sqrt{449}}{32}$$

$$x = \sqrt{\frac{-15 + \sqrt{449}}{32}} \approx 0.4398$$

$$f(0) = 0$$

$$f(1) = 4.7 \text{ Maximum (endpoint)}$$

$$f\left(\sqrt{\frac{-15 + \sqrt{449}}{32}}\right) \approx -1.0613$$

Minimum: (0.4398, -1.0613)

$$43. f(x) = (1 + x^3)^{1/2}, [0, 2]$$

$$f'(x) = \frac{3}{2}x^2(1 + x^3)^{-1/2}$$

$$f''(x) = \frac{3}{4}(x^4 + 4x)(1 + x^3)^{-3/2}$$

$$f'''(x) = -\frac{3}{8}(x^6 + 20x^3 - 8)(1 + x^3)^{-5/2}$$

Setting $f''' = 0$, we have $x^6 + 20x^3 - 8 = 0$.

$$x^3 = \frac{-20 \pm \sqrt{400 - 4(1)(-8)}}{2}$$

$$x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1$$

In the interval $[0, 2]$, choose

$$x = \sqrt[3]{-10 + \sqrt{108}} = \sqrt{3} - 1 \approx 0.732.$$

$$\left| f''\left(\sqrt[3]{-10 + \sqrt{108}}\right) \right| \approx 1.47 \text{ is the maximum value.}$$

$$45. f(x) = (x + 1)^{2/3}, [0, 2]$$

$$f'(x) = \frac{2}{3}(x + 1)^{-1/3}$$

$$f''(x) = -\frac{2}{9}(x + 1)^{-4/3}$$

$$f'''(x) = \frac{8}{27}(x + 1)^{-7/3}$$

$$f^{(4)}(x) = -\frac{56}{81}(x + 1)^{-10/3}$$

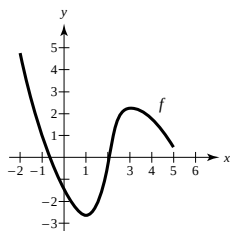
$$f^{(5)}(x) = \frac{560}{243}(x + 1)^{-13/3}$$

$$|f^{(4)}(0)| = \frac{56}{81} \text{ is the maximum value.}$$

47. $f(x) = \tan x$

f is continuous on $[0, \pi/4]$ but not on $[0, \pi]$. $\lim_{x \rightarrow \pi/2^-} \tan x = \infty$.

49.



51. (a) Yes

(b) No

53. (a) No

(b) Yes

55. $P = VI - RI^2 = 12I - 0.5I^2, 0 \leq I \leq 15$

$P = 0$ when $I = 0$.

$P = 67.5$ when $I = 15$.

$P' = 12 - I = 0$

Critical number: $I = 12$ ampsWhen $I = 12$ amps, $P = 72$, the maximum output.

No, a 20-amp fuse would not increase the power output.

 P is decreasing for $I > 12$.

57.

$$S = 6hs + \frac{3s^2}{2} \left(\frac{\sqrt{3} - \cos \theta}{\sin \theta} \right), \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$$

$$\frac{dS}{d\theta} = \frac{3s^2}{2} (-\sqrt{3} \csc \theta \cot \theta + \csc^2 \theta)$$

$$= \frac{3s^2}{2} \csc \theta (-\sqrt{3} \cot \theta + \csc \theta) = 0$$

$\csc \theta = \sqrt{3} \cot \theta$

$\sec \theta = \sqrt{3}$

$\theta = \operatorname{arcsec} \sqrt{3} \approx 0.9553$ radians

$$S\left(\frac{\pi}{6}\right) = 6hs + \frac{3s^2}{2} (\sqrt{3})$$

$$S\left(\frac{\pi}{2}\right) = 6hs + \frac{3s^2}{2} (\sqrt{3})$$

$$S(\operatorname{arcsec} \sqrt{3}) = 6hs + \frac{3s^2}{2} (\sqrt{2})$$

 S is minimum when $\theta = \operatorname{arcsec} \sqrt{3} \approx 0.9553$ radians.

59. (a) $y = ax^2 + bx + c$

$y' = 2ax + b$

The coordinates of B are $(500, 30)$, and those of A are $(-500, 45)$.From the slopes at A and B ,

$-1000a + b = -0.09$

$1000a + b = 0.06$.

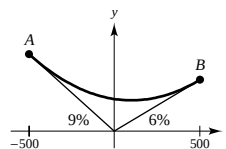
Solving these two equations, you obtain $a = 3/40000$ and $b = -3/200$. From the points $(500, 30)$ and $(-500, 45)$, you obtain

$$30 = \frac{3}{40000} 500^2 + 500 \left(\frac{-3}{200} \right) + c$$

$$45 = \frac{3}{40000} 500^2 - 500 \left(\frac{-3}{200} \right) + c.$$

In both cases, $c = 18.75 = \frac{75}{4}$. Thus,

$$y = \frac{3}{40000} x^2 - \frac{3}{200} x + \frac{75}{4}.$$



—CONTINUED—

59. —CONTINUED—

(b)

x	-500	-400	-300	-200	-100	0	100	200	300	400	500
d	0	.75	3	6.75	12	18.75	12	6.75	3	.75	0

For $-500 \leq x \leq 0$, $d = (ax^2 + bx + c) - (-0.09x)$.For $0 \leq x \leq 500$, $d = (ax^2 + bx + c) - (0.06x)$.

(c) The lowest point on the highway is (100, 18), which is not directly over the point where the two hillsides come together.

61. True. See Exercise 25.

63. True.

Section 3.2 Rolle's Theorem and the Mean Value Theorem

1. Rolle's Theorem does not apply to $f(x) = 1 - |x - 1|$ over $[0, 2]$ since f is not differentiable at $x = 1$.

3. $f(x) = x^2 - x - 2 = (x - 2)(x + 1)$

x -intercepts: $(-1, 0)$, $(2, 0)$

$$f'(x) = 2x - 1 = 0 \text{ at } x = \frac{1}{2}.$$

5. $f(x) = x\sqrt{x+4}$

x -intercepts: $(-4, 0)$, $(0, 0)$

$$f'(x) = x \frac{1}{2}(x+4)^{-1/2} + (x+4)^{1/2}$$

$$= (x+4)^{-1/2} \left(\frac{x}{2} + (x+4) \right)$$

$$f'(x) = \left(\frac{3}{2}x + 4 \right) (x+4)^{-1/2} = 0 \text{ at } x = -\frac{8}{3}$$

7. $f(x) = x^2 - 2x$, $[0, 2]$

$$f(0) = f(2) = 0$$

f is continuous on $[0, 2]$. f is differentiable on $(0, 2)$.
Rolle's Theorem applies.

$$f'(x) = 2x - 2$$

$$2x - 2 = 0 \Rightarrow x = 1$$

c value: 1

9. $f(x) = (x-1)(x-2)(x-3)$, $[1, 3]$

$$f(1) = f(3) = 0$$

f is continuous on $[1, 3]$. f is differentiable on $(1, 3)$.
Rolle's Theorem applies.

$$f(x) = x^3 - 6x^2 + 11x - 6$$

$$f'(x) = 3x^2 - 12x + 11$$

$$3x^2 - 12x + 11 = 0 \Rightarrow x = \frac{6 \pm \sqrt{3}}{3}$$

$$c = \frac{6 - \sqrt{3}}{3}, c = \frac{6 + \sqrt{3}}{3}$$

11. $f(x) = x^{2/3} - 1$, $[-8, 8]$

$$f(-8) = f(8) = 3$$

f is continuous on $[-8, 8]$. f is not differentiable on $(-8, 8)$ since $f'(0)$ does not exist. Rolle's Theorem does not apply.

13. $f(x) = \frac{x^2 - 2x - 3}{x + 2}, [-1, 3]$

$$f(-1) = f(3) = 0$$

f is continuous on $[-1, 3]$. (**Note:** The discontinuity, $x = -2$, is not in the interval.) f is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$f'(x) = \frac{(x+2)(2x-2) - (x^2-2x-3)(1)}{(x+2)^2} = 0$$

$$\frac{x^2 + 4x - 1}{(x+2)^2} = 0$$

$$x = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$$

c value: $-2 + \sqrt{5}$

15. $f(x) = \sin x, [0, 2\pi]$

$$f(0) = f(2\pi) = 0$$

f is continuous on $[0, 2\pi]$. f is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$$f'(x) = \cos x$$

c values: $\frac{\pi}{2}, \frac{3\pi}{2}$

17. $f(x) = \frac{6x}{\pi} - 4 \sin^2 x, \left[0, \frac{\pi}{6}\right]$

$$f(0) = f\left(\frac{\pi}{6}\right) = 0$$

f is continuous on $[0, \pi/6]$. f is differentiable on $(0, \pi/6)$. Rolle's Theorem applies.

$$f'(x) = \frac{6}{\pi} - 8 \sin x \cos x = 0$$

$$\frac{6}{\pi} = 8 \sin x \cos x$$

$$\frac{3}{4\pi} = \frac{1}{2} \sin 2x$$

$$\frac{3}{2\pi} = \sin 2x$$

$$\frac{1}{2} \arcsin\left(\frac{3}{2\pi}\right) = x$$

$$x \approx 0.2489$$

c value: 0.2489

19. $f(x) = \tan x, [0, \pi]$

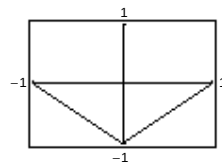
$$f(0) = f(\pi) = 0$$

f is not continuous on $[0, \pi]$ since $f(\pi/2)$ does not exist. Rolle's Theorem does not apply.

21. $f(x) = |x| - 1, [-1, 1]$

$$f(-1) = f(1) = 0$$

f is continuous on $[-1, 1]$. f is not differentiable on $(-1, 1)$ since $f'(0)$ does not exist. Rolle's Theorem does not apply.



$$23. \quad f(x) = 4x - \tan \pi x, \quad \left[-\frac{1}{4}, \frac{1}{4}\right]$$

$$f\left(-\frac{1}{4}\right) = f\left(\frac{1}{4}\right) = 0$$

f is continuous on $[-1/4, 1/4]$. f is differentiable on $(-1/4, 1/4)$. Rolle's Theorem applies.

$$f'(x) = 4 - \pi \sec^2 \pi x = 0$$

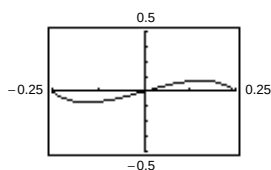
$$\sec^2 \pi x = \frac{4}{\pi}$$

$$\sec \pi x = \pm \frac{2}{\sqrt{\pi}}$$

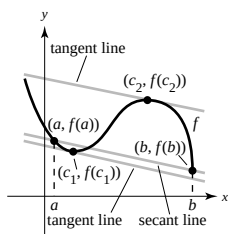
$$x = \pm \frac{1}{\pi} \operatorname{arcsec} \frac{2}{\sqrt{\pi}} = \pm \frac{1}{\pi} \arccos \frac{\sqrt{\pi}}{2}$$

$$\approx \pm 0.1533 \text{ radian}$$

c values: ± 0.1533 radian



27.



31. $f(x) = x^2$ is continuous on $[-2, 1]$ and differentiable on $(-2, 1)$.

$$\frac{f(1) - f(-2)}{1 - (-2)} = \frac{1 - 4}{3} = -1$$

$f'(x) = 2x = -1$ when $x = -\frac{1}{2}$. Therefore,

$$c = -\frac{1}{2}$$

$$25. \quad f(t) = -16t^2 + 48t + 32$$

$$(a) \quad f(1) = f(2) = 64$$

(b) $v = f'(t)$ must be 0 at some time in $(1, 2)$.

$$f'(t) = -32t + 48 = 0$$

$$t = \frac{3}{2} \text{ seconds}$$

$$29. \quad f(x) = \frac{1}{x-3}, \quad [0, 6]$$

f has a discontinuity at $x = 3$.

33. $f(x) = x^{2/3}$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$$\frac{f(1) - f(0)}{1 - 0} = 1$$

$$f'(x) = \frac{2}{3}x^{-1/3} = 1$$

$$x = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

$$c = \frac{8}{27}$$

35. $f(x) = \sqrt{2-x}$ is continuous on $[-7, 2]$ and differentiable on $(-7, 2)$.

$$\frac{f(2) - f(-7)}{2 - (-7)} = \frac{0 - 3}{9} = -\frac{1}{3}$$

$$f'(x) = \frac{-1}{2\sqrt{2-x}} = -\frac{1}{3}$$

$$2\sqrt{2-x} = 3$$

$$\sqrt{2-x} = \frac{3}{2}$$

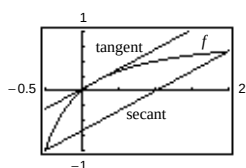
$$2-x = \frac{9}{4}$$

$$x = -\frac{1}{4}$$

$$c = -\frac{1}{4}$$

39. $f(x) = \frac{x}{x+1}$ on $\left[-\frac{1}{2}, 2\right]$.

(a)



(b) Secant line:

$$\text{slope} = \frac{f(2) - f(-1/2)}{2 - (-1/2)} = \frac{2/3 - (-1)}{5/2} = \frac{2}{3}$$

$$y - \frac{2}{3} = \frac{2}{3}(x - 2)$$

$$3y - 2 = 2x - 4$$

$$3y - 2x + 2 = 0$$

37. $f(x) = \sin x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

$$f'(x) = \cos x = 0$$

$$c = \frac{\pi}{2}$$

$$(c) f'(x) = \frac{1}{(x+1)^2} = \frac{2}{3}$$

$$(x+1)^2 = \frac{3}{2}$$

$$x = -1 \pm \sqrt{\frac{3}{2}} = -1 \pm \frac{\sqrt{6}}{2}$$

In the interval $[-1/2, 2]$, $c = -1 + (\sqrt{6}/2)$.

$$f(c) = \frac{-1 + (\sqrt{6}/2)}{[-1 + (\sqrt{6}/2)] + 1} = \frac{-2 + \sqrt{6}}{\sqrt{6}} = \frac{-2}{\sqrt{6}} + 1$$

$$\text{Tangent line: } y - 1 + \frac{2}{\sqrt{6}} = \frac{2}{3}\left(x - \frac{\sqrt{6}}{2} + 1\right)$$

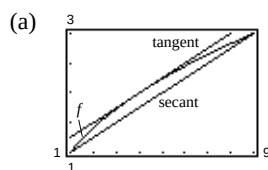
$$y - 1 + \frac{\sqrt{6}}{3} = \frac{2}{3}x - \frac{\sqrt{6}}{3} + \frac{2}{3}$$

$$3y - 2x - 5 + 2\sqrt{6} = 0$$

41. $f(x) = \sqrt{x}$, $[1, 9]$

$(1, 1), (9, 3)$

$$m = \frac{3 - 1}{9 - 1} = \frac{1}{4}$$



(b) Secant line: $y - 1 = \frac{1}{4}(x - 1)$

$$y = \frac{1}{4}x + \frac{3}{4}$$

$$0 = x - 4y + 3$$

(c) $f'(x) = \frac{1}{2\sqrt{x}}$

$$\frac{f(9) - f(1)}{9 - 1} = \frac{1}{4}$$

$$\frac{1}{2\sqrt{c}} = \frac{1}{4}$$

$$\sqrt{c} = 2$$

$$c = 4$$

$$(c, f(c)) = (4, 2)$$

$$m = f'(4) = \frac{1}{4}$$

Tangent line: $y - 2 = \frac{1}{4}(x - 4)$

$$y = \frac{1}{4}x + 1$$

$$0 = x - 4y + 4$$

43. $s(t) = -4.9t^2 + 500$

(a) $V_{\text{avg}} = \frac{s(3) - s(0)}{3 - 0} = \frac{455.9 - 500}{3} = -14.7 \text{ m/sec}$

(b) $s(t)$ is continuous on $[0, 3]$ and differentiable on $(0, 3)$.
Therefore, the Mean Value Theorem applies.

$$v(t) = s'(t) = -9.8t = -14.7 \text{ m/sec}$$

$$t = \frac{-14.7}{-9.8} = 1.5 \text{ seconds}$$

45. No. Let $f(x) = x^2$ on $[-1, 2]$.

$$f'(x) = 2x$$

$f'(0) = 0$ and zero is in the interval $(-1, 2)$ but
 $f(-1) \neq f(2)$.

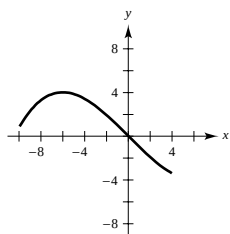
47. Let $S(t)$ be the position function of the plane. If $t = 0$ corresponds to 2 P.M., $S(0) = 0$, $S(5.5) = 2500$ and the Mean Value Theorem says that there exists a time t_0 , $0 < t_0 < 5.5$, such that

$$S'(t_0) = v(t_0) = \frac{2500 - 0}{5.5 - 0} \approx 454.54.$$

Applying the Intermediate Value Theorem to the velocity function on the intervals $[0, t_0]$ and $[t_0, 5.5]$, you see that there are at least two times during the flight when the speed was 400 miles per hour. ($0 < 400 < 454.54$)

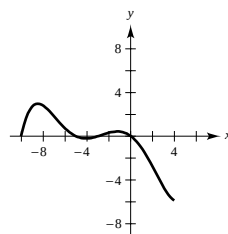
49. (a) f is continuous on $[-10, 4]$ and changes sign, ($f(-8) > 0$, $f(3) < 0$). By the Intermediate Value Theorem, there exists at least one value of x in $[-10, 4]$ satisfying $f(x) = 0$.

(c)



- (b) There exist real numbers a and b such that $-10 < a < b < 4$ and $f(a) = f(b) = 2$. Therefore, by Rolle's Theorem there exists at least one number c in $(-10, 4)$ such that $f'(c) = 0$. This is called a critical number.

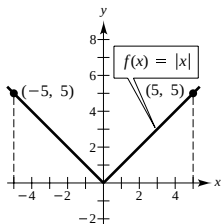
(d)



(e) No, f' did not have to be continuous on $[-10, 4]$.

51. f is continuous on $[-5, 5]$ and does not satisfy the conditions of the Mean Value Theorem.
 $\Rightarrow f$ is not differentiable on $(-5, 5)$.

Example: $f(x) = |x|$



53. False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

55. True. A polynomial is continuous and differentiable everywhere.

57. Suppose that $p(x) = x^{2n+1} + ax + b$ has two real roots x_1 and x_2 . Then by Rolle's Theorem, since $p(x_1) = p(x_2) = 0$, there exists c in (x_1, x_2) such that $p'(c) = 0$. But $p'(x) = (2n+1)x^{2n} + a \neq 0$, since $n > 0$, $a > 0$. Therefore, $p(x)$ cannot have two real roots.

59. If $p(x) = Ax^2 + Bx + C$, then

$$\begin{aligned} p'(x) &= 2Ax + B = \frac{f(b) - f(a)}{b - a} = \frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a} \\ &= \frac{A(b^2 - a^2) + B(b - a)}{b - a} \\ &= \frac{(b - a)[A(b + a) + B]}{b - a} \\ &= A(b + a) + B. \end{aligned}$$

Thus, $2Ax = A(b + a)$ and $x = (b + a)/2$ which is the midpoint of $[a, b]$.

61. $f(x) = \frac{1}{2} \cos x$ differentiable on $(-\infty, \infty)$.

$$f'(x) = -\frac{1}{2} \sin x$$

$$-\frac{1}{2} \leq f'(x) \leq \frac{1}{2} \Rightarrow f'(x) < 1 \text{ for all real numbers.}$$

Thus, from Exercise 60, f has, at most, one fixed point. ($x \approx 0.4502$)

Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

1. $f(x) = x^2 - 6x + 8$

Increasing on: $(3, \infty)$

Decreasing on: $(-\infty, 3)$

5. $f(x) = \frac{1}{x^2} = x^{-2}$

$$f'(x) = \frac{-2}{x^3}$$

Discontinuity: $x = 0$

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on $(-\infty, 0)$

Decreasing on $(0, \infty)$

9. $y = x\sqrt{16 - x^2}$ Domain: $[-4, 4]$

$$y' = \frac{-2(x^2 - 8)}{\sqrt{16 - x^2}} = \frac{-2}{\sqrt{16 - x^2}}(x - 2\sqrt{2})(x + 2\sqrt{2})$$

Critical numbers: $x = \pm 2\sqrt{2}$

Test intervals:	$-4 < x < -2\sqrt{2}$	$-2\sqrt{2} < x < 2\sqrt{2}$	$2\sqrt{2} < x < 4$
Sign of y' :	$y' < 0$	$y' > 0$	$y' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on $(-2\sqrt{2}, 2\sqrt{2})$

Decreasing on $(-4, -2\sqrt{2}), (2\sqrt{2}, 4)$

11. $f(x) = x^2 - 6x$

$$f'(x) = 2x - 6 = 0$$

Critical number: $x = 3$

Test intervals:	$-\infty < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(3, \infty)$

Decreasing on: $(-\infty, 3)$

Relative minimum: $(3, -9)$

3. $y = \frac{x^3}{4} - 3x$

Increasing on: $(-\infty, -2), (2, \infty)$

Decreasing on: $(-2, 2)$

7. $g(x) = x^2 - 2x - 8$

$$g'(x) = 2x - 2$$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $g'(x)$:	$g' < 0$	$g' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(1, \infty)$

Decreasing on: $(-\infty, 1)$

13. $f(x) = -2x^2 + 4x + 3$

$$f'(x) = -4x + 4 = 0$$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, 1)$

Decreasing on: $(1, \infty)$

Relative maximum: $(1, 5)$

15. $f(x) = 2x^3 + 3x^2 - 12x$

$$f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1) = 0$$

Critical numbers: $x = -2, 1$

Test intervals:	$-\infty < x < -2$	$-2 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -2), (1, \infty)$

Decreasing on: $(-2, 1)$

Relative maximum: $(-2, 20)$

Relative minimum: $(1, -7)$

17. $f(x) = x^2(3 - x) = 3x^2 - x^3$

$$f'(x) = 6x - 3x^2 = 3x(2 - x)$$

Critical numbers: $x = 0, 2$

Test intervals:	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(0, 2)$

Decreasing on: $(-\infty, 0), (2, \infty)$

Relative maximum: $(2, 4)$

Relative minimum: $(0, 0)$

19. $f(x) = \frac{x^5 - 5x}{5}$

$$f'(x) = x^4 - 1$$

Critical numbers: $x = -1, 1$

Test intervals:	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -1), (1, \infty)$

Decreasing on: $(-1, 1)$

Relative maximum: $(-1, \frac{4}{5})$

Relative minimum: $(1, -\frac{4}{5})$

21. $f(x) = x^{1/3} + 1$

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

 Critical number: $x = 0$

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

 Increasing on: $(-\infty, \infty)$

No relative extrema

23. $f(x) = (x - 1)^{2/3}$

$$f'(x) = \frac{2}{3(x - 1)^{1/3}}$$

 Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

 Increasing on: $(1, \infty)$

 Decreasing on: $(-\infty, 1)$

 Relative minimum: $(1, 0)$

25. $f(x) = 5 - |x - 5|$

$$f'(x) = -\frac{x - 5}{|x - 5|} = \begin{cases} 1, & x < 5 \\ -1, & x > 5 \end{cases}$$

 Critical number: $x = 5$

Test intervals:	$-\infty < x < 5$	$5 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

 Increasing on: $(-\infty, 5)$

 Decreasing on: $(5, \infty)$

 Relative maximum: $(5, 5)$

27. $f(x) = x + \frac{1}{x}$

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

 Critical numbers: $x = -1, 1$

 Discontinuity: $x = 0$

Test intervals:	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

 Increasing on: $(-\infty, -1), (1, \infty)$

 Decreasing on: $(-1, 0), (0, 1)$

 Relative maximum: $(-1, -2)$

 Relative minimum: $(1, 2)$

29. $f(x) = \frac{x^2}{x^2 - 9}$

$$f'(x) = \frac{(x^2 - 9)(2x) - (x^2)(2x)}{(x^2 - 9)^2} = \frac{-18x}{(x^2 - 9)^2}$$

Critical number: $x = 0$

Discontinuities: $x = -3, 3$

Test intervals:	$-\infty < x < -3$	$-3 < x < 0$	$0 < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$	$f' < 0$	$f' < 0$
Conclusion:	Increasing	Increasing	Decreasing	Decreasing

Increasing on: $(-\infty, -3), (-3, 0)$

Decreasing on: $(0, 3), (3, \infty)$

Relative maximum: $(0, 0)$

31. $f(x) = \frac{x^2 - 2x + 1}{x + 1}$

$$f'(x) = \frac{(x + 1)(2x - 2) - (x^2 - 2x + 1)(1)}{(x + 1)^2} = \frac{x^2 + 2x - 3}{(x + 1)^2} = \frac{(x + 3)(x - 1)}{(x + 1)^2}$$

Critical numbers: $x = -3, 1$

Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -3$	$-3 < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on: $(-\infty, -3), (1, \infty)$

Decreasing on: $(-3, -1), (-1, 1)$

Relative maximum: $(-3, -8)$

Relative minimum: $(1, 0)$

33. $f(x) = \frac{x}{2} + \cos x, 0 < x < 2\pi$

$$f'(x) = \frac{1}{2} - \sin x = 0$$

Critical numbers: $x = \frac{\pi}{6}, \frac{5\pi}{6}$

Test intervals:	$0 < x < \frac{\pi}{6}$	$\frac{\pi}{6} < x < \frac{5\pi}{6}$	$\frac{5\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{6}\right), \left(\frac{5\pi}{6}, 2\pi\right)$

Relative maximum: $\left(\frac{\pi}{6}, \frac{\pi + 6\sqrt{3}}{12}\right)$

Decreasing on: $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$

Relative minimum: $\left(\frac{5\pi}{6}, \frac{5\pi - 6\sqrt{3}}{12}\right)$

35. $f(x) = \sin^2 x + \sin x, 0 < x < 2\pi$

$$f'(x) = 2 \sin x \cos x + \cos x = \cos x(2 \sin x + 1) = 0$$

Critical numbers: $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{7\pi}{6}$	$\frac{7\pi}{6} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < \frac{11\pi}{6}$	$\frac{11\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{2}\right), \left(\frac{7\pi}{6}, \frac{3\pi}{2}\right), \left(\frac{11\pi}{6}, 2\pi\right)$

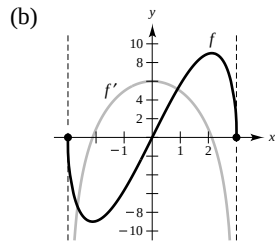
Decreasing on: $\left(\frac{\pi}{2}, \frac{7\pi}{6}\right), \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)$

Relative minima: $\left(\frac{7\pi}{6}, -\frac{1}{4}\right), \left(\frac{11\pi}{6}, -\frac{1}{4}\right)$

Relative maxima: $\left(\frac{\pi}{2}, 2\right), \left(\frac{3\pi}{2}, 0\right)$

37. $f(x) = 2x\sqrt{9-x^2}, [-3, 3]$

(a) $f'(x) = \frac{2(9-2x^2)}{\sqrt{9-x^2}}$



(c) $\frac{2(9-2x^2)}{\sqrt{9-x^2}} = 0$

Critical numbers: $x = \pm \frac{3}{\sqrt{2}} = \pm \frac{3\sqrt{2}}{2}$

(d) Intervals:

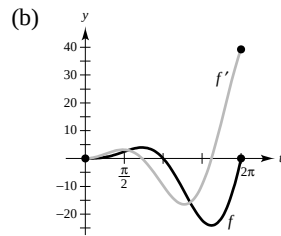
$$\left(-3, -\frac{3\sqrt{2}}{2}\right) \quad \left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right) \quad \left(\frac{3\sqrt{2}}{2}, 3\right)$$

$$\begin{array}{ccc} f'(x) < 0 & f'(x) > 0 & f'(x) < 0 \\ \text{Decreasing} & \text{Increasing} & \text{Decreasing} \end{array}$$

f is increasing when f' is positive and decreasing when f' is negative.

39. $f(t) = t^2 \sin t, [0, 2\pi]$

(a) $f'(t) = t^2 \cos t + 2t \sin t$
 $= t(t \cos t + 2 \sin t)$



(c) $t(t \cos t + 2 \sin t) = 0$

$$t = 0 \text{ or } t = -2 \tan t$$

$$t \cot t = -2$$

$$t \approx 2.2889, 5.0870 \text{ (graphing utility)}$$

Critical numbers: $t = 2.2889, t = 5.0870$

(d) Intervals:

$$(0, 2.2889) \quad (2.2889, 5.0870) \quad (5.0870, 2\pi)$$

$$\begin{array}{ccc} f'(t) > 0 & f'(t) < 0 & f'(t) > 0 \\ \text{Increasing} & \text{Decreasing} & \text{Increasing} \end{array}$$

f is increasing when f' is positive and decreasing when f' is negative.

$$41. f(x) = \frac{x^5 - 4x^3 + 3x}{x^2 - 1} = \frac{(x^2 - 1)(x^3 - 3x)}{x^2 - 1} = x^3 - 3x, x \neq \pm 1$$

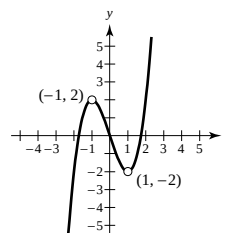
$$f(x) = g(x) = x^3 - 3x \text{ for all } x \neq \pm 1.$$

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1), x \neq \pm 1 \quad f'(x) \neq 0$$

f symmetric about origin

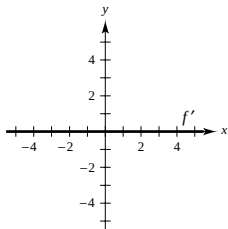
zeros of f : $(0, 0), (\pm\sqrt{3}, 0)$

No relative extrema

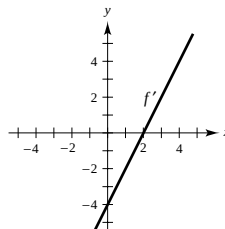


Holes at $(-1, 2)$ and $(1, -2)$

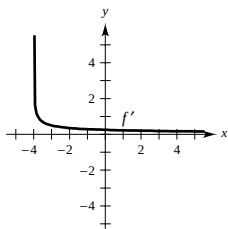
$$43. f(x) = c \text{ is constant} \Rightarrow f'(x) = 0$$



$$45. f \text{ is quadratic} \Rightarrow f' \text{ is a line.}$$



$$47. f \text{ has positive, but decreasing slope}$$



In Exercises 49–53, $f'(x) > 0$ on $(-\infty, -4)$, $f'(x) < 0$ on $(-4, 6)$ and $f'(x) > 0$ on $(6, \infty)$.

$$49. g(x) = f(x) + 5$$

$$g'(x) = f'(x)$$

$$g'(0) = f'(0) < 0$$

$$51. g(x) = -f(x)$$

$$g'(x) = -f'(x)$$

$$g'(-6) = -f'(-6) < 0$$

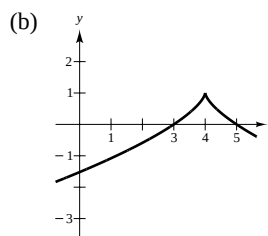
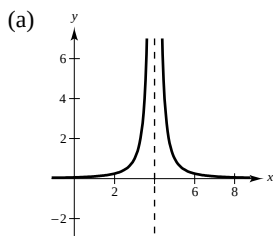
$$53. g(x) = f(x - 10)$$

$$g'(x) = f'(x - 10)$$

$$g'(0) = f'(-10) > 0$$

$$55. f'(x) = \begin{cases} > 0, & x < 4 \Rightarrow f \text{ is increasing on } (-\infty, 4). \\ \text{undefined}, & x = 4 \\ < 0, & x > 4 \Rightarrow f \text{ is decreasing on } (4, \infty). \end{cases}$$

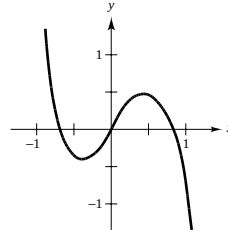
Two possibilities for $f(x)$ are given below.



57. The critical numbers are in intervals $(-0.50, -0.25)$ and $(0.25, 0.50)$ since the sign of f' changes in these intervals. f is decreasing on approximately $(-1, -0.40)$, $(0.48, 1)$, and increasing on $(-0.40, 0.48)$.

Relative minimum when $x \approx -0.40$.

Relative maximum when $x \approx 0.48$.

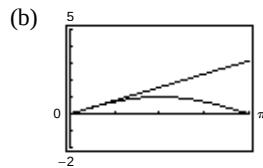


59. $f(x) = x$, $g(x) = \sin x$, $0 < x < \pi$

(a)

x	0.5	1	1.5	2	2.5	3
$f(x)$	0.5	1	1.5	2	2.5	3
$g(x)$	0.479	0.841	0.997	0.909	0.598	0.141

$f(x)$ seems greater than $g(x)$ on $(0, \pi)$.



$x > \sin x$ on $(0, \pi)$

- (c) Let $h(x) = f(x) - g(x) = x - \sin x$

$$h'(x) = 1 - \cos x > 0 \text{ on } (0, \pi).$$

Therefore, $h(x)$ is increasing on $(0, \pi)$. Since $h(0) = 0$, $h(x) > 0$ on $(0, \pi)$. Thus,

$$x - \sin x > 0$$

$$x > \sin x$$

$$f(x) > g(x) \text{ on } (0, \pi).$$

61. $v = k(R - r)r^2 = k(Rr^2 - r^3)$

$$v' = k(2Rr - 3r^2)$$

$$= kr(2R - 3r) = 0$$

$$r = 0 \text{ or } \frac{2}{3}R$$

Maximum when $r = \frac{2}{3}R$.

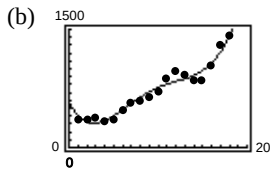
63. $P = \frac{vR_1R_2}{(R_1 + R_2)^2}$, v and R_1 are constant

$$\frac{dP}{dR_2} = \frac{(R_1 + R_2)^2(vR_1) - vR_1R_2[2(R_1 + R_2)(1)]}{(R_1 + R_2)^4}$$

$$= \frac{vR_1(R_1 - R_2)}{(R_1 + R_2)^3} = 0 \Rightarrow R_2 = R_1$$

Maximum when $R_1 = R_2$.

65. (a) $B = 0.1198t^4 - 4.4879t^3 + 56.9909t^2 - 223.0222t + 579.9541$



- (c) $B' = 0$ for $t \approx 2.78$, or 1983, (311.1 thousand bankruptcies)
Actual minimum: 1984 (344.3 thousand bankruptcies)

67. (a) Use a cubic polynomial
 $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.

(b) $f'(x) = 3a_3x^2 + 2a_2x + a_1$.

$(0, 0): \quad 0 = a_0 \quad (f(0) = 0)$

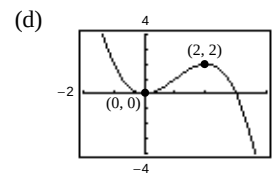
$0 = a_1 \quad (f'(0) = 0)$

$(2, 2): \quad 2 = 8a_3 + 4a_2 \quad (f(2) = 2)$

$0 = 12a_3 + 4a_2 \quad (f'(2) = 0)$

- (c) The solution is $a_0 = a_1 = 0$, $a_2 = \frac{3}{2}$, $a_3 = -\frac{1}{2}$:

$$f(x) = -\frac{1}{2}x^3 + \frac{3}{2}x^2.$$



69. (a) Use a fourth degree polynomial $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

$(0, 0): 0 = a_0 \quad (f(0) = 0)$

$0 = a_1 \quad (f'(0) = 0)$

$(4, 0): 0 = 256a_4 + 64a_3 + 16a_2 \quad (f(4) = 0)$

$0 = 256a_4 + 48a_3 + 8a_2 \quad (f'(4) = 0)$

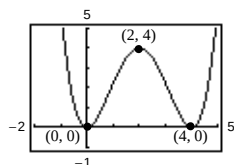
$(2, 4): 4 = 16a_4 + 8a_3 + 4a_2 \quad (f(2) = 4)$

$0 = 32a_4 + 12a_3 + 4a_2 \quad (f'(2) = 0)$

(c) The solution is $a_0 = a_1 = 0, a_2 = 4, a_3 = -2, a_4 = \frac{1}{4}$.

$$f(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2$$

(d)



71. True

Let $h(x) = f(x) + g(x)$ where f and g are increasing. Then $h'(x) = f'(x) + g'(x) > 0$ since $f'(x) > 0$ and $g'(x) > 0$.

73. False

Let $f(x) = x^3$, then $f'(x) = 3x^2$ and f only has one critical number. Or, let $f(x) = x^3 + 3x + 1$, then $f'(x) = 3(x^2 + 1)$ has no critical numbers.

75. False. For example, $f(x) = x^3$ does not have a relative extrema at the critical number $x = 0$.

77. Assume that $f'(x) < 0$ for all x in the interval (a, b) and let $x_1 < x_2$ be any two points in the interval. By the Mean Value Theorem, we know there exists a number c such that $x_1 < c < x_2$, and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(c) < 0$ and $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) < 0$, which implies that $f(x_2) < f(x_1)$. Thus, f is decreasing on the interval.

79. Let $f(x) = (1 + x)^n - nx - 1$. Then

$$\begin{aligned} f'(x) &= n(1 + x)^{n-1} - n \\ &= n[(1 + x)^{n-1} - 1] > 0 \text{ since } x > 0 \text{ and } n > 1. \end{aligned}$$

Thus, $f(x)$ is increasing on $(0, \infty)$. Since $f(0) = 0 \Rightarrow f(x) > 0$ on $(0, \infty)$

$$(1 + x)^n - nx - 1 > 0 \Rightarrow (1 + x)^n > 1 + nx.$$

Section 3.4 Concavity and the Second Derivative Test

1. $y = x^2 - x - 2, y'' = 2$

Concave upward: $(-\infty, \infty)$

3. $f(x) = \frac{24}{x^2 + 12}, y'' = \frac{-144(4 - x^2)}{(x^2 + 12)^3}$

Concave upward: $(-\infty, -2), (2, \infty)$

Concave downward: $(-2, 2)$

5. $f(x) = \frac{x^2 + 1}{x^2 - 1}, y'' = \frac{4(3x^2 + 1)}{(x^2 - 1)^3}$

Concave upward: $(-\infty, -1), (1, \infty)$

Concave downward: $(-1, 1)$

7. $f(x) = 3x^2 - x^3$

$f'(x) = 6x - 3x^2$

$f''(x) = 6 - 6x$

Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

9. $y = 2x - \tan x, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$y' = 2 - \sec^2 x$

$y'' = -2 \sec^2 x \tan x$

Concave upward: $\left(-\frac{\pi}{2}, 0\right)$

Concave downward: $\left(0, \frac{\pi}{2}\right)$

11. $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12$

$f''(x) = 6(x - 2) = 0$ when $x = 2$.

The concavity changes at $x = 2$. $(2, 8)$ is a point of inflection.

Concave upward: $(2, \infty)$

Concave downward: $(-\infty, 2)$

13. $f(x) = \frac{1}{4}x^4 - 2x^2$

$f'(x) = x^3 - 4x$

$f''(x) = 3x^2 - 4$

$f''(x) = 3x^2 - 4 = 0$ when $x = \pm \frac{2}{\sqrt{3}}$.

Test interval:	$-\infty < x < -\frac{2}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}} < x < \infty$
Sign of $f''(x)$:	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Points of inflection: $\left(\pm \frac{2}{\sqrt{3}}, -\frac{20}{9}\right)$

15. $f(x) = x(x - 4)^3$

$$f'(x) = x[3(x - 4)^2] + (x - 4)^3$$

$$= (x - 4)^2(4x - 4)$$

$$f''(x) = 4(x - 1)[2(x - 4)] + 4(x - 4)^2$$

$$= 4(x - 4)[2(x - 1) + (x - 4)]$$

$$= 4(x - 4)(3x - 6) = 12(x - 4)(x - 2)$$

$$f''(x) = 12(x - 4)(x - 2) = 0 \text{ when } x = 2, 4.$$

Test interval:	$-\infty < x < 2$	$2 < x < 4$	$4 < x < \infty$
Sign of $f''(x)$:	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Points of inflection: $(2, -16), (4, 0)$

17. $f(x) = x\sqrt{x+3}$, Domain: $[-3, \infty)$

$$f'(x) = x\left(\frac{1}{2}\right)(x+3)^{-1/2} + \sqrt{x+3} = \frac{3(x+2)}{2\sqrt{x+3}}$$

$$f''(x) = \frac{6\sqrt{x+3} - 3(x+2)(x+3)^{-1/2}}{4(x+3)} = \frac{3(x+4)}{4(x+3)^{3/2}}$$

 $f''(x) > 0$ on the entire domain of f (except for $x = -3$, for which $f''(x)$ is undefined). There are no points of inflection.Concave upward on $(-3, \infty)$

19. $f(x) = \frac{x}{x^2 + 1}$

$$f'(x) = \frac{1 - x^2}{(x^2 + 1)^2}$$

$$f''(x) = \frac{2x(x^2 - 3)}{(x^2 + 1)^3} = 0 \text{ when } x = 0, \pm\sqrt{3}$$

Test intervals:	$-\infty < x < -\sqrt{3}$	$-\sqrt{3} < x < 0$	$0 < x < \sqrt{3}$	$\sqrt{3} < x < \infty$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward	Concave downward	Concave upward

Points of inflection: $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), (0, 0), \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$

21. $f(x) = \sin\left(\frac{x}{2}\right), 0 \leq x \leq 4\pi$

$$f'(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right)$$

$$f''(x) = -\frac{1}{4}\sin\left(\frac{x}{2}\right)$$

$$f''(x) = 0 \text{ when } x = 0, 2\pi, 4\pi.$$

Point of inflection: $(2\pi, 0)$

Test interval:	$0 < x < 2\pi$	$2\pi < x < 4\pi$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward

23. $f(x) = \sec\left(x - \frac{\pi}{2}\right), 0 < x < 4\pi$

$$f'(x) = \sec\left(x - \frac{\pi}{2}\right) \tan\left(x - \frac{\pi}{2}\right)$$

$$f''(x) = \sec^3\left(x - \frac{\pi}{2}\right) + \sec\left(x - \frac{\pi}{2}\right) \tan^2\left(x - \frac{\pi}{2}\right) \neq 0 \text{ for any } x \text{ in the domain of } f.$$

Concave upward: $(0, \pi), (2\pi, 3\pi)$

Concave downward: $(\pi, 2\pi), (3\pi, 4\pi)$

No points of inflection

25. $f(x) = 2 \sin x + \sin 2x, 0 \leq x \leq 2\pi$

$$f'(x) = 2 \cos x + 2 \cos 2x$$

$$f''(x) = -2 \sin x - 4 \sin 2x = -2 \sin x(1 + 4 \cos x)$$

$$f''(x) = 0 \text{ when } x = 0, 1.823, \pi, 4.460.$$

Test interval:	$0 < x < 1.823$	$1.823 < x < \pi$	$\pi < x < 4.460$	$4.460 < x < 2\pi$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward	Concave downward	Concave upward

Points of inflection: $(1.823, 1.452), (\pi, 0), (4.46, -1.452)$

27. $f(x) = x^4 - 4x^3 + 2$

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

Critical numbers: $x = 0, x = 3$

However, $f''(0) = 0$, so we must use the First Derivative Test. $f'(x) < 0$ on the intervals $(-\infty, 0)$ and $(0, 3)$; hence, $(0, 2)$ is not an extremum. $f''(3) > 0$ so $(3, -25)$ is a relative minimum.

31. $f(x) = x^3 - 3x^2 + 3$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

$$f''(x) = 6x - 6 = 6(x - 1)$$

Critical numbers: $x = 0, x = 2$

$$f''(0) = -6 < 0$$

Therefore, $(0, 3)$ is a relative maximum.

$$f''(2) = 6 > 0$$

Therefore, $(2, -1)$ is a relative minimum.

29. $f(x) = (x - 5)^2$

$$f'(x) = 2(x - 5)$$

$$f''(x) = 2$$

Critical number: $x = 5$

$$f''(5) > 0$$

Therefore, $(5, 0)$ is a relative minimum.

33. $g(x) = x^2(6 - x)^3$

$$g'(x) = x(x - 6)^2(12 - 5x)$$

$$g''(x) = 4(6 - x)(5x^2 - 24x + 18)$$

Critical numbers: $x = 0, \frac{12}{5}, 6$

$$g''(0) = 432 > 0$$

Therefore, $(0, 0)$ is a relative minimum.

$$g''\left(\frac{12}{5}\right) = -155.52 < 0$$

Therefore, $\left(\frac{12}{5}, 268.7\right)$ is a relative minimum.

$$g''(6) = 0$$

Test fails by the First Derivative Test, $(6, 0)$ is not an extremum.

35. $f(x) = x^{2/3} - 3$

$$f'(x) = \frac{2}{3x^{1/3}}$$

$$f''(x) = \frac{-2}{9x^{4/3}}$$

Critical number: $x = 0$

However, $f''(0)$ is undefined, so we must use the First Derivative Test. Since $f'(x) < 0$ on $(-\infty, 0)$ and $f'(x) > 0$ on $(0, \infty)$, $(0, -3)$ is a relative minimum.

39. $f(x) = \cos x - x$, $0 \leq x \leq 4\pi$

$$f'(x) = -\sin x - 1 \leq 0$$

Therefore, f is non-increasing and there are no relative extrema.

41. $f(x) = 0.2x^2(x - 3)^3$, $[-1, 4]$

(a) $f'(x) = 0.2x(5x - 6)(x - 3)^2$

$$\begin{aligned} f''(x) &= (x - 3)(4x^2 - 9.6x + 3.6) \\ &= 0.4(x - 3)(10x^2 - 24x + 9) \end{aligned}$$

(b) $f''(0) < 0 \Rightarrow (0, 0)$ is a relative maximum.

$$f''\left(\frac{6}{5}\right) > 0 \Rightarrow (1.2, -1.6796)$$
 is a relative minimum.

Points of inflection:

$$(3, 0), (0.4652, -0.7049), (1.9348, -0.9049)$$

43. $f(x) = \sin x - \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x$, $[0, \pi]$

(a) $f'(x) = \cos x - \cos 3x + \cos 5x$

$$f'(x) = 0 \text{ when } x = \frac{\pi}{6}, x = \frac{\pi}{2}, x = \frac{5\pi}{6}.$$

$$f''(x) = -\sin x + 3\sin 3x - 5\sin 5x$$

$$f''(x) = 0 \text{ when } x = \frac{\pi}{6}, x = \frac{5\pi}{6}, x \approx 1.1731, x \approx 1.9685$$

(b) $f''\left(\frac{\pi}{2}\right) < 0 \Rightarrow \left(\frac{\pi}{2}, 1.53333\right)$ is a relative maximum.

Points of inflection: $\left(\frac{\pi}{6}, 0.2667\right), (1.1731, 0.9638),$

$$(1.9685, 0.9637), \left(\frac{5\pi}{6}, 0.2667\right)$$

Note: $(0, 0)$ and $(\pi, 0)$ are not points of inflection since they are endpoints.

37. $f(x) = x + \frac{4}{x}$

$$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2}$$

$$f''(x) = \frac{8}{x^3}$$

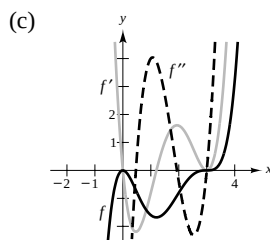
Critical numbers: $x = \pm 2$

$$f''(-2) < 0$$

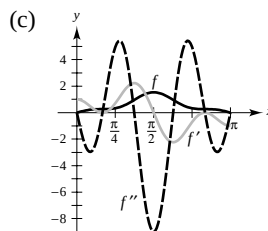
Therefore, $(-2, -4)$ is a relative maximum.

$$f''(2) > 0$$

Therefore, $(2, 4)$ is a relative minimum.

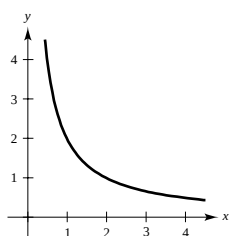


f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.



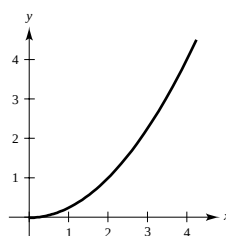
The graph of f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

45. (a)


 $f' < 0$ means f decreasing

 f' increasing means
concave upward

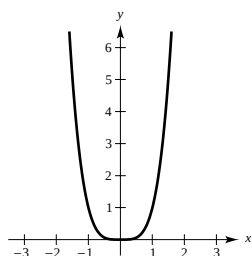
(b)


 $f' > 0$ means f increasing

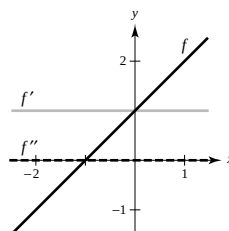
 f' increasing means
concave upward

 47. Let $f(x) = x^4$.

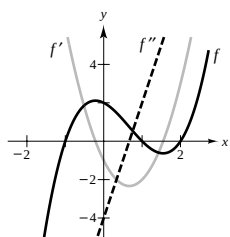
$$f''(x) = 12x^2$$

 $f''(0) = 0$, but $(0, 0)$ is not a point of inflection.


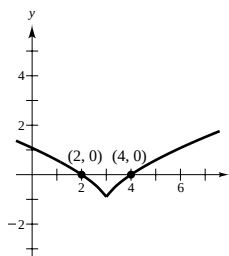
49.



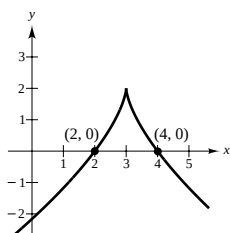
51.



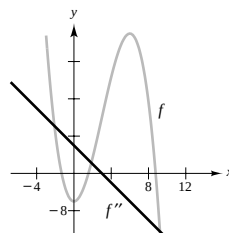
53.



55.



57.


 f'' is linear.

 f' is quadratic.

 f is cubic.

 f concave upwards on $(-\infty, 3)$, downward on $(3, \infty)$.

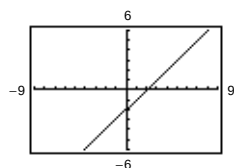
59. (a) $n = 1$:

$$f(x) = x - 2$$

$$f'(x) = 1$$

$$f''(x) = 0$$

No inflection points

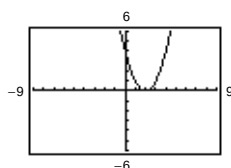
 $n = 2$:

$$f(x) = (x - 2)^2$$

$$f'(x) = 2(x - 2)$$

$$f''(x) = 2$$

No inflection points

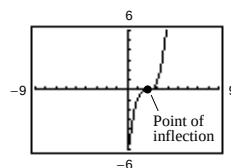
Relative minimum:
(2, 0) $n = 3$:

$$f(x) = (x - 2)^3$$

$$f'(x) = 3(x - 2)^2$$

$$f''(x) = 6(x - 2)$$

Inflection point: (2, 0)

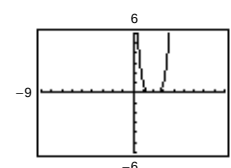
 $n = 4$:

$$f(x) = (x - 2)^4$$

$$f'(x) = 4(x - 2)^3$$

$$f''(x) = 12(x - 2)^2$$

No inflection points:

Relative minimum:
(2, 0)

Conclusion: If $n \geq 3$ and n is odd, then (2, 0) is an inflection point. If $n \geq 2$ and n is even, then (2, 0) is a relative minimum.

(b) Let $f(x) = (x - 2)^n$, $f'(x) = n(x - 2)^{n-1}$, $f''(x) = n(n - 1)(x - 2)^{n-2}$.

For $n \geq 3$ and odd, $n - 2$ is also odd and the concavity changes at $x = 2$.

For $n \geq 4$ and even, $n - 2$ is also even and the concavity does not change at $x = 2$.

Thus, $x = 2$ is an inflection point if and only if $n \geq 3$ is odd.

61. $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: (3, 3)

Relative minimum: (5, 1)

Point of inflection: (4, 2)

$$f'(x) = 3ax^2 + 2bx + c, f''(x) = 6ax + 2b$$

$$\left. \begin{array}{l} f(3) = 27a + 9b + 3c + d = 3 \\ f(5) = 125a + 25b + 5c + d = 1 \end{array} \right\} 98a + 16b + 2c = -2 \Rightarrow 49a + 8b + c = -1$$

$$f'(3) = 27a + 6b + c = 0, f''(4) = 24a + 2b = 0$$

$$49a + 8b + c = -1 \quad 24a + 2b = 0$$

$$27a + 6b + c = 0 \quad 22a + 2b = -1$$

$$22a + 2b = -1 \quad 2a = 1$$

$$a = \frac{1}{2}, b = -6, c = \frac{45}{2}, d = -24$$

$$f(x) = \frac{1}{2}x^3 - 6x^2 + \frac{45}{2}x - 24$$

63. $f(x) = ax^3 + bx^2 + cx + d$

Maximum: $(-4, 1)$

Minimum: $(0, 0)$

(a) $f'(x) = 3ax^2 + 2bx + c$, $f''(x) = 6ax + 2b$

$$f(0) = 0 \Rightarrow d = 0$$

$$f(-4) = 1 \Rightarrow -64a + 16b - 4c = 1$$

$$f'(-4) = 0 \Rightarrow 48a - 8b + c = 0$$

$$f'(0) = 0 \Rightarrow c = 0$$

Solving this system yields $a = \frac{1}{32}$ and $b = 6a = \frac{3}{16}$.

$$f(x) = \frac{1}{32}x^3 + \frac{3}{16}x^2$$

(b) The plane would be descending at the greatest rate at the point of inflection.

$$f''(x) = 6ax + 2b = \frac{3}{16}x + \frac{3}{8} = 0 \Rightarrow x = -2.$$

Two miles from touchdown.

65. $D = 2x^4 - 5Lx^3 + 3L^2x^2$

$$D' = 8x^3 - 15Lx^2 + 6L^2x = x(8x^2 - 15Lx + 6L^2) = 0$$

$$x = 0 \text{ or } x = \frac{15L \pm \sqrt{33}L}{16} = \left(\frac{15 \pm \sqrt{33}}{16}\right)L$$

By the Second Derivative Test, the deflection is maximum when

$$x = \left(\frac{15 - \sqrt{33}}{16}\right)L \approx 0.578L.$$

67. $C = 0.5x^2 + 15x + 5000$

$$\bar{C} = \frac{C}{x} = 0.5x + 15 + \frac{5000}{x}$$

$\bar{C}$ = average cost per unit

$$\frac{d\bar{C}}{dx} = 0.5 - \frac{5000}{x^2} = 0 \text{ when } x = 100$$

By the First Derivative Test, $\bar{C}$ is minimized when $x = 100$ units.

69. $S = \frac{5000t^2}{8 + t^2}$

$$S'(t) = \frac{80,000t}{(8 + t^2)^2}$$

$$S''(t) = \frac{80,000(8 - 3t^2)}{(8 + t^2)^3}$$

$$S''(t) = 0 \text{ for } t = \sqrt{8/3} \approx 1.633.$$

Sales are increasing at the greatest rate at $t = 1.633$ years.

71. $f(x) = 2(\sin x + \cos x)$, $f\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

$$f'(x) = 2(\cos x - \sin x), \quad f'\left(\frac{\pi}{4}\right) = 0$$

$$f''(x) = 2(-\sin x - \cos x), \quad f''\left(\frac{\pi}{4}\right) = -2\sqrt{2}$$

$$P_1(x) = 2\sqrt{2} + 0\left(x - \frac{\pi}{4}\right) = 2\sqrt{2}$$

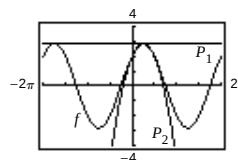
$$P_1'(x) = 0$$

$$P_2(x) = 2\sqrt{2} + 0\left(x - \frac{\pi}{4}\right) + \frac{1}{2}(-2\sqrt{2})\left(x - \frac{\pi}{4}\right)^2 = 2\sqrt{2} - \sqrt{2}\left(x - \frac{\pi}{4}\right)^2$$

$$P_2'(x) = -2\sqrt{2}\left(x - \frac{\pi}{4}\right)$$

$$P_2''(x) = -2\sqrt{2}$$

The values of f , P_1 , P_2 , and their first derivatives are equal at $x = \pi/4$. The values of the second derivatives of f and P_2 are equal at $x = \pi/4$. The approximations worsen as you move away from $x = \pi/4$.



$$73. \quad f(x) = \sqrt{1-x}, \quad f(0) = 1$$

$$f'(x) = -\frac{1}{2\sqrt{1-x}}, \quad f'(0) = -\frac{1}{2}$$

$$f''(x) = -\frac{1}{4(1-x)^{3/2}}, \quad f''(0) = -\frac{1}{4}$$

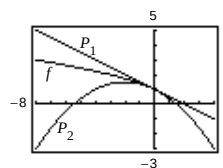
$$P_1(x) = 1 + \left(-\frac{1}{2}\right)(x-0) = 1 - \frac{x}{2}$$

$$P_1'(x) = -\frac{1}{2}$$

$$P_2(x) = 1 + \left(-\frac{1}{2}\right)(x-0) + \frac{1}{2}\left(-\frac{1}{4}\right)(x-0)^2 = 1 - \frac{x}{2} - \frac{x^2}{8}$$

$$P_2'(x) = -\frac{1}{2} - \frac{x}{4}$$

$$P_2''(x) = -\frac{1}{4}$$



The values of f , P_1 , P_2 , and their first derivatives are equal at $x = 0$. The values of the second derivatives of f and P_2 are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

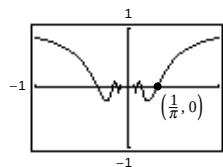
$$75. \quad f(x) = x \sin\left(\frac{1}{x}\right)$$

$$f'(x) = x \left[-\frac{1}{x^2} \cos\left(\frac{1}{x}\right) \right] + \sin\left(\frac{1}{x}\right) = -\frac{1}{x} \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$$

$$f''(x) = -\frac{1}{x} \left[\frac{1}{x^2} \sin\left(\frac{1}{x}\right) \right] + \frac{1}{x^2} \cos\left(\frac{1}{x}\right) - \frac{1}{x^2} \cos\left(\frac{1}{x}\right) = -\frac{1}{x^3} \sin\left(\frac{1}{x}\right) = 0$$

$$x = \frac{1}{\pi}$$

$$\text{Point of inflection: } \left(\frac{1}{\pi}, 0\right)$$



When $x > 1/\pi$, $f'' < 0$, so the graph is concave downward.

77. Assume the zeros of f are all real. Then express the function as $f(x) = a(x - r_1)(x - r_2)(x - r_3)$ where r_1 , r_2 , and r_3 are the distinct zeros of f . From the Product Rule for a function involving three factors, we have

$$f'(x) = a[(x - r_1)(x - r_2) + (x - r_1)(x - r_3) + (x - r_2)(x - r_3)]$$

$$f''(x) = a[(x - r_1) + (x - r_2) + (x - r_1) + (x - r_3) + (x - r_2) + (x - r_3)]$$

$$= a[6x - 2(r_1 + r_2 + r_3)].$$

Consequently, $f''(x) = 0$ if

$$x = \frac{2(r_1 + r_2 + r_3)}{6} = \frac{r_1 + r_2 + r_3}{3} = (\text{Average of } r_1, r_2, \text{ and } r_3).$$

79. True. Let $y = ax^3 + bx^2 + cx + d$, $a \neq 0$. Then $y'' = 6ax + 2b = 0$ when $x = -(b/3a)$, and the concavity changes at this point.

81. False.

$$f(x) = 3 \sin x + 2 \cos x$$

$$f'(x) = 3 \cos x - 2 \sin x$$

$$3 \cos x - 2 \sin x = 0$$

$$3 \cos x = 2 \sin x$$

$$\frac{3}{2} = \tan x$$

$$\text{Critical number: } x = \tan^{-1}\left(\frac{3}{2}\right)$$

$$f\left(\tan^{-1}\left(\frac{3}{2}\right)\right) \approx 3.60555 \text{ is the maximum value of } y.$$

 83. False. Concavity is determined by f'' .

Section 3.5 Limits at Infinity

1. $f(x) = \frac{3x^2}{x^2 + 2}$

No vertical asymptotes

 Horizontal asymptote: $y = 3$

Matches (f)

3. $f(x) = \frac{x}{x^2 + 2}$

No vertical asymptotes

 Horizontal asymptote: $y = 0$

Matches (d)

5. $f(x) = \frac{4 \sin x}{x^2 + 1}$

No vertical asymptotes

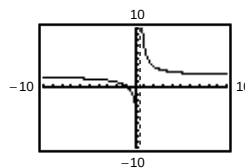
 Horizontal asymptotes: $y = 0$

Matches (b)

7. $f(x) = \frac{4x + 3}{2x - 1}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	7	2.26	2.025	2.0025	2.0003	2	2

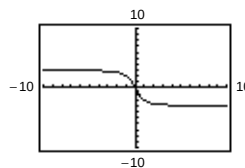
$$\lim_{x \rightarrow \infty} f(x) = 2$$



9. $f(x) = \frac{-6x}{\sqrt{4x^2 + 5}}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	-2	-2.98	-2.9998	-3	-3	-3	-3

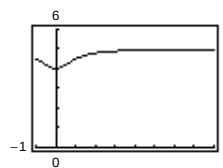
$$\lim_{x \rightarrow \infty} f(x) = -3$$



11. $f(x) = 5 - \frac{1}{x^2 + 1}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	4.5	4.99	4.9999	4.999999	5	5	5

$$\lim_{x \rightarrow \infty} f(x) = 5$$



$$13. (a) h(x) = \frac{f(x)}{x^2} = \frac{5x^3 - 3x^2 + 10}{x^2} = 5x - 3 + \frac{10}{x^2}$$

$$\lim_{x \rightarrow \infty} h(x) = \infty \quad (\text{Limit does not exist})$$

$$(b) h(x) = \frac{f(x)}{x^3} = \frac{5x^3 - 3x^2 + 10}{x^3} = 5 - \frac{3}{x} + \frac{10}{x^3}$$

$$\lim_{x \rightarrow \infty} h(x) = 5$$

$$(c) h(x) = \frac{f(x)}{x^4} = \frac{5x^3 - 3x^2 + 10}{x^4} = \frac{5}{x} - \frac{3}{x^2} + \frac{10}{x^4}$$

$$\lim_{x \rightarrow \infty} h(x) = 0$$

$$17. (a) \lim_{x \rightarrow \infty} \frac{5 - 2x^{3/2}}{3x^2 - 4} = 0$$

$$(b) \lim_{x \rightarrow \infty} \frac{5 - 2x^{3/2}}{3x^{3/2} - 4} = -\frac{2}{3}$$

$$(c) \lim_{x \rightarrow \infty} \frac{5 - 2x^{3/2}}{3x - 4} = -\infty \quad (\text{Limit does not exist})$$

$$21. \lim_{x \rightarrow \infty} \frac{x}{x^2 - 1} = \lim_{x \rightarrow \infty} \frac{1/x}{1 - (1/x^2)} = \frac{0}{1} = 0$$

$$15. (a) \lim_{x \rightarrow \infty} \frac{x^2 + 2}{x^3 - 1} = 0$$

$$(b) \lim_{x \rightarrow \infty} \frac{x^2 + 2}{x^2 - 1} = 1$$

$$(c) \lim_{x \rightarrow \infty} \frac{x^2 + 2}{x - 1} = \infty \quad (\text{Limit does not exist})$$

$$19. \lim_{x \rightarrow \infty} \frac{2x - 1}{3x + 2} = \lim_{x \rightarrow \infty} \frac{2 - (1/x)}{3 + (2/x)} = \frac{2 - 0}{3 + 0} = \frac{2}{3}$$

$$23. \lim_{x \rightarrow -\infty} \frac{5x^2}{x + 3} = \lim_{x \rightarrow -\infty} \frac{5x}{1 + (3/x)} = -\infty$$

Limit does not exist.

$$25. \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 - x}} = \lim_{x \rightarrow -\infty} \frac{1}{\frac{\sqrt{x^2 - x}}{-\sqrt{x^2}}}, \quad (\text{for } x < 0 \text{ we have } x = -\sqrt{x^2})$$

$$= \lim_{x \rightarrow -\infty} \frac{-1}{\sqrt{1 - (1/x)}} = -1$$

$$27. \lim_{x \rightarrow -\infty} \frac{2x + 1}{\sqrt{x^2 - x}} = \lim_{x \rightarrow -\infty} \frac{2 + \frac{1}{x}}{\left(\frac{\sqrt{x^2 - x}}{-\sqrt{x^2}} \right)} \quad (\text{for } x < 0, x = -\sqrt{x^2})$$

$$= \lim_{x \rightarrow -\infty} \frac{-2 - (1/x)}{\sqrt{x + (1/x)}} = -2$$

29. Since $(-1/x) \leq (\sin(2x))/x \leq (1/x)$ for all $x \neq 0$, we have by the Squeeze Theorem,

$$\lim_{x \rightarrow \infty} -\frac{1}{x} \leq \lim_{x \rightarrow \infty} \frac{\sin(2x)}{x} \leq \lim_{x \rightarrow \infty} \frac{1}{x}$$

$$0 \leq \lim_{x \rightarrow \infty} \frac{\sin(2x)}{x} \leq 0.$$

$$\text{Therefore, } \lim_{x \rightarrow \infty} \frac{\sin(2x)}{x} = 0.$$

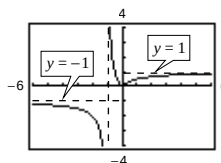
$$31. \lim_{x \rightarrow \infty} \frac{1}{2x + \sin x} = 0$$

33. (a) $f(x) = \frac{|x|}{x+1}$

$$\lim_{x \rightarrow \infty} \frac{|x|}{x+1} = 1$$

$$\lim_{x \rightarrow -\infty} \frac{|x|}{x+1} = -1$$

Therefore, $y = 1$ and $y = -1$ are both horizontal asymptotes.



35. $\lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{t \rightarrow 0^+} \frac{\sin t}{t} = 1$

(Let $x = 1/t$.)

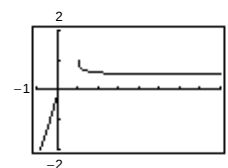
37. $\lim_{x \rightarrow -\infty} (x + \sqrt{x^2 + 3}) = \lim_{x \rightarrow -\infty} \left[(x + \sqrt{x^2 + 3}) \cdot \frac{x - \sqrt{x^2 + 3}}{x - \sqrt{x^2 + 3}} \right] = \lim_{x \rightarrow -\infty} \frac{-3}{x - \sqrt{x^2 + 3}} = 0$

39. $\lim_{x \rightarrow \infty} (x - \sqrt{x^2 + x}) = \lim_{x \rightarrow \infty} \left[(x - \sqrt{x^2 + x}) \cdot \frac{x + \sqrt{x^2 + x}}{x + \sqrt{x^2 + x}} \right]$
 $= \lim_{x \rightarrow \infty} \frac{-x}{x + \sqrt{x^2 + x}} = \lim_{x \rightarrow \infty} \frac{-1}{1 + \sqrt{1 + (1/x)}} = -\frac{1}{2}$

41.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1	0.513	0.501	0.500	0.500	0.500	0.500

$$\begin{aligned} \lim_{x \rightarrow \infty} (x - \sqrt{x(x-1)}) &= \lim_{x \rightarrow \infty} \frac{x - \sqrt{x^2 - x}}{1} \cdot \frac{x + \sqrt{x^2 - x}}{x + \sqrt{x^2 - x}} \\ &= \lim_{x \rightarrow \infty} \frac{x}{x + \sqrt{x^2 - x}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{1 + \sqrt{1 - (1/x)}} \\ &= \frac{1}{2} \end{aligned}$$

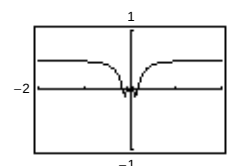


43.

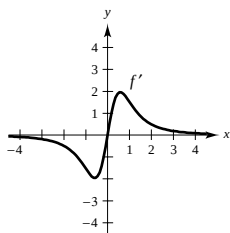
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	0.479	0.500	0.500	0.500	0.500	0.500	0.500

Let $x = 1/t$.

$$\lim_{x \rightarrow \infty} x \sin \left(\frac{1}{2x} \right) = \lim_{t \rightarrow 0^+} \frac{\sin(t/2)}{t} = \lim_{t \rightarrow 0^+} \frac{1}{2} \frac{\sin(t/2)}{t/2} = \frac{1}{2}$$



45. (a)



$$(b) \lim_{x \rightarrow \infty} f(x) = 3 \quad \lim_{x \rightarrow \infty} f'(x) = 0$$

(c) Since $\lim_{x \rightarrow \infty} f(x) = 3$, the graph approaches that of a horizontal line, $\lim_{x \rightarrow \infty} f'(x) = 0$.

$$49. y = \frac{2+x}{1-x}$$

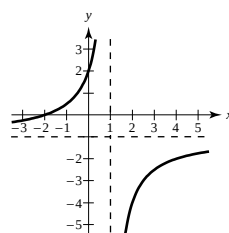
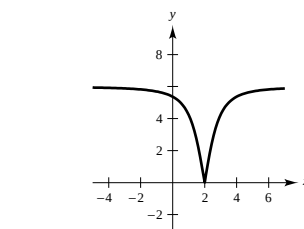
Intercepts: $(-2, 0)$, $(0, 2)$

Symmetry: none

Horizontal asymptote: $y = -1$ since

$$\lim_{x \rightarrow -\infty} \frac{2+x}{1-x} = -1 = \lim_{x \rightarrow \infty} \frac{2+x}{1-x}.$$

Discontinuity: $x = 1$ (Vertical asymptote)



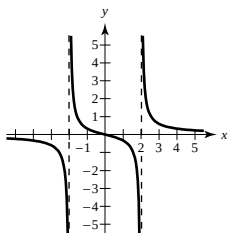
$$51. y = \frac{x}{x^2 - 4}$$

Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = \pm 2$



$$53. y = \frac{x^2}{x^2 + 9}$$

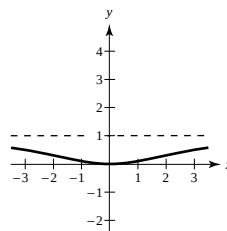
Intercept: $(0, 0)$

Symmetry: y -axis

Horizontal asymptote: $y = 1$ since

$$\lim_{x \rightarrow -\infty} \frac{x^2}{x^2 + 9} = 1 = \lim_{x \rightarrow \infty} \frac{x^2}{x^2 + 9}.$$

Relative minimum: $(0, 0)$



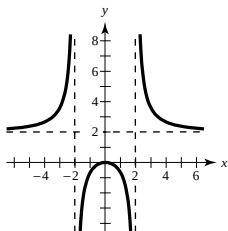
55. $y = \frac{2x^2}{x^2 - 4}$

Intercept: (0, 0)

Symmetry: y-axis

Horizontal asymptote: $y = 2$

Vertical asymptote: $x = \pm 2$



57. $xy^2 = 4$

Domain: $x > 0$

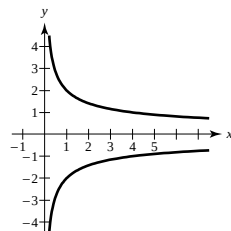
Intercepts: none

Symmetry: x-axis

Horizontal asymptote: $y = 0$ since

$$\lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0 = \lim_{x \rightarrow \infty} -\frac{2}{\sqrt{x}}.$$

Discontinuity: $x = 0$ (Vertical asymptote)



59. $y = \frac{2x}{1 - x}$

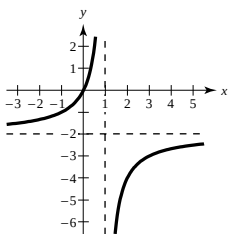
Intercept: (0, 0)

Symmetry: none

Horizontal asymptote: $y = -2$ since

$$\lim_{x \rightarrow -\infty} \frac{2x}{1 - x} = -2 = \lim_{x \rightarrow \infty} \frac{2x}{1 - x}.$$

Discontinuity: $x = 1$ (Vertical asymptote)



61. $y = 2 - \frac{3}{x^2}$

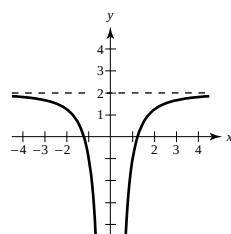
Intercepts: $(\pm\sqrt{3/2}, 0)$

Symmetry: y-axis

Horizontal asymptote: $y = 2$ since

$$\lim_{x \rightarrow -\infty} \left(2 - \frac{3}{x^2}\right) = 2 = \lim_{x \rightarrow \infty} \left(2 - \frac{3}{x^2}\right).$$

Discontinuity: $x = 0$ (Vertical asymptote)



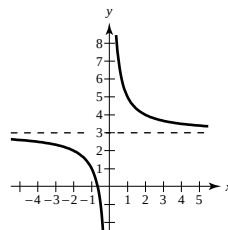
63. $y = 3 + \frac{2}{x}$

Intercept: $y = 0 = 3 + \frac{2}{x} \Rightarrow \frac{2}{x} = -3 \Rightarrow x = -\frac{2}{3} \left(-\frac{2}{3}, 0\right)$

Symmetry: none

Horizontal asymptote: $y = 3$

Vertical asymptote: $x = 0$



65. $y = \frac{x^3}{\sqrt{x^2 - 4}}$

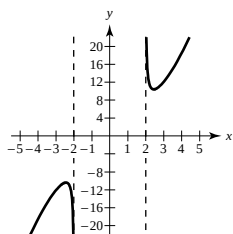
Domain: $(-\infty, -2), (2, \infty)$

Intercepts: none

Symmetry: origin

Horizontal asymptote: none

Vertical asymptotes: $x = \pm 2$ (discontinuities)



67. $f(x) = 5 - \frac{1}{x^2} = \frac{5x^2 - 1}{x^2}$

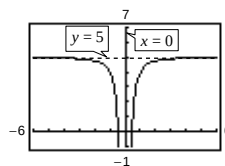
Domain: $(-\infty, 0), (0, \infty)$

$$f'(x) = \frac{2}{x^3} \Rightarrow \text{No relative extrema}$$

$$f''(x) = -\frac{6}{x^4} \Rightarrow \text{No points of inflection}$$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 5$



69. $f(x) = \frac{x}{x^2 - 4}$

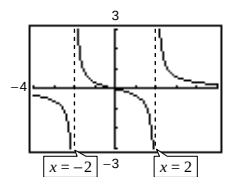
$$\begin{aligned} f'(x) &= \frac{(x^2 - 4) - x(2x)}{(x^2 - 4)^2} \\ &= \frac{-(x^2 + 4)}{(x^2 - 4)^2} \neq 0 \text{ for any } x \text{ in the domain of } f. \end{aligned}$$

$$\begin{aligned} f''(x) &= \frac{(x^2 - 4)^2(-2x) + (x^2 + 4)(2)(x^2 - 4)(2x)}{(x^2 - 4)^4} \\ &= \frac{2x(x^2 + 12)}{(x^2 - 4)^3} = 0 \text{ when } x = 0. \end{aligned}$$

Since $f''(x) > 0$ on $(-2, 0)$ and $f''(x) < 0$ on $(0, 2)$, then $(0, 0)$ is a point of inflection.

Vertical asymptotes: $x = \pm 2$

Horizontal asymptote: $y = 0$



71. $f(x) = \frac{x - 2}{x^2 - 4x + 3} = \frac{x - 2}{(x - 1)(x - 3)}$

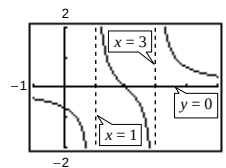
$$f'(x) = \frac{(x^2 - 4x + 3) - (x - 2)(2x - 4)}{(x^2 - 4x + 3)^2} = \frac{-x^2 + 4x - 5}{(x^2 - 4x + 3)^2} \neq 0$$

$$\begin{aligned} f''(x) &= \frac{(x^2 - 4x + 3)^2(-2x + 4) - (-x^2 + 4x - 5)(2)(x^2 - 4x + 3)(2x - 4)}{(x^2 - 4x + 3)^4} \\ &= \frac{2(x^3 - 6x^2 + 15x - 14)}{(x^2 - 4x + 3)^3} = 0 \text{ when } x = 2. \end{aligned}$$

Since $f''(x) > 0$ on $(1, 2)$ and $f''(x) < 0$ on $(2, 3)$, then $(2, 0)$ is a point of inflection.

Vertical asymptote: $x = 1, x = 3$

Horizontal asymptote: $y = 0$



$$73. f(x) = \frac{3x}{\sqrt{4x^2 + 1}}$$

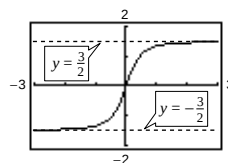
$$f'(x) = \frac{3}{(4x^2 + 1)^{3/2}} \Rightarrow \text{No relative extrema}$$

$$f''(x) = \frac{-36x}{(4x^2 + 1)^{5/2}} = 0 \text{ when } x = 0.$$

Point of inflection: $(0, 0)$

Horizontal asymptotes: $y = \pm \frac{3}{2}$

No vertical asymptotes



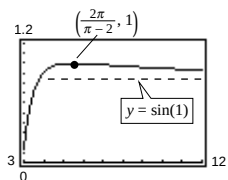
$$75. g(x) = \sin\left(\frac{x}{x-2}\right), \quad 3 < x < \infty$$

$$g'(x) = \frac{-2 \cos\left(\frac{x}{x-2}\right)}{(x-2)^2}$$

Horizontal asymptote: $y = 1$

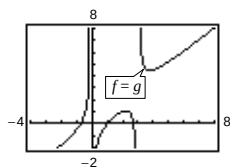
$$\text{Relative maximum: } \frac{x}{x-2} = \frac{\pi}{2} \Rightarrow x = \frac{2\pi}{\pi-2} \approx 5.5039$$

No vertical asymptotes



$$77. f(x) = \frac{x^3 - 3x^2 + 2}{x(x-3)}, g(x) = x + \frac{2}{x(x-3)}$$

(a)

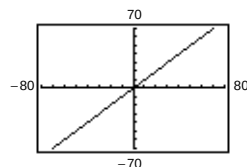


$$(b) f(x) = \frac{x^3 - 3x^2 + 2}{x(x-3)}$$

$$= \frac{x^2(x-3)}{x(x-3)} + \frac{2}{x(x-3)}$$

$$= x + \frac{2}{x(x-3)} = g(x)$$

(c)



The graph appears as the slant asymptote $y = x$.

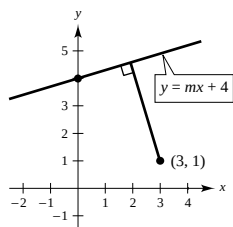
$$79. C = 0.5x + 500$$

$$\bar{C} = \frac{C}{x}$$

$$\bar{C} = 0.5 + \frac{500}{x}$$

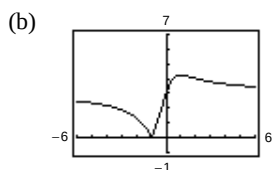
$$\lim_{x \rightarrow \infty} \left(0.5 + \frac{500}{x}\right) = 0.5$$

81. line:
- $mx - y + 4 = 0$



$$(a) d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}}$$

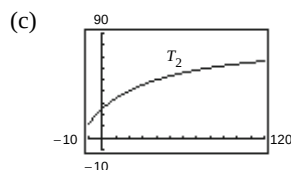
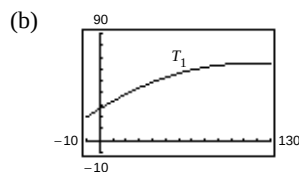
$$= \frac{|3m + 3|}{\sqrt{m^2 + 1}}$$



$$(c) \lim_{m \rightarrow \infty} d(m) = 3 = \lim_{m \rightarrow -\infty} d(m)$$

The line approaches the vertical line $x = 0$. Hence, the distance approaches 3.

83. (a)
- $T_1(t) = -0.003t^2 + 0.677t + 26.564$



$$T_2 = \frac{1451 + 86t}{58 + t}$$

$$(d) T_1(0) \approx 26.6$$

$$T_2(0) \approx 25.0$$

$$(e) \lim_{t \rightarrow \infty} T_2 = \frac{86}{1} = 86$$

- (f) The limiting temperature is 86.
 T_1 has no horizontal asymptote.

85. Answers will vary. See page 195.

87. False. Let
- $f(x) = \frac{2x}{\sqrt{x^2 + 2}}$
- . (See Exercise 2.)

Section 3.6 A Summary of Curve Sketching

- 1.
- f
- has constant negative slope. Matches (D)

5. (a)
- $f'(x) = 0$
- for
- $x = -2$
- and
- $x = 2$

f' is negative for $-2 < x < 2$ (decreasing function).

f' is positive for $x > 2$ and $x < -2$
 (increasing function).

- (b) $f''(x) = 0$ at $x = 0$ (Inflection point).

f'' is positive for $x > 0$ (Concave upwards).

f'' is negative for $x < 0$ (Concave downward).

3. The slope is periodic, and zero at
- $x = 0$
- . Matches (A)

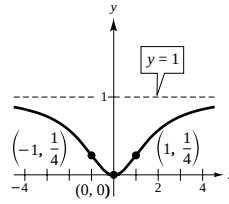
- (c) f' is increasing on $(0, \infty)$. ($f'' > 0$)

- (d) $f'(x)$ is minimum at $x = 0$. The rate of change of f at $x = 0$ is less than the rate of change of f for all other values of x .

7. $y = \frac{x^2}{x^2 + 3}$

$$y' = \frac{6x}{(x^2 + 3)^2} = 0 \text{ when } x = 0.$$

$$y'' = \frac{18(1 - x^2)}{(x^2 + 3)^3} = 0 \text{ when } x = \pm 1.$$

 Horizontal asymptote: $y = 1$


	y	y'	y''	Conclusion
$-\infty < x < -1$		-	-	Decreasing, concave down
$x = -1$	$\frac{1}{4}$	-	0	Point of inflection
$-1 < x < 0$		-	+	Decreasing, concave up
$x = 0$	0	0	+	Relative minimum
$0 < x < 1$		+	+	Increasing, concave up
$x = 1$	$\frac{1}{4}$	+	0	Point of inflection
$1 < x < \infty$		+	-	Increasing, concave down

9. $y = \frac{1}{x-2} - 3$

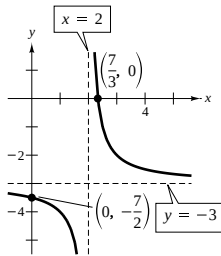
$$y' = -\frac{1}{(x-2)^2} < 0 \text{ when } x \neq 2.$$

$$y'' = \frac{2}{(x-2)^3}$$

No relative extrema, no points of inflection

 Intercepts: $\left(\frac{7}{3}, 0\right), \left(0, -\frac{7}{2}\right)$

 Vertical asymptote: $x = 2$

 Horizontal asymptote: $y = -3$


11. $y = \frac{2x}{x^2 - 1}$

$$y' = \frac{-2(x^2 + 1)}{(x^2 - 1)^2} < 0 \text{ if } x \neq \pm 1.$$

$$y'' = \frac{4x(x^2 + 3)}{(x^2 - 1)^3} = 0 \text{ if } x = 0.$$

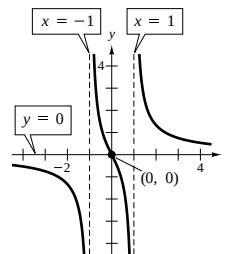
 Inflection point: $(0, 0)$

 Intercept: $(0, 0)$

 Vertical asymptote: $x = \pm 1$

 Horizontal asymptote: $y = 0$

Symmetry with respect to the origin



13. $g(x) = x + \frac{4}{x^2 + 1}$

$$g'(x) = 1 - \frac{8x}{(x^2 + 1)^2} = \frac{x^4 + 2x^2 - 8x + 1}{(x^2 + 1)^2} = 0 \text{ when } x \approx 0.1292, 1.6085$$

$$g''(x) = \frac{8(3x^2 - 1)}{(x^2 + 1)^3} = 0 \text{ when } x = \pm \frac{\sqrt{3}}{3}$$

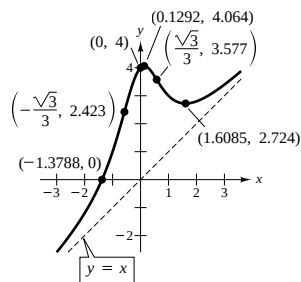
$g''(0.1292) < 0$, therefore, $(0.1292, 4.064)$ is relative maximum.

$g''(1.6085) > 0$, therefore, $(1.6085, 2.724)$ is a relative minimum.

Points of inflection: $\left(-\frac{\sqrt{3}}{3}, 2.423\right), \left(\frac{\sqrt{3}}{3}, 3.577\right)$

Intercepts: $(0, 4), (-1.3788, 0)$

Slant asymptote: $y = x$



15. $f(x) = \frac{x^2 + 1}{x} = x + \frac{1}{x}$

$$f'(x) = 1 - \frac{1}{x^2} = 0 \text{ when } x = \pm 1.$$

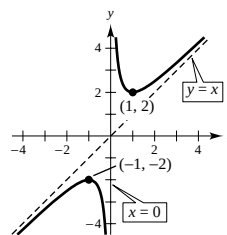
$$f''(x) = \frac{2}{x^3} \neq 0$$

Relative maximum: $(-1, -2)$

Relative minimum: $(1, 2)$

Vertical asymptote: $x = 0$

Slant asymptote: $y = x$



17. $y = \frac{x^2 - 6x + 12}{x - 4} = x - 2 + \frac{4}{x - 4}$

$$y' = 1 - \frac{4}{(x - 4)^2} = \frac{(x - 2)(x - 6)}{(x - 4)^2} = 0 \text{ when } x = 2, 6.$$

$$y'' = \frac{8}{(x - 4)^3}$$

$y'' < 0$ when $x = 2$.

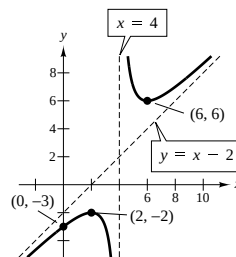
Therefore, $(2, -2)$ is a relative maximum.

$y'' > 0$ when $x = 6$.

Therefore, $(6, 6)$ is a relative minimum.

Vertical asymptote: $x = 4$

Slant asymptote: $y = x - 2$

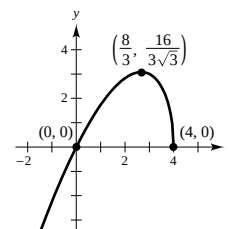


19. $y = x\sqrt{x-4}$,

 Domain: $(-\infty, 4]$

$$y' = \frac{8-3x}{2\sqrt{4-x}} = 0 \text{ when } x = \frac{8}{3} \text{ and undefined when } x = 4.$$

$$y'' = \frac{3x-16}{4(4-x)^{3/2}} = 0 \text{ when } x = \frac{16}{3} \text{ and undefined when } x = 4.$$

Note: $x = \frac{16}{3}$ is not in the domain.


	y	y'	y''	Conclusion
$-\infty < x < \frac{8}{3}$		+	−	Increasing, concave down
$x = \frac{8}{3}$	$\frac{16}{3\sqrt{3}}$	0	−	Relative maximum
$\frac{8}{3} < x < 4$		−	−	Decreasing, concave down
$x = 4$	0	Undefined	Undefined	Endpoint

21. $h(x) = x\sqrt{9-x^2}$ Domain: $-3 \leq x \leq 3$

$$h'(x) = \frac{9-2x^2}{\sqrt{9-x^2}} = 0 \text{ when } x = \pm \frac{3}{\sqrt{2}} = \pm \frac{3\sqrt{2}}{2}$$

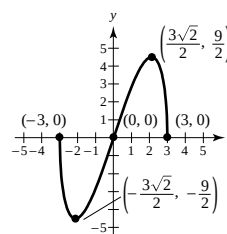
$$h''(x) = \frac{x(2x^2-27)}{(9-x^2)^{3/2}} = 0 \text{ when } x = 0$$

 Relative maximum: $\left(\frac{3\sqrt{2}}{2}, \frac{9}{2}\right)$

 Relative minimum: $\left(-\frac{3\sqrt{2}}{2}, -\frac{9}{2}\right)$

 Intercepts: $(0, 0), (\pm 3, 0)$

Symmetric with respect to the origin

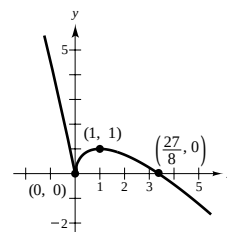
 Point of inflection: $(0, 0)$


23. $y = 3x^{2/3} - 2x$

$$y' = 2x^{-1/3} - 2 = \frac{2(1-x^{1/3})}{x^{1/3}}$$

 $= 0$ when $x = 1$ and undefined when $x = 0$.

$$y'' = \frac{-2}{3x^{4/3}} < 0 \text{ when } x \neq 0.$$



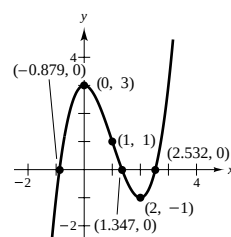
	y	y'	y''	Conclusion
$-\infty < x < 0$		−	−	Decreasing, concave down
$x = 0$	0	Undefined	Undefined	Relative minimum
$0 < x < 1$		+	−	Increasing, concave down
$x = 1$	1	0	−	Relative maximum
$1 < x < \infty$		−	−	Decreasing, concave down

25. $y = x^3 - 3x^2 + 3$

$$y' = 3x^2 - 6x = 3x(x - 2) = 0 \text{ when } x = 0, x = 2$$

$$y'' = 6x - 6 = 6(x - 1) = 0 \text{ when } x = 1$$

	y	y'	y''	Conclusion
$-\infty < x < 0$		+	-	Increasing, concave down
$x = 0$	3	0	-	Relative maximum
$0 < x < 1$		-	-	Decreasing, concave down
$x = 1$	1	-	0	Point of inflection
$1 < x < 2$		-	+	Decreasing, concave up
$x = 2$	-1	0	+	Relative minimum
$2 < x < \infty$		+	+	Increasing, concave up



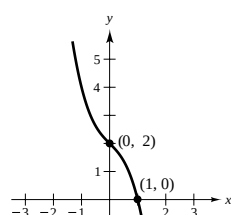
27. $y = 2 - x - x^3$

$$y' = -1 - 3x^2$$

No critical numbers

$$y'' = -6x = 0 \text{ when } x = 0.$$

	y	y'	y''	Conclusion
$-\infty < x < 0$		-	+	Decreasing, concave up
$x = 0$	2	-	0	Point of inflection
$0 < x < \infty$		-	-	Decreasing, concave down

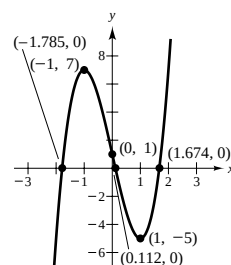


29. $f(x) = 3x^3 - 9x + 1$

$$f'(x) = 9x^2 - 9 = 9(x^2 - 1) = 0 \text{ when } x = \pm 1$$

$$f''(x) = 18x = 0 \text{ when } x = 0$$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < -1$		+	-	Increasing, concave down
$x = -1$	7	0	-	Relative maximum
$-1 < x < 0$		-	-	Decreasing, concave down
$x = 0$	1	-	0	Point of inflection
$0 < x < 1$		-	+	Decreasing, concave up
$x = 1$	-5	0	+	Relative minimum
$1 < x < \infty$		+	+	Increasing, concave up

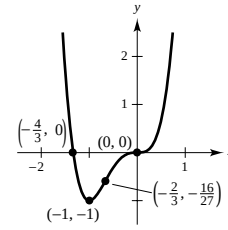


31. $y = 3x^4 + 4x^3$

$$y' = 12x^3 + 12x^2 = 12x^2(x + 1) = 0 \text{ when } x = 0, x = -1.$$

$$y'' = 36x^2 + 24x = 12x(3x + 2) = 0 \text{ when } x = 0, x = -\frac{2}{3}.$$

	y	y'	y''	Conclusion
$-\infty < x < -1$		-	+	Decreasing, concave up
$x = -1$	-1	0	+	Relative minimum
$-1 < x < -\frac{2}{3}$		+	+	Increasing, concave up
$x = -\frac{2}{3}$	$-\frac{16}{27}$	+	0	Point of inflection
$-\frac{2}{3} < x < 0$		+	-	Increasing, concave down
$x = 0$	0	0	0	Point of inflection
$0 < x < \infty$		+	+	Increasing, concave up

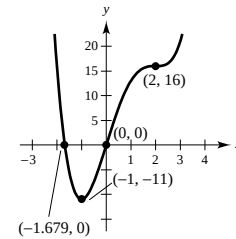


33. $f(x) = x^4 - 4x^3 + 16x$

$$f'(x) = 4x^3 - 12x^2 + 16 = 4(x + 1)(x - 2)^2 = 0 \text{ when } x = -1, x = 2.$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2) = 0 \text{ when } x = 0, x = 2.$$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < -1$		-	+	Decreasing, concave up
$x = -1$	-11	0	+	Relative minimum
$-1 < x < 0$		+	+	Increasing, concave up
$x = 0$	0	+	0	Point of inflection
$0 < x < 2$		+	-	Increasing, concave down
$x = 2$	16	0	0	Point of inflection
$2 < x < \infty$		+	+	Increasing, concave up

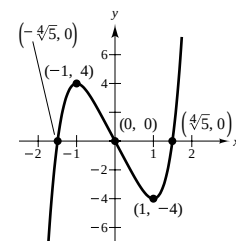


35. $y = x^5 - 5x$

$$y' = 5x^4 - 5 = 5(x^4 - 1) = 0 \text{ when } x = \pm 1.$$

$$y'' = 20x^3 = 0 \text{ when } x = 0.$$

	y	y'	y''	Conclusion
$-\infty < x < -1$		+	-	Increasing, concave down
$x = -1$	4	0	-	Relative maximum
$-1 < x < 0$		-	-	Decreasing, concave down
$x = 0$	0	-	0	Point of inflection
$0 < x < 1$		-	+	Decreasing, concave up
$x = 1$	-4	0	+	Relative minimum
$1 < x < \infty$		+	+	Increasing, concave up

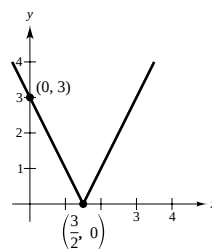


37. $y = |2x - 3|$

$$y' = \frac{2(2x - 3)}{|2x - 3|} \text{ undefined at } x = \frac{3}{2}.$$

$$y'' = 0$$

	y	y'	Conclusion
$-\infty < x < \frac{3}{2}$		$-$	Decreasing
$x = \frac{3}{2}$	0	Undefined	Relative minimum
$\frac{3}{2} < x < \infty$		$+$	Increasing



39. $y = \sin x - \frac{1}{18} \sin 3x, 0 \leq x \leq 2\pi$

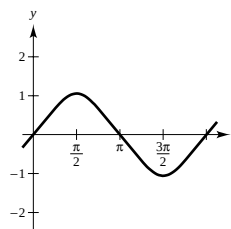
$$y' = \cos x - \frac{1}{6} \cos 3x = 0 \text{ when } x = \frac{\pi}{2}, \frac{3\pi}{2}.$$

$$y'' = -\sin x + \frac{1}{2} \sin 3x = 0 \text{ when } x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi.$$

Relative maximum: $\left(\frac{\pi}{2}, \frac{19}{18}\right)$

Relative minimum: $\left(\frac{3\pi}{2}, -\frac{19}{18}\right)$

Inflection points: $\left(\frac{\pi}{6}, \frac{4}{9}\right), \left(\frac{5\pi}{6}, \frac{4}{9}\right), (\pi, 0), \left(\frac{7\pi}{6}, -\frac{4}{9}\right), \left(\frac{11\pi}{6}, -\frac{4}{9}\right)$



41. $y = 2x - \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$y' = 2 - \sec^2 x = 0 \text{ when } x = \pm \frac{\pi}{4}.$$

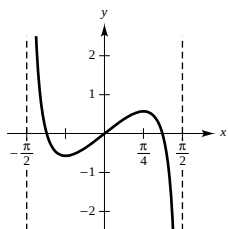
$$y'' = -2\sec^2 x \tan x = 0 \text{ when } x = 0.$$

Relative maximum: $\left(\frac{\pi}{4}, \frac{\pi}{2} - 1\right)$

Relative minimum: $\left(-\frac{\pi}{4}, 1 - \frac{\pi}{2}\right)$

Inflection point: $(0, 0)$

Vertical asymptotes: $x = \pm \frac{\pi}{2}$

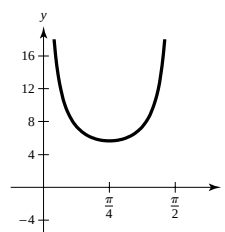


43. $y = 2(\csc x + \sec x), 0 < x < \frac{\pi}{2}$

$$y' = 2(\sec x \tan x - \csc x \cot x) = 0 \Rightarrow x = \pi/4$$

Relative minimum: $\left(\frac{\pi}{4}, 4\sqrt{2}\right)$

Vertical asymptotes: $x = 0, x = \frac{\pi}{2}$



45. $g(x) = x \tan x, -\frac{3\pi}{2} < x < \frac{3\pi}{2}$

$$g'(x) = \frac{x + \sin x \cos x}{\cos^2 x} = 0 \text{ when } x = 0$$

$$g''(x) = \frac{2(\cos x + x \sin x)}{\cos^3 x}$$

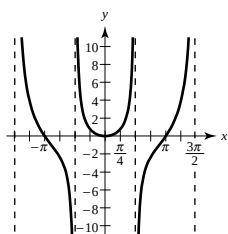
Vertical asymptotes: $x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

Intercepts: $(-\pi, 0), (0, 0), (\pi, 0)$

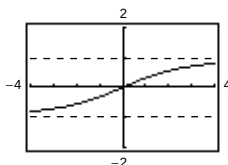
Symmetric with respect to y-axis.

Increasing on $(0, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \frac{3\pi}{2})$

Points of inflection: $(\pm 2.80, 0)$



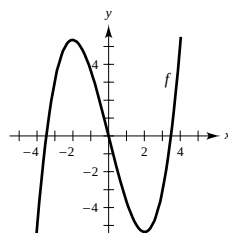
49. $y = \frac{x}{\sqrt{x^2 + 7}}$



$(0, 0)$ point of inflection

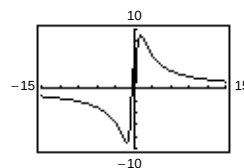
$y = \pm 1$ horizontal asymptotes

53.



(any vertical translate of f will do)

47. $f(x) = \frac{20x}{x^2 + 1} - \frac{1}{x} = \frac{19x^2 - 1}{x(x^2 + 1)}$



$x = 0$ vertical asymptote

$y = 0$ horizontal asymptote

Minimum: $(-1.10, -9.05)$

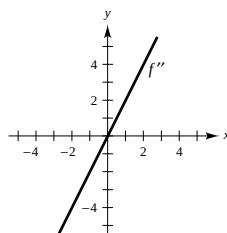
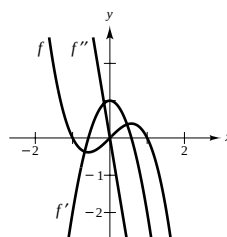
Maximum: $(1.10, 9.05)$

Points of inflection: $(-1.84, -7.86), (1.84, 7.86)$

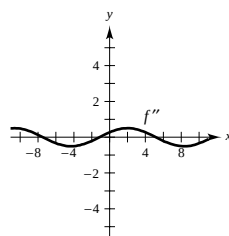
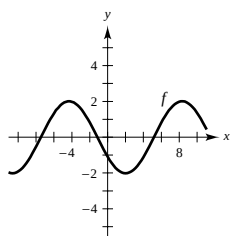
51. f is cubic.

f' is quadratic.

f'' is linear.



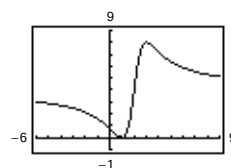
55.

(any vertical translate of f will do)

57. Since the slope is negative, the function is decreasing on $(2, 8)$, and hence $f(3) > f(5)$.

$$59. f(x) = \frac{4(x-1)^2}{x^2 - 4x + 5}$$

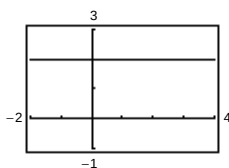
Vertical asymptote: none

Horizontal asymptote: $y = 4$ 

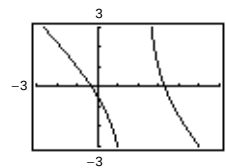
The graph crosses the horizontal asymptote $y = 4$. If a function has a vertical asymptote at $x = c$, the graph would not cross it since $f(c)$ is undefined.

$$61. h(x) = \frac{6-2x}{3-x} = \frac{2(3-x)}{3-x} = \begin{cases} 2, & \text{if } x \neq 3 \\ \text{Undefined,} & \text{if } x = 3 \end{cases}$$

The rational function is not reduced to lowest terms.

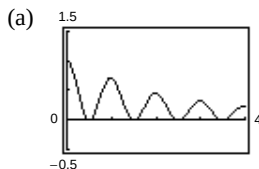
hole at $(3, 2)$

$$63. f(x) = \frac{x^2 - 3x - 1}{x - 2} = -x + 1 + \frac{3}{x - 2}$$



The graph appears to approach the slant asymptote $y = -x + 1$.

$$65. f(x) = \frac{\cos^2 \pi x}{\sqrt{x^2 + 1}}, (0, 4)$$

On $(0, 4)$ there seem to be 7 critical numbers:

0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5

$$(b) f'(x) = \frac{-\cos \pi x (x \cos \pi x + 2\pi(x^2 + 1)\sin \pi x)}{(x^2 + 1)^{3/2}} = 0$$

$$\text{Critical numbers} \approx \frac{1}{2}, 0.97, \frac{3}{2}, 1.98, \frac{5}{2}, 2.98, \frac{7}{2}.$$

The critical numbers where maxima occur appear to be integers in part (a), but approximating them using f' shows that they are not integers.

67. Vertical asymptote: $x = 5$

 Horizontal asymptote: $y = 0$

$$y = \frac{1}{x - 5}$$

71. $f(x) = \frac{ax}{(x - b)^2}$

- (a) The graph has a vertical asymptote at $x = b$. If $a > 0$, the graph approaches ∞ as $x \rightarrow b$. If $a < 0$, the graph approaches $-\infty$ as $x \rightarrow b$. The graph approaches its vertical asymptote faster as $|a| \rightarrow 0$.

73. $f(x) = \frac{3x^n}{x^4 + 1}$

- (a) For n even, f is symmetric about the y -axis. For n odd, f is symmetric about the origin.
- (b) The x -axis will be the horizontal asymptote if the degree of the numerator is less than 4. That is, $n = 0, 1, 2, 3$.
- (c) $n = 4$ gives $y = 3$ as the horizontal asymptote.

 69. Vertical asymptote: $x = 5$

 Slant asymptote: $y = 3x + 2$

$$y = 3x + 2 + \frac{1}{x - 5} = \frac{3x^2 - 13x - 9}{x - 5}$$

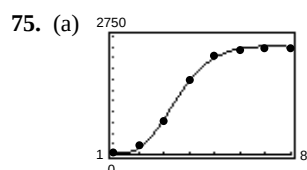
- (b) As b varies, the position of the vertical asymptote changes: $x = b$. Also, the coordinates of the minimum ($a > 0$) or maximum ($a < 0$) are changed.

- (d) There is a slant asymptote $y = 3x$ if $n = 5$:

$$\frac{3x^5}{x^4 + 1} = 3x - \frac{3x}{x^4 + 1}$$

(e)

n	0	1	2	3	4	5
M	1	2	3	2	1	0
N	2	3	4	5	2	3



- (b) When $t = 10$, $N(10) \approx 2434$ bacteria.
- (c) N is a maximum when $t \approx 7.2$ (seventh day).
- (d) $N'(t) = 0$ for $t \approx 3.2$
- (e) $\lim_{t \rightarrow \infty} N(t) = \frac{13,250}{7} \approx 1892.86$

Section 3.7 Optimization Problems

1. (a)

First Number, x	Second Number	Product, P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$

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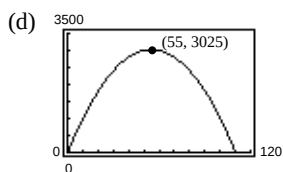
1. —CONTINUED—

(b)

First Number, x	Second Number	Product, P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$
70	$110 - 70$	$70(110 - 70) = 2800$
80	$110 - 80$	$80(110 - 80) = 2400$
90	$110 - 90$	$90(110 - 90) = 1800$
100	$110 - 100$	$100(110 - 100) = 1000$

The maximum is attained near $x = 50$ and 60 .

(c) $P = x(110 - x) = 110x - x^2$



The solution appears to be $x = 55$.

(e) $\frac{dP}{dx} = 110 - 2x = 0$ when $x = 55$.

$$\frac{d^2P}{dx^2} = -2 < 0$$

P is a maximum when $x = 110 - x = 55$.

The two numbers are 55 and 55.

3. Let x and y be two positive numbers such that $xy = 192$.

$$S = x + y = x + \frac{192}{x}$$

$$\frac{dS}{dx} = 1 - \frac{192}{x^2} = 0 \text{ when } x = \sqrt{192}.$$

$$\frac{d^2S}{dx^2} = \frac{384}{x^3} > 0 \text{ when } x = \sqrt{192}.$$

S is a minimum when $x = y = \sqrt{192}$.

7. Let x be the length and y the width of the rectangle.

$$2x + 2y = 100$$

$$y = 50 - x$$

$$A = xy = x(50 - x)$$

$$\frac{dA}{dx} = 50 - 2x = 0 \text{ when } x = 25.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = 25.$$

A is maximum when $x = y = 25$ meters.

5. Let x be a positive number.

$$S = x + \frac{1}{x}$$

$$\frac{dS}{dx} = 1 - \frac{1}{x^2} = 0 \text{ when } x = 1.$$

$$\frac{d^2S}{dx^2} = \frac{2}{x^3} > 0 \text{ when } x = 1.$$

The sum is a minimum when $x = 1$ and $1/x = 1$.

9. Let x be the length and y the width of the rectangle.

$$xy = 64$$

$$y = \frac{64}{x}$$

$$P = 2x + 2y = 2x + 2\left(\frac{64}{x}\right) = 2x + \frac{128}{x}$$

$$\frac{dP}{dx} = 2 - \frac{128}{x^2} = 0 \text{ when } x = 8.$$

$$\frac{d^2P}{dx^2} = \frac{256}{x^3} > 0 \text{ when } x = 8.$$

P is minimum when $x = y = 8$ feet.

$$11. d = \sqrt{(x-4)^2 + (\sqrt{x}-0)^2}$$

$$= \sqrt{x^2 - 7x + 16}$$

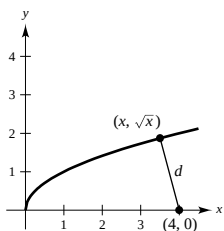
Since d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^2 - 7x + 16.$$

$$f'(x) = 2x - 7 = 0$$

$$x = \frac{7}{2}$$

By the First Derivative Test, the point nearest to $(4, 0)$ is $(7/2, \sqrt{7/2})$.



$$13. d = \sqrt{(x-2)^2 + [x^2 - (1/2)]^2}$$

$$= \sqrt{x^4 - 4x + (17/4)}$$

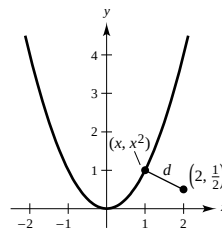
Since d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^4 - 4x + \frac{17}{4}.$$

$$f'(x) = 4x^3 - 4 = 0$$

$$x = 1$$

By the First Derivative Test, the point nearest to $(2, \frac{1}{2})$ is $(1, 1)$.



$$15. \frac{dQ}{dx} = kx(Q_0 - x) = kQ_0x - kx^2$$

$$\frac{d^2Q}{dx^2} = kQ_0 - 2kx$$

$$= k(Q_0 - 2x) = 0 \text{ when } x = \frac{Q_0}{2}.$$

$$\frac{d^3Q}{dx^3} = -2k < 0 \text{ when } x = \frac{Q_0}{2}.$$

dQ/dx is maximum when $x = Q_0/2$.

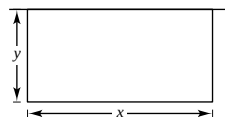
$$17. xy = 180,000 \text{ (see figure)}$$

$S = x + 2y = \left(x + \frac{360,000}{x}\right)$ where S is the length of fence needed.

$$\frac{dS}{dx} = 1 - \frac{360,000}{x^2} = 0 \text{ when } x = 600.$$

$$\frac{d^2S}{dx^2} = \frac{720,000}{x^3} > 0 \text{ when } x = 600.$$

S is a minimum when $x = 600$ meters and $y = 300$ meters.



$$19. (a) A = 4(\text{area of side}) + 2(\text{area of Top})$$

$$(a) A = 4(3)(11) + 2(3)(3) = 150 \text{ square inches}$$

$$(b) A = 4(5)(5) + 2(5)(5) = 150 \text{ square inches}$$

$$(c) A = 4(3.25)(6) + 2(6)(6) = 150 \text{ square inches}$$

$$(c) S = 4xy + 2x^2 = 150 \Rightarrow y = \frac{150 - 2x^2}{4x}$$

$$V = x^2y = x^2\left(\frac{150 - 2x^2}{4x}\right) = \frac{75}{2}x - \frac{1}{2}x^3$$

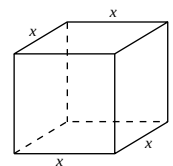
$$V' = \frac{75}{2} - \frac{3}{2}x^2 = 0 \Rightarrow x = \pm 5$$

$$(b) V = (\text{length})(\text{width})(\text{height})$$

$$(a) V = (3)(3)(11) = 99 \text{ cubic inches}$$

$$(b) V = (5)(5)(5) = 125 \text{ cubic inches}$$

$$(c) V = (6)(6)(3.25) = 117 \text{ cubic inches}$$



By the First Derivative Test, $x = 5$ yields the maximum volume. Dimensions: $5 \times 5 \times 5$. (A cube!)

21. (a) $V = x(s - 2x)^2, 0 < x < \frac{s}{2}$

$$\frac{dV}{dx} = 2x(s - 2x)(-2) + (s - 2x)^2$$

$$= (s - 2x)(s - 6x) = 0 \text{ when } x = \frac{s}{2}, \frac{s}{6} \text{ (} s/2 \text{ is not in the domain).}$$

$$\frac{d^2V}{dx^2} = 24x - 8s$$

$$\frac{d^2V}{dx^2} < 0 \text{ when } x = \frac{s}{6}.$$

$$V = \frac{2s^3}{27} \text{ is maximum when } x = \frac{s}{6}.$$

(b) If the length is doubled, $V = \frac{2}{27}(2s)^3 = 8\left(\frac{2}{27}s^3\right)$. Volume is increased by a factor of 8.

23. $16 = 2y + x + \pi\left(\frac{x}{2}\right)$

$$32 = 4y + 2x + \pi x$$

$$y = \frac{32 - 2x - \pi x}{4}$$

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2 = \left(\frac{32 - 2x - \pi x}{4}\right)x + \frac{\pi x^2}{8}$$

$$= 8x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2$$

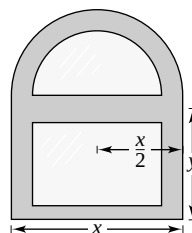
$$\frac{dA}{dx} = 8 - x - \frac{\pi}{2}x + \frac{\pi}{4}x = 8 - x\left(1 + \frac{\pi}{4}\right)$$

$$= 0 \text{ when } x = \frac{8}{1 + (\pi/4)} = \frac{32}{4 + \pi}.$$

$$\frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{4}\right) < 0 \text{ when } x = \frac{32}{4 + \pi}$$

$$y = \frac{32 - 2[32/(4 + \pi)] - \pi[32/(4 + \pi)]}{4} = \frac{16}{4 + \pi}$$

The area is maximum when $y = \frac{16}{4 + \pi}$ feet and $x = \frac{32}{4 + \pi}$ feet.

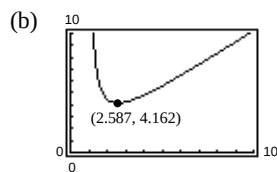


25. (a) $\frac{y-2}{0-1} = \frac{0-2}{x-1}$

$$y = 2 + \frac{2}{x-1}$$

$$L = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(2 + \frac{2}{x-1}\right)^2}$$

$$= \sqrt{x^2 + 4 + \frac{8}{x-1} + \frac{4}{(x-1)^2}}, \quad x > 1$$



L is minimum when $x \approx 2.587$ and $L \approx 4.162$.

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25. —CONTINUED—

$$(c) \text{ Area} = A(x) = \frac{1}{2}xy = \frac{1}{2}x\left(2 + \frac{2}{x-1}\right) = x + \frac{x}{x-1}$$

$$A'(x) = 1 + \frac{(x-1) - x}{(x-1)^2} = 1 - \frac{1}{(x-1)^2} = 0$$

$$(x-1)^2 = 1$$

$$x-1 = \pm 1$$

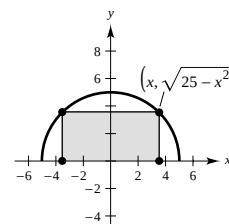
$$x = 0, 2 \text{ (select } x = 2\text{)}$$

Then $y = 4$ and $A = 4$.

Vertices: $(0, 0)$, $(2, 0)$, $(0, 4)$

$$27. A = 2xy = 2x\sqrt{25-x^2} \text{ (see figure)}$$

$$\begin{aligned} \frac{dA}{dx} &= 2x\left(\frac{1}{2}\right)\left(\frac{-2x}{\sqrt{25-x^2}}\right) + 2\sqrt{25-x^2} \\ &= 2\left(\frac{25-2x^2}{\sqrt{25-x^2}}\right) = 0 \text{ when } x = y = \frac{5\sqrt{2}}{2} \approx 3.54. \end{aligned}$$



By the First Derivative Test, the inscribed rectangle of maximum area has vertices

$$\left(\pm \frac{5\sqrt{2}}{2}, 0\right), \left(\pm \frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right).$$

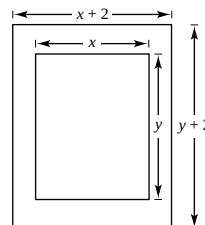
$$\text{Width: } \frac{5\sqrt{2}}{2}; \text{ Length: } 5\sqrt{2}$$

$$29. xy = 30 \Rightarrow y = \frac{30}{x}$$

$$A = (x+2)\left(\frac{30}{x} + 2\right) \text{ (see figure)}$$

$$\frac{dA}{dx} = (x+2)\left(\frac{-30}{x^2}\right) + \left(\frac{30}{x} + 2\right) = \frac{2(x^2-30)}{x^2} = 0 \text{ when } x = \sqrt{30}.$$

$$y = \frac{30}{\sqrt{30}} = \sqrt{30}$$



By the First Derivative Test, the dimensions $(x+2)$ by $(y+2)$ are $(2 + \sqrt{30})$ by $(2 + \sqrt{30})$ (approximately 7.477 by 7.477). These dimensions yield a minimum area.

$$31. V = \pi r^2 h = 22 \text{ cubic inches or } h = \frac{22}{\pi r^2}$$

(a)	Radius, r	Height	Surface Area
	0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$
	0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$
	0.6	$\frac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \frac{22}{\pi(0.6)^2}\right] \approx 75.6$
	0.8	$\frac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \frac{22}{\pi(0.8)^2}\right] \approx 59.0$

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31. —CONTINUED—

(b)

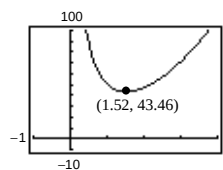
Radius, r	Height	Surface Area
0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\frac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \frac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\frac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \frac{22}{\pi(0.8)^2}\right] \approx 59.0$
1.0	$\frac{22}{\pi(1.0)^2}$	$2\pi(1.0)\left[1.0 + \frac{22}{\pi(1.0)^2}\right] \approx 50.3$
1.2	$\frac{22}{\pi(1.2)^2}$	$2\pi(1.2)\left[1.2 + \frac{22}{\pi(1.2)^2}\right] \approx 45.7$
1.4	$\frac{22}{\pi(1.4)^2}$	$2\pi(1.4)\left[1.4 + \frac{22}{\pi(1.4)^2}\right] \approx 43.7$
1.6	$\frac{22}{\pi(1.6)^2}$	$2\pi(1.6)\left[1.6 + \frac{22}{\pi(1.6)^2}\right] \approx 43.6$
1.8	$\frac{22}{\pi(1.8)^2}$	$2\pi(1.8)\left[1.8 + \frac{22}{\pi(1.8)^2}\right] \approx 44.8$
2.0	$\frac{22}{\pi(2.0)^2}$	$2\pi(2.0)\left[2.0 + \frac{22}{\pi(2.0)^2}\right] \approx 47.1$

The minimum seems to be about 43.6 for $r = 1.6$.

$$(c) S = 2\pi r^2 + 2\pi rh$$

$$= 2\pi r(r + h) = 2\pi r\left[r + \frac{22}{\pi r^2}\right] = 2\pi r^2 + \frac{44}{r}$$

(d)



The minimum seems to be 43.46 for $r \approx 1.52$.

$$(e) \frac{dS}{dr} = 4\pi r - \frac{44}{r^2} = 0 \text{ when } r = \sqrt[3]{11/\pi} \approx 1.52 \text{ in.}$$

$$h = \frac{22}{\pi r^2} \approx 3.04 \text{ in.}$$

Note: Notice that

$$h = \frac{22}{\pi r^2} = \frac{22}{\pi(11/\pi)^{2/3}} = 2\left(\frac{11^{1/3}}{\pi^{1/3}}\right) = 2r.$$

33. Let x be the sides of the square ends and y the length of the package.

$$P = 4x + y = 108 \Rightarrow y = 108 - 4x$$

$$V = x^2y = x^2(108 - 4x) = 108x^2 - 4x^3$$

$$\frac{dV}{dx} = 216x - 12x^2$$

$$= 12x(18 - x) = 0 \text{ when } x = 18.$$

$$\frac{d^2V}{dx^2} = 216 - 24x = -216 < 0 \text{ when } x = 18.$$

The volume is maximum when $x = 18$ inches and $y = 108 - 4(18) = 36$ inches.

35. $V = \frac{1}{3}\pi x^2 h = \frac{1}{3}\pi x^2(r + \sqrt{r^2 - x^2})$ (see figure)

$$\frac{dV}{dx} = \frac{1}{3}\pi \left[\frac{-x^3}{\sqrt{r^2 - x^2}} + 2x(r + \sqrt{r^2 - x^2}) \right] = \frac{\pi x}{3\sqrt{r^2 - x^2}} (2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2) = 0$$

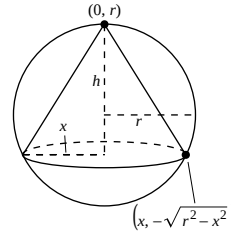
$$2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2 = 0$$

$$2r\sqrt{r^2 - x^2} = 3x^2 - 2r^2$$

$$4r^2(r^2 - x^2) = 9x^4 - 12x^2r^2 + 4r^4$$

$$0 = 9x^4 - 8x^2r^2 = x^2(9x^2 - 8r^2)$$

$$x = 0, \frac{2\sqrt{2}r}{3}$$



By the First Derivative Test, the volume is a maximum when

$$x = \frac{2\sqrt{2}r}{3} \text{ and } h = r + \sqrt{r^2 - x^2} = \frac{4r}{3}.$$

Thus, the maximum volume is

$$V = \frac{1}{3}\pi \left(\frac{8r^2}{9} \right) \left(\frac{4r}{3} \right) = \frac{32\pi r^3}{81} \text{ cubic units.}$$

37. No, there is no minimum area. If the sides are x and y , then $2x + 2y = 20 \Rightarrow y = 10 - x$.
The area is $A(x) = x(10 - x) = 10x - x^2$. This can be made arbitrarily small by selecting $x \approx 0$.

39. $V = 12 = \frac{4}{3}\pi r^3 + \pi r^2 h$

$$h = \frac{12 - (4/3)\pi r^3}{\pi r^2} = \frac{12}{\pi r^2} - \frac{4}{3}r$$

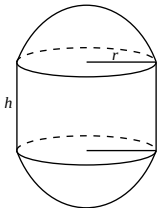
$$S = 4\pi r^2 + 2\pi r h = 4\pi r^2 + 2\pi r \left(\frac{12}{\pi r^2} - \frac{4}{3}r \right)$$

$$= 4\pi r^2 + \frac{24}{r} - \frac{8}{3}\pi r^2 = \frac{4}{3}\pi r^2 + \frac{24}{r}$$

$$\frac{dS}{dr} = \frac{8}{3}\pi r - \frac{24}{r^2} = 0 \text{ when } r = \sqrt[3]{9/\pi} \approx 1.42 \text{ cm.}$$

$$\frac{d^2S}{dr^2} = \frac{8}{3}\pi + \frac{48}{r^3} > 0 \text{ when } r = \sqrt[3]{9/\pi} \text{ cm.}$$

The surface area is minimum when $r = \sqrt[3]{9/\pi}$ cm and $h = 0$. The resulting solid is a sphere of radius $r \approx 1.42$ cm.



41. Let x be the length of a side of the square and y the length of a side of the triangle.

$$4x + 3y = 10$$

$$A = x^2 + \frac{1}{2}y \left(\frac{\sqrt{3}}{2}y \right)$$

$$= \frac{(10 - 3y)^2}{16} + \frac{\sqrt{3}}{4}y^2$$

$$\frac{dA}{dy} = \frac{1}{8}(10 - 3y)(-3) + \frac{\sqrt{3}}{2}y = 0$$

$$-30 + 9y + 4\sqrt{3}y = 0$$

$$y = \frac{30}{9 + 4\sqrt{3}}$$

$$\frac{d^2A}{dy^2} = \frac{9 + 4\sqrt{3}}{8} > 0$$

A is minimum when

$$y = \frac{30}{9 + 4\sqrt{3}} \text{ and } x = \frac{10\sqrt{3}}{9 + 4\sqrt{3}}.$$

43. Let S be the strength and k the constant of proportionality. Given $h^2 + w^2 = 24^2$, $h^2 = 24^2 - w^2$,

$$S = kwh^2$$

$$S = kw(576 - w^2) = k(576w - w^3)$$

$$\frac{dS}{dw} = k(576 - 3w^2) = 0 \text{ when } w = 8\sqrt{3}, h = 8\sqrt{6}.$$

$$\frac{d^2S}{dw^2} = -6kw < 0 \text{ when } w = 8\sqrt{3}.$$

These values yield a maximum.

47. $\sin \alpha = \frac{h}{s} \Rightarrow s = \frac{h}{\sin \alpha}, 0 < \alpha < \frac{\pi}{2}$

$$\tan \alpha = \frac{h}{2} \Rightarrow h = 2 \tan \alpha \Rightarrow s = \frac{2 \tan \alpha}{\sin \alpha} = 2 \sec \alpha$$

$$I = \frac{k \sin \alpha}{s^2} = \frac{k \sin \alpha}{4 \sec^2 \alpha} = \frac{k}{4} \sin \alpha \cos^2 \alpha$$

$$\frac{dI}{d\alpha} = \frac{k}{4} [\sin \alpha (-2 \sin \alpha \cos \alpha) + \cos^2 \alpha (\cos \alpha)]$$

$$= \frac{k}{4} \cos \alpha [\cos^2 \alpha - 2 \sin^2 \alpha]$$

$$= \frac{k}{4} \cos \alpha [1 - 3 \sin^2 \alpha]$$

$$= 0 \text{ when } \alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \text{ or when } \sin \alpha = \pm \frac{1}{\sqrt{3}}.$$

Since α is acute, we have

$$\sin \alpha = \frac{1}{\sqrt{3}} \Rightarrow h = 2 \tan \alpha = 2 \left(\frac{1}{\sqrt{2}} \right) = \sqrt{2} \text{ feet.}$$

Since $(d^2I)/(d\alpha^2) = (k/4) \sin \alpha (9 \sin^2 \alpha - 7) < 0$ when $\sin \alpha = 1/\sqrt{3}$, this yields a maximum.

49. $S = \sqrt{x^2 + 4}, L = \sqrt{1 + (3 - x)^2}$

$$\text{Time} = T = \frac{\sqrt{x^2 + 4}}{2} + \frac{\sqrt{x^2 - 6x + 10}}{4}$$

$$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2 + 4}} + \frac{x - 3}{4\sqrt{x^2 - 6x + 10}} = 0$$

$$\frac{x^2}{x^2 + 4} = \frac{9 - 6x + x^2}{4(x^2 - 6x + 10)}$$

$$x^4 - 6x^3 + 9x^2 + 8x - 12 = 0$$

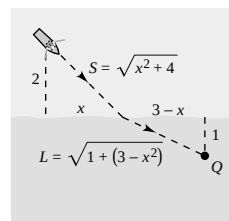
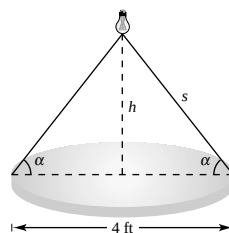
You need to find the roots of this equation in the interval $[0, 3]$. By using a computer or graphics calculator, you can determine that this equation has only one root in this interval ($x = 1$). Testing at this value and at the endpoints, you see that $x = 1$ yields the minimum time. Thus, the man should row to a point 1 mile from the nearest point on the coast.

45. $R = \frac{v_0^2}{g} \sin 2\theta$

$$\frac{dR}{d\theta} = \frac{2v_0^2}{g} \cos 2\theta = 0 \text{ when } \theta = \frac{\pi}{4}, \frac{3\pi}{4}.$$

$$\frac{d^2R}{d\theta^2} = -\frac{4v_0^2}{g} \sin 2\theta < 0 \text{ when } \theta = \frac{\pi}{4}.$$

By the Second Derivative Test, R is maximum when $\theta = \pi/4$.



$$51. T = \frac{\sqrt{x^2 + 4}}{v_1} + \frac{\sqrt{x^2 - 6x + 10}}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2 + 4}} + \frac{x - 3}{v_2\sqrt{x^2 - 6x + 10}} = 0$$

Since

$$\frac{x}{\sqrt{x^2 + 4}} = \sin \theta_1 \text{ and } \frac{x - 3}{\sqrt{x^2 - 6x + 10}} = -\sin \theta_2$$

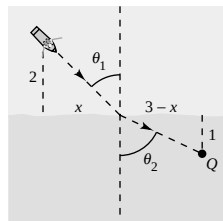
we have

$$\frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0 \Rightarrow \frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.$$

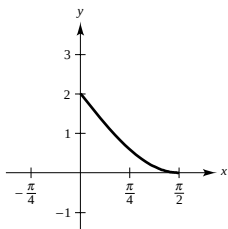
Since

$$\frac{d^2T}{dx^2} = \frac{4}{v_1(x^2 + 4)^{3/2}} + \frac{1}{v_2(x^2 - 6x + 10)^{3/2}} > 0$$

this condition yields a minimum time.

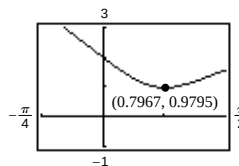


$$53. f(x) = 2 - 2 \sin x$$



- (a) Distance from origin to y-intercept is 2.
Distance from origin to x-intercept is $\pi/2 \approx 1.57$.

$$(b) d = \sqrt{x^2 + y^2} = \sqrt{x^2 + (2 - 2 \sin x)^2}$$



Minimum distance = 0.9795 at $x = 0.7967$.

$$(c) \text{ Let } f(x) = d^2(x) = x^2 + (2 - 2 \sin x)^2.$$

$$f'(x) = 2x + 2(2 - 2 \sin x)(-2 \cos x)$$

Setting $f'(x) = 0$, you obtain $x \approx 0.7967$, which corresponds to $d = 0.9795$.

$$55. F \cos \theta = k(W - F \sin \theta)$$

$$F = \frac{kW}{\cos \theta + k \sin \theta}$$

$$\frac{dF}{d\theta} = \frac{-kW(k \cos \theta - \sin \theta)}{(\cos \theta + k \sin \theta)^2} = 0$$

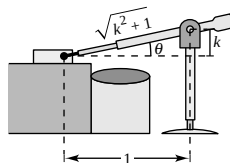
$$k \cos \theta = \sin \theta \Rightarrow k = \tan \theta \Rightarrow \theta = \arctan k$$

Since

$$\cos \theta + k \sin \theta = \frac{1}{\sqrt{k^2 + 1}} + \frac{k^2}{\sqrt{k^2 + 1}} = \sqrt{k^2 + 1},$$

the minimum force is

$$F = \frac{kW}{\cos \theta + k \sin \theta} = \frac{kW}{\sqrt{k^2 + 1}}.$$



57. (a)

Base 1	Base 2	Altitude	Area
8	$8 + 16 \cos 10^\circ$	$8 \sin 10^\circ$	≈ 22.1
8	$8 + 16 \cos 20^\circ$	$8 \sin 20^\circ$	≈ 42.5
8	$8 + 16 \cos 30^\circ$	$8 \sin 30^\circ$	≈ 59.7
8	$8 + 16 \cos 40^\circ$	$8 \sin 40^\circ$	≈ 72.7
8	$8 + 16 \cos 50^\circ$	$8 \sin 50^\circ$	≈ 80.5
8	$8 + 16 \cos 60^\circ$	$8 \sin 60^\circ$	≈ 83.1

(b)

Base 1	Base 2	Altitude	Area
8	$8 + 16 \cos 10^\circ$	$8 \sin 10^\circ$	≈ 22.1
8	$8 + 16 \cos 20^\circ$	$8 \sin 20^\circ$	≈ 42.5
8	$8 + 16 \cos 30^\circ$	$8 \sin 30^\circ$	≈ 59.7
8	$8 + 16 \cos 40^\circ$	$8 \sin 40^\circ$	≈ 72.7
8	$8 + 16 \cos 50^\circ$	$8 \sin 50^\circ$	≈ 80.5
8	$8 + 16 \cos 60^\circ$	$8 \sin 60^\circ$	≈ 83.1
8	$8 + 16 \cos 70^\circ$	$8 \sin 70^\circ$	≈ 80.7
8	$8 + 16 \cos 80^\circ$	$8 \sin 80^\circ$	≈ 74.0
8	$8 + 16 \cos 90^\circ$	$8 \sin 90^\circ$	≈ 64.0

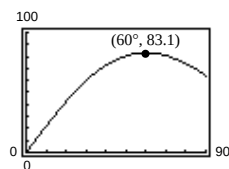
The maximum cross-sectional area is approximately 83.1 square feet.

$$(c) A = (a + b) \frac{h}{2}$$

$$= [8 + (8 + 16 \cos \theta)] \frac{8 \sin \theta}{2}$$

$$= 64(1 + \cos \theta) \sin \theta, 0^\circ < \theta < 90^\circ$$

(e)



$$(d) \frac{dA}{d\theta} = 64(1 + \cos \theta) \cos \theta + (-64 \sin \theta) \sin \theta$$

$$= 64(\cos \theta + \cos^2 \theta - \sin^2 \theta)$$

$$= 64(2 \cos^2 \theta + \cos \theta - 1)$$

$$= 64(2 \cos \theta - 1)(\cos \theta + 1)$$

$$= 0 \text{ when } \theta = 60^\circ, 180^\circ, 300^\circ.$$

The maximum occurs when $\theta = 60^\circ$.

$$59. C = 100 \left(\frac{200}{x^2} + \frac{x}{x+30} \right), 1 \leq x$$

$$C' = 100 \left(-\frac{400}{x^3} + \frac{30}{(x+30)^2} \right)$$

Approximation: $x \approx 40.45$ units, or 4045 units

$$61. S_1 = (4m - 1)^2 + (5m - 6)^2 + (10m - 3)^2$$

$$\frac{dS_1}{dm} = 2(4m - 1)(4) + 2(5m - 6)(5) + 2(10m - 3)(10) = 282m - 128 = 0 \text{ when } m = \frac{64}{141}.$$

$$\text{Line: } y = \frac{64}{141}x$$

$$S = \left| 4 \left(\frac{64}{141} \right) - 1 \right| + \left| 5 \left(\frac{64}{141} \right) - 6 \right| + \left| 10 \left(\frac{64}{141} \right) - 3 \right|$$

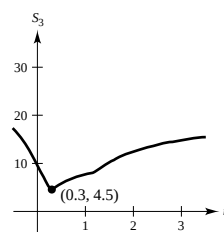
$$= \left| \frac{256}{141} - 1 \right| + \left| \frac{320}{141} - 6 \right| + \left| \frac{640}{141} - 3 \right| = \frac{858}{141} \approx 6.1 \text{ mi}$$

$$63. S_3 = \frac{|4m - 1|}{\sqrt{m^2 + 1}} + \frac{|5m - 6|}{\sqrt{m^2 + 1}} + \frac{|10m - 3|}{\sqrt{m^2 + 1}}$$

Using a graphing utility, you can see that the minimum occurs when $x \approx 0.3$.

$$\text{Line: } y \approx 0.3x$$

$$S_3 = \frac{|4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3|}{\sqrt{(0.3)^2 + 1}} \approx 4.5 \text{ mi.}$$



Section 3.8 Newton's Method

1. $f(x) = x^2 - 3$

$f'(x) = 2x$

$x_1 = 1.7$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.7000	-0.1100	3.4000	-0.0324	1.7324
2	1.7324	0.0012	3.4648	0.0003	1.7321

3. $f(x) = \sin x$

$f'(x) = \cos x$

$x_1 = 3$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.0000	0.1411	-0.9900	-0.1425	3.1425
2	3.1425	-0.0009	-1.0000	0.0009	3.1416

5. $f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1$

Approximation of the zero of f is 0.682.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	-0.3750	1.7500	-0.2143	0.7143
2	0.7143	0.0788	2.5307	0.0311	0.6832
3	0.6832	0.0021	2.4003	0.0009	0.6823

7. $f(x) = 3\sqrt{x-1} - x$

$f'(x) = \frac{3}{2\sqrt{x-1}} - 1$

Approximation of the zero of f is 1.146.

Similarly, the other zero is approximately 7.854.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.2000	0.1416	2.3541	0.0602	1.1398
2	1.1398	-0.0181	3.0118	-0.0060	1.1458
3	1.1458	-0.0003	2.9284	-0.0001	1.1459

9. $f(x) = x^3 + 3$

$f'(x) = 3x^2$

Approximation of the zero of f is -1.442.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.5000	-0.3750	6.7500	-0.0556	-1.4444
2	-1.4444	-0.0134	6.2589	-0.0021	-1.4423
3	-1.4423	-0.0003	6.2407	-0.0001	-1.4422

11. $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$

$f'(x) = 3x^2 - 7.8x + 4.79$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	-0.3360	1.6400	-0.2049	0.7049
2	0.7049	-0.0921	0.7824	-0.1177	0.8226
3	0.8226	-0.0231	0.4037	-0.0573	0.8799
4	0.8799	-0.0045	0.2495	-0.0181	0.8980
5	0.8980	-0.0004	0.2048	-0.0020	0.9000
6	0.9000	0.0000	0.2000	0.0000	0.9000

Approximation of the zero of f is 0.900.

—CONTINUED—

11. —CONTINUED—

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.1	0.0000	-0.1600	-0.0000	1.1000

Approximation of the zero of f is 1.100.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.9	0.0000	0.8000	0.0000	1.9000

Approximation of the zero of f is 1.900.

13. $f(x) = x + \sin(x + 1)$

$f'(x) = 1 + \cos(x + 1)$

Approximation of the zero of f is -0.489.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	-0.0206	1.8776	-0.0110	-0.4890
2	-0.4890	0.0000	1.8723	0.0000	-0.4890

15. $h(x) = f(x) - g(x) = 2x + 1 - \sqrt{x + 4}$

$h'(x) = 2 - \frac{1}{2\sqrt{x + 4}}$

Point of intersection of the graphs of f and g occurs when $x \approx 0.569$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	0.6000	0.0552	1.7669	0.0313	0.5687
2	0.5687	-0.0001	1.7661	0.0000	0.5687

17. $h(x) = f(x) - g(x) = x - \tan x$

$h'(x) = 1 - \sec^2 x$

Point of intersection of the graphs of f and g occurs when $x \approx 4.493$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	4.5000	-0.1373	-21.5048	0.0064	4.4936
2	4.4936	-0.0039	-20.2271	0.0002	4.4934

19. $f(x) = x^2 - a = 0$

$f'(x) = 2x$

$x_{i+1} = x_i - \frac{x_i^2 - a}{2x_i}$

$= \frac{2x_i^2 - x_i^2 + a}{2x_i} = \frac{x_i^2 + a}{2x_i} = \frac{x_i}{2} + \frac{a}{2x_i}$

21. $x_{i+1} = \frac{x_i^2 + 7}{2x_i}$

i	1	2	3	4	5
x_i	2.0000	2.7500	2.6477	2.6458	2.6458

$\sqrt{7} \approx 2.646$

23. $x_{i+1} = \frac{3x_i^4 + 6}{4x_i^3}$

i	1	2	3	4
x_i	1.5000	1.5694	1.5651	1.5651

$\sqrt[4]{6} \approx 1.565$

25. $f(x) = 1 + \cos x$

$$f'(x) = -\sin x$$

Approximation of the zero: 3.141

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.0000	0.0100	-0.1411	-0.0709	3.0709
2	3.0709	0.0025	-0.0706	-0.0354	3.1063
3	3.1063	0.0006	-0.0353	-0.0176	3.1239
4	3.1239	0.0002	-0.0177	-0.0088	3.1327
5	3.1327	0.0000	-0.0089	-0.0044	3.1371
6	3.1371	0.0000	-0.0045	-0.0022	3.1393
7	3.1393	0.0000	-0.0023	-0.0011	3.1404
8	3.1404	0.0000	-0.0012	-0.0006	3.1410

27. $y = 2x^3 - 6x^2 + 6x - 1 = f(x)$

$$y' = 6x^2 - 12x + 6 = f'(x)$$

$$x_1 = 1$$

$f'(x) = 0$; therefore, the method fails.

n	x_n	$f(x_n)$	$f'(x_n)$
1	1	1	0

29. $y = -x^3 + 6x^2 - 10x + 6 = f(x)$

$$y' = -3x^2 + 12x - 10 = f'(x)$$

$$x_1 = 2$$

$$x_2 = 1$$

$$x_3 = 2$$

$x_4 = 1$ and so on.

Fails to converge

31. Answers will vary. See page 222.

Newton's Method uses tangent lines to approximate c such that $f(c) = 0$.

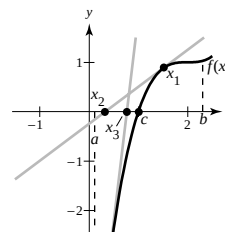
First, estimate an initial x_1 close to c (see graph).

Then determine x_2 by $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

Calculate a third estimate by $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$.

Continue this process until $|x_n - x_{n+1}|$ is within the desired accuracy.

Let x_{n+1} be the final approximation of c .



33. Let $g(x) = f(x) - x = \cos x - x$

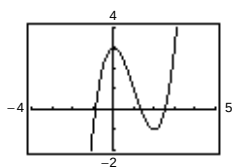
$$g'(x) = -\sin x - 1.$$

The fixed point is approximately 0.74.

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	1.0000	-0.4597	-1.8415	0.2496	0.7504
2	0.7504	-0.0190	-1.6819	0.0113	0.7391
3	0.7391	0.0000	-1.6736	0.0000	0.7391

35. $f(x) = x^3 - 3x^2 + 3$, $f'(x) = 3x^2 - 6x$

(a)



(c) $x_1 = \frac{1}{4}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 2.405$$

Continuing, the zero is 2.532.

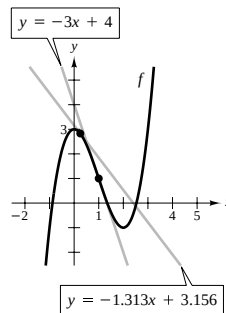
(e) If the initial guess x_1 is not “close to” the desired zero of the function, the x-intercept of the tangent line may approximate another zero of the function.

(b) $x_1 = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.333$$

Continuing, the zero is 1.347.

(d)



The x-intercepts correspond to the values resulting from the first iteration of Newton's Method.

37. $f(x) = \frac{1}{x} - a = 0$

$$f'(x) = -\frac{1}{x^2}$$

$$x_{n+1} = x_n - \frac{(1/x_n) - a}{-1/x_n^2} = x_n + x_n^2 \left(\frac{1}{x_n} - a \right) = x_n + x_n - x_n^2 a = 2x_n - x_n^2 a = x_n(2 - ax_n)$$

39. $f(x) = x \cos x$, $(0, \pi)$

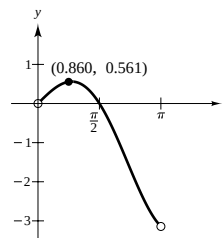
$$f'(x) = -x \sin x + \cos x = 0$$

Letting $F(x) = f'(x)$, we can use Newton's Method as follows.

$$[F'(x) = -2 \sin x + x \cos x]$$

n	x_n	$F(x_n)$	$F'(x_n)$	$\frac{F(x_n)}{F'(x_n)}$	$x_n - \frac{F(x_n)}{F'(x_n)}$
1	0.9000	-0.0834	-2.1261	0.0392	0.8608
2	0.8608	-0.0010	-2.0778	0.0005	0.8603

Approximation to the critical number: 0.860



41. $y = f(x) = 4 - x^2, (1, 0)$

$$d = \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{(x-1)^2 + (4-x^2)^2} = \sqrt{x^4 - 7x^2 - 2x + 17}$$

d is minimized when $D = x^4 - 7x^2 - 2x + 17$ is a minimum.

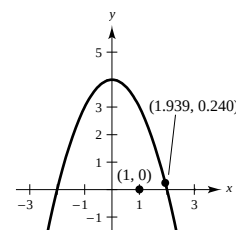
$$g(x) = D' = 4x^3 - 14x - 2$$

$$g'(x) = 12x^2 - 14$$

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	2.0000	2.0000	34.0000	0.0588	1.9412
2	1.9412	0.0830	31.2191	0.0027	1.9385
3	1.9385	-0.0012	31.0934	0.0000	1.9385

$$x \approx 1.939$$

Point closest to $(1, 0)$ is $\approx (1.939, 0.240)$.



43. Minimize: $T = \frac{\text{Distance rowed}}{\text{Rate rowed}} + \frac{\text{Distance walked}}{\text{Rate walked}}$

$$T = \frac{\sqrt{x^2 + 4}}{3} + \frac{\sqrt{x^2 - 6x + 10}}{4}$$

$$T' = \frac{x}{3\sqrt{x^2 + 4}} + \frac{x - 3}{4\sqrt{x^2 - 6x + 10}} = 0$$

$$4x\sqrt{x^2 - 6x + 10} = -3(x - 3)\sqrt{x^2 + 4}$$

$$16x^2(x^2 - 6x + 10) = 9(x - 3)^2(x^2 + 4)$$

$$7x^4 - 42x^3 + 43x^2 + 216x - 324 = 0$$

Let $f(x) = 7x^4 - 42x^3 + 43x^2 + 216x - 324$ and $f'(x) = 28x^3 - 126x^2 + 86x + 216$. Since $f(1) = -100$ and $f(2) = 56$, the solution is in the interval $(1, 2)$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.7000	19.5887	135.6240	0.1444	1.5556
2	1.5556	-1.0480	150.2780	-0.0070	1.5626
3	1.5626	0.0014	49.5591	0.0000	1.5626

Approximation: $x \approx 1.563$ miles

45. $2,500,000 = -76x^3 + 4830x^2 - 320,000$

$$76x^3 - 4830x^2 + 2,820,000 = 0$$

Let $f(x) = 76x^3 - 4830x^2 + 2,820,000$

$$f'(x) = 228x^2 - 9660x.$$

From the graph, choose $x_1 = 40$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	40.0000	-44000.0000	-21600.0000	2.0370	37.9630
2	37.9630	17157.6209	-38131.4039	-0.4500	38.4130
3	38.4130	780.0914	-34642.2263	-0.0225	38.4355
4	38.4355	2.6308	-34465.3435	-0.0001	38.4356

The zero occurs when $x \approx 38.4356$ which corresponds to \$384,356.

47. False. Let $f(x) = (x^2 - 1)/(x - 1)$. $x = 1$ is a discontinuity. It is not a zero of $f(x)$. This statement would be true if $f(x) = p(x)/q(x)$ is given in **reduced** form.

49. True

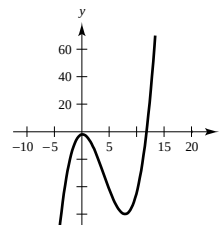
51. $f(x) = \frac{1}{4}x^3 - 3x^2 + \frac{3}{4}x - 2$

$$f'(x) = \frac{3}{4}x^2 - 6x + \frac{3}{4}$$

Let $x_1 = 12$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	12.0000	7.0000	36.7500	0.1905	11.8095
2	11.8095	0.2151	34.4912	0.0062	11.8033
3	11.8033	0.0015	34.4186	0.0000	11.8033

Approximation: $x \approx 11.803$



Section 3.9 Differentials

1. $f(x) = x^2$

$$f'(x) = 2x$$

Tangent line at $(2, 4)$: $y - f(2) = f'(2)(x - 2)$

$$y - 4 = 4(x - 2)$$

$$y = 4x - 4$$

x	1.9	1.99	2	2.01	2.1
$f(x) = x^2$	3.6100	3.9601	4	4.0401	4.4100
$T(x) = 4x - 4$	3.6000	3.9600	4	4.0400	4.4000

3. $f(x) = x^5$

$$f'(x) = 5x^4$$

Tangent line at $(2, 32)$: $y - f(2) = f'(2)(x - 2)$

$$y - 32 = 80(x - 2)$$

$$y = 80x - 128$$

x	1.9	1.99	2	2.01	2.1
$f(x) = x^5$	24.7610	31.2080	32	32.8080	40.8410
$T(x) = 80x - 128$	24.0000	31.2000	32	32.8000	40.0000

5. $f(x) = \sin x$

$$f'(x) = \cos x$$

Tangent line at $(2, \sin 2)$:

$$y - f(2) = f'(2)(x - 2)$$

$$y - \sin 2 = (\cos 2)(x - 2)$$

$$y = (\cos 2)(x - 2) + \sin 2$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \sin x$	0.9463	0.9134	0.9093	0.9051	0.8632
$T(x) = (\cos 2)(x - 2) + \sin 2$	0.9509	0.9135	0.9093	0.9051	0.8677

7. $y = f(x) = \frac{1}{2}x^3$, $f'(x) = \frac{3}{2}x^2$, $x = 2$, $\Delta x = dx = 0.1$

$$\Delta y = f(x + \Delta x) - f(x)$$

$$= f(2.1) - f(2)$$

$$= 0.6305$$

$$dy = f'(x)dx$$

$$= f'(2)(0.1)$$

$$= 6(0.1) = 0.6$$

9. $y = f(x) = x^4 + 1, f'(x) = 4x^3, x = -1, \Delta x = dx = 0.01$

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x) \\ &= f(-0.99) - f(-1) \\ &= [(-0.99)^4 + 1] - [(-1)^4 + 1] \approx -0.0394\end{aligned}$$

$$\begin{aligned}dy &= f'(x) dx \\ &= f'(-1)(0.01) \\ &= (-4)(0.01) = -0.04\end{aligned}$$

11. $y = 3x^2 - 4$

$$dy = 6x dx$$

13. $y = \frac{x+1}{2x-1}$

$$dy = \frac{-3}{(2x-1)^2} dx$$

15. $y = x\sqrt{1-x^2}$

$$dy = \left(x \frac{-x}{\sqrt{1-x^2}} + \sqrt{1-x^2} \right) dx = \frac{1-2x^2}{\sqrt{1-x^2}} dx$$

17. $y = 2x - \cot^2 x$

$$\begin{aligned}dy &= (2 + 2 \cot x \csc^2 x) dx \\ &= (2 + 2 \cot x + 2 \cot^3 x) dx\end{aligned}$$

19. $y = \frac{1}{3} \cos\left(\frac{6\pi x - 1}{2}\right)$

$$dy = -\pi \sin\left(\frac{6\pi x - 1}{2}\right) dx$$

21. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$
 $\approx 1 + (1)(-0.1) = 0.9$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$
 $\approx 1 + (1)(0.04) = 1.04$

23. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$
 $\approx 1 + \left(-\frac{1}{2}\right)(-0.1) = 1.05$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$
 $\approx 1 + \left(-\frac{1}{2}\right)(0.04) = 0.98$

25. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$
 $\approx 8 + \left(-\frac{1}{2}\right)(-0.07) = 8.035$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$
 $\approx 8 + \left(-\frac{1}{2}\right)(0.1) = 7.95$

27. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$
 $\approx 8 + 0(-0.07) = 8$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$
 $\approx 8 + 0(0.1) = 8$

29. $A = x^2$

$$x = 12$$

$$\Delta x = dx = \pm \frac{1}{64}$$

$$dA = 2x dx$$

$$\begin{aligned}\Delta A &\approx dA = 2(12)\left(\pm \frac{1}{64}\right) \\ &= \pm \frac{3}{8} \text{ square inches}\end{aligned}$$

31. $A = \pi r^2$

$$r = 14$$

$$\Delta r = dr = \pm \frac{1}{4}$$

$$\begin{aligned}\Delta A &\approx dA = 2\pi r dr = \pi(28)\left(\pm \frac{1}{4}\right) \\ &= \pm 7\pi \text{ square inches}\end{aligned}$$

33. (a)
- $x = 15$
- centimeter

$$\Delta x = dx = \pm 0.05 \text{ centimeters}$$

$$A = x^2$$

$$dA = 2x dx = 2(15)(\pm 0.05)$$

$$= \pm 1.5 \text{ square centimeters}$$

Percentage error:

$$\frac{dA}{A} = \frac{\pm 1.5}{(15)^2} = 0.00666 \dots = \frac{2}{3}\%$$

$$(b) \frac{dA}{A} = \frac{2x dx}{x^2} = \frac{2 dx}{x} \leq 0.025$$

$$\frac{dx}{x} \leq \frac{0.025}{2} = 0.0125 = 1.25\%$$

- 37.
- $V = \pi r^2 h = 40\pi r^2$
- ,
- $r = 5$
- cm,
- $h = 40$
- cm,
- $dr = 0.2$
- cm

$$\Delta V \approx dV = 80\pi r dr = 80\pi(5)(0.2) = 80\pi \text{ cm}^3$$

39. (a)
- $T = 2\pi\sqrt{L/g}$

$$dT = \frac{\pi}{g\sqrt{L/g}} dL$$

Relative error:

$$\frac{dT}{T} = \frac{(\pi dL)/(g\sqrt{L/g})}{2\pi\sqrt{L/g}}$$

$$= \frac{dL}{2L}$$

$$= \frac{1}{2} (\text{relative error in } L)$$

$$= \frac{1}{2}(0.005) = 0.0025$$

$$\text{Percentage error: } \frac{dT}{T}(100) = 0.25\% = \frac{1}{4}\%$$

- 41.
- $\theta = 26^\circ 45' = 26.75^\circ$

$$d\theta = \pm 15' = \pm 0.25^\circ$$

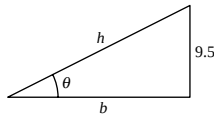
$$(a) h = 9.5 \csc \theta$$

$$dh = -9.5 \csc \theta \cot \theta d\theta$$

$$\frac{dh}{h} = -\cot \theta d\theta$$

$$\left| \frac{dh}{h} \right| = (\cot 26.75^\circ)(0.25^\circ)$$

Converting to radians, $(\cot 0.4669)(0.0044)$
 $\approx 0.0087 = 0.87\%$ (in radians).



- 35.
- $r = 6$
- inches

$$\Delta r = dr = \pm 0.02 \text{ inches}$$

$$(a) V = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2 dr = 4\pi(6)^2(\pm 0.02) = \pm 2.88\pi \text{ cubic inches}$$

$$(b) S = 4\pi r^2$$

$$dS = 8\pi r dr = 8\pi(6)(\pm 0.02) = \pm 0.96\pi \text{ square inches}$$

$$(c) \text{Relative error: } \frac{dV}{V} = \frac{4\pi r^2 dr}{(4/3)\pi r^3} = \frac{3dr}{r}$$

$$= \frac{3}{6}(0.02) = 0.01 = 1\%$$

$$\text{Relative error: } \frac{dS}{S} = \frac{8\pi r dr}{4\pi r^2} = \frac{2dr}{r}$$

$$= \frac{2(0.02)}{6} = 0.00666 \dots = \frac{2}{3}\%$$

$$(b) (0.0025)(3600)(24) = 216 \text{ seconds}$$

$$= 3.6 \text{ minutes}$$

$$(b) \left| \frac{dh}{h} \right| = \cot \theta d\theta \leq 0.02$$

$$\frac{d\theta}{\theta} \leq \frac{0.02}{\theta(\cot \theta)} = \frac{0.02 \tan \theta}{\theta}$$

$$\frac{d\theta}{\theta} \leq \frac{0.02 \tan 26.75^\circ}{26.75^\circ} \approx \frac{0.02 \tan 0.4669}{0.4669}$$

$$\approx 0.0216 = 2.16\% \text{ (in radians)}$$

43. $r = \frac{v_0^2}{32} (\sin 2\theta)$

$v_0 = 2200$ ft/sec

θ changes from 10° to 11°

$dr = \frac{(2200)^2}{16} (\cos 2\theta) d\theta$

$\theta = 10\left(\frac{\pi}{180}\right)$

$d\theta = (11 - 10)\frac{\pi}{180}$

$\Delta r \approx dr$

$= \frac{(2200)^2}{16} \cos\left(\frac{20\pi}{180}\right)\left(\frac{\pi}{180}\right) \approx 4961$ feet

≈ 4961 feet

47. Let $f(x) = \sqrt[4]{x}$, $x = 625$, $dx = -1$.

$f(x + \Delta x) \approx f(x) + f'(x) dx = \sqrt[4]{x} + \frac{1}{4^4 \sqrt{x^3}} dx$

$f(x + \Delta x) = \sqrt[4]{624} \approx \sqrt[4]{625} + \frac{1}{4(\sqrt[4]{625})^3}(-1)$

$= 5 - \frac{1}{500} = 4.998$

Using a calculator, $\sqrt[4]{624} \approx 4.9980$.

45. Let $f(x) = \sqrt{x}$, $x = 100$, $dx = -0.6$.

$f(x + \Delta x) \approx f(x) + f'(x) dx$

$= \sqrt{x} + \frac{1}{2\sqrt{x}} dx$

$f(x + \Delta x) = \sqrt{99.4}$

$\approx \sqrt{100} + \frac{1}{2\sqrt{100}}(-0.6) = 9.97$

Using a calculator: $\sqrt{99.4} \approx 9.96995$

49. Let $f(x) = \sqrt{x}$, $x = 4$, $dx = 0.02$, $f'(x) = 1/(2\sqrt{x})$.

Then

$f(4.02) \approx f(4) + f'(4) dx$

$\sqrt{4.02} \approx \sqrt{4} + \frac{1}{2\sqrt{4}}(0.02) = 2 + \frac{1}{4}(0.02).$

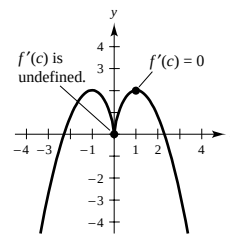
51. In general, when $\Delta x \rightarrow 0$, dy approaches Δy .

53. True

55. True

Review Exercises for Chapter 3

1. A number c in the domain of f is a critical number if $f'(c) = 0$ or f' is undefined at c .



3. $g(x) = 2x + 5 \cos x$, $[0, 2\pi]$

$g'(x) = 2 - 5 \sin x$

$= 0$ when $\sin x = \frac{2}{5}$.

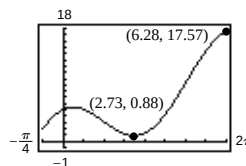
Critical numbers: $x \approx 0.41$, $x \approx 2.73$

Left endpoint: $(0, 5)$

Critical number: $(0.41, 5.41)$

Critical number: $(2.73, 0.88)$ Minimum

Right endpoint: $(2\pi, 17.57)$ Maximum

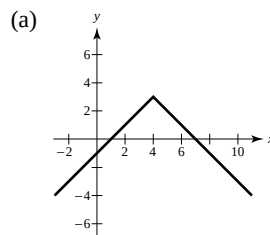


5. Yes. $f(-3) = f(2) = 0$. f is continuous on $[-3, 2]$, differentiable on $(-3, 2)$.

$$f'(x) = (x+3)(3x-1) = 0 \text{ for } x = \frac{1}{3}.$$

$$c = \frac{1}{3} \text{ satisfies } f'(c) = 0.$$

7. $f(x) = 3 - |x - 4|$



$$f(1) = f(7) = 0$$

- (b) f is not differentiable at $x = 4$.

9. $f(x) = x^{2/3}, 1 \leq x \leq 8$

$$f'(x) = \frac{2}{3}x^{-1/3}$$

$$\frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{8 - 1} = \frac{3}{7}$$

$$f'(c) = \frac{2}{3}c^{-1/3} = \frac{3}{7}$$

$$c = \left(\frac{14}{9}\right)^3 = \frac{2744}{729} \approx 3.764$$

13. $f(x) = Ax^2 + Bx + C$

$$f'(x) = 2Ax + B$$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{A(x_2^2 - x_1^2) + B(x_2 - x_1)}{x_2 - x_1}$$

$$= A(x_1 + x_2) + B$$

$$f'(c) = 2Ac + B = A(x_1 + x_2) + B$$

$$2Ac = A(x_1 + x_2)$$

$$c = \frac{x_1 + x_2}{2} = \text{Midpoint of } [x_1, x_2]$$

15. $f(x) = (x-1)^2(x-3)$

$$f'(x) = (x-1)^2(1) + (x-3)(2)(x-1)$$

$$= (x-1)(3x-7)$$

Critical numbers: $x = 1$ and $x = \frac{7}{3}$

Interval:	$-\infty < x < 1$	$1 < x < \frac{7}{3}$	$\frac{7}{3} < x < \infty$
Sign of $f'(x)$:	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion:	Increasing	Decreasing	Increasing

17. $h(x) = \sqrt{x}(x-3) = x^{3/2} - 3x^{1/2}$

Domain: $(0, \infty)$

$$h'(x) = \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}$$

$$= \frac{3}{2}x^{-1/2}(x-1) = \frac{3(x-1)}{2\sqrt{x}}$$

Critical number: $x = 1$

Interval:	$0 < x < 1$	$1 < x < \infty$
Sign of $h'(x)$:	$h'(x) < 0$	$h'(x) > 0$
Conclusion:	Decreasing	Increasing

19. $h(t) = \frac{1}{4}t^4 - 8t$

$h'(t) = t^3 - 8 = 0$ when $t = 2$.

Relative minimum: $(2, -12)$

Test Interval:	$-\infty < t < 2$	$2 < t < \infty$
Sign of $h'(t)$:	$h'(t) < 0$	$h'(t) > 0$
Conclusion:	Decreasing	Increasing

21. $y = \frac{1}{3} \cos(12t) - \frac{1}{4} \sin(12t)$

$v = y' = -4 \sin(12t) - 3 \cos(12t)$

(a) When $t = \frac{\pi}{8}$, $y = \frac{1}{4}$ inch and $v = y' = 4$ inches/second.

(b) $y' = -4 \sin(12t) - 3 \cos(12t) = 0$ when $\frac{\sin(12t)}{\cos(12t)} = -\frac{3}{4} \Rightarrow \tan(12t) = -\frac{3}{4}$.

Therefore, $\sin(12t) = -\frac{3}{5}$ and $\cos(12t) = \frac{4}{5}$. The maximum displacement is

$$y = \left(\frac{1}{3}\right)\left(\frac{4}{5}\right) - \frac{1}{4}\left(-\frac{3}{5}\right) = \frac{5}{12} \text{ inch.}$$

(c) Period: $\frac{2\pi}{12} = \frac{\pi}{6}$

Frequency: $\frac{1}{\pi/6} = \frac{6}{\pi}$

23. $f(x) = x + \cos x$, $0 \leq x \leq 2\pi$

$f'(x) = 1 - \sin x$

$f''(x) = -\cos x = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of inflection: $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

Test Interval:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f''(x)$:	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

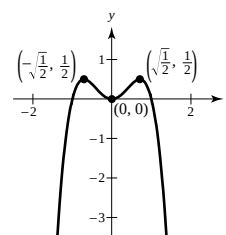
25. $g(x) = 2x^2(1 - x^2)$

$g'(x) = -4x(2x^2 - 1)$ Critical numbers: $x = 0, \pm \frac{1}{\sqrt{2}}$

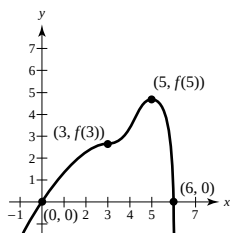
$g''(x) = 4 - 24x^2$

$g''(0) = 4 > 0$ Relative minimum at $(0, 0)$

$g''\left(\pm \frac{1}{\sqrt{2}}\right) = -8 < 0$ Relative maximums at $\left(\pm \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

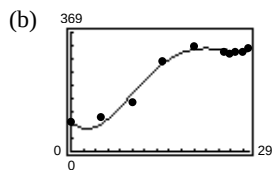


27.



29. The first derivative is positive and the second derivative is negative. The graph is increasing and is concave down.

31. (a) $D = 0.0034t^4 - 0.2352t^3 + 4.9423t^2 - 20.8641t + 94.4025$



(c) Maximum at (21.9, 319.5) (≈ 1992)

Minimum at (2.6, 69.6) (≈ 1972)

(d) Outlays increasing at greatest rate at the point of inflection (9.8, 173.7) (≈ 1979)

33. $\lim_{x \rightarrow \infty} \frac{2x^2}{3x^2 + 5} = \lim_{x \rightarrow \infty} \frac{2}{3 + 5/x^2} = \frac{2}{3}$

35. $\lim_{x \rightarrow \infty} \frac{5 \cos x}{x} = 0$, since $|5 \cos x| \leq 5$.

37. $h(x) = \frac{2x + 3}{x - 4}$

Discontinuity: $x = 4$

$$\lim_{x \rightarrow \infty} \frac{2x + 3}{x - 4} = \lim_{x \rightarrow \infty} \frac{2 + (3/x)}{1 - (4/x)} = 2$$

Vertical asymptote: $x = 4$

Horizontal asymptote: $y = 2$

39. $f(x) = \frac{3}{x} - 2$

Discontinuity: $x = 0$

$$\lim_{x \rightarrow \infty} \left(\frac{3}{x} - 2 \right) = -2$$

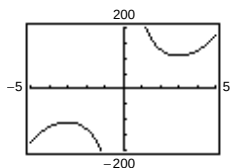
Vertical asymptote: $x = 0$

Horizontal asymptote: $y = -2$

41. $f(x) = x^3 + \frac{243}{x}$

Relative minimum: (3, 108)

Relative maximum: (-3, -108)

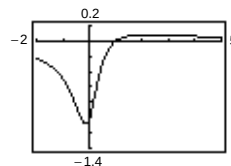


Vertical asymptote: $x = 0$

43. $f(x) = \frac{x - 1}{1 + 3x^2}$

Relative minimum: (-0.155, -1.077)

Relative maximum: (2.155, 0.077)



Horizontal asymptote: $y = 0$

45. $f(x) = 4x - x^2 = x(4 - x)$

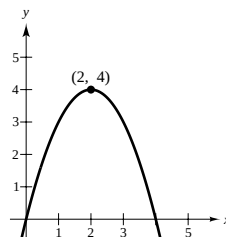
Domain: $(-\infty, \infty)$; Range: $(-\infty, 4)$

$$f'(x) = 4 - 2x = 0 \text{ when } x = 2.$$

$$f''(x) = -2$$

Therefore, (2, 4) is a relative maximum.

Intercepts: (0, 0), (4, 0)



47. $f(x) = x\sqrt{16 - x^2}$, Domain: $[-4, 4]$, Range: $[-8, 8]$

Domain: $[-4, 4]$; Range: $[-8, 8]$

$$f'(x) = \frac{16 - 2x^2}{\sqrt{16 - x^2}} = 0 \text{ when } x = \pm 2\sqrt{2} \text{ and undefined when } x = \pm 4.$$

$$f''(x) = \frac{2x(x^2 - 24)}{(16 - x^2)^{3/2}}$$

$$f'(-2\sqrt{2}) > 0$$

Therefore, $(-2\sqrt{2}, -8)$ is a relative minimum.

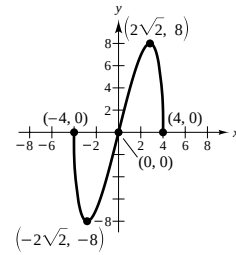
$$f''(2\sqrt{2}) < 0$$

Therefore, $(2\sqrt{2}, 8)$ is a relative maximum.

Point of inflection: $(0, 0)$

Intercepts: $(-4, 0)$, $(0, 0)$, $(4, 0)$

Symmetry with respect to origin



49. $f(x) = (x - 1)^3(x - 3)^2$

Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

$$f'(x) = (x - 1)^2(x - 3)(5x - 11) = 0 \text{ when } x = 1, \frac{11}{5}, 3.$$

$$f''(x) = 4(x - 1)(5x^2 - 22x + 23) = 0 \text{ when } x = 1, \frac{11 \pm \sqrt{6}}{5}.$$

$$f''(3) > 0$$

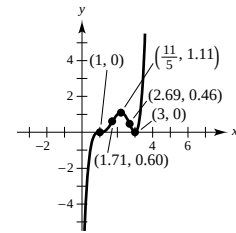
Therefore, $(3, 0)$ is a relative minimum.

$$f''\left(\frac{11}{5}\right) < 0$$

Therefore, $\left(\frac{11}{5}, \frac{3456}{3125}\right)$ is a relative maximum.

Points of inflection: $(1, 0)$, $\left(\frac{11 - \sqrt{6}}{5}, 0.60\right)$, $\left(\frac{11 + \sqrt{6}}{5}, 0.46\right)$

Intercepts: $(0, -9)$, $(1, 0)$, $(3, 0)$



51. $f(x) = x^{1/3}(x + 3)^{2/3}$

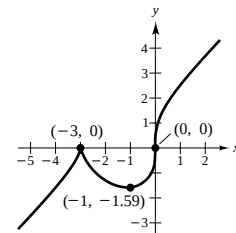
Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

$$f'(x) = \frac{x + 1}{(x + 3)^{1/3}x^{2/3}} = 0 \text{ when } x = -1 \text{ and undefined when } x = -3, 0.$$

$$f''(x) = \frac{-2}{x^{5/3}(x + 3)^{4/3}} \text{ is undefined when } x = 0, -3.$$

By the First Derivative Test $(-3, 0)$ is a relative maximum and $(-1, -\sqrt[3]{4})$ is a relative minimum. $(0, 0)$ is a point of inflection.

Intercepts: $(-3, 0)$, $(0, 0)$



53. $f(x) = \frac{x+1}{x-1}$

Domain: $(-\infty, 1), (1, \infty)$; Range: $(-\infty, 1), (1, \infty)$

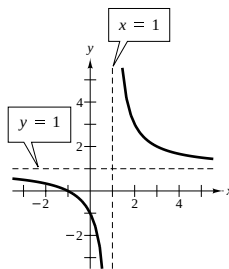
$$f'(x) = \frac{-2}{(x-1)^2} < 0 \text{ if } x \neq 1.$$

$$f''(x) = \frac{4}{(x-1)^3}$$

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 1$

Intercepts: $(-1, 0), (0, -1)$



55. $f(x) = \frac{4}{1+x^2}$

Domain: $(-\infty, \infty)$; Range: $(0, 4]$

$$f'(x) = \frac{-8x}{(1+x^2)^2} = 0 \text{ when } x = 0.$$

$$f''(x) = \frac{-8(1-3x^2)}{(1+x^2)^3} = 0 \text{ when } x = \pm \frac{\sqrt{3}}{3}.$$

$$f''(0) < 0$$

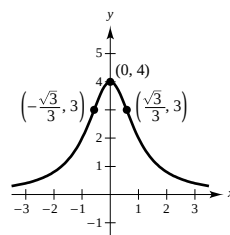
Therefore, $(0, 4)$ is a relative maximum.

Points of inflection: $(\pm \sqrt{3}/3, 3)$

Intercept: $(0, 4)$

Symmetric to the y-axis

Horizontal asymptote: $y = 0$



57. $f(x) = x^3 + x + \frac{4}{x}$

Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, -6], [6, \infty)$

$$f'(x) = 3x^2 + 1 - \frac{4}{x^2} = \frac{3x^4 + x^2 - 4}{x^2} = 0 \text{ when } x = \pm 1.$$

$$f''(x) = 6x + \frac{8}{x^3} = \frac{6x^4 + 8}{x^3} \neq 0$$

$$f''(-1) < 0$$

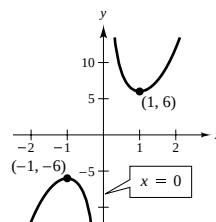
Therefore, $(-1, -6)$ is a relative maximum.

$$f''(1) > 0$$

Therefore, $(1, 6)$ is a relative minimum.

Vertical asymptote: $x = 0$

Symmetric with respect to origin



59. $f(x) = |x^2 - 9|$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$$f'(x) = \frac{2x(x^2 - 9)}{|x^2 - 9|} = 0 \text{ when } x = 0 \text{ and is undefined when } x = \pm 3.$$

$$f''(x) = \frac{2(x^2 - 9)}{|x^2 - 9|} \text{ is undefined at } x = \pm 3.$$

$$f''(0) < 0$$

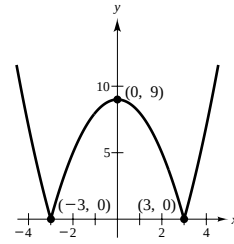
Therefore, $(0, 9)$ is a relative maximum.

Relative minima: $(\pm 3, 0)$

Points of inflection: $(\pm 3, 0)$

Intercepts: $(\pm 3, 0)$, $(0, 9)$

Symmetric to the y-axis



61. $f(x) = x + \cos x$

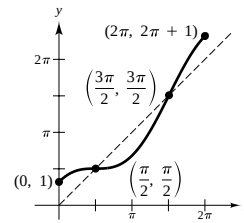
Domain: $[0, 2\pi]$; Range: $[1, 1 + 2\pi]$

$$f'(x) = 1 - \sin x \geq 0, f \text{ is increasing.}$$

$$f''(x) = -\cos x = 0 \text{ when } x = \frac{\pi}{2}, \frac{3\pi}{2}.$$

Points of inflection: $(\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{3\pi}{2}, \frac{3\pi}{2})$

Intercept: $(0, 1)$



63. $x^2 + 4y^2 - 2x - 16y + 13 = 0$

(a) $(x^2 - 2x + 1) + 4(y^2 - 4y + 4) = -13 + 1 + 16$

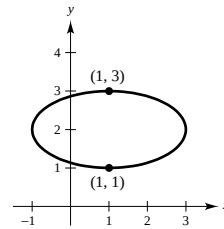
$$(x - 1)^2 + 4(y - 2)^2 = 4$$

$$\frac{(x - 1)^2}{4} + \frac{(y - 2)^2}{1} = 1$$

The graph is an ellipse:

Maximum: $(1, 3)$

Minimum: $(1, 1)$



(b) $x^2 + 4y^2 - 2x - 16y + 13 = 0$

$$2x + 8y \frac{dy}{dx} - 2 - 16 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(8y - 16) = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2 - 2x}{8y - 16} = \frac{1 - x}{4y - 8}$$

The critical numbers are $x = 1$ and $y = 2$. These correspond to the points $(1, 1)$, $(1, 3)$, $(2, -1)$, and $(2, 3)$. Hence, the maximum is $(1, 3)$ and the minimum is $(1, 1)$.

65. Let $t = 0$ at noon.

$$L = d^2 = (100 - 12t)^2 + (-10t)^2 = 10,000 - 2400t + 244t^2$$

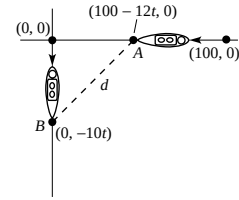
$$\frac{dL}{dt} = -2400 + 488t = 0 \text{ when } t = \frac{300}{61} \approx 4.92 \text{ hr.}$$

Ship A at (40.98, 0); Ship B at (0, -49.18)

$$d^2 = 10,000 - 2400t + 244t^2$$

$$\approx 4098.36 \text{ when } t \approx 4.92 \approx 4:55 \text{ P.M..}$$

$$d \approx 64 \text{ km}$$



67. We have points $(0, y)$, $(x, 0)$, and $(1, 8)$. Thus,

$$m = \frac{y - 8}{0 - 1} = \frac{0 - 8}{x - 1} \text{ or } y = \frac{8x}{x - 1}.$$

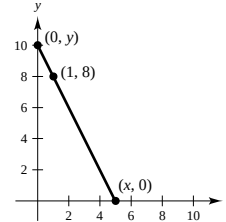
$$\text{Let } f(x) = L^2 = x^2 + \left(\frac{8x}{x - 1}\right)^2.$$

$$f'(x) = 2x + 128\left(\frac{x}{x - 1}\right)\left[\frac{(x - 1) - x}{(x - 1)^2}\right] = 0$$

$$x - \frac{64x}{(x - 1)^3} = 0$$

$$x[(x - 1)^3 - 64] = 0 \text{ when } x = 0, 5 \text{ (minimum).}$$

Vertices of triangle: $(0, 0)$, $(5, 0)$, $(0, 10)$



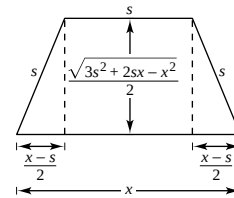
69. $A = (\text{Average of bases})(\text{Height})$

$$= \left(\frac{x + s}{2}\right) \frac{\sqrt{3s^2 + 2sx - x^2}}{2} \text{ (see figure)}$$

$$\frac{dA}{dx} = \frac{1}{4} \left[\frac{(s - x)(s + x)}{\sqrt{3s^2 + 2sx - x^2}} + \sqrt{3s^2 + 2sx - x^2} \right]$$

$$= \frac{2(2s - x)(s + x)}{4\sqrt{3s^2 + 2sx - x^2}} = 0 \text{ when } x = 2s.$$

A is a maximum when $x = 2s$.



71. You can form a right triangle with vertices $(0, 0)$, $(x, 0)$ and $(0, y)$.

Assume that the hypotenuse of length L passes through $(4, 6)$.

$$m = \frac{y - 6}{0 - 4} = \frac{6 - 0}{4 - x} \text{ or } y = \frac{6x}{x - 4}$$

$$\text{Let } f(x) = L^2 = x^2 + y^2 = x^2 + \left(\frac{6x}{x - 4}\right)^2.$$

$$f'(x) = 2x + 72\left(\frac{x}{x - 4}\right)\left[\frac{-4}{(x - 4)^2}\right] = 0$$

$$x[(x - 4)^3 - 144] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{144}.$$

$$L \approx 14.05 \text{ feet}$$

73. $\csc \theta = \frac{L_1}{6}$ or $L_1 = 6 \csc \theta$ (see figure)

$$\csc\left(\frac{\pi}{2} - \theta\right) = \frac{L_2}{9} \text{ or } L_2 = 9 \csc\left(\frac{\pi}{2} - \theta\right)$$

$$L = L_1 + L_2 = 6 \csc \theta + 9 \csc\left(\frac{\pi}{2} - \theta\right) = 6 \csc \theta + 9 \sec \theta$$

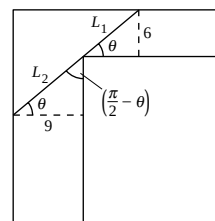
$$\frac{dL}{d\theta} = -6 \csc \theta \cot \theta + 9 \sec \theta \tan \theta = 0$$

$$\tan^3 \theta = \frac{2}{3} \Rightarrow \tan \theta = \frac{\sqrt[3]{2}}{\sqrt[3]{3}}$$

$$\sec \theta = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + \left(\frac{2}{3}\right)^{2/3}} = \frac{\sqrt{3^{2/3} + 2^{2/3}}}{3^{1/3}}$$

$$\csc \theta = \frac{\sec \theta}{\tan \theta} = \frac{\sqrt{3^{2/3} + 2^{2/3}}}{2^{1/3}}$$

$$L = 6 \frac{(3^{2/3} + 2^{2/3})^{1/2}}{2^{1/3}} + 9 \frac{(3^{2/3} + 2^{2/3})^{1/2}}{3^{1/3}} = 3(3^{2/3} + 2^{2/3})^{3/2} \text{ ft} \approx 21.07 \text{ ft (Compare to Exercise 72 using } a = 9 \text{ and } b = 6.)$$



75. Total cost = (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{600} + 5\right)\left(\frac{110}{v}\right) = \frac{11v}{60} + \frac{550}{v}$$

$$\frac{dT}{dv} = \frac{11}{60} - \frac{550}{v^2} = \frac{11v^2 - 33,000}{60v^2}$$

$$= 0 \text{ when } v = \sqrt{3000} = 10\sqrt{30} \approx 54.8 \text{ mph.}$$

$$\frac{d^2T}{dv^2} = \frac{1100}{v^3} > 0 \text{ when } v = 10\sqrt{30} \text{ so this value yields a minimum.}$$

77. $f(x) = x^3 - 3x - 1$

From the graph you can see that $f(x)$ has three real zeros.

$$f'(x) = 3x^2 - 3$$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.5000	0.1250	3.7500	0.0333	-1.5333
2	-1.5333	-0.0049	4.0530	-0.0012	-1.5321

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	0.3750	-2.2500	-0.1667	-0.3333
2	-0.3333	-0.0371	-2.6667	0.0139	-0.3472
3	-0.3472	-0.0003	-2.6384	0.0001	-0.3473

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.9000	0.1590	7.8300	0.0203	1.8797
2	1.8797	0.0024	7.5998	0.0003	1.8794

The three real zeros of $f(x)$ are $x \approx -1.532$, $x \approx -0.347$, and $x \approx 1.879$.

CHAPTER 3

Applications of Differentiation

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CHAPTER 3

Applications of Differentiation

Section 3.1 Extrema on an Interval

Solutions to Even-Numbered Exercises

2. $f(x) = \cos \frac{\pi x}{2}$

$$f'(x) = -\frac{\pi}{2} \sin \frac{\pi x}{2}$$

$$f'(0) = 0$$

$$f'(2) = 0$$

4. $f(x) = -3x\sqrt{x+1}$

$$f'(x) = -3x \left[\frac{1}{2}(x+1)^{-1/2} \right] + \sqrt{x+1}(-3)$$

$$= -\frac{3}{2}(x+1)^{-1/2}[x+2(x+1)]$$

$$= -\frac{3}{2}(x+1)^{-1/2}(3x+2)$$

$$f'\left(-\frac{2}{3}\right) = 0$$

6. Using the limit definition of the derivative,

$$\lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{(4 - |x|) - 4}{x} = 1$$

$$\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{(4 - |x|) - 4}{x} = -1$$

$f'(0)$ does not exist, since the one-sided derivatives are not equal.

8. Critical number: $x = 0$.

$x = 0$: neither

10. Critical numbers: $x = 2, 5$

$x = 2$: neither

$x = 5$: absolute maximum

12. $g(x) = x^2(x^2 - 4) = x^4 - 4x^2$

$$g'(x) = 4x^3 - 8x = 4x(x^2 - 2)$$

Critical numbers: $x = 0, x = \pm\sqrt{2}$

14. $f(x) = \frac{4x}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(4) - (4x)(2x)}{(x^2 + 1)^2} = \frac{4(1 - x^2)}{(x^2 + 1)^2}$$

Critical numbers: $x = \pm 1$

16. $f(\theta) = 2 \sec \theta + \tan \theta, 0 < \theta < 2\pi$

$$f'(\theta) = 2 \sec \theta \tan \theta + \sec^2 \theta$$

$$= \sec \theta (2 \tan \theta + \sec \theta)$$

$$= \sec \theta \left[2 \left(\frac{\sin \theta}{\cos \theta} \right) + \frac{1}{\cos \theta} \right]$$

$$= \sec^2 \theta (2 \sin \theta + 1)$$

On $(0, 2\pi)$, critical numbers: $\theta = \frac{7\pi}{6}, \theta = \frac{11\pi}{6}$

18. $f(x) = \frac{2x+5}{3}, [0, 5]$

$$f'(x) = \frac{2}{3} \Rightarrow \text{No critical numbers}$$

Left endpoint: $\left(0, \frac{5}{3}\right)$ Minimum

Right endpoint: $(5, 5)$ Maximum

22. $f(x) = x^3 - 12x, [0, 4]$

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$

Left endpoint: $(0, 0)$

Critical number: $(2, -16)$ Minimum

Right endpoint: $(4, 16)$ Maximum

Note: $x = -2$ is not in the interval.

20. $f(x) = x^2 + 2x - 4, [-1, 1]$

$$f'(x) = 2x + 2 = 2(x + 1)$$

Left endpoint: $(-1, -5)$ Minimum

Right endpoint: $(1, -1)$ Maximum

24. $g(x) = \sqrt[3]{x}, [-1, 1]$

$$g'(x) = \frac{1}{3x^{2/3}}$$

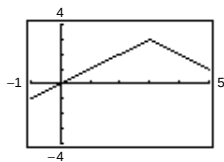
Left endpoint: $(-1, -1)$ Minimum

Critical number: $(0, 0)$

Right endpoint: $(1, 1)$ Maximum

26. $y = 3 - |t - 3|, [-1, 5]$

From the graph, you see that $t = 3$ is a critical number.



Left endpoint: $(-1, -1)$ Minimum

Right endpoint: $(5, 1)$

Critical number: $(3, 3)$ Maximum

28. $h(t) = \frac{t}{t-2}, [3, 5]$

$$h'(t) = \frac{-2}{(t-2)^2}$$

Left endpoint: $(3, 3)$ Maximum

Right endpoint: $\left(5, \frac{5}{3}\right)$ Minimum

30. $g(x) = \sec x, \left[-\frac{\pi}{6}, \frac{\pi}{3}\right]$

$$g'(x) = \sec x \tan x$$

$$\text{Left endpoint: } \left(-\frac{\pi}{6}, \frac{2}{\sqrt{3}}\right) \approx \left(-\frac{\pi}{6}, 1.1547\right)$$

$$\text{Right endpoint: } \left(\frac{\pi}{3}, 2\right) \text{ Maximum}$$

Critical number: $(0, 1)$ Minimum

32. $y = x^2 - 2 - \cos x, [-1, 3]$

$$y' = 2x - \sin x$$

Left endpoint: $(-1, -1.5403)$

Right endpoint: $(3, 7.99)$ Maximum

Critical number: $(0, -3)$ Minimum

34. (a) Minimum: $(4, 1)$

Maximum: $(1, 4)$

(b) Maximum: $(1, 4)$

(c) Minimum: $(4, 1)$

(d) No extrema

36. (a) Minima: $(-2, 0)$ and $(2, 0)$

Maximum: $(0, 2)$

(b) Minimum: $(-2, 0)$

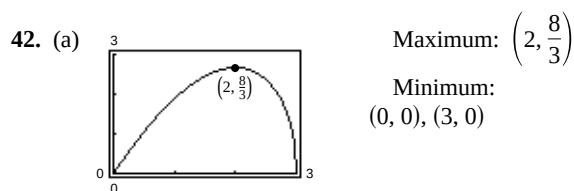
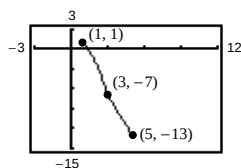
(c) Maximum: $(0, 2)$

(d) Maximum: $(1, \sqrt{3})$

$$38. f(x) = \begin{cases} 2 - x^2, & 1 \leq x < 3 \\ 2 - 3x, & 3 \leq x \leq 5 \end{cases}$$

Left endpoint: (1, 1) Maximum

Right endpoint: (5, -13) Minimum



$$(b) f(x) = \frac{4}{3}x\sqrt{3-x}, [0, 3]$$

$$\begin{aligned} f'(x) &= \frac{4}{3} \left[x \left(\frac{1}{2} \right) (3-x)^{-1/2} (-1) + (3-x)^{1/2} (1) \right] \\ &= \frac{4}{3} (3-x)^{-1/2} \left(\frac{1}{2} \right) [-x + 2(3-x)] \\ &= \frac{2(6-3x)}{3\sqrt{3-x}} = \frac{6(2-x)}{3\sqrt{3-x}} = \frac{2(2-x)}{\sqrt{3-x}} \end{aligned}$$

Critical number: $x = 2$

$f(0) = 0$ Minimum

$f(3) = 0$ Minimum

$$f(2) = \frac{8}{3}$$

Maximum: $\left(2, \frac{8}{3}\right)$

$$46. f(x) = \frac{1}{x^2 + 1}, [-1, 1]$$

$$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4} \quad (\text{See Exercise 44.})$$

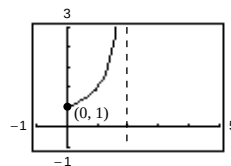
$$f^{(4)}(x) = \frac{24(5x^4 - 10x^2 + 1)}{(x^2 + 1)^5}$$

$$f^{(5)}(x) = \frac{-240x(3x^4 - 10x^2 + 3)}{(x^2 + 1)^6}$$

$|f^{(4)}(0)| = 24$ is the maximum value.

$$40. f(x) = \frac{2}{2-x}, [0, 2)$$

Left endpoint: (0, 1) Minimum



$$44. f(x) = \frac{1}{x^2 + 1}, \left[\frac{1}{2}, 3\right]$$

$$f'(x) = \frac{-2x}{(x^2 + 1)^2}$$

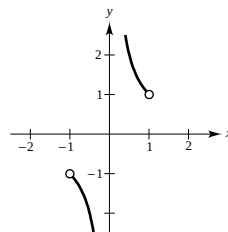
$$f''(x) = \frac{-2(1 - 3x^2)}{(x^2 + 1)^3}$$

$$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4}$$

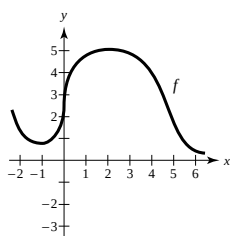
Setting $f''' = 0$, we have $x = 0, \pm 1$.

$|f''(1)| = \frac{1}{2}$ is the maximum value.

48. Let $f(x) = 1/x$. f is continuous on $(0, 1)$ but does not have a maximum. f is also continuous on $(-1, 0)$ but does not have a minimum. This can occur if one of the endpoints is an infinite discontinuity.



50.



52. (a) No

(b) Yes

54. (a) No

(b) Yes

56. $x = \frac{v^2 \sin 2\theta}{32}, \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$

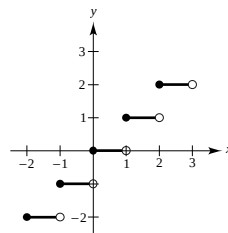
 $\frac{d\theta}{dt}$ is constant.

$$\begin{aligned} \frac{dx}{dt} &= \frac{dx}{d\theta} \frac{d\theta}{dt} \quad (\text{by the Chain Rule}) \\ &= \frac{v^2 \cos 2\theta}{16} \frac{d\theta}{dt} \end{aligned}$$

In the interval $[\pi/4, 3\pi/4]$, $\theta = \pi/4, 3\pi/4$ indicate minimums for dx/dt and $\theta = \pi/2$ indicates a maximum for dx/dt . This implies that the sprinkler waters longest when $\theta = \pi/4$ and $3\pi/4$. Thus, the lawn farthest from the spinkler gets the most water.

60. $f(x) = \llbracket x \rrbracket$

The derivative of f is undefined at every integer and is zero at any noninteger real number. All real numbers are critical numbers.



62. True. This is stated in the Extreme Value Theorem.

 64. False. Let $f(x) = x^2$. $x = 0$ is a critical number of f .

$$\begin{aligned} g(x) &= f(x - k) \\ &= (x - k)^2 \end{aligned}$$

$x = k$ is a critical number of g .

Section 3.2 Rolle's Theorem and the Mean Value Theorem

 2. Rolle's Theorem does not apply to $f(x) = \cot(x/2)$ over $[\pi, 3\pi]$ since f is not continuous at $x = 2\pi$.

4. $f(x) = x(x - 3)$

x -intercepts: $(0, 0), (3, 0)$

$$f'(x) - 2x - 3 = 0 \text{ at } x = \frac{3}{2}.$$

6. $f(x) = -3x\sqrt{x+1}$

x -intercepts: $(-1, 0), (0, 0)$

$$f'(x) = -3x \cdot \frac{1}{2}(x+1)^{-1/2} - 3(x+1)^{1/2} = -3(x+1)^{-1/2} \left(\frac{x}{2} + (x+1) \right)$$

$$f'(x) = -3(x+1)^{-1/2} \left(\frac{3}{2}x + 1 \right) = 0 \text{ at } x = -\frac{2}{3}.$$

8. $f(x) = x^2 - 5x + 4$, $[1, 4]$

$$f(1) = f(4) = 0$$

f is continuous on $[1, 4]$. f is differentiable on $(1, 4)$.

Rolle's Theorem applies.

$$f'(x) = 2x - 5$$

$$2x - 5 = 0 \Rightarrow x = \frac{5}{2}$$

$$c \text{ value: } \frac{5}{2}$$

12. $f(x) = 3 - |x - 3|$, $[0, 6]$

$$f(0) = f(6) = 0$$

f is continuous on $[0, 6]$. f is not differentiable on $(0, 6)$ since $f'(3)$ does not exist. Rolle's Theorem does not apply.

16. $f(x) = \cos x$, $[0, 2\pi]$

$$f(0) = f(2\pi) = 1$$

f is continuous on $[0, 2\pi]$. f is differentiable on $(0, 2\pi)$.

Rolle's Theorem applies.

$$f'(x) = -\sin x$$

$$c \text{ value: } \pi$$

20. $f(x) = \sec x$, $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

$$f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) = \sqrt{2}$$

f is continuous on $[-\pi/4, \pi/4]$. f is differentiable on $(-\pi/4, \pi/4)$. Rolle's Theorem applies.

$$f'(x) = \sec x \tan x$$

$$\sec x \tan x = 0$$

$$x = 0$$

$$c \text{ value: } 0$$

10. $f(x) = (x - 3)(x + 1)^2$, $[-1, 3]$

$$f(-1) = f(3) = 0$$

f is continuous on $[-1, 3]$. f is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$f'(x) = (x - 3)(2)(x + 1) + (x + 1)^2$$

$$= (x + 1)[2x - 6 + x + 1]$$

$$= (x + 1)(3x - 5)$$

$$c \text{ value: } \frac{5}{3}$$

14. $f(x) = \frac{x^2 - 1}{x}$, $[-1, 1]$

$$f(-1) = f(1) = 0$$

f is not continuous on $[-1, 1]$ since $f(0)$ does not exist. Rolle's Theorem does not apply.

18. $f(x) = \cos 2x$, $\left[-\frac{\pi}{12}, \frac{\pi}{6}\right]$

$$f\left(-\frac{\pi}{12}\right) = \frac{\sqrt{3}}{2}$$

$$f\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$f\left(-\frac{\pi}{12}\right) \neq f\left(\frac{\pi}{6}\right)$$

Rolle's Theorem does not apply.

22. $f(x) = x - x^{1/3}$, $[0, 1]$

$$f(0) = f(1) = 0$$

f is continuous on $[0, 1]$. f is differentiable on $(0, 1)$.

(Note: f is not differentiable at $x = 0$.) Rolle's Theorem applies.

$$f'(x) = 1 - \frac{1}{3\sqrt[3]{x^2}} = 0$$

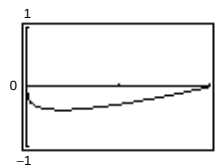
$$1 = \frac{1}{3\sqrt[3]{x^2}}$$

$$\sqrt[3]{x^2} = \frac{1}{3}$$

$$x^2 = \frac{1}{27}$$

$$x = \sqrt{\frac{1}{27}} = \frac{\sqrt{3}}{9}$$

$$c \text{ value: } \frac{\sqrt{3}}{9} \approx 0.1925$$



$$24. f(x) = \frac{x}{2} - \sin \frac{\pi x}{6}, [-1, 0]$$

$$f(-1) = f(0) = 0$$

f is continuous on $[-1, 0]$. f is differentiable on $(-1, 0)$.
Rolle's Theorem applies.

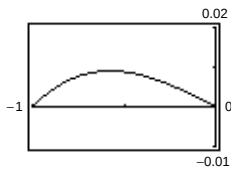
$$f'(x) = \frac{1}{2} - \frac{\pi}{6} \cos \frac{\pi x}{6} = 0$$

$$\cos \frac{\pi x}{6} = \frac{3}{\pi}$$

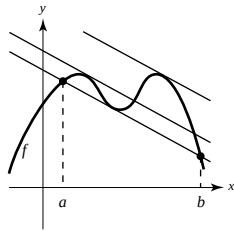
$$x = -\frac{6}{\pi} \arccos \frac{3}{\pi} \quad [\text{Value needed in } (-1, 0).]$$

$$\approx -0.5756 \text{ radian}$$

c value: -0.5756



28.



32. $f(x) = x(x^2 - x - 2)$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$.

$$\frac{f(1) - f(-1)}{1 - (-1)} = -1$$

$$f'(x) = 3x^2 - 2x - 2 = -1$$

$$(3x + 1)(x - 1) = 0$$

$$c = -\frac{1}{3}$$

$$26. C(x) = 10\left(\frac{1}{x} + \frac{x}{x+3}\right)$$

$$(a) C(3) = C(6) = \frac{25}{3}$$

$$(b) C'(x) = 10\left(-\frac{1}{x^2} + \frac{3}{(x+3)^2}\right) = 0$$

$$\frac{3}{x^2 + 6x + 9} = \frac{1}{x^2}$$

$$2x^2 - 6x - 9 = 0$$

$$x = \frac{6 \pm \sqrt{108}}{4}$$

$$= \frac{6 \pm 6\sqrt{3}}{4} = \frac{3 \pm 3\sqrt{3}}{2}$$

$$\text{In the interval } (3, 6): c = \frac{3 + 3\sqrt{3}}{2} \approx 4.098.$$

$$30. f(x) = |x - 3|, [0, 6]$$

f is not differentiable at $x = 3$.

34. $f(x) = (x + 1)/x$ is continuous on $[1/2, 2]$ and differentiable on $(1/2, 2)$.

$$\frac{f(2) - f(1/2)}{2 - (1/2)} = \frac{(3/2) - 3}{3/2} = -1$$

$$f'(x) = \frac{-1}{x^2} = -1$$

$$x^2 = 1$$

$$c = 1$$

36. $f(x) = x^3$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$$\frac{f(1) - f(0)}{1 - 0} = \frac{1 - 0}{1} = 1$$

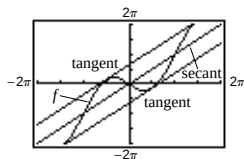
$$f'(x) = 3x^2 = 1$$

$$x = \pm \frac{\sqrt{3}}{3}$$

In the interval $(0, 1)$: $c = \frac{\sqrt{3}}{3}$.

40. $f(x) = x - 2 \sin x$ on $[-\pi, \pi]$

(a)



(b) Secant line:

$$\text{slope} = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)} = \frac{\pi - (-\pi)}{2\pi} = 1$$

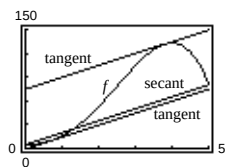
$$y - \pi = 1(x - \pi)$$

$$y = x$$

42. $f(x) = -x^4 + 4x^3 + 8x^2 + 5$, $(0, 5)$, $(5, 80)$

$$m = \frac{80 - 5}{5 - 0} = 15$$

(a)



(b) Secant line: $y - 5 = 15(x - 0)$

$$0 = 15x - y + 5$$

$$f'(x) = -4x^3 + 12x^2 + 16x$$

$$\frac{f(5) - f(1)}{5 - 1} = 15$$

$$-4c^3 + 12c^2 + 16c = 15$$

$$0 = 4c^3 - 12c^2 - 16c + 15$$

$$c \approx 0.67 \text{ or } c \approx 3.79$$

38. $f(x) = 2 \sin x + \sin 2x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

$$f'(x) = 2 \cos x + 2 \cos 2x = 0$$

$$2[\cos x + 2 \cos^2 x - 1] = 0$$

$$2(2 \cos x - 1)(\cos x + 1) = 0$$

$$\cos x = \frac{1}{2}$$

$$\cos x = -1$$

$$x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

In the interval $(0, \pi)$: $c = \frac{\pi}{3}$.

(c) $f'(x) = 1 - 2 \cos x = 1$

$$\cos x = 0$$

$$c = \pm \frac{\pi}{2}, \quad f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 2$$

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + 2$$

$$\text{Tangent lines: } y - \left(\frac{\pi}{2} - 2\right) = 1\left(x - \frac{\pi}{2}\right)$$

$$y = x - 2$$

$$y - \left(-\frac{\pi}{2} + 2\right) = 1\left(x + \frac{\pi}{2}\right)$$

$$y = x + 2$$

(c) First tangent line: $y - f(c) = m(x - c)$

$$y - 9.59 = 15(x - 0.67)$$

$$0 = 15x - y - 0.46$$

Second tangent line: $y - f(c) = m(x - c)$

$$y - 131.35 = 15(x - 3.79)$$

$$0 = 15x - y + 74.5$$

$$44. S(t) = 200\left(5 - \frac{9}{2+t}\right)$$

$$(a) \frac{S(12) - S(0)}{12 - 0} = \frac{200[5 - (9/14)] - 200[5 - (9/2)]}{12} = \frac{450}{7}$$

$$(b) S'(t) = 200\left(\frac{9}{(2+t)^2}\right) = \frac{450}{7}$$

$$\frac{1}{(2+t)^2} = \frac{1}{28}$$

$$2+t = 2\sqrt{7}$$

$$t = 2\sqrt{7} - 2 \approx 3.2915 \text{ months}$$

$S'(t)$ is equal to the average value in April.

46. $f(a) = f(b)$ and $f'(c) = 0$ where c is in the interval (a, b) .

$$(a) g(x) = f(x) + k$$

$$g(a) = g(b) = f(a) + k$$

$$g'(x) = f'(x) \Rightarrow g'(c) = 0$$

Interval: $[a, b]$

Critical number of g : c

$$(b) g(x) = f(x - k)$$

$$g(a+k) = g(b+k) = f(a)$$

$$g'(x) = f'(x - k)$$

$$g'(c+k) = f'(c) = 0$$

Interval: $[a+k, b+k]$

Critical number of g : $c+k$

$$(c) g(x) = f(kx)$$

$$g\left(\frac{a}{k}\right) = g\left(\frac{b}{k}\right) = f(a)$$

$$g'(x) = kf'(kx)$$

$$g'\left(\frac{c}{k}\right) = kf'(c) = 0$$

Interval: $\left[\frac{a}{k}, \frac{b}{k}\right]$

Critical number of g : $\frac{c}{k}$

48. Let $T(t)$ be the temperature of the object. Then $T(0) = 1500^\circ$ and $T(5) = 390^\circ$. The average temperature over the interval $[0, 5]$ is

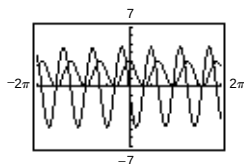
$$\frac{390 - 1500}{5 - 0} = -222^\circ \text{ F/hr.}$$

By the Mean Value Theorem, there exists a time t_0 , $0 < t_0 < 5$, such that $T'(t_0) = -222$.

$$50. f(x) = 3 \cos^2\left(\frac{\pi x}{2}\right), \quad f'(x) = 6 \cos\left(\frac{\pi x}{2}\right)\left(-\sin\left(\frac{\pi x}{2}\right)\right)\left(\frac{\pi}{2}\right)$$

$$= -3\pi \cos\left(\frac{\pi x}{2}\right) \sin\left(\frac{\pi x}{2}\right)$$

(a)



(b) f and f' are both continuous on the entire real line.

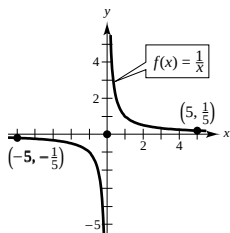
(c) Since $f(-1) = f(1) = 0$, Rolle's Theorem applies on $[-1, 1]$. Since $f(1) = 0$ and $f(2) = 3$, Rolle's Theorem does not apply on $[1, 2]$.

$$(d) \lim_{x \rightarrow 3^-} f'(x) = 0$$

$$\lim_{x \rightarrow 3^+} f'(x) = 0$$

- 52.
- f
- is not continuous on
- $[-5, 5]$
- .

Example: $f(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$



54. False.
- f
- must also be continuous
- and*
- differentiable on each interval. Let

$$f(x) = \frac{x^3 - 4x}{x^2 - 1}.$$

56. True

58. Suppose
- $f(x)$
- is not constant on
- (a, b)
- . Then there exists
- x_1
- and
- x_2
- in
- (a, b)
- such that
- $f(x_1) \neq f(x_2)$
- . Then by the Mean Value Theorem, there exists
- c
- in
- (a, b)
- such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0.$$

This contradicts the fact that $f'(x) = 0$ for all x in (a, b) .

60. Suppose
- $f(x)$
- has two fixed points
- c_1
- and
- c_2
- . Then, by the Mean Value Theorem, there exists
- c
- such that

$$f'(c) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1.$$

This contradicts the fact that $f'(x) < 1$ for all x .

62. Let
- $f(x) = \cos x$
- .
- f
- is continuous and differentiable for all real numbers. By the Mean Value Theorem, for any interval
- $[a, b]$
- , there exists
- c
- in
- (a, b)
- such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\frac{\cos b - \cos a}{b - a} = -\sin c$$

$$\cos b - \cos a = (-\sin c)(b - a)$$

$$|\cos b - \cos a| = |-\sin c| |b - a|$$

$$|\cos b - \cos a| \leq |b - a| \text{ since } |-\sin c| \leq 1.$$

Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

2. $y = -(x + 1)^2$

Increasing on: $(-\infty, -1)$ Decreasing on: $(-1, \infty)$

4. $f(x) = x^4 - 2x^2$

Increasing on: $(-1, 0), (1, \infty)$ Decreasing on: $(-\infty, -1), (0, 1)$

6. $y = \frac{x^2}{x + 1}$

$$y' = \frac{x(x + 2)}{(x + 1)^2}$$

Critical numbers: $x = 0, -2$ Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -2$	$-2 < x < -1$	$-1 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$y' > 0$	$y' < 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on $(-\infty, -2), (0, \infty)$ Decreasing on $(-2, -1), (-1, 0)$

8. $h(x) = 27x - x^3$

$$h'(x) = 27 - 3x^2 = 3(3 - x)(3 + x)$$

$$h'(x) = 0$$

Critical numbers: $x = \pm 3$

Test intervals:	$-\infty < x < -3$	$-3 < x < 3$	$3 < x < \infty$
Sign of $h'(x)$:	$h' < 0$	$h' > 0$	$h' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on $(-3, 3)$ Decreasing on $(-\infty, -3), (3, \infty)$

10. $y = x + \frac{4}{x}$

$$y' = \frac{(x - 2)(x + 2)}{x^2}$$

Critical numbers: $x = \pm 2$ Discontinuity: 0

Test intervals:	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of y' :	$y' > 0$	$y' < 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing: $(-\infty, -2), (2, \infty)$ Decreasing: $(-2, 0), (0, 2)$

12. $f(x) = x^2 + 8x + 10$

$f'(x) = 2x + 8 = 0$

Critical number: $x = -4$

Test intervals:	$-\infty < x < -4$	$-4 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(-4, \infty)$ Decreasing on: $(-\infty, -4)$ Relative minimum: $(-4, -6)$

14. $f(x) = -(x^2 + 8x + 12)$

$f'(x) = -2x - 8 = 0$

Critical number: $x = -4$

Test intervals:	$-\infty < x < -4$	$-4 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, -4)$ Decreasing on: $(-4, \infty)$ Relative maximum: $(-4, 4)$

16. $f(x) = x^3 - 6x^2 + 15$

$f'(x) = 3x^2 - 12x = 3x(x - 4)$

Critical numbers: $x = 0, 4$

Test intervals:	$-\infty < x < 0$	$0 < x < 4$	$4 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on $(-\infty, 0), (4, \infty)$ Decreasing on $(0, 4)$ Relative maximum: $(0, 15)$ Relative minimum: $(4, -17)$

18. $f(x) = (x + 2)^2(x - 1)$

$f'(x) = 3x(x + 2)$

Critical numbers: $x = -2, 0$

Test intervals:	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -2), (0, \infty)$ Decreasing on: $(-2, 0)$ Relative maximum: $(-2, 0)$ Relative minimum: $(0, -4)$

20. $f(x) = x^4 - 32x + 4$

$$f'(x) = 4x^3 - 32 = 4(x^3 - 8)$$

Critical number: $x = 2$

Test intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(2, \infty)$ Decreasing on: $(-\infty, 2)$ Relative minimum: $(2, -44)$

22. $f(x) = x^{2/3} - 4$

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$$

Critical number: $x = 0$

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(0, \infty)$ Decreasing on: $(-\infty, 0)$ Relative minimum: $(0, -4)$

24. $f(x) = (x - 1)^{1/3}$

$$f'(x) = \frac{1}{3(x - 1)^{2/3}}$$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, \infty)$

No relative extrema

26. $f(x) = |x + 3| - 1$

$$f'(x) = \frac{x + 3}{|x + 3|} = \begin{cases} 1, & x > -3 \\ -1, & x < -3 \end{cases}$$

Critical number: $x = -3$

Test intervals:	$-\infty < x < -3$	$-3 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(-3, \infty)$ Decreasing on: $(-\infty, -3)$ Relative minimum: $(-3, -1)$

28. $f(x) = \frac{x}{x + 1}$

$$f'(x) = \frac{(x + 1)(1) - (x)(1)}{(x + 1)^2} = \frac{1}{(x + 1)^2}$$

Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, -1), (-1, \infty)$

No relative extrema

$$30. f(x) = \frac{x+3}{x^2} = \frac{1}{x} + \frac{3}{x^2}$$

$$f'(x) = -\frac{1}{x^2} - \frac{6}{x^3} = \frac{-(x+6)}{x^3}$$

Critical number: $x = -6$

Discontinuity: $x = 0$

Test intervals:	$-\infty < x < -6$	$-6 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(-6, 0)$

Decreasing on: $(-\infty, -6), (0, \infty)$

Relative minimum: $\left(-6, -\frac{1}{12}\right)$

$$32. f(x) = \frac{x^2 - 3x - 4}{x - 2}$$

$$f'(x) = \frac{(x-2)(2x-3) - (x^2-3x-4)(1)}{(x-2)^2} = \frac{x^2 - 4x + 10}{(x-2)^2}$$

Discontinuity: $x = 2$

Test intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, 2), (2, \infty)$

No relative extrema

$$34. f(x) = \sin x \cos x = \frac{1}{2} \sin 2x, 0 < x < 2\pi$$

$$f'(x) = \cos 2x = 0$$

Critical numbers: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Test intervals:	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{7\pi}{4}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right), \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

Relative maxima: $\left(\frac{\pi}{4}, \frac{1}{2}\right), \left(\frac{5\pi}{4}, \frac{1}{2}\right)$

Relative minima: $\left(\frac{3\pi}{4}, -\frac{1}{2}\right), \left(\frac{7\pi}{4}, -\frac{1}{2}\right)$

$$36. f(x) = \frac{\sin x}{1 + \cos^2 x}, \quad 0 < x < 2\pi$$

$$f'(x) = \frac{\cos x(2 + \sin^2 x)}{(1 + \cos^2 x)^2} = 0$$

$$\text{Critical numbers: } x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\text{Increasing on: } \left(0, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, 2\pi\right)$$

$$\text{Decreasing on: } \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

$$\text{Relative maximum: } \left(\frac{\pi}{2}, 1\right)$$

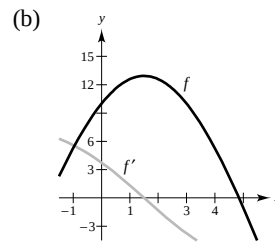
$$\text{Relative minimum: } \left(\frac{3\pi}{2}, -1\right)$$

$$38. f(x) = 10(5 - \sqrt{x^2 - 3x + 16}), [0, 5]$$

$$(a) f'(x) = -\frac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}}$$

$$(c) -\frac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}} = 0$$

$$\text{Critical number: } x = \frac{3}{2}$$



(d) Intervals:

$$\left(0, \frac{3}{2}\right) \quad \left(\frac{3}{2}, 5\right)$$

$$f'(x) > 0 \quad f'(x) < 0$$

Increasing Decreasing

f is increasing when f' is positive and decreasing when f' is negative.

$$40. f(x) = \frac{x}{2} + \cos \frac{x}{2}, [0, 4\pi]$$

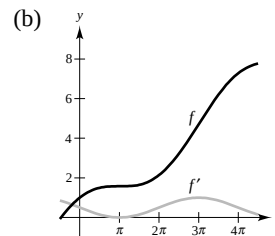
$$(a) f'(x) = \frac{1}{2} - \frac{1}{2} \sin \frac{x}{2}$$

$$(c) \frac{1}{2} - \frac{1}{2} \sin \frac{x}{2} = 0$$

$$\sin \frac{x}{2} = 1$$

$$\frac{x}{2} = \frac{\pi}{2}$$

$$\text{Critical number: } x = \pi$$



(d) Intervals:

$$(0, \pi) \quad (\pi, 4\pi)$$

$$f'(x) > 0 \quad f'(x) > 0$$

Increasing Increasing

f is increasing when f' is positive.

42. $f(t) = \cos^2 t - \sin^2 t = -2 \sin^2 t = g(t)$, $-2 < t < 2$

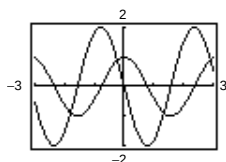
$$f'(t) = -4 \sin t \cos t = -2 \sin 2t$$

f symmetric with respect to y -axis

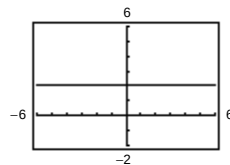
zeros of f : $\pm \frac{\pi}{4}$

Relative maximum: $(0, 1)$

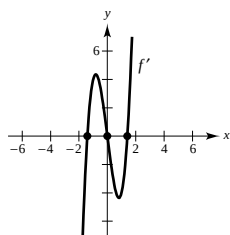
Relative minimum: $\left(-\frac{\pi}{2}, -1\right), \left(\frac{\pi}{2}, -1\right)$



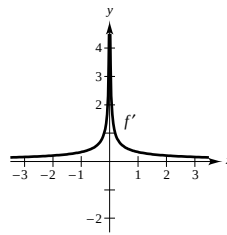
44. $f(x)$ is a line of slope $\approx 2 \Rightarrow f'(x) = 2$.



46. f is a 4th degree polynomial $\Rightarrow f'$ is a cubic polynomial.



48. f has positive slope



In Exercises 50–54, $f'(x) > 0$ on $(-\infty, -4)$, $f'(x) < 0$ on $(-4, 6)$ and $f'(x) > 0$ on $(6, \infty)$.

50. $g(x) = 3f(x) - 3$

$$g'(x) = 3f'(x)$$

$$g'(-5) = 3f'(-5) > 0$$

52. $g(x) = -f(x)$

$$g'(x) = -f'(x)$$

$$g'(0) = -f'(0) > 0$$

54. $g(x) = f(x - 10)$

$$g'(x) = f'(x - 10)$$

$$g'(8) = f'(-2) < 0$$

56. Critical number: $x = 5$

$$f'(4) = -2.5 \Rightarrow f \text{ is decreasing at } x = 4.$$

$$f'(6) = 3 \Rightarrow f \text{ is increasing at } x = 6.$$

$(5, f(5))$ is a relative minimum.

58. $s(t) = 4.9(\sin \theta)t^2$

(a) $v(t) = 9.8(\sin \theta)t$ speed = $|9.8(\sin \theta)t|$

(b) If $\theta = \pi/2$, the speed is maximum,

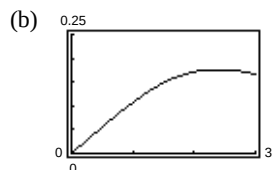
$$v(t) = 9.8t.$$

60. $C = \frac{3t}{27 + t^3}$, $t \geq 0$

(a)

t	0	0.5	1	1.5	2	2.5	3
$C(t)$	0	0.055	0.107	0.148	0.171	0.176	0.167

The concentration seems greater near $t = 2.5$ hours.



The concentration is greatest when $t \approx 2.38$ hours.

(c)
$$C' = \frac{(27 + t^3)(3) - (3t)(3t^2)}{(27 + t^3)^2}$$

$$= \frac{3(27 - 2t^3)}{(27 + t^3)^2}$$

$$C' = 0 \text{ when } t = 3/\sqrt[3]{2} \approx 2.38 \text{ hours.}$$

By the First Derivative Test, this is a maximum.

$$62. P = 2.44x - \frac{x^2}{20,000} - 5000, 0 \leq x \leq 35,000$$

$$P' = 2.44 - \frac{x}{10,000} = 0$$

$$x = 24,400$$

Increasing when $0 < x < 24,400$ hamburgers.

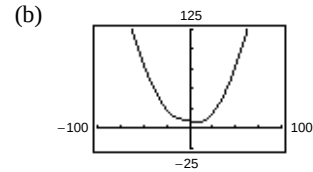
Decreasing when $24,400 < x < 35,000$ hamburgers.

Test intervals:	$0 < x < 24,400$	$24,400 < x < 35,000$
Sign of P' :	$P' > 0$	$P' < 0$

$$64. R = \sqrt{0.001T^4 - 4T + 100}$$

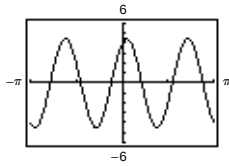
$$(a) R' = \frac{0.004T^3 - 4}{2\sqrt{0.001T^4 - 4T + 100}} = 0$$

$$T = 10^\circ, R \approx 8.3666\Omega$$



The minimum resistance is approximately $R \approx 8.37\Omega$ at $T = 10^\circ$.

$$66. f(x) = 2 \sin(3x) + 4 \cos(3x)$$



The maximum value is approximately 4.472. You could use calculus by finding $f'(x)$ and then observing that the maximum value of f occurs at a point where $f'(x) = 0$. For instance, $f'(0.154) \approx 0$, and $f(0.154) = 4.472$.

68. (a) Use a cubic polynomial

$$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0.$$

$$(b) f'(x) = 3a_3x^2 + 2a_2x + a_1$$

$$(0, 0): \quad 0 = a_0 \quad (f(0) = 0)$$

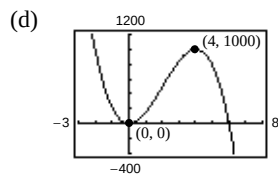
$$0 = a_1 \quad (f'(0) = 0)$$

$$(4, 1000): \quad 1000 = 64a_3 + 16a_2 \quad (f(4) = 1000)$$

$$0 = 48a_3 + 8a_2 \quad (f'(4) = 0)$$

$$(c) \text{ The solution is } a_0 = a_1 = 0, a_2 = \frac{375}{2}, a_3 = \frac{-125}{4}$$

$$f(x) = \frac{-125}{4}x^3 + \frac{375}{2}x^2.$$



70. (a) Use a fourth degree polynomial $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

(1, 2): $2 = a_4 + a_3 + a_2 + a_1 + a_0$ $(f(1) = 2)$

$0 = 4a_4 + 3a_3 + 2a_2 + a_1$ $(f'(1) = 0)$

(-1, 4): $4 = a_4 - a_3 + a_2 - a_1 + a_0$ $(f(-1) = 4)$

$0 = -4a_4 + 3a_3 - 2a_2 + a_1$ $(f'(-1) = 0)$

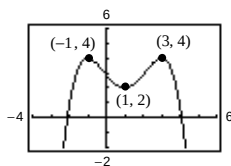
(3, 4): $4 = 81a_4 + 27a_3 + 9a_2 + 3a_1 + a_0$ $(f(3) = 4)$

$0 = 108a_4 + 27a_3 + 6a_2 + a_1$ $(f'(3) = 0)$

(c) The solution is $a_0 = \frac{23}{8}, a_1 = -\frac{3}{2}, a_2 = \frac{1}{4}, a_3 = \frac{1}{2}, a_4 = -\frac{1}{8}$

$$f(x) = -\frac{1}{8}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 - \frac{3}{2}x + \frac{23}{8}.$$

(d)



72. False

Let $h(x) = f(x)g(x)$ where $f(x) = g(x) = x$. Then $h(x) = x^2$ is decreasing on $(-\infty, 0)$.

74. True

If $f(x)$ is an n th-degree polynomial, then the degree of $f'(x)$ is $n - 1$.

76. False.

The function might not be continuous.

78. Suppose $f'(x)$ changes from positive to negative at c . Then there exists a and b in I such that $f'(x) > 0$ for all x in (a, c) and $f'(x) < 0$ for all x in (c, b) . By Theorem 3.5, f is increasing on (a, c) and decreasing on (c, b) . Therefore, $f(c)$ is a maximum of f on (a, b) and thus, a relative maximum of f .

Section 3.4 Concavity and the Second Derivative Test

2. $y = -x^3 + 3x^2 - 2, y'' = -6x + 6$

Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

4. $f(x) = \frac{x^2 - 1}{2x + 1}, y'' = \frac{-6}{(2x + 1)^3}$

Concave upward: $(-\infty, -\frac{1}{2})$

Concave downward: $(-\frac{1}{2}, \infty)$

6. $y = \frac{1}{270}(-3x^5 + 40x^3 + 135x), y'' = \frac{-2}{9}x(x - 2)(x + 2)$

Concave upward: $(-\infty, -2), (0, 2)$

Concave downward: $(-2, 0), (2, \infty)$

8. $h(x) = x^5 - 5x + 2$

$h'(x) = 5x^4 - 5$

$h''(x) = 20x^3$

Concave upward: $(0, \infty)$

Concave downward: $(-\infty, 0)$

10. $y = x + 2 \csc x, \quad (-\pi, \pi)$

$$y' = 1 - 2 \csc x \cot x$$

$$\begin{aligned} y'' &= -2 \csc x(-\csc^2 x) - 2 \cot x(-\csc x \cot x) \\ &= 2(\csc^3 x + \csc x \cot^2 x) \end{aligned}$$

Concave upward: $(0, \pi)$

Concave downward: $(-\pi, 0)$

12. $f(x) = 2x^3 - 3x^2 - 12x + 5$

$$f'(x) = 6x^2 - 6x - 12$$

$$f''(x) = 12x - 6$$

$$f''(x) = 12x - 6 = 0 \text{ when } x = \frac{1}{2}.$$

Test interval	$-\infty < x < \frac{1}{2}$	$\frac{1}{2} < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward

Point of inflection: $(\frac{1}{2}, -\frac{13}{2})$

14. $f(x) = 2x^4 - 8x + 3$

$$f'(x) = 8x^3 - 8$$

$$f''(x) = 24x^2 = 0 \text{ when } x = 0.$$

However, $(0, 3)$ is not a point of inflection since $f''(x) \geq 0$ for all x .

Concave upward on $(-\infty, \infty)$

16. $f(x) = x^3(x - 4)$

$$f'(x) = x^3 + 3x^2(x - 4)$$

$$= x^2[x + 3(x - 4)] = 4x^2(x - 3)$$

$$f''(x) = 4x^2 + 8x(x - 3) = 4x[x + 2(x - 3)] = 12x(x - 2) = 0$$

$$f''(x) = 12x(x - 2) = 0 \text{ when } x = 0, 2.$$

Test interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of $f''(x)$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Points of inflection: $(0, 0), (2, -16)$

18. $f(x) = x\sqrt{x+1}$, Domain: $[-1, \infty)$

$$f'(x) = (x) \frac{1}{2}(x+1)^{-1/2} + \sqrt{x+1} = \frac{3x+2}{2\sqrt{x+1}}$$

$$f''(x) = \frac{6\sqrt{x+1} - (3x+2)(x+1)^{-1/2}}{4(x+1)} = \frac{3x+4}{4(x+1)^{3/2}}$$

$f''(x) > 0$ on the entire domain of f (except for $x = -1$, for which $f''(x)$ is undefined).

There are no points of inflection.

Concave upward on $(-1, \infty)$

20. $f(x) = \frac{x+1}{\sqrt{x}}$ Domain: $x > 0$

$$f'(x) = \frac{x-1}{2x^{3/2}}$$

$$f''(x) = \frac{3-x}{4x^{5/2}}$$

Point of inflection: $(3, \frac{4}{\sqrt{3}}) = (3, \frac{4\sqrt{3}}{3})$

Test intervals	$0 < x < 3$	$3 < x < \infty$
Sign of $f''(x)$	$f'' > 0$	$f'' < 0$
Conclusion	Concave upward	Concave downward

22. $f(x) = 2 \csc \frac{3x}{2}, 0 < x < 2\pi$

$$f'(x) = -3 \csc \frac{3x}{2} \cot \frac{3x}{2}$$

$$f''(x) = \frac{9}{2} \left(\csc^3 \frac{3x}{2} + \csc \frac{3x}{2} \cot^2 \frac{3x}{2} \right) \neq 0 \text{ for any } x \text{ in the domain of } f.$$

$$\text{Concave upward: } \left(0, \frac{2\pi}{3}\right), \left(\frac{4\pi}{3}, 2\pi\right)$$

$$\text{Concave downward: } \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$$

No points of inflection

24. $f(x) = \sin x + \cos x, 0 \leq x \leq 2\pi$

$$f'(x) = \cos x - \sin x$$

$$f''(x) = -\sin x - \cos x$$

$$f''(x) = 0 \text{ when } x = \frac{3\pi}{4}, \frac{7\pi}{4}.$$

Test interval:	$0 < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
Sign of $f''(x)$:	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

$$\text{Points of inflection: } \left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$$

26. $f(x) = x + 2 \cos x, [0, 2\pi]$

$$f'(x) = 1 - 2 \sin x$$

$$f''(x) = -2 \cos x$$

$$f''(x) = 0 \text{ when } x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$	$f'' < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

$$\text{Points of inflection: } \left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$$

28. $f(x) = x^2 + 3x - 8$

$$f'(x) = 2x + 3$$

$$f''(x) = 2$$

$$\text{Critical number: } x = -\frac{3}{2}$$

$$f''\left(-\frac{3}{2}\right) > 0$$

Therefore, $\left(-\frac{3}{2}, -\frac{41}{4}\right)$ is a relative minimum.

30. $f(x) = -(x - 5)^2$

$$f'(x) = -2(x - 5)$$

$$f''(x) = -2$$

$$\text{Critical number: } x = 5$$

$$f''(5) < 0$$

Therefore, $(5, 0)$ is a relative maximum.

32. $f(x) = x^3 - 9x^2 + 27x$

$$f'(x) = 3x^2 - 18x + 27 = 3(x - 3)^2$$

$$f''(x) = 6(x - 3)$$

Critical number: $x = 3$

However, $f''(3) = 0$, so we must use the First Derivative Test. $f'(x) \geq 0$ for all x and, therefore, there are no relative extrema.

34. $g(x) = -\frac{1}{8}(x + 2)^2(x - 4)^2$

$$g'(x) = \frac{-(x - 4)(x - 1)(x + 2)}{2}$$

$$g''(x) = 3 + 3x - \frac{3}{2}x^2$$

Critical numbers: $x = -2, 1, 4$

$$g''(-2) = -9 < 0$$

$(-2, 0)$ is a relative maximum.

$$g''(1) = 9/2 > 0$$

$(1, -10.125)$ is a relative minimum.

$$g''(4) = -9 < 0$$

$(4, 0)$ is a relative maximum.

36. $f(x) = \sqrt{x^2 + 1}$

$$f'(x) = \frac{x}{\sqrt{x^2 + 1}}$$

Critical number: $x = 0$

$$f''(x) = \frac{1}{(x^2 + 1)^{3/2}}$$

$$f''(0) = 1 > 0$$

Therefore, $(0, 1)$ is a relative minimum.

38. $f(x) = \frac{x}{x - 1}$

$$f'(x) = \frac{-1}{(x - 1)^2}$$

There are no critical numbers and $x = 1$ is not in the domain. There are no relative extrema.

40. $f(x) = 2 \sin x + \cos 2x, 0 \leq x \leq 2\pi$

$$f'(x) = 2 \cos x - 2 \sin 2x = 2 \cos x - 4 \sin x \cos x = 2 \cos x(1 - 2 \sin x) = 0 \text{ when } x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}.$$

$$f''(x) = -2 \sin x - 4 \cos 2x$$

$$f''\left(\frac{\pi}{6}\right) < 0$$

$$f''\left(\frac{\pi}{2}\right) > 0$$

$$f''\left(\frac{5\pi}{6}\right) < 0$$

$$f''\left(\frac{3\pi}{2}\right) > 0$$

Relative maxima: $\left(\frac{\pi}{6}, \frac{3}{2}\right), \left(\frac{5\pi}{6}, \frac{3}{2}\right)$

Relative minima: $\left(\frac{\pi}{2}, 1\right), \left(\frac{3\pi}{2}, -3\right)$

42. $f(x) = x^2\sqrt{6-x^2}, [-\sqrt{6}, \sqrt{6}]$

(a) $f'(x) = \frac{3x(4-x^2)}{\sqrt{6-x^2}}$

$f'(x) = 0$ when $x = 0, x = \pm 2$.

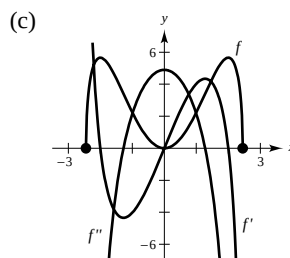
$f''(x) = \frac{6(x^4 - 9x^2 + 12)}{(6-x^2)^{3/2}}$

$f''(x) = 0$ when $x = \pm \sqrt{\frac{9-\sqrt{33}}{2}}$.

(b) $f''(0) > 0 \Rightarrow (0, 0)$ is a relative minimum.

$f''(\pm 2) < 0 \Rightarrow (\pm 2, 4\sqrt{2})$ are relative maxima.

Points of inflection: $(\pm 1.2758, 3.4035)$



The graph of f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

44. $f(x) = \sqrt{2x} \sin x, [0, 2\pi]$

(a) $f'(x) = \sqrt{2x} \cos x + \frac{\sin x}{\sqrt{2x}}$

Critical numbers: $x \approx 1.84, 4.82$

$f''(x) = -\sqrt{2x} \sin x + \frac{\cos x}{\sqrt{2x}} + \frac{\cos x}{\sqrt{2x}} - \frac{\sin x}{2x\sqrt{2x}}$

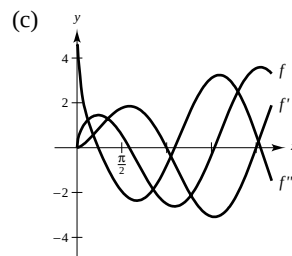
$= \frac{2\cos x}{\sqrt{2x}} - \frac{(4x^2 + 1)\sin x}{2x\sqrt{2x}}$

$= \frac{4x \cos x - (4x^2 + 1)\sin x}{2x\sqrt{2x}}$

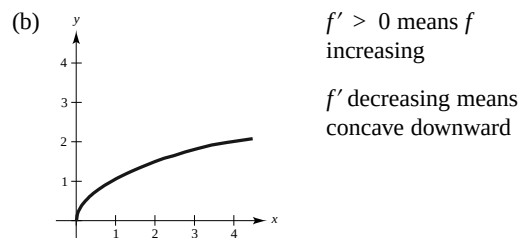
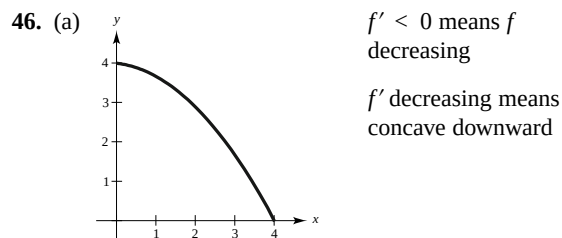
(b) Relative maximum: $(1.84, 1.85)$

Relative minimum: $(4.82, -3.09)$

Points of inflection: $(0.75, 0.83), (3.42, -0.72)$



f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.



48. (a) The rate of change of sales is increasing.
 $S'' > 0$

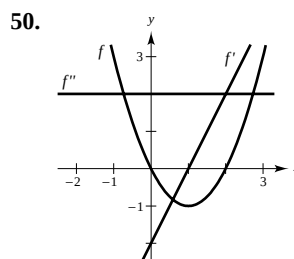
(b) The rate of change of sales is decreasing.
 $S' > 0, S'' < 0$

(c) The rate of change of sales is constant.
 $S' = C, S'' = 0$

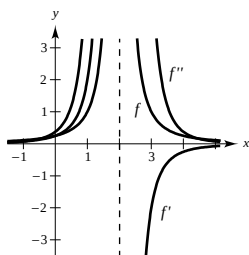
(d) Sales are steady.
 $S = C, S' = 0, S'' = 0$

(e) Sales are declining, but at a lower rate.
 $S' < 0, S'' > 0$

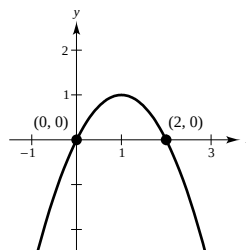
(f) Sales have bottomed out and have started to rise.
 $S' > 0$



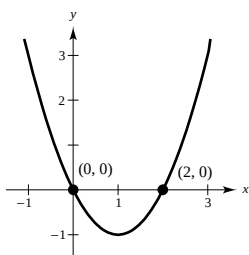
52.



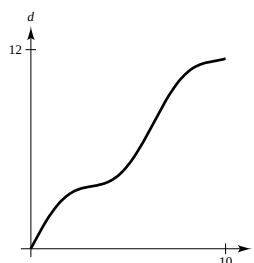
54.



56.



58. (a)



(b) Since the depth d is always increasing, there are no relative extrema. $f'(x) > 0$

(c) The rate of change of d is decreasing until you reach the widest point of the jug, then the rate increases until you reach the narrowest part of the jug's neck, then the rate decreases until you reach the top of the jug.

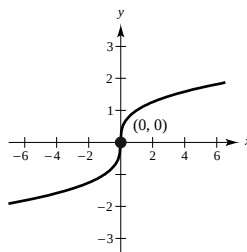
 60. (a) $f(x) = \sqrt[3]{x}$

$$f'(x) = \frac{1}{3}x^{-2/3}$$

$$f''(x) = -\frac{2}{9}x^{-5/3}$$

Inflection point: $(0, 0)$

(b) $f''(x)$ does not exist at $x = 0$.


 62. $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: $(2, 4)$

Relative minimum: $(4, 2)$

Point of inflection: $(3, 3)$

$$f'(x) = 3ax^2 + 2bx + c, f''(x) = 6ax + 2b$$

$$\left. \begin{aligned} f(2) &= 8a + 4b + 2c + d = 4 \\ f(4) &= 64a + 16b + 4c + d = 2 \end{aligned} \right\} 56a + 12b + 2c = -2 \Rightarrow 28a + 6b + c = -1$$

$$f'(2) = 12a + 4b + c = 0, f'(4) = 48a + 8b + c = 0, f''(3) = 18a + 2b = 0$$

$$28a + 6b + c = -1 \quad 18a + 2b = 0$$

$$12a + 4b + c = 0 \quad 16a + 2b = -1$$

$$16a + 2b = -1 \quad 2a = 1$$

$$a = \frac{1}{2}, b = -\frac{9}{2}, c = 12, d = -6$$

$$f(x) = \frac{1}{2}x^3 - \frac{9}{2}x^2 + 12x - 6$$

64. (a) line
- OA
- :
- $y = -0.06x$
- slope:
- -0.06

line CB : $y = 0.04x + 50$ slope: 0.04

$$f(x) = ax^3 + bx^2 + cx + d$$

$$f'(x) = 3ax^2 + 2bx + c$$

$$(-1000, 60): 60 = (-1000)^3a + (1000)^2b - 1000c + d$$

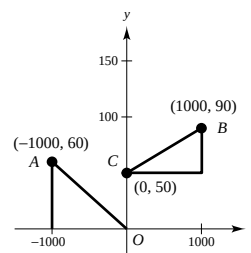
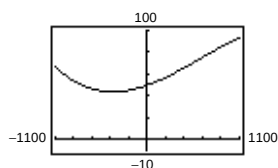
$$-0.06 = (1000)^2 3a - 2000b + c$$

$$(1000, 90): 90 = (1000)^3a + (1000)^2b + 1000c + d$$

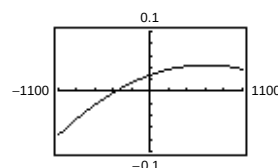
$$0.04 = (1000)^2 3a + 2000b + c$$

The solution to this system of 4 equations is $a = -1.25 \times 10^{-8}$, $b = 0.000025$, $c = 0.0275$, and $d = 50$.

- (b)
- $y = -1.25 \times 10^{-8}x^3 + 0.000025x^2 + 0.0275x + 50$



- (c)



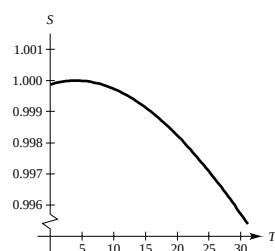
- (d) The steepest part of the road is 6% at the point A.

66. $S = \frac{5.755T^3}{10^8} - \frac{8.521T^2}{10^6} + \frac{0.654T}{10^4} + 0.99987, \quad 0 < T < 25$

- (a) The maximum occurs when
- $T \approx 4^\circ$
- and
- $S \approx 0.999999$
- .

(c) $S(20^\circ) \approx 0.9982$

- (b)



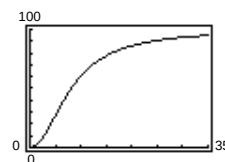
68. $C = 2x + \frac{300,000}{x}$

$$C' = 2 - \frac{300,000}{x^2} = 0 \text{ when } x = 100\sqrt{15} \approx 387$$

By the First Derivative Test, C is minimized when $x \approx 387$ units.

70. $S = \frac{100t^2}{65 + t^2}, \quad t > 0$

- (a)



(b) $S'(t) = \frac{13,000t}{(65 + t^2)^2}$

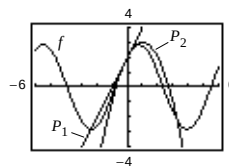
$$S''(t) = \frac{13,000(65 - 3t^2)}{(65 + t^2)^3} = 0 \Rightarrow t = 4.65$$

S is concave upwards on $(0, 4.65)$, concave downwards on $(4.65, 30)$.

- (c)
- $S'(t) > 0$
- for
- $t > 0$
- .

As t increases, the speed increases, but at a slower rate.

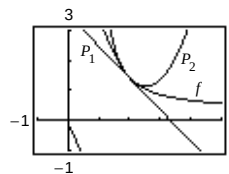
$$\begin{aligned}
 72. \quad f(x) &= 2(\sin x + \cos x), & f(0) &= 2 \\
 f'(x) &= 2(\cos x - \sin x), & f'(0) &= 2 \\
 f''(x) &= 2(-\sin x - \cos x), & f''(0) &= -2 \\
 P_1(x) &= 2 + 2(x - 0) = 2(1 + x)
 \end{aligned}$$



$$\begin{aligned}
 P_1'(x) &= 2 \\
 P_2(x) &= 2 + 2(x - 0) + \frac{1}{2}(-2)(x - 0)^2 = 2 + 2x - x^2 \\
 P_2'(x) &= 2 - 2x \\
 P_2''(x) &= -2
 \end{aligned}$$

The values of f , P_1 , P_2 , and their first derivatives are equal at $x = 0$. The values of the second derivatives of f and P_2 are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

$$\begin{aligned}
 74. \quad f(x) &= \frac{\sqrt{x}}{x-1}, & f(2) &= \sqrt{2} \\
 f'(x) &= \frac{-(x+1)}{2\sqrt{x}(x-1)^2}, & f'(2) &= -\frac{3}{2\sqrt{2}} = -\frac{3\sqrt{2}}{4} \\
 f''(x) &= \frac{3x^2 + 6x - 1}{4x^{3/2}(x-1)^3}, & f''(2) &= \frac{23}{8\sqrt{2}} = \frac{23\sqrt{2}}{16} \\
 P_1(x) &= \sqrt{2} + \left(-\frac{3\sqrt{2}}{4}\right)(x-2) = -\frac{3\sqrt{2}}{4}x + \frac{5\sqrt{2}}{2} \\
 P_1'(x) &= -\frac{3\sqrt{2}}{4}
 \end{aligned}$$



$$\begin{aligned}
 P_2(x) &= \sqrt{2} + \left(-\frac{3\sqrt{2}}{4}\right)(x-2) + \frac{1}{2}\left(\frac{23\sqrt{2}}{16}\right)(x-2)^2 = \sqrt{2} - \frac{3\sqrt{2}}{4}(x-2) + \frac{23\sqrt{2}}{32}(x-2)^2 \\
 P_2'(x) &= -\frac{3\sqrt{2}}{4} + \frac{23\sqrt{2}}{16}(x-2) \\
 P_2''(x) &= \frac{23\sqrt{2}}{16}
 \end{aligned}$$

The values of f , P_1 , P_2 and their first derivatives are equal at $x = 2$. The values of the second derivatives of f and P_2 are equal at $x = 2$. The approximations worsen as you move away from $x = 2$.

$$\begin{aligned}
 76. \quad f(x) &= x(x-6)^2 = x^3 - 12x^2 + 36x \\
 f'(x) &= 3x^2 - 24x + 36 = 3(x-2)(x-6) = 0 \\
 f''(x) &= 6x - 24 = 6(x-4) = 0
 \end{aligned}$$

Relative extrema: (2, 32) and (6, 0)

Point of inflection (4, 16) is midway between the relative extrema of f .

78. $p(x) = ax^3 + bx^2 + cx + d$

$$p'(x) = 3ax^2 + 2bx + c$$

$$p''(x) = 6ax + 2b$$

$$6ax + 2b = 0$$

$$x = -\frac{b}{3a}$$

The sign of $p''(x)$ changes at $x = -b/3a$. Therefore, $(-b/3a, p(-b/3a))$ is a point of inflection.

$$p\left(-\frac{b}{3a}\right) = a\left(-\frac{b^3}{27a^3}\right) + b\left(\frac{b^2}{9a^2}\right) + c\left(-\frac{b}{3a}\right) + d = \frac{2b^3}{27a^2} - \frac{bc}{3a} + d$$

When $p(x) = x^3 - 3x^2 + 2$, $a = 1$, $b = -3$, $c = 0$, and $d = 2$.

$$x_0 = \frac{-(-3)}{3(1)} = 1$$

$$y_0 = \frac{2(-3)^3}{27(1)^2} - \frac{(-3)(0)}{3(1)} + 2 = -2 - 0 + 2 = 0$$

The point of inflection of $p(x) = x^3 - 3x^2 + 2$ is $(x_0, y_0) = (1, 0)$.

80. False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

82. True

$$y = \sin(bx)$$

$$\text{Slope: } y' = b \cos(bx)$$

$$-b \leq y' \leq b \quad (\text{Assume } b > 0)$$

84. False. For example, let $f(x) = (x - 2)^4$.

Section 3.5 Limits at Infinity

2. $f(x) = \frac{2x}{\sqrt{x^2 + 2}}$

No vertical asymptotes

Horizontal asymptotes: $y = \pm 2$

Matches (c)

4. $f(x) = 2 + \frac{x^2}{x^4 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (a)

6. $f(x) = \frac{2x^2 - 3x + 5}{x^2 + 1}$

No vertical asymptotes

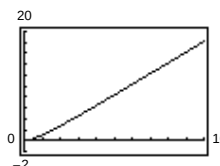
Horizontal asymptote: $y = 2$

Matches (e)

8. $f(x) = \frac{2x^2}{x + 1}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1	18.18	198.02	1998.02	19,998	199,998	1,999,998

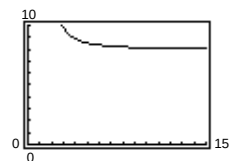
$\lim_{x \rightarrow \infty} f(x) = \infty$ (Limit does not exist.)



$$10. f(x) = \frac{8x}{\sqrt{x^2 - 3}}$$

x	10^1	10^2	10^3	10^4	10^5	10^6	10^7
$f(x)$	8.12	8.001	8	8	8	8	8

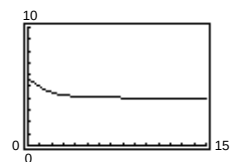
$$\lim_{x \rightarrow \infty} f(x) = 8$$



$$12. f(x) = 4 + \frac{3}{x^2 + 2}$$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	5	4.03	4.0003	4.0	4.0	4	4

$$\lim_{x \rightarrow \infty} f(x) = 4$$



$$14. (a) h(x) = \frac{f(x)}{x} = \frac{5x^2 - 3x + 7}{x} = 5x - 3 + \frac{7}{x}$$

$$\lim_{x \rightarrow \infty} h(x) = \infty \quad (\text{Limit does not exist})$$

$$(b) h(x) = \frac{f(x)}{x^2} = \frac{5x^2 - 3x + 7}{x^2} = 5 - \frac{3}{x} + \frac{7}{x^2}$$

$$\lim_{x \rightarrow \infty} h(x) = 5$$

$$(c) h(x) = \frac{f(x)}{x^3} = \frac{5x^2 - 3x + 7}{x^3} = \frac{5}{x} - \frac{3}{x^2} + \frac{7}{x^3}$$

$$\lim_{x \rightarrow \infty} h(x) = 0$$

$$16. (a) \lim_{x \rightarrow \infty} \frac{3 - 2x}{3x^3 - 1} = 0$$

$$(b) \lim_{x \rightarrow \infty} \frac{3 - 2x}{3x - 1} = -\frac{2}{3}$$

$$(c) \lim_{x \rightarrow \infty} \frac{3 - 2x^2}{3x - 1} = -\infty \quad (\text{Limit does not exist})$$

$$18. (a) \lim_{x \rightarrow \infty} \frac{5x^{3/2}}{4x^2 + 1} = 0$$

$$(b) \lim_{x \rightarrow \infty} \frac{5x^{3/2}}{4x^{3/2} + 1} = \frac{5}{4}$$

$$(c) \lim_{x \rightarrow \infty} \frac{5x^{3/2}}{4\sqrt{x} + 1} = \infty \quad (\text{Limit does not exist})$$

$$20. \lim_{x \rightarrow \infty} \frac{3x^3 + 2}{9x^3 - 2x^2 + 7} = \frac{3}{9} = \frac{1}{3}$$

$$22. \lim_{x \rightarrow \infty} \left(4 + \frac{3}{x} \right) = 4 + 0 = 4$$

$$24. \lim_{x \rightarrow \infty} \left(\frac{1}{2}x - \frac{4}{x^2} \right) = -\infty \quad (\text{Limit does not exist})$$

$$26. \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \rightarrow -\infty} \frac{1}{\left(\frac{\sqrt{x^2 + 1}}{-\sqrt{x^2}} \right)} \quad (\text{for } x < 0, x = -\sqrt{x^2})$$

$$= \lim_{x \rightarrow -\infty} \frac{-1}{\sqrt{x + (1/x)}} = -1$$

$$\begin{aligned}
 28. \lim_{x \rightarrow -\infty} \frac{-3x + 1}{\sqrt{x^2 + x}} &= \lim_{x \rightarrow -\infty} \frac{-3 + (1/x)}{\frac{\sqrt{x^2 + x}}{-\sqrt{x^2}}}, \text{ (for } x < 0 \text{ we have } -\sqrt{x^2} = x) \\
 &= \lim_{x \rightarrow -\infty} \frac{3 - (1/x)}{\sqrt{1 + (1/x)}} = 3
 \end{aligned}$$

$$\begin{aligned}
 30. \lim_{x \rightarrow \infty} \frac{x - \cos x}{x} &= \lim_{x \rightarrow \infty} \left(1 - \frac{\cos x}{x}\right) \\
 &= 1 - 0 = 1
 \end{aligned}$$

$$32. \lim_{x \rightarrow \infty} \cos\left(\frac{1}{x}\right) = \cos 0 = 1$$

Note:

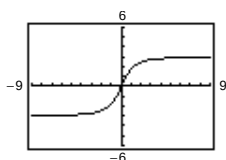
$$\lim_{x \rightarrow \infty} \frac{\cos x}{x} = 0 \text{ by the Squeeze Theorem since}$$

$$-\frac{1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x}.$$

$$34. f(x) = \frac{3x}{\sqrt{x^2 + 2}}$$

$$\lim_{x \rightarrow \infty} f(x) = 3$$

$$\lim_{x \rightarrow -\infty} f(x) = -3$$



$$\begin{aligned}
 36. \lim_{x \rightarrow \infty} x \tan \frac{1}{x} &= \lim_{t \rightarrow 0^+} \frac{\tan t}{t} = \lim_{t \rightarrow 0^+} \left[\frac{\sin t}{t} \cdot \frac{1}{\cos t} \right] \\
 &= (1)(1) = 1
 \end{aligned}$$

(Let $x = 1/t$.)

Therefore, $y = 3$ and $y = -3$ are both horizontal asymptotes.

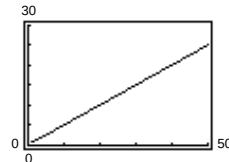
$$38. \lim_{x \rightarrow \infty} (2x - \sqrt{4x^2 + 1}) = \lim_{x \rightarrow \infty} \left[(2x - \sqrt{4x^2 + 1}) \cdot \frac{2x + \sqrt{4x^2 + 1}}{2x + \sqrt{4x^2 + 1}} \right] = \lim_{x \rightarrow \infty} \frac{-1}{2x + \sqrt{4x^2 + 1}} = 0$$

$$\begin{aligned}
 40. \lim_{x \rightarrow -\infty} (3x + \sqrt{9x^2 - x}) &= \lim_{x \rightarrow -\infty} \left[(3x + \sqrt{9x^2 - x}) \cdot \frac{3x - \sqrt{9x^2 - x}}{3x - \sqrt{9x^2 - x}} \right] \\
 &= \lim_{x \rightarrow -\infty} \frac{x}{3x - \sqrt{9x^2 - x}} \\
 &= \lim_{x \rightarrow -\infty} \frac{1}{3 - \frac{\sqrt{9x^2 - x}}{x}} \text{ (for } x < 0 \text{ we have } x = -\sqrt{x^2}) \\
 &= \lim_{x \rightarrow -\infty} \frac{1}{3 + \sqrt{9 - (1/x)}} = \frac{1}{6}
 \end{aligned}$$

42.	x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
	$f(x)$	1.0	5.1	50.1	500.1	5000.1	50,000.1	500,000.1

$$\lim_{x \rightarrow \infty} \frac{x^2 - x\sqrt{x^2 - x}}{1} \cdot \frac{x^2 + x\sqrt{x^2 - x}}{x^2 + x\sqrt{x^2 - x}} = \lim_{x \rightarrow \infty} \frac{x^3}{x^2 + x\sqrt{x^2 - x}} = \infty$$

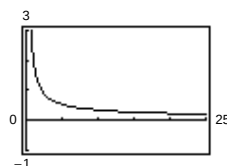
Limit does not exist.



44.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	2.000	0.348	0.101	0.032	0.010	0.003	0.001

$$\lim_{x \rightarrow \infty} \frac{x+1}{x\sqrt{x}} = 0$$



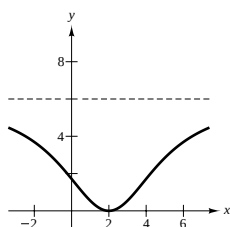
46. $x = 2$ is a critical number.

$$f'(x) < 0 \text{ for } x < 2.$$

$$f'(x) > 0 \text{ for } x > 2.$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow \infty} f(x) = 6$$

For example, let $f(x) = \frac{-6}{0.1(x-2)^2 + 1} + 6$.



48. (a) The function is even: $\lim_{x \rightarrow -\infty} f(x) = 5$

(b) The function is odd: $\lim_{x \rightarrow -\infty} f(x) = -5$

50. $y = \frac{x-3}{x-2}$

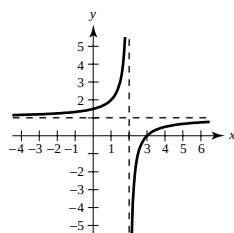
Intercepts: $(3, 0), (0, \frac{3}{2})$

Symmetry: none

Horizontal asymptote: $y = 1$ since

$$\lim_{x \rightarrow -\infty} \frac{x-3}{x-2} = 1 = \lim_{x \rightarrow \infty} \frac{x-3}{x-2}.$$

Discontinuity: $x = 2$ (Vertical asymptote)



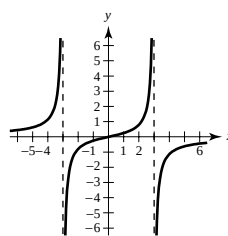
52. $y = \frac{2x}{9-x^2}$

Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = \pm 3$



54. $y = \frac{x^2}{x^2-9}$

Intercept: $(0, 0)$

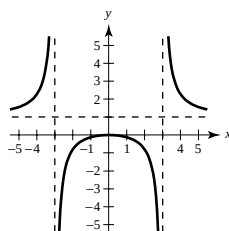
Symmetry: y -axis

Horizontal asymptote: $y = 1$ since

$$\lim_{x \rightarrow -\infty} \frac{x^2}{x^2-9} = 1 = \lim_{x \rightarrow \infty} \frac{x^2}{x^2-9}.$$

Discontinuities: $x = \pm 3$ (Vertical asymptotes)

Relative maximum: $(0, 0)$



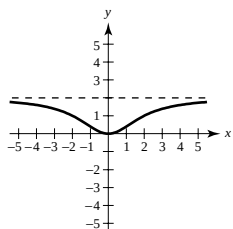
56. $y = \frac{2x^2}{x^2 + 4}$

Intercept: (0, 0)

Symmetry: y-axis

Horizontal asymptote: $y = 2$

Relative minimum: (0, 0)



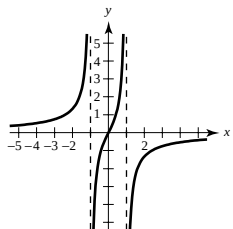
60. $y = \frac{2x}{1 - x^2}$

Intercept: (0, 0)

Symmetry: origin

Horizontal asymptote: $y = 0$ since

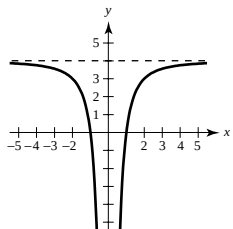
$$\lim_{x \rightarrow -\infty} \frac{2x}{1 - x^2} = 0 = \lim_{x \rightarrow \infty} \frac{2x}{1 - x^2}.$$

Discontinuities: $x = \pm 1$ (Vertical asymptotes)

64. $y = 4\left(1 - \frac{1}{x^2}\right)$

Intercepts: $(\pm 1, 0)$

Symmetry: y-axis

Horizontal asymptote: $y = 4$ Vertical asymptote: $x = 0$ 

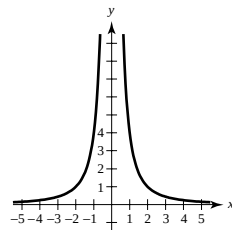
58. $x^2y = 4$

Intercepts: none

Symmetry: y-axis

Horizontal asymptote: $y = 0$ since

$$\lim_{x \rightarrow -\infty} \frac{4}{x^2} = 0 = \lim_{x \rightarrow \infty} \frac{4}{x^2}.$$

Discontinuity: $x = 0$ (Vertical asymptote)

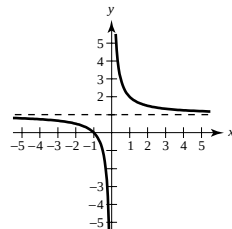
62. $y = 1 + \frac{1}{x}$

Intercept: $(-1, 0)$

Symmetry: none

Horizontal asymptote: $y = 1$ since

$$\lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right) = 1 = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right).$$

Discontinuity: $x = 0$ (Vertical asymptote)

66. $y = \frac{x}{\sqrt{x^2 - 4}}$

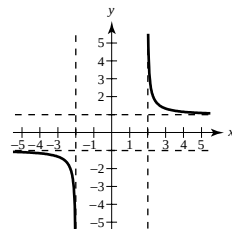
Domain: $(-\infty, -2), (2, \infty)$

Intercepts: none

Symmetry: origin

Horizontal asymptotes: $y = \pm 1$ since

$$\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 - 4}} = 1, \quad \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 - 4}} = -1.$$

Vertical asymptotes: $x = \pm 2$ (discontinuities)

68. $f(x) = \frac{x^2}{x^2 - 1}$

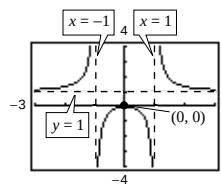
$$f'(x) = \frac{(x^2 - 1)(2x) - x^2(2x)}{(x^2 - 1)^2} = \frac{-2x}{(x^2 - 1)^2} = 0 \text{ when } x = 0.$$

$$f''(x) = \frac{(x^2 - 1)^2(-2) + 2x(2)(x^2 - 1)(2x)}{(x^2 - 1)^4} = \frac{2(3x^2 + 1)}{(x^2 - 1)^3}$$

Since $f''(0) < 0$, then $(0, 0)$ is a relative maximum. Since $f''(x) \neq 0$, nor is it undefined in the domain of f , there are no points of inflection.

Vertical asymptotes: $x = \pm 1$

Horizontal asymptote: $y = 1$



70. $f(x) = \frac{1}{x^2 - x - 2} = \frac{1}{(x + 1)(x - 2)}$

$$f'(x) = \frac{-(2x - 1)}{(x^2 - x - 2)^2} = 0 \text{ when } x = \frac{1}{2}.$$

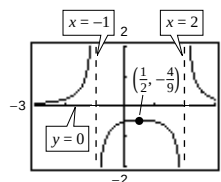
$$f''(x) = \frac{(x^2 - x - 2)^2(-2) + (2x - 1)(2)(x^2 - x - 2)(2x - 1)}{(x^2 - x - 2)^4}$$

$$= \frac{6(x^2 - x + 1)}{(x^2 - x - 2)^3}$$

Since $f''(\frac{1}{2}) < 0$, then $(\frac{1}{2}, -\frac{4}{9})$ is a relative maximum. Since $f''(x) \neq 0$, nor is it undefined in the domain of f , there are no points of inflection.

Vertical asymptotes: $x = -1, x = 2$

Horizontal asymptote: $y = 0$



72. $f(x) = \frac{x + 1}{x^2 + x + 1}$

$$f'(x) = \frac{-x(x + 2)}{(x^2 + x + 1)^2} = 0 \text{ when } x = 0, -2.$$

$$f''(x) = \frac{2(x^3 + 3x^2 - 1)}{(x^2 + x + 1)^3} = 0 \text{ when } x \approx 0.5321, -0.6527, -2.8794.$$

$$f''(0) < 0$$

Therefore, $(0, 1)$ is a relative maximum.

$$f''(-2) > 0$$

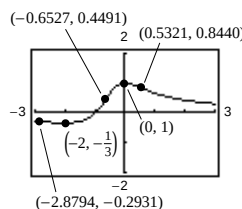
Therefore,

$$\left(-2, -\frac{1}{3}\right)$$

is a relative minimum.

Points of inflection: $(0.5321, 0.8440)$, $(-0.6527, 0.4491)$ and $(-2.8794, -0.2931)$

Horizontal asymptote: $y = 0$



$$74. g(x) = \frac{2x}{\sqrt{3x^2 + 1}}$$

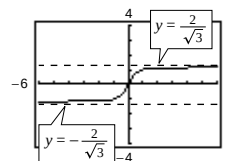
$$g'(x) = \frac{2}{(3x^2 + 1)^{3/2}}$$

$$g''(x) = \frac{-18x}{(3x^2 + 1)^{5/2}}$$

No relative extrema. Point of inflection: $(0, 0)$.

Horizontal asymptotes: $y = \pm \frac{2}{\sqrt{3}}$

No vertical asymptotes



$$76. f(x) = \frac{2 \sin 2x}{x} \quad \text{Hole at } (0, 4)$$

$$f'(x) = \frac{4x \cos 2x - 2 \sin 2x}{x^2}$$

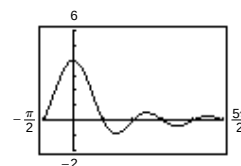
There are an infinite number of relative extrema. In the interval $(-2\pi, 2\pi)$, you obtain the following.

Relative minima: $(\pm 2.25, -0.869)$, $(\pm 5.45, -0.365)$

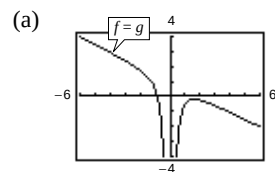
Relative maxima: $(\pm 3.87, 0.513)$

Horizontal asymptote: $y = 0$

No vertical asymptotes



$$78. f(x) = -\frac{x^3 - 2x^2 + 2}{2x^2}, g(x) = -\frac{1}{2}x + 1 - \frac{1}{x^2}$$

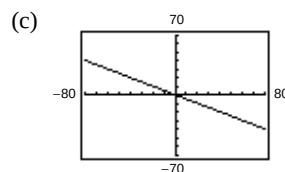


(b)

$$f(x) = -\frac{x^3 - 2x^2 + 2}{2x^2}$$

$$= -\left[\frac{x^3}{2x^2} - \frac{2x^2}{2x^2} + \frac{2}{2x^2} \right]$$

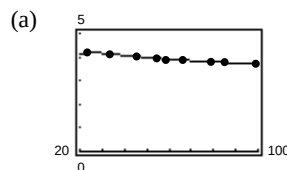
$$= -\frac{1}{2}x + 1 - \frac{1}{x^2} = g(x)$$



The graph appears as the slant asymptote $y = -\frac{1}{2}x + 1$.

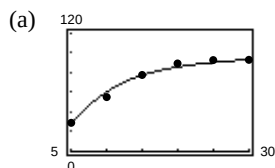
$$80. \lim_{v_1/v_2 \rightarrow \infty} 100 \left[1 - \frac{1}{(v_1/v_2)^c} \right] = 100[1 - 0] = 100\%$$

$$82. y = \frac{3.351t^2 + 42.461t - 543.730}{t^2}$$



(b) Yes. $\lim_{t \rightarrow \infty} y = 3.351$

84. $S = \frac{100t^2}{65 + t^2}, t > 0$



(b) Yes, $\lim_{t \rightarrow \infty} S = \frac{100}{1} = 100$

86. $\lim_{x \rightarrow \infty} \frac{p(x)}{q(x)} = \lim_{x \rightarrow \infty} \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$

Divide $p(x)$ and $q(x)$ by x^m .

Case 1: If $n < m$: $\lim_{x \rightarrow \infty} \frac{p(x)}{q(x)} = \lim_{x \rightarrow \infty} \frac{\frac{a_n}{x^{m-n}} + \cdots + \frac{a_1}{x^{m-1}} + \frac{a_0}{x^m}}{b_m + \cdots + \frac{b_1}{x^{m-1}} + \frac{b_0}{x^m}} = \frac{0 + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \frac{0}{b_m} = 0$

Case 2: If $m = n$: $\lim_{x \rightarrow \infty} \frac{p(x)}{q(x)} = \lim_{x \rightarrow \infty} \frac{\frac{a_n}{x^{m-n}} + \cdots + \frac{a_1}{x^{m-1}} + \frac{a_0}{x^m}}{b_m + \cdots + \frac{b_1}{x^{m-1}} + \frac{b_0}{x^m}} = \frac{a_n + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \frac{a_n}{b_m}$.

Case 3: If $n > m$: $\lim_{x \rightarrow \infty} \frac{p(x)}{q(x)} = \lim_{x \rightarrow \infty} \frac{a_n x^{n-m} + \cdots + \frac{a_1}{x^{m-1}} + \frac{a_0}{x^m}}{b_m + \cdots + \frac{b_1}{x^{m-1}} + \frac{b_0}{x^m}} = \frac{\pm\infty + \cdots + 0}{b_m + \cdots + 0} = \pm\infty$.

88. False. Let $y_1 = \sqrt{x+1}$, then $y_1(0) = 1$. Thus, $y_1' = 1/(2\sqrt{x+1})$ and $y_1'(0) = 1/2$. Finally,

$$y_1'' = -\frac{1}{4(x+1)^{3/2}} \text{ and } y_1''(0) = -\frac{1}{4}.$$

Let $p = ax^2 + bx + 1$, then $p(0) = 1$. Thus, $p' = 2ax + b$ and $p'(0) = \frac{1}{2} \Rightarrow b = \frac{1}{2}$. Finally, $p'' = 2a$ and $p''(0) = -\frac{1}{4} \Rightarrow a = -\frac{1}{8}$. Therefore,

$$f(x) = \begin{cases} (-1/8)x^2 + (1/2)x + 1, & x < 0 \\ \sqrt{x+1}, & x \geq 0 \end{cases} \text{ and } f(0) = 1,$$

$$f'(x) = \begin{cases} (1/2) - (1/4)x, & x < 0 \\ 1/(2\sqrt{x+1}), & x \geq 0 \end{cases} \text{ and } f'(0) = \frac{1}{2}, \text{ and}$$

$$f''(x) = \begin{cases} (-1/4), & x < 0 \\ -1/(4(x+1)^{3/2}), & x \geq 0 \end{cases} \text{ and } f''(0) = -\frac{1}{4}.$$

$f''(x) < 0$ for all real x , but $f(x)$ increases without bound.

Section 3.6 A Summary of Curve Sketching

2. The slope of f approaches ∞ as $x \rightarrow 0^-$, and approaches $-\infty$ as $x \rightarrow 0^+$. Matches (C)

6. (a) x_0, x_2, x_4

(c) x_1

(e) x_2, x_3

4. The slope is positive up to approximately $x = 1.5$. Matches (B)

(b) x_2, x_3

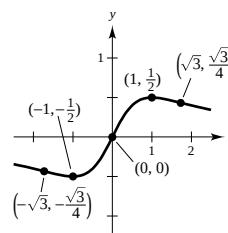
(d) x_1

8. $y = \frac{x}{x^2 + 1}$

$$y' = \frac{1 - x^2}{(x^2 + 1)^2} = \frac{(1 - x)(x + 1)}{(x^2 + 1)^2} = 0 \text{ when } x = \pm 1.$$

$$y'' = -\frac{2x(3 - x^2)}{(x^2 + 1)^3} = 0 \text{ when } x = 0, \pm\sqrt{3}.$$

Horizontal asymptote: $y = 0$



	y	y'	y''	Conclusion
$-\infty < x < -\sqrt{3}$		-	-	Decreasing, concave down
$x = -\sqrt{3}$	$-\frac{\sqrt{3}}{4}$	-	0	Point of inflection
$-\sqrt{3} < x < -1$		-	+	Decreasing, concave up
$x = -1$	$-\frac{1}{2}$	0	+	Relative minimum
$-1 < x < 0$		+	+	Increasing, concave up
$x = 0$	0	+	0	Point of inflection
$0 < x < 1$		+	-	Increasing, concave down
$x = 1$	$\frac{1}{2}$	0	-	Relative maximum
$1 < x < \sqrt{3}$		-	-	Decreasing, concave down
$x = \sqrt{3}$	$\frac{\sqrt{3}}{4}$	-	0	Point of inflection
$\sqrt{3} < x < \infty$		-	+	Decreasing, concave up

$$10. y = \frac{x^2 + 1}{x^2 - 9}$$

$$y' = \frac{-20x}{(x^2 - 9)^2} = 0 \quad \text{when } x = 0$$

$$y'' = \frac{60(x^2 + 3)}{(x^2 - 9)^3} < 0 \quad \text{when } x = 0$$

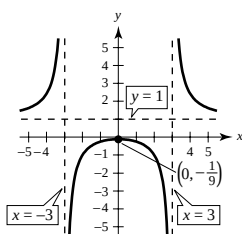
Therefore, $(0, -\frac{1}{9})$ is a relative maximum.

Intercept: $(0, -\frac{1}{9})$

Vertical asymptotes: $x = \pm 3$

Horizontal asymptote: $y = 1$

Symmetric about y-axis



$$12. f(x) = \frac{x+2}{x} = 1 + \frac{2}{x}$$

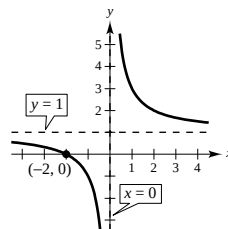
$$f'(x) = \frac{-2}{x^2} < 0 \quad \text{when } x \neq 0.$$

$$f''(x) = \frac{4}{x^3} \neq 0$$

Intercept: $(-2, 0)$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 1$



$$14. f(x) = x + \frac{32}{x^2}$$

$$f'(x) = 1 - \frac{64}{x^3} = \frac{(x-4)(x^2+4x+16)}{x^3} = 0 \quad \text{when } x = 4.$$

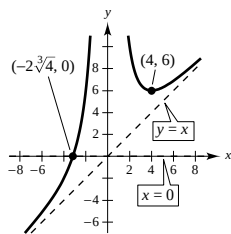
$$f''(x) = \frac{192}{x^4} > 0 \quad \text{if } x \neq 0.$$

Therefore, $(4, 6)$ is a relative minimum.

Intercept: $(-2\sqrt[3]{4}, 0)$

Vertical asymptote: $x = 0$

Slant asymptote: $y = x$



$$16. f(x) = \frac{x^3}{x^2 - 4} = x + \frac{4x}{x^2 - 4}$$

$$f'(x) = \frac{x^2(x^2 - 12)}{(x^2 - 4)^2} = 0 \quad \text{when } x = 0, \pm 2\sqrt{3}$$

$$f''(x) = \frac{8x(x^2 + 12)}{(x^2 - 4)^3} = 0 \quad \text{when } x = 0$$

Intercept: $(0, 0)$

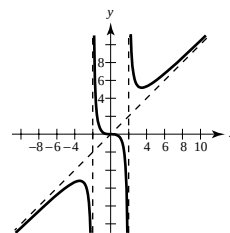
Relative maximum: $(-2\sqrt{3}, -3\sqrt{3})$

Relative minimum: $(2\sqrt{3}, 3\sqrt{3})$

Inflection point: $(0, 0)$

Vertical asymptotes: $x = \pm 2$

Slant asymptote: $y = x$



$$18. y = \frac{2x^2 - 5x + 5}{x - 2} = 2x - 1 + \frac{3}{x - 2}$$

$$y' = 2 - \frac{3}{(x - 2)^2} = \frac{2x^2 - 8x + 5}{(x - 2)^2} = 0 \text{ when } x = \frac{4 \pm \sqrt{6}}{2}.$$

$$y'' = \frac{6}{(x - 2)^3} \neq 0$$

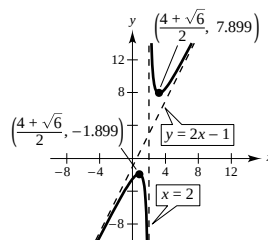
$$\text{Relative maximum: } \left(\frac{4 - \sqrt{6}}{2}, -1.8990 \right)$$

$$\text{Relative minimum: } \left(\frac{4 + \sqrt{6}}{2}, 7.8990 \right)$$

$$\text{Intercept: } (0, -5/2)$$

$$\text{Vertical asymptote: } x = 2$$

$$\text{Slant asymptote: } y = 2x - 1$$



$$20. g(x) = x\sqrt{9 - x} \quad \text{Domain: } x \leq 9$$

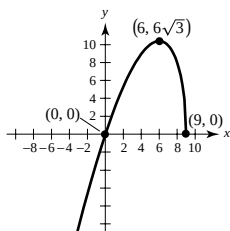
$$g'(x) = \frac{3(6 - x)}{2\sqrt{9 - x}} = 0 \text{ when } x = 6$$

$$g''(x) = \frac{3(x - 12)}{4(9 - x)^{3/2}} < 0 \text{ when } x = 6$$

$$\text{Relative maximum: } (6, 6\sqrt{3})$$

$$\text{Intercepts: } (0, 0), (9, 0)$$

$$\text{Concave downward on } (-\infty, 9)$$



$$22. y = x\sqrt{16 - x^2} \quad \text{Domain: } -4 \leq x \leq 4$$

$$y' = \frac{2(8 - x^2)}{\sqrt{16 - x^2}} = 0 \text{ when } x = \pm 2\sqrt{2}$$

$$y'' = \frac{2x(x^2 - 24)}{(16 - x^2)^{3/2}} = 0 \text{ when } x = 0$$

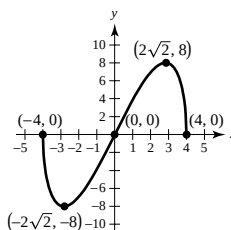
$$\text{Relative maximum: } (2\sqrt{2}, 8)$$

$$\text{Relative minimum: } (-2\sqrt{2}, -8)$$

$$\text{Intercepts: } (0, 0), (\pm 4, 0)$$

$$\text{Symmetric with respect to the origin}$$

$$\text{Point of inflection: } (0, 0)$$



$$24. y = 3(x - 1)^{2/3} - (x - 1)^2$$

$$y' = \frac{2}{(x - 1)^{1/3}} - 2(x - 1) = \frac{2 - 2(x - 1)^{4/3}}{(x - 1)^{1/3}} = 0 \text{ when } x = 0, 2$$

$$(y' \text{ undefined for } x = 1)$$

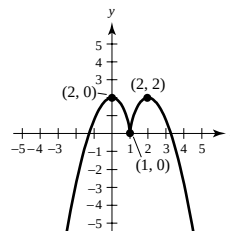
$$y'' = \frac{-2}{3(x - 1)^{4/3}} - 2 < 0 \text{ for all } x \neq 1$$

$$\text{Concave downward on } (-\infty, 1) \text{ and } (1, \infty)$$

$$\text{Relative maximum: } (0, 2), (2, 2)$$

$$\text{Relative minimum: } (1, 0)$$

$$\text{Intercepts: } (0, 2), (1, 0), (-1.280, 0), (3.280, 0)$$

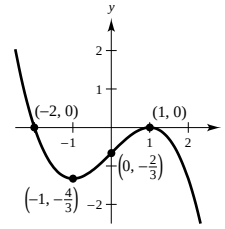


26. $y = -\frac{1}{3}(x^3 - 3x + 2)$

$y' = -x^2 + 1 = 0$ when $x = \pm 1$

$y'' = -2x = 0$ when $x = 0$

	y	y'	y''	Conclusion
$-\infty < x < -1$		-	+	Decreasing, concave up
$x = -1$	$-\frac{4}{3}$	0	+	Relative minimum
$-1 < x < 0$		+	+	Increasing, concave up
$x = 0$	$-\frac{2}{3}$	+	0	Point of inflection
$0 < x < 1$		+	-	Increasing, concave down
$x = 1$	0	0	-	Relative maximum
$1 < x < \infty$		-	-	Decreasing, concave down

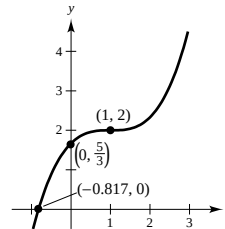


28. $f(x) = \frac{1}{3}(x - 1)^3 + 2$

$f'(x) = (x - 1)^2 = 0$ when $x = 1$.

$f''(x) = 2(x - 1) = 0$ when $x = 1$.

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < 1$		+	-	Increasing, concave down
$x = 1$	2	0	0	Point of inflection
$1 < x < \infty$		+	+	Increasing, concave up



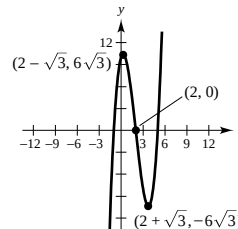
30. $f(x) = (x + 1)(x - 2)(x - 5)$

$f'(x) = (x + 1)(x - 2) + (x + 1)(x - 5) + (x - 2)(x - 5)$

$= 3(x^2 - 4x + 1) = 0$ when $x = 2 \pm \sqrt{3}$.

$f''(x) = 6(x - 2) = 0$ when $x = 2$.

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < 2 - \sqrt{3}$		+	-	Increasing, concave down
$x = 2 - \sqrt{3}$	$6\sqrt{3}$	0	-	Relative maximum
$2 - \sqrt{3} < x < 2$		-	-	Decreasing, concave down
$x = 2$	0	-	0	Point of inflection
$2 < x < 2 + \sqrt{3}$		-	+	Decreasing, concave up
$x = 2 + \sqrt{3}$	$-6\sqrt{3}$	0	+	Relative minimum
$2 + \sqrt{3} < x < \infty$		+	+	Increasing, concave up



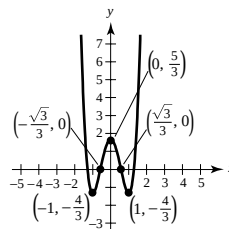
Intercepts: (0, 10), (-1, 0), (2, 0), (5, 0)

32. $y = 3x^4 - 6x^2 + \frac{5}{3}$

$$y' = 12x^3 - 12x = 12x(x^2 - 1) = 0 \text{ when } x = 0, x = \pm 1.$$

$$y'' = 36x^2 - 12 = 12(3x^2 - 1) = 0 \text{ when } x = \pm \frac{\sqrt{3}}{3}.$$

	y	y'	y''	Conclusion
$-\infty < x < -1$		-	+	Decreasing, concave up
$x = -1$	$-4/3$	0	+	Relative minimum
$-1 < x < -\frac{\sqrt{3}}{3}$		+	+	Increasing, concave up
$x = -\frac{\sqrt{3}}{3}$	0	+	0	Point of inflection
$-\frac{\sqrt{3}}{3} < x < 0$		+	-	Increasing, concave down
$x = 0$	$5/3$	0	-	Relative maximum
$0 < x < \frac{\sqrt{3}}{3}$		-	-	Decreasing, concave down
$x = \frac{\sqrt{3}}{3}$	0	-	0	Point of inflection
$\frac{\sqrt{3}}{3} < x < 1$		-	+	Decreasing, concave up
$x = 1$	$-4/3$	0	+	Relative minimum
$1 < x < \infty$		+	+	Increasing, concave up

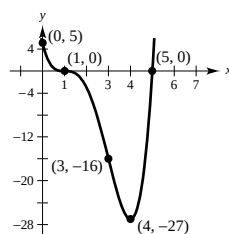


34. $f(x) = x^4 - 8x^3 + 18x^2 - 16x + 5$

$$f'(x) = 4x^3 - 24x^2 + 36x - 16 = 4(x - 4)(x - 1)^2 = 0 \text{ when } x = 1, x = 4.$$

$$f''(x) = 12x^2 - 48x + 36 = 12(x - 3)(x - 1) = 0 \text{ when } x = 3, x = 1.$$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < 1$		-	+	Decreasing, concave up
$x = 1$	0	0	0	Point of inflection
$1 < x < 3$		-	-	Decreasing, concave down
$x = 3$	-16	-	0	Point of inflection
$3 < x < 4$		-	+	Decreasing, concave up
$x = 4$	-27	0	+	Relative minimum
$4 < x < \infty$		+	+	Increasing, concave up

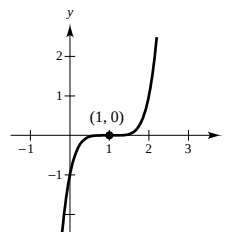


36. $y = (x - 1)^5$

$$y' = 5(x - 1)^4 = 0 \text{ when } x = 1.$$

$$y'' = 20(x - 1)^3 = 0 \text{ when } x = 1.$$

	y	y'	y''	Conclusion
$-\infty < x < 1$		+	-	Increasing, concave down
$x = 1$	0	0	0	Point of inflection
$1 < x < \infty$		+	+	Increasing, concave up

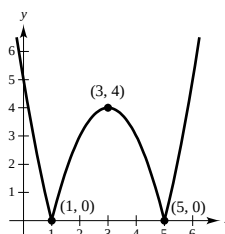


38. $y = |x^2 - 6x + 5|$

$$y' = \frac{2(x - 3)(x^2 - 6x + 5)}{|x^2 - 6x + 5|} = \frac{2(x - 3)(x - 5)(x - 1)}{|(x - 5)(x - 1)|}$$

$$= 0 \text{ when } x = 3 \text{ and undefined when } x = 1, x = 5.$$

$$y'' = \frac{2(x^2 - 6x + 5)}{|x^2 - 6x + 5|} = \frac{2(x - 5)(x - 1)}{|(x - 5)(x - 1)|} \text{ undefined when } x = 1, x = 5.$$



	y	y'	y''	Conclusion
$-\infty < x < 1$		-	+	Decreasing, concave up
$x = 1$	0	Undefined	Undefined	Relative minimum, point of inflection
$1 < x < 3$		+	-	Increasing, concave down
$x = 3$	4	0	-	Relative maximum
$3 < x < 5$		-	-	Decreasing, concave down
$x = 5$	0	Undefined	Undefined	Relative minimum, point of inflection
$5 < x < \infty$		+	+	Increasing, concave up

40. $y = \cos x - \frac{1}{2} \cos 2x, 0 \leq x \leq 2\pi$

$$y' = -\sin x + \sin 2x = -\sin x(1 - 2 \cos x) = 0 \text{ when } x = 0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}.$$

$$y'' = -\cos x + 2 \cos 2x = -\cos x + 2(2 \cos^2 x - 1)$$

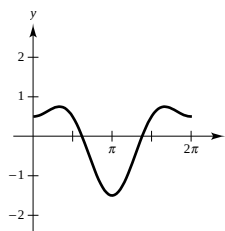
$$= 4 \cos^2 x - \cos x - 2 = 0 \text{ when } \cos x = \frac{1 \pm \sqrt{33}}{8} \approx 0.8431, -0.5931.$$

Therefore, $x \approx 0.5678$ or 5.7154 , $x \approx 2.2057$ or 4.0775 .

Relative maxima: $\left(\frac{\pi}{3}, \frac{3}{4}\right), \left(\frac{5\pi}{3}, \frac{3}{4}\right)$

Relative minimum: $\left(\pi, -\frac{3}{2}\right)$

Inflection points: $(0.5678, 0.6323), (2.2057, -0.4449), (5.7154, 0.6323), (4.0775, -0.4449)$



42. $y = 2(x - 2) + \cot x, 0 < x < \pi$

$$y' = 2 - \csc^2 x = 0 \text{ when } x = \frac{\pi}{4}, \frac{3\pi}{4}$$

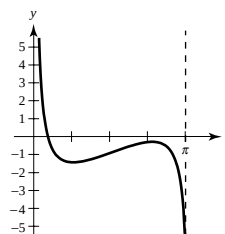
$$y'' = 2 \csc^2 x \cot x = 0 \text{ when } x = \frac{\pi}{2}$$

Relative maximum: $\left(\frac{3\pi}{4}, \frac{3\pi}{2} - 5\right)$

Relative minimum: $\left(\frac{\pi}{4}, \frac{\pi}{2} - 3\right)$

Point of inflection: $\left(\frac{\pi}{2}, \pi - 4\right)$

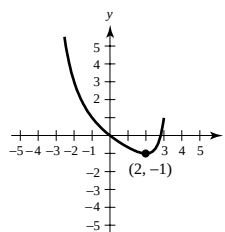
Vertical asymptotes: $x = 0, \pi$



44. $y = \sec^2\left(\frac{\pi x}{8}\right) - 2 \tan\left(\frac{\pi x}{8}\right) - 1, -3 < x < 3$

$$y' = 2 \sec^2\left(\frac{\pi x}{8}\right) \tan\left(\frac{\pi x}{8}\right) \left(\frac{\pi}{8}\right) - 2 \sec^2\left(\frac{\pi x}{8}\right) \left(\frac{\pi}{8}\right) = 0 \Rightarrow x = 2$$

Relative minimum: $(2, -1)$



46. $g(x) = x \cot x, -2\pi < x < 2\pi$

$$g'(x) = \frac{\sin x \cos x - x}{\sin^2 x}$$

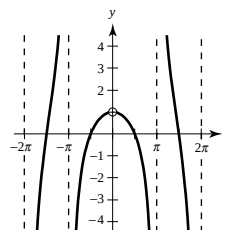
$$g'(0) \text{ does not exist. But } \lim_{x \rightarrow 0} x \cot x = \lim_{x \rightarrow 0} \frac{x}{\tan x} = 1.$$

Vertical asymptotes: $x = \pm 2\pi, \pm \pi$

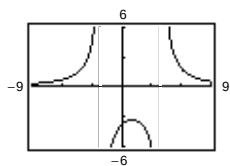
Intercepts: $\left(-\frac{3\pi}{2}, 0\right), \left(-\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right)$

Symmetric with respect to y-axis.

Decreasing on $(0, \pi)$ and $(\pi, 2\pi)$



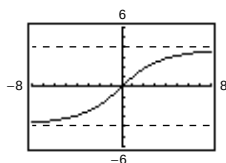
48. $f(x) = 5\left(\frac{1}{x-4} - \frac{1}{x+2}\right)$



$x = -2, 4$ vertical asymptote

$y = 0$ horizontal asymptote

50. $f(x) = \frac{4x}{\sqrt{x^2 + 15}}$



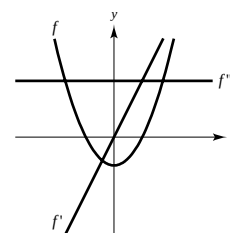
$y = \pm 4$ horizontal asymptotes

$(0, 0)$ point of inflection

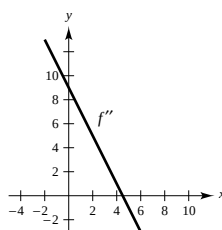
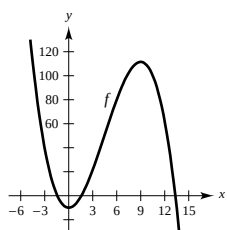
52. f'' is constant.

f' is linear.

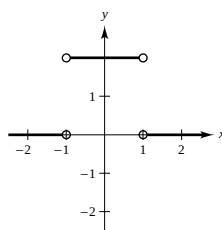
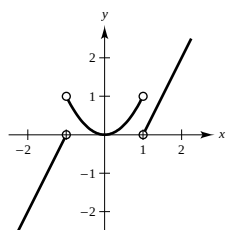
f is quadratic.



54.


 (any vertical translate of f will do)

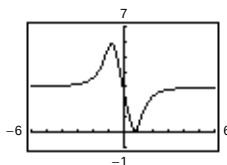
56.


 (any vertical translate of the 3 segments of f will do)

58. If $f'(x) = 2$ in $[-5, 5]$, then $f(x) = 2x + 3$ and $f(2) = 7$ is the least possible value of $f(2)$. If $f'(x) = 4$ in $[-5, 5]$, then $f(x) = 4x + 3$ and $f(2) = 11$ is the greatest possible value of $f(2)$.

$$60. g(x) = \frac{3x^4 - 5x + 3}{x^4 + 1}$$

Vertical asymptote: none

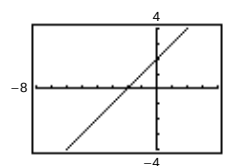
 Horizontal asymptote: $y = 3$


The graph crosses the horizontal asymptote $y = 3$. If a function has a vertical asymptote at $x = c$, the graph would not cross it since $f(c)$ is undefined.

$$62. g(x) = \frac{x^2 + x - 2}{x - 1}$$

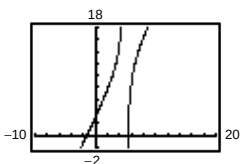
$$= \frac{(x + 2)(x - 1)}{x - 1} = \begin{cases} x + 2, & \text{if } x \neq 1 \\ \text{Undefined,} & \text{if } x = 1 \end{cases}$$

The rational function is not reduced to lowest terms.



hole at $(1, 3)$

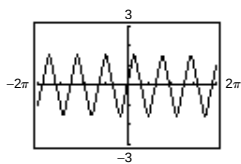
$$64. g(x) = \frac{2x^2 - 8x - 15}{x - 5} = 2x + 2 - \frac{5}{x - 5}$$



The graph appears to approach the slant asymptote $y = 2x + 2$.

66. $f(x) = \tan(\sin \pi x)$

(a)



(c) Periodic with period 2

(e) On $(0, 1)$, the graph of f is concave downward.

68. Vertical asymptote: $x = -3$

Horizontal asymptote: none

$$y = \frac{x^2}{x+3}$$

72. $f(x) = \frac{1}{2}(ax)^2 - (ax) = \frac{1}{2}(ax)(ax - 2), a \neq 0$

$$f'(x) = a^2x - a = a(ax - 1) = 0 \text{ when } x = \frac{1}{a}.$$

$$f''(x) = a^2 > 0 \text{ for all } x.$$

(a) Intercepts: $(0, 0), \left(\frac{2}{a}, 0\right)$

Relative minimum: $\left(\frac{1}{a}, -\frac{1}{2}\right)$

Points of inflection: none

74. Tangent line at P : $y - y_0 = f'(x_0)(x - x_0)$

(a) Let $y = 0$: $-y_0 = f'(x_0)(x - x_0)$

$$f'(x_0)x = x_0 f'(x_0) - y_0$$

$$x = x_0 - \frac{y_0}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

x-intercept: $\left(x_0 - \frac{f(x_0)}{f'(x_0)}, 0\right)$

(c) Normal line: $y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

Let $y = 0$: $-y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

$$-y_0 f'(x_0) = -x + x_0$$

$$x = x_0 + y_0 f'(x_0) = x_0 + f(x_0) f'(x_0)$$

x-intercept: $(x_0 + f(x_0) f'(x_0), 0)$

(e) $|BC| = \left|x_0 - \frac{f(x_0)}{f'(x_0)} - x_0\right| = \left|\frac{f(x_0)}{f'(x_0)}\right|$

(g) $|AB| = |x_0 - (x_0 + f(x_0) f'(x_0))| = |f(x_0) f'(x_0)|$

(b) $f(-x) = \tan(\sin(-\pi x)) = \tan(-\sin \pi x)$

$$= -\tan(\sin \pi x) = -f(x)$$

Symmetry with respect to the origin

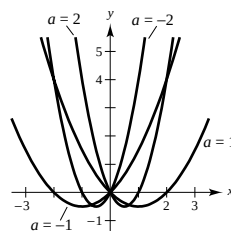
(d) On $(-1, 1)$, there is a relative maximum at $\left(\frac{1}{2}, \tan 1\right)$ and a relative minimum at $\left(-\frac{1}{2}, -\tan 1\right)$.

70. Vertical asymptote: $x = 0$

Slant asymptote: $y = -x$

$$y = -x + \frac{1}{x} = \frac{1-x^2}{x}$$

(b)



(b) Let $x = 0$: $y - y_0 = f'(x_0)(-x_0)$

$$y = y_0 - x_0 f'(x_0)$$

$$y = f(x_0) - x_0 f'(x_0)$$

y-intercept: $(0, f(x_0) - x_0 f'(x_0))$

(d) Let $x = 0$: $y - y_0 = \frac{-1}{f'(x_0)}(-x_0)$

$$y = y_0 + \frac{x_0}{f'(x_0)}$$

y-intercept: $\left(0, y_0 + \frac{x_0}{f'(x_0)}\right)$

(f) $|PC|^2 = y_0^2 + \left(\frac{f(x_0)}{f'(x_0)}\right)^2 = \frac{f(x_0)^2 f'(x_0)^2 + f(x_0)^2}{f'(x_0)^2}$

$$|PC|^2 = \left|\frac{f(x_0) \sqrt{1 + [f'(x_0)]^2}}{f'(x_0)}\right|^2$$

(h) $|AP|^2 = f(x_0)^2 f'(x_0)^2 + y_0^2$

$$|AP| = |f(x_0)| \sqrt{1 + [f'(x_0)]^2}$$

Section 3.7 Optimization Problems

2. Let x and y be two positive numbers such that $x + y = S$.

$$P = xy = x(S - x) = Sx - x^2$$

$$\frac{dP}{dx} = S - 2x = 0 \text{ when } x = \frac{S}{2}.$$

$$\frac{d^2P}{dx^2} = -2 < 0 \text{ when } x = \frac{S}{2}.$$

P is a maximum when $x = y = S/2$.

6. Let x and y be two positive numbers such that $x + 2y = 100$.

$$P = xy = y(100 - 2y) = 100y - 2y^2$$

$$\frac{dP}{dy} = 100 - 4y = 0 \text{ when } y = 25.$$

$$\frac{d^2P}{dy^2} = -4 < 0 \text{ when } y = 25.$$

P is a maximum when $x = 50$ and $y = 25$.

8. Let x be the length and y the width of the rectangle.

$$2x + 2y = P$$

$$y = \frac{P - 2x}{2} = \frac{P}{2} - x$$

$$A = xy = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2$$

$$\frac{dA}{dx} = \frac{P}{2} - 2x = 0 \text{ when } x = \frac{P}{4}.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = \frac{P}{4}.$$

A is maximum when $x = y = P/4$ units. (A square!)

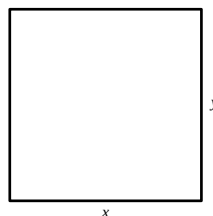
4. Let x and y be two positive numbers such that $xy = 192$.

$$S = x + 3y = \frac{192}{y} + 3y$$

$$\frac{dS}{dy} = 3 - \frac{192}{y^2} = 0 \text{ when } y = 8.$$

$$\frac{d^2S}{dy^2} = \frac{384}{y^3} > 0 \text{ when } y = 8.$$

S is minimum when $y = 8$ and $x = 24$.



10. Let x be the length and y the width of the rectangle.

$$xy = A$$

$$y = \frac{A}{x}$$

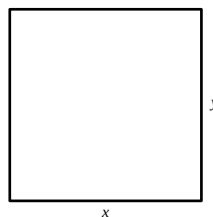
$$P = 2x + 2y = 2x + 2\left(\frac{A}{x}\right) = 2x + \frac{2A}{x}$$

$$\frac{dP}{dx} = 2 - \frac{2A}{x^2} = 0 \text{ when } x = \sqrt{A}.$$

$$\frac{d^2P}{dx^2} = \frac{4A}{x^3} > 0 \text{ when } x = \sqrt{A}.$$

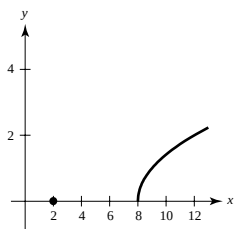
P is minimum when $x = y = \sqrt{A}$ centimeters.

(A square!)



12. $f(x) = \sqrt{x-8}$, $(2, 0)$

From the graph, it is clear that $(8, 0)$ is the closest point on the graph of f to $(2, 0)$.



16. $F = \frac{v}{22 + 0.02v^2}$

$$\frac{dF}{dv} = \frac{22 - 0.02v^2}{(22 + 0.02v^2)^2}$$

$$= 0 \text{ when } v = \sqrt{1100} \approx 33.166.$$

By the First Derivative Test, the flow rate on the road is maximized when $v \approx 33$ mph.

14. $f(x) = (x+1)^2$, $(5, 3)$

$$\begin{aligned} d &= \sqrt{(x-5)^2 + [(x+1)^2 - 3]^2} \\ &= \sqrt{x^2 - 10x + 25 + (x^2 + 2x - 2)^2} \\ &= \sqrt{x^2 - 10x + 25 + x^4 + 4x^3 - 8x + 4} \\ &= \sqrt{x^4 + 4x^3 + x^2 - 18x + 29} \end{aligned}$$

Since d is smallest when the expression inside the radical is smallest, you need to find the critical numbers of

$$g(x) = x^4 + 4x^3 + x^2 - 18x + 29$$

$$g'(x) = 4x^3 + 12x^2 + 2x - 18$$

$$= 2(x-1)(2x^2 + 8x + 9)$$

By the First Derivative Test, $x = 1$ yields a minimum. Hence, $(1, 4)$ is closest to $(5, 3)$.

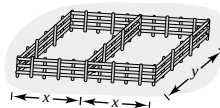
18. $4x + 3y = 200$ is the perimeter. (see figure)

$$A = 2xy = 2x\left(\frac{200-4x}{3}\right) = \frac{8}{3}(50x - x^2)$$

$$\frac{dA}{dx} = \frac{8}{3}(50 - 2x) = 0 \text{ when } x = 25.$$

$$\frac{d^2A}{dx^2} = -\frac{16}{3} < 0 \text{ when } x = 25.$$

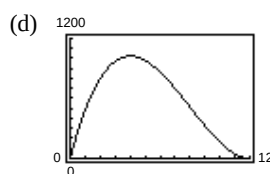
A is a maximum when $x = 25$ feet and $y = \frac{100}{3}$ feet.



20. (a)

Height, x	Length & Width	Volume
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$
3	$24 - 2(3)$	$3[24 - 2(3)]^2 = 972$
4	$24 - 2(4)$	$4[24 - 2(4)]^2 = 1024$
5	$24 - 2(5)$	$5[24 - 2(5)]^2 = 980$
6	$24 - 2(6)$	$6[24 - 2(6)]^2 = 864$

(b) $V = x(24 - 2x)^2$, $0 < x < 12$



The maximum volume seems to be 1024.

$$(c) \quad \frac{dV}{dx} = 2x(24 - 2x)(-2) + (24 - 2x)^2 = (24 - 2x)(24 - 6x)$$

$$= 12(12 - x)(4 - x) = 0 \text{ when } x = 12, 4 \text{ (12 is not in the domain).}$$

$$\frac{d^2V}{dx^2} = 12(2x - 16)$$

$$\frac{d^2V}{dx^2} < 0 \text{ when } x = 4.$$

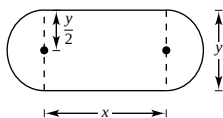
When $x = 4$, $V = 1024$ is maximum.

22. (a) $P = 2x + 2\pi r$

$$= 2x + 2\pi\left(\frac{y}{2}\right)$$

$$= 2x + \pi y = 200$$

$$\Rightarrow y = \frac{200 - 2x}{\pi} = \frac{2}{\pi}(100 - x)$$

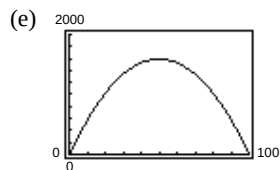


(b)

Length, x	Width, y	Area, xy
10	$\frac{2}{\pi}(100 - 10)$	$(10)\frac{2}{\pi}(100 - 10) \approx 573$
20	$\frac{2}{\pi}(100 - 20)$	$(20)\frac{2}{\pi}(100 - 20) \approx 1019$
30	$\frac{2}{\pi}(100 - 30)$	$(30)\frac{2}{\pi}(100 - 30) \approx 1337$
40	$\frac{2}{\pi}(100 - 40)$	$(40)\frac{2}{\pi}(100 - 40) \approx 1528$
50	$\frac{2}{\pi}(100 - 50)$	$(50)\frac{2}{\pi}(100 - 50) \approx 1592$
60	$\frac{2}{\pi}(100 - 60)$	$(60)\frac{2}{\pi}(100 - 60) \approx 1528$

The maximum area of the rectangle is approximately 1592 m².

(c) $A = xy = x \frac{2}{\pi}(100 - x) = \frac{2}{\pi}(100x - x^2)$



Maximum area is approximately

$$1591.55 \text{ m}^2 \text{ (} x = 50 \text{ m).}$$

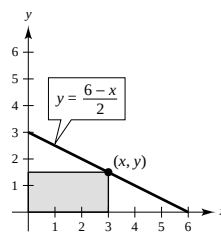
24. You can see from the figure that $A = xy$ and $y = \frac{6 - x}{2}$.

$$A = x\left(\frac{6 - x}{2}\right) = \frac{1}{2}(6x - x^2).$$

$$\frac{dA}{dx} = \frac{1}{2}(6 - 2x) = 0 \text{ when } x = 3.$$

$$\frac{d^2A}{dx^2} = -1 < 0 \text{ when } x = 3.$$

A is a maximum when $x = 3$ and $y = 3/2$.



26. (a) $A = \frac{1}{2} \text{ base} \times \text{height}$

$$= \frac{1}{2}(2\sqrt{16-h^2})(4+h)$$

$$= \sqrt{16-h^2}(4+h)$$

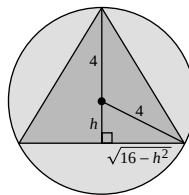
$$\frac{dA}{dh} = \frac{1}{2}(16-h^2)^{-1/2}(-2h)(4+h) + (16-h^2)^{1/2}$$

$$= (16-h^2)^{-1/2}[-h(4+h) + (16-h^2)]$$

$$= \frac{-2(h^2+2h-8)}{\sqrt{16-h^2}} = \frac{-2(h+4)(h-2)}{\sqrt{16-h^2}}$$

$$\frac{dA}{dh} = 0 \text{ when } h = 2, \text{ which is a maximum by the First Derivative Test.}$$

Hence, the sides are $2\sqrt{16-h^2} = 4\sqrt{3}$, an equilateral triangle. Area = $12\sqrt{3}$ sq. units.



(b) $\cos \alpha = \frac{4+h}{\sqrt{8}\sqrt{4+h}} = \frac{\sqrt{4+h}}{\sqrt{8}}$

$$\tan \alpha = \frac{\sqrt{16-h^2}}{4+h}$$

$$\text{Area} = 2\left(\frac{1}{2}\right)(\sqrt{16-h^2})(4+h)$$

$$= (4+h)^2 \tan \alpha$$

$$= 64 \cos^4 \alpha \tan \alpha$$

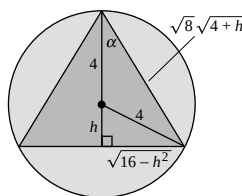
$$A'(\alpha) = 64[\cos^4 \alpha \sec^2 \alpha + 4 \cos^3 \alpha (-\sin \alpha) \tan \alpha] = 0$$

$$\Rightarrow \cos^4 \alpha \sec^2 \alpha = 4 \cos^3 \alpha \sin \alpha \tan \alpha$$

$$1 = 4 \cos \alpha \sin \alpha \tan \alpha$$

$$\frac{1}{4} = \sin^2 \alpha$$

$$\sin \alpha = \frac{1}{2} \Rightarrow \alpha = 30^\circ \text{ and } A = 12\sqrt{3}.$$

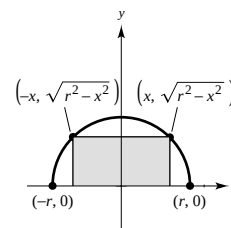


(c) Equilateral triangle

28. $A = 2xy = 2x\sqrt{r^2-x^2}$ (see figure)

$$\frac{dA}{dx} = \frac{2(r^2-2x^2)}{\sqrt{r^2-x^2}} = 0 \text{ when } x = \frac{\sqrt{2}r}{2}.$$

By the First Derivative Test, A is maximum when the rectangle has dimensions $\sqrt{2}r$ by $(\sqrt{2}r)/2$.



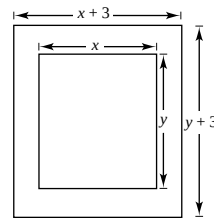
$$30. \quad xy = 36 \Rightarrow y = \frac{36}{x}$$

$$A = (x+3)(y+3) = (x+3)\left(\frac{36}{x} + 3\right)$$

$$= 36 + \frac{108}{x} + 3x + 9$$

$$\frac{dA}{dx} = \frac{-108}{x^2} + 3 = 0 \Rightarrow 3x^2 = 108 \Rightarrow x = 6, y = 6$$

Dimensions: 9×9



$$32. \quad V = \pi r^2 h = V_0 \text{ cubic units or } h = \frac{V_0}{\pi r^2}$$

$$S = 2\pi r^2 + 2\pi r h = 2\left(\pi r^2 + \frac{V_0}{r}\right)$$

$$\frac{dS}{dr} = 2\left(2\pi r - \frac{V_0}{r^2}\right) = 0 \text{ when } r = \sqrt[3]{\frac{V_0}{2\pi}} \text{ units.}$$

$$h = \frac{V_0}{\pi\left(\sqrt[3]{V_0/2\pi}\right)^2} = \frac{V_0(2\pi)^{2/3}}{\pi V_0^{2/3}} = \frac{2V_0^{1/3}}{(2\pi)^{1/3}} = 2r$$

By the First Derivative Test, this will yield the minimum surface area.

$$34. \quad V = \pi r^2 x$$

$$x + 2\pi r = 108 \Rightarrow x = 108 - 2\pi r \quad (\text{see figure})$$

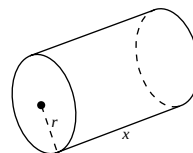
$$V = \pi r^2(108 - 2\pi r) = \pi(108r^2 - 2\pi r^3)$$

$$\frac{dV}{dr} = \pi(216r - 6\pi r^2) = 6\pi r(36 - \pi r)$$

$$= 0 \text{ when } r = \frac{36}{\pi} \text{ and } x = 36.$$

$$\frac{d^2V}{dr^2} = \pi(216 - 12\pi r) < 0 \text{ when } r = \frac{36}{\pi}.$$

Volume is maximum when $x = 36$ inches and $r = 36/\pi \approx 11.459$ inches.



$$36. \quad V = \pi x^2 h = \pi x^2(2\sqrt{r^2 - x^2}) = 2\pi x^2\sqrt{r^2 - x^2} \quad (\text{see figure})$$

$$\frac{dV}{dx} = 2\pi\left[x^2\left(\frac{1}{2}\right)(r^2 - x^2)^{-1/2}(-2x) + 2x\sqrt{r^2 - x^2}\right]$$

$$= \frac{2\pi x}{\sqrt{r^2 - x^2}}(2r^2 - 3x^2)$$

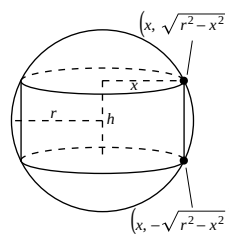
$$= 0 \text{ when } x = 0 \text{ and } x^2 = \frac{2r^2}{3} \Rightarrow x = \frac{\sqrt{6}r}{3}.$$

By the First Derivative Test, the volume is a maximum when

$$x = \frac{\sqrt{6}r}{3} \text{ and } h = \frac{2r}{\sqrt{3}}.$$

Thus, the maximum volume is

$$V = \pi\left(\frac{2}{3}r^2\right)\left(\frac{2r}{\sqrt{3}}\right) = \frac{4\pi r^3}{3\sqrt{3}}.$$



38. No. The volume will change because the shape of the container changes when squeezed.

$$40. V = 3000 = \frac{4}{3}\pi r^3 + \pi r^2 h$$

$$h = \frac{3000}{\pi r^2} - \frac{4}{3}r$$

Let k = cost per square foot of the surface area of the sides, then $2k$ = cost per square foot of the hemispherical ends.

$$C = 2k(4\pi r^2) + k(2\pi r h) = k\left[8\pi r^2 + 2\pi r\left(\frac{3000}{\pi r^2} - \frac{4}{3}r\right)\right] = k\left[\frac{16}{3}\pi r^2 + \frac{6000}{r}\right]$$

$$\frac{dC}{dr} = k\left[\frac{32}{3}\pi r - \frac{6000}{r^2}\right] = 0 \text{ when } r = \sqrt[3]{\frac{1125}{2\pi}} \approx 5.636 \text{ feet and } h \approx 22.545 \text{ feet.}$$

By the Second Derivative Test, we have

$$\frac{d^2C}{dr^2} = k\left[\frac{32}{3}\pi + \frac{12,000}{r^3}\right] > 0 \text{ when } r = \sqrt[3]{\frac{1125}{2\pi}}.$$

Therefore, these dimensions will produce a minimum cost.

42. (a) Let x be the side of the triangle and y the side of the square.

$$A = \frac{3}{4}\left(\cot \frac{\pi}{3}\right)x^2 + \frac{4}{4}\left(\cot \frac{\pi}{4}\right)y^2 \text{ where } 3x + 4y = 20$$

$$= \frac{\sqrt{3}}{4}x^2 + \left(5 - \frac{3}{4}x\right)^2, \quad 0 \leq x \leq \frac{20}{3}.$$

$$A' = \frac{\sqrt{3}}{2}x + 2\left(5 - \frac{3}{4}x\right)\left(-\frac{3}{4}\right) = 0$$

$$x = \frac{60}{4\sqrt{3} + 9}$$

When $x = 0$, $A = 25$, when $x = 60/(4\sqrt{3} + 9)$, $A \approx 10.847$, and when $x = 20/3$, $A \approx 19.245$. Area is maximum when all 20 feet are used on the square.

(c) Let x be the side of the pentagon and y the side of the hexagon.

$$A = \frac{5}{4}\left(\cot \frac{\pi}{5}\right)x^2 + \frac{6}{4}\left(\cot \frac{\pi}{6}\right)y^2 \text{ where } 5x + 6y = 20$$

$$= \frac{5}{4}\left(\cot \frac{\pi}{5}\right)x^2 + \frac{3}{2}(\sqrt{3})\left(\frac{20 - 5x}{6}\right)^2, \quad 0 \leq x \leq 4.$$

$$A' = \frac{5}{2}\left(\cot \frac{\pi}{5}\right)x + 3\sqrt{3}\left(-\frac{5}{6}\right)\left(\frac{20 - 5x}{6}\right) = 0$$

$$x \approx 2.0475$$

When $x = 0$, $A \approx 28.868$, when $x \approx 2.0475$, $A \approx 14.091$, and when $x = 4$, $A \approx 27.528$. Area is maximum when all 20 feet are used on the hexagon

(b) Let x be the side of the square and y the side of the pentagon.

$$A = \frac{4}{4}\left(\cot \frac{\pi}{4}\right)x^2 + \frac{5}{4}\left(\cot \frac{\pi}{5}\right)y^2 \text{ where } 4x + 5y = 20$$

$$= x^2 + 1.7204774\left(4 - \frac{4}{5}x\right)^2, \quad 0 \leq x \leq 5.$$

$$A' = 2x - 2.75276384\left(4 - \frac{4}{5}x\right) = 0$$

$$x \approx 2.62$$

When $x = 0$, $A \approx 27.528$, when $x \approx 2.62$, $A \approx 13.102$, and when $x = 5$, $A \approx 25$. Area is maximum when all 20 feet are used on the pentagon.

(d) Let x be the side of the hexagon and r the radius of the circle.

$$A = \frac{6}{4}\left(\cot \frac{\pi}{6}\right)x^2 + \pi r^2 \text{ where } 6x + 2\pi r = 20$$

$$= \frac{3\sqrt{3}}{2}x^2 + \pi\left(\frac{10}{\pi} - \frac{3x}{\pi}\right)^2, \quad 0 \leq x \leq \frac{10}{3}.$$

$$A' = 3\sqrt{3} - 6\left(\frac{10}{\pi} - \frac{3x}{\pi}\right) = 0$$

$$x \approx 1.748$$

When $x = 0$, $A \approx 31.831$, when $x \approx 1.748$, $A \approx 15.138$, and when $x = 10/3$, $A \approx 28.868$. Area is maximum when all 20 feet are used on the circle.

In general, using all of the wire for the figure with more sides will enclose the most area.

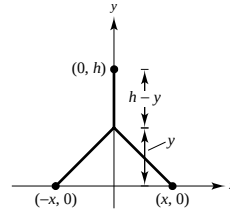
44. Let A be the amount of the power line.

$$A = h - y + 2\sqrt{x^2 + y^2}$$

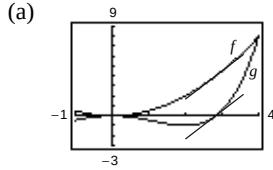
$$\frac{dA}{dy} = -1 + \frac{2y}{\sqrt{x^2 + y^2}} = 0 \text{ when } y = \frac{x}{\sqrt{3}}.$$

$$\frac{d^2A}{dy^2} = \frac{2x^2}{(x^2 + y^2)^{3/2}} > 0 \text{ for } y = \frac{x}{\sqrt{3}}.$$

The amount of power line is minimum when $y = x/\sqrt{3}$.



46. $f(x) = \frac{1}{2}x^2$ $g(x) = \frac{1}{16}x^4 - \frac{1}{2}x^2$ on $[0, 4]$



(b) $d(x) = f(x) - g(x) = \frac{1}{2}x^2 - \left(\frac{1}{16}x^4 - \frac{1}{2}x^2\right) = x^2 - \frac{1}{16}x^4$
 $d'(x) = 2x - \frac{1}{4}x^3 = 0 \Rightarrow 8x = x^3$

$$\Rightarrow x = 0, 2\sqrt{2} \text{ (in } [0, 4])$$

The maximum distance is $d = 4$ when $x = 2\sqrt{2}$.

- (c) $f'(x) = x$, Tangent line at $(2\sqrt{2}, 4)$ is

$$y - 4 = 2\sqrt{2}(x - 2\sqrt{2})$$

$$y = 2\sqrt{2}x - 4.$$

$g'(x) = \frac{1}{4}x^3 - x$, Tangent line at $(2\sqrt{2}, 0)$ is

$$y - 0 = \left(\frac{1}{4}(2\sqrt{2})^3 - 2\sqrt{2}\right)(x - 2\sqrt{2})$$

$$y = 2\sqrt{2}x - 8.$$

The tangent lines are parallel and 4 vertical units apart.

- (d) The tangent lines will be parallel. If $d(x) = f(x) - g(x)$, then $d'(x) = 0 = f'(x) - g'(x)$ implies that $f'(x) = g'(x)$ at the point x where the distance is maximum.

48. Let F be the illumination at point P which is x units from source 1.

$$F = \frac{kI_1}{x^2} + \frac{kI_2}{(d-x)^2}$$

$$\frac{dF}{dx} = \frac{-2kI_1}{x^3} + \frac{2kI_2}{(d-x)^3} = 0 \text{ when } \frac{2kI_1}{x^3} = \frac{2kI_2}{(d-x)^3}.$$

$$\frac{\sqrt[3]{I_1}}{\sqrt[3]{I_2}} = \frac{x}{d-x}$$

$$(d-x)\sqrt[3]{I_1} = x\sqrt[3]{I_2}$$

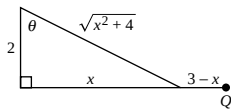
$$d\sqrt[3]{I_1} = x(\sqrt[3]{I_1} + \sqrt[3]{I_2})$$

$$x = \frac{d\sqrt[3]{I_1}}{\sqrt[3]{I_1} + \sqrt[3]{I_2}}$$

$$\frac{d^2F}{dx^2} = \frac{6kI_1}{x^4} + \frac{6kI_2}{(d-x)^4} > 0 \text{ when } x = \frac{d\sqrt[3]{I_1}}{\sqrt[3]{I_1} + \sqrt[3]{I_2}}.$$

This is the minimum point.

50. (a) $T = \frac{\sqrt{x^2 + 4}}{2} + \frac{(3-x)}{4}$



(b) $\frac{dT}{dx} = \frac{x}{2\sqrt{x^2 + 4}} - \frac{1}{4} = 0$

$$\frac{x}{\sqrt{x^2 + 4}} = \frac{1}{2}$$

$$2x^2 = x^2 + 4$$

$$x^2 = 4$$

$$x = 2$$

$$T(2) = \sqrt{2} + \frac{1}{4} \text{ hours}$$

—CONTINUED—

50. —CONTINUED—

$$(c) \quad T = \frac{\sqrt{x^2 + 4}}{v_1} + \frac{(3-x)}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2 + 4}} - \frac{1}{v_2} = 0$$

$$\frac{x}{\sqrt{x^2 + 4}} = \frac{v_1}{v_2}$$

$$\sin \theta = \frac{v_1}{v_2}$$

θ depends on $\frac{v_1}{v_2}$ only.

$$(d) \text{ Cost} = \sqrt{x^2 + 4} C_1 + (3-x)C_2$$

$$= \frac{\sqrt{x^2 + 4}}{(1/C_1)} + \frac{(3-x)}{(1/C_2)}$$

$$\text{From above, } \sin \theta = \frac{1/C_1}{1/C_2} = \frac{C_2}{C_1}$$

$$52. \quad T = \frac{\sqrt{x^2 + d_1^2}}{v_1} + \frac{\sqrt{d_2^2 + (a-x)^2}}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2 + d_1^2}} + \frac{x-a}{v_2\sqrt{d_2^2 + (a-x)^2}} = 0$$

Since

$$\frac{x}{\sqrt{x^2 + d_1^2}} = \sin \theta_1 \text{ and } \frac{x-a}{\sqrt{d_2^2 + (a-x)^2}} = -\sin \theta_2$$

we have

$$\frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0 \Rightarrow \frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.$$

Since

$$\frac{d^2T}{dx^2} = \frac{d_1^2}{v_1(x^2 + d_1^2)^{3/2}} + \frac{d_2^2}{v_2[d_2^2 + (a-x)^2]^{3/2}} > 0$$

this condition yields a minimum time.

$$54. \quad C(x) = 2k\sqrt{x^2 + 4} + k(4-x)$$

$$C'(x) = \frac{2xk}{\sqrt{x^2 + 4}} - k = 0$$

$$2x = \sqrt{x^2 + 4}$$

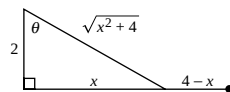
$$4x^2 = x^2 + 4$$

$$3x^2 = 4$$

$$x = \frac{2}{\sqrt{3}}$$

$$\text{Or, use Exercise 50(d): } \sin \theta = \frac{C_2}{C_1} = \frac{1}{2} \Rightarrow \theta = 30^\circ.$$

$$\text{Thus, } x = \frac{2}{\sqrt{3}}.$$



$$56. \quad V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 \sqrt{144 - r^2}$$

$$\frac{dV}{dr} = \frac{1}{3}\pi \left[r^2 \left(\frac{1}{2} \right) (144 - r^2)^{-1/2} (-2r) + 2r \sqrt{144 - r^2} \right]$$

$$= \frac{1}{3}\pi \left[\frac{288r - 3r^3}{\sqrt{144 - r^2}} \right] = \pi \left[\frac{r(96 - r^2)}{\sqrt{144 - r^2}} \right] = 0 \text{ when } r = 0, 4\sqrt{6}.$$

By the First Derivative Test, V is maximum when $r = 4\sqrt{6}$ and $h = 4\sqrt{3}$.

$$\text{Area of circle: } A = \pi(12)^2 = 144\pi$$

$$\text{Lateral surface area of cone: } S = \pi(4\sqrt{6})\sqrt{(4\sqrt{6})^2 + (4\sqrt{3})^2} = 48\sqrt{6}\pi$$

$$\text{Area of sector: } 144\pi - 48\sqrt{6}\pi = \frac{1}{2}\theta r^2 = 72\theta$$

$$\theta = \frac{144\pi - 48\sqrt{6}\pi}{72} = \frac{2\pi}{3}(3 - \sqrt{6}) \approx 1.153 \text{ radians or } 66^\circ$$

58. Let d be the amount deposited in the bank, i be the interest rate paid by the bank, and P be the profit.

$$P = (0.12)d - id$$

$$d = ki^2 \text{ (since } d \text{ is proportional to } i^2\text{)}$$

$$P = (0.12)(ki^2) - i(ki^2) = k(0.12i^2 - i^3)$$

$$\frac{dP}{di} = k(0.24i - 3i^2) = 0 \text{ when } i = \frac{0.24}{3} = 0.08.$$

$$\frac{d^2P}{di^2} = k(0.24 - 6i) < 0 \text{ when } i = 0.08 \text{ (Note: } k > 0\text{)}.$$

The profit is a maximum when $i = 8\%$.

60. $P = -\frac{1}{10}s^3 + 6s^2 + 400$

(a) $\frac{dP}{ds} = -\frac{3}{10}s^2 + 12s = -\frac{3}{10}s(s - 40) = 0$ when $s = 0$, $s = 40$.

$$\frac{d^2P}{ds^2} = -\frac{3}{5}s + 12$$

$$\frac{d^2P}{ds^2}(0) > 0 \Rightarrow s = 0 \text{ yields a minimum.}$$

$$\frac{d^2P}{ds^2}(40) < 0 \Rightarrow s = 40 \text{ yields a maximum.}$$

The maximum profit occurs when $s = 40$, which corresponds to \$40,000 ($P = \$3,600,000$).

(b) $\frac{d^2P}{ds^2} = -\frac{3}{5}s + 12 = 0$ when $s = 20$.

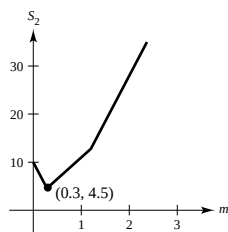
The point of diminishing returns occurs when $s = 20$, which corresponds to \$20,000 being spent on advertising.

62. $S_2 = |4m - 1| + |5m - 6| + |10m - 3|$

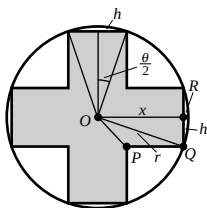
Using a graphing utility, you can see that the minimum occurs when $m = 0.3$.

Line $y = 0.3x$

$$S_2 = |4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3| = 4.7 \text{ mi.}$$



64. (a) Label the figure so that $r^2 = x^2 + h^2$.
Then, the area A is 8 times the area of the region given by $OPQR$:



$$\begin{aligned} A &= 8 \left[\frac{1}{2} h^2 + (x - h)h \right] \\ &= 8 \left[\frac{1}{2} (r^2 - x^2) + (x - \sqrt{r^2 - x^2}) \sqrt{r^2 - x^2} \right] \\ &= 8x\sqrt{r^2 - x^2} + 4x^2 - 4r^2 \\ A'(x) &= 8\sqrt{r^2 - x^2} - \frac{8x^2}{\sqrt{r^2 - x^2}} + 8x = 0 \end{aligned}$$

$$\begin{aligned} \frac{8x^2}{\sqrt{r^2 - x^2}} &= 8x + 8\sqrt{r^2 - x^2} \\ x^2 &= x\sqrt{r^2 - x^2} + (r^2 - x^2) \\ 2x^2 - r^2 &= x\sqrt{r^2 - x^2} \end{aligned}$$

$$4x^4 - 4x^2r^2 + r^4 = x^2(r^2 - x^2)$$

$$5x^4 - 5x^2r^2 + r^4 = 0 \quad \text{Quadratic in } x^2.$$

$$x^2 = \frac{5r^2 \pm \sqrt{25r^4 - 20r^4}}{10} = \frac{r^2}{10} [5 \pm \sqrt{5}].$$

Take positive value.

$$x = r \sqrt{\frac{5 + \sqrt{5}}{10}} \approx 0.85065r \quad \text{Critical number}$$

- (c) Note that $x^2 = \frac{r^2}{10}(5 + \sqrt{5})$ and $r^2 - x^2 = \frac{r^2}{10}(5 - \sqrt{5})$.

$$\begin{aligned} A(x) &= 8x\sqrt{r^2 - x^2} + 4x^2 - 4r^2 \\ &= 8 \left[\frac{r^2}{10}(5 + \sqrt{5}) \frac{r^2}{10}(5 - \sqrt{5}) \right]^{1/2} + 4 \frac{r^2}{10}(5 + \sqrt{5}) - 4r^2 \\ &= 8 \left[\frac{r^4}{10}(20) \right]^{1/2} + 2r^2 + \frac{2}{5}\sqrt{5}r^2 - 4r^2 \\ &= \frac{8}{5}r^2\sqrt{5} - 2r^2 + \frac{2\sqrt{5}}{5}r^2 \\ &= 2r^2 \left[\frac{4}{5}\sqrt{5} - 1 + \frac{\sqrt{5}}{5} \right] = 2r^2(\sqrt{5} - 1) \end{aligned}$$

Using the angle approach, note that $\tan \theta = 2$, $\sin \theta = \frac{2}{\sqrt{5}}$ and $\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1 - \cos \theta) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)$.

$$\begin{aligned} \text{Thus, } A(\theta) &= 4r^2 \left(\sin \theta - \sin^2 \frac{\theta}{2} \right) \\ &= 4r^2 \left(\frac{2}{\sqrt{5}} - \frac{1}{2} \left(1 - \frac{1}{\sqrt{5}} \right) \right) \\ &= \frac{4r^2(\sqrt{5} - 1)}{2} = 2r^2(\sqrt{5} - 1) \end{aligned}$$

- (b) Note that $\sin \frac{\theta}{2} = \frac{h}{r}$ and $\cos \frac{\theta}{2} = \frac{x}{r}$. The area A of the cross equals the sum of two large rectangles minus the common square in the middle.

$$\begin{aligned} A &= 2(2x)(2h) - 4h^2 = 8xh - 4h^2 \\ &= 8r^2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} - 4r^2 \sin^2 \frac{\theta}{2} \\ &= 4r^2 \left(\sin \theta - \sin^2 \frac{\theta}{2} \right) \end{aligned}$$

$$A'(\theta) = 4r^2 \left(\cos \theta - \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right) = 0$$

$$\cos \theta = \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 2 \sin \theta$$

$$\tan \theta = 2$$

$$\theta = \arctan(2) \approx 1.10715 \quad \text{or} \quad 63.4^\circ$$

Section 3.8 Newton's Method

2. $f(x) = 2x^2 - 3$

$$f'(x) = 4x$$

$$x_1 = 1$$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1	-1	4	$-\frac{1}{4}$	$\frac{5}{4}$
2	$\frac{5}{4} = 1.25$	0.125	5.0	0.025	1.225

4. $f(x) = \tan x$

$$f'(x) = \sec^2 x$$

$$x_1 = 0.1$$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.1000	0.1003	1.0101	0.0993	0.0007
2	0.0007	0.0007	1.0000	0.0007	0.0000

6. $f(x) = x^5 + x - 1$

$$f'(x) = 5x^4 + 1$$

Approximation of the zero of f is 0.755.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	-0.4688	1.3125	-0.3571	0.8571
2	0.8571	0.3196	3.6983	0.0864	0.7707
3	0.7707	0.0426	2.7641	0.0154	0.7553
4	0.7553	0.0011	2.6272	0.0004	0.7549

8. $f(x) = x - 2\sqrt{x+1}$

$$f'(x) = 1 - \frac{1}{\sqrt{x+1}}$$

Approximation of the zero of f is 4.8284.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	5	0.1010	0.5918	0.1707	4.8293
2	4.8293	0.0005	0.5858	.00085	4.8284

10. $f(x) = 1 - 2x^3$

$$f'(x) = -6x^2$$

Approximation of the zero of f is 0.7937.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1	-1	-6	0.1667	0.8333
2	0.8333	-0.1573	-4.1663	0.0378	0.7955
3	0.7955	-0.0068	-3.7969	0.0018	0.7937
4	0.7937	0.0000	-3.7798	0.0000	0.7937

12. $f(x) = \frac{1}{2}x^4 - 3x - 3$

$$f'(x) = 2x^3 - 3$$

Approximation of the zero of f is -0.8937 .

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1	0.5	-5	-0.1	-0.9
2	-0.9	0.0281	-4.458	-0.0063	-0.8937
3	-0.8937	0.0001	-4.4276	0.0000	-0.8937

Approximation of the zero of f is 2.0720.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	2	-1	13	-0.0769	2.0769
2	2.0769	0.0725	14.9175	0.0049	2.0720
3	2.0720	-0.0003	14.7910	0.0000	2.0720

14. $f(x) = x^3 - \cos x$

$$f'(x) = 3x^2 + \sin x$$

Approximation of the zero of f is 0.866.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.9000	0.1074	3.2133	0.0334	0.8666
2	0.8666	0.0034	3.0151	0.0011	0.8655
3	0.8655	0.0001	3.0087	0.0000	0.8655

16. $h(x) = f(x) - g(x) = 3 - x - \frac{1}{x^2 + 1}$

$$h'(x) = -1 + \frac{2x}{(x^2 + 1)^2}$$

Point of intersection of the graphs of f and g occurs when $x \approx 2.893$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	2.9000	-0.0063	-0.9345	0.0067	2.8933
2	2.8933	0.0000	-0.9341	0.0000	2.8933

18. $h(x) = f(x) - g(x) = x^2 - \cos x$

$$h'(x) = 2x + \sin x$$

One point of intersection of the graphs of f and g occurs when $x \approx 0.824$. Since $f(x) = x^2$ and $g(x) = \cos x$ are both symmetric with respect to the y -axis, the other point of intersection occurs when $x \approx -0.824$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	0.8000	-0.0567	2.3174	-0.0245	0.8245
2	0.8245	0.0009	2.3832	0.0004	0.8241

20. $f(x) = x^n - a = 0$

$$f'(x) = nx^{n-1}$$

$$\begin{aligned} x_{i+1} &= x_i - \frac{x_i^n - a}{nx_i^{n-1}} \\ &= \frac{nx_i^n - x_i^n + a}{nx_i^{n-1}} = \frac{(n-1)x_i^n + a}{nx_i^{n-1}} \end{aligned}$$

22. $x_{i+1} = \frac{x_i^2 + 5}{2x_i}$

i	1	2	3	4
x_i	2.0000	2.2500	2.2361	2.2361

$$\sqrt{5} \approx 2.236$$

26. $f(x) = \tan x$

$$f'(x) = \sec^2 x$$

Approximation of the zero: 3.142

28. $y = 4x^3 - 12x^2 + 12x - 3 = f(x)$

$$y' = 12x^2 - 24x + 12 = f'(x)$$

$$x_1 = \frac{3}{2}$$

$f'(x_2) = 0$; therefore, the method fails.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	$\frac{3}{2}$	$\frac{3}{2}$	3	$\frac{1}{2}$	1
2	1	1	0	—	—

24. $x_{i+1} = \frac{2x_i^3 + 15}{3x_i^2}$

i	1	2	3	4
x_i	2.5000	2.4667	2.4662	2.4662

$$\sqrt[3]{15} \approx 2.466$$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.0000	-0.1425	1.0203	-0.1397	3.1397
2	3.1397	-0.0019	1.0000	-0.0019	3.1416
3	3.1416	0.0000	1.0000	0.0000	3.1416

30. $f(x) = 2 \sin x + \cos 2x$

$$f'(x) = 2 \cos x - 2 \sin 2x$$

$$x_1 = \frac{3\pi}{2}$$

Fails because $f'(x_1) = 0$.

n	x_n	$f(x_n)$	$f'(x_n)$
1	$\frac{3\pi}{2}$	-3	0

32. Newton's Method could fail if $f'(c) \approx 0$, or if the initial value x_1 is far from c .

34. Let $g(x) = f(x) - x = \cot x - x$

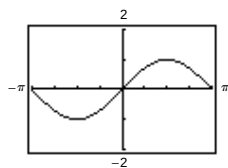
$$g'(x) = -\csc^2 x - 1.$$

The fixed point is approximately 0.86.

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	1.0000	-0.3579	-2.4123	0.1484	0.8516
2	0.8516	0.0240	-2.7668	-0.0087	0.8603
3	0.8603	0.0001	-2.7403	0.0000	0.8603

36. $f(x) = \sin x$, $f'(x) = \cos x$

(a)



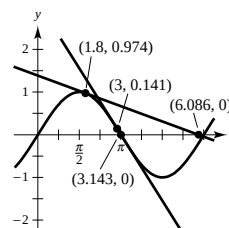
(b) $x_1 = 1.8$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 6.086$$

(c) $x_1 = 3$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 3.143$$

(d)



The x -intercepts correspond to the values resulting from the first iteration of Newton's Method.

(e) If the initial guess x_1 is not "close to" the desired zero of the function, the x -intercept of the tangent line may approximate another zero of the function.

38. (a) $x_{n+1} = x_n(2 - 3x_n)$

i	1	2	3	4
x_i	0.3000	0.3300	0.3333	0.3333

$$\frac{1}{3} \approx 0.333$$

(b) $x_{n+1} = x_n(2 - 11x_n)$

i	1	2	3	4
x_i	0.1000	0.0900	0.0909	0.0909

$$\frac{1}{11} \approx 0.091$$

40. $f(x) = x \sin x, (0, \pi)$

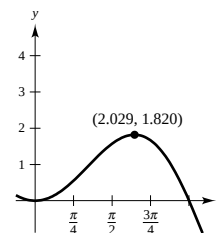
$$f'(x) = x \cos x + \sin x = 0$$

Letting $F(x) = f'(x)$, we can use Newton's Method as follows.

$$[F'(x) = 2 \cos x - x \sin x]$$

n	x_n	$F(x_n)$	$F'(x_n)$	$\frac{F(x_n)}{F'(x_n)}$	$x_n - \frac{F(x_n)}{F'(x_n)}$
1	2.0000	0.0770	-2.6509	-0.0290	2.0290
2	2.0290	-0.0007	-2.7044	0.0002	2.0288

Approximation to the critical number: 2.029



42. $y = f(x) = x^2, (4, -3)$

$$d = \sqrt{(x-4)^2 + (y+3)^2} = \sqrt{(x-4)^2 + (x^2+3)^2} = \sqrt{x^4 + 7x^2 - 8x + 25}$$

d is minimum when $D = x^4 + 7x^2 - 8x + 25$ is minimum.

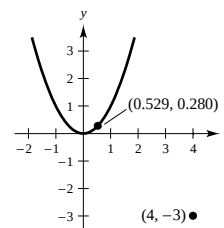
$$g(x) = D' = 4x^3 + 14x - 8$$

$$g'(x) = 12x^2 + 14$$

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	0.5000	-0.5000	17.0000	-0.0294	0.5294
2	0.5294	0.0051	17.3632	0.0003	0.5291
3	0.5291	-0.0001	17.3594	0.0000	0.5291

$$x \approx 0.529$$

Point closest to $(4, -3)$ is approximately $(0.529, 0.280)$.



44. Maximize: $C = \frac{3t^2 + t}{50 + t^3}$

$$C' = \frac{-3t^4 - 2t^3 + 300t + 50}{(50 + t^3)^2} = 0$$

Let $f(x) = 3t^4 + 2t^3 - 300t - 50$

$$f'(x) = 12t^3 + 6t^2 - 300.$$

Since $f(4) = -354$ and $f(5) = 575$, the solution is in the interval $(4, 5)$.

Approximation: $t \approx 4.486$ hours

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	4.5000	12.4375	915.0000	0.0136	4.4864
2	4.4864	0.0658	904.3822	0.0001	4.4863

46. $170 = 0.808x^3 - 17.974x^2 + 71.248x + 110.843$, $1 \leq x \leq 5$

Let $f(x) = 0.808x^3 - 17.974x^2 + 71.248x - 59.157$

$f'(x) = 2.424x^2 - 35.948x + 71.248$.

From the graph, choose $x_1 = 1$ and $x_1 = 3.5$. Apply Newton's Method.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.0000	-5.0750	37.7240	-0.1345	1.1345
2	1.1345	-0.2805	33.5849	-0.0084	1.1429
3	1.1429	0.0006	33.3293	0.0000	1.1429

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.5000	4.6725	-24.8760	-0.1878	3.6878
2	3.6878	-0.3286	-28.3550	0.0116	3.6762
3	3.6762	-0.0009	-28.1450	0.0000	3.6762

The zeros occur when $x \approx 1.1429$ and $x \approx 3.6762$. These approximately correspond to engine speeds of 1143 rev/min and 3676 rev/min.

48. True

50. True

52. $f(x) = \sqrt{4 - x^2} \sin(x - 2)$

Domain: $[-2, 2]$

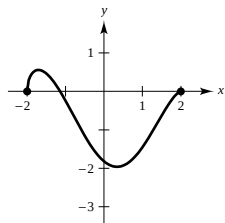
$x = -2$ and $x = 2$ are both zeros.

$$f'(x) = \sqrt{4 - x^2} \cos(x - 2) - \frac{x}{\sqrt{4 - x^2}} \sin(x - 2)$$

Let $x_1 = -1$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.0000	-0.2444	-1.7962	0.1361	-1.1361
2	-1.1361	-0.0090	-1.6498	0.0055	-1.1416
3	-1.1416	0.0000	-1.6422	0.0000	-1.1416

Zeros: $x = \pm 2$, $x \approx -1.142$



Section 3.9 Differentials

2. $f(x) = \frac{6}{x^2} = 6x^{-2}$

$$f'(x) = -12x^{-3} = \frac{-12}{x^3}$$

Tangent line at $\left(2, \frac{3}{2}\right)$:

$$y - \frac{3}{2} = \frac{-12}{8}(x - 2) = \frac{-3}{2}(x - 2)$$

$$y = -\frac{3}{2}x + \frac{9}{2}$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \frac{6}{x^2}$	1.6620	1.5151	1.5	1.4851	1.3605
$T(x) = -\frac{3}{2}x + \frac{9}{2}$	1.65	1.515	1.5	1.485	1.35

4. $f(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Tangent line at $(2, \sqrt{2})$:

$$y - f(2) = f'(2)(x - 2)$$

$$y - \sqrt{2} = \frac{1}{2\sqrt{2}}(x - 2)$$

$$y = \frac{x}{2\sqrt{2}} + \frac{1}{\sqrt{2}}$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \sqrt{x}$	1.3784	1.4107	1.4142	1.4177	1.4491
$T(x) = \frac{x}{2\sqrt{2}} + \frac{1}{\sqrt{2}}$	1.3789	1.4107	1.4142	1.4177	1.4496

6. $f(x) = \csc x$

$$f'(x) = -\csc x \cot x$$

Tangent line at $(2, \csc 2)$: $y - f(2) = f'(2)(x - 2)$

$$y - \csc 2 = (-\csc 2 \cot 2)(x - 2)$$

$$y = (-\csc 2 \cot 2)(x - 2) + \csc 2$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \csc x$	1.0567	1.0948	1.0998	1.1049	1.1585
$T(x) = (-\csc 2 \cot 2)(x - 2) + \csc 2$	1.0494	1.0947	1.0998	1.1048	1.1501

8. $y = f(x) = 1 - 2x^2, f'(x) = -4x, x = 0, \Delta x = dx = -0.1$

$$\Delta y = f(x + \Delta x) - f(x)$$

$$= f(-0.1) - f(0)$$

$$= [1 - 2(-0.1)^2] - [1 - 2(0)^2] = -0.02$$

$$dy = f'(x) dx$$

$$= f'(0)(-0.1)$$

$$= (0)(-0.1) = 0$$

10. $y = f(x) = 2x + 1, f'(x) = 2, x = 2, \Delta x = dx = 0.01$

$$\Delta y = f(x + \Delta x) - f(x)$$

$$= f(2.01) - f(2)$$

$$= [2(2.01) + 1] - [2(2) + 1] = 0.02$$

$$dy = f'(x) dx$$

$$= f'(2)(0.01)$$

$$= 2(0.01) = 0.02$$

12. $y = 3x^{2/3}$

$$dy = 2x^{-1/3}dx = \frac{2}{x^{1/3}}dx$$

16. $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

$$dy = \left(\frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}} \right) dx = \frac{x-1}{2x\sqrt{x}} dx$$

20. $y = \frac{\sec^2 x}{x^2 + 1}$

$$dy = \left[\frac{(x^2 + 1)2 \sec^2 x \tan x - \sec^2 x(2x)}{(x^2 + 1)^2} \right] dx$$

$$= \left[\frac{2 \sec^2 x(x^2 \tan x + \tan x - x)}{(x^2 + 1)^2} \right] dx$$

24. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$$\approx 1 + 0(-0.1) = 1$$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$$\approx 1 + 0(0.04) = 1$$

28. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$$\approx 8 + 5(-0.07) = 7.65$$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$$\approx 8 + 5(0.1) = 8.5$$

32. $x = 12$ inches

$$\Delta x = dx = \pm 0.03 \text{ inch}$$

(a) $V = x^3$

$$dV = 3x^2 dx = 3(12)^2(\pm 0.03)$$

$$= \pm 12.96 \text{ cubic inches}$$

(b) $S = 6x^2$

$$dS = 12x dx = 12(12)(\pm 0.03)$$

$$= \pm 4.32 \text{ square inches}$$

14. $y = \sqrt{9 - x^2}$

$$dy = \frac{1}{2}(9 - x^2)^{-1/2}(-2x)dx = \frac{-x}{\sqrt{9 - x^2}} dx$$

18. $y = x \sin x$

$$dy = (x \cos x + \sin x) dx$$

22. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$$\approx 1 + (-1)(-0.1) = 1.1$$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$$\approx 1 + (-1)(0.04) = 0.96$$

26. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$$\approx 8 + (3)(-0.07) = 7.79$$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$$\approx 8 + (3)(0.1) = 8.3$$

30. $A = \frac{1}{2}bh$, $b = 36$, $h = 50$

$$db = dh = \pm 0.25$$

$$dA = \frac{1}{2}b dh + \frac{1}{2}h db$$

$$\Delta A \approx dA = \frac{1}{2}(36)(\pm 0.25) + \frac{1}{2}(50)(\pm 0.25)$$

$$= \pm 10.75 \text{ square centimeters}$$

34. (a) $C = 56$ centimeters

$$\Delta C = dC = \pm 1.2 \text{ centimeters}$$

$$C = 2\pi r \Rightarrow r = \frac{C}{2\pi}$$

$$A = \pi r^2 = \pi \left(\frac{C}{2\pi} \right)^2 = \frac{1}{4\pi} C^2$$

$$dA = \frac{1}{2\pi} C dC = \frac{1}{2\pi} (56)(\pm 1.2) = \frac{33.6}{\pi}$$

$$\frac{dA}{A} = \frac{33.6/\pi}{[1/(4\pi)](56)^2} \approx 0.042857 = 4.2857\%$$

(b) $\frac{dA}{A} = \frac{(1/2\pi)C dC}{(1/4\pi)C^2} = \frac{2dC}{C} \leq 0.03$

$$\frac{dC}{C} \leq \frac{0.03}{2} = 0.015 = 1.5\%$$

36. $P = (500x - x^2) - \left(\frac{1}{2}x^2 - 77x + 3000\right)$, x changes from 115 to 120

$$dP = (500 - 2x - x + 77)dx = (577 - 3x)dx = [577 - 3(115)](120 - 115) = 1160$$

Approximate percentage change: $\frac{dP}{P}(100) = \frac{1160}{43517.50}(100) \approx 2.7\%$

38. $V = \frac{4}{3}\pi r^3$, $r = 100$ cm, $dr = 0.2$ cm

$$\Delta V \approx dV = 4\pi r^2 dr = 4\pi(100)^2(0.2) = 8000\pi \text{ cm}^3$$

40. $E = IR$

$$R = \frac{E}{I}$$

$$dR = -\frac{E}{I^2}dI$$

$$\frac{dR}{R} = \frac{-(E/I^2)dI}{E/I} = -\frac{dI}{I}$$

$$\left|\frac{dR}{R}\right| = \left|-\frac{dI}{I}\right| = \left|\frac{dI}{I}\right|$$

42. See Exercise 41.

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(9.5 \cot \theta)(9.5) = 45.125 \cot \theta$$

$$dA = -45.125 \csc^2 \theta d\theta$$

$$\left|\frac{dA}{A}\right| = \frac{\csc^2 \theta d\theta}{\cot \theta} = \frac{d\theta}{\sin \theta \cos \theta}$$

$$= \frac{0.25^\circ}{(\sin 26.75^\circ)(\cos 26.75^\circ)}$$

$$\approx \frac{0.0044}{(\sin 0.4669)(\cos 0.4669)}$$

$$\approx 0.0109 = 1.09\% \text{ (in radians)}$$

44. $h = 50 \tan \theta$

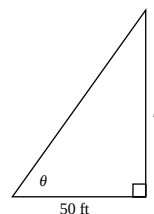
$$\theta = 71.5^\circ = 1.2479 \text{ radians}$$

$$dh = 50 \sec^2 \theta \cdot d\theta$$

$$\left|\frac{dh}{h}\right| = \left|\frac{50 \sec^2(1.2479)}{50 \tan(1.2479)}d\theta\right| \leq 0.06$$

$$\left|\frac{9.9316}{2.9886}d\theta\right| \leq 0.06$$

$$|d\theta| \leq 0.018$$



46. Let $f(x) = \sqrt[3]{x}$, $x = 27$, $dx = -1$

$$f(x + \Delta x) \approx f(x) + f'(x)dx = \sqrt[3]{x} + \frac{1}{3\sqrt[3]{x^2}}dx$$

$$\sqrt[3]{26} \approx \sqrt[3]{27} + \frac{1}{3\sqrt[3]{27^2}}(-1) = 3 - \frac{1}{27} \approx 2.9630$$

Using a calculator, $\sqrt[3]{26} \approx 2.9625$

48. Let $f(x) = x^3$, $x = 3$, $dx = -0.01$.

$$f(x + \Delta x) \approx f(x) + f'(x)dx = x^3 + 3x^2dx$$

$$f(x + \Delta x) = (2.99)^3 \approx 3^3 + 3(3)^2(-0.01) = 27 - 0.27 = 26.73$$

Using a calculator: $(2.99)^3 \approx 26.7309$

50. Let $f(x) = \tan x$, $x = 0$, $dx = 0.05$, $f'(x) = \sec^2 x$.

Then

$$f(0.05) \approx f(0) + f'(0)dx$$

$$\tan 0.05 \approx \tan 0 + \sec^2 0(0.05) = 0 + 1(0.05).$$

54. True, $\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = a$

52. Propagated error $= f(x + \Delta x) - f(x)$,

$$\text{relative error} = \left| \frac{dy}{y} \right|, \text{ and the percent error} = \left| \frac{dy}{y} \right| \times 100.$$

56. False

Let $f(x) = \sqrt{x}$, $x = 1$, and $\Delta x = dx = 3$. Then

$$\Delta y = f(x + \Delta x) - f(x) = f(4) - f(1) = 1$$

and

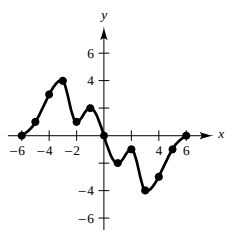
$$dy = f'(x) dx = \frac{1}{2\sqrt{1}}(3) = \frac{3}{2}.$$

Thus, $dy > \Delta y$ in this example.

Review Exercises for Chapter 3

2. (a) $f(4) = -f(-4) = -3$

(c)



At least six critical numbers on $(-6, 6)$.

4. $f(x) = \frac{x}{\sqrt{x^2 + 1}}$, $[0, 2]$

$$\begin{aligned} f'(x) &= x \left[-\frac{1}{2}(x^2 + 1)^{-3/2}(2x) \right] + (x^2 + 1)^{-1/2} \\ &= \frac{1}{(x^2 + 1)^{3/2}} \end{aligned}$$

No critical numbers

Left endpoint: $(0, 0)$ Minimum

Right endpoint: $(2, 2/\sqrt{5})$ Maximum

- (b) $f(-3) = -f(3) = -(-4) = 4$

- (d) Yes. Since $f(-2) = -f(2) = -(-1) = 1$ and $f(1) = -f(-1) = -2$, the Mean Value says that there exists at least one value c in $(-2, 1)$ such that

$$f'(c) = \frac{f(1) - f(-2)}{1 - (-2)} = \frac{-2 - 1}{1 + 2} = -1.$$

- (e) No, $\lim_{x \rightarrow 0} f(x)$ exists because f is continuous at $(0, 0)$.

- (f) Yes, f is differentiable at $x = 2$.

6. No. f is not differentiable at $x = 2$.

8. No; the function is discontinuous at $x = 0$ which is in the interval $[-2, 1]$.

10. $f(x) = \frac{1}{x}$, $1 \leq x \leq 4$

$$f'(x) = -\frac{1}{x^2}$$

$$\frac{f(b) - f(a)}{b - a} = \frac{(1/4) - 1}{4 - 1} = \frac{-3/4}{3} = -\frac{1}{4}$$

$$f'(c) = \frac{-1}{c^2} = -\frac{1}{4}$$

$$c = 2$$

12. $f(x) = \sqrt{x} - 2x$, $0 \leq x \leq 4$

$$f'(x) = \frac{1}{2\sqrt{x}} - 2$$

$$\frac{f(b) - f(a)}{b - a} = \frac{-6 - 0}{4 - 0} = -\frac{3}{2}$$

$$f'(c) = \frac{1}{2\sqrt{c}} - 2 = -\frac{3}{2}$$

$$c = 1$$

14. $f(x) = 2x^2 - 3x + 1$

$f'(x) = 4x - 3$

$$\frac{f(b) - f(a)}{b - a} = \frac{21 - 1}{4 - 0} = 5$$

$f'(c) = 4c - 3 = 5$

$c = 2 = \text{Midpoint of } [0, 4]$

16. $g(x) = (x + 1)^3$

$g'(x) = 3(x + 1)^2$

Critical number: $x = -1$

Interval	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $g'(x)$	$g'(x) > 0$	$g'(x) > 0$
Conclusion	Increasing	Increasing

18. $f(x) = \sin x + \cos x, \quad 0 \leq x \leq 2\pi$

$f'(x) = \cos x - \sin x$

Critical numbers: $x = \frac{\pi}{4}, x = \frac{5\pi}{4}$

Interval	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < 2\pi$
Sign of $f'(x)$	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion	Increasing	Decreasing	Increasing

20. $g(x) = \frac{3}{2} \sin\left(\frac{\pi x}{2} - 1\right), [0, 4]$

$g'(x) = \frac{3}{2} \left(\frac{\pi}{2}\right) \cos\left(\frac{\pi x}{2} - 1\right)$

$= 0 \text{ when } x = 1 + \frac{2}{\pi}, 3 + \frac{2}{\pi}$

Relative maximum: $\left(1 + \frac{2}{\pi}, \frac{3}{2}\right)$

Relative minimum: $\left(3 + \frac{2}{\pi}, -\frac{3}{2}\right)$

Test Interval	$0 < x < 1 + \frac{2}{\pi}$	$1 + \frac{2}{\pi} < x < 3 + \frac{2}{\pi}$	$3 + \frac{2}{\pi} < x < 4$
Sign of $g'(x)$	$g'(x) > 0$	$g'(x) < 0$	$g'(x) > 0$
Conclusion	Increasing	Decreasing	Increasing

22. (a) $y = A \sin(\sqrt{k/m}t) + B \cos(\sqrt{k/m}t)$

$y' = A\sqrt{k/m} \cos(\sqrt{k/m}t) - B\sqrt{k/m} \sin(\sqrt{k/m}t)$

$= 0 \text{ when } \frac{\sin \sqrt{k/m}t}{\cos \sqrt{k/m}t} = \frac{A}{B} \Rightarrow \tan(\sqrt{k/m}t) = \frac{A}{B}.$

Therefore,

$$\sin(\sqrt{k/m}t) = \frac{A}{\sqrt{A^2 + B^2}}$$

$$\cos(\sqrt{k/m}t) = \frac{B}{\sqrt{A^2 + B^2}}.$$

When $v = y' = 0$,

$$y = A\left(\frac{A}{\sqrt{A^2 + B^2}}\right) + B\left(\frac{B}{\sqrt{A^2 + B^2}}\right) = \sqrt{A^2 + B^2}.$$

(b) Period: $\frac{2\pi}{\sqrt{k/m}}$

Frequency: $\frac{1}{2\pi/\sqrt{k/m}} = \frac{1}{2\pi} \sqrt{k/m}$

24. $f(x) = (x + 2)^2(x - 4) = x^3 - 12x - 16$

$$f'(x) = 3x^2 - 12$$

$$f''(x) = 6x = 0 \text{ when } x = 0.$$

Point of inflection: $(0, -16)$

26. $h(t) = t - 4\sqrt{t+1}$ Domain: $[-1, \infty)$

$$h'(t) = 1 - \frac{2}{\sqrt{t+1}} = 0 \Rightarrow t = 3$$

$$h''(t) = \frac{1}{(t+1)^{3/2}}$$

$$h''(3) = \frac{1}{8} > 0 \quad (3, -5) \text{ is a relative minimum.}$$

30. $C = \left(\frac{Q}{x}\right)s + \left(\frac{x}{2}\right)r$

$$\frac{dC}{dx} = -\frac{Qs}{x^2} + \frac{r}{2} = 0$$

$$\frac{Qs}{x^2} = \frac{r}{2}$$

$$x^2 = \frac{2Qs}{r}$$

$$x = \sqrt{\frac{2Qs}{r}}$$

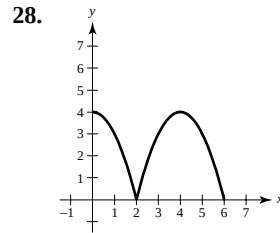
34. $\lim_{x \rightarrow \infty} \frac{2x}{3x^2 + 5} = \lim_{x \rightarrow \infty} \frac{2/x}{3 + 5/x^2} = 0$

38. $g(x) = \frac{5x^2}{x^2 + 2}$

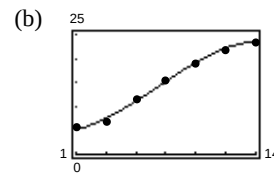
$$\lim_{x \rightarrow \infty} \frac{5x^2}{x^2 + 2} = \lim_{x \rightarrow \infty} \frac{5}{1 + (2/x^2)} = 5$$

Horizontal asymptote: $y = 5$

Test Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward



32. (a) $S = -0.1222t^3 + 1.3655t^2 - 0.9052t + 4.8429$



(c) $S'(t) = 0$ when $t = 3.7$. This is a maximum by the First Derivative Test.

(d) No, because the t^3 coefficient term is negative.

36. $\lim_{x \rightarrow \infty} \frac{3x}{\sqrt{x^2 + 4}} = \lim_{x \rightarrow \infty} \frac{3}{\sqrt{1 + 4/x^2}} = 3$

40. $f(x) = \frac{3x}{\sqrt{x^2 + 2}}$

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{3x}{\sqrt{x^2 + 2}} &= \lim_{x \rightarrow \infty} \frac{3x/x}{\sqrt{x^2 + 2}/\sqrt{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{3}{\sqrt{1 + (2/x^2)}} = 3 \end{aligned}$$

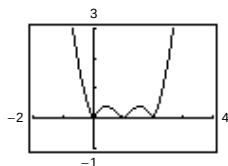
$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{3x}{\sqrt{x^2 + 2}} &= \lim_{x \rightarrow -\infty} \frac{3x/x}{\sqrt{x^2 + 2}/(-\sqrt{x^2})} \\ &= \lim_{x \rightarrow -\infty} \frac{3}{-\sqrt{1 + (2/x^2)}} = -3 \end{aligned}$$

Horizontal asymptotes: $y = \pm 3$

42. $f(x) = |x^3 - 3x^2 + 2x| = |x(x-1)(x-2)|$

Relative minima: $(0, 0)$, $(1, 0)$, $(2, 0)$

Relative maxima: $(1.577, 0.38)$, $(0.423, 0.38)$



46. $f(x) = 4x^3 - x^4 = x^3(4 - x)$

Domain: $(-\infty, \infty)$; Range: $(-\infty, 27)$

$$f'(x) = 12x^2 - 4x^3 = 4x^2(3 - x) = 0 \text{ when } x = 0, 3.$$

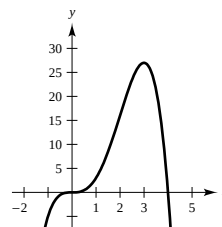
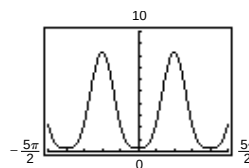
$$f''(x) = 24x - 12x^2 = 12x(2 - x) = 0 \text{ when } x = 0, 2.$$

$$f''(3) < 0$$

Therefore, $(3, 27)$ is a relative maximum.

Points of inflection: $(0, 0)$, $(2, 16)$

Intercepts: $(0, 0)$, $(4, 0)$



48. $f(x) = (x^2 - 4)^2$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$$f'(x) = 4x(x^2 - 4) = 0 \text{ when } x = 0, \pm 2.$$

$$f''(x) = 4(3x^2 - 4) = 0 \text{ when } x = \pm \frac{2\sqrt{3}}{3}.$$

$$f''(0) < 0$$

Therefore, $(0, 16)$ is a relative maximum.

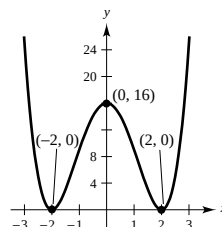
$$f''(\pm 2) > 0$$

Therefore, $(\pm 2, 0)$ are relative minima.

Points of inflection: $(\pm 2\sqrt{3}/3, 64/9)$

Intercepts: $(-2, 0)$, $(0, 16)$, $(2, 0)$

Symmetry with respect to y-axis



50. $f(x) = (x - 3)(x + 2)^3$

Domain: $(-\infty, \infty)$; Range: $[-\frac{16,875}{256}, \infty)$

$$f'(x) = (x - 3)(3)(x + 2)^2 + (x + 2)^3$$

$$= (4x - 7)(x + 2)^2 = 0 \text{ when } x = -2, \frac{7}{4}.$$

$$f''(x) = (4x - 7)(2)(x + 2) + (x + 2)^2(4)$$

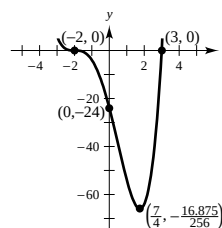
$$= 6(2x - 1)(x + 2) = 0 \text{ when } x = -2, \frac{1}{2}.$$

$$f''(\frac{7}{4}) > 0$$

Therefore, $(\frac{7}{4}, -\frac{16,875}{256})$ is a relative minimum.

Points of inflection: $(-2, 0)$, $(\frac{1}{2}, -\frac{625}{16})$

Intercepts: $(-2, 0)$, $(0, -24)$, $(3, 0)$



52. $f(x) = (x - 2)^{1/3}(x + 1)^{2/3}$

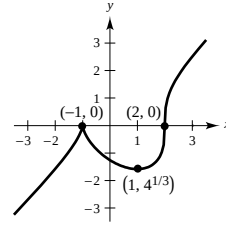
Graph of Exercise 39 translated 2 units to the right (x replaces by $x - 2$).

$(-1, 0)$ is a relative maximum.

$(1, -\sqrt[3]{4})$ is a relative minimum.

$(2, 0)$ is a point of inflection.

Intercepts: $(-1, 0), (2, 0)$



54. $f(x) = \frac{2x}{1+x^2}$

Domain: $(-\infty, \infty)$; Range: $[-1, 1]$

$$f'(x) = \frac{2(1-x)(1+x)}{(1+x^2)^2} = 0 \text{ when } x = \pm 1.$$

$$f''(x) = \frac{-2x(3-x^2)}{(1+x^2)^3} = 0 \text{ when } x = 0, \pm\sqrt{3}.$$

$$f''(1) < 0$$

Therefore, $(1, 1)$ is a relative maximum.

$$f''(-1) > 0$$

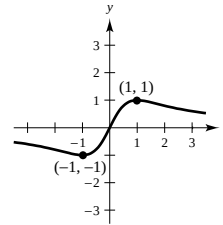
Therefore, $(-1, -1)$ is a relative minimum.

Points of inflection: $(-\sqrt{3}, -\sqrt{3}/2), (0, 0), (\sqrt{3}, \sqrt{3}/2)$

Intercept: $(0, 0)$

Symmetric with respect to the origin

Horizontal asymptote: $y = 0$



56. $f(x) = \frac{x^2}{1+x^4}$

Domain: $(-\infty, \infty)$; Range: $\left[0, \frac{1}{2}\right]$

$$f'(x) = \frac{(1+x^4)(2x) - x^2(4x^3)}{(1+x^4)^2} = \frac{2x(1-x)(1+x)(1+x^2)}{(1+x^4)^2} = 0 \text{ when } x = 0, \pm 1.$$

$$f''(x) = \frac{(1+x^4)^2(2-10x^4) - (2x-2x^5)(2)(1+x^4)(4x^3)}{(1+x^4)^4} = \frac{2(1-12x^4+3x^8)}{(1+x^4)^3} = 0 \text{ when } x = \pm \sqrt[4]{\frac{6 \pm \sqrt{33}}{3}}.$$

$$f''(\pm 1) < 0$$

Therefore, $\left(\pm 1, \frac{1}{2}\right)$ are relative maxima.

$$f''(0) > 0$$

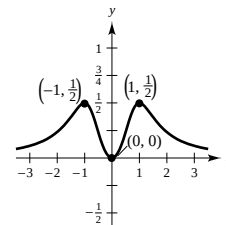
Therefore, $(0, 0)$ is a relative minimum.

Points of inflection: $\left(\pm \sqrt[4]{\frac{6 - \sqrt{33}}{3}}, 0.29\right), \left(\pm \sqrt[4]{\frac{6 + \sqrt{33}}{3}}, 0.40\right)$

Intercept: $(0, 0)$

Symmetric to the y-axis

Horizontal asymptote: $y = 0$



58. $f(x) = x^2 + \frac{1}{x} = \frac{x^3 + 1}{x}$

Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, \infty)$

$$f'(x) = 2x - \frac{1}{x^2} = \frac{2x^3 - 1}{x^2} = 0 \text{ when } x = \frac{1}{\sqrt[3]{2}}.$$

$$f''(x) = 2 + \frac{2}{x^3} = \frac{2(x^3 + 1)}{x^3} = 0 \text{ when } x = -1.$$

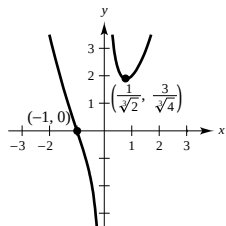
$$f''\left(\frac{1}{\sqrt[3]{2}}\right) > 0$$

Therefore, $\left(\frac{1}{\sqrt[3]{2}}, \frac{3}{\sqrt[3]{4}}\right)$ is a relative minimum.

Point of inflection: $(-1, 0)$

Intercept: $(-1, 0)$

Vertical asymptote: $x = 0$

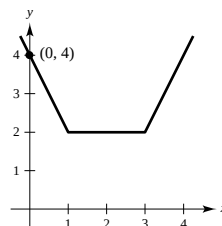


60. $f(x) = |x - 1| + |x - 3| = \begin{cases} -2x + 4, & x \leq 1 \\ 2, & 1 < x \leq 3 \\ 2x - 4, & x > 3 \end{cases}$

Domain: $(-\infty, \infty)$

Range: $[2, \infty)$

Intercept: $(0, 4)$



62. $f(x) = \frac{1}{\pi}(2 \sin \pi x - \sin 2\pi x)$

Domain: $[-1, 1]$; Range: $\left[\frac{-3\sqrt{3}}{2\pi}, \frac{3\sqrt{3}}{2\pi}\right]$

$$f'(x) = 2(\cos \pi x - \cos 2\pi x) = -2(2 \cos \pi x + 1)(\cos \pi x - 1) = 0$$

Critical Numbers: $x = \pm \frac{2}{3}, 0$

$$f''(x) = 2\pi(-\sin \pi x + 2 \sin 2\pi x) = 2\pi \sin \pi x(-1 + 4 \cos \pi x) = 0 \text{ when } x = 0, \pm 1, \pm 0.420.$$

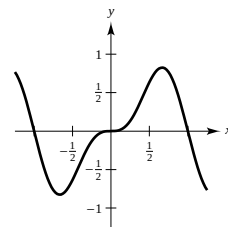
By the First Derivative Test: $\left(-\frac{2}{3}, \frac{-3\sqrt{3}}{2\pi}\right)$ is a relative minimum.

$\left(\frac{2}{3}, \frac{3\sqrt{3}}{2\pi}\right)$ is a relative maximum.

Points of inflection: $(-0.420, -0.462), (0.420, 0.462), (\pm 1, 0), (0, 0)$

Intercepts: $(-1, 0), (0, 0), (1, 0)$

Symmetric with respect to the origin



64. $f(x) = x^n$, n is a positive integer.

(a) $f'(x) = nx^{n-1}$

The function has a relative minimum at $(0, 0)$ when n is even.

(b) $f''(x) = n(n-1)x^{n-2}$

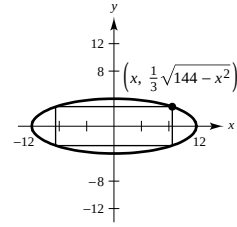
The function has a point of inflection at $(0, 0)$ when n is odd and $n \geq 3$.

66. Ellipse: $\frac{x^2}{144} + \frac{y^2}{16} = 1$, $y = \frac{1}{3}\sqrt{144 - x^2}$

$$A = (2x)\left(\frac{2}{3}\sqrt{144 - x^2}\right) = \frac{4}{3}x\sqrt{144 - x^2}$$

$$\begin{aligned}\frac{dA}{dx} &= \frac{4}{3}\left[\frac{-x^2}{\sqrt{144 - x^2}} + \sqrt{144 - x^2}\right] \\ &= \frac{4}{3}\left[\frac{144 - 2x^2}{\sqrt{144 - x^2}}\right] = 0 \text{ when } x = \sqrt{72} = 6\sqrt{2}.\end{aligned}$$

The dimensions of the rectangle are $2x = 12\sqrt{2}$ by $y = \frac{2}{3}\sqrt{144 - 72} = 4\sqrt{2}$.



68. We have points $(0, y)$, $(x, 0)$, and $(4, 5)$. Thus,

$$m = \frac{y - 5}{0 - 4} = \frac{5 - 0}{4 - x} \text{ or } y = \frac{5x}{x - 4}.$$

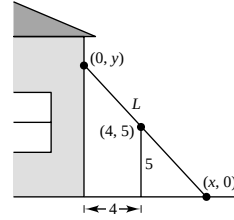
$$\text{Let } f(x) = L^2 = x^2 + \left(\frac{5x}{x - 4}\right)^2$$

$$f'(x) = 2x + 50\left(\frac{x}{x - 4}\right)\left[\frac{x - 4 - x}{(x - 4)^2}\right] = 0$$

$$x - \frac{100x}{(x - 4)^3} = 0$$

$$x[(x - 4)^3 - 100] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{100}.$$

$$L = \sqrt{x^2 + \frac{25x^2}{(x - 4)^2}} = \frac{x}{x - 4} \sqrt{(x - 4)^2 + 25} = \frac{\sqrt[3]{100} + 4}{\sqrt[3]{100}} \sqrt{100^{2/3} + 25} \approx 12.7 \text{ feet}$$



70. Label triangle with vertices $(0, 0)$, $(a, 0)$, and (b, c) . The equations of the sides of the triangle are $y = (c/b)x$ and $y = [c/(b - a)](x - a)$. Let $(x, 0)$ be a vertex of the inscribed rectangle. The coordinates of the upper left vertex are $(x, (c/b)x)$. The y -coordinate of the upper right vertex of the rectangle is $(c/b)x$. Solving for the x -coordinate $\bar{x}$ of the rectangle's upper right vertex, you get

$$\frac{c}{b}x = \frac{c}{b - a}(\bar{x} - a)$$

$$(b - a)x = b(\bar{x} - a)$$

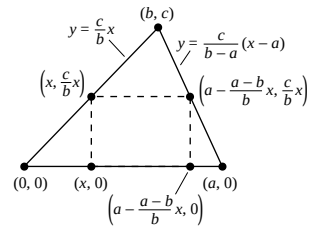
$$\bar{x} = \frac{b - a}{b}x + a = a - \frac{a - b}{b}x.$$

Finally, the lower right vertex is

$$\left(a - \frac{a - b}{b}x, 0\right).$$

Width of rectangle: $a - \frac{a - b}{b}x - x$

Height of rectangle: $\frac{c}{b}x$ (see figure)



$$A = (\text{Width})(\text{Height}) = \left(a - \frac{a - b}{b}x - x\right)\left(\frac{c}{b}x\right) = \left(a - \frac{a}{b}x\right)\frac{c}{b}x$$

$$\frac{dA}{dx} = \left(a - \frac{a}{b}x\right)\frac{c}{b} + \left(\frac{c}{b}x\right)\left(-\frac{a}{b}\right) = \frac{ac}{b} - \frac{2ac}{b^2}x = 0 \text{ when } x = \frac{b}{2}.$$

$$A\left(\frac{b}{2}\right) = \left(a - \frac{a}{b}\frac{b}{2}\right)\left(\frac{c}{b}\frac{b}{2}\right) = \left(\frac{a}{2}\right)\left(\frac{c}{2}\right) = \frac{1}{4}ac = \frac{1}{2}\left(\frac{1}{2}ac\right) = \frac{1}{2}(\text{Area of triangle})$$

72. You can form a right triangle with vertices $(0, y)$, $(0, 0)$, and $(x, 0)$. Choosing a point (a, b) on the hypotenuse (assuming the triangle is in the first quadrant), the slope is

$$m = \frac{y - b}{0 - a} = \frac{b - 0}{a - x} \Rightarrow y = \frac{-bx}{a - x}.$$

$$\text{Let } f(x) = L^2 = x^2 + y^2 = x^2 + \left(\frac{-bx}{a - x}\right)^2.$$

$$f'(x) = 2x + 2\left(\frac{-bx}{a - x}\right)\left[\frac{-ab}{(a - x)^2}\right]$$

$$\frac{2x[(a - x)^3 + ab^2]}{(a - x)^3} = 0 \text{ when } x = 0, a + \sqrt[3]{ab^2}.$$

Choosing the nonzero value, we have $y = b + \sqrt[3]{a^2b}$.

$$\begin{aligned} L &= \sqrt{(a + \sqrt[3]{ab^2})^2 + (b + \sqrt[3]{a^2b})^2} \\ &= (a^2 + 3a^{4/3}b^{2/3} + 3a^{2/3}b^{4/3} + b^2)^{1/2} \\ &= (a^{2/3} + b^{2/3})^{3/2} \text{ meters} \end{aligned}$$

74. Using Exercise 73 as a guide we have $L_1 = a \csc \theta$ and $L_2 = b \sec \theta$. Then $dL/d\theta = -a \csc \theta \cot \theta + b \sec \theta \tan \theta = 0$ when

$$\tan \theta = \sqrt[3]{a/b}, \sec \theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{b^{1/3}}, \csc \theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}} \text{ and}$$

$$L = L_1 + L_2 = a \csc \theta + b \sec \theta = a \frac{(a^{2/3} + b^{2/3})^{1/2}}{a^{1/3}} + b \frac{(a^{2/3} + b^{2/3})^{1/2}}{b^{1/3}} = (a^{2/3} + b^{2/3})^{3/2}.$$

This matches the result of Exercise 72.

76. Total cost = (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{500} + 7.50\right)\left(\frac{110}{v}\right) = \frac{11v}{50} + \frac{825}{v}$$

$$\begin{aligned} \frac{dT}{dv} &= \frac{11}{50} - \frac{825}{v^2} = \frac{11v^2 - 41,250}{50v^2} \\ &= 0 \text{ when } v = \sqrt{3750} = 25\sqrt{6} \approx 61.2 \text{ mph.} \end{aligned}$$

$$\frac{d^2T}{dv^2} = \frac{1650}{v^3} > 0 \text{ when } v = 25\sqrt{6} \text{ so this value yields a minimum.}$$

78. $f(x) = x^3 + 2x + 1$

From the graph, you can see that $f(x)$ has one real zero.

$$f'(x) = 3x^2 + 2$$

f changes sign in $[-1, 0]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	-0.1250	2.7500	-0.0455	-0.4545
2	-0.4545	-0.0029	2.6197	-0.0011	-0.4534

On the interval $[-1, 0]$: $x \approx -0.453$.

43. $r = \frac{v_0^2}{32} (\sin 2\theta)$

$v_0 = 2200$ ft/sec

θ changes from 10° to 11°

$dr = \frac{(2200)^2}{16} (\cos 2\theta) d\theta$

$\theta = 10 \left(\frac{\pi}{180} \right)$

$d\theta = (11 - 10) \frac{\pi}{180}$

$\Delta r \approx dr$

$= \frac{(2200)^2}{16} \cos \left(\frac{20\pi}{180} \right) \left(\frac{\pi}{180} \right) \approx 4961$ feet

≈ 4961 feet

47. Let $f(x) = \sqrt[4]{x}$, $x = 625$, $dx = -1$.

$f(x + \Delta x) \approx f(x) + f'(x) dx = \sqrt[4]{x} + \frac{1}{4^4 \sqrt{x^3}} dx$

$f(x + \Delta x) = \sqrt[4]{624} \approx \sqrt[4]{625} + \frac{1}{4(\sqrt[4]{625})^3} (-1)$

$= 5 - \frac{1}{500} = 4.998$

Using a calculator, $\sqrt[4]{624} \approx 4.9980$.

45. Let $f(x) = \sqrt{x}$, $x = 100$, $dx = -0.6$.

$f(x + \Delta x) \approx f(x) + f'(x) dx$

$= \sqrt{x} + \frac{1}{2\sqrt{x}} dx$

$f(x + \Delta x) = \sqrt{99.4}$

$\approx \sqrt{100} + \frac{1}{2\sqrt{100}} (-0.6) = 9.97$

Using a calculator: $\sqrt{99.4} \approx 9.96995$

49. Let $f(x) = \sqrt{x}$, $x = 4$, $dx = 0.02$, $f'(x) = 1/(2\sqrt{x})$.

Then

$f(4.02) \approx f(4) + f'(4) dx$

$\sqrt{4.02} \approx \sqrt{4} + \frac{1}{2\sqrt{4}} (0.02) = 2 + \frac{1}{4} (0.02).$

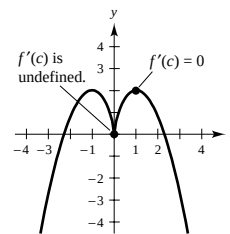
51. In general, when $\Delta x \rightarrow 0$, dy approaches Δy .

53. True

55. True

Review Exercises for Chapter 3

1. A number c in the domain of f is a critical number if $f'(c) = 0$ or f' is undefined at c .



3. $g(x) = 2x + 5 \cos x$, $[0, 2\pi]$

$g'(x) = 2 - 5 \sin x$

$= 0$ when $\sin x = \frac{2}{5}$.

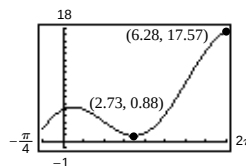
Critical numbers: $x \approx 0.41$, $x \approx 2.73$

Left endpoint: $(0, 5)$

Critical number: $(0.41, 5.41)$

Critical number: $(2.73, 0.88)$ Minimum

Right endpoint: $(2\pi, 17.57)$ Maximum



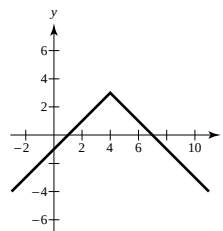
5. Yes. $f(-3) = f(2) = 0$. f is continuous on $[-3, 2]$, differentiable on $(-3, 2)$.

$$f'(x) = (x+3)(3x-1) = 0 \text{ for } x = \frac{1}{3}.$$

$$c = \frac{1}{3} \text{ satisfies } f'(c) = 0.$$

7. $f(x) = 3 - |x - 4|$

(a)



$$f(1) = f(7) = 0$$

(b) f is not differentiable at $x = 4$.

9. $f(x) = x^{2/3}, 1 \leq x \leq 8$

$$f'(x) = \frac{2}{3}x^{-1/3}$$

$$\frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{8 - 1} = \frac{3}{7}$$

$$f'(c) = \frac{2}{3}c^{-1/3} = \frac{3}{7}$$

$$c = \left(\frac{14}{9}\right)^3 = \frac{2744}{729} \approx 3.764$$

13. $f(x) = Ax^2 + Bx + C$

$$f'(x) = 2Ax + B$$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{A(x_2^2 - x_1^2) + B(x_2 - x_1)}{x_2 - x_1}$$

$$= A(x_1 + x_2) + B$$

$$f'(c) = 2Ac + B = A(x_1 + x_2) + B$$

$$2Ac = A(x_1 + x_2)$$

$$c = \frac{x_1 + x_2}{2} = \text{Midpoint of } [x_1, x_2]$$

15. $f(x) = (x-1)^2(x-3)$

$$f'(x) = (x-1)^2(1) + (x-3)(2)(x-1)$$

$$= (x-1)(3x-7)$$

Critical numbers: $x = 1$ and $x = \frac{7}{3}$

17. $h(x) = \sqrt{x}(x-3) = x^{3/2} - 3x^{1/2}$

Domain: $(0, \infty)$

$$h'(x) = \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}$$

$$= \frac{3}{2}x^{-1/2}(x-1) = \frac{3(x-1)}{2\sqrt{x}}$$

Critical number: $x = 1$

11. $f(x) = x - \cos x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$$f'(x) = 1 + \sin x$$

$$\frac{f(b) - f(a)}{b - a} = \frac{(\pi/2) - (-\pi/2)}{(\pi/2) - (-\pi/2)} = 1$$

$$f'(c) = 1 + \sin c = 1$$

$$c = 0$$

Interval:	$-\infty < x < 1$	$1 < x < \frac{7}{3}$	$\frac{7}{3} < x < \infty$
Sign of $f'(x)$:	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion:	Increasing	Decreasing	Increasing

Interval:	$0 < x < 1$	$1 < x < \infty$
Sign of $h'(x)$:	$h'(x) < 0$	$h'(x) > 0$
Conclusion:	Decreasing	Increasing

19. $h(t) = \frac{1}{4}t^4 - 8t$

$h'(t) = t^3 - 8 = 0$ when $t = 2$.

Relative minimum: $(2, -12)$

Test Interval:	$-\infty < t < 2$	$2 < t < \infty$
Sign of $h'(t)$:	$h'(t) < 0$	$h'(t) > 0$
Conclusion:	Decreasing	Increasing

21. $y = \frac{1}{3} \cos(12t) - \frac{1}{4} \sin(12t)$

$v = y' = -4 \sin(12t) - 3 \cos(12t)$

(a) When $t = \frac{\pi}{8}$, $y = \frac{1}{4}$ inch and $v = y' = 4$ inches/second.

(b) $y' = -4 \sin(12t) - 3 \cos(12t) = 0$ when $\frac{\sin(12t)}{\cos(12t)} = -\frac{3}{4} \Rightarrow \tan(12t) = -\frac{3}{4}$.

Therefore, $\sin(12t) = -\frac{3}{5}$ and $\cos(12t) = \frac{4}{5}$. The maximum displacement is

$$y = \left(\frac{1}{3}\right)\left(\frac{4}{5}\right) - \frac{1}{4}\left(-\frac{3}{5}\right) = \frac{5}{12} \text{ inch.}$$

(c) Period: $\frac{2\pi}{12} = \frac{\pi}{6}$

Frequency: $\frac{1}{\pi/6} = \frac{6}{\pi}$

23. $f(x) = x + \cos x$, $0 \leq x \leq 2\pi$

$f'(x) = 1 - \sin x$

$f''(x) = -\cos x = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of inflection: $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

Test Interval:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f''(x)$:	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

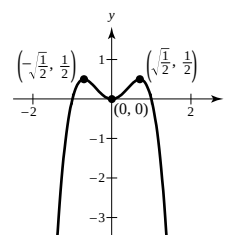
25. $g(x) = 2x^2(1 - x^2)$

$g'(x) = -4x(2x^2 - 1)$ Critical numbers: $x = 0, \pm \frac{1}{\sqrt{2}}$

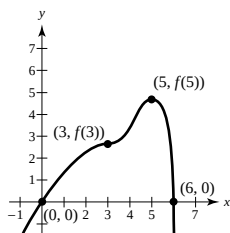
$g''(x) = 4 - 24x^2$

$g''(0) = 4 > 0$ Relative minimum at $(0, 0)$

$g''\left(\pm \frac{1}{\sqrt{2}}\right) = -8 < 0$ Relative maximums at $\left(\pm \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

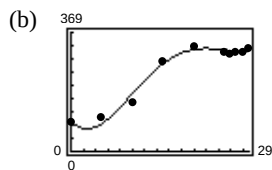


27.



29. The first derivative is positive and the second derivative is negative. The graph is increasing and is concave down.

31. (a) $D = 0.0034t^4 - 0.2352t^3 + 4.9423t^2 - 20.8641t + 94.4025$



(c) Maximum at (21.9, 319.5) (≈ 1992)

Minimum at (2.6, 69.6) (≈ 1972)

(d) Outlays increasing at greatest rate at the point of inflection (9.8, 173.7) (≈ 1979)

33. $\lim_{x \rightarrow \infty} \frac{2x^2}{3x^2 + 5} = \lim_{x \rightarrow \infty} \frac{2}{3 + 5/x^2} = \frac{2}{3}$

35. $\lim_{x \rightarrow \infty} \frac{5 \cos x}{x} = 0$, since $|5 \cos x| \leq 5$.

37. $h(x) = \frac{2x + 3}{x - 4}$

Discontinuity: $x = 4$

$$\lim_{x \rightarrow \infty} \frac{2x + 3}{x - 4} = \lim_{x \rightarrow \infty} \frac{2 + (3/x)}{1 - (4/x)} = 2$$

Vertical asymptote: $x = 4$

Horizontal asymptote: $y = 2$

39. $f(x) = \frac{3}{x} - 2$

Discontinuity: $x = 0$

$$\lim_{x \rightarrow \infty} \left(\frac{3}{x} - 2 \right) = -2$$

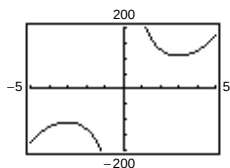
Vertical asymptote: $x = 0$

Horizontal asymptote: $y = -2$

41. $f(x) = x^3 + \frac{243}{x}$

Relative minimum: (3, 108)

Relative maximum: (-3, -108)

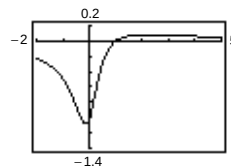


Vertical asymptote: $x = 0$

43. $f(x) = \frac{x - 1}{1 + 3x^2}$

Relative minimum: (-0.155, -1.077)

Relative maximum: (2.155, 0.077)



Horizontal asymptote: $y = 0$

45. $f(x) = 4x - x^2 = x(4 - x)$

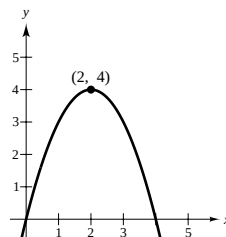
Domain: $(-\infty, \infty)$; Range: $(-\infty, 4)$

$$f'(x) = 4 - 2x = 0 \text{ when } x = 2.$$

$$f''(x) = -2$$

Therefore, (2, 4) is a relative maximum.

Intercepts: (0, 0), (4, 0)



47. $f(x) = x\sqrt{16 - x^2}$, Domain: $[-4, 4]$, Range: $[-8, 8]$

Domain: $[-4, 4]$; Range: $[-8, 8]$

$$f'(x) = \frac{16 - 2x^2}{\sqrt{16 - x^2}} = 0 \text{ when } x = \pm 2\sqrt{2} \text{ and undefined when } x = \pm 4.$$

$$f''(x) = \frac{2x(x^2 - 24)}{(16 - x^2)^{3/2}}$$

$$f'(-2\sqrt{2}) > 0$$

Therefore, $(-2\sqrt{2}, -8)$ is a relative minimum.

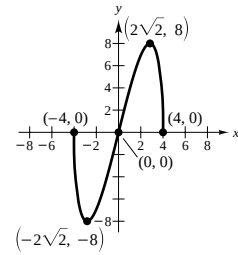
$$f'(2\sqrt{2}) < 0$$

Therefore, $(2\sqrt{2}, 8)$ is a relative maximum.

Point of inflection: $(0, 0)$

Intercepts: $(-4, 0)$, $(0, 0)$, $(4, 0)$

Symmetry with respect to origin



49. $f(x) = (x - 1)^3(x - 3)^2$

Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

$$f'(x) = (x - 1)^2(x - 3)(5x - 11) = 0 \text{ when } x = 1, \frac{11}{5}, 3.$$

$$f''(x) = 4(x - 1)(5x^2 - 22x + 23) = 0 \text{ when } x = 1, \frac{11 \pm \sqrt{6}}{5}.$$

$$f''(3) > 0$$

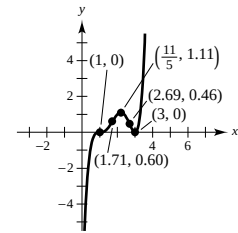
Therefore, $(3, 0)$ is a relative minimum.

$$f''\left(\frac{11}{5}\right) < 0$$

Therefore, $\left(\frac{11}{5}, \frac{3456}{3125}\right)$ is a relative maximum.

Points of inflection: $(1, 0)$, $\left(\frac{11 - \sqrt{6}}{5}, 0.60\right)$, $\left(\frac{11 + \sqrt{6}}{5}, 0.46\right)$

Intercepts: $(0, -9)$, $(1, 0)$, $(3, 0)$



51. $f(x) = x^{1/3}(x + 3)^{2/3}$

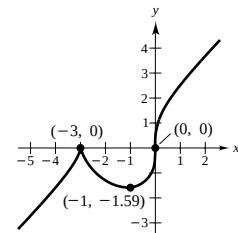
Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

$$f'(x) = \frac{x + 1}{(x + 3)^{1/3}x^{2/3}} = 0 \text{ when } x = -1 \text{ and undefined when } x = -3, 0.$$

$$f''(x) = \frac{-2}{x^{5/3}(x + 3)^{4/3}} \text{ is undefined when } x = 0, -3.$$

By the First Derivative Test $(-3, 0)$ is a relative maximum and $(-1, -\sqrt[3]{4})$ is a relative minimum. $(0, 0)$ is a point of inflection.

Intercepts: $(-3, 0)$, $(0, 0)$



53. $f(x) = \frac{x+1}{x-1}$

Domain: $(-\infty, 1), (1, \infty)$; Range: $(-\infty, 1), (1, \infty)$

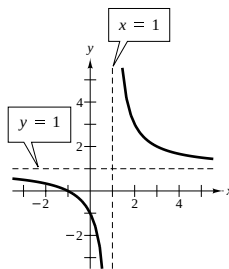
$$f'(x) = \frac{-2}{(x-1)^2} < 0 \text{ if } x \neq 1.$$

$$f''(x) = \frac{4}{(x-1)^3}$$

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 1$

Intercepts: $(-1, 0), (0, -1)$



55. $f(x) = \frac{4}{1+x^2}$

Domain: $(-\infty, \infty)$; Range: $(0, 4]$

$$f'(x) = \frac{-8x}{(1+x^2)^2} = 0 \text{ when } x = 0.$$

$$f''(x) = \frac{-8(1-3x^2)}{(1+x^2)^3} = 0 \text{ when } x = \pm \frac{\sqrt{3}}{3}.$$

$$f''(0) < 0$$

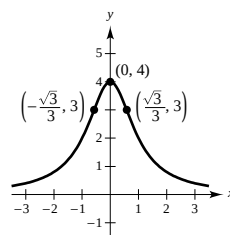
Therefore, $(0, 4)$ is a relative maximum.

Points of inflection: $(\pm \sqrt{3}/3, 3)$

Intercept: $(0, 4)$

Symmetric to the y-axis

Horizontal asymptote: $y = 0$



57. $f(x) = x^3 + x + \frac{4}{x}$

Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, -6], [6, \infty)$

$$f'(x) = 3x^2 + 1 - \frac{4}{x^2} = \frac{3x^4 + x^2 - 4}{x^2} = 0 \text{ when } x = \pm 1.$$

$$f''(x) = 6x + \frac{8}{x^3} = \frac{6x^4 + 8}{x^3} \neq 0$$

$$f''(-1) < 0$$

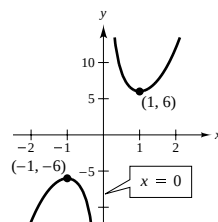
Therefore, $(-1, -6)$ is a relative maximum.

$$f''(1) > 0$$

Therefore, $(1, 6)$ is a relative minimum.

Vertical asymptote: $x = 0$

Symmetric with respect to origin



59. $f(x) = |x^2 - 9|$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$$f'(x) = \frac{2x(x^2 - 9)}{|x^2 - 9|} = 0 \text{ when } x = 0 \text{ and is undefined when } x = \pm 3.$$

$$f''(x) = \frac{2(x^2 - 9)}{|x^2 - 9|} \text{ is undefined at } x = \pm 3.$$

$$f''(0) < 0$$

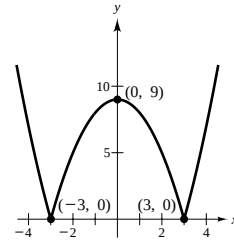
Therefore, $(0, 9)$ is a relative maximum.

Relative minima: $(\pm 3, 0)$

Points of inflection: $(\pm 3, 0)$

Intercepts: $(\pm 3, 0)$, $(0, 9)$

Symmetric to the y-axis



61. $f(x) = x + \cos x$

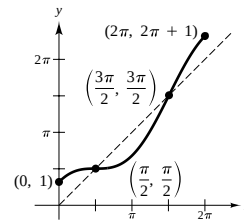
Domain: $[0, 2\pi]$; Range: $[1, 1 + 2\pi]$

$$f'(x) = 1 - \sin x \geq 0, f \text{ is increasing.}$$

$$f''(x) = -\cos x = 0 \text{ when } x = \frac{\pi}{2}, \frac{3\pi}{2}.$$

Points of inflection: $(\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{3\pi}{2}, \frac{3\pi}{2})$

Intercept: $(0, 1)$



63. $x^2 + 4y^2 - 2x - 16y + 13 = 0$

(a) $(x^2 - 2x + 1) + 4(y^2 - 4y + 4) = -13 + 1 + 16$

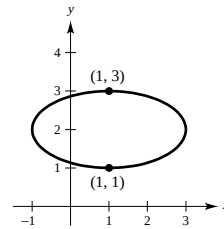
$$(x - 1)^2 + 4(y - 2)^2 = 4$$

$$\frac{(x - 1)^2}{4} + \frac{(y - 2)^2}{1} = 1$$

The graph is an ellipse:

Maximum: $(1, 3)$

Minimum: $(1, 1)$



(b) $x^2 + 4y^2 - 2x - 16y + 13 = 0$

$$2x + 8y \frac{dy}{dx} - 2 - 16 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(8y - 16) = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2 - 2x}{8y - 16} = \frac{1 - x}{4y - 8}$$

The critical numbers are $x = 1$ and $y = 2$. These correspond to the points $(1, 1)$, $(1, 3)$, $(2, -1)$, and $(2, 3)$. Hence, the maximum is $(1, 3)$ and the minimum is $(1, 1)$.

65. Let $t = 0$ at noon.

$$L = d^2 = (100 - 12t)^2 + (-10t)^2 = 10,000 - 2400t + 244t^2$$

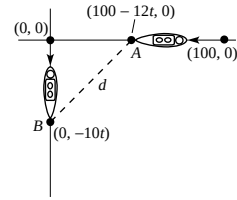
$$\frac{dL}{dt} = -2400 + 488t = 0 \text{ when } t = \frac{300}{61} \approx 4.92 \text{ hr.}$$

Ship A at (40.98, 0); Ship B at (0, -49.18)

$$d^2 = 10,000 - 2400t + 244t^2$$

$$\approx 4098.36 \text{ when } t \approx 4.92 \approx 4:55 \text{ P.M..}$$

$$d \approx 64 \text{ km}$$



67. We have points $(0, y)$, $(x, 0)$, and $(1, 8)$. Thus,

$$m = \frac{y - 8}{0 - 1} = \frac{0 - 8}{x - 1} \text{ or } y = \frac{8x}{x - 1}.$$

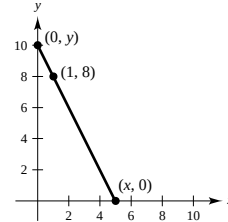
$$\text{Let } f(x) = L^2 = x^2 + \left(\frac{8x}{x - 1}\right)^2.$$

$$f'(x) = 2x + 128\left(\frac{x}{x - 1}\right)\left[\frac{(x - 1) - x}{(x - 1)^2}\right] = 0$$

$$x - \frac{64x}{(x - 1)^3} = 0$$

$$x[(x - 1)^3 - 64] = 0 \text{ when } x = 5 \text{ (minimum).}$$

Vertices of triangle: $(0, 0)$, $(5, 0)$, $(0, 10)$



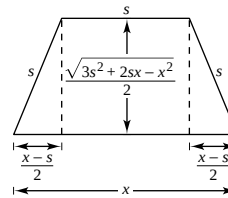
69. $A = (\text{Average of bases})(\text{Height})$

$$= \left(\frac{x + s}{2}\right) \frac{\sqrt{3s^2 + 2sx - x^2}}{2} \text{ (see figure)}$$

$$\frac{dA}{dx} = \frac{1}{4} \left[\frac{(s - x)(s + x)}{\sqrt{3s^2 + 2sx - x^2}} + \sqrt{3s^2 + 2sx - x^2} \right]$$

$$= \frac{2(2s - x)(s + x)}{4\sqrt{3s^2 + 2sx - x^2}} = 0 \text{ when } x = 2s.$$

A is a maximum when $x = 2s$.



71. You can form a right triangle with vertices $(0, 0)$, $(x, 0)$ and $(0, y)$.

Assume that the hypotenuse of length L passes through $(4, 6)$.

$$m = \frac{y - 6}{0 - 4} = \frac{6 - 0}{4 - x} \text{ or } y = \frac{6x}{x - 4}$$

$$\text{Let } f(x) = L^2 = x^2 + y^2 = x^2 + \left(\frac{6x}{x - 4}\right)^2.$$

$$f'(x) = 2x + 72\left(\frac{x}{x - 4}\right)\left[\frac{-4}{(x - 4)^2}\right] = 0$$

$$x[(x - 4)^3 - 144] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{144}.$$

$$L \approx 14.05 \text{ feet}$$

73. $\csc \theta = \frac{L_1}{6}$ or $L_1 = 6 \csc \theta$ (see figure)

$$\csc\left(\frac{\pi}{2} - \theta\right) = \frac{L_2}{9} \text{ or } L_2 = 9 \csc\left(\frac{\pi}{2} - \theta\right)$$

$$L = L_1 + L_2 = 6 \csc \theta + 9 \csc\left(\frac{\pi}{2} - \theta\right) = 6 \csc \theta + 9 \sec \theta$$

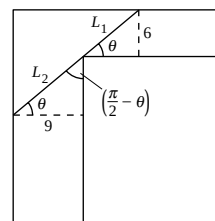
$$\frac{dL}{d\theta} = -6 \csc \theta \cot \theta + 9 \sec \theta \tan \theta = 0$$

$$\tan^3 \theta = \frac{2}{3} \Rightarrow \tan \theta = \frac{\sqrt[3]{2}}{\sqrt[3]{3}}$$

$$\sec \theta = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + \left(\frac{2}{3}\right)^{2/3}} = \frac{\sqrt{3^{2/3} + 2^{2/3}}}{3^{1/3}}$$

$$\csc \theta = \frac{\sec \theta}{\tan \theta} = \frac{\sqrt{3^{2/3} + 2^{2/3}}}{2^{1/3}}$$

$$L = 6 \frac{(3^{2/3} + 2^{2/3})^{1/2}}{2^{1/3}} + 9 \frac{(3^{2/3} + 2^{2/3})^{1/2}}{3^{1/3}} = 3(3^{2/3} + 2^{2/3})^{3/2} \text{ ft} \approx 21.07 \text{ ft (Compare to Exercise 72 using } a = 9 \text{ and } b = 6.)$$



75. Total cost = (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{600} + 5\right)\left(\frac{110}{v}\right) = \frac{11v}{60} + \frac{550}{v}$$

$$\frac{dT}{dv} = \frac{11}{60} - \frac{550}{v^2} = \frac{11v^2 - 33,000}{60v^2}$$

$$= 0 \text{ when } v = \sqrt{3000} = 10\sqrt{30} \approx 54.8 \text{ mph.}$$

$$\frac{d^2T}{dv^2} = \frac{1100}{v^3} > 0 \text{ when } v = 10\sqrt{30} \text{ so this value yields a minimum.}$$

77. $f(x) = x^3 - 3x - 1$

From the graph you can see that $f(x)$ has three real zeros.

$$f'(x) = 3x^2 - 3$$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.5000	0.1250	3.7500	0.0333	-1.5333
2	-1.5333	-0.0049	4.0530	-0.0012	-1.5321

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	0.3750	-2.2500	-0.1667	-0.3333
2	-0.3333	-0.0371	-2.6667	0.0139	-0.3472
3	-0.3472	-0.0003	-2.6384	0.0001	-0.3473

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.9000	0.1590	7.8300	0.0203	1.8797
2	1.8797	0.0024	7.5998	0.0003	1.8794

The three real zeros of $f(x)$ are $x \approx -1.532$, $x \approx -0.347$, and $x \approx 1.879$.

79. Find the zeros of
- $f(x) = x^4 - x - 3$
- .

$$f'(x) = 4x^3 - 1$$

From the graph you can see that $f(x)$ has two real zeros.

f changes sign in $[-2, -1]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.2000	0.2736	-7.9120	-0.0346	-1.1654
2	-1.1654	0.0100	-7.3312	-0.0014	-1.1640

On the interval $[-2, -1]$: $x \approx -1.164$.

f changes sign in $[1, 2]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.5000	0.5625	12.5000	0.0450	1.4550
2	1.4550	0.0268	11.3211	0.0024	1.4526
3	1.4526	-0.0003	11.2602	0.0000	1.4526

On the interval $[1, 2]$: $x \approx 1.453$.

- 81.
- $y = x(1 - \cos x) = x - x \cos x$

$$\frac{dy}{dx} = 1 + x \sin x - \cos x$$

$$dy = (1 + x \sin x - \cos x) dx$$

- 83.
- $S = 4\pi r^2$
- ,
- $dr = \Delta r = \pm 0.025$

$$dS = 8\pi r dr = 8\pi(9)(\pm 0.025)$$

$$= \pm 1.8\pi \text{ square cm}$$

$$\begin{aligned} \frac{dS}{S}(100) &= \frac{8\pi r dr}{4\pi r^2}(100) = \frac{2 dr}{r}(100) \\ &= \frac{2(\pm 0.025)}{9}(100) \approx \pm 0.56\% \end{aligned}$$

$$V = \frac{4}{3}\pi r^3$$

$$\begin{aligned} dV &= 4\pi r^2 dr = 4\pi(9)^2(\pm 0.025) \\ &= \pm 8.1\pi \text{ cubic cm} \end{aligned}$$

$$\begin{aligned} \frac{dV}{V}(100) &= \frac{4\pi r^2 dr}{(4/3)\pi r^3}(100) = \frac{3 dr}{r}(100) \\ &= \frac{3(\pm 0.025)}{9}(100) \approx \pm 0.83\% \end{aligned}$$

Problem Solving for Chapter 3

1. Assume $y_1 < d < y_2$. Let $g(x) = f(x) - d(x - a)$. g is continuous on $[a, b]$ and therefore has a minimum $(c, g(c))$ on $[a, b]$. The point c cannot be an endpoint of $[a, b]$ because

$$g'(a) = f'(a) - d = y_1 - d < 0$$

$$g'(b) = f'(b) - d = y_2 - d > 0$$

Hence, $a < c < b$ and $g'(c) = 0 \Rightarrow f'(c) = d$.

3. (a) For $a = -3, -2, -1, 0$, p has a relative maximum at $(0, 0)$.

For $a = 1, 2, 3$, p has a relative maximum at $(0, 0)$ and 2 relative minima.

$$(b) \quad p'(x) = 4ax^3 - 12x = 4x(ax^2 - 3) = 0 \Rightarrow x = 0, \pm\sqrt{\frac{3}{a}}$$

$$p''(x) = 12ax^2 - 12 = 12(ax^2 - 1)$$

For $x = 0$, $p''(0) = -12 < 0 \Rightarrow p$ has a relative maximum at $(0, 0)$.

- (c) If $a > 0$, $x = \pm\sqrt{\frac{3}{a}}$ are the remaining critical numbers.

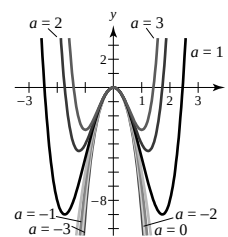
$$p''\left(\pm\sqrt{\frac{3}{a}}\right) = 12\left(\frac{3}{a}\right) - 12 = 24 > 0 \Rightarrow p \text{ has relative minima for } a > 0.$$

- (d) $(0, 0)$ lies on $y = -3x^2$.

Let $x = \pm\sqrt{\frac{3}{a}}$. Then

$$p(x) = a\left(\frac{3}{a}\right)^2 - 6\left(\frac{3}{a}\right) = \frac{9}{a} - \frac{18}{a} = -\frac{9}{a}.$$

Thus, $y = -\frac{9}{a} = -3\left(\pm\sqrt{\frac{3}{a}}\right)^2 = -3x^2$ is satisfied by all the relative extrema of p .



5. $p(x) = x^4 + ax^2 + 1$

$$(a) \quad p'(x) = 4x^3 + 2ax = 2x(2x^2 + a)$$

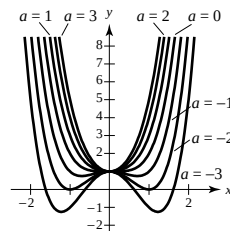
$$p''(x) = 16x^2 + 2a$$

For $a \geq 0$, there is one relative minimum at $(0, 0)$.

- (b) For $a < 0$, there is a relative maximum at $(0, 1)$.

- (c) For $a < 0$, there are two relative minima at $x = \pm\sqrt{-\frac{a}{2}}$.

- (d) There are either 1 or 3 critical points. The above analysis shows that there cannot be exactly two relative extrema.



7. $f(x) = \frac{c}{x} + x^2$

$$f'(x) = -\frac{c}{x^2} + 2x = 0 \Rightarrow \frac{c}{x^2} = 2x \Rightarrow x^3 = \frac{c}{2} \Rightarrow x = \sqrt[3]{\frac{c}{2}}$$

$$f''(x) = \frac{2c}{x^3} + 2$$

If $c = 0$, $f(x) = x^2$ has a relative minimum, but no relative maximum.

If $c > 0$, $x = \sqrt[3]{\frac{c}{2}}$ is a relative minimum, because $f''\left(\sqrt[3]{\frac{c}{2}}\right) > 0$.

If $c < 0$, $x = \sqrt[3]{\frac{c}{2}}$ is a relative minimum too.

Answer: all c .

9. Set $\frac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = k$.

Define $F(x) = f(x) - f(a) - f'(a)(x - a) - k(x - a)^2$.

$F(a) = 0, F(b) = f(b) - f(a) - f'(a)(b - a) - k(b - a)^2 = 0$

F is continuous on $[a, b]$ and differentiable on (a, b) .

There exists $c_1, a < c_1 < b$, satisfying $F'(c_1) = 0$.

$F'(x) = f'(x) - f'(a) - 2k(x - a)$ satisfies the hypothesis of Rolle's Theorem on $[a, c_1]$:

$F'(a) = 0, F'(c_1) = 0$.

There exists $c_2, a < c_2 < c_1$ satisfying $F''(c_2) = 0$.

Finally, $F''(x) = f''(x) - 2k$ and $F''(c_2) = 0$ implies that

$k = \frac{f''(c_2)}{2}$.

Thus, $k = \frac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = \frac{f''(c_2)}{2} \Rightarrow f(b) = f(a) + f'(a)(b - a) + \frac{1}{2}f''(c_2)(b - a)^2$.

11. $E(\phi) = \frac{\tan \phi(1 - 0.1 \tan \phi)}{0.1 + \tan \phi} = \frac{10 \tan \phi - \tan^2 \phi}{1 + 10 \tan \phi}$

$E'(\phi) = \frac{(1 + 10 \tan \phi)(10 \sec^2 \phi - 2 \tan \phi \sec^2 \phi) - (10 \tan \phi - \tan^2 \phi)10 \sec^2 \phi}{(1 + 10 \tan \phi)^2} = 0$

$\Rightarrow (1 + 10 \tan \phi)(10 \sec^2 \phi - 2 \tan \phi \sec^2 \phi) = (10 \tan \phi - \tan^2 \phi)10 \sec^2 \phi$

$\Rightarrow 10 \sec^2 \phi - 2 \tan \phi \sec^2 \phi + 100 \tan \phi \sec^2 \phi - 20 \tan^2 \phi \sec^2 \phi$

$= 100 \tan \phi \sec^2 \phi - 10 \tan^2 \phi \sec^2 \phi$

$\Rightarrow 10 - 2 \tan \phi = 10 \tan^2 \phi$

$\Rightarrow 10 \tan^2 \phi + 2 \tan \phi - 10 = 0$

$\tan \phi = \frac{-2 \pm \sqrt{4 + 400}}{20} \approx 0.90499, -1.10499$

Using the positive value, $\phi \approx 0.7356$, or 42.14° .

13. $v = -2400\pi \sin \theta$

$v' = -2400\pi \cos \theta = 0$

$\theta = \frac{\pi}{2} + 2n\pi, \frac{3\pi}{2} + 2n\pi, n \text{ an integer}$

15. The line has equation $\frac{x}{3} + \frac{y}{4} = 1$ or $y = -\frac{4}{3}x + 4$.

Rectangle:

$$\text{Area} = A = xy = x\left(-\frac{4}{3}x + 4\right) = -\frac{4}{3}x^2 + 4x.$$

$$A'(x) = -\frac{8}{3}x + 4 = 0 \Rightarrow \frac{8}{3}x = 4 \Rightarrow x = \frac{3}{2}$$

Dimensions: $\frac{3}{2} \times 2$ Calculus was helpful.

Circle: The distance from the center (r, r) to the line $\frac{x}{3} + \frac{y}{4} - 1 = 0$ must be r :

$$r = \frac{\left|\frac{r}{3} + \frac{r}{4} - 1\right|}{\sqrt{\frac{1}{9} + \frac{1}{16}}} = \frac{12|7r - 12|}{5|12|} = \frac{|7r - 12|}{5}$$

$$5r = |7r - 12| \Rightarrow r = 1 \text{ or } r = 6.$$

Clearly, $r = 1$.

Semicircle: The center lies on the line $\frac{x}{3} + \frac{y}{4} = 1$ and satisfies $x = y = r$.

Thus $\frac{r}{3} + \frac{r}{4} = 1 \Rightarrow \frac{7}{12}r = 1 \Rightarrow r = \frac{12}{7}$. No calculus necessary.

17. $y = (1 + x^2)^{-1}$

$$y' = \frac{-2x}{(1 + x^2)^2}$$

$$y'' = \frac{2(3x^2 - 1)}{(x^2 + 1)^3} = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}} = \pm \frac{\sqrt{3}}{3}$$

$$y'': \begin{array}{c} + + + \quad - - - \quad - - - \quad + + + \\ \hline -\frac{\sqrt{3}}{3} \quad 0 \quad \frac{\sqrt{3}}{3} \end{array}$$

The tangent line has greatest slope at $\left(-\frac{\sqrt{3}}{3}, \frac{3}{4}\right)$ and least slope at $\left(\frac{\sqrt{3}}{3}, \frac{3}{4}\right)$.

19. (a)

x	0.1	0.2	0.3	0.4	0.5	1.0
$\sin x$	0.09983	0.19867	0.29552	0.38942	0.47943	0.84147

$$\sin x \leq x$$

- (b) Let $f(x) = \sin x$. Then $f'(x) = \cos x$ and on $[0, x]$ you have by the Mean Value Theorem,

$$f'(c) = \frac{f(x) - f(0)}{x - 0}, \quad 0 < c < x$$

$$\cos(c) = \frac{\sin x}{x}$$

$$\text{Hence, } \left|\frac{\sin x}{x}\right| = |\cos(c)| \leq 1$$

$$\Rightarrow |\sin x| \leq |x|$$

$$\Rightarrow \sin x \leq x$$

50. Let $f(x) = \tan x$, $x = 0$, $dx = 0.05$, $f'(x) = \sec^2 x$.

Then

$$f(0.05) \approx f(0) + f'(0)dx$$

$$\tan 0.05 \approx \tan 0 + \sec^2 0(0.05) = 0 + 1(0.05).$$

54. True, $\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = a$

52. Propagated error $= f(x + \Delta x) - f(x)$,

$$\text{relative error} = \left| \frac{dy}{y} \right|, \text{ and the percent error} = \left| \frac{dy}{y} \right| \times 100.$$

56. False

Let $f(x) = \sqrt{x}$, $x = 1$, and $\Delta x = dx = 3$. Then

$$\Delta y = f(x + \Delta x) - f(x) = f(4) - f(1) = 1$$

and

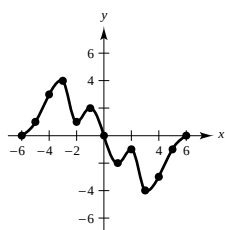
$$dy = f'(x) dx = \frac{1}{2\sqrt{1}}(3) = \frac{3}{2}.$$

Thus, $dy > \Delta y$ in this example.

Review Exercises for Chapter 3

2. (a) $f(4) = -f(-4) = -3$

(c)



At least six critical numbers on $(-6, 6)$.

4. $f(x) = \frac{x}{\sqrt{x^2 + 1}}$, $[0, 2]$

$$\begin{aligned} f'(x) &= x \left[-\frac{1}{2}(x^2 + 1)^{-3/2}(2x) \right] + (x^2 + 1)^{-1/2} \\ &= \frac{1}{(x^2 + 1)^{3/2}} \end{aligned}$$

No critical numbers

Left endpoint: $(0, 0)$ Minimum

Right endpoint: $(2, 2/\sqrt{5})$ Maximum

- (b) $f(-3) = -f(3) = -(-4) = 4$

- (d) Yes. Since $f(-2) = -f(2) = -(-1) = 1$ and $f(1) = -f(-1) = -2$, the Mean Value says that there exists at least one value c in $(-2, 1)$ such that

$$f'(c) = \frac{f(1) - f(-2)}{1 - (-2)} = \frac{-2 - 1}{1 + 2} = -1.$$

- (e) No, $\lim_{x \rightarrow 0} f(x)$ exists because f is continuous at $(0, 0)$.

- (f) Yes, f is differentiable at $x = 2$.

6. No. f is not differentiable at $x = 2$.

8. No; the function is discontinuous at $x = 0$ which is in the interval $[-2, 1]$.

10. $f(x) = \frac{1}{x}$, $1 \leq x \leq 4$

$$f'(x) = -\frac{1}{x^2}$$

$$\frac{f(b) - f(a)}{b - a} = \frac{(1/4) - 1}{4 - 1} = \frac{-3/4}{3} = -\frac{1}{4}$$

$$f'(c) = \frac{-1}{c^2} = -\frac{1}{4}$$

$$c = 2$$

12. $f(x) = \sqrt{x} - 2x$, $0 \leq x \leq 4$

$$f'(x) = \frac{1}{2\sqrt{x}} - 2$$

$$\frac{f(b) - f(a)}{b - a} = \frac{-6 - 0}{4 - 0} = -\frac{3}{2}$$

$$f'(c) = \frac{1}{2\sqrt{c}} - 2 = -\frac{3}{2}$$

$$c = 1$$

14. $f(x) = 2x^2 - 3x + 1$

$f'(x) = 4x - 3$

$$\frac{f(b) - f(a)}{b - a} = \frac{21 - 1}{4 - 0} = 5$$

$f'(c) = 4c - 3 = 5$

$c = 2 = \text{Midpoint of } [0, 4]$

16. $g(x) = (x + 1)^3$

$g'(x) = 3(x + 1)^2$

Critical number: $x = -1$

Interval	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $g'(x)$	$g'(x) > 0$	$g'(x) > 0$
Conclusion	Increasing	Increasing

18. $f(x) = \sin x + \cos x, \quad 0 \leq x \leq 2\pi$

$f'(x) = \cos x - \sin x$

Critical numbers: $x = \frac{\pi}{4}, x = \frac{5\pi}{4}$

Interval	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < 2\pi$
Sign of $f'(x)$	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion	Increasing	Decreasing	Increasing

20. $g(x) = \frac{3}{2} \sin\left(\frac{\pi x}{2} - 1\right), [0, 4]$

$g'(x) = \frac{3}{2} \left(\frac{\pi}{2}\right) \cos\left(\frac{\pi x}{2} - 1\right)$

$= 0 \text{ when } x = 1 + \frac{2}{\pi}, 3 + \frac{2}{\pi}$

Relative maximum: $\left(1 + \frac{2}{\pi}, \frac{3}{2}\right)$

Relative minimum: $\left(3 + \frac{2}{\pi}, -\frac{3}{2}\right)$

Test Interval	$0 < x < 1 + \frac{2}{\pi}$	$1 + \frac{2}{\pi} < x < 3 + \frac{2}{\pi}$	$3 + \frac{2}{\pi} < x < 4$
Sign of $g'(x)$	$g'(x) > 0$	$g'(x) < 0$	$g'(x) > 0$
Conclusion	Increasing	Decreasing	Increasing

22. (a) $y = A \sin(\sqrt{k/m}t) + B \cos(\sqrt{k/m}t)$

$y' = A\sqrt{k/m} \cos(\sqrt{k/m}t) - B\sqrt{k/m} \sin(\sqrt{k/m}t)$

$= 0 \text{ when } \frac{\sin \sqrt{k/m}t}{\cos \sqrt{k/m}t} = \frac{A}{B} \Rightarrow \tan(\sqrt{k/m}t) = \frac{A}{B}.$

Therefore,

$$\sin(\sqrt{k/m}t) = \frac{A}{\sqrt{A^2 + B^2}}$$

$$\cos(\sqrt{k/m}t) = \frac{B}{\sqrt{A^2 + B^2}}.$$

When $v = y' = 0$,

$$y = A\left(\frac{A}{\sqrt{A^2 + B^2}}\right) + B\left(\frac{B}{\sqrt{A^2 + B^2}}\right) = \sqrt{A^2 + B^2}.$$

(b) Period: $\frac{2\pi}{\sqrt{k/m}}$

Frequency: $\frac{1}{2\pi/\sqrt{k/m}} = \frac{1}{2\pi} \sqrt{k/m}$

24. $f(x) = (x + 2)^2(x - 4) = x^3 - 12x - 16$

$$f'(x) = 3x^2 - 12$$

$$f''(x) = 6x = 0 \text{ when } x = 0.$$

Point of inflection: $(0, -16)$

26. $h(t) = t - 4\sqrt{t+1}$ Domain: $[-1, \infty)$

$$h'(t) = 1 - \frac{2}{\sqrt{t+1}} = 0 \Rightarrow t = 3$$

$$h''(t) = \frac{1}{(t+1)^{3/2}}$$

$$h''(3) = \frac{1}{8} > 0 \quad (3, -5) \text{ is a relative minimum.}$$

30. $C = \left(\frac{Q}{x}\right)s + \left(\frac{x}{2}\right)r$

$$\frac{dC}{dx} = -\frac{Qs}{x^2} + \frac{r}{2} = 0$$

$$\frac{Qs}{x^2} = \frac{r}{2}$$

$$x^2 = \frac{2Qs}{r}$$

$$x = \sqrt{\frac{2Qs}{r}}$$

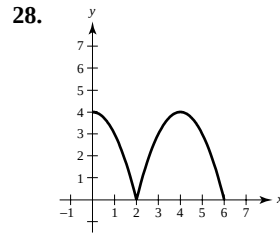
34. $\lim_{x \rightarrow \infty} \frac{2x}{3x^2 + 5} = \lim_{x \rightarrow \infty} \frac{2/x}{3 + 5/x^2} = 0$

38. $g(x) = \frac{5x^2}{x^2 + 2}$

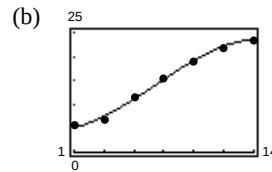
$$\lim_{x \rightarrow \infty} \frac{5x^2}{x^2 + 2} = \lim_{x \rightarrow \infty} \frac{5}{1 + (2/x^2)} = 5$$

Horizontal asymptote: $y = 5$

Test Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward



32. (a) $S = -0.1222t^3 + 1.3655t^2 - 0.9052t + 4.8429$



(c) $S'(t) = 0$ when $t = 3.7$. This is a maximum by the First Derivative Test.

(d) No, because the t^3 coefficient term is negative.

36. $\lim_{x \rightarrow \infty} \frac{3x}{\sqrt{x^2 + 4}} = \lim_{x \rightarrow \infty} \frac{3}{\sqrt{1 + 4/x^2}} = 3$

40. $f(x) = \frac{3x}{\sqrt{x^2 + 2}}$

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{3x}{\sqrt{x^2 + 2}} &= \lim_{x \rightarrow \infty} \frac{3x/x}{\sqrt{x^2 + 2}/\sqrt{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{3}{\sqrt{1 + (2/x^2)}} = 3 \end{aligned}$$

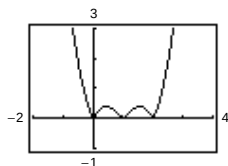
$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{3x}{\sqrt{x^2 + 2}} &= \lim_{x \rightarrow -\infty} \frac{3x/x}{\sqrt{x^2 + 2}/(-\sqrt{x^2})} \\ &= \lim_{x \rightarrow -\infty} \frac{3}{-\sqrt{1 + (2/x^2)}} = -3 \end{aligned}$$

Horizontal asymptotes: $y = \pm 3$

42. $f(x) = |x^3 - 3x^2 + 2x| = |x(x-1)(x-2)|$

Relative minima: $(0, 0), (1, 0), (2, 0)$

Relative maxima: $(1.577, 0.38), (0.423, 0.38)$



46. $f(x) = 4x^3 - x^4 = x^3(4 - x)$

Domain: $(-\infty, \infty)$; Range: $(-\infty, 27)$

$$f'(x) = 12x^2 - 4x^3 = 4x^2(3 - x) = 0 \text{ when } x = 0, 3.$$

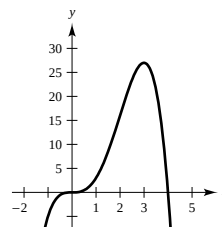
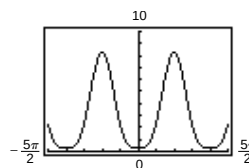
$$f''(x) = 24x - 12x^2 = 12x(2 - x) = 0 \text{ when } x = 0, 2.$$

$$f''(3) < 0$$

Therefore, $(3, 27)$ is a relative maximum.

Points of inflection: $(0, 0), (2, 16)$

Intercepts: $(0, 0), (4, 0)$



48. $f(x) = (x^2 - 4)^2$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$$f'(x) = 4x(x^2 - 4) = 0 \text{ when } x = 0, \pm 2.$$

$$f''(x) = 4(3x^2 - 4) = 0 \text{ when } x = \pm \frac{2\sqrt{3}}{3}.$$

$$f''(0) < 0$$

Therefore, $(0, 16)$ is a relative maximum.

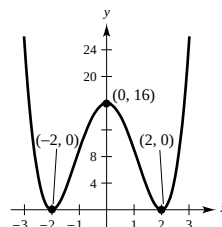
$$f''(\pm 2) > 0$$

Therefore, $(\pm 2, 0)$ are relative minima.

Points of inflection: $(\pm 2\sqrt{3}/3, 64/9)$

Intercepts: $(-2, 0), (0, 16), (2, 0)$

Symmetry with respect to y-axis



50. $f(x) = (x - 3)(x + 2)^3$

Domain: $(-\infty, \infty)$; Range: $[-\frac{16,875}{256}, \infty)$

$$f'(x) = (x - 3)(3)(x + 2)^2 + (x + 2)^3$$

$$= (4x - 7)(x + 2)^2 = 0 \text{ when } x = -2, \frac{7}{4}.$$

$$f''(x) = (4x - 7)(2)(x + 2) + (x + 2)^2(4)$$

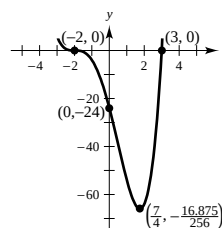
$$= 6(2x - 1)(x + 2) = 0 \text{ when } x = -2, \frac{1}{2}.$$

$$f''(\frac{7}{4}) > 0$$

Therefore, $(\frac{7}{4}, -\frac{16,875}{256})$ is a relative minimum.

Points of inflection: $(-2, 0), (\frac{1}{2}, -\frac{625}{16})$

Intercepts: $(-2, 0), (0, -24), (3, 0)$



52. $f(x) = (x - 2)^{1/3}(x + 1)^{2/3}$

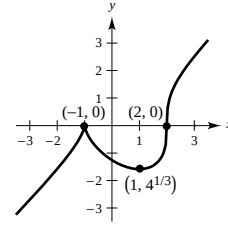
Graph of Exercise 39 translated 2 units to the right (x replaces by $x - 2$).

$(-1, 0)$ is a relative maximum.

$(1, -\sqrt[3]{4})$ is a relative minimum.

$(2, 0)$ is a point of inflection.

Intercepts: $(-1, 0), (2, 0)$



54. $f(x) = \frac{2x}{1+x^2}$

Domain: $(-\infty, \infty)$; Range: $[-1, 1]$

$$f'(x) = \frac{2(1-x)(1+x)}{(1+x^2)^2} = 0 \text{ when } x = \pm 1.$$

$$f''(x) = \frac{-2x(3-x^2)}{(1+x^2)^3} = 0 \text{ when } x = 0, \pm\sqrt{3}.$$

$$f''(1) < 0$$

Therefore, $(1, 1)$ is a relative maximum.

$$f''(-1) > 0$$

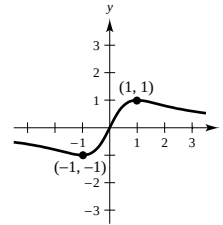
Therefore, $(-1, -1)$ is a relative minimum.

Points of inflection: $(-\sqrt{3}, -\sqrt{3}/2), (0, 0), (\sqrt{3}, \sqrt{3}/2)$

Intercept: $(0, 0)$

Symmetric with respect to the origin

Horizontal asymptote: $y = 0$



56. $f(x) = \frac{x^2}{1+x^4}$

Domain: $(-\infty, \infty)$; Range: $\left[0, \frac{1}{2}\right]$

$$f'(x) = \frac{(1+x^4)(2x) - x^2(4x^3)}{(1+x^4)^2} = \frac{2x(1-x)(1+x)(1+x^2)}{(1+x^4)^2} = 0 \text{ when } x = 0, \pm 1.$$

$$f''(x) = \frac{(1+x^4)^2(2-10x^4) - (2x-2x^5)(2)(1+x^4)(4x^3)}{(1+x^4)^4} = \frac{2(1-12x^4+3x^8)}{(1+x^4)^3} = 0 \text{ when } x = \pm \sqrt[4]{\frac{6 \pm \sqrt{33}}{3}}.$$

$$f''(\pm 1) < 0$$

Therefore, $\left(\pm 1, \frac{1}{2}\right)$ are relative maxima.

$$f''(0) > 0$$

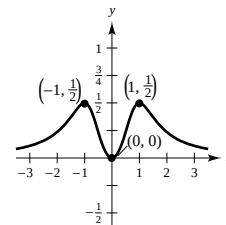
Therefore, $(0, 0)$ is a relative minimum.

Points of inflection: $\left(\pm \sqrt[4]{\frac{6 - \sqrt{33}}{3}}, 0.29\right), \left(\pm \sqrt[4]{\frac{6 + \sqrt{33}}{3}}, 0.40\right)$

Intercept: $(0, 0)$

Symmetric to the y-axis

Horizontal asymptote: $y = 0$



58. $f(x) = x^2 + \frac{1}{x} = \frac{x^3 + 1}{x}$

Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, \infty)$

$$f'(x) = 2x - \frac{1}{x^2} = \frac{2x^3 - 1}{x^2} = 0 \text{ when } x = \frac{1}{\sqrt[3]{2}}.$$

$$f''(x) = 2 + \frac{2}{x^3} = \frac{2(x^3 + 1)}{x^3} = 0 \text{ when } x = -1.$$

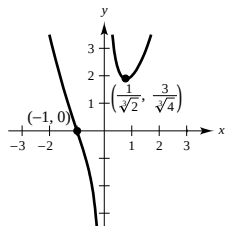
$$f''\left(\frac{1}{\sqrt[3]{2}}\right) > 0$$

Therefore, $\left(\frac{1}{\sqrt[3]{2}}, \frac{3}{\sqrt[3]{4}}\right)$ is a relative minimum.

Point of inflection: $(-1, 0)$

Intercept: $(-1, 0)$

Vertical asymptote: $x = 0$

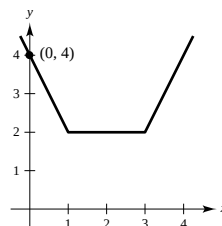


60. $f(x) = |x - 1| + |x - 3| = \begin{cases} -2x + 4, & x \leq 1 \\ 2, & 1 < x \leq 3 \\ 2x - 4, & x > 3 \end{cases}$

Domain: $(-\infty, \infty)$

Range: $[2, \infty)$

Intercept: $(0, 4)$



62. $f(x) = \frac{1}{\pi}(2 \sin \pi x - \sin 2\pi x)$

Domain: $[-1, 1]$; Range: $\left[\frac{-3\sqrt{3}}{2\pi}, \frac{3\sqrt{3}}{2\pi}\right]$

$$f'(x) = 2(\cos \pi x - \cos 2\pi x) = -2(2 \cos \pi x + 1)(\cos \pi x - 1) = 0$$

Critical Numbers: $x = \pm \frac{2}{3}, 0$

$$f''(x) = 2\pi(-\sin \pi x + 2 \sin 2\pi x) = 2\pi \sin \pi x(-1 + 4 \cos \pi x) = 0 \text{ when } x = 0, \pm 1, \pm 0.420.$$

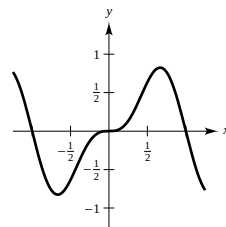
By the First Derivative Test: $\left(-\frac{2}{3}, \frac{-3\sqrt{3}}{2\pi}\right)$ is a relative minimum.

$\left(\frac{2}{3}, \frac{3\sqrt{3}}{2\pi}\right)$ is a relative maximum.

Points of inflection: $(-0.420, -0.462), (0.420, 0.462), (\pm 1, 0), (0, 0)$

Intercepts: $(-1, 0), (0, 0), (1, 0)$

Symmetric with respect to the origin



64. $f(x) = x^n$, n is a positive integer.

(a) $f'(x) = nx^{n-1}$

The function has a relative minimum at $(0, 0)$ when n is even.

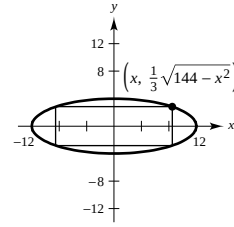
(b) $f''(x) = n(n-1)x^{n-2}$

The function has a point of inflection at $(0, 0)$ when n is odd and $n \geq 3$.

66. Ellipse: $\frac{x^2}{144} + \frac{y^2}{16} = 1$, $y = \frac{1}{3}\sqrt{144 - x^2}$

$$A = (2x)\left(\frac{2}{3}\sqrt{144 - x^2}\right) = \frac{4}{3}x\sqrt{144 - x^2}$$

$$\begin{aligned}\frac{dA}{dx} &= \frac{4}{3}\left[\frac{-x^2}{\sqrt{144 - x^2}} + \sqrt{144 - x^2}\right] \\ &= \frac{4}{3}\left[\frac{144 - 2x^2}{\sqrt{144 - x^2}}\right] = 0 \text{ when } x = \sqrt{72} = 6\sqrt{2}.\end{aligned}$$



The dimensions of the rectangle are $2x = 12\sqrt{2}$ by $y = \frac{2}{3}\sqrt{144 - 72} = 4\sqrt{2}$.

68. We have points $(0, y)$, $(x, 0)$, and $(4, 5)$. Thus,

$$m = \frac{y - 5}{0 - 4} = \frac{5 - 0}{4 - x} \text{ or } y = \frac{5x}{x - 4}.$$

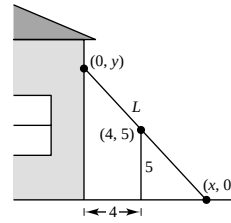
$$\text{Let } f(x) = L^2 = x^2 + \left(\frac{5x}{x - 4}\right)^2$$

$$f'(x) = 2x + 50\left(\frac{x}{x - 4}\right)\left[\frac{x - 4 - x}{(x - 4)^2}\right] = 0$$

$$x - \frac{100x}{(x - 4)^3} = 0$$

$$x[(x - 4)^3 - 100] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{100}.$$

$$L = \sqrt{x^2 + \frac{25x^2}{(x - 4)^2}} = \frac{x}{x - 4} \sqrt{(x - 4)^2 + 25} = \frac{\sqrt[3]{100} + 4}{\sqrt[3]{100}} \sqrt{100^{2/3} + 25} \approx 12.7 \text{ feet}$$



70. Label triangle with vertices $(0, 0)$, $(a, 0)$, and (b, c) . The equations of the sides of the triangle are $y = (c/b)x$ and $y = [c/(b - a)](x - a)$. Let $(x, 0)$ be a vertex of the inscribed rectangle. The coordinates of the upper left vertex are $(x, (c/b)x)$. The y-coordinate of the upper right vertex of the rectangle is $(c/b)x$. Solving for the x-coordinate $\bar{x}$ of the rectangle's upper right vertex, you get

$$\frac{c}{b}x = \frac{c}{b - a}(\bar{x} - a)$$

$$(b - a)x = b(\bar{x} - a)$$

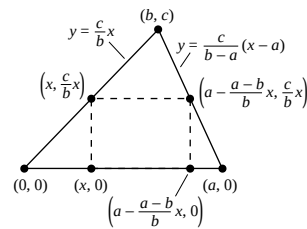
$$\bar{x} = \frac{b - a}{b}x + a = a - \frac{a - b}{b}x.$$

Finally, the lower right vertex is

$$\left(a - \frac{a - b}{b}x, 0\right).$$

Width of rectangle: $a - \frac{a - b}{b}x - x$

Height of rectangle: $\frac{c}{b}x$ (see figure)



$$A = (\text{Width})(\text{Height}) = \left(a - \frac{a - b}{b}x - x\right)\left(\frac{c}{b}x\right) = \left(a - \frac{a + b}{b}x\right)\frac{c}{b}x$$

$$\frac{dA}{dx} = \left(a - \frac{a + b}{b}x\right)\frac{c}{b} + \left(\frac{c}{b}x\right)\left(-\frac{a + b}{b}\right) = \frac{ac}{b} - \frac{2ac}{b^2}x = 0 \text{ when } x = \frac{b}{2}.$$

$$A\left(\frac{b}{2}\right) = \left(a - \frac{a + b}{b}\frac{b}{2}\right)\left(\frac{c}{b}\frac{b}{2}\right) = \left(\frac{a}{2}\right)\left(\frac{c}{2}\right) = \frac{1}{4}ac = \frac{1}{2}\left(\frac{1}{2}ac\right) = \frac{1}{2}(\text{Area of triangle})$$

72. You can form a right triangle with vertices $(0, y)$, $(0, 0)$, and $(x, 0)$. Choosing a point (a, b) on the hypotenuse (assuming the triangle is in the first quadrant), the slope is

$$m = \frac{y - b}{0 - a} = \frac{b - 0}{a - x} \Rightarrow y = \frac{-bx}{a - x}.$$

$$\text{Let } f(x) = L^2 = x^2 + y^2 = x^2 + \left(\frac{-bx}{a - x}\right)^2.$$

$$f'(x) = 2x + 2\left(\frac{-bx}{a - x}\right)\left[\frac{-ab}{(a - x)^2}\right]$$

$$\frac{2x[(a - x)^3 + ab^2]}{(a - x)^3} = 0 \text{ when } x = 0, a + \sqrt[3]{ab^2}.$$

Choosing the nonzero value, we have $y = b + \sqrt[3]{a^2b}$.

$$\begin{aligned} L &= \sqrt{(a + \sqrt[3]{ab^2})^2 + (b + \sqrt[3]{a^2b})^2} \\ &= (a^2 + 3a^{4/3}b^{2/3} + 3a^{2/3}b^{4/3} + b^2)^{1/2} \\ &= (a^{2/3} + b^{2/3})^{3/2} \text{ meters} \end{aligned}$$

74. Using Exercise 73 as a guide we have $L_1 = a \csc \theta$ and $L_2 = b \sec \theta$. Then $dL/d\theta = -a \csc \theta \cot \theta + b \sec \theta \tan \theta = 0$ when

$$\tan \theta = \sqrt[3]{a/b}, \sec \theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{b^{1/3}}, \csc \theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}} \text{ and}$$

$$L = L_1 + L_2 = a \csc \theta + b \sec \theta = a \frac{(a^{2/3} + b^{2/3})^{1/2}}{a^{1/3}} + b \frac{(a^{2/3} + b^{2/3})^{1/2}}{b^{1/3}} = (a^{2/3} + b^{2/3})^{3/2}.$$

This matches the result of Exercise 72.

76. Total cost = (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{500} + 7.50\right)\left(\frac{110}{v}\right) = \frac{11v}{50} + \frac{825}{v}$$

$$\begin{aligned} \frac{dT}{dv} &= \frac{11}{50} - \frac{825}{v^2} = \frac{11v^2 - 41,250}{50v^2} \\ &= 0 \text{ when } v = \sqrt{3750} = 25\sqrt{6} \approx 61.2 \text{ mph.} \end{aligned}$$

$$\frac{d^2T}{dv^2} = \frac{1650}{v^3} > 0 \text{ when } v = 25\sqrt{6} \text{ so this value yields a minimum.}$$

78. $f(x) = x^3 + 2x + 1$

From the graph, you can see that $f(x)$ has one real zero.

$$f'(x) = 3x^2 + 2$$

f changes sign in $[-1, 0]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	-0.1250	2.7500	-0.0455	-0.4545
2	-0.4545	-0.0029	2.6197	-0.0011	-0.4534

On the interval $[-1, 0]$: $x \approx -0.453$.

80. Find the zeros of $f(x) = \sin \pi x + x - 1$.

$$f'(x) = \pi \cos \pi x + 1$$

From the graph you can see that $f(x)$ has three real zeros.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.2000	-0.2122	3.5416	-0.0599	0.2599
2	0.2599	-0.0113	3.1513	-0.0036	0.2635
3	0.2635	0.0000	3.1253	0.0000	0.2635

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.0000	0.0000	-2.1416	0.0000	1.0000

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.8000	0.2122	3.5416	0.0599	1.7401
2	1.7401	0.0113	3.1513	0.0036	1.7365
3	1.7365	0.0000	3.1253	0.0000	1.7365

The three real zeros of $f(x)$ are $x \approx 0.264$, $x = 1$, and $x \approx 1.737$.

82. $y = \sqrt{36 - x^2}$

$$\frac{dy}{dx} = \frac{1}{2}(36 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{36 - x^2}}$$

$$dy = \frac{-x}{\sqrt{36 - x^2}} dx$$

84. $p = 75 - \frac{1}{4}x$

$$\Delta p = p(8) - p(7)$$

$$= \left(75 - \frac{8}{4}\right) - \left(75 - \frac{7}{4}\right) = -\frac{1}{4}$$

$$dp = -\frac{1}{4}dx = -\frac{1}{4}(1) = -\frac{1}{4}$$

$[\Delta p = dp \text{ because } p \text{ is linear}]$

Problem Solving for Chapter 3

2. (a) $dV = 3x^2 dx = 3x^2 \Delta x$

$$\Delta V = (x + \Delta x)^3 - x^3 = 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

$$\Delta V - dV = 3x(\Delta x)^2 + (\Delta x)^3 = \underbrace{[3x\Delta x + (\Delta x)^2]}_{\varepsilon} \Delta x$$

$$= \varepsilon \Delta x, \text{ where } \varepsilon \rightarrow 0 \text{ as } \Delta x \rightarrow 0.$$

- (b) Let $\varepsilon = \frac{\Delta y}{\Delta x} - f'(x)$. Then $\varepsilon \rightarrow 0$ as $\Delta x \rightarrow 0$.

$$\text{Furthermore, } \Delta y - dy = \Delta y - f'(x)dx = \varepsilon \Delta x.$$

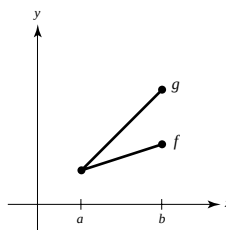
4. Let $h(x) = g(x) - f(x)$, which is continuous on $[a, b]$ and differentiable on (a, b) . $h(a) = 0$ and $h(b) = g(b) - f(b)$.

By the Mean Value Theorem, there exists c in (a, b) such that

$$h'(c) = \frac{h(b) - h(a)}{b - a} = \frac{g(b) - f(b)}{b - a}.$$

Since $h'(c) = g'(c) - f'(c) > 0$ and $b - a > 0$,

$$g(b) - f(b) > 0 \Rightarrow g(b) > f(b).$$



6. (a) $f' = 2ax + b$, $f'' = 2a \neq 0$. No points of inflection.

(b) $f' = 3ax^2 - 2bx + c$, $f'' = 6ax + 2b = 0 \Rightarrow x = \frac{-b}{3a}$. One point of inflection.

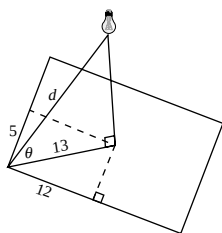
(c) $y' = ky(L - y) = kLy - ky^2$

$$y'' = kLy' - 2kyy' = ky'(L - 2y)$$

If $y = \frac{L}{2}$, then $y'' = 0$ and this is a point of inflection because of the analysis below.

$$y'': \begin{array}{c} ++++++ \quad - - - - - \\ | \\ y = \frac{L}{2} \end{array}$$

8.



$$d = \sqrt{13^2 + x^2}, \sin \theta = \frac{x}{d}.$$

Let A be the amount of illumination at one of the corners, as indicated in the figure. Then

$$A = \frac{kI}{(13^2 + x^2)} \sin \theta = \frac{kIx}{(13^2 + x^2)^{3/2}}$$

$$A'(x) = kI \frac{(x^2 + 169)^{3/2}(1) - x\left(\frac{3}{2}\right)(x^2 + 169)^{1/2}(2x)}{(169 + x^2)^3} = 0$$

$$\Rightarrow (x^2 + 169)^{3/2} = 3x^2(x^2 + 169)^{1/2}$$

$$x^2 + 169 = 3x^2$$

$$2x^2 = 169$$

$$x = \frac{13}{\sqrt{2}} \approx 9.19 \text{ feet}$$

By the First Derivative Test, this is a maximum.

10. Let T be the intersection of PQ and RS . Let MN be the perpendicular to SQ and PR passing through T .

Let $TM = x$ and $TN = b - x$.

$$\frac{SN}{b-x} = \frac{MR}{x} \Rightarrow SN = \frac{b-x}{x}MR$$

$$\frac{NQ}{b-x} = \frac{PM}{x} \Rightarrow NQ = \frac{b-x}{x}PM$$

$$SQ = \frac{b-x}{x}(MR + PM) = \frac{b-x}{x}d$$

$$A(x) = \text{Area} = \frac{1}{2}dx + \frac{1}{2}\left(\frac{b-x}{x}d\right)(b-x) = \frac{1}{2}d\left[x + \frac{(b-x)^2}{x}\right] = \frac{1}{2}d\left[\frac{2x^2 - 2bx + b^2}{x}\right]$$

$$A'(x) = \frac{1}{2}d\left[\frac{x(4x - 2b) - (2x^2 - 2bx + b^2)}{x^2}\right]$$

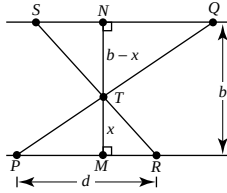
$$A'(x) = 0 \Rightarrow 4x^2 - 2xb = 2x^2 - 2bx + b^2$$

$$2x^2 = b^2$$

$$x = \frac{b}{\sqrt{2}}$$

$$\text{Hence, we have } SQ = \frac{b-x}{x}d = \frac{b - (b/\sqrt{2})}{b/\sqrt{2}}d = (\sqrt{2} - 1)d.$$

Using the Second Derivative Test, this is a minimum. There is no maximum.



12. (a) Let $M > 0$ be given. Take $N = \sqrt{M}$. Then whenever $x > N = \sqrt{M}$, you have

$$f(x) = x^2 > M.$$

- (b) Let $\varepsilon > 0$ be given. Let $M = \sqrt{\frac{1}{\varepsilon}}$. Then whenever $x > M = \sqrt{\frac{1}{\varepsilon}}$, you have

$$x^2 > \frac{1}{\varepsilon} \Rightarrow \frac{1}{x^2} < \varepsilon \Rightarrow \left| \frac{1}{x^2} - 0 \right| < \varepsilon.$$

- (c) Let $\varepsilon > 0$ be given. There exists $N > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > N$.

$$\text{Let } \delta = \frac{1}{N}. \text{ Let } x = \frac{1}{y}.$$

$$\text{If } 0 < y < \delta = \frac{1}{N}, \text{ then } \frac{1}{x} < \frac{1}{N} \Rightarrow x > N \text{ and}$$

$$|f(x) - L| = \left| f\left(\frac{1}{y}\right) - L \right| < \varepsilon.$$

$$14. \text{Distance} = \sqrt{4^2 + x^2} + \sqrt{(4-x)^2 + 4^2} = f(x)$$

$$f'(x) = \frac{x}{\sqrt{4^2 + x^2}} + \frac{4-x}{\sqrt{(4-x)^2 + 4^2}} = 0$$

$$x\sqrt{(4-x)^2 + 4^2} = (x-4)\sqrt{4^2 + x^2}$$

$$x^2[16 - 8x + x^2 + 16] = (x^2 - 8x + 16)(16 + x^2)$$

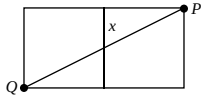
$$32x^2 - 8x^3 + x^4 = x^4 - 8x^3 + 32x^2 - 128x + 256$$

$$128x = 256$$

$$x = 2$$

The bug should head towards the midpoint of the opposite side.

Without Calculus: Imagine opening up the cube:

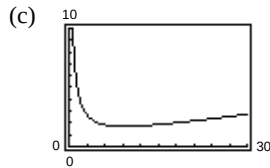


The shortest distance is the line PQ , passing through the midpoint.

$$16. (a) s = \frac{v \frac{\text{km}}{\text{hr}} \left(1000 \frac{\text{m}}{\text{km}} \right)}{\left(3600 \frac{\text{sec}}{\text{hr}} \right)} = \frac{5}{18} v$$

v	20	40	60	80	100
s	5.56	11.11	16.67	22.22	27.78
d	5.1	13.7	27.2	44.2	66.4

$$d(t) = 0.071s^2 + 0.389s + 0.727$$



$$T = \frac{1}{s}(0.071s^2 + 0.389s + 0.727) + \frac{5.5}{s}$$

The minimum is attained when $s \approx 9.365$ m/sec.

- (b) The distance between the back of the first vehicle and the front of the second vehicle is $d(t)$, the safe stopping distance. The first vehicle passes the given point in $5.5/s$ seconds, and the second vehicle takes $d(s)/s$ more seconds. Hence,

$$T = \frac{d(s)}{s} + \frac{5.5}{s}.$$

$$(d) \quad T(s) = 0.071s + 0.389 + \frac{6.227}{s}$$

$$T'(s) = 0.071 - \frac{6.227}{s^2} \Rightarrow s^2 = \frac{6.227}{0.071}$$

$$\Rightarrow s \approx 9.365 \text{ m/sec}$$

$$T(9.365) \approx 1.719 \text{ seconds}$$

$$9.365 \text{ m/sec} \cdot \frac{3600}{1000} = 3.37 \text{ km/hr}$$

$$(e) d(9.365) = 10.597 \text{ m}$$

18. (a)

x	0	0.5	1	2
$\sqrt{1+x}$	1	1.2247	1.4142	1.7321
$\frac{1}{2}x + 1$	1	1.25	1.5	2

- (b) Let $f(x) = \sqrt{1+x}$. Using the Mean Value Theorem on the interval $[0, x]$, there exists c , $0 < c < x$, satisfying

$$f'(c) = \frac{1}{2\sqrt{1+c}} = \frac{f(x) - f(0)}{x - 0} = \frac{\sqrt{1+x} - 1}{x}.$$

$$\text{Thus } \sqrt{1+x} = \frac{x}{2\sqrt{1+c}} + 1 < \frac{x}{2} + 1 \text{ (because } \sqrt{1+c} > 1).$$

CHAPTER 4

Integration

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CHAPTER 4

Integration

Section 4.1 Antiderivatives and Indefinite Integration

Solutions to Odd-Numbered Exercises

$$1. \frac{d}{dx}\left(\frac{3}{x^3} + C\right) = \frac{d}{dx}(3x^{-3} + C) = -9x^{-4} = \frac{-9}{x^4}$$

$$3. \frac{d}{dx}\left(\frac{1}{3}x^3 - 4x + C\right) = x^2 - 4 = (x - 2)(x + 2)$$

$$5. \frac{dy}{dt} = 3t^2$$

$$y = t^3 + C$$

$$\text{Check: } \frac{d}{dt}[t^3 + C] = 3t^2$$

$$7. \frac{dy}{dx} = x^{3/2}$$

$$y = \frac{2}{5}x^{5/2} + C$$

$$\text{Check: } \frac{d}{dx}\left[\frac{2}{5}x^{5/2} + C\right] = x^{3/2}$$

Given

Rewrite

Integrate

Simplify

$$9. \int \sqrt[3]{x} \, dx$$

$$\int x^{1/3} \, dx$$

$$\frac{x^{4/3}}{4/3} + C$$

$$\frac{3}{4}x^{4/3} + C$$

$$11. \int \frac{1}{x\sqrt{x}} \, dx$$

$$\int x^{-3/2} \, dx$$

$$\frac{x^{-1/2}}{-1/2} + C$$

$$-\frac{2}{\sqrt{x}} + C$$

$$13. \int \frac{1}{2x^3} \, dx$$

$$\frac{1}{2} \int x^{-3} \, dx$$

$$\frac{1}{2} \left(\frac{x^{-2}}{-2} \right) + C$$

$$-\frac{1}{4x^2} + C$$

$$15. \int (x + 3) \, dx = \frac{x^2}{2} + 3x + C$$

$$\text{Check: } \frac{d}{dx}\left[\frac{x^2}{2} + 3x + C\right] = x + 3$$

$$17. \int (2x - 3x^2) \, dx = x^2 - x^3 + C$$

$$\text{Check: } \frac{d}{dx}[x^2 - x^3 + C] = 2x - 3x^2$$

$$19. \int (x^3 + 2) \, dx = \frac{1}{4}x^4 + 2x + C$$

$$\text{Check: } \frac{d}{dx}\left(\frac{1}{4}x^4 + 2x + C\right) = x^3 + 2$$

$$21. \int (x^{3/2} + 2x + 1) \, dx = \frac{2}{5}x^{5/2} + x^2 + x + C$$

$$\text{Check: } \frac{d}{dx}\left(\frac{2}{5}x^{5/2} + x^2 + x + C\right) = x^{3/2} + 2x + 1$$

$$23. \int \sqrt[3]{x^2} \, dx = \int x^{2/3} \, dx = \frac{x^{5/3}}{5/3} + C = \frac{3}{5}x^{5/3} + C$$

$$\text{Check: } \frac{d}{dx}\left(\frac{3}{5}x^{5/3} + C\right) = x^{2/3} = \sqrt[3]{x^2}$$

$$25. \int \frac{1}{x^3} \, dx = \int x^{-3} \, dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$$

$$\text{Check: } \frac{d}{dx}\left(-\frac{1}{2x^2} + C\right) = \frac{1}{x^3}$$

$$27. \int \frac{x^2 + x + 1}{\sqrt{x}} dx = \int (x^{3/2} + x^{1/2} + x^{-1/2}) dx = \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C = \frac{2}{15}x^{1/2}(3x^2 + 5x + 15) + C$$

Check: $\frac{d}{dx}\left(\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C\right) = x^{3/2} + x^{1/2} + x^{-1/2} = \frac{x^2 + x + 1}{\sqrt{x}}$

$$29. \int (x + 1)(3x - 2) dx = \int (3x^2 + x - 2) dx$$

$$= x^3 + \frac{1}{2}x^2 - 2x + C$$

Check: $\frac{d}{dx}\left(x^3 + \frac{1}{2}x^2 - 2x + C\right) = 3x^2 + x - 2$

$$= (x + 1)(3x - 2)$$

$$31. \int y^2 \sqrt{y} dy = \int y^{5/2} dy = \frac{2}{7}y^{7/2} + C$$

Check: $\frac{d}{dy}\left(\frac{2}{7}y^{7/2} + C\right) = y^{5/2} = y^2 \sqrt{y}$

$$33. \int dx = \int 1 dx = x + C$$

Check: $\frac{d}{dx}(x + C) = 1$

$$35. \int (2 \sin x + 3 \cos x) dx = -2 \cos x + 3 \sin x + C$$

Check: $\frac{d}{dx}(-2 \cos x + 3 \sin x + C) = 2 \sin x + 3 \cos x$

$$37. \int (1 - \csc t \cot t) dt = t + \csc t + C$$

Check: $\frac{d}{dt}(t + \csc t + C) = 1 - \csc t \cot t$

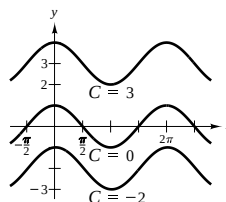
$$39. \int (\sec^2 \theta - \sin \theta) d\theta = \tan \theta + \cos \theta + C$$

Check: $\frac{d}{d\theta}(\tan \theta + \cos \theta + C) = \sec^2 \theta - \sin \theta$

$$41. \int (\tan^2 y + 1) dy = \int \sec^2 y dy = \tan y + C$$

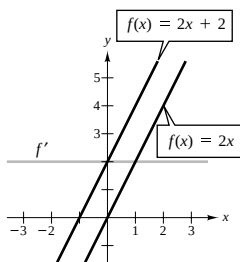
Check: $\frac{d}{dy}(\tan y + C) = \sec^2 y = \tan^2 y + 1$

$$43. f(x) = \cos x$$



$$45. f'(x) = 2$$

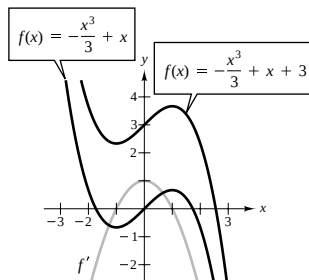
$$f(x) = 2x + C$$



Answers will vary.

$$47. f'(x) = 1 - x^2$$

$$f(x) = x - \frac{x^3}{3} + C$$



Answers will vary.

$$49. \frac{dy}{dx} = 2x - 1, (1, 1)$$

$$y = \int (2x - 1) dx = x^2 - x + C$$

$$1 = (1)^2 - (1) + C \Rightarrow C = 1$$

$$y = x^2 - x + 1$$

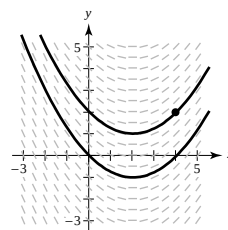
51. $\frac{dy}{dx} = \cos x, (0, 4)$

$$y = \int \cos x \, dx = \sin x + C$$

$$4 = \sin 0 + C \Rightarrow C = 4$$

$$y = \sin x + 4$$

53. (a) Answers will vary.



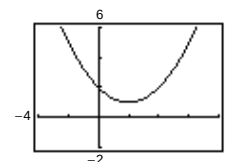
(b) $\frac{dy}{dx} = \frac{1}{2}x - 1, (4, 2)$

$$y = \frac{x^2}{4} - x + C$$

$$2 = \frac{4^2}{4} - 4 + C$$

$$2 = C$$

$$y = \frac{x^2}{4} - x + 2$$



55. $f'(x) = 4x, f(0) = 6$

$$f(x) = \int 4x \, dx = 2x^2 + C$$

$$f(0) = 6 = 2(0)^2 + C \Rightarrow C = 6$$

$$f(x) = 2x^2 + 6$$

57. $h'(t) = 8t^3 + 5, h(1) = -4$

$$h(t) = \int (8t^3 + 5) dt = 2t^4 + 5t + C$$

$$h(1) = -4 = 2 + 5 + C \Rightarrow C = -11$$

$$h(t) = 2t^4 + 5t - 11$$

59. $f''(x) = 2$

$$f'(2) = 5$$

$$f(2) = 10$$

$$f'(x) = \int 2 \, dx = 2x + C_1$$

$$f'(2) = 4 + C_1 = 5 \Rightarrow C_1 = 1$$

$$f'(x) = 2x + 1$$

$$f(x) = \int (2x + 1) \, dx = x^2 + x + C_2$$

$$f(2) = 6 + C_2 = 10 \Rightarrow C_2 = 4$$

$$f(x) = x^2 + x + 4$$

61. $f''(x) = x^{-3/2}$

$$f'(4) = 2$$

$$f(0) = 0$$

$$f'(x) = \int x^{-3/2} \, dx = -2x^{-1/2} + C_1 = -\frac{2}{\sqrt{x}} + C_1$$

$$f'(4) = -\frac{2}{2} + C_1 = 2 \Rightarrow C_1 = 3$$

$$f'(x) = -\frac{2}{\sqrt{x}} + 3$$

$$f(x) = \int \left(-2x^{-1/2} + 3\right) \, dx = -4x^{1/2} + 3x + C_2$$

$$f(0) = 0 + 0 + C_2 = 0 \Rightarrow C_2 = 0$$

$$f(x) = -4x^{1/2} + 3x = -4\sqrt{x} + 3x$$

63. (a) $h(t) = \int (1.5t + 5) \, dt = 0.75t^2 + 5t + C$

$$h(0) = 0 + 0 + C = 12 \Rightarrow C = 12$$

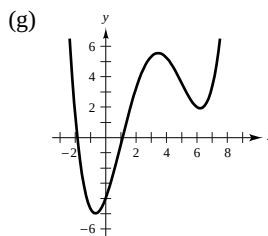
$$h(t) = 0.75t^2 + 5t + 12$$

(b) $h(6) = 0.75(6)^2 + 5(6) + 12 = 69 \text{ cm}$

65. $f(0) = -4$. Graph of f' is given.

- (a) $f'(4) \approx -1.0$
 (b) No. The slopes of the tangent lines are greater than 2 on $[0, 2]$. Therefore, f must increase more than 4 units on $[0, 4]$.
 (c) No, $f(5) < f(4)$ because f is decreasing on $[4, 5]$.
 (d) f is an maximum at $x = 3.5$ because $f'(3.5) \approx 0$ and the first derivative test.
 (e) f is concave upward when f' is increasing on $(-\infty, 1)$ and $(5, \infty)$. f is concave downward on $(1, 5)$. Points of inflection at $x = 1, 5$.

(f) f'' is a minimum at $x = 3$.



67. $a(t) = -32 \text{ ft/sec}^2$

$$v(t) = \int -32 \, dt = -32t + C_1$$

$$v(0) = 60 = C_1$$

$$s(t) = \int (-32t + 60) \, dt = -16t^2 + 60t + C_2$$

$$s(0) = 6 = C_2$$

$$s(t) = -16t^2 + 60t + 6 \text{ Position function}$$

The ball reaches its maximum height when

$$v(t) = -32t + 60 = 0$$

$$32t = 60$$

$$t = \frac{15}{8} \text{ seconds}$$

$$s\left(\frac{15}{8}\right) = -16\left(\frac{15}{8}\right)^2 + 60\left(\frac{15}{8}\right) + 6 \approx 62.26 \text{ feet}$$

71. $a(t) = -9.8$

$$v(t) = \int -9.8 \, dt = -9.8t + C_1$$

$$v(0) = v_0 = C_1 \Rightarrow v(t) = -9.8t + v_0$$

$$f(t) = \int (-9.8t + v_0) \, dt = -4.9t^2 + v_0t + C_2$$

$$f(0) = s_0 = C_2 \Rightarrow f(t) = -4.9t^2 + v_0t + s_0$$

75. $a = -1.6$

$$v(t) = \int -1.6 \, dt = -1.6t + v_0 = -1.6t, \text{ since the stone was dropped, } v_0 = 0.$$

$$s(t) = \int (-1.6t) \, dt = -0.8t^2 + s_0$$

$$s(20) = 0 \Rightarrow -0.8(20)^2 + s_0 = 0$$

$$s_0 = 320$$

Thus, the height of the cliff is 320 meters.

$$v(t) = -1.6t$$

$$v(20) = -32 \text{ m/sec}$$

69. From Exercise 68, we have:

$$s(t) = -16t^2 + v_0t$$

$s'(t) = -32t + v_0 = 0$ when $t = \frac{v_0}{32} = \text{time to reach maximum height.}$

$$s\left(\frac{v_0}{32}\right) = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = 550$$

$$-\frac{v_0^2}{64} + \frac{v_0^2}{32} = 550$$

$$v_0^2 = 35,200$$

$$v_0 \approx 187.617 \text{ ft/sec}$$

73. From Exercise 71, $f(t) = -4.9t^2 + 10t + 2$.

$$v(t) = -9.8t + 10 = 0 \text{ (Maximum height when } v = 0.)$$

$$9.8t = 10$$

$$t = \frac{10}{9.8}$$

$$f\left(\frac{10}{9.8}\right) \approx 7.1 \text{ m}$$

77. $x(t) = t^3 - 6t^2 + 9t - 2 \quad 0 \leq t \leq 5$

(a) $v(t) = x'(t) = 3t^2 - 12t + 9$
 $= 3(t^2 - 4t + 3) = 3(t-1)(t-3)$

$a(t) = v'(t) = 6t - 12 = 6(t-2)$

(b) $v(t) > 0$ when $0 < t < 1$ or $3 < t < 5$.

(c) $a(t) = 6(t-2) = 0$ when $t = 2$.

$v(2) = 3(1)(-1) = -3$

79. $v(t) = \frac{1}{\sqrt{t}} = t^{-1/2} \quad t > 0$

$x(t) = \int v(t) dt = 2t^{1/2} + C$

$x(1) = 4 = 2(1) + C \Rightarrow C = 2$

$x(t) = 2t^{1/2} + 2$ position function

$a(t) = v'(t) = -\frac{1}{2}t^{-3/2} = \frac{-1}{2t^{3/2}}$ acceleration

81. (a) $v(0) = 25 \text{ km/hr} = 25 \cdot \frac{1000}{3600} = \frac{250}{36} \text{ m/sec}$

$v(13) = 80 \text{ km/hr} = 80 \cdot \frac{1000}{3600} = \frac{800}{36} \text{ m/sec}$

$a(t) = a$ (constant acceleration)

$v(t) = at + C$

$v(0) = \frac{250}{36} \Rightarrow v(t) = at + \frac{250}{36}$

$v(13) = \frac{800}{36} = 13a + \frac{250}{36}$

$\frac{550}{36} = 13a$

$a = \frac{550}{468} = \frac{275}{234} \approx 1.175 \text{ m/sec}^2$

(b) $s(t) = a\frac{t^2}{2} + \frac{250}{36}t \quad (s(0) = 0)$

$s(13) = \frac{275}{234} \frac{(13)^2}{2} + \frac{250}{36}(13) \approx 189.58 \text{ m}$

85. $\frac{(1 \text{ mi/hr})(5280 \text{ ft/mi})}{(3600 \text{ sec/hr})} = \frac{22}{15} \text{ ft/sec}$

(a)	t	0	5	10	15	20	25	30
	$V_1(\text{ft/sec})$	0	3.67	10.27	23.47	42.53	66	95.33
	$V_2(\text{ft/sec})$	0	30.8	55.73	74.8	88	93.87	95.33

(c) $S_1(t) = \int V_1(t) dt = \frac{0.1068}{3}t^3 - \frac{0.0416}{2}t^2 + 0.3679t$

$S_2(t) = \int V_2(t) dt = -\frac{0.1208t^3}{3} + \frac{6.7991t^2}{2} - 0.0707t$

[In both cases, the constant of integration is 0 because $S_1(0) = S_2(0) = 0$]

$S_1(30) \approx 953.5 \text{ feet}$

$S_2(30) \approx 1970.3 \text{ feet}$

The second car was going faster than the first until the end.

83. Truck: $v(t) = 30$

$s(t) = 30t$ (Let $s(0) = 0$.)

Automobile: $a(t) = 6$

$v(t) = 6t$ (Let $v(0) = 0$.)

$s(t) = 3t^2$ (Let $s(0) = 0$.)

At the point where the automobile overtakes the truck:

$30t = 3t^2$

$0 = 3t^2 - 30t$

$0 = 3t(t-10)$ when $t = 10 \text{ sec}$.

(a) $s(10) = 3(10)^2 = 300 \text{ ft}$

(b) $v(10) = 6(10) = 60 \text{ ft/sec} \approx 41 \text{ mph}$

(b) $V_1(t) = 0.1068t^2 - 0.0416t + 0.3679$

$V_2(t) = -0.1208t^2 + 6.7991t - 0.0707$

87. $a(t) = k$

$v(t) = kt$

$s(t) = \frac{k}{2}t^2$ since $v(0) = s(0) = 0$.

At the time of lift-off, $kt = 160$ and $(k/2)t^2 = 0.7$. Since $(k/2)t^2 = 0.7$,

$$t = \sqrt{\frac{1.4}{k}}$$

$$v\left(\sqrt{\frac{1.4}{k}}\right) = k\sqrt{\frac{1.4}{k}} = 160$$

$$1.4k = 160^2 \Rightarrow k = \frac{160^2}{1.4}$$

$$\approx 18,285.714 \text{ mi/hr}^2$$

$$\approx 7.45 \text{ ft/sec}^2.$$

89. True

91. True

93. False. For example, $\int x \cdot x \, dx \neq \int x \, dx \cdot \int x \, dx$ because $\frac{x^3}{3} + C \neq \left(\frac{x^2}{2} + C_1\right)\left(\frac{x^2}{2} + C_2\right)$

95. $f'(x) = \begin{cases} 1, & 0 \leq x < 2 \\ 3x, & 2 \leq x \leq 5 \end{cases}$

$$f(x) = \begin{cases} x + C_1, & 0 \leq x < 2 \\ \frac{3x^2}{2} + C_2, & 2 \leq x \leq 5 \end{cases}$$

$$f(1) = 3 \Rightarrow 1 + C_1 = 3 \Rightarrow C_1 = 2$$

f is continuous: Values must agree at $x = 2$:

$$4 = 6 + C_2 \Rightarrow C_2 = -2$$

$$f(x) = \begin{cases} x + 2, & 0 \leq x < 2 \\ \frac{3x^2}{2} - 2, & 2 \leq x \leq 5 \end{cases}$$

The left and right hand derivatives at $x = 2$ do not agree. Hence f is not differentiable at $x = 2$.

Section 4.2 Area

1. $\sum_{i=1}^5 (2i + 1) = 2 \sum_{i=1}^5 i + \sum_{i=1}^5 1 = 2(1 + 2 + 3 + 4 + 5) + 5 = 35$

3. $\sum_{k=0}^4 \frac{1}{k^2 + 1} = 1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} = \frac{158}{85}$

5. $\sum_{k=1}^4 c = c + c + c + c = 4c$

7. $\sum_{i=1}^9 \frac{1}{3i}$

9. $\sum_{j=1}^8 \left[5\left(\frac{j}{8}\right) + 3 \right]$

11. $\frac{2}{n} \sum_{i=1}^n \left[\left(\frac{2i}{n}\right)^3 - \left(\frac{2i}{n}\right) \right]$

13. $\frac{3}{n} \sum_{i=1}^n \left[2\left(1 + \frac{3i}{n}\right)^2 \right]$

15. $\sum_{i=1}^{20} 2i = 2 \sum_{i=1}^{20} i = 2 \left[\frac{20(21)}{2} \right] = 420$

$$17. \sum_{i=1}^{20} (i - 1)^2 = \sum_{i=1}^{19} i^2$$

$$= \left[\frac{19(20)(39)}{6} \right] = 2470$$

$$\begin{aligned}
 19. \sum_{i=1}^{15} i(i-1)^2 &= \sum_{i=1}^{15} i^3 - 2 \sum_{i=1}^{15} i^2 + \sum_{i=1}^{15} i \\
 &= \frac{15^2(16)^2}{4} - 2 \frac{15(16)(31)}{6} + \frac{15(16)}{2} \\
 &= 14,400 - 2,480 + 120 \\
 &= 12,040
 \end{aligned}$$

$$21. \text{sum seq}(x \sqcap 2 + 3, x, 1, 20, 1) = 2930 \quad (TI-82)$$

$$\begin{aligned}
 \sum_{i=1}^{20} (i^2 + 3) &= \frac{20(20+1)(2(20)+1)}{6} + 3(20) \\
 &= \frac{(20)(21)(41)}{6} + 60 = 2930
 \end{aligned}$$

$$\begin{aligned}
 23. S &= [3 + 4 + \frac{9}{2} + 5](1) = \frac{33}{2} = 16.5 \\
 s &= [1 + 3 + 4 + \frac{9}{2}](1) = \frac{25}{2} = 12.5
 \end{aligned}$$

$$\begin{aligned}
 25. S &= [3 + 3 + 5](1) = 11 \\
 s &= [2 + 2 + 3](1) = 7
 \end{aligned}$$

$$\begin{aligned}
 27. S(4) &= \sqrt{\frac{1}{4}\left(\frac{1}{4}\right)} + \sqrt{\frac{1}{2}\left(\frac{1}{4}\right)} + \sqrt{\frac{3}{4}\left(\frac{1}{4}\right)} + \sqrt{1\left(\frac{1}{4}\right)} = \frac{1 + \sqrt{2} + \sqrt{3} + 2}{8} \approx 0.768 \\
 s(4) &= 0\left(\frac{1}{4}\right) + \sqrt{\frac{1}{4}\left(\frac{1}{4}\right)} + \sqrt{\frac{1}{2}\left(\frac{1}{4}\right)} + \sqrt{\frac{3}{4}\left(\frac{1}{4}\right)} = \frac{1 + \sqrt{2} + \sqrt{3}}{8} \approx 0.518
 \end{aligned}$$

$$\begin{aligned}
 29. S(5) &= 1\left(\frac{1}{5}\right) + \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \approx 0.746 \\
 s(5) &= \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5}\right) = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \approx 0.646
 \end{aligned}$$

$$31. \lim_{n \rightarrow \infty} \left[\left(\frac{81}{n^4} \right) \frac{n^2(n+1)^2}{4} \right] = \frac{81}{4} \lim_{n \rightarrow \infty} \left[\frac{n^4 + 2n^3 + n^2}{n^4} \right] = \frac{81}{4}(1) = \frac{81}{4}$$

$$33. \lim_{n \rightarrow \infty} \left[\left(\frac{18}{n^2} \right) \frac{n(n+1)}{2} \right] = \frac{18}{2} \lim_{n \rightarrow \infty} \left[\frac{n^2 + n}{n^2} \right] = \frac{18}{2}(1) = 9$$

$$\begin{aligned}
 35. \sum_{i=1}^n \frac{2i+1}{n^2} &= \frac{1}{n^2} \sum_{i=1}^n (2i+1) = \frac{1}{n^2} \left[2 \frac{n(n+1)}{2} + n \right] = \frac{n+2}{n} = S(n) \\
 S(10) &= \frac{12}{10} = 1.2
 \end{aligned}$$

$$S(100) = 1.02$$

$$S(1000) = 1.002$$

$$S(10,000) = 1.0002$$

$$\begin{aligned}
 37. \sum_{k=1}^n \frac{6k(k-1)}{n^3} &= \frac{6}{n^3} \sum_{k=1}^n (k^2 - k) = \frac{6}{n^3} \left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \right] \\
 &= \frac{6}{n^3} \left[\frac{2n^2 + 3n + 1 - 3n - 3}{6} \right] = \frac{1}{n^2} [2n^2 - 2] = S(n)
 \end{aligned}$$

$$S(10) = 1.98$$

$$S(100) = 1.9998$$

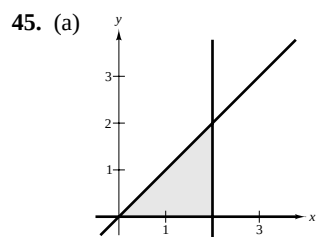
$$S(1000) = 1.999998$$

$$S(10,000) = 1.99999998$$

$$39. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{16i}{n^2} \right) = \lim_{n \rightarrow \infty} \frac{16}{n^2} \sum_{i=1}^n i = \lim_{n \rightarrow \infty} \frac{16}{n^2} \left(\frac{n(n+1)}{2} \right) = \lim_{n \rightarrow \infty} \left[8 \left(\frac{n^2 + n}{n^2} \right) \right] = 8 \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = 8$$

$$\begin{aligned}
 41. \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n^3} (i-1)^2 &= \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{i=1}^{n-1} i^2 = \lim_{n \rightarrow \infty} \frac{1}{n^3} \left[\frac{(n-1)(n)(2n-1)}{6} \right] \\
 &= \lim_{n \rightarrow \infty} \frac{1}{6} \left[\frac{2n^3 - 3n^2 + n}{n^3} \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{6} \left(2 - \frac{3}{n} + \frac{1}{n^2} \right) \right] = \frac{1}{3}
 \end{aligned}$$

$$43. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + \frac{i}{n} \right) \left(\frac{2}{n} \right) = 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\sum_{i=1}^n 1 + \frac{1}{n} \sum_{i=1}^n i \right] = 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{1}{n} \left(\frac{n(n+1)}{2} \right) \right] = 2 \lim_{n \rightarrow \infty} \left[1 + \frac{n^2 + n}{2n^2} \right] = 2 \left(1 + \frac{1}{2} \right) = 3$$



(b) $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

Endpoints:

$$0 < 1\left(\frac{2}{n}\right) < 2\left(\frac{2}{n}\right) < \cdots < (n-1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right) = 2$$

(c) Since $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$\begin{aligned}
 s(n) &= \sum_{i=1}^n f(x_{i-1}) \Delta x \\
 &= \sum_{i=1}^n f\left(\frac{2i-2}{n}\right) \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[(i-1) \left(\frac{2}{n}\right) \right] \left(\frac{2}{n}\right)
 \end{aligned}$$

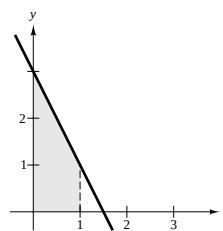
(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n f\left(\frac{2i}{n}\right) \frac{2}{n} = \sum_{i=1}^n \left[i \left(\frac{2}{n}\right) \right] \left(\frac{2}{n}\right)$$

47. $y = -2x + 3$ on $[0, 1]$. (Note: $\Delta x = \frac{1-0}{n} = \frac{1}{n}$)

$$\begin{aligned}
 s(n) &= \sum_{i=1}^n f\left(\frac{i}{n}\right) \left(\frac{1}{n}\right) = \sum_{i=1}^n \left[-2\left(\frac{i}{n}\right) + 3 \right] \left(\frac{1}{n}\right) \\
 &= 3 - \frac{2}{n^2} \sum_{i=1}^n i = 3 - \frac{2(n+1)n}{2n^2} = 2 - \frac{1}{n}
 \end{aligned}$$

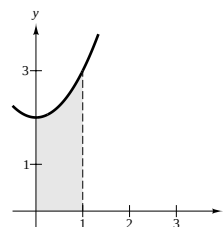
Area = $\lim_{n \rightarrow \infty} s(n) = 2$



49. $y = x^2 + 2$ on $[0, 1]$. (Note: $\Delta x = \frac{1}{n}$)

$$\begin{aligned}
 S(n) &= \sum_{i=1}^n f\left(\frac{i}{n}\right) \left(\frac{1}{n}\right) = \sum_{i=1}^n \left[\left(\frac{i}{n}\right)^2 + 2 \right] \left(\frac{1}{n}\right) \\
 &= \left[\frac{1}{n^3} \sum_{i=1}^n i^2 \right] + 2 = \frac{n(n+1)(2n+1)}{6n^3} + 2 = \frac{1}{6} \left(2 + \frac{3}{n} + \frac{1}{n^2} \right) + 2
 \end{aligned}$$

Area = $\lim_{n \rightarrow \infty} S(n) = \frac{7}{3}$



(e)

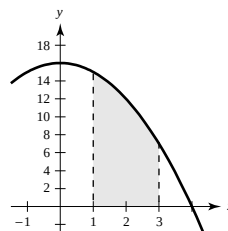
x	5	10	50	100
$s(n)$	1.6	1.8	1.96	1.98
$S(n)$	2.4	2.2	2.04	2.02

$$\begin{aligned}
 (f) \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[(i-1) \left(\frac{2}{n}\right) \right] \left(\frac{2}{n}\right) &= \lim_{n \rightarrow \infty} \frac{4}{n^2} \sum_{i=1}^n (i-1) \\
 &= \lim_{n \rightarrow \infty} \frac{4}{n^2} \left[\frac{n(n+1)}{2} - n \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{2(n+1)}{n} - \frac{4}{n} \right] = 2 \\
 \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[i \left(\frac{2}{n}\right) \right] \left(\frac{2}{n}\right) &= \lim_{n \rightarrow \infty} \frac{4}{n^2} \sum_{i=1}^n i \\
 &= \lim_{n \rightarrow \infty} \left(\frac{4}{n^2} \right) \frac{n(n+1)}{2} \\
 &= \lim_{n \rightarrow \infty} \frac{2(n+1)}{n} = 2
 \end{aligned}$$

51. $y = 16 - x^2$ on $[1, 3]$. (Note: $\Delta x = \frac{2}{n}$)

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^n \left[16 - \left(1 + \frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right) \\ &= \frac{2}{n} \sum_{i=1}^n \left[15 - \frac{4i^2}{n^2} - \frac{4i}{n}\right] \\ &= \frac{2}{n} \left[15n - \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} - \frac{4}{n} \frac{n(n+1)}{2}\right] \\ &= 30 - \frac{8}{6n^2}(n+1)(2n+1) - \frac{4}{n}(n+1) \end{aligned}$$

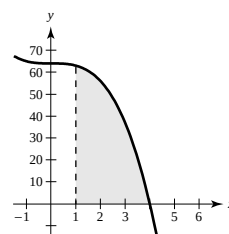
$$\text{Area} = \lim_{n \rightarrow \infty} s(n) = 30 - \frac{8}{3} - 4 = \frac{70}{3} = 23\frac{1}{3}$$



53. $y = 64 - x^3$ on $[1, 4]$. (Note: $\Delta x = \frac{4-1}{n} = \frac{3}{n}$)

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^n \left[64 - \left(1 + \frac{3i}{n}\right)^3\right]\left(\frac{3}{n}\right) \\ &= \frac{3}{n} \sum_{i=1}^n \left[63 - \frac{27i^3}{n^3} - \frac{27i^2}{n^2} - \frac{9i}{n}\right] \\ &= \frac{3}{n} \left[63n - \frac{27}{n^3} \frac{n^2(n+1)^2}{4} - \frac{27}{n^2} \frac{n(n+1)(2n+1)}{6} - \frac{9}{n} \frac{n(n+1)}{2}\right] \\ &= 189 - \frac{81}{4n^2}(n+1)^2 - \frac{81}{6n^2}(n+1)(2n+1) - \frac{27}{2} \frac{n+1}{n} \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} s(n) = 189 - \frac{81}{4} - 27 - \frac{27}{2} = \frac{513}{4} = 128.25$$

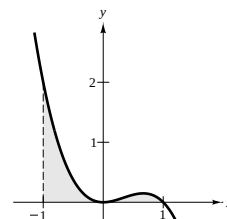


55. $y = x^2 - x^3$ on $[-1, 1]$. (Note: $\Delta x = \frac{1 - (-1)}{n} = \frac{2}{n}$)

Again, $T(n)$ is neither an upper nor a lower sum.

$$\begin{aligned} T(n) &= \sum_{i=1}^n f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^n \left[\left(-1 + \frac{2i}{n}\right)^2 - \left(-1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n \left[\left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right)\right]\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n \left[2 - \frac{10i}{n} + \frac{16i^2}{n^2} - \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right) = \frac{4}{n} \sum_{i=1}^n 1 - \frac{20}{n^2} \sum_{i=1}^n i + \frac{32}{n^3} \sum_{i=1}^n i^2 - \frac{16}{n^4} \sum_{i=1}^n i^3 \\ &= \frac{4}{n}(n) - \frac{20}{n^2} \cdot \frac{n(n+1)}{2} + \frac{32}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\ &= 4 - 10\left(1 + \frac{1}{n}\right) + \frac{16}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} T(n) = 4 - 10 + \frac{32}{3} - 4 = \frac{2}{3}$$

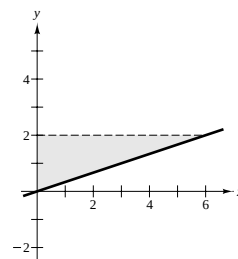


57. $f(y) = 3y, 0 \leq y \leq 2$ (Note: $\Delta y = \frac{2-0}{n} = \frac{2}{n}$)

$$S(n) = \sum_{i=1}^n f(m_i) \Delta y = \sum_{i=1}^n f\left(\frac{2i}{n}\right) \left(\frac{2}{n}\right) = \sum_{i=1}^n 3\left(\frac{2i}{n}\right) \left(\frac{2}{n}\right)$$

$$= \frac{12}{n^2} \sum_{i=1}^n i = \left(\frac{12}{n^2}\right) \cdot \frac{n(n+1)}{2} = \frac{6(n+1)}{n} = 6 + \frac{6}{n}$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = \lim_{n \rightarrow \infty} \left(6 + \frac{6}{n}\right) = 6$$

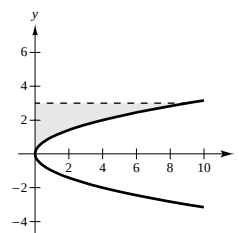


59. $f(y) = y^2, 0 \leq y \leq 3$ (Note: $\Delta y = \frac{3-0}{n} = \frac{3}{n}$)

$$S(n) = \sum_{i=1}^n f\left(\frac{3i}{n}\right) \left(\frac{3}{n}\right) = \sum_{i=1}^n \left(\frac{3i}{n}\right)^2 \left(\frac{3}{n}\right) = \frac{27}{n^3} \sum_{i=1}^n i^2$$

$$= \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{9}{n^2} \left(\frac{2n^2 + 3n + 1}{2}\right) = 9 + \frac{27}{2n} + \frac{9}{2n^2}$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = \lim_{n \rightarrow \infty} \left(9 + \frac{27}{2n} + \frac{9}{2n^2}\right) = 9$$



61. $g(y) = 4y^2 - y^3, 1 \leq y \leq 3$ (Note: $\Delta y = \frac{3-1}{n} = \frac{2}{n}$)

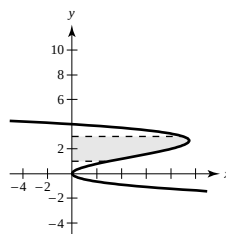
$$S(n) = \sum_{i=1}^n g\left(1 + \frac{2i}{n}\right) \left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^n \left[4\left(1 + \frac{2i}{n}\right)^2 - \left(1 + \frac{2i}{n}\right)^3 \right] \frac{2}{n}$$

$$= \frac{2}{n} \sum_{i=1}^n \left[4\left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(1 + \frac{6i}{n} + \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right) \right]$$

$$= \frac{2}{n} \sum_{i=1}^n \left[3 + \frac{10i}{n} + \frac{4i^2}{n^2} - \frac{8i^3}{n^3} \right] = \frac{2}{n} \left[3n + \frac{10}{n} \frac{n(n+1)}{2} + \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} - \frac{8}{n^2} \frac{n^2(n+1)^2}{4} \right]$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 6 + 10 + \frac{8}{3} - 4 = \frac{44}{3}$$



63. $f(x) = x^2 + 3, 0 \leq x \leq 2, n = 4$

$$\text{Let } c_i = \frac{x_i + x_{i-1}}{2}.$$

$$\Delta x = \frac{1}{2}, c_1 = \frac{1}{4}, c_2 = \frac{3}{4}, c_3 = \frac{5}{4}, c_4 = \frac{7}{4}$$

$$\text{Area} \approx \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^4 [c_i^2 + 3] \left(\frac{1}{2}\right)$$

$$= \frac{1}{2} \left[\left(\frac{1}{4}\right)^2 + 3 + \left(\frac{3}{4}\right)^2 + 3 + \left(\frac{5}{4}\right)^2 + 3 + \left(\frac{7}{4}\right)^2 + 3 \right]$$

$$= \frac{69}{8}$$

65. $f(x) = \tan x, 0 \leq x \leq \frac{\pi}{4}, n = 4$

$$\text{Let } c_i = \frac{x_i + x_{i-1}}{2}.$$

$$\Delta x = \frac{\pi}{16}, c_1 = \frac{\pi}{32}, c_2 = \frac{3\pi}{32}, c_3 = \frac{5\pi}{32}, c_4 = \frac{7\pi}{32}$$

$$\text{Area} \approx \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^4 (\tan c_i) \left(\frac{\pi}{16}\right)$$

$$= \frac{\pi}{16} \left(\tan \frac{\pi}{32} + \tan \frac{3\pi}{32} + \tan \frac{5\pi}{32} + \tan \frac{7\pi}{32} \right) \approx 0.345$$

67. $f(x) = \sqrt{x}$ on $[0, 4]$.

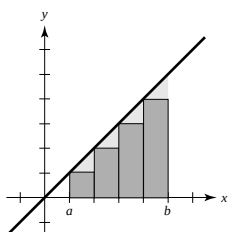
n	4	8	12	16	20
Approximate area	5.3838	5.3523	5.3439	5.3403	5.3384

(Exact value is $16/3$)

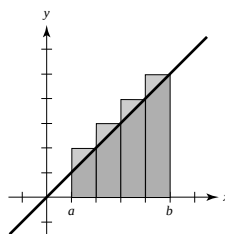
69. $f(x) = \tan\left(\frac{\pi x}{8}\right)$ on $[1, 3]$.

n	4	8	12	16	20
Approximate area	2.2223	2.2387	2.2418	2.2430	2.2435

71. We can use the line $y = x$ bounded by $x = a$ and $x = b$. The sum of the areas of these inscribed rectangles is the lower sum.



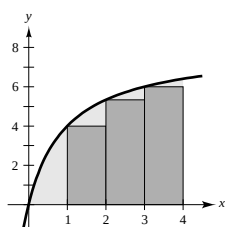
The sum of the areas of these circumscribed rectangles is the upper sum.



We can see that the rectangles do not contain all of the area in the first graph and the rectangles in the second graph cover more than the area of the region.

The exact value of the area lies between these two sums.

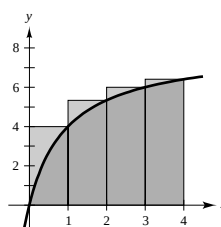
73. (a)



Lower sum:

$$s(4) = 0 + 4 + 5\frac{1}{3} + 6 = 15\frac{1}{3} = \frac{46}{3} \approx 15.333$$

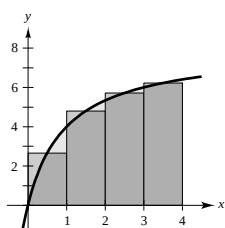
- (b)



Upper sum:

$$S(4) = 4 + 5\frac{1}{3} + 6 + 6\frac{2}{5} = 21\frac{11}{15} = \frac{326}{15} \approx 21.733$$

- (c)



Midpoint Rule:

$$M(4) = 2\frac{2}{3} + 4\frac{4}{5} + 5\frac{5}{7} + 6\frac{2}{9} = \frac{6112}{315} \approx 19.403$$

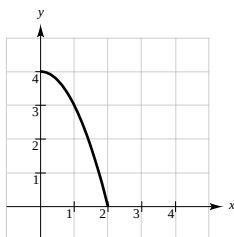
- (d) In each case, $\Delta x = 4/n$. The lower sum uses left endpoints, $(i-1)(4/n)$. The upper sum uses right endpoints, $i(4/n)$. The Midpoint Rule uses midpoints, $(i-\frac{1}{2})(4/n)$.

- (e)

n	4	8	20	100	200
$s(n)$	15.333	17.368	18.459	18.995	19.06
$S(n)$	21.733	20.568	19.739	19.251	19.188
$M(n)$	19.403	19.201	19.137	19.125	19.125

- (f) $s(n)$ increases because the lower sum approaches the exact value as n increases. $S(n)$ decreases because the upper sum approaches the exact value as n increases. Because of the shape of the graph, the lower sum is always smaller than the exact value, whereas the upper sum is always larger.

75.

b. $A \approx 6$ square units

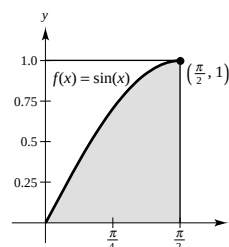
79. $f(x) = \sin x, \left[0, \frac{\pi}{2}\right]$

Let A_1 = area bounded by $f(x) = \sin x$, the x -axis, $x = 0$ and $x = \pi/2$. Let A_2 = area of the rectangle bounded by $y = 1$, $y = 0$, $x = 0$, and $x = \pi/2$. Thus, $A_2 = (\pi/2)(1) \approx 1.570796$.

In this program, the computer is generating N_2 pairs of random points in the rectangle whose area is represented by A_2 . It is keeping track of how many of these points, N_1 , lie in the region whose area is represented by A_1 . Since the points are randomly generated, we assume that

$$\frac{A_1}{A_2} \approx \frac{N_1}{N_2} \Rightarrow A_1 \approx \frac{N_1}{N_2} A_2.$$

The larger N_2 is the better the approximation to A_1 .



81. Suppose there are n rows in the figure. The stars on the left total $1 + 2 + \cdots + n$, as do the stars on the right. There are $n(n + 1)$ stars in total, hence

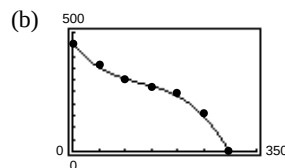
$$2[1 + 2 + \cdots + n] = n(n + 1)$$

$$1 + 2 + \cdots + n = \frac{1}{2}n(n + 1).$$

83. (a) $y = (-4.09 \times 10^{-5})x^3 + 0.016x^2 - 2.67x + 452.9$

(c) Using the integration capability of a graphing utility, you obtain

$$A \approx 76,897.5 \text{ ft}^2.$$



Section 4.3 Riemann Sums and Definite Integrals

1. $f(x) = \sqrt{x}, y = 0, x = 0, x = 3, c_i = \frac{3i^2}{n^2}$

$$\Delta x_i = \frac{3i^2}{n^2} - \frac{3(i-1)^2}{n^2} = \frac{3}{n^2}(2i-1)$$

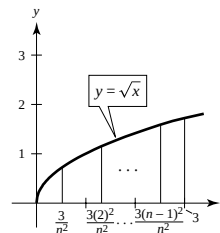
$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{\frac{3i^2}{n^2}} \frac{3}{n^2} (2i-1)$$

$$= \lim_{n \rightarrow \infty} \frac{3\sqrt{3}}{n^3} \sum_{i=1}^n (2i^2 - i)$$

$$= \lim_{n \rightarrow \infty} \frac{3\sqrt{3}}{n^3} \left[2 \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \right]$$

$$= \lim_{n \rightarrow \infty} 3\sqrt{3} \left[\frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{2n^2} \right]$$

$$= 3\sqrt{3} \left[\frac{2}{3} - 0 \right] = 2\sqrt{3} \approx 3.464$$



3. $y = 6$ on $[4, 10]$. $\left(\text{Note: } \Delta x = \frac{10 - 4}{n} = \frac{6}{n}, \|\Delta\| \rightarrow 0 \text{ as } n \rightarrow \infty\right)$

$$\sum_{i=1}^n f(c_i) \Delta x_i = \sum_{i=1}^n f\left(4 + \frac{6i}{n}\right) \left(\frac{6}{n}\right) = \sum_{i=1}^n 6 \left(\frac{6}{n}\right) = \sum_{i=1}^n \frac{36}{n} = 36$$

$$\int_4^{10} 6 \, dx = \lim_{n \rightarrow \infty} 36 = 36$$

5. $y = x^3$ on $[-1, 1]$. $\left(\text{Note: } \Delta x = \frac{1 - (-1)}{n} = \frac{2}{n}, \|\Delta\| \rightarrow 0 \text{ as } n \rightarrow \infty\right)$

$$\sum_{i=1}^n f(c_i) \Delta x_i = \sum_{i=1}^n f\left(-1 + \frac{2i}{n}\right) \left(\frac{2}{n}\right) = \sum_{i=1}^n \left(-1 + \frac{2i}{n}\right)^3 \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right] \left(\frac{2}{n}\right)$$

$$= -2 + \frac{12}{n^2} \sum_{i=1}^n i - \frac{24}{n^3} \sum_{i=1}^n i^2 + \frac{16}{n^4} \sum_{i=1}^n i^3$$

$$= -2 + 6\left(1 + \frac{1}{n}\right) - 4\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{2}{n}$$

$$\int_{-1}^1 x^3 \, dx = \lim_{n \rightarrow \infty} \frac{2}{n} = 0$$

7. $y = x^2 + 1$ on $[1, 2]$. $\left(\text{Note: } \Delta x = \frac{2 - 1}{n} = \frac{1}{n}, \|\Delta\| \rightarrow 0 \text{ as } n \rightarrow \infty\right)$

$$\sum_{i=1}^n f(c_i) \Delta x_i = \sum_{i=1}^n f\left(1 + \frac{i}{n}\right) \left(\frac{1}{n}\right) = \sum_{i=1}^n \left[\left(1 + \frac{i}{n}\right)^2 + 1\right] \left(\frac{1}{n}\right) = \sum_{i=1}^n \left[1 + \frac{2i}{n} + \frac{i^2}{n^2} + 1\right] \left(\frac{1}{n}\right)$$

$$= 2 + \frac{2}{n^2} \sum_{i=1}^n i + \frac{1}{n^3} \sum_{i=1}^n i^2 = 2 + \left(1 + \frac{1}{n}\right) + \frac{1}{6} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}$$

$$\int_1^2 (x^2 + 1) \, dx = \lim_{n \rightarrow \infty} \left(\frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}\right) = \frac{10}{3}$$

9. $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (3c_i + 10) \Delta x_i = \int_{-1}^5 (3x + 10) \, dx$

on the interval $[-1, 5]$.

11. $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \sqrt{c_i^2 + 4} \Delta x_i = \int_0^3 \sqrt{x^2 + 4} \, dx$

on the interval $[0, 3]$.

13. $\int_0^5 3 \, dx$

15. $\int_{-4}^4 (4 - |x|) \, dx$

17. $\int_{-2}^2 (4 - x^2) \, dx$

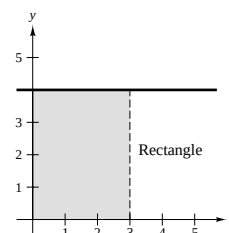
19. $\int_0^\pi \sin x \, dx$

21. $\int_0^2 y^3 \, dy$

23. Rectangle

$$A = bh = 3(4)$$

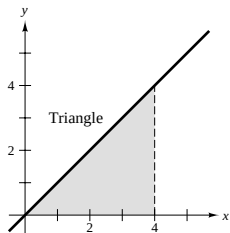
$$A = \int_0^3 4 \, dx = 12$$



25. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(4)(4)$$

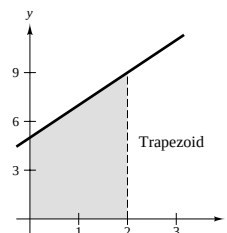
$$A = \int_0^4 x \, dx = 8$$



27. Trapezoid

$$A = \frac{b_1 + b_2}{2}h = \left(\frac{5 + 9}{2}\right)2$$

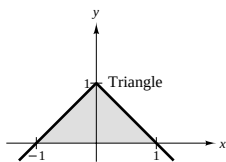
$$A = \int_0^2 (2x + 5) \, dx = 14$$



29. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(2)(1)$$

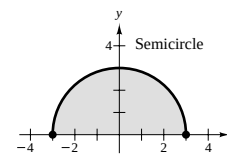
$$A = \int_{-1}^1 (1 - |x|) \, dx = 1$$



31. Semicircle

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2$$

$$A = \int_{-3}^3 \sqrt{9 - x^2} \, dx = \frac{9\pi}{2}$$



In Exercises 33–39, $\int_2^4 x^3 \, dx = 60$, $\int_2^4 x \, dx = 6$, $\int_2^4 dx = 2$

$$33. \int_4^2 x \, dx = -\int_2^4 x \, dx = -6$$

$$37. \int_2^4 (x - 8) \, dx = \int_2^4 x \, dx - 8 \int_2^4 dx = 6 - 8(2) = -10$$

$$41. (a) \int_0^7 f(x) \, dx = \int_0^5 f(x) \, dx + \int_5^7 f(x) \, dx = 10 + 3 = 13$$

$$(b) \int_5^0 f(x) \, dx = -\int_0^5 f(x) \, dx = -10$$

$$(c) \int_5^5 f(x) \, dx = 0$$

$$(d) \int_0^5 3f(x) \, dx = 3 \int_0^5 f(x) \, dx = 3(10) = 30$$

$$35. \int_2^4 4x \, dx = 4 \int_2^4 x \, dx = 4(6) = 24$$

$$39. \int_2^4 \left(\frac{1}{2}x^3 - 3x + 2\right) dx = \frac{1}{2} \int_2^4 x^3 \, dx - 3 \int_2^4 x \, dx + 2 \int_2^4 dx$$

$$= \frac{1}{2}(60) - 3(6) + 2(2) = 16$$

$$43. (a) \int_2^6 [f(x) + g(x)] \, dx = \int_2^6 f(x) \, dx + \int_2^6 g(x) \, dx$$

$$= 10 + (-2) = 8$$

$$(b) \int_2^6 [g(x) - f(x)] \, dx = \int_2^6 g(x) \, dx - \int_2^6 f(x) \, dx$$

$$= -2 - 10 = -12$$

$$(c) \int_2^6 2g(x) \, dx = 2 \int_2^6 g(x) \, dx = 2(-2) = -4$$

$$(d) \int_2^6 3f(x) \, dx = 3 \int_2^6 f(x) \, dx = 3(10) = 30$$

$$45. (a) \text{ Quarter circle below } x\text{-axis: } -\frac{1}{4}\pi r^2 = -\frac{1}{4}\pi(2)^2 = -\pi$$

$$(b) \text{ Triangle: } \frac{1}{2}bh = \frac{1}{2}(4)(2) = 4$$

$$(c) \text{ Triangle + Semicircle below } x\text{-axis: } -\frac{1}{2}(2)(1) - \frac{1}{2}\pi(2)^2 = -(1 + 2\pi)$$

$$(d) \text{ Sum of parts (b) and (c): } 4 - (1 + 2\pi) = 3 - 2\pi$$

$$(e) \text{ Sum of absolute values of (b) and (c): } 4 + (1 + 2\pi) = 5 + 2\pi$$

$$(f) \text{ Answer to (d) plus } 2(10) = 20: (3 - 2\pi) + 20 = 23 - 2\pi$$

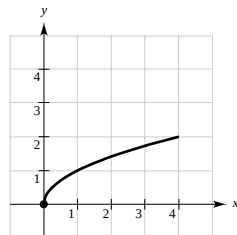
47. The left endpoint approximation will be greater than the actual area: $>$

49. Because the curve is concave upward, the midpoint approximation will be less than the actual area: $<$

51. $f(x) = \frac{1}{x-4}$

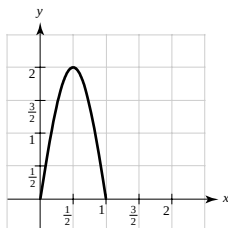
is not integrable on the interval $[3, 5]$ and f has a discontinuity at $x = 4$.

53.



a. $A \approx 5$ square units

55.



d. $\int_0^1 2 \sin \pi x \, dx \approx \frac{1}{2}(1)(2) \approx 1$

57. $\int_0^3 x\sqrt{3-x} \, dx$

n	4	8	12	16	20
$L(n)$	3.6830	3.9956	4.0707	4.1016	4.1177
$M(n)$	4.3082	4.2076	4.1838	4.1740	4.1690
$R(n)$	3.6830	3.9956	4.0707	4.1016	4.1177

59. $\int_0^{\pi/2} \sin^2 x \, dx$

n	4	8	12	16	20
$L(n)$	0.5890	0.6872	0.7199	0.7363	0.7461
$M(n)$	0.7854	0.7854	0.7854	0.7854	0.7854
$R(n)$	0.9817	0.8836	0.8508	0.8345	0.8247

61. True

63. True

65. False

$$\int_0^2 (-x) \, dx = -2$$

67. $f(x) = x^2 + 3x, [0, 8]$

$$x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 7, x_4 = 8$$

$$\Delta x_1 = 1, \Delta x_2 = 2, \Delta x_3 = 4, \Delta x_4 = 1$$

$$c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 8$$

$$\begin{aligned} \sum_{i=1}^4 f(c_i) \Delta x &= f(1) \Delta x_1 + f(2) \Delta x_2 + f(5) \Delta x_3 + f(8) \Delta x_4 \\ &= (4)(1) + (10)(2) + (40)(4) + (88)(1) = 272 \end{aligned}$$

69. $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

is not integrable on the interval $[0, 1]$. As $\|\Delta\| \rightarrow 0$, $f(c_i) = 1$ or $f(c_i) = 0$ in each subinterval since there are an infinite number of both rational and irrational numbers in any interval, no matter how small.

71. Let $f(x) = x^2$, $0 \leq x \leq 1$, and $\Delta x_i = 1/n$. The appropriate Riemann Sum is

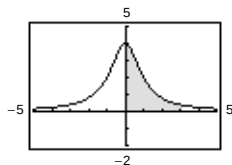
$$\sum_{i=1}^n f(c_i) \Delta x_i = \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^n i^2.$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^3} [1^2 + 2^2 + 3^2 + \cdots + n^2] &= \lim_{n \rightarrow \infty} \frac{1}{n^3} \cdot \frac{n(2n+1)(n+1)}{6} \\ &= \lim_{n \rightarrow \infty} \frac{2n^2 + 3n + 1}{6n^2} = \lim_{n \rightarrow \infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right) = \frac{1}{3} \end{aligned}$$

Section 4.4 The Fundamental Theorem of Calculus

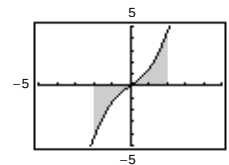
1. $f(x) = \frac{4}{x^2 + 1}$

$\int_0^\pi \frac{4}{x^2 + 1} dx$ is positive.



3. $f(x) = x\sqrt{x^2 + 1}$

$\int_{-2}^2 x\sqrt{x^2 + 1} dx = 0$



5. $\int_0^1 2x dx = \left[x^2 \right]_0^1 = 1 - 0 = 1$

7. $\int_{-1}^0 (x - 2) dx = \left[\frac{x^2}{2} - 2x \right]_{-1}^0 = 0 - \left(\frac{1}{2} + 2 \right) = -\frac{5}{2}$

9. $\int_{-1}^1 (t^2 - 2) dt = \left[\frac{t^3}{3} - 2t \right]_{-1}^1 = \left(\frac{1}{3} - 2 \right) - \left(-\frac{1}{3} + 2 \right) = -\frac{10}{3}$

11. $\int_0^1 (2t - 1)^2 dt = \int_0^1 (4t^2 - 4t + 1) dt = \left[\frac{4}{3}t^3 - 2t^2 + t \right]_0^1 = \frac{4}{3} - 2 + 1 = \frac{1}{3}$

13. $\int_1^2 \left(\frac{3}{x^2} - 1 \right) dx = \left[-\frac{3}{x} - x \right]_1^2 = \left(-\frac{3}{2} - 2 \right) - (-3 - 1) = \frac{1}{2}$

15. $\int_1^4 \frac{u - 2}{\sqrt{u}} du = \int_1^4 (u^{1/2} - 2u^{-1/2}) du = \left[\frac{2}{3}u^{3/2} - 4u^{1/2} \right]_1^4 = \left[\frac{2}{3}(\sqrt{4})^3 - 4\sqrt{4} \right] - \left[\frac{2}{3} - 4 \right] = \frac{2}{3}$

17. $\int_{-1}^1 (\sqrt[3]{t} - 2) dt = \left[\frac{3}{4}t^{4/3} - 2t \right]_{-1}^1 = \left(\frac{3}{4} - 2 \right) - \left(\frac{3}{4} + 2 \right) = -4$

19. $\int_0^1 \frac{x - \sqrt{x}}{3} dx = \frac{1}{3} \int_0^1 (x - x^{1/2}) dx = \frac{1}{3} \left[\frac{x^2}{2} - \frac{2}{3}x^{3/2} \right]_0^1 = \frac{1}{3} \left(\frac{1}{2} - \frac{2}{3} \right) = -\frac{1}{18}$

21. $\int_{-1}^0 (t^{1/3} - t^{2/3}) dt = \left[\frac{3}{4}t^{4/3} - \frac{3}{5}t^{5/3} \right]_{-1}^0 = 0 - \left(\frac{3}{4} + \frac{3}{5} \right) = -\frac{27}{20}$

23. $\begin{aligned} \int_0^3 |2x - 3| dx &= \int_0^{3/2} (3 - 2x) dx + \int_{3/2}^3 (2x - 3) dx \quad \left(\text{split up the integral at the zero } x = \frac{3}{2} \right) \\ &= \left[3x - x^2 \right]_0^{3/2} + \left[x^2 - 3x \right]_{3/2}^3 = \left(\frac{9}{2} - \frac{9}{4} \right) - 0 + (9 - 9) - \left(\frac{9}{4} - \frac{9}{2} \right) = 2 \left(\frac{9}{2} - \frac{9}{4} \right) = \frac{9}{2} \end{aligned}$

$$\begin{aligned}
 25. \int_0^3 |x^2 - 4| dx &= \int_0^2 (4 - x^2) dx + \int_2^3 (x^2 - 4) dx \\
 &= \left[4x - \frac{x^3}{3} \right]_0^2 + \left[\frac{x^3}{3} - 4x \right]_2^3 \\
 &= \left(8 - \frac{8}{3} \right) + (9 - 12) - \left(\frac{8}{3} - 8 \right) \\
 &= \frac{23}{3}
 \end{aligned}$$

$$27. \int_0^\pi (1 + \sin x) dx = \left[x - \cos x \right]_0^\pi = (\pi + 1) - (0 - 1) = 2 + \pi$$

$$29. \int_{-\pi/6}^{\pi/6} \sec^2 x dx = \left[\tan x \right]_{-\pi/6}^{\pi/6} = \frac{\sqrt{3}}{3} - \left(-\frac{\sqrt{3}}{3} \right) = \frac{2\sqrt{3}}{3}$$

$$31. \int_{-\pi/3}^{\pi/3} 4 \sec \theta \tan \theta d\theta = \left[4 \sec \theta \right]_{-\pi/3}^{\pi/3} = 4(2) - 4(2) = 0$$

$$33. \int_0^3 10,000(t - 6) dt = 10,000 \left[\frac{t^2}{2} - 6t \right]_0^3 = -\$135,000$$

$$35. A = \int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}$$

$$37. A = \int_0^3 (3 - x)\sqrt{x} dx = \int_0^3 (3x^{1/2} - x^{3/2}) dx = \left[2x^{3/2} - \frac{2}{5}x^{5/2} \right]_0^3 = \left[\frac{x\sqrt{x}}{5}(10 - 2x) \right]_0^3 = \frac{12\sqrt{3}}{5}$$

$$39. A = \int_0^{\pi/2} \cos x dx = \left[\sin x \right]_0^{\pi/2} = 1$$

41. Since $y \geq 0$ on $[0, 2]$,

$$A = \int_0^2 (3x^2 + 1) dx = \left[x^3 + x \right]_0^2 = 8 + 2 = 10.$$

43. Since $y \geq 0$ on $[0, 2]$,

$$A = \int_0^2 (x^3 + x) dx = \left[\frac{x^4}{4} + \frac{x^2}{2} \right]_0^2 = 4 + 2 = 6.$$

$$45. \int_0^2 (x - 2\sqrt{x}) dx = \left[\frac{x^2}{2} - \frac{4x^{3/2}}{3} \right]_0^2 = 2 - \frac{8\sqrt{2}}{3}$$

$$f(c)(2 - 0) = \frac{6 - 8\sqrt{2}}{3}$$

$$c - 2\sqrt{c} = \frac{3 - 4\sqrt{2}}{3}$$

$$c - 2\sqrt{c} + 1 = \frac{3 - 4\sqrt{2}}{3} + 1$$

$$(\sqrt{c} - 1)^2 = \frac{6 - 4\sqrt{2}}{3}$$

$$\sqrt{c} - 1 = \pm \sqrt{\frac{6 - 4\sqrt{2}}{3}}$$

$$c = \left[1 \pm \sqrt{\frac{6 - 4\sqrt{2}}{3}} \right]^2$$

$$c \approx 0.4380 \text{ or } c \approx 1.7908$$

$$47. \int_{-\pi/4}^{\pi/4} 2 \sec^2 x \, dx = \left[2 \tan x \right]_{-\pi/4}^{\pi/4} = 2(1) - 2(-1) = 4$$

$$f(c) \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right] = 4$$

$$2 \sec^2 c = \frac{8}{\pi}$$

$$\sec^2 c = \frac{4}{\pi}$$

$$\sec c = \pm \frac{2}{\sqrt{\pi}}$$

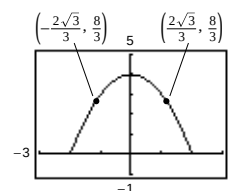
$$c = \pm \operatorname{arcsec} \left(\frac{2}{\sqrt{\pi}} \right)$$

$$= \pm \arccos \frac{\sqrt{\pi}}{2} \approx \pm 0.4817$$

$$49. \frac{1}{2 - (-2)} \int_{-2}^2 (4 - x^2) \, dx = \frac{1}{4} \left[4x - \frac{1}{3}x^3 \right]_{-2}^2 = \frac{1}{4} \left[\left(8 - \frac{8}{3} \right) - \left(-8 + \frac{8}{3} \right) \right] = \frac{8}{3}$$

$$\text{Average value} = \frac{8}{3}$$

$$4 - x^2 = \frac{8}{3} \text{ when } x^2 = 4 - \frac{8}{3} \text{ or } x = \pm \frac{2\sqrt{3}}{3} \approx \pm 1.155.$$

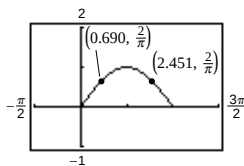


$$51. \frac{1}{\pi - 0} \int_0^{\pi} \sin x \, dx = \left[-\frac{1}{\pi} \cos x \right]_0^{\pi} = \frac{2}{\pi}$$

$$\text{Average value} = \frac{2}{\pi}$$

$$\sin x = \frac{2}{\pi}$$

$$x \approx 0.690, 2.451$$



53. If f is continuous on $[a, b]$ and $F'(x) = f(x)$ on $[a, b]$,

$$\text{then } \int_a^b f(x) \, dx = F(b) - F(a).$$

$$55. \int_0^2 f(x) \, dx = -(\text{area of region A}) = -1.5$$

$$57. \int_0^6 |f(x)| \, dx = -\int_0^2 f(x) \, dx + \int_2^6 f(x) \, dx = 1.5 + 5.0 = 6.5$$

$$59. \int_0^6 [2 + f(x)] \, dx = \int_0^6 2 \, dx + \int_0^6 f(x) \, dx \\ = 12 + 3.5 = 15.5$$

$$61. (a) F(x) = k \sec^2 x$$

$$F(0) = k = 500$$

$$F(x) = 500 \sec^2 x$$

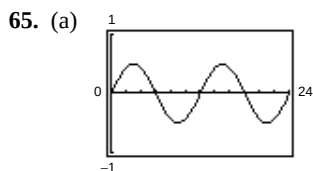
$$(b) \frac{1}{\pi/3 - 0} \int_0^{\pi/3} 500 \sec^2 x \, dx = \frac{1500}{\pi} \left[\tan x \right]_0^{\pi/3}$$

$$= \frac{1500}{\pi} (\sqrt{3} - 0)$$

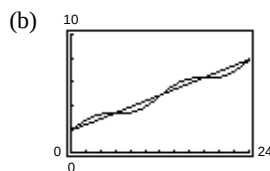
$$\approx 826.99 \text{ newtons}$$

$$\approx 827 \text{ newtons}$$

$$63. \frac{1}{5 - 0} \int_0^5 (0.1729t + 0.1552t^2 - 0.0374t^3) \, dt \approx \frac{1}{5} \left[0.08645t^2 + 0.05073t^3 - 0.00935t^4 \right]_0^5 \approx 0.5318 \text{ liter}$$

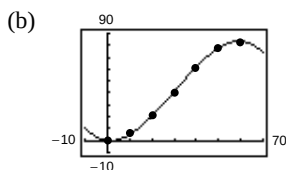


The area above the x -axis equals the area below the x -axis. Thus, the average value is zero.



The average value of S appears to be g .

67. (a) $v = -8.61 \times 10^{-4}t^3 + 0.0782t^2 - 0.208t + 0.0952$



(c) $\int_0^{60} v(t) dt = \left[\frac{-8.61 \times 10^{-4}t^4}{4} + \frac{0.0782t^3}{3} - \frac{0.208t^2}{2} + 0.0952t \right]_0^{60} \approx 2476 \text{ meters}$

69. $F(x) = \int_0^x (t - 5) dt = \left[\frac{t^2}{2} - 5t \right]_0^x = \frac{x^2}{2} - 5x$

$$F(2) = \frac{4}{2} - 5(2) = -8$$

$$F(5) = \frac{25}{2} - 5(5) = -\frac{25}{2}$$

$$F(8) = \frac{64}{2} - 5(8) = -8$$

71. $F(x) = \int_1^x \frac{10}{v^2} dv = \int_1^x 10v^{-2} dv = \left[\frac{-10}{v} \right]_1^x$

$$= -\frac{10}{x} + 10 = 10 \left(1 - \frac{1}{x} \right)$$

$$F(2) = 10 \left(\frac{1}{2} \right) = 5$$

$$F(5) = 10 \left(\frac{4}{5} \right) = 8$$

$$F(8) = 10 \left(\frac{7}{8} \right) = \frac{35}{4}$$

73. $F(x) = \int_1^x \cos \theta d\theta = \left[\sin \theta \right]_1^x = \sin x - \sin 1$

$$F(2) = \sin 2 - \sin 1 = 0.0678$$

$$F(5) = \sin 5 - \sin 1 \approx -1.8004$$

$$F(8) = \sin 8 - \sin 1 \approx 0.1479$$

75. (a) $\int_0^x (t + 2) dt = \left[\frac{t^2}{2} + 2t \right]_0^x = \frac{1}{2}x^2 + 2x$

(b) $\frac{d}{dx} \left[\frac{1}{2}x^2 + 2x \right] = x + 2$

77. (a) $\int_8^x \sqrt[3]{t} dt = \left[\frac{3}{4}t^{4/3} \right]_8^x = \frac{3}{4}(x^{4/3} - 16) = \frac{3}{4}x^{4/3} - 12$

(b) $\frac{d}{dx} \left[\frac{3}{4}x^{4/3} - 12 \right] = x^{1/3} = \sqrt[3]{x}$

79. (a) $\int_{x/4}^x \sec^2 t dt = \left[\tan t \right]_{x/4}^x = \tan x - 1$

(b) $\frac{d}{dx} [\tan x - 1] = \sec^2 x$

81. $F(x) = \int_{-2}^x (t^2 - 2t) dt$

$$F'(x) = x^2 - 2x$$

83. $F(x) = \int_{-1}^x \sqrt{t^4 + 1} dt$

$$F'(x) = \sqrt{x^4 + 1}$$

85. $F(x) = \int_0^x t \cos t dt$

$$F'(x) = x \cos x$$

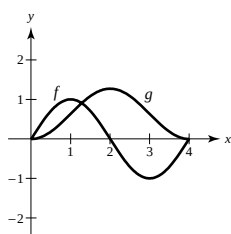
$$\begin{aligned}
 87. \quad F(x) &= \int_x^{x+2} (4t + 1) dt \\
 &= \left[2t^2 + t \right]_x^{x+2} \\
 &= [2(x+2)^2 + (x+2)] - [2x^2 + x] \\
 &= 8x + 10 \\
 F'(x) &= 8
 \end{aligned}$$

$$\begin{aligned}
 89. \quad F(x) &= \int_0^{\sin x} \sqrt{t} dt = \left[\frac{2}{3} t^{3/2} \right]_0^{\sin x} = \frac{2}{3} (\sin x)^{3/2} \\
 F'(x) &= (\sin x)^{1/2} \cos x = \cos x \sqrt{\sin x}
 \end{aligned}$$

Alternate solution

$$\begin{aligned}
 F(x) &= \int_0^{\sin x} \sqrt{t} dt \\
 F'(x) &= \sqrt{\sin x} \frac{d}{dx}(\sin x) = \sqrt{\sin x}(\cos x)
 \end{aligned}$$

$$\begin{aligned}
 93. \quad g(x) &= \int_0^x f(t) dt \\
 g(0) &= 0, g(1) \approx \frac{1}{2}, g(2) \approx 1, g(3) \approx \frac{1}{2}, g(4) = 0
 \end{aligned}$$



g has a relative maximum at $x = 2$.

97. True

$$101. \quad f(x) = \int_0^{1/x} \frac{1}{t^2 + 1} dt + \int_0^x \frac{1}{t^2 + 1} dt$$

By the Second Fundamental Theorem of Calculus, we have

$$\begin{aligned}
 f'(x) &= \frac{1}{(1/x)^2 + 1} \left(-\frac{1}{x^2} \right) + \frac{1}{x^2 + 1} \\
 &= -\frac{1}{1 + x^2} + \frac{1}{x^2 + 1} = 0.
 \end{aligned}$$

Since $f'(x) = 0$, $f(x)$ must be constant.

Alternate solution:

$$\begin{aligned}
 F(x) &= \int_x^{x+2} (4t + 1) dt \\
 &= \int_x^0 (4t + 1) dt + \int_0^{x+2} (4t + 1) dt \\
 &= -\int_0^x (4t + 1) dt + \int_0^{x+2} (4t + 1) dt \\
 F'(x) &= -(4x + 1) + 4(x + 2) + 1 = 8
 \end{aligned}$$

$$\begin{aligned}
 91. \quad F(x) &= \int_0^{x^3} \sin t^2 dt \\
 F'(x) &= \sin(x^3)^2 \cdot 3x^2 = 3x^2 \sin x^6
 \end{aligned}$$

$$\begin{aligned}
 95. \quad (a) \quad C(x) &= 5000 \left(25 + 3 \int_0^x t^{1/4} dt \right) \\
 &= 5000 \left(25 + 3 \left[\frac{4}{5} t^{5/4} \right]_0^x \right) \\
 &= 5000 \left(25 + \frac{12}{5} x^{5/4} \right) = 1000(125 + 12x^{5/4})
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad C(1) &= 1000(125 + 12(1)) = \$137,000 \\
 C(5) &= 1000(125 + 12(5)^{5/4}) \approx \$214,721 \\
 C(10) &= 1000(125 + 12(10)^{5/4}) \approx \$338,394
 \end{aligned}$$

$$99. \quad \text{False; } \int_{-1}^1 x^{-2} dx = \int_{-1}^0 x^{-2} dx + \int_0^1 x^{-2} dx$$

Each of these integrals is infinite. $f(x) = x^{-2}$ has a nonremovable discontinuity at $x = 0$.

103. $x(t) = t^3 - 6t^2 + 9t - 2$

$$x'(t) = 3t^2 - 12t + 9$$

$$= 3(t^2 - 4t + 3)$$

$$= 3(t - 3)(t - 1)$$

$$\begin{aligned} \text{Total distance} &= \int_0^5 |x'(t)| dt \\ &= \int_0^5 3|(t - 3)(t - 1)| dt \\ &= 3 \int_0^1 (t^2 - 4t + 3) dt - 3 \int_1^3 (t^2 - 4t + 3) dt + 3 \int_3^5 (t^2 - 4t + 3) dt \\ &= 4 + 4 + 20 \\ &= 28 \text{ units} \end{aligned}$$

105. $\text{Total distance} = \int_1^4 |x'(t)| dt$

$$\begin{aligned} &= \int_1^4 |v(t)| dt \\ &= \int_1^4 \frac{1}{\sqrt{t}} dt \\ &= \left[2t^{1/2} \right]_1^4 \\ &= 2(2 - 1) = 2 \text{ units} \end{aligned}$$

Section 4.5 Integration by Substitution

$\int f(g(x))g'(x) dx$	$u = g(x)$	$du = g'(x) dx$
1. $\int (5x^2 + 1)^2(10x) dx$	$5x^2 + 1$	$10x dx$
3. $\int \frac{x}{\sqrt{x^2 + 1}} dx$	$x^2 + 1$	$2x dx$
5. $\int \tan^2 x \sec^2 x dx$	$\tan x$	$\sec^2 x dx$
7. $\int (1 + 2x)^4 2 dx = \frac{(1 + 2x)^5}{5} + C$		
Check: $\frac{d}{dx} \left[\frac{(1 + 2x)^5}{5} + C \right] = 2(1 + 2x)^4$		
9. $\int (9 - x^2)^{1/2}(-2x) dx = \frac{(9 - x^2)^{3/2}}{3/2} + C = \frac{2}{3}(9 - x^2)^{3/2} + C$		
Check: $\frac{d}{dx} \left[\frac{2}{3}(9 - x^2)^{3/2} + C \right] = \frac{2}{3} \cdot \frac{3}{2}(9 - x^2)^{1/2}(-2x) = \sqrt{9 - x^2}(-2x)$		

$$11. \int x^3(x^4 + 3)^2 dx = \frac{1}{4} \int (x^4 + 3)^2(4x^3) dx = \frac{1}{4} \frac{(x^4 + 3)^3}{3} + C = \frac{(x^4 + 3)^3}{12} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(x^4 + 3)^3}{12} + C \right] = \frac{3(x^4 + 3)^2}{12} (4x^3) = (x^4 + 3)^2(x^3)$$

$$13. \int x^2(x^3 - 1)^4 dx = \frac{1}{3} \int (x^3 - 1)^4(3x^2) dx = \frac{1}{3} \left[\frac{(x^3 - 1)^5}{5} \right] + C = \frac{(x^3 - 1)^5}{15} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(x^3 - 1)^5}{15} + C \right] = \frac{5(x^3 - 1)^4(3x^2)}{15} = x^2(x^3 - 1)^4$$

$$15. \int t\sqrt{t^2 + 2} dt = \frac{1}{2} \int (t^2 + 2)^{1/2}(2t) dt = \frac{1}{2} \frac{(t^2 + 2)^{3/2}}{3/2} + C = \frac{(t^2 + 2)^{3/2}}{3} + C$$

$$\text{Check: } \frac{d}{dt} \left[\frac{(t^2 + 2)^{3/2}}{3} + C \right] = \frac{3/2(t^2 + 2)^{1/2}(2t)}{3} = (t^2 + 2)^{1/2}t$$

$$17. \int 5x(1 - x^2)^{1/3} dx = -\frac{5}{2} \int (1 - x^2)^{1/3}(-2x) dx = -\frac{5}{2} \cdot \frac{(1 - x^2)^{4/3}}{4/3} + C = -\frac{15}{8}(1 - x^2)^{4/3} + C$$

$$\text{Check: } \frac{d}{dx} \left[-\frac{15}{8}(1 - x^2)^{4/3} + C \right] = -\frac{15}{8} \cdot \frac{4}{3}(1 - x^2)^{1/3}(-2x) = 5x(1 - x^2)^{1/3} = 5x\sqrt[3]{1 - x^2}$$

$$19. \int \frac{x}{(1 - x^2)^3} dx = -\frac{1}{2} \int (1 - x^2)^{-3}(-2x) dx = -\frac{1}{2} \frac{(1 - x^2)^{-2}}{-2} + C = \frac{1}{4(1 - x^2)^2} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{1}{4(1 - x^2)^2} + C \right] = \frac{1}{4}(-2)(1 - x^2)^{-3}(-2x) = \frac{x}{(1 - x^2)^3}$$

$$21. \int \frac{x^2}{(1 + x^3)^2} dx = \frac{1}{3} \int (1 + x^3)^{-2}(3x^2) dx = \frac{1}{3} \left[\frac{(1 + x^3)^{-1}}{-1} \right] + C = -\frac{1}{3(1 + x^3)} + C$$

$$\text{Check: } \frac{d}{dx} \left[-\frac{1}{3(1 + x^3)} + C \right] = -\frac{1}{3}(-1)(1 + x^3)^{-2}(3x^2) = \frac{x^2}{(1 + x^3)^2}$$

$$23. \int \frac{x}{\sqrt{1 - x^2}} dx = -\frac{1}{2} \int (1 - x^2)^{-1/2}(-2x) dx = -\frac{1}{2} \frac{(1 - x^2)^{1/2}}{1/2} + C = -\sqrt{1 - x^2} + C$$

$$\text{Check: } \frac{d}{dx} [-\sqrt{1 - x^2} + C] = -\frac{1}{2}(1 - x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{1 - x^2}}$$

$$25. \int \left(1 + \frac{1}{t}\right)^3 \left(\frac{1}{t^2}\right) dt = - \int \left(1 + \frac{1}{t}\right)^3 \left(-\frac{1}{t^2}\right) dt = -\frac{[1 + (1/t)]^4}{4} + C$$

$$\text{Check: } \frac{d}{dt} \left[-\frac{[1 + (1/t)]^4}{4} + C \right] = -\frac{1}{4}(4)\left(1 + \frac{1}{t}\right)^3 \left(-\frac{1}{t^2}\right) = \frac{1}{t^2} \left(1 + \frac{1}{t}\right)^3$$

$$27. \int \frac{1}{\sqrt{2x}} dx = \frac{1}{2} \int (2x)^{-1/2} 2 dx = \frac{1}{2} \left[\frac{(2x)^{1/2}}{1/2} \right] + C = \sqrt{2x} + C$$

$$\text{Check: } \frac{d}{dx} [\sqrt{2x} + C] = \frac{1}{2}(2x)^{-1/2}(2) = \frac{1}{\sqrt{2x}}$$

$$29. \int \frac{x^2 + 3x + 7}{\sqrt{x}} dx = \int (x^{3/2} + 3x^{1/2} + 7x^{-1/2}) dx = \frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C = \frac{2}{5}\sqrt{x}(x^2 + 5x + 35) + C$$

Check: $\frac{d}{dx} \left[\frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C \right] = \frac{x^2 + 3x + 7}{\sqrt{x}}$

$$31. \int t^2 \left(t - \frac{2}{t} \right) dt = \int (t^3 - 2t) dt = \frac{1}{4}t^4 - t^2 + C$$

Check: $\frac{d}{dt} \left[\frac{1}{4}t^4 - t^2 + C \right] = t^3 - 2t = t^2 \left(t - \frac{2}{t} \right)$

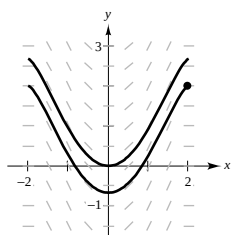
$$33. \int (9 - y)\sqrt{y} dy = \int (9y^{1/2} - y^{3/2}) dy = 9 \left(\frac{2}{3}y^{3/2} \right) - \frac{2}{5}y^{5/2} + C = \frac{2}{5}y^{3/2}(15 - y) + C$$

Check: $\frac{d}{dy} \left[\frac{2}{5}y^{3/2}(15 - y) + C \right] = \frac{d}{dy} \left[6y^{3/2} - \frac{2}{5}y^{5/2} + C \right] = 9y^{1/2} - y^{3/2} = (9 - y)\sqrt{y}$

$$\begin{aligned} 35. y &= \int \left[4x + \frac{4x}{\sqrt{16 - x^2}} \right] dx \\ &= 4 \int x dx - 2 \int (16 - x^2)^{-1/2} (-2x) dx \\ &= 4 \left(\frac{x^2}{2} \right) - 2 \left[\frac{(16 - x^2)^{1/2}}{1/2} \right] + C \\ &= 2x^2 - 4\sqrt{16 - x^2} + C \end{aligned}$$

$$\begin{aligned} 37. y &= \int \frac{x + 1}{(x^2 + 2x - 3)^2} dx \\ &= \frac{1}{2} \int (x^2 + 2x - 3)^{-2} (2x + 2) dx \\ &= \frac{1}{2} \left[\frac{(x^2 + 2x - 3)^{-1}}{-1} \right] + C \\ &= -\frac{1}{2(x^2 + 2x - 3)} + C \end{aligned}$$

39. (a)

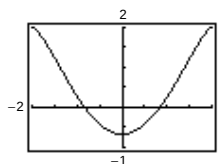


(b) $\frac{dy}{dx} = x\sqrt{4 - x^2}, (2, 2)$

$$\begin{aligned} y &= \int x\sqrt{4 - x^2} dx = -\frac{1}{2} \int (4 - x^2)^{1/2} (-2x dx) \\ &= -\frac{1}{2} \cdot \frac{2}{3} (4 - x^2)^{3/2} + C = -\frac{1}{3} (4 - x^2)^{3/2} + C \end{aligned}$$

$$(2, 2): 2 = -\frac{1}{3} (4 - 2^2)^{3/2} + C \Rightarrow C = 2$$

$$y = -\frac{1}{3} (4 - x^2)^{3/2} + 2$$



$$41. \int \pi \sin \pi x dx = -\cos \pi x + C$$

$$43. \int \sin 2x dx = \frac{1}{2} \int (\sin 2x)(2x) dx = -\frac{1}{2} \cos 2x + C$$

$$45. \int \frac{1}{\theta^2} \cos \frac{1}{\theta} d\theta = -\int \cos \frac{1}{\theta} \left(-\frac{1}{\theta^2} \right) d\theta = -\sin \frac{1}{\theta} + C$$

$$47. \int \sin 2x \cos 2x \, dx = \frac{1}{2} \int (\sin 2x)(2 \cos 2x) \, dx = \frac{1}{2} \frac{(\sin 2x)^2}{2} + C = \frac{1}{4} \sin^2 2x + C \quad \text{OR}$$

$$\int \sin 2x \cos 2x \, dx = -\frac{1}{2} \int (\cos 2x)(-2 \sin 2x) \, dx = -\frac{1}{2} \frac{(\cos 2x)^2}{2} + C_1 = -\frac{1}{4} \cos^2 2x + C_1 \quad \text{OR}$$

$$\int \sin 2x \cos 2x \, dx = \frac{1}{2} \int 2 \sin 2x \cos 2x \, dx = \frac{1}{2} \int \sin 4x \, dx = \frac{1}{8} \cos 4x + C_2$$

$$49. \int \tan^4 x \sec^2 x \, dx = \frac{\tan^5 x}{5} + C = \frac{1}{5} \tan^5 x + C$$

$$\begin{aligned} 51. \int \frac{\csc^2 x}{\cot^3 x} \, dx &= - \int (\cot x)^{-3} (-\csc^2 x) \, dx \\ &= -\frac{(\cot x)^{-2}}{-2} + C = \frac{1}{2 \cot^2 x} + C = \frac{1}{2} \tan^2 x + C = \frac{1}{2} (\sec^2 x - 1) + C = \frac{1}{2} \sec^2 x + C_1 \end{aligned}$$

$$53. \int \cot^2 x \, dx = \int (\csc^2 x - 1) \, dx = -\cot x - x + C$$

$$55. f(x) = \int \cos \frac{x}{2} \, dx = 2 \sin \frac{x}{2} + C$$

Since $f(0) = 3 = 2 \sin 0 + C$, $C = 3$. Thus,

$$f(x) = 2 \sin \frac{x}{2} + 3.$$

$$57. u = x + 2, x = u - 2, dx = du$$

$$\begin{aligned} \int x \sqrt{x+2} \, dx &= \int (u-2) \sqrt{u} \, du \\ &= \int (u^{3/2} - 2u^{1/2}) \, du \\ &= \frac{2}{5} u^{5/2} - \frac{4}{3} u^{3/2} + C \\ &= \frac{2u^{3/2}}{15} (3u - 10) + C \\ &= \frac{2}{15} (x+2)^{3/2} [3(x+2) - 10] + C \\ &= \frac{2}{15} (x+2)^{3/2} (3x-4) + C \end{aligned}$$

$$59. u = 1 - x, x = 1 - u, dx = -du$$

$$\begin{aligned} \int x^2 \sqrt{1-x} \, dx &= - \int (1-u)^2 \sqrt{u} \, du \\ &= - \int (u^{1/2} - 2u^{3/2} + u^{5/2}) \, du \\ &= - \left(\frac{2}{3} u^{3/2} - \frac{4}{5} u^{5/2} + \frac{2}{7} u^{7/2} \right) + C \\ &= -\frac{2u^{3/2}}{105} (35 - 42u + 15u^2) + C \\ &= -\frac{2}{105} (1-x)^{3/2} [35 - 42(1-x) + 15(1-x)^2] + C \\ &= -\frac{2}{105} (1-x)^{3/2} (15x^2 + 12x + 8) + C \end{aligned}$$

61. $u = 2x - 1, x = \frac{1}{2}(u + 1), dx = \frac{1}{2} du$

$$\begin{aligned}
 \int \frac{x^2 - 1}{\sqrt{2x - 1}} dx &= \int \frac{[(1/2)(u + 1)]^2 - 1}{\sqrt{u}} \frac{1}{2} du \\
 &= \frac{1}{8} \int u^{-1/2} [(u^2 + 2u + 1) - 4] du \\
 &= \frac{1}{8} \int (u^{3/2} + 2u^{1/2} - 3u^{-1/2}) du \\
 &= \frac{1}{8} \left(\frac{2}{5} u^{5/2} + \frac{4}{3} u^{3/2} - 6u^{1/2} \right) + C \\
 &= \frac{u^{1/2}}{60} (3u^2 + 10u - 45) + C \\
 &= \frac{\sqrt{2x - 1}}{60} [3(2x - 1)^2 + 10(2x - 1) - 45] + C \\
 &= \frac{1}{60} \sqrt{2x - 1} (12x^2 + 8x - 52) + C \\
 &= \frac{1}{15} \sqrt{2x - 1} (3x^2 + 2x - 13) + C
 \end{aligned}$$

63. $u = x + 1, x = u - 1, dx = du$

$$\begin{aligned}
 \int \frac{-x}{(x + 1) - \sqrt{x + 1}} dx &= \int \frac{-(u - 1)}{u - \sqrt{u}} du \\
 &= - \int \frac{(\sqrt{u} + 1)(\sqrt{u} - 1)}{\sqrt{u}(\sqrt{u} - 1)} du \\
 &= - \int (1 + u^{-1/2}) du \\
 &= -(u + 2u^{1/2}) + C \\
 &= -u - 2\sqrt{u} + C \\
 &= -(x + 1) - 2\sqrt{x + 1} + C \\
 &= -x - 2\sqrt{x + 1} - 1 + C \\
 &= -(x + 2\sqrt{x + 1}) + C_1
 \end{aligned}$$

where $C_1 = -1 + C$.

65. Let $u = x^2 + 1, du = 2x dx$.

$$\int_{-1}^1 x(x^2 + 1)^3 dx = \frac{1}{2} \int_{-1}^1 (x^2 + 1)^3 (2x) dx = \left[\frac{1}{8} (x^2 + 1)^4 \right]_{-1}^1 = 0$$

67. Let $u = x^3 + 1, du = 3x^2 dx$

$$\begin{aligned}
 \int_1^2 2x^2 \sqrt{x^3 + 1} dx &= 2 \cdot \frac{1}{3} \int_1^2 (x^3 + 1)^{1/2} (3x^2) dx \\
 &= \left[\frac{2}{3} \frac{(x^3 + 1)^{3/2}}{3/2} \right]_1^2 \\
 &= \frac{4}{9} \left[(x^3 + 1)^{3/2} \right]_1^2 \\
 &= \frac{4}{9} [27 - 2\sqrt{2}] = 12 - \frac{8}{9}\sqrt{2}
 \end{aligned}$$

69. Let
- $u = 2x + 1$
- ,
- $du = 2 dx$
- .

$$\int_0^4 \frac{1}{\sqrt{2x+1}} dx = \frac{1}{2} \int_0^4 (2x+1)^{-1/2} (2) dx = \left[\sqrt{2x+1} \right]_0^4 = \sqrt{9} - \sqrt{1} = 2$$

71. Let
- $u = 1 + \sqrt{x}$
- ,
- $du = \frac{1}{2\sqrt{x}} dx$
- .

$$\int_1^9 \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx = 2 \int_1^9 (1+\sqrt{x})^{-2} \left(\frac{1}{2\sqrt{x}} \right) dx = \left[-\frac{2}{1+\sqrt{x}} \right]_1^9 = -\frac{1}{2} + 1 = \frac{1}{2}$$

- 73.
- $u = 2 - x$
- ,
- $x = 2 - u$
- ,
- $dx = -du$

When $x = 1$, $u = 1$. When $x = 2$, $u = 0$.

$$\int_1^2 (x-1)\sqrt{2-x} dx = \int_1^0 -[(2-u)-1]\sqrt{u} du = \int_1^0 (u^{3/2} - u^{1/2}) du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_1^0 = -\left[\frac{2}{5} - \frac{2}{3} \right] = \frac{4}{15}$$

- 75.
- $\int_0^{\pi/2} \cos\left(\frac{2}{3}x\right) dx = \left[\frac{3}{2} \sin\left(\frac{2}{3}x\right) \right]_0^{\pi/2} = \frac{3}{2} \left(\frac{\sqrt{3}}{2} \right) = \frac{3\sqrt{3}}{4}$

- 77.
- $u = x + 1$
- ,
- $x = u - 1$
- ,
- $dx = du$

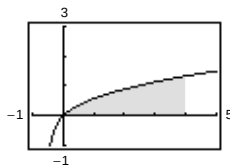
When $x = 0$, $u = 1$. When $x = 7$, $u = 8$.

$$\begin{aligned} \text{Area} &= \int_0^7 x \sqrt[3]{x+1} dx = \int_1^8 (u-1) \sqrt[3]{u} du \\ &= \int_1^8 (u^{4/3} - u^{1/3}) du = \left[\frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3} \right]_1^8 = \left(\frac{384}{7} - 12 \right) - \left(\frac{3}{7} - \frac{3}{4} \right) = \frac{1209}{28} \end{aligned}$$

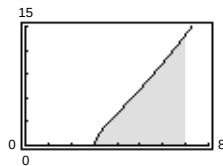
- 79.
- $A = \int_0^\pi (2 \sin x + \sin 2x) dx = -\left[2 \cos x + \frac{1}{2} \cos 2x \right]_0^\pi = 4$

- 81.
- $\text{Area} = \int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right) dx = 2 \int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right) \left(\frac{1}{2}\right) dx = \left[2 \tan\left(\frac{x}{2}\right) \right]_{\pi/2}^{2\pi/3} = 2(\sqrt{3} - 1)$

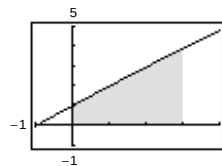
- 83.
- $\int_0^4 \frac{x}{\sqrt{2x+1}} dx \approx 3.333 = \frac{10}{3}$



- 85.
- $\int_3^7 x\sqrt{x-3} dx \approx 28.8 = \frac{144}{5}$



- 87.
- $\int_0^3 \left(\theta + \cos \frac{\theta}{6} \right) d\theta \approx 7.377$



- 89.
- $\int (2x-1)^2 dx = \frac{1}{2} \int (2x-1)^2 (2) dx = \frac{1}{6} (2x-1)^3 + C_1 = \frac{4}{3}x^3 - 2x^2 + x - \frac{1}{6} + C_1$

$$\int (2x-1)^2 dx = \int (4x^2 - 4x + 1) dx = \frac{4}{3}x^3 - 2x^2 + x + C_2$$

They differ by a constant: $C_2 = C_1 - \frac{1}{6}$.

91. $f(x) = x^2(x^2 + 1)$ is even.

$$\begin{aligned}\int_{-2}^2 x^2(x^2 + 1) dx &= 2 \int_0^2 (x^4 + x^2) dx = 2 \left[\frac{x^5}{5} + \frac{x^3}{3} \right]_0^2 \\ &= 2 \left[\frac{32}{5} + \frac{8}{3} \right] = \frac{272}{15}\end{aligned}$$

93. $f(x) = x(x^2 + 1)^3$ is odd.

$$\int_{-2}^2 x(x^2 + 1)^3 dx = 0$$

95. $\int_0^2 x^2 dx = \left[\frac{x^3}{3} \right]_0^2 = \frac{8}{3}$; the function x^2 is an even function.

$$(a) \int_{-2}^0 x^2 dx = \int_0^2 x^2 dx = \frac{8}{3}$$

$$(c) \int_0^2 (-x^2) dx = - \int_0^2 x^2 dx = -\frac{8}{3}$$

$$(b) \int_{-2}^2 x^2 dx = 2 \int_0^2 x^2 dx = \frac{16}{3}$$

$$(d) \int_{-2}^0 3x^2 dx = 3 \int_0^2 x^2 dx = 8$$

$$97. \int_{-4}^4 (x^3 + 6x^2 - 2x - 3) dx = \int_{-4}^4 (x^3 - 2x) dx + \int_{-4}^4 (6x^2 - 3) dx = 0 + 2 \int_0^4 (6x^2 - 3) dx = 2 \left[2x^3 - 3x \right]_0^4 = 232$$

99. Answers will vary. See "Guidelines for Making a Change of Variables" on page 292.

101. $f(x) = x(x^2 + 1)^2$ is odd. Hence, $\int_{-2}^2 x(x^2 + 1)^2 dx = 0$.

$$103. \frac{dV}{dt} = \frac{k}{(t+1)^2}$$

$$V(t) = \int \frac{k}{(t+1)^2} dt = -\frac{k}{t+1} + C$$

$$V(0) = -k + C = 500,000$$

$$V(1) = -\frac{1}{2}k + C = 400,000$$

Solving this system yields $k = -200,000$ and $C = 300,000$. Thus,

$$V(t) = \frac{200,000}{t+1} + 300,000.$$

When $t = 4$, $V(4) = \$340,000$.

$$105. \frac{1}{b-a} \int_a^b \left[74.50 + 43.75 \sin \frac{\pi t}{6} \right] dt = \frac{1}{b-a} \left[74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_a^b$$

$$(a) \frac{1}{3} \left[74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_0^3 = \frac{1}{3} \left(223.5 + \frac{262.5}{\pi} \right) \approx 102.352 \text{ thousand units}$$

$$(b) \frac{1}{3} \left[74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_3^6 = \frac{1}{3} \left(447 + \frac{262.5}{\pi} - 223.5 \right) \approx 102.352 \text{ thousand units}$$

$$(c) \frac{1}{12} \left[74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_0^{12} = \frac{1}{12} \left(894 - \frac{262.5}{\pi} + \frac{262.5}{\pi} \right) = 74.5 \text{ thousand units}$$

107. $\frac{1}{b-a} \int_a^b [2 \sin(60\pi t) + \cos(120\pi t)] dt = \frac{1}{b-a} \left[-\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_a^b$

$$(a) \frac{1}{(1/60) - 0} \left[-\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_0^{1/60} = 60 \left[\left(\frac{1}{30\pi} + 0 \right) - \left(-\frac{1}{30\pi} \right) \right] = \frac{4}{\pi} \approx 1.273 \text{ amps}$$

$$\begin{aligned} \text{(b)} \quad \frac{1}{(1/240) - 0} \left[-\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_0^{1/240} &= 240 \left[\left(-\frac{1}{30\sqrt{2}\pi} + \frac{1}{120\pi} \right) - \left(-\frac{1}{30\pi} \right) \right] \\ &= \frac{2}{\pi} (5 - 2\sqrt{2}) \approx 1.382 \text{ amps} \end{aligned}$$

(c) $\frac{1}{(1/30) - 0} \left[-\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_0^{1/30} = 30 \left[\left(\frac{1}{30\pi} \right) - \left(-\frac{1}{30\pi} \right) \right] = 0 \text{ amps}$

109. False

$$\int (2x + 1)^2 dx = \frac{1}{2} \int (2x + 1)^2 2 dx = \frac{1}{6} (2x + 1)^3 + C$$

111. True

$$\int_{-10}^{10} (ax^3 + bx^2 + cx + d) \, dx = \underbrace{\int_{-10}^{10} (ax^3 + cx) \, dx}_{\text{Odd}} + \underbrace{\int_{-10}^{10} (bx^2 + d) \, dx}_{\text{Even}} = 0 + 2 \int_0^{10} (bx^2 + d) \, dx$$

113. True

$$4 \int \sin x \cos x \, dx = 2 \int \sin 2x \, dx = -\cos 2x + C$$

115. Let $u = x + h$, then $du = dx$. When $x = a$, $u = a + h$. When $x = b$, $u = b + h$. Thus,

$$\int_a^b f(x+h) dx = \int_{a+h}^{b+h} f(u) du = \int_{a+h}^{b+h} f(x) dx.$$

Section 4.6 Numerical Integration

1. Exact: $\int_0^2 x^2 dx = \left[\frac{1}{3}x^3 \right]_0^2 = \frac{8}{3} \approx 2.6667$

Trapezoidal: $\int_0^2 x^2 dx \approx \frac{1}{4} \left[0 + 2\left(\frac{1}{2}\right)^2 + 2(1)^2 + 2\left(\frac{3}{2}\right)^2 + (2)^2 \right] = \frac{11}{4} = 2.7500$

Simpson's: $\int_0^2 x^2 dx \approx \frac{1}{6} \left[0 + 4\left(\frac{1}{2}\right)^2 + 2(1)^2 + 4\left(\frac{3}{2}\right)^2 + (2)^2 \right] = \frac{8}{3} \approx 2.6667$

3. Exact: $\int_0^2 x^3 dx = \left[\frac{x^4}{4} \right]_0^2 = 4.000$

Trapezoidal: $\int_0^2 x^3 dx \approx \frac{1}{4} \left[0 + 2\left(\frac{1}{2}\right)^3 + 2(1)^3 + 2\left(\frac{3}{2}\right)^3 + (2)^3 \right] = \frac{17}{4} = 4.2500$

Simpson's: $\int_0^2 x^3 dx \approx \frac{1}{6} \left[0 + 4\left(\frac{1}{2}\right)^3 + 2(1)^3 + 4\left(\frac{3}{2}\right)^3 + (2)^3 \right] = \frac{24}{6} = 4.0000$

5. Exact: $\int_0^2 x^3 dx = \left[\frac{1}{4}x^4 \right]_0^2 = 4.0000$

Trapezoidal: $\int_0^2 x^3 dx \approx \frac{1}{8} \left[0 + 2\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 2\left(\frac{3}{4}\right)^3 + 2(1)^3 + 2\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 2\left(\frac{7}{4}\right)^3 + 8 \right] = 4.0625$

Simpson's: $\int_0^2 x^3 dx \approx \frac{1}{12} \left[0 + 4\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 4\left(\frac{3}{4}\right)^3 + 2(1)^3 + 4\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 4\left(\frac{7}{4}\right)^3 + 8 \right] = 4.0000$

7. Exact: $\int_4^9 \sqrt{x} dx = \left[\frac{2}{3}x^{3/2} \right]_4^9 = 18 - \frac{16}{3} = \frac{38}{3} \approx 12.6667$

Trapezoidal: $\int_4^9 \sqrt{x} dx \approx \frac{5}{16} \left[2 + 2\sqrt{\frac{37}{8}} + 2\sqrt{\frac{21}{4}} + 2\sqrt{\frac{47}{8}} + 2\sqrt{\frac{26}{4}} + 2\sqrt{\frac{57}{8}} + 2\sqrt{\frac{31}{4}} + 2\sqrt{\frac{67}{8}} + 3 \right]$
 ≈ 12.6640

Simpson's: $\int_4^9 \sqrt{x} dx \approx \frac{5}{24} \left[2 + 4\sqrt{\frac{37}{8}} + \sqrt{21} + 4\sqrt{\frac{47}{8}} + \sqrt{26} + 4\sqrt{\frac{57}{8}} + \sqrt{31} + 4\sqrt{\frac{67}{8}} + 3 \right] \approx 12.6667$

9. Exact: $\int_1^2 \frac{1}{(x+1)^2} dx = \left[-\frac{1}{x+1} \right]_1^2 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6} \approx 0.1667$

Trapezoidal: $\int_1^2 \frac{1}{(x+1)^2} dx \approx \frac{1}{8} \left[\frac{1}{4} + 2\left(\frac{1}{((5/4)+1)^2}\right) + 2\left(\frac{1}{((3/2)+1)^2}\right) + 2\left(\frac{1}{((7/4)+1)^2}\right) + \frac{1}{9} \right]$
 $= \frac{1}{8} \left(\frac{1}{4} + \frac{32}{81} + \frac{8}{25} + \frac{32}{121} + \frac{1}{9} \right) \approx 0.1676$

Simpson's: $\int_1^2 \frac{1}{(x+1)^2} dx \approx \frac{1}{12} \left[\frac{1}{4} + 4\left(\frac{1}{((5/4)+1)^2}\right) + 2\left(\frac{1}{((3/2)+1)^2}\right) + 4\left(\frac{1}{((7/4)+1)^2}\right) + \frac{1}{9} \right]$
 $= \frac{1}{12} \left(\frac{1}{4} + \frac{64}{81} + \frac{8}{25} + \frac{64}{121} + \frac{1}{9} \right) \approx 0.1667$

11. Trapezoidal: $\int_0^2 \sqrt{1+x^3} dx \approx \frac{1}{4} [1 + 2\sqrt{1+(1/8)} + 2\sqrt{2} + 2\sqrt{1+(27/8)} + 3] \approx 3.283$

Simpson's: $\int_0^2 \sqrt{1+x^3} dx \approx \frac{1}{6} [1 + 4\sqrt{1+(1/8)} + 2\sqrt{2} + 4\sqrt{1+(27/8)} + 3] \approx 3.240$

Graphing utility: 3.241

13. $\int_0^1 \sqrt{x}\sqrt{1-x} dx = \int_0^1 \sqrt{x(1-x)} dx$

Trapezoidal: $\int_0^1 \sqrt{x(1-x)} dx \approx \frac{1}{8} \left[0 + 2\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 2\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)} \right] \approx 0.342$

Simpson's: $\int_0^1 \sqrt{x(1-x)} dx \approx \frac{1}{12} \left[0 + 4\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 4\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)} \right] \approx 0.372$

Graphing utility: 0.393

15. Trapezoidal: $\int_0^{\sqrt{\pi/2}} \cos(x^2) dx \approx \frac{\sqrt{\pi/2}}{8} \left[\cos 0 + 2 \cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2 \cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 2 \cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2 \right]$
 ≈ 0.957

Simpson's: $\int_0^{\sqrt{\pi/2}} \cos(x^2) dx \approx \frac{\sqrt{\pi/2}}{12} \left[\cos 0 + 4 \cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2 \cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 4 \cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2 \right]$
 ≈ 0.978

Graphing utility: 0.977

17. Trapezoidal: $\int_1^{1.1} \sin x^2 dx \approx \frac{1}{80} [\sin(1) + 2 \sin(1.025)^2 + 2 \sin(1.05)^2 + 2 \sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$

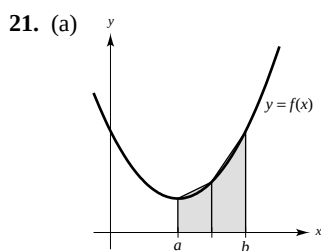
Simpson's: $\int_1^{1.1} \sin x^2 dx \approx \frac{1}{120} [\sin(1) + 4 \sin(1.025)^2 + 2 \sin(1.05)^2 + 4 \sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$

Graphing utility: 0.089

19. Trapezoidal: $\int_0^{\pi/4} x \tan x dx \approx \frac{\pi}{32} \left[0 + 2\left(\frac{\pi}{16}\right) \tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right) \tan\left(\frac{2\pi}{16}\right) + 2\left(\frac{3\pi}{16}\right) \tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4} \right] \approx 0.194$

Simpson's: $\int_0^{\pi/4} x \tan x dx \approx \frac{\pi}{48} \left[0 + 4\left(\frac{\pi}{16}\right) \tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right) \tan\left(\frac{2\pi}{16}\right) + 4\left(\frac{3\pi}{16}\right) \tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4} \right] \approx 0.186$

Graphing utility: 0.186



The Trapezoidal Rule overestimates the area if the graph of the integrand is concave up.

23. $f(x) = x^3$
 $f'(x) = 3x^2$
 $f''(x) = 6x$
 $f'''(x) = 6$
 $f^{(4)}(x) = 0$

(a) Trapezoidal: Error $\leq \frac{(2-0)^3}{12(4^2)}(12) = 0.5$ since

$f''(x)$ is maximum in $[0, 2]$ when $x = 2$.

(b) Simpson's: Error $\leq \frac{(2-0)^5}{180(4^4)}(0) = 0$ since

$f^{(4)}(x) = 0$.

25. $f''(x) = \frac{2}{x^3}$ in $[1, 3]$.

(a) $|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| = 2$.

Trapezoidal: Error $\leq \frac{2^3}{12n^2}(2) < 0.00001$, $n^2 > 133,333.33$, $n > 365.15$; let $n = 366$.

$f^{(4)}(x) = \frac{24}{x^5}$ in $[1, 3]$

(b) $|f^{(4)}(x)|$ is maximum when $x = 1$ and when $|f^{(4)}(1)| = 24$.

Simpson's: Error $\leq \frac{2^5}{180n^4}(24) < 0.00001$, $n^4 > 426,666.67$, $n > 25.56$; let $n = 26$.

27. $f(x) = \sqrt{1+x}$

(a) $f''(x) = -\frac{1}{4(1+x)^{3/2}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \frac{1}{4}$.

Trapezoidal: Error $\leq \frac{8}{12n^2} \left(\frac{1}{4}\right) < 0.00001$, $n^2 > 16,666.67$, $n > 129.10$; let $n = 130$.

(b) $f^{(4)}(x) = \frac{-15}{16(1+x)^{7/2}}$ in $[0, 2]$

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \frac{15}{16}$.

Simpson's: Error $\leq \frac{32}{180n^4} \left(\frac{15}{16}\right) < 0.00001$, $n^4 > 16,666.67$, $n > 11.36$; let $n = 12$.

29. $f(x) = \tan(x^2)$

(a) $f''(x) = 2 \sec^2(x^2)[1 + 4x^2 \tan(x^2)]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 49.5305$.

Trapezoidal: Error $\leq \frac{(1-0)^3}{12n^2} (49.5305) < 0.00001$, $n^2 > 412,754.17$, $n > 642.46$; let $n = 643$.

(b) $f^{(4)}(x) = 8 \sec^2(x^2)[12x^2 + (3 + 32x^4) \tan(x^2) + 36x^2 \tan^2(x^2) + 48x^4 \tan^3(x^2)]$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x = 1$ and $|f^{(4)}(1)| \approx 9184.4734$.

Simpson's: Error $\leq \frac{(1-0)^5}{180n^4} (9184.4734) < 0.00001$, $n^4 > 5,102,485.22$, $n > 47.53$; let $n = 48$.

31. Let $f(x) = Ax^3 + Bx^2 + Cx + D$. Then $f^{(4)}(x) = 0$.

Simpson's: Error $\leq \frac{(b-a)^5}{180n^4} (0) = 0$

Therefore, Simpson's Rule is exact when approximating the integral of a cubic polynomial.

Example: $\int_0^1 x^3 dx = \frac{1}{6} \left[0 + 4 \left(\frac{1}{2} \right)^3 + 1 \right] = \frac{1}{4}$

This is the exact value of the integral.

33. $f(x) = \sqrt{2 + 3x^2}$ on $[0, 4]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	12.7771	15.3965	18.4340	15.6055	15.4845
8	14.0868	15.4480	16.9152	15.5010	15.4662
10	14.3569	15.4544	16.6197	15.4883	15.4658
12	14.5386	15.4578	16.4242	15.4814	15.4657
16	14.7674	15.4613	16.1816	15.4745	15.4657
20	14.9056	15.4628	16.0370	15.4713	15.4657

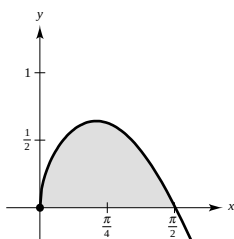
35. $f(x) = \sin \sqrt{x}$ on $[0, 4]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	2.8163	3.5456	3.7256	3.2709	3.3996
8	3.1809	3.5053	3.6356	3.4083	3.4541
10	3.2478	3.4990	3.6115	3.4296	3.4624
12	3.2909	3.4952	3.5940	3.4425	3.4674
16	3.3431	3.4910	3.5704	3.4568	3.4730
20	3.3734	3.4888	3.5552	3.4643	3.4759

37. $A = \int_0^{\pi/2} \sqrt{x} \cos x \, dx$

Simpson's Rule: $n = 14$

$$\int_0^{\pi/2} \sqrt{x} \cos x \, dx \approx \frac{\pi}{84} \left[\sqrt{0} \cos 0 + 4 \sqrt{\frac{\pi}{28}} \cos \frac{\pi}{28} + 2 \sqrt{\frac{\pi}{14}} \cos \frac{\pi}{14} + 4 \sqrt{\frac{3\pi}{28}} \cos \frac{3\pi}{28} + \cdots + \sqrt{\frac{\pi}{2}} \cos \frac{\pi}{2} \right] \\ \approx 0.701$$



39. $W = \int_0^5 100x \sqrt{125 - x^3} \, dx$

Simpson's Rule: $n = 12$

$$\int_0^5 100x \sqrt{125 - x^3} \, dx \approx \frac{5}{3(12)} \left[0 + 400 \left(\frac{5}{12} \right) \sqrt{125 - \left(\frac{5}{12} \right)^3} + 200 \left(\frac{10}{12} \right) \sqrt{125 - \left(\frac{10}{12} \right)^3} \right. \\ \left. + 400 \left(\frac{15}{12} \right) \sqrt{125 - \left(\frac{15}{12} \right)^3} + \cdots + 0 \right] \approx 10,233.58 \text{ ft} \cdot \text{lb}$$

41. $\int_0^{1/2} \frac{6}{\sqrt{1-x^2}} \, dx$ Simpson's Rule, $n = 6$

$$\pi \approx \frac{\left(\frac{1}{2} - 0 \right)}{3(6)} [6 + 4(6.0209) + 2(6.0851) + 4(6.1968) + 2(6.3640) + 4(6.6002) + 6.9282] \\ \approx \frac{1}{36} [113.098] \approx 3.1416$$

43. Area $\approx \frac{1000}{2(10)} [125 + 2(125) + 2(120) + 2(112) + 2(90) + 2(90) + 2(95) + 2(88) + 2(75) + 2(35)] = 89,250 \text{ sq m}$

45. $\int_0^t \sin \sqrt{x} \, dx = 2, n = 10$

By trial and error, we obtain $t \approx 2.477$.

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Integration

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CHAPTER 4

Integration

Section 4.1 Antiderivatives and Indefinite Integration

Solutions to Even-Numbered Exercises

$$2. \frac{d}{dx} \left(x^4 + \frac{1}{x} + C \right) = 4x^3 - \frac{1}{x^2}$$

$$4. \frac{d}{dx} \left(\frac{2(x^2 + 3)}{3\sqrt{x}} + C \right) = \frac{d}{dx} \left(\frac{2}{3}x^{3/2} + 2x^{-1/2} + C \right) \\ = x^{1/2} - x^{-3/2} = \frac{x^2 - 1}{x^{3/2}}$$

$$6. \frac{dr}{d\theta} = \pi$$

$$r = \pi\theta + C$$

$$\text{Check: } \frac{d}{d\theta} [\pi\theta + C] = \pi$$

$$8. \frac{dy}{dx} = 2x^{-3}$$

$$y = \frac{2x^{-2}}{-2} + C = \frac{-1}{x^2} + c$$

$$\text{Check: } \frac{d}{dx} \left[\frac{-1}{x^2} + C \right] = 2x^{-3}$$

Given

Rewrite

Integrate

Simplify

$$10. \int \frac{1}{x^2} dx$$

$$\int x^{-2} dx$$

$$\frac{x^{-1}}{-1} + C$$

$$-\frac{1}{x} + C$$

$$12. \int x(x^2 + 3) dx$$

$$\int (x^3 + 3x) dx$$

$$\frac{x^4}{4} + 3\left(\frac{x^2}{2}\right) + C$$

$$\frac{1}{4}x^4 + \frac{3}{2}x^2 + C$$

$$14. \int \frac{1}{(3x^2)} dx$$

$$\frac{1}{9} \int x^{-2} dx$$

$$\frac{1}{9} \left(\frac{x^{-1}}{-1} \right) + C$$

$$\frac{-1}{9x} + C$$

$$16. \int (5 - x) dx = 5x - \frac{x^2}{2} + C$$

$$\text{Check: } \frac{d}{dx} \left[5x - \frac{x^2}{2} + C \right] = 5 - x$$

$$18. \int (4x^3 + 6x^2 - 1) dx = x^4 + 2x^3 - x + C$$

$$\text{Check: } \frac{d}{dx} [x^4 + 2x^3 - x + C] = 4x^3 + 6x^2 - 1$$

$$20. \int (x^3 - 4x + 2) dx = \frac{x^4}{4} - 2x^2 + 2x + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{x^4}{4} - 2x^2 + 2x + C \right] = x^3 - 4x + 2$$

$$22. \int \left(\sqrt{x} + \frac{1}{2\sqrt{x}} \right) dx = \int \left(x^{1/2} + \frac{1}{2}x^{-1/2} \right) dx = \frac{x^{3/2}}{3/2} + \frac{1}{2} \left(\frac{x^{1/2}}{1/2} \right) + C = \frac{2}{3}x^{3/2} + x^{1/2} + C$$

$$\text{Check: } \frac{d}{dx} \left(\frac{2}{3}x^{3/2} + x^{1/2} + C \right) = x^{1/2} + \frac{1}{2}x^{-1/2} = \sqrt{x} + \frac{1}{2\sqrt{x}}$$

$$24. \int (\sqrt[4]{x^3} + 1) dx = \int (x^{3/4} + 1) dx = \frac{4}{7}x^{7/4} + x + C$$

$$\text{Check: } \frac{d}{dx} \left(\frac{4}{7}x^{7/4} + x + C \right) = x^{3/4} + 1 = \sqrt[4]{x^3} + 1$$

$$26. \int \frac{1}{x^4} dx = \int x^{-4} dx = \frac{x^{-3}}{-3} + C = -\frac{1}{3x^3} + C$$

$$\text{Check: } \frac{d}{dx} \left(-\frac{1}{3x^3} + C \right) = \frac{1}{x^4}$$

$$28. \int \frac{x^2 + 2x - 3}{x^4} dx = \int (x^{-2} + 2x^{-3} - 3x^{-4}) dx$$

$$= \frac{x^{-1}}{-1} + \frac{2x^{-2}}{-2} - \frac{3x^{-3}}{-3} + C$$

$$= -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C$$

$$\text{Check: } \frac{d}{dx} \left[-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C \right] = x^{-2} + 2x^{-3} - 3x^{-4} \\ = \frac{x^2 + 2x - 3}{x^4}$$

$$30. \int (2t^2 - 1)^2 dt = \int (4t^4 - 4t^2 + 1) dt$$

$$= \frac{4}{5}t^5 - \frac{4}{3}t^3 + t + C$$

$$\text{Check: } \frac{d}{dt} \left(\frac{4}{5}t^5 - \frac{4}{3}t^3 + t + C \right) = 4t^4 - 4t^2 + 1 \\ = (2t^2 - 1)^2$$

$$32. \int (1 + 3t)t^2 dt = \int (t^2 + 3t^3) dt = \frac{1}{3}t^3 + \frac{3}{4}t^4 + C$$

$$\text{Check: } \frac{d}{dt} \left(\frac{1}{3}t^3 + \frac{3}{4}t^4 + C \right) = t^2 + 3t^3 = (1 + 3t)t^2$$

$$34. \int 3 dt = 3t + C$$

$$\text{Check: } \frac{d}{dt} (3t + C) = 3$$

$$36. \int (t^2 - \sin t) dt = \frac{1}{3}t^3 + \cos t + C$$

$$\text{Check: } \frac{d}{dt} \left(\frac{1}{3}t^3 + \cos t + C \right) = t^2 - \sin t$$

$$38. \int (\theta^2 + \sec^2 \theta) d\theta = \frac{1}{3}\theta^3 + \tan \theta + C$$

$$\text{Check: } \frac{d}{d\theta} \left(\frac{1}{3}\theta^3 + \tan \theta + C \right) = \theta^2 + \sec^2 \theta$$

$$40. \int \sec y (\tan y - \sec y) dy = \int (\sec y \tan y - \sec^2 y) dy$$

$$= \sec y - \tan y + C$$

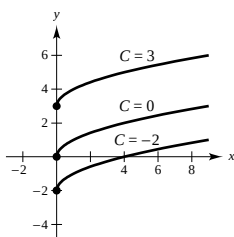
$$\text{Check: } \frac{d}{dy} (\sec y - \tan y + C) = \sec y \tan y - \sec^2 y \\ = \sec y (\tan y - \sec y)$$

$$42. \int \frac{\cos x}{1 - \cos^2 x} dx = \int \frac{\cos x}{\sin^2 x} dx = \int \left(\frac{1}{\sin x} \right) \left(\frac{\cos x}{\sin x} \right) dx$$

$$= \int \csc x \cot x dx = -\csc x + C$$

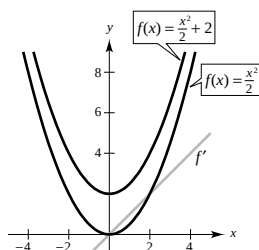
$$\text{Check: } \frac{d}{dx} [-\csc x + C] = \csc x \cot x + \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} \\ = \frac{\cos x}{1 - \cos^2 x}$$

$$44. f(x) = \sqrt{x}$$



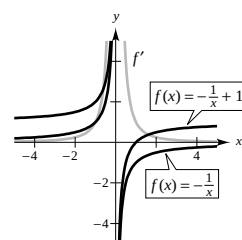
$$46. f'(x) = x$$

$$f(x) = \frac{x^2}{2} + C$$



$$48. f'(x) = \frac{1}{x^2}$$

$$f(x) = -\frac{1}{x} + C$$



50. $\frac{dy}{dx} = 2(x - 1) = 2x - 2, (3, 2)$

$$y = \int 2(x - 1) dx = x^2 - 2x + C$$

$$2 = (3)^2 - 2(3) + C \Rightarrow C = -1$$

$$y = x^2 - 2x - 1$$

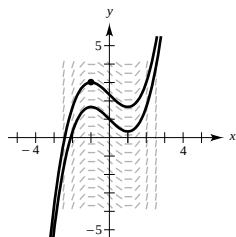
52. $\frac{dy}{dx} = -\frac{1}{x^2} = -x^{-2}, (1, 3)$

$$y = \int -x^{-2} dx = \frac{1}{x} + C$$

$$3 = \frac{1}{1} + C \Rightarrow C = 2$$

$$y = \frac{1}{x} + 2, x > 0$$

54. (a)



(b) $\frac{dy}{dx} = x^2 - 1, (-1, 3)$

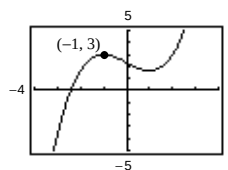
$$y = \frac{x^3}{3} - x + C$$

$$3 = \frac{(-1)^3}{3} - (-1) + C$$

$$3 = -\frac{1}{3} + 1 + C$$

$$C = \frac{7}{3}$$

$$y = \frac{x^3}{3} - x + \frac{7}{3}$$



56. $g'(x) = 6x^2, g(0) = -1$

$$g(x) = \int 6x^2 dx = 2x^3 + C$$

$$g(0) = -1 = 2(0)^3 + C \Rightarrow C = -1$$

$$g(x) = 2x^3 - 1$$

58. $f'(s) = 6s - 8s^3, f(2) = 3$

$$f(s) = \int (6s - 8s^3) ds = 3s^2 - 2s^4 + C$$

$$f(2) = 3 = 3(2)^2 - 2(2)^4 + C = 12 - 32 + C \Rightarrow C = 23$$

$$f(s) = 3s^2 - 2s^4 + 23$$

60. $f''(x) = x^2$

$$f'(0) = 6$$

$$f(0) = 3$$

$$f'(x) = \int x^2 dx = \frac{1}{3}x^3 + C_1$$

$$f'(0) = 0 + C_1 = 6 \Rightarrow C_1 = 6$$

$$f'(x) = \frac{1}{3}x^3 + 6$$

$$f(x) = \int \left(\frac{1}{3}x^3 + 6 \right) dx = \frac{1}{12}x^4 + 6x + C_2$$

$$f(0) = 0 + 0 + C_2 = 3 \Rightarrow C_2 = 3$$

$$f(x) = \frac{1}{12}x^4 + 6x + 3$$

62. $f''(x) = \sin x$

$$f'(0) = 1$$

$$f(0) = 6$$

$$f'(x) = \int \sin x dx = -\cos x + C_1$$

$$f'(0) = -1 + C_1 = 1 \Rightarrow C_1 = 2$$

$$f'(x) = -\cos x + 2$$

$$f(x) = \int (-\cos x + 2) dx = -\sin x + 2x + C_2$$

$$f(0) = 0 + 0 + C_2 = 6 \Rightarrow C_2 = 6$$

$$f(x) = -\sin x + 2x + 6$$

$$64. \frac{dP}{dt} = k\sqrt{t}, \quad 0 \leq t \leq 10$$

$$P(t) = \int kt^{1/2} dt = \frac{2}{3}kt^{3/2} + C$$

$$P(0) = 0 + C = 500 \Rightarrow C = 500$$

$$P(1) = \frac{2}{3}k + 500 = 600 \Rightarrow k = 150$$

$$P(t) = \frac{2}{3}(150)t^{3/2} + 500 = 100t^{3/2} + 500$$

$$P(7) = 100(7)^{3/2} + 500 \approx 2352 \text{ bacteria}$$

$$68. f''(t) = a(t) = -32 \text{ ft/sec}^2$$

$$f'(0) = v_0$$

$$f(0) = s_0$$

$$f'(t) = v(t) = \int -32 dt = -32t + C_1$$

$$f'(0) = 0 + C_1 = v_0 \Rightarrow C_1 = v_0$$

$$f'(t) = -32t + v_0$$

$$f(t) = s(t) = \int (-32t + v_0) dt = -16t^2 + v_0t + C_2$$

$$f(0) = 0 + 0 + C_2 = s_0 \Rightarrow C_2 = s_0$$

$$f(t) = -16t^2 + v_0t + s_0$$

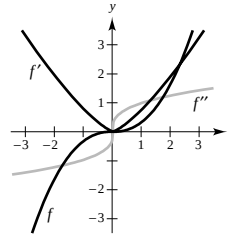
$$72. \text{ From Exercise 71, } f(t) = -4.9t^2 + 1600. \text{ (Using the canyon floor as position 0.)}$$

$$f(t) = 0 = -4.9t^2 + 1600$$

$$4.9t^2 = 1600$$

$$t^2 = \frac{1600}{4.9} \Rightarrow t \approx \sqrt{326.53} \approx 18.1 \text{ sec}$$

66. Since f'' is negative on $(-\infty, 0)$, f' is decreasing on $(-\infty, 0)$. Since f'' is positive on $(0, \infty)$, f' is increasing on $(0, \infty)$. f' has a relative minimum at $(0, 0)$. Since f' is positive on $(-\infty, \infty)$, f is increasing on $(-\infty, \infty)$.



$$70. v_0 = 16 \text{ ft/sec}$$

$$s_0 = 64 \text{ ft}$$

$$(a) \quad s(t) = -16t^2 + 16t + 64 = 0$$

$$-16(t^2 - t - 4) = 0$$

$$t = \frac{1 \pm \sqrt{17}}{2}$$

Choosing the positive value,

$$t = \frac{1 + \sqrt{17}}{2} \approx 2.562 \text{ seconds.}$$

$$(b) \quad v(t) = s'(t) = -32t + 16$$

$$v\left(\frac{1 + \sqrt{17}}{2}\right) = -32\left(\frac{1 + \sqrt{17}}{2}\right) + 16$$

$$= -16\sqrt{17} \approx -65.970 \text{ ft/sec}$$

$$74. \text{ From Exercise 71, } f(t) = -4.9t^2 + v_0t + 2. \text{ If}$$

$$f(t) = 200 = -4.9t^2 + v_0t + 2,$$

then

$$v(t) = -9.8t + v_0 = 0$$

for this t value. Hence, $t = v_0/9.8$ and we solve

$$-4.9\left(\frac{v_0}{9.8}\right)^2 + v_0\left(\frac{v_0}{9.8}\right) + 2 = 200$$

$$\frac{-4.9 v_0^2}{(9.8)^2} + \frac{v_0^2}{9.8} = 198$$

$$-4.9 v_0^2 + 9.8 v_0^2 = (9.8)^2 198$$

$$4.9 v_0^2 = (9.8)^2 198$$

$$v_0^2 = 3880.8 \Rightarrow v_0 \approx 62.3 \text{ m/sec.}$$

$$76. \int v \, dv = -GM \int \frac{1}{y^2} \, dy$$

$$\frac{1}{2}v^2 = \frac{GM}{y} + C$$

When $y = R$, $v = v_0$.

$$\frac{1}{2}v_0^2 = \frac{GM}{R} + C$$

$$C = \frac{1}{2}v_0^2 - \frac{GM}{R}$$

$$\frac{1}{2}v^2 = \frac{GM}{y} + \frac{1}{2}v_0^2 - \frac{GM}{R}$$

$$v^2 = \frac{2GM}{y} + v_0^2 - \frac{2GM}{R}$$

$$v^2 = v_0^2 + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$$

$$80. (a) a(t) = \cos t$$

$$v(t) = \int a(t) \, dt = \int \cos t \, dt = \sin t + C_1 = \sin t \quad (\text{since } v_0 = 0)$$

$$f(t) = \int v(t) \, dt = \int \sin t \, dt = -\cos t + C_2$$

$$f(0) = 3 = -\cos(0) + C_2 = -1 + C_2 \Rightarrow C_2 = 4$$

$$f(t) = -\cos t + 4$$

$$(b) v(t) = 0 = \sin t \text{ for } t = k\pi, k = 0, 1, 2, \dots$$

$$82. \quad v(0) = 45 \text{ mph} = 66 \text{ ft/sec}$$

$$30 \text{ mph} = 44 \text{ ft/sec}$$

$$15 \text{ mph} = 22 \text{ ft/sec}$$

$$a(t) = -a$$

$$v(t) = -at + 66$$

$$s(t) = -\frac{a}{2}t^2 + 66t \quad (\text{Let } s(0) = 0.)$$

$$v(t) = 0 \text{ after car moves 132 ft.}$$

$$-at + 66 = 0 \text{ when } t = \frac{66}{a}.$$

$$s\left(\frac{66}{a}\right) = -\frac{a}{2}\left(\frac{66}{a}\right)^2 + 66\left(\frac{66}{a}\right)$$

$$= 132 \text{ when } a = \frac{33}{2} = 16.5.$$

$$a(t) = -16.5$$

$$v(t) = -16.5t + 66$$

$$s(t) = -8.25t^2 + 66t$$

$$78. x(t) = (t-1)(t-3)^2 \quad 0 \leq t \leq 5$$

$$= t^3 - 7t^2 + 15t - 9$$

$$(a) v(t) = x'(t) = 3t^2 - 14t + 15 = (3t-5)(t-3)$$

$$a(t) = v'(t) = 6t - 14$$

$$(b) v(t) > 0 \text{ when } 0 < t < \frac{5}{3} \text{ and } 3 < t < 5.$$

$$(c) a(t) = 6t - 14 = 0 \text{ when } t = \frac{7}{3}.$$

$$v\left(\frac{7}{3}\right) = \left(3\left(\frac{7}{3}\right) - 5\right)\left(\frac{7}{3} - 3\right) = 2\left(-\frac{2}{3}\right) = -\frac{4}{3}$$

$$(a) -16.5t + 66 = 44$$

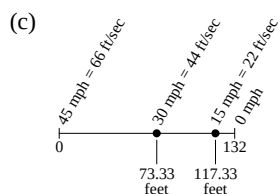
$$t = \frac{22}{16.5} \approx 1.333$$

$$s\left(\frac{22}{16.5}\right) \approx 73.33 \text{ ft}$$

$$(b) -16.5t + 66 = 22$$

$$t = \frac{44}{16.5} \approx 2.667$$

$$s\left(\frac{44}{16.5}\right) \approx 117.33 \text{ ft}$$



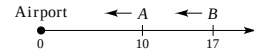
It takes 1.333 seconds to reduce the speed from 45 mph to 30 mph, 1.333 seconds to reduce the speed from 30 mph to 15 mph, and 1.333 seconds to reduce the speed from 15 mph to 0 mph. Each time, less distance is needed to reach the next speed reduction.

84. No, car 2 will be ahead of car 1. If $v_1(t)$ and $v_2(t)$ are the respective velocities, then $\int_0^{30} |v_2(t)| dt > \int_0^{30} |v_1(t)| dt$.

86. (a) $v = 0.6139t^3 - 5.525t^2 + 0.0492t + 65.9881$ (b) $s(t) = \int v(t)dt = \frac{0.6139t^4}{4} - \frac{5.525t^3}{3} + \frac{0.0492t^2}{2} + 65.9881t$
 (Note: Assume $s(0) = 0$ is initial position)
 $s(6) \approx 196.1$ feet

88. Let the aircrafts be located 10 and 17 miles away from the airport, as indicated in the figure.

$$\begin{aligned} v_A(t) &= k_A t - 150 & v_B &= k_B t - 250 \\ s_A(t) &= \frac{1}{2}k_A t^2 - 150t + 10 & s_B &= \frac{1}{2}k_B t^2 - 250t + 17 \end{aligned}$$



(a) When aircraft A lands at time t_A you have

$$v_A(t_A) = k_A t_A - 150 = -100 \Rightarrow k_A = \frac{50}{t_A}$$

$$s_A(t_A) = \frac{1}{2}k_A t_A^2 - 150t_A + 10 = 0$$

$$\frac{1}{2}\left(\frac{50}{t_A}\right)t_A^2 - 150t_A = -10$$

$$125t_A = 10$$

$$t_A = \frac{10}{125}.$$

$$k_A = \frac{50}{t_A} = 50\left(\frac{125}{10}\right) = 625 \Rightarrow S_A(t) = S_1(t) = \frac{625}{2}t^2 - 150t + 10$$

Similarly, when aircraft B lands at time t_B you have

$$v_B(t_B) = k_B t_B - 250 = -115 \Rightarrow k_B = \frac{135}{t_B}$$

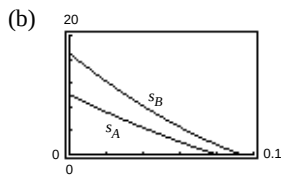
$$s_B(t_B) = \frac{1}{2}k_B t_B^2 - 250t_B + 17 = 0$$

$$\frac{1}{2}\left(\frac{135}{t_B}\right)t_B^2 - 250t_B = -17$$

$$\frac{365}{2}t_B = 17$$

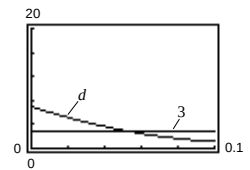
$$t_B = \frac{34}{365}.$$

$$k_B = \frac{135}{t_B} = 135\left(\frac{365}{34}\right) = \frac{49,275}{34} \Rightarrow S_B(t) = S_2(t) = \frac{49,275}{68}t^2 - 250t + 17$$



(c) $d = s_B(t) - s_A(t)$

Yes, $d < 3$ for $t > 0.0505$.



90. True

92. True

94. False. f has an infinite number of antiderivatives, each differing by a constant.

$$\begin{aligned} 96. \frac{d}{dx} [s(x)]^2 + [c(x)]^2 &= 2s(x)s'(x) + 2c(x)c'(x) \\ &= 2s(x)c(x) - 2c(x)s(x) \\ &= 0 \end{aligned}$$

Thus, $[s(x)]^2 + [c(x)]^2 = k$ for some constant k . Since, $s(0) = 0$ and $c(0) = 1$, $k = 1$.

Therefore,

$$[s(x)]^2 + [c(x)]^2 = 1.$$

[Note that $s(x) = \sin x$ and $c(x) = \cos x$ satisfy these properties.]

Section 4.2 Area

$$2. \sum_{k=3}^6 k(k-2) = 3(1) + 4(2) + 5(3) + 6(4) = 50$$

$$4. \sum_{j=3}^5 \frac{1}{j} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$$

$$6. \sum_{i=1}^4 [(i-1)^2 + (i+1)^3] = (0+8) + (1+27) + (4+64) + (9+125) = 238$$

$$8. \sum_{i=1}^{15} \frac{5}{1+i}$$

$$10. \sum_{j=1}^4 \left[1 - \left(\frac{j}{4} \right)^2 \right]$$

$$12. \frac{2}{n} \sum_{i=1}^n \left[1 - \left(\frac{2i}{n} - 1 \right)^2 \right]$$

$$14. \frac{1}{n} \sum_{i=0}^{n-1} \sqrt{1 - \left(\frac{i}{n} \right)^2}$$

$$\begin{aligned} 16. \sum_{i=1}^{15} (2i-3) &= 2 \sum_{i=1}^{15} i - 3(15) \\ &= 2 \left[\frac{15(16)}{2} \right] - 45 = 195 \end{aligned}$$

$$\begin{aligned} 18. \sum_{i=1}^{10} (i^2 - 1) &= \sum_{i=1}^{10} i^2 - \sum_{i=1}^{10} 1 \\ &= \left[\frac{10(11)(21)}{6} \right] - 10 = 375 \end{aligned}$$

$$\begin{aligned} 20. \sum_{i=1}^{10} i(i^2 + 1) &= \sum_{i=1}^{10} i^3 + \sum_{i=1}^{10} i \\ &= \frac{10^2(11)^2}{4} + \left[\frac{10(11)}{2} \right] = 3080 \end{aligned}$$

$$22. \text{sum seq}(x \sqcap 3 - 2x, x, 1, 15, 1) = 14,160 \quad (TI-82)$$

$$\begin{aligned} \sum_{i=1}^{15} (i^3 - 2i) &= \frac{(15)^2(15+1)^2}{4} - 2 \frac{15(15+1)}{2} \\ &= \frac{(15)^2(16)^2}{4} - 15(16) = 14,160 \end{aligned}$$

$$\begin{aligned} 24. S &= [5 + 5 + 4 + 2](1) = 16 \\ s &= [4 + 4 + 2 + 0](1) = 10 \end{aligned}$$

$$\begin{aligned} 26. S &= \left[5 + 2 + 1 + \frac{2}{3} + \frac{1}{2} \right] = \frac{55}{6} \\ s &= \left[2 + 1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{3} \right] = \frac{9}{2} = 4.5 \end{aligned}$$

$$\begin{aligned} 28. S(8) &= \left(\sqrt{\frac{1}{4}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{3}{4}} + 2 \right) \frac{1}{4} + (\sqrt{1} + 2) \frac{1}{4} \\ &\quad + \left(\sqrt{\frac{5}{4}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{3}{2}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{7}{4}} + 2 \right) \frac{1}{4} + (\sqrt{2} + 2) \frac{1}{4} \\ &= \frac{1}{4} \left(16 + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{5}}{2} + \frac{\sqrt{6}}{2} + \frac{\sqrt{7}}{2} + \sqrt{2} \right) \approx 6.038 \\ s(8) &= (0+2) \frac{1}{4} + \left(\sqrt{\frac{1}{4}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2 \right) \frac{1}{4} + \cdots + \left(\sqrt{\frac{7}{4}} + 2 \right) \frac{1}{4} \approx 5.685 \end{aligned}$$

$$\begin{aligned}
30. S(5) &= 1\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) \\
&= \frac{1}{5}\left[1 + \frac{\sqrt{24}}{5} + \frac{\sqrt{21}}{5} + \frac{\sqrt{16}}{5} + \frac{\sqrt{9}}{5}\right] \approx 0.859 \\
s(5) &= \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) + 0 \approx 0.659
\end{aligned}$$

$$32. \lim_{n \rightarrow \infty} \left[\left(\frac{64}{n^3} \right) \frac{n(n+1)(2n+1)}{6} \right] = \frac{64}{6} \lim_{n \rightarrow \infty} \left[\frac{2n^3 + 3n^2 + n}{n^3} \right] = \frac{64}{6}(2) = \frac{64}{3}$$

$$34. \lim_{n \rightarrow \infty} \left[\left(\frac{1}{n^2} \right) \frac{n(n+1)}{2} \right] = \frac{1}{2} \lim_{n \rightarrow \infty} \left[\frac{n^2 + n}{n^2} \right] = \frac{1}{2}(1) = \frac{1}{2}$$

$$\begin{aligned}
36. \sum_{j=1}^n \frac{4j+3}{n^2} &= \frac{1}{n^2} \sum_{j=1}^n (4j+3) = \frac{1}{n^2} \left[\frac{4n(n+1)}{2} + 3n \right] = \frac{2n+5}{n} = S(n) \\
S(10) &= \frac{25}{10} = 2.5
\end{aligned}$$

$$S(100) = 2.05$$

$$S(1000) = 2.005$$

$$S(10,000) = 2.0005$$

$$\begin{aligned}
38. \sum_{i=1}^n \frac{4i^2(i-1)}{n^4} &= \frac{4}{n^4} \sum_{i=1}^n (i^3 - i^2) = \frac{4}{n^4} \left[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6} \right] \\
&= \frac{4}{n^3} \left[\frac{n^3 + 2n^2 + n}{4} - \frac{2n^2 + 3n + 1}{6} \right] \\
&= \frac{1}{3n^3} [3n^3 + 6n^2 + 3n - 4n^2 - 6n - 2] \\
&= \frac{1}{3n^3} [3n^3 + 2n^2 - 3n - 2] = S(n)
\end{aligned}$$

$$S(10) = 1.056$$

$$S(100) = 1.006566$$

$$S(1000) = 1.00066567$$

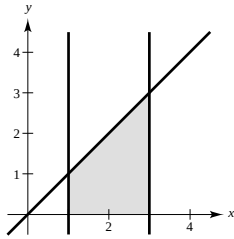
$$S(10,000) = 1.000066657$$

$$40. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2i}{n} \right) \left(\frac{2}{n} \right) = \lim_{n \rightarrow \infty} \frac{4}{n^2} \sum_{i=1}^n i = \lim_{n \rightarrow \infty} \frac{4}{n^2} \left(\frac{n(n+1)}{2} \right) = \lim_{n \rightarrow \infty} \frac{4}{2} \left(1 + \frac{1}{n} \right) = 2$$

$$\begin{aligned}
42. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + \frac{2i}{n} \right)^2 \left(\frac{2}{n} \right) &= \lim_{n \rightarrow \infty} \frac{2}{n^3} \sum_{i=1}^n (n+2i)^2 \\
&= \lim_{n \rightarrow \infty} \frac{2}{n^3} \left[\sum_{i=1}^n n^2 + 4n \sum_{i=1}^n i + 4 \sum_{i=1}^n i^2 \right] \\
&= \lim_{n \rightarrow \infty} \frac{2}{n^3} \left[n^3 + (4n) \left(\frac{n(n+1)}{2} \right) + \frac{4(n)(n+1)(2n+1)}{6} \right] \\
&= 2 \lim_{n \rightarrow \infty} \left[1 + 2 + \frac{2}{n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^2} \right] \\
&= 2 \left(1 + 2 + \frac{4}{3} \right) = \frac{26}{3}
\end{aligned}$$

$$\begin{aligned}
44. \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + \frac{2i}{n}\right)^3 \left(\frac{2}{n}\right) &= 2 \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{i=1}^n (n + 2i)^3 \\
&= 2 \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{i=1}^n (n^3 + 6n^2i + 12ni^2 + 8i^3) \\
&= 2 \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[n^4 + 6n^2 \left(\frac{n(n+1)}{2}\right) + 12n \left(\frac{n(n+1)(2n+1)}{6}\right) + 8 \left(\frac{n^2(n+1)^2}{4}\right) \right] \\
&= 2 \lim_{n \rightarrow \infty} \left(1 + 3 + \frac{3}{n} + 4 + \frac{6}{n} + \frac{2}{n^2} + 2 + \frac{4}{n} + \frac{2}{n^2}\right) \\
&= 2 \lim_{n \rightarrow \infty} \left(10 + \frac{13}{n} + \frac{4}{n^2}\right) = 20
\end{aligned}$$

46. (a)



$$(b) \quad \Delta x = \frac{3-1}{n} = \frac{2}{n}$$

Endpoints:

$$1 < 1 + \frac{2}{n} < 1 + \frac{4}{n} < \dots < 1 + \frac{2n}{n} = 3$$

$$1 < 1 + 1\left(\frac{2}{n}\right) < 1 + 2\left(\frac{2}{n}\right) < \dots < 1 + (n-1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right)$$

(c) Since $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$\begin{aligned}
s(n) &= \sum_{i=1}^n f(x_{i-1}) \Delta x \\
&= \sum_{i=1}^n f\left[1 + (i-1)\left(\frac{2}{n}\right)\right] \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[1 + (i-1)\left(\frac{2}{n}\right)\right] \left(\frac{2}{n}\right)
\end{aligned}$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n f\left[1 + i\left(\frac{2}{n}\right)\right] \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[1 + i\left(\frac{2}{n}\right)\right] \left(\frac{2}{n}\right)$$

(e)

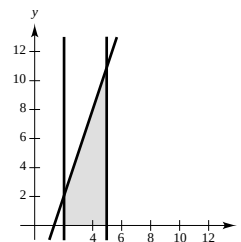
x	5	10	50	100
$s(n)$	3.6	3.8	3.96	3.98
$S(n)$	4.4	4.2	4.04	4.02

$$\begin{aligned}
(f) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[1 + (i-1)\left(\frac{2}{n}\right)\right] \left(\frac{2}{n}\right) &= \lim_{n \rightarrow \infty} \left(\frac{2}{n}\right) \left[n + \frac{2}{n} \left(\frac{n(n+1)}{2} - n\right)\right] \\
&= \lim_{n \rightarrow \infty} \left[2 + \frac{2n+2}{n} - \frac{4}{n}\right] = \lim_{n \rightarrow \infty} \left[4 - \frac{2}{n}\right] = 4 \\
\lim_{n \rightarrow \infty} \sum_{i=1}^n \left[1 + i\left(\frac{2}{n}\right)\right] \left(\frac{2}{n}\right) &= \lim_{n \rightarrow \infty} \frac{2}{n} \left[n + \left(\frac{2}{n}\right) \frac{n(n+1)}{2}\right] \\
&= \lim_{n \rightarrow \infty} \left[2 + \frac{2(n+1)}{n}\right] = \lim_{n \rightarrow \infty} \left[4 + \frac{2}{n}\right] = 4
\end{aligned}$$

48. $y = 3x - 4$ on $[2, 5]$. (Note: $\Delta x = \frac{5-2}{n} = \frac{3}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n f\left(2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\ &= \sum_{i=1}^n \left[3\left(2 + \frac{3i}{n}\right) - 4\right]\left(\frac{3}{n}\right) = 18 + 3\left(\frac{3}{n}\right)^2 \sum_{i=1}^n i - 12 \\ &= 6 + \frac{27}{n^2} \left(\frac{(n+1)n}{2}\right) = 6 + \frac{27}{2} \left(1 + \frac{1}{n}\right) \end{aligned}$$

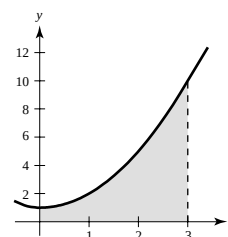
$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 6 + \frac{27}{2} = \frac{39}{2}$$



50. $y = x^2 + 1$ on $[0, 3]$. (Note: $\Delta x = \frac{3-0}{n} = \frac{3}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^n \left[\left(\frac{3i}{n}\right)^2 + 1\right]\left(\frac{3}{n}\right) \\ &= \frac{27}{n^3} \sum_{i=1}^n i^2 + \frac{3}{n} \sum_{i=1}^n 1 \\ &= \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{3}{n}(n) = \frac{9}{2} \frac{2n^2 + 3n + 1}{n^2} + 3 \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = \frac{9}{2}(2) + 3 = 12$$

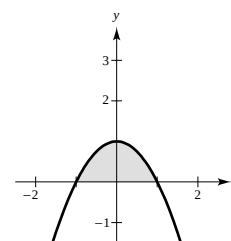


52. $y = 1 - x^2$ on $[-1, 1]$. Find area of region over the interval $[0, 1]$. (Note: $\Delta x = \frac{1}{n}$)

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^n \left[1 - \left(\frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right) \\ &= 1 - \frac{1}{n^3} \sum_{i=1}^n i^2 = 1 - \frac{n(n+1)(2n+1)}{6n^3} = 1 - \frac{1}{6} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \end{aligned}$$

$$\frac{1}{2} \text{Area} = \lim_{n \rightarrow \infty} s(n) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\text{Area} = \frac{4}{3}$$

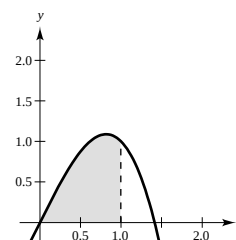


54. $y = 2x - x^3$ on $[0, 1]$. (Note: $\Delta x = \frac{1-0}{n} = \frac{1}{n}$)

Since y both increases and decreases on $[0, 1]$, $T(n)$ is neither an upper nor lower sum.

$$\begin{aligned} T(n) &= \sum_{i=1}^n f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^n \left[2\left(\frac{i}{n}\right) - \left(\frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right) \\ &= \frac{2}{n^2} \sum_{i=1}^n i - \frac{1}{n^4} \sum_{i=1}^n i^3 = \frac{n(n+1)}{n^2} - \frac{1}{n^4} \left[\frac{n^2(n+1)^2}{4}\right] \\ &= 1 + \frac{1}{n} - \frac{1}{4} - \frac{2}{4n} - \frac{1}{4n^2} \end{aligned}$$

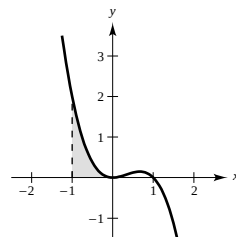
$$\text{Area} = \lim_{n \rightarrow \infty} T(n) = 1 - \frac{1}{4} = \frac{3}{4}$$



56. $y = x^2 - x^3$ on $[-1, 0]$. (Note: $\Delta x = \frac{0 - (-1)}{n} = \frac{1}{n}$)

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(-1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^n \left[\left(-1 + \frac{i}{n}\right)^2 - \left(-1 + \frac{i}{n}\right)^3 \right] \left(\frac{1}{n}\right) \\ &= \sum_{i=1}^n \left[\left(2 - \frac{5i}{n} + \frac{4i^2}{n^2} - \frac{i^3}{n^3}\right) \left(\frac{1}{n}\right) \right] = 2 - \frac{5}{n^2} \sum_{i=1}^n i + \frac{4}{n^3} \sum_{i=1}^n i^2 - \frac{1}{n^4} \sum_{i=1}^n i^3 \\ &= 2 - \frac{5}{2} - \frac{5}{2n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^3} - \frac{1}{4} - \frac{1}{2n} - \frac{1}{4n^2} \end{aligned}$$

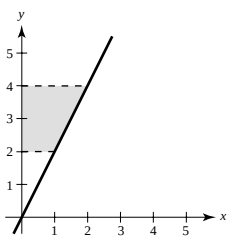
$$\text{Area} = \lim_{n \rightarrow \infty} s(n) = 2 - \frac{5}{2} + \frac{4}{3} - \frac{1}{4} = \frac{7}{12}$$



58. $g(y) = \frac{1}{2}y$, $2 \leq y \leq 4$. (Note: $\Delta y = \frac{4 - 2}{n} = \frac{2}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n g\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n \frac{1}{2} \left(2 + \frac{2i}{n}\right) \left(\frac{2}{n}\right) = \frac{2}{n} \sum_{i=1}^n \left(1 + \frac{i}{n}\right) \\ &= \frac{2}{n} \left[n + \frac{1}{n} \frac{n(n+1)}{2} \right] = 2 + \frac{n+1}{n} \end{aligned}$$

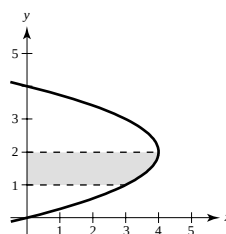
$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 2 + 1 = 3$$



60. $f(y) = 4y - y^2$, $1 \leq y \leq 2$. (Note: $\Delta y = \frac{2 - 1}{n} = \frac{1}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) \\ &= \frac{1}{n} \sum_{i=1}^n \left[4\left(1 + \frac{i}{n}\right) - \left(1 + \frac{i}{n}\right)^2 \right] \\ &= \frac{1}{n} \sum_{i=1}^n \left(4 + \frac{4i}{n} - 1 - \frac{2i}{n} - \frac{i^2}{n^2} \right) \\ &= \frac{1}{n} \sum_{i=1}^n \left(3 + \frac{2i}{n} - \frac{i^2}{n^2} \right) \\ &= \frac{1}{n} \left[3n + \frac{2}{n} \frac{n(n+1)}{2} - \frac{1}{n^2} \frac{n(n+1)(2n+1)}{6} \right] \\ &= 3 + \frac{n+1}{n} - \frac{(n+1)(2n+1)}{6n} \end{aligned}$$

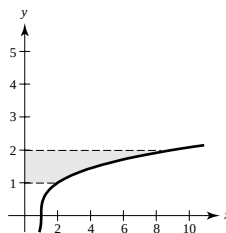
$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 3 + 1 - \frac{1}{3} = \frac{11}{3}$$



62. $h(y) = y^3 + 1$, $1 \leq y \leq 2$ (Note: $\Delta y = \frac{1}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n h\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) \\ &= \sum_{i=1}^n \left[\left(1 + \frac{i}{n}\right)^3 + 1 \right] \frac{1}{n} \\ &= \frac{1}{n} \sum_{i=1}^n \left(2 + \frac{i^3}{n^3} + \frac{3i^2}{n^2} + \frac{3i}{n} \right) \\ &= \frac{1}{n} \left[2n + \frac{1}{n^3} \frac{n^2(n+1)^2}{4} + \frac{3}{n^2} \frac{n(n+1)(2n+1)}{6} + \frac{3}{n} \frac{n(n+1)}{2} \right] \\ &= 2 + \frac{(n+1)^2}{n^2} + \frac{1}{2} \frac{(n+1)(2n+1)}{n^2} + \frac{3(n+1)}{2n} \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 2 + \frac{1}{4} + 1 + \frac{3}{2} = \frac{19}{4}$$



64. $f(x) = x^2 + 4x$, $0 \leq x \leq 4$, $n = 4$

Let $c_i = \frac{x_i + x_{i-1}}{2}$.

$\Delta x = 1$, $c_1 = \frac{1}{2}$, $c_2 = \frac{3}{2}$, $c_3 = \frac{5}{2}$, $c_4 = \frac{7}{2}$

$$\begin{aligned} \text{Area} &\approx \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^4 [c_i^2 + 4c_i](1) \\ &= \left[\left(\frac{1}{4} + 2 \right) + \left(\frac{9}{4} + 6 \right) + \left(\frac{25}{4} + 10 \right) + \left(\frac{49}{4} + 14 \right) \right] \\ &= 53 \end{aligned}$$

66. $f(x) = \sin x$, $0 \leq x \leq \frac{\pi}{2}$, $n = 4$

Let $c_i = \frac{x_i + x_{i-1}}{2}$.

$\Delta x = \frac{\pi}{8}$, $c_1 = \frac{\pi}{16}$, $c_2 = \frac{3\pi}{16}$, $c_3 = \frac{5\pi}{16}$, $c_4 = \frac{7\pi}{16}$

$$\begin{aligned} \text{Area} &\approx \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^4 (\sin c_i) \left(\frac{\pi}{8} \right) \\ &= \frac{\pi}{8} \left(\sin \frac{\pi}{16} + \sin \frac{3\pi}{16} + \sin \frac{5\pi}{16} + \sin \frac{7\pi}{16} \right) \approx 1.006 \end{aligned}$$

68. $f(x) = \frac{8}{x^2 + 1}$ on $[2, 6]$.

n	4	8	12	16	20
Approximate area	2.3397	2.3755	2.3824	2.3848	2.3860

70. $f(x) = \cos \sqrt{x}$ on $[0, 2]$.

n	4	8	12	16	20
Approximate area	1.1041	1.1053	1.1055	1.1056	1.1056

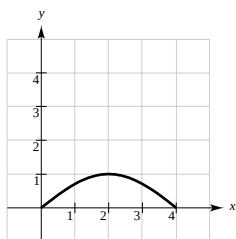
72. See the Definition of Area. Page 259.

74. $f(x) = \sqrt[3]{x}$, $0 \leq x \leq 8$

n	10	20	50	100	200
$s(n)$	10.998	11.519	11.816	11.910	11.956
$S(n)$	12.598	12.319	12.136	12.070	12.036
$M(n)$	12.040	12.016	12.005	12.002	12.001

(Note: exact answer is 12.)

76.



a. $A \approx 3$ square units

78. True. (Theorem 4.3)

80. (a) $\theta = \frac{2\pi}{n}$

(b) $\sin \theta = \frac{h}{r}$

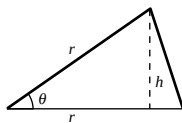
$$h = r \sin \theta$$

$$A = \frac{1}{2}bh = \frac{1}{2}r(r \sin \theta) = \frac{1}{2}r^2 \sin \theta$$

(c) $A_n = n \left(\frac{1}{2}r^2 \sin \frac{2\pi}{n} \right) = \frac{r^2 n}{2} \sin \frac{2\pi}{n} = \pi r^2 \left(\frac{\sin(2\pi/n)}{2\pi/n} \right)$

Let $x = 2\pi/n$. As $n \rightarrow \infty$, $x \rightarrow 0$.

$$\lim_{n \rightarrow \infty} A_n = \lim_{x \rightarrow 0} \pi r^2 \left(\frac{\sin x}{x} \right) = \pi r^2(1) = \pi r^2$$



82. (a) $\sum_{i=1}^n 2i = n(n+1)$

The formula is true for $n = 1$: $2 = 1(1+1) = 2$

Assume that the formula is true for $n = k$:

$$\sum_{i=1}^k 2i = k(k+1).$$

$$\begin{aligned} \text{Then we have } \sum_{i=1}^{k+1} 2i &= \sum_{i=1}^k 2i + 2(k+1) \\ &= k(k+1) + 2(k+1) \\ &= (k+1)(k+2) \end{aligned}$$

Which shows that the formula is true for $n = k+1$.

(b) $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$

The formula is true for $n = 1$ because

$$1^3 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$$

Assume that the formula is true for $n = k$:

$$\sum_{i=1}^k i^3 = \frac{k^2(k+1)^2}{4}$$

$$\begin{aligned} \text{Then we have } \sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^k i^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{(k+1)^2}{4} [k^2 + 4(k+1)] \\ &= \frac{(k+1)^2}{4} (k+2)^2 \end{aligned}$$

which shows that the formula is true for $n = k+1$.

Section 4.3 Riemann Sums and Definite Integrals

2. $f(x) = \sqrt[3]{x}$, $y = 0$, $x = 0$, $x = 1$, $c_i = \frac{i^3}{n^3}$

$$\Delta x_i = \frac{i^3}{n^3} - \frac{(i-1)^3}{n^3} = \frac{3i^2 - 3i + 1}{n^3}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x_i &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt[3]{\frac{i^3}{n^3}} \left[\frac{3i^2 - 3i + 1}{n^3} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{i=1}^n (3i^3 - 3i^2 + i) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[3 \left(\frac{n^2(n+1)^2}{4} \right) - 3 \left(\frac{n(n+1)(2n+1)}{6} \right) + \frac{n(n+1)}{2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[\frac{3n^4 + 6n^3 + 3n^2}{4} - \frac{2n^3 + 3n^2 + n}{2} + \frac{n^2 + n}{2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[\frac{3n^4}{4} + \frac{n^3}{2} - \frac{n^2}{4} \right] = \lim_{n \rightarrow \infty} \left[\frac{3}{4} + \frac{1}{2n} - \frac{1}{4n^2} \right] = \frac{3}{4} \end{aligned}$$

4. $y = x$ on $[-2, 3]$. (Note: $\Delta x = \frac{3 - (-2)}{n} = \frac{5}{n}$, $\|\Delta\| \rightarrow 0$ as $n \rightarrow \infty$)

$$\begin{aligned}\sum_{i=1}^n f(c_i) \Delta x_i &= \sum_{i=1}^n f\left(-2 + \frac{5i}{n}\right) \left(\frac{5}{n}\right) = \sum_{i=1}^n \left(-2 + \frac{5i}{n}\right) \left(\frac{5}{n}\right) = -10 + \frac{25}{n^2} \sum_{i=1}^n i \\ &= -10 + \left(\frac{25}{n^2}\right) \frac{n(n+1)}{2} = -10 + \frac{25}{2} \left(1 + \frac{1}{n}\right) = \frac{5}{2} + \frac{25}{2n} \\ \int_{-2}^3 x \, dx &= \lim_{n \rightarrow \infty} \left(\frac{5}{2} + \frac{25}{2n}\right) = \frac{5}{2}\end{aligned}$$

6. $y = 3x^2$ on $[1, 3]$. (Note: $\Delta x = \frac{3 - 1}{n} = \frac{2}{n}$, $\|\Delta\| \rightarrow 0$ as $n \rightarrow \infty$)

$$\begin{aligned}\sum_{i=1}^n f(c_i) \Delta x_i &= \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right) \left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n 3 \left(1 + \frac{2i}{n}\right)^2 \left(\frac{2}{n}\right) \\ &= \frac{6}{n} \sum_{i=1}^n \left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right) \\ &= \frac{6}{n} \left[n + \frac{4}{n} \frac{n(n+1)}{2} + \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} \right] \\ &= 6 + 12 \frac{n+1}{n} + 4 \frac{(n+1)(2n+1)}{n^2} \\ \int_1^3 3x^2 \, dx &= \lim_{n \rightarrow \infty} \left[6 + \frac{12(n+1)}{n} + \frac{4(n+1)(2n+1)}{n^2} \right] \\ &= 6 + 12 + 8 = 26\end{aligned}$$

8. $y = 3x^2 + 2$ on $[-1, 2]$. (Note: $\Delta x = \frac{2 - (-1)}{n} = \frac{3}{n}$, $\|\Delta\| \rightarrow 0$ as $n \rightarrow \infty$)

$$\begin{aligned}\sum_{i=1}^n f(c_i) \Delta x_i &= \sum_{i=1}^n f\left(-1 + \frac{3i}{n}\right) \left(\frac{3}{n}\right) \\ &= \frac{3}{n} \sum_{i=1}^n \left[3 \left(-1 + \frac{3i}{n}\right)^2 + 2 \right] \\ &= \frac{3}{n} \sum_{i=1}^n \left[3 \left(1 - \frac{6i}{n} + \frac{9i^2}{n^2}\right) + 2 \right] \\ &= \frac{3}{n} \left[3n - \frac{18}{n} \frac{n(n+1)}{2} + \frac{27}{n^2} \frac{n(n+1)(2n+1)}{6} + 2n \right] \\ &= 15 - \frac{27(n+1)}{n} + \frac{27}{2} \frac{(n+1)(2n+1)}{n^2} \\ \int_{-1}^2 (3x^2 + 2) \, dx &= \lim_{n \rightarrow \infty} \left[15 - 27 \frac{(n+1)}{n} + \frac{27}{2} \frac{(n+1)(2n+1)}{n^2} \right] \\ &= 15 - 27 + 27 = 15\end{aligned}$$

10. $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n 6c_i(4 - c_i)^2 \Delta x_i = \int_0^4 6x(4 - x)^2 \, dx$
on the interval $[0, 4]$.

12. $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \left(\frac{3}{c_i^2}\right) \Delta x_i = \int_1^3 \frac{3}{x^2} \, dx$
on the interval $[1, 3]$.

14. $\int_0^2 (4 - 2x) \, dx$

16. $\int_0^2 x^2 \, dx$

18. $\int_{-1}^1 \frac{1}{x^2 + 1} \, dx$

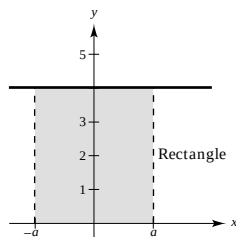
20. $\int_0^{\pi/4} \tan x \, dx$

22. $\int_0^2 (y - 2)^2 dy$

24. Rectangle

$$A = bh = 2(4)(a)$$

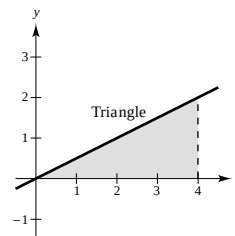
$$A = \int_{-a}^a 4 dx = 8a$$



26. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(4)(2)$$

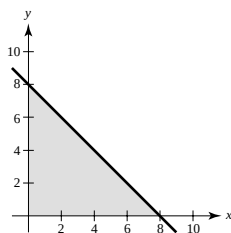
$$A = \int_0^4 \frac{x}{2} dx = 4$$



28. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(8)(8) = 32$$

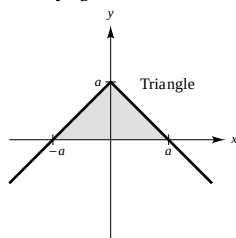
$$A = \int_0^8 (8 - x) dx = 32$$



30. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(2a)a$$

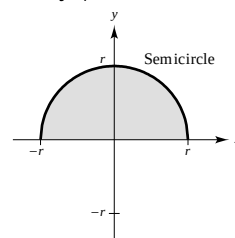
$$A = \int_{-a}^a (a - |x|) dx = a^2$$



32. Semicircle

$$A = \frac{1}{2}\pi r^2$$

$$A = \int_{-r}^r \sqrt{r^2 - x^2} dx = \frac{1}{2}\pi r^2$$



In Exercises 34–40, $\int_2^4 x^3 dx = 60$, $\int_2^4 x dx = 6$, $\int_2^4 dx = 2$.

34. $\int_2^2 x^3 dx = 0$

36. $\int_2^4 15 dx = 15 \int_2^4 dx = 15(2) = 30$

38. $\int_2^4 (x^3 + 4) dx = \int_2^4 x^3 dx + 4 \int_2^4 dx = 60 + 4(2) = 68$

40. $\int_2^4 (6 + 2x - x^3) dx = 6 \int_2^4 dx + 2 \int_2^4 x dx - \int_2^4 x^3 dx$
 $= 6(2) + 2(6) - 60 = -36$

42. (a) $\int_0^6 f(x) dx = \int_0^3 f(x) dx + \int_3^6 f(x) dx = 4 + (-1) = 3$

44. (a) $\int_{-1}^0 f(x) dx = \int_{-1}^1 f(x) dx - \int_0^1 f(x) dx = 0 - 5 = -5$

(b) $\int_6^3 f(x) dx = -\int_3^6 f(x) dx = -(-1) = 1$

(b) $\int_0^1 f(x) dx - \int_1^0 f(x) dx = 5 - (-5) = 10$

(c) $\int_3^3 f(x) dx = 0$

(c) $\int_{-1}^1 3f(x) dx = 3 \int_{-1}^1 f(x) dx = 3(0) = 0$

(d) $\int_3^6 -5f(x) dx = -5 \int_3^6 f(x) dx = -5(-1) = 5$

(d) $\int_0^1 3f(x) dx = 3 \int_0^1 f(x) dx = 3(5) = 15$

46. (a) $\int_0^5 [f(x) + 2] dx = \int_0^5 f(x) dx + \int_0^5 2 dx = 4 + 10 = 14$ (b) $\int_{-2}^3 f(x+2) dx = \int_0^5 f(x) dx = 4$ (Let $u = x + 2$.)
- (c) $\int_{-5}^5 f(x) dx = 2 \int_0^5 f(x) dx = 2(4) = 8$ (f even) (d) $\int_{-5}^5 f(x) dx = 0$ (f odd)

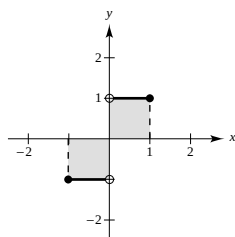
48. The right endpoint approximation will be less than the actual area: $<$

50. The average of Exercise 39 and Exercise 40 consists of a trapezoidal approximation, and is greater than the exact area: $>$

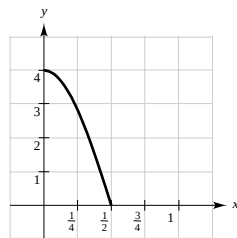
52. $f(x) = |x|/x$ is integrable on $[-1, 1]$, but is not continuous on $[-1, 1]$. There is discontinuity at $x = 0$. To see that

$$\int_{-1}^1 \frac{|x|}{x} dx$$

is integrable, sketch a graph of the region bounded by $f(x) = |x|/x$ and the x -axis for $-1 \leq x \leq 1$. You see that the integral equals 0.

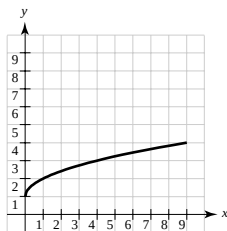


54.



b. $A \approx \frac{4}{3}$ square units

56.



c. Area ≈ 27 .

$$58. \int_0^3 \frac{5}{x^2 + 1} dx$$

n	4	8	12	16	20
$L(n)$	7.9224	7.0855	6.8062	6.6662	6.5822
$M(n)$	6.2485	6.2470	7.2460	6.2457	6.2455
$R(n)$	4.5474	5.3980	5.6812	5.8225	5.9072

$$60. \int_0^3 x \sin x dx$$

n	4	8	12	16	20
$L(n)$	2.8186	2.9985	3.0434	3.0631	3.0740
$M(n)$	3.1784	3.1277	3.1185	3.1152	3.1138
$R(n)$	3.1361	3.1573	3.1493	3.1425	3.1375

62. False

64. True

66. False

$$\int_0^1 x\sqrt{x} dx \neq \left(\int_0^1 x dx \right) \left(\int_0^1 \sqrt{x} dx \right)$$

$$\int_{-2}^4 x dx = 6$$

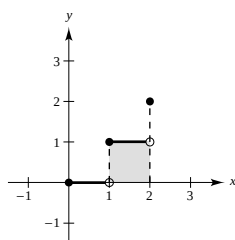
68. $f(x) = \sin x, [0, 2\pi]$

$$x_0 = 0, x_1 = \frac{\pi}{4}, x_2 = \frac{\pi}{3}, x_3 = \pi, x_4 = 2\pi$$

$$\Delta x_1 = \frac{\pi}{4}, \Delta x_2 = \frac{\pi}{12}, \Delta x_3 = \frac{2\pi}{3}, \Delta x_4 = \pi$$

$$c_1 = \frac{\pi}{6}, c_2 = \frac{\pi}{3}, c_3 = \frac{2\pi}{3}, c_4 = \frac{3\pi}{2}$$

$$\begin{aligned} \sum_{i=1}^4 f(c_i) \Delta x_i &= f\left(\frac{\pi}{6}\right) \Delta x_1 + f\left(\frac{\pi}{3}\right) \Delta x_2 + f\left(\frac{2\pi}{3}\right) \Delta x_3 + f\left(\frac{3\pi}{2}\right) \Delta x_4 \\ &= \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{12}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{2\pi}{3}\right) + (-1)(\pi) \approx -0.708 \end{aligned}$$

70. To find $\int_0^2 \llbracket x \rrbracket dx$, use a geometric approach.

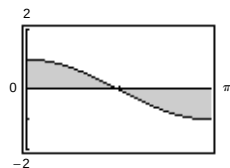
Thus,

$$\int_0^2 \llbracket x \rrbracket dx = 1(2 - 1) = 1.$$

Section 4.4 The Fundamental Theorem of Calculus

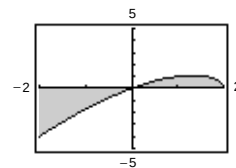
2. $f(x) = \cos x$

$$\int_0^{\pi} \cos x \, dx = 0$$



4. $f(x) = x\sqrt{2-x}$

$$\int_{-2}^2 x\sqrt{2-x} \, dx \text{ is negative.}$$



6. $\int_2^7 3 \, dv = \left[3v \right]_2^7 = 3(7) - 3(2) = 15$

8. $\int_2^5 (-3v + 4) \, dv = \left[-\frac{3}{2}v^2 + 4v \right]_2^5 = \left(-\frac{75}{2} + 20 \right) - (-6 + 8) = -\frac{39}{2}$

10.
$$\begin{aligned} \int_1^3 (3x^2 + 5x - 4) \, dx &= \left[x^3 + \frac{5x^2}{2} - 4x \right]_1^3 = \left(27 + \frac{45}{2} - 12 \right) - \left(1 + \frac{5}{2} - 4 \right) \\ &= 38 \end{aligned}$$

12. $\int_{-1}^1 (t^3 - 9t) \, dt = \left[\frac{1}{4}t^4 - \frac{9}{2}t^2 \right]_{-1}^1 = \left(\frac{1}{4} - \frac{9}{2} \right) - \left(\frac{1}{4} - \frac{9}{2} \right) = 0$

14. $\int_{-2}^{-1} \left(u - \frac{1}{u^2} \right) \, du = \left[\frac{u^2}{2} + \frac{1}{u} \right]_{-2}^{-1} = \left(\frac{1}{2} - 1 \right) - \left(2 - \frac{1}{2} \right) = -2$

$$16. \int_{-3}^3 v^{1/3} dv = \left[\frac{3}{4} v^{4/3} \right]_{-3}^3 = \frac{3}{4} [(\sqrt[3]{-3})^4] - (\sqrt[3]{-3})^4 = 0$$

$$18. \int_1^8 \sqrt{\frac{2}{x}} dx = \sqrt{2} \int_1^8 x^{-1/2} dx = \left[\sqrt{2}(2)x^{1/2} \right]_1^8 = \left[2\sqrt{2x} \right]_1^8 = 8 - 2\sqrt{2}$$

$$20. \int_0^2 (2-t)\sqrt{t} dt = \int_0^2 (2t^{1/2} - t^{3/2}) dt = \left[\frac{4}{3} t^{3/2} - \frac{2}{5} t^{5/2} \right]_0^2 = \left[\frac{t\sqrt{t}}{15} (20 - 6t) \right]_0^2 = \frac{2\sqrt{2}}{15} (20 - 12) = \frac{16\sqrt{2}}{15}$$

$$22. \int_{-8}^{-1} \frac{x - x^2}{2\sqrt[3]{x}} dx = \frac{1}{2} \int_{-8}^{-1} (x^{2/3} - x^{5/3}) dx$$

$$= \frac{1}{2} \left[\frac{3}{5} x^{5/3} - \frac{3}{8} x^{8/3} \right]_{-8}^{-1} = \left[\frac{x^{5/3}}{80} (24 - 15x) \right]_{-8}^{-1} = -\frac{1}{80} (39) + \frac{32}{80} (144) = \frac{4569}{80}$$

$$24. \int_1^4 (3 - 1x - 31) dx = \int_1^3 [3 + (x - 3)] dx + \int_3^4 [3 - (x - 3)] dx$$

$$= \int_1^3 x dx + \int_3^4 (6 - x) dx$$

$$= \left[\frac{x^2}{2} \right]_1^3 + \left[6x - \frac{x^2}{2} \right]_3^4$$

$$= \left(\frac{9}{2} - \frac{1}{2} \right) + \left[(24 - 8) - \left(18 - \frac{9}{2} \right) \right]$$

$$= 4 + 16 - 18 + \frac{9}{2} = \frac{13}{2}$$

$$26. \int_0^4 |x^2 - 4x + 3| dx = \int_0^1 (x^2 - 4x + 3) dx - \int_1^3 (x^2 - 4x + 3) dx + \int_3^4 (x^2 - 4x + 3) dx \quad (\text{split up the integral at the zeros } x = 1, 3)$$

$$= \left[\frac{x^3}{3} - 2x^2 + 3x \right]_0^1 - \left[\frac{x^3}{3} - 2x^2 + 3x \right]_1^3 + \left[\frac{x^3}{3} - 2x^2 + 3x \right]_3^4$$

$$= \left(\frac{1}{3} - 2 + 3 \right) - (9 - 18 + 9) + \left(\frac{1}{3} - 2 + 3 \right) + \left(\frac{64}{3} - 32 + 12 \right) - (9 - 18 + 9)$$

$$= \frac{4}{3} - 0 + \frac{4}{3} + \frac{4}{3} - 0 = 4$$

$$28. \int_0^{\pi/4} \frac{1 - \sin^2 \theta}{\cos^2 \theta} d\theta = \int_0^{\pi/4} d\theta = \left[\theta \right]_0^{\pi/4} = \frac{\pi}{4}$$

$$30. \int_{\pi/4}^{\pi/2} (2 - \csc^2 x) dx = \left[2x + \cot x \right]_{\pi/4}^{\pi/2} = (\pi + 0) - \left(\frac{\pi}{2} + 1 \right) = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$$

$$32. \int_{-\pi/2}^{\pi/2} (2t + \cos t) dt = \left[t^2 + \sin t \right]_{-\pi/2}^{\pi/2} = \left(\frac{\pi^2}{4} + 1 \right) - \left(\frac{\pi^2}{4} - 1 \right) = 2$$

$$34. P = \frac{2}{\pi} \int_0^{\pi/2} \sin \theta d\theta = \left[-\frac{2}{\pi} \cos \theta \right]_0^{\pi/2} = -\frac{2}{\pi} (0 - 1) = \frac{2}{\pi} \approx 63.7\%$$

$$36. A = \int_{-1}^1 (1 - x^4) dx = \left[x - \frac{1}{5} x^5 \right]_{-1}^1 = \frac{8}{5}$$

$$38. A = \int_1^2 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$

$$40. A = \int_0^{\pi} (x + \sin x) dx = \left[\frac{x^2}{2} - \cos x \right]_0^{\pi} = \frac{\pi^2}{2} + 2 = \frac{\pi^2 + 4}{2}$$

42. Since
- $y \geq 0$
- on
- $[0, 8]$
- ,

$$\text{Area} = \int_0^8 (1 + x^{1/3}) dx = \left[x + \frac{3}{4}x^{4/3} \right]_0^8 = 8 + \frac{3}{4}(16) = 20$$

44. Since
- $y \geq 0$
- on
- $[0, 3]$
- ,

$$A = \int_0^3 (3x - x^2) dx = \left[\frac{3}{2}x^2 - \frac{x^3}{3} \right]_0^3 = \frac{9}{2}.$$

$$46. \int_1^3 \frac{9}{x^3} dx = \left[-\frac{9}{2x^2} \right]_1^3 = -\frac{1}{2} + \frac{9}{2} = 4$$

$$f(c)(3 - 1) = 4$$

$$\frac{9}{c^3} = 2$$

$$c^3 = \frac{9}{2}$$

$$c = \sqrt[3]{\frac{9}{2}} \approx 1.6510$$

$$48. \int_{-\pi/3}^{\pi/3} \cos x dx = \left[\sin x \right]_{-\pi/3}^{\pi/3} = \sqrt{3}$$

$$f(c) \left[\frac{\pi}{3} - \left(-\frac{\pi}{3} \right) \right] = \sqrt{3}$$

$$\cos c = \frac{3\sqrt{3}}{2\pi}$$

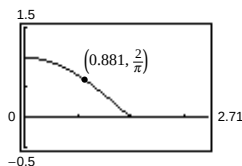
$$c \approx \pm 0.5971$$

$$52. \frac{1}{(\pi/2) - 0} \int_0^{\pi/2} \cos x dx = \left[\frac{2}{\pi} \sin x \right]_0^{\pi/2} = \frac{2}{\pi}$$

$$\text{Average value} = \frac{2}{\pi}$$

$$\cos x = \frac{2}{\pi}$$

$$x \approx 0.881$$

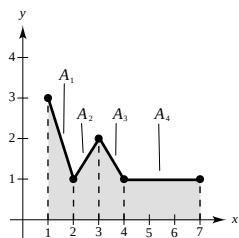


$$54. (a) \int_1^7 f(x) dx = \text{Sum of the areas}$$

$$= A_1 + A_2 + A_3 + A_4$$

$$= \frac{1}{2}(3 + 1) + \frac{1}{2}(1 + 2) + \frac{1}{2}(2 + 1) + (3)(1)$$

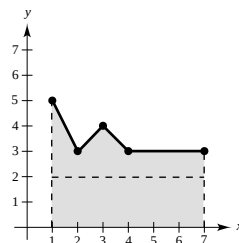
$$= 8$$



$$(b) \text{Average value} = \frac{\int_1^7 f(x) dx}{7 - 1} = \frac{8}{6} = \frac{4}{3}$$

$$(c) A = 8 + (6)(2) = 20$$

$$\text{Average value} = \frac{20}{6} = \frac{10}{3}$$



$$56. \int_2^6 f(x) dx = (\text{area or region } B) = \int_0^6 f(x) dx - \int_0^2 f(x) dx \\ = 3.5 - (-1.5) = 5.0$$

$$58. \int_0^2 -2f(x) dx = -2 \int_0^2 f(x) dx = -2(-1.5) = 3.0$$

$$60. \text{Average value} = \frac{1}{6} \int_0^6 f(x) dx = \frac{1}{6}(3.5) = 0.5833$$

$$62. \frac{1}{R-0} \int_0^R k(R^2 - r^2) dr = \frac{k}{R} \left[R^2 r - \frac{r^3}{3} \right]_0^R = \frac{2kR^2}{3}$$

$$64. P = 5(\sqrt{t} + 30)$$

(a)

t	1	2	3	4	5	6
P	155	157.071	158.660	160	161.180	162.247

$$\text{Average profit} \approx \frac{1}{6}(155 + 157.071 + 158.660 + 160 + 161.180 + 162.247) = \frac{954.158}{6} \approx 159.026$$

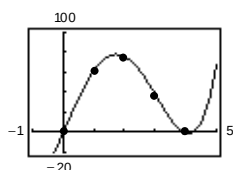
$$(b) \frac{1}{6} \int_{0.5}^{6.5} 5(\sqrt{t} + 30) dt = \frac{1}{6} \left[5 \left(\frac{2}{3} t^{3/2} + 30t \right) \right]_{0.5}^{6.5} \approx \frac{954.061}{6} \approx 159.010$$

(c) The definite integral yields a better approximation.

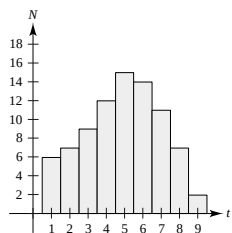
$$66. (a) R = 2.33t^4 - 14.67t^3 + 3.67t^2 + 70.67t$$

$$(c) \int_0^4 R(t) dt = \left[\frac{2.33t^5}{5} - \frac{14.67t^4}{4} + \frac{3.67t^3}{3} + \frac{70.67t^2}{2} \right]_0^4 \\ = 181.957$$

(b)



68. (a) histogram

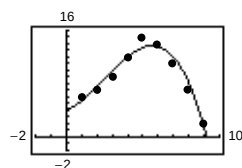


$$(b) [6 + 7 + 9 + 12 + 15 + 14 + 11 + 7 + 2]60 = (83)60 = 4980 \text{ customers}$$

(c) Using a graphing utility, you obtain

$$N(t) = -0.084175t^3 + 0.63492t^2 + 0.79052t + 4.10317.$$

(d)



$$(e) \int_0^9 N(t) dt \approx 85.162$$

The estimated number of customers is $(85.162)(60) \approx 5110$.

$$(f) \text{Between 3 P.M. and 7 P.M., the number of customers is approximately } \left(\int_3^7 N(t) dt \right) (60) \approx (50.28)(60) \approx 3017.$$

Hence, $3017/240 \approx 12.6$ per minute.

$$\begin{aligned}
 70. F(x) &= \int_2^x (t^3 + 2t - 2) dt = \left[\frac{t^4}{4} + t^2 - 2t \right]_2^x \\
 &= \left(\frac{x^4}{4} + x^2 - 2x \right) - (4 + 4 - 4) \\
 &= \frac{x^4}{4} + x^2 - 2x - 4
 \end{aligned}$$

$$F(2) = 4 + 4 - 4 - 4 = 0 \quad \left[\text{Note: } F(2) = \int_2^2 (t^3 + 2t - 2) dt = 0 \right]$$

$$F(5) = \frac{625}{4} + 25 - 10 - 4 = 167.25$$

$$F(8) = \frac{8^4}{4} + 64 - 16 - 4 = 1068$$

$$72. F(x) = \int_2^x \frac{-2}{t^3} dt = - \int_2^x 2t^{-3} dt = \left[\frac{1}{t^2} \right]_2^x = \frac{1}{x^2} - \frac{1}{4}$$

$$F(2) = \frac{1}{4} - \frac{1}{4} = 0$$

$$F(5) = \frac{1}{25} - \frac{1}{4} = -\frac{21}{100} = -0.21$$

$$F(8) = \frac{1}{64} - \frac{1}{4} = -\frac{15}{64}$$

$$74. F(x) = \int_0^x \sin \theta d\theta = -\cos \theta \Big|_0^x = -\cos x + \cos 0 = 1 - \cos x$$

$$F(2) = 1 - \cos 2 \approx 1.4161$$

$$F(5) = 1 - \cos 5 \approx 0.7163$$

$$F(8) = 1 - \cos 8 \approx 1.1455$$

$$76. (a) \int_0^x t(t^2 + 1) dt = \int_0^x (t^3 + t) dt = \left[\frac{1}{4}t^4 + \frac{1}{2}t^2 \right]_0^x = \frac{1}{4}x^4 + \frac{1}{2}x^2 = \frac{x^2}{4}(x^2 + 2)$$

$$(b) \frac{d}{dx} \left[\frac{1}{4}x^4 + \frac{1}{2}x^2 \right] = x^3 + x = x(x^2 + 1)$$

$$78. (a) \int_4^x \sqrt{t} dt = \left[\frac{2}{3}t^{3/2} \right]_4^x = \frac{2}{3}x^{3/2} - \frac{16}{3} = \frac{2}{3}(x^{3/2} - 8)$$

$$(b) \frac{d}{dx} \left[\frac{2}{3}x^{3/2} - \frac{16}{3} \right] = x^{1/2} = \sqrt{x}$$

$$80. (a) \int_{\pi/3}^x \sec t \tan t dt = \left[\sec t \right]_{\pi/3}^x = \sec x - 2$$

$$(b) \frac{d}{dx} [\sec x - 2] = \sec x \tan x$$

$$82. F(x) = \int_1^x \frac{t^2}{t^2 + 1} dt$$

$$F'(x) = \frac{x^2}{x^2 + 1}$$

$$84. F(x) = \int_1^x \sqrt[4]{t} dt$$

$$F'(x) = \sqrt[4]{x}$$

$$86. F(x) = \int_0^x \sec^3 t dt$$

$$F'(x) = \sec^3 x$$

$$88. F(x) = \int_{-x}^x t^3 dt = \left[\frac{t^4}{4} \right]_{-x}^x = 0$$

$$F'(x) = 0$$

Alternate solution

$$\begin{aligned} F(x) &= \int_{-x}^x t^3 dt \\ &= \int_{-x}^0 t^3 dt + \int_0^x t^3 dt \\ &= -\int_0^{-x} t^3 dt + \int_0^x t^3 dt \\ F'(x) &= -(-x)^3(-1) + (x^3) = 0 \end{aligned}$$

$$90. F(x) = \int_2^{x^2} t^{-3} dt = \left[\frac{t^{-2}}{-2} \right]_2^{x^2} = \left[-\frac{1}{2t^2} \right]_2^{x^2} = \frac{-1}{2x^4} + \frac{1}{8} \Rightarrow F'(x) = 2x^{-5}$$

Alternate solution: $F'(x) = (x^2)^{-3}(2x) = 2x^{-5}$

$$92. F(x) = \int_0^{x^2} \sin \theta^2 d\theta$$

$$F'(x) = \sin (x^2)^2 (2x) = 2x \sin x^4$$

94. (a)

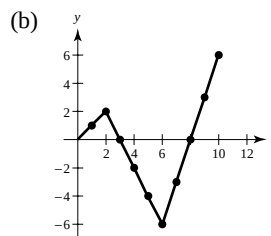
x	1	2	3	4	5	6	7	8	9	10
$g(x)$	1	2	0	-2	-4	-6	-3	0	3	6

(c) Minimum of g at $(6, -6)$.

(d) Minimum at $(10, 6)$. Relative maximum at $(2, 2)$.

(e) On $[6, 10]$ g increases at a rate of $\frac{12}{4} = 3$.

(f) Zeros of g : $x = 3, x = 8$.



$$96. (a) g(t) = 4 - \frac{4}{t^2}$$

$$\lim_{t \rightarrow \infty} g(t) = 4$$

Horizontal asymptote: $y = 4$

$$\begin{aligned} (b) A(x) &= \int_1^x \left(4 - \frac{4}{t^2} \right) dt \\ &= \left[4t + \frac{4}{t} \right]_1^x = 4x + \frac{4}{x} - 8 \\ &= \frac{4x^2 - 8x + 4}{x} = \frac{4(x-1)^2}{x} \end{aligned}$$

$$\lim_{x \rightarrow \infty} A(x) = \lim_{x \rightarrow \infty} \left(4x + \frac{4}{x} - 8 \right) = \infty + 0 - 8 = \infty$$

The graph of $A(x)$ does not have a horizontal asymptote.

98. True

100. Let $F(t)$ be an antiderivative of $f(t)$. Then,

$$\begin{aligned} \int_{u(x)}^{v(x)} f(t) dt &= \left[F(t) \right]_{u(x)}^{v(x)} = F(v(x)) - F(u(x)) \\ \frac{d}{dx} \left[\int_{u(x)}^{v(x)} f(t) dt \right] &= \frac{d}{dx} [F(v(x)) - F(u(x))] \\ &= F'(v(x))v'(x) - F'(u(x))u'(x) \\ &= f(v(x))v'(x) - f(u(x))u'(x). \end{aligned}$$

$$102. G(x) = \int_0^x \left[s \int_0^s f(t) dt \right] ds$$

$$(a) G(0) = \int_0^0 \left[s \int_0^s f(t) dt \right] ds = 0$$

$$(c) G''(x) = x \cdot f(x) + \int_0^x f(t) dt$$

$$(d) G''(0) = 0 \cdot f(0) + \int_0^0 f(t) dt = 0$$

$$(b) \text{ Let } F(s) = s \int_0^s f(t) dt.$$

$$G(x) = \int_0^x F(s) ds$$

$$G'(x) = F(x) = x \int_0^x f(t) dt$$

$$G'(0) = 0 \int_0^0 f(t) dt = 0$$

$$104. x(t) = (t-1)(t-3)^2 = t^3 - 7t^2 + 15t - 9$$

$$x'(t) = 3t^2 - 14t + 15$$

Using a graphing utility,

$$\text{Total distance} = \int_0^5 |x'(t)| dt \approx 27.37 \text{ units}$$

Section 4.5 Integration by Substitution

$$\frac{\int f(g(x))g'(x) dx}{\quad} \quad \frac{u = g(x)}{\quad} \quad \frac{du = g'(x) dx}{\quad}$$

$$2. \int x^2 \sqrt{x^3 + 1} dx \quad x^3 + 1 \quad 3x^2 dx$$

$$4. \int \sec 2x \tan 2x dx \quad 2x \quad 2 dx$$

$$6. \int \frac{\cos x}{\sin^2 x} dx \quad \sin x \quad \cos x dx$$

$$8. \int (x^2 - 9)^3 (2x) dx = \frac{(x^2 - 9)^4}{4} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(x^2 - 9)^4}{4} + C \right] = \frac{4(x^2 - 9)^3}{4} (2x) = (x^2 - 9)^3 (2x)$$

$$10. \int (1 - 2x^2)^{1/3} (-4x) dx = \frac{3}{4} (1 - 2x^2)^{4/3} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{3}{4} (1 - 2x^2)^{4/3} + C \right] = \frac{3}{4} \cdot \frac{4}{3} (1 - 2x^2)^{1/3} (-4x) = (1 - 2x^2)^{1/3} (-4x)$$

$$12. \int x^2 (x^3 + 5)^4 dx = \frac{1}{3} \int (x^3 + 5)^4 (3x^2) dx = \frac{1}{3} \frac{(x^3 + 5)^5}{5} + C = \frac{(x^3 + 5)^5}{15} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(x^3 + 5)^5}{15} + C \right] = \frac{5(x^3 + 5)^4 (3x^2)}{15} = (x^3 + 5)^4 x^2$$

$$14. \int x(4x^2 + 3)^3 dx = \frac{1}{8} \int (4x^2 + 3)^3 (8x) dx = \frac{1}{8} \left[\frac{(4x^2 + 3)^4}{4} \right] + C = \frac{(4x^2 + 3)^4}{32} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(4x^2 + 3)^4}{32} + C \right] = \frac{4(4x^2 + 3)^3 (8x)}{32} = x(4x^2 + 3)^3$$

$$16. \int t^3 \sqrt{t^4 + 5} \, dt = \frac{1}{4} \int (t^4 + 5)^{1/2} (4t^3) \, dt = \frac{1}{4} \frac{(t^4 + 5)^{3/2}}{3/2} + C = \frac{1}{6} (t^4 + 5)^{3/2} + C$$

$$\text{Check: } \frac{d}{dt} \left[\frac{1}{6} (t^4 + 5)^{3/2} + C \right] = \frac{1}{6} \cdot \frac{3}{2} (t^4 + 5)^{1/2} (4t^3) = (t^4 + 5)^{1/2} (t^3)$$

$$18. \int u^2 \sqrt{u^3 + 2} \, du = \frac{1}{3} \int (u^3 + 2)^{1/2} (3u^2) \, du = \frac{1}{3} \frac{(u^3 + 2)^{3/2}}{3/2} + C = \frac{2(u^3 + 2)^{3/2}}{9} + C$$

$$\text{Check: } \frac{d}{du} \left[\frac{2(u^3 + 2)^{3/2}}{9} + C \right] = \frac{2}{9} \cdot \frac{3}{2} (u^3 + 2)^{1/2} (3u^2) = (u^3 + 2)^{1/2} (u^2)$$

$$20. \int \frac{x^3}{(1 + x^4)^2} \, dx = \frac{1}{4} \int (1 + x^4)^{-2} (4x^3) \, dx = -\frac{1}{4} (1 + x^4)^{-1} + C = \frac{-1}{4(1 + x^4)} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{-1}{4(1 + x^4)} + C \right] = \frac{1}{4} (1 + x^4)^{-2} (4x^3) = \frac{x^3}{(1 + x^4)^2}$$

$$22. \int \frac{x^2}{(16 - x^3)^2} \, dx = -\frac{1}{3} \int (16 - x^3)^{-2} (-3x^2) \, dx = -\frac{1}{3} \left[\frac{(16 - x^3)^{-1}}{-1} \right] + C = \frac{1}{3(16 - x^3)} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{1}{3(16 - x^3)} + C \right] = \frac{1}{3} (-1) (16 - x^3)^{-2} (-3x^2) = \frac{x^2}{(16 - x^3)^2}$$

$$24. \int \frac{x^3}{\sqrt{1 + x^4}} \, dx = \frac{1}{4} \int (1 + x^4)^{-1/2} (4x^3) \, dx = \frac{1}{4} \frac{(1 + x^4)^{1/2}}{1/2} + C = \frac{\sqrt{1 + x^4}}{2} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{\sqrt{1 + x^4}}{2} + C \right] = \frac{1}{2} \cdot \frac{1}{2} (1 + x^4)^{-1/2} (4x^3) = \frac{x^3}{\sqrt{1 + x^4}}$$

$$26. \int \left[x^2 + \frac{1}{(3x)^2} \right] \, dx = \int \left(x^2 + \frac{1}{9} x^{-2} \right) \, dx = \frac{x^3}{3} + \frac{1}{9} \left(\frac{x^{-1}}{-1} \right) + C = \frac{x^3}{3} - \frac{1}{9x} + C = \frac{3x^4 - 1}{9x} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{1}{3} x^3 - \frac{1}{9} x^{-1} + C \right] = x^2 + \frac{1}{9} x^{-2} = x^2 + \frac{1}{(3x)^2}$$

$$28. \int \frac{1}{2\sqrt{x}} \, dx = \frac{1}{2} \int x^{-1/2} \, dx = \frac{1}{2} \left(\frac{x^{1/2}}{1/2} \right) + C = \sqrt{x} + C$$

$$\text{Check: } \frac{d}{dx} [\sqrt{x} + C] = \frac{1}{2\sqrt{x}}$$

$$30. \int \frac{t + 2t^2}{\sqrt{t}} \, dt = \int (t^{1/2} + 2t^{3/2}) \, dt = \frac{2}{3} t^{3/2} + \frac{4}{5} t^{5/2} + C = \frac{2}{15} t^{3/2} (5 + 6t) + C$$

$$\text{Check: } \frac{d}{dt} \left[\frac{2}{3} t^{3/2} + \frac{4}{5} t^{5/2} + C \right] = t^{1/2} + 2t^{3/2} = \frac{t + 2t^2}{\sqrt{t}}$$

$$32. \int \left(\frac{t^3}{3} + \frac{1}{4t^2} \right) \, dt = \int \left(\frac{1}{3} t^3 + \frac{1}{4} t^{-2} \right) \, dt = \frac{1}{3} \left(\frac{t^4}{4} \right) + \frac{1}{4} \left(\frac{t^{-1}}{-1} \right) + C = \frac{1}{12} t^4 - \frac{1}{4t} + C$$

$$\text{Check: } \frac{d}{dt} \left[\frac{1}{12} t^4 - \frac{1}{4t} + C \right] = \frac{1}{3} t^3 + \frac{1}{4t^2}$$

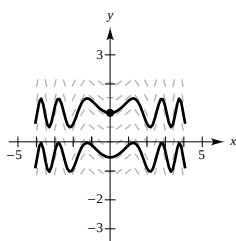
$$34. \int 2\pi y(8 - y^{3/2}) dy = 2\pi \int (8y - y^{5/2}) dy = 2\pi \left(4y^2 - \frac{2}{7}y^{7/2} \right) + C = \frac{4\pi y^2}{7}(14 - y^{3/2}) + C$$

Check: $\frac{d}{dy} \left[\frac{4\pi y^2}{7}(14 - y^{3/2}) + C \right] = \frac{d}{dy} \left[2\pi \left(4y^2 - \frac{2}{7}y^{7/2} \right) + C \right] = 16\pi y - 2\pi y^{5/2} = (2\pi y)(8 - y^{3/2})$

$$\begin{aligned} 36. y &= \int \frac{10x^2}{\sqrt{1+x^3}} dx \\ &= \frac{10}{3} \int (1+x^3)^{-1/2} (3x^2) dx \\ &= \frac{10}{3} \left[\frac{(1+x^3)^{1/2}}{1/2} \right] + C \\ &= \frac{20}{3} \sqrt{1+x^3} + C \end{aligned}$$

$$\begin{aligned} 38. y &= \int \frac{x-4}{\sqrt{x^2-8x+1}} dx \\ &= \frac{1}{2} \int (x^2-8x+1)^{-1/2} (2x-8) dx \\ &= \frac{1}{2} \left[\frac{(x^2-8x+1)^{1/2}}{1/2} \right] + C \\ &= \sqrt{x^2-8x+1} + C \end{aligned}$$

40. (a)



(b) $\frac{dy}{dx} = x \cos x^2, (0, 1)$

$$\begin{aligned} y &= \int x \cos x^2 dx = \frac{1}{2} \int \cos(x^2) 2x dx \\ &= \frac{1}{2} \sin(x^2) + C \end{aligned}$$

$(0, 1): 1 = \frac{1}{2} \sin(0) + C \Rightarrow C = 1$

$$y = \frac{1}{2} \sin(x^2) + 1$$

$$42. \int 4x^3 \sin x^4 dx = \int \sin x^4 (4x^3) dx = -\cos x^4 + C$$

$$44. \int \cos 6x dx = \frac{1}{6} \int (\cos 6x)(6) dx = \frac{1}{6} \sin 6x + C$$

$$46. \int x \sin x^2 dx = \frac{1}{2} \int (\sin x^2)(2x) dx = -\frac{1}{2} \cos x^2 + C$$

$$48. \int \sec(1-x) \tan(1-x) dx = - \int [\sec(1-x) \tan(1-x)](-1) dx = -\sec(1-x) + C$$

$$50. \int \sqrt{\tan x} \sec^2 x dx = \frac{(\tan x)^{3/2}}{3/2} + C = \frac{2}{3} (\tan x)^{3/2} + C$$

$$52. \int \frac{\sin x}{\cos^3 x} dx = - \int (\cos x)^{-3} (-\sin x) dx = -\frac{(\cos x)^{-2}}{-2} + C = \frac{1}{2 \cos^2 x} + C = \frac{1}{2} \sec^2 x + C$$

$$54. \int \csc^2\left(\frac{x}{2}\right) dx = 2 \int \csc^2\left(\frac{x}{2}\right) \left(\frac{1}{2}\right) dx = -2 \cot\left(\frac{x}{2}\right) + C$$

$$56. f(x) = \int \pi \sec \pi x \tan \pi x dx = \sec \pi x + C$$

Since $f(1/3) = 1 = \sec(\pi/3) + C$, $C = -1$. Thus

$$f(x) = \sec \pi x - 1.$$

58. $u = 2x + 1, x = \frac{1}{2}(u - 1), dx = \frac{1}{2} du$

$$\begin{aligned}\int x\sqrt{2x+1} dx &= \int \frac{1}{2}(u-1)\sqrt{u} \frac{1}{2} du \\ &= \frac{1}{4} \int (u^{3/2} - u^{1/2}) du \\ &= \frac{1}{4} \left(\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) + C \\ &= \frac{u^{3/2}}{30} (3u - 5) + C \\ &= \frac{1}{30} (2x+1)^{3/2} [3(2x+1) - 5] + C \\ &= \frac{1}{30} (2x+1)^{3/2} (6x-2) + C \\ &= \frac{1}{15} (2x+1)^{3/2} (3x-1) + C\end{aligned}$$

60. $u = 2 - x, x = 2 - u, dx = -du$

$$\begin{aligned}\int (x+1)\sqrt{2-x} dx &= -\int (3-u)\sqrt{u} du \\ &= -\int (3u^{1/2} - u^{3/2}) du \\ &= -\left(2u^{3/2} - \frac{2}{5} u^{5/2} \right) + C \\ &= -\frac{2u^{3/2}}{5} (5-u) + C \\ &= -\frac{2}{5} (2-x)^{3/2} [5 - (2-x)] + C \\ &= -\frac{2}{5} (2-x)^{3/2} (x+3) + C\end{aligned}$$

62. Let $u = x + 4, x = u - 4, du = dx$.

$$\begin{aligned}\int \frac{2x+1}{\sqrt{x+4}} dx &= \int \frac{2(u-4)+1}{\sqrt{u}} du \\ &= \int (2u^{1/2} - 7u^{-1/2}) du \\ &= \frac{4}{3} u^{3/2} - 14u^{1/2} + C \\ &= \frac{2}{3} u^{1/2} (2u - 21) + C \\ &= \frac{2}{3} \sqrt{x+4} [2(x+4) - 21] + C \\ &= \frac{2}{3} \sqrt{x+4} (2x-13) + C\end{aligned}$$

64. $u = t - 4, t = u + 4, dt = du$

$$\begin{aligned}\int t \sqrt[3]{t-4} dt &= \int (u+4)u^{1/3} du \\ &= \int (u^{4/3} + 4u^{1/3}) du \\ &= \frac{3}{7} u^{7/3} + 3u^{4/3} + C \\ &= \frac{3u^{4/3}}{7} (u+7) + C \\ &= \frac{3}{7} (t-4)^{4/3} [(t-4) + 7] + C \\ &= \frac{3}{7} (t-4)^{4/3} (t+3) + C\end{aligned}$$

66. Let $u = x^3 + 8, du = 3x^2 dx$.

$$\begin{aligned}\int_{-2}^4 x^2(x^3+8)^2 dx &= \frac{1}{3} \int_{-2}^4 (x^3+8)^2 (3x^2) dx = \left[\frac{1}{3} \frac{(x^3+8)^3}{3} \right]_{-2}^4 \\ &= \frac{1}{9} [(64+8)^3 - (-8+8)^3] = 41,472\end{aligned}$$

68. Let $u = 1 - x^2, du = -2x dx$.

$$\int_0^1 x\sqrt{1-x^2} dx = -\frac{1}{2} \int_0^1 (1-x^2)^{1/2} (-2x) dx = \left[-\frac{1}{3} (1-x^2)^{3/2} \right]_0^1 = 0 + \frac{1}{3} = \frac{1}{3}$$

70. Let $u = 1 + 2x^2, du = 4x dx$.

$$\int_0^2 \frac{x}{\sqrt{1+2x^2}} dx = \frac{1}{4} \int_0^2 (1+2x^2)^{-1/2} (4x) dx = \left[\frac{1}{2} \sqrt{1+2x^2} \right]_0^2 = \frac{3}{2} - \frac{1}{2} = 1$$

72. Let $u = 4 + x^2$, $du = 2x \, dx$.

$$\int_0^2 x \sqrt[3]{4 + x^2} \, dx = \frac{1}{2} \int_0^2 (4 + x^2)^{1/3} (2x) \, dx = \left[\frac{3}{8} (4 + x^2)^{4/3} \right]_0^2 = \frac{3}{8} (8^{4/3} - 4^{4/3}) = 6 - \frac{3}{2} \sqrt[3]{4} \approx 3.619$$

74. Let $u = 2x - 1$, $du = 2 \, dx$, $x = \frac{1}{2}(u + 1)$.

When $x = 1$, $u = 1$. When $x = 5$, $u = 9$.

$$\begin{aligned} \int_1^5 \frac{x}{\sqrt{2x-1}} \, dx &= \int_1^9 \frac{1/2(u+1)}{\sqrt{u}} \frac{1}{2} \, du = \frac{1}{4} \int_1^9 (u^{1/2} + u^{-1/2}) \, du \\ &= \frac{1}{4} \left[\frac{2}{3} u^{3/2} + 2u^{1/2} \right]_1^9 \\ &= \frac{1}{4} \left[\left(\frac{2}{3} (27) + 2(3) \right) - \left(\frac{2}{3} + 2 \right) \right] \\ &= \frac{16}{3} \end{aligned}$$

76. $\int_{\pi/3}^{\pi/2} (x + \cos x) \, dx = \left[\frac{x^2}{2} + \sin x \right]_{\pi/3}^{\pi/2} = \left(\frac{\pi^2}{8} + 1 \right) - \left(\frac{\pi^2}{18} + \frac{\sqrt{3}}{2} \right) = \frac{5\pi^2}{72} + \frac{2 - \sqrt{3}}{2}$

78. $u = x + 2$, $x = u - 2$, $dx = du$

When $x = -2$, $u = 0$. When $x = 6$, $u = 8$.

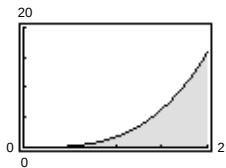
$$\text{Area} = \int_{-2}^6 x^2 \sqrt[3]{x+2} \, dx = \int_0^8 (u-2)^2 \sqrt[3]{u} \, du = \int_0^8 (u^{7/3} - 4u^{4/3} + 4u^{1/3}) \, du = \left[\frac{3}{10} u^{10/3} - \frac{12}{7} u^{7/3} + 3u^{4/3} \right]_0^8 = \frac{4752}{35}$$

80. $A = \int_0^\pi (\sin x + \cos 2x) \, dx = \left[-\cos x + \frac{1}{2} \sin 2x \right]_0^\pi = 2$

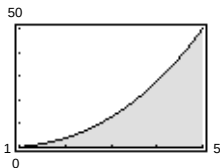
82. Let $u = 2x$, $du = 2 \, dx$.

$$\text{Area} = \int_{\pi/12}^{\pi/4} \csc 2x \cot 2x \, dx = \frac{1}{2} \int_{\pi/12}^{\pi/4} \csc 2x \cot 2x (2) \, dx = \left[-\frac{1}{2} \csc 2x \right]_{\pi/12}^{\pi/4} = \frac{1}{2}$$

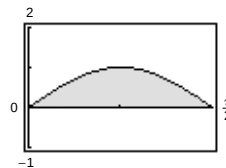
84. $\int_0^2 x^3 \sqrt{x+2} \, dx \approx 7.581$



86. $\int_1^5 x^2 \sqrt{x-1} \, dx \approx 67.505$



88. $\int_0^{\pi/2} \sin 2x \, dx \approx 1.0$



90. $\int \sin x \cos x \, dx = \int (\sin x)^1 (\cos x) \, dx = \frac{\sin^2 x}{2} + C_1$

$$\int \sin x \cos x \, dx = - \int (\cos x)^1 (-\sin x) \, dx = -\frac{\cos^2 x}{2} + C_2$$

$$-\frac{\cos^2 x}{2} + C_2 = -\frac{(1 - \sin^2 x)}{2} + C_2 = \frac{\sin^2 x}{2} - \frac{1}{2} + C_2$$

They differ by a constant: $C_2 = C_1 + \frac{1}{2}$.

92. $f(x) = \sin^2 x \cos x$ is even.

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x \, dx &= \int_0^{\pi/2} \sin^2 x (\cos x) \, dx \\ &= 2 \left[\frac{\sin^3 x}{3} \right]_0^{\pi/2} \\ &= \frac{2}{3}\end{aligned}$$

94. $f(x) = \sin x \cos x$ is odd.

$$\int_{-\pi/2}^{\pi/2} \sin x \cos x \, dx = 0$$

96. (a) $\int_{-\pi/4}^{\pi/4} \sin x \, dx = 0$ since $\sin x$ is symmetric to the origin.

(b) $\int_{-\pi/4}^{\pi/4} \cos x \, dx = 2 \int_0^{\pi/4} \cos x \, dx = \left[2 \sin x \right]_0^{\pi/4} = \sqrt{2}$ since $\cos x$ is symmetric to the y-axis.

(c) $\int_{-\pi/2}^{\pi/2} \cos x \, dx = 2 \int_0^{\pi/2} \cos x \, dx = \left[2 \sin x \right]_0^{\pi/2} = 2$

(d) $\int_{-\pi/2}^{\pi/2} \sin x \cos x \, dx = 0$ since $\sin(-x) \cos(-x) = -\sin x \cos x$ and hence, is symmetric to the origin.

98. $\int_{-\pi}^{\pi} (\sin 3x + \cos 3x) \, dx = \int_{-\pi}^{\pi} \sin 3x \, dx + \int_{-\pi}^{\pi} \cos 3x \, dx = 0 + 2 \int_0^{\pi} \cos 3x \, dx = \left[\frac{2}{3} \sin 3x \right]_0^{\pi} = 0$

100. If $u = 5 - x^2$, then $du = -2x \, dx$ and $\int x(5 - x^2)^3 \, dx = -\frac{1}{2} \int (5 - x^2)^3 (-2x) \, dx = -\frac{1}{2} \int u^3 \, du$.

102. $\frac{dQ}{dt} = k(100 - t)^2$

$$Q(t) = \int k(100 - t)^2 \, dt = -\frac{k}{3}(100 - t)^3 + C$$

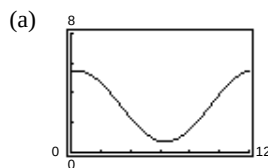
$$Q(100) = C = 0$$

$$Q(t) = -\frac{k}{3}(100 - t)^3$$

$$Q(0) = -\frac{k}{3}(100)^3 = 2,000,000 \Rightarrow k = -6$$

Thus, $Q(t) = 2(100 - t)^3$. When $t = 50$, $Q(50) = \$250,000$.

104. $R = 3.121 + 2.399 \sin(0.524t + 1.377)$

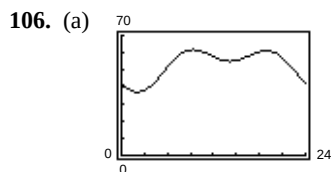


Relative minimum: (6.4, 0.7) or June

Relative maximum: (0.4, 5.5) or January

(b) $\int_0^{12} R(t) \, dt \approx 37.47$ inches

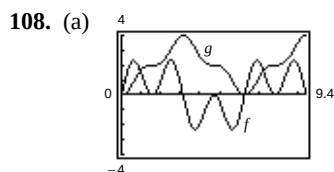
(c) $\frac{1}{3} \int_9^{12} R(t) \, dt \approx \frac{1}{3}(13) = 4.33$ inches



Maximum flow: $R \approx 61.713$ at $t = 9.36$.

[18.861, 61.178] is a relative maximum.]

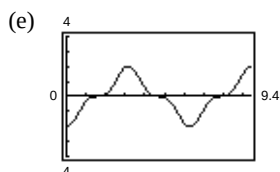
(b) Volume = $\int_0^{24} R(t) \, dt \approx 1272$ (5 thousand of gallons)



(b) g is nonnegative because the graph of f is positive at the beginning, and generally has more positive sections than negative ones.

(c) The points on g that correspond to the extrema of f are points of inflection of g .

(d) No, some zeros of f , like $x = \pi/2$, do not correspond to an extrema of g . The graph of g continues to increase after $x = \pi/2$ because f remains above the x -axis.



The graph of h is that of g shifted 2 units downward.

$$\begin{aligned} g(t) &= \int_0^t f(x) dx \\ &= \int_0^{\pi/2} f(x) dx + \int_{\pi/2}^t f(x) dx = 2 + h(t). \end{aligned}$$

110. False

$$\int x(x^2 + 1)^2 dx = \frac{1}{2} \int (x^2 + 1)(2x) dx = \frac{1}{4}(x^2 + 1)^2 + C$$

112. True

$$\int_a^b \sin x dx = [-\cos x]_a^b = -\cos b + \cos a = -\cos(b + 2\pi) + \cos a = \int_a^{b+2\pi} \sin x dx$$

114. False

$$\int \sin^2 2x \cos 2x dx = \frac{1}{2} \int (\sin 2x)^2 (2 \cos 2x) dx = \frac{1}{2} \frac{(\sin 2x)^3}{3} + C = \frac{1}{6} \sin^3 2x + C$$

116. Because f is odd, $f(-x) = -f(x)$. Then

$$\begin{aligned} \int_{-a}^a f(x) dx &= \int_{-a}^0 f(x) dx + \int_0^a f(x) dx \\ &= -\int_0^{-a} f(x) dx + \int_0^a f(x) dx. \end{aligned}$$

Let $x = -u$, $dx = -du$ in the first integral.

When $x = 0$, $u = 0$. When $x = -a$, $u = a$.

$$\begin{aligned} \int_{-a}^a f(x) dx &= -\int_0^a f(-u)(-du) + \int_0^a f(x) dx \\ &= -\int_0^a f(u) du + \int_0^a f(x) dx = 0 \end{aligned}$$

Section 4.6 Numerical Integration

2. Exact: $\int_0^1 \left(\frac{x^2}{2} + 1 \right) dx = \left[\frac{x^3}{6} + x \right]_0^1 = \frac{7}{6} \approx 1.1667$

Trapezoidal: $\int_0^1 \left(\frac{x^2}{2} + 1 \right) dx \approx \frac{1}{8} \left[1 + 2 \left(\frac{(1/4)^2}{2} + 1 \right) + 2 \left(\frac{(1/2)^2}{2} + 1 \right) + 2 \left(\frac{(3/4)^2}{2} + 1 \right) + \left(\frac{1^2}{2} + 1 \right) \right] = \frac{75}{64} \approx 1.1719$

Simpson's: $\int_0^1 \left(\frac{x^2}{2} + 1 \right) dx \approx \frac{1}{12} \left[1 + 4 \left(\frac{(1/4)^2}{2} + 1 \right) + 2 \left(\frac{(1/2)^2}{2} + 1 \right) + 4 \left(\frac{(3/4)^2}{2} + 1 \right) + \left(\frac{1^2}{2} + 1 \right) \right] = \frac{7}{6} \approx 1.1667$

4. Exact: $\int_1^2 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^2 = 0.5000$

Trapezoidal: $\int_1^2 \frac{1}{x^2} dx \approx \frac{1}{8} \left[1 + 2 \left(\frac{4}{5} \right)^2 + 2 \left(\frac{4}{6} \right)^2 + 2 \left(\frac{4}{7} \right)^2 + \frac{1}{4} \right] \approx 0.5090$

Simpson's: $\int_1^2 \frac{1}{x^2} dx \approx \frac{1}{12} \left[1 + 4 \left(\frac{4}{5} \right)^2 + 2 \left(\frac{4}{6} \right)^2 + 4 \left(\frac{4}{7} \right)^2 + \frac{1}{4} \right] \approx 0.5004$

6. Exact: $\int_0^8 \sqrt[3]{x} dx = \left[\frac{3}{4} x^{4/3} \right]_0^8 = 12.0000$

Trapezoidal: $\int_0^8 \sqrt[3]{x} dx \approx \frac{1}{2} [0 + 2 + 2\sqrt[3]{2} + 2\sqrt[3]{3} + 2\sqrt[3]{4} + 2\sqrt[3]{5} + 2\sqrt[3]{6} + 2\sqrt[3]{7} + 2] \approx 11.7296$

Simpson's: $\int_0^8 \sqrt[3]{x} dx \approx \frac{1}{3} [0 + 4 + 2\sqrt[3]{2} + 4\sqrt[3]{3} + 2\sqrt[3]{4} + 4\sqrt[3]{5} + 2\sqrt[3]{6} + 4\sqrt[3]{7} + 2] \approx 11.8632$

8. Exact: $\int_1^3 (4 - x^2) dx = \left[4x - \frac{x^3}{3} \right]_1^3 = 3 - \frac{11}{3} = -\frac{2}{3} \approx -0.6667$

Trapezoidal: $\int_1^3 (4 - x^2) dx \approx \frac{1}{4} \left\{ 3 + 2 \left[4 - \left(\frac{3}{2} \right)^2 \right] + 2(0) + 2 \left[4 - \left(\frac{5}{2} \right)^2 \right] - 5 \right\} = -0.7500$

Simpson's: $\int_1^3 (4 - x^2) dx \approx \frac{1}{6} \left[3 + 4 \left(4 - \frac{9}{4} \right) + 0 + 4 \left(4 - \frac{25}{4} \right) - 5 \right] \approx -0.6667$

10. Exact: $\int_0^2 x \sqrt{x^2 + 1} dx = \frac{1}{3} \left[(x^2 + 1)^{3/2} \right]_0^2 = \frac{1}{3} (5^{3/2} - 1) \approx 3.393$

Trapezoidal: $\int_0^2 x \sqrt{x^2 + 1} dx \approx \frac{1}{4} \left[0 + 2 \left(\frac{1}{2} \right) \sqrt{(1/2)^2 + 1} + 2(1) \sqrt{1^2 + 1} + 2 \left(\frac{3}{2} \right) \sqrt{(3/2)^2 + 1} + 2 \sqrt{2^2 + 1} \right] \approx 3.457$

Simpson's: $\int_0^2 x \sqrt{x^2 + 1} dx \approx \frac{1}{6} \left[0 + 4 \left(\frac{1}{2} \right) \sqrt{(1/2)^2 + 1} + 2(1) \sqrt{1^2 + 1} + 4 \left(\frac{3}{2} \right) \sqrt{(3/2)^2 + 1} + 2 \sqrt{2^2 + 1} \right] \approx 3.392$

12. Trapezoidal: $\int_0^2 \frac{1}{\sqrt{1+x^3}} dx \approx \frac{1}{4} \left[1 + 2 \left(\frac{1}{\sqrt{1+(1/2)^3}} \right) + 2 \left(\frac{1}{\sqrt{1+1^3}} \right) + 2 \left(\frac{1}{\sqrt{1+(3/2)^3}} \right) + \frac{1}{3} \right] \approx 1.397$

Simpson's: $\int_0^2 \frac{1}{\sqrt{1+x^3}} dx \approx \frac{1}{6} \left[1 + 4 \left(\frac{1}{\sqrt{1+(1/2)^3}} \right) + 2 \left(\frac{1}{\sqrt{1+1^3}} \right) + 4 \left(\frac{1}{\sqrt{1+(3/2)^3}} \right) + \frac{1}{3} \right] \approx 1.405$

Graphing utility: 1.402

14. Trapezoidal: $\int_{\pi/2}^{\pi} \sqrt{x} \sin x \, dx \approx \frac{\pi}{16} \left[\sqrt{\frac{\pi}{2}}(1) + 2\sqrt{\frac{5\pi}{8}} \sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}} \sin\left(\frac{3\pi}{4}\right) + 2\sqrt{\frac{7\pi}{8}} \sin\left(\frac{7\pi}{8}\right) + 0 \right] \approx 1.430$

Simpson's: $\int_{\pi/2}^{\pi} \sqrt{x} \sin x \, dx \approx \frac{\pi}{24} \left[\sqrt{\frac{\pi}{2}} + 4\sqrt{\frac{5\pi}{8}} \sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}} \sin\left(\frac{3\pi}{4}\right) + 4\sqrt{\frac{7\pi}{8}} \sin\left(\frac{7\pi}{8}\right) + 0 \right] \approx 1.458$

Graphing utility: 1.458

16. Trapezoidal: $\int_0^{\sqrt{\pi/4}} \tan(x^2) \, dx \approx \frac{\sqrt{\pi/4}}{8} \left[\tan 0 + 2 \tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2 \tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 2 \tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2 \right]$
 ≈ 0.271

Simpson's: $\int_0^{\sqrt{\pi/4}} \tan(x^2) \, dx \approx \frac{\sqrt{\pi/4}}{12} \left[\tan 0 + 4 \tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2 \tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 4 \tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2 \right]$
 ≈ 0.257

Graphing utility: 0.256

18. Trapezoidal: $\int_0^{\pi/2} \sqrt{1 + \cos^2 x} \, dx \approx \frac{\pi}{16} \left[\sqrt{2} + 2\sqrt{1 + \cos^2(\pi/8)} + 2\sqrt{1 + \cos^2(\pi/4)} + 2\sqrt{1 + \cos^2(3\pi/8)} + 1 \right] \approx 1.910$

Simpson's: $\int_0^{\pi/2} \sqrt{1 + \cos^2 x} \, dx \approx \frac{\pi}{24} \left[\sqrt{2} + 4\sqrt{1 + \cos^2(\pi/8)} + 2\sqrt{1 + \cos^2(\pi/4)} + 4\sqrt{1 + \cos^2(3\pi/8)} + 1 \right] \approx 1.910$

Graphing utility: 1.910

20. Trapezoidal: $\int_0^{\pi} \frac{\sin x}{x} \, dx \approx \frac{\pi}{8} \left[1 + \frac{2 \sin(\pi/4)}{\pi/4} + \frac{2 \sin(\pi/2)}{\pi/2} + \frac{2 \sin(3\pi/4)}{3\pi/4} + 0 \right] \approx 1.836$

Simpson's: $\int_0^{\pi} \frac{\sin x}{x} \, dx \approx \frac{\pi}{12} \left[1 + \frac{4 \sin(\pi/4)}{\pi/4} + \frac{2 \sin(\pi/2)}{\pi/2} + \frac{4 \sin(3\pi/4)}{3\pi/4} + 0 \right] \approx 1.852$

Graphing utility: 1.852

22. Trapezoidal: Linear polynomials

Simpson's: Quadratic polynomials

24. $f(x) = \frac{1}{x+1}$

$$f'(x) = \frac{-1}{(x+1)^2}$$

$$f''(x) = \frac{2}{(x+1)^3}$$

$$f'''(x) = \frac{-6}{(x+1)^4}$$

$$f^{(4)}(x) = \frac{24}{(x+1)^5}$$

(a) Trapezoidal: Error $\leq \frac{(1-0)^2}{12(4^2)}(2) = \frac{1}{96} \approx 0.01$ since

$f''(x)$ is maximum in $[0, 1]$ when $x = 0$.

(b) Simpson's: Error $\leq \frac{(1-0)^5}{180(4^4)}(24) = \frac{1}{1920} \approx 0.0005$

since $f^{(4)}(x)$ is maximum in $[0, 1]$ when $x = 0$.

26. $f''(x) = \frac{2}{(1+x)^3}$ in $[0, 1]$.

(a) $|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = 2$.

Trapezoidal: Error $\leq \frac{1}{12n^2}(2) < 0.00001$, $n^2 > 16,666.67$, $n > 129.10$; let $n = 130$.

$$f^{(4)}(x) = \frac{24}{(1+x)^5} \text{ in } [0, 1]$$

(b) $|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = 24$.

Simpson's: Error $\leq \frac{1}{180n^4}(24) < 0.00001$, $n^4 > 13,333.33$, $n > 10.75$; let $n = 12$. (In Simpson's Rule n must be even.)

28. $f(x) = (x+1)^{2/3}$

(a) $f''(x) = -\frac{2}{9(x+1)^{4/3}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \frac{2}{9}$.

Trapezoidal: Error $\leq \frac{8}{12n^4}\left(\frac{2}{9}\right) < 0.00001$, $n^2 > 14,814.81$, $n > 121.72$; let $n = 122$.

(b) $f^{(4)}(x) = -\frac{56}{81(x+1)^{10/3}}$ in $[0, 2]$

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \frac{56}{81}$.

Simpson's: Error $\leq \frac{32}{180n^4}\left(\frac{56}{81}\right) < 0.00001$, $n^4 > 12,290.81$, $n > 10.53$; let $n = 12$. (In Simpson's Rule n must be even.)

30. $f(x) = \sin(x^2)$

(a) $f''(x) = 2[-2x^2 \sin(x^2) + \cos(x^2)]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 2.2853$.

Trapezoidal: Error $\leq \frac{(1-0)^3}{12n^2}(2.2853) < 0.00001$, $n^2 > 19,044.17$, $n > 138.00$; let $n = 139$.

(b) $f^{(4)}(x) = (16x^4 - 12) \sin(x^2) - 48x^2 \cos(x^2)$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x \approx 0.852$ and $|f^{(4)}(0.852)| \approx 28.4285$.

Simpson's: Error $\leq \frac{(1-0)^5}{180n^4}(28.4285) < 0.00001$, $n^4 > 15,793.61$, $n > 11.21$; let $n = 12$.

32. The program will vary depending upon the computer or programmable calculator that you use.

34. $f(x) = \sqrt{1-x^2}$ on $[0, 1]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	0.8739	0.7960	0.6239	0.7489	0.7709
8	0.8350	0.7892	0.7100	0.7725	0.7803
10	0.8261	0.7881	0.7261	0.7761	0.7818
12	0.8200	0.7875	0.7367	0.7783	0.7826
16	0.8121	0.7867	0.7496	0.7808	0.7836
20	0.8071	0.7864	0.7571	0.7821	0.7841

36. $f(x) = \frac{\sin x}{x}$ on $[1, 2]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	0.7070	0.6597	0.6103	0.6586	0.6593
8	0.6833	0.6594	0.6350	0.6592	0.6593
10	0.6786	0.6594	0.6399	0.6592	0.6593
12	0.6754	0.6594	0.6431	0.6593	0.6593
16	0.6714	0.6594	0.6472	0.6593	0.6593
20	0.6690	0.6593	0.6496	0.6593	0.6593

38. Simpson's Rule: $n = 8$

$$8\sqrt{3} \int_0^{\pi/2} \sqrt{1 - \frac{2}{3} \sin^2 \theta} d\theta \approx \frac{\sqrt{3}\pi}{6} \left[\sqrt{1 - \frac{2}{3} \sin^2 0} + 4\sqrt{1 - \frac{2}{3} \sin^2 \frac{\pi}{16}} + 2\sqrt{1 - \frac{2}{3} \sin^2 \frac{\pi}{8}} + \cdots + \sqrt{1 - \frac{2}{3} \sin^2 \frac{\pi}{2}} \right] \\ \approx 17.476$$

40. (a) Trapezoidal:

$$\int_0^2 f(x) dx \approx \frac{2}{2(8)} [4.32 + 2(4.36) + 2(4.58) + 2(5.79) + 2(6.14) + 2(7.25) + 2(7.64) + 2(8.08) + 8.14] \approx 12.518$$

Simpson's:

$$\int_0^2 f(x) dx \approx \frac{2}{3(8)} [4.32 + 4(4.36) + 2(4.58) + 4(5.79) + 2(6.14) + 4(7.25) + 2(7.64) + 4(8.08) + 8.14] \approx 12.592$$

(b) Using a graphing utility,

$$y = -1.3727x^3 + 4.0092x^2 - 0.6202x + 4.2844$$

$$\text{Integrating, } \int_0^2 y dx \approx 12.53$$

42. Simpson's Rule: $n = 6$

$$\pi = 4 \int_0^1 \frac{1}{1+x^2} dx \approx \frac{4}{3(6)} \left[1 + \frac{4}{1+(1/6)^2} + \frac{2}{1+(2/6)^2} + \frac{4}{1+(3/6)^2} + \frac{2}{1+(4/6)^2} + \frac{4}{1+(5/6)^2} + \frac{1}{2} \right] \\ \approx 3.14159$$

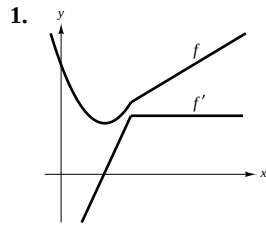
44. Area $\approx \frac{120}{2(12)} [75 + 2(81) + 2(84) + 2(76) + 2(67) + 2(68) + 2(69) + 2(72) + 2(68) + 2(56) + 2(42) + 2(23) + 0]$
 $= 7435 \text{ sq m}$

46. The quadratic polynomial

$$p(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} y_3$$

passes through the three points.

Review Exercises for Chapter 4



$$3. \int (2x^2 + x - 1) dx = \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + C$$

$$5. \int \frac{x^3 + 1}{x^2} dx = \int \left(x + \frac{1}{x^2} \right) dx = \frac{1}{2}x^2 - \frac{1}{x} + C$$

$$7. \int (4x - 3 \sin x) dx = 2x^2 + 3 \cos x + C$$

$$9. f'(x) = -2x, (-1, 1)$$

$$f(x) = \int -2x dx = -x^2 + C$$

When $x = -1$:

$$y = -1 + C = 1$$

$$C = 2$$

$$y = 2 - x^2$$

$$11. a(t) = a$$

$$v(t) = \int a dt = at + C_1$$

$$v(0) = 0 + C_1 = 0 \text{ when } C_1 = 0.$$

$$v(t) = at$$

$$s(t) = \int at dt = \frac{a}{2}t^2 + C_2$$

$$s(0) = 0 + C_2 = 0 \text{ when } C_2 = 0.$$

$$s(t) = \frac{a}{2}t^2$$

$$s(30) = \frac{a}{2}(30)^2 = 3600 \text{ or}$$

$$a = \frac{2(3600)}{(30)^2} = 8 \text{ ft/sec}^2.$$

$$v(30) = 8(30) = 240 \text{ ft/sec}$$

$$13. a(t) = -32$$

$$v(t) = -32t + 96$$

$$s(t) = -16t^2 + 96t$$

$$(a) v(t) = -32t + 96 = 0 \text{ when } t = 3 \text{ sec.}$$

$$(b) s(3) = -144 + 288 = 144 \text{ ft}$$

$$(c) v(t) = -32t + 96 = \frac{96}{2} \text{ when } t = \frac{3}{2} \text{ sec.}$$

$$(d) s\left(\frac{3}{2}\right) = -16\left(\frac{9}{4}\right) + 96\left(\frac{3}{2}\right) = 108 \text{ ft}$$

$$15. (a) \sum_{i=1}^{10} (2i - 1)$$

$$(b) \sum_{i=1}^n i^3$$

$$(c) \sum_{i=1}^{10} (4i + 2)$$

17. $y = \frac{10}{x^2 + 1}, \Delta x = \frac{1}{2}, n = 4$

$$S(n) = S(4) = \frac{1}{2} \left[\frac{10}{1} + \frac{10}{(1/2)^2 + 1} + \frac{10}{(1)^2 + 1} + \frac{10}{(3/2)^2 + 1} \right]$$

$$\approx 13.0385$$

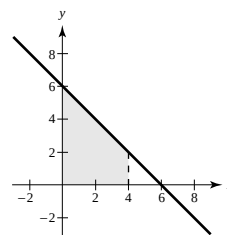
$$s(n) = s(4) = \frac{1}{2} \left[\frac{10}{(1/2)^2 + 1} + \frac{10}{1 + 1} + \frac{10}{(3/2)^2 + 1} + \frac{10}{2^2 + 1} \right]$$

$$\approx 9.0385$$

$$9.0385 < \text{Area of Region} < 13.0385$$

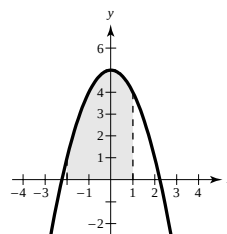
19. $y = 6 - x, \Delta x = \frac{4}{n}, \text{right endpoints}$

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(ci) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(6 - \frac{4i}{n} \right) \frac{4}{n} \\ &= \lim_{n \rightarrow \infty} \frac{4}{n} \left[6n - \frac{4}{n} \frac{n(n+1)}{2} \right] \\ &= \lim_{n \rightarrow \infty} \left[24 - 8 \frac{n+1}{n} \right] = 24 - 8 = 16 \end{aligned}$$



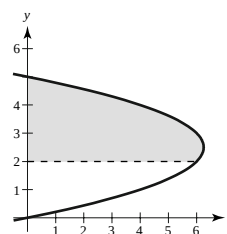
21. $y = 5 - x^2, \Delta x = \frac{3}{n}$

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(ci) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[5 - \left(-2 + \frac{3i}{n} \right)^2 \right] \left(\frac{3}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left[1 + \frac{12i}{n} - \frac{9i^2}{n^2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \left[n + \frac{12}{n} \frac{n(n+1)}{2} - \frac{9}{n^2} \frac{n(n+1)(2n+1)}{6} \right] \\ &= \lim_{n \rightarrow \infty} \left[3 + 18 \frac{n+1}{n} - \frac{9}{2} \frac{(n+1)(2n+1)}{n^2} \right] \\ &= 3 + 18 - 9 = 12 \end{aligned}$$



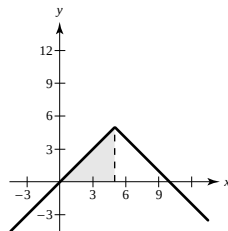
23. $x = 5y - y^2, 2 \leq y \leq 5, \Delta y = \frac{3}{n}$

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[5 \left(2 + \frac{3i}{n} \right) - \left(2 + \frac{3i}{n} \right)^2 \right] \left(\frac{3}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left[10 + \frac{15i}{n} - 4 - 12 \frac{i}{n} - \frac{9i^2}{n^2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left[6 + \frac{3i}{n} - \frac{9i^2}{n^2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \left[6n + \frac{3}{n} \frac{n(n+1)}{2} - \frac{9}{n^2} \frac{n(n+1)(2n+1)}{6} \right] \\ &= \left[18 + \frac{9}{2} - 9 \right] = \frac{27}{2} \end{aligned}$$



$$25. \lim_{\|\Delta\| \rightarrow \infty} \sum_{i=1}^n (2ci - 3) \Delta xi = \int_4^6 (2x - 3) dx$$

27.



$$\int_0^5 (5 - |x - 5|) dx = \int_0^5 (5 - (5 - x)) dx = \int_0^5 x dx = \frac{25}{2}$$

(triangle)

$$29. (a) \int_2^6 [f(x) + g(x)] dx = \int_2^6 f(x) dx + \int_2^6 g(x) dx = 10 + 3 = 13$$

$$(b) \int_2^6 [f(x) - g(x)] dx = \int_2^6 f(x) dx - \int_2^6 g(x) dx = 10 - 3 = 7$$

$$(c) \int_2^6 [2f(x) - 3g(x)] dx = 2 \int_2^6 f(x) dx - 3 \int_2^6 g(x) dx = 2(10) - 3(3) = 11$$

$$(d) \int_2^6 5f(x) dx = 5 \int_2^6 f(x) dx = 5(10) = 50$$

$$31. \int_1^8 (\sqrt[3]{x} + 1) dx = \left[\frac{3}{4} x^{4/3} + x \right]_1^8 = \left[\frac{3}{4}(16) + 8 \right] - \left[\frac{3}{4} + 1 \right] = \frac{73}{4} \quad (c)$$

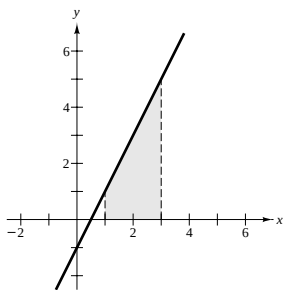
$$33. \int_0^4 (2 + x) dx = \left[2x + \frac{x^2}{2} \right]_0^4 = 8 + \frac{16}{2} = 16$$

$$35. \int_{-1}^1 (4t^3 - 2t) dt = \left[t^4 - t^2 \right]_{-1}^1 = 0$$

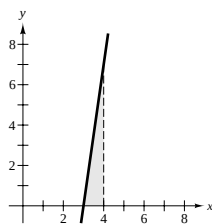
$$37. \int_4^9 x \sqrt{x} dx = \int_4^9 x^{3/2} dx = \left[\frac{2}{5} x^{5/2} \right]_4^9 = \frac{2}{5} [(\sqrt{9})^5 - (\sqrt{4})^5] = \frac{2}{5} (243 - 32) = \frac{422}{5}$$

$$39. \int_0^{3\pi/4} \sin \theta d\theta = \left[-\cos \theta \right]_0^{3\pi/4} = -\left(-\frac{\sqrt{2}}{2} \right) + 1 = 1 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2} + 2}{2}$$

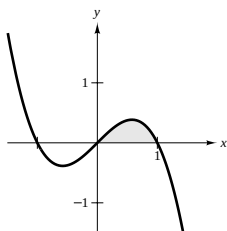
$$41. \int_1^3 (2x - 1) dx = \left[x^2 - x \right]_1^3 = 6$$



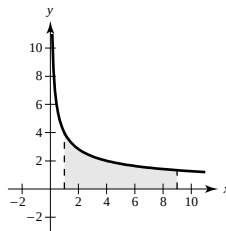
$$\begin{aligned} 43. \int_3^4 (x^2 - 9) dx &= \left[\frac{x^3}{3} - 9x \right]_3^4 \\ &= \left(\frac{64}{3} - 36 \right) - (9 - 27) \\ &= \frac{64}{3} - \frac{54}{3} = \frac{10}{3} \end{aligned}$$



$$45. \int_0^1 (x - x^3) dx = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$



$$47. \text{Area} = \int_1^9 \frac{4}{\sqrt{x}} dx = \left[\frac{4x^{1/2}}{(1/2)} \right]_1^9 = 8(3 - 1) = 16$$

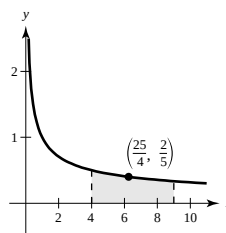


$$49. \frac{1}{9-4} \int_4^9 \frac{1}{\sqrt{x}} dx = \left[\frac{1}{5} 2\sqrt{x} \right]_4^9 = \frac{2}{5}(3 - 2) = \frac{2}{5} \text{ Average value}$$

$$\frac{2}{5} = \frac{1}{\sqrt{x}}$$

$$\sqrt{x} = \frac{5}{2}$$

$$x = \frac{25}{4}$$



$$51. F'(x) = x^2 \sqrt{1+x^3}$$

$$53. F'(x) = x^2 + 3x + 2$$

$$55. \int (x^2 + 1)^3 dx = \int (x^6 + 3x^4 + 3x^2 + 1) dx = \frac{x^7}{7} + \frac{3}{5}x^5 + x^3 + x + C$$

$$57. u = x^3 + 3, du = 3x^2 dx$$

$$\int \frac{x^2}{\sqrt{x^3 + 3}} dx = \int (x^3 + 3)^{-1/2} x^2 dx = \frac{1}{3} \int (x^3 + 3)^{-1/2} 3x^2 dx = \frac{2}{3} (x^3 + 3)^{1/2} + C$$

$$59. u = 1 - 3x^2, du = -6x dx$$

$$\int x(1 - 3x^2)^4 dx = -\frac{1}{6} \int (1 - 3x^2)^4 (-6x dx) = -\frac{1}{30} (1 - 3x^2)^5 + C = \frac{1}{30} (3x^2 - 1)^5 + C$$

$$61. \int \sin^3 x \cos x dx = \frac{1}{4} \sin^4 x + C$$

$$63. \int \frac{\sin \theta}{\sqrt{1 - \cos \theta}} d\theta = \int (1 - \cos \theta)^{-1/2} \sin \theta d\theta = 2(1 - \cos \theta)^{1/2} + C = 2\sqrt{1 - \cos \theta} + C$$

$$65. \int \tan^n x \sec^2 x dx = \frac{\tan^{n+1} x}{n+1} + C, n \neq -1$$

$$67. \int (1 + \sec \pi x)^2 \sec \pi x \tan \pi x dx = \frac{1}{\pi} \int (1 + \sec \pi x)^2 (\pi \sec \pi x \tan \pi x) dx = \frac{1}{3\pi} (1 + \sec \pi x)^3 + C$$

$$69. \int_{-1}^2 x(x^2 - 4) dx = \frac{1}{2} \int_{-1}^2 (x^2 - 4)(2x) dx = \frac{1}{2} \left[\frac{(x^2 - 4)^2}{2} \right]_{-1}^2 = \frac{1}{4} [0 - 9] = -\frac{9}{4}$$

$$71. \int_0^3 \frac{1}{\sqrt{1+x}} dx = \int_0^3 (1+x)^{-1/2} dx = \left[2(1+x)^{1/2} \right]_0^3 = 4 - 2 = 2$$

$$73. u = 1 - y, y = 1 - u, dy = -du$$

When $y = 0$, $u = 1$. When $y = 1$, $u = 0$.

$$\begin{aligned} 2\pi \int_0^1 (y+1)\sqrt{1-y} dy &= 2\pi \int_1^0 -[(1-u)+1]\sqrt{u} du \\ &= 2\pi \int_1^0 (u^{3/2} - 2u^{1/2}) du = 2\pi \left[\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} \right]_1^0 = \frac{28\pi}{15} \end{aligned}$$

$$75. \int_0^\pi \cos\left(\frac{x}{2}\right) dx = 2 \int_0^\pi \cos\left(\frac{x}{2}\right) \frac{1}{2} dx = \left[2 \sin\left(\frac{x}{2}\right) \right]_0^\pi = 2$$

$$77. u = 1 - x, x = 1 - u, dx = -du$$

When $x = a$, $u = 1 - a$. When $x = b$, $u = 1 - b$.

$$\begin{aligned} P_{a,b} &= \int_a^b \frac{15}{4} x \sqrt{1-x} dx = \frac{15}{4} \int_{1-a}^{1-b} -(1-u)\sqrt{u} du \\ &= \frac{15}{4} \int_{1-a}^{1-b} (u^{3/2} - u^{1/2}) du = \frac{15}{4} \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_{1-a}^{1-b} = \frac{15}{4} \left[\frac{2u^{3/2}}{15} (3u - 5) \right]_{1-a}^{1-b} = \left[-\frac{(1-x)^{3/2}}{2} (3x+2) \right]_a^b \end{aligned}$$

$$(a) P_{0.50, 0.75} = \left[-\frac{(1-x)^{3/2}}{2} (3x+2) \right]_{0.50}^{0.75} = 0.353 = 35.3\%$$

$$(b) P_{0,b} = \left[-\frac{(1-x)^{3/2}}{2} (3x+2) \right]_0^b = -\frac{(1-b)^{3/2}}{2} (3b+2) + 1 = 0.5$$

$$(1-b)^{3/2} (3b+2) = 1$$

$$b \approx 0.586 = 58.6\%$$

$$79. p = 1.20 + 0.04t$$

$$C = \frac{15,000}{M} \int_t^{t+1} p ds$$

(a) 2000 corresponds to $t = 10$.

(b) 2005 corresponds to $t = 15$.

$$\begin{aligned} C &= \frac{15,000}{M} \int_{10}^{11} [1.20 + 0.04t] dt \\ &= \frac{15,000}{M} \left[1.20t + 0.02t^2 \right]_{10}^{11} = \frac{24,300}{M} \end{aligned}$$

$$C = \frac{15,000}{M} \left[1.20t + 0.02t^2 \right]_{15}^{16} = \frac{27,300}{M}$$

$$81. \text{Trapezoidal Rule } (n = 4): \int_1^2 \frac{1}{1+x^3} dx \approx \frac{1}{8} \left[\frac{1}{1+1^3} + \frac{2}{1+(1.25)^3} + \frac{2}{1+(1.5)^3} + \frac{2}{1+(1.75)^3} + \frac{1}{1+2^3} \right] \approx 0.257$$

$$\text{Simpson's Rule } (n = 4): \int_1^2 \frac{1}{1+x^3} dx \approx \frac{1}{12} \left[\frac{1}{1+1^3} + \frac{4}{1+(1.25)^3} + \frac{2}{1+(1.5)^3} + \frac{4}{1+(1.75)^3} + \frac{1}{1+2^3} \right] \approx 0.254$$

Graphing utility: 0.254

$$83. \text{Trapezoidal Rule } (n = 4): \int_0^{\pi/2} \sqrt{x} \cos x dx \approx 0.637$$

Simpson's Rule ($n = 4$): 0.685

Graphing Utility: 0.704

$$85. (a) R < I < T < L$$

$$(b) S(4) = \frac{4-0}{3(4)} [f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)]$$

$$\approx \frac{1}{3} \left[4 + 4(2) + 2(1) + 4\left(\frac{1}{2}\right) + \frac{1}{4} \right] \approx 5.417$$

Problem Solving for Chapter 4

1. (a) $L(1) = \int_1^1 \frac{1}{t} dt = 0$

(b) $L'(x) = \frac{1}{x}$ by the Second Fundamental Theorem of Calculus.

$$L'(1) = 1$$

(c) $L(x) = 1 = \int_1^x \frac{1}{t} dt$ for $x \approx 2.718$

$$\int_1^{2.718} \frac{1}{t} dt = 0.999896 \quad (\text{Note: The exact value of } x \text{ is } e, \text{ the base of the natural logarithm function.})$$

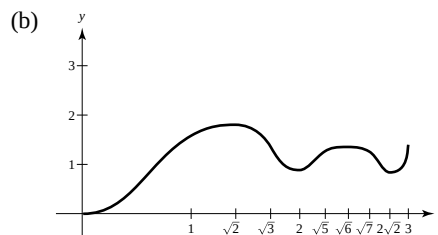
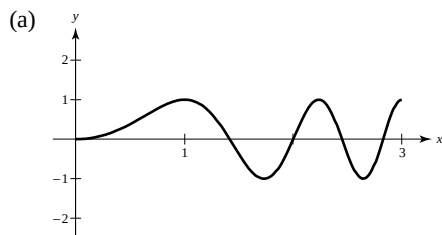
(d) We first show that $\int_1^{x_1} \frac{1}{t} dt = \int_{1/x_1}^1 \frac{1}{t} dt$.

To see this, let $u = \frac{t}{x_1}$ and $du = \frac{1}{x_1} dt$.

Then $\int_1^{x_1} \frac{1}{t} dt = \int_{1/x_1}^1 \frac{1}{ux_1} (x_1 du) = \int_{1/x_1}^1 \frac{1}{u} du = \int_{1/x_1}^1 \frac{1}{t} dt$.

Now,
$$\begin{aligned} L(x_1 x_2) &= \int_1^{x_1 x_2} \frac{1}{t} dt = \int_{1/x_1}^{x_2} \frac{1}{u} du \quad (\text{using } u = \frac{t}{x_1}) \\ &= \int_{1/x_1}^1 \frac{1}{u} du + \int_1^{x_2} \frac{1}{u} du \\ &= \int_1^{x_1} \frac{1}{u} du + \int_1^{x_2} \frac{1}{u} du \\ &= L(x_1) + L(x_2). \end{aligned}$$

3. $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$



The zeros of $y = \sin \frac{\pi x^2}{2}$ correspond to the relative extrema of $S(x)$.

(c) $S'(x) = \sin \frac{\pi x^2}{2} = 0 \Rightarrow \frac{\pi x^2}{2} = n\pi \Rightarrow x^2 = 2n \Rightarrow x = \sqrt{2n}, n \text{ integer.}$

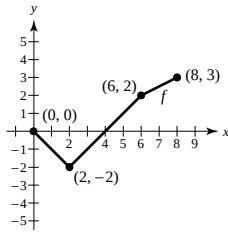
Relative maximum at $x = \sqrt{2} \approx 1.4142$ and $x = \sqrt{6} \approx 2.4495$

Relative minimum at $x = 2$ and $x = \sqrt{8} \approx 2.8284$

(d) $S''(x) = \cos\left(\frac{\pi x^2}{2}\right)(\pi x) = 0 \Rightarrow \frac{\pi x^2}{2} = \frac{\pi}{2} + n\pi \Rightarrow x^2 = 1 + 2n \Rightarrow x = \sqrt{1 + 2n}, n \text{ integer}$

Points of inflection at $x = 1, \sqrt{3}, \sqrt{5}, \text{ and } \sqrt{7}$.

5. (a)



$$(c) f(x) = \begin{cases} -x, & 0 \leq x < 2 \\ x - 4, & 2 \leq x < 6 \\ \frac{1}{2}x - 1, & 6 \leq x \leq 8 \end{cases}$$

$$F(x) = \int_0^x f(t) dt = \begin{cases} (-x^2/2), & 0 \leq x < 2 \\ (x^2/2) - 4x + 4, & 2 \leq x < 6 \\ (1/4)x^2 - x - 5, & 6 \leq x \leq 8 \end{cases}$$

$F'(x) = f(x)$. F is decreasing on $(0, 4)$ and increasing on $(4, 8)$. Therefore, the minimum is -4 at $x = 4$, and the maximum is 3 at $x = 8$.

$$7. (a) \int_{-1}^1 \cos x \, dx \approx \cos\left(-\frac{1}{\sqrt{3}}\right) + \cos\left(\frac{1}{\sqrt{3}}\right) = 2 \cos\left(\frac{1}{\sqrt{3}}\right) \approx 1.6758$$

$$\int_{-1}^1 \cos x \, dx = \sin x \Big|_{-1}^1 = 2 \sin(1) \approx 1.6829$$

$$\text{Error. } |1.6829 - 1.6758| = 0.0071$$

$$(b) \int_{-1}^1 \frac{1}{1+x^2} dx \approx \frac{1}{1+(1/3)} + \frac{1}{1+(1/3)} = \frac{3}{2}$$

(Note: exact answer is $\pi/2 \approx 1.5708$)

9. Consider $F(x) = [f(x)]^2 \Rightarrow F'(x) = 2f(x)f'(x)$. Thus,

$$\begin{aligned} \int_a^b f(x)f'(x) dx &= \int_a^b \frac{1}{2}F'(x) dx \\ &= \left[\frac{1}{2}F(x)\right]_a^b \\ &= \frac{1}{2}[F(b) - F(a)] \\ &= \frac{1}{2}[f(b)^2 - f(a)^2] \end{aligned}$$

(b)

x	0	1	2	3	4	5	6	7	8
$F(x)$	0	$-\frac{1}{2}$	-2	$-\frac{7}{2}$	-4	$-\frac{7}{2}$	-2	$\frac{1}{4}$	3

$$(c) F''(x) = f'(x) = \begin{cases} -1, & 0 < x < 2 \\ 1, & 2 < x < 6 \\ \frac{1}{2}, & 6 < x < 8 \end{cases}$$

$x = 2$ is a point of inflection, whereas $x = 6$ is not. (f is not continuous at $x = 6$.)

(c) Let $p(x) = ax^3 + bx^2 + cx + d$.

$$\int_{-1}^1 p(x) dx = \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_{-1}^1 = \frac{2b}{3} + 2d$$

$$p\left(-\frac{1}{\sqrt{3}}\right) + p\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{b}{3} + d\right) + \left(\frac{b}{3} + d\right) = \frac{2b}{3} + 2d$$

$$11. \text{ Consider } \int_0^1 x^5 dx = \left[\frac{x^6}{6}\right]_0^1 = \frac{1}{6}.$$

The corresponding Riemann Sum using right endpoints is

$$\begin{aligned} S(n) &= \frac{1}{n} \left[\left(\frac{1}{n}\right)^5 + \left(\frac{2}{n}\right)^5 + \cdots + \left(\frac{n}{n}\right)^5 \right] \\ &= \frac{1}{n^6} [1^5 + 2^5 + \cdots + n^5] \end{aligned}$$

$$\text{Thus, } \lim_{n \rightarrow \infty} S(n) = \lim_{n \rightarrow \infty} \frac{1^5 + 2^5 + \cdots + n^5}{n^6} = \frac{1}{6}.$$

13. By Theorem 4.8, $0 < f(x) \leq M \Rightarrow \int_a^b f(x) \, dx \leq \int_a^b M \, dx = M(b - a)$.

Similarly, $m \leq f(x) \Rightarrow m(b - a) = \int_a^b m \, dx \leq \int_a^b f(x) \, dx$.

Thus, $m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a)$. On the interval $[0, 1]$, $1 \leq \sqrt{1 + x^4} \leq \sqrt{2}$ and $b - a = 1$.

Thus, $1 \leq \int_0^1 \sqrt{1 + x^4} \, dx \leq \sqrt{2}$. (Note: $\int_0^1 \sqrt{1 + x^4} \, dx \approx 1.0894$)

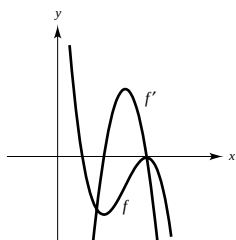
15. Since $-|f(x)| \leq f(x) \leq |f(x)|$,

$$-\int_a^b |f(x)| \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx \Rightarrow \left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$$

17. $\frac{1}{365} \int_0^{365} 100,000 \left[1 + \sin \frac{2\pi(t - 60)}{365} \right] dt = \frac{100,000}{365} \left[t - \frac{365}{2\pi} \cos \frac{2\pi(t - 60)}{365} \right]_0^{365} = 100,000 \text{ lbs.}$

Review Exercises for Chapter 4

2.



$$\begin{aligned} 6. \int \frac{x^3 - 2x^2 + 1}{x^2} dx &= \int (x - 2 + x^{-2}) dx \\ &= \frac{1}{2}x^2 - 2x - \frac{1}{x} + C \end{aligned}$$

$$10. f''(x) = 6(x - 1)$$

$$f'(x) = \int 6(x - 1) dx = 3(x - 1)^2 + C_1$$

Since the slope of the tangent line at (2, 1) is 3, it follows that $f'(2) = 3 + C_1 = 3$ when $C_1 = 0$.

$$f'(x) = 3(x - 1)^2$$

$$f(x) = \int 3(x - 1)^2 dx = (x - 1)^3 + C_2$$

$$f(2) = 1 + C_2 = 1 \text{ when } C_2 = 0.$$

$$f(x) = (x - 1)^3$$

$$4. u = 3x$$

$$du = 3 dx$$

$$\int \frac{2}{\sqrt[3]{3x}} dx = \frac{2}{3} \int (3x)^{-1/3} (3) dx = (3x)^{2/3} + C$$

$$8. \int (5 \cos x - 2 \sec^2 x) dx = 5 \sin x - 2 \tan x + C$$

$$12. 45 \text{ mph} = 66 \text{ ft/sec}$$

$$30 \text{ mph} = 44 \text{ ft/sec}$$

$$a(t) = -a$$

$$v(t) = -at + 66 \text{ since } v(0) = 66 \text{ ft/sec.}$$

$$s(t) = -\frac{a}{2}t^2 + 66t \text{ since } s(0) = 0.$$

Solving the system

$$v(t) = -at + 66 = 44$$

$$s(t) = -\frac{a}{2}t^2 + 66t = 264$$

we obtain $t = 24/5$ and $a = 55/12$. We now solve $-(55/12)t + 66 = 0$ and get $t = 72/5$. Thus,

$$s\left(\frac{72}{5}\right) = -\frac{55/12}{2}\left(\frac{72}{5}\right)^2 + 66\left(\frac{72}{5}\right) \approx 475.2 \text{ ft.}$$

Stopping distance from 30 mph to rest is

$$475.2 - 264 = 211.2 \text{ ft.}$$

$$14. a(t) = -9.8 \text{ m/sec}^2$$

$$v(t) = -9.8t + v_0 = -9.8t + 40$$

$$s(t) = -4.9t^2 + 40t \quad (s(0) = 0)$$

$$(a) v(t) = -9.8t + 40 = 0 \text{ when } t = \frac{40}{9.8} \approx 4.08 \text{ sec.}$$

$$(b) s(4.08) \approx 81.63 \text{ m}$$

$$(c) v(t) = -9.8t + 40 = 20 \text{ when } t = \frac{20}{9.8} \approx 2.04 \text{ sec.}$$

$$(d) s(2.04) \approx 61.2 \text{ m}$$

16. $x_1 = 2, x_2 = -1, x_3 = 5, x_4 = 3, x_5 = 7$

(a) $\frac{1}{5} \sum_{i=1}^5 x_i = \frac{1}{5}(2 - 1 + 5 + 3 + 7) = \frac{16}{5}$

(b) $\sum_{i=1}^5 \frac{1}{x_i} = \frac{1}{2} - 1 + \frac{1}{5} + \frac{1}{3} + \frac{1}{7} = \frac{37}{210}$

(c) $\sum_{i=1}^5 (2x_i - x_i^2) = [2(2) - (2)^2] + [2(-1) - (-1)^2] + [2(5) - (5)^2] + [2(3) - (3)^2] + [2(7) - (7)^2] = -56$

(d) $\sum_{i=2}^5 (x_i - x_{i-1}) = (-1 - 2) + [5 - (-1)] + (3 - 5) + (7 - 3) = 5$

18. $y = 9 - \frac{1}{4}x^2, \Delta x = 1, n = 4$

$$S(4) = 1 \left[\left(9 - \frac{1}{4}(4) \right) + \left(9 - \frac{1}{4}(9) \right) + \left(9 - \frac{1}{4}(16) \right) + 9 - \frac{1}{4}(25) \right]$$

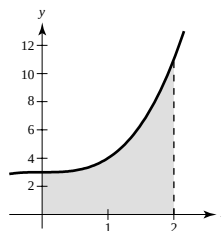
$$\approx 22.5$$

$$s(4) = 1 \left[\left(9 - \frac{1}{4}(9) \right) + \left(9 - \frac{1}{4}(16) \right) + \left(9 - \frac{1}{4}(25) \right) + (9 - 9) \right]$$

$$\approx 14.5$$

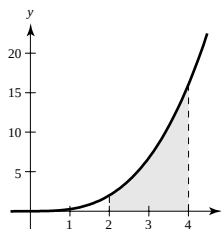
20. $y = x^2 + 3, \Delta x = \frac{2}{n}$ right endpoints

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(ci) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(\frac{2i}{n} \right)^2 + 3 \right] \left(\frac{2}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n \left[\frac{4i^2}{n^2} + 3 \right] \\ &= \lim_{n \rightarrow \infty} \frac{2}{n} \left[\frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} + 3n \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{4}{3} \frac{(n+1)(2n+1)}{n^2} + 6 \right] = \frac{8}{3} + 6 = \frac{26}{3} \end{aligned}$$



22. $y = \frac{1}{4}x^3, \Delta x = \frac{2}{n}$

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(ci) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{4} \left(2 + \frac{2i}{n} \right)^3 \left(\frac{2}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2n} \sum_{i=1}^n \left[8 + \frac{24i}{n} + \frac{24i^2}{n^2} + \frac{8i^3}{n^3} \right] \\ &= \lim_{n \rightarrow \infty} \frac{4}{n} \sum_{i=1}^n \left[1 + \frac{3i}{n} + \frac{3i^2}{n^2} + \frac{i^3}{n^3} \right] \\ &= \lim_{n \rightarrow \infty} \frac{4}{n} \left[n + \frac{3}{n} \frac{n(n+1)}{2} + \frac{3}{n^2} \frac{n(n+1)(2n+1)}{6} + \frac{1}{n^3} \frac{n^2(n+1)^2}{4} \right] \\ &= 4 + 6 + 4 + 1 = 15 \end{aligned}$$



$$24. (a) S = m\left(\frac{b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{2b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{3b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{4b}{4}\right)\left(\frac{b}{4}\right) = \frac{mb^2}{16}(1 + 2 + 3 + 4) = \frac{5mb^2}{8}$$

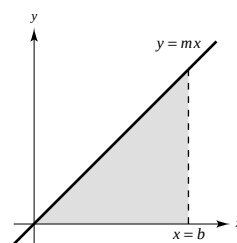
$$s = m(0)\left(\frac{b}{4}\right) + m\left(\frac{b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{2b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{3b}{4}\right)\left(\frac{b}{4}\right) = \frac{mb^2}{16}(1 + 2 + 3) = \frac{3mb^2}{8}$$

$$(b) S(n) = \sum_{i=1}^n f\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = \sum_{i=1}^n \left(\frac{mbi}{n}\right)\left(\frac{b}{n}\right) = m\left(\frac{b}{n}\right)^2 \sum_{i=1}^n i = \frac{mb^2}{n^2} \left(\frac{n(n+1)}{2}\right) = \frac{mb^2(n+1)}{2n}$$

$$s(n) = \sum_{i=0}^{n-1} f\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = \sum_{i=0}^{n-1} m\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = m\left(\frac{b}{n}\right)^2 \sum_{i=0}^{n-1} i = \frac{mb^2}{n^2} \left(\frac{(n-1)n}{2}\right) = \frac{mb^2(n-1)}{2n}$$

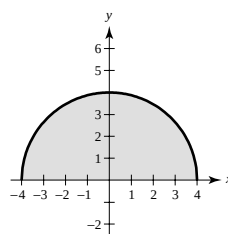
$$(c) \text{Area} = \lim_{n \rightarrow \infty} \frac{mb^2(n+1)}{2n} = \lim_{n \rightarrow \infty} \frac{mb^2(n-1)}{2n} = \frac{1}{2}mb^2 = \frac{1}{2}(b)(mb) = \frac{1}{2}(\text{base})(\text{height})$$

$$(d) \int_0^b mx \, dx = \left[\frac{1}{2}mx^2 \right]_0^b = \frac{1}{2}mb^2$$



$$26. \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n 3ci(9 - ci^2) \Delta xi = \int_1^3 3x(9 - x^2) \, dx$$

28.



$$\int_{-4}^4 \sqrt{16 - x^2} \, dx = \frac{1}{2} \pi (4)^2 = 8\pi \quad (\text{semicircle})$$

$$30. (a) \int_0^6 f(x) \, dx = \int_0^3 f(x) \, dx + \int_3^6 f(x) \, dx = 4 + (-1) = 3$$

$$(b) \int_6^3 f(x) \, dx = -\int_3^6 f(x) \, dx = -(-1) = 1$$

$$(c) \int_4^4 f(x) \, dx = 0$$

$$(d) \int_3^6 -10 f(x) \, dx = -10 \int_3^6 f(x) \, dx = -10(-1) = 10$$

$$32. \int_1^3 \frac{12}{x^3} \, dx = \left[\frac{12x^{-2}}{-2} \right]_1^3 = \left[\frac{-6}{x^2} \right]_1^3 = \frac{-6}{9} + 6 = \frac{16}{3}, (d)$$

$$34. \int_{-1}^1 (t^2 + 2) \, dt = \left[\frac{t^3}{3} + 2t \right]_{-1}^1 = \frac{14}{3}$$

$$36. \int_{-2}^2 (x^4 + 2x^2 - 5) \, dx = \left[\frac{x^5}{5} + \frac{2x^3}{3} - 5x \right]_{-2}^2$$

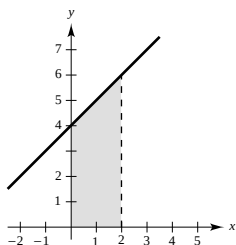
$$= \left(\frac{32}{5} + \frac{16}{3} - 10 \right) - \left(-\frac{32}{5} - \frac{16}{3} + 10 \right)$$

$$= \frac{52}{15}$$

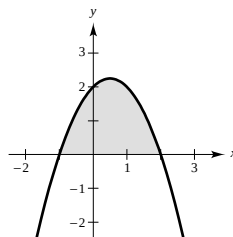
$$38. \int_1^2 \left(\frac{1}{x^2} - \frac{1}{x^3} \right) \, dx = \int_1^2 (x^{-2} - x^{-3}) \, dx = \left[-\frac{1}{x} + \frac{1}{2x^2} \right]_1^2 = \left(-\frac{1}{2} + \frac{1}{8} \right) - \left(-1 + \frac{1}{2} \right) = \frac{1}{8}$$

$$40. \int_{-\pi/4}^{\pi/4} \sec^2 t \, dt = \left[\tan t \right]_{-\pi/4}^{\pi/4} = 1 - (-1) = 2$$

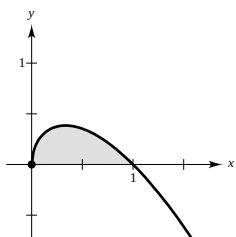
$$42. \int_0^2 (x + 4) dx = \left[\frac{x^2}{2} + 4x \right]_0^2 = 10$$



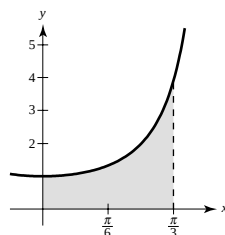
$$\begin{aligned} 44. \int_{-1}^2 (-x^2 + x + 2) dx &= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_{-1}^2 \\ &= \left(-\frac{8}{3} + 2 + 4 \right) - \left(\frac{1}{3} + \frac{1}{2} - 2 \right) \\ &= \frac{10}{3} + \frac{7}{6} = \frac{9}{2} \end{aligned}$$



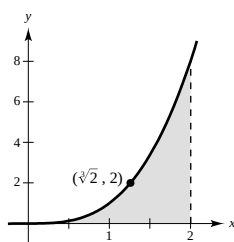
$$\begin{aligned} 46. \int_0^1 \sqrt{x}(1-x) dx &= \int_0^1 (x^{1/2} - x^{3/2}) dx \\ &= \left[\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} \right]_0^1 \\ &= \frac{2}{3} - \frac{2}{5} = \frac{4}{15} \end{aligned}$$



$$\begin{aligned} 48. \text{Area} &= \int_0^{\pi/3} \sec^2 x dx \\ &= \tan x \Big|_0^{\pi/3} = \sqrt{3} \end{aligned}$$



$$\begin{aligned} 50. \frac{1}{2-0} \int_0^2 x^3 dx &= \left[\frac{x^4}{8} \right]_0^2 = 2 \\ x^3 &= 2 \\ x &= \sqrt[3]{2} \end{aligned}$$



$$52. F'(x) = \frac{1}{x^2}$$

$$54. F'(x) = \csc^2 x$$

$$\begin{aligned} 56. \int \left(x + \frac{1}{x} \right)^2 dx &= \int (x^2 + 2 + x^{-2}) dx \\ &= \frac{x^3}{3} + 2x - \frac{1}{x} + C \end{aligned}$$

58. $u = x^3 + 3, du = 3x^2 dx$

$$\int x^2 \sqrt{x^3 + 3} dx = \frac{1}{3} \int (x^3 + 3)^{1/2} 3x^2 dx = \frac{2}{9} (x^3 + 3)^{3/2} + C$$

60. $u = x^2 + 6x - 5, du = (2x + 6) dx$

$$\int \frac{x + 3}{(x^2 + 6x - 5)^2} dx = \frac{1}{2} \int \frac{2x + 6}{(x^2 + 6x - 5)^2} dx = \frac{-1}{2} (x^2 + 6x - 5)^{-1} + C = \frac{-1}{2(x^2 + 6x - 5)} + C$$

62. $\int x \sin 3x^2 dx = \frac{1}{6} \int (\sin 3x^2)(6x) dx = -\frac{1}{6} \cos 3x^2 + C$

64. $\int \frac{\cos x}{\sqrt{\sin x}} dx = \int (\sin x)^{-1/2} \cos x dx = 2(\sin x)^{1/2} + C = 2\sqrt{\sin x} + C$

66. $\int \sec 2x \tan 2x dx = \frac{1}{2} \int (\sec 2x \tan 2x)(2) dx = \frac{1}{2} \sec 2x + C$

68. $\int \cot^4 \alpha \csc^2 \alpha d\alpha = - \int (\cot \alpha)^4 (-\csc^2 \alpha) d\alpha = -\frac{1}{5} \cot^5 \alpha + C$

70. $\int_0^1 x^2 (x^3 + 1)^3 dx = \frac{1}{3} \int_0^1 (x^3 + 1)^3 (3x^2) dx = \frac{1}{12} [(x^3 + 1)^4]_0^1 = \frac{1}{12} (16 - 1) = \frac{5}{4}$

72. $\int_3^6 \frac{x}{3\sqrt{x^2 - 8}} dx = \frac{1}{6} \int_3^6 (x^2 - 8)^{-1/2} (2x) dx = \left[\frac{1}{3} (x^2 - 8)^{1/2} \right]_3^6 = \frac{1}{3} (2\sqrt{7} - 1)$

74. $u = x + 1, x = u - 1, dx = du$

When $x = -1, u = 0$. When $x = 0, u = 1$.

$$\begin{aligned} 2\pi \int_{-1}^0 x^2 \sqrt{x+1} dx &= 2\pi \int_0^1 (u-1)^2 \sqrt{u} du \\ &= 2\pi \int_0^1 (u^{5/2} - 2u^{3/2} + u^{1/2}) du = 2\pi \left[\frac{2}{7} u^{7/2} - \frac{4}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right]_0^1 = \frac{32\pi}{105} \end{aligned}$$

76. $\int_{-\pi/4}^{\pi/4} \sin 2x dx = 0$ since $\sin 2x$ is an odd function.

78. $u = 1 - x, x = 1 - u, dx = -du$

When $x = a, u = 1 - a$. When $x = b, u = 1 - b$.

$$\begin{aligned} P_{a,b} &= \int_a^b \frac{1155}{32} x^3 (1-x)^{3/2} dx = \frac{1155}{32} \int_{1-a}^{1-b} -(1-u)^3 u^{3/2} du \\ &= \frac{1155}{32} \int_{1-a}^{1-b} (u^{9/2} - 3u^{7/2} + 3u^{5/2} - u^{3/2}) du = \frac{1155}{32} \left[\frac{2}{11} u^{11/2} - \frac{2}{3} u^{9/2} + \frac{6}{7} u^{7/2} - \frac{2}{5} u^{5/2} \right]_{1-a}^{1-b} \\ &= \frac{1155}{32} \left[\frac{2u^{5/2}}{1155} (105u^3 - 385u^2 + 495u - 231) \right]_{1-a}^{1-b} = \left[\frac{u^{5/2}}{16} (105u^3 - 385u^2 + 495u - 231) \right]_{1-a}^{1-b} \\ \text{(a) } P_{0,0.25} &= \left[\frac{u^{5/2}}{16} (105u^3 - 385u^2 + 495u - 231) \right]_1^{0.75} \approx 0.025 = 2.5\% \\ \text{(b) } P_{0.5,1} &= \left[\frac{u^{5/2}}{16} (105u^3 - 385u^2 + 495u - 231) \right]_{0.5}^0 \approx 0.736 = 73.6\% \end{aligned}$$

$$80. \int_0^2 1.75 \sin \frac{\pi t}{2} dt = -\frac{2}{\pi} \left[1.75 \cos \frac{\pi t}{2} \right] = -\frac{2}{\pi} (1.75)(-1 - 1) = \frac{7}{\pi} \approx 2.2282 \text{ liters}$$

Increase is

$$\frac{7}{\pi} - \frac{5.1}{\pi} = \frac{1.9}{\pi} \approx 0.6048 \text{ liters.}$$

$$82. \text{ Trapezoidal Rule } (n = 4): \int_0^1 \frac{x^{3/2}}{3 - x^2} dx \approx \frac{1}{8} \left[0 + \frac{2(1/4)^{3/2}}{3 - (1/4)^2} + \frac{2(1/2)^{3/2}}{3 - (1/2)^2} + \frac{2(3/4)^{3/2}}{3 - (3/4)^2} + \frac{1}{2} \right] \approx 0.172$$

$$\text{Simpson's Rule } (n = 4): \int_0^1 \frac{x^{3/2}}{3 - x^2} dx \approx \frac{1}{12} \left[0 + \frac{4(1/4)^{3/2}}{3 - (1/4)^2} + \frac{2(1/2)^{3/2}}{3 - (1/2)^2} + \frac{4(3/4)^{3/2}}{3 - (3/4)^2} + \frac{1}{2} \right] \approx 0.166$$

Graphing utility: 0.166

$$84. \text{ Trapezoidal Rule } (n = 4): \int_0^{\pi} \sqrt{1 + \sin^2 x} dx \approx 3.820$$

Simpson's Rule ($n = 4$): 3.820

Graphing utility: 3.820

Problem Solving for Chapter 4

$$2. (a) F(x) = \int_2^x \sin t^2 dt$$

x	0	1.0	1.5	1.9	2.0	2.1	2.5	3.0	4.0	5.0
$F(x)$	-0.8048	-0.4945	-0.0265	0.0611	0	-0.0867	-0.3743	-0.0312	-0.0576	-0.2769

$$(b) G(x) = \frac{1}{x-2} \int_2^x \sin t^2 dt$$

x	1.9	1.95	1.99	2.01	2.05	2.1
$G(x)$	-0.6106	-0.6873	-0.7436	-0.7697	-0.8174	-0.8671

$$\lim_{x \rightarrow 2} G(x) \approx -0.75$$

$$(c) F'(2) = \lim_{x \rightarrow 2} \frac{F(x) - F(2)}{x - 2}$$

$$= \lim_{x \rightarrow 2} \frac{1}{x-2} \int_2^x \sin t^2 dt$$

$$= \lim_{x \rightarrow 2} G(x)$$

$$\text{Since } F'(x) = \sin x^2, F'(2) = \sin 4 = \lim_{x \rightarrow 2} G(x).$$

(Note: $\sin 4 \approx -0.7568$)

4. Let d be the distance traversed and a be the uniform acceleration.

We can assume that $v(0) = 0$ and $s(0) = 0$. Then

$$a(t) = a$$

$$v(t) = at$$

$$s(t) = \frac{1}{2}at^2.$$

$$s(t) = d \text{ when } t = \sqrt{\frac{2d}{a}}.$$

$$\text{The highest speed is } v = a\sqrt{\frac{2d}{a}} = \sqrt{2ad}.$$

The lowest speed is $v = 0$.

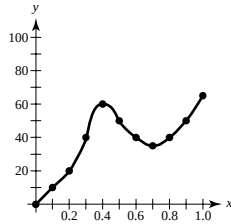
$$\text{The mean speed is } \frac{1}{2}(\sqrt{2ad} + 0) = \sqrt{\frac{ad}{2}}.$$

The time necessary to traverse the distance d at the mean speed is

$$t = \frac{d}{\sqrt{ad/2}} = \sqrt{\frac{2d}{a}}$$

which is the same as the time calculated above.

6. (a)



- (b) v is increasing (positive acceleration) on $(0, 0.4)$ and $(0.7, 1.0)$.

$$(c) \text{ Average acceleration} = \frac{v(0.4) - v(0)}{0.4 - 0} = \frac{60 - 0}{0.4} = 150 \text{ mi/hr}^2$$

- (d) This integral is the total distance traveled in miles.

$$\int_0^1 v(t) dt \approx \frac{1}{10}[0 + 2(20) + 2(60) + 2(40) + 2(40) + 65] = \frac{385}{10} = 38.5 \text{ miles}$$

- (e) One approximation is

$$a(0.8) \approx \frac{v(0.9) - v(0.8)}{0.9 - 0.8} = \frac{50 - 40}{0.1} = 100 \text{ mi/hr}^2$$

(other answers possible)

$$8. \int_0^x f(t)(x-t) dt = \int_0^x x f(t) dt - \int_0^x t f(t) dt = x \int_0^x f(t) dt - \int_0^x t f(t) dt$$

$$\text{Thus, } \frac{d}{dx} \int_0^x f(t)(x-t) dt = x f(x) + \int_0^x f(t) dt - x f(x) = \int_0^x f(t) dt$$

Differentiating the other integral,

$$\frac{d}{dx} \int_0^x \left(\int_0^x f(v) dv \right) dt = \int_0^x f(v) dv.$$

Thus, the two original integrals have equal derivatives,

$$\int_0^x f(t)(x-t) dt = \int_0^x \left(\int_0^t f(v) dv \right) dt + C$$

Letting $x = 0$, we see that $C = 0$.

10. Consider $\int_0^1 \sqrt{x} dx = \left[\frac{2}{3} x^{3/2} \right]_0^1 = \frac{2}{3}$. The corresponding

Riemann Sum using right-hand endpoints is

$$\begin{aligned} S(n) &= \frac{1}{n} \left[\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}} \right] \\ &= \frac{1}{n^{3/2}} [\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}] \end{aligned}$$

$$\text{Thus, } \lim_{n \rightarrow \infty} \frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^{3/2}} = \frac{2}{3}.$$

$$\begin{aligned} 12. (a) \text{ Area} &= \int_{-3}^3 (9 - x^2) dx = 2 \int_0^3 (9 - x^2) dx \\ &= 2 \left[9x - \frac{x^3}{3} \right]_0^3 \\ &= 2[27 - 9] = 36 \end{aligned}$$

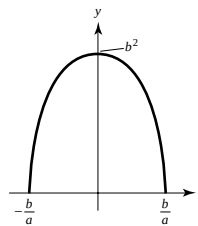
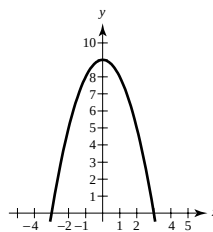
$$(b) \text{ Base} = 6, \text{ height} = 9. \text{ Area} = \frac{2}{3}bh = \frac{2}{3}(6)(9) = 36.$$

(c) Let the parabola be given by $y = b^2 - a^2x^2$, $a, b > 0$.

$$\begin{aligned} \text{Area} &= 2 \int_0^{b/a} (b^2 - a^2x^2) dx \\ &= 2 \left[b^2x - a^2 \frac{x^3}{3} \right]_0^{b/a} \\ &= 2 \left[b^2 \left(\frac{b}{a} \right) - \frac{a^2}{3} \left(\frac{b}{a} \right)^3 \right] \\ &= 2 \left[\frac{b^3}{a} - \frac{1}{3} \frac{b^3}{a} \right] = \frac{4}{3} \frac{b^3}{a} \end{aligned}$$

$$\text{Base} = \frac{2b}{a}, \text{ height} = b^2$$

$$\text{Archimedes' Formula: Area} = \frac{2}{3} \left(\frac{2b}{a} \right) (b^2) = \frac{4}{3} \frac{b^3}{a}$$



$$14. (a) (1+i)^3 = 1 + 3i + 3i^2 + i^3 \Rightarrow (1+i)^3 - i^3 = 3i^2 + 3i + 1$$

$$(b) \quad 3i^2 + 3i + 1 = (i+1)^3 - i^3$$

$$\begin{aligned} \sum_{i=1}^n (3i^2 + 3i + 1) &= \sum_{i=1}^n [(i+1)^3 - i^3] \\ &= (2^3 - 1^3) + (3^3 - 2^3) + \cdots + [(n+1)^3 - n^3] \\ &= (n+1)^3 - 1 \end{aligned}$$

$$\text{Hence, } (n+1)^3 = \sum_{i=1}^n (3i^2 + 3i + 1) + 1$$

$$(c) (n+1)^3 - 1 = \sum_{i=1}^n (3i^2 + 3i + 1) = \sum_{i=1}^n 3i^2 + \frac{3(n)(n+1)}{2} + n$$

$$\begin{aligned} \Rightarrow \sum_{i=1}^n 3i^2 &= n^3 + 3n^2 + 3n - \frac{3n(n+1)}{2} - n \\ &= \frac{2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n}{2} \\ &= \frac{2n^3 + 3n^2 + n}{2} \\ &= \frac{n(n+1)(2n+1)}{2} \\ \Rightarrow \sum_{i=1}^n i^2 &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

$$16. (a) C = 0.1 \int_8^{20} \left[12 \sin \frac{\pi(t-8)}{12} \right] dt = \left[-\frac{14.4}{\pi} \cos \frac{\pi(t-8)}{12} \right]_8^{20} = \frac{-14.4}{\pi} (-1 - 1) \approx \$9.17$$

$$\begin{aligned} (b) C &= 0.1 \int_{10}^{18} \left[12 \sin \frac{\pi(t-8)}{12} - 6 \right] dt = \left[-\frac{14.4}{\pi} \cos \frac{\pi(t-8)}{12} - 0.6t \right]_{10}^{18} \\ &= \left[\frac{-14.4}{\pi} \left(\frac{-\sqrt{3}}{2} \right) - 10.8 \right] - \left[\frac{-14.4}{\pi} \left(\frac{\sqrt{3}}{2} \right) - 6 \right] \approx \$3.14 \end{aligned}$$

$$\text{Savings} \approx 9.17 - 3.14 = \$6.03.$$

$$18. (a) \text{ Let } A = \int_0^b \frac{f(x)}{f(x) + f(b-x)} dx.$$

$$\text{Let } u = b - x, du = -dx.$$

$$\begin{aligned} A &= \int_b^0 \frac{f(b-u)}{f(b-u) + f(u)} (-du) \\ &= \int_0^b \frac{f(b-u)}{f(b-u) + f(u)} du \\ &= \int_0^b \frac{f(b-x)}{f(b-x) + f(x)} dx \end{aligned}$$

Then,

$$\begin{aligned} 2A &= \int_0^b \frac{f(x)}{f(x) + f(b-x)} dx + \int_0^b \frac{f(b-x)}{f(b-x) + f(x)} dx \\ &= \int_0^b 1 dx = b. \end{aligned}$$

$$\text{Thus, } A = \frac{b}{2}.$$

$$(b) b = 1 \Rightarrow \int_0^1 \frac{\sin x}{\sin(1-x) + \sin x} dx = \frac{1}{2}$$

CHAPTER 5

Logarithmic, Exponential, and Other Transcendental Functions

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CHAPTER 5

Logarithmic, Exponential, and Other Transcendental Functions

Section 5.1 The Natural Logarithmic Function: Differentiation

Solutions to Odd-Numbered Exercises

1. Simpson's Rule: $n = 10$

x	0.5	1.5	2	2.5	3	3.5	4
$\int_1^x \frac{1}{t} dt$	-0.6932	0.4055	0.6932	0.9163	1.0987	1.2529	1.3865

Note: $\int_1^{0.5} \frac{1}{t} dt = -\int_{0.5}^1 \frac{1}{t} dt$

3. (a) $\ln 45 \approx 3.8067$

(b) $\int_1^{45} \frac{1}{t} dt \approx 3.8067$

5. (a) $\ln 0.8 \approx -0.2231$

(b) $\int_1^{0.8} \frac{1}{t} dt \approx -0.2231$

7. $f(x) = \ln x + 2$

Vertical shift 2 units upward

Matches (b)

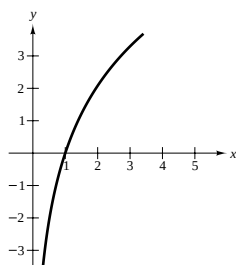
9. $f(x) = \ln(x - 1)$

Horizontal shift 1 unit to the right

Matches (a)

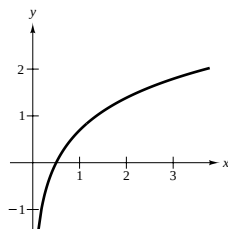
11. $f(x) = 3 \ln x$

Domain: $x > 0$



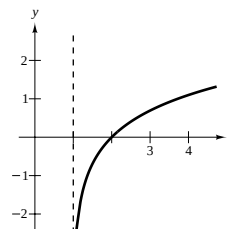
13. $f(x) = \ln 2x$

Domain: $x > 0$



15. $f(x) = \ln(x - 1)$

Domain: $x > 1$



17. (a) $\ln 6 = \ln 2 + \ln 3 \approx 1.7917$

(b) $\ln \frac{2}{3} = \ln 2 - \ln 3 \approx -0.4055$

(c) $\ln 81 = \ln 3^4 = 4 \ln 3 \approx 4.3944$

(d) $\ln \sqrt{3} = \ln 3^{1/2} = \frac{1}{2} \ln 3 \approx 0.5493$

19. $\ln \frac{2}{3} = \ln 2 - \ln 3$

21. $\ln \frac{xy}{z} = \ln x + \ln y - \ln z$

23. $\ln \sqrt[3]{a^2 + 1} = \ln(a^2 + 1)^{1/3} = \frac{1}{3} \ln(a^2 + 1)$

$$25. \ln\left(\frac{x^2-1}{x^3}\right)^3 = 3[\ln(x^2-1) - \ln x^3]$$

$$= 3[\ln(x+1) + \ln(x-1) - 3 \ln x]$$

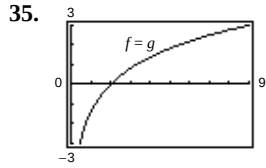
$$27. \ln z(z-1)^2 = \ln z + \ln(z-1)^2$$

$$= \ln z + 2 \ln(z-1)$$

$$29. \ln(x-2) - \ln(x+2) = \ln \frac{x-2}{x+2}$$

$$31. \frac{1}{3}[2 \ln(x+3) + \ln x - \ln(x^2-1)] = \frac{1}{3} \ln \frac{x(x+3)^2}{x^2-1} = \ln \sqrt[3]{\frac{x(x+3)^2}{x^2-1}}$$

$$33. 2 \ln 3 - \frac{1}{2} \ln(x^2+1) = \ln 9 - \ln \sqrt{x^2+1} = \ln \frac{9}{\sqrt{x^2+1}}$$



$$37. \lim_{x \rightarrow 3^+} \ln(x-3) = -\infty$$

$$39. \lim_{x \rightarrow 2^-} \ln[x^2(3-x)] = \ln 4 \approx 1.3863$$

$$41. y = \ln x^3 = 3 \ln x$$

$$y' = \frac{3}{x}$$

$$\text{At } (1, 0), y' = 3.$$

$$43. y = \ln x^2 = 2 \ln x$$

$$y' = \frac{2}{x}$$

$$\text{At } (1, 0), y' = 2.$$

$$45. g(x) = \ln x^2 = 2 \ln x$$

$$g'(x) = \frac{2}{x}$$

$$47. y = (\ln x)^4$$

$$\frac{dy}{dx} = 4(\ln x)^3 \left(\frac{1}{x}\right) = \frac{4(\ln x)^3}{x}$$

$$49. y = \ln x \sqrt{x^2-1} = \ln x + \frac{1}{2} \ln(x^2-1)$$

$$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2} \left(\frac{2x}{x^2-1} \right) = \frac{2x^2-1}{x(x^2-1)}$$

$$51. f(x) = \ln \frac{x}{x^2+1} = \ln x - \ln(x^2+1)$$

$$f'(x) = \frac{1}{x} - \frac{2x}{x^2+1} = \frac{1-x^2}{x(x^2+1)}$$

$$53. g(t) = \frac{\ln t}{t^2}$$

$$g'(t) = \frac{t^2(1/t) - 2t \ln t}{t^4} = \frac{1-2 \ln t}{t^3}$$

$$55. y = \ln(\ln x^2)$$

$$\frac{dy}{dx} = \frac{1}{\ln x^2} \frac{d}{dx}(\ln x^2) = \frac{(2x/x^2)}{\ln x^2} = \frac{2}{x \ln x^2} = \frac{1}{x \ln x}$$

$$57. y = \ln \sqrt{\frac{x+1}{x-1}} = \frac{1}{2} [\ln(x+1) - \ln(x-1)]$$

$$\frac{dy}{dx} = \frac{1}{2} \left[\frac{1}{x+1} - \frac{1}{x-1} \right] = \frac{1}{1-x^2}$$

$$59. f(x) = \ln \frac{\sqrt{4+x^2}}{x} = \frac{1}{2} \ln(4+x^2) - \ln x$$

$$f'(x) = \frac{x}{4+x^2} - \frac{1}{x} = \frac{-4}{x(x^2+4)}$$

$$61. y = \frac{-\sqrt{x^2+1}}{x} + \ln(x + \sqrt{x^2+1})$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-x(x/\sqrt{x^2+1}) + \sqrt{x^2+1}}{x^2} + \left(\frac{1}{x + \sqrt{x^2+1}}\right)\left(1 + \frac{x}{\sqrt{x^2+1}}\right) \\ &= \frac{1}{x^2\sqrt{x^2+1}} + \left(\frac{1}{x + \sqrt{x^2+1}}\right)\left(\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}}\right) = \frac{1}{x^2\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} = \frac{1+x^2}{x^2\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}}{x^2} \end{aligned}$$

$$63. y = \ln|\sin x|$$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

$$65. y = \ln\left|\frac{\cos x}{\cos x - 1}\right|$$

$$= \ln|\cos x| - \ln|\cos x - 1|$$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} - \frac{-\sin x}{\cos x - 1} = -\tan x + \frac{\sin x}{\cos x - 1}$$

$$67. y = \ln\left|\frac{-1 + \sin x}{2 + \sin x}\right|$$

$$= \ln|-1 + \sin x| - \ln|2 + \sin x|$$

$$\frac{dy}{dx} = \frac{\cos x}{-1 + \sin x} - \frac{\cos x}{2 + \sin x}$$

$$= \frac{3 \cos x}{(\sin x - 1)(\sin x + 2)}$$

$$69. f(x) = \sin 2x \ln x^2 = 2 \sin 2x \ln x$$

$$f'(x) = (2 \sin 2x)\left(\frac{1}{x}\right) + 4 \cos 2x \ln x$$

$$= \frac{2}{x}(\sin 2x + 2x \cos 2x \ln x)$$

$$= \frac{2}{x}(\sin 2x + x \cos 2x \ln x^2)$$

$$71. (a) y = 3x^2 - \ln x, (1, 3)$$

$$\frac{dy}{dx} = 6x - \frac{1}{x}$$

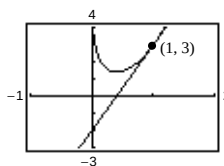
$$\text{When } x = 1, \frac{dy}{dx} = 5.$$

$$\text{Tangent line: } y - 3 = 5(x - 1)$$

$$y = 5x - 2$$

$$0 = 5x - y - 2$$

(b)



$$73. x^2 - 3 \ln y + y^2 = 10$$

$$2x - \frac{3}{y} \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$$

$$2x = \frac{dy}{dx} \left(\frac{3}{y} - 2y \right)$$

$$\frac{dy}{dx} = \frac{2x}{(3/y) - 2y} = \frac{2xy}{3 - 2y^2}$$

$$75. y = 2(\ln x) + 3$$

$$y' = \frac{2}{x}$$

$$y'' = -\frac{2}{x^2}$$

$$xy'' + y' = x\left(-\frac{2}{x^2}\right) + \frac{2}{x} = 0$$

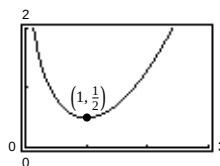
$$77. y = \frac{x^2}{2} - \ln x$$

$$\text{Domain: } x > 0$$

$$y' = x - \frac{1}{x} = \frac{(x+1)(x-1)}{x} = 0 \text{ when } x = 1.$$

$$y'' = 1 + \frac{1}{x^2} > 0$$

$$\text{Relative minimum: } \left(1, \frac{1}{2}\right)$$



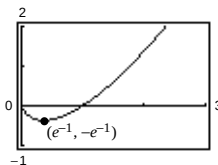
79. $y = x \ln x$

Domain: $x > 0$

$$y' = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x = 0 \text{ when } x = e^{-1}.$$

$$y'' = \frac{1}{x} > 0$$

Relative minimum: $(e^{-1}, -e^{-1})$



81. $y = \frac{x}{\ln x}$

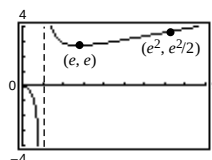
Domain: $0 < x < 1, x > 1$

$$y' = \frac{(\ln x)(1) - (x)(1/x)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} = 0 \text{ when } x = e.$$

$$y'' = \frac{(\ln x)^2(1/x) - (\ln x - 1)(2/x) \ln x}{(\ln x)^4} = \frac{2 - \ln x}{x(\ln x)^3} = 0 \text{ when } x = e^2.$$

Relative minimum: (e, e)

Point of inflection: $(e^2, e^2/2)$



83. $f(x) = \ln x, \quad f(1) = 0$

$$f'(x) = \frac{1}{x}, \quad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2}, \quad f''(1) = -1$$

$$P_1(x) = f(1) + f'(1)(x - 1) = x - 1, \quad P_1(1) = 0$$

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2$$

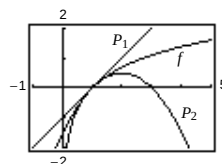
$$= (x - 1) - \frac{1}{2}(x - 1)^2, \quad P_2(1) = 0$$

$$P_1'(x) = 1, \quad P_1'(1) = 1$$

$$P_2'(x) = 1 - (x - 1) = 2 - x, \quad P_2'(1) = 1$$

$$P_2''(x) = -1, \quad P_2''(1) = -1$$

The values of f, P_1, P_2 , and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



85. Find x such that $\ln x = -x$.

$$f(x) = (\ln x) + x = 0$$

$$f'(x) = \frac{1}{x} + 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n \left[\frac{1 - \ln x_n}{1 + x_n} \right]$$

n	1	2	3
x_n	0.5	0.5644	0.5671
$f(x_n)$	-0.1931	-0.0076	-0.0001

Approximate root: $x = 0.567$

87. $y = x\sqrt{x^2 - 1}$

$$\ln y = \ln x + \frac{1}{2} \ln(x^2 - 1)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{x} + \frac{x}{x^2 - 1}$$

$$\frac{dy}{dx} = y \left[\frac{2x^2 - 1}{x(x^2 - 1)} \right] = \frac{2x^2 - 1}{\sqrt{x^2 - 1}}$$

$$89. \quad y = \frac{x^2 \sqrt{3x-2}}{(x-1)^2}$$

$$\ln y = 2 \ln x + \frac{1}{2} \ln(3x-2) - 2 \ln(x-1)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x-1}$$

$$\frac{dy}{dx} = y \left[\frac{3x^2 - 15x + 8}{2x(3x-2)(x-1)} \right]$$

$$= \frac{3x^3 - 15x^2 + 8x}{2(x-1)^3 \sqrt{3x-2}}$$

$$91. \quad y = \frac{x(x-1)^{3/2}}{\sqrt{x+1}}$$

$$\ln y = \ln x + \frac{3}{2} \ln(x-1) - \frac{1}{2} \ln(x+1)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{x} + \frac{3}{2} \left(\frac{1}{x-1} \right) - \frac{1}{2} \left(\frac{1}{x+1} \right)$$

$$\frac{dy}{dx} = \frac{y}{2} \left[\frac{2}{x} + \frac{3}{x-1} - \frac{1}{x+1} \right]$$

$$= \frac{y}{2} \left[\frac{4x^2 + 4x - 2}{x(x^2 - 1)} \right] = \frac{(2x^2 + 2x - 1)\sqrt{x-1}}{(x+1)^{3/2}}$$

93. Answers will vary. See Theorem 5.1 and 5.2.

95. $\ln e^x = x$ because $f(x) = \ln x$ and $g(x) = e^x$ are inverse functions.

97. (a) $f(1) \neq f(3)$

(b) $f'(x) = 1 - \frac{2}{x} = 0$ for $x = 2$.

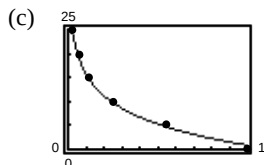
$$99. \quad \beta = 10 \log_{10} \left(\frac{I}{10^{-16}} \right) = \frac{10}{\ln 10} \ln \left(\frac{I}{10^{-16}} \right) = \frac{10}{\ln 10} [\ln I + 16 \ln 10] = 160 + 10 \log_{10} I$$

$$\beta(10^{-10}) = \frac{10}{\ln 10} [\ln 10^{-10} + 16 \ln 10] = \frac{10}{\ln 10} [-10 \ln 10 + 16 \ln 10] = \frac{10}{\ln 10} [6 \ln 10] = 60 \text{ decibels}$$

101. (a) You get an error message because $\ln h$ does not exist for $h = 0$.

(b) Reversing the data, you obtain

$$h = 0.8627 - 6.4474 \ln p.$$



(d) If $p = 0.75$, $h \approx 2.72$ km.

(e) If $h = 13$ km, $p \approx 0.15$ atmosphere.

(f) $h = 0.8627 - 6.4474 \ln p$

$$1 = -6.4474 \frac{1}{p} \frac{dp}{dh} \quad (\text{implicit differentiation})$$

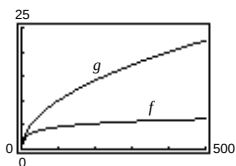
$$\frac{dp}{dh} = \frac{p}{-6.4474}$$

For $h = 5$, $p = 0.55$ and $dp/dh = -0.0853$ atmos/km.

For $h = 20$, $p = 0.06$ and $dp/dh = -0.00931$ atmos/km.

As the altitude increases, the rate of change of pressure decreases.

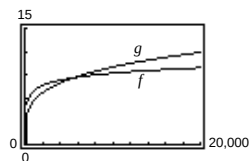
103. (a) $f(x) = \ln x$, $g(x) = \sqrt{x}$



$$f'(x) = \frac{1}{x}, \quad g'(x) = \frac{1}{2\sqrt{x}}$$

For $x > 4$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for “large” values of x .

(b) $f(x) = \ln x$, $g(x) = \sqrt[4]{x}$



$$f'(x) = \frac{1}{x}, \quad g'(x) = \frac{1}{4\sqrt[4]{x^3}}$$

For $x > 256$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for “large” values of x . $f(x) = \ln x$ increases very slowly for “large” values of x .

105. False

$$\ln x + \ln 25 = \ln(25x) \neq \ln(x + 25)$$

Section 5.2 The Natural Logarithmic Function: Integration

$$1. \int \frac{5}{x} dx = 5 \int \frac{1}{x} dx = 5 \ln |x| + C$$

$$3. u = x + 1, du = dx$$

$$\int \frac{1}{x+1} dx = \ln|x+1| + C$$

$$5. u = 3 - 2x, du = -2 dx$$

$$\begin{aligned} \int \frac{1}{3-2x} dx &= -\frac{1}{2} \int \frac{1}{3-2x} (-2) dx \\ &= -\frac{1}{2} \ln|3-2x| + C \end{aligned}$$

$$7. u = x^2 + 1, du = 2x dx$$

$$\begin{aligned} \int \frac{x}{x^2+1} dx &= \frac{1}{2} \int \frac{1}{x^2+1} (2x) dx \\ &= \frac{1}{2} \ln(x^2+1) + C \\ &= \ln \sqrt{x^2+1} + C \end{aligned}$$

$$\begin{aligned} 9. \int \frac{x^2-4}{x} dx &= \int \left(x - \frac{4}{x} \right) dx \\ &= \frac{x^2}{2} - 4 \ln|x| + C \end{aligned}$$

$$11. u = x^3 + 3x^2 + 9x, du = 3(x^2 + 2x + 3) dx$$

$$\begin{aligned} \int \frac{x^2+2x+3}{x^3+3x^2+9x} dx &= \frac{1}{3} \int \frac{3(x^2+2x+3)}{x^3+3x^2+9x} dx \\ &= \frac{1}{3} \ln|x^3+3x^2+9x| + C \end{aligned}$$

$$\begin{aligned} 13. \int \frac{x^2-3x+2}{x+1} dx &= \int \left(x - 4 + \frac{6}{x+1} \right) dx \\ &= \frac{x^2}{2} - 4x + 6 \ln|x+1| + C \end{aligned}$$

$$\begin{aligned} 15. \int \frac{x^3-3x^2+5}{x-3} dx &= \int \left(x^2 + \frac{5}{x-3} \right) dx \\ &= \frac{x^3}{3} + 5 \ln|x-3| + C \end{aligned}$$

$$\begin{aligned} 17. \int \frac{x^4+x-4}{x^2+2} dx &= \int \left(x^2 - 2 + \frac{x}{x^2+2} \right) dx \\ &= \frac{x^3}{3} - 2x + \frac{1}{2} \ln(x^2+2) + C \end{aligned}$$

$$19. u = \ln x, du = \frac{1}{x} dx$$

$$\int \frac{(\ln x)^2}{x} dx = \frac{1}{3} (\ln x)^3 + C$$

$$21. u = x + 1, du = dx$$

$$\begin{aligned} \int \frac{1}{\sqrt{x+1}} dx &= \int (x+1)^{-1/2} dx \\ &= 2(x+1)^{1/2} + C \\ &= 2\sqrt{x+1} + C \end{aligned}$$

$$\begin{aligned} 23. \int \frac{2x}{(x-1)^2} dx &= \int \frac{2x-2+2}{(x-1)^2} dx \\ &= \int \frac{2(x-1)}{(x-1)^2} dx + 2 \int \frac{1}{(x-1)^2} dx \\ &= 2 \int \frac{1}{x-1} dx + 2 \int \frac{1}{(x-1)^2} dx \\ &= 2 \ln|x-1| - \frac{2}{(x-1)} + C \end{aligned}$$

25. $u = 1 + \sqrt{2x}$, $du = \frac{1}{\sqrt{2x}} dx \Rightarrow (u - 1) du = dx$

$$\begin{aligned}\int \frac{1}{1 + \sqrt{2x}} dx &= \int \frac{(u - 1)}{u} du = \int \left(u - \frac{1}{u}\right) du \\ &= u - \ln|u| + C_1 \\ &= (1 + \sqrt{2x}) - \ln|1 + \sqrt{2x}| + C_1 \\ &= \sqrt{2x} - \ln(1 + \sqrt{2x}) + C\end{aligned}$$

where $C = C_1 + 1$.

27. $u = \sqrt{x} - 3$, $du = \frac{1}{2\sqrt{x}} dx \Rightarrow 2(u + 3) du = dx$

$$\begin{aligned}\int \frac{\sqrt{x}}{\sqrt{x} - 3} dx &= 2 \int \frac{(u + 3)^2}{u} du = 2 \int \frac{u^2 + 6u + 9}{u} du = 2 \int \left(u + 6 + \frac{9}{u}\right) du \\ &= 2 \left[\frac{u^2}{2} + 6u + 9 \ln|u| \right] + C_1 = u^2 + 12u + 18 \ln|u| + C_1 \\ &= (\sqrt{x} - 3)^2 + 12(\sqrt{x} - 3) + 18 \ln|\sqrt{x} - 3| + C_1 \\ &= x + 6\sqrt{x} + 18 \ln|\sqrt{x} - 3| + C \text{ where } C = C_1 - 27.\end{aligned}$$

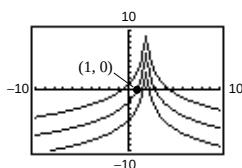
29. $\int \frac{\cos \theta}{\sin \theta} d\theta = \ln|\sin \theta| + C$
 $(u = \sin \theta, du = \cos \theta d\theta)$

31. $\int \csc 2x dx = \frac{1}{2} \int (\csc 2x)(2) dx$
 $= -\frac{1}{2} \ln|\csc 2x + \cot 2x| + C$

33. $\int \frac{\cos t}{1 + \sin t} dt = \ln|1 + \sin t| + C$

35. $\int \frac{\sec x \tan x}{\sec x - 1} dx = \ln|\sec x - 1| + C$

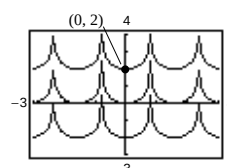
37. $y = \int \frac{3}{2 - x} dx$
 $= -3 \int \frac{1}{x - 2} dx$
 $= -3 \ln|x - 2| + C$



$(1, 0): 0 = -3 \ln|1 - 2| + C \Rightarrow C = 0$

$y = -3 \ln|x - 2|$

39. $s = \int \tan(2\theta) d\theta$
 $= \frac{1}{2} \int \tan(2\theta)(2 d\theta)$
 $= -\frac{1}{2} \ln|\cos 2\theta| + C$

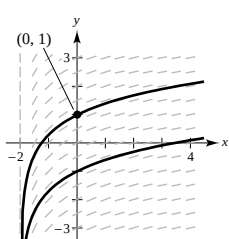


$(0, 2): 2 = -\frac{1}{2} \ln|\cos(0)| + C \Rightarrow C = 2$

$s = -\frac{1}{2} \ln|\cos 2\theta| + 2$

41. $\frac{dy}{dx} = \frac{1}{x + 2}$, $(0, 1)$

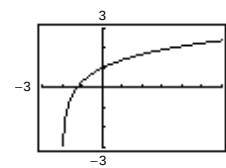
(a)



(b) $y = \int \frac{1}{x + 2} dx = \ln|x + 2| + C$

$y(0) = 1 \Rightarrow 1 = \ln 2 + C \Rightarrow C = 1 - \ln 2$

Hence, $y = \ln|x + 2| + 1 - \ln 2 = \ln\left|\frac{x + 2}{2}\right| + 1$.



$$\begin{aligned}
 43. \int_0^4 \frac{5}{3x+1} dx &= \left[\frac{5}{3} \ln|3x+1| \right]_0^4 \\
 &= \frac{5}{3} \ln 13 \approx 4.275
 \end{aligned}$$

$$\begin{aligned}
 45. u &= 1 + \ln x, du = \frac{1}{x} dx \\
 \int_1^e \frac{(1 + \ln x)^2}{x} dx &= \left[\frac{1}{3} (1 + \ln x)^3 \right]_1^e = \frac{7}{3}
 \end{aligned}$$

$$\begin{aligned}
 47. \int_0^2 \frac{x^2 - 2}{x + 1} dx &= \int_0^2 \left(x - 1 - \frac{1}{x + 1} \right) dx \\
 &= \left[\frac{1}{2} x^2 - x - \ln|x + 1| \right]_0^2 = -\ln 3
 \end{aligned}$$

$$\begin{aligned}
 49. \int_1^2 \frac{1 - \cos \theta}{\theta - \sin \theta} d\theta &= \left[\ln|\theta - \sin \theta| \right]_1^2 \\
 &= \ln \left| \frac{2 - \sin 2}{1 - \sin 1} \right| \approx 1.929
 \end{aligned}$$

$$51. -\ln|\cos x| + C = \ln \left| \frac{1}{\cos x} \right| + C = \ln|\sec x| + C$$

$$\begin{aligned}
 53. \ln|\sec x + \tan x| + C &= \ln \left| \frac{(\sec x + \tan x)(\sec x - \tan x)}{(\sec x - \tan x)} \right| + C = \ln \left| \frac{\sec^2 x - \tan^2 x}{\sec x - \tan x} \right| + C \\
 &= \ln \left| \frac{1}{\sec x - \tan x} \right| + C = -\ln|\sec x - \tan x| + C
 \end{aligned}$$

$$\begin{aligned}
 55. \int \frac{1}{1 + \sqrt{x}} dx &= 2(1 + \sqrt{x}) - 2 \ln(1 + \sqrt{x}) + C_1 \\
 &= 2[\sqrt{x} - \ln(1 + \sqrt{x})] + C \text{ where } C = C_1 + 2.
 \end{aligned}$$

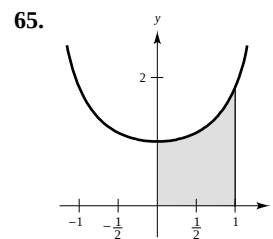
$$57. \int \cos(1 - x) dx = -\sin(1 - x) + C$$

$$59. \int_{\pi/4}^{\pi/2} (\csc x - \sin x) dx = \left[-\ln|\csc x + \cot x| + \cos x \right]_{\pi/4}^{\pi/2} = \ln(\sqrt{2} + 1) - \frac{\sqrt{2}}{2} \approx 0.174$$

Note: In Exercises 61 and 63, you can use the Second Fundamental Theorem of Calculus or integrate the function.

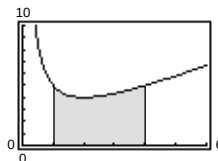
$$\begin{aligned}
 61. F(x) &= \int_1^x \frac{1}{t} dt \\
 F'(x) &= \frac{1}{x}
 \end{aligned}$$

$$\begin{aligned}
 63. F(x) &= \int_x^{3x} \frac{1}{t} dt = \int_1^{3x} \frac{1}{t} dt - \int_1^x \frac{1}{t} dt \\
 F'(x) &= \frac{3}{3x} - \frac{1}{x} = 0
 \end{aligned}$$

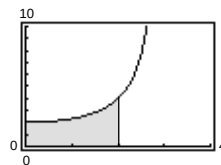


$A \approx 1.25$
Matches (d)

$$\begin{aligned}
 67. A &= \int_1^4 \frac{x^2 + 4}{x} dx = \int_1^4 \left(x + \frac{4}{x} \right) dx \\
 &= \left[\frac{x^2}{2} + 4 \ln x \right]_1^4 = (8 + 4 \ln 4) - \frac{1}{2} \\
 &= \frac{15}{2} + 8 \ln 2 \approx 13.045 \text{ square units}
 \end{aligned}$$



$$\begin{aligned}
 69. \int_0^2 2 \sec \frac{\pi x}{6} dx &= \frac{12}{\pi} \int_0^2 \sec \left(\frac{\pi x}{6} \right) \frac{\pi}{6} dx \\
 &= \left[\frac{12}{\pi} \ln \left| \sec \frac{\pi x}{6} + \tan \frac{\pi x}{6} \right| \right]_0^2 \\
 &= \frac{12}{\pi} \ln \left| \sec \frac{\pi}{3} + \tan \frac{\pi}{3} \right| - \frac{12}{\pi} \ln |1 + 0| \\
 &= \frac{12}{\pi} \ln(2 + \sqrt{3}) \approx 5.03041
 \end{aligned}$$



71. Power Rule

73. Substitution: ($u = x^2 + 4$)
and Log Rule

75. Divide the polynomials:

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$$

$$\begin{aligned}
 77. \text{Average value} &= \frac{1}{4-2} \int_2^4 \frac{8}{x^2} dx = 4 \int_2^4 x^{-2} dx \\
 &= \left[-4 \frac{1}{x} \right]_2^4 \\
 &= -4 \left(\frac{1}{4} - \frac{1}{2} \right) = 1
 \end{aligned}$$

$$\begin{aligned}
 79. \text{Average value} &= \frac{1}{e-1} \int_1^e \frac{\ln x}{x} dx = \frac{1}{e-1} \left[\frac{(\ln x)^2}{2} \right]_1^e \\
 &= \frac{1}{e-1} \left(\frac{1}{2} \right) \\
 &= \frac{1}{2e-2} \approx 0.291
 \end{aligned}$$

$$81. P(t) = \int \frac{3000}{1+0.25t} dt = (3000)(4) \int \frac{0.25}{1+0.25t} dt = 12,000 \ln|1+0.25t| + C$$

$$P(0) = 12,000 \ln|1+0.25(0)| + C = 1000$$

$$C = 1000$$

$$P(t) = 12,000 \ln|1+0.25t| + 1000 = 1000[12 \ln|1+0.25t| + 1]$$

$$P(3) = 1000[12(\ln 1.75) + 1] \approx 7715$$

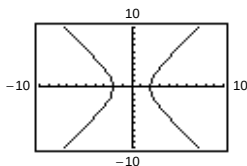
$$83. \frac{1}{50-40} \int_{40}^{50} \frac{90,000}{400+3x} dx = \left[3000 \ln|400+3x| \right]_{40}^{50} \approx \$168.27$$

$$85. (a) 2x^2 - y^2 = 8$$

$$y^2 = 2x^2 - 8$$

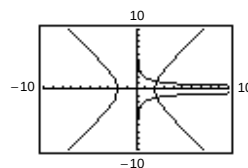
$$y_1 = \sqrt{2x^2 - 8}$$

$$y_2 = -\sqrt{2x^2 - 8}$$



$$(b) y^2 = e^{-\int (1/x) dx} = e^{-\ln x + C} = e^{\ln(1/x)} (e^C) = \frac{1}{x} k$$

$$\text{Let } k = 4 \text{ and graph } y^2 = \frac{4}{x} \quad \left(\begin{array}{l} y_1 = 2/\sqrt{x} \\ y_2 = -2/\sqrt{x} \end{array} \right)$$



$$(c) \text{ In part (a), } 2x^2 - y^2 = 8$$

$$4x - 2yy' = 0$$

$$y' = \frac{2x}{y}$$

$$\text{In part (b), } y^2 = \frac{4}{x} = 4x^{-1}$$

$$2yy' = \frac{-4}{x^2}$$

$$y' = \frac{-2}{yx^2} = \frac{-2y}{y^2 x^2} = \frac{-2y}{4x} = \frac{-y}{2x}$$

Using a graphing utility the graphs intersect at (2.214, 1.344). The slopes are 3.295 and $-0.304 = (-1)/3.295$, respectively.

87. False

$$\frac{1}{2}(\ln x) = \ln(x^{1/2}) \neq (\ln x)^{1/2}$$

89. True

$$\begin{aligned}\int \frac{1}{x} dx &= \ln|x| + C_1 \\ &= \ln|x| + \ln|C| = \ln|Cx|, \quad C \neq 0\end{aligned}$$

Section 5.3 Inverse Functions

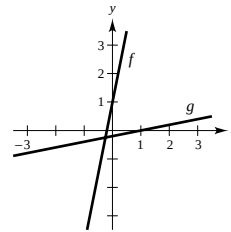
1. (a) $f(x) = 5x + 1$

$$g(x) = \frac{x-1}{5}$$

$$f(g(x)) = f\left(\frac{x-1}{5}\right) = 5\left(\frac{x-1}{5}\right) + 1 = x$$

$$g(f(x)) = g(5x + 1) = \frac{(5x + 1) - 1}{5} = x$$

(b)



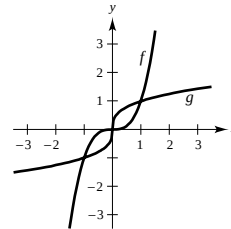
3. (a) $f(x) = x^3$

$$g(x) = \sqrt[3]{x}$$

$$f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$$

$$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$

(b)



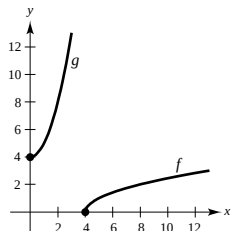
5. (a) $f(x) = \sqrt{x-4}$

$$g(x) = x^2 + 4, \quad x \geq 0$$

$$\begin{aligned}f(g(x)) &= f(x^2 + 4) \\ &= \sqrt{(x^2 + 4) - 4} = \sqrt{x^2} = x\end{aligned}$$

$$\begin{aligned}g(f(x)) &= g(\sqrt{x-4}) \\ &= (\sqrt{x-4})^2 + 4 = x - 4 + 4 = x\end{aligned}$$

(b)



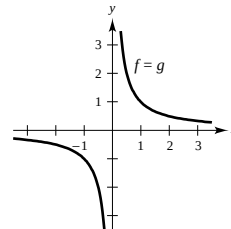
7. (a) $f(x) = \frac{1}{x}$

$$g(x) = \frac{1}{x}$$

$$f(g(x)) = \frac{1}{1/x} = x$$

$$g(f(x)) = \frac{1}{1/x} = x$$

(b)

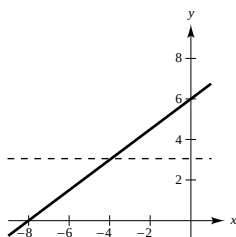


9. Matches (c)

11. Matches (a)

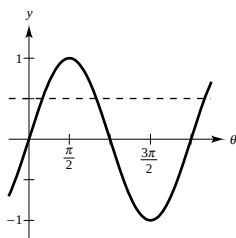
13. $f(x) = \frac{3}{4}x + 6$

One-to-one; has an inverse



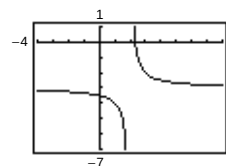
15. $f(\theta) = \sin \theta$

Not one-to-one; does not have an inverse



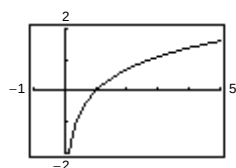
17. $h(s) = \frac{1}{s-2} - 3$

One-to-one; has an inverse



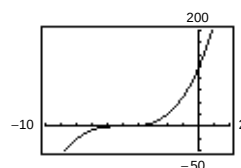
19. $f(x) = \ln x$

One-to-one; has an inverse



21. $g(x) = (x + 5)^3$

One-to-one; has an inverse



23. $f(x) = (x + a)^3 + b$

$$f'(x) = 3(x + a)^2 \geq 0 \text{ for all } x.$$

 f is increasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

25. $f(x) = \frac{x^4}{4} - 2x^2$

$$f'(x) = x^3 - 4x = 0 \text{ when } x = 0, 2, -2.$$

 f is not strictly monotonic on $(-\infty, \infty)$. Therefore, f does not have an inverse.

27. $f(x) = 2 - x - x^3$

$$f'(x) = -1 - 3x^2 < 0 \text{ for all } x.$$

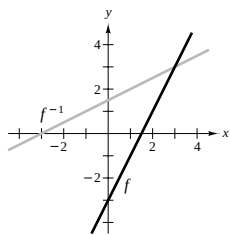
 f is decreasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

29. $f(x) = 2x - 3 = y$

$$x = \frac{y+3}{2}$$

$$y = \frac{x+3}{2}$$

$$f^{-1}(x) = \frac{x+3}{2}$$

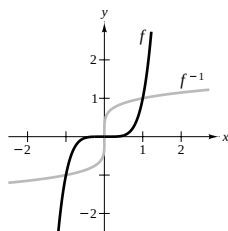


31. $f(x) = x^5 = y$

$$x = \sqrt[5]{y}$$

$$y = \sqrt[5]{x}$$

$$f^{-1}(x) = \sqrt[5]{x} = x^{1/5}$$

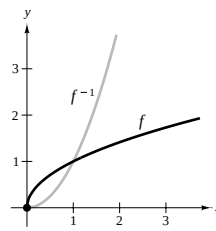


33. $f(x) = \sqrt{x} = y$

$$x = y^2$$

$$y = x^2$$

$$f^{-1}(x) = x^2, \quad x \geq 0$$

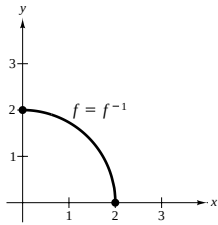


35. $f(x) = \sqrt{4 - x^2} = y, \quad 0 \leq x \leq 2$

$$x = \sqrt{4 - y^2}$$

$$y = \sqrt{4 - x^2}$$

$$f^{-1}(x) = \sqrt{4 - x^2}, \quad 0 \leq x \leq 2$$

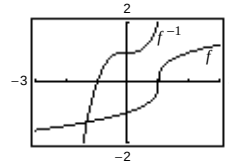


37. $f(x) = \sqrt[3]{x - 1} = y$

$$x = y^3 + 1$$

$$y = x^3 + 1$$

$$f^{-1}(x) = x^3 + 1$$



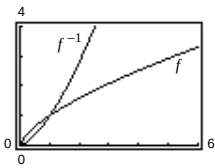
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

39. $f(x) = x^{2/3} = y, \quad x \geq 0$

$$x = y^{3/2}$$

$$y = x^{3/2}$$

$$f^{-1}(x) = x^{3/2}, \quad x \geq 0$$



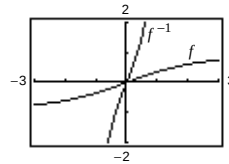
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

41. $f(x) = \frac{x}{\sqrt{x^2 + 7}} = y$

$$x = \frac{\sqrt{7}y}{\sqrt{1 - y^2}}$$

$$y = \frac{\sqrt{7}x}{\sqrt{1 - x^2}}$$

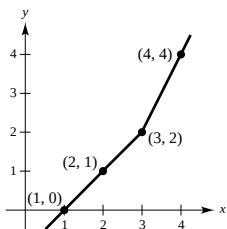
$$f^{-1}(x) = \frac{\sqrt{7}x}{\sqrt{1 - x^2}}, \quad -1 < x < 1$$



The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

 43.

x	1	2	3	4
$f^{-1}(x)$	0	1	2	4



45. (a) Let
- x
- be the number of pounds of the commodity costing 1.25 per pound. Since there are 50 pounds total, the amount of the second commodity is
- $50 - x$
- . The total cost is

$$y = 1.25x + 1.60(50 - x)$$

$$= -0.35x + 80 \quad 0 \leq x \leq 50.$$

- (b) We find the inverse of the original function:

$$y = -0.35x + 80$$

$$0.35x = 80 - y$$

$$x = \frac{100}{35}(80 - y)$$

$$\text{Inverse: } y = \frac{100}{35}(80 - x) = \frac{20}{7}(80 - x).$$

x represents cost and y represents pounds.

- (c) Domain of inverse is
- $62.5 \leq x \leq 80$
- .

- (d) If
- $x = 73$
- in the inverse function,

$$y = \frac{100}{35}(80 - 73) = \frac{100}{5} = 20 \text{ pounds.}$$

47. $f(x) = (x - 4)^2$ on $[4, \infty)$

$$f'(x) = 2(x - 4) > 0 \text{ on } (4, \infty)$$

f is increasing on $[4, \infty)$. Therefore, f is strictly monotonic and has an inverse.

51. $f(x) = \cos x$ on $[0, \pi]$

$$f'(x) = -\sin x < 0 \text{ on } (0, \pi)$$

f is decreasing on $[0, \pi]$. Therefore, f is strictly monotonic and has an inverse.

53. $f(x) = \frac{x}{x^2 - 4} = y$ on $(-2, 2)$

$$x^2y - 4y = x$$

$$x^2y - x - 4y = 0$$

$$a = y, b = -1, c = -4y$$

$$x = \frac{1 \pm \sqrt{1 - 4(y)(-4y)}}{2y} = \frac{1 \pm \sqrt{1 + 16y^2}}{2y}$$

$$y = f^{-1}(x) = \begin{cases} (1 - \sqrt{1 + 16x^2})/2x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

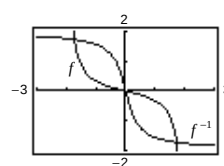
49. $f(x) = \frac{4}{x^2}$ on $(0, \infty)$

$$f'(x) = -\frac{8}{x^3} < 0 \text{ on } (0, \infty)$$

f is decreasing on $(0, \infty)$. Therefore, f is strictly monotonic and has an inverse.

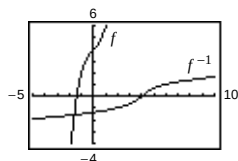
Domain: all x

Range: $-2 < y < 2$



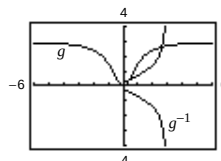
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

55. (a), (b)



(c) Yes, f is one-to-one and has an inverse. The inverse relation is an inverse function.

57. (a), (b)



(c) g is not one-to-one and does not have an inverse. The inverse relation is not an inverse function.

59. $f(x) = \sqrt{x - 2}$, Domain: $x \geq 2$

$$f'(x) = \frac{1}{2\sqrt{x - 2}} > 0 \text{ for } x > 2.$$

f is one-to-one; has an inverse

$$\sqrt{x - 2} = y$$

$$x - 2 = y^2$$

$$x = y^2 + 2$$

$$y = \sqrt{x - 2}$$

$$f^{-1}(x) = x^2 + 2, x \geq 0$$

61. $f(x) = |x - 2|, x \leq 2$

$$= -(x - 2)$$

$$= 2 - x$$

f is one-to-one; has an inverse

$$2 - x = y$$

$$2 - y = x$$

$$f^{-1}(x) = 2 - x, x \geq 0$$

63. $f(x) = (x - 3)^2$ is one-to-one for $x \geq 3$.

$$(x - 3)^2 = y$$

$$x - 3 = \sqrt{y}$$

$$x = \sqrt{y} + 3$$

$$y = \sqrt{x - 3}$$

$$f^{-1}(x) = \sqrt{x} + 3, x \geq 0$$

(Answer is not unique)

65. $f(x) = |x + 3|$ is one-to-one for $x \geq -3$.

$$x + 3 = y$$

$$x = y - 3$$

$$y = x + 3$$

$$f^{-1}(x) = x - 3, x \geq 0$$

(Answer is not unique)

67. Yes, the volume is an increasing function, and hence one-to-one. The inverse function gives the time t corresponding to the volume V .

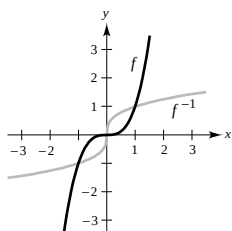
$$\begin{aligned} 71. \quad f(x) &= x^3 + 2x - 1, \quad f(1) = 2 = a \\ f'(x) &= 3x^2 + 2 \\ (f^{-1})'(2) &= \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{3(1)^2 + 2} = \frac{1}{5} \end{aligned}$$

$$\begin{aligned} 75. \quad f(x) &= x^3 - \frac{4}{x}, \quad f(2) = 6 = a \\ f'(x) &= 3x^2 + \frac{4}{x^2} \\ (f^{-1})'(6) &= \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(2)} = \frac{1}{3(2)^2 + (4/2^2)} = \frac{1}{13} \end{aligned}$$

77. (a) Domain f = Domain $f^{-1} = (-\infty, \infty)$

- (b) Range f = Range $f^{-1} = (-\infty, \infty)$

(c)



$$(d) \quad f(x) = x^3, \quad \left(\frac{1}{2}, \frac{1}{8}\right)$$

$$f'(x) = 3x^2$$

$$f'\left(\frac{1}{2}\right) = \frac{3}{4}$$

$$f^{-1}(x) = \sqrt[3]{x}, \quad \left(\frac{1}{8}, \frac{1}{2}\right)$$

$$(f^{-1})'(x) = \frac{1}{3\sqrt[3]{x^2}}$$

$$(f^{-1})'\left(\frac{1}{8}\right) = \frac{4}{3}$$

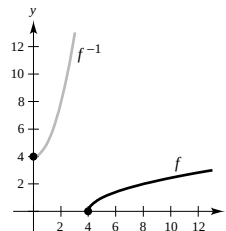
69. No, $C(t)$ is not one-to-one because long distance costs are step functions. A call lasting 2.1 minutes costs the same as one lasting 2.2 minutes.

$$\begin{aligned} 73. \quad f(x) &= \sin x, \quad f\left(\frac{\pi}{6}\right) = \frac{1}{2} = a \\ f'(x) &= \cos x \\ (f^{-1})'\left(\frac{1}{2}\right) &= \frac{1}{f'(f^{-1}(1/2))} = \frac{1}{f'(\pi/6)} = \frac{1}{\cos(\pi/6)} \\ &= \frac{1}{\sqrt{3}/2} = \frac{2\sqrt{3}}{3} \end{aligned}$$

79. (a) Domain $f = [4, \infty)$, Domain $f^{-1} = [0, \infty)$

- (b) Range $f = [0, \infty)$, Range $f^{-1} = [4, \infty)$

(c)



$$(d) \quad f(x) = \sqrt{x-4}, \quad (5, 1)$$

$$f'(x) = \frac{1}{2\sqrt{x-4}}$$

$$f'(5) = \frac{1}{2}$$

$$f^{-1}(x) = x^2 + 4, \quad (1, 5)$$

$$(f^{-1})'(x) = 2x$$

$$(f^{-1})'(1) = 2$$

81. $x = y^3 - 7y^2 + 2$

$$1 = 3y^2 \frac{dy}{dx} - 14y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3y^2 - 14y}. \text{ At } (-4, 1), \frac{dy}{dx} = \frac{1}{3 - 14} = \frac{-1}{11}.$$

Alternate solution: let $f(x) = x^3 - 7x^2 + 2$.Then $f'(x) = 3x^2 - 14x$ and $f'(1) = -11$.

$$\text{Hence, } \frac{dy}{dx} = \frac{1}{-11} = \frac{-1}{11}.$$

In Exercises 83 and 85, use the following.

$$f(x) = \frac{1}{8}x - 3 \text{ and } g(x) = x^3$$

$$f^{-1}(x) = 8(x + 3) \text{ and } g^{-1}(x) = \sqrt[3]{x}$$

83. $(f^{-1} \circ g^{-1})(1) = f^{-1}(g^{-1}(1)) = f^{-1}(1) = 32$

85. $(f^{-1} \circ f^{-1})(6) = f^{-1}(f^{-1}(6)) = f^{-1}(72) = 600$

In Exercises 87 and 89, use the following.

$$f(x) = x + 4 \text{ and } g(x) = 2x - 5$$

$$f^{-1}(x) = x - 4 \text{ and } g^{-1}(x) = \frac{x + 5}{2}$$

87. $(g^{-1} \circ f^{-1})(x) = g^{-1}(f^{-1}(x))$

$$= g^{-1}(x - 4)$$

$$= \frac{(x - 4) + 5}{2}$$

$$= \frac{x + 1}{2}$$

89. $(f \circ g)(x) = f(g(x))$

$$= f(2x - 5)$$

$$= (2x - 5) + 4$$

$$= 2x - 1$$

$$\text{Hence, } (f \circ g)^{-1}(x) = \frac{x + 1}{2}$$

$$(\text{Note: } (f \circ g)^{-1} = g^{-1} \circ f^{-1})$$

91. Answers will vary. See page 335 and Example 3.

93. $y = x^2$ on $(-\infty, \infty)$ does not have an inverse.95. f is not one-to-one because many different x -values yield the same y -value.

$$\text{Example: } f(0) = f(\pi) = 0$$

Not continuous at $\frac{(2n-1)\pi}{2}$, where n is an integer97. Let $(f \circ g)(x) = y$ then $x = (f \circ g)^{-1}(y)$. Also,

$$(f \circ g)(x) = y$$

$$f(g(x)) = y$$

$$g(x) = f^{-1}(y)$$

$$x = g^{-1}(f^{-1}(y))$$

$$= (g^{-1} \circ f^{-1})(y)$$

Since f and g are one-to-one functions,
 $(f \circ g)^{-1} = g^{-1} \circ f^{-1}.$ 99. Suppose $g(x)$ and $h(x)$ are both inverses of $f(x)$. Then the graph of $f(x)$ contains the point (a, b) if and only if the graphs of $g(x)$ and $h(x)$ contain the point (b, a) . Since the graphs of $g(x)$ and $h(x)$ are the same, $g(x) = h(x)$. Therefore, the inverse of $f(x)$ is unique.

101. False

$$\text{Let } f(x) = x^2.$$

105. Not true

$$\text{Let } f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 1 - x, & 1 < x \leq 2 \end{cases}$$

 f is one-to-one, but not strictly monotonic.

103. True

$$107. \quad f(x) = \int_2^x \frac{dt}{\sqrt{1+t^4}}, \quad f(2) = 0$$

$$f'(x) = \frac{1}{\sqrt{1+x^4}}$$

$$(f^{-1})'(0) = \frac{1}{f'(2)} = \frac{1}{1/\sqrt{17}} = \sqrt{17}$$

Section 5.4 Exponential Functions: Differentiation and Integration

1. $e^0 = 1$

$\ln 1 = 0$

3. $\ln 2 = 0.6931$

$e^{0.6931 \dots} = 2$

5. $e^{\ln x} = x$

$x = 4$

7. $e^x = 12$

$x = \ln 12 \approx 2.485$

9. $9 - 2e^x = 7$

$2e^x = 7$

$e^x = 1$

$x = 0$

11. $50e^{-x} = 30$

$e^{-x} = \frac{3}{5}$

$-x = \ln\left(\frac{3}{5}\right)$

$x = \ln\left(\frac{5}{3}\right) \approx 0.511$

13. $\ln x = 2$

$x = e^2 \approx 7.3891$

15. $\ln(x - 3) = 2$

$x - 3 = e^2$

$x = 3 + e^2 \approx 10.389$

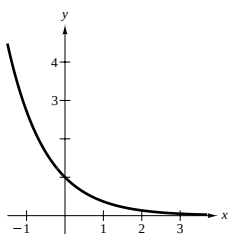
17. $\ln\sqrt{x+2} = 1$

$\sqrt{x+2} = e^1 = e$

$x + 2 = e^2$

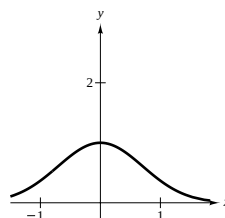
$x = e^2 - 2 \approx 5.389$

19. $y = e^{-x}$

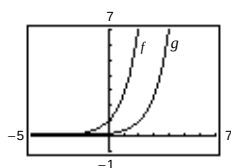


21. $y = e^{-x^2}$

Symmetric with respect to the y-axis

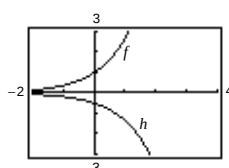
 Horizontal asymptote: $y = 0$


23. (a)



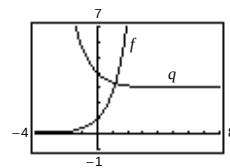
Horizontal shift 2 units to the right

(b)



A reflection in the x -axis and a vertical shrink

(c)



Vertical shift 3 units upward and a reflection in the y -axis

25. $y = Ce^{ax}$

Horizontal asymptote: $y = 0$

Matches (c)

27. $y = C(1 - e^{-ax})$

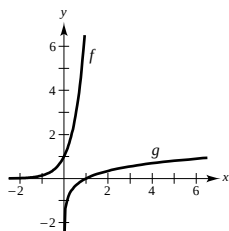
Vertical shift C units

Reflection in both the x - and y -axes

Matches (a)

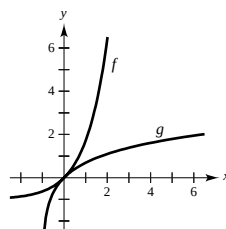
29. $f(x) = e^{2x}$

$$g(x) = \ln \sqrt{x} = \frac{1}{2} \ln x$$

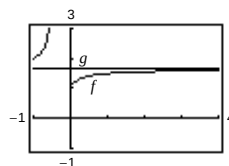


31. $f(x) = e^x - 1$

$$g(x) = \ln(x + 1)$$



33.



As $x \rightarrow \infty$, the graph of f approaches the graph of g .

$$\lim_{x \rightarrow \infty} \left(1 + \frac{0.5}{x}\right)^x = e^{0.5}$$

35. $\left(1 + \frac{1}{1,000,000}\right)^{1,000,000} \approx 2.718280469$

$$e \approx 2.718281828$$

$$e > \left(1 + \frac{1}{1,000,000}\right)^{1,000,000}$$

37. (a) $y = e^{3x}$

$$y' = 3e^{3x}$$

At $(0, 1)$, $y' = 3$.

(b) $y = e^{-3x}$

$$y' = -3e^{-3x}$$

At $(0, 1)$, $y' = -3$.

39. $f(x) = e^{2x}$

$$f'(x) = 2e^{2x}$$

41. $f(x) = e^{-2x+x^2}$

$$\frac{dy}{dx} = 2(x - 1)e^{-2x+x^2}$$

43. $y = e^{\sqrt{x}}$

$$\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

45. $g(t) = (e^{-t} + e^t)^3$

$$g'(t) = 3(e^{-t} + e^t)^2(e^t - e^{-t})$$

47. $y = \ln e^{x^2} = x^2$

$$\frac{dy}{dx} = 2x$$

49. $y = \ln(1 + e^{2x})$

$$\frac{dy}{dx} = \frac{2e^{2x}}{1 + e^{2x}}$$

$$51. \quad y = \frac{2}{e^x + e^{-x}} = 2(e^x + e^{-x})^{-1}$$

$$\begin{aligned} \frac{dy}{dx} &= -2(e^x + e^{-x})^{-2}(e^x - e^{-x}) \\ &= \frac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2} \end{aligned}$$

$$55. \quad f(x) = e^{-x} \ln x$$

$$f'(x) = e^{-x} \left(\frac{1}{x} \right) - e^{-x} \ln x = e^{-x} \left(\frac{1}{x} - \ln x \right)$$

$$59. \quad xe^y - 10x + 3y = 0$$

$$xe^y \frac{dy}{dx} + e^y - 10 + 3 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} (xe^y + 3) = 10 - e^y$$

$$\frac{dy}{dx} = \frac{10 - e^y}{xe^y + 3}$$

$$63. \quad y = e^x (\cos \sqrt{2}x + \sin \sqrt{2}x)$$

$$y' = e^x (-\sqrt{2} \sin \sqrt{2}x + \sqrt{2} \cos \sqrt{2}x) + e^x (\cos \sqrt{2}x + \sin \sqrt{2}x)$$

$$= e^x [(1 + \sqrt{2}) \cos \sqrt{2}x + (1 - \sqrt{2}) \sin \sqrt{2}x]$$

$$y'' = e^x [-(\sqrt{2} + 2) \sin \sqrt{2}x + (\sqrt{2} - 2) \cos \sqrt{2}x] + e^x [(1 + \sqrt{2}) \cos \sqrt{2}x + (1 - \sqrt{2}) \sin \sqrt{2}x]$$

$$= e^x [(-1 - 2\sqrt{2}) \sin \sqrt{2}x + (-1 + 2\sqrt{2}) \cos \sqrt{2}x]$$

$$-2y' + 3y = -2e^x [(1 + \sqrt{2}) \cos \sqrt{2}x + (1 - \sqrt{2}) \sin \sqrt{2}x] + 3e^x [\cos \sqrt{2}x + \sin \sqrt{2}x]$$

$$= e^x [(1 - 2\sqrt{2}) \cos \sqrt{2}x + (1 + 2\sqrt{2}) \sin \sqrt{2}x] = -y''$$

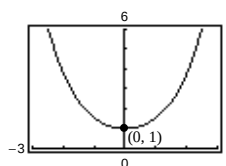
Therefore, $-2y' + 3y = -y'' \Rightarrow y'' - 2y' + 3y = 0$.

$$65. \quad f(x) = \frac{e^x + e^{-x}}{2}$$

$$f'(x) = \frac{e^x - e^{-x}}{2} = 0 \text{ when } x = 0.$$

$$f''(x) = \frac{e^x + e^{-x}}{2} > 0$$

Relative minimum: $(0, 1)$



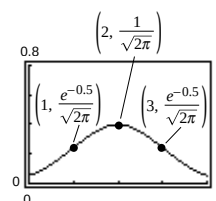
$$67. \quad g(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-2)^2/2}$$

$$g'(x) = \frac{-1}{\sqrt{2\pi}} (x - 2) e^{-(x-2)^2/2}$$

$$g''(x) = \frac{1}{\sqrt{2\pi}} (x - 1)(x - 3) e^{-(x-2)^2/2}$$

Relative maximum: $\left(2, \frac{1}{\sqrt{2\pi}}\right) \approx (2, 0.399)$

Points of inflection: $\left(1, \frac{1}{\sqrt{2\pi}} e^{-1/2}\right), \left(3, \frac{1}{\sqrt{2\pi}} e^{-1/2}\right) \approx (1, 0.242), (3, 0.242)$



$$53. \quad y = x^2 e^x - 2x e^x + 2e^x = e^x (x^2 - 2x + 2)$$

$$\frac{dy}{dx} = e^x (2x - 2) + e^x (x^2 - 2x + 2) = x^2 e^x$$

$$57. \quad y = e^x (\sin x + \cos x)$$

$$\frac{dy}{dx} = e^x (\cos x - \sin x) + (\sin x + \cos x) e^x$$

$$= e^x (2 \cos x) = 2e^x \cos x$$

$$61. \quad f(x) = (3 + 2x)e^{-3x}$$

$$f'(x) = (3 + 2x)(-3e^{-3x}) + 2e^{-3x}$$

$$= (-7 - 6x)e^{-3x}$$

$$f''(x) = (-7 - 6x)(-3e^{-3x}) - 6e^{-3x}$$

$$= 3(6x + 5)e^{-3x}$$

69. $f(x) = x^2 e^{-x}$

$$f'(x) = -x^2 e^{-x} + 2x e^{-x} = x e^{-x}(2 - x) = 0 \text{ when } x = 0, 2.$$

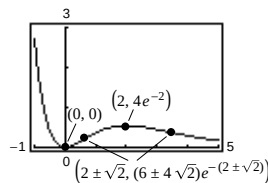
$$f''(x) = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x)$$

$$= e^{-x}(x^2 - 4x + 2) = 0 \text{ when } x = 2 \pm \sqrt{2}.$$

Relative minimum: $(0, 0)$ Relative maximum: $(2, 4e^{-2})$

$$x = 2 \pm \sqrt{2}$$

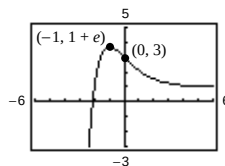
$$y = (2 \pm \sqrt{2})^2 e^{-(2 \pm \sqrt{2})}$$

Points of inflection: $(3.414, 0.384), (0.586, 0.191)$ 

71. $g(t) = 1 + (2 + t)e^{-t}$

$$g'(t) = (1 + t)e^{-t}$$

$$g''(t) = te^{-t}$$

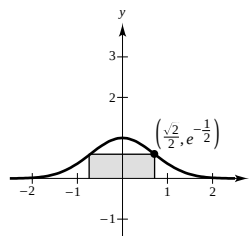
Relative maximum: $(-1, 1 + e) \approx (-1, 3.718)$ Point of inflection: $(0, 3)$ 

73. $A = (\text{base})(\text{height}) = 2xe^{-x^2}$

$$\frac{dA}{dx} = -4x^2 e^{-x^2} + 2e^{-x^2}$$

$$= 2e^{-x^2}(1 - 2x^2) = 0 \text{ when } x = \frac{\sqrt{2}}{2}.$$

$$A = \sqrt{2}e^{-1/2}$$



75. $y = \frac{L}{1 + ae^{-x/b}}, a > 0, b > 0, L > 0$

$$y' = \frac{-L\left(-\frac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^2} = \frac{\frac{aL}{b}e^{-x/b}}{(1 + ae^{-x/b})^2}$$

$$y'' = \frac{(1 + ae^{-x/b})^2 \left(\frac{-aL}{b^2}e^{-x/b}\right) - \left(\frac{aL}{b}e^{-x/b}\right) 2(1 + ae^{-x/b}) \left(\frac{-a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^4}$$

$$= \frac{(1 + ae^{-x/b}) \left(\frac{-aL}{b^2}e^{-x/b}\right) + 2\left(\frac{aL}{b}e^{-x/b}\right) \left(\frac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^3}$$

$$= \frac{La e^{-x/b} [ae^{-x/b} - 1]}{(1 + ae^{-x/b})^3 b^2}$$

$$y'' = 0 \text{ if } ae^{-x/b} = 1 \Rightarrow \frac{-x}{b} = \ln\left(\frac{1}{a}\right) \Rightarrow x = b \ln a$$

$$y(b \ln a) = \frac{L}{1 + ae^{-(b \ln a)/b}} = \frac{L}{1 + a(1/a)} = \frac{L}{2}$$

Therefore, the y-coordinate of the inflection point is $L/2$.

77. $e^{-x} = x \Rightarrow f(x) = x - e^{-x}$

$$f'(x) = 1 + e^{-x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}}$$

$$x_1 = 1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 0.5379$$

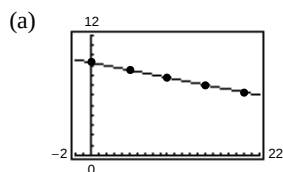
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 0.5670$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 0.5671$$

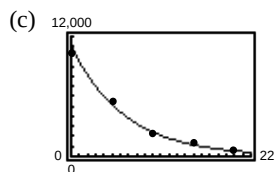
We approximate the root of f to be $x = 0.567$.

 81.

h	0	5	10	15	20
P	10,332	5,583	2,376	1,240	517
$\ln P$	9.243	8.627	7.773	7.123	6.248



$y = -0.1499h + 9.3018$ is the regression line for data $(h, \ln P)$.



83. $f(x) = e^{x/2}, f(0) = 1$

$$f'(x) = \frac{1}{2}e^{x/2}, f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{1}{4}e^{x/2}, f''(0) = \frac{1}{4}$$

$$P_1(x) = 1 + \frac{1}{2}(x - 0) = \frac{x}{2} + 1, P_1(0) = 1$$

$$P_1'(x) = \frac{1}{2}, P_1'(0) = \frac{1}{2}$$

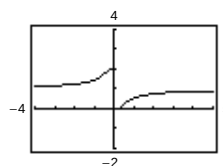
$$P_2(x) = 1 + \frac{1}{2}(x - 0) + \frac{1}{8}(x - 0)^2 = \frac{x^2}{8} + \frac{x}{2} + 1, P_2(0) = 1$$

$$P_2'(x) = \frac{1}{4}x + \frac{1}{2}, P_2'(0) = \frac{1}{2}$$

$$P_2''(x) = \frac{1}{4}, P_2''(0) = \frac{1}{4}$$

The values of f, P_1, P_2 and their first derivatives agree at $x = 0$. The values of the second derivatives of f and P_2 agree at $x = 0$.

79. (a)



(b) When x increases without bound, $1/x$ approaches zero, and $e^{1/x}$ approaches 1. Therefore, $f(x)$ approaches $2/(1 + 1) = 1$. Thus, $f(x)$ has a horizontal asymptote at $y = 1$. As x approaches zero from the right, $1/x$ approaches ∞ , $e^{1/x}$ approaches ∞ and $f(x)$ approaches zero. As x approaches zero from the left, $1/x$ approaches $-\infty$, $e^{1/x}$ approaches zero, and $f(x)$ approaches 2. The limit does not exist since the left limit does not equal the right limit. Therefore, $x = 0$ is a nonremovable discontinuity.

(b) $\ln P = ah + b$

$$P = e^{ah+b} = e^b e^{ah}$$

$$P = C e^{ah}, C = e^b$$

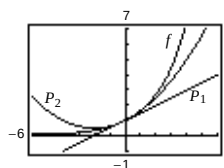
For our data, $a = -0.1499$ and $C = e^{9.3018} = 10,957.7$

$$P = 10,957.7 e^{-0.1499h}$$

(d) $\frac{dP}{dh} = (10,957.71)(-0.1499)e^{-0.1499h}$

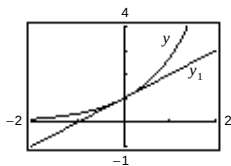
$$= -1642.56 e^{-0.1499h}$$

For $h = 5$, $\frac{dP}{dh} = -776.3$. For $h = 18$, $\frac{dP}{dh} \approx -110.6$.



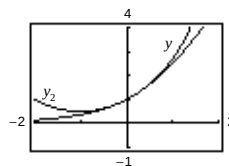
85. (a) $y = e^x$

$$y_1 = 1 + x$$



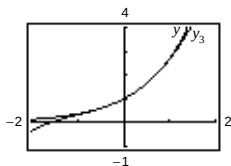
(b) $y = e^x$

$$y_2 = 1 + x + \left(\frac{x^2}{2}\right)$$



(c) $y = e^x$

$$y_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$



87. Let $u = 5x$, $du = 5 dx$.

$$\int e^{5x} 5dx = e^{5x} + C$$

89. Let $u = -2x$, $du = -2 dx$.

$$\begin{aligned} \int_0^1 e^{-2x} dx &= -\frac{1}{2} \int_0^1 e^{-2x} (-2) dx = \left[-\frac{1}{2} e^{-2x} \right]_0^1 \\ &= \frac{1}{2} (1 - e^{-2}) = \frac{e^2 - 1}{2e^2} \end{aligned}$$

91. $\int x e^{-x^2} dx = -\frac{1}{2} \int e^{-x^2} (-2x) dx = -\frac{1}{2} e^{-x^2} + C$

93. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int e^{\sqrt{x}} \left(\frac{1}{2\sqrt{x}} \right) dx = 2e^{\sqrt{x}} + C$

95. Let $u = 1 + e^{-x}$, $du = -e^{-x} dx$.

$$\int \frac{e^{-x}}{1 + e^{-x}} dx = -\int \frac{-e^{-x}}{1 + e^{-x}} dx = -\ln(1 + e^{-x}) + C = \ln\left(\frac{e^x}{e^x + 1}\right) + C = x - \ln(e^x + 1) + C$$

97. Let $u = \frac{3}{x}$, $du = -\frac{3}{x^2} dx$.

$$\begin{aligned} \int_1^3 \frac{e^{3/x}}{x^2} dx &= -\frac{1}{3} \int_1^3 e^{3/x} \left(-\frac{3}{x^2} \right) dx \\ &= \left[-\frac{1}{3} e^{3/x} \right]_1^3 = \frac{e}{3} (e^2 - 1) \end{aligned}$$

99. Let $u = 1 - e^x$, $du = -e^x dx$.

$$\begin{aligned} \int e^x \sqrt{1 - e^x} dx &= -\int (1 - e^x)^{1/2} (-e^x) dx \\ &= -\frac{2}{3} (1 - e^x)^{3/2} + C \end{aligned}$$

101. Let $u = e^x - e^{-x}$, $du = (e^x + e^{-x}) dx$.

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \ln|e^x - e^{-x}| + C$$

$$\begin{aligned} 103. \int \frac{5 - e^x}{e^{2x}} dx &= \int 5e^{-2x} dx - \int e^{-x} dx \\ &= -\frac{5}{2} e^{-2x} + e^{-x} + C \end{aligned}$$

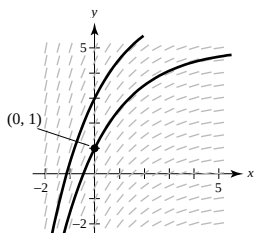
$$\begin{aligned} 105. \int e^{\sin \pi x} \cos \pi x dx &= \frac{1}{\pi} \int e^{\sin \pi x} (\pi \cos \pi x) dx \\ &= \frac{1}{\pi} e^{\sin \pi x} + C \end{aligned}$$

$$\begin{aligned} 107. \int e^{-x} \tan(e^{-x}) dx &= -\int [\tan(e^{-x})] (-e^{-x}) dx \\ &= \ln|\cos(e^{-x})| + C \end{aligned}$$

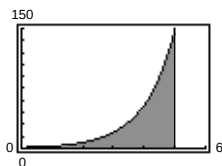
109. Let $u = ax^2$, $du = 2ax \, dx$. (Assume $a \neq 0$)

$$\begin{aligned} y &= \int x e^{ax^2} \, dx \\ &= \frac{1}{2a} \int e^{ax^2} (2ax) \, dx = \frac{1}{2a} e^{ax^2} + C \end{aligned}$$

113. (a)



115. $\int_0^5 e^x \, dx = \left[e^x \right]_0^5 = e^5 - 1 \approx 147.413$



119. (a) $f(u - v) = e^{u-v} = (e^u)(e^{-v}) = \frac{e^u}{e^v} = \frac{f(u)}{f(v)}$

(b) $f(kx) = e^{kx} = (e^x)^k = [f(x)]^k$.

123. $\int_0^x e^t \, dt \geq \int_0^x 1 \, dt$

$$\left[e^t \right]_0^x \geq \left[t \right]_0^x$$

$$e^x - 1 \geq x \Rightarrow e^x \geq 1 + x \text{ for } x \geq 0$$

127. Yes. $f(x) = Ce^x$, C a constant.

111. $f'(x) = \int \frac{1}{2}(e^x + e^{-x}) \, dx = \frac{1}{2}(e^x - e^{-x}) + C_1$

$$f'(0) = C_1 = 0$$

$$f(x) = \int \frac{1}{2}(e^x - e^{-x}) \, dx = \frac{1}{2}(e^x + e^{-x}) + C_2$$

$$f(0) = 1 + C_2 = 1 \Rightarrow C_2 = 0$$

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

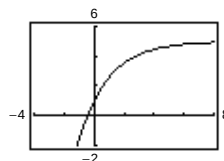
(b) $\frac{dy}{dx} = 2e^{-x/2}$, $(0, 1)$

$$y = \int 2e^{-x/2} \, dx = -4 \int e^{-x/2} \left(-\frac{1}{2} dx \right)$$

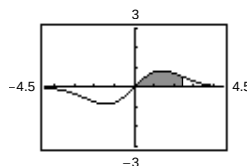
$$= -4e^{-x/2} + C$$

$$(0, 1): 1 = -4e^0 + C = -4 + C \Rightarrow C = 5$$

$$y = -4e^{-x/2} + 5$$



117. $\int_0^{\sqrt{6}} x e^{-x^2/4} \, dx = \left[-2e^{-x^2/4} \right]_0^{\sqrt{6}}$
 $= -2e^{-3/2} + 2 \approx 1.554$



121. $0.0665 \int_{48}^{60} e^{-0.0139(t-48)^2} \, dt$

Graphing Utility: $0.4772 = 47.72\%$

125. $f(x) = e^x$. Domain is $(-\infty, \infty)$ and range is $(0, \infty)$.
 f is continuous, increasing, one-to-one, and concave upwards on its entire domain.

$$\lim_{x \rightarrow -\infty} e^x = 0 \text{ and } \lim_{x \rightarrow \infty} e^x = \infty.$$

129. $e^{-x} > 0 \Rightarrow \int_0^2 e^{-x} \, dx > 0$.

131. $f(x) = \frac{\ln x}{x}$

(a) $f'(x) = \frac{1 - \ln x}{x^2} = 0$ when $x = e$.

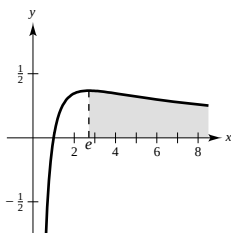
On $(0, e)$, $f'(x) > 0 \Rightarrow f$ is increasing.On (e, ∞) , $f'(x) < 0 \Rightarrow f$ is decreasing.(b) For $e \leq A < B$, we have:

$$\frac{\ln A}{A} > \frac{\ln B}{B}$$

$$B \ln A > A \ln B$$

$$\ln A^B > \ln B^A$$

$$A^B > B^A.$$

(c) Since $e < \pi$, from part (b) we have $e^\pi > \pi^e$.Section 5.5 Bases Other than e and Applications

1. $y = \left(\frac{1}{2}\right)^{t/3}$

At $t_0 = 6$, $y = \left(\frac{1}{2}\right)^{6/3} = \frac{1}{4}$

3. $y = \left(\frac{1}{2}\right)^{t/7}$

At $t_0 = 10$, $y = \left(\frac{1}{2}\right)^{10/7} \approx 0.3715$

5. $\log_2 \frac{1}{8} = \log_2 2^{-3} = -3$

7. $\log_7 1 = 0$

9. (a) $2^3 = 8$

$\log_2 8 = 3$

(b) $3^{-1} = \frac{1}{3}$

$\log_3 \frac{1}{3} = -1$

11. (a) $\log_{10} 0.01 = -2$

$10^{-2} = 0.01$

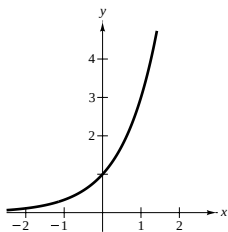
(b) $\log_{0.5} 8 = -3$

$0.5^{-3} = 8$

$\left(\frac{1}{2}\right)^{-3} = 8$

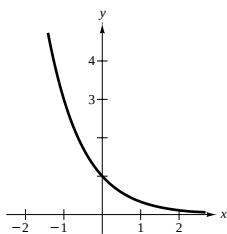
13. $y = 3^x$

x	-2	-1	0	1	2
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9



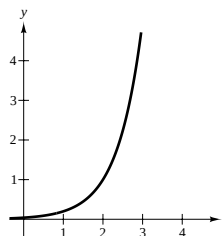
15. $y = \left(\frac{1}{3}\right)^x = 3^{-x}$

x	-2	-1	0	1	2
y	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$



17. $h(x) = 5^{x-2}$

x	-1	0	1	2	3
y	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5



19. (a) $\log_{10} 1000 = x$

$$10^x = 1000$$

$$x = 3$$

(b) $\log_{10} 0.1 = x$

$$10^x = 0.1$$

$$x = -1$$

23. (a) $x^2 - x = \log_5 25$

$$x^2 - x = \log_5 5^2 = 2$$

$$x^2 - x - 2 = 0$$

$$(x + 1)(x - 2) = 0$$

$$x = -1 \text{ OR } x = 2$$

25. $3^{2x} = 75$

$$2x \ln 3 = \ln 75$$

$$x = \frac{1}{2} \frac{\ln 75}{\ln 3} \approx 1.965$$

29. $\left(1 + \frac{0.09}{12}\right)^{12t} = 3$

$$12t \ln\left(1 + \frac{0.09}{12}\right) = \ln 3$$

$$t = \frac{1}{12} \frac{\ln 3}{\ln\left(1 + \frac{0.09}{12}\right)} \approx 12.253$$

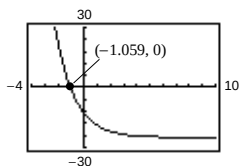
33. $\log_3 x^2 = 4.5$

$$x^2 = 3^{4.5}$$

$$x = \pm \sqrt{3^{4.5}} \approx \pm 11.845$$

35. $g(x) = 6(2^{1-x}) - 25$

$$\text{Zero: } x \approx -1.059$$



21. (a) $\log_3 x = -1$

$$3^{-1} = x$$

$$x = \frac{1}{3}$$

(b) $\log_2 x = -4$

$$2^{-4} = x$$

$$x = \frac{1}{16}$$

(b) $3x + 5 = \log_2 64$

$$3x + 5 = \log_2 2^6 = 6$$

$$3x = 1$$

$$x = \frac{1}{3}$$

27. $2^{3-x} = 625$

$$(3 - x) \ln 2 = \ln 625$$

$$3 - x = \frac{\ln 625}{\ln 2}$$

$$x = 3 - \frac{\ln 625}{\ln 2} \approx -6.288$$

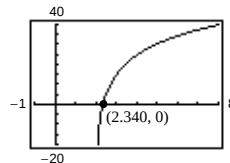
31. $\log_2(x - 1) = 5$

$$x - 1 = 2^5 = 32$$

$$x = 33$$

37. $h(s) = 32 \log_{10}(s - 2) + 15$

$$\text{Zero: } s \approx 2.340$$

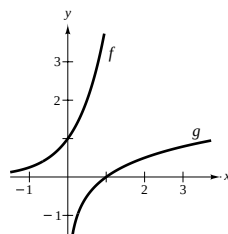


39. $f(x) = 4^x$

$g(x) = \log_4 x$

x	-2	-1	0	$\frac{1}{2}$	1
$f(x)$	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4

x	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4
$g(x)$	-2	-1	0	$\frac{1}{2}$	1



41. $f(x) = 4^x$

$f'(x) = (\ln 4) 4^x$

43. $y = 5^{x-2}$

$\frac{dy}{dx} = (\ln 5) 5^{x-2}$

45. $g(t) = t^2 2^t$

$$\begin{aligned} g'(t) &= t^2 (\ln 2) 2^t + (2t) 2^t \\ &= t 2^t (t \ln 2 + 2) \\ &= 2^t t(2 + t \ln 2) \end{aligned}$$

47. $h(\theta) = 2^{-\theta} \cos \pi \theta$

$$\begin{aligned} h'(\theta) &= 2^{-\theta}(-\pi \sin \pi \theta) - (\ln 2) 2^{-\theta} \cos \pi \theta \\ &= -2^{-\theta}[(\ln 2) \cos \pi \theta + \pi \sin \pi \theta] \end{aligned}$$

49. $y = \log_3 x$

$\frac{dy}{dx} = \frac{1}{x \ln 3}$

51. $f(x) = \log_2 \frac{x^2}{x-1}$

$= 2 \log_2 x - \log_2 (x-1)$

$f'(x) = \frac{2}{x \ln 2} - \frac{1}{(x-1) \ln 2}$

$= \frac{x-2}{(\ln 2)x(x-1)}$

53. $y = \log_5 \sqrt{x^2 - 1} = \frac{1}{2} \log_5 (x^2 - 1)$

$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{2x}{(x^2 - 1) \ln 5} = \frac{x}{(x^2 - 1) \ln 5}$

55. $g(t) = \frac{10 \log_4 t}{t} = \frac{10}{\ln 4} \left(\frac{\ln t}{t} \right)$

$g'(t) = \frac{10}{\ln 4} \left[\frac{t(1/t) - \ln t}{t^2} \right]$

$= \frac{10}{t^2 \ln 4} [1 - \ln t] = \frac{5}{t^2 \ln 2} (1 - \ln t)$

57. $y = x^{2/x}$

$\ln y = \frac{2}{x} \ln x$

$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{2}{x} \left(\frac{1}{x} \right) + \ln x \left(-\frac{2}{x^2} \right) = \frac{2}{x^2} (1 - \ln x)$

$\frac{dy}{dx} = \frac{2y}{x^2} (1 - \ln x) = 2x^{(2/x)-2} (1 - \ln x)$

59. $y = (x-2)^{x+1}$

$\ln y = (x+1) \ln(x-2)$

$\frac{1}{y} \left(\frac{dy}{dx} \right) = (x+1) \left(\frac{1}{x-2} \right) + \ln(x-2)$

$\frac{dy}{dx} = y \left[\frac{x+1}{x-2} + \ln(x-2) \right]$

$= (x-2)^{x+1} \left[\frac{x+1}{x-2} + \ln(x-2) \right]$

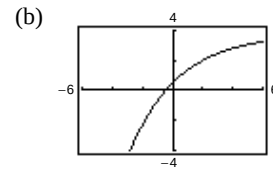
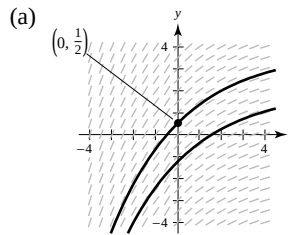
61. $\int 3^x dx = \frac{3^x}{\ln 3} + C$

$$\begin{aligned}
 63. \int_{-1}^2 2^x dx &= \left[\frac{2^x}{\ln 2} \right]_{-1}^2 \\
 &= \frac{1}{\ln 2} \left[4 - \frac{1}{2} \right] \\
 &= \frac{7}{2 \ln 2} = \frac{7}{\ln 4}
 \end{aligned}$$

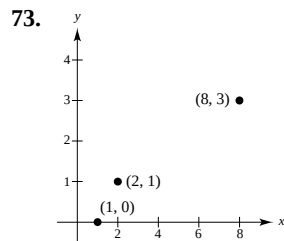
$$\begin{aligned}
 65. \int x 5^{-x^2} dx &= -\frac{1}{2} \int 5^{-x^2} (-2x) dx \\
 &= -\left(\frac{1}{2} \right) \frac{5^{-x^2}}{\ln 5} + C \\
 &= -\frac{1}{2 \ln 5} (5^{-x^2}) + C
 \end{aligned}$$

$$\begin{aligned}
 67. \int \frac{3^{2x}}{1 + 3^{2x}} dx, u = 1 + 3^{2x}, du &= 2(\ln 3) 3^{2x} dx \\
 \frac{1}{2 \ln 3} \int \frac{(2 \ln 3) 3^{2x}}{1 + 3^{2x}} dx &= \frac{1}{2 \ln 3} \ln(1 + 3^{2x}) + C
 \end{aligned}$$

$$\begin{aligned}
 69. \frac{dy}{dx} &= 0.4^{x/3}, \left(0, \frac{1}{2} \right) \\
 y &= \int 0.4^{x/3} dx = 3 \int 0.4^{x/3} \left(\frac{1}{3} dx \right) \\
 &= \frac{3}{\ln 0.4} 0.4^{x/3} + C = 3(\ln 2.5)(0.4)^{x/3} + C \\
 y &= 3 \ln 2.5 (0.4)^{x/3} + \frac{1}{2} - 3 \ln 2.5 \\
 &= \frac{3(1 - 0.4^{x/3})}{\ln 2.5} + \frac{1}{2}
 \end{aligned}$$



71. Answers will vary. Example: Growth and decay problems.



x	1	2	8
y	0	1	3

- (a) y is an exponential function of x : False
 (b) y is a logarithmic function of x : True; $y = \log_2 x$
 (c) x is an exponential function of y : True, $2^y = x$
 (d) y is a linear function of x : False

$$\begin{aligned}
 75. f(x) &= \log_2 x \Rightarrow f'(x) = \frac{1}{x \ln 2} \\
 g(x) &= x^x \Rightarrow g'(x) = x^x(1 + \ln x)
 \end{aligned}$$

[Note: Let $y = g(x)$. Then: $\ln y = \ln x^x = x \ln x$

$$\begin{aligned}
 \frac{1}{y} y' &= x \cdot \frac{1}{x} + \ln x \\
 y' &= y(1 + \ln x) \\
 y' &= x^x(1 + \ln x) = g'(x).]
 \end{aligned}$$

$$h(x) = x^2 \Rightarrow h'(x) = 2x$$

$$k(x) = 2^x \Rightarrow k'(x) = (\ln 2) 2^x$$

From greatest to smallest rate of growth:

$$g(x), k(x), h(x), f(x)$$

$$77. C(t) = P(1.05)^t$$

$$\begin{aligned}
 (a) C(10) &= 24.95(1.05)^{10} \\
 &\approx \$40.64
 \end{aligned}$$

$$(b) \frac{dC}{dt} = P(\ln 1.05)(1.05)^t$$

$$\text{When } t = 1: \frac{dC}{dt} \approx 0.051P$$

$$\text{When } t = 8: \frac{dC}{dt} \approx 0.072P$$

$$\begin{aligned}
 (c) \frac{dC}{dt} &= (\ln 1.05)[P(1.05)^t] \\
 &= (\ln 1.05)C(t)
 \end{aligned}$$

The constant of proportionality is $\ln 1.05$.

79. $P = \$1000$, $r = 3\frac{1}{2}\% = 0.035$, $t = 10$

$$A = 1000\left(1 + \frac{0.035}{n}\right)^{10n}$$

$$A = 1000e^{(0.035)(10)} = 1419.07$$

n	1	2	4	12	365	Continuous
A	1410.60	1414.78	1416.91	1418.34	1419.04	1419.07

81. $P = \$1000$, $r = 5\% = 0.05$, $t = 30$

$$A = 1000\left(1 + \frac{0.05}{n}\right)^{30n}$$

$$A = 1000e^{(0.05)(30)} = 4481.69$$

n	1	2	4	12	365	Continuous
A	4321.94	4399.79	4440.21	4467.74	4481.23	4481.69

83. $100,000 = Pe^{0.05t} \Rightarrow P = 100,000e^{-0.05t}$

t	1	10	20	30	40	50
P	95,122.94	60,653.07	36,787.94	22,313.02	13,583.53	8208.50

85. $100,000 = P\left(1 + \frac{0.05}{12}\right)^{12t} \Rightarrow P = 100,000\left(1 + \frac{0.05}{12}\right)^{-12t}$

t	1	10	20	30	40	50
P	95,132.82	60,716.10	36,864.45	22,382.66	13,589.88	8251.24

87. (a) $A = 20,000\left(1 + \frac{0.06}{365}\right)^{(365)(8)} \approx \$32,320.21$

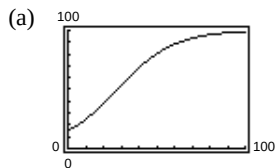
(b) $A = \$30,000$

(c) $A = 8000\left(1 + \frac{0.06}{365}\right)^{(365)(8)} + 20,000\left(1 + \frac{0.06}{365}\right)^{(365)(4)}$
 $\approx \$12,928.09 + 25,424.48 = \$38,352.57$

(d) $A = 9000\left[\left(1 + \frac{0.06}{365}\right)^{(365)(8)} + \left(1 + \frac{0.06}{365}\right)^{(365)(4)} + 1\right]$
 $\approx \$34,985.11$

Take option (c).

91. $y = \frac{300}{3 + 17e^{-0.0625x}}$



(b) If $x = 2$ (2000 egg masses), $y \approx 16.67 \approx 16.7\%$.

89. (a) $\lim_{t \rightarrow \infty} 6.7e^{(-48.1)/t} = 6.7e^0 = 6.7$ million ft^3

(b) $V' = \frac{322.27}{t^2} e^{-(48.1)/t}$

$V'(20) \approx 0.073$ million ft^3/yr

$V'(60) \approx 0.040$ million ft^3/yr

(c) If $y = 66.67\%$, then $x \approx 38.8$ or 38,800 egg masses.

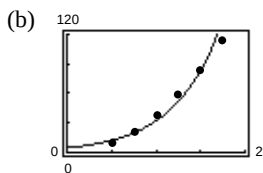
(d) $y = 300(3 + 17e^{-0.0625x})^{-1}$

$$y' = \frac{318.75e^{-0.0625x}}{(3 + 17e^{-0.0625x})^2}$$

$$y'' = \frac{19.921875e^{-0.0625x}(17e^{-0.0625x} - 3)}{(3 + 17e^{-0.0625x})^3}$$

$$17e^{-0.0625x} - 3 = 0 \Rightarrow x \approx 27.8 \text{ or } 27,800 \text{ egg masses.}$$

93. (a) $B = 4.7539(6.7744)^d = 4.7539e^{1.9132d}$



(c) $B'(d) = 9.0952e^{1.9132d}$

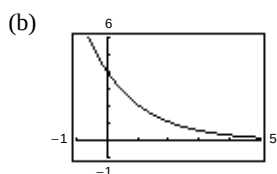
$B'(0.8) \approx 42.03 \text{ tons/inch}$

$B'(1.5) \approx 160.38 \text{ tons/inch}$

95. (a) $\int_0^4 f(t) dt \approx 5.67$

$\int_0^4 g(t) dt \approx 5.67$

$\int_0^4 h(t) dt \approx 5.67$



97.
$$P = \int_0^{10} 2000e^{-0.06t} dt$$

$$= \left[\frac{2000}{-0.06} e^{-0.06t} \right]_0^{10}$$

$$\approx \$15,039.61$$

 (c) The functions appear to be equal: $f(t) = g(t) = h(t)$

Analytically,

$$f(t) = 4\left(\frac{3}{8}\right)^{2t/3} = 4\left[\left(\frac{3}{8}\right)^{2/3}\right]^t = 4\left(\frac{9^{1/3}}{4}\right)^t = g(t)$$

$$h(t) = 4e^{-0.653886t} = 4[e^{-0.653886}]^t = 4(0.52002)^t$$

$$g(t) = 4\left(\frac{9^{1/3}}{4}\right)^t = 4(0.52002)^t$$

No. The definite integrals over a given interval may be equal when the functions are not equal.

99.

t	0	1	2	3	4
y	1200	720	432	259.20	155.52

$y = C(k^t)$

When $t = 0, y = 1200 \Rightarrow C = 1200$.

$y = 1200(k^t)$

$$\frac{720}{1200} = 0.6, \frac{432}{720} = 0.6, \frac{259.20}{432} = 0.6, \frac{155.52}{259.20} = 0.6$$

 Let $k = 0.6$.

$y = 1200(0.6)^t$

 101. False. e is an irrational number.

103. True.

$$f(g(x)) = 2 + e^{\ln(x-2)}$$

$$= 2 + x - 2 = x$$

$$g(f(x)) = \ln(2 + e^x - 2)$$

$$= \ln e^x = x$$

105. True.

$$\frac{d}{dx}[e^x] = e^x \text{ and } \frac{d}{dx}[e^{-x}] = -e^{-x}$$

$$e^x = e^{-x} \text{ when } x = 0.$$

$$(e^0)(-e^{-0}) = -1$$

$$\begin{aligned}
 107. \quad \frac{dy}{dt} &= \frac{8}{25}y\left(\frac{5}{4} - y\right), y(0) = 1 \\
 \frac{dy}{y[(5/4) - y]} &= \frac{8}{25}dt \Rightarrow \frac{4}{5} \int \left(\frac{1}{y} + \frac{1}{(5/4) - y}\right) dy = \int \frac{8}{25} dt \Rightarrow \\
 \ln y - \ln\left(\frac{5}{4} - y\right) &= \frac{2}{5}t + C \\
 \ln\left(\frac{y}{(5/4) - y}\right) &= \frac{2}{5}t + C \\
 \frac{y}{(5/4) - y} &= e^{(2/5)t + C} = C_1 e^{(2/5)t} \\
 y(0) = 1 &\Rightarrow C_1 = 4 \Rightarrow 4e^{(2/5)t} = \frac{y}{(5/4) - y} \\
 \Rightarrow 4e^{(2/5)t}\left(\frac{5}{4} - y\right) &= y \Rightarrow 5e^{(2/5)t} = 4e^{(2/5)t}y + y = (4e^{(2/5)t} + 1)y \\
 \Rightarrow y &= \frac{5e^{(2/5)t}}{4e^{(2/5)t} + 1} = \frac{5}{4 + e^{-0.4t}} = \frac{1.25}{1 + 0.25e^{-0.4t}}
 \end{aligned}$$

Section 5.6 Differential Equations: Growth and Decay

$$1. \quad \frac{dy}{dx} = x + 2$$

$$y = \int (x + 2)dx = \frac{x^2}{2} + 2x + C$$

$$3. \quad \frac{dy}{dx} = y + 2$$

$$\frac{dy}{y + 2} = dx$$

$$\int \frac{1}{y + 2} dy = \int dx$$

$$\ln|y + 2| = x + C_1$$

$$y + 2 = e^{x + C_1} = Ce^x$$

$$y = Ce^x - 2$$

$$5. \quad y' = \frac{5x}{y}$$

$$yy' = 5x$$

$$\int yy' dx = \int 5x dx$$

$$\int y dy = \int 5x dx$$

$$\frac{1}{2}y^2 = \frac{5}{2}x^2 + C_1$$

$$y^2 - 5x^2 = C$$

$$7. \quad y' = \sqrt{x}y$$

$$\frac{y'}{y} = \sqrt{x}$$

$$\int \frac{y'}{y} dx = \int \sqrt{x} dx$$

$$\int \frac{dy}{y} = \int \sqrt{x} dx$$

$$\ln y = \frac{2}{3}x^{3/2} + C_1$$

$$y = e^{(2/3)x^{3/2} + C_1}$$

$$= e^{C_1} e^{(2/3)x^{3/2}}$$

$$= Ce^{(2/3)x^{3/2}}$$

9. $(1 + x^2)y' - 2xy = 0$

$$y' = \frac{2xy}{1 + x^2}$$

$$\frac{y'}{y} = \frac{2x}{1 + x^2}$$

$$\int \frac{y'}{y} dx = \int \frac{2x}{1 + x^2} dx$$

$$\int \frac{dy}{y} = \int \frac{2x}{1 + x^2} dx$$

$$\ln y = \ln(1 + x^2) + C_1$$

$$\ln y = \ln(1 + x^2) + \ln C$$

$$\ln y = \ln C(1 + x^2)$$

$$y = C(1 + x^2)$$

11. $\frac{dQ}{dt} = \frac{k}{t^2}$

$$\int \frac{dQ}{dt} dt = \int \frac{k}{t^2} dt$$

$$\int dQ = -\frac{k}{t} + C$$

$$Q = -\frac{k}{t} + C$$

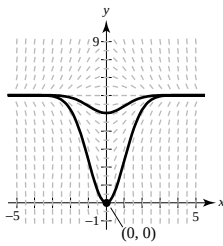
13. $\frac{dN}{ds} = k(250 - s)$

$$\int \frac{dN}{ds} ds = \int k(250 - s) ds$$

$$\int dN = -\frac{k}{2}(250 - s)^2 + C$$

$$N = -\frac{k}{2}(250 - s)^2 + C$$

15. (a)



(b) $\frac{dy}{dx} = x(6 - y), (0, 0)$

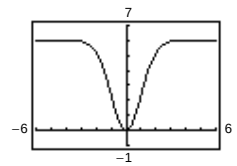
$$\frac{dy}{y - 6} = -x$$

$$\ln|y - 6| = \frac{-x^2}{2} + C$$

$$y - 6 = e^{-x^2/2 + C} = C_1 e^{-x^2/2}$$

$$y = 6 + C_1 e^{-x^2/2}$$

$$(0, 0): 0 = 6 + C_1 \Rightarrow C_1 = -6 \Rightarrow y = 6 - 6e^{-x^2/2}$$



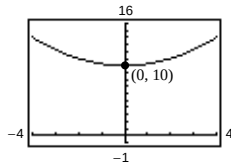
17. $\frac{dy}{dt} = \frac{1}{2}t, (0, 10)$

$$\int dy = \int \frac{1}{2}t dt$$

$$y = \frac{1}{4}t^2 + C$$

$$10 = \frac{1}{4}(0)^2 + C \Rightarrow C = 10$$

$$y = \frac{1}{4}t^2 + 10$$



19. $\frac{dy}{dt} = -\frac{1}{2}y, (0, 10)$

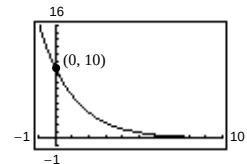
$$\int \frac{dy}{y} = \int -\frac{1}{2} dt$$

$$\ln y = -\frac{1}{2}t + C_1$$

$$y = e^{-(t/2) + C_1} = e^{C_1} e^{-t/2} = C e^{-t/2}$$

$$10 = C e^0 \Rightarrow C = 10$$

$$y = 10e^{-t/2}$$



21. $\frac{dy}{dx} = ky$

$$y = Ce^{kx} \text{ (Theorem 5.16)}$$

$$(0, 4): 4 = Ce^0 = C$$

$$(3, 10): 10 = 4e^{3k} \Rightarrow k = \frac{1}{3} \ln\left(\frac{5}{2}\right)$$

$$\text{When } x = 6, y = 4e^{1/3 \ln(5/2)(6)} = 4e^{\ln(5/2)^2}$$

$$= 4\left(\frac{5}{2}\right)^2 = 25$$

23. $\frac{dV}{dt} = kV$

$$V = Ce^{kt} \text{ (Theorem 5.16)}$$

$$(0, 20,000): C = 20,000$$

$$(4, 12,500): 12,500 = 20,000e^{4k} \Rightarrow k = \frac{1}{4} \ln\left(\frac{5}{8}\right)$$

$$\text{When } t = 6, V = 20,000e^{1/4 \ln(5/8)(6)} = 20,000e^{\ln(5/8)^{3/2}}$$

$$= 20,000\left(\frac{5}{8}\right)^{3/2} \approx 9882.118$$

25. $y = Ce^{kt}, \left(0, \frac{1}{2}\right), (5, 5)$

$$C = \frac{1}{2}$$

$$y = \frac{1}{2}e^{kt}$$

$$5 = \frac{1}{2}e^{5k}$$

$$k = \frac{\ln 10}{5} \approx 0.4605$$

$$y = \frac{1}{2}e^{0.4605t}$$

27. $y = Ce^{kt}, (1, 1), (5, 5)$

$$1 = Ce^k$$

$$5 = Ce^{5k}$$

$$5Ce^k = Ce^{5k}$$

$$5e^k = e^{5k}$$

$$5 = e^{4k}$$

$$k = \frac{\ln 5}{4} \approx 0.4024$$

$$y = Ce^{0.4024t}$$

$$1 = Ce^{0.4024}$$

$$C \approx 0.6687$$

$$y = 0.6687e^{0.4024t}$$

29. A differential equation in x and y is an equation that involves x , y and derivatives of y .

31. $\frac{dy}{dx} = \frac{1}{2}xy$

$$\frac{dy}{dx} > 0 \text{ when } xy > 0. \text{ Quadrants I and III.}$$

33. Since the initial quantity is 10 grams, $y = 10e^{[\ln(1/2)/1620]t}$. When $t = 1000$, $y = 10e^{[\ln(1/2)/1620](1000)} \approx 6.52$ grams. When $t = 10,000$, $y = 10e^{[\ln(1/2)/1620](10,000)} \approx 0.14$ gram.

35. Since $y = Ce^{[\ln(1/2)/1620]t}$, we have $0.5 = Ce^{[\ln(1/2)/1620](10,000)} \Rightarrow C \approx 36.07$.

Initial quantity: 36.07 grams.

When $t = 1000$, we have $y = Ce^{[\ln(1/2)/1620](1000)} \approx 23.51$ grams.

37. Since the initial quantity is 5 grams, we have $y = 5.0e^{[\ln(1/2)/5730]t}$.

When $t = 1000$, $y \approx 4.43$ g.

When $t = 10,000$, $y \approx 1.49$ g.

39. Since $y = Ce^{[\ln(1/2)/24,360]t}$, we have $2.1 = Ce^{[\ln(1/2)/24,360](1000)} \Rightarrow C \approx 2.16$. Thus, the initial quantity is 2.16 grams. When $t = 10,000$, $y = 2.16e^{[\ln(1/2)/24,360](10,000)} \approx 1.63$ grams.

41. Since $\frac{dy}{dx} = ky$, $y = Ce^{kt}$ or $y = y_0e^{kt}$.

$$\frac{1}{2}y_0 = y_0e^{1620k}$$

$$k = \frac{-\ln 2}{1620}$$

$$y = y_0e^{-(\ln 2)t/1620}$$

When $t = 100$, $y = y_0e^{-(\ln 2)/16.2} \approx y_0(0.9581)$.

Therefore, 95.81% of the present amount still exists.

43. Since $A = 1000e^{0.06t}$, the time to double is given by $2000 = 1000e^{0.06t}$ and we have

$$2 = e^{0.06t}$$

$$\ln 2 = 0.06t$$

$$t = \frac{\ln 2}{0.06} \approx 11.55 \text{ years.}$$

Amount after 10 years: $A = 1000e^{(0.06)(10)} \approx \$1822.12$

45. Since $A = 750e^{rt}$ and $A = 1500$ when $t = 7.75$, we have the following.

$$1500 = 750e^{7.75r}$$

$$r = \frac{\ln 2}{7.75} \approx 0.0894 = 8.94\%$$

Amount after 10 years: $A = 750e^{0.0894(10)} \approx \$1833.67$

47. Since $A = 500e^{rt}$ and $A = 1292.85$ when $t = 10$, we have the following.

$$1292.85 = 500e^{10r}$$

$$r = \frac{\ln(1292.85/500)}{10} \approx 0.0950 = 9.50\%$$

The time to double is given by

$$1000 = 500e^{0.0950t}$$

$$t = \frac{\ln 2}{0.095} \approx 7.30 \text{ years.}$$

49. $500,000 = P\left(1 + \frac{0.075}{12}\right)^{(12)(20)}$

$$P = 500,000\left(1 + \frac{0.075}{12}\right)^{-240}$$

$$\approx \$112,087.09$$

51. $500,000 = P\left(1 + \frac{0.08}{12}\right)^{(12)(35)}$

$$P = 500,000\left(1 + \frac{0.08}{12}\right)^{-420}$$

$$= \$30,688.87$$

53. (a) $2000 = 1000(1 + 0.07)^t$

$$2 = 1.07^t$$

$$\ln 2 = t \ln 1.07$$

$$t = \frac{\ln 2}{\ln 1.07} \approx 10.24 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.07}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.07}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.07}{12}\right)$$

$$t = \frac{\ln 2}{12 \ln(1 + (0.07/12))} \approx 9.93 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.07}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.07}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.07}{365}\right)$$

$$t = \frac{\ln 2}{365 \ln(1 + (0.07/365))} \approx 9.90 \text{ years}$$

(d) $2000 = 1000e^{(0.07)t}$

$$2 = e^{0.07t}$$

$$\ln 2 = 0.07t$$

$$t = \frac{\ln 2}{0.07} \approx 9.90 \text{ years}$$

55. (a) $2000 = 1000(1 + 0.085)^t$

$$2 = 1.085^t$$

$$\ln 2 = t \ln 1.085$$

$$t = \frac{\ln 2}{\ln 1.085} \approx 8.50 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.085}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.085}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.085}{12}\right)$$

$$t = \frac{1}{12} \frac{\ln 2}{\ln\left(1 + \frac{0.085}{12}\right)} \approx 8.18 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.085}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.085}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.085}{365}\right)$$

$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \frac{0.085}{365}\right)} \approx 8.16 \text{ years}$$

(d) $2000 = 1000e^{0.085t}$

$$2 = e^{0.085t}$$

$$\ln 2 = 0.085t$$

$$t = \frac{\ln 2}{0.085} \approx 8.15 \text{ years}$$

57. $P = Ce^{kt} = Ce^{-0.009t}$

$$P(-1) = 8.2 = Ce^{-0.009(-1)} \Rightarrow C = 8.1265$$

$$P = 8.1265e^{-0.009t}$$

$$P(10) \approx 7.43 \quad \text{or} \quad 7,430,000 \text{ people in 2010}$$

61. If $k < 0$, the population decreases.If $k > 0$, the population increases.

59. $P = Ce^{kt} = Ce^{0.036t}$

$$P(-1) = 4.6 = Ce^{0.036(-1)} \Rightarrow C = 4.7686$$

$$P = 4.7686e^{0.036t}$$

$$P(10) \approx 6.83 \quad \text{or} \quad 6,830,000 \text{ people in 2010}$$

63. $P = Ce^{kx}, (0, 760), (1000, 672.71)$

$$C = 760$$

$$672.71 = 760e^{1000k}$$

$$x = \frac{\ln(672.71/760)}{1000} \approx -0.000122$$

$$P \approx 760e^{-0.000122x}$$

When $x = 3000$, $P \approx 527.06$ mm Hg.

65. (a) $19 = 30(1 - e^{20k})$

$$30e^{20k} = 11$$

$$k = \frac{\ln(11/30)}{20} \approx -0.0502$$

$$N \approx 30(1 - e^{-0.0502t})$$

(b) $25 = 30(1 - e^{-0.0502t})$

$$e^{-0.0502t} = \frac{1}{6}$$

$$t = \frac{-\ln 6}{-0.0502} \approx 36 \text{ days}$$

67. $S = Ce^{k/t}$

(a) $S = 5$ when $t = 1$

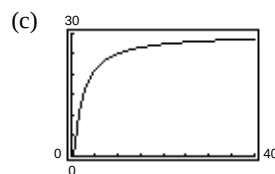
$$5 = Ce^k$$

$$\lim_{t \rightarrow \infty} Ce^{k/t} = C = 30$$

$$5 = 30e^k$$

$$k = \ln \frac{1}{6} \approx -1.7918$$

$$S \approx 30e^{-1.7918/t}$$

(b) When $t = 5$, $S \approx 20.9646$ which is 20,965 units.

69. $A(t) = V(t)e^{-0.10t} = 100,000e^{0.8\sqrt{t}}e^{-0.10t} = 100,000e^{0.8\sqrt{t}-0.10t}$

$$\frac{dA}{dt} = 100,000 \left(\frac{0.4}{\sqrt{t}} - 0.10 \right) e^{0.8\sqrt{t}-0.10t} = 0 \text{ when } 16.$$

The timber should be harvested in the year 2014, $(1998 + 16)$. **Note:** You could also use a graphing utility to graph $A(t)$ and find the maximum of $A(t)$. Use the viewing rectangle $0 \leq x \leq 30$ and $0 \leq y \leq 600,000$.

71. $\beta(I) = 10 \log_{10} \frac{I}{I_0}, I_0 = 10^{-16}$

(a) $\beta(10^{-14}) = 10 \log_{10} \frac{10^{-14}}{10^{-16}} = 20 \text{ decibels}$

(b) $\beta(10^{-9}) = 10 \log_{10} \frac{10^{-9}}{10^{-16}} = 70 \text{ decibels}$

(c) $\beta(10^{-6.5}) = 10 \log_{10} \frac{10^{-6.5}}{10^{-16}} = 95 \text{ decibels}$

(d) $\beta(10^{-4}) = 10 \log_{10} \frac{10^{-4}}{10^{-16}} = 120 \text{ decibels}$

73. $R = \frac{\ln I - 0}{\ln 10}, I = e^{R \ln 10} = 10^R$

(a) $8.3 = \frac{\ln I - 0}{\ln 10}$

$$I = 10^{8.3} \approx 199,526,231.5$$

(b) $2R = \frac{\ln I - 0}{\ln 10}$

$$I = e^{2R \ln 10} = e^{2 \ln 10} = (e^{\ln 10})^2 = (10^R)^2$$

Increases by a factor of $e^{2R \ln 10}$ or 10^R .

(c) $\frac{dR}{dI} = \frac{1}{I \ln 10}$

75. False. If $y = Ce^{kt}$, $y' = Cke^{kt} \neq \text{constant}$.

77. True

Section 5.7 Differential Equations: Separation of Variables

1. Differential equation: $y' = 4y$

Solution: $y = Ce^{4x}$

Check: $y' = 4Ce^{4x} = 4y$

3. Differential equation: $y'' + y = 0$

Solution: $y = C_1 \cos x + C_2 \sin x$

Check: $y' = -C_1 \sin x + C_2 \cos x$

$y'' = -C_1 \cos x - C_2 \sin x$

$y'' + y = -C_1 \cos x - C_2 \sin x + C_1 \cos x + C_2 \sin x = 0$

5. $y = -\cos x \ln|\sec x + \tan x|$

$$y' = (-\cos x) \frac{1}{\sec x + \tan x} (\sec x \cdot \tan x + \sec^2 x) + \sin x \ln|\sec x + \tan x|$$

$$= \frac{(-\cos x)}{\sec x + \tan x} (\sec x)(\tan x + \sec x) + \sin x \ln|\sec x + \tan x|$$

$$= -1 + \sin x \ln|\sec x + \tan x|$$

$$y'' = (\sin x) \frac{1}{\sec x + \tan x} (\sec x \cdot \tan x + \sec^2 x) + \cos x \ln|\sec x + \tan x|$$

$$= (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x|$$

Substituting,

$$y'' + y = (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x| - \cos x \ln|\sec x + \tan x|$$

$$= \tan x.$$

In Exercises 7–11, the differential equation is $y^{(4)} - 16y = 0$.

7. $y = 3 \cos x$

$$y^{(4)} = 3 \cos x$$

$$y^{(4)} - 16y = -45 \cos x \neq 0,$$

No.

9. $y = e^{-2x}$

$$y^{(4)} = 16e^{-2x}$$

$$y^{(4)} - 16y = 16e^{-2x} - 16e^{-2x} = 0,$$

Yes.

11. $y = C_1 e^{2x} + C_2 e^{-2x} + C_3 \sin 2x + C_4 \cos 2x$

$$y^{(4)} = 16C_1 e^{2x} + 16C_2 e^{-2x} + 16C_3 \sin 2x + 16C_4 \cos 2x$$

$$y^{(4)} - 16y = 0,$$

Yes.

In 13–17, the differential equation is $xy' - 2y = x^3e^x$.

13. $y = x^2, y' = 2x$

$$xy' - 2y = x(2x) - 2(x^2) = 0 \neq x^3e^x$$

No.

17. $y = \ln x, y' = \frac{1}{x}$

$$xy' - 2y = x\left(\frac{1}{x}\right) - 2\ln x \neq x^3e^x, \quad \text{No.}$$

19. $y = Ce^{kx}$

$$\frac{dy}{dx} = Cke^{kx}$$

Since $dy/dx = 0.07y$, we have $Cke^{kx} = 0.07Ce^{kx}$.
Thus, $k = 0.07$.

15. $y = x^2(2 + e^x), y' = x^2(e^x) + 2x(2 + e^x)$

$$xy' - 2y = x[x^2e^x + 2xe^x + 4x] - 2[x^2e^x + 2x^2] = x^3e^x,$$

Yes.

21. $y^2 = Cx^3$ passes through (4, 4)

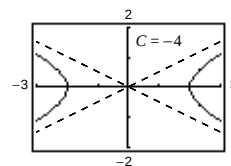
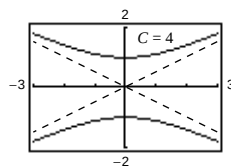
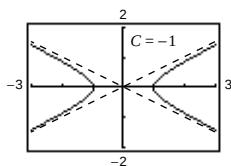
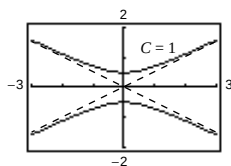
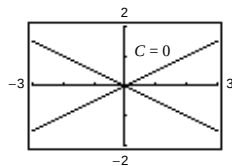
$$16 = C(64) \Rightarrow C = \frac{1}{4}$$

Particular solution: $y^2 = \frac{1}{4}x^3$ or $4y^2 = x^3$

23. Differential equation: $4yy' - x = 0$

General solution: $4y^2 - x^2 = C$

Particular solutions: $C = 0$, Two intersecting lines
 $C = \pm 1$, $C = \pm 4$, Hyperbolas



25. Differential equation: $y' + 2y = 0$

General Solution: $y = Ce^{-2x}$

$$y' + 2y = C(-2)e^{-2x} + 2(Ce^{-2x}) = 0$$

Initial condition: $y(0) = 3, 3 = Ce^0 = C$

Particular solution: $y = 3e^{-2x}$

27. Differential equation: $y'' + 9y = 0$

General solution: $y = C_1 \sin 3x + C_2 \cos 3x$

$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x,$$

$$y'' = -9C_1 \sin 3x - 9C_2 \cos 3x$$

$$y'' + 9y = (-9C_1 \sin 3x - 9C_2 \cos 3x) +$$

$$9(C_1 \sin 3x + C_2 \cos 3x) = 0$$

Initial conditions: $y\left(\frac{\pi}{6}\right) = 2, y'\left(\frac{\pi}{6}\right) = 1$

$$2 = C_1 \sin\left(\frac{\pi}{2}\right) + C_2 \cos\left(\frac{\pi}{2}\right) \Rightarrow C_1 = 2$$

$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x$$

$$1 = 3C_1 \cos\left(\frac{\pi}{2}\right) - 3C_2 \sin\left(\frac{\pi}{2}\right)$$

$$= -3C_2 \Rightarrow C_2 = -\frac{1}{3}$$

Particular solution: $y = 2 \sin 3x - \frac{1}{3} \cos 3x$

29. Differential equation: $x^2y'' - 3xy' + 3y = 0$

General solution: $y = C_1x + C_2x^3$

$y' = C_1 + 3C_2x^2, y'' = 6C_2x$

$x^2y'' - 3xy' + 3y = x^2(6C_2x) - 3x(C_1 + 3C_2x^2) +$

$3(C_1x + C_2x^3) = 0$

Initial conditions: $y(2) = 0, y'(2) = 4$

$0 = 2C_1 + 8C_2$

$y' = C_1 + 3C_2x^2$

$4 = C_1 + 12C_2$

$\left. \begin{array}{l} C_1 + 4C_2 = 0 \\ C_1 + 12C_2 = 4 \end{array} \right\} C_2 = \frac{1}{2}, C_1 = -2$

Particular solution: $y = -2x + \frac{1}{2}x^3$

31. $\frac{dy}{dx} = 3x^2$

$y = \int 3x^2 dx = x^3 + C$

33. $\frac{dy}{dx} = \frac{x}{1+x^2}$

$y = \int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C$

$(u = 1+x^2, du = 2x dx)$

35. $\frac{dy}{dx} = \frac{x-2}{x} = 1 - \frac{2}{x}$

$y = \int \left[1 - \frac{2}{x} \right] dx$

$= x - 2 \ln|x| + C = x - \ln x^2 + C$

37. $\frac{dy}{dx} = \sin 2x$

$y = \int \sin 2x dx = -\frac{1}{2} \cos 2x + C$

$(u = 2x, du = 2dx)$

39. $\frac{dy}{dx} = x\sqrt{x-3}$ Let $u = \sqrt{x-3}$, then $x = u^2 + 3$ and $dx = 2u du$.

$y = \int x\sqrt{x-3} dx = \int (u^2 + 3)(u)(2u) du$

$= 2 \int (u^4 + 3u^2) du = 2 \left(\frac{u^5}{5} + u^3 \right) + C = \frac{2}{5} (x-3)^{5/2} + 2(x-3)^{3/2} + C$

41. $\frac{dy}{dx} = xe^{x^2}$

$y = \int xe^{x^2} dx = \frac{1}{2} e^{x^2} + C$

$(u = x^2, du = 2x dx)$

43. $\frac{dy}{dx} = \frac{x}{y}$

$\int y dy = \int x dx$

$\frac{y^2}{2} = \frac{x^2}{2} + C_1$

$y^2 - x^2 = C$

45. $\frac{dr}{ds} = 0.05r$

$\int \frac{dr}{r} = \int 0.05 ds$

$\ln |r| = 0.05s + C_1$

$r = e^{0.05s + C_1} = Ce^{0.05s}$

47. $(2+x)y' = 3y$

$\int \frac{dy}{y} = \int \frac{3}{2+x} dx$

$\ln y = 3 \ln(2+x) + \ln C = \ln C(2+x)^3$

$y = C(x+2)^3$

49. $yy' = \sin x$

$$\int y \, dy = \int \sin x \, dx$$

$$\frac{y^2}{2} = -\cos x + C_1$$

$$y^2 = -2 \cos x + C$$

53. $y \ln x - xy' = 0$

$$\int \frac{dy}{y} = \int \frac{\ln x}{x} dx \quad \left(u = \ln x, du = \frac{dx}{x} \right)$$

$$\ln y = \frac{1}{2}(\ln x)^2 + C_1$$

$$y = e^{(1/2)(\ln x)^2 + C_1} = Ce^{(\ln x)^2/2}$$

57. $y(x+1) + y' = 0$

$$\int \frac{dy}{y} = -\int (x+1) \, dx$$

$$\ln y = -\frac{(x+1)^2}{2} + C_1$$

$$y = Ce^{-(x+1)^2/2}$$

Initial condition: $y(-2) = 1, 1 = Ce^{-1/2}, C = e^{1/2}$

Particular solution: $y = e^{[1-(x+1)^2]/2} = e^{-(x^2+2x)/2}$

61. $\frac{du}{dv} = uv \sin v^2$

$$\int \frac{du}{u} = \int v \sin v^2 \, dv$$

$$\ln u = -\frac{1}{2} \cos v^2 + C_1$$

$$u = Ce^{-(\cos v^2)/2}$$

Initial condition: $u(0) = 1, C = \frac{1}{e^{-1/2}} = e^{1/2}$

Particular solution: $u = e^{(1-\cos v^2)/2}$

51. $\sqrt{1-4x^2} \frac{dy}{dx} = x$

$$dy = \frac{x}{\sqrt{1-4x^2}} dx$$

$$\begin{aligned} \int dy &= \int \frac{x}{\sqrt{1-4x^2}} dx \\ &= -\frac{1}{8} \int (1-4x^2)^{-1/2} (-8x \, dx) \end{aligned}$$

$$y = -\frac{1}{4}(1-4x^2)^{1/2} + C$$

55. $yy' - e^x = 0$

$$\int y \, dy = \int e^x \, dx$$

$$\frac{y^2}{2} = e^x + C_1$$

$$y^2 = 2e^x + C$$

Initial condition: $y(0) = 4, 16 = 2 + C, C = 14$

Particular solution: $y^2 = 2e^x + 14$

59. $y(1+x^2) \frac{dy}{dx} = x(1+y^2)$

$$\frac{y}{1+y^2} dy = \frac{x}{1+x^2} dx$$

$$\frac{1}{2} \ln(1+y^2) = \frac{1}{2} \ln(1+x^2) + C_1$$

$$\ln(1+y^2) = \ln(1+x^2) + \ln C = \ln[C(1+x^2)]$$

$$1+y^2 = C(1+x^2)$$

$$y(0) = \sqrt{3}: 1+3 = C \Rightarrow C = 4$$

$$1+y^2 = 4(1+x^2)$$

$$y^2 = 3 + 4x^2$$

63. $dP - kP \, dt = 0$

$$\int \frac{dP}{P} = k \int dt$$

$$\ln P = kt + C_1$$

$$P = Ce^{kt}$$

Initial condition: $P(0) = P_0, P_0 = Ce^0 = C$

Particular solution: $P = P_0 e^{kt}$

$$65. \quad \frac{dy}{dx} = \frac{-9x}{16y}$$

$$\int 16y \, dy = -\int 9x \, dx$$

$$8y^2 = \frac{-9}{2}x^2 + C$$

$$\text{Initial condition: } y(1) = 1, \quad 8 = -\frac{9}{2} + C, \quad C = \frac{25}{2}$$

$$\text{Particular solution: } 8y^2 = \frac{-9}{2}x^2 + \frac{25}{2},$$

$$16y^2 + 9x^2 = 25$$

$$69. \quad f(x, y) = x^3 - 4xy^2 + y^3$$

$$\begin{aligned} f(tx, ty) &= t^3 x^3 - 4t x t^2 y^2 + t^3 y^3 \\ &= t^3(x^3 - 4xy^2 + y^3) \end{aligned}$$

Homogeneous of degree 3

$$73. \quad f(x, y) = 2 \ln xy$$

$$\begin{aligned} f(tx, ty) &= 2 \ln tx ty \\ &= 2 \ln t^2 xy = 2(\ln t^2 + \ln xy) \end{aligned}$$

Not homogeneous

$$77. \quad y' = \frac{x+y}{2x}, \quad y = vx$$

$$v + x \frac{dv}{dx} = \frac{x + vx}{2x}$$

$$x \frac{dv}{dx} = \frac{1+v}{2} - v$$

$$2 \int \frac{dv}{1-v} = \int \frac{dx}{x}$$

$$-\ln(1-v)^2 = \ln|x| + \ln C = \ln|Cx|$$

$$\frac{1}{(1-v^2)} = |Cx|$$

$$\frac{1}{[1-(y/x)]^2} = |Cx|$$

$$\frac{x^2}{(x-y)^2} = |Cx|$$

$$|x| = C(x-y)^2$$

$$67. \quad m = \frac{dy}{dx} = \frac{0-y}{(x+2)-x} = -\frac{y}{2}$$

$$\int \frac{dy}{y} = \int -\frac{1}{2} dx$$

$$\ln y = -\frac{1}{2}x + C_1$$

$$y = Ce^{-x/2}$$

$$71. \quad f(x, y) = \frac{x^2 y^2}{\sqrt{x^2 + y^2}}$$

$$f(tx, ty) = \frac{t^4 x^2 y^2}{\sqrt{t^2 x^2 + t^2 y^2}} = t^3 \frac{x^2 y^2}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 3

$$75. \quad f(x, y) = 2 \ln \frac{x}{y}$$

$$f(tx, ty) = 2 \ln \frac{tx}{ty} = 2 \ln \frac{x}{y}$$

Homogeneous degree 0

$$79. \quad y' = \frac{x-y}{x+y}, \quad y = vx$$

$$v + x \frac{dv}{dx} = \frac{x - xv}{x + xv}$$

$$v \, dx + x \, dv = \frac{1-v}{1+v} dx$$

$$\int \frac{v+1}{v^2+2v-1} dv = -\int \frac{dx}{x}$$

$$\frac{1}{2} \ln|v^2+2v-1| = -\ln|x| + \ln C_1 = \ln \left| \frac{C_1}{x} \right|$$

$$|v^2+2v-1| = \frac{C}{x^2}$$

$$\left| \frac{y^2}{x^2} + 2\frac{y}{x} - 1 \right| = \frac{C}{x^2}$$

$$|y^2 + 2xy - x^2| = C$$

81. $y' = \frac{xy}{x^2 - y^2}, y = vx$

$$v + x \frac{dv}{dx} = \frac{x^2 v}{x^2 - x^2 v^2}$$

$$v dx + x dv = \frac{v}{1 - v^2} dx$$

$$\int \frac{1 - v^2}{v^3} dv = \int \frac{dx}{x}$$

$$-\frac{1}{2v^2} - \ln|v| = \ln|x| + \ln C_1 = \ln|C_1 x|$$

$$\frac{-1}{2v^2} = \ln|C_1 x v|$$

$$\frac{-x^2}{2y^2} = \ln|C_1 y|$$

$$y = C e^{-x^2/2y^2}$$

85. $\left(x \sec \frac{y}{x} + y\right) dx - x dy = 0, y = vx$

$$(x \sec v + xv) dx - x(v dx + x dv) = 0$$

$$(\sec v + v) dx = v dx + x dv$$

$$\int \cos v dv = \int \frac{dx}{x}$$

$$\sin v = \ln x + \ln C_1$$

$$x = C e^{\sin v}$$

$$= C e^{\sin(y/x)}$$

Initial condition: $y(1) = 0, 1 = C e^0 = C$

Particular solution: $x = e^{\sin(y/x)}$

83. $x dy - (2xe^{-y/x} + y) dx = 0, y = vx$

$$x(v dx + x dv) - (2xe^{-v} + vx) dx = 0$$

$$\int e^v dv = \int \frac{2}{x} dx$$

$$e^v = \ln C_1 x^2$$

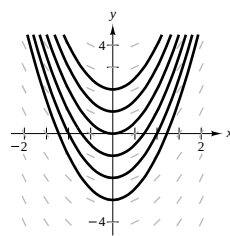
$$e^{y/x} = \ln C_1 + \ln x^2$$

$$e^{y/x} = C + \ln x^2$$

Initial condition: $y(1) = 0, 1 = C$

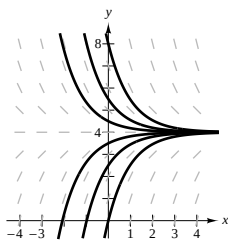
Particular solution: $e^{y/x} = 1 + \ln x^2$

87. $\frac{dy}{dx} = x$



$$y = \int x dx = \frac{1}{2}x^2 + C$$

89. $\frac{dy}{dx} = 4 - y$



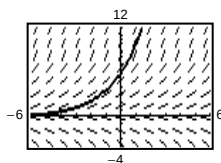
$$\int \frac{dy}{4 - y} = \int dx$$

$$\ln|4 - y| = -x + C_1$$

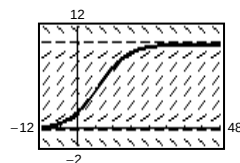
$$4 - y = e^{-x + C_1}$$

$$y = 4 + C e^{-x}$$

91. $\frac{dy}{dx} = 0.5y, y(0) = 6$



93. $\frac{dy}{dx} = 0.02y(10 - y), y(0) = 2$



95. $\frac{dy}{dt} = ky, y = Ce^{kt}$

Initial conditions: $y(0) = y_0$

$$y(1620) = \frac{y_0}{2}$$

$$C = y_0$$

$$\frac{y_0}{2} = y_0 e^{1620k}$$

$$k = \frac{\ln(1/2)}{1620}$$

Particular solution: $y = y_0 e^{-t(\ln 2)/1620}$

When $t = 25$, $y \approx 0.989y_0$, $y = 98.9\%$ of y_0 .

99. $\frac{dy}{dx} = ky(y - 4)$

The direction field satisfies $(dy/dx) = 0$ along $y = 0$ and $y = 4$. Matches (c).

101. $\frac{dw}{dt} = k(1200 - w)$

$$\int \frac{dw}{1200 - w} = \int k dt$$

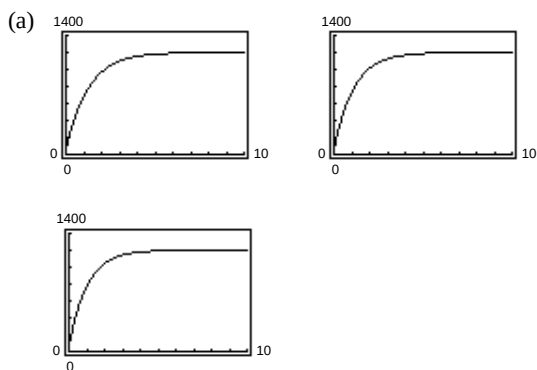
$$\ln(1200 - w) = -kt + C_1$$

$$1200 - w = e^{-kt + C_1} = Ce^{-kt}$$

$$w = 1200 - Ce^{-kt}$$

$$w(0) = 60 = 1200 - C \Rightarrow C = 1200 - 60 = 1140$$

$$w = 1200 - 1140e^{-kt}$$



(b) $k = 0.8$: $t = 1.31$ years

$k = 0.9$: $t = 1.16$ years

$k = 1.0$: $t = 1.05$ years

(c) Maximum weight: 1200 pounds

$$\lim_{t \rightarrow \infty} w = 1200$$

97. $\frac{dy}{dx} = k(y - 4)$

The direction field satisfies $(dy/dx) = 0$ along $y = 4$; but not along $y = 0$. Matches (a).

103. (a) $\frac{dv}{dt} = k(W - v)$

$$\int \frac{dv}{W - v} = \int k dt$$

$$-\ln(W - v) = kt + C_1$$

$$v = W - Ce^{-kt}$$

Initial conditions:

$W = 20, v = 0$ when $t = 0$, and

$v = 5$ when $t = 1$.

$$C = 20, k = -\ln(3/4)$$

Particular solution:

$$v = 20(1 - e^{\ln(3/4)t}) \approx 20(1 - e^{-0.2877t})$$

$$(b) s = \int 20(1 - e^{-0.2877t}) dt \\ \approx 20[t + 3.4761e^{-0.2877t}] + C$$

Since $S(0) = 0$, $C \approx -69.5$ and we have

$$s \approx 20t + 69.5(e^{-0.2877t} - 1).$$

105. Given family (circles):
- $x^2 + y^2 = C$

$$2x + 2yy' = 0$$

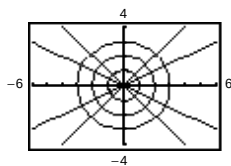
$$y' = -\frac{x}{y}$$

Orthogonal trajectory (lines): $y' = \frac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + \ln K$$

$$y = Kx$$



107. Given family (parabolas):
- $x^2 = Cy$

$$2x = Cy'$$

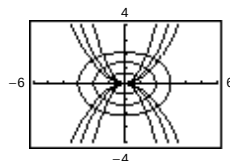
$$y' = \frac{2x}{C} = \frac{2x}{x^2/y} = \frac{2y}{x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{x}{2y}$

$$2 \int y \, dy = - \int x \, dx$$

$$y^2 = -\frac{x^2}{2} + K_1$$

$$x^2 + 2y^2 = K$$



109. Given family:
- $y^2 = Cx^3$

$$2yy' = 3Cx^2$$

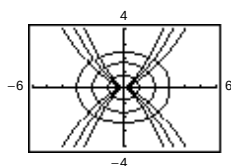
$$y' = \frac{3Cx^2}{2y} = \frac{3x^2}{2y} \left(\frac{y^2}{x^3} \right) = \frac{3y}{2x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{2x}{3y}$

$$3 \int y \, dy = -2 \int x \, dx$$

$$\frac{3y^2}{2} = -x^2 + K_1$$

$$3y^2 + 2x^2 = K$$



111. A general solution of order
- n
- has
- n
- arbitrary constants while in a particular solution initial conditions are given in order to solve for all these constants.

- 113.
- $M(x, y)dx + N(x, y)dy = 0$
- , where
- M
- and
- N
- are homogeneous functions of the same degree.

115. False. Consider Example 2.
- $y = x^3$
- is a solution to
- $xy' - 3y = 0$
- , but
- $y = x^3 + 1$
- is not a solution.

117. False

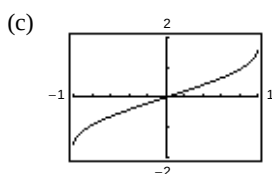
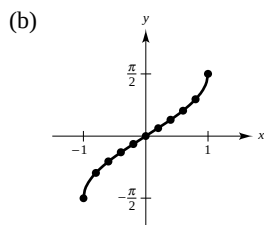
$$\begin{aligned} f(tx, ty) &= t^2x^2 + t^2xy + 2 \\ &\neq t^2f(x, y) \end{aligned}$$

Section 5.8 Inverse Trigonometric Functions: Differentiation

1. $y = \arcsin x$

(a)

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y	-1.571	-0.927	-0.644	-0.412	-0.201	0	0.201	0.412	0.644	0.927	1.571



(d) Symmetric about origin:
 $\arcsin(-x) = -\arcsin x$
 Intercept: (0, 0)

3. False.

$$\arccos \frac{1}{2} = \frac{\pi}{3}$$

since the range is $[0, \pi]$.

5. $\arcsin \frac{1}{2} = \frac{\pi}{6}$

7. $\arccos \frac{1}{2} = \frac{\pi}{3}$

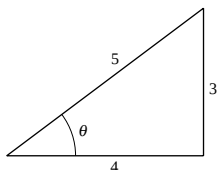
9. $\arctan \frac{\sqrt{3}}{3} = \frac{\pi}{6}$

11. $\operatorname{arccsc}(-\sqrt{2}) = -\frac{\pi}{4}$

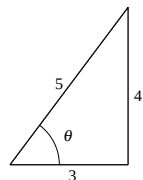
13. $\arccos(-0.8) \approx 2.50$

15. $\operatorname{arcsec}(1.269) = \arccos\left(\frac{1}{1.269}\right) \approx 0.66$

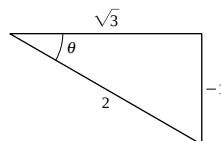
17. (a) $\sin(\arctan \frac{3}{4}) = \frac{3}{5}$



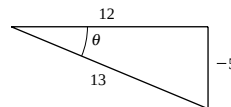
(b) $\sec(\arcsin \frac{4}{5}) = \frac{5}{3}$



19. (a) $\cot\left[\arcsin\left(-\frac{1}{2}\right)\right] = \cot\left(-\frac{\pi}{6}\right) = -\sqrt{3}$



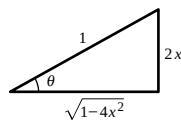
(b) $\csc\left[\arctan\left(-\frac{5}{12}\right)\right] = -\frac{13}{5}$



21. $y = \cos(\arcsin 2x)$

$$\theta = \arcsin 2x$$

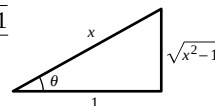
$$y = \cos \theta = \sqrt{1 - 4x^2}$$



23. $y = \sin(\operatorname{arcsec} x)$

$$\theta = \operatorname{arcsec} x, 0 \leq \theta \leq \pi, \theta \neq \frac{\pi}{2}$$

$$y = \sin \theta = \frac{\sqrt{x^2 - 1}}{|x|}$$

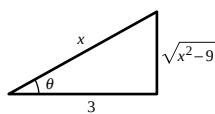


The absolute value bars on x are necessary because of the restriction $0 \leq \theta \leq \pi, \theta \neq \pi/2$, and $\sin \theta$ for this domain must always be nonnegative.

25. $y = \tan\left(\operatorname{arcsec} \frac{x}{3}\right)$

$$\theta = \operatorname{arcsec} \frac{x}{3}$$

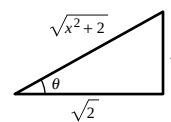
$$y = \tan \theta = \frac{\sqrt{x^2 - 9}}{3}$$



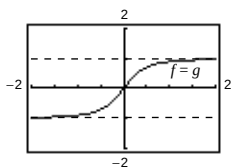
27. $y = \csc\left(\arctan \frac{x}{\sqrt{2}}\right)$

$$\theta = \arctan \frac{x}{\sqrt{2}}$$

$$y = \csc \theta = \frac{\sqrt{x^2 + 2}}{x}$$



29. $\sin(\arctan 2x) = \frac{2x}{\sqrt{1 + 4x^2}}$

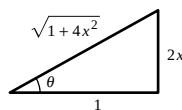


Asymptotes: $y = \pm 1$

$$\arctan 2x = \theta$$

$$\tan \theta = 2x$$

$$\sin \theta = \frac{2x}{\sqrt{1 + 4x^2}}$$



33. $\arcsin \sqrt{2x} = \arccos \sqrt{x}$

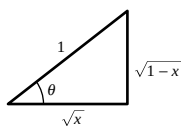
$$\sqrt{2x} = \sin(\arccos \sqrt{x})$$

$$\sqrt{2x} = \sqrt{1 - x}, 0 \leq x \leq 1$$

$$2x = 1 - x$$

$$3x = 1$$

$$x = \frac{1}{3}$$



35. (a) $\operatorname{arccsc} x = \arcsin \frac{1}{x}, |x| \geq 1$

Let $y = \operatorname{arccsc} x$. Then for

$$-\frac{\pi}{2} \leq y < 0 \text{ and } 0 < y \leq \frac{\pi}{2},$$

$$\csc y = x \Rightarrow \sin y = 1/x. \text{ Thus, } y = \arcsin(1/x).$$

Therefore, $\operatorname{arccsc} x = \arcsin(1/x)$.

(b) $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}, x > 0$

Let $y = \arctan x + \arctan(1/x)$. Then,

$$\begin{aligned} \tan y &= \frac{\tan(\arctan x) + \tan[\arctan(1/x)]}{1 - \tan(\arctan x) \tan[\arctan(1/x)]} \\ &= \frac{x + (1/x)}{1 - x(1/x)} \\ &= \frac{x + (1/x)}{0} \text{ (which is undefined).} \end{aligned}$$

Thus, $y = \pi/2$. Therefore, $\arctan x + \arctan(1/x) = \pi/2$.

37. $f(x) = \arcsin(x - 1)$

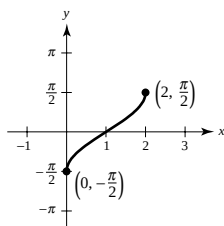
$$x - 1 = \sin y$$

$$x = 1 + \sin y$$

$$\text{Domain: } [0, 2]$$

$$\text{Range: } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$f(x)$ is the graph of $\arcsin x$ shifted 1 unit to the right.



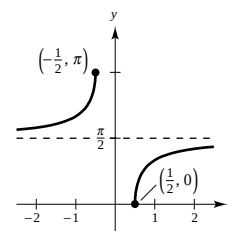
39. $f(x) = \operatorname{arcsec} 2x$

$$2x = \sec y$$

$$x = \frac{1}{2} \sec y$$

$$\text{Domain: } \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$$

$$\text{Range: } \left[0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right]$$



$$41. f(x) = 2 \arcsin(x - 1)$$

$$f'(x) = \frac{2}{\sqrt{1 - (x - 1)^2}} = \frac{2}{\sqrt{2x - x^2}}$$

$$45. f(x) = \arctan \frac{x}{a}$$

$$f'(x) = \frac{1/a}{1 + (x^2/a^2)} = \frac{a}{a^2 + x^2}$$

$$49. h(t) = \sin(\arccos t) = \sqrt{1 - t^2}$$

$$h'(t) = \frac{1}{2}(1 - t^2)^{-1/2}(-2t) = \frac{-t}{\sqrt{1 - t^2}}$$

$$53. y = \frac{1}{2} \left(\frac{1}{2} \ln \frac{x+1}{x-1} + \arctan x \right)$$

$$= \frac{1}{4} [\ln(x+1) - \ln(x-1)] + \frac{1}{2} \arctan x$$

$$\frac{dy}{dx} = \frac{1}{4} \left(\frac{1}{x+1} - \frac{1}{x-1} \right) + \frac{1/2}{1+x^2} = \frac{1}{1-x^4}$$

$$57. y = 8 \arcsin \frac{x}{4} - \frac{x\sqrt{16-x^2}}{2}$$

$$\begin{aligned} y' &= 2 \frac{1}{\sqrt{1 - (x/4)^2}} - \frac{\sqrt{16-x^2}}{2} - \frac{x}{4}(16-x^2)^{-1/2}(-2x) \\ &= \frac{8}{\sqrt{16-x^2}} - \frac{\sqrt{16-x^2}}{2} + \frac{x^2}{2\sqrt{16-x^2}} \\ &= \frac{16 - (16-x^2) + x^2}{2\sqrt{16-x^2}} = \frac{x^2}{\sqrt{16-x^2}} \end{aligned}$$

$$61. f(x) = \arcsin x, a = \frac{1}{2}$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f''(x) = \frac{x}{(1-x^2)^{3/2}}$$

$$P_1(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$$

$$P_2(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{1}{2}f''\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)^2 = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right) + \frac{2\sqrt{3}}{9}\left(x - \frac{1}{2}\right)^2$$

$$43. g(x) = 3 \arccos \frac{x}{2}$$

$$g'(x) = \frac{-3(1/2)}{\sqrt{1 - (x^2/4)}} = \frac{-3}{\sqrt{4-x^2}}$$

$$47. g(x) = \frac{\arcsin 3x}{x}$$

$$\begin{aligned} g'(x) &= \frac{x(3/\sqrt{1-9x^2}) - \arcsin 3x}{x^2} \\ &= \frac{3x - \sqrt{1-9x^2} \arcsin 3x}{x^2 \sqrt{1-9x^2}} \end{aligned}$$

$$51. y = x \arccos x - \sqrt{1-x^2}$$

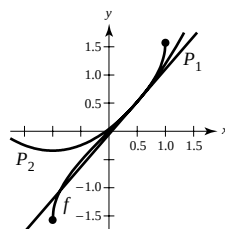
$$\begin{aligned} y' &= \arccos x - \frac{x}{\sqrt{1-x^2}} - \frac{1}{2}(1-x^2)^{-1/2}(-2x) \\ &= \arccos x \end{aligned}$$

$$55. y = x \arcsin x + \sqrt{1-x^2}$$

$$\frac{dy}{dx} = x \left(\frac{1}{\sqrt{1-x^2}} \right) + \arcsin x - \frac{x}{\sqrt{1-x^2}} = \arcsin x$$

$$59. y = \arctan x + \frac{x}{1+x^2}$$

$$\begin{aligned} y' &= \frac{1}{1+x^2} + \frac{(1+x^2) - x(2x)}{(1+x^2)^2} \\ &= \frac{(1+x^2) + (1-x^2)}{(1+x^2)^2} \\ &= \frac{2}{(1+x^2)^2} \end{aligned}$$



$$63. \quad f(x) = \operatorname{arcsec} x - x$$

$$f'(x) = \frac{1}{|x|\sqrt{x^2 - 1}} - 1$$

$$= 0 \text{ when } |x|\sqrt{x^2 - 1} = 1.$$

$$x^2(x^2 - 1) = 1$$

$$x^4 - x^2 - 1 = 0 \text{ when } x^2 = \frac{1 + \sqrt{5}}{2} \text{ or}$$

$$x = \pm \sqrt{\frac{1 + \sqrt{5}}{2}} = \pm 1.272$$

Relative maximum: (1.272, -0.606)

Relative minimum: (-1.272, 3.747)

67. The trigonometric functions are not one-to-one on $(-\infty, \infty)$, so their domains must be restricted to intervals on which they are one-to-one.

$$71. \text{ (a) } \cot \theta = \frac{x}{5}$$

$$\theta = \operatorname{arccot}\left(\frac{x}{5}\right)$$

$$73. \text{ (a) } h(t) = -16t^2 + 256$$

$$-16t^2 + 256 = 0 \text{ when } t = 4 \text{ sec.}$$

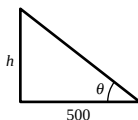
$$\text{(b) } \tan \theta = \frac{h}{500} = \frac{-16t^2 + 256}{500}$$

$$\theta = \arctan\left[\frac{16}{500}(-t^2 + 16)\right]$$

$$\frac{d\theta}{dt} = \frac{-8t/125}{1 + [(4/125)(-t^2 + 16)]^2} = \frac{-1000t}{15,625 + 16(16 - t^2)^2}$$

When $t = 1$, $d\theta/dt \approx -0.0520$ rad/sec.

When $t = 2$, $d\theta/dt \approx -0.1116$ rad/sec.



$$65. \quad f(x) = \arctan x - \arctan(x - 4)$$

$$f'(x) = \frac{1}{1 + x^2} - \frac{1}{1 + (x - 4)^2} = 0$$

$$1 + x^2 = 1 + (x - 4)^2$$

$$0 = -8x + 16$$

$$x = 2$$

By the First Derivative Test, (2, 2.214) is a relative maximum.

$$69. \quad y = \operatorname{arccot} x, 0 < y < \pi$$

$$x = \cot y$$

$$\tan y = \frac{1}{x}$$

So, graph the function

$$y = \arctan\left(\frac{1}{x}\right) \text{ for } x > 0 \text{ and } y = \arctan\left(\frac{1}{x}\right) + \pi \text{ for } x < 0.$$

$$\text{(b) } \frac{d\theta}{dt} = \frac{-\frac{1}{5}}{1 + \left(\frac{x}{5}\right)^2} \frac{dx}{dt} = \frac{-5}{x^2 + 25} \frac{dx}{dt}$$

$$\text{If } \frac{dx}{dt} = -400 \text{ and } x = 10, \frac{d\theta}{dt} = 16 \text{ rad/hr.}$$

$$\text{If } \frac{dx}{dt} = -400 \text{ and } x = 3, \frac{d\theta}{dt} \approx 58.824 \text{ rad/hr.}$$

$$75. \tan(\arctan x + \arctan y) = \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x) \tan(\arctan y)} = \frac{x + y}{1 - xy}, xy \neq 1$$

Therefore,

$$\arctan x + \arctan y = \arctan\left(\frac{x + y}{1 - xy}\right), xy \neq 1.$$

Let $x = \frac{1}{2}$ and $y = \frac{1}{3}$.

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan \frac{(1/2) + (1/3)}{1 - [(1/2) \cdot (1/3)]} = \arctan \frac{5/6}{1 - (1/6)} = \arctan \frac{5/6}{5/6} = \arctan 1 = \frac{\pi}{4}$$

$$77. f(x) = kx + \sin x$$

$$f'(x) = k + \cos x \geq 0 \text{ for } k \geq 1$$

$$f'(x) = k + \cos x \leq 0 \text{ for } k \leq -1$$

Therefore, $f(x) = kx + \sin x$ is strictly monotonic and has an inverse for $k \leq -1$ or $k \geq 1$.

79. True

$$\frac{d}{dx}[\arctan x] = \frac{1}{1 + x^2} > 0 \text{ for all } x.$$

81. True

$$\frac{d}{dx}[\arctan(\tan x)] = \frac{\sec^2 x}{1 + \tan^2 x} = \frac{\sec^2 x}{\sec^2 x} = 1$$

Section 5.9 Inverse Trigonometric Functions: Integration

$$1. \int \frac{5}{\sqrt{9 - x^2}} dx = 5 \arcsin\left(\frac{x}{3}\right) + C$$

$$3. \text{ Let } u = 3x, du = 3 dx.$$

$$\int_0^{1/6} \frac{1}{\sqrt{1 - 9x^2}} dx = \frac{1}{3} \int_0^{1/6} \frac{1}{\sqrt{1 - (3x)^2}} (3) dx = \left[\frac{1}{3} \arcsin(3x) \right]_0^{1/6} = \frac{\pi}{18}$$

$$5. \int \frac{7}{16 + x^2} dx = \frac{7}{4} \arctan\left(\frac{x}{4}\right) + C$$

$$7. \text{ Let } u = 2x, du = 2 dx.$$

$$\int_0^{\sqrt{3}/2} \frac{1}{1 + 4x^2} dx = \frac{1}{2} \int_0^{\sqrt{3}/2} \frac{2}{1 + (2x)^2} dx = \left[\frac{1}{2} \arctan(2x) \right]_0^{\sqrt{3}/2} = \frac{\pi}{6}$$

$$9. \int \frac{1}{x\sqrt{4x^2 - 1}} dx = \int \frac{2}{2x\sqrt{(2x)^2 - 1}} dx = \operatorname{arcsec}|2x| + C$$

$$11. \int \frac{x^3}{x^2 + 1} dx = \int \left[x - \frac{x}{x^2 + 1} \right] dx = \int x dx - \frac{1}{2} \int \frac{2x}{x^2 + 1} dx = \frac{1}{2} x^2 - \frac{1}{2} \ln(x^2 + 1) + C \quad (\text{Use long division.})$$

$$13. \int \frac{1}{\sqrt{1 - (x + 1)^2}} dx = \arcsin(x + 1) + C$$

$$15. \text{ Let } u = t^2, du = 2t dt.$$

$$\int \frac{t}{\sqrt{1 - t^4}} dt = \frac{1}{2} \int \frac{1}{\sqrt{1 - (t^2)^2}} (2t) dt = \frac{1}{2} \arcsin(t^2) + C$$

17. Let $u = \arcsin x$, $du = \frac{1}{\sqrt{1-x^2}} dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}} dx = \left[\frac{1}{2} \arcsin^2 x \right]_0^{1/\sqrt{2}} = \frac{\pi^2}{32} \approx 0.308$$

19. Let $u = 1 - x^2$, $du = -2x dx$.

$$\begin{aligned} \int_{-1/2}^0 \frac{x}{\sqrt{1-x^2}} dx &= -\frac{1}{2} \int_{-1/2}^0 (1-x^2)^{-1/2} (-2x) dx \\ &= \left[-\sqrt{1-x^2} \right]_{-1/2}^0 = \frac{\sqrt{3}-2}{2} \\ &\approx -0.134 \end{aligned}$$

21. Let $u = e^{2x}$, $du = 2e^{2x} dx$.

$$\int \frac{e^{2x}}{4 + e^{4x}} dx = \frac{1}{2} \int \frac{2e^{2x}}{4 + (e^{2x})^2} dx = \frac{1}{4} \arctan \frac{e^{2x}}{2} + C$$

23. Let $u = \cos x$, $du = -\sin x dx$.

$$\begin{aligned} \int_{\pi/2}^{\pi} \frac{\sin x}{1 + \cos^2 x} dx &= -\int_{\pi/2}^{\pi} \frac{-\sin x}{1 + \cos^2 x} dx \\ &= \left[-\arctan(\cos x) \right]_{\pi/2}^{\pi} = \frac{\pi}{4} \end{aligned}$$

25. $\int \frac{1}{\sqrt{x}\sqrt{1-x}} dx$. $u = \sqrt{x}$, $x = u^2$, $dx = 2u du$

$$\begin{aligned} \int \frac{1}{u\sqrt{1-u^2}} (2u du) &= 2 \int \frac{du}{\sqrt{1-u^2}} = 2 \arcsin u + C \\ &= 2 \arcsin \sqrt{x} + C \end{aligned}$$

27. $\int \frac{x-3}{x^2+1} dx = \frac{1}{2} \int \frac{2x}{x^2+1} dx - 3 \int \frac{1}{x^2+1} dx$

$$= \frac{1}{2} \ln(x^2+1) - 3 \arctan x + C$$

29. $\int \frac{x+5}{\sqrt{9-(x-3)^2}} dx = \int \frac{(x-3)}{\sqrt{9-(x-3)^2}} dx + \int \frac{8}{\sqrt{9-(x-3)^2}} dx$

$$= -\sqrt{9-(x-3)^2} - 8 \arcsin\left(\frac{x-3}{3}\right) + C$$

$$= -\sqrt{6x-x^2} + 8 \arcsin\left(\frac{x}{3} - 1\right) + C$$

31. $\int_0^2 \frac{1}{x^2-2x+2} dx = \int_0^2 \frac{1}{1+(x-1)^2} dx = \left[\arctan(x-1) \right]_0^2 = \frac{\pi}{2}$

33. $\int \frac{2x}{x^2+6x+13} dx = \int \frac{2x+6}{x^2+6x+13} dx - 6 \int \frac{1}{x^2+6x+13} dx = \int \frac{2x+6}{x^2+6x+13} dx - 6 \int \frac{1}{4+(x+3)^2} dx$

$$= \ln|x^2+6x+13| - 3 \arctan\left(\frac{x+3}{2}\right) + C$$

35. $\int \frac{1}{\sqrt{-x^2-4x}} dx = \int \frac{1}{\sqrt{4-(x+2)^2}} dx = \arcsin\left(\frac{x+2}{2}\right) + C$

37. Let $u = -x^2 - 4x$, $du = (-2x - 4) dx$.

$$\int \frac{x+2}{\sqrt{-x^2-4x}} dx = -\frac{1}{2} \int (-x^2-4x)^{-1/2} (-2x-4) dx = -\sqrt{-x^2-4x} + C$$

39. $\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}} dx = \int_2^3 \frac{2x-4}{\sqrt{4x-x^2}} dx + \int_2^3 \frac{1}{\sqrt{4x-x^2}} dx = -\int_2^3 (4x-x^2)^{-1/2} (4-2x) dx + \int_2^3 \frac{1}{\sqrt{4-(x-2)^2}} dx$

$$= \left[-2\sqrt{4x-x^2} + \arcsin\left(\frac{x-2}{2}\right) \right]_2^3 = 4 - 2\sqrt{3} + \frac{\pi}{6} \approx 1.059$$

41. Let $u = x^2 + 1$, $du = 2x dx$.

$$\int \frac{x}{x^4 + 2x^2 + 2} dx = \frac{1}{2} \int \frac{2x}{(x^2 + 1)^2 + 1} dx = \frac{1}{2} \arctan(x^2 + 1) + C$$

43. Let $u = \sqrt{e^t - 3}$. Then $u^2 + 3 = e^t$, $2u du = e^t dt$, and $\frac{2u du}{u^2 + 3} = dt$.

$$\begin{aligned} \int \sqrt{e^t - 3} dt &= \int \frac{2u^2}{u^2 + 3} du = \int 2 du - \int 6 \frac{1}{u^2 + 3} du \\ &= 2u - 2\sqrt{3} \arctan \frac{u}{\sqrt{3}} + C = 2\sqrt{e^t - 3} - 2\sqrt{3} \arctan \sqrt{\frac{e^t - 3}{3}} + C \end{aligned}$$

45. A perfect square trinomial is an expression in x with three terms that factor as a perfect square.

Example: $x^2 + 6x + 9 = (x + 3)^2$

47. (a) $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$, $u = x$ (b) $\int \frac{x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} + C$, $u = 1-x^2$

- (c) $\int \frac{1}{x\sqrt{1-x^2}} dx$ cannot be evaluated using the basic integration rules.

49. (a) $\int \sqrt{x-1} dx = \frac{2}{3}(x-1)^{3/2} + C$, $u = x-1$

- (b) Let $u = \sqrt{x-1}$. Then $x = u^2 + 1$ and $dx = 2u du$.

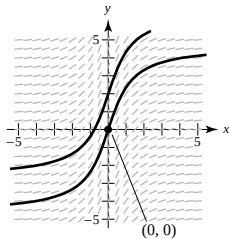
$$\begin{aligned} \int x\sqrt{x-1} dx &= \int (u^2 + 1)(u)(2u) du = 2 \int (u^4 + u^2) du = 2 \left(\frac{u^5}{5} + \frac{u^3}{3} \right) + C \\ &= \frac{2}{15} u^3(3u^2 + 5) + C = \frac{2}{15} (x-1)^{3/2} [3(x-1) + 5] + C = \frac{2}{15} (x-1)^{3/2} (3x+2) + C \end{aligned}$$

- (c) Let $u = \sqrt{x-1}$. Then $x = u^2 + 1$ and $dx = 2u du$.

$$\int \frac{x}{\sqrt{x-1}} dx = \int \frac{u^2 + 1}{u} (2u) du = 2 \int (u^2 + 1) du = 2 \left(\frac{u^3}{3} + u \right) + C = \frac{2}{3} u(u^2 + 3) + C = \frac{2}{3} \sqrt{x-1} (x+2) + C$$

Note: In (b) and (c), substitution was necessary *before* the basic integration rules could be used.

51. (a)

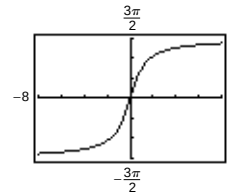


- (b) $\frac{dy}{dx} = \frac{3}{1+x^2}$, $(0, 0)$

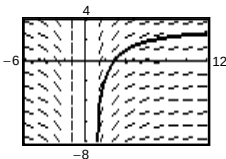
$$y = 3 \int \frac{dx}{1+x^2} = 3 \arctan x + C$$

$$(0, 0): 0 = 3 \arctan(0) + C \Rightarrow C = 0$$

$$y = 3 \arctan x$$



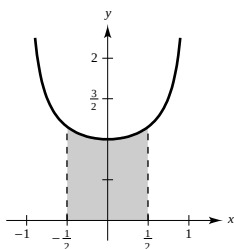
53. $\frac{dy}{dx} = \frac{10}{x\sqrt{x^2-1}}$, $y(3) = 0$



$$\begin{aligned} 55. A &= \int_1^3 \frac{1}{x^2 - 2x + 1 + 4} dx = \int_1^3 \frac{1}{(x-1)^2 + 2^2} dx \\ &= \left[\frac{1}{2} \arctan \left(\frac{x-1}{2} \right) \right]_1^3 = \frac{1}{2} \arctan(1) = \frac{\pi}{8} \approx 0.3927 \end{aligned}$$

57. Area $\approx (1)(1) = 1$

Matches (c)



59. (a) $\int_0^1 \frac{4}{1+x^2} dx = \left[4 \arctan x \right]_0^1 = 4 \arctan 1 - 4 \arctan 0 = 4\left(\frac{\pi}{4}\right) - 4(0) = \pi$

(b) Let $n = 6$.

$$4 \int_0^1 \frac{4}{1+x^2} dx \approx 4\left(\frac{1}{36}\right) \left[1 + \frac{4}{1+(1/36)} + \frac{2}{1+(1/9)} + \frac{4}{1+(1/4)} + \frac{2}{1+(4/9)} + \frac{4}{1+(25/36)} + \frac{1}{2} \right] \approx 3.1415918$$

(c) 3.1415927

61. (a) $\frac{d}{dx} \left[\arcsin\left(\frac{u}{a}\right) + C \right] = \frac{1}{\sqrt{1-(u^2/a^2)}} \left(\frac{u'}{a} \right) = \frac{u'}{\sqrt{a^2-u^2}}$

Thus, $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$.

(b) $\frac{d}{dx} \left[\frac{1}{a} \arctan \frac{u}{a} + C \right] = \frac{1}{a} \left[\frac{u'/a}{1+(u/a)^2} \right] = \frac{1}{a^2} \left[\frac{u'}{(a^2+u^2)/a^2} \right] = \frac{u'}{a^2+u^2}$

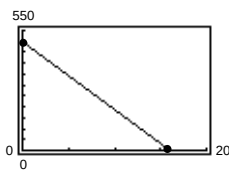
Thus, $\int \frac{du}{a^2+u^2} = \int \frac{u'}{a^2+u^2} dx = \frac{1}{a} \arctan \frac{u}{a} + C$.

(c) Assume $u > 0$.

$$\frac{d}{dx} \left[\frac{1}{a} \operatorname{arccsc} \frac{u}{a} + C \right] = \frac{1}{a} \left[\frac{u'/a}{(u/a)\sqrt{(u/a)^2-1}} \right] = \frac{1}{a} \left[\frac{u'}{u\sqrt{u^2-a^2}/a^2} \right] = \frac{u'}{u\sqrt{u^2-a^2}}. \text{ The case } u < 0 \text{ is handled in a similar manner.}$$

Thus, $\int \frac{du}{u\sqrt{u^2-a^2}} = \int \frac{u'}{u\sqrt{u^2-a^2}} dx = \frac{1}{a} \operatorname{arccsc} \frac{|u|}{a} + C$.

63. (a) $v(t) = -32t + 500$



(b) $s(t) = \int v(t) dt = \int (-32t + 500) dt$

$$= -16t^2 + 500t + C$$

$$s(0) = -16(0) + 500(0) + C = 0 \Rightarrow C = 0$$

$$s(t) = -16t^2 + 500t$$

When the object reaches its maximum height,

$$v(t) = 0.$$

$$v(t) = -32t + 500 = 0$$

$$-32t = -500$$

$$t = 15.625$$

$$s(15.625) = -16(15.625)^2 + 500(15.625)$$

$$= 3906.25 \text{ ft (Maximum height)}$$

—CONTINUED—

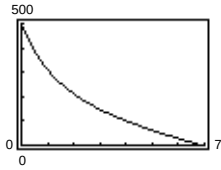
63. —CONTINUED—

$$\begin{aligned}
 \text{(c)} \quad \int \frac{1}{32 + kv^2} dv &= -\int dt \\
 \frac{1}{\sqrt{32k}} \arctan\left(\sqrt{\frac{k}{32}} v\right) &= -t + C_1 \\
 \arctan\left(\sqrt{\frac{k}{32}} v\right) &= -\sqrt{32k} t + C \\
 \sqrt{\frac{k}{32}} v &= \tan(C - \sqrt{32k} t) \\
 v &= \sqrt{\frac{32}{k}} \tan(C - \sqrt{32k} t)
 \end{aligned}$$

When $t = 0$, $v = 500$, $C = \arctan(500\sqrt{k/32})$, and we have

$$v(t) = \sqrt{\frac{32}{k}} \tan\left[\arctan\left(500\sqrt{\frac{k}{32}}\right) - \sqrt{32k} t\right].$$

$$\text{(d) When } k = 0.001, v(t) = \sqrt{32,000} \tan\left[\arctan(500\sqrt{0.00003125}) - \sqrt{0.032} t\right].$$



$v(t) = 0$ when $t_0 \approx 6.86$ sec.

$$\text{(e) } h = \int_0^{6.86} \sqrt{32,000} \tan\left[\arctan(500\sqrt{0.00003125}) - \sqrt{0.032} t\right] dt$$

Simpson's Rule: $n = 10$; $h \approx 1088$ feet

(f) Air resistance lowers the maximum height.

Section 5.10 Hyperbolic Functions

$$1. \text{ (a) } \sinh 3 = \frac{e^3 - e^{-3}}{2} \approx 10.018$$

$$\text{(b) } \tanh(-2) = \frac{\sinh(-2)}{\cosh(-2)} = \frac{e^{-2} - e^2}{e^{-2} + e^2} \approx -0.964$$

$$3. \text{ (a) } \operatorname{csch}(\ln 2) = \frac{2}{e^{\ln 2} - e^{-\ln 2}} = \frac{2}{2 - (1/2)} = \frac{4}{3}$$

$$\begin{aligned}
 \text{(b) } \coth(\ln 5) &= \frac{\cosh(\ln 5)}{\sinh(\ln 5)} = \frac{e^{\ln 5} + e^{-\ln 5}}{e^{\ln 5} - e^{-\ln 5}} \\
 &= \frac{5 + (1/5)}{5 - (1/5)} = \frac{13}{12}
 \end{aligned}$$

$$5. \text{ (a) } \cosh^{-1}(2) = \ln(2 + \sqrt{3}) \approx 1.317$$

$$\text{(b) } \operatorname{sech}^{-1}\left(\frac{2}{3}\right) = \ln\left(\frac{1 + \sqrt{1 - (4/9)}}{2/3}\right) \approx 0.962$$

$$7. \tanh^2 x + \operatorname{sech}^2 x = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 + \left(\frac{2}{e^x + e^{-x}}\right)^2 = \frac{e^{2x} - 2 + e^{-2x} + 4}{(e^x + e^{-x})^2} = \frac{e^{2x} + 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} = 1$$

$$\begin{aligned}
9. \sinh x \cosh y + \cosh x \sinh y &= \left(\frac{e^x - e^{-x}}{2} \right) \left(\frac{e^y + e^{-y}}{2} \right) + \left(\frac{e^x + e^{-x}}{2} \right) \left(\frac{e^y - e^{-y}}{2} \right) \\
&= \frac{1}{4} [e^{x+y} - e^{-x+y} + e^{x-y} - e^{-(x+y)} + e^{x+y} + e^{-x+y} - e^{x-y} - e^{-(x+y)}] \\
&= \frac{1}{4} [2(e^{x+y} - e^{-(x+y)})] = \frac{e^{(x+y)} - e^{-(x+y)}}{2} = \sinh(x+y)
\end{aligned}$$

$$\begin{aligned}
11. 3 \sinh x + 4 \sinh^3 x &= \sinh x (3 + 4 \sinh^2 x) = \left(\frac{e^x - e^{-x}}{2} \right) \left[3 + 4 \left(\frac{e^x - e^{-x}}{2} \right)^2 \right] \\
&= \left(\frac{e^x - e^{-x}}{2} \right) [3 + e^{2x} - 2 + e^{-2x}] = \frac{1}{2} (e^x - e^{-x})(e^{2x} + e^{-2x} + 1) \\
&= \frac{1}{2} [e^{3x} + e^{-x} + e^x - e^x - e^{-3x} - e^{-x}] = \frac{e^{3x} - e^{-3x}}{2} = \sinh(3x)
\end{aligned}$$

$$13. \quad \sinh x = \frac{3}{2}$$

$$\cosh^2 x - \left(\frac{3}{2} \right)^2 = 1 \Rightarrow \cosh^2 x = \frac{13}{4} \Rightarrow \cosh x = \frac{\sqrt{13}}{2}$$

$$\tanh x = \frac{3/2}{\sqrt{13}/2} = \frac{3\sqrt{13}}{13}$$

$$\operatorname{csch} x = \frac{1}{3/2} = \frac{2}{3}$$

$$\operatorname{sech} x = \frac{1}{\sqrt{13}/2} = \frac{2\sqrt{13}}{13}$$

$$\coth x = \frac{1}{3/\sqrt{13}} = \frac{\sqrt{13}}{3}$$

$$15. y = \sinh(1 - x^2)$$

$$y' = -2x \cosh(1 - x^2)$$

$$17. f(x) = \ln(\sinh x)$$

$$f'(x) = \frac{1}{\sinh}(\cosh x) = \coth x$$

$$19. y = \ln\left(\tanh \frac{x}{2}\right)$$

$$\begin{aligned}
y' &= \frac{1/2}{\tanh(x/2)} \operatorname{sech}^2\left(\frac{x}{2}\right) = \frac{1}{2 \sinh(x/2) \cosh(x/2)} \\
&= \frac{1}{\sinh x} = \operatorname{csch} x
\end{aligned}$$

$$21. h(x) = \frac{1}{4} \sinh(2x) - \frac{x}{2}$$

$$h'(x) = \frac{1}{2} \cosh(2x) - \frac{1}{2} = \frac{\cosh(2x) - 1}{2} = \sinh^2 x$$

$$23. f(t) = \arctan(\sinh t)$$

$$\begin{aligned}
f'(t) &= \frac{1}{1 + \sinh^2 t} (\cosh t) \\
&= \frac{\cosh t}{\cosh^2 t} = \operatorname{sech} t
\end{aligned}$$

$$25. \text{ Let } y = g(x).$$

$$y = x^{\cosh x}$$

$$\ln y = \cosh x \ln x$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{\cosh x}{x} + \sinh x \ln x$$

$$\frac{dy}{dx} = \frac{y}{x} [\cosh x + x(\sinh x) \ln x]$$

$$= \frac{x^{\cosh x}}{x} [\cosh x + x(\sinh x) \ln x]$$

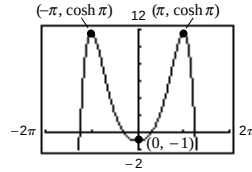
27. $y = (\cosh x - \sinh x)^2$

$$\begin{aligned} y' &= 2(\cosh x - \sinh x)(\sinh x - \cosh x) \\ &= -2(\cosh x - \sinh x)^2 = -2e^{-2x} \end{aligned}$$

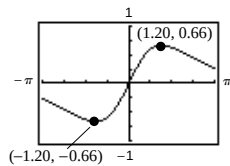
29. $f(x) = \sin x \sinh x - \cos x \cosh x, -4 \leq x \leq 4$

$$\begin{aligned} f'(x) &= \sin x \cosh x + \cos x \sinh x - \cos x \sinh x + \sin x \cosh x \\ &= 2 \sin x \cosh x = 0 \text{ when } x = 0, \pm\pi. \end{aligned}$$

 Relative maxima: $(\pm\pi, \cosh \pi)$

 Relative minimum: $(0, -1)$


31. $g(x) = x \operatorname{sech} x = \frac{x}{\cosh x}$


 Relative maximum: $(1.20, 0.66)$

 Relative minimum: $(-1.20, -0.66)$

33. $y = a \sinh x$

$y' = a \cosh x$

$y'' = a \sinh x$

$y''' = a \cosh x$

 Therefore, $y''' - y' = 0$.

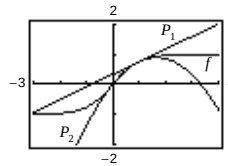
35. $f(x) = \tanh x$ $f(1) = \tanh(1) \approx 0.7616$

$f'(x) = \operatorname{sech}^2 x$ $f'(1) = \frac{1}{\cosh^2(1)} \approx 0.4200$

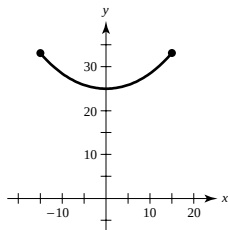
$f''(x) = -2 \operatorname{sech}^2 x \cdot \tanh x$ $f''(1) \approx -0.6397$

$P_1(x) = f(1) + f'(1)(x - 1) = 0.7616 + 0.42(x - 1)$

$P_2(x) = 0.7616 + 0.42(x - 1) - \frac{0.6397}{2}(x - 1)^2$



37. (a) $y = 10 + 15 \cosh \frac{x}{15}, -15 \leq x \leq 15$



(b) At $x = \pm 15, y = 10 + 15 \cosh(1) \approx 33.146$.

At $x = 0, y = 10 + 15 \cosh(0) = 25$.

(c) $y' = \sinh \frac{x}{15}$. At $x = 15, y' = \sinh(1) \approx 1.175$

39. Let $u = 1 - 2x, du = -2 dx$.

$$\begin{aligned} \int \sinh(1 - 2x) dx &= -\frac{1}{2} \int \sinh u du \\ &= -\frac{1}{2} \cosh u + C \\ &= -\frac{1}{2} \cosh(1 - 2x) + C \end{aligned}$$

41. Let $u = \cosh(x - 1), du = \sinh(x - 1) dx$.

$$\int \cosh^2(x - 1) \sinh(x - 1) dx = \frac{1}{3} \cosh^3(x - 1) + C$$

43. Let $u = \sinh x$, $du = \cosh x \, dx$.

$$\int \frac{\cosh x}{\sinh x} dx = \ln|\sinh x| + C$$

45. Let $u = \frac{x^2}{2}$, $du = x \, dx$.

$$\int x \operatorname{csch}^2 \frac{x^2}{2} dx = \int \left(\operatorname{csch}^2 \frac{x^2}{2} \right) x dx = -\coth \frac{x^2}{2} + C$$

47. Let $u = \frac{1}{x}$, $du = -\frac{1}{x^2} dx$.

$$\int \frac{\operatorname{csch}(1/x) \coth(1/x)}{x^2} dx = -\int \operatorname{csch} \frac{1}{x} \coth \frac{1}{x} \left(-\frac{1}{x^2} \right) dx = \operatorname{csch} \frac{1}{x} + C$$

49. $\int_0^4 \frac{1}{25 - x^2} dx = \left[\frac{1}{10} \ln \left| \frac{5+x}{5-x} \right| \right]_0^4 = \frac{1}{10} \ln 9 = \frac{1}{5} \ln 3$

51. Let $u = 2x$, $du = 2 \, dx$.

$$\int_0^{\sqrt{2}/4} \frac{1}{\sqrt{1 - (2x)^2}} (2) dx = \left[\arcsin(2x) \right]_0^{\sqrt{2}/4} = \frac{\pi}{4}$$

53. Let $u = x^2$, $du = 2x \, dx$.

$$\int \frac{x}{x^4 + 1} dx = \frac{1}{2} \int \frac{2x}{(x^2)^2 + 1} dx = \frac{1}{2} \arctan(x^2) + C$$

55. $y = \cosh^{-1}(3x)$

$$y' = \frac{3}{\sqrt{9x^2 - 1}}$$

57. $y = \sinh^{-1}(\tan x)$

$$y' = \frac{1}{\sqrt{\tan^2 x + 1}} (\sec^2 x) = |\sec x|$$

59. $y = \coth^{-1}(\sin 2x)$

$$y' = \frac{1}{1 - \sin^2 2x} (2 \cos 2x) = 2 \sec 2x$$

61. $y = 2x \sinh^{-1}(2x) - \sqrt{1 + 4x^2}$

$$y' = 2x \left(\frac{2}{\sqrt{1 + 4x^2}} \right) + 2 \sinh^{-1}(2x) - \frac{4x}{\sqrt{1 + 4x^2}} = 2 \sinh^{-1}(2x)$$

63. See page 395.

65. $y = a \operatorname{sech}^{-1} \left(\frac{x}{a} \right) - \sqrt{a^2 - x^2}$

$$\frac{dy}{dx} = \frac{-1}{(x/a)\sqrt{1 - (x^2/a^2)}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{-a^2}{x\sqrt{a^2 - x^2}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{x^2 - a^2}{x\sqrt{a^2 - x^2}} = \frac{-\sqrt{a^2 - x^2}}{x}$$

67. $\int \frac{1}{\sqrt{1 + e^{2x}}} dx = \int \frac{e^x}{e^x \sqrt{1 + (e^x)^2}} dx = -\operatorname{csch}^{-1}(e^x) + C = -\ln \left(\frac{1 + \sqrt{1 + e^{2x}}}{e^x} \right) + C$

69. Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$.

$$\int \frac{1}{\sqrt{x}\sqrt{1+x}} dx = 2 \int \frac{1}{\sqrt{1 + (\sqrt{x})^2}} \left(\frac{1}{2\sqrt{x}} \right) dx = 2 \sinh^{-1} \sqrt{x} + C = 2 \ln(\sqrt{x} + \sqrt{1+x}) + C$$

71. $\int \frac{-1}{4x - x^2} dx = \int \frac{1}{(x-2)^2 - 4} dx = \frac{1}{4} \ln \left| \frac{(x-2) - 2}{(x-2) + 2} \right| = \frac{1}{4} \ln \left| \frac{x-4}{x} \right| + C$

$$\begin{aligned}
 73. \int \frac{1}{1-4x-2x^2} dx &= \int \frac{1}{3-2(x+1)^2} dx = \frac{-1}{\sqrt{2}} \int \frac{\sqrt{2}}{[\sqrt{2}(x+1)]^2 - (\sqrt{3})^2} dx \\
 &= \frac{-1}{2\sqrt{6}} \ln \left| \frac{\sqrt{2}(x+1) - \sqrt{3}}{\sqrt{2}(x+1) + \sqrt{3}} \right| + C = \frac{1}{2\sqrt{6}} \ln \left| \frac{\sqrt{2}(x+1) + \sqrt{3}}{\sqrt{2}(x+1) - \sqrt{3}} \right| + C
 \end{aligned}$$

75. Let $u = 4x - 1$, $du = 4 dx$.

$$y = \int \frac{1}{\sqrt{80+8x-16x^2}} dx = \frac{1}{4} \int \frac{4}{\sqrt{81-(4x-1)^2}} dx = \frac{1}{4} \arcsin\left(\frac{4x-1}{9}\right) + C$$

$$\begin{aligned}
 77. y &= \int \frac{x^3 - 21x}{5 + 4x - x^2} dx = \int \left(-x - 4 + \frac{20}{5 + 4x - x^2}\right) dx = \int (-x - 4) dx + 20 \int \frac{1}{3^2 - (x-2)^2} dx \\
 &= -\frac{x^2}{2} - 4x + \frac{20}{6} \ln \left| \frac{(x-2) + 3}{(x-2) - 3} \right| + C = -\frac{x^2}{2} - 4x + \frac{10}{3} \ln \left| \frac{x+1}{x-5} \right| + C = -\frac{x^2}{2} - 4x - \frac{10}{3} \ln \left| \frac{x-5}{x+1} \right| + C
 \end{aligned}$$

$$\begin{aligned}
 79. A &= 2 \int_0^4 \operatorname{sech} \frac{x}{2} dx \\
 &= 2 \int_0^4 \frac{2}{e^{x/2} + e^{-x/2}} dx \\
 &= 4 \int_0^4 \frac{e^{x/2}}{(e^{x/2})^2 + 1} dx \\
 &= \left[8 \arctan(e^{x/2}) \right]_0^4 \\
 &= 8 \arctan(e^2) - 2\pi \approx 5.207
 \end{aligned}$$

$$\begin{aligned}
 81. A &= \int_0^2 \frac{5x}{\sqrt{x^4+1}} dx \\
 &= \frac{5}{2} \int_0^2 \frac{2x}{\sqrt{(x^2)^2+1}} dx \\
 &= \left[\frac{5}{2} \ln(x^2 + \sqrt{x^4+1}) \right]_0^2 \\
 &= \frac{5}{2} \ln(4 + \sqrt{17}) \approx 5.237
 \end{aligned}$$

$$\begin{aligned}
 83. \int \frac{3k}{16} dt &= \int \frac{1}{x^2 - 12x + 32} dx \\
 \frac{3kt}{16} &= \int \frac{1}{(x-6)^2 - 4} dx = \frac{1}{2(2)} \ln \left| \frac{(x-6) - 2}{(x-6) + 2} \right| + C = \frac{1}{4} \ln \left| \frac{x-8}{x-4} \right| + C
 \end{aligned}$$

When $x = 0$: $t = 0$

$$C = -\frac{1}{4} \ln(2)$$

When $x = 1$: $t = 10$

$$\frac{30k}{16} = \frac{1}{4} \ln \left| \frac{-7}{-3} \right| - \frac{1}{4} \ln(2) = \frac{1}{4} \ln\left(\frac{7}{6}\right)$$

$$k = \frac{2}{15} \ln\left(\frac{7}{6}\right)$$

When $t = 20$: $\left(\frac{3}{16}\right)\left(\frac{2}{15}\right) \ln\left(\frac{7}{6}\right)(20) = \frac{1}{4} \ln \frac{x-8}{2x-8}$

$$\ln\left(\frac{7}{6}\right)^2 = \ln \frac{x-8}{2x-8}$$

$$\frac{49}{36} = \frac{x-8}{2x-8}$$

$$62x = 104$$

$$x = \frac{104}{62} = \frac{52}{31} \approx 1.677 \text{ kg}$$

85. As k increases, the time required for the object to reach the ground increases.

$$87. y = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$y' = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$89. y = \cosh^{-1} x$$

$$\cosh y = x$$

$$(\sinh y)(y') = 1$$

$$y' = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

$$91. y = \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

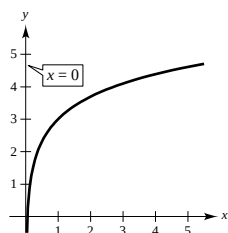
$$y' = -2(e^x + e^{-x})^{-2}(e^x - e^{-x}) = \left(\frac{-2}{e^x + e^{-x}}\right)\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = -\operatorname{sech} x \tanh x$$

Review Exercises for Chapter 5

$$1. f(x) = \ln x + 3$$

Vertical shift 3 units upward

Vertical asymptote: $x = 0$



$$3. \ln \sqrt[5]{\frac{4x^2 - 1}{4x^2 + 1}} = \frac{1}{5} \ln \frac{(2x - 1)(2x + 1)}{4x^2 + 1} = \frac{1}{5} [\ln(2x - 1) + \ln(2x + 1) - \ln(4x^2 + 1)]$$

$$5. \ln 3 + \frac{1}{3} \ln(4 - x^2) - \ln x = \ln 3 + \ln \sqrt[3]{4 - x^2} - \ln x = \ln \left(\frac{3\sqrt[3]{4 - x^2}}{x} \right)$$

$$7. \ln \sqrt{x + 1} = 2$$

$$\sqrt{x + 1} = e^2$$

$$x + 1 = e^4$$

$$x = e^4 - 1 \approx 53.598$$

$$9. g(x) = \ln \sqrt{x} = \frac{1}{2} \ln x$$

$$g'(x) = \frac{1}{2x}$$

$$11. f(x) = x\sqrt{\ln x}$$

$$f'(x) = \left(\frac{x}{2}\right)(\ln x)^{-1/2}\left(\frac{1}{x}\right) + \sqrt{\ln x}$$

$$= \frac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \frac{1 + 2\ln x}{2\sqrt{\ln x}}$$

$$13. y = \frac{1}{b^2} \left[\ln(a + bx) + \frac{a}{a + bx} \right]$$

$$\frac{dy}{dx} = \frac{1}{b^2} \left[\frac{b}{a + bx} - \frac{ab}{(a + bx)^2} \right] = \frac{x}{(a + bx)^2}$$

$$15. y = -\frac{1}{a} \ln \left(\frac{a + bx}{x} \right) = -\frac{1}{a} [\ln(a + bx) - \ln x]$$

$$\frac{dy}{dx} = -\frac{1}{a} \left(\frac{b}{a + bx} - \frac{1}{x} \right) = \frac{1}{x(a + bx)}$$

$$17. u = 7x - 2, du = 7dx$$

$$\int \frac{1}{7x - 2} dx = \frac{1}{7} \int \frac{1}{7x - 2} (7) dx = \frac{1}{7} \ln |7x - 2| + C$$

$$\begin{aligned}
 19. \int \frac{\sin x}{1 + \cos x} dx &= - \int \frac{-\sin x}{1 + \cos x} dx \\
 &= -\ln|1 + \cos x| + C
 \end{aligned}$$

$$23. \int_0^{\pi/3} \sec \theta d\theta = \left[\ln|\sec \theta + \tan \theta| \right]_0^{\pi/3} = \ln(2 + \sqrt{3})$$

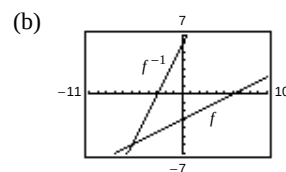
$$\begin{aligned}
 25. (a) \quad f(x) &= \frac{1}{2}x - 3 \\
 y &= \frac{1}{2}x - 3 \\
 2(y + 3) &= x \\
 2(x + 3) &= y \\
 f^{-1}(x) &= 2x + 6
 \end{aligned}$$

$$\begin{aligned}
 27. (a) \quad f(x) &= \sqrt{x+1} \\
 y &= \sqrt{x+1} \\
 y^2 - 1 &= x \\
 x^2 - 1 &= y \\
 f^{-1}(x) &= x^2 - 1, x \geq 0
 \end{aligned}$$

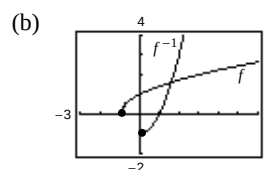
$$\begin{aligned}
 29. (a) \quad f(x) &= \sqrt[3]{x+1} \\
 y &= \sqrt[3]{x+1} \\
 y^3 - 1 &= x \\
 x^3 - 1 &= y \\
 f^{-1}(x) &= x^3 - 1
 \end{aligned}$$

$$\begin{aligned}
 31. \quad f(x) &= x^3 + 2 \\
 f^{-1}(x) &= (x - 2)^{1/3} \\
 (f^{-1})'(x) &= \frac{1}{3}(x - 2)^{-2/3} \\
 (f^{-1})'(-1) &= \frac{1}{3}(-1 - 2)^{-2/3} = \frac{1}{3(-3)^{2/3}} \\
 &= \frac{1}{3^{5/3}} \approx 0.160
 \end{aligned}$$

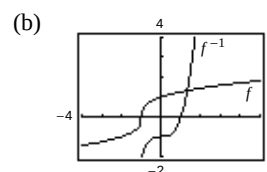
$$21. \int_1^4 \frac{x+1}{x} dx = \int_1^4 \left(1 + \frac{1}{x}\right) dx = \left[x + \ln|x| \right]_1^4 = 3 + \ln 4$$



$$\begin{aligned}
 (c) \quad f^{-1}(f(x)) &= f^{-1}\left(\frac{1}{2}x - 3\right) = 2\left(\frac{1}{2}x - 3\right) + 6 = x \\
 f(f^{-1}(x)) &= f(2x + 6) = \frac{1}{2}(2x + 6) - 3 = x
 \end{aligned}$$



$$\begin{aligned}
 (c) \quad f^{-1}(f(x)) &= f^{-1}(\sqrt{x+1}) = \sqrt{(x^2 - 1)^2} - 1 = x \\
 f(f^{-1}(x)) &= f(x^2 - 1) = \sqrt{(x^2 - 1) + 1} \\
 &= \sqrt{x^2} = x \text{ for } x \geq 0.
 \end{aligned}$$



$$\begin{aligned}
 (c) \quad f^{-1}(f(x)) &= f^{-1}(\sqrt[3]{x+1}) = (\sqrt[3]{x+1})^3 - 1 = x \\
 f(f^{-1}(x)) &= f(x^3 - 1) = \sqrt[3]{(x^3 - 1) + 1} = x
 \end{aligned}$$

$$\begin{aligned}
 33. \quad f(x) &= \tan x \\
 f\left(\frac{\pi}{6}\right) &= \frac{\sqrt{3}}{3} \\
 f'(x) &= \sec^2 x \\
 f'\left(\frac{\pi}{6}\right) &= \frac{4}{3} \\
 (f^{-1})'\left(\frac{\sqrt{3}}{3}\right) &= \frac{1}{f'(\pi/6)} = \frac{3}{4}
 \end{aligned}$$

35. (a) $f(x) = \ln \sqrt{x}$

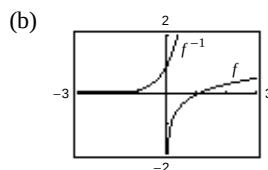
$$y = \ln \sqrt{x}$$

$$e^y = \sqrt{x}$$

$$e^{2y} = x$$

$$e^{2x} = y$$

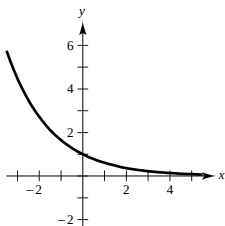
$$f^{-1}(x) = e^{2x}$$



(c) $f^{-1}(f(x)) = f^{-1}(\ln \sqrt{x}) = e^{2 \ln \sqrt{x}} = e^{\ln x} = x$

$$f(f^{-1}(x)) = f(e^{2x}) = \ln \sqrt{e^{2x}} = \ln e^x = x$$

37. $y = e^{-x/2}$



39. $f(x) = \ln(e^{-x^2}) = -x^2$

$$f'(x) = -2x$$

41. $g(t) = t^2 e^t$

$$g'(x) = t^2 e^t + 2te^t = te^t(t + 2)$$

43. $y = \sqrt{e^{2x} + e^{-2x}}$

$$y' = \frac{1}{2}(e^{2x} + e^{-2x})^{-1/2}(2e^{2x} - 2e^{-2x}) = \frac{e^{2x} - e^{-2x}}{\sqrt{e^{2x} + e^{-2x}}}$$

45. $g(x) = \frac{x^2}{e^x}$

$$g'(x) = \frac{e^x(2x) - x^2 e^x}{e^{2x}} = \frac{x(2 - x)}{e^x}$$

47. $y(\ln x) + y^2 = 0$

$$y\left(\frac{1}{x}\right) + (\ln x)\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 0$$

$$(2y + \ln x)\frac{dy}{dx} = \frac{-y}{x}$$

$$\frac{dy}{dx} = \frac{-y}{x(2y + \ln x)}$$

49. Let $u = -3x^2$, $du = -6x dx$.

$$\int x e^{-3x^2} dx = -\frac{1}{6} \int e^{-3x^2} (-6x) dx = -\frac{1}{6} e^{-3x^2} + C$$

51. $\int \frac{e^{4x} - e^{2x} + 1}{e^x} dx = \int (e^{3x} - e^x + e^{-x}) dx$

$$= \frac{1}{3} e^{3x} - e^x - e^{-x} + C$$

$$= \frac{e^{4x} - 3e^{2x} - 3}{3e^x} + C$$

53. $\int x e^{1-x^2} dx = -\frac{1}{2} \int e^{1-x^2} (-2x) dx$

$$= -\frac{1}{2} e^{1-x^2} + C$$

55. Let $u = e^x - 1$, $du = e^x dx$.

$$\int \frac{e^x}{e^x - 1} dx = \ln|e^x - 1| + C$$

57. $y = e^x(a \cos 3x + b \sin 3x)$

$$y' = e^x(-3a \sin 3x + 3b \cos 3x) + e^x(a \cos 3x + b \sin 3x)$$

$$= e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x]$$

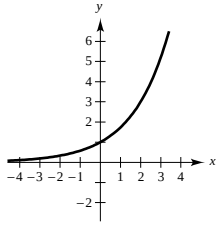
$$y'' = e^x[3(-3a + b) \cos 3x - 3(a + 3b) \sin 3x] + e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x]$$

$$= e^x[(-6a - 8b) \sin 3x + (-8a + 6b) \cos 3x]$$

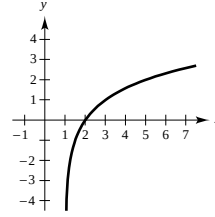
$$y'' - 2y' + 10y = e^x[(-6a - 8b) - 2(-3a + b) + 10b] \sin 3x + [(-8a + 6b) - 2(a + 3b) + 10a] \cos 3x = 0$$

$$59. \text{Area} = \int_0^4 x e^{-x^2} dx = \left[-\frac{1}{2} e^{-x^2} \right]_0^4 = -\frac{1}{2} (e^{-16} - 1) \approx 0.500$$

$$61. y = 3^{3/2}$$



$$63. y = \log_2(x - 1)$$



$$65. f(x) = 3^{x-1}$$

$$f'(x) = 3^{x-1} \ln 3$$

$$67. y = x^{2x+1}$$

$$\ln y = (2x + 1) \ln x$$

$$\frac{y'}{y} = \frac{2x + 1}{x} + 2 \ln x$$

$$y' = y \left(\frac{2x + 1}{x} + 2 \ln x \right) = x^{2x+1} \left(\frac{2x + 1}{x} + 2 \ln x \right)$$

$$69. g(x) = \log_3 \sqrt{1-x} = \frac{1}{2} \log_3(1-x)$$

$$g'(x) = \frac{1}{2} \frac{-1}{(1-x) \ln 3} = \frac{1}{2(x-1) \ln 3}$$

$$71. \int (x+1) 5^{(x+1)^2} dx = \frac{1}{2} \frac{1}{\ln 5} 5^{(x+1)^2} + C$$

$$73. (a) y = x^a$$

$$y' = ax^{a-1}$$

$$(b) y = a^x$$

$$y' = (\ln a) a^x$$

$$(c) y = x^x$$

$$\ln y = x \ln x$$

$$(d) y = a^a$$

$$y' = 0$$

$$\frac{1}{y} y' = x \cdot \frac{1}{x} + (1) \ln x$$

$$y' = y(1 + \ln x)$$

$$y' = x^x(1 + \ln x)$$

$$75. 10,000 = P e^{(0.07)(15)}$$

$$P = \frac{10,000}{e^{1.05}} \approx \$3499.38$$

$$77. P(h) = 30e^{kh}$$

$$P(18,000) = 30e^{18,000k} = 15$$

$$k = \frac{\ln(1/2)}{18,000} = \frac{-\ln 2}{18,000}$$

$$P(h) = 30e^{-(h \ln 2)/18,000}$$

$$P(35,000) = 30e^{-(35,000 \ln 2)/18,000} \approx 7.79 \text{ inches}$$

$$79. P = Ce^{0.015t}$$

$$2C = Ce^{0.015t}$$

$$2 = e^{0.015t}$$

$$\ln 2 = 0.015t$$

$$t = \frac{\ln 2}{0.015} \approx 46.21 \text{ years}$$

$$81. \frac{dy}{dx} = \frac{x^2 + 3}{x}$$

$$\int dy = \int \left(x + \frac{3}{x} \right) dx$$

$$y = \frac{x^2}{2} + 3 \ln|x| + C$$

83. $y' - 2xy = 0$

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{1}{y} dy = \int 2x dx$$

$$\ln|y| = x^2 + C_1$$

$$e^{x^2+C_1} = y$$

$$y = Ce^{x^2}$$

85. $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$ (homogeneous differential equation)

$$(x^2 + y^2) dx - 2xy dy = 0$$

Let $y = vx$, $dy = x dv + v dx$.

$$(x^2 + v^2x^2) dx - 2x(vx)(x dv + v dx) = 0$$

$$(x^2 + v^2x^2 - 2x^2v^2) dx - 2x^3v dv = 0$$

$$(x^2 - x^2v^2) dx = 2x^3v dv$$

$$(1 - v^2) dx = 2x dv$$

$$\int \frac{dx}{x} = \int \frac{2v}{1 - v^2} dv$$

$$\ln|x| = -\ln|1 - v^2| + C_1 = -\ln|1 - v^2| + \ln C$$

$$x = \frac{C}{1 - v^2} = \frac{C}{1 - (y/x)^2} = \frac{Cx^2}{x^2 - y^2}$$

$$1 = \frac{Cx}{x^2 - y^2} \quad \text{or} \quad C_1 = \frac{x}{x^2 - y^2}$$

87. $y = C_1x + C_2x^3$

$$y' = C_1 + 3C_2x^2$$

$$y'' = 6C_2x$$

$$\begin{aligned} x^2y'' - 3xy' + 3y &= x^2(6C_2x) - 3x(C_1 + 3C_2x^2) + (C_1x + C_2x^3) \\ &= 6C_2x^3 - 3C_1x - 9C_2x^3 + 3C_1x + 3C_2x^3 = 0 \end{aligned}$$

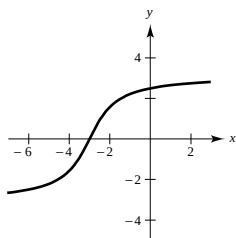
$$x = 2, y = 0: 0 = 2C_1 + 8C_2 \Rightarrow C_1 = -4C_2$$

$$x = 2, y' = 4: 4 = C_1 + 12C_2$$

$$4 = (-4C_2) + 12C_2 = 8C_2 \Rightarrow C_2 = \frac{1}{2}, C_1 = -2$$

$$y = -2x + \frac{1}{2}x^3$$

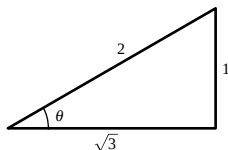
89. $f(x) = 2 \arctan(x + 3)$



91. (a) Let $\theta = \arcsin \frac{1}{2}$

$$\sin \theta = \frac{1}{2}$$

$$\sin\left(\arcsin \frac{1}{2}\right) = \sin \theta = \frac{1}{2}.$$



(b) Let $\theta = \arcsin \frac{1}{2}$

$$\sin \theta = \frac{1}{2}$$

$$\cos\left(\arcsin \frac{1}{2}\right) = \cos \theta = \frac{\sqrt{3}}{2}.$$

93. $y = \tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}$

$$y' = \frac{(1-x^2)^{1/2} + x^2(1-x^2)^{-1/2}}{1-x^2} = (1-x^2)^{-3/2}$$

95. $y = x \operatorname{arcsec} x$

$$y' = \frac{x}{|x|\sqrt{x^2-1}} + \operatorname{arcsec} x$$

97. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2} \arcsin x$

$$y' = \frac{2x \arcsin x}{\sqrt{1-x^2}} + (\arcsin x)^2 - 2 + \frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}} - \frac{2x}{\sqrt{1-x^2}} \arcsin x = (\arcsin x)^2$$

99. Let $u = e^{2x}$, $du = 2e^{2x} dx$.

$$\int \frac{1}{e^{2x} + e^{-2x}} dx = \int \frac{e^{2x}}{1 + e^{4x}} dx = \frac{1}{2} \int \frac{1}{1 + (e^{2x})^2} (2e^{2x}) dx = \frac{1}{2} \arctan(e^{2x}) + C$$

101. Let $u = x^2$, $du = 2x dx$.

$$\int \frac{x}{\sqrt{1-x^4}} dx = \frac{1}{2} \int \frac{1}{\sqrt{1-(x^2)^2}} (2x) dx = \frac{1}{2} \arcsin x^2 + C$$

103. Let $u = 16 + x^2$, $du = 2x dx$.

$$\int \frac{x}{16+x^2} dx = \frac{1}{2} \int \frac{1}{16+x^2} (2x) dx = \frac{1}{2} \ln(16+x^2) + C$$

105. Let $u = \arctan\left(\frac{x}{2}\right)$, $du = \frac{2}{4+x^2} dx$.

$$\int \frac{\arctan(x/2)}{4+x^2} dx = \frac{1}{2} \int \left(\arctan \frac{x}{2}\right) \left(\frac{2}{4+x^2}\right) dx = \frac{1}{4} \left(\arctan \frac{x}{2}\right)^2 + C$$

107. $\int \frac{dy}{\sqrt{A^2 - y^2}} = \int \sqrt{\frac{k}{m}} dt$
 $\arcsin\left(\frac{y}{A}\right) = \sqrt{\frac{k}{m}} t + C$

109. $y = 2x - \cosh \sqrt{x}$

$$y' = 2 - \frac{1}{2\sqrt{x}} (\sinh \sqrt{x}) = 2 - \frac{\sinh \sqrt{x}}{2\sqrt{x}}$$

Since $y = 0$ when $t = 0$, you have $C = 0$. Thus,

$$\sin\left(\sqrt{\frac{k}{m}} t\right) = \frac{y}{A}$$

$$y = A \sin\left(\sqrt{\frac{k}{m}} t\right)$$

111. Let $u = x^2$, $du = 2x dx$.

$$\int \frac{x}{\sqrt{x^4-1}} dx = \frac{1}{2} \int \frac{1}{\sqrt{(x^2)^2-1}} (2x) dx = \frac{1}{2} \ln(x^2 + \sqrt{x^4-1}) + C$$

CHAPTER 5

Logarithmic, Exponential, and Other Transcendental Functions

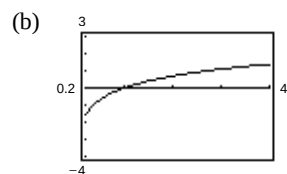
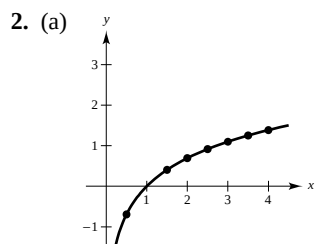
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CHAPTER 5

Logarithmic, Exponential, and Other Transcendental Functions

Section 5.1 The Natural Logarithmic Function: Differentiation

Solutions to Even-Numbered Exercises



The graphs are identical.

4. (a) $\ln 8.3 \approx 2.1163$

(b) $\int_1^{8.3} \frac{1}{t} dt \approx 2.1163$

6. (a) $\ln 0.6 \approx -0.5108$

(b) $\int_1^{0.6} \frac{1}{t} dt \approx -0.5108$

8. $f(x) = -\ln x$

Reflection in the x -axis

Matches (d)

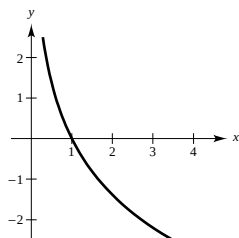
10. $f(x) = -\ln(-x)$

Reflection in the y -axis and the x -axis

Matches (c)

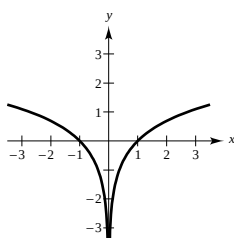
12. $f(x) = -2 \ln x$

Domain: $x > 0$



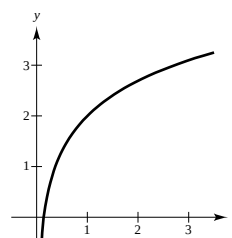
14. $f(x) = \ln|x|$

Domain: $x \neq 0$



16. $g(x) = 2 + \ln x$

Domain: $x > 0$



18. (a) $\ln 0.25 = \ln \frac{1}{4} = \ln 1 - 2 \ln 2 \approx -1.3862$

(b) $\ln 24 = 3 \ln 2 + \ln 3 \approx 3.1779$

(c) $\ln \sqrt[3]{12} = \frac{1}{3}(2 \ln 2 + \ln 3) \approx 0.8283$

(d) $\ln \frac{1}{72} = \ln 1 - (3 \ln 2 + 2 \ln 3) \approx -4.2765$

20. $\ln \sqrt{2^3} = \ln 2^{3/2} = \frac{3}{2} \ln 2$

22. $\ln xyz = \ln x + \ln y + \ln z$

24. $\ln \sqrt{a-1} = \ln(a-1)^{1/2} = \left(\frac{1}{2}\right) \ln(a-1)$

26. $\ln 3e^2 = \ln 3 + 2 \ln e = 2 + \ln 3$

28. $\ln \frac{1}{e} = \ln 1 - \ln e = -1$

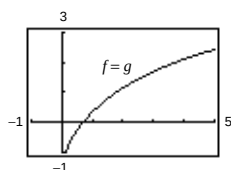
$$30. 3 \ln x + 2 \ln y - 4 \ln z = \ln x^3 + \ln y^2 - \ln z^4$$

$$= \ln \frac{x^3 y^2}{z^4}$$

$$32. 2[\ln x - \ln(x+1) - \ln(x-1)] = 2 \ln \frac{x}{(x+1)(x-1)} = \ln \left(\frac{x}{x^2-1} \right)^2$$

$$34. \frac{3}{2}[\ln(x^2+1) - \ln(x+1) - \ln(x-1)] = \frac{3}{2} \ln \frac{x^2+1}{(x+1)(x-1)} = \ln \sqrt{\left(\frac{x^2+1}{x^2-1} \right)^3}$$

36.



$$38. \lim_{x \rightarrow 6^-} \ln(6-x) = -\infty$$

$$40. \lim_{x \rightarrow 5^+} \ln \frac{x}{\sqrt{x-4}} = \ln 5 \approx 1.6094$$

$$42. y = \ln x^{3/2} = \frac{3}{2} \ln x$$

$$y' = \frac{3}{2x}$$

$$\text{At } (1, 0), y' = \frac{3}{2}.$$

$$44. y = \ln x^{1/2} = \frac{1}{2} \ln x$$

$$y' = \frac{1}{2x}$$

$$\text{At } (1, 0), y' = \frac{1}{2}.$$

$$46. h(x) = \ln(2x^2 + 1)$$

$$h'(x) = \frac{1}{2x^2+1}(4x) = \frac{4x}{2x^2+1}$$

$$48. y = x \ln x$$

$$\frac{dy}{dx} = x \left(\frac{1}{x} \right) + \ln x = 1 + \ln x$$

$$50. y = \ln \sqrt{x^2-4} = \frac{1}{2} \ln(x^2-4)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{2x}{x^2-4} \right) = \frac{x}{x^2-4}$$

$$52. f(x) = \ln \left(\frac{2x}{x+3} \right) = \ln 2x - \ln(x+3)$$

$$f'(x) = \frac{1}{x} - \frac{1}{x+3} = \frac{3}{x(x+3)}$$

$$54. h(t) = \frac{\ln t}{t}$$

$$h'(t) = \frac{t(1/t) - \ln t}{t^2} = \frac{1 - \ln t}{t^2}$$

$$56. y = \ln(\ln x)$$

$$\frac{dy}{dx} = \frac{1/x}{\ln x} = \frac{1}{x \ln x}$$

$$58. y = \ln \sqrt[3]{\frac{x-1}{x+1}} = \frac{1}{3} [\ln(x-1) - \ln(x+1)]$$

$$y' = \frac{1}{3} \left[\frac{1}{x-1} - \frac{1}{x+1} \right] = \frac{1}{3} \frac{2}{x^2-1} = \frac{2}{3(x^2-1)}$$

$$60. f(x) = \ln(x + \sqrt{4+x^2})$$

$$\begin{aligned} f'(x) &= \frac{1}{x + \sqrt{4+x^2}} \left(1 + \frac{x}{\sqrt{4+x^2}} \right) \\ &= \frac{1}{\sqrt{4+x^2}} \end{aligned}$$

$$62. \quad y = \frac{-\sqrt{x^2+4}}{2x^2} - \frac{1}{4} \ln\left(\frac{2+\sqrt{x^2+4}}{x}\right) = \frac{-\sqrt{x^2+4}}{2x^2} - \frac{1}{4} \ln(2+\sqrt{x^2+4}) + \frac{1}{4} \ln x$$

$$\frac{dy}{dx} = \frac{-2x^2(x/\sqrt{x^2+4}) + 4x\sqrt{x^2+4}}{4x^4} - \frac{1}{4} \left(\frac{1}{2+\sqrt{x^2+4}} \right) \left(\frac{x}{\sqrt{x^2+4}} \right) + \frac{1}{4x}$$

Note that:

$$\frac{1}{2+\sqrt{x^2+4}} = \frac{1}{2+\sqrt{x^2+4}} \cdot \frac{2-\sqrt{x^2+4}}{2-\sqrt{x^2+4}} = \frac{2-\sqrt{x^2+4}}{-x^2}$$

$$\begin{aligned} \text{Hence, } \frac{dy}{dx} &= \frac{-1}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} - \frac{1}{4} \left(\frac{2-\sqrt{x^2+4}}{-x^2} \right) \left(\frac{x}{\sqrt{x^2+4}} \right) + \frac{1}{4x} \\ &= \frac{-1 + (1/2)(2-\sqrt{x^2+4})}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x} \\ &= \frac{-\sqrt{x^2+4}}{4x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x} = \frac{\sqrt{x^2+4}}{x^3} \end{aligned}$$

$$64. \quad y = \ln|\csc x|$$

$$y' = \frac{-\csc x \cdot \cot x}{\csc x} = -\cot x$$

$$66. \quad y = \ln|\sec x + \tan x|$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \\ &= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x \end{aligned}$$

$$68. \quad y = \ln\sqrt{1+\sin^2 x} = \frac{1}{2} \ln(1+\sin^2 x)$$

$$\frac{dy}{dx} = \left(\frac{1}{2}\right) \frac{2 \sin x \cos x}{1+\sin^2 x} = \frac{\sin x \cos x}{1+\sin^2 x}$$

$$70. \quad g(x) = \int_1^{\ln x} (t^2 + 3) dt$$

$$g'(x) = [(\ln x)^2 + 3] \frac{d}{dx}(\ln x) = \frac{(\ln x)^2 + 3}{x}$$

(Second Fundamental Theorem of Calculus)

$$72. \quad (a) \quad y = 4 - x^2 - \ln\left(\frac{1}{2}x + 1\right), \quad (0, 4)$$

$$\begin{aligned} \frac{dy}{dx} &= -2x - \frac{1}{(1/2)x + 1} \left(\frac{1}{2}\right) \\ &= -2x - \frac{1}{x+2} \end{aligned}$$

$$\text{When } x = 0, \frac{dy}{dx} = -\frac{1}{2}.$$

$$\text{Tangent line: } y - 4 = -\frac{1}{2}(x - 0)$$

$$y = -\frac{1}{2}x + 4$$

$$74. \quad \ln(xy) + 5x = 30$$

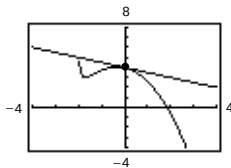
$$\ln x + \ln y + 5x = 30$$

$$\frac{1}{x} + \frac{1}{y} \frac{dy}{dx} + 5 = 0$$

$$\frac{1}{y} \frac{dy}{dx} = -\frac{1}{x} - 5$$

$$\frac{dy}{dx} = -\frac{y}{x} - 5y = -\left(\frac{y+5xy}{x}\right)$$

(b)



76. $y = x(\ln x) - 4x$

$$y' = x\left(\frac{1}{x}\right) + \ln x - 4 = -3 + \ln x$$

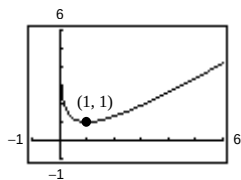
$$(x + y) - xy' = x + x \ln x - 4x - x(-3 + \ln x) = 0$$

78. $y = x - \ln x$

Domain: $x > 0$

$$y' = 1 - \frac{1}{x} = 0 \text{ when } x = 1.$$

$$y'' = \frac{1}{x^2} > 0$$

 Relative minimum: $(1, 1)$


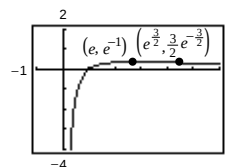
80. $y = \frac{\ln x}{x}$

Domain: $x > 0$

$$y' = \frac{x(1/x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2} = 0 \text{ when } x = e.$$

$$y'' = \frac{x^2(-1/x) - (1 - \ln x)(2x)}{x^4} = \frac{2(\ln x) - 3}{x^3} = 0 \text{ when } x = e^{3/2}.$$

 Relative maximum: (e, e^{-1})

 Point of inflection: $(e^{3/2}, \frac{3}{2}e^{-3/2})$


82. $y = x^2 \ln \frac{x}{4}$. Domain $x > 0$

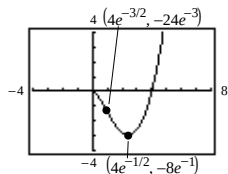
$$y' = x^2\left(\frac{1}{x}\right) + 2x \ln \frac{x}{4} = x\left(1 + 2 \ln \frac{x}{4}\right) = 0 \text{ when}$$

$$-1 = 2 \ln \frac{x}{4} \Rightarrow \ln \frac{x}{4} = -\frac{1}{2} \Rightarrow x = 4e^{-1/2}$$

$$y'' = 1 + 2 \ln \frac{x}{4} + 2x\left(\frac{1}{x}\right) = 3 + 2 \ln \frac{x}{4}$$

$$y'' = 0 \text{ when } x = 4e^{-3/2}$$

 Relative minimum: $(4e^{-1/2}, -8e^{-1})$

 Point of inflection: $(4e^{-3/2}, -24e^{-3})$


84. $f(x) = x \ln x$, $f(1) = 0$

$$f'(x) = 1 + \ln x$$
, $f'(1) = 1$

$$f''(x) = \frac{1}{x}$$
, $f''(1) = 1$

$$P_1(x) = f(1) + f'(1)(x - 1) = x - 1$$
, $P_1(1) = 0$

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2$$

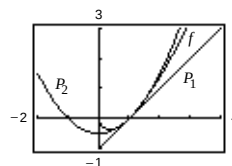
$$= (x - 1) + \frac{1}{2}(x - 1)^2$$
, $P_2(1) = 0$

$$P_1'(x) = 1$$
, $P_1'(1) = 1$

$$P_2'(x) = 1 + (x - 1) = x$$
, $P_2'(1) = 1$

$$P_2''(x) = 1$$
, $P_2''(1) = 1$

The values of f , P_1 , P_2 , and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



86. Find
- x
- such that
- $\ln x = 3 - x$
- .

$$f(x) = x + (\ln x) - 3 = 0$$

$$f'(x) = 1 + \frac{1}{x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n \left[\frac{4 - \ln x_n}{1 + x_n} \right]$$

n	1	2	3
x_n	2	2.2046	2.2079
$f(x_n)$	-0.3069	-0.0049	0.0000

Approximate root: $x = 2.208$

$$90. \quad y = \sqrt{\frac{x^2 - 1}{x^2 + 1}}$$

$$\ln y = \frac{1}{2} [\ln(x^2 - 1) - \ln(x^2 + 1)]$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \left[\frac{2x}{x^2 - 1} - \frac{2x}{x^2 + 1} \right]$$

$$\begin{aligned} \frac{dy}{dx} &= \sqrt{\frac{x^2 - 1}{x^2 + 1}} \left[\frac{2x}{x^4 - 1} \right] \\ &= \frac{(x^2 - 1)^{1/2} 2x}{(x^2 + 1)^{1/2} (x^2 - 1)(x^2 + 1)} \\ &= \frac{2x}{(x^2 + 1)^{3/2} (x^2 - 1)^{1/2}} \end{aligned}$$

94. The base of the natural logarithmic function is
- e
- .

$$96. \quad g(x) = \ln f(x), f(x) > 0$$

$$g'(x) = \frac{f'(x)}{f(x)}$$

- (a) Yes. If the graph of g is increasing, then $g'(x) > 0$. Since $f(x) > 0$, you know that $f'(x) = g'(x)f(x)$ and thus, $f'(x) > 0$. Therefore, the graph of f is increasing.

$$88. \quad y = \sqrt{(x-1)(x-2)(x-3)}$$

$$\ln y = \frac{1}{2} [\ln(x-1) + \ln(x-2) + \ln(x-3)]$$

$$\begin{aligned} \frac{1}{y} \left(\frac{dy}{dx} \right) &= \frac{1}{2} \left[\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3} \right] \\ &= \frac{1}{2} \left[\frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)} \right] \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{3x^2 - 12x + 11}{2y} \\ &= \frac{3x^2 - 12x + 11}{2\sqrt{(x-1)(x-2)(x-3)}} \end{aligned}$$

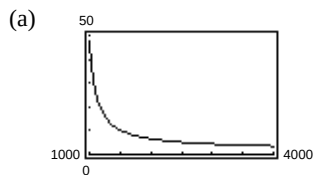
$$92. \quad y = \frac{(x+1)(x+2)}{(x-1)(x-2)}$$

$$\ln y = \ln(x+1) + \ln(x+2) - \ln(x-1) - \ln(x-2)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x-1} - \frac{1}{x-2}$$

$$\begin{aligned} \frac{dy}{dx} &= y \left[\frac{-2}{x^2 - 1} + \frac{-4}{x^2 - 4} \right] = y \left[\frac{-6x^2 + 12}{(x^2 - 1)(x^2 - 4)} \right] \\ &= \frac{(x+1)(x+2)}{(x-1)(x-2)} \cdot \frac{-6(x^2 - 2)}{(x+1)(x-1)(x+2)(x-2)} \\ &= -\frac{6(x^2 - 2)}{(x-1)^2(x-2)^2} \end{aligned}$$

$$98. \quad t = \frac{5.315}{-6.7968 + \ln x}, 1000 < x$$



(b) $t(1167.41) \approx 20$ years

$$T = (1167.41)(20)(12) = \$280,178.40$$

(c) $t(1068.45) \approx 30$ years

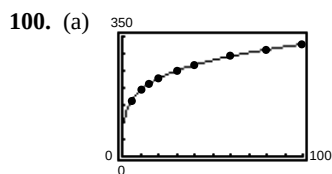
$$T = (1068.45)(30)(12) = \$384,642.00$$

$$\begin{aligned} \text{(d)} \quad \frac{dt}{dx} &= -5.315(-6.7968 + \ln x)^{-2} \left(\frac{1}{x} \right) \\ &= -\frac{5.315}{x(-6.7968 + \ln x)^2} \end{aligned}$$

When $x = 1167.41$, $dt/dx \approx -0.0645$. When $x = 1068.45$, $dt/dx \approx -0.1585$.

- (e) There are two obvious benefits to paying a higher monthly payment:

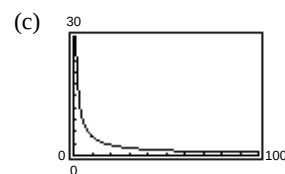
1. The term is lower
2. The total amount paid is lower.



(b) $T'(p) = \frac{34.96}{p} + \frac{3.955}{\sqrt{p}}$

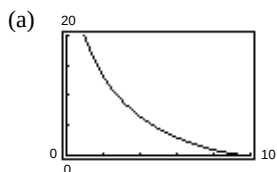
$T'(10) \approx 4.75 \text{ deg/lb/in}^2$

$T'(70) \approx 0.97 \text{ deg/lb/in}^2$



$\lim_{p \rightarrow \infty} T'(p) = 0$

102. $y = 10 \ln \left(\frac{10 + \sqrt{100 - x^2}}{x} \right) - \sqrt{100 - x^2} = 10 [\ln(10 + \sqrt{100 - x^2}) - \ln x] - \sqrt{100 - x^2}$



(c) $\lim_{x \rightarrow 10^-} \frac{dy}{dx} = 0$

(b)
$$\begin{aligned} \frac{dy}{dx} &= 10 \left[\frac{-x}{\sqrt{100 - x^2}(10 + \sqrt{100 - x^2})} - \frac{1}{x} \right] + \frac{x}{\sqrt{100 - x^2}} \\ &= \frac{x}{\sqrt{100 - x^2}} \left[\frac{-10}{10 + \sqrt{100 - x^2}} \right] - \frac{10}{x} + \frac{x}{\sqrt{100 - x^2}} \\ &= \frac{x}{\sqrt{100 - x^2}} \left[\frac{-10}{10 + \sqrt{100 - x^2}} + 1 \right] - \frac{10}{x} \\ &= \frac{x}{\sqrt{100 - x^2}} \left[\frac{\sqrt{100 - x^2}}{10 + \sqrt{100 - x^2}} \right] - \frac{10}{x} \\ &= \frac{x}{10 + \sqrt{100 - x^2}} - \frac{10}{x} \\ &= \frac{x(10 - \sqrt{100 - x^2})}{x^2} - \frac{10}{x} = -\frac{\sqrt{100 - x^2}}{x} \end{aligned}$$

When $x = 5$, $dy/dx = -\sqrt{3}$. When $x = 9$, $dy/dx = -\sqrt{19}/9$.

104. $y = \ln x$

$y' = \frac{1}{x} > 0$ for $x > 0$.

Since $\ln x$ is increasing on its entire domain $(0, \infty)$, it is a strictly monotonic function and therefore, is one-to-one.

106. False

π is a constant.

$\frac{d}{dx} [\ln \pi] = 0$

Section 5.2 The Natural Logarithmic Function: Integration

2. $\int \frac{10}{x} dx = 10 \int \frac{1}{x} dx = 10 \ln|x| + C$

4. $u = x - 5$, $du = dx$

$\int \frac{1}{x - 5} dx = \ln|x - 5| + C$

6. $\int \frac{1}{3x + 2} dx = \frac{1}{3} \int \frac{1}{3x + 2} (3) dx$
 $= \frac{1}{3} \ln|3x + 2| + C$

8. $u = 3 - x^3$, $du = -3x^2 dx$

$\int \frac{x^2}{3 - x^3} dx = -\frac{1}{3} \int \frac{1}{3 - x^3} (-3x^2) dx$
 $= -\frac{1}{3} \ln|3 - x^3| + C$

10. $u = 9 - x^2, du = -2x dx$

$$\int \frac{x}{\sqrt{9 - x^2}} dx = -\frac{1}{2} \int (9 - x^2)^{-1/2} (-2x) dx = -\sqrt{9 - x^2} + C$$

12. $\int \frac{x(x+2)}{x^3 + 3x^2 - 4} dx = \frac{1}{3} \int \frac{3x^2 + 6x}{x^3 + 3x^2 - 4} dx \quad (u = x^3 + 3x^2 - 4)$

$$= \frac{1}{3} \ln|x^3 + 3x^2 - 4| + C$$

14. $\int \frac{2x^2 + 7x - 3}{x - 2} dx = \int \left(2x + 11 + \frac{19}{x - 2} \right) dx$

$$= x^2 + 11x + 19 \ln|x - 2| + C$$

16. $\int \frac{x^3 - 6x - 20}{x + 5} dx = \int \left(x^2 - 5x + 19 - \frac{115}{x + 5} \right) dx$

$$= \frac{x^3}{3} - \frac{5x^2}{2} + 19x - 115 \ln|x + 5| + C$$

18. $\int \frac{x^3 - 3x^2 + 4x - 9}{x^2 + 3} dx = \int \left(-3 + x + \frac{x}{x^2 + 3} \right) dx$

$$= -3x + \frac{x^2}{2} + \frac{1}{2} \ln(x^2 + 3) + C$$

20. $\int \frac{1}{x \ln(x^3)} dx = \frac{1}{3} \int \frac{1}{\ln x} \cdot \frac{1}{x} dx$

$$= \frac{1}{3} \ln|\ln|x|| + C$$

22. $u = 1 + x^{1/3}, du = \frac{1}{3x^{2/3}} dx$

$$\int \frac{1}{x^{2/3}(1 + x^{1/3})} dx = 3 \int \frac{1}{1 + x^{1/3}} \left(\frac{1}{3x^{2/3}} \right) dx$$

$$= 3 \ln|1 + x^{1/3}| + C$$

24. $\int \frac{x(x-2)}{(x-1)^3} dx = \int \frac{x^2 - 2x + 1 - 1}{(x-1)^3} dx$

$$= \int \frac{(x-1)^2}{(x-1)^3} dx - \int \frac{1}{(x-1)^3} dx$$

$$= \int \frac{1}{x-1} dx - \int \frac{1}{(x-1)^3} dx$$

$$= \ln|x-1| + \frac{1}{2(x-1)^2} + C$$

26. $u = 1 + \sqrt{3x}, du = \frac{3}{2\sqrt{3x}} dx \Rightarrow dx = \frac{2}{3}(u-1) du$

$$\int \frac{1}{1 + \sqrt{3x}} dx = \int \frac{1}{u} \frac{2}{3}(u-1) du$$

$$= \frac{2}{3} \int \left(1 - \frac{1}{u} \right) du$$

$$= \frac{2}{3} [u - \ln|u|] + C$$

$$= \frac{2}{3} [1 + \sqrt{3x} - \ln(1 + \sqrt{3x})] + C$$

$$= \frac{2}{3} \sqrt{3x} - \frac{2}{3} \ln(1 + \sqrt{3x}) + C_1$$

$$28. u = x^{1/3} - 1, du = \frac{1}{3x^{2/3}} dx \Rightarrow dx = 3(u + 1)^2 du$$

$$\begin{aligned} \int \frac{\sqrt[3]{x}}{\sqrt[3]{x} - 1} dx &= \int \frac{u + 1}{u} 3(u + 1)^2 du \\ &= 3 \int \frac{u + 1}{u} (u^2 + 2u + 1) du \\ &= 3 \int \left(u^2 + 3u + 3 + \frac{1}{u} \right) du \\ &= 3 \left[\frac{u^3}{3} + \frac{3u^2}{2} + 3u + \ln|u| \right] + C \\ &= 3 \left[\frac{(x^{1/3} - 1)^3}{3} + \frac{3(x^{1/3} - 1)^2}{2} + 3(x^{1/3} - 1) + \ln|x^{1/3} - 1| \right] + C \\ &= 3 \ln|x^{1/3} - 1| + \frac{3x^{2/3}}{2} + 3x^{1/3} + x + C_1 \end{aligned}$$

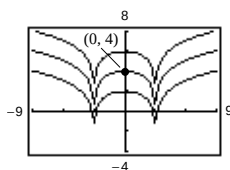
$$\begin{aligned} 30. \int \tan 5\theta d\theta &= \frac{1}{5} \int \frac{5 \sin 5\theta}{\cos 5\theta} d\theta \\ &= -\frac{1}{5} \ln|\cos 5\theta| + C \end{aligned}$$

$$\begin{aligned} 32. \int \sec \frac{x}{2} dx &= 2 \int \sec \frac{x}{2} \left(\frac{1}{2} \right) dx \\ &= 2 \ln \left| \sec \frac{x}{2} + \tan \frac{x}{2} \right| + C \end{aligned}$$

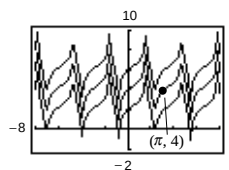
$$\begin{aligned} 34. u = \cot t, du &= -\csc^2 t dt \\ \int \frac{\csc^2 t}{\cot t} dt &= -\ln|\cot t| + C \end{aligned}$$

$$\begin{aligned} 36. \int (\sec t + \tan t) dt &= \ln|\sec t + \tan t| - \ln|\cos t| + C \\ &= \ln \left| \frac{\sec t + \tan t}{\cos t} \right| + C \\ &= \ln|\sec t(\sec t + \tan t)| + C \end{aligned}$$

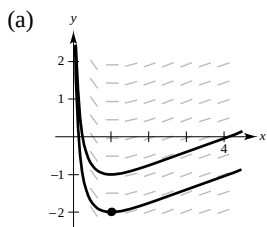
$$\begin{aligned} 38. y &= \int \frac{2x}{x^2 - 9} dx \\ &= \ln|x^2 - 9| + C \\ (0, 4): 4 &= \ln|0 - 9| + C \Rightarrow C = 4 - \ln 9 \\ y &= \ln|x^2 - 9| + 4 - \ln 9 \end{aligned}$$



$$\begin{aligned} 40. r &= \int \frac{\sec^2 t}{\tan t + 1} dt \\ &= \ln|\tan t + 1| + C \\ (\pi, 4): 4 &= \ln|0 + 1| + C \Rightarrow C = 4 \\ r &= \ln|\tan t + 1| + 4 \end{aligned}$$



$$42. \frac{dy}{dx} = \frac{\ln x}{x}, (1, -2)$$



$$\begin{aligned} (b) \quad y &= \int \frac{\ln x}{x} dx = \frac{(\ln x)^2}{2} + C \\ y(1) &= -2 \Rightarrow -2 = \frac{(\ln 1)^2}{2} + C \Rightarrow C = -2 \\ \text{Hence, } y &= \frac{(\ln x)^2}{2} - 2. \end{aligned}$$

$$44. \int_{-1}^1 \frac{1}{x+2} dx = \left[\ln|x+2| \right]_{-1}^1 \\ = \ln 3 - \ln 1 = \ln 3$$

$$46. u = \ln x, du = \frac{1}{x} dx \\ \int_e^{e^2} \frac{1}{x \ln x} dx = \int_e^{e^2} \left(\frac{1}{\ln x} \right) \frac{1}{x} dx = \left[\ln|\ln|x|| \right]_e^{e^2} = \ln 2$$

$$48. \int_0^1 \frac{x-1}{x+1} dx = \int_0^1 1 dx + \int_0^1 \frac{-2}{x+1} dx \\ = \left[x - 2 \ln|x+1| \right]_0^1 = 1 - 2 \ln 2$$

$$50. \int_{0.1}^{0.2} (\csc 2\theta - \cot 2\theta)^2 d\theta = \int_{0.1}^{0.2} (\csc^2 2\theta - 2 \csc 2\theta \cot 2\theta + \cot^2 2\theta) d\theta \\ = \int_{0.1}^{0.2} (2 \csc^2 2\theta - 2 \csc 2\theta \cot 2\theta - 1) d\theta \\ = \left[-\cot 2\theta + \csc 2\theta - \theta \right]_{0.1}^{0.2} \approx 0.0024$$

$$52. \ln|\sin x| + C = \ln \left| \frac{1}{\csc x} \right| + C = -\ln|\csc x| + C$$

$$54. -\ln|\csc x + \cot x| + C = -\ln \left| \frac{(\csc x + \cot x)(\csc x - \cot x)}{(\csc x - \cot x)} \right| + C = -\ln \left| \frac{\csc^2 x - \cot^2 x}{\csc x - \cot x} \right| + C \\ = -\ln \left| \frac{1}{\csc x - \cot x} \right| + C = \ln|\csc x - \cot x| + C$$

$$56. \int \frac{1-\sqrt{x}}{1+\sqrt{x}} dx = -(1+\sqrt{x})^2 + 6(1+\sqrt{x}) - 4 \ln(1+\sqrt{x}) + C_1 \\ = 4\sqrt{x} - x - 4 \ln(1+\sqrt{x}) + C \text{ where } C = C_1 + 5.$$

$$58. \int \frac{\tan^2 2x}{\sec 2x} dx = \frac{1}{2} [\ln|\sec 2x + \tan 2x| - \sin 2x] + C$$

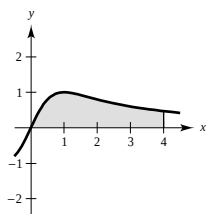
$$60. \int_{-\pi/4}^{\pi/4} \frac{\sin^2 x - \cos^2 x}{\cos x} dx = \left[\ln|\sec x + \tan x| - 2 \sin x \right]_{-\pi/4}^{\pi/4} \\ = \ln \left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right) - 2\sqrt{2} \approx -1.066$$

Note: In Exercises 62 and 64, you can use the Second Fundamental Theorem of Calculus or integrate the function.

$$62. F(x) = \int_0^x \tan t dt \\ F'(x) = \tan x$$

$$64. F(x) = \int_1^{x^2} \frac{1}{t} dt \\ F'(x) = \frac{2x}{x^2} = \frac{2}{x}$$

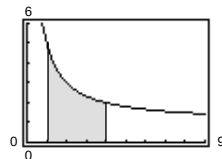
66.



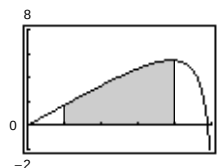
$$A \approx 3$$

Matches (a)

$$\begin{aligned} 68. A &= \int_1^4 \frac{x+4}{x} dx = \int_1^4 \left(1 + \frac{4}{x}\right) dx \\ &= \left[x + 4 \ln x \right]_1^4 \\ &= 4 + 4 \ln 4 - 1 \\ &= 3 + 4 \ln 4 \approx 8.5452 \end{aligned}$$



$$\begin{aligned} 70. \int_1^4 (2x - \tan(0.3x)) dx &= \left[x^2 + \frac{10}{3} \ln |\cos(0.3x)| \right]_1^4 \\ &= \left[16 + \frac{10}{3} \ln \cos(1.2) \right] - \left[1 + \frac{10}{3} \ln \cos(0.3) \right] \approx 11.7686 \end{aligned}$$

72. Substitution: ($u = x^2 + 4$) and Power Rule74. Substitution: ($u = \tan x$) and Log Rule

76. Answers will vary.

$$\begin{aligned} 78. \text{Average value} &= \frac{1}{4-2} \int_2^4 \frac{4(x+1)}{x^2} dx \\ &= 2 \int_2^4 \left(\frac{1}{x} + \frac{1}{x^2} \right) dx \\ &= 2 \left[\ln x - \frac{1}{x} \right]_2^4 \\ &= 2 \left[\ln 4 - \frac{1}{4} - \ln 2 + \frac{1}{2} \right] \\ &= 2 \left[\ln 2 + \frac{1}{4} \right] = \ln 4 + \frac{1}{2} \approx 1.8863 \end{aligned}$$

$$\begin{aligned} 80. \text{Average value} &= \frac{1}{2-0} \int_0^2 \sec \frac{\pi x}{6} dx \\ &= \left[\frac{1}{2} \left(\frac{6}{\pi} \right) \ln \left| \sec \frac{\pi x}{6} + \tan \frac{\pi x}{6} \right| \right]_0^2 \\ &= \frac{3}{\pi} [\ln(2 + \sqrt{3}) - \ln(1 + 0)] \\ &= \frac{3}{\pi} \ln(2 + \sqrt{3}) \end{aligned}$$

$$\begin{aligned} 82. t &= \frac{10}{\ln 2} \int_{250}^{300} \frac{1}{T-100} dT \\ &= \frac{10}{\ln 2} [\ln(T-100)]_{250}^{300} = \frac{10}{\ln 2} [\ln 200 - \ln 150] = \frac{10}{\ln 2} \left[\ln \left(\frac{4}{3} \right) \right] \approx 4.1504 \text{ units of time} \end{aligned}$$

$$84. \frac{dS}{dt} = \frac{k}{t}$$

$$S(t) = \int \frac{k}{t} dt = k \ln |t| + C = k \ln t + C \text{ since } t > 1.$$

$$S(2) = k \ln 2 + C = 200$$

$$S(4) = k \ln 4 + C = 300$$

Solving this system yields $k = 100/\ln 2$ and $C = 100$. Thus,

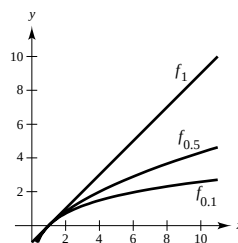
$$S(t) = \frac{100 \ln t}{\ln 2} + 100 = 100 \left[\frac{\ln t}{\ln 2} + 1 \right].$$

86. $k = 1$: $f_1(x) = x - 1$

$$k = 0.5: f_{0.5}(x) = \frac{\sqrt{x} - 1}{0.5} = 2(\sqrt{x} - 1)$$

$$k = 0.1: f_{0.1}(x) = \frac{\sqrt[10]{x} - 1}{0.1} = 10(\sqrt[10]{x} - 1)$$

$$\lim_{k \rightarrow 0^+} f_k(x) = \ln x$$



88. False

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

90. False; the integrand has a nonremovable discontinuity at $x = 0$.

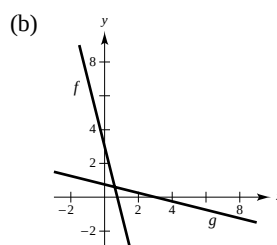
Section 5.3 Inverse Functions

2. (a) $f(x) = 3 - 4x$

$$g(x) = \frac{3 - x}{4}$$

$$f(g(x)) = f\left(\frac{3 - x}{4}\right) = 3 - 4\left(\frac{3 - x}{4}\right) = x$$

$$g(f(x)) = g(3 - 4x) = \frac{3 - (3 - 4x)}{4} = x$$

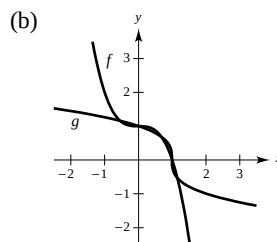


4. (a) $f(x) = 1 - x^3$

$$g(x) = \sqrt[3]{1 - x}$$

$$\begin{aligned} f(g(x)) &= f(\sqrt[3]{1 - x}) = 1 - (\sqrt[3]{1 - x})^3 \\ &= 1 - (1 - x) = x \end{aligned}$$

$$\begin{aligned} g(f(x)) &= g(1 - x^3) \\ &= \sqrt[3]{1 - (1 - x^3)} = \sqrt[3]{x^3} = x \end{aligned}$$

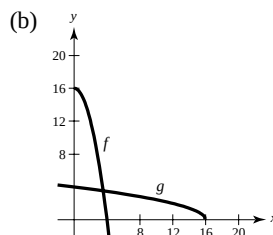


6. (a) $f(x) = 16 - x^2$, $x \geq 0$

$$g(x) = \sqrt{16 - x}$$

$$\begin{aligned} f(g(x)) &= f(\sqrt{16 - x}) = 16 - (\sqrt{16 - x})^2 \\ &= 16 - (16 - x) = x \end{aligned}$$

$$\begin{aligned} g(f(x)) &= g(16 - x^2) = \sqrt{16 - (16 - x^2)} \\ &= \sqrt{x^2} = x \end{aligned}$$

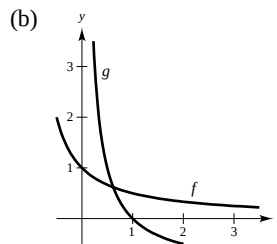


8. (a) $f(x) = \frac{1}{1 + x}$, $x \geq 0$

$$g(x) = \frac{1 - x}{x}$$
, $0 < x \leq 1$

$$f(g(x)) = f\left(\frac{1 - x}{x}\right) = \frac{1}{1 + \frac{1 - x}{x}} = \frac{1}{\frac{1 + x}{x}} = \frac{x}{1 + x} = x$$

$$g(f(x)) = g\left(\frac{1}{1 + x}\right) = \frac{1 - \frac{1}{1 + x}}{\frac{1}{1 + x}} = \frac{\frac{1 + x - 1}{1 + x}}{\frac{1}{1 + x}} = \frac{x}{1 + x} \cdot \frac{1 + x}{1} = x$$

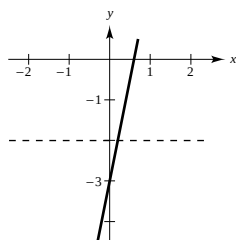


10. Matches (b)

12. Matches (d)

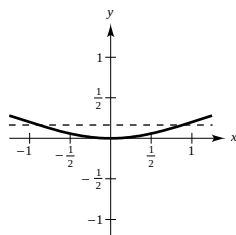
14. $f(x) = 5x - 3$

One-to-one; has an inverse



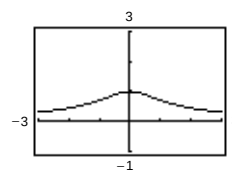
16. $F(x) = \frac{x^2}{x^2 + 4}$

Not one-to-one; does not have an inverse



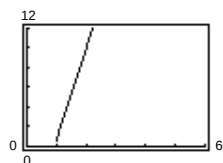
18. $g(t) = \frac{1}{\sqrt{t^2 + 1}}$

Not one-to-one; does not have an inverse



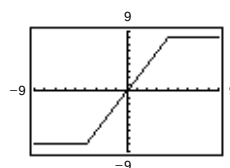
20. $f(x) = 5x\sqrt{x-1}$

One-to-one; has an inverse



22. $h(x) = |x + 4| - |x - 4|$

Not one-to-one; does not have an inverse



24. $f(x) = \cos \frac{3x}{2}$

$$f'(x) = -\frac{3}{2} \sin \frac{3x}{2} = 0 \text{ when } x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \dots$$

 f is not strictly monotonic on $(-\infty, \infty)$. Therefore, f does not have an inverse.

26. $f(x) = x^3 - 6x^2 + 12x$

$$f'(x) = 3x^2 - 12x + 12 = 3(x - 2)^2 \geq 0 \text{ for all } x.$$

 f is increasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

28. $f(x) = \ln(x - 3), x > 3$

$$f'(x) = \frac{1}{x - 3} > 0 \text{ for } x > 3.$$

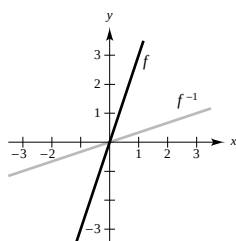
 f is increasing on $(3, \infty)$. Therefore, f is strictly monotonic and has an inverse.

30. $f(x) = 3x = y$

$$x = \frac{y}{3}$$

$$y = \frac{x}{3}$$

$$f^{-1}(x) = \frac{x}{3}$$

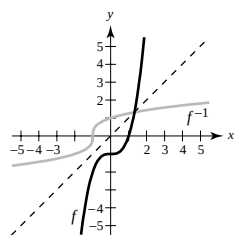


32. $f(x) = x^3 - 1 = y$

$$x = \sqrt[3]{y + 1}$$

$$y = \sqrt[3]{x + 1}$$

$$f^{-1}(x) = \sqrt[3]{x + 1}$$

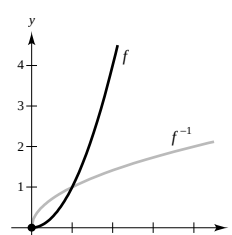


34. $f(x) = x^2 = y, 0 \leq x$

$$x = \sqrt{y}$$

$$y = \sqrt{x}$$

$$f^{-1}(x) = \sqrt{x}$$

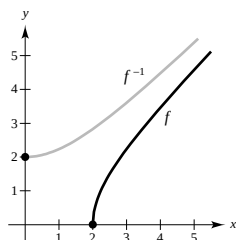


36. $f(x) = \sqrt{x^2 - 4} = y, x \geq 2$

$$x = \sqrt{y^2 + 4}$$

$$y = \sqrt{x^2 - 4}$$

$$f^{-1}(x) = \sqrt{x^2 + 4}, x \geq 0$$

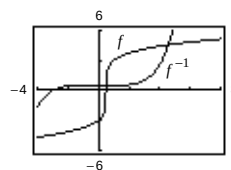


38. $f(x) = 3\sqrt[5]{2x - 1} = y$

$$x = \frac{y^5 + 243}{486}$$

$$y = \frac{x^5 + 243}{486}$$

$$f^{-1}(x) = \frac{x^5 + 243}{486}$$



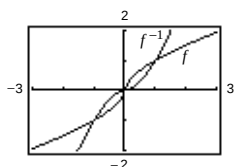
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

40. $f(x) = x^{3/5} = y$

$$x = y^{5/3}$$

$$y = x^{3/5}$$

$$f^{-1}(x) = x^{5/3}$$



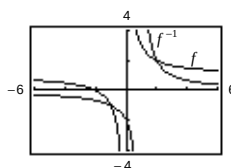
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

42. $f(x) = \frac{x + 2}{x} = y$

$$x = \frac{2}{y - 1}$$

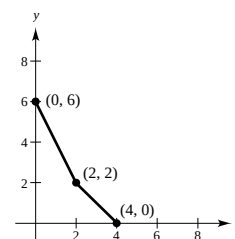
$$y = \frac{2}{x - 1}$$

$$f^{-1}(x) = \frac{2}{x - 1}$$



The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

x	0	2	4
$f^{-1}(x)$	6	2	0



46. $f(x) = k(2 - x - x^3)$ is one-to-one for all $k \neq 0$. Since $f^{-1}(3) = -2$, $f(-2) = 3 = k(2 - (-2) - (-2)^3) = 12k \Rightarrow k = \frac{1}{4}$.

48. $f(x) = |x + 2|$ on $[-2, \infty)$

$$f'(x) = \frac{|x + 2|}{x + 2}(1) = 1 > 0 \text{ on } (-2, \infty)$$

f is increasing on $[-2, \infty)$. Therefore, f is strictly monotonic and has an inverse.

50. $f(x) = \cot x$ on $(0, \pi)$

$$f'(x) = -\csc^2 x < 0 \text{ on } (0, \pi)$$

f is decreasing on $(0, \pi)$. Therefore, f is strictly monotonic and has an inverse.

52. $f(x) = \sec x$ on $\left[0, \frac{\pi}{2}\right)$

$$f'(x) = \sec x \tan x > 0 \text{ on } \left(0, \frac{\pi}{2}\right)$$

f is increasing on $\left[0, \frac{\pi}{2}\right)$. Therefore, f is strictly monotonic and has an inverse.

54. $f(x) = 2 - \frac{3}{x^2} = y$ on $(0, 10)$

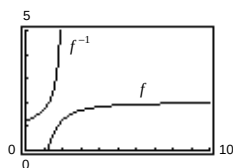
$$2x^2 - 3 = x^2y$$

$$x^2(2 - y) = 3$$

$$x = \pm \sqrt{\frac{3}{2 - y}}$$

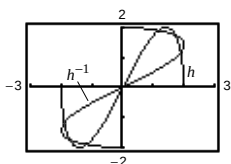
$$y = \pm \sqrt{\frac{3}{2 - x}}$$

$$f^{-1}(x) = \sqrt{\frac{3}{2 - x}}, x < 2$$



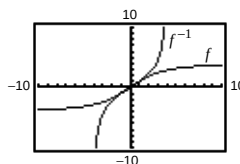
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

56. (a), (b)



(c) h is not one-to-one and does not have an inverse. The inverse relation is not an inverse function.

58. (a), (b)



(c) Yes, f is one-to-one and has an inverse. The inverse relation is an inverse function.

60. $f(x) = -3$

Not one-to-one; does not have an inverse

62. $f(x) = ax + b$

f is one-to-one; has an inverse

$$ax + b = y$$

$$x = \frac{y - b}{a}$$

$$y = \frac{x - b}{a}$$

$$f^{-1}(x) = \frac{x - b}{a}, a \neq 0$$

64. $f(x) = 16 - x^4$ is one-to-one for $x \geq 0$.

$$16 - x^4 = y$$

$$16 - y = x^4$$

$$\sqrt[4]{16 - y} = x$$

$$\sqrt[4]{16 - x} = y$$

$$f^{-1}(x) = \sqrt[4]{16 - x}, x \leq 16$$

66. $f(x) = |x - 3|$ is one-to-one for $x \geq 3$.

$$x - 3 = y$$

$$x = y + 3$$

$$y = x - 3$$

$$f^{-1}(x) = x + 3, x \geq 0$$

68. No, there could be two times $t_1 \neq t_2$ for which $h(t_1) = h(t_2)$.

70. Yes, the area function is increasing and hence one-to-one. The inverse function gives the radius r corresponding to the area A .

72. $f(x) = \frac{1}{27}(x^5 + 2x^3); f(-3) = \frac{1}{27}(-243 - 54) = -11 = a$.

$$f'(x) = \frac{1}{27}(5x^4 + 6x^2)$$

$$(f^{-1})'(-11) = \frac{1}{f'(f^{-1}(-11))} = \frac{1}{f'(-3)} = \frac{27}{5(-3)^4 + 6(-3)^2} = \frac{1}{17}$$

74. $f(x) = \cos 2x, f(0) = 1 = a$

$$f'(x) = -2 \sin 2x$$

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{-2 \sin 0} = \frac{1}{0} \text{ which is undefined.}$$

76. $f(x) = \sqrt{x-4}, f(8) = 2 = a$

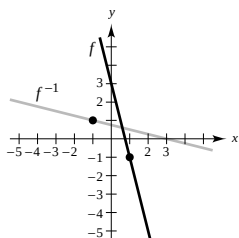
$$f'(x) = \frac{1}{2\sqrt{x-4}}$$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(8)} = \frac{1}{1/(2\sqrt{8-4})} = \frac{1}{1/4} = 4$$

78. (a) Domain $f = \text{Domain } f^{-1} = (-\infty, \infty)$

(b) Range $f = \text{Range } f^{-1} = (-\infty, \infty)$

(c)



(d) $f(x) = 3 - 4x, (1, -1)$

$$f'(x) = -4$$

$$f'(1) = -4$$

$$f^{-1}(x) = \frac{3-x}{4}, (-1, 1)$$

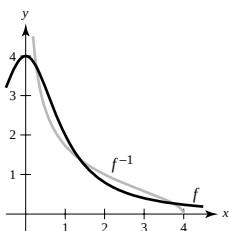
$$(f^{-1})'(x) = -\frac{1}{4}$$

$$(f^{-1})'(-1) = -\frac{1}{4}$$

80. (a) Domain $f = [0, \infty)$, Domain $f^{-1} = (0, 4]$

(b) Range $f = (0, 4]$, Range $f^{-1} = [0, \infty)$

(c)



(d) $f(x) = \frac{4}{1+x^2}$

$$f'(x) = \frac{-8x}{(x^2+1)^2}, f'(1) = -2$$

$$f^{-1}(x) = \sqrt{\frac{4-x}{x}}$$

$$(f^{-1})'(x) = \frac{-2}{x^2 \sqrt{\frac{4-x}{x}}}, (f^{-1})'(2) = -\frac{1}{2}$$

82. $x = 2 \ln(y^2 - 3)$

$$1 = 2 \frac{1}{y^2 - 3} 2y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y^2 - 3}{4y}. \text{ At } (0, 4), \frac{dy}{dx} = \frac{16 - 3}{16} = \frac{13}{16}.$$

In Exercises 84 and 86, use the following.

$$f(x) = \frac{1}{8}x - 3 \text{ and } g(x) = x^3$$

$$f^{-1}(x) = 8(x+3) \text{ and } g^{-1}(x) = \sqrt[3]{x}$$

84. $(g^{-1} \circ f^{-1})(-3) = g^{-1}(f^{-1}(-3)) = g^{-1}(0) = 0$

86. $(g^{-1} \circ g^{-1})(-4) = g^{-1}(g^{-1}(-4)) = g^{-1}(\sqrt[3]{-4})$
 $= \sqrt[3]{\sqrt[3]{-4}} = -\sqrt[9]{4}$

In Exercises 88 and 90, use the following.

$$f(x) = x + 4 \text{ and } g(x) = 2x - 5$$

$$f^{-1}(x) = x - 4 \text{ and } g^{-1}(x) = \frac{x + 5}{2}$$

$$88. (f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x))$$

$$= f^{-1}\left(\frac{x + 5}{2}\right)$$

$$= \frac{x + 5}{2} - 4$$

$$= \frac{x - 3}{2}$$

$$90. (g \circ f)(x) = g(f(x))$$

$$= g(x + 4)$$

$$= 2(x + 4) - 5$$

$$= 2x + 3$$

$$\text{Hence, } (g \circ f)^{-1}(x) = \frac{x - 3}{2}$$

$$(\text{Note: } (g \circ f)^{-1} = f^{-1} \circ g^{-1})$$

92. The graphs of f and f^{-1} are mirror images with respect to the line $y = x$.

94. Theorem 5.9: Let f be differentiable on an interval I . If f has an inverse g , then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0$$

96. f is not one-to-one because different x -values yield the same y -value.

$$\text{Example: } f(3) = f\left(-\frac{4}{3}\right) = \frac{3}{5}$$

Not continuous at ± 2 .

98. If f has an inverse, then f and f^{-1} are both one-to-one. Let $(f^{-1})^{-1}(x) = y$ then $x = f^{-1}(y)$ and $f(x) = y$. Thus, $(f^{-1})^{-1} = f$.

100. If f has an inverse and $f(x_1) = f(x_2)$, then $f^{-1}(f(x_1)) = f^{-1}(f(x_2)) \Rightarrow x_1 = x_2$. Therefore, f is one-to-one. If $f(x)$ is one-to-one, then for every value b in the range, there corresponds exactly one value a in the domain. Define $g(x)$ such that the domain of g equals the range of f and $g(b) = a$. By the reflexive property of inverses, $g = f^{-1}$.

102. True; if f has a y -intercept.

104. False

$$\text{Let } f(x) = x \text{ or } g(x) = 1/x.$$

106. From Theorem 5.9, we have:

$$g'(x) = \frac{1}{f'(g(x))}$$

$$g''(x) = \frac{f'(g(x))(0) - f''(g(x))g'(x)}{[f'(g(x))]^2}$$

$$= -\frac{f''(g(x)) \cdot [1/f'(g(x))]}{[f'(g(x))]^2}$$

$$= -\frac{f''(g(x))}{[f'(g(x))]^3}$$

If f is increasing and concave down, then $f' > 0$ and $f'' < 0$ which implies that g is increasing and concave up.

Section 5.4 Exponential Functions: Differentiation and Integration

$$2. \quad e^{-2} = 0.1353 \dots$$

$$\ln 0.1353 \dots = -2$$

$$4. \quad \ln 0.5 = -0.6931 \dots$$

$$e^{-0.6931 \dots} = \frac{1}{2}$$

$$6. \quad e^{\ln 2x} = 12$$

$$2x = 12$$

$$x = 6$$

$$8. \quad 4e^x = 83$$

$$e^x = \frac{83}{4}$$

$$x = \ln\left(\frac{83}{4}\right) \approx 3.033$$

$$10. \quad -6 + 3e^x = 8$$

$$3e^x = 14$$

$$e^x = \frac{14}{3}$$

$$x = \ln\left(\frac{14}{3}\right) \approx 1.540$$

$$12. \quad 200e^{-4x} = 15$$

$$e^{-4x} = \frac{15}{200} = \frac{3}{40}$$

$$-4x = \ln\left(\frac{3}{40}\right)$$

$$x = \frac{1}{4} \ln\left(\frac{40}{3}\right) \approx 0.648$$

$$14. \quad \ln x^2 = 10$$

$$x^2 = e^{10}$$

$$x = \pm e^5 \approx \pm 148.4132$$

$$16. \quad \ln 4x = 1$$

$$4x = e^1 = e$$

$$x = \frac{e}{4} \approx 0.680$$

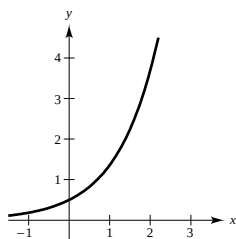
$$18. \quad \ln(x - 2)^2 = 12$$

$$(x - 2)^2 = e^{12}$$

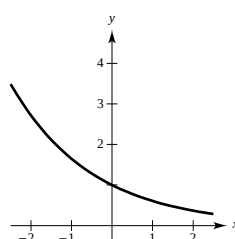
$$x - 2 = e^6$$

$$x = 2 + e^6 \approx 405.429$$

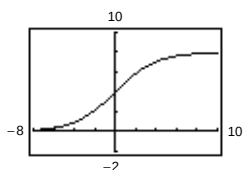
$$20. \quad y = \frac{1}{2}e^x$$



$$22. \quad y = e^{-x/2}$$

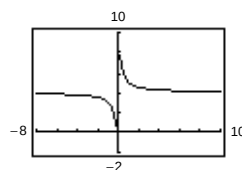


24. (a)



Horizontal asymptotes: $y = 0$ and $y = 8$

(b)



Horizontal asymptote: $y = 4$

26. $y = Ce^{-ax}$

Horizontal asymptote: $y = 0$ Reflection in the y -axis

Matches (d)

28. $y = \frac{C}{1 + e^{-ax}}$

$$\lim_{x \rightarrow \infty} \frac{C}{1 + e^{-ax}} = C$$

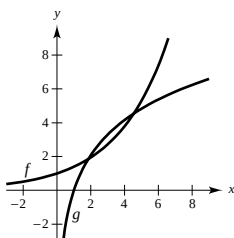
$$\lim_{x \rightarrow -\infty} \frac{C}{1 + e^{-ax}} = 0$$

Horizontal asymptotes: $y = C$ and $y = 0$

Matches (b)

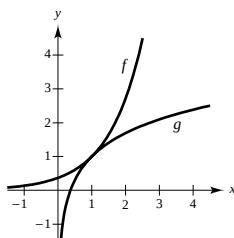
30. $f(x) = e^{x/3}$

$g(x) = \ln x^3 = 3 \ln x$



32. $f(x) = e^{x-1}$

$g(x) = 1 + \ln x$



34. In the same way,

$$\lim_{x \rightarrow \infty} \left(1 + \frac{r}{x}\right)^x = e^r \text{ for } r > 0.$$

36. $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} = 2.71825396$

$e \approx 2.718281828$

$e > 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040}$

38. (a) $y = e^{2x}$

$y' = 2e^{2x}$

At $(0, 1)$, $y' = 2$.

(b) $y = e^{-2x}$

$y' = -2e^{-2x}$

At $(0, 1)$, $y' = -2$.

40. $f(x) = e^{1-x}$

$f'(x) = -e^{1-x}$

42. $y = e^{-x^2}$

$\frac{dy}{dx} = -2xe^{-x^2}$

44. $y = x^2e^{-x}$

$\frac{dy}{dx} = -x^2e^{-x} + 2xe^{-x}$

$= xe^{-x}(2 - x)$

46. $g(t) = e^{-3/t^2}$

$g'(t) = e^{-3/t^2}(6t^{-3}) = \frac{6}{t^3e^{3/t^2}}$

48. $y = \ln\left(\frac{1 + e^x}{1 - e^x}\right)$

$= \ln(1 + e^x) - \ln(1 - e^x)$

$\frac{dy}{dx} = \frac{e^x}{1 + e^x} + \frac{e^x}{1 - e^x}$

$= \frac{2e^x}{1 - e^{2x}}$

50. $y = \ln\left(\frac{e^x + e^{-x}}{2}\right)$

$= \ln(e^x + e^{-x}) - \ln 2$

$\frac{dy}{dx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$= \frac{e^{2x} - 1}{e^{2x} + 1}$

52. $y = \frac{e^x - e^{-x}}{2}$

$\frac{dy}{dx} = \frac{e^x + e^{-x}}{2}$

54. $y = xe^x - e^x = e^x(x - 1)$

$\frac{dy}{dx} = e^x + e^x(x - 1) = xe^x$

56. $f(x) = e^3 \ln x$

$$f'(x) = \frac{e^3}{x}$$

58. $y = \ln e^x = x$

$$\frac{dy}{dx} = 1$$

60. $e^{xy} + x^2 - y^2 = 10$

$$\left(x \frac{dy}{dx} + y\right)e^{xy} + 2x - 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(xe^{xy} - 2y) = -ye^{xy} - 2x$$

$$\frac{dy}{dx} = \frac{-ye^{xy} - 2x}{xe^{xy} - 2y}$$

62. $g(x) = \sqrt{x} + e^x \ln x$

$$g'(x) = \frac{1}{2\sqrt{x}} + \frac{e^x}{x} + e^x \ln x$$

$$\begin{aligned} g''(x) &= -\frac{1}{4x^{3/2}} + \frac{xe^x - e^x}{x^2} + \frac{e^x}{x} + e^x \ln x \\ &= -\frac{1}{4x\sqrt{x}} + \frac{e^x(2x - 1)}{x^2} + e^x \ln x \end{aligned}$$

64. $y = e^x(3 \cos 2x - 4 \sin 2x)$

$$\begin{aligned} y' &= e^x(-6 \sin 2x - 8 \cos 2x) + e^x(3 \cos 2x - 4 \sin 2x) \\ &= e^x(-10 \sin 2x - 5 \cos 2x) = -5e^x(2 \sin 2x + \cos 2x) \end{aligned}$$

$$y'' = -5e^x(4 \cos 2x - 2 \sin 2x) - 5e^x(2 \sin 2x + \cos 2x) = -5e^x(5 \cos 2x) = -25e^x \cos 2x$$

$$y'' - 2y' = -25e^x \cos 2x - 2(-5e^x)(2 \sin 2x + \cos 2x) = -5e^x(3 \cos 2x - 4 \sin 2x) = -5y$$

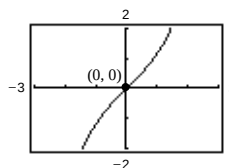
$$\text{Therefore, } y'' - 2y' = -5y \Rightarrow y'' - 2y' + 5y = 0.$$

66. $f(x) = \frac{e^x - e^{-x}}{2}$

$$f'(x) = \frac{e^x + e^{-x}}{2} > 0$$

$$f''(x) = \frac{e^x - e^{-x}}{2} = 0 \text{ when } x = 0.$$

Point of inflection: $(0, 0)$



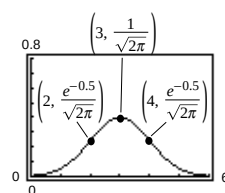
68. $g(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-3)^2/2}$

$$g'(x) = \frac{-1}{\sqrt{2\pi}}(x-3)e^{-(x-3)^2/2}$$

$$g''(x) = \frac{1}{\sqrt{2\pi}}(x-2)(x-4)e^{-(x-3)^2/2}$$

$$\text{Relative maximum: } \left(3, \frac{1}{\sqrt{2\pi}}\right) \approx (3, 0.399)$$

$$\text{Points of inflection: } \left(2, \frac{1}{\sqrt{2\pi}}e^{-1/2}\right), \left(4, \frac{1}{\sqrt{2\pi}}e^{-1/2}\right) \approx (2, 0.242), (4, 0.242)$$



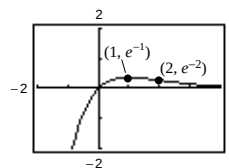
70. $f(x) = xe^{-x}$

$$f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x) = 0 \text{ when } x = 1.$$

$$f''(x) = -e^{-x} + (-e^{-x})(1-x) = e^{-x}(x-2) = 0 \text{ when } x = 2.$$

Relative maximum: $(1, e^{-1})$

Point of inflection: $(2, 2e^{-2})$



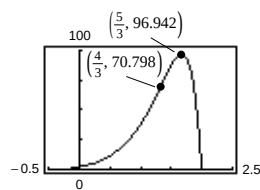
72. $f(x) = -2 + e^{3x}(4 - 2x)$

$$f'(x) = e^{3x}(-2) + 3e^{3x}(4 - 2x) = e^{3x}(10 - 6x) = 0 \text{ when } x = \frac{5}{3}.$$

$$f''(x) = e^{3x}(-6) + 3e^{3x}(10 - 6x) = e^{3x}(24 - 18x) = 0 \text{ when } x = \frac{4}{3}.$$

Relative maximum: $(\frac{5}{3}, 96.942)$

Point of inflection: $(\frac{4}{3}, 70.798)$



74. (a) $f(c) = f(c + x)$

$$10ce^{-c} = 10(c + x)e^{-(c+x)}$$

$$\frac{c}{e^c} = \frac{c + x}{e^{c+x}}$$

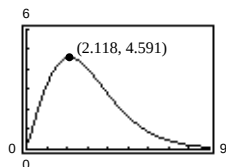
$$ce^{c+x} = (c + x)e^c$$

$$ce^x = c + x$$

$$ce^x - c = x$$

$$c = \frac{x}{e^x - 1}$$

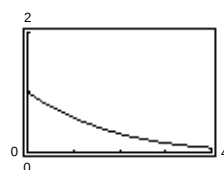
(c) $A(x) = \frac{10x^2}{e^x - 1} e^{x/(1-e^x)}$



The maximum area is 4.591 for $x = 2.118$ and $f(x) = 2.547$.

(b) $A(x) = xf(c) = x \left[10 \left(\frac{x}{e^x - 1} \right) e^{-(x/(e^x - 1))} \right]$
 $= \frac{10x^2}{e^x - 1} e^{x/(1-e^x)}$

(d) $c = \frac{x}{e^x - 1}$



$$\lim_{x \rightarrow 0^+} c = 1$$

$$\lim_{x \rightarrow \infty} c = 0$$

76. Let (x_0, y_0) be the desired point on $y = e^{-x}$.

$$y = e^{-x}$$

$$y' = -e^{-x} \quad (\text{Slope of tangent line})$$

$$-\frac{1}{y'} = e^x \quad (\text{Slope of normal line})$$

$$y - e^{-x_0} = e^{x_0}(x - x_0)$$

We want $(0, 0)$ to satisfy the equation:

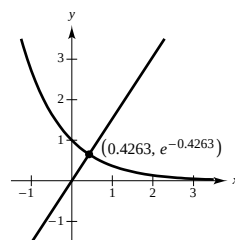
$$-e^{-x_0} = -x_0 e^{x_0}$$

$$1 = x_0 e^{2x_0}$$

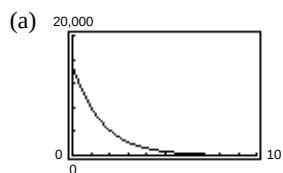
$$x_0 e^{2x_0} - 1 = 0$$

Solving by Newton's Method or using a computer, the solution is $x_0 \approx 0.4263$.

$$(0.4263, e^{-0.4263})$$



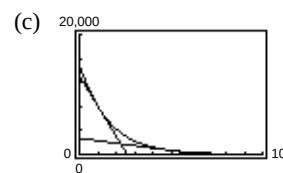
78. $V = 15,000e^{-0.6286t}, 0 \leq t \leq 10$



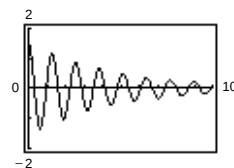
(b) $\frac{dV}{dt} = -9429e^{-0.6286t}$

When $t = 1$, $\frac{dV}{dt} \approx -5028.84$.

When $t = 5$, $\frac{dV}{dt} \approx -406.89$.

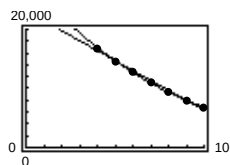


80. $1.56e^{-0.22t} \cos 4.9t \leq 0.25$ (3 inches equals one-fourth foot.) Using a graphing utility or Newton's Method, we have $t \geq 7.79$ seconds.



82. (a) $V_1 = -1686.79t + 23,181.79$

$V_2 = 109.52t^2 - 3220.12t + 28,110.36$



- (b) The slope represents the rate of decrease in value of the car.

(c) $V_3 = 31,450.77(0.8592)^t = 31,450.77e^{-0.1518t}$

(d) Horizontal asymptote: $\lim_{t \rightarrow \infty} V_3(t) = 0$

As $t \rightarrow \infty$, the value of the car approaches 0.

(e) $\frac{dV_3}{dt} = -4774.2e^{-0.1518t}$

For $t = 5$, $\frac{dV_3}{dt} \approx -2235$ dollars/year.

For $t = 9$, $\frac{dV_3}{dt} \approx -1218$ dollars/year.

84. $f(x) = e^{-x^2/2}, f(0) = 1$

$f'(x) = -xe^{-x^2/2}, f'(0) = 0$

$f''(x) = x^2e^{-x^2/2} - e^{-x^2/2} = e^{-x^2/2}(x^2 - 1), f''(0) = -1$

$P_1(x) = 1 + 0(x - 0) = 1, P_1(0) = 1$

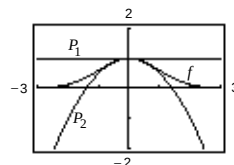
$P_1'(x) = 0, P_1'(0) = 0$

$P_2(x) = 1 + 0(x - 0) - \frac{1}{2}(x - 0)^2 = 1 - \frac{x^2}{2}, P_2(0) = 1$

$P_2'(x) = -x, P_2'(0) = 0$

$P_2''(x) = -1, P_2''(0) = -1$

The values of f, P_1, P_2 and their first derivatives agree at $x = 0$. The values of the second derivatives of f and P_2 agree at $x = 0$.



86. n^{th} term is $x^n/n!$ in polynomial:

$$y_4 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$\text{Conjecture: } e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

90. $\int_3^4 e^{3-x} dx = \left[-e^{3-x} \right]_3^4 = -e^{-1} + 1 = 1 - \frac{1}{e}$

94. $\int \frac{e^{1/x^2}}{x^3} dx = -\frac{1}{2} \int e^{1/x^2} \left(\frac{-2}{x^3} \right) dx = -\frac{1}{2} e^{1/x^2} + C$

98. Let $u = \frac{-x^2}{2}$, $du = -x dx$.

$$\int_0^{\sqrt{2}} x e^{-x^2/2} dx = - \int_0^{\sqrt{2}} e^{-x^2/2} (-x) dx = \left[-e^{-x^2/2} \right]_0^{\sqrt{2}} = 1 - e^{-1} = \frac{e-1}{e}$$

100. Let $u = e^x + e^{-x}$, $du = (e^x - e^{-x})dx$.

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \ln(e^x + e^{-x}) + C$$

104. $\int \frac{e^{2x} + 2e^x + 1}{e^x} dx = \int (e^x + 2 + e^{-x}) dx$
 $= e^x + 2x - e^{-x} + C$

108. $\int \ln(e^{2x-1}) dx = \int (2x - 1) dx$
 $= x^2 - x + C$

112. $f'(x) = \int (\sin x + e^{2x}) dx = -\cos x + \frac{1}{2} e^{2x} + C_1$

$$f'(0) = -1 + \frac{1}{2} + C_1 = \frac{1}{2} \Rightarrow C_1 = 1$$

$$f'(x) = -\cos x + \frac{1}{2} e^{2x} + 1$$

$$f(x) = \int \left(-\cos x + \frac{1}{2} e^{2x} + 1 \right) dx$$

$$= -\sin x + \frac{1}{4} e^{2x} + x + C_2$$

$$f(0) = \frac{1}{4} + C_2 = \frac{1}{4} \Rightarrow C_2 = 0$$

$$f(x) = x - \sin x + \frac{1}{4} e^{2x}$$

88. Let $u = -x^4$, $du = -4x^3 dx$.

$$\int e^{-x^4} (-4x^3) dx = e^{-x^4} + C$$

92. $\int x^2 e^{x^3/2} dx = \frac{2}{3} \int e^{x^3/2} \left(\frac{3x^2}{2} \right) dx = \frac{2}{3} e^{x^3/2} + C$

96. Let $u = 1 + e^{2x}$, $du = 2e^{2x} dx$.

$$\int \frac{e^{2x}}{1 + e^{2x}} dx = \frac{1}{2} \int \frac{2e^{2x}}{1 + e^{2x}} dx = \frac{1}{2} \ln(1 + e^{2x}) + C$$

102. Let $u = e^x + e^{-x}$, $du = (e^x - e^{-x}) dx$.

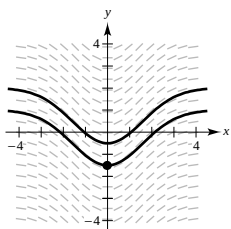
$$\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2} dx = 2 \int (e^x + e^{-x})^{-2} (e^x - e^{-x}) dx$$

$$= \frac{-2}{e^x + e^{-x}} + C$$

106. $\int e^{\sec 2x} \sec 2x \tan 2x dx = \frac{1}{2} e^{\sec 2x} + C$
 $(u = \sec 2x, du = 2 \sec 2x \tan 2x)$

110. $y = \int (e^x - e^{-x})^2 dx$
 $= \int (e^{2x} - 2 + e^{-2x}) dx$
 $= \frac{1}{2} e^{2x} - 2x - \frac{1}{2} e^{-2x} + C$

114. (a)



(b) $\frac{dy}{dx} = xe^{-0.2x^2}, \left(0, -\frac{3}{2}\right)$

$$y = \int xe^{-0.2x^2} dx = \frac{1}{-0.4} \int e^{-0.2x^2} (-0.4x) dx$$

$$= -\frac{1}{0.4} e^{-0.2x^2} + C = -2.5e^{-0.2x^2} + C$$

$$\left(0, -\frac{3}{2}\right): -\frac{3}{2} = -2.5e^0 + C = -2.5 + C \Rightarrow C = 1$$

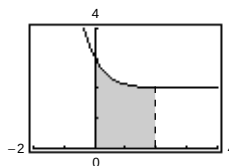
$$y = -2.5e^{-0.2x^2} + 1$$

116. $\int_a^b e^{-x} dx = \left[-e^{-x}\right]_a^b = e^{-a} - e^{-b}$



118. $\int_0^2 (e^{-2x} + 2) dx = \left[-\frac{1}{2}e^{-2x} + 2x\right]_0^2$

$$= -\frac{1}{2}e^{-4} + 4 + \frac{1}{2} \approx 4.491$$



120. (a) $\int_0^4 \sqrt{x} e^x dx, n = 12$

Midpoint Rule: 92.1898

Trapezoidal Rule: 93.8371

Simpson's Rule: 92.7385

Graphing Utility: 92.7437

(b) $\int_0^2 2xe^{-x} dx, n = 12$

Midpoint Rule: 1.1906

Trapezoidal Rule: 1.1827

Simpson's Rule: 1.1880

Graphing Utility: 1.18799

122. $\int_0^x 0.3^{-0.3t} dt = \frac{1}{2}$

$$\left[-e^{-0.3t}\right]_0^x = \frac{1}{2}$$

$$-e^{-0.3x} + 1 = \frac{1}{2}$$

$$e^{-0.3x} = \frac{1}{2}$$

$$-0.3x = \ln \frac{1}{2} = -\ln 2$$

$$x = \frac{\ln 2}{0.3} \approx 2.31 \text{ minutes}$$

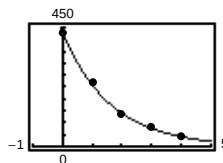
124.

t	0	1	2	3	4
R	425	240	118	71	36
$\ln R$	6.052	5.481	4.771	4.263	3.584

(a) $\ln R = -0.6155t + 6.0609$

$$R = e^{-0.6155t + 6.0609} = 428.78e^{-0.6155t}$$

(b)



(c) $\int_0^4 R(t) dt = \int_0^4 428.78e^{-0.6155t} dt \approx 637.2 \text{ liters}$

126. The graphs of $f(x) = \ln x$ and $g(x) = e^x$ are mirror images across the line $y = x$.

128. (a) Log Rule: ($u = e^x + 1$)

(b) Substitution: ($u = x^2$)

130. $\ln \frac{e^a}{e^b} = \ln e^a - \ln e^b = a - b$

$$\ln e^{a-b} = a - b$$

Therefore, $\ln \frac{e^a}{e^b} = \ln e^{a-b}$ and since $y = \ln x$ is one-to-one, we have $\frac{e^a}{e^b} = e^{a-b}$.

Section 5.5 Bases Other than e and Applications

2. $y = \left(\frac{1}{2}\right)^{t/8}$

At $t_0 = 16$, $y = \left(\frac{1}{2}\right)^{16/8} = \frac{1}{4}$

4. $y = \left(\frac{1}{2}\right)^{t/5}$

At $t_0 = 2$, $y = \left(\frac{1}{2}\right)^{2/5} \approx 0.7579$

6. $\log_{27} 9 = \log_{27} 27^{2/3} = \frac{2}{3}$

8. $\log_a \frac{1}{a} = \log_a 1 - \log_a a = -1$

10. (a) $27^{2/3} = 9$

$$\log_{27} 9 = \frac{2}{3}$$

(b) $16^{3/4} = 8$

$$\log_{16} 8 = \frac{3}{4}$$

12. (a) $\log_3 \frac{1}{9} = -2$

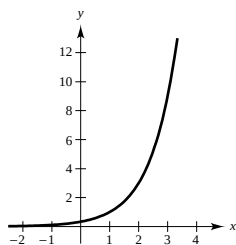
$$3^{-2} = \frac{1}{9}$$

(b) $49^{1/2} = 7$

$$\log_{49} 7 = \frac{1}{2}$$

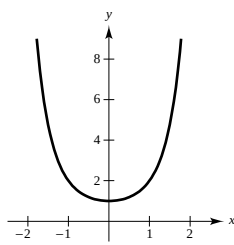
14. $y = 3^{x-1}$

x	-1	0	1	2	3
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9



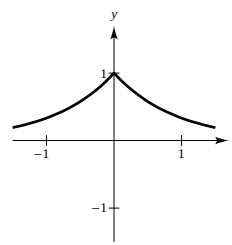
16. $y = 2^{x^2}$

x	-2	-1	0	1	2
y	16	2	1	2	16



18. $y = 3^{-|x|}$

x	0	± 1	± 2
y	1	$\frac{1}{3}$	$\frac{1}{9}$



20. (a) $\log_3 \frac{1}{81} = x$

$$3^x = \frac{1}{81}$$

$$x = -4$$

(b) $\log_6 36 = x$

$$6^x = 36$$

$$x = 2$$

22. (a) $\log_b 27 = 3$

$$b^3 = 27$$

$$b = 3$$

(b) $\log_b 125 = 3$

$$b^3 = 125$$

$$b = 5$$

24. (a) $\log_3 x + \log_3(x - 2) = 1$

$$\log_3[x(x - 2)] = 1$$

$$x(x - 2) = 3^1$$

$$x^2 - 2x - 3 = 0$$

$$(x + 1)(x - 3) = 0$$

$$x = -1 \text{ OR } x = 3$$

$x = 3$ is the only solution since the domain of the logarithmic function is the set of all *positive* real numbers.

(b) $\log_{10}(x + 3) - \log_{10} x = 1$

$$\log_{10} \frac{x + 3}{x} = 1$$

$$\frac{x + 3}{x} = 10^1$$

$$x + 3 = 10x$$

$$3 = 9x$$

$$x = \frac{1}{3}$$

26. $5^{6x} = 8320$

$$6x \ln 5 = \ln 8320$$

$$x = \frac{\ln 8320}{6 \ln 5} \approx 0.935$$

28. $3(5^{x-1}) = 86$

$$5^{x-1} = \frac{86}{3}$$

$$(x - 1) \ln 5 = \ln\left(\frac{86}{3}\right)$$

$$x - 1 = \frac{\ln\left(\frac{86}{3}\right)}{\ln 5}$$

$$x = 1 + \frac{\ln\left(\frac{86}{3}\right)}{\ln 5} \approx 3.085$$

30. $\left(1 + \frac{0.10}{365}\right)^{365t} = 2$

$$365t \ln\left(1 + \frac{0.10}{365}\right) = \ln 2$$

$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \frac{0.10}{365}\right)} \approx 6.932$$

32. $\log_{10}(t - 3) = 2.6$

$$t - 3 = 10^{2.6}$$

$$t = 3 + 10^{2.6} \approx 401.107$$

34. $\log_5 \sqrt{x - 4} = 3.2$

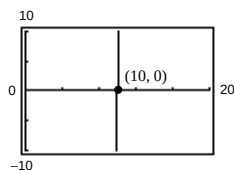
$$\sqrt{x - 4} = 5^{3.2}$$

$$x - 4 = (5^{3.2})^2 = 5^{6.4}$$

$$x = 4 + 5^{6.4} \approx 29,748.593$$

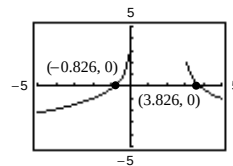
36. $f(t) = 300(1.0075^{12t}) - 735.41$

Zero: $t \approx 10$



38. $g(x) = 1 - 2 \log_{10}[x(x - 3)]$

Zeros: $x \approx -0.826, 3.826$

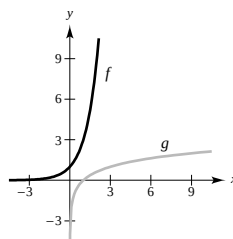


40. $f(x) = 3^x$

$g(x) = \log_3 x$

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$g(x)$	-2	-1	0	1	2



42. $g(x) = 2^{-x}$

$g'(x) = -(\ln 2) 2^{-x}$

44. $y = x(6^{-2x})$

$$\frac{dy}{dx} = x[-2(\ln 6)6^{-2x}] + 6^{-2x}$$

$$= 6^{-2x}[-2x(\ln 6) + 1]$$

$$= 6^{-2x}(1 - 2x \ln 6)$$

46. $f(t) = \frac{3^{2t}}{t}$

$$f'(t) = \frac{t(2 \ln 3) 3^{2t} - 3^{2t}}{t^2}$$

$$= \frac{3^{2t}(2t \ln 3 - 1)}{t^2}$$

48. $g(\alpha) = 5^{-\alpha/2} \sin 2\alpha$

$$g'(\alpha) = 5^{-\alpha/2} 2 \cos 2\alpha - \frac{1}{2}(\ln 5) 5^{-\alpha/2} \sin 2\alpha$$

50. $y = \log_{10}(2x) = \log_{10} 2 + \log_{10} x$

$$\frac{dy}{dx} = 0 + \frac{1}{x \ln 10} = \frac{1}{x \ln 10}$$

52. $h(x) = \log_3 \frac{x\sqrt{x-1}}{2}$

$$= \log_3 x + \frac{1}{2} \log_3 (x-1) - \log_3 2$$

$$h'(x) = \frac{1}{x \ln 3} + \frac{1}{2} \cdot \frac{1}{(x-1) \ln 3} - 0$$

$$= \frac{1}{\ln 3} \left[\frac{1}{x} + \frac{1}{2(x-1)} \right]$$

$$= \frac{1}{\ln 3} \left[\frac{3x-2}{2x(x-1)} \right]$$

54. $y = \log_{10} \frac{x^2-1}{x}$

$$= \log_{10}(x^2-1) - \log_{10} x$$

$$\frac{dy}{dx} = \frac{2x}{(x^2-1) \ln 10} - \frac{1}{x \ln 10}$$

$$= \frac{1}{\ln 10} \left[\frac{2x}{x^2-1} - \frac{1}{x} \right]$$

$$= \frac{1}{\ln 10} \left[\frac{x^2+1}{x(x^2-1)} \right]$$

56. $f(t) = t^{3/2} \log_2 \sqrt{t+1} = t^{3/2} \frac{1}{2} \frac{\ln(t+1)}{\ln 2}$

$$f'(t) = \frac{1}{2 \ln 2} \left[t^{3/2} \frac{1}{t+1} + \frac{3}{2} t^{1/2} \ln(t+1) \right]$$

58. $y = x^{x-1}$

$$\ln y = (x-1)(\ln x)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = (x-1) \left(\frac{1}{x} \right) + \ln x$$

$$\frac{dy}{dx} = y \left[\frac{x-1}{x} + \ln x \right]$$

$$= x^{x-2} (x-1 + x \ln x)$$

60. $y = (1+x)^{1/x}$

$$\ln y = \frac{1}{x} \ln(1+x)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{x} \left(\frac{1}{1+x} \right) + \ln(1+x) \left(-\frac{1}{x^2} \right)$$

$$\frac{dy}{dx} = \frac{y}{x} \left[\frac{1}{x+1} - \frac{\ln(x+1)}{x} \right]$$

$$= \frac{(1+x)^{1/x}}{x} \left[\frac{1}{x+1} - \frac{\ln(x+1)}{x} \right]$$

62. $\int 5^{-x} dx = \frac{-5^{-x}}{\ln 5} + C$

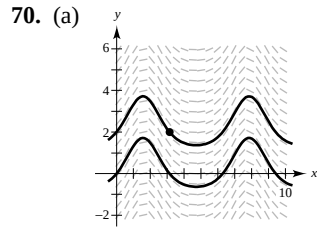
64. $\int_{-2}^0 (3^3 - 5^2) dx = \int_{-2}^0 (27 - 25) dx$

$$= \int_{-2}^0 2 dx$$

$$= \left[2x \right]_{-2}^0 = 4$$

$$\begin{aligned}
 66. \int (3-x) 7^{(3-x)^2} dx &= -\frac{1}{2} \int -2(3-x) 7^{(3-x)^2} dx \\
 &= -\frac{1}{2 \ln 7} [7^{(3-x)^2}] + C
 \end{aligned}$$

$$\begin{aligned}
 68. \int 2^{\sin x} \cos x dx, u &= \sin x, du = \cos x dx \\
 &= \frac{1}{\ln 2} 2^{\sin x} + C
 \end{aligned}$$



$$\begin{aligned}
 (b) \frac{dy}{dx} &= e^{\sin x} \cos x \quad (\pi, 2) \\
 y &= \int e^{\sin x} \cos x dx = e^{\sin x} + C \\
 (\pi, 2): 2 &= e^{\sin \pi} + C = 1 + C \Rightarrow C = 1 \\
 y &= e^{\sin x} + 1
 \end{aligned}$$

$$72. \log_b x = \frac{\ln x}{\ln b} = \frac{\log_{10} x}{\log_{10} b}$$

$$74. f(x) = \log_{10} x$$

$$(a) \text{ Domain: } x > 0$$

$$(b) y = \log_{10} x$$

$$10^y = x$$

$$f^{-1}(x) = 10^x$$

$$(c) \log_{10} 1000 = \log_{10} 10^3 = 3$$

$$\log_{10} 10,000 = \log_{10} 10^4 = 4$$

$$\text{If } 1000 \leq x \leq 10,000, \text{ then } 3 \leq f(x) \leq 4.$$

$$(d) \text{ If } f(x) < 0, \text{ then } 0 < x < 1.$$

$$(e) f(x) + 1 = \log_{10} x + \log_{10} 10$$

$$= \log_{10}(10x)$$

x must have been increased by a factor of 10.

$$(f) \log_{10} \left(\frac{x_1}{x_2} \right) = \log_{10} x_1 - \log_{10} x_2$$

$$= 3n - n = 2n$$

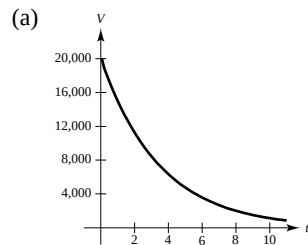
$$\text{Thus, } x_1/x_2 = 10^{2n} = 100^n.$$

$$76. f(x) = a^x$$

$$(a) f(u+v) = a^{u+v} = a^u a^v = f(u) f(v)$$

$$(b) f(2x) = a^{2x} = (a^x)^2 = [f(x)]^2$$

$$78. V(t) = 20,000 \left(\frac{3}{4} \right)^t$$

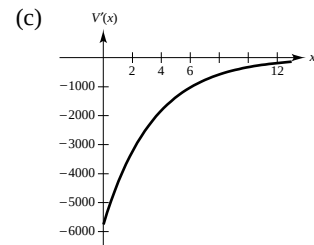


$$V(2) = 20,000 \left(\frac{3}{4} \right)^2 = \$11,250$$

$$(b) \frac{dV}{dt} = 20,000 \left(\ln \frac{3}{4} \right) \left(\frac{3}{4} \right)^t$$

$$\text{When } t = 1: \frac{dV}{dt} \approx -4315.23$$

$$\text{When } t = 4: \frac{dV}{dt} \approx -1820.49$$



Horizontal asymptote: $v' = 0$

As the car ages, it is worth less each year and depreciates less each year, but the value of the car will never reach \$0.

80. $P = \$2500$, $r = 6\% = 0.06$, $t = 20$

$$A = 2500 \left(1 + \frac{0.06}{n} \right)^{20n}$$

$$A = 2500e^{(0.06)(20)} = 8300.29$$

n	1	2	4	12	365	Continuous
A	8017.84	8155.09	8226.66	8275.51	8299.47	8300.29

82. $P = \$5000$, $r = 7\% = 0.07$, $t = 25$

$$A = 5000 \left(1 + \frac{0.07}{n} \right)^{25n}$$

$$A = 5000e^{0.07(25)}$$

n	1	2	4	12	365	Continuous
A	27,137.16	27,924.63	28,340.78	28,627.09	28,768.19	28,773.01

84. $100,000 = Pe^{0.06t} \Rightarrow P = 100,000e^{-0.06t}$

t	1	10	20	30	40	50
P	94,176.45	54,881.16	30,119.42	16,529.89	9071.80	4978.71

86. $100,000 = P \left(1 + \frac{0.07}{365} \right)^{365t} \Rightarrow P = 100,000 \left(1 + \frac{0.07}{365} \right)^{-365t}$

t	1	10	20	30	40	50
P	93,240.01	49,661.86	24,663.01	12,248.11	6082.64	3020.75

88. Let $P = \$100$, $0 \leq t \leq 20$.

(a) $A = 100e^{0.03t}$

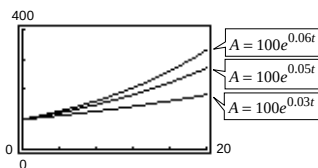
$A(20) \approx 182.21$

(b) $A = 100e^{0.05t}$

$A(20) \approx 271.83$

(c) $A = 100e^{0.06t}$

$A(20) \approx 332.01$



90. (a) $\lim_{n \rightarrow \infty} \frac{0.86}{1 + e^{-0.25n}} = 0.86$ or 86%

(b) $P' = \frac{-0.86(-0.25)(e^{-0.25n})}{(1 + e^{-0.25n})} = \frac{0.215e^{-0.25n}}{(1 + e^{-0.25n})}$

$P'(3) \approx 0.069$

$P'(10) \approx 0.016$

92. (a)

(b) Limiting size: 10,000 fish

(c) $p(t) = \frac{10,000}{1 + 19e^{-t/5}}$

$$p'(t) = \frac{e^{-t/5}}{(1 + 19e^{t/5})^2} \left(\frac{19}{5} \right) (10,000)$$

$$= \frac{38,000e^{-t/5}}{(1 + 19e^{-t/5})^2}$$

$p'(1) \approx 113.5$ fish/month

$p'(10) \approx 403.2$ fish/month

(d) $p''(t) = -\frac{38,000}{5}(e^{-t/5}) \left[\frac{1 - 19e^{-t/5}}{(1 + 19e^{-t/5})^3} \right] = 0$

$19e^{-t/5} = 1$

$\frac{t}{5} = \ln 19$

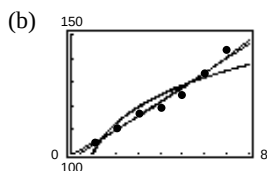
$t = 5 \ln 19 \approx 14.72$

94. (a) $y_1 = 6.0536x + 97.5571$

$$y_2 = 100.0751 + 17.8148 \ln x$$

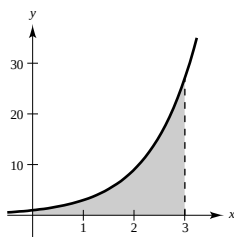
$$y_3 = 99.4557(1.0506)^x$$

$$y_4 = 101.2875x^{0.1471}$$



y_3 seems best.

96. $A = \int_0^3 3^x dx = \left[\frac{3^x}{\ln 3} \right]_0^3 = \frac{26}{\ln 3} \approx 23.666$



(c) The slope of 6.0536 is the annual rate of change in the amount given to philanthropy.

(d) For 1996, $x = 6$ and $y_1' = 6.0536$, $y_2' \approx 2.9691$,

$$y_3' \approx 6.6015, y_4' \approx 3.2321.$$

y_3 is increasing at the greatest rate in 1996.

98.

x	1	10^{-1}	10^{-2}	10^{-4}	10^{-6}
$(1+x)^{1/x}$	2	2.594	2.705	2.718	2.718

100.

t	0	1	2	3	4
y	600	630	661.50	694.58	729.30

$$y = C(k^t)$$

When $t = 0$, $y = 600 \Rightarrow C = 600$.

$$y = 600(k^t)$$

$$\frac{630}{600} = 1.05, \frac{661.50}{630} = 1.05, \frac{694.58}{661.50} \approx 1.05, \frac{729.30}{694.58} \approx 1.05$$

Let $k = 1.05$.

$$y = 600(1.05)^t$$

102. True.

$$\begin{aligned} f(e^{n+1}) - f(e^n) &= \ln e^{n+1} - \ln e^n \\ &= n + 1 - n \\ &= 1 \end{aligned}$$

104. True.

$$\begin{aligned} \frac{d^n y}{dx^n} &= Ce^x \\ &= y \text{ for } n = 1, 2, 3, \dots \end{aligned}$$

106. True.

$$\begin{aligned} f(x) &= g(x)e^x = 0 \Rightarrow \\ g(x) &= 0 \text{ since } e^x > 0 \text{ for all } x. \end{aligned}$$

108. $y = x^{\sin x}$

$$\ln y = \ln x^{\sin x} = \sin x \cdot \ln x$$

$$\frac{y'}{y} = \sin x \left(\frac{1}{x} \right) + \cos x \cdot \ln x$$

$$y' = x^{\sin x} \left[\frac{\sin x}{x} + \cos x \cdot \ln x \right]$$

$$\text{At } \left(\frac{\pi}{2}, \frac{\pi}{2} \right),$$

$$y' = \left(\frac{\pi}{2} \right)^{\sin(\pi/2)} \left[\frac{\sin(\pi/2)}{\pi/2} + \cos\left(\frac{\pi}{2}\right) \ln\left(\frac{\pi}{2}\right) \right]$$

$$= \frac{\pi}{2} \left[\frac{2}{\pi} + 0 \right] = 1$$

$$\text{Tangent line: } y - \frac{\pi}{2} = 1 \left(x - \frac{\pi}{2} \right)$$

$$y = x$$

Section 5.6 Differential Equations: Growth and Decay

2. $\frac{dy}{dx} = 4 - x$

$$y = \int (4 - x) dx = 4x - \frac{x^2}{2} + C$$

4. $\frac{dy}{dx} = 4 - y$

$$\frac{dy}{4 - y} = dx$$

$$\int \frac{-1}{4 - y} dy = \int -dx$$

$$\ln|4 - y| dy = -x + C_1$$

$$4 - y = e^{-x+C_1} = Ce^{-x}$$

$$y = 4 - Ce^{-x}$$

6. $y' = \frac{\sqrt{x}}{3y}$

$$3yy' = \sqrt{x}$$

$$\int 3yy' dx = \int \sqrt{x} dx$$

$$\frac{3y^2}{2} = \frac{2}{3}x^{3/2} + C_1$$

$$9y^2 - 4x^{3/2} = C$$

8. $y' = x(1 + y)$

$$\frac{y'}{1 + y} = x$$

$$\int \frac{y'}{1 + y} dx = \int x dx$$

$$\int \frac{dy}{1 + y} = \int x dx$$

$$\ln(1 + y) = \frac{x^2}{2} + C_1$$

$$1 + y = e^{(x^2/2)+C_1}$$

$$y = e^{C_1} e^{x^2/2} - 1$$

$$= Ce^{x^2/2} - 1$$

10. $xy + y' = 100x$

$$y' = 100x + xy = x(100 - y)$$

$$\frac{y'}{100 - y} = x$$

$$\int \frac{y'}{100 - y} dx = \int x dx$$

$$\int \frac{1}{100 - y} dy = \int x dx$$

$$-\ln(100 - y) = \frac{x^2}{2} + C_1$$

$$\ln(100 - y) = -\frac{x^2}{2} - C_1$$

$$100 - y = e^{-(x^2/2)-C_1}$$

$$-y = e^{-C_1} e^{-x^2/2} - 100$$

$$y = 100 - Ce^{-x^2/2}$$

$$12. \quad \frac{dP}{dt} = k(10 - t)$$

$$\int \frac{dP}{dt} dt = \int k(10 - t) dt$$

$$\int dP = -\frac{k}{2}(10 - t)^2 + C$$

$$P = -\frac{k}{2}(10 - t)^2 + C$$

$$14. \quad \frac{dy}{dx} = kx(L - y)$$

$$\frac{1}{L - y} \frac{dy}{dx} = kx$$

$$\int \frac{1}{L - y} \frac{dy}{dx} dx = \int kx dx$$

$$\int \frac{1}{L - y} dy = \frac{kx^2}{2} + C_1$$

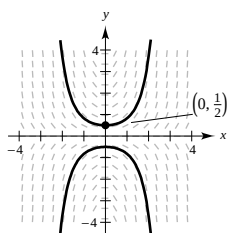
$$-\ln(L - y) = \frac{kx^2}{2} + C_1$$

$$L - y = e^{-(kx^2/2) - C_1}$$

$$-y = -L + e^{-C_1} e^{-kx^2/2}$$

$$y = L - Ce^{-kx^2/2}$$

16. (a)



$$(b) \quad \frac{dy}{dx} = xy, \quad \left(0, \frac{1}{2}\right)$$

$$\frac{dy}{y} = x dx$$

$$\ln|y| = \frac{x^2}{2} + C$$

$$y = e^{x^2/2 + C} = C_1 e^{x^2/2}$$

$$\left(0, \frac{1}{2}\right): \frac{1}{2} = C_1 e^0 \Rightarrow C_1 = \frac{1}{2} \Rightarrow y = \frac{1}{2} e^{x^2/2}$$

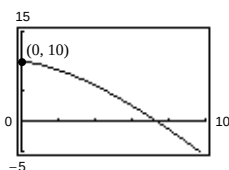
$$18. \quad \frac{dy}{dt} = -\frac{3}{4}\sqrt{t}, \quad (0, 10)$$

$$\int dy = \int -\frac{3}{4}\sqrt{t} dt$$

$$y = -\frac{1}{2}t^{3/2} + C$$

$$10 = -\frac{1}{2}(0)^{3/2} + C \Rightarrow C = 10$$

$$y = -\frac{1}{2}t^{3/2} + 10$$



$$20. \quad \frac{dy}{dt} = \frac{3}{4}y, \quad (0, 10)$$

$$\int \frac{dy}{y} = \int \frac{3}{4} dt$$

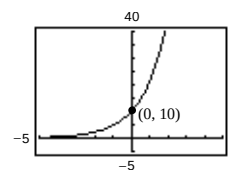
$$\ln y = \frac{3}{4}t + C_1$$

$$y = e^{(3/4)t + C_1}$$

$$= e^{C_1} e^{(3/4)t} = Ce^{3t/4}$$

$$10 = Ce^0 \Rightarrow C = 10$$

$$y = 10e^{3t/4}$$



$$22. \quad \frac{dN}{dt} = kN$$

$$N = Ce^{kt} \quad (\text{Theorem 5.16})$$

$$(0, 250): C = 250$$

$$(1, 400): 400 = 250e^k \Rightarrow k = \ln \frac{400}{250} = \ln \frac{8}{5}$$

$$\text{When } t = 4, N = 250e^{4 \ln(8/5)} = 250e^{\ln(8/5)^4}$$

$$= 250 \left(\frac{8}{5}\right)^4 = \frac{8192}{5}$$

$$24. \quad \frac{dP}{dt} = kP$$

$$P = Ce^{kt} \quad (\text{Theorem 5.16})$$

$$(0, 5000): C = 5000$$

$$(1, 4750): 4750 = 5000e^k \Rightarrow k = \ln \left(\frac{19}{20}\right)$$

$$\text{When } t = 5, P = 5000e^{\ln(19/20)(5)}$$

$$= 5000 \left(\frac{19}{20}\right)^5 \approx 3868.905$$

26. $y = Ce^{kt}$, $(0, 4)$, $\left(5, \frac{1}{2}\right)$

$$C = 4$$

$$y = 4e^{kt}$$

$$\frac{1}{2} = 4e^{5k}$$

$$k = \frac{\ln(1/8)}{5} \approx -0.4159$$

$$y = 4e^{-0.4159t}$$

28. $y = Ce^{kt}$, $\left(3, \frac{1}{2}\right)$, $(4, 5)$

$$\frac{1}{2} = Ce^{3k}$$

$$5 = Ce^{4k}$$

$$2Ce^{3k} = \frac{1}{5}Ce^{4k}$$

$$10e^{3k} = e^{4k}$$

$$10 = e^k$$

$$k = \ln 10 \approx 2.3026$$

$$y = Ce^{2.3026t}$$

$$5 = Ce^{2.3026(4)}$$

$$C \approx 0.0005$$

$$y = 0.0005e^{2.3026t}$$

30. $y' = \frac{dy}{dt} = ky$

32. $\frac{dy}{dx} = \frac{1}{2}x^2y$

$$\frac{dy}{dx} > 0 \text{ when } y > 0. \text{ Quadrants I and II.}$$

34. Since $y = Ce^{[\ln(1/2)/1620]t}$, we have $1.5 = Ce^{[\ln(1/2)/1620](1000)} \Rightarrow C \approx 2.30$ which implies that the initial quantity is 2.30 grams. When $t = 10,000$, we have $y = 2.30e^{[\ln(1/2)/1620](10,000)} \approx 0.03$ gram.

36. Since $y = Ce^{[\ln(1/2)/5730]t}$, we have $2.0 = Ce^{[\ln(1/2)/5730](10,000)} \Rightarrow C \approx 6.70$ which implies that the initial quantity is 6.70 grams. When $t = 1000$, we have $y = 6.70e^{[\ln(1/2)/5730](1000)} \approx 5.94$ grams.

38. Since $y = Ce^{[\ln(1/2)/5730]t}$, we have $3.2 = Ce^{[\ln(1/2)/5730]1000} \Rightarrow C \approx 3.61$.

Initial quantity: 3.61 grams.

When $t = 10,000$, we have $y \approx 1.08$ grams.

40. Since $y = Ce^{[\ln(1/2)/24,360]t}$, we have $0.4 = Ce^{[\ln(1/2)/24,360](10,000)} \Rightarrow C \approx 0.53$ which implies that the initial quantity is 0.53 gram. When $t = 1000$, we have $y = 0.53e^{[\ln(1/2)/24,360](1000)} \approx 0.52$ gram.

42. Since $\frac{dy}{dx} = ky$, $y = Ce^{kt}$ or $y = y_0e^{kt}$.

$$\frac{1}{2}y_0 = y_0e^{5730k}$$

$$k = -\frac{\ln 2}{5730}$$

$$0.15y_0 = y_0e^{(-\ln 2/5730)t}$$

$$\ln 0.15 = -\frac{(\ln 2)t}{5730}$$

$$t = -\frac{5730 \ln 0.15}{\ln 2} \approx 15,682.8 \text{ years.}$$

44. Since $A = 20,000e^{0.055t}$, the time to double is given by $40,000 = 20,000e^{0.055t}$ and we have

$$2 = e^{0.055t}$$

$$\ln 2 = 0.055t$$

$$t = \frac{\ln 2}{0.055} \approx 12.6 \text{ years.}$$

Amount after 10 years:

$$A = 20,000e^{(0.055)(10)} \approx \$34,665.06$$

46. Since $A = 10,000e^{rt}$ and $A = 20,000$ when $t = 5$, we have the following.

$$20,000 = 10,000e^{5r}$$

$$r = \frac{\ln 2}{5} \approx 0.1386 = 13.86\%$$

$$\text{Amount after 10 years: } A = 10,000e^{[(\ln 2)/5](10)} = \$40,000$$

48. Since $A = 2000e^{rt}$ and $A = 5436.56$ when $t = 10$, we have the following.

$$5436.56 = 2000e^{10r}$$

$$r = \frac{\ln(5436.56/2000)}{10} \approx 0.10 = 10\%$$

The time to double is given by

$$4000 = 2000e^{0.10t}$$

$$t = \frac{\ln 2}{0.10} \approx 6.93 \text{ years.}$$

50. $500,000 = P\left(1 + \frac{0.06}{12}\right)^{(12)(40)}$

$$P = 500,000(1.005)^{-480} \approx \$45,631.04$$

52. $500,000 = P\left(1 + \frac{0.09}{12}\right)^{(12)(25)}$

$$P = 500,000\left(1 + \frac{0.09}{12}\right)^{-300} \approx \$53,143.92$$

54. (a) $2000 = 1000(1 + 0.6)^t$

$$2 = 1.06^t$$

$$\ln 2 = t \ln 1.06$$

$$t = \frac{\ln 2}{\ln 1.06} \approx 11.90 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.06}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.06}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.06}{12}\right)$$

$$t = \frac{1}{12} \frac{\ln 2}{\ln\left(1 + \frac{0.06}{12}\right)} \approx 11.58 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.06}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.06}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.06}{365}\right)$$

$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \frac{0.06}{365}\right)} \approx 11.55 \text{ years}$$

(d) $2000 = 1000e^{0.06t}$

$$2 = e^{0.06t}$$

$$\ln 2 = 0.06t$$

$$t = \frac{\ln 2}{0.06} \approx 11.55 \text{ years}$$

56. (a) $2000 = 1000(1 + 0.055)^t$

$$2 = 1.055^t$$

$$\ln 2 = t \ln 1.055$$

$$t = \frac{\ln 2}{\ln 1.055} \approx 12.95 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.055}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.055}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.055}{12}\right)$$

$$t = \frac{1}{12} \frac{\ln 2}{\ln\left(1 + \frac{0.055}{12}\right)} \approx 12.63 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.055}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.055}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.055}{365}\right)$$

$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \frac{0.055}{365}\right)} \approx 12.60 \text{ years}$$

(d) $2000 = 1000e^{0.055t}$

$$2 = e^{0.055t}$$

$$\ln 2 = 0.055t$$

$$t = \frac{\ln 2}{0.055} \approx 12.60 \text{ years}$$

58. $P = Ce^{kt} = Ce^{0.031t}$

$$P(-1) = 11.6 = Ce^{0.031(-1)} \Rightarrow C = 11.9652$$

$$P = 11.9652e^{0.031t}$$

$$P(10) \approx 16.31 \text{ or } 16,310,000 \text{ people in 2010}$$

60. $P = Ce^{kt} = Ce^{-0.004t}$

$$P(-1) = 3.6 = Ce^{-0.004(-1)} \Rightarrow C = 3.5856$$

$$P = 3.5856e^{-0.004t}$$

$$P(10) \approx 3.45 \text{ or } 3,450,000 \text{ people in 2010}$$

62. (a) $N = 100.1596(1.2455)^t$

(b) $N = 400$ when $t = 6.3$ hours (graphing utility)

Analytically,

$$400 = 100.1596(1.2455)^t$$

$$1.2455^t = \frac{400}{100.1596} = 3.9936$$

$$t \ln 1.2455 = \ln 3.9936$$

$$t = \frac{\ln 3.9936}{\ln 1.2455} \approx 6.3 \text{ hours.}$$

64. $y = Ce^{kt}$, $(0, 742,000)$, $(2, 632,000)$

$$C = 742,000$$

$$632,000 = 742,000e^{2k}$$

$$k = \frac{\ln(632/742)}{2} \approx -0.0802$$

$$y \approx 742,000e^{-0.0802t}$$

When $t = 4$, $y \approx \$538,372$.

66. (a) $20 = 30(1 - e^{30k})$

$$30e^{30k} = 10$$

$$k = \frac{\ln(1/3)}{30} = \frac{-\ln 3}{30} \approx -0.0366$$

$$N \approx 30(1 - e^{-0.0366t})$$

(b) $25 = 30(1 - e^{-0.0366t})$

$$e^{-0.0366t} = \frac{1}{6}$$

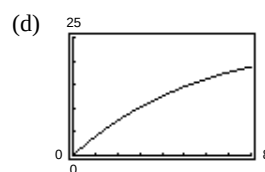
$$t = \frac{-\ln 6}{-0.0366} \approx 49 \text{ days}$$

68. $S = 25(1 - e^{kt})$

(a) $4 = 25(1 - e^{k(1)}) \Rightarrow 1 - e^k = \frac{4}{25} \Rightarrow e^k = \frac{21}{25} \Rightarrow k = \ln\left(\frac{21}{25}\right) \approx -0.1744$

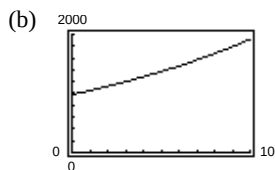
(b) 25,000 units $\left(\lim_{t \rightarrow \infty} S = 25\right)$

(c) When $t = 5$, $S \approx 14.545$ which is 14,545 units.

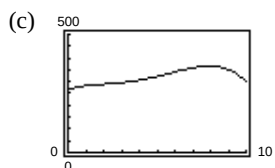


70. (a) $R = 979.3993(1.0694)^t = 979.3993e^{0.0671t}$

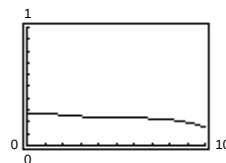
$$I = -0.1385t^4 + 2.1770t^3 - 9.9755t^2 + 23.8513t + 266.4923$$



Rate of growth $= R'(t) = 65.7e^{0.0671t}$



(d) $P(t) = \frac{I}{R}$



72. $93 = 10 \log_{10} \frac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$$-6.7 = \log_{10} I \Rightarrow I = 10^{-6.7}$$

$$80 = 10 \log_{10} \frac{I}{10^{-16}} = 10(\log_{10} I + 16)$$

$$-8 = \log_{10} I \Rightarrow I = 10^{-8}$$

$$\text{Percentage decrease: } \left(\frac{10^{-6.7} - 10^{-8}}{10^{-6.7}} \right) (100) \approx 95\%$$

74. Since $\frac{dy}{dt} = k(y - 80)$

$$\int \frac{1}{y - 80} dy = \int k dt$$

$$\ln(y - 80) = kt + C.$$

When $t = 0, y = 1500$. Thus, $C = \ln 1420$.

When $t = 1, y = 1120$. Thus,

$$k(1) + \ln 1420 = \ln(1120 - 80)$$

$$k = \ln 1040 - \ln 1420 = \ln \frac{104}{142}.$$

Thus, $y = 1420e^{[\ln(104/142)]t} + 80$.

When $t = 5, y \approx 379.2^\circ$.

76. True

78. True

Section 5.7 Differential Equations: Separation of Variables

2. Differential equation: $y' = \frac{2xy}{x^2 - y^2}$

Solution: $x^2 + y^2 = Cy$

Check: $2x + 2yy' = Cy'$

$$y' = \frac{-2x}{(2y - C)}$$

$$y' = \frac{-2xy}{2y^2 - Cy} = \frac{-2xy}{2y^2 - (x^2 + y^2)} = \frac{-2xy}{y^2 - x^2} = \frac{2xy}{x^2 - y^2}$$

4. Differential equation: $y'' + 2y' + 2y = 0$

Solution: $y = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x$

Check: $y' = -(C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x$

$$y'' = 2C_1 e^{-x} \sin x - 2C_2 e^{-x} \cos x$$

$$y'' + 2y' + 2y = 2C_1 e^{-x} \sin x - 2C_2 e^{-x} \cos x +$$

$$2(-(C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x) + 2(C_1 e^{-x} \cos x + C_2 e^{-x} \sin x)$$

$$= (2C_1 - 2C_1 - 2C_2 + 2C_2)e^{-x} \sin x + (-2C_2 - 2C_1 + 2C_2 + 2C_1)e^{-x} \cos x = 0$$

6. $y = \frac{2}{3}(e^{-2x} + e^x)$

$$y' = \frac{2}{3}(-2e^{-2x} + e^x)$$

$$y'' = \frac{2}{3}(4e^{-2x} + e^x)$$

Substituting, $y'' + 2y' = \frac{2}{3}(4e^{-2x} + e^x) + 2\left(\frac{2}{3}\right)(-2e^{-2x} + e^x) = 2e^x$.

In Exercises 8–12, the differential equation is $y^{(4)} - 16y = 0$.

8. $y = 3 \cos 2x$

$$y^{(4)} = 48 \cos 2x$$

$$y^{(4)} - 16y = 48 \cos 2x - 48 \cos 2x = 0, \quad \text{Yes.}$$

10. $y = 5 \ln x$

$$y^{(4)} = -\frac{30}{x^4}$$

$$y^{(4)} - 16y = -\frac{30}{x^4} - 80 \ln x \neq 0, \quad \text{No.}$$

12. $y = 3e^{2x} - 4 \sin 2x$

$$y^{(4)} = 48e^{2x} - 64 \sin 2x$$

$$y^{(4)} - 16y = (48e^{2x} - 64 \sin 2x) - 16(3e^{2x} - 4 \sin 2x) = 0, \quad \text{Yes}$$

In 14–18, the differential equation is $xy' - 2y = x^3e^x$.

14. $y = x^2e^x, y' = x^2e^x + 2xe^x = e^x(x^2 + 2x)$

$$xy' - 2y = x(e^x(x^2 + 2x)) - 2(x^2e^x) = x^3e^x, \quad \text{Yes.}$$

16. $y = \sin x, y' = \cos x$

$$xy' - 2y = x(\cos x) - 2(\sin x) \neq x^3e^x, \quad \text{No.}$$

18. $y = x^2e^x - 5x^2, y' = x^2e^x + 2xe^x - 10x$

$$xy' - 2y = x[x^2e^x + 2xe^x - 10x] - 2[x^2e^x - 5x^2] = x^3e^x, \quad \text{Yes.}$$

20. $y = A \sin \omega t$

$$\frac{d^2y}{dt^2} = -A\omega^2 \sin \omega t$$

Since $(d^2y/dt^2) + 16y = 0$, we have

$$-A\omega^2 \sin \omega t + 16A \sin \omega t = 0.$$

Thus, $\omega^2 = 16$ and $\omega = \pm 4$.

22. $2x^2 - y^2 = C$ passes through (3, 4)

$$2(9) - 16 = C \Rightarrow C = 2$$

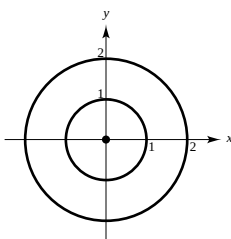
$$\text{Particular solution: } 2x^2 - y^2 = 2$$

24. Differential equation: $yy' + x = 0$

$$\text{General solution: } x^2 + y^2 = C$$

$$\text{Particular solutions: } C = 0, \text{ Point}$$

$$C = 1, C = 4, \text{ Circles}$$



26. Differential equation: $3x + 2yy' = 0$

$$\text{General solution: } 3x^2 + 2y^2 = C$$

$$6x + 4yy' = 0$$

$$2(3x + 2yy') = 0$$

$$3x + 2yy' = 0$$

Initial condition:

$$y(1) = 3: 3(1)^2 + 2(3)^2 = 3 + 18 = 21 = C$$

$$\text{Particular solution: } 3x^2 + 2y^2 = 21$$

28. Differential equation: $xy'' + y' = 0$ General solution: $y = C_1 + C_2 \ln x$

$$y' = C_2 \left(\frac{1}{x} \right), y'' = -C_2 \left(\frac{1}{x^2} \right)$$

$$xy'' + y' = x \left(-C_2 \frac{1}{x^2} \right) + C_2 \frac{1}{x} = 0$$

Initial conditions: $y(2) = 0, y'(2) = \frac{1}{2}$

$$0 = C_1 + C_2 \ln 2$$

$$y' = \frac{C_2}{x}$$

$$\frac{1}{2} = \frac{C_2}{2} \Rightarrow C_2 = 1, C_1 = -\ln 2$$

Particular solution: $y = -\ln 2 + \ln x = \ln \frac{x}{2}$ 30. Differential equation: $9y'' - 12y' + 4y = 0$ General solution: $y = e^{2x/3}(C_1 + C_2 x)$

$$y' = \frac{2}{3}e^{2x/3}(C_1 + C_2 x) + C_2 e^{2x/3} = e^{2x/3} \left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2 x \right)$$

$$y'' = \frac{2}{3}e^{2x/3} \left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2 x \right) + e^{2x/3} \frac{2}{3}C_2 = \frac{2}{3}e^{2x/3} \left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2 x \right)$$

$$9y'' - 12y' + 4y = 9 \left(\frac{2}{3}e^{2x/3} \right) \left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2 x \right) - 12(e^{2x/3}) \left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2 x \right) + 4(e^{2x/3})(C_1 + C_2 x) = 0$$

Initial conditions: $y(0) = 4, y(3) = 0$

$$0 = e^2(C_1 + 3C_2)$$

$$4 = (1)(C_1 + 0) \Rightarrow C_1 = 4$$

$$0 = e^2(4 + 3C_2) \Rightarrow C_2 = -\frac{4}{3}$$

Particular solution: $y = e^{2x/3} \left(4 - \frac{4}{3}x \right)$ 32. $\frac{dy}{dx} = x^3 - 4x$

$$y = \int (x^3 - 4x) dx = \frac{x^4}{4} - 2x^2 + C$$

34. $\frac{dy}{dx} = \frac{e^x}{1 + e^x}$

$$y = \int \frac{e^x}{1 + e^x} dx = \ln(1 + e^x) + C$$

36. $\frac{dy}{dx} = x \cos x^2$

$$y = \int x \cos(x^2) dx = \frac{1}{2} \sin(x^2) + C$$

$$(u = x^2, du = 2x dx)$$

38. $\frac{dy}{dx} = \tan^2 x = \sec^2 x - 1$

$$y = \int (\sec^2 x - 1) dx = \tan x - x + C$$

40. $\frac{dy}{dx} = x\sqrt{5-x}$. Let $u = \sqrt{5-x}, u^2 = 5-x, dx = -2u du$

$$y = \int x\sqrt{5-x} dx = \int (5 - u^2)u(-2u) du$$

$$= \int (-10u^2 + 2u^4) du$$

$$= -\frac{10u^3}{3} + \frac{2u^5}{5} + C$$

$$= -\frac{10}{3}(5-x)^{3/2} + \frac{2}{5}(5-x)^{5/2} + C$$

42. $\frac{dy}{dx} = 5e^{-x/2}$

$$y = \int 5e^{-x/2} dx = 5(-2) \int e^{-x/2} \left(-\frac{1}{2}\right) dx$$

$$= -10e^{-x/2} + C$$

44. $\frac{dy}{dx} = \frac{x^2 + 2}{3y^2}$

$$\int 3y^2 dy = \int (x^2 + 2) dx$$

$$y^3 = \frac{x^3}{3} + 2x + C$$

46. $\frac{dr}{ds} = 0.05s$

$$\int dr = \int 0.05s ds$$

$$r = 0.025s^2 + C$$

48. $xy' = y$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + \ln C = \ln Cx$$

$$y = Cx$$

50. $y \frac{dy}{dx} = 6 \cos \pi x$

$$\int y dy = \int 6 \cos \pi x dx$$

$$\frac{y^2}{2} = \frac{6}{\pi} \sin \pi x + C_1$$

$$y^2 = \frac{12}{\pi} \sin \pi x + C$$

52. $\sqrt{x^2 - 9} \frac{dy}{dx} = 5x$

$$\int dy = \int \frac{5x}{\sqrt{x^2 - 9}} dx$$

$$y = 5(x^2 - 9)^{1/2} + C$$

54. $4y \frac{dy}{dx} = 3e^x$

$$\int 4y dy = \int 3e^x dx$$

$$2y^2 = 3e^x + C$$

56. $\sqrt{x} + \sqrt{y} y' = 0$

$$\int y^{1/2} dy = - \int x^{1/2} dx$$

$$\frac{2}{3} y^{3/2} = -\frac{2}{3} x^{3/2} + C_1$$

$$y^{3/2} + x^{3/2} = C$$

Initial condition: $y(1) = 4$,
 $(4)^{3/2} + (1)^{3/2} = 8 + 1 = 9 = C$

Particular solution: $y^{3/2} + x^{3/2} = 9$

58. $2xy' - \ln x^2 = 0$

$$2x \frac{dy}{dx} = 2 \ln x$$

$$\int dy = \int \frac{\ln x}{x} dx$$

$$y = \frac{(\ln x)^2}{2} + C$$

$y(1) = 2$: $2 = C$

$y = \frac{1}{2}(\ln x)^2 + 2$

60. $y\sqrt{1-x^2} \frac{dy}{dx} = x\sqrt{1-y^2}$

$$\int (1-y^2)^{-1/2} y dy = \int (1-x^2)^{-1/2} x dx$$

$$-(1-y^2)^{1/2} = -(1-x^2)^{1/2} + C$$

$y(0) = 1$: $0 = -1 + C \Rightarrow C = 1$

$$\sqrt{1-y^2} = \sqrt{1-x^2} - 1$$

62. $\frac{dr}{ds} = e^{r-2s}$

$$\int e^{-r} dr = \int e^{-2s} ds$$

$$-e^{-r} = -\frac{1}{2}e^{-2s} + C$$

$r(0) = 0$: $-1 = -\frac{1}{2} + C \Rightarrow C = -\frac{1}{2}$

$$-e^{-r} = -\frac{1}{2}e^{-2s} - \frac{1}{2}$$

$$e^{-r} = \frac{1}{2}e^{-2s} + \frac{1}{2}$$

$$-r = \ln\left(\frac{1}{2}e^{-2s} + \frac{1}{2}\right) = \ln\left(\frac{1+e^{-2s}}{2}\right)$$

$$r = \ln\left(\frac{2}{1+e^{-2s}}\right)$$

64. $dT + k(T - 70) dt = 0$

$$\int \frac{dT}{T - 70} = -k \int dt$$

$$\ln(T - 70) = -kt + C_1$$

$$T - 70 = Ce^{-kt}$$

Initial condition: $T(0) = 140$;

$$140 - 70 = 70 = Ce^0 = C$$

Particular solution: $T - 70 = 70e^{-kt}$, $T = 70(1 + e^{-kt})$

66. $\frac{dy}{dx} = \frac{2y}{3x}$

$$\int \frac{3}{y} dy = \int \frac{2}{x} dx$$

$$\ln y^3 = \ln x^2 + \ln C$$

$$y^3 = Cx^2$$

Initial condition: $y(8) = 2$, $2^3 = C(8^2)$, $C = \frac{1}{8}$

Particular solution: $8y^3 = x^2$, $y = \frac{1}{2}x^{2/3}$

68. $m = \frac{dy}{dx} = \frac{y-0}{x-0} = \frac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + C_1 = \ln x + \ln C = \ln Cx$$

$$y = Cx$$

70. $f(x, y) = x^3 + 3x^2y^2 - 2y^2$

$$f(tx, ty) = t^3x^3 + 3t^4x^2y^2 - 2t^2y^2$$

Not homogeneous

72. $f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{tx ty}{\sqrt{t^2 x^2 + t^2 y^2}}$$

$$= \frac{t^2 xy}{t\sqrt{x^2 + y^2}} = t \frac{xy}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 1

74. $f(x, y) = \tan(x + y)$

$$f(tx, ty) = \tan(tx + ty) = \tan[t(x + y)]$$

Not homogeneous

76. $f(x, y) = \tan \frac{y}{x}$

$$f(tx, ty) = \tan \frac{ty}{tx} = \tan \frac{y}{x}$$

Homogeneous of degree 0

78. $y' = \frac{(x^3 + y^3)}{xy^2}$

$$xy^2 dy = (x^3 + y^3) dx$$

$$y = vx, \quad dy = x dv + v dx$$

$$x(vx)^2(x dv + v dx) = (x^3 + (vx)^3) dx$$

$$x^4 v^2 dv + x^3 v^3 dx = x^3 dx + v^3 x^3 dx$$

$$xv^2 dv = dx$$

$$\int v^2 dv = \int \frac{1}{x} dx$$

$$\frac{v^3}{3} = \ln|x| + C$$

$$\left(\frac{y}{x}\right)^3 = 3 \ln|x| + C$$

$$y^3 = 3x^3 \ln|x| + Cx^3$$

80. $y' = \frac{x^2 + y^2}{2xy}$, $y = vx$

$$v + x \frac{dv}{dx} = \frac{x^2 + v^2 x^2}{2x^2 v}$$

$$2v dx + 2x dv = \frac{1 + v^2}{v} dx$$

$$\int \frac{2v}{v^2 - 1} dv = - \int \frac{dx}{x}$$

$$\ln(v^2 - 1) = -\ln x + \ln C = \ln \frac{C}{x}$$

$$v^2 - 1 = \frac{C}{x}$$

$$\frac{y^2}{x^2} - 1 = \frac{C}{x}$$

$$y^2 - x^2 = Cx$$

82. $y' = \frac{2x + 3y}{x}, y = vx$

$$v + x \frac{dv}{dx} = \frac{2x + 3vx}{x} = 2 + 3v$$

$$x \frac{dv}{dx} = 2 + 2v \Rightarrow \int \frac{dv}{1 + v} = 2 \int \frac{dx}{x}$$

$$\ln|1 + v| = \ln x^2 + \ln C = \ln x^2 C$$

$$1 + v = x^2 C$$

$$1 + \frac{y}{x} = x^2 C$$

$$\frac{y}{x} = x^2 C - 1$$

$$y = Cx^3 - x$$

84. $-y^2 dx + x(x + y) dy = 0, y = vx$

$$-x^2 v^2 dx + (x^2 + x^2 v)(v dx + x dv) = 0$$

$$\int \frac{1 + v}{v} dv = - \int \frac{dx}{x}$$

$$v + \ln v = -\ln x + \ln C_1 = \ln \frac{C_1}{x}$$

$$v = \ln \frac{C_1}{xv}$$

$$\frac{C_1}{vx} = e^v$$

$$\frac{C_1}{y} = e^{y/x}$$

$$y = Ce^{-y/x}$$

Initial condition: $y(1) = 1, 1 = Ce^{-1} \Rightarrow C = e$

Particular solution: $y = e^{1-y/x}$

86. $(2x^2 + y^2) dx + xy dy = 0$

Let $y = vx, dy = x dv + v dx$.

$$(2x^2 + v^2 x^2) dx + x(vx)(x dv + v dx) = 0$$

$$(2x^2 + 2x^2 v^2) dx + x^3 v dv = 0$$

$$(2 + 2v^2) dx = -xv dv$$

$$\frac{-2}{x} dx = \frac{v}{1 + v^2} dv$$

$$-2 \ln x = \frac{1}{2} \ln(1 + v^2) + C_1$$

$$\ln x^{-2} = \ln(1 + v^2)^{1/2} + \ln C$$

$$x^{-2} = C(1 + v^2)^{1/2}$$

$$\frac{1}{x^2} = C \left(1 + \frac{y^2}{x^2}\right)^{1/2} = \frac{C}{x} (x^2 + y^2)^{1/2}$$

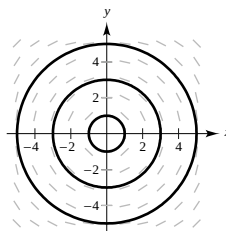
$$\frac{1}{x} = C(x^2 + y^2)^{1/2}$$

$y(1) = 0: 1 = C(1 + 0) \Rightarrow C = 1$

$$\frac{1}{x} = \sqrt{x^2 + y^2}$$

$$1 = x\sqrt{x^2 + y^2}$$

88. $\frac{dy}{dx} = -\frac{x}{y}$



$$y dy = -x dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C_1$$

$$y^2 + x^2 = C$$

90. $\frac{dy}{dx} = 0.25x(4 - y)$

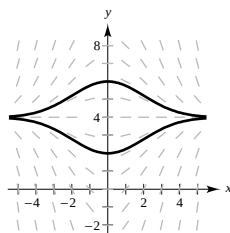
$$\frac{dy}{4 - y} = 0.25x \, dx$$

$$\begin{aligned} \int \frac{dy}{y - 4} &= \int -0.25x \, dx \\ &= -\frac{1}{4} \int x \, dx \end{aligned}$$

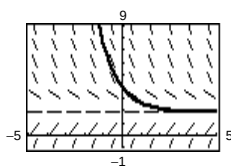
$$\ln |y - 4| = -\frac{1}{8}x^2 + C_1$$

$$y - 4 = e^{C_1 - (1/8)x^2} = Ce^{-(1/8)x^2}$$

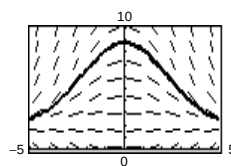
$$y = 4 + Ce^{-(1/8)x^2}$$



92. $\frac{dy}{dx} = 2 - y, y(0) = 4$



94. $\frac{dy}{dx} = 0.2x(2 - y), y(0) = 9$



96. $\frac{dy}{dt} = ky, y = Ce^{kt}$

Initial conditions: $y(0) = 20, y(1) = 16$

$$20 = Ce^0 = C$$

$$16 = 20e^k$$

$$k = \ln \frac{4}{5}$$

Particular solution: $y = 20e^{t \ln(4/5)}$

When 75% has been changed:

$$5 = 20e^{t \ln(4/5)}$$

$$\frac{1}{4} = e^{t \ln(4/5)}$$

$$t = \frac{\ln(1/4)}{\ln(4/5)} \approx 6.2 \text{ hr}$$

98. $\frac{dy}{dx} = k(x - 4)$

The direction field satisfies $(dy/dx) = 0$ along $x = 4$:
Matches (b).

100. $\frac{dy}{dx} = ky^2$

The direction field satisfies $(dy/dx) = 0$ along $y = 0$, and grows more positive as y increases. Matches (d).

102. From Exercise 101,

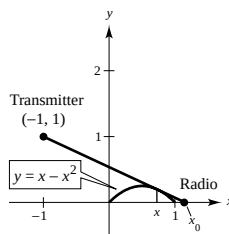
$$w = 1200 - Ce^{-kt}, k = 1$$

$$w = 1200 - Ce^{-t}$$

$$w(0) = w_0 = 1200 - C \Rightarrow C = 1200 - w_0$$

$$w = 1200 - (1200 - w_0)e^{-t}$$

104. Let the radio receiver be located at $(x_0, 0)$.
The tangent line to $y = x - x^2$ joins $(-1, 1)$
and $(x_0, 0)$.



- (a) If (x, y) is the point of tangency on the $y = x - x^2$, then

$$1 - 2x = \frac{y - 1}{x + 1} = \frac{x - x^2 - 1}{x + 1}$$

$$x - 2x^2 + 1 - 2x = x - x^2 - 1$$

$$x^2 + 2x - 2 = 0$$

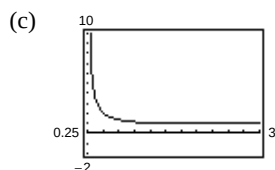
$$x = \left(\frac{-2 \pm \sqrt{4 + 8}}{2} \right) = -1 + \sqrt{3}$$

$$y = x - x^2 = 3\sqrt{3} - 5$$

$$\text{Then } \frac{1 - 0}{-1 - x_0} = \frac{1 - 3\sqrt{3} + 5}{-1 + 1 - \sqrt{3}} = \frac{6 - 3\sqrt{3}}{-\sqrt{3}}$$

$$\begin{aligned} \sqrt{3} &= (1 + x_0)(6 - 3\sqrt{3}) \\ &= 6 - 3\sqrt{3} + x_0(6 - 3\sqrt{3}) \end{aligned}$$

$$x_0 = \frac{4\sqrt{3} - 6}{6 - 3\sqrt{3}} \approx 1.155$$



There is a vertical asymptote at $h = \frac{1}{4}$, which is the height of the mountain.

- (b) Now let the transmitter be located at $(-1, h)$.

$$1 - 2x = \frac{y - h}{x + 1} = \frac{x - x^2 - h}{x + 1}$$

$$x - 2x^2 + 1 - 2x = x - x^2 - h$$

$$x^2 + 2x - h - 1 = 0$$

$$x = \frac{(-2 \pm \sqrt{4 + 4(h + 1)})}{2}$$

$$= -1 + \sqrt{2 + h}$$

$$y = x - x^2$$

$$= 3\sqrt{2 + h} - h - 4$$

$$\text{Then, } \frac{h - 0}{-1 - x_0} = \frac{h - (3\sqrt{2 + h} - h - 4)}{-1 - (-1 + \sqrt{2 + h})}$$

$$= \frac{2h + 4 - 3\sqrt{2 + h}}{-\sqrt{2 + h}}$$

$$\frac{x_0 + 1}{h} = \frac{\sqrt{2 + h}}{2h + 4 - 3\sqrt{2 + h}}$$

$$x_0 = \frac{h\sqrt{2 + h}}{2h + 4 - 3\sqrt{2 + h}} - 1$$

106. Given family (hyperbolas): $x^2 - 2y^2 = C$

$$2x - 4yy' = 0$$

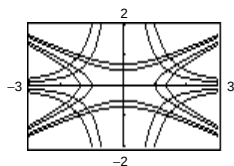
$$y' = \frac{x}{2y}$$

Orthogonal trajectory: $y' = \frac{-2y}{x}$

$$\int \frac{dy}{y} = -\int \frac{2}{x} dx$$

$$\ln y = -2 \ln x + \ln k$$

$$y = kx^{-2} = \frac{k}{x^2}$$



108. Given family (parabolas): $y^2 = 2Cx$

$$2yy' = 2C$$

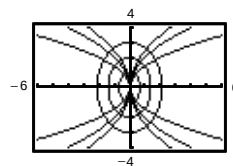
$$y' = \frac{C}{y} = \frac{y^2}{2x} \left(\frac{1}{y} \right) = \frac{y}{2x}$$

Orthogonal trajectory (ellipse): $y' = -\frac{2x}{y}$

$$\int y dy = -\int 2x dx$$

$$\frac{y^2}{2} = -x^2 + K_1$$

$$2x^2 + y^2 = K$$



110. Given family (exponential functions): $y = Ce^x$

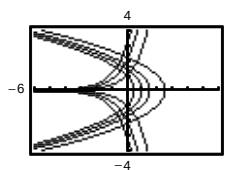
$$y' = Ce^x = y$$

Orthogonal trajectory (parabolas): $y' = -\frac{1}{y}$

$$\int y \, dy = -\int dx$$

$$\frac{y^2}{2} = -x + K_1$$

$$y^2 = -2x + K$$



112. The number of initial conditions matches the number of constants in the general solution.

114. Two families of curves are mutually orthogonal if each curve in the first family intersects each curve in the second family at right angles.

116. True

$$\frac{dy}{dx} = (x - 2)(y + 1)$$

118. True

$$x^2 + y^2 = 2Cy$$

$$x^2 + y^2 = 2Kx$$

$$\frac{dy}{dx} = \frac{x}{C - y}$$

$$\frac{dy}{dx} = \frac{K - x}{y}$$

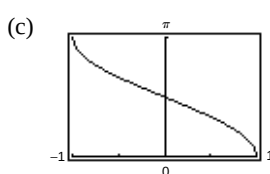
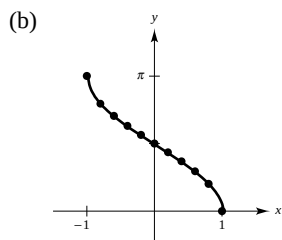
$$\frac{x}{C - y} \cdot \frac{K - x}{y} = \frac{Kx - x^2}{Cy - y^2} = \frac{2Kx - 2x^2}{2Cy - 2y^2} = \frac{x^2 + y^2 - 2x^2}{x^2 + y^2 - 2y^2} = \frac{y^2 - x^2}{x^2 - y^2} = -1$$

Section 5.8 Inverse Trigonometric Functions: Differentiation

2. $y = \arccos x$

(a)

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y	3.142	2.499	2.214	1.982	1.772	1.571	1.369	1.159	0.927	0.634	0



(d) Intercepts: $\left(0, \frac{\pi}{2}\right)$ and $(1, 0)$

No symmetry

4. $\left(\_, \frac{\pi}{4}\right) = \left(1, \frac{\pi}{4}\right)$

$$\left(\_, -\frac{\pi}{6}\right) = \left(-\frac{\sqrt{3}}{3}, -\frac{\pi}{6}\right)$$

$$\left(-\sqrt{3}, \_\right) = \left(-\sqrt{3}, -\frac{\pi}{3}\right)$$

6. $\arcsin 0 = 0$

8. $\arccos 0 = \frac{\pi}{2}$

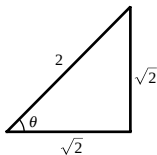
10. $\operatorname{arc} \cot(-\sqrt{3}) = \frac{5\pi}{6}$

12. $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

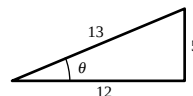
14. $\arcsin(-0.39) \approx -0.40$

16. $\arctan(-3) \approx -1.25$

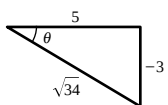
18. (a) $\tan\left(\arccos \frac{\sqrt{2}}{2}\right) = \tan\left(\frac{\pi}{4}\right) = 1$



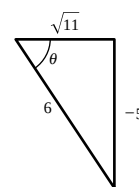
(b) $\cos\left(\arcsin \frac{5}{13}\right) = \frac{12}{13}$



20. (a) $\sec\left[\arctan\left(-\frac{3}{5}\right)\right] = \frac{\sqrt{34}}{5}$



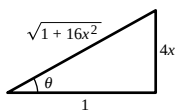
(b) $\tan\left[\arcsin\left(-\frac{5}{6}\right)\right] = -\frac{5\sqrt{11}}{11}$



22. $y = \sec(\arctan 4x)$

$\theta = \arctan 4x$

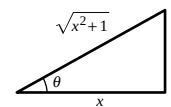
$y = \sec \theta = \sqrt{1 + 16x^2}$



24. $y = \cos(\operatorname{arccot} x)$

$\theta = \operatorname{arccot} x$

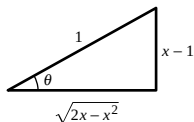
$y = \cos \theta = \frac{x}{\sqrt{x^2 + 1}}$



26. $y = \sec[\arcsin(x - 1)]$

$\theta = \arcsin(x - 1)$

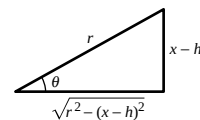
$y = \sec \theta = \frac{1}{\sqrt{2x - x^2}}$



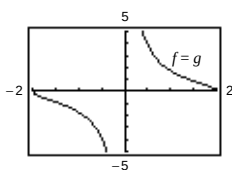
28. $y = \cos\left(\arcsin \frac{x - h}{r}\right)$

$\theta = \arcsin \frac{x - h}{r}$

$y = \cos \theta = \frac{\sqrt{r^2 - (x - h)^2}}{r}$



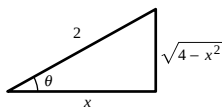
30.

Asymptote: $x = 0$

$\arccos \frac{x}{2} = \theta$

$\cos \theta = \frac{x}{2}$

$\tan \theta = \frac{\sqrt{4 - x^2}}{x}$



32. $\arctan(2x - 5) = -1$

$2x - 5 = \tan(-1)$

$x = \frac{1}{2}(\tan(-1) + 5) \approx 1.721$

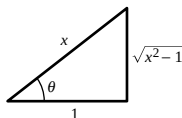
34. $\arccos x = \operatorname{arcsec} x$

$$x = \cos(\operatorname{arcsec} x)$$

$$x = \frac{1}{x}$$

$$x^2 = 1$$

$$x = \pm 1$$



36. (a) $\arcsin(-x) = -\arcsin x, |x| \leq 1.$

Let $y = \arcsin(-x)$. Then,

$$-x = \sin y \Rightarrow x = -\sin y \Rightarrow x = \sin(-y).$$

Thus, $-y = \arcsin x \Rightarrow y = -\arcsin x$. Therefore, $\arcsin(-x) = -\arcsin x$.

(b) $\arccos(-x) = \pi - \arccos x, |x| \leq 1.$

Let $y = \arccos(-x)$. Then,

$$-x = \cos y \Rightarrow x = -\cos y \Rightarrow x = \cos(\pi - y).$$

Thus, $\pi - y = \arccos x \Rightarrow y = \pi - \arccos x$. Therefore, $\arccos(-x) = \pi - \arccos x$.

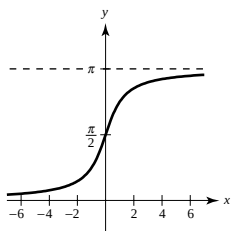
38. $f(x) = \arctan x + \frac{\pi}{2}$

$$x = \tan\left(y - \frac{\pi}{2}\right)$$

Domain: $(-\infty, \infty)$

Range: $(0, \pi)$

$f(x)$ is the graph of $\arctan x$ shifted $\pi/2$ units upward.



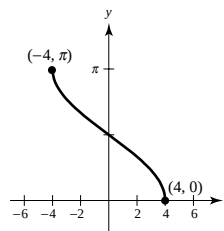
40. $f(x) = \arccos\left(\frac{x}{4}\right)$

$$\frac{x}{4} = \cos y$$

$$x = 4 \cos y$$

Domain: $[-4, 4]$

Range: $[0, \pi]$



42. $f(t) = \arcsin t^2$

$$f'(t) = \frac{2t}{\sqrt{1-t^4}}$$

44. $f(x) = \operatorname{arcsec} 2x$

$$f'(x) = \frac{2}{|2x|\sqrt{4x^2-1}} = \frac{1}{|x|\sqrt{4x^2-1}}$$

46. $f(x) = \arctan \sqrt{x}$

$$f'(x) = \left(\frac{1}{1+x}\right)\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{x}(1+x)}$$

48. $h(x) = x^2 \arctan x$

$$h'(x) = 2x \arctan x + \frac{x^2}{1+x^2}$$

50. $f(x) = \arcsin x + \arccos x = \frac{\pi}{2}$

$$f'(x) = 0$$

52. $y = \ln(t^2 + 4) - \frac{1}{2} \arctan \frac{t}{2}$

$$\begin{aligned} y' &= \frac{2t}{t^2+4} - \frac{1}{2} \cdot \frac{1}{1+\left(\frac{t}{2}\right)^2} \left(\frac{1}{2}\right) \\ &= \frac{2t}{t^2+4} - \frac{1}{t^2+4} = \frac{2t-1}{t^2+4} \end{aligned}$$

54. $y = \frac{1}{2} \left[x\sqrt{4-x^2} + 4 \arcsin\left(\frac{x}{2}\right) \right]$

$$\begin{aligned} y' &= \frac{1}{2} \left[x \frac{1}{2} (4-x^2)^{-1/2} (-2x) + \sqrt{4-x^2} + 2 \frac{1}{\sqrt{1-(x/2)^2}} \right] \\ &= \frac{1}{2} \left[\frac{-x^2}{\sqrt{4-x^2}} + \sqrt{4-x^2} + \frac{4}{\sqrt{4-x^2}} \right] \\ &= \sqrt{4-x^2} \end{aligned}$$

56. $y = x \arctan 2x - \frac{1}{4} \ln(1+4x^2)$

$$\frac{dy}{dx} = \frac{2x}{1+4x^2} + \arctan(2x) - \frac{1}{4} \left(\frac{8x}{1+4x^2} \right) = \arctan(2x)$$

58. $y = 25 \arcsin \frac{x}{5} - x\sqrt{25 - x^2}$

$$\begin{aligned} y' &= 5 \frac{1}{\sqrt{1 - (x/5)^2}} - \sqrt{25 - x^2} - x \frac{1}{2} (25 - x^2)^{-1/2} (-2x) \\ &= \frac{25}{\sqrt{25 - x^2}} - \frac{(25 - x^2)}{\sqrt{25 - x^2}} + \frac{x^2}{\sqrt{25 - x^2}} \\ &= \frac{2x^2}{\sqrt{25 - x^2}} \end{aligned}$$

60. $y = \arctan \frac{x}{2} - \frac{1}{2(x^2 + 4)}$

$$\begin{aligned} y' &= \frac{1}{2} \frac{1}{1 + (x/2)^2} + \frac{1}{2} (x^2 + 4)^{-2} (2x) \\ &= \frac{2}{x^2 + 4} + \frac{x}{(x^2 + 4)^2} \\ &= \frac{2x^2 + 8 + x}{(x^2 + 4)^2} \end{aligned}$$

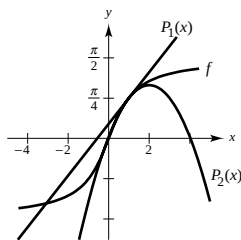
62. $f(x) = \arctan x$, $a = 1$

$$f'(x) = \frac{1}{1 + x^2}$$

$$f''(x) = \frac{-2x}{(1 + x^2)^2}$$

$$P_1(x) = f(1) + f'(1)(x - 1) = \frac{\pi}{4} + \frac{1}{2}(x - 1)$$

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2} f''(1)(x - 1)^2 = \frac{\pi}{4} + \frac{1}{2}(x - 1) - \frac{1}{4}(x - 1)^2$$



64. $f(x) = \arcsin x - 2x$

$$f'(x) = \frac{1}{\sqrt{1 - x^2}} - 2 = 0 \text{ when } \sqrt{1 - x^2} = \frac{1}{2} \text{ or}$$

$$x = \pm \frac{\sqrt{3}}{2}.$$

$$f''(x) = \frac{x}{(1 - x^2)^{3/2}}$$

$$f''\left(\frac{\sqrt{3}}{2}\right) > 0$$

$$\text{Relative minimum: } \left(\frac{\sqrt{3}}{2}, -0.68\right)$$

$$f''\left(-\frac{\sqrt{3}}{2}\right) < 0$$

$$\text{Relative maximum: } \left(-\frac{\sqrt{3}}{2}, 0.68\right)$$

66. $f(x) = \arcsin x - 2 \arctan x$

$$f'(x) = \frac{1}{\sqrt{1 - x^2}} - \frac{2}{1 + x^2} = 0$$

$$1 + x^2 = 2\sqrt{1 - x^2}$$

$$1 + 2x^2 + x^4 = 4(1 - x^2)$$

$$x^4 + 6x^2 - 3 = 0$$

$$x = \pm 0.681$$

By the First Derivative Test, $(-0.681, 0.447)$ is a relative maximum and $(0.681, -0.447)$ is a relative minimum.

68. $\arctan 0 = 0$. π is not in the range of $y = \arctan x$.

70. The derivatives are algebraic. See Theorem 5.18.

72. (a) $\cot \theta = \frac{x}{3}$

$$\theta = \operatorname{arccot}\left(\frac{x}{3}\right)$$

(b) $\frac{d\theta}{dt} = \frac{-3}{x^2 + 9} \frac{dx}{dt}$

If $x = 10$, $\frac{d\theta}{dt} \approx 11.001 \text{ rad/hr.}$

If $x = 3$, $\frac{d\theta}{dt} \approx 66.667 \text{ rad/hr.}$

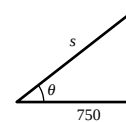
A lower altitude results in a greater rate of change of θ .

74. $\cos \theta = \frac{750}{s}$

$$\theta = \arccos\left(\frac{750}{s}\right)$$

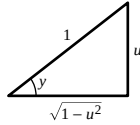
$$\frac{d\theta}{dt} = \frac{d\theta}{ds} \cdot \frac{ds}{dt} = \frac{-1}{\sqrt{1 - (750/s)^2}} \left(\frac{-750}{s^2}\right) \frac{ds}{dt}$$

$$= \frac{750}{s\sqrt{s^2 - 750^2}} \frac{ds}{dt}$$



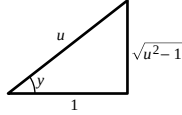
76. (a) Let
- $y = \arcsin u$
- . Then

$$\begin{aligned}\sin y &= u \\ \cos y \cdot y' &= u' \\ \frac{dy}{dx} &= \frac{u'}{\cos y} = \frac{u'}{\sqrt{1-u^2}}.\end{aligned}$$



- (c) Let
- $y = \operatorname{arcsec} u$
- . Then

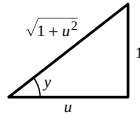
$$\begin{aligned}\sec y &= u \\ \sec y \tan y \frac{dy}{dx} &= u' \\ \frac{dy}{dx} &= \frac{u'}{\sec y \tan y} = \frac{u'}{|u|\sqrt{u^2-1}}.\end{aligned}$$



Note: The absolute value sign in the formula for the derivative of $\operatorname{arcsec} u$ is necessary because the inverse secant function has a positive slope at every value in its domain.

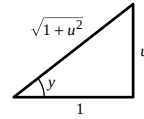
- (e) Let
- $y = \operatorname{arccot} u$
- . Then

$$\begin{aligned}\cot y &= u \\ -\csc^2 y \frac{dy}{dx} &= u' \\ \frac{dy}{dx} &= \frac{u'}{-\csc^2 y} = -\frac{u'}{1+u^2}.\end{aligned}$$



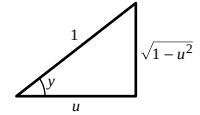
- (b) Let
- $y = \arctan u$
- . Then

$$\begin{aligned}\tan y &= u \\ \sec^2 y \frac{dy}{dx} &= u' \\ \frac{dy}{dx} &= \frac{u'}{\sec^2 y} = \frac{u'}{1+u^2}.\end{aligned}$$



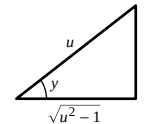
- (d) Let
- $y = \arccos u$
- . Then

$$\begin{aligned}\cos y &= u \\ -\sin y \frac{dy}{dx} &= u' \\ \frac{dy}{dx} &= -\frac{u'}{\sin y} = -\frac{u'}{\sqrt{1-u^2}}.\end{aligned}$$



- (f) Let
- $y = \operatorname{arccsc} u$
- . Then

$$\begin{aligned}\csc y &= u \\ -\csc y \cot y \frac{dy}{dx} &= u' \\ \frac{dy}{dx} &= \frac{u'}{-\csc y \cot y} = -\frac{u'}{|u|\sqrt{u^2-1}}.\end{aligned}$$



Note: The absolute value sign in the formula for the derivative of $\operatorname{arccsc} u$ is necessary because the inverse cosecant function has a negative slope at every value in its domain.

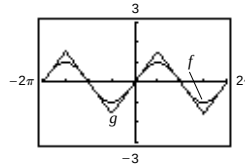
- 78.
- $f(x) = \sin x$

$$g(x) = \arcsin(\sin x)$$

- (a) The range of
- $y = \arcsin x$
- is
- $-\pi/2 \leq y \leq \pi/2$
- .

- (b) Maximum:
- $\pi/2$

$$\text{Minimum: } -\pi/2$$



80. False

$$\text{The range of } y = \arcsin x \text{ is } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

82. False

$$\arcsin^2 0 + \arccos^2 0 = 0 + \left(\frac{\pi}{2}\right)^2 \neq 1$$

Section 5.9 Inverse Trigonometric Functions: Integration

$$2. \int \frac{3}{\sqrt{1-4x^2}} dx = \frac{3}{2} \int \frac{2}{\sqrt{1-4x^2}} dx = \frac{3}{2} \arcsin(2x) + C$$

$$4. \int_0^1 \frac{1}{\sqrt{4-x^2}} dx = \left[\arcsin \frac{x}{2} \right]_0^1 = \frac{\pi}{6}$$

$$6. \int \frac{4}{1+9x^2} dx = \frac{4}{3} \int \frac{3}{1+9x^2} dx = \frac{4}{3} \arctan(3x) + C$$

$$8. \int_{\sqrt{3}}^3 \frac{1}{9+x^2} dx = \left[\frac{1}{3} \arctan \frac{x}{3} \right]_{\sqrt{3}}^3 = \frac{\pi}{36}$$

$$10. \int \frac{1}{4+(x-1)^2} dx = \frac{1}{2} \arctan\left(\frac{x-1}{2}\right) + C$$

$$12. \int \frac{x^4-1}{x^2+1} dx = \int (x^2-1) dx = \frac{1}{3}x^3 - x + C$$

14. Let $u = t^2$, $du = 2t dt$.

$$\int \frac{t}{t^4 + 16} dt = \frac{1}{2} \int \frac{1}{(4)^2 + (t^2)^2} (2t) dt = \frac{1}{8} \arctan \frac{t^2}{4} + C$$

16. Let $u = x^2$, $du = 2x dx$.

$$\begin{aligned} \int \frac{1}{x\sqrt{x^4 - 4}} dx &= \frac{1}{2} \int \frac{1}{x^2 \sqrt{(x^2)^2 - 2^2}} (2x) dx \\ &= \frac{1}{4} \operatorname{arcsec} \frac{x^2}{2} + C \end{aligned}$$

18. Let $u = \arccos x$, $du = -\frac{1}{\sqrt{1-x^2}} dx$.

$$\begin{aligned} \int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1-x^2}} dx &= - \int_0^{1/\sqrt{2}} \frac{-\arccos x}{\sqrt{1-x^2}} dx \\ &= \left[-\frac{1}{2} \arccos^2 x \right]_0^{1/\sqrt{2}} = \frac{3\pi^2}{32} \approx 0.925 \end{aligned}$$

20. Let $u = 1 + x^2$, $du = 2x dx$.

$$\begin{aligned} \int_{-\sqrt{3}}^0 \frac{x}{1+x^2} dx &= \frac{1}{2} \int_{-\sqrt{3}}^0 \frac{1}{1+x^2} (2x) dx \\ &= \left[\frac{1}{2} \ln(1+x^2) \right]_{-\sqrt{3}}^0 = -\ln 2 \end{aligned}$$

22. $\int_1^2 \frac{1}{3+(x-2)^2} dx = \int_1^2 \frac{1}{(\sqrt{3})^2 + (x-2)^2} dx = \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{x-2}{\sqrt{3}}\right) \right]_1^2 = \frac{\sqrt{3}\pi}{18}$

24. $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x} dx = \arctan(\sin x) \Big|_0^{\pi/2} = \frac{\pi}{4}$

26. $\int \frac{3}{2\sqrt{x}(1+x)} dx$. $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$, $dx = 2u du$

$$\begin{aligned} \frac{3}{2} \int \frac{2u du}{u(1+u^2)} &= 3 \int \frac{du}{1+u^2} = 3 \arctan u + C \\ &= 3 \arctan \sqrt{x} + C \end{aligned}$$

28. $\int \frac{4x+3}{\sqrt{1-x^2}} dx = (-2) \int \frac{-2x}{\sqrt{1-x^2}} dx + 3 \int \frac{1}{\sqrt{1-x^2}} dx = -4\sqrt{1-x^2} + 3 \arcsin x + C$

30. $\int \frac{x-2}{(x+1)^2+4} dx = \frac{1}{2} \int \frac{2x+2}{(x+1)^2+4} dx - \int \frac{3}{(x+1)^2+4} dx$

$$= \frac{1}{2} \ln(x^2+2x+5) - \frac{3}{2} \arctan\left(\frac{x+1}{2}\right) + C$$

32. $\int_{-2}^2 \frac{dx}{x^2+4x+13} = \int_{-2}^2 \frac{dx}{(x+2)^2+9} = \left[\frac{1}{3} \arctan\left(\frac{x+2}{3}\right) \right]_{-2}^2 = \frac{1}{3} \arctan\left(\frac{4}{3}\right)$

34. $\int \frac{2x-5}{x^2+2x+2} dx = \int \frac{2x+2}{x^2+2x+2} dx - 7 \int \frac{1}{1+(x+1)^2} dx = \ln|x^2+2x+2| - 7 \arctan(x+1) + C$

36. $\int \frac{2}{\sqrt{-x^2+4x}} dx = \int \frac{2}{\sqrt{4-(x^2-4x+4)}} dx$

$$\begin{aligned} &= \int \frac{2}{\sqrt{4-(x-2)^2}} dx \\ &= 2 \arcsin\left(\frac{x-2}{2}\right) + C \end{aligned}$$

38. Let $u = x^2 - 2x$, $du = (2x-2) dx$.

$$\begin{aligned} \int \frac{x-1}{\sqrt{x^2-2x}} dx &= \frac{1}{2} \int (x^2-2x)^{-1/2} (2x-2) dx \\ &= \sqrt{x^2-2x} + C \end{aligned}$$

40. $\int \frac{1}{(x-1)\sqrt{x^2-2x}} dx = \int \frac{1}{(x-1)\sqrt{(x-1)^2-1}} dx = \operatorname{arcsec}|x-1| + C$

42. Let $u = x^2 - 4$, $du = 2x dx$.

$$\int \frac{x}{\sqrt{9 + 8x^2 - x^4}} dx = \frac{1}{2} \int \frac{2x}{\sqrt{25 - (x^2 - 4)^2}} dx = \frac{1}{2} \arcsin\left(\frac{x^2 - 4}{5}\right) + C$$

44. Let $u = \sqrt{x - 2}$, $u^2 + 2 = x$, $2u du = dx$

$$\begin{aligned} \int \frac{\sqrt{x-2}}{x+1} dx &= \int \frac{2u^2}{u^2+3} du = \int \frac{2u^2+6-6}{u^2+3} du = 2 \int du - 6 \int \frac{1}{u^2+3} du \\ &= 2u - \frac{6}{\sqrt{3}} \arctan \frac{u}{\sqrt{3}} + C = 2\sqrt{x-2} - 2\sqrt{3} \arctan \sqrt{\frac{x-2}{3}} + C \end{aligned}$$

46. The term is $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$; $x^2 + 3x = x^2 + 3x + \frac{9}{4} - \frac{9}{4} = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4}$

48. (a) $\int e^{x^2} dx$ cannot be evaluated using the basic integration rules.

(b) $\int x e^{x^2} dx = \frac{1}{2} e^{x^2} + C$, $u = x^2$

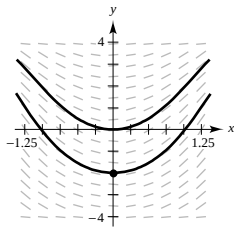
(c) $\int \frac{1}{x^2} e^{1/x} dx = -e^{1/x} + C$, $u = \frac{1}{x}$

50. (a) $\int \frac{1}{1+x^4} dx$ cannot be evaluated using the basic integration rules.

(b) $\int \frac{x}{1+x^4} dx = \frac{1}{2} \int \frac{2x}{1+(x^2)^2} dx = \frac{1}{2} \arctan(x^2) + C$, $u = x^2$

(c) $\int \frac{x^3}{1+x^4} dx = \frac{1}{4} \int \frac{4x^3}{1+x^4} dx = \frac{1}{4} \ln(1+x^4) + C$, $u = 1+x^4$

52. (a)



(b) $\frac{dy}{dx} = x\sqrt{16-y^2}$, $(0, -2)$

$$\frac{dy}{\sqrt{16-y^2}} = x dx$$

$$\arcsin\left(\frac{y}{4}\right) = \frac{x^2}{2} + C$$

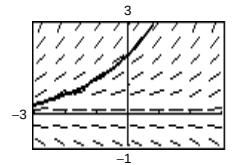
$$(0, -2): \arcsin\left(-\frac{2}{4}\right) = C \Rightarrow C = -\frac{\pi}{6}$$

$$\arcsin\left(\frac{y}{4}\right) = \frac{x^2}{2} - \frac{\pi}{6}$$

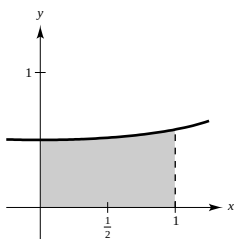
$$\frac{y}{4} = \sin\left(\frac{x^2}{2} - \frac{\pi}{6}\right)$$

$$y = 4 \sin\left(\frac{x^2}{2} - \frac{\pi}{6}\right)$$

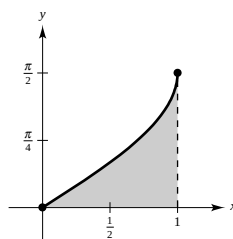
54. $\frac{dy}{dx} = \frac{2y}{\sqrt{16-x^2}}$, $y(0) = 2$



$$56. A = \int_0^1 \frac{1}{\sqrt{4-x^2}} dx = \left[\arcsin \frac{x}{2} \right]_0^1 = \frac{\pi}{6}$$



$$58. \int_0^1 \arcsin x \, dx \approx 0.571$$



$$60. F(x) = \frac{1}{2} \int_x^{x+2} \frac{2}{t^2 + 1} dt$$

(a) $F(x)$ represents the average value of $f(x)$ over the interval $[x, x+2]$. Maximum at $x = -1$, since the graph is greatest on $[-1, 1]$.

$$(b) F(x) = \left[\arctan t \right]_x^{x+2} = \arctan(x+2) - \arctan x$$

$$F'(x) = \frac{1}{1+(x+2)^2} - \frac{1}{1+x^2} = \frac{(1+x^2) - (x^2+4x+5)}{(x^2+1)(x^2+4x+5)} = \frac{-4(x+1)}{(x^2+1)(x^2+4x+5)} = 0 \text{ when } x = -1.$$

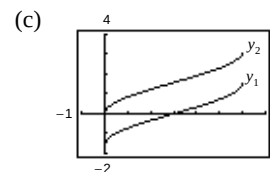
$$62. \int \frac{1}{\sqrt{6x-x^2}} dx$$

$$(a) 6x - x^2 = 9 - (x^2 - 6x + 9) = 9 - (x-3)^2$$

$$\int \frac{1}{\sqrt{6x-x^2}} dx = \int \frac{dx}{\sqrt{9-(x-3)^2}} = \arcsin\left(\frac{x-3}{3}\right) + C$$

$$(b) u = \sqrt{x}, u^2 = x, 2u \, du = dx$$

$$\int \frac{1}{\sqrt{6u^2-u^4}} (2u \, du) = \int \frac{2}{\sqrt{6-u^2}} du = 2 \arcsin\left(\frac{u}{\sqrt{6}}\right) + C = 2 \arcsin\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$



The antiderivatives differ by a constant, $\pi/2$.

Domain: $[0, 6]$

$$64. \text{ Let } f(x) = \arctan x - \frac{x}{1+x^2}$$

$$f'(x) = \frac{1}{1+x^2} - \frac{1-x^2}{(1+x^2)^2} = \frac{2x^2}{(1+x^2)^2} > 0 \text{ for } x > 0.$$

Since $f(0) = 0$ and f is increasing for $x > 0$, $\arctan x - \frac{x}{1+x^2} > 0$ for $x > 0$. Thus,

$$\arctan x > \frac{x}{1+x^2}.$$

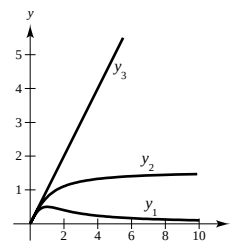
$$\text{Let } g(x) = x - \arctan x$$

$$g'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} > 0 \text{ for } x > 0.$$

Since $g(0) = 0$ and g is increasing for $x > 0$, $x - \arctan x > 0$ for $x > 0$. Thus, $x > \arctan x$.

Therefore,

$$\frac{x}{1+x^2} < \arctan x < x.$$



Section 5.10 Hyperbolic Functions

$$2. (a) \cosh(0) = \frac{e^0 + e^0}{2} = 1$$

$$(b) \operatorname{sech}(1) = \frac{2}{e + e^{-1}} \approx 0.648$$

$$6. (a) \operatorname{csch}^{-1}(2) = \ln\left(\frac{1 + \sqrt{5}}{2}\right) \approx 0.481$$

$$(b) \operatorname{coth}^{-1}(3) = \frac{1}{2} \ln\left(\frac{4}{2}\right) \approx 0.347$$

$$8. \frac{1 + \cosh 2x}{2} = \frac{1 + (e^{2x} + e^{-2x})/2}{2} = \frac{e^{2x} + 2 + e^{-2x}}{4} = \left(\frac{e^x + e^{-x}}{2}\right)^2 = \cosh^2 x$$

$$10. 2 \sinh x \cosh x = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x$$

$$\begin{aligned} 12. 2 \cosh\left(\frac{x+y}{2}\right) \cosh\left(\frac{x-y}{2}\right) &= 2 \left[\frac{e^{(x+y)/2} + e^{-(x+y)/2}}{2} \right] \left[\frac{e^{(x-y)/2} + e^{-(x-y)/2}}{2} \right] \\ &= 2 \left[\frac{e^x + e^y + e^{-y} + e^{-x}}{4} \right] = \frac{e^x + e^{-x}}{2} + \frac{e^y + e^{-y}}{2} \\ &= \cosh x + \cosh y \end{aligned}$$

$$14. \quad \tanh x = \frac{1}{2}$$

$$\left(\frac{1}{2}\right)^2 + \operatorname{sech}^2 x = 1 \Rightarrow \operatorname{sech}^2 x = \frac{3}{4} \Rightarrow \operatorname{sech} x = \frac{\sqrt{3}}{2}$$

$$\cosh x = \frac{1}{\sqrt{3}/2} = \frac{2\sqrt{3}}{3}$$

$$\coth x = \frac{1}{1/2} = 2$$

$$\sinh x = \tanh x \cosh x = \left(\frac{1}{2}\right)\left(\frac{2\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3}$$

$$\operatorname{csch} x = \frac{1}{\sqrt{3}/3} = \sqrt{3}$$

Putting these in order:

$$\sinh x = \frac{\sqrt{3}}{3} \quad \operatorname{csch} x = \sqrt{3}$$

$$\cosh x = \frac{2\sqrt{3}}{3} \quad \operatorname{sech} x = \frac{\sqrt{3}}{2}$$

$$\tanh x = \frac{1}{2} \quad \coth x = 2$$

$$16. y = \coth(3x)$$

$$y' = -3 \operatorname{csch}^2(3x)$$

$$18. g(x) = \ln(\cosh x)$$

$$g'(x) = \frac{1}{\cosh x} (\sinh x) = \tanh x$$

$$20. y = x \cosh x - \sinh x$$

$$y' = x \sinh x + \cosh x - \cosh x = x \sinh x$$

$$22. h(t) = t - \coth t$$

$$h'(t) = 1 + \operatorname{csch}^2 t = \coth^2 t$$

24. $g(x) = \operatorname{sech}^2 3x$

$$g'(x) = -2 \operatorname{sech}(3x) \operatorname{sech}(3x) \tanh(3x)(3) \\ = -6 \operatorname{sech}^2 3x \tanh 3x$$

26. $f(x) = e^{\sinh x}$

$f'(x) = (\cosh x)(e^{\sinh x})$

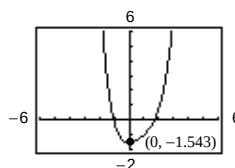
28. $y = \operatorname{sech}(x + 1)$

$y' = -\operatorname{sech}(x + 1) \tanh(x + 1)$

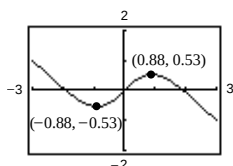
30. $f(x) = x \sinh(x - 1) - \cosh(x - 1)$

$f'(x) = x \cosh(x - 1) + \sinh(x - 1) - \sinh(x - 1) = x \cosh(x - 1)$

$f'(x) = 0$ for $x = 0$. By the First Derivative Test, $(0, -\cosh(-1)) \approx (0, -1.543)$ is a relative minimum.



32. $h(x) = 2 \tanh x - x$



Relative maximum: $(0.88, 0.53)$

Relative minimum: $(-0.88, -0.53)$

34. $y = a \cosh x$

$y' = a \sinh x$

$y'' = a \cosh x$

Therefore, $y'' - y = 0$.

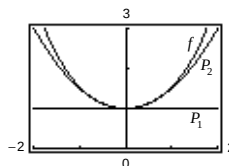
36. $f(x) = \cosh x$ $f(1) = \cosh(0) \approx 1$

$f'(x) = \sinh x$ $f'(1) = \sinh(0) \approx 0$

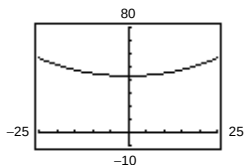
$f''(x) = \cosh x$ $f''(1) = \cosh(0) \approx 1$

$P_1(x) = f(0) + f'(0)(x - 0) = 1$

$P_2(x) = 1 + \frac{1}{2}x^2$



38. (a) $y = 18 + 25 \cosh \frac{x}{25}$, $-25 \leq x \leq 25$



(b) At $x = \pm 25$, $y = 18 + 25 \cosh(1) \approx 56.577$.

At $x = 0$, $y = 18 + 25 = 43$.

(c) $y' = \sinh \frac{x}{25}$. At $x = 25$, $y' = \sinh(1) \approx 1.175$

40. Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$.

$$\int \frac{\cosh \sqrt{x}}{\sqrt{x}} dx = 2 \int \cosh \sqrt{x} \left(\frac{1}{2\sqrt{x}} \right) dx = 2 \sinh \sqrt{x} + C$$

42. Let $u = \cosh x$, $du = \sinh x dx$.

$$\int \frac{\sinh x}{1 + \sinh^2 x} dx = \int \frac{\sinh x}{\cosh^2 x} dx = \frac{-1}{\cosh x} + C \\ = -\operatorname{sech} x + C$$

44. Let $u = 2x - 1$, $du = 2 dx$.

$$\int \operatorname{sech}^2(2x - 1) dx = \frac{1}{2} \int \operatorname{sech}^2(2x - 1)(2) dx \\ = \frac{1}{2} \tanh(2x - 1) + C$$

46. Let $u = \operatorname{sech} x$, $du = -\operatorname{sech} x \tanh x dx$.

$$\int \operatorname{sech}^3 x \tanh x dx = - \int \operatorname{sech}^2 x (-\operatorname{sech} x \tanh x) dx \\ = -\frac{1}{3} \operatorname{sech}^3 x + C$$

$$\begin{aligned}
 48. \int \cosh^2 x \, dx &= \int \frac{1 + \cosh 2x}{2} \, dx \\
 &= \frac{1}{2} \left[x + \frac{\sinh 2x}{2} \right] + C \\
 &= \frac{1}{2}x + \frac{1}{4} \sinh 2x + C
 \end{aligned}$$

$$50. \int_0^4 \frac{1}{\sqrt{25-x^2}} \, dx = \left[\arcsin \frac{x}{5} \right]_0^4 = \arcsin \frac{4}{5}$$

$$52. \int \frac{2}{x\sqrt{1+4x^2}} \, dx = 2 \int \frac{1}{(2x)\sqrt{1+(2x)^2}} (2) \, dx = -2 \ln \left(\frac{1 + \sqrt{1+4x^2}}{|2x|} \right) + C$$

$$54. \text{ Let } u = \sinh x, \, du = \cosh x \, dx.$$

$$\begin{aligned}
 \int \frac{\cosh x}{\sqrt{9 - \sinh^2 x}} \, dx &= \arcsin \left(\frac{\sinh x}{3} \right) + C \\
 &= \arcsin \left(\frac{e^x - e^{-x}}{6} \right) + C
 \end{aligned}$$

$$56. \, y = \tanh^{-1} \left(\frac{x}{2} \right)$$

$$y' = \frac{1}{1 - (x/2)^2} \left(\frac{1}{2} \right) = \frac{2}{4 - x^2}$$

$$58. \, y = \operatorname{sech}^{-1}(\cos 2x), \, 0 < x < \frac{\pi}{4}$$

$$y' = \frac{-1}{\cos 2x \sqrt{1 - \cos^2 2x}} (-2 \sin 2x) = \frac{2 \sin 2x}{\cos 2x |\sin 2x|} = \frac{2}{\cos 2x} = 2 \sec 2x,$$

since $\sin 2x \geq 0$ for $0 < x < \pi/4$.

$$60. \, y = (\operatorname{csch}^{-1} x)^2$$

$$y' = 2 \operatorname{csch}^{-1} x \left(\frac{-1}{|x| \sqrt{1+x^2}} \right) = \frac{-2 \operatorname{csch}^{-1} x}{|x| \sqrt{1+x^2}}$$

$$62. \, y = x \tanh^{-1} x + \ln \sqrt{1-x^2} = x \tanh^{-1} x + \frac{1}{2} \ln(1-x^2) \quad 64. \text{ See page 401, Theorem 5.22.}$$

$$y' = x \left(\frac{1}{1-x^2} \right) + \tanh^{-1} x + \frac{-x}{1-x^2} = \tanh^{-1} x$$

$$66. \text{ Equation of tangent line through } P = (x_0, y_0):$$

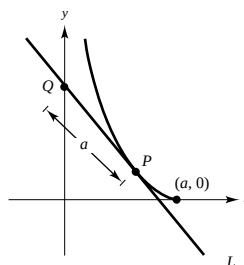
$$y - a \operatorname{sech}^{-1} \frac{x_0}{a} + \sqrt{a^2 - x_0^2} = -\frac{\sqrt{a^2 - x_0^2}}{x_0} (x - x_0)$$

When $x = 0$,

$$y = a \operatorname{sech}^{-1} \frac{x_0}{a} - \sqrt{a^2 - x_0^2} + \sqrt{a^2 - x_0^2} = a \operatorname{sech}^{-1} \frac{x_0}{a}.$$

Hence, Q is the point $[0, a \operatorname{sech}^{-1}(x_0/a)]$.

Distance from P to Q : $d = \sqrt{x_0^2 + (-\sqrt{a^2 - x_0^2})^2} = a$



$$\begin{aligned}
 68. \int \frac{x}{9-x^4} \, dx &= -\frac{1}{2} \int \frac{-2x}{9-(x^2)^2} \, dx = -\frac{1}{2} \left(\frac{1}{6} \right) \ln \left| \frac{3-x^2}{3+x^2} \right| + C \\
 &= -\frac{1}{12} \ln \left| \frac{3-x^2}{3+x^2} \right| + C
 \end{aligned}$$

70. Let $u = x^{3/2}$, $du = \frac{3}{2}\sqrt{x} dx$.

$$\int \frac{\sqrt{x}}{\sqrt{1+x^3}} dx = \frac{2}{3} \int \frac{1}{\sqrt{1+(x^{3/2})^2}} \left(\frac{3}{2}\sqrt{x}\right) dx = \frac{2}{3} \sinh^{-1}(x^{3/2}) + C = \frac{2}{3} \ln(x^{3/2} + \sqrt{1+x^3}) + C$$

72.
$$\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}} = \int \frac{dx}{(x+2)\sqrt{(x+2)^2+4}}$$
$$= -\frac{1}{2} \ln\left(\frac{2 + \sqrt{(x+2)^2+4}}{|x+2|}\right) + C$$

74.
$$\int \frac{1}{(x+1)\sqrt{2x^2+4x+8}} dx = \int \frac{1}{(x+1)\sqrt{2(x+1)^2+6}} dx$$
$$= \frac{1}{\sqrt{2}} \int \frac{1}{(x+1)\sqrt{(x+1)^2+(\sqrt{3})^2}} dx = -\frac{1}{\sqrt{6}} \ln\left(\frac{\sqrt{3} + \sqrt{(x+1)^2+3}}{x+1}\right) + C$$

76. Let $u = 2(x-1)$, $du = 2 dx$.

$$y = \int \frac{1}{(x-1)\sqrt{-4x^2+8x-1}} dx = \int \frac{2}{2(x-1)\sqrt{(\sqrt{3})^2-[2(x-1)]^2}} dx = -\frac{1}{\sqrt{3}} \ln\left|\frac{\sqrt{3} + \sqrt{-4x^2+8x-1}}{2(x-1)}\right| + C$$

78.
$$y = \int \frac{1-2x}{4x-x^2} dx = \int \frac{4-2x}{4x-x^2} dx + 3 \int \frac{1}{(x-2)^2-4} dx$$
$$= \ln|4x-x^2| + \frac{3}{4} \ln\left|\frac{(x-2)-2}{(x-2)+2}\right| + C = \ln|4x-x^2| + \frac{3}{4} \ln\left|\frac{x-4}{x}\right| + C$$

80.
$$A = \int_0^2 \tanh 2x dx$$
$$= \int_0^2 \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx$$
$$= \frac{1}{2} \int_0^2 \frac{1}{e^{2x} + e^{-2x}} (2)(e^{2x} - e^{-2x}) dx$$
$$= \left[\frac{1}{2} \ln(e^{2x} + e^{-2x}) \right]_0^2$$
$$= \frac{1}{2} \ln(e^4 + e^{-4}) - \frac{1}{2} \ln 2$$
$$= \ln \sqrt{\frac{e^4 + e^{-4}}{2}} \approx 1.654$$

82.
$$A = \int_3^5 \frac{6}{\sqrt{x^2-4}} dx$$
$$= \left[6 \ln(x + \sqrt{x^2-4}) \right]_3^5$$
$$= 6 \ln(5 + \sqrt{21}) - 6 \ln(3 + \sqrt{5})$$
$$= 6 \ln\left(\frac{5 + \sqrt{21}}{3 + \sqrt{5}}\right) \approx 3.626$$

84. (a) $v(t) = -32t$

(b)
$$s(t) = \int v(t) dt = \int (-32t) dt = -16t^2 + C$$
$$s(0) = -16(0)^2 + C = 400 \Rightarrow C = 400$$
$$s(t) = -16t^2 + 400$$

—CONTINUED—

84. —CONTINUED—

$$(c) \quad \frac{dv}{dt} = -32 + kv^2$$

$$\int \frac{dv}{kv^2 - 32} = \int dt$$

$$\int \frac{dv}{32 - kv^2} = - \int dt$$

$$\text{Let } u = \sqrt{k}v, \text{ then } du = \sqrt{k}dv.$$

$$\frac{1}{\sqrt{k}} \cdot \frac{1}{2\sqrt{32}} \ln \left| \frac{\sqrt{32} + \sqrt{k}v}{\sqrt{32} - \sqrt{k}v} \right| = -t + C$$

$$\text{Since } v(0) = 0, C = 0.$$

$$\ln \left| \frac{\sqrt{32} + \sqrt{k}v}{\sqrt{32} - \sqrt{k}v} \right| = -2\sqrt{32k}t$$

$$\frac{\sqrt{32} + \sqrt{k}v}{\sqrt{32} - \sqrt{k}v} = e^{-2\sqrt{32k}t}$$

$$\sqrt{32} + \sqrt{k}v = e^{-2\sqrt{32k}t}(\sqrt{32} - \sqrt{k}v)$$

$$v(\sqrt{k} + \sqrt{k}e^{-2\sqrt{32k}t}) = \sqrt{32}(e^{-2\sqrt{32k}t} - 1)$$

$$v = \frac{\sqrt{32}(e^{-2\sqrt{32k}t} - 1)}{\sqrt{k}(e^{-2\sqrt{32k}t} + 1)} \cdot \frac{e^{\sqrt{32k}t}}{e^{\sqrt{32k}t}}$$

$$= \frac{\sqrt{32}}{\sqrt{k}} \left[\frac{-(e^{\sqrt{32k}t} - e^{-\sqrt{32k}t})}{e^{\sqrt{32k}t} + e^{-\sqrt{32k}t}} \right]$$

$$= -\frac{\sqrt{32}}{\sqrt{k}} \tanh(\sqrt{32k}t)$$

$$(d) \quad \lim_{t \rightarrow \infty} \left[-\frac{\sqrt{32}}{\sqrt{k}} \tanh(\sqrt{32k}t) \right] = -\frac{\sqrt{32}}{\sqrt{k}}.$$

The velocity is bounded by $-\sqrt{32}/\sqrt{k}$.

(e) Since $\int \tanh(ct) dt = (1/c) \ln \cosh(ct)$ (which can be verified by differentiation), then

$$\begin{aligned} s(t) &= \int -\frac{\sqrt{32}}{\sqrt{k}} \tanh(\sqrt{32k}t) dt \\ &= -\frac{\sqrt{32}}{\sqrt{k}} \frac{1}{\sqrt{32k}} \ln[\cosh(\sqrt{32k}t)] + C \\ &= -\frac{1}{k} \ln[\cosh(\sqrt{32k}t)] + C. \end{aligned}$$

When $t = 0$,

$$s(0) = C$$

$$= 400 \Rightarrow 400 - (1/k) \ln[\cosh(\sqrt{32k}t)].$$

When $k = 0.01$,

$$s_2(t) = 400 - 100 \ln(\cosh \sqrt{0.32}t)$$

$$s_1(t) = -16t^2 + 400.$$

$s_1(t) = 0$ when $t = 5$ seconds.

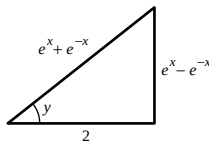
$s_2(t) = 0$ when $t \approx 8.3$ seconds

When air resistance is not neglected, it takes approximately 3.3 more seconds to reach the ground.

86. Let $y = \arcsin(\tanh x)$. Then,

$$\sin y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ and}$$

$$\tan y = \frac{e^x - e^{-x}}{2} = \sinh x.$$



Thus, $y = \arctan(\sinh x)$. Therefore,

$$\arctan(\sinh x) = \arcsin(\tanh x).$$

88. $y = \operatorname{sech}^{-1} x$

$$\operatorname{sech} y = x$$

$$-(\operatorname{sech} y)(\tanh y)y' = 1$$

$$y' = \frac{-1}{(\operatorname{sech} y)(\tanh y)} = \frac{-1}{(\operatorname{sech} y)\sqrt{1 - \operatorname{sech}^2 y}} = \frac{-1}{x\sqrt{1 - x^2}}$$

90. $y = \sinh^{-1} x$

$$\sinh y = x$$

$$(\cosh y)y' = 1$$

$$y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{\sinh^2 y + 1}} = \frac{1}{\sqrt{x^2 + 1}}$$

85. As k increases, the time required for the object to reach the ground increases.

$$87. y = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$y' = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$89. y = \cosh^{-1} x$$

$$\cosh y = x$$

$$(\sinh y)(y') = 1$$

$$y' = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

$$91. y = \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

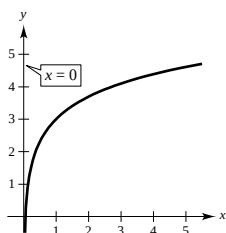
$$y' = -2(e^x + e^{-x})^{-2}(e^x - e^{-x}) = \left(\frac{-2}{e^x + e^{-x}}\right)\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = -\operatorname{sech} x \tanh x$$

Review Exercises for Chapter 5

1. $f(x) = \ln x + 3$

Vertical shift 3 units upward

Vertical asymptote: $x = 0$



$$3. \ln \sqrt[5]{\frac{4x^2 - 1}{4x^2 + 1}} = \frac{1}{5} \ln \frac{(2x - 1)(2x + 1)}{4x^2 + 1} = \frac{1}{5} [\ln(2x - 1) + \ln(2x + 1) - \ln(4x^2 + 1)]$$

$$5. \ln 3 + \frac{1}{3} \ln(4 - x^2) - \ln x = \ln 3 + \ln \sqrt[3]{4 - x^2} - \ln x = \ln \left(\frac{3\sqrt[3]{4 - x^2}}{x} \right)$$

$$7. \ln \sqrt{x + 1} = 2$$

$$\sqrt{x + 1} = e^2$$

$$x + 1 = e^4$$

$$x = e^4 - 1 \approx 53.598$$

$$9. g(x) = \ln \sqrt{x} = \frac{1}{2} \ln x$$

$$g'(x) = \frac{1}{2x}$$

$$11. f(x) = x\sqrt{\ln x}$$

$$f'(x) = \left(\frac{x}{2}\right)(\ln x)^{-1/2}\left(\frac{1}{x}\right) + \sqrt{\ln x}$$

$$= \frac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \frac{1 + 2\ln x}{2\sqrt{\ln x}}$$

$$13. y = \frac{1}{b^2} \left[\ln(a + bx) + \frac{a}{a + bx} \right]$$

$$\frac{dy}{dx} = \frac{1}{b^2} \left[\frac{b}{a + bx} - \frac{ab}{(a + bx)^2} \right] = \frac{x}{(a + bx)^2}$$

$$15. y = -\frac{1}{a} \ln \left(\frac{a + bx}{x} \right) = -\frac{1}{a} [\ln(a + bx) - \ln x]$$

$$\frac{dy}{dx} = -\frac{1}{a} \left(\frac{b}{a + bx} - \frac{1}{x} \right) = \frac{1}{x(a + bx)}$$

$$17. u = 7x - 2, du = 7dx$$

$$\int \frac{1}{7x - 2} dx = \frac{1}{7} \int \frac{1}{7x - 2} (7) dx = \frac{1}{7} \ln |7x - 2| + C$$

$$\begin{aligned}
 19. \int \frac{\sin x}{1 + \cos x} dx &= -\int \frac{-\sin x}{1 + \cos x} dx \\
 &= -\ln|1 + \cos x| + C
 \end{aligned}$$

$$23. \int_0^{\pi/3} \sec \theta d\theta = \left[\ln|\sec \theta + \tan \theta| \right]_0^{\pi/3} = \ln(2 + \sqrt{3})$$

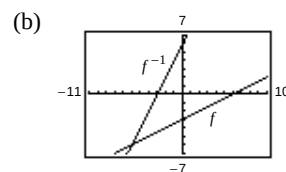
$$\begin{aligned}
 25. (a) \quad f(x) &= \frac{1}{2}x - 3 \\
 y &= \frac{1}{2}x - 3 \\
 2(y + 3) &= x \\
 2(x + 3) &= y \\
 f^{-1}(x) &= 2x + 6
 \end{aligned}$$

$$\begin{aligned}
 27. (a) \quad f(x) &= \sqrt{x+1} \\
 y &= \sqrt{x+1} \\
 y^2 - 1 &= x \\
 x^2 - 1 &= y \\
 f^{-1}(x) &= x^2 - 1, x \geq 0
 \end{aligned}$$

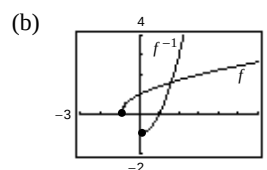
$$\begin{aligned}
 29. (a) \quad f(x) &= \sqrt[3]{x+1} \\
 y &= \sqrt[3]{x+1} \\
 y^3 - 1 &= x \\
 x^3 - 1 &= y \\
 f^{-1}(x) &= x^3 - 1
 \end{aligned}$$

$$\begin{aligned}
 31. \quad f(x) &= x^3 + 2 \\
 f^{-1}(x) &= (x - 2)^{1/3} \\
 (f^{-1})'(x) &= \frac{1}{3}(x - 2)^{-2/3} \\
 (f^{-1})'(-1) &= \frac{1}{3}(-1 - 2)^{-2/3} = \frac{1}{3(-3)^{2/3}} \\
 &= \frac{1}{3^{5/3}} \approx 0.160
 \end{aligned}$$

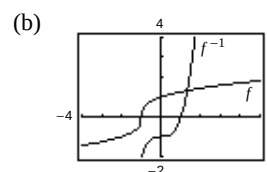
$$21. \int_1^4 \frac{x+1}{x} dx = \int_1^4 \left(1 + \frac{1}{x}\right) dx = \left[x + \ln|x| \right]_1^4 = 3 + \ln 4$$



$$\begin{aligned}
 (c) \quad f^{-1}(f(x)) &= f^{-1}\left(\frac{1}{2}x - 3\right) = 2\left(\frac{1}{2}x - 3\right) + 6 = x \\
 f(f^{-1}(x)) &= f(2x + 6) = \frac{1}{2}(2x + 6) - 3 = x
 \end{aligned}$$



$$\begin{aligned}
 (c) \quad f^{-1}(f(x)) &= f^{-1}(\sqrt{x+1}) = \sqrt{(x^2 - 1)^2} - 1 = x \\
 f(f^{-1}(x)) &= f(x^2 - 1) = \sqrt{(x^2 - 1) + 1} \\
 &= \sqrt{x^2} = x \text{ for } x \geq 0.
 \end{aligned}$$



$$\begin{aligned}
 (c) \quad f^{-1}(f(x)) &= f^{-1}(\sqrt[3]{x+1}) = (\sqrt[3]{x+1})^3 - 1 = x \\
 f(f^{-1}(x)) &= f(x^3 - 1) = \sqrt[3]{(x^3 - 1) + 1} = x
 \end{aligned}$$

$$\begin{aligned}
 33. \quad f(x) &= \tan x \\
 f\left(\frac{\pi}{6}\right) &= \frac{\sqrt{3}}{3} \\
 f'(x) &= \sec^2 x \\
 f'\left(\frac{\pi}{6}\right) &= \frac{4}{3} \\
 (f^{-1})'\left(\frac{\sqrt{3}}{3}\right) &= \frac{1}{f'(\pi/6)} = \frac{3}{4}
 \end{aligned}$$

35. (a) $f(x) = \ln \sqrt{x}$

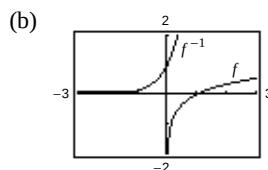
$$y = \ln \sqrt{x}$$

$$e^y = \sqrt{x}$$

$$e^{2y} = x$$

$$e^{2x} = y$$

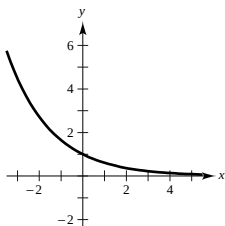
$$f^{-1}(x) = e^{2x}$$



(c) $f^{-1}(f(x)) = f^{-1}(\ln \sqrt{x}) = e^{2 \ln \sqrt{x}} = e^{\ln x} = x$

$$f(f^{-1}(x)) = f(e^{2x}) = \ln \sqrt{e^{2x}} = \ln e^x = x$$

37. $y = e^{-x/2}$



39. $f(x) = \ln(e^{-x^2}) = -x^2$

$$f'(x) = -2x$$

41. $g(t) = t^2 e^t$

$$g'(x) = t^2 e^t + 2te^t = te^t(t + 2)$$

43. $y = \sqrt{e^{2x} + e^{-2x}}$

$$y' = \frac{1}{2}(e^{2x} + e^{-2x})^{-1/2}(2e^{2x} - 2e^{-2x}) = \frac{e^{2x} - e^{-2x}}{\sqrt{e^{2x} + e^{-2x}}}$$

45. $g(x) = \frac{x^2}{e^x}$

$$g'(x) = \frac{e^x(2x) - x^2 e^x}{e^{2x}} = \frac{x(2 - x)}{e^x}$$

47. $y(\ln x) + y^2 = 0$

$$y\left(\frac{1}{x}\right) + (\ln x)\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 0$$

$$(2y + \ln x)\frac{dy}{dx} = \frac{-y}{x}$$

$$\frac{dy}{dx} = \frac{-y}{x(2y + \ln x)}$$

49. Let $u = -3x^2$, $du = -6x dx$.

$$\int x e^{-3x^2} dx = -\frac{1}{6} \int e^{-3x^2} (-6x) dx = -\frac{1}{6} e^{-3x^2} + C$$

51. $\int \frac{e^{4x} - e^{2x} + 1}{e^x} dx = \int (e^{3x} - e^x + e^{-x}) dx$

$$= \frac{1}{3} e^{3x} - e^x - e^{-x} + C$$

$$= \frac{e^{4x} - 3e^{2x} - 3}{3e^x} + C$$

53. $\int x e^{1-x^2} dx = -\frac{1}{2} \int e^{1-x^2} (-2x) dx$

$$= -\frac{1}{2} e^{1-x^2} + C$$

55. Let $u = e^x - 1$, $du = e^x dx$.

$$\int \frac{e^x}{e^x - 1} dx = \ln|e^x - 1| + C$$

57. $y = e^x(a \cos 3x + b \sin 3x)$

$$y' = e^x(-3a \sin 3x + 3b \cos 3x) + e^x(a \cos 3x + b \sin 3x)$$

$$= e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x]$$

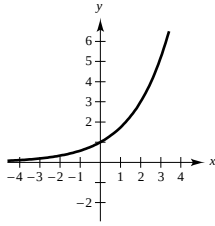
$$y'' = e^x[3(-3a + b) \cos 3x - 3(a + 3b) \sin 3x] + e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x]$$

$$= e^x[(-6a - 8b) \sin 3x + (-8a + 6b) \cos 3x]$$

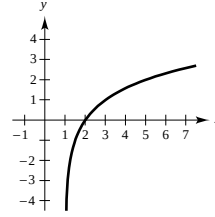
$$y'' - 2y' + 10y = e^x[(-6a - 8b) - 2(-3a + b) + 10b] \sin 3x + [(-8a + 6b) - 2(a + 3b) + 10a] \cos 3x = 0$$

$$59. \text{Area} = \int_0^4 x e^{-x^2} dx = \left[-\frac{1}{2} e^{-x^2} \right]_0^4 = -\frac{1}{2} (e^{-16} - 1) \approx 0.500$$

$$61. y = 3^{3/2}$$



$$63. y = \log_2(x - 1)$$



$$65. f(x) = 3^{x-1}$$

$$f'(x) = 3^{x-1} \ln 3$$

$$67. y = x^{2x+1}$$

$$\ln y = (2x + 1) \ln x$$

$$\frac{y'}{y} = \frac{2x + 1}{x} + 2 \ln x$$

$$y' = y \left(\frac{2x + 1}{x} + 2 \ln x \right) = x^{2x+1} \left(\frac{2x + 1}{x} + 2 \ln x \right)$$

$$69. g(x) = \log_3 \sqrt{1-x} = \frac{1}{2} \log_3(1-x)$$

$$g'(x) = \frac{1}{2} \frac{-1}{(1-x) \ln 3} = \frac{1}{2(x-1) \ln 3}$$

$$71. \int (x+1) 5^{(x+1)^2} dx = \frac{1}{2} \frac{1}{\ln 5} 5^{(x+1)^2} + C$$

$$73. (a) y = x^a$$

$$y' = ax^{a-1}$$

$$(b) y = a^x$$

$$y' = (\ln a) a^x$$

$$(c) y = x^x$$

$$\ln y = x \ln x$$

$$(d) y = a^a$$

$$y' = 0$$

$$\frac{1}{y} y' = x \cdot \frac{1}{x} + (1) \ln x$$

$$y' = y(1 + \ln x)$$

$$y' = x^x(1 + \ln x)$$

$$75. 10,000 = P e^{(0.07)(15)}$$

$$P = \frac{10,000}{e^{1.05}} \approx \$3499.38$$

$$77. P(h) = 30e^{kh}$$

$$P(18,000) = 30e^{18,000k} = 15$$

$$k = \frac{\ln(1/2)}{18,000} = \frac{-\ln 2}{18,000}$$

$$P(h) = 30e^{-(h \ln 2)/18,000}$$

$$P(35,000) = 30e^{-(35,000 \ln 2)/18,000} \approx 7.79 \text{ inches}$$

$$79. P = Ce^{0.015t}$$

$$2C = Ce^{0.015t}$$

$$2 = e^{0.015t}$$

$$\ln 2 = 0.015t$$

$$t = \frac{\ln 2}{0.015} \approx 46.21 \text{ years}$$

$$81. \frac{dy}{dx} = \frac{x^2 + 3}{x}$$

$$\int dy = \int \left(x + \frac{3}{x} \right) dx$$

$$y = \frac{x^2}{2} + 3 \ln|x| + C$$

83. $y' - 2xy = 0$

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{1}{y} dy = \int 2x dx$$

$$\ln|y| = x^2 + C_1$$

$$e^{x^2+C_1} = y$$

$$y = Ce^{x^2}$$

85. $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$ (homogeneous differential equation)

$$(x^2 + y^2) dx - 2xy dy = 0$$

Let $y = vx$, $dy = x dv + v dx$.

$$(x^2 + v^2x^2) dx - 2x(vx)(x dv + v dx) = 0$$

$$(x^2 + v^2x^2 - 2x^3v) dx - 2x^3v dv = 0$$

$$(x^2 - x^2v^2) dx = 2x^3v dv$$

$$(1 - v^2) dx = 2x dv$$

$$\int \frac{dx}{x} = \int \frac{2v}{1 - v^2} dv$$

$$\ln|x| = -\ln|1 - v^2| + C_1 = -\ln|1 - v^2| + \ln C$$

$$x = \frac{C}{1 - v^2} = \frac{C}{1 - (y/x)^2} = \frac{Cx^2}{x^2 - y^2}$$

$$1 = \frac{Cx}{x^2 - y^2} \quad \text{or} \quad C_1 = \frac{x}{x^2 - y^2}$$

87. $y = C_1x + C_2x^3$

$$y' = C_1 + 3C_2x^2$$

$$y'' = 6C_2x$$

$$\begin{aligned} x^2y'' - 3xy' + 3y &= x^2(6C_2x) - 3x(C_1 + 3C_2x^2) + (C_1x + C_2x^3) \\ &= 6C_2x^3 - 3C_1x - 9C_2x^3 + 3C_1x + 3C_2x^3 = 0 \end{aligned}$$

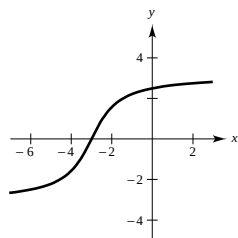
$$x = 2, y = 0: 0 = 2C_1 + 8C_2 \Rightarrow C_1 = -4C_2$$

$$x = 2, y' = 4: 4 = C_1 + 12C_2$$

$$4 = (-4C_2) + 12C_2 = 8C_2 \Rightarrow C_2 = \frac{1}{2}, C_1 = -2$$

$$y = -2x + \frac{1}{2}x^3$$

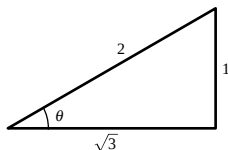
89. $f(x) = 2 \arctan(x + 3)$



91. (a) Let $\theta = \arcsin \frac{1}{2}$

$$\sin \theta = \frac{1}{2}$$

$$\sin\left(\arcsin \frac{1}{2}\right) = \sin \theta = \frac{1}{2}.$$



(b) Let $\theta = \arcsin \frac{1}{2}$

$$\sin \theta = \frac{1}{2}$$

$$\cos\left(\arcsin \frac{1}{2}\right) = \cos \theta = \frac{\sqrt{3}}{2}.$$

93. $y = \tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}$

$$y' = \frac{(1-x^2)^{1/2} + x^2(1-x^2)^{-1/2}}{1-x^2} = (1-x^2)^{-3/2}$$

95. $y = x \operatorname{arcsec} x$

$$y' = \frac{x}{|x|\sqrt{x^2-1}} + \operatorname{arcsec} x$$

97. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2} \arcsin x$

$$y' = \frac{2x \arcsin x}{\sqrt{1-x^2}} + (\arcsin x)^2 - 2 + \frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}} - \frac{2x}{\sqrt{1-x^2}} \arcsin x = (\arcsin x)^2$$

99. Let $u = e^{2x}$, $du = 2e^{2x} dx$.

$$\int \frac{1}{e^{2x} + e^{-2x}} dx = \int \frac{e^{2x}}{1 + e^{4x}} dx = \frac{1}{2} \int \frac{1}{1 + (e^{2x})^2} (2e^{2x}) dx = \frac{1}{2} \arctan(e^{2x}) + C$$

101. Let $u = x^2$, $du = 2x dx$.

$$\int \frac{x}{\sqrt{1-x^4}} dx = \frac{1}{2} \int \frac{1}{\sqrt{1-(x^2)^2}} (2x) dx = \frac{1}{2} \arcsin x^2 + C$$

103. Let $u = 16 + x^2$, $du = 2x dx$.

$$\int \frac{x}{16+x^2} dx = \frac{1}{2} \int \frac{1}{16+x^2} (2x) dx = \frac{1}{2} \ln(16+x^2) + C$$

105. Let $u = \arctan\left(\frac{x}{2}\right)$, $du = \frac{2}{4+x^2} dx$.

$$\int \frac{\arctan(x/2)}{4+x^2} dx = \frac{1}{2} \int \left(\arctan \frac{x}{2}\right) \left(\frac{2}{4+x^2}\right) dx = \frac{1}{4} \left(\arctan \frac{x}{2}\right)^2 + C$$

107. $\int \frac{dy}{\sqrt{A^2 - y^2}} = \int \sqrt{\frac{k}{m}} dt$
 $\arcsin\left(\frac{y}{A}\right) = \sqrt{\frac{k}{m}} t + C$

109. $y = 2x - \cosh \sqrt{x}$

$$y' = 2 - \frac{1}{2\sqrt{x}} (\sinh \sqrt{x}) = 2 - \frac{\sinh \sqrt{x}}{2\sqrt{x}}$$

Since $y = 0$ when $t = 0$, you have $C = 0$. Thus,

$$\sin\left(\sqrt{\frac{k}{m}} t\right) = \frac{y}{A}$$

$$y = A \sin\left(\sqrt{\frac{k}{m}} t\right)$$

111. Let $u = x^2$, $du = 2x dx$.

$$\int \frac{x}{\sqrt{x^4-1}} dx = \frac{1}{2} \int \frac{1}{\sqrt{(x^2)^2-1}} (2x) dx = \frac{1}{2} \ln(x^2 + \sqrt{x^4-1}) + C$$

Problem Solving for Chapter 5

1. $\tan \theta_1 = \frac{3}{x}$

$$\tan \theta_2 = \frac{6}{10-x}$$

Minimize $\theta_1 + \theta_2$:

$$f(x) = \theta_1 + \theta_2 = \arctan\left(\frac{3}{x}\right) + \arctan\left(\frac{6}{10-x}\right)$$

$$f'(x) = \frac{1}{1 + \frac{9}{x^2}} \left(\frac{-3}{x^2}\right) + \frac{1}{1 + \frac{36}{(10-x)^2}} \left(\frac{6}{(10-x)^2}\right) = 0$$

$$\frac{3}{x^2 + 9} = \frac{6}{(10-x)^2 + 36}$$

$$(10-x)^2 + 36 = 2(x^2 + 9)$$

$$100 - 20x + x^2 + 36 = 2x^2 + 18$$

$$x^2 + 20x - 118 = 0$$

$$x = \frac{-20 \pm \sqrt{20^2 - 4(-118)}}{2} = -10 \pm \sqrt{218}$$

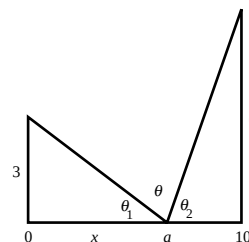
$$a = -10 + \sqrt{218} \approx 4.7648 \quad f(a) \approx 1.4153$$

$$\theta = \pi - (\theta_1 + \theta_2) \approx 1.7263 \quad \text{or} \quad 98.9^\circ$$

$$\text{Endpoints: } a = 0: \theta \approx 1.0304$$

$$a = 10: \theta \approx 1.2793$$

$$\text{Maximum is } 1.7263 \text{ at } a = -10 + \sqrt{218} \approx 4.7648.$$



3. $f(x) = \sin(\ln x)$

(a) Domain: $x > 0$ or $(0, \infty)$

(b) $f(x) = 1 = \sin(\ln x) \Rightarrow \ln x = \frac{\pi}{2} + 2k\pi$.

Two values are $x = e^{\pi/2}, e^{(\pi/2)+2\pi}$.

(c) $f(x) = -1 = \sin(\ln x) \Rightarrow \ln x = \frac{3\pi}{2} + 2k\pi$.

Two values are $x = e^{-\pi/2}, e^{3\pi/2}$.

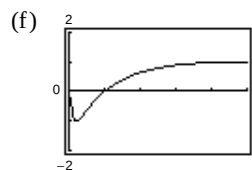
(d) Since the range of the sine function is $[-1, 1]$, parts (b) and (c) show that the range of f is $[-1, 1]$.

(e) $f'(x) = \frac{1}{x} \cos(\ln x)$

$$f'(x) = 0 \Rightarrow \cos(\ln x) = 0 \Rightarrow \ln x = \frac{\pi}{2} + k\pi \Rightarrow$$

$$x = e^{\pi/2} \text{ on } [1, 10]$$

$$\left. \begin{array}{l} f(e^{\pi/2}) = 1 \\ f(1) = 0 \\ f(10) \approx 0.7440 \end{array} \right\} \text{Maximum is } 1 \text{ at } x = e^{\pi/2} \approx 4.8105$$



$$\lim_{x \rightarrow 0^+} f(x) \text{ seems to be } -\frac{1}{2}. \text{ (This is incorrect.)}$$

(g) For the points $x = e^{\pi/2}, e^{-3\pi/2}, e^{-7\pi/2}, \dots$

we have $f(x) = 1$.

For the points $x = e^{-\pi/2}, e^{-5\pi/2}, e^{-9\pi/2}, \dots$

we have $f(x) = -1$.

That is, as $x \rightarrow 0^+$, there is an infinite number of points where $f(x) = 1$, and an infinite number where $f(x) = -1$. Thus $\lim_{x \rightarrow 0^+} \sin(\ln x)$ does not exist.

You can verify this by graphing $f(x)$ on small intervals close to the origin.

$$5. (a) \frac{\text{Area sector}}{\text{Area circle}} = \frac{t}{2\pi} \Rightarrow \text{Area sector} = \frac{t}{2\pi}(\pi) = \frac{t}{2}$$

$$(b) \text{Area } AOP = \frac{1}{2}(\text{base})(\text{height}) - \int_1^{\cosh t} \sqrt{x^2 - 1} dx$$

$$A(t) = \frac{1}{2} \cosh t \cdot \sinh t - \int_1^{\cosh t} \sqrt{x^2 - 1} dx$$

$$A'(t) = \frac{1}{2}[\cosh^2 t + \sinh^2 t] - \sqrt{\cosh^2 t - 1} \sinh t$$

$$= \frac{1}{2}[\cosh^2 t + \sinh^2 t] - \sinh^2 t$$

$$= \frac{1}{2}[\cosh^2 t - \sinh^2 t] = \frac{1}{2}$$

$$A(t) = \frac{1}{2}t + C. \text{ But, } A(0) = C = 0 \Rightarrow C = 0$$

$$\text{Thus, } A(t) = \frac{1}{2}t \text{ or } t = 2A(t).$$

$$7. y = \ln x$$

$$y' = \frac{1}{x}$$

$$y - b = \frac{1}{a}(x - a)$$

$$y = \frac{1}{a}x + b - 1 \quad \text{Tangent line}$$

$$\text{If } x = 0, c = b - 1. \text{ Thus, } b - c = b - (b - 1) = 1.$$

$$9. \text{ Let } u = 1 + \sqrt{x}, \sqrt{x} = u - 1, x = u^2 - 2u + 1,$$

$$dx = (2u - 2)du.$$

$$\text{Area} = \int_1^4 \frac{1}{\sqrt{x} + x} dx = \int_2^3 \frac{2u - 2}{(u - 1) + (u^2 - 2u + 1)} du$$

$$= \int_2^3 \frac{2(u - 1)}{u^2 - u} du$$

$$= \int_2^3 \frac{2}{u} du$$

$$= \left[2 \ln u \right]_2^3$$

$$= 2 \ln 3 - 2 \ln 2 = 2 \ln \left(\frac{3}{2} \right)$$

$$\approx 0.8109$$

$$11. (a) \frac{dy}{dt} = y^{1.01}$$

$$\int y^{-1.01} dy = \int dt$$

$$\frac{y^{-0.01}}{-0.01} = t + C_1$$

$$\frac{1}{y^{0.01}} = -0.01t + C$$

$$y^{0.01} = \frac{1}{C - 0.01t}$$

$$y = \frac{1}{(C - 0.01t)^{100}}$$

$$y(0) = 1: 1 = \frac{1}{C^{100}} \Rightarrow C = 1$$

$$\text{Hence, } y = \frac{1}{(1 - 0.01t)^{100}}.$$

$$\text{For } T = 100, \lim_{t \rightarrow T^-} y = \infty.$$

$$(b) \int y^{-(1+\varepsilon)} dy = \int k dt$$

$$\frac{y^{-\varepsilon}}{-\varepsilon} = kt + C_1$$

$$y^{-\varepsilon} = -\varepsilon kt + C$$

$$y = \frac{1}{(C - \varepsilon kt)^{1/\varepsilon}}$$

$$y(0) = y_0 = \frac{1}{C^{1/\varepsilon}} \Rightarrow C^{1/\varepsilon} = \frac{1}{y_0} \Rightarrow C = \left(\frac{1}{y_0} \right)^\varepsilon$$

$$\text{Hence, } y = \frac{1}{\left(\frac{1}{y_0^\varepsilon} - \varepsilon kt \right)^{1/\varepsilon}}.$$

$$\text{For } t \rightarrow \frac{1}{y_0^\varepsilon \varepsilon k}, y \rightarrow \infty.$$

13. Since $\frac{dy}{dt} = k(y - 20)$,

$$\int \frac{1}{y - 20} dy = \int k dt$$

$$\ln(y - 20) = kt + C$$

$$y = Ce^{kt} + 20.$$

When $t = 0$, $y = 72$. Therefore, $C = 52$.

When $t = 1$, $y = 48$. Therefore, $48 = 52e^k + 20$, $e^k = (28/52) = (7/13)$, and $k = \ln(7/13)$.

Thus, $y = 52e^{\ln(7/13)t} + 20$.

When $t = 5$, $y = 52e^{5 \ln(7/13)} + 20 \approx 22.35^\circ$.

15. (a) $\frac{dS}{dt} = k_1 S(L - S)$

$S = \frac{L}{1 + Ce^{-kt}}$ is a solution because

$$\frac{dS}{dt} = -L(1 + Ce^{-kt})^{-2}(-Cke^{-kt})$$

$$= \frac{LCke^{-kt}}{(1 + Ce^{-kt})^2}$$

$$= \left(\frac{k}{L}\right) \frac{L}{1 + Ce^{-kt}} \cdot \frac{CLe^{-kt}}{1 + Ce^{-kt}}$$

$$= \left(\frac{k}{L}\right) \frac{L}{1 + Ce^{-kt}} \cdot \left(L - \frac{L}{1 + Ce^{-kt}}\right)$$

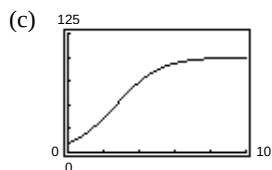
$$= k_1 S(L - S), \text{ where } k_1 = \frac{k}{L}.$$

$L = 100$. Also, $S = 10$ when $t = 0 \Rightarrow C = 9$. And,

$S = 20$ when $t = 1 \Rightarrow k = -\ln(4/9)$.

$$\text{Particular Solution. } S = \frac{100}{1 + 9e^{\ln(4/9)t}}$$

$$= \frac{100}{1 + 9e^{-0.8109t}}$$



(b) $\frac{dS}{dt} = \ln\left(\frac{4}{9}\right) S(100 - S)$

$$\frac{d^2S}{dt^2} = \ln\left(\frac{4}{9}\right) \left[S\left(-\frac{dS}{dt}\right) + (100 - S) \frac{dS}{dt} \right]$$

$$= \ln\left(\frac{4}{9}\right) (100 - 2S) \frac{dS}{dt}$$

$$= 0 \text{ when } S = 50 \text{ or } \frac{dS}{dt} = 0.$$

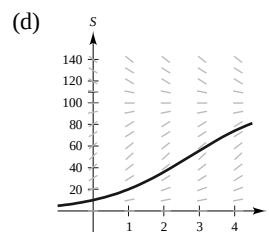
Choosing $S = 50$, we have:

$$50 = \frac{100}{1 + 9e^{\ln(4/9)t}}$$

$$2 = 1 + 9e^{\ln(4/9)t}$$

$$\frac{\ln(1/9)}{\ln(4/9)} = t$$

$$t \approx 2.7 \text{ months}$$

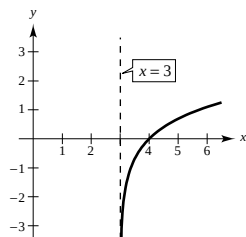


(e) Sales will decrease toward the line $S = L$.

Review Exercises for Chapter 5

2. $f(x) = \ln(x - 3)$

Horizontal shift 3 units to the right

Vertical asymptote: $x = 3$ 

4. $\ln[(x^2 + 1)(x - 1)] = \ln(x^2 + 1) + \ln(x - 1)$

6. $3[\ln x - 2 \ln(x^2 + 1)] + 2 \ln 5 = 3 \ln x - 6 \ln(x^2 + 1) + \ln 5^2 = \ln x^3 - \ln(x^2 + 1)^6 + \ln 25 = \ln \left[\frac{25x^3}{(x^2 + 1)^6} \right]$

8. $\ln x + \ln(x - 3) = 0$

$$\ln x(x - 3) = 0$$

$$x(x - 3) = e^0$$

$$x^2 - 3x - 1 = 0$$

$$x = \frac{3 \pm \sqrt{13}}{2}$$

$$x = \frac{3 + \sqrt{13}}{2} \text{ only since } \frac{3 - \sqrt{13}}{2} < 0.$$

10. $h(x) = \ln \frac{x(x-1)}{x-2} = \ln x + \ln(x-1) - \ln(x-2)$

$$h'(x) = \frac{1}{x} + \frac{1}{x-1} - \frac{1}{x-2} = \frac{x^2 - 4x + 2}{x^3 - 3x^2 + 2x}$$

12. $f(x) = \ln[x(x^2 - 2)^{2/3}] = \ln x + \frac{2}{3} \ln(x^2 - 2)$

$$f'(x) = \frac{1}{x} + \frac{2}{3} \left(\frac{2x}{x^2 - 2} \right) = \frac{7x^2 - 6}{3x^3 - 6x}$$

14. $y = \frac{1}{b^2}[a + bx - a \ln(a + bx)]$

$$\frac{dy}{dx} = \frac{1}{b^2} \left(b - \frac{ab}{a + bx} \right) = \frac{x}{a + bx}$$

16. $y = -\frac{1}{ax} + \frac{b}{a^2} \ln \frac{a + bx}{x}$

$$= -\frac{1}{ax} + \frac{b}{a^2} [\ln(a + bx) - \ln x]$$

$$\frac{dy}{dx} = -\frac{1}{a} \left(-\frac{1}{x^2} \right) + \frac{b}{a^2} \left[\frac{b}{a + bx} - \frac{1}{x} \right]$$

$$= \frac{1}{ax^2} + \frac{b}{a^2} \left[\frac{-a}{x(a + bx)} \right] = \frac{1}{ax^2} - \frac{b}{ax(a + bx)}$$

$$= \frac{(a + bx) - bx}{ax^2(a + bx)} = \frac{1}{x^2(a + bx)}$$

18. $u = x^2 - 1, du = 2x dx$

$$\int \frac{x}{x^2 - 1} dx = \frac{1}{2} \int \frac{2x}{x^2 - 1} dx = \frac{1}{2} \ln|x^2 - 1| + C$$

20. $u = \ln x, du = \frac{1}{x} dx$

$$\int \frac{\ln \sqrt{x}}{x} dx = \frac{1}{2} \int (\ln x) \left(\frac{1}{x} \right) dx = \frac{1}{4} (\ln x)^2 + C$$

22. $\int_1^e \frac{\ln x}{x} dx = \int_1^e (\ln x)^1 \left(\frac{1}{x} \right) dx = \left[\frac{1}{2} (\ln x)^2 \right]_1^e = \frac{1}{2}$

24. $\int_0^{\pi/4} \tan\left(\frac{\pi}{4} - x\right) dx = \left[\ln \left| \cos\left(\frac{\pi}{4} - x\right) \right| \right]_0^{\pi/4}$

$$= 0 - \ln\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} \ln 2$$

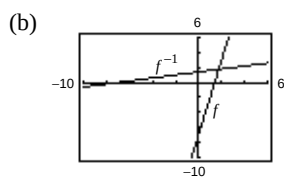
26. (a) $f(x) = 5x - 7$

$$y = 5x - 7$$

$$\frac{y+7}{5} = x$$

$$\frac{x+7}{5} = y$$

$$f^{-1}(x) = \frac{x+7}{5}$$



(c) $f^{-1}(f(x)) = f^{-1}(5x - 7) = \frac{(5x - 7) + 7}{5} = x$

$$f(f^{-1}(x)) = f\left(\frac{x+7}{5}\right) = 5\left(\frac{x+7}{5}\right) - 7 = x$$

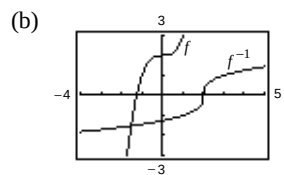
28. (a) $f(x) = x^3 + 2$

$$y = x^3 + 2$$

$$\sqrt[3]{y-2} = x$$

$$\sqrt[3]{x-2} = y$$

$$f^{-1}(x) = \sqrt[3]{x-2}$$



(c) $f^{-1}(f(x)) = f^{-1}(x^3 + 2) = \sqrt[3]{(x^3 + 2) - 2} = x$

$$f(f^{-1}(x)) = f(\sqrt[3]{x-2}) = (\sqrt[3]{x-2})^3 + 2 = x$$

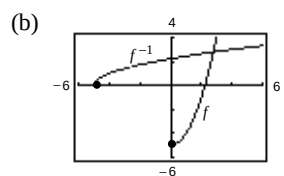
30. (a) $f(x) = x^2 - 5, x \geq 0$

$$y = x^2 - 5$$

$$\sqrt{y+5} = x$$

$$\sqrt{x+5} = y$$

$$f^{-1}(x) = \sqrt{x+5}$$



(c) $f^{-1}(f(x)) = f^{-1}(x^2 - 5) = \sqrt{(x^2 - 5) + 5} = x$ for $x \geq 0$.

$$f(f^{-1}(x)) = f(\sqrt{x+5}) = (\sqrt{x+5})^2 - 5 = x$$

32. $f(x) = x\sqrt{x-3}$

$$f(4) = 4$$

$$f'(x) = \sqrt{x-3} + \frac{1}{2}x(x-3)^{-1/2}$$

$$f'(4) = 1 + 2 = 3$$

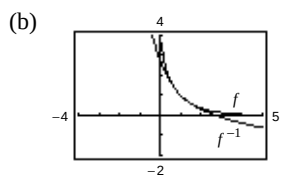
$$(f^{-1})'(4) = \frac{1}{f'(4)} = \frac{1}{3}$$

34. $f(x) = \ln x$

$$f^{-1}(x) = e^x$$

$$(f^{-1})'(x) = e^x$$

$$(f^{-1})'(0) = e^0 = 1$$



(c) $f^{-1}(f(x)) = f^{-1}(\ln x) = e^{\ln x} = x$

$$= 1 - (1 - x) = x$$

$$f(f^{-1}(x)) = f(1 - \ln x) = e^{1 - (1 - \ln x)} = e^{\ln x} = x$$

36. (a) $f(x) = e^{1-x}$

$$y = e^{1-x}$$

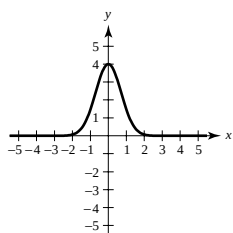
$$\ln y = 1 - x$$

$$x = 1 - \ln y$$

$$y = 1 - \ln x$$

$$f^{-1}(x) = 1 - \ln x$$

38. $y = 4e^{-x^2}$



40. $g(x) = \ln\left(\frac{e^x}{1 + e^x}\right)$

$$= \ln e^x - \ln(1 + e^x) = x - \ln(1 + e^x)$$

$$g'(x) = 1 - \frac{e^x}{1 + e^x} = \frac{1}{1 + e^x}$$

42. $h(z) = e^{-z^2/2}$

$$h'(z) = -ze^{-z^2/2}$$

44. $y = 3e^{-3/t}$

$$y' = 3e^{-3/t}(3t^{-2}) = \frac{9e^{-3/t}}{t^2}$$

46. $f(\theta) = \frac{1}{2}e^{\sin 2\theta}$

$$f'(\theta) = \cos 2\theta e^{\sin 2\theta}$$

48. $\cos x^2 = xe^y$

$$-2x \sin x^2 = xe^y \frac{dy}{dx} + e^y$$

$$\frac{dy}{dx} = -\frac{2x \sin x^2 + e^y}{xe^y}$$

50. Let $u = \frac{1}{x}$, $du = -\frac{1}{x^2} dx$.

$$\int \frac{e^{1/x}}{x^2} dx = -\int e^{1/x} \left(-\frac{1}{x^2}\right) dx = -e^{1/x} + C$$

52. Let $u = e^{2x} + e^{-2x}$, $du = (2e^{2x} - e^{-2x}) dx$.

$$\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx = \frac{1}{2} \int \frac{2e^{2x} - 2e^{-2x}}{e^{2x} + e^{-2x}} dx$$

$$= \frac{1}{2} \ln(e^{2x} + e^{-2x}) + C$$

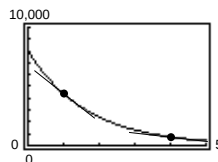
54. Let $u = x^3 + 1$, $du = 3x^2 dx$.

$$\int x^2 e^{x^3+1} dx = \frac{1}{3} \int e^{x^3+1} (3x^2) dx = \frac{1}{3} e^{x^3+1} + C$$

56. $\int \frac{e^{2x}}{e^{2x} + 1} dx = \frac{1}{2} \int \frac{1}{e^{2x} + 1} 2e^{2x} dx$

$$= \frac{1}{2} \ln(e^{2x} + 1) + C$$

58. (a), (c)



(b) $V = 8000e^{-0.6t}$, $0 \leq t \leq 5$

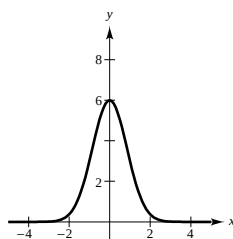
$$V'(t) = -4800e^{-0.6t}$$

$$V'(1) = -2634.3 \text{ dollars/year}$$

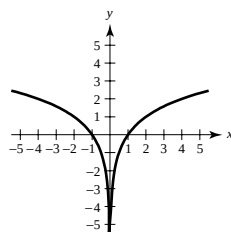
$$V'(4) = -435.4 \text{ dollars/year}$$

60. Area $= \int_0^2 2e^{-x} dx = \left[-2e^{-x}\right]_0^2 = -2e^{-2} + 2 = 2 - \frac{2}{e^2} \approx 1.729$

62. $g(x) = 6(2^{-x^2})$



64. $y = \log_4 x^2$



66. $f(x) = 4^x e^x$

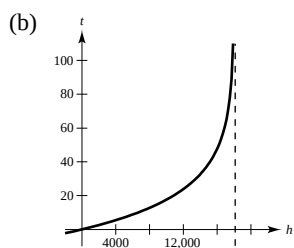
$$f'(x) = 4^x e^x + (\ln 4) 4^x e^x = 4^x e^x (1 + \ln 4)$$

70. $h(x) = \log_5 \frac{x}{x-1} = \log_5 x - \log_5 (x-1)$

$$h'(x) = \frac{1}{\ln 5} \left[\frac{1}{x} - \frac{1}{x-1} \right] = \frac{1}{\ln 5} \left[\frac{-1}{x(x-1)} \right]$$

74. $t = 50 \log_{10} \left(\frac{18,000}{18,000 - h} \right)$

(a) Domain: $0 \leq h < 18,000$



Vertical asymptote: $h = 18,000$

76. $2P = Pe^{10r}$

$$2 = e^{10r}$$

$$\ln 2 = 10r$$

$$r = \frac{\ln 2}{10} \approx 6.93\%$$

80. (a) $\frac{dy}{ds} = -0.012y, s > 50$

$$\frac{-1}{0.012} \int \frac{dy}{y} = \int ds$$

$$\frac{-1}{0.012} \ln y = s + C_1$$

$$y = Ce^{-0.012s}$$

$$\text{When } s = 50, y = 28 = Ce^{-0.012(50)} \Rightarrow C = 28e^{0.6}$$

$$y = 28e^{0.6 - 0.012s}, s > 50$$

82. $\frac{dy}{dx} = \frac{e^{-2x}}{1 + e^{-2x}}$

$$\int dy = \int \frac{e^{-2x}}{1 + e^{-2x}} dx = -\frac{1}{2} \int \frac{-2e^{-2x}}{1 + e^{-2x}} dx$$

$$y = -\frac{1}{2} \ln(1 + e^{-2x}) + C$$

68. $y = x(4^{-x})$

$$y' = 4^{-x} - x \cdot 4^{-x} \ln 4$$

72. $\int \frac{2^{-1/t}}{t^2} dt = \frac{1}{\ln 2} 2^{-1/t} + C$

(c) $t = 50 \log_{10} \left(\frac{18,000}{18,000 - h} \right)$

$$10^{t/50} = \frac{18,000}{18,000 - h}$$

$$18,000 - h = 18,000(10^{-t/50})$$

$$h = 18,000(1 - 10^{-t/50})$$

As $h \rightarrow 18,000, t \rightarrow \infty$.

(d) $t = 50 \log_{10} 18,000 - 50 \log_{10}(18,000 - h)$

$$\frac{dt}{dh} = \frac{50}{(\ln 10)(18,000 - h)}$$

$$\frac{d^2t}{dh^2} = \frac{50}{(\ln 10)(18,000 - h)^2}$$

No critical numbers

As t increases, the rate of change of the altitude is increasing.

78. $y = 5 \left(\frac{1}{2} \right)^{t/1620}$

$$y(600) = 5 \left(\frac{1}{2} \right)^{600/1620} \approx 3.868 \text{ grams}$$

(b)

Speed(s)	50	55	60	65	70
Miles per Gallon (y)	28	26.4	24.8	23.4	22.0

84. $y' - e^y \sin x = 0$

$$\frac{dy}{dx} = e^y \sin x$$

$$\int e^{-y} dy = \int \sin x dx$$

$$-e^{-y} = -\cos x + C_1$$

$$e^y = \frac{1}{\cos x + C} \quad (C = -C_1)$$

$$y = \ln \left| \frac{1}{\cos x + C} \right| = -\ln |\cos x + C|$$

86. $\frac{dy}{dx} = \frac{3(x+y)}{x}$ (homogeneous differential equation)

$$3(x+y) dx - x dy = 0$$

Let $y = vx$, $dy = x dv + v dx$.

$$3(x+vx) dx - x(x dv + v dx) = 0$$

$$(3x + 2vx) dx - x^2 dv = 0$$

$$(3 + 2v) dx = x dv$$

$$\int \frac{1}{x} dx = \int \frac{1}{3 + 2v} dv$$

$$\ln|x| = \frac{1}{2} \ln|3 + 2v| + C_1 = \ln(3 + 2v)^{1/2} + \ln C_2$$

$$x = C_2(3 + 2v)^{1/2}$$

$$x^2 = C(3 + 2v) = C\left(3 + 2\left(\frac{y}{x}\right)\right)$$

$$x^3 = C(3x + 2y) = 3Cx + 2Cy$$

$$y = \frac{x^3 - 3Cx}{2C}$$

88. $\frac{dv}{dt} = kv - 9.8$

(a) $\int \frac{dv}{kv - 9.8} = \int dt$

$$\frac{1}{k} \ln|kv - 9.8| = t + C_1$$

$$\ln|kv - 9.8| = kt + C_2$$

$$kv - 9.8 = e^{kt+C_2} = C_3 e^{kt}$$

$$v = \frac{1}{k} [9.8 + C_3 e^{kt}]$$

At $t = 0$, $v_0 = \frac{1}{k}(9.8 + C_3) \Rightarrow C_3 = kv_0 - 9.8$

$$v = \frac{1}{k} [9.8 + (kv_0 - 9.8)e^{kt}]$$

Note that $k < 0$ since the object is moving downward.

(b) $\lim_{t \rightarrow \infty} v(t) = \frac{9.8}{k}$

(c) $s(t) = \int \frac{1}{k} [9.8 + (kv_0 - 9.8)e^{kt}] dt$

$$= \frac{1}{k} \left[9.8t + \frac{1}{k} (kv_0 - 9.8)e^{kt} \right] + C$$

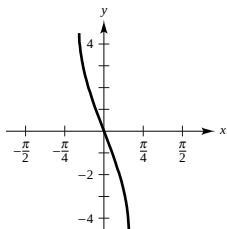
$$= \frac{9.8t}{k} + \frac{1}{k^2} (kv_0 - 9.8)e^{kt} + C$$

$$s(0) = \frac{1}{k^2} (kv_0 - 9.8) + C \Rightarrow C = s_0 - \frac{1}{k^2} (kv_0 - 9.8)$$

$$s(t) = \frac{9.8t}{k} + \frac{1}{k^2} (kv_0 - 9.8)e^{kt} + s_0 - \frac{1}{k^2} (kv_0 - 9.8)$$

$$= \frac{9.8t}{k} + \frac{1}{k^2} (kv_0 - 9.8)(e^{kt} - 1) + s_0$$

90. $h(x) = -3 \arcsin(2x)$



94. $y = \arctan(x^2 - 1)$

$$y' = \frac{2x}{1 + (x^2 - 1)^2} = \frac{2x}{x^4 - 2x^2 + 2}$$

98. $y = \sqrt{x^2 - 4} - 2 \operatorname{arcsec} \frac{x}{2}, 2 < x < 4$

$$y' = \frac{x}{\sqrt{x^2 - 4}} - \frac{1}{(|x|/2)\sqrt{(x/2)^2 - 1}} = \frac{x}{\sqrt{x^2 - 4}} - \frac{4}{|x|\sqrt{x^2 - 4}} = \frac{x^2 - 4}{|x|\sqrt{x^2 - 4}} = \frac{\sqrt{x^2 - 4}}{x}$$

100. Let $u = 5x, du = 5 dx$.

$$\int \frac{1}{3 + 25x^2} dx = \frac{1}{5} \int \frac{1}{(\sqrt{3})^2 + (5x)^2} (5) dx = \frac{1}{5\sqrt{3}} \arctan \frac{5x}{\sqrt{3}} + C$$

102. $\int \frac{1}{16 + x^2} dx = \frac{1}{4} \arctan \frac{x}{4} + C$

104. $\int \frac{4 - x}{\sqrt{4 - x^2}} dx = 4 \int \frac{1}{\sqrt{4 - x^2}} dx + \frac{1}{2} \int (4 - x^2)^{-1/2} (-2x) dx = 4 \arcsin \frac{x}{2} + \sqrt{4 - x^2} + C$

106. Let $u = \arcsin x, du = \frac{1}{\sqrt{1 - x^2}} dx$.

$$\int \frac{\arcsin x}{\sqrt{1 - x^2}} dx = \frac{1}{2} (\arcsin x)^2 + C$$

110. $y = x \tanh^{-1} 2x$

$$y' = x \left(\frac{2}{1 - 4x^2} \right) + \tanh^{-1} 2x = \frac{2x}{1 - 4x^2} + \tanh^{-1} 2x$$

92. (a) Let $\theta = \operatorname{arccot} 2$

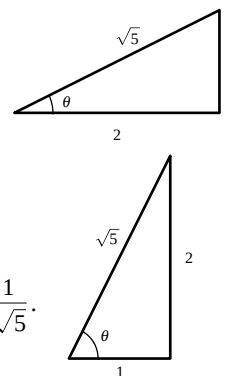
$$\cot \theta = 2$$

$$\tan(\operatorname{arccot} 2) = \tan \theta = \frac{1}{2}.$$

(b) Let $\theta = \operatorname{arcsec} \sqrt{5}$

$$\sec \theta = \sqrt{5}$$

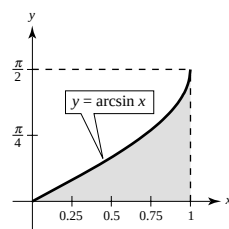
$$\cos(\operatorname{arcsec} \sqrt{5}) = \cos \theta = \frac{1}{\sqrt{5}}.$$



96. $y = \frac{1}{2} \arctan e^{2x}$

$$y' = \frac{1}{2} \left(\frac{1}{1 + e^{4x}} \right) (2e^{2x}) = \frac{e^{2x}}{1 + e^{4x}}$$

108.



Since the area of region A is $1 \left(\int_0^{\pi/2} \sin y dy \right)$,

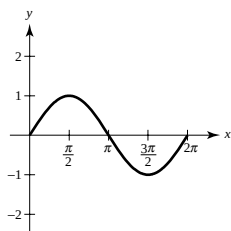
the shaded area is $\int_0^1 \arcsin x dx = \frac{\pi}{2} - 1 \approx 0.571$.

112. Let $u = x^3, du = 3x^2 dx$.

$$\int x^2 (\operatorname{sech} x^3)^2 dx = \frac{1}{3} \int (\operatorname{sech} x^3)^2 (3x^2) dx = \frac{1}{3} \tanh x^3 + C$$

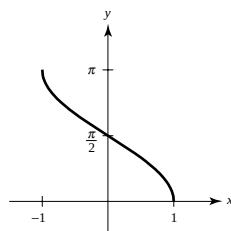
Problem Solving for Chapter 5

2. (a)



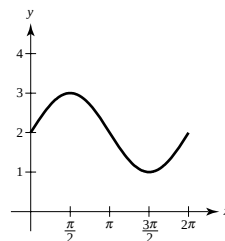
$$\int_0^{\pi} \sin x \, dx = -\int_{\pi}^{2\pi} \sin x \, dx \Rightarrow \int_0^{2\pi} \sin x \, dx = 0$$

(c)



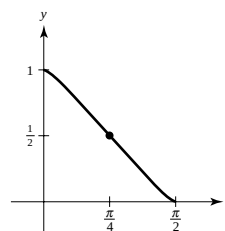
$$\int_{-1}^1 \arccos x \, dx = 2\left(\frac{\pi}{2}\right) = \pi$$

(b)



$$\int_0^{2\pi} (\sin x + 2) \, dx = 2(2\pi) = 4\pi$$

(d)

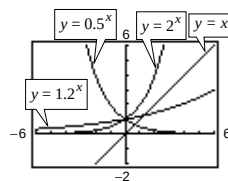


$$y = \frac{1}{1 + (\tan x)\sqrt{2}} \text{ is symmetric with respect to the point } \left(\frac{\pi}{4}, \frac{1}{2}\right). \quad \int_0^{\pi/2} \frac{1}{1 + (\tan x)\sqrt{2}} \, dx = \frac{\pi}{2} \left(\frac{1}{2}\right) = \frac{\pi}{4}$$

4. $y = 0.5^x$ and $y = 1.2^x$ intersect $y = x$. $y = 2^x$ does not intersect $y = x$.Suppose $y = x$ is tangent to $y = a^x$ at (x, y) .

$$a^x = x \Rightarrow a = x^{1/x}.$$

$$y' = a^x \ln a = 1 \Rightarrow x \ln x^{1/x} = 1 \Rightarrow \ln x = 1 \Rightarrow x = e, a = e^{1/e}$$

For $0 < a \leq e^{1/e} \approx 1.445$, the curve $y = a^x$ intersects $y = x$.6. (a) $y = f(x) = \arcsin x$

$$\sin y = x$$

$$\text{Area } A = \int_{\pi/6}^{\pi/4} \sin y \cdot dy = -\cos y \Big|_{\pi/6}^{\pi/4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} = \frac{\sqrt{3} - \sqrt{2}}{2} \approx 0.1589$$

$$\text{Area } B = \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) = \frac{\pi}{12} \approx 0.2618$$

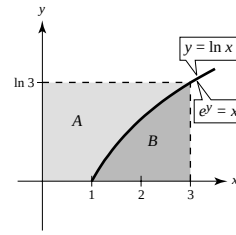
$$\begin{aligned} \text{(b)} \quad \int_{1/2}^{\sqrt{2}/2} \arcsin x \, dx &= \text{Area}(C) = \left(\frac{\pi}{4}\right)\left(\frac{\sqrt{2}}{2}\right) - A - B \\ &= \frac{\pi\sqrt{2}}{8} - \frac{\sqrt{3} - \sqrt{2}}{2} - \frac{\pi}{12} \\ &= \pi\left(\frac{\sqrt{2}}{8} - \frac{1}{12}\right) + \frac{\sqrt{2} - \sqrt{3}}{2} \approx 0.1346 \end{aligned}$$

—CONTINUED—

6. —CONTINUED—

$$\begin{aligned} \text{(c) Area } A &= \int_0^{\ln 3} e^y dy \\ &= e^y \Big|_0^{\ln 3} = 3 - 1 = 2 \end{aligned}$$

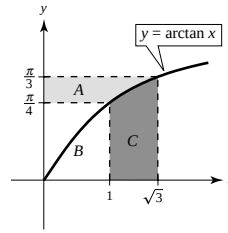
$$\text{Area } B = \int_1^3 \ln x dx = 3(\ln 3) - A = 3 \ln 3 - 2 = \ln 27 - 2 \approx 1.2958$$



$$\text{(d) } \tan y = x$$

$$\begin{aligned} \text{Area } A &= \int_{\pi/4}^{\pi/3} \tan y dy \\ &= -\ln|\cos y| \Big|_{\pi/4}^{\pi/3} \\ &= -\ln \frac{1}{2} + \ln \frac{\sqrt{2}}{2} = \ln \sqrt{2} = \frac{1}{2} \ln 2 \end{aligned}$$

$$\begin{aligned} \text{Area } C &= \int_1^{\sqrt{3}} \arctan x dx = \left(\frac{\pi}{3}\right)(\sqrt{3}) - \frac{1}{2} \ln 2 - \left(\frac{\pi}{4}\right)(1) \\ &= \frac{\pi}{12}(4\sqrt{3} - 3) - \frac{1}{2} \ln 2 \approx 0.6818 \end{aligned}$$



$$\begin{aligned} 8. \quad y &= e^x \\ y' &= e^x \\ y - b &= e^a(x - a) \\ y &= e^a x - ae^a + b \quad \text{Tangent line} \\ \text{If } y &= 0, \\ e^a x &= ae^a - b \\ bx &= ab - b \quad (b = e^a) \\ x &= a - 1 \\ c &= a - 1 \end{aligned}$$

$$\text{Thus, } a - c = a - (a - 1) = 1.$$

$$10. \text{ Let } u = \tan x, du = \sec^2 x dx$$

$$\begin{aligned} \text{Area} &= \int_0^{\pi/4} \frac{1}{\sin^2 x + 4 \cos^2 x} dx = \int_0^{\pi/4} \frac{\sec^2 x}{\tan^2 x + 4} dx \\ &= \int_0^1 \frac{du}{u^2 + 4} \\ &= \left[\frac{1}{2} \arctan\left(\frac{u}{2}\right) \right]_0^1 \\ &= \frac{1}{2} \arctan\left(\frac{1}{2}\right) \end{aligned}$$

$$12. \text{ (a) } \frac{dy}{dt} = y(1 - y), y(0) = \frac{1}{4}$$

$$\int \left(\frac{1}{y} + \frac{1}{1 - y} \right) dy = \int dt$$

$$\ln|y| - \ln|1 - y| = t + C$$

$$\ln \left| \frac{y}{1 - y} \right| = t + C$$

$$\frac{y}{1 - y} = e^{t+C} = C_1 e^t$$

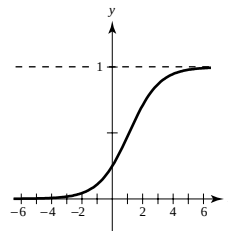
$$y = C_1 e^t - y C_1 e^t$$

$$y = \frac{C_1 e^t}{1 + C_1 e^t} = \frac{1}{1 + C_2 e^{-t}}$$

$$y(0) = \frac{1}{4} = \frac{1}{1 + C_2} \Rightarrow C_2 = 3$$

$$\text{Hence, } y = \frac{1}{1 + 3e^{-t}}.$$

(b)



$$\frac{dy}{dt} = y(1 - y) = y - y^2$$

$$\frac{d^2y}{dt^2} = y'' = y' - 2yy' \Rightarrow y'' = 0 \text{ for } y = \frac{1}{2}$$

$$\frac{d^2y}{dt^2} > 0 \text{ if } 0 < y < \frac{1}{2} \text{ and } \frac{d^2y}{dt^2} < 0 \text{ if } \frac{1}{2} < y < 1.$$

Thus, the rate of growth is maximum at $y = \frac{1}{2}$, the point of inflection.

12. —CONTINUED—

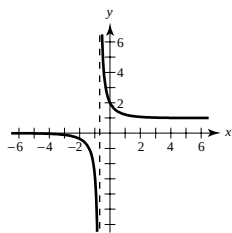
$$(c) \quad y' = y(1 - y), y(0) = 2$$

$$\text{As before, } y = \frac{1}{1 + C_2 e^{-t}}$$

$$y(0) = 2 = \frac{1}{1 + C_2} \Rightarrow C_2 = -\frac{1}{2}$$

$$\text{Thus, } y = \frac{1}{1 - \frac{1}{2}e^{-t}} = \frac{2}{2 - e^{-t}}.$$

The graph is different:



$$14. (a) \quad u = 985.93 - \left(985.93 - \frac{(120,000)(0.095)}{12} \right) \left(1 + \frac{0.095}{12} \right)^{12t}$$

$$v = \left(985.93 - \frac{(120,000)(0.095)}{12} \right) \left(1 + \frac{0.095}{12} \right)^{12t}$$

(b) The larger part goes for interest. The curves intersect when $t \approx 27.7$ years.

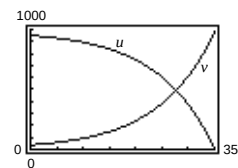
(c) The slopes are negatives of each other. Analytically,

$$u = 985.93 - v \Rightarrow \frac{du}{dt} = -\frac{dv}{dt}$$

$$u'(15) = -v'(15) = -14.06.$$

(d) $t = 12.7$ years

Again, the larger part goes for interest.



PART I

CHAPTER 6

Applications of Integration

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CHAPTER 6

Applications of Integration

Section 6.1 Area of a Region Between Two Curves

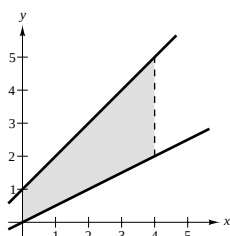
Solutions to Odd-Numbered Exercises

$$1. A = \int_0^6 [0 - (x^2 - 6x)] dx = - \int_0^6 (x^2 - 6x) dx$$

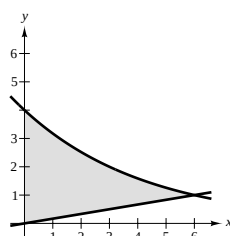
$$3. A = \int_0^3 [(-x^2 + 2x + 3) - (x^2 - 4x + 3)] dx = \int_0^3 (-2x^2 + 6x) dx$$

$$5. A = 2 \int_{-1}^0 3(x^3 - x) dx = 6 \int_{-1}^0 (x^3 - x) dx \quad \text{or} \quad -6 \int_0^1 (x^3 - x) dx$$

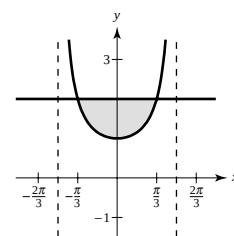
$$7. \int_0^4 \left[(x + 1) - \frac{x}{2} \right] dx$$



$$9. \int_0^6 \left[4(2^{-x/3}) - \frac{x}{6} \right] dx$$



$$11. \int_{-\pi/3}^{\pi/3} [2 - \sec x] dx$$

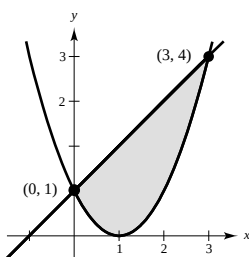


$$13. f(x) = x + 1$$

$$g(x) = (x - 1)^2$$

$$A \approx 4$$

Matches (d)

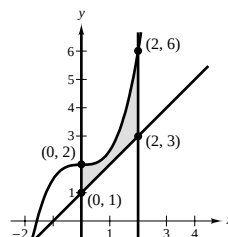


$$15. A = \int_0^2 \left[\left(\frac{1}{2}x^3 + 2 \right) - (x + 1) \right] dx$$

$$= \int_0^2 \left(\frac{1}{2}x^3 - x + 1 \right) dx$$

$$= \left[\frac{x^4}{8} - \frac{x^2}{2} + x \right]_0^2$$

$$= \left(\frac{16}{8} - \frac{4}{2} + 2 \right) - 0 = 2$$

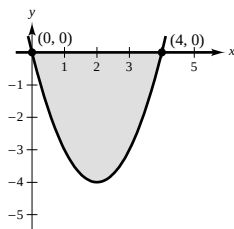


17. The points of intersection are given by:

$$x^2 - 4x = 0$$

$$x(x - 4) = 0 \quad \text{when } x = 0, 4$$

$$\begin{aligned} A &= \int_0^4 [g(x) - f(x)] dx \\ &= -\int_0^4 (x^2 - 4x) dx \\ &= -\left[\frac{x^3}{3} - 2x^2\right]_0^4 \\ &= \frac{32}{3} \end{aligned}$$

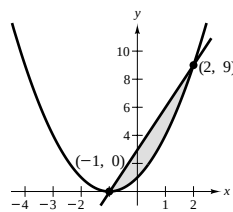


19. The points of intersection are given by:

$$x^2 + 2x + 1 = 3x + 3$$

$$(x - 2)(x + 1) = 0 \quad \text{when } x = -1, 2$$

$$\begin{aligned} A &= \int_{-1}^2 [g(x) - f(x)] dx \\ &= \int_{-1}^2 [(3x + 3) - (x^2 + 2x + 1)] dx \\ &= \int_{-1}^2 (2 + x - x^2) dx \\ &= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2 = \frac{9}{2} \end{aligned}$$



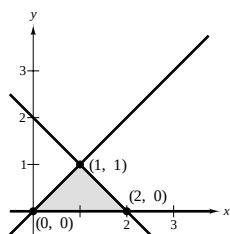
21. The points of intersection are given by:

$$x = 2 - x \quad \text{and} \quad x = 0 \quad \text{and} \quad 2 - x = 0$$

$$x = 1 \quad x = 0 \quad x = 2$$

$$A = \int_0^1 [(2 - y) - (y)] dy = \left[2y - y^2\right]_0^1 = 1$$

Note that if we integrate with respect to x , we need two integrals. Also, note that the region is a triangle.

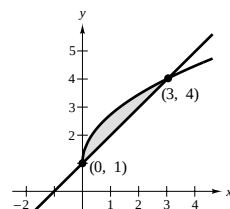


23. The points of intersection are given by:

$$\sqrt{3x} + 1 = x + 1$$

$$\sqrt{3x} = x \quad \text{when } x = 0, 3$$

$$\begin{aligned} A &= \int_0^3 [f(x) - g(x)] dx \\ &= \int_0^3 [(\sqrt{3x} + 1) - (x + 1)] dx \\ &= \int_0^3 [(3x)^{1/2} - x] dx \\ &= \left[\frac{2}{9} (3x)^{3/2} - \frac{x^2}{2}\right]_0^3 = \frac{3}{2} \end{aligned}$$

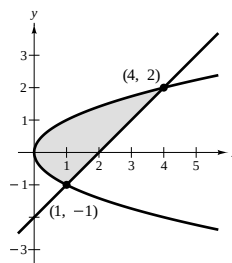


25. The points of intersection are given by:

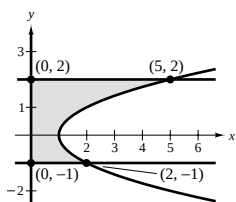
$$y^2 = y + 2$$

$$(y - 2)(y + 1) = 0 \quad \text{when } y = -1, 2$$

$$\begin{aligned} A &= \int_{-1}^2 [g(y) - f(y)] dy \\ &= \int_{-1}^2 [(y + 2) - y^2] dy \\ &= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_{-1}^2 = \frac{9}{2} \end{aligned}$$

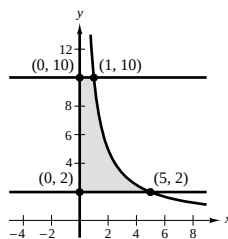


$$\begin{aligned}
 27. A &= \int_{-1}^2 [f(y) - g(y)] dy \\
 &= \int_{-1}^2 [(y^2 + 1) - 0] dy \\
 &= \left[\frac{y^3}{3} + y \right]_{-1}^2 = 6
 \end{aligned}$$



$$29. y = \frac{10}{x} \Rightarrow x = \frac{10}{y}$$

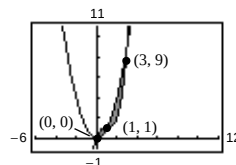
$$\begin{aligned}
 A &= \int_2^{10} \frac{10}{y} dy \\
 &= \left[10 \ln y \right]_2^{10} \\
 &= 10(\ln 10 - \ln 2) \\
 &= 10 \ln 5 \approx 16.0944
 \end{aligned}$$



31. The points of intersection are given by:

$$\begin{aligned}
 x^3 - 3x^2 + 3x &= x^2 \\
 x(x-1)(x-3) &= 0 \quad \text{when } x = 0, 1, 3 \\
 A &= \int_0^1 [f(x) - g(x)] dx + \int_1^3 [g(x) - f(x)] dx \\
 &= \int_0^1 [(x^3 - 3x^2 + 3x) - x^2] dx + \int_1^3 [x^2 - (x^3 - 3x^2 + 3x)] dx \\
 &= \int_0^1 (x^3 - 4x^2 + 3x) dx + \int_1^3 (-x^3 + 4x^2 - 3x) dx \\
 &= \left[\frac{x^4}{4} - \frac{4}{3}x^3 + \frac{3}{2}x^2 \right]_0^1 + \left[-\frac{x^4}{4} + \frac{4}{3}x^3 - \frac{3}{2}x^2 \right]_1^3 = \frac{37}{12}
 \end{aligned}$$

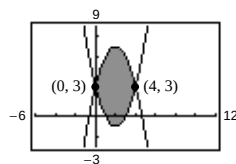
Numerical Approximation: $0.417 + 2.667 \approx 3.083$



33. The points of intersection are given by:

$$\begin{aligned}
 x^2 - 4x + 3 &= 3 + 4x - x^2 \\
 2x(x-4) &= 0 \quad \text{when } x = 0, 4 \\
 A &= \int_0^4 [(3 + 4x - x^2) - (x^2 - 4x + 3)] dx \\
 &= \int_0^4 (-2x^2 + 8x) dx \\
 &= \left[-\frac{2x^3}{3} + 4x^2 \right]_0^4 = \frac{64}{3}
 \end{aligned}$$

Numerical Approximation: 21.333



35. $f(x) = x^4 - 4x^2$, $g(x) = x^2 - 4$

The points of intersection are given by:

$$x^4 - 4x^2 = x^2 - 4$$

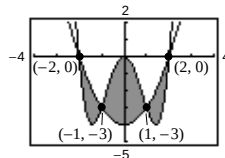
$$x^4 - 5x^2 + 4 = 0$$

$$(x^2 - 4)(x^2 - 1) = 0 \quad \text{when } x = \pm 2, \pm 1$$

By symmetry,

$$\begin{aligned} A &= 2 \int_0^1 [(x^4 - 4x^2) - (x^2 - 4)] dx + 2 \int_1^2 [(x^2 - 4) - (x^4 - 4x^2)] dx \\ &= 2 \int_0^1 (x^4 - 5x^2 + 4) dx + 2 \int_1^2 (-x^4 + 5x^2 - 4) dx \\ &= 2 \left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x \right]_0^1 + 2 \left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x \right]_1^2 \\ &= 2 \left[\frac{1}{5} - \frac{5}{3} + 4 \right] + 2 \left[\left(-\frac{32}{5} + \frac{40}{3} - 8 \right) - \left(-\frac{1}{5} + \frac{5}{3} - 4 \right) \right] = 8. \end{aligned}$$

Numerical Approximation: $5.067 + 2.933 = 8.0$



37. The points of intersection are given by:

$$\frac{1}{1+x^2} = \frac{x^2}{2}$$

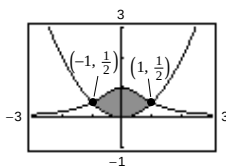
$$x^4 + x^2 - 2 = 0$$

$$(x^2 + 2)(x^2 - 1) = 0$$

$$x = \pm 1$$

$$\begin{aligned} A &= 2 \int_0^1 [f(x) - g(x)] dx \\ &= 2 \int_0^1 \left[\frac{1}{1+x^2} - \frac{x^2}{2} \right] dx \\ &= 2 \left[\arctan x - \frac{x^3}{6} \right]_0^1 \\ &= 2 \left(\frac{\pi}{4} - \frac{1}{6} \right) = \frac{\pi}{2} - \frac{1}{3} \approx 1.237 \end{aligned}$$

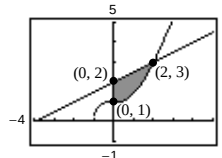
Numerical Approximation: 1.237



39. $\sqrt{1+x^3} \leq \frac{1}{2}x + 2$ on $[0, 2]$

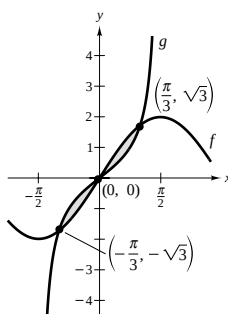
Numerical approximation: 1.759

$$A = \int_0^2 \left[\frac{1}{2}x + 2 - \sqrt{1+x^3} \right] dx \approx 1.759$$

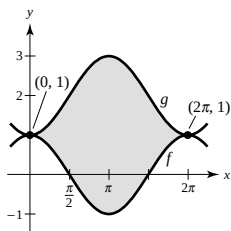


41. $A = 2 \int_0^{\pi/3} [f(x) - g(x)] dx$

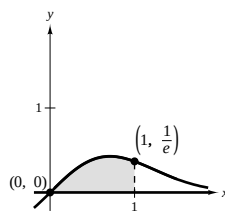
$$\begin{aligned} &= 2 \int_0^{\pi/3} (2 \sin x - \tan x) dx \\ &= 2 \left[-2 \cos x + \ln |\cos x| \right]_0^{\pi/3} \\ &= 2(1 - \ln 2) \approx 0.614 \end{aligned}$$



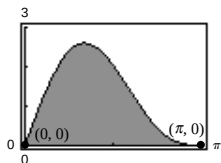
$$\begin{aligned}
 43. A &= \int_0^{2\pi} [(2 - \cos x) - \cos x] dx \\
 &= 2 \int_0^{2\pi} (1 - \cos x) dx \\
 &= 2 \left[x - \sin x \right]_0^{2\pi} = 4\pi \approx 12.566
 \end{aligned}$$



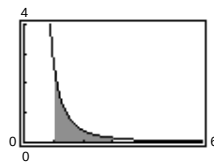
$$\begin{aligned}
 45. A &= \int_0^1 [xe^{-x^2} - 0] dx \\
 &= \left[-\frac{1}{2}e^{-x^2} \right]_0^1 = \frac{1}{2} \left(1 - \frac{1}{e} \right) \approx 0.316
 \end{aligned}$$



$$\begin{aligned}
 47. A &= \int_0^{\pi} [(2 \sin x + \sin 2x) - 0] dx \\
 &= \left[-2 \cos x - \frac{1}{2} \cos 2x \right]_0^{\pi} = 4.0
 \end{aligned}$$



$$\begin{aligned}
 49. A &= \int_1^3 \left[\frac{1}{x^2} e^{1/x} - 0 \right] dx \\
 &= \left[-e^{1/x} \right]_1^3 = e - e^{1/3} \approx 1.323
 \end{aligned}$$

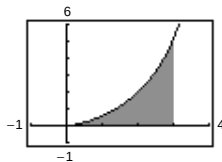


$$51. (a) y = \sqrt{\frac{x^3}{4-x}}, \quad y = 0, \quad x = 3$$

$$(b) A = \int_0^3 \sqrt{\frac{x^3}{4-x}} dx,$$

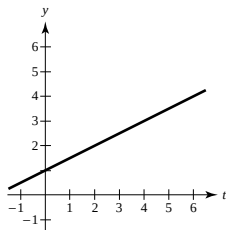
No, it cannot be evaluated by hand.

$$(c) 4.7721$$

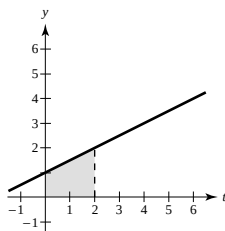


$$53. F(x) = \int_0^x \left(\frac{1}{2}t + 1 \right) dt = \left[\frac{t^2}{4} + t \right]_0^x = \frac{x^2}{4} + x$$

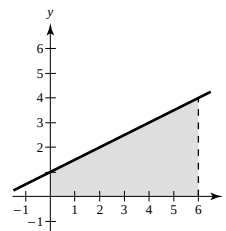
$$(a) F(0) = 0$$



$$(b) F(2) = \frac{2^2}{4} + 2 = 3$$

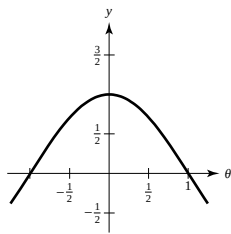


$$(c) F(6) = \frac{6^2}{4} + 6 = 15$$

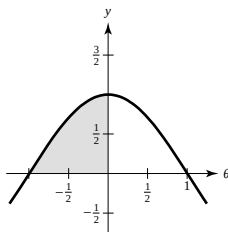


$$55. F(\alpha) = \int_{-1}^{\alpha} \cos \frac{\pi\theta}{2} d\theta = \left[\frac{2}{\pi} \sin \frac{\pi\theta}{2} \right]_{-1}^{\alpha} = \frac{2}{\pi} \sin \frac{\pi\alpha}{2} + \frac{2}{\pi}$$

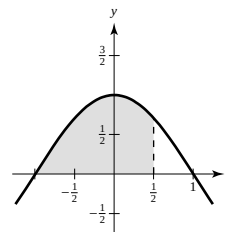
$$(a) F(-1) = 0$$



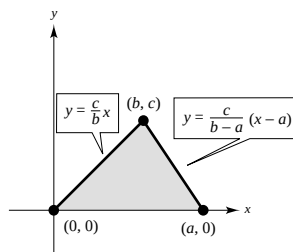
$$(b) F(0) = \frac{2}{\pi} \approx 0.6366$$



$$(c) F\left(\frac{1}{2}\right) = \frac{2 + \sqrt{2}}{\pi} \approx 1.0868$$



$$\begin{aligned} 57. A &= \int_0^c \left[\left(\frac{b-a}{c}y + a \right) - \frac{b}{c}y \right] dy \\ &= \int_0^c \left(-\frac{a}{c}y + a \right) dy \\ &= \left[-\frac{a}{2c}y^2 + ay \right]_0^c \\ &= -\frac{ac}{2} + ac = \frac{ac}{2} \quad \left(= \frac{1}{2}(\text{base})(\text{height}) \right) \end{aligned}$$



$$59. f(x) = x^3$$

$$f'(x) = 3x^2$$

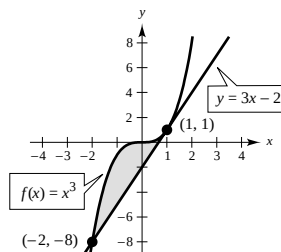
$$\text{At } (1, 1), f'(1) = 3.$$

Tangent line:

$$y - 1 = 3(x - 1) \text{ or } y = 3x - 2$$

The tangent line intersects $f(x) = x^3$ at $x = -2$.

$$A = \int_{-2}^1 [x^3 - (3x - 2)] dx = \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x \right]_{-2}^1 = \frac{27}{4}$$

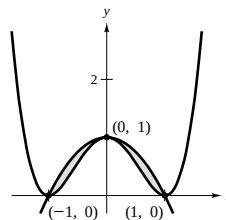


61. The variable is y .

$$63. x^4 - 2x^2 + 1 \leq 1 - x^2 \text{ on } [-1, 1]$$

$$\begin{aligned} A &= \int_{-1}^1 [(1 - x^2) - (x^4 - 2x^2 + 1)] dx \\ &= \int_{-1}^1 (x^2 - x^4) dx \\ &= \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_{-1}^1 = \frac{4}{15} \end{aligned}$$

You can use a single integral because $x^4 - 2x^2 + 1 \leq 1 - x^2$ on $[-1, 1]$.



65. Offer 2 is better because the accumulated salary (area under the curve) is larger.

$$67. \quad A = \int_{-3}^3 (9 - x^2) dx = 36$$

$$\int_{-\sqrt{9-b}}^{\sqrt{9-b}} [(9 - x^2) - b] dx = 18$$

$$\int_0^{\sqrt{9-b}} [(9 - b) - x^2] dx = 9$$

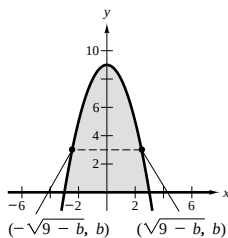
$$\left[(9 - b)x - \frac{x^3}{3} \right]_0^{\sqrt{9-b}} = 9$$

$$\frac{2}{3} (9 - b)^{3/2} = 9$$

$$(9 - b)^{3/2} = \frac{27}{2}$$

$$9 - b = \frac{9}{\sqrt[3]{4}}$$

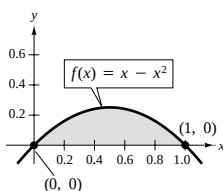
$$b = 9 - \frac{9}{\sqrt[3]{4}} \approx 3.330$$



$$69. \quad \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (x_i - x_i^2) \Delta x$$

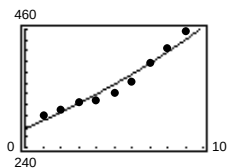
where $x_i = \frac{i}{n}$ and $\Delta x = \frac{1}{n}$ is the same as

$$\int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}.$$

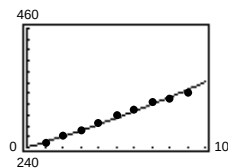


$$71. \quad \int_0^5 [(7.21 + 0.58t) - (7.21 + 0.45t)] dt = \int_0^5 0.13t dt = \left[\frac{0.13t^2}{2} \right]_0^5 = \$1.625 \text{ billion}$$

$$73. \quad (a) \quad y_1 = (275.0675)(1.0537)^t = (275.0675)e^{0.0523t}$$



$$(b) \quad y_2 = (239.9407)(1.0417)^t = (239.9407)e^{0.0408t}$$



$$(c) \quad \int_{10}^{15} (y_1 - y_2) dt \approx 649.5 \text{ billion dollars}$$

(d) No, model $y_1 > y_2$ forever because $1.0537 > 1.0417$.

No, these models are not accurate. According to news reports, $E > R$ eventually.

75. The total area is 8 times the area of the shaded region to the right. A point (x, y) is on the upper boundary of the region if

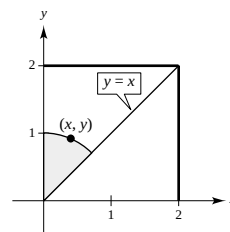
$$\sqrt{x^2 + y^2} = 2 - y$$

$$x^2 + y^2 = 4 - 4y + y^2$$

$$x^2 = 4 - 4y$$

$$4y = 4 - x^2$$

$$y = 1 - \frac{x^2}{4}$$



We now determine where this curve intersects the line $y = x$.

$$x = 1 - \frac{x^2}{4}$$

$$x^2 + 4x - 4 = 0$$

$$x = \frac{-4 \pm \sqrt{16 + 16}}{2} = -2 \pm 2\sqrt{2} \Rightarrow x = -2 + 2\sqrt{2}$$

$$\text{Total area} = 8 \int_0^{-2+2\sqrt{2}} \left(1 - \frac{x^2}{4} - x\right) dx$$

$$= 8 \left[x - \frac{x^3}{12} - \frac{x^2}{2} \right]_0^{-2+2\sqrt{2}} = \frac{16}{3}(4\sqrt{2} - 5) \approx 8(0.4379) = 3.503$$

$$\begin{aligned} 77. \text{ (a) } A &= 2 \left[\int_0^5 \left(1 - \frac{1}{3} \sqrt{5-x}\right) dx + \int_5^{5.5} (1-0) dx \right] \\ &= 2 \left(\left[x + \frac{2}{9} (5-x)^{3/2} \right]_0^5 + \left[x \right]_5^{5.5} \right) = 2 \left(5 - \frac{10\sqrt{5}}{9} + 5.5 - 5 \right) \approx 6.031 \text{ m}^2 \end{aligned}$$

$$\text{(b) } V = 2A \approx 2(6.031) \approx 12.062 \text{ m}^3$$

$$\text{(c) } 5000 V \approx 5000(12.062) = 60,310 \text{ pounds}$$

79. True

81. False. Let $f(x) = x$ and $g(x) = 2x - x^2$. f and g intersect at $(1, 1)$, the midpoint of $[0, 2]$. But

$$\int_a^b [f(x) - g(x)] dx = \int_0^2 [x - (2x - x^2)] dx = \frac{2}{3} \neq 0.$$

Section 6.2 Volume: The Disk Method

$$1. V = \pi \int_0^1 (-x + 1)^2 dx = \pi \int_0^1 (x^2 - 2x + 1) dx = \pi \left[\frac{x^3}{3} - x^2 + x \right]_0^1 = \frac{\pi}{3}$$

$$3. V = \pi \int_1^4 (\sqrt{x})^2 dx = \pi \int_1^4 x dx = \pi \left[\frac{x^2}{2} \right]_1^4 = \frac{15\pi}{2}$$

$$5. V = \pi \int_0^1 [(x^2)^2 - (x^3)^2] dx = \pi \int_0^1 (x^4 - x^6) dx = \pi \left[\frac{x^5}{5} - \frac{x^7}{7} \right]_0^1 = \frac{2\pi}{35}$$

$$7. y = x^2 \Rightarrow x = \sqrt{y}$$

$$V = \pi \int_0^4 (\sqrt{y})^2 dy = \pi \int_0^4 y dy$$

$$= \pi \left[\frac{y^2}{2} \right]_0^4 = 8\pi$$

$$9. y = x^{2/3} \Rightarrow x = y^{3/2}$$

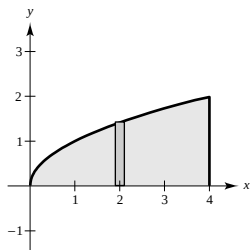
$$V = \pi \int_0^1 (y^{3/2})^2 dy = \pi \int_0^1 y^3 dy = \pi \left[\frac{y^4}{4} \right]_0^1 = \frac{\pi}{4}$$

11. $y = \sqrt{x}$, $y = 0$, $x = 4$

(a) $R(x) = \sqrt{x}$, $r(x) = 0$

$$V = \pi \int_0^4 (\sqrt{x})^2 dx$$

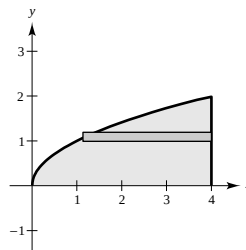
$$= \pi \int_0^4 x dx = \left[\frac{\pi}{2} x^2 \right]_0^4 = 8\pi$$



(b) $R(y) = 4$, $r(y) = y^2$

$$V = \pi \int_0^2 (16 - y^4) dy$$

$$= \pi \left[16y - \frac{1}{5} y^5 \right]_0^2 = \frac{128\pi}{5}$$

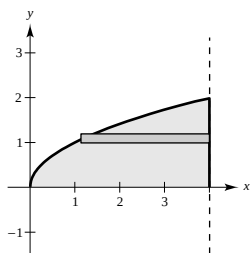


(c) $R(y) = 4 - y^2$, $r(y) = 0$

$$V = \pi \int_0^2 (4 - y^2)^2 dy$$

$$= \pi \int_0^2 (16 - 8y^2 + y^4) dy$$

$$= \pi \left[16y - \frac{8}{3} y^3 + \frac{1}{5} y^5 \right]_0^2 = \frac{256\pi}{15}$$

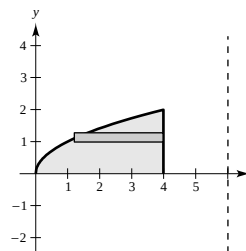


(d) $R(y) = 6 - y^2$, $r(y) = 2$

$$V = \pi \int_0^2 [(6 - y^2)^2 - 4] dy$$

$$= \pi \int_0^2 (32 - 12y^2 + y^4) dy$$

$$= \pi \left[32y - 4y^3 + \frac{1}{5} y^5 \right]_0^2 = \frac{192\pi}{5}$$



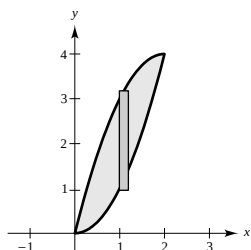
13. $y = x^2$, $y = 4x - x^2$ intersect at $(0, 0)$ and $(2, 4)$.

(a) $R(x) = 4x - x^2$, $r(x) = x^2$

$$V = \pi \int_0^2 [(4x - x^2)^2 - x^4] dx$$

$$= \pi \int_0^2 (16x^2 - 8x^3) dx$$

$$= \pi \left[\frac{16}{3} x^3 - 2x^4 \right]_0^2 = \frac{32\pi}{3}$$

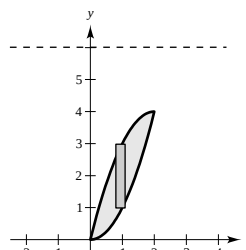


(b) $R(x) = 6 - x^2$, $r(x) = 6 - (4x - x^2)$

$$V = \pi \int_0^2 [(6 - x^2)^2 - (6 - 4x + x^2)^2] dx$$

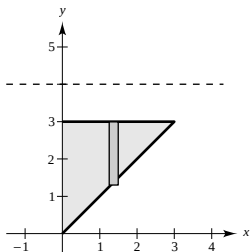
$$= 8\pi \int_0^2 (x^3 - 5x^2 + 6x) dx$$

$$= 8\pi \left[\frac{x^4}{4} - \frac{5}{3} x^3 + 3x^2 \right]_0^2 = \frac{64\pi}{3}$$



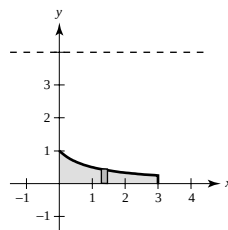
15. $R(x) = 4 - x$, $r(x) = 1$

$$\begin{aligned} V &= \pi \int_0^3 [(4 - x)^2 - (1)^2] dx \\ &= \pi \int_0^3 (x^2 - 8x + 15) dx \\ &= \pi \left[\frac{x^3}{3} - 4x^2 + 15x \right]_0^3 = 18\pi \end{aligned}$$



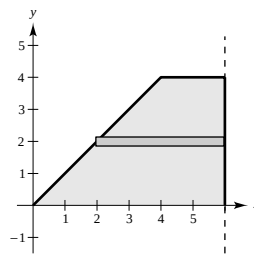
17. $R(x) = 4$, $r(x) = 4 - \frac{1}{1+x}$

$$\begin{aligned} V &= \pi \int_0^3 \left[4^2 - \left(4 - \frac{1}{1+x} \right)^2 \right] dx \\ &= \pi \int_0^3 \left[\frac{8}{1+x} - \frac{1}{(1+x)^2} \right] dx \\ &= \pi \left[8 \ln(1+x) + \frac{1}{1+x} \right]_0^3 \\ &= \pi \left[8 \ln 4 + \frac{1}{4} - 1 \right] \\ &= \left(8 \ln 4 - \frac{3}{4} \right) \pi \approx 32.485 \end{aligned}$$



19. $R(y) = 6 - y$, $r(y) = 0$

$$\begin{aligned} V &= \pi \int_0^4 (6 - y)^2 dy \\ &= \pi \int_0^4 (y^2 - 12y + 36) dy \\ &= \pi \left[\frac{y^3}{3} - 6y^2 + 36y \right]_0^4 \\ &= \frac{208\pi}{3} \end{aligned}$$

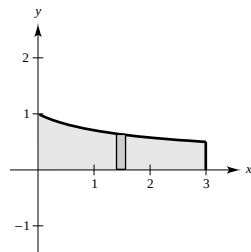


21. $R(y) = 6 - y^2$, $r(y) = 2$

$$\begin{aligned} V &= \pi \int_{-2}^2 [(6 - y^2)^2 - (2)^2] dy \\ &= 2\pi \int_0^2 (y^4 - 12y^2 + 32) dy \\ &= 2\pi \left[\frac{y^5}{5} - 4y^3 + 32y \right]_0^2 \\ &= \frac{384\pi}{5} \end{aligned}$$

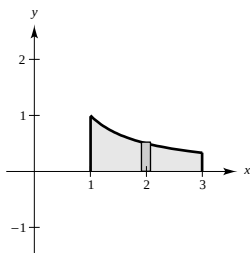
23. $R(x) = \frac{1}{\sqrt{x+1}}$, $r(x) = 0$

$$\begin{aligned} V &= \pi \int_0^3 \left(\frac{1}{\sqrt{x+1}} \right)^2 dx \\ &= \pi \int_0^3 \frac{1}{x+1} dx \\ &= \left[\pi \ln|x+1| \right]_0^3 = \pi \ln 4 \end{aligned}$$



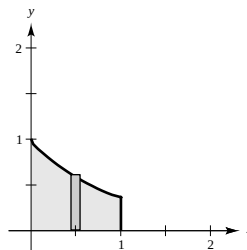
25. $R(x) = \frac{1}{x}$, $r(x) = 0$

$$\begin{aligned} V &= \pi \int_1^4 \left(\frac{1}{x}\right)^2 dx \\ &= \pi \left[-\frac{1}{x} \right]_1^4 \\ &= \frac{3\pi}{4} \end{aligned}$$



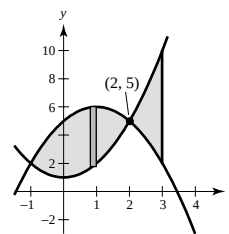
27. $R(x) = e^{-x}$, $r(x) = 0$

$$\begin{aligned} V &= \pi \int_0^1 (e^{-x})^2 dx \\ &= \pi \int_0^1 e^{-2x} dx \\ &= \left[-\frac{\pi}{2} e^{-2x} \right]_0^1 \\ &= \frac{\pi}{2} (1 - e^{-2}) \approx 1.358 \end{aligned}$$



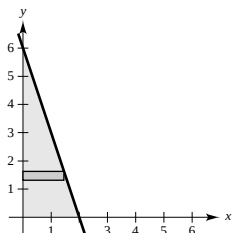
29. $V = \pi \int_0^2 [(5 + 2x - x^2)^2 - (x^2 + 1)^2] dx + \pi \int_2^3 [(x^2 + 1)^2 - (5 + 2x - x^2)^2] dx$

$$\begin{aligned} &= \pi \int_0^2 (-4x^3 - 8x^2 + 20x + 24) dx + \pi \int_2^3 (4x^3 + 8x^2 - 20x - 24) dx \\ &= \pi \left[-x^4 - \frac{8}{3}x^3 + 10x^2 + 24x \right]_0^2 + \pi \left[x^4 + \frac{8}{3}x^3 - 10x^2 - 24x \right]_2^3 \\ &= \pi \frac{152}{3} + \pi \frac{125}{3} = \frac{277\pi}{3} \end{aligned}$$

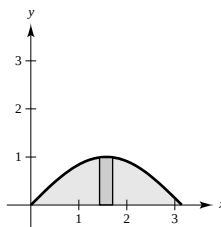


31. $y = 6 - 3x \Rightarrow x = \frac{1}{3}(6 - y)$

$$\begin{aligned} V &= \pi \int_0^6 \left[\frac{1}{3}(6 - y) \right]^2 dy \\ &= \frac{\pi}{9} \int_0^6 [36 - 12y + y^2] dy \\ &= \frac{\pi}{9} \left[36y - 6y^2 + \frac{y^3}{3} \right]_0^6 \\ &= \frac{\pi}{9} \left[216 - 216 + \frac{216}{3} \right] \\ &= 8\pi \end{aligned}$$



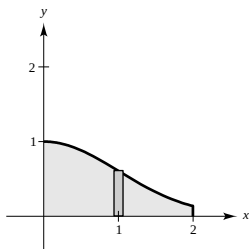
33. $V = \pi \int_0^\pi [\sin x]^2 dx \approx 4.9348$



$$35. V = \pi \int_0^2 [e^{-x^2}]^2 dx \approx 1.9686$$

$$39. A \approx 3$$

Matches (a)



$$37. V = \pi \int_{-1}^2 [e^{x/2} + e^{-x/2}]^2 dx \approx 49.0218$$

41. Disk Method:

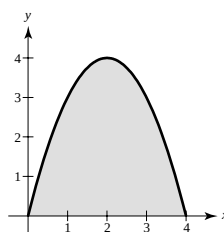
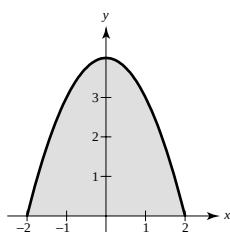
$$V = \pi \int_a^b [R(x)]^2 dx \quad \text{or} \quad V = \pi \int_c^d [R(y)]^2 dy$$

Washer Method:

$$V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2) dx \quad \text{or}$$

$$V = \pi \int_c^d ([R(y)]^2 - [r(y)]^2) dy$$

43.



The volumes are the same because the solid has been translated horizontally.

$$45. R(x) = \frac{1}{2}x, \quad r(x) = 0$$

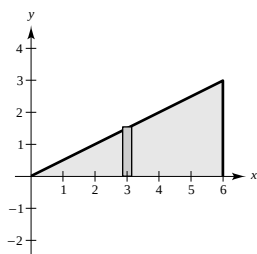
$$V = \pi \int_0^6 \frac{1}{4}x^2 dx$$

$$= \left[\frac{\pi}{12}x^3 \right]_0^6 = 18\pi$$

$$\text{Note: } V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi(3^2)6$$

$$= 18\pi$$



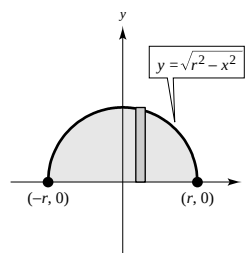
$$47. R(x) = \sqrt{r^2 - x^2}, \quad r(x) = 0$$

$$V = \pi \int_{-r}^r (r^2 - x^2) dx$$

$$= 2\pi \int_0^r (r^2 - x^2) dx$$

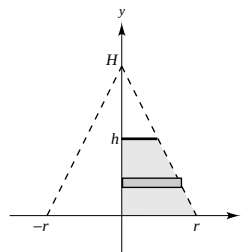
$$= 2\pi \left[r^2x - \frac{1}{3}x^3 \right]_0^r$$

$$= 2\pi \left(r^3 - \frac{1}{3}r^3 \right) = \frac{4}{3}\pi r^3$$



49. $x = r - \frac{r}{H}y = r\left(1 - \frac{y}{H}\right)$, $R(y) = r\left(1 - \frac{y}{H}\right)$, $r(y) = 0$

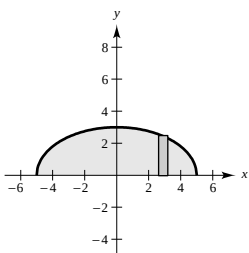
$$\begin{aligned} V &= \pi \int_0^h \left[r\left(1 - \frac{y}{H}\right) \right]^2 dy = \pi r^2 \int_0^h \left(1 - \frac{2}{H}y + \frac{1}{H^2}y^2 \right) dy \\ &= \pi r^2 \left[y - \frac{1}{H}y^2 + \frac{1}{3H^2}y^3 \right]_0^h \\ &= \pi r^2 \left(h - \frac{h^2}{H} + \frac{h^3}{3H^2} \right) \\ &= \pi r^2 h \left(1 - \frac{h}{H} + \frac{h^2}{3H^2} \right) \end{aligned}$$



51. $V = \pi \int_0^2 \left(\frac{1}{8}x^2 \sqrt{2-x} \right)^2 dx = \frac{\pi}{64} \int_0^2 x^4(2-x) dx = \frac{\pi}{64} \left[\frac{2x^5}{5} - \frac{x^6}{6} \right]_0^2 = \frac{\pi}{30}$

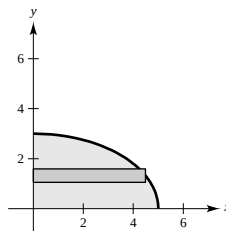
53. (a) $R(x) = \frac{3}{5} \sqrt{25-x^2}$, $r(x) = 0$

$$\begin{aligned} V &= \frac{9\pi}{25} \int_{-5}^5 (25-x^2) dx \\ &= \frac{18\pi}{25} \int_0^5 (25-x^2) dx \\ &= \frac{18\pi}{25} \left[25x - \frac{x^3}{3} \right]_0^5 = 60\pi \end{aligned}$$



(b) $R(y) = \frac{5}{3} \sqrt{9-y^2}$, $r(y) = 0$, $x \geq 0$

$$\begin{aligned} V &= \frac{25\pi}{9} \int_0^3 (9-y^2) dy \\ &= \frac{25\pi}{9} \left[9y - \frac{y^3}{3} \right]_0^3 = 50\pi \end{aligned}$$



55. Total volume: $V = \frac{4\pi(50)^3}{3} = \frac{500,000}{3} \text{ ft}^3$

Volume of water in the tank:

$$\begin{aligned} \pi \int_{-50}^{y_0} (\sqrt{2500-y^2})^2 dy &= \pi \int_{-50}^{y_0} (2500-y^2) dy \\ &= \pi \left[2500y - \frac{y^3}{3} \right]_{-50}^{y_0} \\ &= \pi \left(2500y_0 - \frac{y_0^3}{3} + \frac{250,000}{3} \right) \end{aligned}$$

When the tank is one-fourth of its capacity:

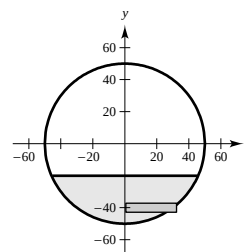
$$\begin{aligned} \frac{1}{4} \left(\frac{500,000\pi}{3} \right) &= \pi \left(2500y_0 - \frac{y_0^3}{3} + \frac{250,000}{3} \right) \\ 125,000 &= 7500y_0 - y_0^3 + 250,000 \end{aligned}$$

$$y_0^3 - 7500y_0 - 125,000 = 0$$

$$y_0 \approx -17.36$$

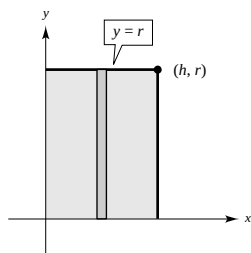
Depth: $-17.36 - (-50) = 32.64$ feet

When the tank is three-fourths of its capacity the depth is $100 - 32.64 = 67.36$ feet.



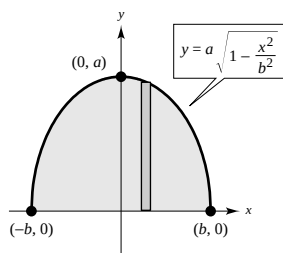
57. (a) $\pi \int_0^h r^2 dx$ (ii)

is the volume of a right circular cylinder with radius r and height h .



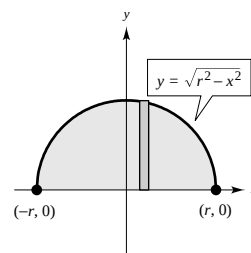
(b) $\pi \int_{-b}^b \left(a \sqrt{1 - \frac{x^2}{b^2}} \right)^2 dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.



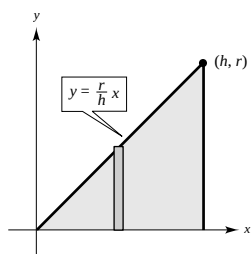
(c) $\pi \int_{-r}^r (\sqrt{r^2 - x^2})^2 dx$ (iii)

is the volume of a sphere with radius r .



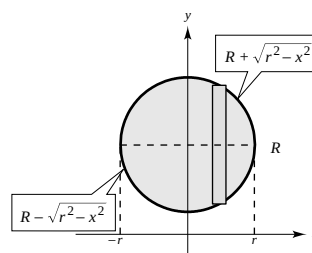
(d) $\pi \int_0^h \left(\frac{rx}{h} \right)^2 dx$ (i)

is the volume of a right circular cone with the radius of the base as r and height h .

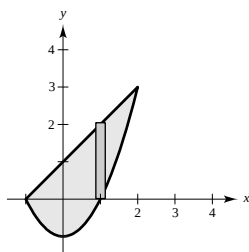


(e) $\pi \int_{-r}^r \left[(R + \sqrt{r^2 - x^2})^2 - (R - \sqrt{r^2 - x^2})^2 \right] dx$ (v)

is the volume of a torus with the radius of its circular cross section as r and the distance from the axis of the torus to the center of its cross section as R .



59.



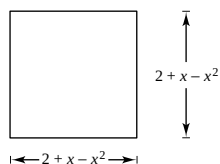
$$\text{Base of Cross Section} = (x + 1) - (x^2 - 1) = 2 + x - x^2$$

(a) $A(x) = b^2 = (2 + x - x^2)^2$

$$= 4 + 4x - 3x^2 - 2x^3 + x^4$$

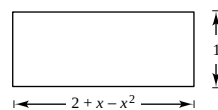
$$V = \int_{-1}^2 (4 + 4x - 3x^2 - 2x^3 + x^4) dx$$

$$= \left[4x + 2x^2 - x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5 \right]_{-1}^2 = \frac{81}{10}$$

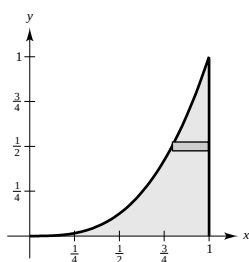


(b) $A(x) = bh = (2 + x - x^2)1$

$$V = \int_{-1}^2 (2 + x - x^2) dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^2 = \frac{9}{2}$$



61.

Base of Cross Section = $1 - \sqrt[3]{y}$

$$(b) A(y) = \frac{1}{2} \pi r^2 = \frac{1}{2} \pi \left(\frac{1 - \sqrt[3]{y}}{2} \right)^2 = \frac{1}{8} \pi (1 - \sqrt[3]{y})^2$$

$$V = \frac{1}{8} \pi \int_0^1 (1 - \sqrt[3]{y})^2 dy = \frac{\pi}{8} \left(\frac{1}{10} \right) = \frac{\pi}{80}$$

$$(c) A(y) = \frac{1}{2} bh = \frac{1}{2} (1 - \sqrt[3]{y}) \left(\frac{\sqrt{3}}{2} \right) (1 - \sqrt[3]{y})$$

$$= \frac{\sqrt{3}}{4} (1 - \sqrt[3]{y})^2$$

$$V = \frac{\sqrt{3}}{4} \int_0^1 (1 - \sqrt[3]{y})^2 dy = \frac{\sqrt{3}}{4} \left(\frac{1}{10} \right) = \frac{\sqrt{3}}{40}$$

$$(d) A(y) = \frac{1}{2} \pi ab = \frac{\pi}{2} (2) (1 - \sqrt[3]{y}) \frac{1 - \sqrt[3]{y}}{2}$$

$$= \frac{\pi}{2} (1 - \sqrt[3]{y})^2$$

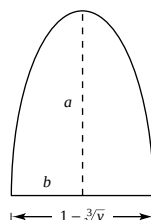
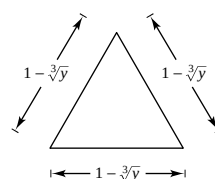
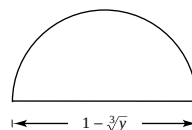
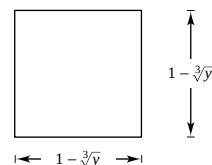
$$V = \frac{\pi}{2} \int_0^1 (1 - \sqrt[3]{y})^2 dy = \frac{\pi}{2} \left(\frac{1}{10} \right) = \frac{\pi}{20}$$

$$(a) A(y) = b^2 = (1 - \sqrt[3]{y})^2$$

$$V = \int_0^1 (1 - \sqrt[3]{y})^2 dy$$

$$= \int_0^1 (1 - 2y^{1/3} + y^{2/3}) dy$$

$$= \left[y - \frac{3}{2} y^{4/3} + \frac{3}{5} y^{5/3} \right]_0^1 = \frac{1}{10}$$



63. Let $A_1(x)$ and $A_2(x)$ equal the areas of the cross sections of the two solids for $a \leq x \leq b$. Since $A_1(x) = A_2(x)$, we have

$$V_1 = \int_a^b A_1(x) dx = \int_a^b A_2(x) dx = V_2$$

Thus, the volumes are the same.

$$65. \frac{4}{3} \pi (25 - r^2)^{3/2} = \frac{1}{2} \left(\frac{4}{3} \right) \pi (125)$$

$$(25 - r^2)^{3/2} = \frac{125}{2}$$

$$25 - r^2 = \left(\frac{125}{2} \right)^{2/3}$$

$$25 - \frac{25}{(2^{2/3})} = r^2$$

$$25(1 - 2^{-2/3}) = r^2$$

$$r = 5\sqrt{1 - 2^{-2/3}} \approx 3.0415$$

67. (a) Since the cross sections are isosceles right triangles:

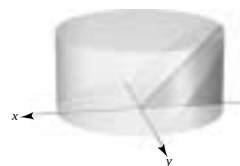
$$A(x) = \frac{1}{2} bh = \frac{1}{2} (\sqrt{r^2 - y^2}) (\sqrt{r^2 - y^2}) = \frac{1}{2} (r^2 - y^2)$$

$$V = \frac{1}{2} \int_{-r}^r (r^2 - y^2) dy = \int_0^r (r^2 - y^2) dy = \left[r^2 y - \frac{y^3}{3} \right]_0^r = \frac{2}{3} r^3$$

$$(b) A(x) = \frac{1}{2} bh = \frac{1}{2} \sqrt{r^2 - y^2} (\sqrt{r^2 - y^2} \tan \theta) = \frac{\tan \theta}{2} (r^2 - y^2)$$

$$V = \frac{\tan \theta}{2} \int_{-r}^r (r^2 - y^2) dy = \tan \theta \int_0^r (r^2 - y^2) dy = \tan \theta \left[r^2 y - \frac{y^3}{3} \right]_0^r = \frac{2}{3} r^3 \tan \theta$$

As $\theta \rightarrow 90^\circ$, $V \rightarrow \infty$.



Section 6.3 Volume: The Shell Method

1. $p(x) = x$

$h(x) = x$

$$V = 2\pi \int_0^2 x(x) dx = \left[\frac{2\pi x^3}{3} \right]_0^2 = \frac{16\pi}{3}$$

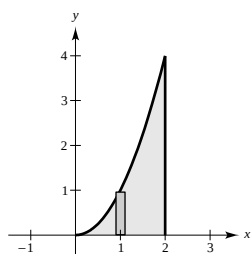
$$= \left[\frac{2\pi x^3}{3} \right]_0^2 = \frac{16\pi}{3}$$

5. $p(x) = x$

$h(x) = x^2$

$$V = 2\pi \int_0^2 x^3 dx$$

$$= \left[\frac{\pi x^4}{2} \right]_0^2 = 8\pi$$



3. $p(x) = x$

$h(x) = \sqrt{x}$

$$V = 2\pi \int_0^4 x\sqrt{x} dx$$

$$= 2\pi \int_0^4 x^{3/2} dx$$

$$= \left[\frac{4\pi}{5} x^{5/2} \right]_0^4 = \frac{128\pi}{5}$$

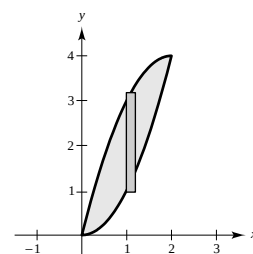
7. $p(x) = x$

$h(x) = (4x - x^2) - x^2 = 4x - 2x^2$

$$V = 2\pi \int_0^2 x(4x - 2x^2) dx$$

$$= 4\pi \int_0^2 (2x^2 - x^3) dx$$

$$= 4\pi \left[\frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_0^2 = \frac{16\pi}{3}$$

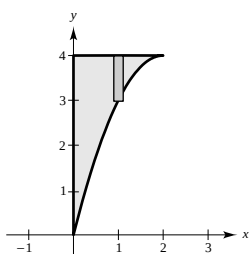


9. $p(x) = x$

$h(x) = 4 - (4x - x^2) = x^2 - 4x + 4$

$$V = 2\pi \int_0^2 (x^3 - 4x^2 + 4x) dx$$

$$= 2\pi \left[\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2 \right]_0^2 = \frac{8\pi}{3}$$



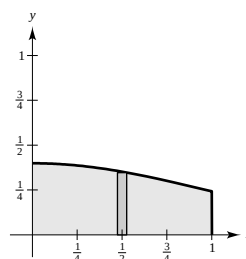
11. $p(x) = x$

$h(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$V = 2\pi \int_0^1 x \left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right) dx$$

$$= \sqrt{2\pi} \int_0^1 e^{-x^2/2} x dx$$

$$= \left[-\sqrt{2\pi} e^{-x^2/2} \right]_0^1 = \sqrt{2\pi} \left(1 - \frac{1}{\sqrt{e}} \right) \approx 0.986$$



13. $p(y) = y$

$h(y) = 2 - y$

$$V = 2\pi \int_0^2 y(2 - y) dy$$

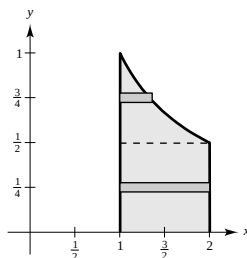
$$= 2\pi \int_0^2 (2y - y^2) dy$$

$$= 2\pi \left[y^2 - \frac{y^3}{3} \right]_0^2 = \frac{8\pi}{3}$$

15. $p(y) = y$ and $h(y) = 1$ if $0 \leq y < \frac{1}{2}$.

$$p(y) = y \text{ and } h(y) = \frac{1}{y} - 1 \text{ if } \frac{1}{2} \leq y \leq 1.$$

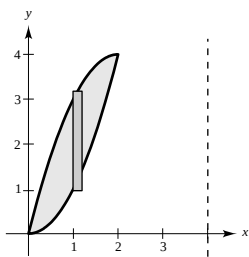
$$\begin{aligned} V &= 2\pi \int_0^{1/2} y \, dy + 2\pi \int_{1/2}^1 (1 - y) \, dy \\ &= 2\pi \left[\frac{y^2}{2} \right]_0^{1/2} + 2\pi \left[y - \frac{y^2}{2} \right]_{1/2}^1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} \end{aligned}$$



17. $p(x) = 4 - x$

$$h(x) = 4x - x^2 - x^2 = 4x - 2x^2$$

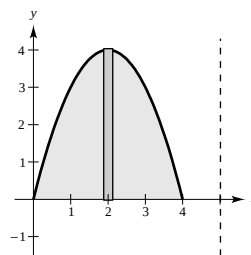
$$\begin{aligned} V &= 2\pi \int_0^2 (4 - x)(4x - 2x^2) \, dx \\ &= 2\pi(2) \int_0^2 (x^3 - 6x^2 + 8x) \, dx \\ &= 4\pi \left[\frac{x^4}{4} - 2x^3 + 4x^2 \right]_0^2 = 16\pi \end{aligned}$$



19. $p(x) = 5 - x$

$$h(x) = 4x - x^2$$

$$\begin{aligned} V &= 2\pi \int_0^4 (5 - x)(4x - x^2) \, dx \\ &= 2\pi \int_0^4 (x^3 - 9x^2 + 20x) \, dx \\ &= 2\pi \left[\frac{x^4}{4} - 3x^3 + 10x^2 \right]_0^4 = 64\pi \end{aligned}$$

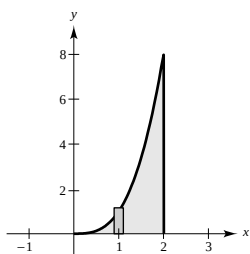


21. (a) Disk

$$R(x) = x^3$$

$$r(x) = 0$$

$$V = \pi \int_0^2 x^6 \, dx = \pi \left[\frac{x^7}{7} \right]_0^2 = \frac{128\pi}{7}$$

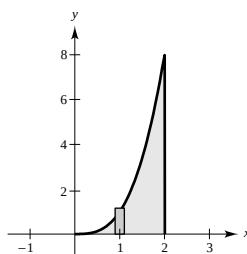


- (b) Shell

$$p(x) = x$$

$$h(x) = x^3$$

$$V = 2\pi \int_0^2 x^4 \, dx = 2\pi \left[\frac{x^5}{5} \right]_0^2 = \frac{64\pi}{5}$$

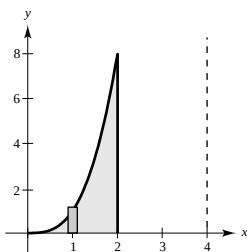


- (c) Shell

$$p(x) = 4 - x$$

$$h(x) = x^3$$

$$\begin{aligned} V &= 2\pi \int_0^2 (4 - x)x^3 \, dx \\ &= 2\pi \int_0^2 (4x^3 - x^4) \, dx \\ &= 2\pi \left[x^4 - \frac{1}{5}x^5 \right]_0^2 = \frac{96\pi}{5} \end{aligned}$$

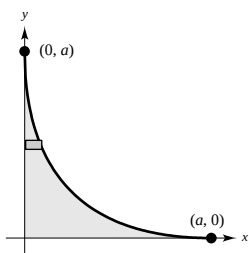


23. (a) Shell

$$p(y) = y$$

$$h(y) = (a^{1/2} - y^{1/2})^2$$

$$\begin{aligned} V &= 2\pi \int_0^a y(a - 2a^{1/2}y^{1/2} + y) dy \\ &= 2\pi \int_0^a (ay - 2a^{1/2}y^{3/2} + y^2) dy \\ &= 2\pi \left[\frac{a}{2}y^2 - \frac{4a^{1/2}}{5}y^{5/2} + \frac{y^3}{3} \right]_0^a \\ &= 2\pi \left[\frac{a^3}{2} - \frac{4a^3}{5} + \frac{a^3}{3} \right] = \frac{\pi a^3}{15} \end{aligned}$$



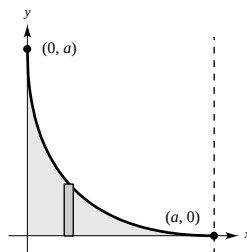
(b) Same as part (a) by symmetry

(c) Shell

$$p(x) = a - x$$

$$h(x) = (a^{1/2} - x^{1/2})^2$$

$$\begin{aligned} V &= 2\pi \int_0^a (a - x)(a^{1/2} - x^{1/2})^2 dx \\ &= 2\pi \int_0^a (a^2 - 2a^{3/2}x^{1/2} + 2a^{1/2}x^{3/2} - x^2) dx \\ &= 2\pi \left[a^2x - \frac{4}{3}a^{3/2}x^{3/2} + \frac{4}{5}a^{1/2}x^{5/2} - \frac{1}{3}x^3 \right]_0^a = \frac{4\pi a^3}{15} \end{aligned}$$



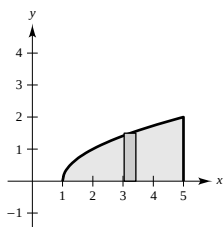
25. $V = 2\pi \int_x^d p(y)h(y) dy \quad \text{or} \quad V = 2\pi \int_a^b p(x)h(x) dx$

27. $\pi \int_1^5 (x - 1) dx = \pi \int_1^5 (\sqrt{x - 1})^2 dx$

This integral represents the volume of the solid generated by revolving the region bounded by $y = \sqrt{x - 1}$, $y = 0$, and $x = 5$ about the x -axis by using the Disk Method.

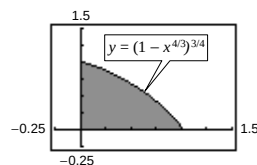
$$2\pi \int_0^2 y[5 - (y^2 + 1)] dy$$

represents this same volume by using the Shell Method.



Disk Method

29. (a)

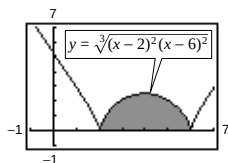


(b) $x^{4/3} + y^{4/3} = 1, x = 0, y = 0$

$$y = (1 - x^{4/3})^{3/4}$$

$$V = 2\pi \int_0^1 x(1 - x^{4/3})^{3/4} dx \approx 1.5056$$

31. (a)



$$(b) V = 2\pi \int_2^6 x \sqrt[3]{(x-2)^2(x-6)^2} dx \approx 187.249$$

35. $p(x) = x$

$$h(x) = 2 - \frac{1}{2}x^2$$

$$V = 2\pi \int_0^2 x \left(2 - \frac{1}{2}x^2\right) dx = 2\pi \int_0^2 \left(2x - \frac{1}{2}x^3\right) dx = 2\pi \left[x^2 - \frac{1}{8}x^4\right]_0^2 = 4\pi \text{ (total volume)}$$

Now find x_0 such that

$$\pi = 2\pi \int_0^{x_0} \left(2x - \frac{1}{2}x^3\right) dx$$

$$1 = 2 \left[x^2 - \frac{1}{8}x^4 \right]_0^{x_0}$$

$$1 = 2x_0^2 - \frac{1}{4}x_0^4$$

$$x_0^4 - 8x_0^2 + 4 = 0$$

$$x_0^2 = 4 \pm 2\sqrt{3} \quad (\text{Quadratic Formula})$$

Take $x_0 = \sqrt{4 - 2\sqrt{3}}$ since the other root is too large.

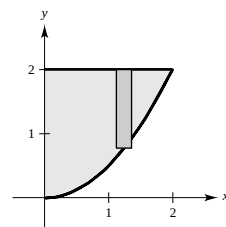
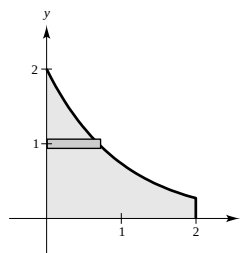
$$\text{Diameter: } 2\sqrt{4 - 2\sqrt{3}} \approx 1.464$$

$$\begin{aligned} 37. V &= 4\pi \int_{-1}^1 (2-x)\sqrt{1-x^2} dx \\ &= 8\pi \int_{-1}^1 \sqrt{1-x^2} dx - 4\pi \int_{-1}^1 x\sqrt{1-x^2} dx \\ &= 8\pi \left(\frac{\pi}{2}\right) + 2\pi \int_{-1}^1 x(1-x^2)^{1/2}(-2) dx \\ &= 4\pi^2 + \left[2\pi \left(\frac{2}{3}\right) (1-x^2)^{3/2} \right]_{-1}^1 = 4\pi^2 \end{aligned}$$

33. $y = 2e^{-x}, y = 0, x = 0, x = 2$

$$\text{Volume} \approx 7.5$$

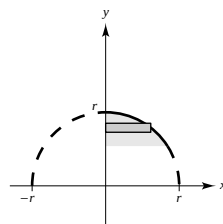
Matches (d)

**39. Disk Method**

$$R(y) = \sqrt{r^2 - y^2}$$

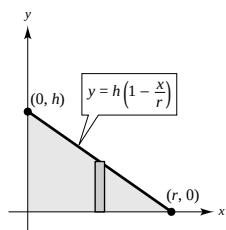
$$r(y) = 0$$

$$\begin{aligned} V &= \pi \int_{r-h}^r (r^2 - y^2) dy \\ &= \pi \left[r^2 y - \frac{y^3}{3} \right]_{r-h}^r = \frac{1}{3} \pi h^2 (3r - h) \end{aligned}$$



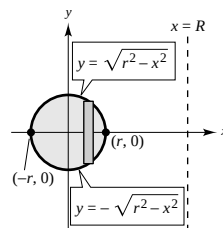
41. (a) $2\pi \int_0^r hx \left(1 - \frac{x}{r}\right) dx$ (ii)

is the volume of a right circular cone with the radius of the base as r and height h .

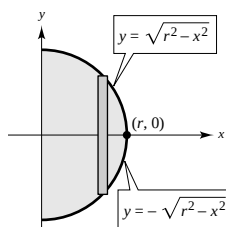


(b) $2\pi \int_{-r}^r (R - x)(2\sqrt{r^2 - x^2}) dx$ (v)

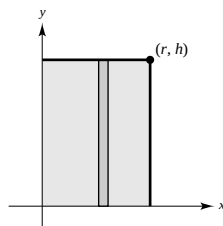
is the volume of a torus with the radius of its circular cross section as r and the distance from the axis of the torus to the center of its cross section as R .



(c) $2\pi \int_0^r 2x\sqrt{r^2 - x^2} dx$ (iii) is the volume of a sphere with radius r .

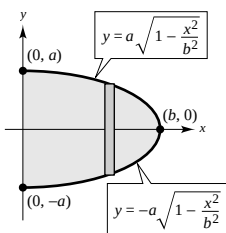


(d) $2\pi \int_0^r hx dx$ (i) is the volume of a right circular cylinder with a radius of r and a height of h .



(e) $2\pi \int_0^b 2ax\sqrt{1 - (x^2/b^2)} dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.

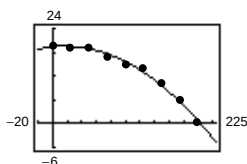


43. (a) $V = 2\pi \int_0^{200} xf(x) dx$

$$\approx \frac{2\pi(200)}{3(8)} [0 + 4(25)(19) + 2(50)(19) + 4(75)(17) + 2(100)15 + 4(125)(14) + 2(150)(10) + 4(175)(6) + 0]$$

$$\approx 1,366,593 \text{ cubic feet}$$

(b) $d = -0.000561x^2 + 0.0189x + 19.39$



(c) $V \approx 2\pi \int_0^{200} xd(x) dx \approx 2\pi(213,800) = 1,343,345 \text{ cubic feet}$

(d) Number gallons $\approx V(7.48) = 10,048,221 \text{ gallons}$

Section 6.4 Arc Length and Surfaces of Revolution

1. $(0, 0), (5, 12)$

(a) $d = \sqrt{(5-0)^2 + (12-0)^2} = 13$

(b) $y = \frac{12}{5}x$

$y' = \frac{12}{5}$

$s = \int_0^5 \sqrt{1 + \left(\frac{12}{5}\right)^2} dx = \left[\frac{13}{5}x\right]_0^5 = 13$

3. $y = \frac{2}{3}x^{3/2} + 1$

$y' = x^{1/2}, [0, 1]$

$s = \int_0^1 \sqrt{1+x} dx$

$= \left[\frac{2}{3}(1+x)^{3/2}\right]_0^1$

$= \frac{2}{3}(\sqrt{8}-1) \approx 1.219$

5. $y = \frac{3}{2}x^{2/3}$

$y' = \frac{1}{x^{1/3}}, [1, 8]$

$s = \int_1^8 \sqrt{1 + \left(\frac{1}{x^{1/3}}\right)^2} dx$

$= \int_1^8 \sqrt{\frac{x^{2/3} + 1}{x^{2/3}}} dx$

$= \frac{3}{2} \int_1^8 \sqrt{x^{2/3} + 1} \left(\frac{2}{3x^{1/3}}\right) dx$

$= \frac{3}{2} \left[\frac{2}{3}(x^{2/3} + 1)^{3/2}\right]_1^8$

$= 5\sqrt{5} - 2\sqrt{2} \approx 8.352$

7. $y = \frac{x^4}{8} + \frac{1}{4x^2}$

$y' = \frac{1}{2}x^3 - \frac{1}{2x^3}, [1, 2]$

$1 + (y')^2 = \left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right)^2, [1, 2]$

$s = \int_a^b \sqrt{1 + (y')^2} dx$

$= \int_1^2 \left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right) dx$

$= \left[\frac{1}{8}x^4 - \frac{1}{4x^2}\right]_1^2 = \frac{33}{16} \approx 2.063$

9. $y = \ln(\sin x), \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

$y' = \frac{1}{\sin x} \cos x = \cot x$

$1 + (y')^2 = 1 + \cot^2 x = \csc^2 x$

$s = \int_{\pi/4}^{3\pi/4} \csc x dx$

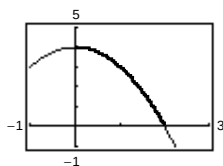
$= \left[\ln|\csc x - \cot x|\right]_{\pi/4}^{3\pi/4}$

$= \ln(\sqrt{2} + 1) - \ln(\sqrt{2} - 1) \approx 1.763$

11. (a) $y = 4 - x^2, 0 \leq x \leq 2$

(b) $y' = -2x$

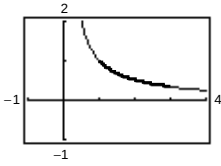
(c) $L \approx 4.647$



$1 + (y')^2 = 1 + 4x^2$

$L = \int_0^2 \sqrt{1 + 4x^2} dx$

13. (a) $y = \frac{1}{x}, 1 \leq x \leq 3$



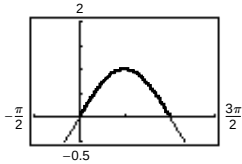
(b) $y' = -\frac{1}{x^2}$

(c) $L \approx 2.147$

$$1 + (y')^2 = 1 + \frac{1}{x^4}$$

$$L = \int_1^3 \sqrt{1 + \frac{1}{x^4}} dx$$

15. (a) $y = \sin x, 0 \leq x \leq \pi$



(b) $y' = \cos x$

(c) $L \approx 3.820$

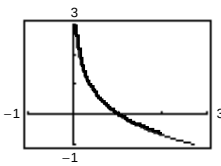
$$1 + (y')^2 = 1 + \cos^2 x$$

$$L = \int_0^\pi \sqrt{1 + \cos^2 x} dx$$

17. (a) $x = e^{-y}, 0 \leq y \leq 2$

$$y = -\ln x$$

$$1 \geq x \geq e^{-2} \approx 0.135$$



(b) $y' = -\frac{1}{x}$

(c) $L \approx 2.221$

$$1 + (y')^2 = 1 + \frac{1}{x^2}$$

$$L = \int_{e^{-2}}^1 \sqrt{1 + \frac{1}{x^2}} dx$$

Alternatively, you can do all the computations with respect to y .

(a) $x = e^{-y}, 0 \leq y \leq 2$

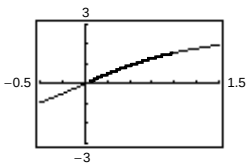
(b) $\frac{dx}{dy} = -e^{-y}$

(c) $L \approx 2.221$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + e^{-2y}$$

$$L = \int_0^2 \sqrt{1 + e^{-2y}} dy$$

19. (a) $y = 2 \arctan x, 0 \leq x \leq 1$



(b) $y' = \frac{2}{1+x^2}$

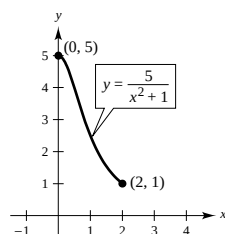
(c) $L \approx 1.871$

$$L = \int_0^1 \sqrt{1 + \frac{4}{(1+x^2)^2}} dx$$

21. $\int_0^2 \sqrt{1 + \left[\frac{d}{dx} \left(\frac{5}{x^2 + 1} \right) \right]^2} dx$

$s \approx 5$

Matches (b)



23. $y = x^3, [0, 4]$

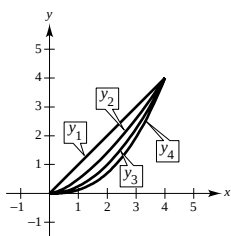
(a) $d = \sqrt{(4 - 0)^2 + (64 - 0)^2} \approx 64.125$

(b) $d = \sqrt{(1 - 0)^2 + (1 - 0)^2} + \sqrt{(2 - 1)^2 + (8 - 1)^2} + \sqrt{(3 - 2)^2 + (27 - 8)^2} + \sqrt{(4 - 3)^2 + (64 - 27)^2}$
 ≈ 64.525

(c) $s = \int_0^4 \sqrt{1 + (3x^2)^2} dx = \int_0^4 \sqrt{1 + 9x^4} dx \approx 64.666$

(d) 64.672

25. (a)



(b) y_1, y_2, y_3, y_4

(c) $y_1' = 1, L_1 = \int_0^4 \sqrt{2} dx \approx 5.657$

$y_2' = \frac{3}{4}x^{1/2}, L_2 = \int_0^4 \sqrt{1 + \frac{9x}{16}} dx \approx 5.759$

$y_3' = \frac{1}{2}x, L_3 = \int_0^4 \sqrt{1 + \frac{x^2}{4}} dx \approx 5.916$

$y_4' = \frac{5}{16}x^{3/2}, L_4 = \int_0^4 \sqrt{1 + \frac{25}{256}x^3} dx \approx 6.063$

27. $y = \frac{1}{3}[x^{3/2} - 3x^{1/2} + 2]$

When $x = 0, y = \frac{2}{3}$. Thus, the fleeing object has traveled $\frac{2}{3}$ units when it is caught.

$$y' = \frac{1}{3} \left[\frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2} \right] = \left(\frac{1}{2} \right) \frac{x - 1}{x^{1/2}}$$

$$1 + (y')^2 = 1 + \frac{(x - 1)^2}{4x} = \frac{(x + 1)^2}{4x}$$

$$s = \int_0^1 \frac{x + 1}{2x^{1/2}} dx = \frac{1}{2} \int_0^1 (x^{1/2} + x^{-1/2}) dx = \frac{1}{2} \left[\frac{2}{3}x^{3/2} + 2x^{1/2} \right]_0^1 = \frac{4}{3} = 2 \left(\frac{2}{3} \right)$$

The pursuer has traveled twice the distance that the fleeing object has traveled when it is caught.

29. $y = 20 \cosh \frac{x}{20}, -20 \leq x \leq 20$

$$y' = \sinh \frac{x}{20}$$

$$1 + (y')^2 = 1 + \sinh^2 \frac{x}{20} = \cosh^2 \frac{x}{20}$$

$$L = \int_{-20}^{20} \cosh \frac{x}{20} dx = 2 \int_0^{20} \cosh \frac{x}{20} dx = 2(20) \sinh \frac{x}{20} \Big|_0^{20}$$

$$= 40 \sinh(1) \approx 47.008 \text{ m.}$$

$$31. \quad y = \sqrt{9 - x^2}$$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$1 + (y')^2 = \frac{9}{9 - x^2}$$

$$s = \int_0^2 \sqrt{\frac{9}{9 - x^2}} dx$$

$$= \int_0^2 \frac{3}{\sqrt{9 - x^2}} dx$$

$$= \left[3 \arcsin \frac{x}{3} \right]_0^2$$

$$= 3 \left(\arcsin \frac{2}{3} - \arcsin 0 \right)$$

$$= 3 \arcsin \frac{2}{3} \approx 2.1892$$

$$35. \quad y = \frac{x^3}{6} + \frac{1}{2x}$$

$$y' = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$1 + (y')^2 = \left(\frac{x^2}{2} + \frac{1}{2x^2} \right)^2, [1, 2]$$

$$S = 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x} \right) \left(\frac{x^2}{2} + \frac{1}{2x^2} \right) dx$$

$$= 2\pi \int_1^2 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3} \right) dx$$

$$= 2\pi \left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2} \right]_1^2 = \frac{47\pi}{16}$$

$$33. \quad y = \frac{x^3}{3}$$

$$y' = x^2, [0, 3]$$

$$S = 2\pi \int_0^3 \frac{x^3}{3} \sqrt{1 + x^4} dx$$

$$= \frac{\pi}{6} \int_0^3 (1 + x^4)^{1/2} (4x^3) dx$$

$$= \left[\frac{\pi}{9} (1 + x^4)^{3/2} \right]_0^3$$

$$= \frac{\pi}{9} (82\sqrt{82} - 1) \approx 258.85$$

$$37. \quad y = \sqrt[3]{x} + 2$$

$$y' = \frac{1}{3x^{2/3}}, [1, 8]$$

$$S = 2\pi \int_1^8 x \sqrt{1 + \frac{1}{9x^{4/3}}} dx$$

$$= \frac{2\pi}{3} \int_1^8 x^{1/3} \sqrt{9x^{4/3} + 1} dx$$

$$= \frac{\pi}{18} \int_1^8 (9x^{4/3} + 1)^{1/2} (12x^{1/3}) dx$$

$$= \left[\frac{\pi}{27} (9x^{4/3} + 1)^{3/2} \right]_1^8$$

$$= \frac{\pi}{27} (145\sqrt{145} - 10\sqrt{10}) \approx 199.48$$

$$39. \quad y = \sin x$$

$$y' = \cos x, [0, \pi]$$

$$S = 2\pi \int_0^\pi \sin x \sqrt{1 + \cos^2 x} dx$$

$$\approx 14.4236$$

41. A rectifiable curve is one that has a finite arc length.

43. The precalculus formula is the surface area formula for the lateral surface of the frustum of a right circular cone. The representative element is

$$2\pi f(di) \sqrt{\Delta x_i^2 + \Delta y_i^2} = 2\pi f(di) \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i} \right)^2} \Delta x_i.$$

45. $y = \frac{hx}{r}$

$$y' = \frac{h}{r}$$

$$1 + (y')^2 = \frac{r^2 + h^2}{r^2}$$

$$\begin{aligned} S &= 2\pi \int_0^r x \sqrt{\frac{r^2 + h^2}{r^2}} dx \\ &= \left[\frac{2\pi \sqrt{r^2 + h^2}}{r} \left(\frac{x^2}{2} \right) \right]_0^r = \pi r \sqrt{r^2 + h^2} \end{aligned}$$

47. $y = \sqrt{9 - x^2}$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + (y')^2} = \frac{3}{\sqrt{9 - x^2}}$$

$$\begin{aligned} S &= 2\pi \int_0^2 \frac{3x}{\sqrt{9 - x^2}} dx \\ &= -3\pi \int_0^2 \frac{-2x}{\sqrt{9 - x^2}} dx \\ &= \left[-6\pi \sqrt{9 - x^2} \right]_0^2 \\ &= 6\pi(3 - \sqrt{5}) \approx 14.40 \end{aligned}$$

See figure in Exercise 48.

49. $y = \frac{1}{3}x^{1/2} - x^{3/2}$

$$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{6}(x^{-1/2} - 9x^{1/2})$$

$$1 + (y')^2 = 1 + \frac{1}{36}(x^{-1} - 18 + 81x) = \frac{1}{36}(x^{-1/2} + 9x^{1/2})^2$$

$$\begin{aligned} S &= 2\pi \int_0^{1/3} \left(\frac{1}{3}x^{1/2} - x^{3/2} \right) \sqrt{\frac{1}{36}(x^{-1/2} + 9x^{1/2})^2} dx = \frac{2\pi}{6} \int_0^{1/3} \left(\frac{1}{3}x^{1/2} - x^{3/2} \right) (x^{-1/2} + 9x^{1/2}) dx \\ &= \frac{\pi}{3} \int_0^{1/3} \left(\frac{1}{3} + 2x - 9x^2 \right) dx = \frac{\pi}{3} \left[\frac{1}{3}x + x^2 - 3x^3 \right]_0^{1/3} = \frac{\pi}{27} \text{ ft}^2 \approx 0.1164 \text{ ft}^2 \approx 16.8 \text{ in}^2 \end{aligned}$$

Amount of glass needed: $V = \frac{\pi}{27} \left(\frac{0.015}{12} \right) \approx 0.00015 \text{ ft}^3 \approx 0.25 \text{ in}^3$

51. (a) $y = f(x) = 0.0000001953x^4 - 0.0001804x^3 + 0.0496x^2 - 4.8323x + 536.9270$

(b) $\text{Area} = \int_0^{400} f(x) dx \approx 131,734.5 \text{ square feet}$

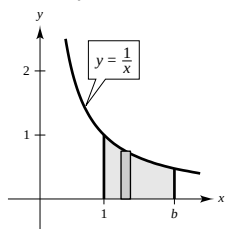
$$\approx 3.0 \text{ acres}$$

(Answers will vary.)

(c) $L = \int_0^{400} \sqrt{1 + f'(x)^2} dx \approx 794.9 \text{ feet}$

(Answers will vary.)

53. (a) $V = \pi \int_1^b \frac{1}{x^2} dx = \left[-\frac{\pi}{x} \right]_1^b = \pi \left(1 - \frac{1}{b} \right)$



(b) $\begin{aligned} S &= 2\pi \int_1^b \frac{1}{x} \sqrt{1 + \left(-\frac{1}{x^2} \right)^2} dx \\ &= 2\pi \int_1^b \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx \\ &= 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3} dx \end{aligned}$

—CONTINUED—

53. —CONTINUED—

$$(c) \lim_{b \rightarrow \infty} V = \lim_{b \rightarrow \infty} \pi \left(1 - \frac{1}{b} \right) = \pi$$

(d) Since

$$\frac{\sqrt{x^4 + 1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x} > 0 \text{ on } [1, b]$$

we have

$$\int_1^b \frac{\sqrt{x^4 + 1}}{x^3} dx > \int_1^b \frac{1}{x} dx = \left[\ln x \right]_1^b = \ln b$$

and $\lim_{b \rightarrow \infty} \ln b \rightarrow \infty$. Thus,

$$\lim_{b \rightarrow \infty} 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3} dx = \infty.$$

55. (a) Area of circle with radius L : $A = \pi L^2$ Area of sector with central angle θ (in radians)

$$S = \frac{\theta}{2\pi} A = \frac{\theta}{2\pi} (\pi L^2) = \frac{1}{2} L^2 \theta$$

(b) Let s be the arc length of the sector, which is the circumference of the base of the cone. Here, $s = L\theta = 2\pi r$, and you have

$$S = \frac{1}{2} L^2 \theta = \frac{1}{2} L^2 \left(\frac{s}{L} \right) = \frac{1}{2} L s = \frac{1}{2} L (2\pi r) = \pi r L$$

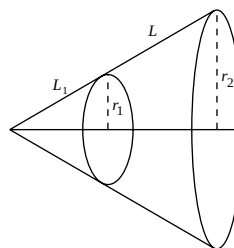
(c) The lateral surface area of the frustum is the difference of the large cone and the small one.

$$\begin{aligned} S &= \pi r_2(L + L_1) - \pi r_1 L_1 \\ &= \pi r_2 L + \pi L_1(r_2 - r_1) \end{aligned}$$

$$\text{By similar triangles, } \frac{L + L_1}{r_2} = \frac{L_1}{r_1} \Rightarrow L r_1 = L_1(r_2 - r_1)$$

Hence,

$$\begin{aligned} S &= \pi r_2 L + \pi L_1(r_2 - r_1) = \pi r_2 L + \pi L r_1 \\ &= \pi L(r_1 + r_2). \end{aligned}$$



Section 6.5 Work

$$1. W = Fd = (100)(10) = 1000 \text{ ft} \cdot \text{lb}$$

$$5. \text{ Work equals force times distance, } W = FD.$$

$$9. F(x) = kx$$

$$5 = k(4)$$

$$k = \frac{5}{4}$$

$$W = \int_0^7 \frac{5}{4} x dx = \left[\frac{5}{8} x^2 \right]_0^7$$

$$= \frac{245}{8} \text{ in} \cdot \text{lb}$$

$$= 30.625 \text{ in} \cdot \text{lb} \approx 2.55 \text{ ft} \cdot \text{lb}$$

$$3. W = Fd = (112)(4) = 448 \text{ joules (newton-meters)}$$

7. Since the work equals the area under the force function, you have $(c) < (d) < (a) < (b)$.

$$11. F(x) = kx$$

$$250 = k(30) \Rightarrow k = \frac{25}{3}$$

$$W = \int_{20}^{50} F(x) dx = \int_{20}^{50} \frac{25}{3} x dx = \left[\frac{25x^2}{6} \right]_{20}^{50}$$

$$= 8750 \text{ n} \cdot \text{cm} = 87.5 \text{ joules or Nm}$$

13. $F(x) = kx$

$20 = k(9)$

$k = \frac{20}{9}$

$$W = \int_0^{12} \frac{20}{9}x \, dx = \left[\frac{10}{9}x^2 \right]_0^{12} = \frac{40}{3} \text{ ft} \cdot \text{lb}$$

15. $W = 18 = \int_0^{1/3} kx \, dx = \left[\frac{kx^2}{2} \right]_0^{1/3} = \frac{k}{18} \Rightarrow k = 324$

$$W = \int_{1/3}^{7/12} 324x \, dx = 162x^2 \Big|_{1/3}^{7/12} = 37.125 \text{ ft} \cdot \text{lbs}$$

[Note: 4 inches = $\frac{1}{3}$ foot]

17. Assume that Earth has a radius of 4000 miles.

$F(x) = \frac{k}{x^2}$

(a) $W = \int_{4000}^{4100} \frac{80,000,000}{x^2} \, dx = \left[-\frac{80,000,000}{x} \right]_{4000}^{4100} \approx 487.8 \text{ mi} \cdot \text{tons}$

$s = \frac{k}{(4000)^2}$

$\approx 5.15 \times 10^9 \text{ ft} \cdot \text{lb}$

$k = 80,000,000$

(b) $W = \int_{4000}^{4300} \frac{80,000,000}{x^2} \, dx \approx 1395.3 \text{ mi} \cdot \text{ton}$

$F(x) = \frac{80,000,000}{x^2}$

$\approx 1.47 \times 10^{10} \text{ ft} \cdot \text{lb}$

19. Assume that the earth has a radius of 4000 miles.

$F(x) = \frac{k}{x^2}$

(a) $W = \int_{4000}^{15,000} \frac{160,000,000}{x^2} \, dx = \left[-\frac{160,000,000}{x} \right]_{4000}^{15,000} \approx -10,666.667 + 40,000$

$10 = \frac{k}{(4000)^2}$

$= 29,333.333 \text{ mi} \cdot \text{ton}$

$\approx 2.93 \times 10^4 \text{ mi} \cdot \text{ton}$

$k = 160,000,000$

$\approx 3.10 \times 10^{11} \text{ ft} \cdot \text{lb}$

$F(x) = \frac{160,000,000}{x^2}$

(b) $W = \int_{4000}^{26,000} \frac{160,000,000}{x^2} \, dx = \left[-\frac{160,000,000}{x} \right]_{4000}^{26,000} \approx -6,153.846 + 40,000$

$= 33,846.154 \text{ mi} \cdot \text{ton}$

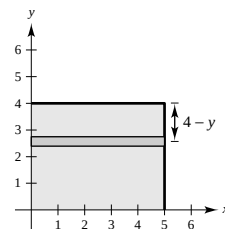
$\approx 3.38 \times 10^4 \text{ mi} \cdot \text{ton}$

$\approx 3.57 \times 10^{11} \text{ ft} \cdot \text{lb}$

21. Weight of each layer: $62.4(20) \Delta y$ Distance: $4 - y$

(a) $W = \int_2^4 62.4(20)(4 - y) \, dy = \left[4992y - 624y^2 \right]_2^4 = 2496 \text{ ft} \cdot \text{lb}$

(b) $W = \int_0^4 62.4(20)(4 - y) \, dy = \left[4992y - 624y^2 \right]_0^4 = 9984 \text{ ft} \cdot \text{lb}$

23. Volume of disk: $\pi(2)^2 \Delta y = 4\pi \Delta y$ Weight of disk of water: $9800(4\pi) \Delta y$ Distance the disk of water is moved: $5 - y$

$$W = \int_0^4 (5 - y)(9800)4\pi \, dy = 39,200\pi \int_0^4 (5 - y) \, dy$$

$$= 39,200\pi \left[5y - \frac{y^2}{2} \right]_0^4$$

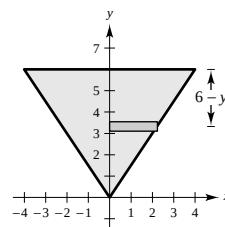
$$= 39,200\pi(12) = 470,400\pi \text{ newton-meters}$$

25. Volume of disk: $\pi\left(\frac{2}{3}y\right)^2 \Delta y$

Weight of disk: $62.4\pi\left(\frac{2}{3}y\right)^2 \Delta y$

Distance: $6 - y$

$$W = \frac{4(62.4)\pi}{9} \int_0^6 (6 - y)y^2 dy = \frac{4}{9}(62.4)\pi \left[2y^3 - \frac{1}{4}y^4 \right]_0^6 = 2995.2\pi \text{ ft} \cdot \text{lb}$$

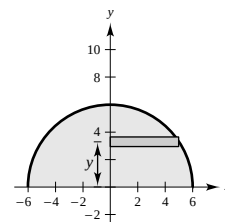


27. Volume of disk: $\pi(\sqrt{36 - y^2})^2 \Delta y$

Weight of disk: $62.4\pi(36 - y^2) \Delta y$

Distance: y

$$\begin{aligned} W &= 62.4\pi \int_0^6 y(36 - y^2) dy \\ &= 62.4\pi \int_0^6 (36y - y^3) dy = 62.4\pi \left[18y^2 - \frac{1}{4}y^4 \right]_0^6 \\ &= 20,217.6\pi \text{ ft} \cdot \text{lb} \end{aligned}$$



29. Volume of layer: $V = lwh = 4(2)\sqrt{(9/4) - y^2} \Delta y$

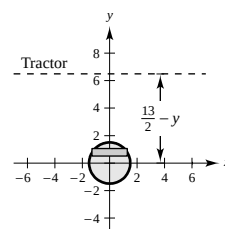
Weight of layer: $W = 42(8)\sqrt{(9/4) - y^2} \Delta y$

Distance: $\frac{13}{2} - y$

$$\begin{aligned} W &= \int_{-1.5}^{1.5} 42(8)\sqrt{(9/4) - y^2} \left(\frac{13}{2} - y \right) dy \\ &= 336 \left[\frac{13}{2} \int_{-1.5}^{1.5} \sqrt{(9/4) - y^2} dy - \int_{-1.5}^{1.5} \sqrt{(9/4) - y^2} y dy \right] \end{aligned}$$

The second integral is zero since the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius $\frac{3}{2}$. Thus, the work is

$$W = 336 \left(\frac{13}{2} \right) \pi \left(\frac{3}{2} \right)^2 \left(\frac{1}{2} \right) = 2457\pi \text{ ft} \cdot \text{lb}$$



31. Weight of section of chain: $3 \Delta y$

Distance: $15 - y$

$$\begin{aligned} W &= 3 \int_0^{15} (15 - y) dy \\ &= \left[-\frac{3}{2}(15 - y)^2 \right]_0^{15} \\ &= 337.5 \text{ ft} \cdot \text{lb} \end{aligned}$$

33. The lower 5 feet of chain are raised 10 feet with a constant force.

$$W_1 = 3(5)(10) = 150 \text{ ft} \cdot \text{lb}$$

The top 10 feet of chain are raised with a variable force.

Weight per section: $3 \Delta y$

Distance: $10 - y$

$$\begin{aligned} W_2 &= 3 \int_0^{10} (10 - y) dy = \left[-\frac{3}{2}(10 - y)^2 \right]_0^{10} \\ &= 150 \text{ ft} \cdot \text{lb} \end{aligned}$$

$$W = W_1 + W_2 = 300 \text{ ft} \cdot \text{lb}$$

35. Weight of section of chain: $3 \Delta y$ Distance: $15 - 2y$

$$W = 3 \int_0^{7.5} (15 - 2y) dy = \left[-\frac{3}{4}(15 - 2y)^2 \right]_0^{7.5} \\ = \frac{3}{4}(15)^2 = 168.75 \text{ ft} \cdot \text{lb}$$

37. Work to pull up the ball: $W_1 = 500(15) = 7500 \text{ ft} \cdot \text{lb}$

Work to wind up the top 15 feet of cable: force is variable

Weight per section: $1 \Delta y$ Distance: $15 - x$

$$W_2 = \int_0^{15} (15 - x) dx = \left[-\frac{1}{2}(15 - x)^2 \right]_0^{15} \\ = 112.5 \text{ ft} \cdot \text{lb}$$

Work to lift the lower 25 feet of cable with a constant force:

$$W_3 = (1)(25)(15) = 375 \text{ ft} \cdot \text{lb}$$

$$W = W_1 + W_2 + W_3 = 7500 + 112.5 + 375 \\ = 7987.5 \text{ ft} \cdot \text{lb}$$

$$39. \quad p = \frac{k}{V}$$

$$1000 = \frac{k}{2}$$

$$k = 2000$$

$$W = \int_2^3 \frac{2000}{V} dV = \left[2000 \ln |V| \right]_2^3 \\ = 2000 \ln \left(\frac{3}{2} \right) \approx 810.93 \text{ ft} \cdot \text{lb}$$

$$41. F(x) = \frac{k}{(2 - x)^2}$$

$$W = \int_{-2}^1 \frac{k}{(2 - x)^2} dx = \left[\frac{k}{2 - x} \right]_{-2}^1 = k \left(1 - \frac{1}{4} \right) \\ = \frac{3k}{4} (\text{units of work})$$

$$43. W = \int_0^5 1000[1.8 - \ln(x + 1)] dx \approx 3249.44 \text{ ft} \cdot \text{lb}$$

$$45. W = \int_0^5 100x\sqrt{125 - x^3} dx \approx 10,330.3 \text{ ft} \cdot \text{lb}$$

Section 6.6 Moments, Centers of Mass, and Centroids

$$1. \quad \bar{x} = \frac{6(-5) + 3(1) + 5(3)}{6 + 3 + 5} = -\frac{6}{7}$$

$$3. \quad \bar{x} = \frac{1(7) + 1(8) + 1(12) + 1(15) + 1(18)}{1 + 1 + 1 + 1 + 1} = 12$$

$$5. (a) \quad \bar{x} = \frac{(7 + 5) + (8 + 5) + (12 + 5) + (15 + 5) + (18 + 5)}{5} = 17 = 12 + 5$$

$$(b) \quad \bar{x} = \frac{12(-6 - 3) + 1(-4 - 3) + 6(-2 - 3) + 3(0 - 3) + 11(8 - 3)}{12 + 1 + 6 + 3 + 11} = \frac{-99}{33} = -3$$

$$7. \quad 50x = 75(L - x) = 75(10 - x)$$

$$50x = 750 - 75x$$

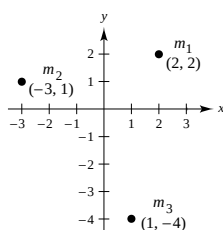
$$125x = 750$$

$$x = 6 \text{ feet}$$

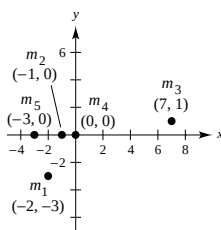
$$9. \quad \bar{x} = \frac{5(2) + 1(-3) + 3(1)}{5 + 1 + 3} = \frac{10}{9}$$

$$\bar{y} = \frac{5(2) + 1(1) + 3(-4)}{5 + 1 + 3} = -\frac{1}{9}$$

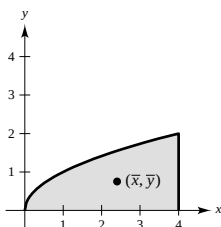
$$(\bar{x}, \bar{y}) = \left(\frac{10}{9}, -\frac{1}{9} \right)$$



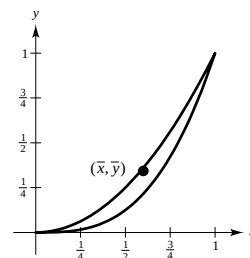
$$\begin{aligned}
 11. \quad \bar{x} &= \frac{3(-2) + 4(-1) + 2(7) + 1(0) + 6(-3)}{3 + 4 + 2 + 1 + 6} = -\frac{7}{8} \\
 \bar{y} &= \frac{3(-3) + 4(0) + 2(1) + 1(0) + 6(0)}{3 + 4 + 2 + 1 + 6} = -\frac{7}{16} \\
 (\bar{x}, \bar{y}) &= \left(-\frac{7}{8}, -\frac{7}{16}\right)
 \end{aligned}$$



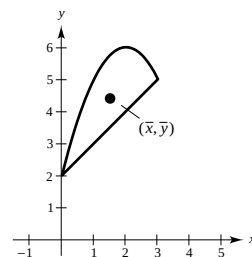
$$\begin{aligned}
 13. \quad m &= \rho \int_0^4 \sqrt{x} \, dx = \left[\frac{2\rho}{3} x^{3/2} \right]_0^4 = \frac{16\rho}{3} \\
 M_x &= \rho \int_0^4 \frac{\sqrt{x}}{2} (\sqrt{x}) \, dx = \left[\rho \frac{x^2}{4} \right]_0^4 = 4\rho \\
 \bar{y} &= \frac{M_x}{m} = 4\rho \left(\frac{3}{16\rho} \right) = \frac{3}{4} \\
 M_y &= \rho \int_0^4 x \sqrt{x} \, dx = \left[\rho \frac{2}{5} x^{5/2} \right]_0^4 = \frac{64\rho}{5} \\
 \bar{x} &= \frac{M_y}{m} = \frac{64\rho}{5} \left(\frac{3}{16\rho} \right) = \frac{12}{5} \\
 (\bar{x}, \bar{y}) &= \left(\frac{12}{5}, \frac{3}{4} \right)
 \end{aligned}$$



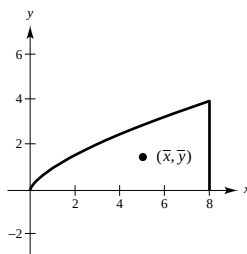
$$\begin{aligned}
 15. \quad m &= \rho \int_0^1 (x^2 - x^3) \, dx = \rho \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{\rho}{12} \\
 M_x &= \rho \int_0^1 \frac{(x^2 + x^3)}{2} (x^2 - x^3) \, dx = \frac{\rho}{2} \int_0^1 (x^4 - x^6) \, dx = \frac{\rho}{2} \left[\frac{x^5}{5} - \frac{x^7}{7} \right]_0^1 = \frac{\rho}{35} \\
 \bar{y} &= \frac{M_x}{m} = \frac{\rho}{35} \left(\frac{12}{\rho} \right) = \frac{12}{35} \\
 M_y &= \rho \int_0^1 x(x^2 - x^3) \, dx = \rho \int_0^1 (x^3 - x^4) \, dx = \rho \left[\frac{x^4}{4} - \frac{x^5}{5} \right]_0^1 = \frac{\rho}{20} \\
 \bar{x} &= \frac{M_y}{m} = \frac{\rho}{20} \left(\frac{12}{\rho} \right) = \frac{3}{5} \\
 (\bar{x}, \bar{y}) &= \left(\frac{3}{5}, \frac{12}{35} \right)
 \end{aligned}$$



$$\begin{aligned}
 17. \quad m &= \rho \int_0^3 [(-x^2 + 4x + 2) - (x + 2)] \, dx = -\rho \left[\frac{x^3}{3} + \frac{3x^2}{2} \right]_0^3 = \frac{9\rho}{2} \\
 M_x &= \rho \int_0^3 \left[\frac{(-x^2 + 4x + 2) + (x + 2)}{2} \right] [(-x^2 + 4x + 2) - (x + 2)] \, dx \\
 &= \frac{\rho}{2} \int_0^3 (-x^2 + 5x + 4)(-x^2 + 3x) \, dx = \frac{\rho}{2} \int_0^3 (x^4 - 8x^3 + 11x^2 + 12x) \, dx \\
 &= \frac{\rho}{2} \left[\frac{x^5}{5} - 2x^4 + \frac{11x^3}{3} + 6x^2 \right]_0^3 = \frac{99\rho}{5} \\
 \bar{y} &= \frac{M_x}{m} = \frac{99\rho}{5} \left(\frac{2}{9\rho} \right) = \frac{22}{5} \\
 M_y &= \rho \int_0^3 x[(-x^2 + 4x - 2) - (x + 2)] \, dx = \rho \int_0^3 (-x^3 + 3x^2) \, dx = \rho \left[-\frac{x^4}{4} + x^3 \right]_0^3 = \frac{27\rho}{4} \\
 \bar{x} &= \frac{M_y}{m} = \frac{27\rho}{4} \left(\frac{2}{9\rho} \right) = \frac{3}{2} \\
 (\bar{x}, \bar{y}) &= \left(\frac{3}{2}, \frac{22}{5} \right)
 \end{aligned}$$



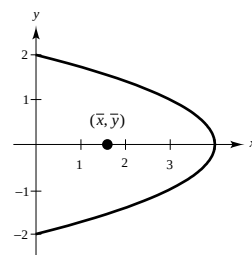
$$\begin{aligned}
 19. \quad m &= \rho \int_0^8 x^{2/3} dx = \rho \left[\frac{3}{5} x^{5/3} \right]_0^8 = \frac{96\rho}{5} \\
 M_x &= \rho \int_0^8 \frac{x^{2/3}}{2} (x^{2/3}) dx = \frac{\rho}{2} \left[\frac{3}{7} x^{7/3} \right]_0^8 = \frac{192\rho}{7} \\
 \bar{y} &= \frac{M_x}{m} = \frac{192\rho}{7} \left(\frac{5}{96\rho} \right) = \frac{10}{7} \\
 M_y &= \rho \int_0^8 x(x^{2/3}) dx = \rho \left[\frac{3}{8} x^{8/3} \right]_0^8 = 96\rho \\
 \bar{x} &= \frac{M_y}{m} = 96\rho \left(\frac{5}{96\rho} \right) = 5 \\
 (\bar{x}, \bar{y}) &= \left(5, \frac{10}{7} \right)
 \end{aligned}$$



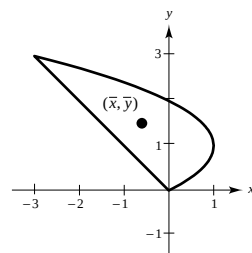
$$\begin{aligned}
 21. \quad m &= 2\rho \int_0^2 (4 - y^2) dy = 2\rho \left[4y - \frac{y^3}{3} \right]_0^2 = \frac{32\rho}{3} \\
 M_y &= 2\rho \int_0^2 \left(\frac{4 - y^2}{2} \right) (4 - y^2) dy = \rho \left[16y - \frac{8}{3}y^3 + \frac{y^5}{5} \right]_0^2 = \frac{256\rho}{15} \\
 \bar{x} &= \frac{M_y}{m} = \frac{256\rho}{15} \left(\frac{3}{32\rho} \right) = \frac{8}{5}
 \end{aligned}$$

By symmetry, M_x and $\bar{y} = 0$.

$$(\bar{x}, \bar{y}) = \left(\frac{8}{5}, 0 \right)$$



$$\begin{aligned}
 23. \quad m &= \rho \int_0^3 [(2y - y^2) - (-y)] dy = \rho \left[\frac{3y^2}{2} - \frac{y^3}{3} \right]_0^3 = \frac{9\rho}{2} \\
 M_y &= \rho \int_0^3 \frac{[(2y - y^2) + (-y)]}{2} [(2y - y^2) - (-y)] dy = \frac{\rho}{2} \int_0^3 (y - y^2)(3y - y^2) dy \\
 &= \frac{\rho}{2} \int_0^3 (y^4 - 4y^3 + 3y^2) dy = \frac{\rho}{2} \left[\frac{y^5}{5} - y^4 + y^3 \right]_0^3 = -\frac{27\rho}{10} \\
 \bar{x} &= \frac{M_y}{m} = -\frac{27\rho}{10} \left(\frac{2}{9\rho} \right) = -\frac{3}{5} \\
 M_x &= \rho \int_0^3 y[(2y - y^2) - (-y)] dy = \rho \int_0^3 (3y^2 - y^3) dy = \rho \left[y^3 - \frac{y^4}{4} \right]_0^3 = \frac{27\rho}{4} \\
 \bar{y} &= \frac{M_x}{m} = \frac{27\rho}{4} \left(\frac{2}{9\rho} \right) = \frac{3}{2} \\
 (\bar{x}, \bar{y}) &= \left(-\frac{3}{5}, \frac{3}{2} \right)
 \end{aligned}$$



$$\begin{aligned}
 25. \quad A &= \int_0^1 (x - x^2) dx = \left[\frac{1}{2}x^2 - \frac{x^3}{3} \right]_0^1 = \frac{1}{6} \\
 M_x &= \frac{1}{2} \int_0^1 (x^2 - x^4) dx = \frac{1}{2} \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{1}{15} \\
 M_y &= \int_0^1 (x^2 - x^3) dx = \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{1}{12}
 \end{aligned}$$

$$27. \quad A = \int_0^3 (2x + 4) dx = \left[x^2 + 4x \right]_0^3 = 9 + 12 = 21$$

$$M_x = \frac{1}{2} \int_0^3 (2x + 4)^2 dx = \int_0^3 (2x^2 + 8x + 8) dx = \left[\frac{2x^3}{3} + 4x^2 + 8x \right]_0^3 = 18 + 36 + 24 = 78$$

$$M_y = \int_0^3 (2x^2 + 4x) dx = \left[\frac{2x^3}{3} + 2x^2 \right]_0^3 = 18 + 18 = 36$$

$$29. \quad m = \rho \int_0^5 10x \sqrt{125 - x^3} dx \approx 1033.0\rho$$

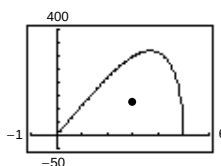
$$M_x = \rho \int_0^5 \left(\frac{10x \sqrt{125 - x^3}}{2} \right) (10x \sqrt{125 - x^3}) dx = 50\rho \int_0^5 x^2 (125 - x^3) dx = \frac{3,124,375\rho}{24} \approx 130,208\rho$$

$$M_y = \rho \int_0^5 10x^2 \sqrt{125 - x^3} dx = -\frac{10\rho}{3} \int_0^5 \sqrt{125 - x^3} (-3x^2) dx = \frac{12,500\sqrt{5}\rho}{9} \approx 3105.6\rho$$

$$\bar{x} = \frac{M_y}{m} \approx 3.0$$

$$\bar{y} = \frac{M_x}{m} \approx 126.0$$

Therefore, the centroid is (3.0, 126.0).



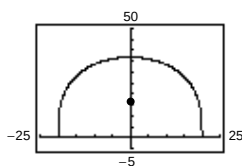
$$31. \quad m = \rho \int_{-20}^{20} 5 \sqrt[3]{400 - x^2} dx \approx 1239.76\rho$$

$$M_x = \rho \int_{-20}^{20} \frac{5 \sqrt[3]{400 - x^2}}{2} (5 \sqrt[3]{400 - x^2}) dx$$

$$= \frac{25\rho}{2} \int_{-20}^{20} (400 - x^2)^{2/3} dx \approx 20064.27$$

$$\bar{y} = \frac{M_x}{m} \approx 16.18$$

$\bar{x} = 0$ by symmetry. Therefore, the centroid is (0, 16.2).



$$33. \quad A = \frac{1}{2}(2a)c = ac$$

$$\frac{1}{A} = \frac{1}{ac}$$

$$\bar{x} = \left(\frac{1}{ac} \right) \frac{1}{2} \int_0^c \left[\left(\frac{b-a}{c}y + a \right)^2 - \left(\frac{b+a}{c}y - a \right)^2 \right] dy$$

$$= \frac{1}{2ac} \int_0^c \left[\frac{4ab}{c}y - \frac{4ab}{c^2}y^2 \right] dy$$

$$= \frac{1}{2ac} \left[\frac{2ab}{c}y^2 - \frac{4ab}{3c^2}y^3 \right]_0^c = \frac{1}{2ac} \left(\frac{2}{3}abc \right) = \frac{b}{3}$$

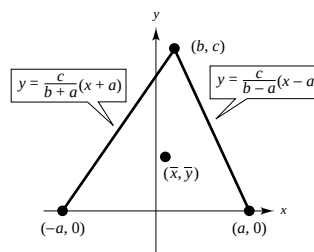
$$\bar{y} = \frac{1}{ac} \int_0^c y \left[\left(\frac{b-a}{c}y + a \right) - \left(\frac{b+a}{c}y - a \right) \right] dy$$

$$= \frac{1}{ac} \int_0^c y \left(-\frac{2a}{c}y + 2a \right) dy = \frac{2}{c} \int_0^c \left(y - \frac{y^2}{c} \right) dy$$

$$= \frac{2}{c} \left[\frac{y^2}{2} - \frac{y^3}{3c} \right]_0^c = \frac{c}{3}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{c}{3} \right)$$

In Exercise 566 of Section P.2, you found that $(b/3, c/3)$ is the point of intersection of the medians.



35. $A = \frac{c}{2}(a + b)$

$$\frac{1}{A} = \frac{2}{c(a + b)}$$

$$\bar{x} = \frac{2}{c(a + b)} \int_0^c x \left(\frac{b-a}{c}x + a \right) dx = \frac{2}{c(a + b)} \int_0^c \left(\frac{b-a}{c}x^2 + ax \right) dx = \frac{2}{c(a + b)} \left[\frac{b-a}{c} \frac{x^3}{3} + \frac{ax^2}{2} \right]_0^c$$

$$= \frac{2}{c(a + b)} \left[\frac{(b-a)c^2}{3} + \frac{ac^2}{2} \right] = \frac{2}{c(a + b)} \left[\frac{2bc^2 - 2ac^2 + 3ac^2}{6} \right] = \frac{c(2b + a)}{3(a + b)} = \frac{(a + 2b)c}{3(a + b)}$$

$$\bar{y} = \frac{2}{c(a + b)} \frac{1}{2} \int_0^c \left(\frac{b-a}{c}x + a \right)^2 dx = \frac{1}{c(a + b)} \int_0^c \left[\left(\frac{b-a}{c} \right)^2 x^2 + \frac{2a(b-a)}{c}x + a^2 \right] dx$$

$$= \frac{1}{c(a + b)} \left[\left(\frac{b-a}{c} \right)^2 \frac{x^3}{3} + \frac{2a(b-a)}{c} \frac{x^2}{2} + a^2x \right]_0^c = \frac{1}{c(a + b)} \left[\frac{(b-a)^2c}{3} + ac(b-a) + a^2c \right]$$

$$= \frac{1}{3c(a + b)} [(b^2 - 2ab + a^2)c + 3ac(b-a) + 3a^2c]$$

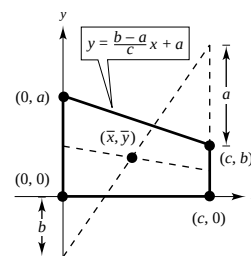
$$= \frac{1}{3(a + b)} [b^2 - 2ab + a^2 + 3ab - 3a^2 + 3a^2] = \frac{a^2 + ab + b^2}{3(a + b)}$$

Thus, $(\bar{x}, \bar{y}) = \left(\frac{(a + 2b)c}{3(a + b)}, \frac{a^2 + ab + b^2}{3(a + b)} \right)$.

The one line passes through $(0, a/2)$ and $(c, b/2)$. Its equation is $y = \frac{b-a}{2c}x + \frac{a}{2}$.

The other line passes through $(0, -b)$ and $(c, a + b)$. Its equation is $y = \frac{a + 2b}{c}x - b$.

$(\bar{x}, \bar{y})$ is the point of intersection of these two lines.



37. $\bar{x} = 0$ by symmetry

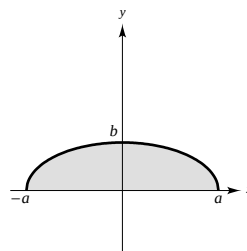
$$A = \frac{1}{2}\pi ab$$

$$\frac{1}{A} = \frac{2}{\pi ab}$$

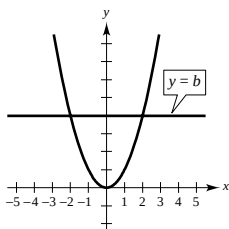
$$\bar{y} = \frac{2}{\pi ab} \frac{1}{2} \int_{-a}^a \left(\frac{b}{a} \sqrt{a^2 - x^2} \right)^2 dx$$

$$= \frac{1}{\pi ab} \left(\frac{b^2}{a^2} \right) \left[a^2x - \frac{x^3}{3} \right]_{-a}^a = \frac{b}{\pi a^3} \left[\frac{4a^3}{3} \right] = \frac{4b}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4b}{3\pi} \right)$$



39. (a)



(b) $\bar{x} = 0$ by symmetry

(c) $M_y = \int_{-\sqrt{b}}^{\sqrt{b}} x(b - x^2) dx = 0$ because $bx - x^3$ is odd

(d) $\bar{y} > \frac{b}{2}$ since there is more area above $y = \frac{b}{2}$ than below

(e) $M_x = \int_{-\sqrt{b}}^{\sqrt{b}} \frac{(b + x^2)(b - x^2)}{2} dx$

$$= \int_{-\sqrt{b}}^{\sqrt{b}} \frac{b^2 - x^4}{2} dx = \frac{1}{2} \left[b^2x - \frac{x^5}{5} \right]_{-\sqrt{b}}^{\sqrt{b}}$$

$$= b^2\sqrt{b} - \frac{b^2\sqrt{b}}{5} = \frac{4b^2\sqrt{b}}{5}$$

$$A = \int_{-\sqrt{b}}^{\sqrt{b}} (b - x^2) dx = \left[bx - \frac{x^3}{3} \right]_{-\sqrt{b}}^{\sqrt{b}}$$

$$= \left(b\sqrt{b} - \frac{b\sqrt{b}}{3} \right) 2 = 4 \frac{b\sqrt{b}}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{4b^2\sqrt{b}/5}{4b\sqrt{b}/3} = \frac{3}{5}b.$$

41. (a)
- $\bar{x} = 0$
- by symmetry

$$A = 2 \int_0^{40} f(x) dx = \frac{2(40)}{3(4)} [30 + 4(29) + 2(26) + 4(20) + 0] = \frac{20}{3} (278) = \frac{5560}{3}$$

$$M_x = \int_{-40}^{40} \frac{f(x)^2}{2} dx = \frac{40}{3(4)} [30^2 + 4(29)^2 + 2(26)^2 + 4(20)^2 + 0] = \frac{10}{3} (7216) = \frac{72160}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{72160/3}{5560/3} = \frac{72160}{5560} \approx 12.98$$

$$(\bar{x}, \bar{y}) = (0, 12.98)$$

(b) $y = (-1.02 \times 10^{-5})x^4 - 0.0019x^2 + 29.28$

(c) $\bar{y} = \frac{M_x}{A} \approx \frac{23697.68}{1843.54} \approx 12.85$

$$(\bar{x}, \bar{y}) = (0, 12.85)$$

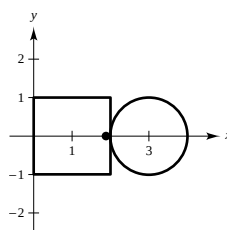
43. Centroids of the given regions: (1, 0) and (3, 0)

Area: $A = 4 + \pi$

$$\bar{x} = \frac{4(1) + \pi(3)}{4 + \pi} = \frac{4 + 3\pi}{4 + \pi}$$

$$\bar{y} = \frac{4(0) + \pi(0)}{4 + \pi} = 0$$

$$(\bar{x}, \bar{y}) = \left(\frac{4 + 3\pi}{4 + \pi}, 0 \right) \approx (1.88, 0)$$



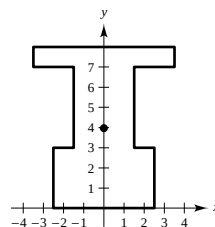
45. Centroids of the given regions:
- $\left(0, \frac{3}{2}\right)$
- , (0, 5), and
- $\left(0, \frac{15}{2}\right)$

Area: $A = 15 + 12 + 7 = 34$

$$\bar{x} = \frac{15(0) + 12(0) + 7(0)}{34} = 0$$

$$\bar{y} = \frac{15(3/2) + 12(5) + 7(15/2)}{34} = \frac{135}{34}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{135}{34}\right)$$



47. Centroids of the given regions: (1, 0) and (3, 0)

Mass: $4 + 2\pi$

$$\bar{x} = \frac{4(1) + 2\pi(3)}{4 + 2\pi} = \frac{2 + 3\pi}{2 + \pi}$$

$$\bar{y} = 0$$

$$(\bar{x}, \bar{y}) = \left(\frac{2 + 3\pi}{2 + \pi}, 0 \right) \approx (2.22, 0)$$

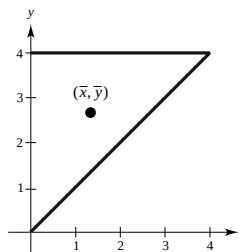
49. $V = 2\pi rA = 2\pi(5)(16\pi) = 160\pi^2 \approx 1579.14$

$$51. A = \frac{1}{2}(4)(4) = 8$$

$$\bar{y} = \left(\frac{1}{8}\right)\frac{1}{2}\int_0^4 (4+x)(4-x) dx = \frac{1}{16}\left[16x - \frac{x^3}{3}\right]_0^4 = \frac{8}{3}$$

$$r = \bar{y} = \frac{8}{3}$$

$$V = 2\pi rA = 2\pi\left(\frac{8}{3}\right)(8) = \frac{128\pi}{3} \approx 134.04$$



$$53. m = m_1 + \cdots + m_n$$

$$M_y = m_1x_1 + \cdots + m_nx_n$$

$$M_x = m_1y_1 + \cdots + m_ny_n$$

$$\bar{x} = \frac{M_y}{m}, \bar{y} = \frac{M_x}{m}$$

$$55. (a) \text{ Yes. } (\bar{x}, \bar{y}) = \left(\frac{5}{6}, \frac{5}{18} + 2\right) = \left(\frac{5}{6}, \frac{41}{18}\right)$$

$$(b) \text{ Yes. } (\bar{x}, \bar{y}) = \left(\frac{5}{6} + 2, \frac{5}{18}\right) = \left(\frac{17}{6}, \frac{5}{18}\right)$$

$$(c) \text{ Yes. } (\bar{x}, \bar{y}) = \left(\frac{5}{6}, -\frac{5}{18}\right)$$

$$(d) \text{ No.}$$

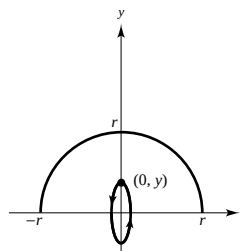
57. The surface area of the sphere is $S = 4\pi r^2$. The arc length of C is $s = \pi r$. The distance traveled by the centroid is

$$d = \frac{S}{s} = \frac{4\pi r^2}{\pi r} = 4r.$$

This distance is also the circumference of the circle of radius y .

$$d = 2\pi y$$

Thus, $2\pi y = 4r$ and we have $y = 2r/\pi$. Therefore, the centroid of the semicircle $y = \sqrt{r^2 - x^2}$ is $(0, 2r/\pi)$.



$$59. A = \int_0^1 x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}$$

$$m = \rho A = \frac{\rho}{n+1}$$

$$M_x = \frac{\rho}{2} \int_0^1 (x^n)^2 dx = \left[\frac{\rho}{2} \cdot \frac{x^{2n+1}}{2n+1}\right]_0^1 = \frac{\rho}{2(2n+1)}$$

$$M_y = \rho \int_0^1 x(x^n) dx = \left[\rho \cdot \frac{x^{n+2}}{n+2}\right]_0^1 = \frac{\rho}{n+2}$$

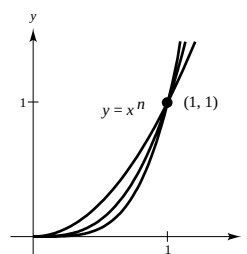
$$\bar{x} = \frac{M_y}{m} = \frac{n+1}{n+2}$$

$$\bar{y} = \frac{M_x}{m} = \frac{n+1}{2(2n+1)} = \frac{n+1}{4n+2}$$

$$\text{Centroid: } \left(\frac{n+1}{n+2}, \frac{n+1}{4n+2}\right)$$

$$\text{As } n \rightarrow \infty, (\bar{x}, \bar{y}) \rightarrow \left(1, \frac{1}{4}\right).$$

The graph approaches the x -axis and the line $x = 1$ as $n \rightarrow \infty$.



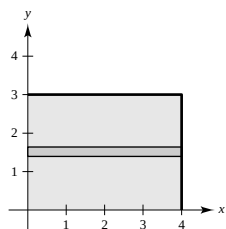
Section 6.7 Fluid Pressure and Fluid Force

$$1. F = PA = [62.4(5)](3) = 936 \text{ lb}$$

$$5. h(y) = 3 - y$$

$$L(y) = 4$$

$$\begin{aligned} F &= 62.4 \int_0^3 (3 - y)(4) dy \\ &= 249.6 \int_0^3 (3 - y) dy \\ &= 249.6 \left[3y - \frac{y^2}{2} \right]_0^3 = 1123.2 \text{ lb} \end{aligned}$$

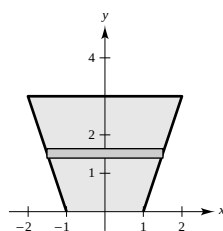


$$\begin{aligned} 3. F &= 62.4(h + 2)(6) - (62.4)(h)(6) \\ &= 62.4(2)(6) = 748.8 \text{ lb} \end{aligned}$$

$$7. h(y) = 3 - y$$

$$L(y) = 2\left(\frac{y}{3} + 1\right)$$

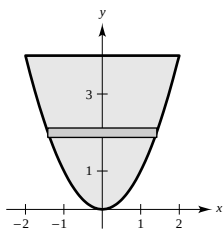
$$\begin{aligned} F &= 2(62.4) \int_0^3 (3 - y)\left(\frac{y}{3} + 1\right) dy \\ &= 124.8 \int_0^3 \left(3 - \frac{y^2}{3}\right) dy \\ &= 124.8 \left[3y - \frac{y^3}{9} \right]_0^3 = 748.8 \text{ lb} \end{aligned}$$



$$9. h(y) = 4 - y$$

$$L(y) = 2\sqrt{y}$$

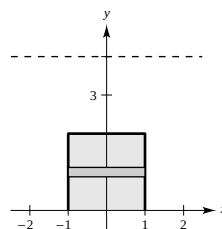
$$\begin{aligned} F &= 2(62.4) \int_0^4 (4 - y)\sqrt{y} dy \\ &= 124.8 \int_0^4 (4y^{1/2} - y^{3/2}) dy \\ &= 124.8 \left[\frac{8y^{3/2}}{3} - \frac{2y^{5/2}}{5} \right]_0^4 = 1064.96 \text{ lb} \end{aligned}$$



$$11. h(y) = 4 - y$$

$$L(y) = 2$$

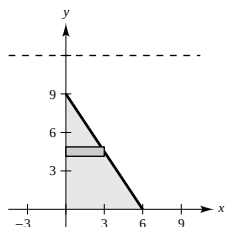
$$\begin{aligned} F &= 9800 \int_0^2 2(4 - y) dy \\ &= 9800 \left[8y - y^2 \right]_0^2 = 117,600 \text{ Newtons} \end{aligned}$$



13. $h(y) = 12 - y$

$$L(y) = 6 - \frac{2y}{3}$$

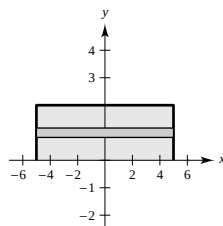
$$\begin{aligned}
 F &= 9800 \int_0^9 (12 - y) \left(6 - \frac{2y}{3} \right) dy \\
 &= 9800 \left[72y - 7y^2 + \frac{2y^3}{9} \right]_0^9 = 2,381,400 \text{ Newtons}
 \end{aligned}$$



15. $h(y) = 2 - y$

$$L(y) = 10$$

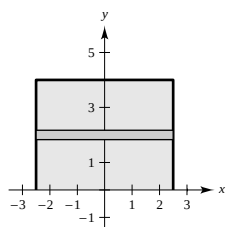
$$\begin{aligned}
 F &= 140.7 \int_0^2 (2 - y)(10) dy \\
 &= 1407 \int_0^2 (2 - y) dy \\
 &= 1407 \left[2y - \frac{y^2}{2} \right]_0^2 = 2814 \text{ lb}
 \end{aligned}$$



17. $h(y) = 4 - y$

$$L(y) = 6$$

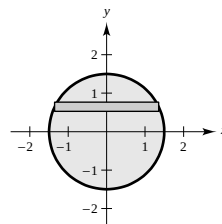
$$\begin{aligned}
 F &= 140.7 \int_0^4 (4 - y)(6) dy \\
 &= 844.2 \int_0^4 (4 - y) dy \\
 &= 844.2 \left[4y - \frac{y^2}{2} \right]_0^4 = 6753.6 \text{ lb}
 \end{aligned}$$



19. $h(y) = -y$

$$L(y) = 2 \left(\frac{1}{2} \right) \sqrt{9 - 4y^2}$$

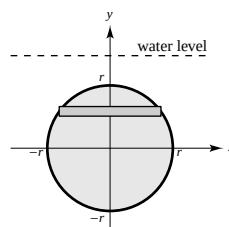
$$\begin{aligned}
 F &= 42 \int_{-3/2}^0 (-y) \sqrt{9 - 4y^2} dy \\
 &= \frac{42}{8} \int_{-3/2}^0 (9 - 4y^2)^{1/2} (-8y) dy \\
 &= \left[\left(\frac{21}{4} \right) \left(\frac{2}{3} \right) (9 - 4y^2)^{3/2} \right]_{-3/2}^0 = 94.5 \text{ lb}
 \end{aligned}$$



21. $h(y) = k - y$

$$L(y) = 2\sqrt{r^2 - y^2}$$

$$\begin{aligned}
 F &= w \int_{-r}^r (k - y) \sqrt{r^2 - y^2} (2) dy \\
 &= w \left[2k \int_{-r}^r \sqrt{r^2 - y^2} dy + \int_{-r}^r \sqrt{r^2 - y^2} (-2y) dy \right]
 \end{aligned}$$



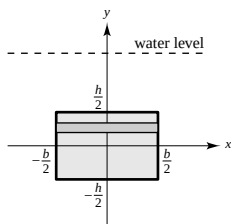
The second integral is zero since its integrand is odd and the limits of integration are symmetric to the origin. The first integral is the area of a semicircle with radius r .

$$F = w \left[(2k) \frac{\pi r^2}{2} + 0 \right] = wk\pi r^2$$

23. $h(y) = k - y$

$$L(y) = b$$

$$\begin{aligned} F &= w \int_{-h/2}^{h/2} (k - y)b \, dy \\ &= wb \left[ky - \frac{y^2}{2} \right]_{-h/2}^{h/2} = wb(hk) = wkhb \end{aligned}$$



27. $h(y) = 4 - y$

$$F = 62.4 \int_0^4 (4 - y)L(y) \, dy$$

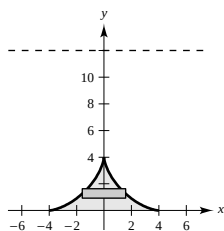
Using Simpson's Rule with $n = 8$ we have:

$$\begin{aligned} F &\approx 62.4 \left(\frac{4 - 0}{3(8)} \right) [0 + 4(3.5)(3) + 2(3)(5) + 4(2.5)(8) + 2(2)(9) + 4(1.5)(10) + 2(1)(10.25) + 4(0.5)(10.5) + 0] \\ &= 3010.8 \text{ lb} \end{aligned}$$

29. $h(y) = 12 - y$

$$L(y) = 2(4^{2/3} - y^{2/3})^{3/2}$$

$$\begin{aligned} F &= 62.4 \int_0^4 2(12 - y)(4^{2/3} - y^{2/3})^{3/2} \, dy \\ &\approx 6448.73 \text{ lb} \end{aligned}$$



25. From Exercise 22:

$$F = 64(15)(1)(1) = 960 \text{ lb}$$

 31. (a) If the fluid force is one half of 1123.2 lb, and the height of the water is b , then

$$h(y) = b - y$$

$$L(y) = 4$$

$$F = 62.4 \int_0^b (b - y)(4) \, dy = \frac{1}{2}(1123.2)$$

$$\int_0^b (b - y) \, dy = 2.25$$

$$\left[by - \frac{y^2}{2} \right]_0^b = 2.25$$

$$b^2 - \frac{b^2}{2} = 2.25$$

$$b^2 = 4.5 \Rightarrow b \approx 2.12 \text{ ft.}$$

(b) The pressure increases with increasing depth.

33. $F = F_w = w \int_c^d h(y)L(y) \, dy$, see page 471.

PART II

CHAPTER 6

Applications of Integration

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CHAPTER 6

Applications of Integration

Section 6.1 Area of a Region Between Two Curves

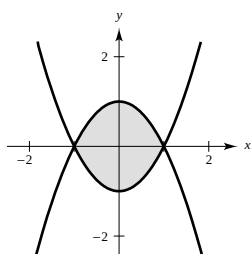
Solutions to Even-Numbered Exercises

$$2. A = \int_{-2}^2 [(2x + 5) - (x^2 + 2x + 1)] dx = \int_{-2}^2 (-x^2 + 4) dx$$

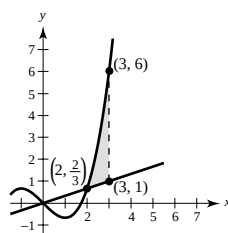
$$4. A = \int_0^1 (x^2 - x^3) dx$$

$$6. A = 2 \int_0^1 [(x - 1)^3 - (x - 1)] dx$$

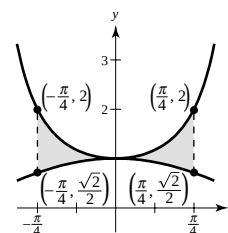
$$8. \int_{-1}^1 [(1 - x^2) - (x^2 - 1)] dx$$



$$10. \int_2^3 \left[\left(\frac{x^3}{3} - x \right) - \frac{x}{3} \right] dx$$



$$12. \int_{-\pi/4}^{\pi/4} (\sec^2 x - \cos x) dx$$

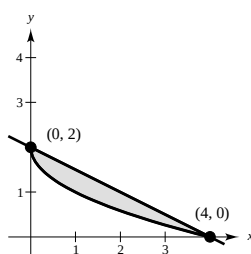


$$14. f(x) = 2 - \frac{1}{2}x$$

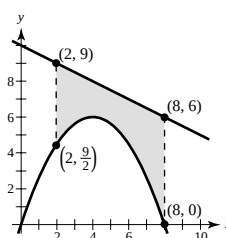
$$g(x) = 2 - \sqrt{x}$$

$$A \approx 1$$

Matches (a)



$$\begin{aligned} 16. A &= \int_2^8 \left[\left(10 - \frac{1}{2}x \right) - \left(-\frac{3}{8}x(x - 8) \right) \right] dx \\ &= \int_2^8 \left(\frac{3}{8}x^2 - \frac{7}{2}x + 10 \right) dx \\ &= \left[\frac{x^3}{8} - \frac{7x^2}{4} + 10x \right]_2^8 \\ &= (64 - 112 + 80) - (1 - 7 + 20) = 18 \end{aligned}$$



18. The points of intersection are given by

$$-x^2 + 4x + 1 = x + 1$$

$$-x^2 + 3x = 0$$

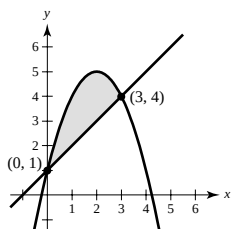
$$x^2 = 3x \text{ when } x = 0, 3$$

$$A = \int_0^3 [(-x^2 + 4x + 1) - (x + 1)] dx$$

$$= \int_0^3 (-x^2 + 3x) dx$$

$$= \left[-\frac{x^3}{3} + \frac{3x^2}{2} \right]_0^3$$

$$= -9 + \frac{27}{2} = \frac{9}{2}$$



20. The points of intersection are given by:

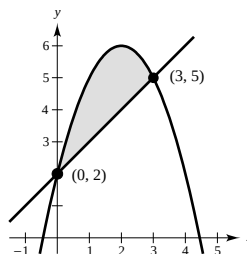
$$-x^2 + 4x + 2 = x + 2$$

$$x(3 - x) = 0 \text{ when } x = 0, 3$$

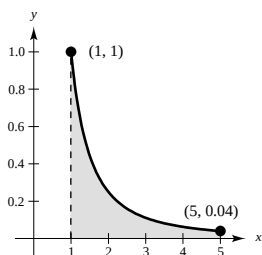
$$A = \int_0^3 [f(x) - g(x)] dx$$

$$= \int_0^3 [(-x^2 + 4x + 2) - (x + 2)] dx$$

$$= \int_0^3 (-x^2 + 3x) dx = \left[-\frac{x^3}{3} + \frac{3}{2}x^2 \right]_0^3 = \frac{9}{2}$$



$$22. A = \int_1^5 \left(\frac{1}{x^2} - 0 \right) dx = \left[-\frac{1}{x} \right]_1^5 = \frac{4}{5}$$



24. The points of intersection are given by

$$\sqrt[3]{x-1} = x-1$$

$$x-1 = (x-1)^3 = x^3 - 3x^2 + 3x - 1$$

$$x^3 - 3x^2 + 2x = 0$$

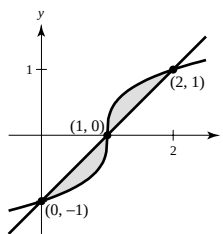
$$x(x^2 - 3x + 2) = 0$$

$$x(x-2)(x-1) = 0 \Rightarrow x = 0, 1, 2$$

$$A = 2 \int_0^1 [(x-1) - \sqrt[3]{x-1}] dx$$

$$= 2 \left[\frac{x^2}{2} - x - \frac{3}{4}(x-1)^{4/3} \right]_0^1$$

$$= 2 \left[\left(\frac{1}{2} - 1 - 0 \right) - \left(-\frac{3}{4} \right) \right] = \frac{1}{2}$$

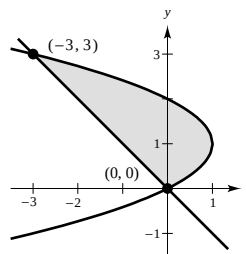


26. The points of intersection are given by:

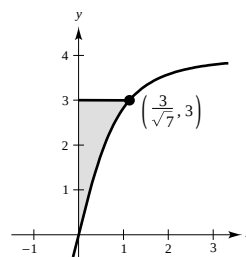
$$2y - y^2 = -y$$

$$y(y - 3) = 0 \quad \text{when } y = 0, 3$$

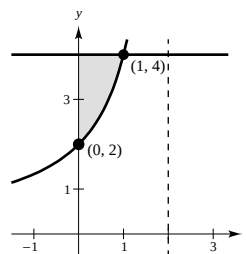
$$\begin{aligned} A &= \int_0^3 [f(y) - g(y)] dy \\ &= \int_0^3 [(2y - y^2) - (-y)] dy \\ &= \int_0^3 (3y - y^2) dy = \left[\frac{3}{2}y^2 - \frac{1}{3}y^3 \right]_0^3 = \frac{9}{2} \end{aligned}$$



$$\begin{aligned} 28. A &= \int_0^3 [f(y) - g(y)] dy \\ &= \int_0^3 \left[\frac{y}{\sqrt{16 - y^2}} - 0 \right] dy \\ &= -\frac{1}{2} \int_0^3 (16 - y^2)^{-1/2} (-2y) dy \\ &= \left[-\sqrt{16 - y^2} \right]_0^3 = 4 - \sqrt{7} \approx 1.354 \end{aligned}$$



$$\begin{aligned} 30. A &= \int_0^1 \left(4 - \frac{4}{2 - x} \right) dx \\ &= \left[4x + 4 \ln |2 - x| \right]_0^1 \\ &= 4 - 4 \ln 2 \\ &\approx 1.227 \end{aligned}$$



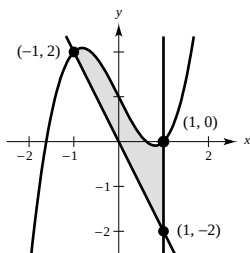
32. The point of intersection is given by:

$$x^3 - 2x + 1 = -2x$$

$$x^3 + 1 = 0 \quad \text{when } x = -1$$

$$\begin{aligned} A &= \int_{-1}^1 [f(x) - g(x)] dx \\ &= \int_{-1}^1 [(x^3 - 2x + 1) - (-2x)] dx \\ &= \int_{-1}^1 (x^3 + 1) dx = \left[\frac{x^4}{4} + x \right]_{-1}^1 = 2 \end{aligned}$$

Numerical Approximation: 2.0



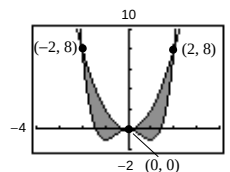
34. The points of intersection are given by:

$$x^4 - 2x^2 = 2x^2$$

$$x^2(x^2 - 4) = 0 \quad \text{when } x = 0, \pm 2$$

$$\begin{aligned} A &= 2 \int_0^2 [2x^2 - (x^4 - 2x^2)] dx \\ &= 2 \int_0^2 (4x^2 - x^4) dx \\ &= 2 \left[\frac{4x^3}{3} - \frac{x^5}{5} \right]_0^2 = \frac{128}{15} \end{aligned}$$

Numerical Approximation: 8.533



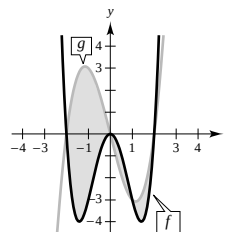
36. $f(x) = x^4 - 4x^2$, $g(x) = x^3 - 4x$

The points of intersection are given by:

$$x^4 - 4x^2 = x^3 - 4x$$

$$x^4 - x^3 - 4x^2 + 4x = 0$$

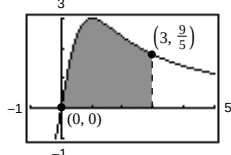
$$x(x-1)(x+2)(x-2) = 0 \quad \text{when } x = -2, 0, 1, 2$$



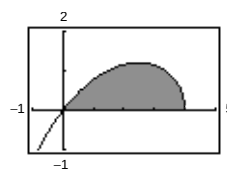
$$\begin{aligned} A &= \int_{-2}^0 [(x^3 - 4x) - (x^4 - 4x^2)] dx + \int_0^1 [(x^4 - 4x^2) - (x^3 - 4x)] dx + \int_1^2 [(x^3 - 4x) - (x^4 - 4x^2)] dx \\ &= \frac{248}{30} + \frac{37}{60} + \frac{53}{60} = \frac{293}{30} \end{aligned}$$

 Numerical Approximation: $8.267 + 0.617 + 0.883 \approx 9.767$

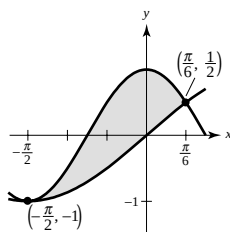
38.
$$\begin{aligned} A &= \int_0^3 \left[\frac{6x}{x^2 + 1} - 0 \right] dx \\ &= \left[3 \ln(x^2 + 1) \right]_0^3 \\ &= 3 \ln 10 \\ &\approx 6.908 \end{aligned}$$



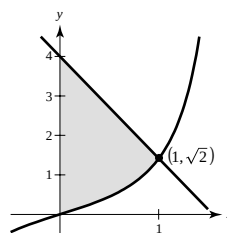
40.
$$A = \int_0^4 x \sqrt{\frac{4-x}{4+x}} dx \approx 3.434$$



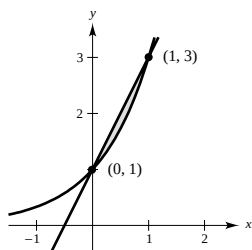
42.
$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/6} (\cos 2x - \sin x) dx \\ &= \left[\frac{1}{2} \sin 2x + \cos x \right]_{-\pi/2}^{\pi/6} \\ &= \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} \right) - (0) = \frac{3\sqrt{3}}{4} \approx 1.299 \end{aligned}$$



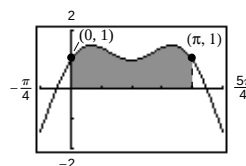
44.
$$\begin{aligned} A &= \int_0^1 \left[(\sqrt{2} - 4)x + 4 - \sec \frac{\pi x}{4} \tan \frac{\pi x}{4} \right] dx \\ &= \left[\frac{\sqrt{2} - 4}{2} x^2 + 4x - \frac{4}{\pi} \sec \frac{\pi x}{4} \right]_0^1 \\ &= \left(\frac{\sqrt{2} - 4}{2} + 4 - \frac{4}{\pi} \sqrt{2} \right) - \left(-\frac{4}{\pi} \right) \\ &= \frac{\sqrt{2}}{2} + 2 + \frac{4}{\pi} (1 - \sqrt{2}) \approx 2.1797 \end{aligned}$$


 46. From the graph we see that f and g intersect twice at $x = 0$ and $x = 1$.

$$\begin{aligned} A &= \int_0^1 [g(x) - f(x)] dx \\ &= \int_0^1 [(2x + 1) - 3^x] dx \\ &= \left[x^2 + x - \frac{1}{\ln 3} (3^x) \right]_0^1 \\ &= 2 \left(1 - \frac{1}{\ln 3} \right) \approx 0.180 \end{aligned}$$

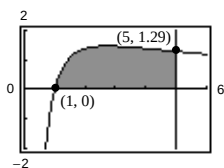


48.
$$\begin{aligned} A &= \int_0^{\pi} [(2 \sin x + \cos 2x) - 0] dx \\ &= \left[-2 \cos x + \frac{1}{2} \sin 2x \right]_0^{\pi} = 4 \end{aligned}$$



$$50. A = \int_1^5 \left[\frac{4 \ln x}{x} - 0 \right] dx$$

$$= \left[2(\ln x)^2 \right]_1^5 = 2(\ln 5)^2 \approx 5.181$$

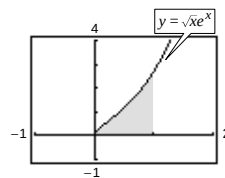


$$52. (a) y = \sqrt{x} e^x, y = 0, x = 0, x = 1$$

$$(b) A = \int_0^1 \sqrt{x} e^x dx.$$

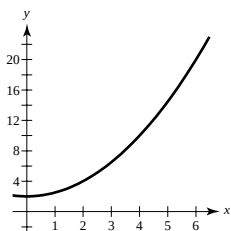
No, it cannot be evaluated by hand.

$$(c) 1.2556$$

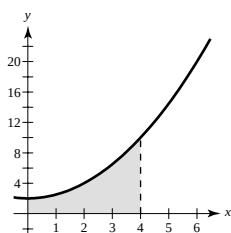


$$54. F(x) = \int_0^x \left(\frac{1}{2} t^2 + 2 \right) dt = \left[\frac{1}{6} t^3 + 2t \right]_0^x = \frac{x^3}{6} + 2x$$

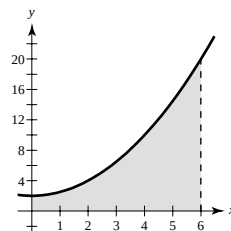
$$(a) F(0) = 0$$



$$(b) F(4) = \frac{4^3}{6} + 2(4) = \frac{56}{3}$$

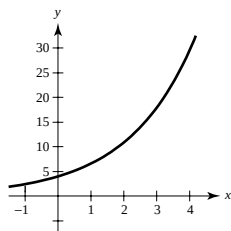


$$(c) F(6) = 36 + 12 = 48$$

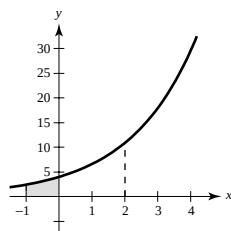


$$56. F(y) = \int_{-1}^y 4e^{x/2} dx = \left[8e^{x/2} \right]_{-1}^y = 8e^{y/2} - 8e^{-1/2}$$

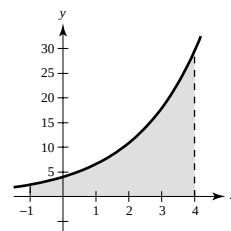
$$(a) F(-1) = 0$$



$$(b) F(0) = 8 - 8e^{-1/2} \approx 3.1478$$



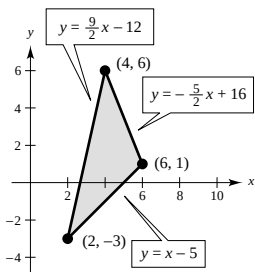
$$(c) F(4) = 8e^2 - 8e^{-1/2} \approx 54.2602$$



$$58. A = \int_2^4 \left[\left(\frac{9}{2}x - 12 \right) - (x - 5) \right] dx + \int_4^6 \left[\left(-\frac{5}{2}x + 16 \right) - (x - 5) \right] dx$$

$$= \int_2^4 \left(\frac{7}{2}x - 7 \right) dx + \int_4^6 \left(-\frac{7}{2}x + 21 \right) dx$$

$$= \left[\frac{7}{4}x^2 - 7x \right]_2^4 + \left[-\frac{7}{4}x^2 + 21x \right]_4^6 = 7 + 7 = 14$$



60. $f(x) = \frac{1}{x^2 + 1}$

$$f'(x) = -\frac{2x}{(x^2 + 1)^2}$$

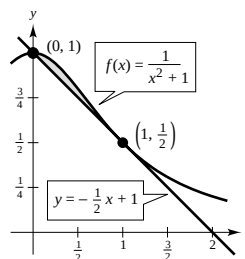
At $\left(1, \frac{1}{2}\right)$, $f'(1) = -\frac{1}{2}$.

Tangent line:

$$y - \frac{1}{2} = -\frac{1}{2}(x - 1) \text{ or } y = -\frac{1}{2}x + 1$$

The tangent line intersects $f(x) = \frac{1}{x^2 + 1}$ at $x = 0$.

$$A = \int_0^1 \left[\frac{1}{x^2 + 1} - \left(-\frac{1}{2}x + 1 \right) \right] dx = \left[\arctan x + \frac{x^2}{4} - x \right]_0^1 = \frac{\pi - 3}{4} \approx 0.0354$$



62. Answers will vary. See page 417.

64. $x^3 \geq x$ on $[-1, 0]$

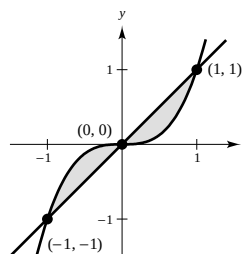
$x^3 \leq x$ on $[0, 1]$

Both functions symmetric to origin

$$\int_{-1}^0 (x^3 - x) dx = -\int_0^1 (x^3 - x) dx.$$

Thus, $\int_{-1}^1 (x^3 - x) dx = 0$.

$$A = 2 \int_0^1 (x - x^3) dx = 2 \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2}$$



66. Proposal 2 is better, since the cumulative deficit (the area under the curve) is less.

68. $A = 2 \int_0^9 (9 - x) dx = 2 \left[9x - \frac{x^2}{2} \right]_0^9 = 81$

$$2 \int_0^{9-b} [(9 - x) - b] dx = \frac{81}{2}$$

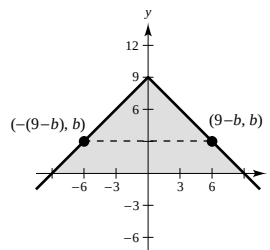
$$2 \int_0^{9-b} [(9 - b) - x] dx = \frac{81}{2}$$

$$2 \left[(9 - b)x - \frac{x^2}{2} \right]_0^{9-b} = \frac{81}{2}$$

$$(9 - b)(9 - b) = \frac{81}{2}$$

$$9 - b = \frac{9}{\sqrt{2}}$$

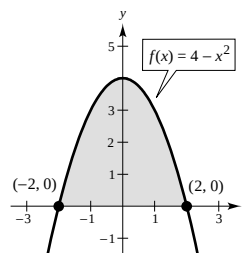
$$b = 9 - \frac{9}{\sqrt{2}} \approx 2.636$$



70. $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (4 - x_i^2) \Delta x$

where $x_i = -2 + \frac{4i}{n}$ and $\Delta x = \frac{4}{n}$ is the same as

$$\int_{-2}^2 (4 - x^2) dx = \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{32}{3}.$$



$$\begin{aligned} 72. \int_0^5 [(7.21 + 0.26t + 0.02t^2) - (7.21 + 0.1t + 0.01t^2)] dt &= \int_0^5 (0.01t^2 + 0.16t) dt \\ &= \left[\frac{0.01t^3}{3} + \frac{0.16t^2}{2} \right]_0^5 \\ &= \frac{29}{12} \text{ billion} \approx \$2.417 \text{ billion} \end{aligned}$$

74. 5% : $P_1 = 893,000 e^{(0.05)t}$

$3\frac{1}{2}\%$: $P_2 = 893,000 e^{(0.035)t}$

Difference in profits over 5 years:

$$\begin{aligned} \int_0^5 [893,000e^{0.05t} - 893,000e^{0.035t}] dt &= 893,000 \left[\frac{e^{0.05t}}{0.05} - \frac{e^{0.035t}}{0.035} \right]_0^5 \\ &\approx 893,000[(25.6805 - 34.0356) - (20 - 28.5714)] \\ &\approx 893,000(0.2163) \approx \$193,156 \end{aligned}$$

Note: Using a graphing utility you obtain \$193,183.

76. The curves intersect at the point where the slope of y_2 equals that of y_1 , 1.

$$y_2 = 0.08x^2 + k \Rightarrow y'_2 = 0.16x = 1 \Rightarrow x = \frac{1}{.16} = 6.25$$

(a) The value of k is given by

$$y_1 = y_2$$

$$6.25 = (0.08)(6.25)^2 + k$$

$$k = 3.125.$$

$$\begin{aligned} \text{(b) Area} &= 2 \int_0^{6.25} (y_2 - y_1) dx \\ &= 2 \int_0^{6.25} (0.08x^2 + 3.125 - x) dx \\ &= 2 \left[\frac{0.08x^3}{3} + 3.125x - \frac{x^2}{2} \right]_0^{6.25} \\ &= 2(6.510417) \approx 13.02083 \end{aligned}$$

78. (a) $A \approx 6.031 - 2 \left[\pi \left(\frac{1}{16} \right)^2 \right] - 2 \left[\pi \left(\frac{1}{8} \right)^2 \right] \approx 5.908$

(b) $V = 2A \approx 2(5.908) \approx 11.816 \text{ m}^3$

(c) $5000V \approx 5000(11.816) = 59,082 \text{ pounds}$

80. True

Section 6.2 Volume: The Disk Method

$$2. V = \pi \int_0^2 (4 - x^2)^2 dx = \pi \int_0^2 (x^4 - 8x^2 + 16) dx = \pi \left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x \right]_0^2 = \frac{256\pi}{15}$$

$$4. V = \pi \int_0^3 (\sqrt{9 - x^2})^2 dx = \pi \int_0^3 (9 - x^2) dx \\ = \pi \left[9x - \frac{x^3}{3} \right]_0^3 = 18\pi$$

$$6. \begin{aligned} 2 &= 4 - \frac{x^2}{4} & V &= \pi \int_{-2\sqrt{2}}^{2\sqrt{2}} \left[\left(4 - \frac{x^2}{4} \right)^2 - (2)^2 \right] dx \\ 8 &= 16 - x^2 & &= 2\pi \int_0^{2\sqrt{2}} \left[\frac{x^4}{16} - 2x^2 + 12 \right] dx \\ x^2 &= 8 & &= 2\pi \left[\frac{x^5}{80} - \frac{2x^3}{3} + 12x \right]_0^{2\sqrt{2}} \\ x &= \pm 2\sqrt{2} & &= 2\pi \left[\frac{128\sqrt{2}}{80} - \frac{32\sqrt{2}}{3} + 24\sqrt{2} \right] \\ & & &= \frac{448\sqrt{2}}{15}\pi \approx 132.69 \end{aligned}$$

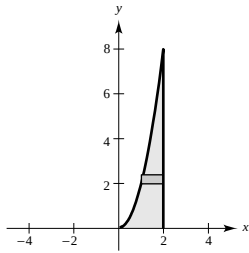
$$8. y = \sqrt{16 - x^2} \Rightarrow x = \sqrt{16 - y^2} \\ V = \pi \int_0^4 (\sqrt{16 - y^2})^2 dy = \pi \int_0^4 (16 - y^2) dy \\ = \pi \left[16y - \frac{y^3}{3} \right]_0^4 = \frac{128\pi}{3}$$

$$10. V = \pi \int_1^4 (-y^2 + 4y)^2 dy = \pi \int_1^4 (y^4 - 8y^3 + 16y^2) dy \\ = \pi \left[\frac{y^5}{5} - 2y^4 + \frac{16y^3}{3} \right]_1^4 \\ = \frac{459\pi}{15} = \frac{153\pi}{5}$$

$$12. y = 2x^2, y = 0, x = 2$$

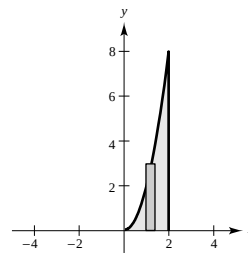
$$(a) R(y) = 2, r(y) = \sqrt{y/2}$$

$$V = \pi \int_0^8 \left(4 - \frac{y}{2} \right) dy = \pi \left[4y - \frac{y^2}{4} \right]_0^8 = 16\pi$$



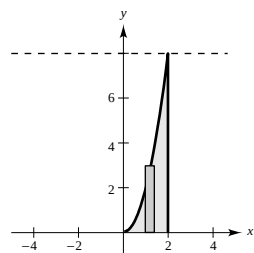
$$(b) R(x) = 2x^2, r(x) = 0$$

$$V = \pi \int_0^2 4x^4 dx = \pi \left[\frac{4x^5}{5} \right]_0^2 = \frac{128\pi}{5}$$



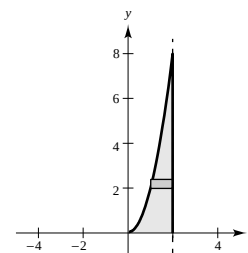
$$(c) R(x) = 8, r(x) = 8 - 2x^2$$

$$V = \pi \int_0^2 [64 - (64 - 32x^2 + 4x^4)] dx \\ = \pi \int_0^2 (32x^2 - 4x^4) dx = 4\pi \int_0^2 (8x^2 - x^4) dx \\ = 4\pi \left[\frac{8}{3}x^3 - \frac{1}{5}x^5 \right]_0^2 = \frac{896\pi}{15}$$



$$(d) R(y) = 2 - \sqrt{y/2}, r(y) = 0$$

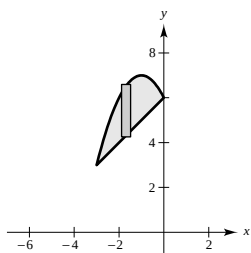
$$V = \pi \int_0^8 \left(2 - \sqrt{\frac{y}{2}} \right)^2 dy \\ = \pi \int_0^8 \left(4 - 4\sqrt{\frac{y}{2}} + \frac{y}{2} \right) dy \\ = \pi \left[4y - \frac{4\sqrt{2}}{3}y^{3/2} + \frac{y^2}{4} \right]_0^8 = \frac{16\pi}{3}$$



14. $y = 6 - 2x - x^2$, $y = x + 6$ intersect at $(-3, 3)$ and $(0, 6)$.

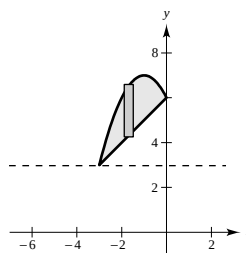
(a) $R(x) = 6 - 2x - x^2$, $r(x) = x + 6$

$$\begin{aligned} V &= \pi \int_{-3}^0 [(6 - 2x - x^2)^2 - (x + 6)^2] dx \\ &= \pi \int_{-3}^0 (x^4 + 4x^3 - 9x^2 - 36x) dx \\ &= \pi \left[\frac{1}{5}x^5 + x^4 - 3x^3 - 18x^2 \right]_{-3}^0 = \frac{243\pi}{5} \end{aligned}$$



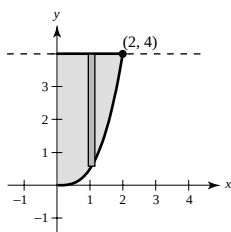
(b) $R(x) = (6 - 2x - x^2) - 3$, $r(x) = (x + 6) - 3$

$$\begin{aligned} V &= \pi \int_{-3}^0 [(3 - 2x - x^2)^2 - (x + 3)^2] dx \\ &= \pi \int_{-3}^0 (x^4 + 4x^3 - 3x^2 - 18x) dx \\ &= \pi \left[\frac{1}{5}x^5 + x^4 - x^3 - 9x^2 \right]_{-3}^0 = \frac{108\pi}{5} \end{aligned}$$



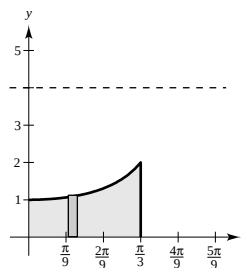
16. $R(x) = 4 - \frac{x^3}{2}$, $r(x) = 0$

$$\begin{aligned} V &= \pi \int_0^2 \left(4 - \frac{x^3}{2} \right)^2 dx \\ &= \pi \int_0^2 \left[16 - 4x^3 + \frac{x^6}{4} \right] dx \\ &= \pi \left[16x - x^4 + \frac{x^7}{28} \right]_0^2 \\ &= \pi \left[32 - 16 + \frac{128}{28} \right] = \frac{144}{7}\pi \end{aligned}$$



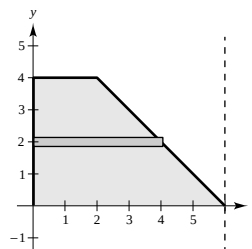
18. $R(x) = 4$, $r(x) = 4 - \sec x$

$$\begin{aligned} V &= \pi \int_0^{\pi/3} [(4)^2 - (4 - \sec x)^2] dx \\ &= \pi \int_0^{\pi/3} (8 \sec x - \sec^2 x) dx \\ &= \pi \left[8 \ln |\sec x + \tan x| - \tan x \right]_0^{\pi/3} \\ &= \pi \left[(8 \ln |2 + \sqrt{3}| - \sqrt{3}) - (8 \ln |1 + 0| - 0) \right] \\ &= \pi [8 \ln(2 + \sqrt{3}) - \sqrt{3}] \approx 27.66 \end{aligned}$$



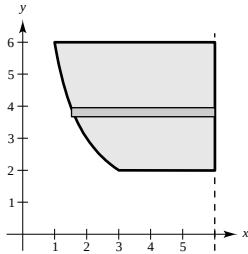
20. $R(y) = 6$, $r(y) = 6 - (6 - y) = y$

$$\begin{aligned} V &= \pi \int_0^4 [(6)^2 - (y)^2] dy \\ &= \pi \left[36y - \frac{y^3}{3} \right]_0^4 = \frac{368\pi}{3} \end{aligned}$$



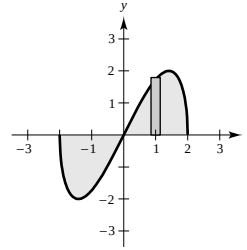
$$22. R(y) = 6 - \frac{6}{y}, \quad r(y) = 0$$

$$\begin{aligned} V &= \pi \int_2^6 \left(6 - \frac{6}{y}\right)^2 dy \\ &= 36\pi \int_2^6 \left(1 - \frac{2}{y} + \frac{1}{y^2}\right) dy \\ &= 36\pi \left[y - 2 \ln|y| - \frac{1}{y} \right]_2^6 \\ &= 36\pi \left[\left(\frac{35}{6} - 2 \ln 6\right) - \left(\frac{3}{2} - 2 \ln 2\right) \right] \\ &= 36\pi \left(\frac{13}{3} + 2 \ln \frac{1}{3} \right) = 12\pi(13 - 6 \ln 3) \approx 241.59 \end{aligned}$$



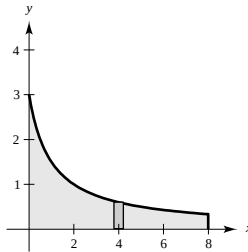
$$24. R(x) = x\sqrt{4-x^2}, \quad r(x) = 0$$

$$\begin{aligned} V &= 2\pi \int_0^2 [x\sqrt{4-x^2}]^2 dx \\ &= 2\pi \int_0^2 (4x^2 - x^4) dx \\ &= 2\pi \left[\frac{4x^3}{3} - \frac{x^5}{5} \right]_0^2 \\ &= \frac{128\pi}{15} \end{aligned}$$



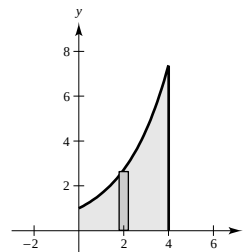
$$26. R(x) = \frac{3}{x+1}, \quad r(x) = 0$$

$$\begin{aligned} V &= \pi \int_0^8 \left(\frac{3}{x+1}\right)^2 dx \\ &= 9\pi \int_0^8 (x+1)^{-2} dx \\ &= 9\pi \left[-\frac{1}{x+1} \right]_0^8 = 8\pi \end{aligned}$$

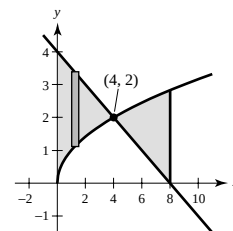


$$28. R(x) = e^{x/2}, \quad r(x) = 0$$

$$\begin{aligned} V &= \pi \int_0^4 (e^{x/2})^2 dx \\ &= \pi \int_0^4 e^x dx \\ &= \left[\pi e^x \right]_0^4 \\ &= \pi(e^4 - 1) \approx 168.38 \end{aligned}$$



$$\begin{aligned} 30. V &= \pi \int_0^4 \left[\left(4 - \frac{1}{2}x\right)^2 - (\sqrt{x})^2 \right] dx + \pi \int_4^8 \left[(\sqrt{x})^2 - \left(4 - \frac{1}{2}x\right)^2 \right] dx \\ &= \pi \int_0^4 \left(\frac{x^2}{4} - 5x + 16 \right) dx + \pi \int_4^8 \left(-\frac{x^2}{4} + 5x - 16 \right) dx \\ &= \pi \left[\frac{x^3}{12} - \frac{5x^2}{2} + 16x \right]_0^4 + \pi \left[-\frac{x^3}{12} + \frac{5x^2}{2} - 16x \right]_4^8 \\ &= \frac{88}{3}\pi + \frac{56}{3}\pi = 48\pi \end{aligned}$$



32. $y = 9 - x^2$, $y = 0$, $x = 2$, $x = 3$

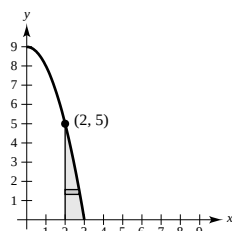
$$x = \sqrt{9 - y}$$

$$V = \pi \int_0^5 [(\sqrt{9 - y})^2 - 2^2] dy$$

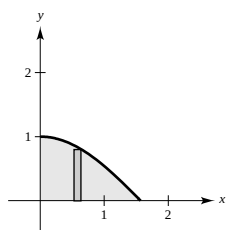
$$= \pi \int_0^5 (5 - y) dy$$

$$= \pi \left[5y - \frac{y^2}{2} \right]_0^5$$

$$= \pi \left(25 - \frac{25}{2} \right) = \frac{25\pi}{2}$$



34. $V = \pi \int_0^{\pi/2} [\cos x]^2 dx \approx 2.4674$

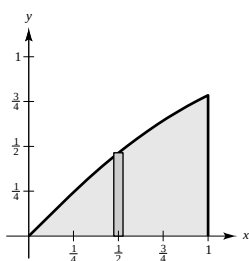


36. $V = \pi \int_1^3 [\ln x]^2 dx \approx 3.2332$

38. $V = \pi \int_0^5 [2 \arctan(0.2x)]^2 dx \approx 15.4115$

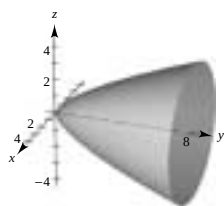
40. $A \approx \frac{3}{4}$

Matches (b)

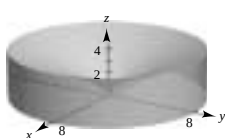


42. $V = \int_a^b A(x) dx$ or $V = \int_c^d A(y) dy$

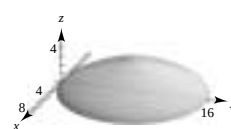
44. (a)



(b)



(c)



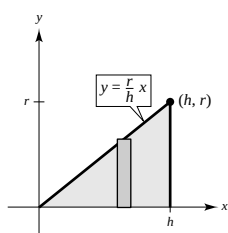
$$a < c < b.$$

46. $R(x) = \frac{r}{h}x$, $r(x) = 0$

$$V = \pi \int_0^h \frac{r^2}{h^2} x^2 dx$$

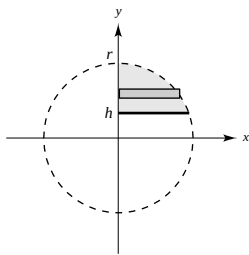
$$= \left[\frac{r^2 \pi}{3h^2} x^3 \right]_0^h$$

$$= \frac{r^2 \pi}{3h^2} h^3 = \frac{1}{3} \pi r^2 h$$

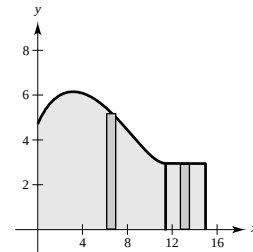


$$48. x = \sqrt{r^2 - y^2}, R(y) = \sqrt{r^2 - y^2}, r(y) = 0$$

$$\begin{aligned} V &= \pi \int_h^r (\sqrt{r^2 - y^2})^2 dy \\ &= \pi \int_h^r (r^2 - y^2) dy \\ &= \pi \left[r^2 y - \frac{y^3}{3} \right]_h^r \\ &= \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(r^2 h - \frac{h^3}{3} \right) \right] \\ &= \pi \left(\frac{2r^3}{3} - r^2 h + \frac{h^3}{3} \right) \\ &= \frac{\pi}{3} (2r^3 - 3r^2 h + h^3) \end{aligned}$$

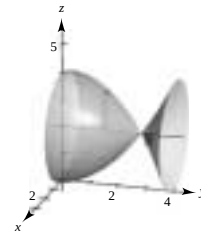


$$\begin{aligned} 52. y &= \begin{cases} \sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}, & 0 \leq x \leq 11.5 \\ 2.95, & 11.5 < x \leq 15 \end{cases} \\ V &= \pi \int_0^{11.5} (\sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2})^2 dx + \pi \int_{11.5}^{15} 2.95^2 dx \\ &= \pi \left[\frac{0.1x^4}{4} - \frac{2.2x^3}{3} + \frac{10.9x^2}{2} + 22.2x \right]_0^{11.5} + \pi \left[2.95^2 x \right]_{11.5}^{15} \\ &\approx 1031.9016 \text{ cubic centimeters} \end{aligned}$$



54. (a) First find where $y = b$ intersects the parabola:

$$\begin{aligned} b &= 4 - \frac{x^2}{4} \\ x^2 &= 16 - 4b = 4(4 - b) \\ x &= 2\sqrt{4 - b} \\ V &= \int_0^{2\sqrt{4-b}} \pi \left[4 - \frac{x^2}{4} - b \right]^2 dx + \int_{2\sqrt{4-b}}^4 \pi \left[b - 4 + \frac{x^2}{4} \right]^2 dx \\ &= \int_0^4 \pi \left[4 - \frac{x^2}{4} - b \right]^2 dx \\ &= \pi \int_0^4 \left[\frac{x^4}{16} - 2x^2 + \frac{bx^2}{2} + b^2 - 8b + 16 \right] dx \\ &= \pi \left[\frac{x^5}{80} - \frac{2x^3}{3} + \frac{bx^3}{6} + b^2x - 8bx + 16x \right]_0^4 \\ &= \pi \left[\frac{64}{5} - \frac{128}{3} + \frac{32}{3}b + 4b^2 - 32b + 64 \right] = \pi \left[4b^2 - \frac{64}{3}b + \frac{512}{15} \right] \end{aligned}$$



$$50. (a) V = \pi \int_0^4 (\sqrt{x})^2 dx = \pi \int_0^4 x dx = \left[\frac{\pi x^2}{2} \right]_0^4 = 8\pi$$

Let $0 < c < 4$ and set

$$\pi \int_0^c x dx = \left[\frac{\pi x^2}{2} \right]_0^c = \frac{\pi c^2}{2} = 4\pi.$$

$$c^2 = 8$$

$$c = \sqrt{8} = 2\sqrt{2}$$

Thus, when $x = 2\sqrt{2}$, the solid is divided into two parts of equal volume.

(b) Set $\pi \int_0^c x dx = \frac{8\pi}{3}$ (one third of the volume). Then

$$\frac{\pi c^2}{2} = \frac{8\pi}{3}, c^2 = \frac{16}{3}, c = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}.$$

To find the other value, set $\pi \int_0^d x dx = \frac{16\pi}{3}$ (two thirds of the volume). Then

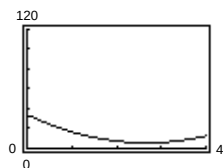
$$\frac{\pi d^2}{2} = \frac{16\pi}{3}, d^2 = \frac{32}{3}, d = \frac{\sqrt{32}}{\sqrt{3}} = \frac{4\sqrt{6}}{3}.$$

The x -values that divide the solid into three parts of equal volume are $x = (4\sqrt{3})/3$ and $x = (4\sqrt{6})/3$.

—CONTINUED—

54. —CONTINUED—

(b) graph of $V(b) = \pi \left[4b^2 - \frac{64}{3}b + \frac{512}{15} \right]$



Minimum Volume is 17.87 for $b = 2.67$

(c) $V'(b) = \pi \left[8b - \frac{64}{3} \right] = 0 \Rightarrow b = \frac{64/3}{8} = \frac{8}{3} = 2\frac{2}{3}$

$V''(b) = 8\pi > 0 \Rightarrow b = \frac{8}{3}$ is a relative minimum.

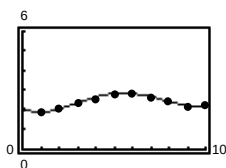
56. (a) $V = \int_0^{10} \pi [f(x)]^2 dx$

Simpson's Rule: $b - a = 10 - 0 = 10$, $n = 10$

$$V \approx \frac{\pi}{3} [(2.1)^2 + 4(1.9)^2 + 2(2.1)^2 + 4(2.35)^2 + 2(2.6)^2 + 4(2.85)^2 + 2(2.9)^2 + 4(2.7)^2 + 2(2.45)^2 + 4(2.2)^2 + (2.3)^2]$$

$$\approx \frac{\pi}{3} [178.405] \approx 186.83 \text{ cm}^3$$

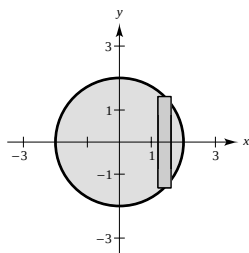
(b) $f(x) = 0.00249x^4 - 0.0529x^3 + 0.3314x^2 - 0.4999x + 2.112$



(c) $V \approx \int_0^{10} \pi f(x)^2 dx \approx 186.35 \text{ cm}^3$

58. $V = \frac{1}{2}(10)(2)(3) = 30 \text{ m}^3$

60.

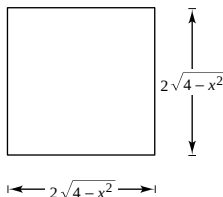


Base of Cross Section $= 2\sqrt{4 - x^2}$

(a) $A(x) = b^2 = (2\sqrt{4 - x^2})^2$

$$V = \int_{-2}^2 4(4 - x^2) dx$$

$$= 4 \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{128}{3}$$

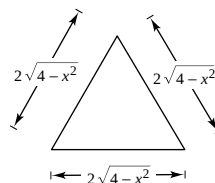


(b) $A(x) = \frac{1}{2}bh = \frac{1}{2}(2\sqrt{4 - x^2})(\sqrt{3}\sqrt{4 - x^2})$

$$= \sqrt{3}(4 - x^2)$$

$$V = \sqrt{3} \int_{-2}^2 (4 - x^2) dx$$

$$= \sqrt{3} \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{32\sqrt{3}}{3}$$



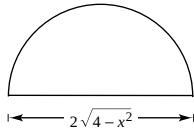
—CONTINUED—

60. —CONTINUED—

$$(c) A(x) = \frac{1}{2} \pi r^2$$

$$= \frac{\pi}{2} (\sqrt{4-x^2})^2 = \frac{\pi}{2} (4-x^2)$$

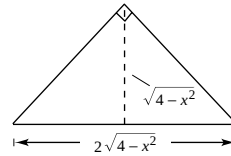
$$V = \frac{\pi}{2} \int_{-2}^2 (4-x^2) dx = \frac{\pi}{2} \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{16\pi}{3}$$



$$(d) A(x) = \frac{1}{2}bh$$

$$= \frac{1}{2} (2\sqrt{4-x^2})(\sqrt{4-x^2}) = 4-x^2$$

$$V = \int_{-2}^2 (4-x^2) dx = \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{32}{3}$$



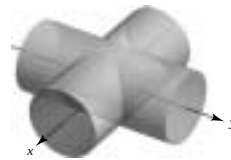
62. The cross sections are squares. By symmetry, we can set up an integral for an eighth of the volume and multiply by 8.

$$A(y) = b^2 = (\sqrt{r^2 - y^2})^2$$

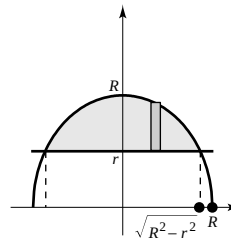
$$V = 8 \int_0^r (r^2 - y^2) dy$$

$$= 8 \left[r^2 y - \frac{1}{3} y^3 \right]_0^r$$

$$= \frac{16}{3} r^3$$



$$\begin{aligned} 64. V &= \pi \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} [(\sqrt{R^2-x^2})^2 - r^2] dx \\ &= 2\pi \int_0^{\sqrt{R^2-r^2}} (R^2 - r^2 - x^2) dx \\ &= 2\pi \left[(R^2 - r^2)x - \frac{x^3}{3} \right]_0^{\sqrt{R^2-r^2}} \\ &= 2\pi \left[(R^2 - r^2)^{3/2} - \frac{(R^2 - r^2)^{3/2}}{3} \right] \\ &= \frac{4}{3} \pi (R^2 - r^2)^{3/2} \end{aligned}$$



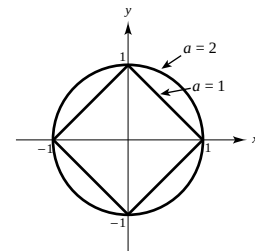
66. (a) When $a = 1$: $|x| + |y| = 1$ represents a square.

When $a = 2$: $|x|^2 + |y|^2 = 1$ represents a circle.

- (b) $|y| = (1 - |x|^a)^{1/a}$

$$A = 2 \int_{-1}^1 (1 - |x|^a)^{1/a} dx = 4 \int_0^1 (1 - x^a)^{1/a} dx$$

To approximate the volume of the solid, form n slices, each of whose area is approximated by the integral above. Then sum the volumes of these n slices.



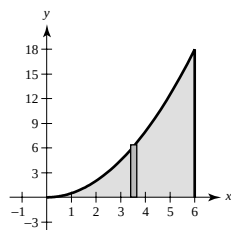
Section 6.3 Volume: The Shell Method

2. $p(x) = x$

$h(x) = 1 - x$

$$V = 2\pi \int_0^1 x(1 - x) dx$$

$$= 2\pi \int_0^1 (x - x^2) dx = 2\pi \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{\pi}{3}$$



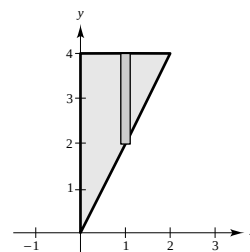
4. $p(x) = x$

$h(x) = 8 - (x^2 + 4) = 4 - x^2$

$$V = 2\pi \int_0^2 x(4 - x^2) dx$$

$$= 2\pi \int_0^2 (4x - x^3) dx$$

$$= 2\pi \left[2x^2 - \frac{x^4}{4} \right]_0^2 = 8\pi$$



6. $p(x) = x$

$h(x) = \frac{1}{2}x^2$

$$V = 2\pi \int_0^6 \frac{1}{2}x^3 dx$$

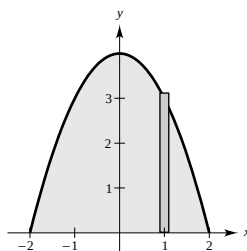
$$= \left[\pi \frac{x^4}{4} \right]_0^6 = 324\pi$$

8. $p(x) = x$

$h(x) = 4 - x^2$

$$V = 2\pi \int_0^2 (4x - x^3) dx$$

$$= 2\pi \left[2x^2 - \frac{1}{4}x^4 \right]_0^2 = 8\pi$$



10. $p(x) = x$

$h(x) = 4 - 2x$

$$V = 2\pi \int_0^2 x(4 - 2x) dx$$

$$= 2\pi \int_0^2 (4x - 2x^2) dx$$

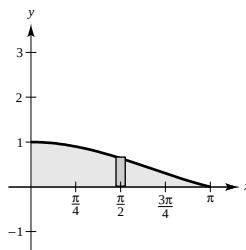
$$= 2\pi \left[2x^2 - \frac{2}{3}x^3 \right]_0^2 = \frac{16\pi}{3}$$

12. $p(x) = x$

$h(x) = \frac{\sin x}{x}$

$$V = 2\pi \int_0^\pi x \left[\frac{\sin x}{x} \right] dx$$

$$= 2\pi \int_0^\pi \sin x dx = \left[-2\pi \cos x \right]_0^\pi = 4\pi$$



14. $p(y) = -y$ ($p(y) \geq 0$ on $[-2, 0]$)

$h(y) = 4 - (2 - y) = 2 + y$

$$V = 2\pi \int_{-2}^0 (-y)(2 + y) dy$$

$$= 2\pi \int_{-2}^0 (-2y - y^2) dy$$

$$= 2\pi \left[-y^2 - \frac{y^3}{3} \right]_{-2}^0 = \frac{8\pi}{3}$$

16. $p(y) = y$

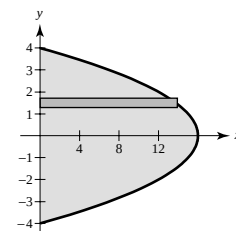
$h(y) = 16 - y^2$

$$V = 2\pi \int_0^4 y(16 - y^2) dy$$

$$= 2\pi \int_0^4 (16y - y^3) dy$$

$$= 2\pi \left[8y^2 - \frac{y^4}{4} \right]_0^4$$

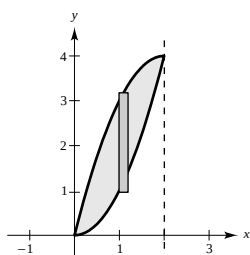
$$= 2\pi[128 - 64] = 128\pi$$



18. $p(x) = 2 - x$

$$h(x) = 4x - x^2 - x^2 = 4x - 2x^2$$

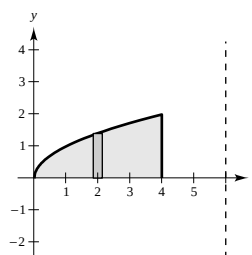
$$\begin{aligned} V &= 2\pi \int_0^2 (2-x)(4x-2x^2) dx \\ &= 2\pi \int_0^2 (8x - 8x^2 + 2x^3) dx \\ &= 2\pi \left[4x^2 - \frac{8}{3}x^3 + \frac{1}{2}x^4 \right]_0^2 = \frac{16\pi}{3} \end{aligned}$$



20. $p(x) = 6 - x$

$$h(x) = \sqrt{x}$$

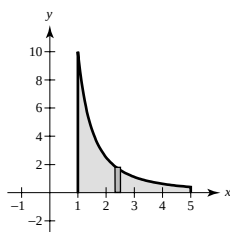
$$\begin{aligned} V &= 2\pi \int_0^4 (6-x)\sqrt{x} dx \\ &= 2\pi \int_0^4 (6x^{1/2} - x^{3/2}) dx \\ &= 2\pi \left[4x^{3/2} - \frac{2}{5}x^{5/2} \right]_0^4 = \frac{192\pi}{5} \end{aligned}$$



22. (a) Disk

$$R(x) = \frac{10}{x^2}, r(x) = 0$$

$$\begin{aligned} V &= \pi \int_1^5 \left(\frac{10}{x^2} \right)^2 dx \\ &= 100\pi \int_1^5 x^{-4} dx \\ &= 100\pi \left[\frac{x^{-3}}{-3} \right]_1^5 \\ &= -\frac{100\pi}{3} \left[\frac{1}{125} - 1 \right] = \frac{496\pi}{15} \end{aligned}$$



(b) Shell

$$R(x) = x, r(x) = 0$$

$$\begin{aligned} V &= 2\pi \int_1^5 x \left(\frac{10}{x^2} \right) dx \\ &= 20\pi \int_1^5 \frac{1}{x} dx \\ &= 20\pi \left[\ln|x| \right]_1^5 = 20\pi \ln 5 \end{aligned}$$

(c) Disk

$$R(x) = 10, r(x) = 10 - \frac{10}{x^2}$$

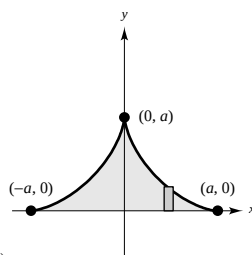
$$\begin{aligned} V &= \pi \int_1^5 \left[10^2 - \left(10 - \frac{10}{x^2} \right)^2 \right] dx \\ &= \pi \left[\frac{100}{3x^3} - \frac{200}{x} \right]_1^5 = \frac{1904\pi}{15} \end{aligned}$$

24. (a) Disk

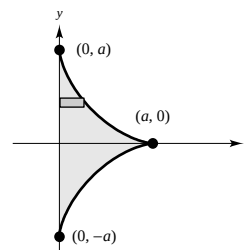
$$R(x) = (a^{2/3} - x^{2/3})^{3/2}$$

$$r(x) = 0$$

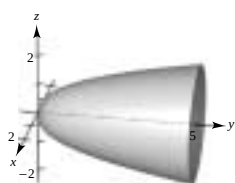
$$\begin{aligned} V &= \pi \int_{-a}^a (a^{2/3} - x^{2/3})^3 dx \\ &= 2\pi \int_0^a (a^2 - 3a^{4/3}x^{2/3} + 3a^{2/3}x^{4/3} - x^2) dx \\ &= 2\pi \left[a^2x - \frac{9}{5}a^{4/3}x^{5/3} + \frac{9}{7}a^{2/3}x^{7/3} - \frac{1}{3}x^3 \right]_0^a \\ &= 2\pi \left(a^3 - \frac{9}{5}a^3 + \frac{9}{7}a^3 - \frac{1}{3}a^3 \right) = \frac{32\pi a^3}{105} \end{aligned}$$



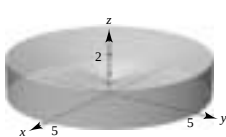
(b) Same as part a by symmetry



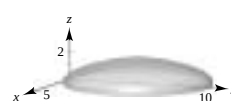
26. (a)



(b)



(c)



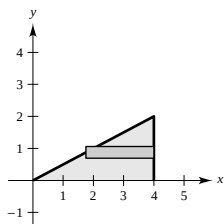
$$a < c < b$$

$$28. 2\pi \int_0^4 x \left(\frac{x}{2} \right) dx$$

represents the volume of the solid generated by revolving the region bounded by $y = x/2$, $y = 0$, and $x = 4$ about the y -axis by using the Shell Method.

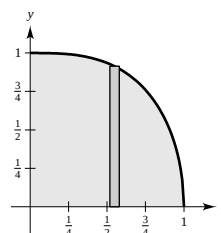
$$\pi \int_0^2 [16 - (2y)^2] dy = \pi \int_0^2 [(4)^2 - (2y)^2] dy$$

represents this same volume by using the Disk Method.



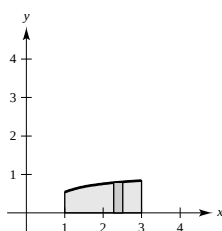
Disk Method

30. (a)



$$(b) V = 2\pi \int_0^1 x \sqrt{1 - x^3} dx \approx 2.3222$$

32. (a)

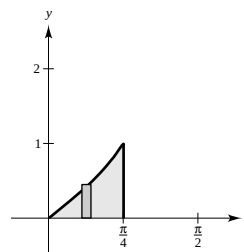


$$(b) V = 2\pi \int_1^3 \frac{2x}{1 + e^{1/x}} dx \approx 19.0162$$

$$34. y = \tan x, y = 0, x = 0, x = \frac{\pi}{4}$$

Volume ≈ 1

Matches (e)



36. Total volume of the hemisphere is $\frac{1}{2}(\frac{4}{3})\pi r^3 = \frac{2}{3}\pi(3)^3 = 18\pi$. By the Shell Method, $p(x) = x$, $h(x) = \sqrt{9 - x^2}$. Find x_0 such that

$$6\pi = 2\pi \int_0^{x_0} x \sqrt{9 - x^2} dx$$

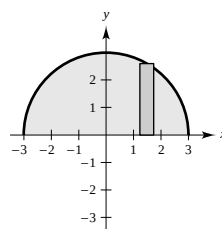
$$6 = - \int_0^{x_0} (9 - x^2)^{1/2} (-2x) dx$$

$$= \left[-\frac{2}{3} (9 - x^2)^{3/2} \right]_0^{x_0} = 18 - \frac{2}{3} (9 - x_0^2)^{3/2}$$

$$(9 - x_0^2)^{3/2} = 18$$

$$x_0 = \sqrt{9 - 18^{2/3}} \approx 1.460.$$

$$\text{Diameter: } 2\sqrt{9 - 18^{2/3}} \approx 2.920$$



$$\begin{aligned}
 38. V &= 4\pi \int_{-r}^r (R-x)\sqrt{r^2-x^2} dx \\
 &= 4\pi R \int_{-r}^r \sqrt{r^2-x^2} dx - 4\pi \int_{-r}^r x\sqrt{r^2-x^2} dx \\
 &= 4\pi R \left(\frac{\pi r^2}{2} \right) + \left[2\pi \left(\frac{2}{3} \right) (r^2-x^2)^{3/2} \right]_{-r}^r \\
 &= 2\pi^2 r^2 R
 \end{aligned}$$

$$\begin{aligned}
 40. (a) \text{ Area region} &= \int_0^b [ab^n - ax^n] dx \\
 &= \left[ab^n x - a \frac{x^{n+1}}{n+1} \right]_0^b \\
 &= ab^{n+1} - a \frac{b^{n+1}}{n+1} \\
 &= ab^{n+1} \left(1 - \frac{1}{n+1} \right) = ab^{n+1} \left(\frac{n}{n+1} \right) \\
 R_1(n) &= \frac{ab^{n+1} \left(\frac{n}{n+1} \right)}{(ab^n)b} = \frac{n}{n+1}
 \end{aligned}$$

$$\begin{aligned}
 (b) \lim_{n \rightarrow \infty} R_1(n) &= \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \\
 \lim_{n \rightarrow \infty} (ab^n)b &= \infty
 \end{aligned}$$

(c) **Disk Method:**

$$\begin{aligned}
 V &= 2\pi \int_0^b x(ab^n - ax^n) dx \\
 &= 2\pi a \int_0^b (xb^n - x^{n+1}) dx \\
 &= 2\pi a \left[\frac{b^n}{2} x^2 - \frac{x^{n+2}}{n+2} \right]_0^b \\
 &= 2\pi a \left[\frac{b^{n+2}}{2} - \frac{b^{n+2}}{n+2} \right] = \pi ab^{n+2} \left(\frac{n}{n+2} \right) \\
 R_2(n) &= \frac{\pi ab^{n+2} \left(\frac{n}{n+2} \right)}{(\pi b^2)(ab^n)} = \left(\frac{n}{n+2} \right)
 \end{aligned}$$

$$(d) \lim_{n \rightarrow \infty} R_2(n) = \lim_{n \rightarrow \infty} \left(\frac{n}{n+2} \right) = 1$$

$$\lim_{n \rightarrow \infty} (\pi b^2)(ab^n) = \infty$$

(e) As $n \rightarrow \infty$, the graph approaches the line $x = 1$.

$$\begin{aligned}
 42. (a) V &= 2\pi \int_0^4 x f(x) dx \\
 &= \frac{2\pi(40)}{3(4)} [0 + 4(10)(45) + 2(20)(40) + 4(30)(20) + 0] \\
 &= \frac{20\pi}{3} [5800] \approx 121,475 \text{ cubic feet}
 \end{aligned}$$

$$(b) \text{ Top line: } y - 50 = \frac{40 - 50}{20 - 0}(x - 0) = -\frac{1}{2}x \Rightarrow y = -\frac{1}{2}x + 50$$

$$\text{Bottom line: } y - 40 = \frac{0 - 40}{40 - 20}(x - 20) = -2(x - 20) \Rightarrow y = -2x + 80$$

$$\begin{aligned}
 V &= 2\pi \int_0^{20} x \left(-\frac{1}{2}x + 50 \right) dx + 2\pi \int_{20}^{40} x(-2x + 80) dx \\
 &= 2\pi \int_0^{20} \left(-\frac{1}{2}x^2 + 50x \right) dx + 2\pi \int_{20}^{40} (-2x^2 + 80x) dx \\
 &= 2\pi \left[-\frac{x^3}{6} + 25x^2 \right]_0^{20} + 2\pi \left[-\frac{2x^3}{3} + 40x^2 \right]_{20}^{40} \\
 &= 2\pi \left[\frac{26,000}{3} \right] + 2\pi \left[\frac{32,000}{3} \right] \\
 &\approx 121,475 \text{ cubic feet}
 \end{aligned}$$

(Note that Simpson's Rule is exact for this problem.)

Section 6.4 Arc Length and Surfaces of Revolution

2. (1, 2), (7, 10)

(a) $d = \sqrt{(7-1)^2 + (10-2)^2} = 10$

(b) $y = \frac{4}{3}x + \frac{2}{3}$

$y' = \frac{4}{3}$

$$s = \int_1^7 \sqrt{1 + \left(\frac{4}{3}\right)^2} dx = \left[\frac{5}{3}x\right]_1^7 = 10$$

6. $y = \frac{3}{2}x^{2/3} + 4$

$y' = x^{-1/3}, [1, 27]$

$$\begin{aligned}
 s &= \int_1^{27} \sqrt{1 + \left(\frac{1}{x^{1/3}}\right)^2} dx \\
 &= \int_1^{27} \sqrt{\frac{x^{2/3} + 1}{x^{2/3}}} dx \\
 &= \frac{3}{2} \int_1^{27} \sqrt{x^{2/3} + 1} \left(\frac{2}{3x^{1/3}}\right) dx \\
 &= \left[\frac{3}{2} \cdot \frac{2}{3} (x^{2/3} + 1)^{3/2}\right]_1^{27} \\
 &= 10^{3/2} - 2^{3/2} \approx 28.794
 \end{aligned}$$

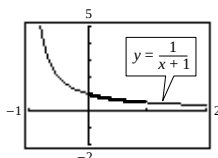
10. $y = \frac{1}{2}(e^x + e^{-x})$

$y' = \frac{1}{2}(e^x - e^{-x}), [0, 2]$

$1 + (y')^2 = \left[\frac{1}{2}(e^x + e^{-x})\right]^2, [0, 2]$

$$\begin{aligned}
 s &= \int_0^2 \sqrt{\left[\frac{1}{2}(e^x + e^{-x})\right]^2} dx \\
 &= \frac{1}{2} \int_0^2 (e^x + e^{-x}) dx \\
 &= \frac{1}{2} \left[e^x - e^{-x}\right]_0^2 = \frac{1}{2} \left(e^2 - \frac{1}{e^2}\right) \approx 3.627
 \end{aligned}$$

14. (a) $y = \frac{1}{1+x}, 0 \leq x \leq 1$



(b) $y' = -\frac{1}{(1+x)^2}$

(c) $L \approx 1.132$

$1 + (y')^2 = 1 + \frac{1}{(1+x)^4}$

$$L = \int_0^1 \sqrt{1 + \frac{1}{(1+x)^4}} dx$$

4. $y = 2x^{3/2} + 3$

$y' = 3x^{1/2}, [0, 9]$

$$\begin{aligned}
 s &= \int_0^9 \sqrt{1 + 9x} dx \\
 &= \left[\frac{2}{27}(1 + 9x)^{3/2}\right]_0^9 \\
 &= \frac{2}{27}(82^{3/2} - 1) \approx 54.929
 \end{aligned}$$

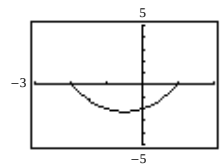
8. $y = \frac{x^5}{10} + \frac{1}{6x^3}$

$y' = \frac{1}{2}x^4 - \frac{1}{2x^4}$

$1 + (y')^2 = \left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right)^2, [1, 2]$

$$\begin{aligned}
 s &= \int_a^b \sqrt{1 + (y')^2} dx \\
 &= \int_1^2 \sqrt{\left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right)^2} dx \\
 &= \int_1^2 \left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right) dx \\
 &= \left[\frac{1}{10}x^5 - \frac{1}{6x^3}\right]_1^2 = \frac{779}{240} \approx 3.246
 \end{aligned}$$

12. (a) $y = x^2 + x - 2, -2 \leq x \leq 1$



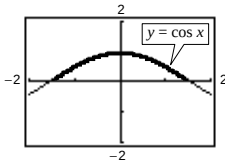
(b) $y' = 2x + 1$

$1 + (y')^2 = 1 + 4x^2 + 4x + 1$

$$L = \int_{-2}^1 \sqrt{2 + 4x + 4x^2} dx$$

(c) $L \approx 5.653$

16. (a) $y = \cos x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$



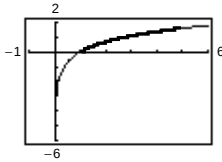
(b) $y' = -\sin x$

(c) 3.820

$$1 + (y')^2 = 1 + \sin^2 x$$

$$L = \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin^2 x} \, dx$$

18. (a) $y = \ln x, 1 \leq x \leq 5$



(b) $y' = \frac{1}{x}$

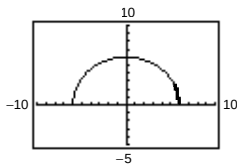
(c) $L \approx 4.367$

$$1 + (y')^2 = 1 + \frac{1}{x^2}$$

$$L = \int_1^5 \sqrt{1 + \frac{1}{x^2}} \, dx$$

20. (a) $x = \sqrt{36 - y^2}, 0 \leq y \leq 3$

$$y = \sqrt{36 - x^2}, 3\sqrt{3} \leq x \leq 6$$



(b) $\frac{dx}{dy} = \frac{1}{2}(36 - y^2)^{-1/2}(-2y)$

(c) $L \approx 3.142 (\pi!)$

$$= \frac{-y}{\sqrt{36 - y^2}}$$

$$L = \int_0^3 \sqrt{1 + \frac{y^2}{36 - y^2}} \, dy$$

$$= \int_0^3 \frac{6}{\sqrt{36 - y^2}} \, dy$$

Alternatively, you can convert to a function of x .

$$y = \sqrt{36 - x^2}$$

$$y' = \frac{dy}{dx} = -\frac{x}{\sqrt{36 - x^2}}$$

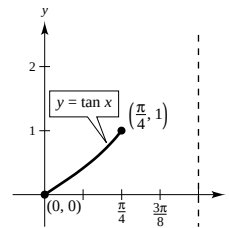
$$L = \int_{3\sqrt{3}}^6 \sqrt{1 + \frac{x^2}{36 - x^2}} \, dx = \int_{3\sqrt{3}}^6 \frac{6}{\sqrt{36 - x^2}} \, dx$$

Although this integral is undefined at $x = 0$, a graphing utility still gives $L \approx 3.142$.

22. $\int_0^{\pi/4} \sqrt{1 + \left[\frac{d}{dx}(\tan x)\right]^2} \, dx$

$$s \approx 1$$

Matches (e)



24. $f(x) = (x^2 - 4)^2, [0, 4]$

(a) $d = \sqrt{(4 - 0)^2 + (144 - 16)^2} \approx 128.062$

(b) $d = \sqrt{(1 - 0)^2 + (9 - 16)^2} + \sqrt{(2 - 1)^2 + (0 - 9)^2} + \sqrt{(3 - 2)^2 + (25 - 0)^2} + \sqrt{(4 - 3)^2 + (144 - 25)^2}$
 ≈ 160.151

(c) $s = \int_0^4 \sqrt{1 + [4x(x^2 - 4)]^2} \, dx \approx 159.087$

(d) 160.287

26. Let $y = \ln x$, $1 \leq x \leq e$, $y' = \frac{1}{x}$ and $L_1 = \int_1^e \sqrt{1 + \frac{1}{x^2}} dx$.

Equivalently, $x = e^y$, $0 \leq y \leq 1$, $\frac{dx}{dy} = e^y$, and $L_2 = \int_0^1 \sqrt{1 + e^{2y}} dy = \int_0^1 \sqrt{1 + e^{2x}} dx$.

Numerically, both integrals yield $L = 2.0035$

28. $y = 31 - 10(e^{x/20} + e^{-x/20})$

$$y' = -\frac{1}{2}(e^{x/20} - e^{-x/20})$$

$$1 + (y')^2 = 1 + \frac{1}{4}(e^{x/10} - 2 + e^{-x/10}) = \left[\frac{1}{2}(e^{x/20} + e^{-x/20}) \right]^2$$

$$\begin{aligned} s &= \int_{-20}^{20} \sqrt{\left[\frac{1}{2}(e^{x/20} + e^{-x/20}) \right]^2} dx \\ &= \frac{1}{2} \int_{-20}^{20} (e^{x/20} + e^{-x/20}) dx = \left[10(e^{x/20} - e^{-x/20}) \right]_{-20}^{20} = 20 \left(e - \frac{1}{e} \right) \approx 47 \text{ ft} \end{aligned}$$

Thus, there are $100(47) = 4700$ square feet of roofing on the barn.

30. $y = 693.8597 - 68.7672 \cosh 0.0100333x$

$$y' = -0.6899619478 \sinh 0.0100333x$$

$$s = \int_{-299.2239}^{299.2239} \sqrt{1 + (-0.6899619478 \sinh 0.0100333x)^2} dx \approx 1480$$

(Use Simpson's Rule with $n = 100$ or a graphing utility.)

32. $y = \sqrt{25 - x^2}$

$$y' = \frac{-x}{\sqrt{25 - x^2}}$$

$$1 + (y')^2 = \frac{25}{25 - x^2}$$

$$\begin{aligned} s &= \int_{-3}^4 \sqrt{\frac{25}{25 - x^2}} dx \\ &= \int_{-3}^4 \frac{5}{\sqrt{25 - x^2}} dx \\ &= \left[5 \arcsin \frac{x}{5} \right]_{-3}^4 \\ &= 5 \left[\arcsin \frac{4}{5} - \arcsin \left(-\frac{3}{5} \right) \right] \approx 7.8540 \end{aligned}$$

$$\frac{1}{4}[2\pi(5)] \approx 7.8540 = s$$

34. $y = 2\sqrt{x}$

$$y' = \frac{1}{\sqrt{x}}, [4, 9]$$

$$\begin{aligned} S &= 2\pi \int_4^9 2\sqrt{x} \sqrt{1 + \frac{1}{x}} dx \\ &= 4\pi \int_4^9 \sqrt{x+1} dx \\ &= \frac{8}{3} \pi (x+1)^{3/2} \Big|_4^9 \\ &= \frac{8\pi}{3} (10^{3/2} - 5^{3/2}) \approx 171.258 \end{aligned}$$

36. $y = \frac{x}{2}$

$$y' = \frac{1}{2}$$

$$1 + (y')^2 = \frac{5}{4}, [0, 6]$$

$$\begin{aligned} S &= 2\pi \int_0^6 \frac{x}{2} \sqrt{\frac{5}{4}} dx \\ &= \left[\frac{2\pi\sqrt{5}}{8} x^2 \right]_0^6 = 9\sqrt{5} \pi \end{aligned}$$

40. $y = \ln x$

$$y' = \frac{1}{x}$$

$$1 + (y')^2 = \frac{x^2 + 1}{x^2}, [1, e]$$

$$\begin{aligned} S &= 2\pi \int_1^e x \sqrt{\frac{x^2 + 1}{x^2}} dx \\ &= 2\pi \int_1^e \sqrt{x^2 + 1} dx \approx 22.943 \end{aligned}$$

44. The surface of revolution given by f_1 will be larger. $r(x)$ is larger for f_1 .

46. $y = \sqrt{r^2 - x^2}$

$$y' = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$1 + (y')^2 = \frac{r^2}{r^2 - x^2}$$

$$\begin{aligned} S &= 2\pi \int_{-r}^r \sqrt{r^2 - x^2} \sqrt{\frac{r^2}{r^2 - x^2}} dx \\ &= 2\pi \int_{-r}^r r dx = \left[2\pi r x \right]_{-r}^r = 4\pi r^2 \end{aligned}$$

38. $y = 9 - x^2, [0, 3]$

$$y' = -2x$$

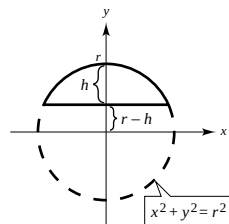
$$\begin{aligned} S &= 2\pi \int_0^3 x \sqrt{1 + 4x^2} dx \\ &= \frac{\pi}{4} \int_0^3 (1 + 4x^2)^{1/2} (8x) dx \\ &= \left[\frac{\pi}{6} (1 + 4x^2)^{3/2} \right]_0^3 \\ &= \frac{\pi}{6} (37^{3/2} - 1) \approx 117.319 \end{aligned}$$

42. The precalculus formula is the distance formula between two points. The representative element is

$$\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i} \right)^2} \Delta x_i.$$

48. From Exercise 47 we have:

$$\begin{aligned} S &= 2\pi \int_0^a \frac{rx}{\sqrt{r^2 - x^2}} dx \\ &= -r\pi \int_0^a \frac{-2x dx}{\sqrt{r^2 - x^2}} \\ &= \left[-2r\pi \sqrt{r^2 - x^2} \right]_0^a \\ &= 2r^2\pi - 2r\pi \sqrt{r^2 - a^2} \\ &= 2r\pi(r - \sqrt{r^2 - a^2}) \\ &= 2\pi rh \text{ (where } h \text{ is the height of the zone)} \end{aligned}$$

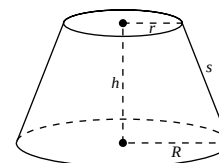


50. (a) We approximate the volume by summing 6 disks of thickness 3 and circumference C_i equal to the average of the given circumferences:

$$\begin{aligned} V &\approx \sum_{i=1}^6 \pi r_i^2(3) = \sum_{i=1}^6 \pi \left(\frac{C_i}{2\pi} \right)^2 (3) = \frac{3}{4\pi} \sum_{i=1}^6 C_i^2 \\ &= \frac{3}{4\pi} \left[\left(\frac{50 + 65.5}{2} \right)^2 + \left(\frac{65.5 + 70}{2} \right)^2 + \left(\frac{70 + 66}{2} \right)^2 + \left(\frac{66 + 58}{2} \right)^2 + \left(\frac{58 + 51}{2} \right)^2 + \left(\frac{51 + 48}{2} \right)^2 \right] \\ &= \frac{3}{4\pi} [57.75^2 + 67.75^2 + 68^2 + 62^2 + 54.5^2 + 49.5^2] \\ &= \frac{3}{4\pi} [21813.625] = 5207.62 \text{ cubic inches} \end{aligned}$$

- (b) The lateral surface area of a frustum of a right circular cone is $\pi s(R + r)$. For the first frustum,

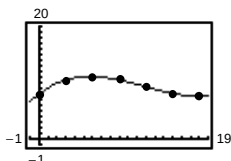
$$\begin{aligned} S_1 &\approx \pi \left[3^2 + \left(\frac{65.5 - 50}{2\pi} \right)^2 \right]^{1/2} \left[\frac{50}{2\pi} + \frac{65.5}{2\pi} \right] \\ &= \left(\frac{50 + 65.5}{2} \right) \left[9 + \left(\frac{65.5 - 50}{2\pi} \right)^2 \right]^{1/2}. \end{aligned}$$



Adding the six frustums together,

$$\begin{aligned} S &\approx \left(\frac{50 + 65.5}{2} \right) \left[9 + \left(\frac{15.5}{2\pi} \right)^2 \right]^{1/2} + \left(\frac{65.5 + 70}{2} \right) \left[9 + \left(\frac{4.5}{2\pi} \right)^2 \right]^{1/2} + \\ &\quad \left(\frac{70 + 66}{2} \right) \left[9 + \left(\frac{4}{2\pi} \right)^2 \right]^{1/2} + \left(\frac{66 + 58}{2} \right) \left[9 + \left(\frac{8}{2\pi} \right)^2 \right]^{1/2} + \\ &\quad \left(\frac{58 + 51}{2} \right) \left[9 + \left(\frac{7}{2\pi} \right)^2 \right]^{1/2} + \left(\frac{51 + 48}{2} \right) \left[9 + \left(\frac{3}{2\pi} \right)^2 \right]^{1/2} \\ &\approx 224.30 + 208.96 + 208.54 + 202.06 + 174.41 + 150.37 \\ &= 1168.64 \end{aligned}$$

- (c) $r = 0.00401y^3 - 0.1416y^2 + 1.232y + 7.943$



- (d) $V = \int_0^{18} \pi r^2 dy \approx 5275.9$ cubic inches

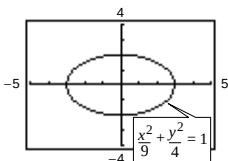
$$\begin{aligned} S &= \int_0^{18} 2\pi r(y) \sqrt{1 + r'(y)^2} dy \\ &\approx 1179.5 \text{ square inches} \end{aligned}$$

52. Individual project, see Exercise 50, 51.

54. (a) $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Ellipse: $y_1 = 2\sqrt{1 - \frac{x^2}{9}}$

$y_2 = -2\sqrt{1 - \frac{x^2}{9}}$



- (b) $y = 2\sqrt{1 - \frac{x^2}{9}}, 0 \leq x \leq 3$

$$\begin{aligned} y' &= 2 \left(\frac{1}{2} \right) \left(1 - \frac{x^2}{9} \right)^{-1/2} \left(-\frac{2x}{9} \right) \\ &= \frac{-2x}{9\sqrt{1 - \frac{x^2}{9}}} = \frac{-2x}{3\sqrt{9 - x^2}} \end{aligned}$$

$$L = \int_0^3 \sqrt{1 + \frac{4x^2}{81 - 9x^2}} dx$$

- (c) You cannot evaluate this definite integral, since the integrand is not defined at $x = 3$. Simpson's Rule will not work for the same reason. Also, the integrand does not have an elementary antiderivative.

56. Essay

Section 6.5 Work

2. $W = Fd = (2800)(4) = 11,200 \text{ ft} \cdot \text{lb}$

4. $W = Fd = [9(2000)]\left[\frac{1}{2}(5280)\right] = 47,520,000 \text{ ft} \cdot \text{lb}$

6. $W = \int_a^b F(x) dx$ is the work done by a force F moving an object along a straight line from $x = a$ to $x = b$.

8. (a) $W = \int_0^9 6 dx = 54 \text{ ft} \cdot \text{lbs}$

(b) $W = \int_0^7 20 dx + \int_7^9 (-10x + 90) dx = 140 + 20$
 $= 160 \text{ ft} \cdot \text{lbs}$

(c) $W = \int_0^9 \frac{1}{27} x^2 dx = \left[\frac{x^3}{81}\right]_0^9 = 9 \text{ ft} \cdot \text{lbs}$

(d) $W = \int_0^9 \sqrt{x} dx = \left[\frac{2}{3} x^{3/2}\right]_0^9 = \frac{2}{3}(27) = 18 \text{ ft} \cdot \text{lbs}$

10. $W = \int_0^{10} \frac{5}{4} x dx = \left[\frac{5}{8} x^2\right]_0^{10}$
 $= 40 \text{ in} \cdot \text{lb} \approx 3.33 \text{ ft} \cdot \text{lb}$

12. $F(x) = kx$

$$800 = k(70) \Rightarrow k = \frac{80}{7}$$

$$W = \int_0^{70} F(x) dx = \int_0^{70} \frac{80}{7} x dx = \left[\frac{40x^2}{7}\right]_0^{70}$$

 $= 28000 \text{ n} \cdot \text{cm} = 280 \text{ Nm}$

14. $F(x) = kx$

$$15 = k(1) = k$$

$$W = 2 \int_0^4 15x dx = \left[15x^2\right]_0^4$$

 $= 240 \text{ ft} \cdot \text{lb}$

16. $W = 7.5 = \int_0^{1/6} kx dx = \left[\frac{kx^2}{2}\right]_0^{1/6} = \frac{k}{72} \Rightarrow k = 540$

$$W = \int_{1/6}^{5/24} 540x dx = 270x^2 \Big|_{1/6}^{5/24} = 4.21875 \text{ ft} \cdot \text{lbs}$$

18. $W = \int_{4000}^h \frac{80,000,000}{x^2} dx = \left[-\frac{80,000,000}{x}\right]_{4000}^h$
 $= \frac{-80,000,000}{h} + 20,000$

$$\lim_{h \rightarrow \infty} W = 20,000 \text{ mi/ton} \approx 2.1 \times 10^{11} \text{ ft} \cdot \text{lb}$$

20. Weight on surface of moon: $\frac{1}{6}(12) = 2 \text{ tons}$

Weight varies inversely as the square of distance from the center of the moon. Therefore,

$$F(x) = \frac{k}{x^2}$$

$$2 = \frac{k}{(1100)^2}$$

$$k = 2.42 \times 10^6$$

$$W = \int_{1100}^{1150} \frac{2.42 \times 10^6}{x^2} dx = \left[\frac{-2.42 \times 10^6}{x}\right]_{1100}^{1150} = 2.42 \times 10^6 \left(\frac{1}{1100} - \frac{1}{1150}\right)$$

 $\approx 95.652 \text{ mi} \cdot \text{ton} \approx 1.01 \times 10^9 \text{ ft} \cdot \text{lb}$

22. The bottom half had to be pumped a greater distance than the top half.

24. Volume of disk: $4\pi \Delta y$

Weight of disk: $9800(4\pi) \Delta y$

Distance the disk of water is moved: y

$$\begin{aligned} W &= \int_{10}^{12} y(9800)(4\pi) dy = 39,200\pi \left[\frac{y^2}{2} \right]_{10}^{12} \\ &= 39,200\pi(22) \\ &= 862,400\pi \text{ newton-meters} \end{aligned}$$

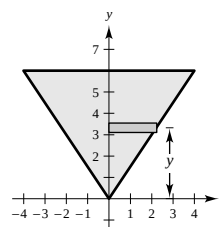
26. Volume of disk: $\pi \left(\frac{2}{3}y \right)^2 \Delta y$

Weight of disk: $62.4\pi \left(\frac{2}{3}y \right)^2 \Delta y$

Distance: y

$$(a) \quad W = \frac{4}{9}(62.4)\pi \int_0^2 y^3 dy = \left[\frac{4}{9}(62.4)\pi \left(\frac{1}{4}y^4 \right) \right]_0^2 \approx 110.9\pi \text{ ft} \cdot \text{lb}$$

$$(b) \quad W = \frac{4}{9}(62.4)\pi \int_4^6 y^3 dy = \left[\frac{4}{9}(62.4)\pi \left(\frac{1}{4}y^4 \right) \right]_4^6 \approx 7210.7\pi \text{ ft} \cdot \text{lb}$$

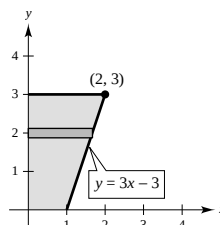


28. Volume of each layer: $\frac{y+3}{3}(3) \Delta y = (y+3) \Delta y$

Weight of each layer: $55.6(y+3) \Delta y$

Distance: $6-y$

$$\begin{aligned} W &= \int_0^3 55.6(6-y)(y+3) dy = 55.6 \int_0^3 (18+3y-y^2) dy \\ &= 55.6 \left[18y + \frac{3y^2}{2} - \frac{y^3}{3} \right]_0^3 \\ &= 3252.6 \text{ ft} \cdot \text{lb} \end{aligned}$$



30. Volume of layer: $V = 12(2)\sqrt{(25/4) - y^2} \Delta y$

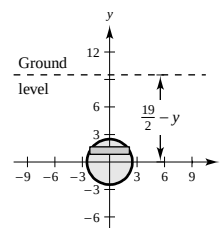
Weight of layer: $W = 42(24)\sqrt{(25/4) - y^2} \Delta y$

Distance: $\frac{19}{2} - y$

$$\begin{aligned} W &= \int_{-2.5}^{2.5} 42(24)\sqrt{\frac{25}{4} - y^2} \left(\frac{19}{2} - y \right) dy \\ &= 1008 \left[\frac{19}{2} \int_{-2.5}^{2.5} \sqrt{\frac{25}{4} - y^2} dy + \int_{-2.5}^{2.5} \sqrt{\frac{25}{4} - y^2} (-y) dy \right] \end{aligned}$$

The second integral is zero since the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius $\frac{5}{2}$. Thus, the work is

$$W = 1008 \left(\frac{19}{2} \right) \pi \left(\frac{5}{2} \right)^2 \left(\frac{1}{2} \right) = 29,925\pi \text{ ft} \cdot \text{lb} \approx 94,012.16 \text{ ft} \cdot \text{lb}.$$



32. The lower 10 feet of chain are raised 5 feet with a constant force.

$$W_1 = 3(10)5 = 150 \text{ ft} \cdot \text{lb}$$

The top 5 feet will be raised with variable force.

Weight of section: $3 \Delta y$

Distance: $5 - y$

$$W_2 = 3 \int_0^5 (5 - y) dy = \left[-\frac{3}{2}(5 - y)^2 \right]_0^5 = \frac{75}{2} \text{ ft} \cdot \text{lb}$$

$$W = W_1 + W_2 = 150 + \frac{75}{2} = \frac{375}{2} \text{ ft} \cdot \text{lb}$$

34. The work required to lift the chain is $337.5 \text{ ft} \cdot \text{lb}$ (from Exercise 31). The work required to lift the 500-pound load is $W = (500)(15) = 7500$. The work required to lift the chain with a 100-pound load attached is

$$W = 337.5 + 7500 = 7837.5 \text{ ft} \cdot \text{lbs}$$

$$36. W = 3 \int_0^6 (12 - 2y) dy = \left[-\frac{3}{4}(12 - 2y)^2 \right]_0^6 = \frac{3}{4}(12)^2 = 108 \text{ ft} \cdot \text{lb}$$

38. Work to pull up the ball: $W_1 = 500(40) = 20,000 \text{ ft} \cdot \text{lb}$

Work to pull up the cable: force is variable

Weight per section: $1 \Delta y$

Distance: $40 - x$

$$W_2 = \int_0^{40} (40 - x) dx = \left[-\frac{1}{2}(40 - x)^2 \right]_0^{40}$$

$$= 800 \text{ ft} \cdot \text{lb}$$

$$W = W_1 + W_2 = 20,000 + 800 = 20,800 \text{ ft} \cdot \text{lb}$$

$$40. p = \frac{k}{V}$$

$$2500 = \frac{k}{1} \Rightarrow k = 2500$$

$$W = \int_1^3 \frac{2500}{V} dV = \left[2500 \ln V \right]_1^3$$

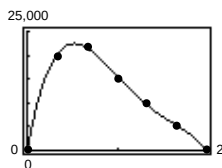
$$= 2500 \ln 3 \approx 2746.53 \text{ ft} \cdot \text{lb}$$

42. (a) $W = FD = (8000\pi)(2) = 16,000\pi \text{ ft} \cdot \text{lbs}$

$$(b) W \approx \frac{2 - 0}{3(6)} [0 + 4(20,000) + 2(22,000) + 4(15,000) + 2(10,000) + 4(5000) + 0]$$

$$\approx 24,88.889 \text{ ft} \cdot \text{lb}$$

$$(c) F(x) = -16,261.36x^4 + 85,295.45x^3 - 157,738.64x^2 + 104,386.36x - 32.4675$$



- (d) $F(x) = 0$ when $x \approx 0.524$ feet. $F(x)$ is a maximum when $x \approx 0.524$ feet.

$$(e) W = \int_0^2 F(x) dx \approx 25,180.5 \text{ ft} \cdot \text{lbs}$$

$$44. W = \int_0^4 \left(\frac{e^{x^2} - 1}{100} \right) dx \approx 11,494 \text{ ft} \cdot \text{lb}$$

$$46. W = \int_0^2 1000 \sinh x dx \approx 2762.2 \text{ ft} \cdot \text{lb}$$

Section 6.6 Moments, Centers of Mass, and Centroids

$$2. \bar{x} = \frac{7(-3) + 4(-2) + 3(5) + 8(6)}{7 + 4 + 3 + 8} = \frac{17}{11}$$

6. The center of mass is translated k units as well.

$$4. \bar{x} = \frac{12(-6) + 1(-4) + 6(-2) + 3(0) + 11(8)}{12 + 1 + 6 + 3 + 11} = 0$$

$$8. 200x = 550(5 - x) \text{ (Person on left)}$$

$$200x = 2750 - 550x$$

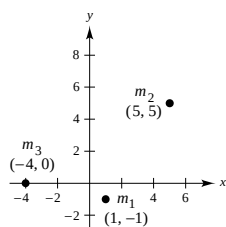
$$750x = 2750$$

$$x = 3\frac{2}{3} \text{ feet}$$

$$10. \bar{x} = \frac{10(1) + 2(5) + 5(-4)}{10 + 2 + 5} = 0$$

$$\bar{y} = \frac{10(-1) + 2(5) + 5(0)}{10 + 2 + 5} = 0$$

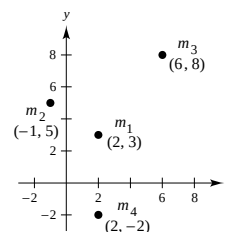
$$(\bar{x}, \bar{y}) = (0, 0)$$



$$12. \bar{x} = \frac{12(2) + 6(-1) + \frac{15}{2}(6) + 15(2)}{12 + 6 + \frac{15}{2} + 15} = \frac{93}{40.5} = \frac{62}{27}$$

$$\bar{y} = \frac{12(3) + 6(5) + \frac{15}{2}(8) + 15(-2)}{12 + 6 + \frac{15}{2} + 15} = \frac{96}{40.5} = \frac{64}{27}$$

$$(\bar{x}, \bar{y}) = \left(\frac{62}{27}, \frac{64}{27}\right)$$



$$14. m = \rho \int_0^2 \frac{1}{2}x^2 dx = \left[\rho \frac{x^3}{6}\right]_0^2 = \frac{4}{3}\rho$$

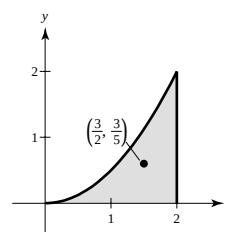
$$M_x = \rho \int_0^2 \frac{1}{2} \left(\frac{1}{2}x^2\right) \left(\frac{1}{2}x^2\right) dx = \frac{\rho}{8} \int_0^2 x^4 dx = \left[\frac{\rho}{40}x^5\right]_0^2 = \frac{32}{40}\rho = \frac{4}{5}\rho$$

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{4}{5}\rho}{\frac{4}{3}\rho} = \frac{3}{5}$$

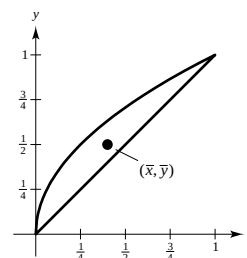
$$M_y = \rho \int_0^2 x \left(\frac{1}{2}x^2\right) dx = \frac{1}{2}\rho \int_0^2 x^3 dx = \left[\frac{\rho}{8}x^4\right]_0^2 = 2\rho$$

$$\bar{x} = \frac{M_y}{m} = \frac{2\rho}{\frac{4}{3}\rho} = \frac{3}{2}$$

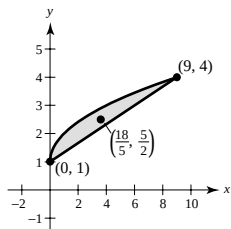
$$(\bar{x}, \bar{y}) = \left(\frac{3}{2}, \frac{3}{5}\right)$$



$$\begin{aligned}
 16. \quad m &= \rho \int_0^1 (\sqrt{x} - x) dx = \rho \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2} \right]_0^1 = \frac{\rho}{6} \\
 M_x &= \rho \int_0^1 \frac{(\sqrt{x} + x)}{2} (\sqrt{x} - x) dx = \frac{\rho}{2} \int_0^1 (x - x^2) dx = \frac{\rho}{2} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{\rho}{12} \\
 \bar{y} &= \frac{M_x}{m} = \frac{\rho}{12} \left(\frac{6}{\rho} \right) = \frac{1}{2} \\
 M_y &= \rho \int_0^1 x(\sqrt{x} - x) dx = \rho \int_0^1 (x^{3/2} - x^2) dx = \rho \left[\frac{2}{5} x^{5/2} - \frac{x^3}{3} \right]_0^1 = \frac{\rho}{15} \\
 \bar{x} &= \frac{M_y}{m} = \frac{\rho}{15} \left(\frac{6}{\rho} \right) = \frac{2}{5} \\
 (\bar{x}, \bar{y}) &= \left(\frac{2}{5}, \frac{1}{2} \right)
 \end{aligned}$$



$$\begin{aligned}
 18. \quad m &= \rho \int_0^9 \left[(\sqrt{x} + 1) - \left(\frac{1}{3}x + 1 \right) \right] dx = \rho \int_0^9 \left(\sqrt{x} - \frac{1}{3}x \right) dx \\
 &= \rho \left[\frac{2}{3} x^{3/2} - \frac{x^2}{6} \right]_0^9 = \rho \left(18 - \frac{27}{2} \right) = \frac{9}{2} \rho \\
 M_x &= \rho \int_0^9 \frac{\sqrt{x} + 1 + \frac{1}{3}x + 1}{2} \left(\sqrt{x} + 1 - \frac{1}{3}x - 1 \right) dx = \frac{\rho}{2} \int_0^9 \left(\sqrt{x} + \frac{1}{3}x + 2 \right) \left(\sqrt{x} - \frac{1}{3}x \right) dx \\
 &= \frac{\rho}{2} \int_0^9 \left(x - \frac{1}{3}x^{3/2} + \frac{1}{3}x^{3/2} - \frac{1}{9}x^2 + 2\sqrt{x} - \frac{2}{3}x \right) dx = \frac{\rho}{2} \int_0^9 \left(\frac{1}{3}x - \frac{1}{9}x^2 + 2\sqrt{x} \right) dx \\
 &= \frac{\rho}{2} \left[\frac{x^2}{6} - \frac{x^3}{27} + \frac{4}{3}x^{3/2} \right]_0^9 = \frac{\rho}{2} \left[\frac{27}{2} - 27 + 36 \right] = \frac{45}{3} \rho \\
 M_y &= \rho \int_0^9 x \left[\sqrt{x} + 1 - \frac{1}{3}x - 1 \right] dx = \rho \int_0^9 \left(x^{3/2} - \frac{1}{3}x^2 \right) dx = \rho \left[\frac{2}{5} x^{5/2} - \frac{1}{9}x^3 \right]_0^9 \\
 &= \rho \left[\frac{486}{5} - 81 \right] = \frac{81}{5} \rho \\
 \bar{x} &= \frac{M_y}{m} = \frac{\frac{81}{5} \rho}{\frac{9}{2} \rho} = \frac{18}{5}; \quad \bar{y} = \frac{M_x}{m} = \frac{\frac{45}{3} \rho}{\frac{9}{2} \rho} = \frac{5}{2} \\
 (\bar{x}, \bar{y}) &= \left(\frac{18}{5}, \frac{5}{2} \right)
 \end{aligned}$$



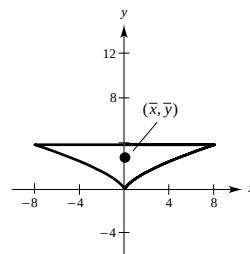
$$20. \quad m = 2\rho \int_0^8 (4 - x^{2/3}) dx = 2\rho \left[4x - \frac{3}{5}x^{5/3} \right]_0^8 = \frac{128\rho}{5}$$

By symmetry, M_y and $\bar{x} = 0$.

$$M_x = 2\rho \int_0^8 \left(\frac{4 + x^{2/3}}{2} \right) (4 - x^{2/3}) dx = \rho \left[16x - \frac{3}{7}x^{7/3} \right]_0^8 = \frac{512\rho}{7}$$

$$\bar{y} = \frac{512\rho}{7} \left(\frac{5}{128\rho} \right) = \frac{20}{7}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{20}{7} \right)$$



$$22. \quad m = \rho \int_0^2 (2y - y^2) dy = \rho \left[y^2 - \frac{y^3}{3} \right]_0^2 = \frac{4\rho}{3}$$

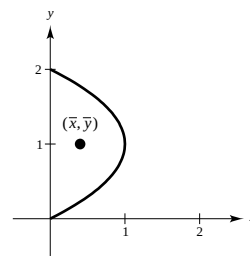
$$M_y = \rho \int_0^2 \left(\frac{2y - y^2}{2} \right) (2y - y^2) dy = \frac{\rho}{2} \left[\frac{4y^3}{3} - y^4 + \frac{y^5}{5} \right]_0^2 = \frac{8\rho}{15}$$

$$\bar{x} = \frac{M_y}{m} = \frac{8\rho}{15} \left(\frac{3}{4\rho} \right) = \frac{2}{5}$$

$$M_x = \rho \int_0^2 y(2y - y^2) dy = \rho \left[\frac{2y^3}{3} - \frac{y^4}{4} \right]_0^2 = \frac{4\rho}{3}$$

$$\bar{y} = \frac{M_x}{m} = \frac{4\rho}{3} \left(\frac{3}{4\rho} \right) = 1$$

$$(\bar{x}, \bar{y}) = \left(\frac{2}{5}, 1 \right)$$



$$24. \quad m = \rho \int_{-1}^2 [(y+2) - y^2] dy = \rho \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^2 = \frac{9\rho}{2}$$

$$M_y = \rho \int_{-1}^2 \frac{[(y+2) + y^2]}{2} [(y+2) - y^2] dy$$

$$= \frac{\rho}{2} \int_{-1}^2 [(y+2)^2 - y^4] dy = \frac{\rho}{2} \left[\frac{(y+2)^3}{3} - \frac{y^5}{5} \right]_{-1}^2 = \frac{36\rho}{5}$$

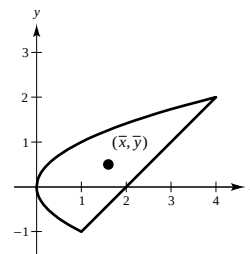
$$\bar{x} = \frac{M_y}{m} = \frac{36\rho}{5} \left(\frac{2}{9\rho} \right) = \frac{8}{5}$$

$$M_x = \rho \int_{-1}^2 y[(y+2) - y^2] dy$$

$$= \rho \int_{-1}^2 (2y + y^2 - y^3) dy = \rho \left[y^2 + \frac{y^3}{3} - \frac{y^4}{4} \right]_{-1}^2 = \frac{9\rho}{4}$$

$$\bar{y} = \frac{M_x}{m} = \frac{9\rho}{4} \left(\frac{2}{9\rho} \right) = \frac{1}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{5}, \frac{1}{2} \right)$$



$$26. \quad A = \int_1^4 \frac{1}{x} dx = \left[\ln|x| \right]_1^4 = \ln 4$$

$$M_x = \frac{1}{2} \int_1^4 \frac{1}{x^2} dx = \left[\frac{1}{2} \left(-\frac{1}{x} \right) \right]_1^4 = \left(-\frac{1}{8} + \frac{1}{2} \right) = \frac{3}{8}$$

$$M_y = \int_1^4 x \left(\frac{1}{x} \right) dx = \left[x \right]_1^4 = 3$$

$$28. \quad A = \int_{-2}^2 -(x^2 - 4) dx = 2 \int_0^2 (4 - x^2) dx = \left[8x - \frac{2x^3}{3} \right]_0^2 = 16 - \frac{16}{3} = \frac{32}{3}$$

$$\begin{aligned} M_x &= \frac{1}{2} \int_{-2}^2 (x^2 - 4)(4 - x^2) dx = -\frac{1}{2} \int_{-2}^2 (x^4 - 8x^2 + 16) dx \\ &= -\frac{1}{2} \left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x \right]_{-2}^2 = -\left[\frac{32}{5} - \frac{64}{3} + 32 \right] = -\frac{256}{15} \end{aligned}$$

$M_y = 0$ by symmetry.

$$30. \quad m = \rho \int_0^4 x e^{-x/2} dx \approx 2.3760\rho$$

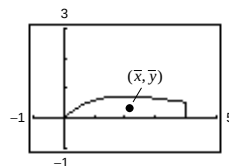
$$M_x = \rho \int_0^4 \left(\frac{x e^{-x/2}}{2} \right) (x e^{-x/2}) dx = \frac{\rho}{2} \int_0^4 x^2 e^{-x} dx \approx 0.7619\rho$$

$$M_y = \rho \int_0^4 x^2 e^{-x/2} dx \approx 5.1732\rho$$

$$\bar{x} = \frac{M_y}{m} \approx 2.2$$

$$\bar{y} = \frac{M_x}{m} \approx 0.3$$

Therefore, the centroid is (2.2, 0.3).

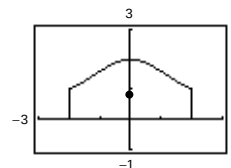


$$32. \quad m = \rho \int_{-2}^2 \frac{8}{x^2 + 4} dx \approx 6.2832\rho$$

$$M_x = \rho \int_{-2}^2 \frac{1}{2} \left(\frac{8}{x^2 + 4} \right) \left(\frac{8}{x^2 + 4} \right) dx = 32\rho \int_{-2}^2 \frac{1}{(x^2 + 4)^2} dx \approx 5.14149\rho$$

$$\bar{y} = \frac{M_x}{m} \approx 0.8$$

$\bar{x} = 0$ by symmetry. Therefore, the centroid is (0, 0.8).



$$34. \quad A = bh = ac$$

$$\frac{1}{A} = \frac{1}{ac}$$

$$\bar{x} = \frac{1}{ac} \frac{1}{2} \int_0^c \left[\left(\frac{b}{c}y + a \right)^2 - \left(\frac{b}{c}y \right)^2 \right] dy$$

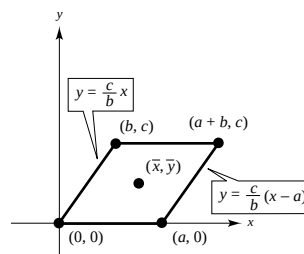
$$= \frac{1}{2ac} \int_0^c \left(\frac{2ab}{c}y + a^2 \right) dy$$

$$= \frac{1}{2ac} \left[\frac{ab}{c}y^2 + a^2y \right]_0^c = \frac{1}{2ac} [abc + a^2c] = \frac{1}{2}(b + a)$$

$$\bar{y} = \frac{1}{ac} \int_0^c y \left[\left(\frac{b}{c}y + a \right) - \left(\frac{b}{c}y \right) \right] dy = \left[\frac{1}{c} \frac{y^2}{2} \right]_0^c = \frac{c}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b + a}{2}, \frac{c}{2} \right)$$

This is the point of intersection of the diagonals.



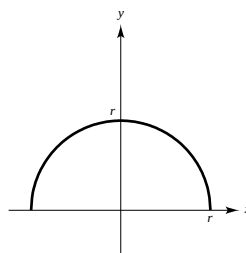
36. $\bar{x} = 0$ by symmetry

$$A = \frac{1}{2}\pi r^2$$

$$\frac{1}{A} = \frac{2}{\pi r^2}$$

$$\begin{aligned}\bar{y} &= \frac{2}{\pi r^2} \frac{1}{2} \int_{-r}^r (\sqrt{r^2 - x^2})^2 dx \\ &= \frac{1}{\pi r^2} \left[r^2 x - \frac{x^3}{3} \right]_{-r}^r = \frac{1}{\pi r^2} \left[\frac{4r^3}{3} \right] = \frac{4r}{3\pi}\end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4r}{3\pi} \right)$$



$$38. \quad A = \int_0^1 [1 - (2x - x^2)] dx = \frac{1}{3}$$

$$\frac{1}{A} = 3$$

$$\bar{x} = 3 \int_0^1 x[1 - (2x - x^2)] dx = 3 \int_0^1 [x - 2x^2 + x^3] dx = 3 \left[\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4} \right]_0^1 = \frac{1}{4}$$

$$\begin{aligned}\bar{y} &= 3 \int_0^1 \frac{[1 + (2x - x^2)]}{2} [1 - (2x - x^2)] dx = \frac{3}{2} \int_0^1 [1 - (2x - x^2)^2] dx \\ &= \frac{3}{2} \int_0^1 [1 - 4x^2 + 4x^3 - x^4] dx = \frac{3}{2} \left[x - \frac{4}{3}x^3 + x^4 - \frac{x^5}{5} \right]_0^1 = \frac{7}{10}\end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(\frac{1}{4}, \frac{7}{10} \right)$$

40. (a) $M_y = 0$ by symmetry

$$M_y = \int_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}} x(b - x^{2n}) dx = 0$$

because $bx - x^{2n+1}$ is an odd function.

$$(c) \quad M_x = \int_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}} \frac{(b + x^{2n})(b - x^{2n})}{2} dx = \int_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}} \frac{1}{2}(b^2 - x^{4n}) dx$$

$$= \frac{1}{2} \left(b^2 x - \frac{x^{4n+1}}{4n+1} \right) \Big|_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}}$$

$$= b^2 b^{1/2n} - \frac{b^{(4n+1)/2n}}{4n+1} = \frac{4n}{4n+1} b^{(4n+1)/2n}$$

$$A = \int_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}} (b - x^{2n}) dx = 2 \left[bx - \frac{x^{2n+1}}{2n+1} \right]_0^{2\sqrt[n]{b}}$$

$$= 2 \left[b \cdot b^{1/2n} - \frac{b^{(2n+1)/2n}}{2n+1} \right] = \frac{4n}{2n+1} b^{(2n+1)/2n}$$

$$\bar{y} = \frac{M_x}{A} = \frac{4n b^{(4n+1)/2n} / (4n+1)}{4n b^{(2n+1)/2n} / (2n+1)} = \frac{2n+1}{4n+1} b$$

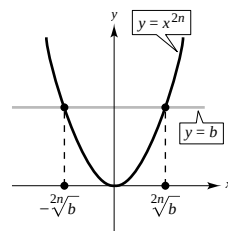
(b) $\bar{y} > \frac{b}{2}$ because there is more area above $y = \frac{b}{2}$ than below.

(d)

n	1	2	3	4
$\bar{y}$	$\frac{3}{5}b$	$\frac{5}{9}b$	$\frac{7}{13}b$	$\frac{9}{17}b$

$$(e) \quad \lim_{n \rightarrow \infty} \bar{y} = \lim_{n \rightarrow \infty} \frac{2n+1}{4n+1} b = \frac{1}{2} b$$

(f) As $n \rightarrow \infty$, the figure gets narrower.

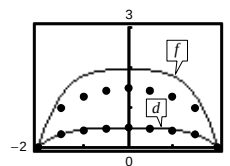


42. Let $f(x)$ be the top curve, given by $l + d$. The bottom curve is $d(x)$.

x	0	0.5	1.0	1.5	2.0
f	2.0	1.93	1.73	1.32	0
d	0.50	0.48	0.43	0.33	0

$$\begin{aligned}
 \text{(a) Area} &= 2 \int_0^2 [f(x) - d(x)] dx \\
 &\approx 2 \frac{2}{3(4)} [1.50 + 4(1.45) + 2(1.30) + 4(.99) + 0] \\
 &= \frac{1}{3} [13.86] = 4.62 \\
 M_x &= \int_{-2}^2 \frac{f(x) + d(x)}{2} (f(x) - d(x)) dx \\
 &= \int_0^2 [f(x)^2 - d(x)^2] dx \\
 &= \frac{2}{3(4)} [3.75 + 4(3.4945) + 2(2.808) + 4(1.6335) + 0] \\
 &= \frac{1}{6} [29.878] = 4.9797 \\
 \bar{y} &= \frac{M_x}{A} = \frac{4.9797}{4.62} = 1.078 \\
 (\bar{x}, \bar{y}) &= (0, 1.078)
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } f(x) &= -0.1061x^4 - 0.06126x^2 + 1.9527 \\
 d(x) &= -0.02648x^4 - 0.01497x^2 + .4862 \\
 \text{(c) } \bar{y} &= \frac{M_x}{A} \approx \frac{4.9133}{4.59998} = 1.068 \\
 (\bar{x}, \bar{y}) &= (0, 1.068)
 \end{aligned}$$



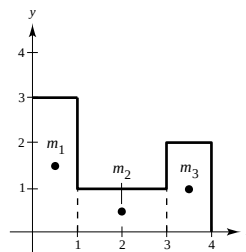
44. Centroids of the given regions: $\left(\frac{1}{2}, \frac{3}{2}\right)$, $\left(2, \frac{1}{2}\right)$, and $\left(\frac{7}{2}, 1\right)$

$$\text{Area: } A = 3 + 2 + 2 = 7$$

$$\bar{x} = \frac{3(1/2) + 2(2) + 2(7/2)}{7} = \frac{25/2}{7} = \frac{25}{14}$$

$$\bar{y} = \frac{3(3/2) + 2(1/2) + 2(1)}{7} = \frac{15/2}{7} = \frac{15}{14}$$

$$(\bar{x}, \bar{y}) = \left(\frac{25}{14}, \frac{15}{14}\right)$$



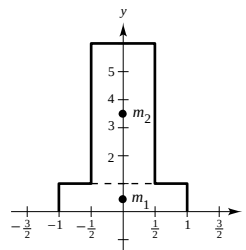
$$46. m_1 = \frac{7}{8}(2) = \frac{7}{4}, P_1 = \left(0, \frac{7}{16}\right)$$

$$m_2 = \frac{7}{8}\left(6 - \frac{7}{8}\right) = \frac{287}{64}, P_2 = \left(0, \frac{55}{16}\right)$$

By symmetry, $\bar{x} = 0$.

$$\bar{y} = \frac{(7/4)(7/16) + (287/64)(55/16)}{(7/4) + (287/64)} = \frac{16,569}{6384} = \frac{5523}{2128}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{5523}{2128}\right) \approx (0, 2.595)$$



48. Centroids of the given regions: (3, 0) and (1, 0)

Mass: $8 + \pi$

$$\bar{y} = 0$$

$$\bar{x} = \frac{8(1) + \pi(3)}{8 + \pi} = \frac{8 + 3\pi}{8 + \pi}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8 + 3\pi}{8 + \pi}, 0 \right) \approx (1.56, 0)$$

50. $V = 2\pi rA = 2\pi(3)(4\pi) = 24\pi^2$

52. $A = \int_2^6 2\sqrt{x-2} \, dx = \frac{4}{3}(x-2)^{3/2} \Big|_2^6 = \frac{32}{3}$

$$M_y = \int_2^6 (x)2\sqrt{x-2} \, dx = 2 \int_2^6 x\sqrt{x-2} \, dx$$

Let $u = x - 2$, $x = u + 2$, $du = dx$:

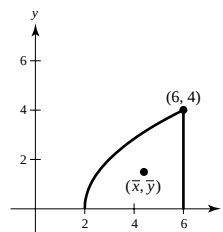
$$M_y = 2 \int_0^4 (u+2)\sqrt{u} \, du = 2 \int_0^4 (u^{3/2} + 2u^{1/2}) \, du = 2 \left[\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2} \right]_0^4$$

$$= 2 \left[\frac{64}{5} + \frac{32}{3} \right] = \frac{704}{15}$$

$$\bar{x} = \frac{M_y}{A} = \frac{704/15}{32/3} = \frac{22}{5}$$

$$r = \bar{x} = \frac{22}{5}$$

$$V = 2\pi rA = 2\pi \left(\frac{22}{5} \right) \left(\frac{32}{3} \right) = \frac{1408\pi}{15} \approx 294.89$$



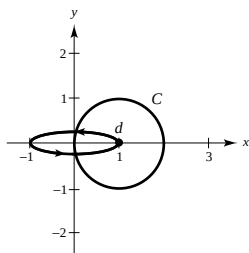
54. A planar lamina is a thin flat plate of constant density. The center of mass
- $(\bar{x}, \bar{y})$
- is the balancing point on the lamina.

56. Let
- R
- be a region in a plane and let
- L
- be a line such that
- L
- does not intersect the interior of
- R
- . If
- r
- is the distance between the centroid of
- R
- and
- L
- , then the volume
- V
- of the solid of revolution formed by revolving
- R
- about
- L
- is

$$V = 2\pi rA$$

where A is the area of R .

58. The centroid of the circle is (1, 0). The distance traveled by the centroid is
- 2π
- . The arc length of the circle is also
- 2π
- . Therefore,
- $S = (2\pi)(2\pi) = 4\pi^2$
- .



Section 6.7 Fluid Pressure and Fluid Force

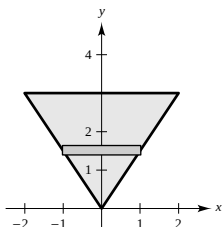
$$2. F = PA = [62.4(5)](16) = 4992 \text{ lb}$$

$$6. h(y) = 3 - y$$

$$L(y) = \frac{4}{3}y$$

$$\begin{aligned} F &= 62.4 \int_0^3 (3 - y) \left(\frac{4}{3}y \right) dy \\ &= \frac{4}{3}(62.4) \int_0^3 (3y - y^2) dy \\ &= \frac{4}{3}(62.4) \left[\frac{3y^2}{2} - \frac{y^3}{3} \right]_0^3 = 374.4 \text{ lb} \end{aligned}$$

Force is one-third that of Exercise 5.

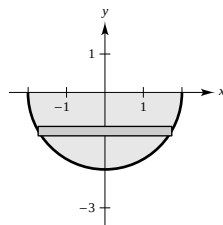


$$\begin{aligned} 4. F &= 62.4(h + 4)(48) - (62.4)(h)(48) \\ &= 62.4(4)(48) = 11,980.8 \text{ lb} \end{aligned}$$

$$8. h(y) = -y$$

$$L(y) = 2\sqrt{4 - y^2}$$

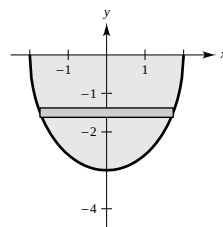
$$\begin{aligned} F &= 62.4 \int_{-2}^0 (-y)(2)\sqrt{4 - y^2} dy \\ &= \left[62.4 \left(\frac{2}{3} \right) (4 - y^2)^{3/2} \right]_{-2}^0 = 332.8 \text{ lb} \end{aligned}$$



$$10. h(y) = -y$$

$$L(y) = \frac{4}{3}\sqrt{9 - y^2}$$

$$\begin{aligned} F &= 62.4 \int_{-3}^0 (-y) \frac{4}{3} \sqrt{9 - y^2} dy \\ &= 62.4 \left(\frac{2}{3} \right) \int_{-3}^0 (9 - y^2)^{1/2} (-2y) dy \\ &= \left[62.4 \left(\frac{4}{9} \right) (9 - y^2)^{3/2} \right]_{-3}^0 = 748.8 \text{ lb} \end{aligned}$$

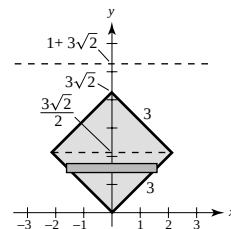


$$12. h(y) = (1 + 3\sqrt{2}) - y$$

$$L_1(y) = 2y \quad (\text{lower part})$$

$$L_2(y) = 2(3\sqrt{2} - y) \quad (\text{upper part})$$

$$\begin{aligned} F &= 2(9800) \left[\int_0^{3\sqrt{2}/2} (1 + 3\sqrt{2} - y)y dy + \int_{3\sqrt{2}/2}^{3\sqrt{2}} (1 + 3\sqrt{2} - y)(3\sqrt{2} - y) dy \right] \\ &= 19,600 \left[\left[\frac{y^2}{2} - 3\sqrt{2}y - \frac{y^3}{3} \right]_0^{3\sqrt{2}/2} + \left[3\sqrt{2}y + 18y + \frac{y^3}{3} - \frac{6\sqrt{2} + 1}{2}y \right]_{3\sqrt{2}/2}^{3\sqrt{2}} \right] \\ &= 19,600 \left[\frac{9(2\sqrt{2} + 1)}{4} + \frac{9(\sqrt{2} + 1)}{4} \right] \\ &= 44,100(3\sqrt{2} + 2) \text{ Newtons} \end{aligned}$$

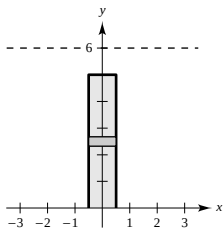


14. $h(y) = 6 - y$

$L(y) = 1$

$$F = 9800 \int_0^5 1(6 - y) dy$$

$$= 9800 \left[6y - \frac{y^2}{2} \right]_0^5 = 171,500 \text{ Newtons}$$



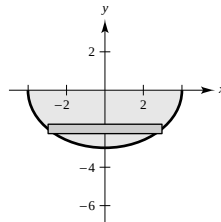
16. $h(y) = -y$

$L(y) = 2\left(\frac{4}{3}\sqrt{9 - y^2}\right)$

$$F = 140.7 \int_{-3}^0 (-y)(2)\left(\frac{4}{3}\sqrt{9 - y^2}\right) dy$$

$$= \frac{(140.7)(4)}{3} \int_{-3}^0 \sqrt{9 - y^2} (-2y) dy$$

$$= \left[\frac{(140.7)(4)}{3} \left(\frac{2}{3}\right)(9 - y^2)^{3/2} \right]_{-3}^0 = 3376.8 \text{ lb}$$



18. $h(y) = -y$

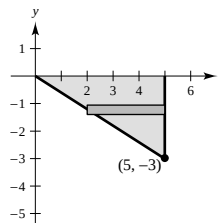
$L(y) = 5 + \frac{5}{3}y$

$$F = 140.7 \int_{-3}^0 (-y)\left(5 + \frac{5}{3}y\right) dy$$

$$= 140.7 \int_{-3}^0 \left(-5y - \frac{5}{3}y^2\right) dy$$

$$= 140.7 \left[-\frac{5}{2}y^2 - \frac{5}{9}y^3 \right]_{-3}^0$$

$$= 140.7 \left[\frac{45}{2} - 15 \right] = 1055.25 \text{ lb}$$



20. $h(y) = \frac{3}{2} - y$

$L(y) = 2\left(\frac{1}{2}\right)\sqrt{9 - 4y^2}$

$$F = 42 \int_{-3/2}^{3/2} \left(\frac{3}{2} - y\right)\sqrt{9 - 4y^2} dy = 63 \int_{-3/2}^{3/2} \sqrt{9 - 4y^2} dy + \frac{21}{4} \int_{-3/2}^{3/2} \sqrt{9 - 4y^2} (-8y) dy$$

The second integral is zero since it is an odd function and the limits of integration are symmetric to the origin. The first integral is twice the area of a semicircle of radius $\frac{3}{2}$.

$$(\sqrt{9 - 4y^2} = 2\sqrt{(9/4) - y^2})$$

Thus, the force is $63\left(\frac{9}{4}\pi\right) = 141.75\pi \approx 445.32 \text{ lb}$.

22. (a) $F = wk\pi r^2 = (62.4)(7)(\pi 2^2) = 1747.2\pi \text{ lbs}$

(b) $F = wk\pi r^2 = (62.4)(5)(\pi 3^2) = 2808\pi \text{ lbs}$

24. (a) $F = wkhb = (62.4)\left(\frac{11}{2}\right)(3)(5) = 5148 \text{ lbs}$

(b) $F = wkhb = (62.4)\left(\frac{17}{5}\right)(5)(10) = 10,608 \text{ lbs}$

26. From Exercise 21:

$$F = 64(15)\pi\left(\frac{1}{2}\right)^2 \approx 753.98 \text{ lb}$$

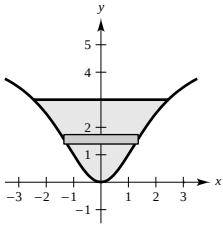
28. $h(y) = 3 - y$

Solving $y = 5x^2/(x^2 + 4)$ for x , you obtain

$$x = \sqrt{4y/(5 - y)}.$$

$$L(y) = 2\sqrt{\frac{4y}{5 - y}}$$

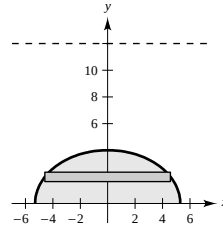
$$\begin{aligned} F &= 62.4(2) \int_0^3 (3 - y) \sqrt{\frac{4y}{5 - y}} dy \\ &= 2(124.8) \int_0^3 (3 - y) \sqrt{\frac{y}{5 - y}} dy \approx 546.265 \text{ lb} \end{aligned}$$



30. $h(y) = 12 - y$

$$L(y) = 2 \frac{\sqrt{7(16 - y^2)}}{2} = \sqrt{7(16 - y^2)}$$

$$\begin{aligned} F &= 62.4 \int_0^4 (12 - y) \sqrt{7(16 - y^2)} dy \\ &= 62.4\sqrt{7} \int_0^4 (12 - y) \sqrt{16 - y^2} dy \approx 21373.7 \text{ lb} \end{aligned}$$

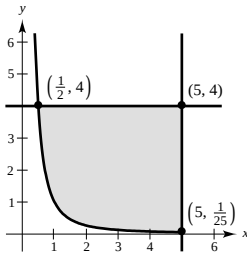


32. Fluid pressure is the force per unit of area exerted by a fluid over the surface of a body.

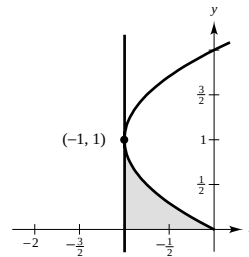
34. The left window experiences the greater fluid force because its centroid is lower.

Review Exercises for Chapter 6

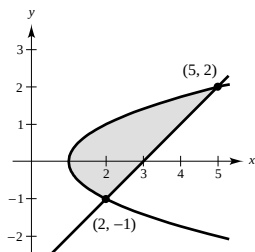
$$\begin{aligned} 2. A &= \int_{1/2}^5 \left(4 - \frac{1}{x^2}\right) dx \\ &= \left[4x + \frac{1}{x}\right]_{1/2}^5 = \frac{81}{5} \end{aligned}$$



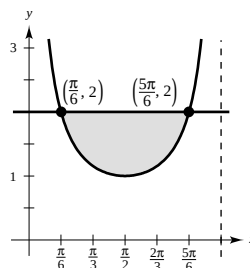
$$\begin{aligned} 4. A &= \int_0^1 [(y^2 - 2y) - (-1)] dy \\ &= \int_0^1 (y^2 - 2y + 1) dy \\ &= \int_0^1 (y - 1)^2 dy \\ &= \left[\frac{(y - 1)^3}{3}\right]_0^1 = \frac{1}{3} \end{aligned}$$



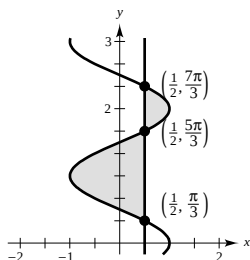
$$\begin{aligned}
 6. A &= \int_{-1}^2 [(y+3) - (y^2+1)] dy \\
 &= \int_{-1}^2 (2+y-y^2) dy \\
 &= \left[2y + \frac{1}{2}y^2 - \frac{1}{3}y^3 \right]_{-1}^2 = \frac{9}{2}
 \end{aligned}$$



$$\begin{aligned}
 8. A &= 2 \int_{\pi/6}^{\pi/2} (2 - \csc x) dx \\
 &= 2 \left[2x - \ln |\csc x - \cot x| \right]_{\pi/6}^{\pi/2} \\
 &= 2 \left([\pi - 0] - \left[\frac{\pi}{3} - \ln(2 - \sqrt{3}) \right] \right) \\
 &= 2 \left[\frac{2\pi}{3} + \ln(2 - \sqrt{3}) \right] \approx 1.555
 \end{aligned}$$



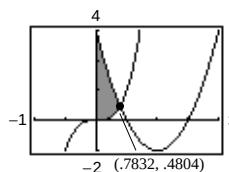
$$\begin{aligned}
 10. A &= \int_{\pi/3}^{5\pi/3} \left(\frac{1}{2} - \cos y \right) dy + \int_{5\pi/3}^{7\pi/3} \left(\cos y - \frac{1}{2} \right) dy \\
 &= \left[\frac{y}{2} - \sin y \right]_{\pi/3}^{5\pi/3} + \left[\sin y - \frac{y}{2} \right]_{5\pi/3}^{7\pi/3} \\
 &= \frac{\pi}{3} + 2\sqrt{3}
 \end{aligned}$$



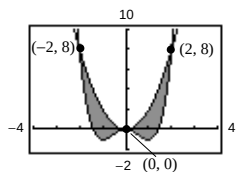
12. Point of intersection is given by:

$$x^3 - x^2 + 4x - 3 = 0 \Rightarrow x \approx 0.783.$$

$$\begin{aligned}
 A &\approx \int_0^{0.783} (3 - 4x + x^2 - x^3) dx \\
 &= \left[3x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^{0.783} \\
 &\approx 1.189
 \end{aligned}$$



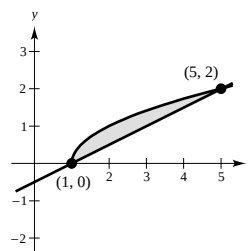
$$\begin{aligned}
 14. A &= 2 \int_0^2 [2x^2 - (x^4 - 2x^2)] dx \\
 &= 2 \int_0^2 (4x^2 - x^4) dx \\
 &= 2 \left[\frac{4}{3}x^3 - \frac{1}{5}x^5 \right]_0^2 = \frac{128}{15} \approx 8.5333
 \end{aligned}$$



$$16. y = \sqrt{x-1} \Rightarrow x = y^2 + 1$$

$$y = \frac{x-1}{2} \Rightarrow x = 2y + 1$$

$$\begin{aligned}
 A &= \int_0^2 [(2y+1) - (y^2+1)] dy \\
 &= \int_1^5 \left[\sqrt{x-1} - \frac{x-1}{2} \right] dx \\
 &= \left[\frac{2}{3}(x-1)^{3/2} - \frac{1}{4}(x-1)^2 \right]_1^5 = \frac{4}{3}
 \end{aligned}$$

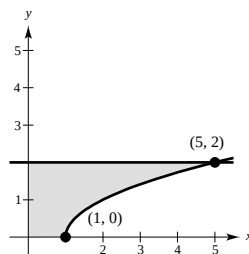


$$18. A = \int_0^1 2 \, dx + \int_1^5 [2 - \sqrt{x-1}] \, dx$$

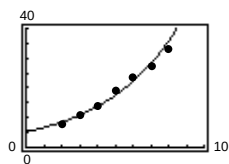
$$x = y^2 + 1$$

$$A = \int_0^2 (y^2 + 1) \, dy$$

$$= \left[\frac{1}{3}y^3 + y \right]_0^2 = \frac{14}{3}$$



$$20. (a) R_1(t) = 5.2834(1.2701)^t = 5.2834 e^{0.2391t}$$

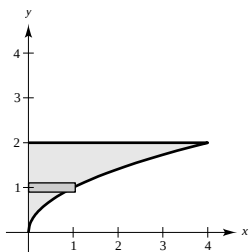


$$(b) R_2(t) = 10 + 5.28 e^{0.2t}$$

$$\text{Difference} = \int_{10}^{15} [R_1(t) - R_2(t)] \, dt \approx 171.25 \text{ billion dollars}$$

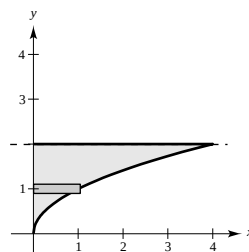
22. (a) Shell

$$V = 2\pi \int_0^2 y^3 \, dy = \left[\frac{\pi}{2} y^4 \right]_0^2 = 8\pi$$



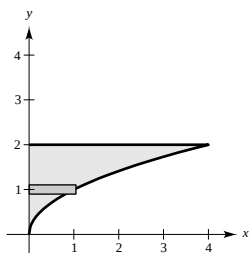
(b) Shell

$$\begin{aligned} V &= 2\pi \int_0^2 (2-y)y^2 \, dy \\ &= 2\pi \int_0^2 (2y^2 - y^3) \, dy \\ &= 2\pi \left[\frac{2}{3}y^3 - \frac{1}{4}y^4 \right]_0^2 = \frac{8\pi}{3} \end{aligned}$$



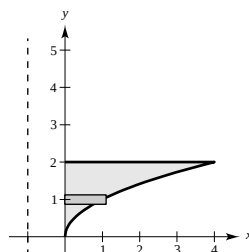
(c) Disk

$$V = \pi \int_0^2 y^4 \, dy = \left[\frac{\pi}{5} y^5 \right]_0^2 = \frac{32\pi}{5}$$



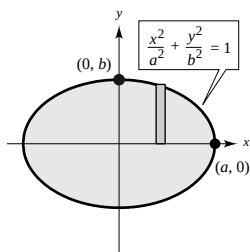
(d) Disk

$$\begin{aligned} V &= \pi \int_0^2 [(y^2 + 1)^2 - 1^2] \, dy \\ &= \pi \int_0^2 (y^4 + 2y^2) \, dy \\ &= \pi \left[\frac{1}{5}y^5 + \frac{2}{3}y^3 \right]_0^2 = \frac{176\pi}{15} \end{aligned}$$



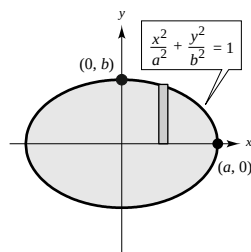
24. (a) Shell

$$\begin{aligned}
 V &= 4\pi \int_0^a (x) \frac{b}{a} \sqrt{a^2 - x^2} dx \\
 &= \frac{-2\pi b}{a} \int_0^a (a^2 - x^2)^{1/2} (-2x) dx \\
 &= \left[\frac{-4\pi b}{3a} (a^2 - x^2)^{3/2} \right]_0^a = \frac{4}{3} \pi a^2 b
 \end{aligned}$$



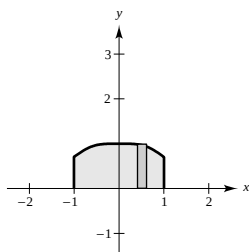
(b) Disk

$$\begin{aligned}
 V &= 2\pi \int_0^a \frac{b^2}{a^2} (a^2 - x^2) dx \\
 &= \frac{2\pi b^2}{a^2} \left[a^2 x - \frac{1}{3} x^3 \right]_0^a \\
 &= \frac{4}{3} \pi a b^2
 \end{aligned}$$



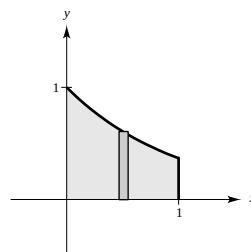
26. Disk

$$\begin{aligned}
 V &= 2\pi \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \right]^2 dx \\
 &= \left[2\pi \arctan x \right]_0^1 \\
 &= 2\pi \left(\frac{\pi}{4} - 0 \right) \\
 &= \frac{\pi^2}{2}
 \end{aligned}$$



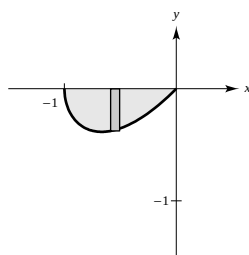
28. Disk

$$\begin{aligned}
 V &= \pi \int_0^1 (e^{-x})^2 dx \\
 &= \pi \int_0^1 e^{-2x} dx = \left[-\frac{\pi}{2} e^{-2x} \right]_0^1 \\
 &= \left(\frac{-\pi}{2e^2} + \frac{\pi}{2} \right) = \frac{\pi}{2} \left(1 - \frac{1}{e^2} \right)
 \end{aligned}$$



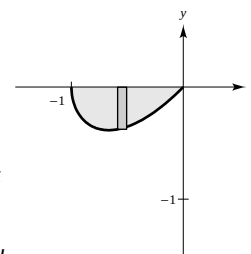
30. (a) Disk

$$\begin{aligned}
 V &= \pi \int_{-1}^0 x^2(x+1) dx \\
 &= \pi \int_{-1}^0 (x^3 + x^2) dx \\
 &= \pi \left[\frac{x^4}{4} + \frac{x^3}{3} \right]_{-1}^0 = \frac{\pi}{12}
 \end{aligned}$$



(b) Shell

$$\begin{aligned}
 u &= \sqrt{x+1} \\
 x &= u^2 - 1 \\
 dx &= 2u du \\
 V &= 2\pi \int_{-1}^0 x^2 \sqrt{x+1} dx \\
 &= 4\pi \int_0^1 (u^2 - 1)^2 u^2 du \\
 &= 4\pi \int_0^1 (u^6 - 2u^4 + u^2) du \\
 &= 4\pi \left[\frac{1}{7} u^7 - \frac{2}{5} u^5 + \frac{1}{3} u^3 \right]_0^1 = \frac{32\pi}{105}
 \end{aligned}$$

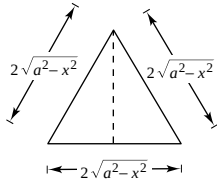


$$\begin{aligned}
 32. A(x) &= \frac{1}{2}bh = \frac{1}{2}(2\sqrt{a^2 - x^2})(\sqrt{3}\sqrt{a^2 - x^2}) \\
 &= \sqrt{3}(a^2 - x^2)
 \end{aligned}$$

$$\begin{aligned}
 V &= \sqrt{3} \int_{-a}^a (a^2 - x^2) dx = \sqrt{3} \left[a^2x - \frac{x^3}{3} \right]_{-a}^a \\
 &= \sqrt{3} \left(\frac{4a^3}{3} \right)
 \end{aligned}$$

Since $(4\sqrt{3}a^3)/3 = 10$, we have $a^3 = (5\sqrt{3})/2$. Thus,

$$a = \sqrt[3]{\frac{5\sqrt{3}}{2}} \approx 1.630 \text{ meters.}$$



36. Since $f(x) = \tan x$ has $f'(x) = \sec^2 x$, this integral represents the length of the graph of $\tan x$ from $x = 0$ to $x = \pi/4$. This length is a little over 1 unit. Answers (b).

40. $F = kx$

$$50 = k(9) \Rightarrow k = \frac{50}{9}$$

$$F = \frac{50}{9}x$$

$$\begin{aligned}
 W &= \int_0^9 \frac{50}{9}x dx = \left[\frac{25}{9}x^2 \right]_0^9 \\
 &= 225 \text{ in} \cdot \text{lb} = 18.75 \text{ ft} \cdot \text{lb}
 \end{aligned}$$

$$34. y = \frac{x^3}{6} + \frac{1}{2x}$$

$$y' = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

$$1 + (y')^2 = \left(\frac{1}{2}x^2 + \frac{1}{2x^2} \right)^2$$

$$s = \int_1^3 \left(\frac{1}{2}x^2 + \frac{1}{2x^2} \right) dx = \left[\frac{1}{6}x^3 - \frac{1}{2x} \right]_1^3 = \frac{14}{3}$$

38. $y = 2\sqrt{x}$

$$y' = \frac{1}{\sqrt{x}}$$

$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$\begin{aligned}
 S &= 2\pi \int_0^3 2\sqrt{x} \sqrt{\frac{x+1}{x}} dx = 4\pi \int_0^3 \sqrt{x+1} dx \\
 &= 4\pi \left[\left(\frac{2}{3} \right) (x+1)^{3/2} \right]_0^3 = \frac{56\pi}{3}
 \end{aligned}$$

42. We know that

$$\frac{dV}{dt} = \frac{4 \text{ gal/min} - 12 \text{ gal/min}}{7.481 \text{ gal/ft}^3} = -\frac{8}{7.481} \text{ ft}^3/\text{min}$$

$$V = \pi r^2 h = \pi \left(\frac{1}{9} \right) h$$

$$\frac{dV}{dt} = \frac{\pi}{9} \left(\frac{dh}{dt} \right)$$

$$\frac{dh}{dt} = \frac{9}{\pi} \left(\frac{dV}{dt} \right) = \frac{9}{\pi} \left(-\frac{8}{7.481} \right) \approx -3.064 \text{ ft/min.}$$

Depth of water: $-3.064t + 150$

Time to drain well: $t = \frac{150}{3.064} \approx 49 \text{ minutes}$

$(49)(12) = 588 \text{ gallons pumped}$

Volume of water pumped in Exercise 41: 391.7 gallons

$$\frac{391.7}{52\pi} = \frac{588}{x\pi}$$

$$x = \frac{588(52)}{391.7} \approx 78$$

Work $\approx 78\pi \text{ ft} \cdot \text{ton}$

44. (a) Weight of section of cable:
- $4 \Delta x$

Distance: $200 - x$

$$W = 4 \int_0^{200} (200 - x) dx = \left[-2(200 - x)^2 \right]_0^{200} = 80,000 \text{ ft} \cdot \text{lb} = 40 \text{ ft} \cdot \text{ton}$$

- (b) Work to move 300 pounds 200 feet vertically:
- $200(300) = 60,000 \text{ ft} \cdot \text{lb} = 30 \text{ ft} \cdot \text{ton}$

Total work = work for drawing up the cable + work of lifting the load

$$= 40 \text{ ft} \cdot \text{ton} + 30 \text{ ft} \cdot \text{ton} = 70 \text{ ft} \cdot \text{ton}$$

46. $W = \int_a^b F(x) dx$

$$F(x) = \begin{cases} -(2/9)x + 6, & 0 \leq x \leq 9 \\ -(4/3)x + 16, & 9 \leq x \leq 12 \end{cases}$$

$$\begin{aligned} W &= \int_0^9 \left(-\frac{2}{9}x + 6 \right) dx + \int_9^{12} \left(-\frac{4}{3}x + 16 \right) dx \\ &= \left[-\frac{1}{9}x^2 + 6x \right]_0^9 + \left[-\frac{2}{3}x^2 + 16x \right]_9^{12} \\ &= (-9 + 54) + (-96 + 192 + 54 - 144) = 51 \text{ ft} \cdot \text{lbs} \end{aligned}$$

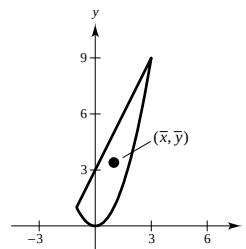
48. $A = \int_{-1}^3 [(2x + 3) - x^2] dx = \left[x^2 + 3x - \frac{1}{3}x^3 \right]_{-1}^3 = \frac{32}{3}$

$$\frac{1}{A} = \frac{3}{32}$$

$$\bar{x} = \frac{3}{32} \int_{-1}^3 x(2x + 3 - x^2) dx = \frac{3}{32} \int_{-1}^3 (3x + 2x^2 - x^3) dx = \frac{3}{32} \left[\frac{3}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_{-1}^3 = 1$$

$$\begin{aligned} \bar{y} &= \left(\frac{3}{32} \right) \frac{1}{2} \int_{-1}^3 [(2x + 3)^2 - x^4] dx = \frac{3}{64} \int_{-1}^3 (9 + 12x + 4x^2 - x^4) dx \\ &= \frac{3}{64} \left[9x + 6x^2 + \frac{4}{3}x^3 - \frac{1}{5}x^5 \right]_{-1}^3 = \frac{17}{5} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(1, \frac{17}{5} \right)$$



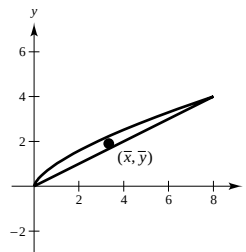
50. $A = \int_0^8 \left(x^{2/3} - \frac{1}{2}x \right) dx = \left[\frac{3}{5}x^{5/3} - \frac{1}{4}x^2 \right]_0^8 = \frac{16}{5}$

$$\frac{1}{A} = \frac{5}{16}$$

$$\begin{aligned} \bar{x} &= \frac{5}{16} \int_0^8 x \left(x^{2/3} - \frac{1}{2}x \right) dx \\ &= \frac{5}{16} \left[\frac{3}{8}x^{8/3} - \frac{1}{6}x^3 \right]_0^8 = \frac{10}{3} \end{aligned}$$

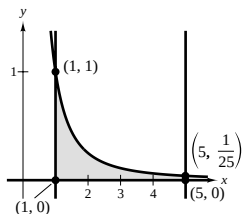
$$\begin{aligned} \bar{y} &= \left(\frac{5}{16} \right) \frac{1}{2} \int_0^8 \left(x^{4/3} - \frac{1}{4}x^2 \right) dx \\ &= \frac{1}{2} \left(\frac{5}{16} \right) \left[\frac{3}{7}x^{7/3} - \frac{1}{12}x^3 \right]_0^8 = \frac{40}{21} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(\frac{10}{3}, \frac{40}{21} \right)$$

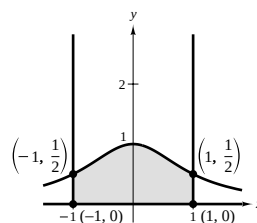


Review Exercises for Chapter 6

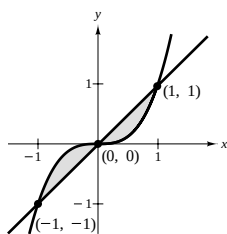
$$1. A = \int_1^5 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^5 = \frac{4}{5}$$



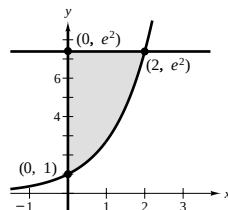
$$\begin{aligned} 3. A &= \int_{-1}^1 \frac{1}{x^2 + 1} dx \\ &= \left[\arctan x \right]_{-1}^1 \\ &= \frac{\pi}{4} - \left(-\frac{\pi}{4} \right) = \frac{\pi}{2} \end{aligned}$$



$$\begin{aligned} 5. A &= 2 \int_0^1 (x - x^3) dx \\ &= 2 \left[\frac{1}{2}x^2 - \frac{1}{4}x^4 \right]_0^1 \\ &= \frac{1}{2} \end{aligned}$$

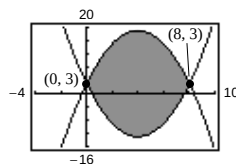


$$\begin{aligned} 7. A &= \int_0^2 (e^2 - e^x) dx \\ &= \left[xe^2 - e^x \right]_0^2 \\ &= e^2 + 1 \end{aligned}$$



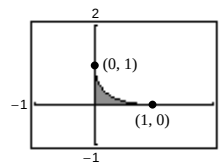
$$\begin{aligned} 9. A &= \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx \\ &= \left[-\cos x - \sin x \right]_{\pi/4}^{5\pi/4} \\ &= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \\ &= \frac{4}{\sqrt{2}} = 2\sqrt{2} \end{aligned}$$

$$\begin{aligned} 11. A &= \int_0^8 [(3 + 8x - x^2) - (x^2 - 8x + 3)] dx \\ &= \int_0^8 (16x - 2x^2) dx \\ &= \left[8x^2 - \frac{2}{3}x^3 \right]_0^8 = \frac{512}{3} \approx 170.667 \end{aligned}$$



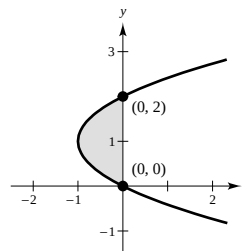
13. $y = (1 - \sqrt{x})^2$

$$\begin{aligned} A &= \int_0^1 (1 - \sqrt{x})^2 dx \\ &= \int_0^1 (1 - 2x^{1/2} + x) dx \\ &= \left[x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2 \right]_0^1 = \frac{1}{6} \approx 0.1667 \end{aligned}$$



15. $x = y^2 - 2y \Rightarrow x + 1 = (y - 1)^2 \Rightarrow y = 1 \pm \sqrt{x + 1}$

$$\begin{aligned} A &= \int_{-1}^0 [(1 + \sqrt{x+1}) - (1 - \sqrt{x+1})] dx = \int_{-1}^0 2\sqrt{x+1} dx \\ A &= \int_0^2 [0 - (y^2 - 2y)] dy = \int_0^2 (2y - y^2) dy = \left[y^2 - \frac{1}{3}y^3 \right]_0^2 = \frac{4}{3} \end{aligned}$$



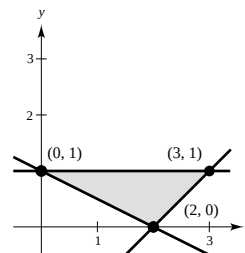
17. $A = \int_0^2 \left[1 - \left(1 - \frac{x}{2} \right) \right] dx + \int_2^3 [1 - (x - 2)] dx$

$$= \int_0^2 \frac{x}{2} dx + \int_2^3 (3 - x) dx$$

$$y = 1 - \frac{x}{2} \Rightarrow x = 2 - 2y$$

$$y = x - 2 \Rightarrow x = y + 2, y = 1$$

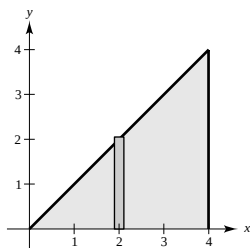
$$\begin{aligned} A &= \int_0^1 [(y + 2) - (2 - 2y)] dy \\ &= \int_0^1 3y dy = \left[\frac{3}{2}y^2 \right]_0^1 = \frac{3}{2} \end{aligned}$$



19. Job 1 is better. The salary for Job 1 is greater than the salary for Job 2 for all the years except the first and 10th years.

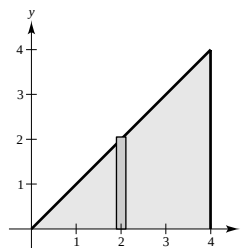
21. (a) **Disk**

$$V = \pi \int_0^4 x^2 dx = \left[\frac{\pi x^3}{3} \right]_0^4 = \frac{64\pi}{3}$$



(b) **Shell**

$$V = 2\pi \int_0^4 x^2 dx = \left[\frac{2\pi}{3} x^3 \right]_0^4 = \frac{128\pi}{3}$$

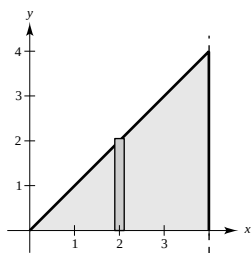


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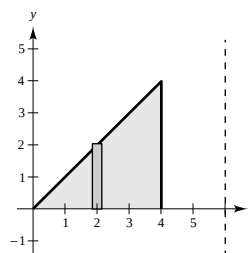
21. —CONTINUED—

(c) **Shell**

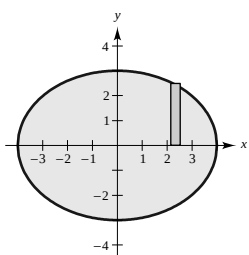
$$\begin{aligned}
 V &= 2\pi \int_0^4 (4-x)x \, dx \\
 &= 2\pi \int_0^4 (4x - x^2) \, dx \\
 &= 2\pi \left[2x^2 - \frac{x^3}{3} \right]_0^4 = \frac{64\pi}{3}
 \end{aligned}$$

(d) **Shell**

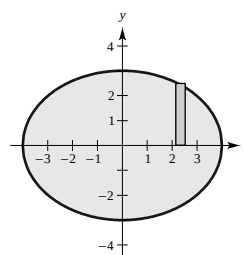
$$\begin{aligned}
 V &= 2\pi \int_0^4 (6-x)x \, dx \\
 &= 2\pi \int_0^4 (6x - x^2) \, dx \\
 &= 2\pi \left[3x^2 - \frac{1}{3}x^3 \right]_0^4 = \frac{160\pi}{3}
 \end{aligned}$$

23. (a) **Shell**

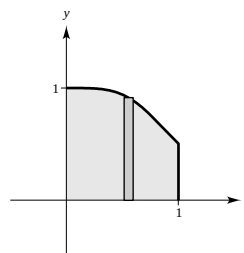
$$\begin{aligned}
 V &= 4\pi \int_0^4 x \left(\frac{3}{4} \right) \sqrt{16 - x^2} \, dx \\
 &= \left[3\pi \left(-\frac{1}{2} \right) \left(\frac{2}{3} \right) (16 - x^2)^{3/2} \right]_0^4 = 64\pi
 \end{aligned}$$

(b) **Disk**

$$\begin{aligned}
 V &= 2\pi \int_0^4 \left[\frac{3}{4} \sqrt{16 - x^2} \right]^2 dx \\
 &= \frac{9\pi}{8} \left[16x - \frac{x^3}{3} \right]_0^4 = 48\pi
 \end{aligned}$$

25. **Shell**

$$\begin{aligned}
 V &= 2\pi \int_0^1 \frac{x}{x^4 + 1} \, dx \\
 &= \pi \int_0^1 \frac{(2x)}{(x^2)^2 + 1} \, dx \\
 &= \left[\pi \arctan(x^2) \right]_0^1 \\
 &= \pi \left[\frac{\pi}{4} - 0 \right] = \frac{\pi^2}{4}
 \end{aligned}$$



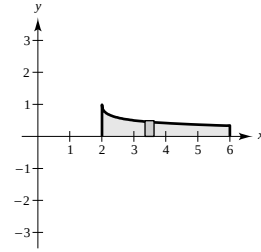
27. Shell

$$u = \sqrt{x-2}$$

$$x = u^2 + 2$$

$$dx = 2u \, du$$

$$\begin{aligned} V &= 2\pi \int_2^6 \frac{x}{1 + \sqrt{x-2}} dx = 4\pi \int_0^2 \frac{(u^2 + 2)u}{1 + u} du \\ &= 4\pi \int_0^2 \frac{u^3 + 2u}{1 + u} du = 4\pi \int_0^2 \left(u^2 - u + 3 - \frac{3}{1+u} \right) du \\ &= 4\pi \left[\frac{1}{3}u^3 - \frac{1}{2}u^2 + 3u - 3 \ln(1+u) \right]_0^2 = \frac{4\pi}{3}(20 - 9 \ln 3) \approx 42.359 \end{aligned}$$



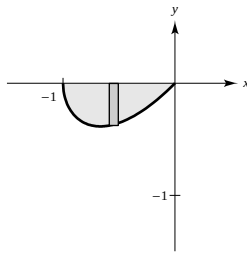
29. Since $y \leq 0$, $A = -\int_{-1}^0 x\sqrt{x+1} \, dx$.

$$u = x + 1$$

$$x = u - 1$$

$$dx = du$$

$$\begin{aligned} A &= -\int_0^1 (u-1)\sqrt{u} \, du = -\int_0^1 (u^{3/2} - u^{1/2}) \, du \\ &= -\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_0^1 = \frac{4}{15} \end{aligned}$$



31. From Exercise 23(a) we have: $V = 64\pi \text{ ft}^3$

$$\frac{1}{4}V = 16\pi$$

Disk: $\pi \int_{-3}^{y_0} \frac{16}{9}(9 - y^2) \, dy = 16\pi$

$$\frac{1}{9} \int_{-3}^{y_0} (9 - y^2) \, dy = 1$$

$$\left[9y - \frac{1}{3}y^3 \right]_{-3}^{y_0} = 9$$

$$\left(9y_0 - \frac{1}{3}y_0^3 \right) - (-27 + 9) = 9$$

$$y_0^3 - 27y_0 - 27 = 0$$

By Newton's Method, $y_0 \approx -1.042$ and the depth of the gasoline is $3 - 1.042 = 1.958 \text{ ft}$.

33. $f(x) = \frac{4}{5}x^{5/4}$

$$f'(x) = x^{1/4}$$

$$1 + [f'(x)]^2 = 1 + \sqrt{x}$$

$$u = 1 + \sqrt{x}$$

$$x = (u-1)^2$$

$$dx = 2(u-1) \, du$$

$$\begin{aligned} s &= \int_0^4 \sqrt{1 + \sqrt{x}} \, dx = 2 \int_1^3 \sqrt{u}(u-1) \, du \\ &= 2 \int_1^3 (u^{3/2} - u^{1/2}) \, du \\ &= 2 \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_1^3 = \frac{4}{15} \left[u^{3/2}(3u-5) \right]_1^3 \\ &= \frac{8}{15}(1 + 6\sqrt{3}) \approx 6.076 \end{aligned}$$

35. $y = 300 \cosh\left(\frac{x}{2000}\right) - 280, -2000 \leq x \leq 2000$

$$y' = \frac{3}{20} \sinh\left(\frac{x}{2000}\right)$$

$$s = \int_{-2000}^{2000} \sqrt{1 + \left[\frac{3}{20} \sinh\left(\frac{x}{2000}\right) \right]^2} \, dx$$

$$= \frac{1}{20} \int_{-2000}^{2000} \sqrt{400 + 9 \sinh^2\left(\frac{x}{2000}\right)} \, dx$$

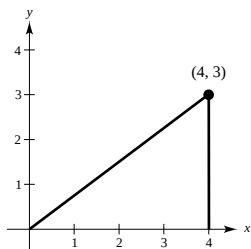
$$\approx 4018.2 \text{ ft (by Simpson's Rule or graphing utility)}$$

$$37. y = \frac{3}{4}x$$

$$y' = \frac{3}{4}$$

$$1 + (y')^2 = \frac{25}{16}$$

$$S = 2\pi \int_0^4 \left(\frac{3}{4}x\right) \sqrt{\frac{25}{16}} dx = \left[\left(\frac{15\pi}{8}\right)\frac{x^2}{2}\right]_0^4 = 15\pi$$



$$39. F = kx$$

$$4 = k(1)$$

$$F = 4x$$

$$W = \int_0^5 4x dx = \left[2x^2\right]_0^5 = 50 \text{ in} \cdot \text{lb} \approx 4.167 \text{ ft} \cdot \text{lb}$$

$$41. \text{Volume of disk: } \pi\left(\frac{1}{3}\right)^2 \Delta y$$

$$\text{Weight of disk: } 62.4\pi\left(\frac{1}{3}\right)^2 \Delta y$$

$$\text{Distance: } 175 - y$$

$$W = \frac{62.4\pi}{9} \int_0^{150} (175 - y) dy = \frac{62.4\pi}{9} \left[175y - \frac{y^2}{2}\right]_0^{150} = 104,000\pi \text{ ft} \cdot \text{lb} \approx 163.4 \text{ ft} \cdot \text{ton}$$

$$43. \text{Weight of section of chain: } 5 \Delta x$$

$$\text{Distance moved: } 10 - x$$

$$W = 5 \int_0^{10} (10 - x) dx = \left[-\frac{5}{2}(10 - x)^2\right]_0^{10} = 250 \text{ ft} \cdot \text{lb}$$

$$45. W = \int_a^b F(x) dx$$

$$80 = \int_0^4 ax^2 dx = \left[\frac{ax^3}{3}\right]_0^4 = \frac{64}{3}a$$

$$a = \frac{3(80)}{64} = \frac{15}{4} = 3.75$$

$$47. A = \int_0^a (\sqrt{a} - \sqrt{x})^2 dx = \int_0^a (a - 2\sqrt{a}x^{1/2} + x) dx = \left[ax - \frac{4}{3}\sqrt{a}x^{3/2} + \frac{1}{2}x^2\right]_0^a = \frac{a^2}{6}$$

$$\frac{1}{A} = \frac{6}{a^2}$$

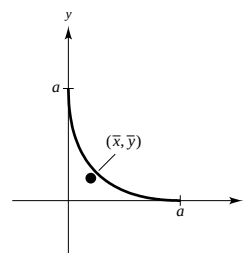
$$\bar{x} = \frac{6}{a^2} \int_0^a x(\sqrt{a} - \sqrt{x})^2 dx = \frac{6}{a^2} \int_0^a (ax - 2\sqrt{a}x^{3/2} + x^2) dx$$

$$\bar{y} = \left(\frac{6}{a^2}\right) \frac{1}{2} \int_0^a (\sqrt{a} - \sqrt{x})^4 dx$$

$$= \frac{3}{a^2} \int_0^a (a^2 - 4a^{3/2}x^{1/2} + 6ax - 4a^{1/2}x^{3/2} + x^2) dx$$

$$= \frac{3}{a^2} \left[a^2x - \frac{8}{3}a^{3/2}x^{3/2} + 3ax^2 - \frac{8}{5}a^{1/2}x^{5/2} + \frac{1}{3}x^3\right]_0^a = \frac{a}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{a}{5}, \frac{a}{5}\right)$$



49. By symmetry, $\bar{x} = 0$.

$$A = 2 \int_0^1 (a^2 - x^2) dx = 2 \left[a^2x - \frac{x^3}{3} \right]_0^1 = \frac{4a^3}{3}$$

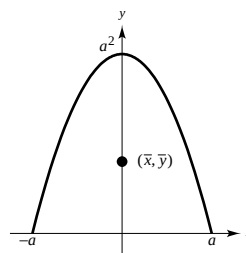
$$\frac{1}{A} = \frac{3}{4a^3}$$

$$\begin{aligned} \bar{y} &= \left(\frac{3}{4a^3} \right) \frac{1}{2} \int_{-a}^a (a^2 - x^2)^2 dx \\ &= \frac{6}{8a^3} \int_0^a (a^4 - 2a^2x^2 + x^4) dx \end{aligned}$$

$$= \frac{6}{8a^3} \left[a^4x - \frac{2a^2}{3}x^3 + \frac{1}{5}x^5 \right]_0^a$$

$$= \frac{6}{8a^3} \left(a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5 \right) = \frac{2a^2}{5}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{2a^2}{5} \right)$$

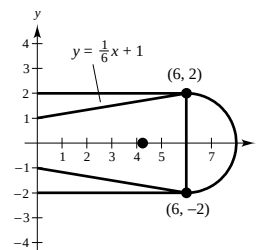


51. $\bar{y} = 0$ by symmetry

For the trapezoid:

$$m = [(4)(6) - (1)(6)]\rho = 18\rho$$

$$\begin{aligned} M_y &= \rho \int_0^6 x \left[\left(\frac{1}{6}x + 1 \right) - \left(-\frac{1}{6}x - 1 \right) \right] dx \\ &= \rho \int_0^6 \left(\frac{1}{3}x^2 + 2x \right) dx = \rho \left[\frac{x^3}{9} + x^2 \right]_0^6 = 60\rho \end{aligned}$$



For the semicircle:

$$m = \left(\frac{1}{2} \right) (\pi)(2)^2 \rho = 2\pi\rho$$

$$M_y = \rho \int_6^8 x \left[\sqrt{4 - (x-6)^2} - (-\sqrt{4 - (x-6)^2}) \right] dx = 2\rho \int_6^8 x \sqrt{4 - (x-6)^2} dx$$

Let $u = x - 6$, then $x = u + 6$ and $dx = du$. When $x = 6$, $u = 0$. When $x = 8$, $u = 2$.

$$\begin{aligned} M_y &= 2\rho \int_0^2 (u+6) \sqrt{4-u^2} du = 2\rho \int_0^2 u \sqrt{4-u^2} du + 12\rho \int_0^2 \sqrt{4-u^2} du \\ &= 2\rho \left[\left(-\frac{1}{2} \right) \left(\frac{2}{3} \right) (4-u^2)^{3/2} \right]_0^2 + 12\rho \left[\frac{\pi(2)^2}{4} \right] = \frac{16\rho}{3} + 12\pi\rho = \frac{4\rho(4+9\pi)}{3} \end{aligned}$$

Thus, we have:

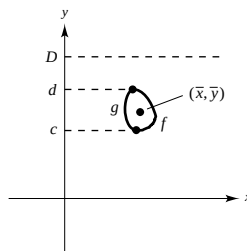
$$\bar{x}(18\rho + 2\pi\rho) = 60\rho + \frac{4\rho(4+9\pi)}{3}$$

$$\bar{x} = \frac{180\rho + 4\rho(4+9\pi)}{3} \cdot \frac{1}{2\rho(9+\pi)} = \frac{2(9\pi+49)}{3(\pi+9)}$$

The centroid of the blade is $\left(\frac{2(9\pi+49)}{3(\pi+9)}, 0 \right)$.

53. Let D = surface of liquid; ρ = weight per cubic volume.

$$\begin{aligned}
 F &= \rho \int_c^d (D - y)[f(y) - g(y)] dy \\
 &= \rho \left[\int_c^d D[f(y) - g(y)] dy - \int_c^d y[f(y) - g(y)] dy \right] \\
 &= \rho \left[\int_c^d [f(y) - g(y)] dy \right] \left[D - \frac{\int_c^d y[f(y) - g(y)] dy}{\int_c^d [f(y) - g(y)] dy} \right] \\
 &= \rho(\text{Area})(D - \bar{y}) \\
 &= \rho(\text{Area})(\text{depth of centroid})
 \end{aligned}$$



Problem Solving for Chapter 6

1. $T = \frac{1}{2}c(c^2) = \frac{1}{2}c^3$

$$R = \int_0^c (cx - x^2) dx = \left[\frac{cx^2}{2} - \frac{x^3}{3} \right]_0^c = \frac{c^3}{2} - \frac{c^3}{3} = \frac{c^3}{6}$$

$$\lim_{c \rightarrow 0^+} \frac{T}{R} = \lim_{c \rightarrow 0^+} \frac{\frac{1}{2}c^3}{\frac{1}{6}c^3} = 3$$

3. (a) $\frac{1}{2}V = \int_0^1 [\pi(2 + \sqrt{1 - y^2})^2 - \pi(2 - \sqrt{1 - y^2})^2] dy$

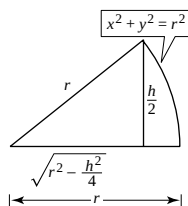
$$\begin{aligned}
 &= \pi \int_0^1 [(4 + 4\sqrt{1 - y^2} + (1 - y^2)) - (4 - 4\sqrt{1 - y^2} + (1 - y^2))] dy \\
 &= 8\pi \int_0^1 \sqrt{1 - y^2} dy \quad (\text{Integral represents } 1/4 \text{ (area of circle)}) \\
 &= 8\pi \left(\frac{\pi}{4} \right) = 2\pi^2 \Rightarrow V = 4\pi^2
 \end{aligned}$$

(b) $(x - R)^2 + y^2 = r^2 \Rightarrow x = R \pm \sqrt{r^2 - y^2}$

$$\begin{aligned}
 \frac{1}{2}V &= \int_0^r [\pi(R + \sqrt{r^2 - y^2})^2 - \pi(R - \sqrt{r^2 - y^2})^2] dy \\
 &= \pi \int_0^r 4R\sqrt{r^2 - y^2} dy \\
 &= \pi(4R) \frac{1}{4} \pi r^2 - \pi^2 r^2 R \\
 V &= 2\pi^2 r^2 R
 \end{aligned}$$

5. $V = 2(2\pi) \int_{\sqrt{r^2 - (h^2/4)}}^r x\sqrt{r^2 - x^2} dx$

$$\begin{aligned}
 &= -2\pi \left[\frac{2}{3}(r^2 - x^2)^{3/2} \right]_{\sqrt{r^2 - (h^2/4)}}^r \\
 &= \frac{-4\pi}{3} \left[-\frac{h^3}{8} \right] = \frac{\pi h^3}{6} \text{ which does not depend on } r!
 \end{aligned}$$



7. (a) Tangent at A:
- $y = x^3, y' = 3x^2$

$$y - 1 = 3(x - 1)$$

$$y = 3x - 2$$

To find point B: $x^3 = 3x - 2$

$$x^3 - 3x + 2 = 0$$

$$(x - 1)^2(x + 2) = 0 \Rightarrow B = (-2, -8)$$

Tangent at B: $y = x^3, y' = 3x^2$

$$y + 8 = 12(x + 2)$$

$$y = 12x + 16$$

To find point C: $x^3 = 12x + 16$

$$x^3 - 12x - 16 = 0$$

$$(x + 2)^2(x - 4) = 0 \Rightarrow C = (4, 64)$$

$$\text{Area of } R = \int_{-2}^1 (x^3 - 3x + 2) dx = \frac{27}{4}$$

$$\text{Area of } S = \int_{-2}^4 (12x + 16 - x^3) dx = 108$$

$$\text{Area of } S = 16(\text{area of } R) \quad \left[\frac{\text{area } S}{\text{area } R} = 16 \right]$$

- (b) Tangent at A(a, a
- ³
-):
- $y - a^3 = 3a^2(x - a)$

$$y = 3a^2x - 2a^3$$

To find point B: $x^3 - 3a^2x + 2a^3 = 0$

$$(x - a)^2(x + 2a) = 0 \Rightarrow$$

$$B = (-2a, -8a^3)$$

Tangent at B: $y + 8a^3 = 12a^2(x + 2a)$

$$y = 12a^2x + 16a^3$$

To find point C: $x^3 - 12a^2x - 16a^3 = 0$

$$(x + 2a)^2(x - 4a) = 0 \Rightarrow$$

$$C = (4a, 64a^3)$$

$$\text{Area of } R = \int_{-2a}^a [x^3 - 3a^2x + 2a^3] dx = \frac{27}{4}a^4$$

$$\text{Area of } S = \int_{-2a}^{4a} [12a^2x + 16a^3 - x^3] dx = 108a^4$$

$$\text{Area of } S = 16(\text{area of } R)$$

9. $s(x) = \int_{\alpha}^x \sqrt{1 + f'(t)^2} dt$

(a) $s'(x) = \frac{ds}{dx} = \sqrt{1 + f'(x)^2}$

(b) $ds = \sqrt{1 + f'(x)^2} dx$

$$(ds)^2 = [1 + f'(x)^2](dx)^2 = \left[1 + \left(\frac{dy}{dx} \right)^2 \right] (dx)^2 = (dx)^2 + (dy)^2$$

(c) $s(x) = \int_1^x \sqrt{1 + \left(\frac{3}{2}t^{1/2} \right)^2} dt = \int_1^x \sqrt{1 + \frac{9}{4}t} dt$

(d) $s(2) = \int_1^2 \sqrt{1 + \frac{9}{4}t} dt = \left[\frac{8}{27} \left(1 + \frac{9}{4}t \right)^{3/2} \right]_1^2 = \frac{22}{27}\sqrt{22} - \frac{13}{27}\sqrt{13} \approx 2.0858$

This is the length of the curve $y = x^{3/2}$ from $x = 1$ to $x = 2$.

11. (a)
- $\bar{y} = 0$
- by symmetry

$$M_y = \int_1^6 x \left(\frac{1}{x^3} - \left(-\frac{1}{x^3} \right) \right) dx = \int_1^6 \frac{2}{x^2} dx = \left[-2\frac{1}{x} \right]_1^6 = \frac{5}{3}$$

$$m = 2 \int_1^6 \frac{1}{x^3} dx = \left[-\frac{1}{x^2} \right]_1^6 = \frac{35}{36}$$

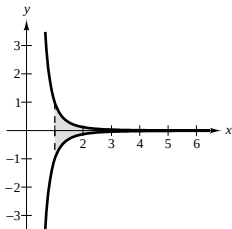
$$\bar{x} = \frac{5/3}{35/36} = \frac{12}{7} \quad (\bar{x}, \bar{y}) = \left(\frac{12}{7}, 0 \right)$$

(b) $m = 2 \int_1^b \frac{1}{x^3} dx = \frac{b^2 - 1}{b^2}$

$$M_y = 2 \int_1^b \frac{1}{x^2} dx = \frac{2(b - 1)}{b}$$

$$\bar{x} = \frac{2(b - 1)/b}{(b^2 - 1)/b^2} = \frac{2b}{b + 1} \quad (\bar{x}, \bar{y}) = \left(\frac{2b}{b + 1}, 0 \right)$$

(c) $\lim_{b \rightarrow \infty} \bar{x} = \lim_{b \rightarrow \infty} \frac{2b}{b + 1} = 2 \quad (\bar{x}, \bar{y}) = (2, 0)$



13. (a) $W = \text{area} = 2 + 4 + 6 = 12$

(b) $W = \text{area} = 3 + (1 + 1) + 2 + \frac{1}{2} = 7\frac{1}{2}$

17. (a) Wall at shallow end

From Exercise 22: $F = 62.4(2)(4)(20) = 9984 \text{ lb}$

(b) Wall at deep end

From Exercise 22: $F = 62.4(4)(8)(20) = 39,936 \text{ lb}$

(c) Side wall

From Exercise 22: $F_1 = 62.4(2)(4)(40) = 19,968 \text{ lb}$

$$\begin{aligned} F_2 &= 62.4 \int_0^4 (8 - y)(10y) dy \\ &= 624 \int_0^4 (8y - y^2) dy = 624 \left[4y^2 - \frac{y^3}{3} \right]_0^4 \\ &= 26,624 \text{ lb} \end{aligned}$$

Total force: $F_1 + F_2 = 46,592 \text{ lb}$

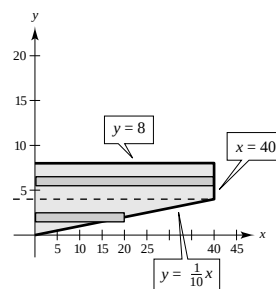
15. Point of equilibrium: $50 - 0.5x = 0.125x$

$$x = 80, p = 10$$

$$(P_0, x_0) = (10, 80)$$

$$\text{Consumer surplus} = \int_0^{80} [(50 - 0.5x) - 10] dx = 1600$$

$$\text{Producer surplus} = \int_0^{80} [10 - 0.125x] dx = 400$$



26. From Exercise 21:

$$F = 64(15)\pi\left(\frac{1}{2}\right)^2 \approx 753.98 \text{ lb}$$

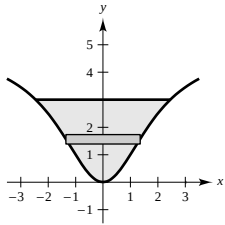
28. $h(y) = 3 - y$

Solving $y = 5x^2/(x^2 + 4)$ for x , you obtain

$$x = \sqrt{4y/(5 - y)}.$$

$$L(y) = 2\sqrt{\frac{4y}{5 - y}}$$

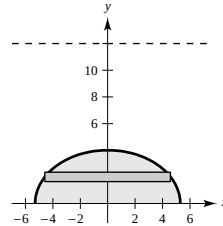
$$\begin{aligned} F &= 62.4(2) \int_0^3 (3 - y) \sqrt{\frac{4y}{5 - y}} dy \\ &= 2(124.8) \int_0^3 (3 - y) \sqrt{\frac{y}{5 - y}} dy \approx 546.265 \text{ lb} \end{aligned}$$



30. $h(y) = 12 - y$

$$L(y) = 2 \frac{\sqrt{7(16 - y^2)}}{2} = \sqrt{7(16 - y^2)}$$

$$\begin{aligned} F &= 62.4 \int_0^4 (12 - y) \sqrt{7(16 - y^2)} dy \\ &= 62.4\sqrt{7} \int_0^4 (12 - y) \sqrt{16 - y^2} dy \approx 21373.7 \text{ lb} \end{aligned}$$

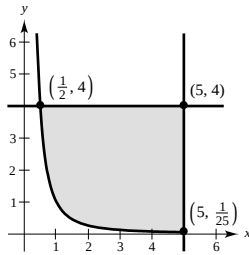


32. Fluid pressure is the force per unit of area exerted by a fluid over the surface of a body.

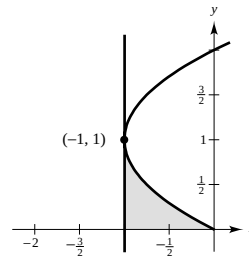
34. The left window experiences the greater fluid force because its centroid is lower.

Review Exercises for Chapter 6

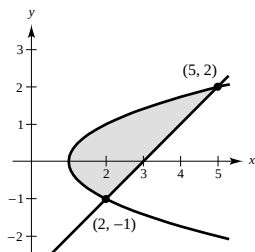
$$\begin{aligned} 2. A &= \int_{1/2}^5 \left(4 - \frac{1}{x^2}\right) dx \\ &= \left[4x + \frac{1}{x}\right]_{1/2}^5 = \frac{81}{5} \end{aligned}$$



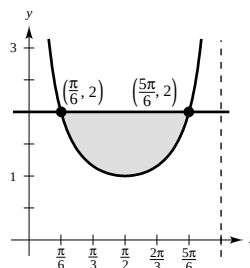
$$\begin{aligned} 4. A &= \int_0^1 [(y^2 - 2y) - (-1)] dy \\ &= \int_0^1 (y^2 - 2y + 1) dy \\ &= \int_0^1 (y - 1)^2 dy \\ &= \left[\frac{(y - 1)^3}{3}\right]_0^1 = \frac{1}{3} \end{aligned}$$



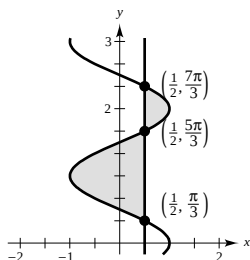
$$\begin{aligned}
 6. A &= \int_{-1}^2 [(y+3) - (y^2+1)] dy \\
 &= \int_{-1}^2 (2+y-y^2) dy \\
 &= \left[2y + \frac{1}{2}y^2 - \frac{1}{3}y^3 \right]_{-1}^2 = \frac{9}{2}
 \end{aligned}$$



$$\begin{aligned}
 8. A &= 2 \int_{\pi/6}^{\pi/2} (2 - \csc x) dx \\
 &= 2 \left[2x - \ln |\csc x - \cot x| \right]_{\pi/6}^{\pi/2} \\
 &= 2 \left([\pi - 0] - \left[\frac{\pi}{3} - \ln(2 - \sqrt{3}) \right] \right) \\
 &= 2 \left[\frac{2\pi}{3} + \ln(2 - \sqrt{3}) \right] \approx 1.555
 \end{aligned}$$



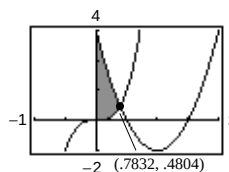
$$\begin{aligned}
 10. A &= \int_{\pi/3}^{5\pi/3} \left(\frac{1}{2} - \cos y \right) dy + \int_{5\pi/3}^{7\pi/3} \left(\cos y - \frac{1}{2} \right) dy \\
 &= \left[\frac{y}{2} - \sin y \right]_{\pi/3}^{5\pi/3} + \left[\sin y - \frac{y}{2} \right]_{5\pi/3}^{7\pi/3} \\
 &= \frac{\pi}{3} + 2\sqrt{3}
 \end{aligned}$$



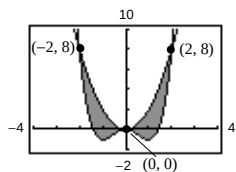
12. Point of intersection is given by:

$$x^3 - x^2 + 4x - 3 = 0 \Rightarrow x \approx 0.783.$$

$$\begin{aligned}
 A &\approx \int_0^{0.783} (3 - 4x + x^2 - x^3) dx \\
 &= \left[3x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^{0.783} \\
 &\approx 1.189
 \end{aligned}$$



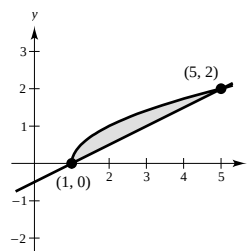
$$\begin{aligned}
 14. A &= 2 \int_0^2 [2x^2 - (x^4 - 2x^2)] dx \\
 &= 2 \int_0^2 (4x^2 - x^4) dx \\
 &= 2 \left[\frac{4}{3}x^3 - \frac{1}{5}x^5 \right]_0^2 = \frac{128}{15} \approx 8.5333
 \end{aligned}$$



$$16. y = \sqrt{x-1} \Rightarrow x = y^2 + 1$$

$$y = \frac{x-1}{2} \Rightarrow x = 2y + 1$$

$$\begin{aligned}
 A &= \int_0^2 [(2y+1) - (y^2+1)] dy \\
 &= \int_1^5 \left[\sqrt{x-1} - \frac{x-1}{2} \right] dx \\
 &= \left[\frac{2}{3}(x-1)^{3/2} - \frac{1}{4}(x-1)^2 \right]_1^5 = \frac{4}{3}
 \end{aligned}$$

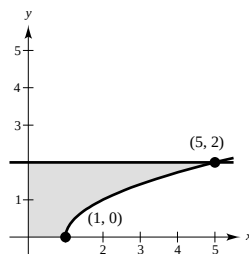


$$18. A = \int_0^1 2 \, dx + \int_1^5 [2 - \sqrt{x-1}] \, dx$$

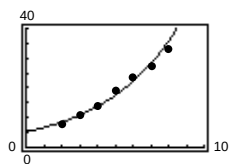
$$x = y^2 + 1$$

$$A = \int_0^2 (y^2 + 1) \, dy$$

$$= \left[\frac{1}{3}y^3 + y \right]_0^2 = \frac{14}{3}$$



$$20. (a) R_1(t) = 5.2834(1.2701)^t = 5.2834 e^{0.2391t}$$

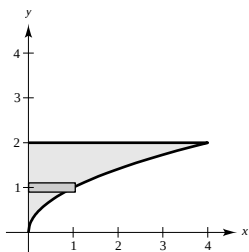


$$(b) R_2(t) = 10 + 5.28 e^{0.2t}$$

$$\text{Difference} = \int_{10}^{15} [R_1(t) - R_2(t)] \, dt \approx 171.25 \text{ billion dollars}$$

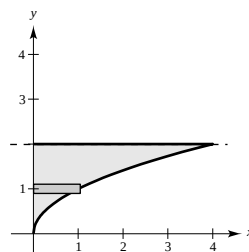
22. (a) Shell

$$V = 2\pi \int_0^2 y^3 \, dy = \left[\frac{\pi}{2} y^4 \right]_0^2 = 8\pi$$



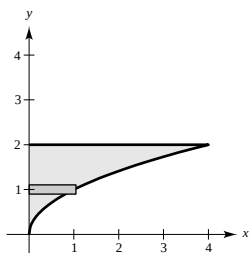
(b) Shell

$$\begin{aligned} V &= 2\pi \int_0^2 (2-y)y^2 \, dy \\ &= 2\pi \int_0^2 (2y^2 - y^3) \, dy \\ &= 2\pi \left[\frac{2}{3}y^3 - \frac{1}{4}y^4 \right]_0^2 = \frac{8\pi}{3} \end{aligned}$$



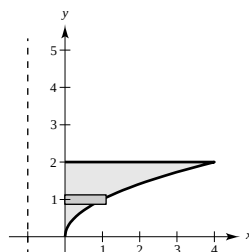
(c) Disk

$$V = \pi \int_0^2 y^4 \, dy = \left[\frac{\pi}{5} y^5 \right]_0^2 = \frac{32\pi}{5}$$



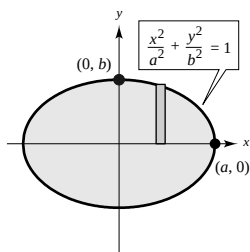
(d) Disk

$$\begin{aligned} V &= \pi \int_0^2 [(y^2 + 1)^2 - 1^2] \, dy \\ &= \pi \int_0^2 (y^4 + 2y^2) \, dy \\ &= \pi \left[\frac{1}{5}y^5 + \frac{2}{3}y^3 \right]_0^2 = \frac{176\pi}{15} \end{aligned}$$



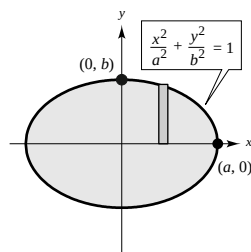
24. (a) Shell

$$\begin{aligned}
 V &= 4\pi \int_0^a (x) \frac{b}{a} \sqrt{a^2 - x^2} dx \\
 &= \frac{-2\pi b}{a} \int_0^a (a^2 - x^2)^{1/2} (-2x) dx \\
 &= \left[\frac{-4\pi b}{3a} (a^2 - x^2)^{3/2} \right]_0^a = \frac{4}{3} \pi a^2 b
 \end{aligned}$$



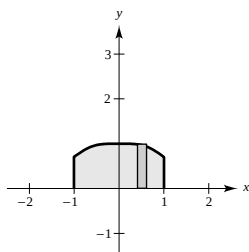
(b) Disk

$$\begin{aligned}
 V &= 2\pi \int_0^a \frac{b^2}{a^2} (a^2 - x^2) dx \\
 &= \frac{2\pi b^2}{a^2} \left[a^2 x - \frac{1}{3} x^3 \right]_0^a \\
 &= \frac{4}{3} \pi a b^2
 \end{aligned}$$



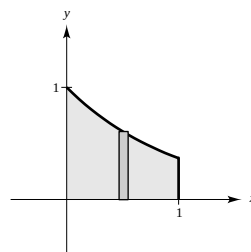
26. Disk

$$\begin{aligned}
 V &= 2\pi \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \right]^2 dx \\
 &= \left[2\pi \arctan x \right]_0^1 \\
 &= 2\pi \left(\frac{\pi}{4} - 0 \right) \\
 &= \frac{\pi^2}{2}
 \end{aligned}$$



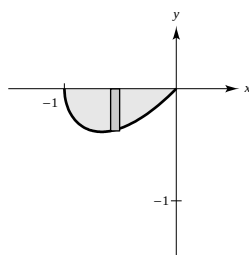
28. Disk

$$\begin{aligned}
 V &= \pi \int_0^1 (e^{-x})^2 dx \\
 &= \pi \int_0^1 e^{-2x} dx = \left[-\frac{\pi}{2} e^{-2x} \right]_0^1 \\
 &= \left(\frac{-\pi}{2e^2} + \frac{\pi}{2} \right) = \frac{\pi}{2} \left(1 - \frac{1}{e^2} \right)
 \end{aligned}$$



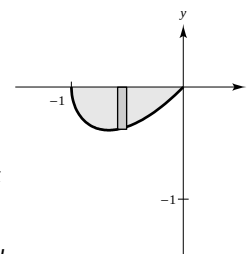
30. (a) Disk

$$\begin{aligned}
 V &= \pi \int_{-1}^0 x^2(x+1) dx \\
 &= \pi \int_{-1}^0 (x^3 + x^2) dx \\
 &= \pi \left[\frac{x^4}{4} + \frac{x^3}{3} \right]_{-1}^0 = \frac{\pi}{12}
 \end{aligned}$$



(b) Shell

$$\begin{aligned}
 u &= \sqrt{x+1} \\
 x &= u^2 - 1 \\
 dx &= 2u du \\
 V &= 2\pi \int_{-1}^0 x^2 \sqrt{x+1} dx \\
 &= 4\pi \int_0^1 (u^2 - 1)^2 u^2 du \\
 &= 4\pi \int_0^1 (u^6 - 2u^4 + u^2) du \\
 &= 4\pi \left[\frac{1}{7} u^7 - \frac{2}{5} u^5 + \frac{1}{3} u^3 \right]_0^1 = \frac{32\pi}{105}
 \end{aligned}$$



$$32. A(x) = \frac{1}{2}bh = \frac{1}{2}(2\sqrt{a^2 - x^2})(\sqrt{3}\sqrt{a^2 - x^2})$$

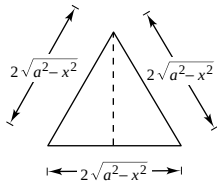
$$= \sqrt{3}(a^2 - x^2)$$

$$V = \sqrt{3} \int_{-a}^a (a^2 - x^2) dx = \sqrt{3} \left[a^2x - \frac{x^3}{3} \right]_{-a}^a$$

$$= \sqrt{3} \left(\frac{4a^3}{3} \right)$$

Since $(4\sqrt{3}a^3)/3 = 10$, we have $a^3 = (5\sqrt{3})/2$. Thus,

$$a = \sqrt[3]{\frac{5\sqrt{3}}{2}} \approx 1.630 \text{ meters.}$$



36. Since $f(x) = \tan x$ has $f'(x) = \sec^2 x$, this integral represents the length of the graph of $\tan x$ from $x = 0$ to $x = \pi/4$. This length is a little over 1 unit. Answers (b).

40. $F = kx$

$$50 = k(9) \Rightarrow k = \frac{50}{9}$$

$$F = \frac{50}{9}x$$

$$W = \int_0^9 \frac{50}{9}x dx = \left[\frac{25}{9}x^2 \right]_0^9$$

$$= 225 \text{ in} \cdot \text{lb} = 18.75 \text{ ft} \cdot \text{lb}$$

34. $y = \frac{x^3}{6} + \frac{1}{2x}$

$$y' = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

$$1 + (y')^2 = \left(\frac{1}{2}x^2 + \frac{1}{2x^2} \right)^2$$

$$s = \int_1^3 \left(\frac{1}{2}x^2 + \frac{1}{2x^2} \right) dx = \left[\frac{1}{6}x^3 - \frac{1}{2x} \right]_1^3 = \frac{14}{3}$$

38. $y = 2\sqrt{x}$

$$y' = \frac{1}{\sqrt{x}}$$

$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$S = 2\pi \int_0^3 2\sqrt{x} \sqrt{\frac{x+1}{x}} dx = 4\pi \int_0^3 \sqrt{x+1} dx$$

$$= 4\pi \left[\left(\frac{2}{3} \right) (x+1)^{3/2} \right]_0^3 = \frac{56\pi}{3}$$

42. We know that

$$\frac{dV}{dt} = \frac{4 \text{ gal/min} - 12 \text{ gal/min}}{7.481 \text{ gal/ft}^3} = -\frac{8}{7.481} \text{ ft}^3/\text{min}$$

$$V = \pi r^2 h = \pi \left(\frac{1}{9} \right) h$$

$$\frac{dV}{dt} = \frac{\pi}{9} \left(\frac{dh}{dt} \right)$$

$$\frac{dh}{dt} = \frac{9}{\pi} \left(\frac{dV}{dt} \right) = \frac{9}{\pi} \left(-\frac{8}{7.481} \right) \approx -3.064 \text{ ft/min.}$$

Depth of water: $-3.064t + 150$

Time to drain well: $t = \frac{150}{3.064} \approx 49 \text{ minutes}$

$(49)(12) = 588 \text{ gallons pumped}$

Volume of water pumped in Exercise 41: 391.7 gallons

$$\frac{391.7}{52\pi} = \frac{588}{x\pi}$$

$$x = \frac{588(52)}{391.7} \approx 78$$

Work $\approx 78\pi \text{ ft} \cdot \text{ton}$

44. (a) Weight of section of cable:
- $4 \Delta x$

Distance: $200 - x$

$$W = 4 \int_0^{200} (200 - x) dx = \left[-2(200 - x)^2 \right]_0^{200} = 80,000 \text{ ft} \cdot \text{lb} = 40 \text{ ft} \cdot \text{ton}$$

- (b) Work to move 300 pounds 200 feet vertically:
- $200(300) = 60,000 \text{ ft} \cdot \text{lb} = 30 \text{ ft} \cdot \text{ton}$

Total work = work for drawing up the cable + work of lifting the load

$$= 40 \text{ ft} \cdot \text{ton} + 30 \text{ ft} \cdot \text{ton} = 70 \text{ ft} \cdot \text{ton}$$

46. $W = \int_a^b F(x) dx$

$$F(x) = \begin{cases} -(2/9)x + 6, & 0 \leq x \leq 9 \\ -(4/3)x + 16, & 9 \leq x \leq 12 \end{cases}$$

$$\begin{aligned} W &= \int_0^9 \left(-\frac{2}{9}x + 6 \right) dx + \int_9^{12} \left(-\frac{4}{3}x + 16 \right) dx \\ &= \left[-\frac{1}{9}x^2 + 6x \right]_0^9 + \left[-\frac{2}{3}x^2 + 16x \right]_9^{12} \\ &= (-9 + 54) + (-96 + 192 + 54 - 144) = 51 \text{ ft} \cdot \text{lbs} \end{aligned}$$

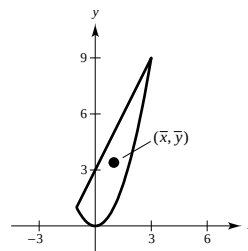
48. $A = \int_{-1}^3 [(2x + 3) - x^2] dx = \left[x^2 + 3x - \frac{1}{3}x^3 \right]_{-1}^3 = \frac{32}{3}$

$$\frac{1}{A} = \frac{3}{32}$$

$$\bar{x} = \frac{3}{32} \int_{-1}^3 x(2x + 3 - x^2) dx = \frac{3}{32} \int_{-1}^3 (3x + 2x^2 - x^3) dx = \frac{3}{32} \left[\frac{3}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_{-1}^3 = 1$$

$$\begin{aligned} \bar{y} &= \left(\frac{3}{32} \right) \frac{1}{2} \int_{-1}^3 [(2x + 3)^2 - x^4] dx = \frac{3}{64} \int_{-1}^3 (9 + 12x + 4x^2 - x^4) dx \\ &= \frac{3}{64} \left[9x + 6x^2 + \frac{4}{3}x^3 - \frac{1}{5}x^5 \right]_{-1}^3 = \frac{17}{5} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(1, \frac{17}{5} \right)$$



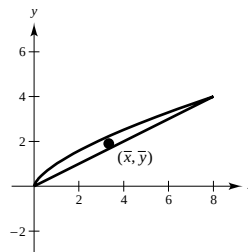
50. $A = \int_0^8 \left(x^{2/3} - \frac{1}{2}x \right) dx = \left[\frac{3}{5}x^{5/3} - \frac{1}{4}x^2 \right]_0^8 = \frac{16}{5}$

$$\frac{1}{A} = \frac{5}{16}$$

$$\begin{aligned} \bar{x} &= \frac{5}{16} \int_0^8 x \left(x^{2/3} - \frac{1}{2}x \right) dx \\ &= \frac{5}{16} \left[\frac{3}{8}x^{8/3} - \frac{1}{6}x^3 \right]_0^8 = \frac{10}{3} \end{aligned}$$

$$\begin{aligned} \bar{y} &= \left(\frac{5}{16} \right) \frac{1}{2} \int_0^8 \left(x^{4/3} - \frac{1}{4}x^2 \right) dx \\ &= \frac{1}{2} \left(\frac{5}{16} \right) \left[\frac{3}{7}x^{7/3} - \frac{1}{12}x^3 \right]_0^8 = \frac{40}{21} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(\frac{10}{3}, \frac{40}{21} \right)$$



52. Wall at shallow end:

$$F = 62.4 \int_0^5 y(20) dy = \left[(1248) \frac{y^2}{2} \right]_0^5 = 15,600 \text{ lb}$$

Wall at deep end:

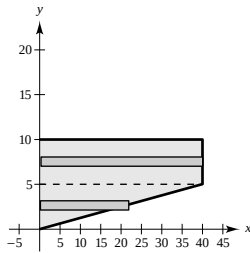
$$F = 62.4 \int_0^{10} y(20) dy = \left[(624) y^2 \right]_0^{10} = 62,400 \text{ lb}$$

Side wall:

$$F_1 = 62.4 \int_0^5 y(40) dy = \left[(1248) y^2 \right]_0^5 = 31,200 \text{ lb}$$

$$F_2 = 62.4 \int_0^5 (10 - y)8y dy = 62.4 \int_0^5 (80y - 8y^2) dy$$

$$F = F_1 + F_2 = 72,800 \text{ lb}$$



Problem Solving for Chapter 6

$$2. R = \int_0^1 x(1-x) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Let (c, mc) be the intersection of the line and the parabola.

Then, $mc = c(1-c) \Rightarrow m = 1-c$ or $c = 1-m$.

$$\frac{1}{2} \left(\frac{1}{6} \right) = \int_0^{1-m} (x - x^2 - mx) dx$$

$$\begin{aligned} \frac{1}{12} &= \left[\frac{x^2}{2} - \frac{x^3}{3} - m \frac{x^2}{2} \right]_0^{1-m} \\ &= \frac{(1-m)^2}{2} - \frac{(1-m)^3}{3} - m \frac{(1-m)^2}{2} \end{aligned}$$

$$1 = 6(1-m)^2 - 4(1-m)^3 - 6m(1-m)^2$$

$$= (1-m)^2(6 - 4(1-m) - 6m)$$

$$= (1-m)^2(2 - 2m)$$

$$\frac{1}{2} = (1-m)^3$$

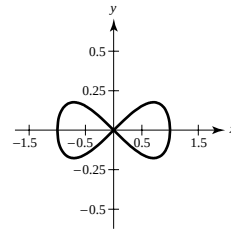
$$\left(\frac{1}{2} \right)^{1/3} = 1-m$$

$$m = 1 - \left(\frac{1}{2} \right)^{1/3} \approx 0.2063$$

$$54. F = 62.4(16\pi)5 = 4992\pi \text{ lb}$$

$$4. 8y^2 = x^2(1-x^2)$$

$$y = \pm \frac{|x|\sqrt{1-x^2}}{2\sqrt{2}}$$



$$\text{For } x > 0, y' = \frac{1-2x^2}{2\sqrt{2}\sqrt{1-x^2}}$$

$$\begin{aligned} S &= 2(2\pi) \int_0^1 x \sqrt{1 + \left(\frac{1-2x^2}{2\sqrt{2}\sqrt{1-x^2}} \right)^2} dx \\ &= \frac{5\sqrt{2}\pi}{3} \end{aligned}$$

6. By the Theorem of Pappus,

$$\begin{aligned} V &= 2\pi r A \\ &= 2\pi \left[d + \frac{1}{2} \sqrt{w^2 + l^2} \right] lw \end{aligned}$$

- 8.
- $f'(x)^2 = e^x$

$$f'(x) = e^{x/2}$$

$$f(x) = 2e^{x/2} + C$$

$$f(0) = 0 \Rightarrow C = -2$$

$$f(x) = 2e^{x/2} - 2$$

10. Let
- ρ_f
- be the density of the fluid and
- ρ_0
- the density of the iceberg. The buoyant force is

$$F = \rho_f g \int_{-h}^0 A(y) dy$$

where $A(y)$ is a typical cross section and g is the acceleration due to gravity. The weight of the object is

$$W = \rho_0 g \int_{-h}^{L-h} A(y) dy.$$

$$F = W$$

$$\rho_f g \int_{-h}^0 A(y) dy = \rho_0 g \int_{-h}^{L-h} A(y) dy$$

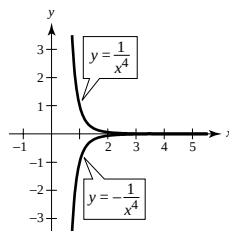
$$\frac{\rho_0}{\rho_f} = \frac{\text{submerged volume}}{\text{total volume}} = \frac{0.92 \times 10^3}{1.03 \times 10^3} = 0.893 \text{ or } 89.3\%$$

12. (a)
- $\bar{y} = 0$
- by symmetry

$$M_y = 2 \int_1^6 x \frac{1}{x^4} dx = 2 \int_1^6 \frac{1}{x^3} dx = \frac{35}{36}$$

$$m = 2 \int_1^6 \frac{1}{x^4} dx = \frac{215}{324}$$

$$\bar{x} = \frac{35/36}{215/324} = \frac{63}{43} \quad (\bar{x}, \bar{y}) = \left(\frac{63}{43}, 0 \right)$$



$$(b) M_y = 2 \int_1^b \frac{1}{x^3} dx = \frac{b^2 - 1}{b^2}$$

$$m = 2 \int_1^b \frac{1}{x^4} dx = \frac{2(b^3 - 1)}{3b^3}$$

$$\bar{x} = \frac{(b^2 - 1)/b^2}{2(b^3 - 1)/3b^3} = \frac{3b(b + 1)}{2(b^2 + b + 1)} \quad (\bar{x}, \bar{y}) = \left(\frac{3b(b + 1)}{2(b^2 + b + 1)}, 0 \right)$$

$$\lim_{b \rightarrow \infty} \bar{x} = \frac{3}{2} \quad (\bar{x}, \bar{y}) = \left(\frac{3}{2}, 0 \right)$$

14. (a) Trapezoidal: Area
- $\approx \frac{160}{2(8)}[0 + 2(50) + 2(54) + 2(82) + 2(82) + 2(73) + 2(75) + 2(80) + 0] = 9920$
- sq ft

$$(b) \text{ Simpson's: Area } \approx \frac{160}{3(8)}[0 + 4(50) + 2(54) + 4(82) + 2(82) + 4(73) + 2(75) + 4(80) + 0] = 10,413\frac{1}{3} \text{ sq ft}$$

16. Point of equilibrium:
- $1000 - 0.4x^2 = 42x$

$$x = 20, p = 840$$

$$(P_0, x_0) = (840, 20)$$

$$\text{Consumer surplus} = \int_0^{20} [(1000 - 0.4x^2) - 840] dx = 2133.33$$

$$\text{Producer surplus} = \int_0^{20} [840 - 42x] dx = 8400$$

C H A P T E R 7

Integration Techniques, L'Hôpital's Rule, and Improper Integrals

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CHAPTER 7

Integration Techniques, L'Hôpital's Rule, and Improper Integrals

Section 7.1 Basic Integration Rules

Solutions to Odd-Numbered Exercises

$$1. (a) \frac{d}{dx}[2\sqrt{x^2+1} + C] = 2\left(\frac{1}{2}\right)(x^2+1)^{-1/2}(2x) = \frac{2x}{\sqrt{x^2+1}}$$

$$(b) \frac{d}{dx}[\sqrt{x^2+1} + C] = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}$$

$$(c) \frac{d}{dx}\left[\frac{1}{2}\sqrt{x^2+1} + C\right] = \frac{1}{2}\left(\frac{1}{2}\right)(x^2+1)^{-1/2}(2x) = \frac{x}{2\sqrt{x^2+1}}$$

$$(d) \frac{d}{dx}[\ln(x^2+1) + C] = \frac{2x}{x^2+1}$$

$$\int \frac{x}{\sqrt{x^2+1}} dx \text{ matches (b).}$$

$$3. (a) \frac{d}{dx}[\ln\sqrt{x^2+1} + C] = \frac{1}{2}\left(\frac{2x}{x^2+1}\right) = \frac{x}{x^2+1}$$

$$(b) \frac{d}{dx}\left[\frac{2x}{(x^2+1)^2} + C\right] = \frac{(x^2+1)^2(2) - (2x)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2(1-3x^2)}{(x^2+1)^3}$$

$$(c) \frac{d}{dx}[\arctan x + C] = \frac{1}{1+x^2}$$

$$(d) \frac{d}{dx}[\ln(x^2+1) + C] = \frac{2x}{x^2+1}$$

$$\int \frac{1}{x^2+1} dx \text{ matches (c).}$$

$$5. \int (3x-2)^4 dx$$

$$u = 3x-2, du = 3 dx, n = 4$$

$$\text{Use } \int u^n du.$$

$$7. \int \frac{1}{\sqrt{x}(1-2\sqrt{x})} dx$$

$$u = 1-2\sqrt{x}, du = -\frac{1}{\sqrt{x}} dx$$

$$\text{Use } \int \frac{du}{u}.$$

$$9. \int \frac{3}{\sqrt{1-t^2}} dt$$

$$u = t, du = dt, a = 1$$

$$\text{Use } \int \frac{du}{\sqrt{a^2-u^2}}$$

$$11. \int t \sin t^2 dt$$

$$u = t^2, du = 2t dt$$

$$\text{Use } \int \sin u du.$$

$$13. \int \cos x e^{\sin x} dx$$

$$u = \sin x, du = \cos x dx$$

$$\text{Use } \int e^u du.$$

15. Let
- $u = -2x + 5$
- ,
- $du = -2 dx$
- .

$$\begin{aligned}\int (-2x + 5)^{3/2} dx &= -\frac{1}{2} \int (-2x + 5)^{3/2} (-2) dx \\ &= -\frac{1}{5} (-2x + 5)^{5/2} + C\end{aligned}$$

17. Let
- $u = z - 4$
- ,
- $du = dz$

$$\begin{aligned}\int \frac{5}{(z-4)^5} dz &= 5 \int (z-4)^{-5} dz = 5 \frac{(z-4)^{-4}}{-4} + C \\ &= \frac{-5}{4(z-4)^4} + C\end{aligned}$$

19. Let
- $u = t^3 - 1$
- ,
- $du = 3t^2 dt$
- .

$$\begin{aligned}\int t^2 \sqrt[3]{t^3 - 1} dt &= \frac{1}{3} \int (t^3 - 1)^{1/3} (3t^2) dt \\ &= \frac{1}{3} \frac{(t^3 - 1)^{4/3}}{4/3} + C \\ &= \frac{(t^3 - 1)^{4/3}}{4} + C\end{aligned}$$

$$\begin{aligned}21. \int \left[v + \frac{1}{(3v-1)^3} \right] dv &= \int v dv + \frac{1}{3} \int (3v-1)^{-3} (3) dv \\ &= \frac{1}{2} v^2 - \frac{1}{6(3v-1)^2} + C\end{aligned}$$

23. Let
- $u = -t^3 + 9t + 1$
- ,
- $du = (-3t^2 + 9) dt = -3(t^2 - 3) dt$
- .

$$\int \frac{t^2 - 3}{-t^3 + 9t + 1} dt = -\frac{1}{3} \int \frac{-3(t^2 - 3)}{-t^3 + 9t + 1} dt = -\frac{1}{3} \ln |-t^3 + 9t + 1| + C$$

$$\begin{aligned}25. \int \frac{x^2}{x-1} dx &= \int (x+1) dx + \int \frac{1}{x-1} dx \\ &= \frac{1}{2} x^2 + x + \ln|x-1| + C\end{aligned}$$

27. Let
- $u = 1 + e^x$
- ,
- $du = e^x dx$
- .

$$\int \frac{e^x}{1 + e^x} dx = \ln(1 + e^x) + C$$

$$29. \int (1 + 2x^2)^2 dx = \int (4x^4 + 4x^2 + 1) dx = \frac{4}{5} x^5 + \frac{4}{3} x^3 + x + C = \frac{x}{15} (12x^4 + 20x^2 + 15) + C$$

31. Let
- $u = 2\pi x^2$
- ,
- $du = 4\pi x dx$
- .

$$\begin{aligned}\int x(\cos 2\pi x^2) dx &= \frac{1}{4\pi} \int (\cos 2\pi x^2) (4\pi x) dx \\ &= \frac{1}{4\pi} \sin 2\pi x^2 + C\end{aligned}$$

33. Let
- $u = \pi x$
- ,
- $du = \pi dx$
- .

$$\int \csc(\pi x) \cot(\pi x) dx = \frac{1}{\pi} \int \csc(\pi x) \cot(\pi x) \pi dx = -\frac{1}{\pi} \csc(\pi x) + C$$

35. Let
- $u = 5x$
- ,
- $du = 5 dx$
- .

$$\int e^{5x} dx = \frac{1}{5} \int e^{5x} (5) dx = \frac{1}{5} e^{5x} + C$$

37. Let
- $u = 1 + e^x$
- ,
- $du = e^x dx$
- .

$$\begin{aligned}\int \frac{2}{e^{-x} + 1} dx &= 2 \int \left(\frac{1}{e^{-x} + 1} \right) \left(\frac{e^x}{e^x} \right) dx \\ &= 2 \int \frac{e^x}{1 + e^x} dx = 2 \ln(1 + e^x) + C\end{aligned}$$

$$39. \int \frac{\ln x^2}{x} dx = 2 \int (\ln x) \frac{1}{x} dx = 2 \frac{(\ln x)^2}{2} + C = (\ln x)^2 + C$$

$$41. \int \frac{1 + \sin x}{\cos x} dx = \int (\sec x + \tan x) dx = \ln|\sec x + \tan x| + \ln|\sec x| + C = \ln|\sec x(\sec x + \tan x)| + C$$

$$43. \frac{1}{\cos \theta - 1} = \frac{1}{\cos \theta - 1} \cdot \frac{\cos \theta + 1}{\cos \theta + 1} = \frac{\cos \theta + 1}{\cos^2 \theta - 1} = \frac{\cos \theta + 1}{-\sin^2 \theta}$$

$$= -\csc \theta \cdot \cot \theta - \csc^2 \theta$$

$$\int \frac{1}{\cos \theta - 1} d\theta = \int (-\csc \theta \cot \theta - \csc^2 \theta) d\theta$$

$$= \csc \theta + \cot \theta + C$$

$$= \frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta} + C$$

$$= \frac{1 + \cos \theta}{\sin \theta} + C$$

$$45. \int \frac{3z + 2}{z^2 + 9} dz = \frac{3}{2} \int \frac{2z}{z^2 + 9} dz + 2 \int \frac{dz}{z^2 + 9}$$

$$= \frac{3}{2} \ln(z^2 + 9) + \frac{2}{3} \arctan\left(\frac{z}{3}\right) + C$$

$$47. \text{ Let } u = 2t - 1, du = 2 dt.$$

$$\int \frac{-1}{\sqrt{1 - (2t - 1)^2}} dt = -\frac{1}{2} \int \frac{2}{\sqrt{1 - (2t - 1)^2}} dt$$

$$= -\frac{1}{2} \arcsin(2t - 1) + C$$

$$49. \text{ Let } u = \cos\left(\frac{2}{t}\right), du = \frac{2 \sin(2/t)}{t^2} dt.$$

$$\int \frac{\tan(2/t)}{t^2} dt = \frac{1}{2} \int \frac{1}{\cos(2/t)} \left[\frac{2 \sin(2/t)}{t^2} \right] dt$$

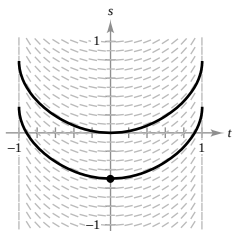
$$= \frac{1}{2} \ln \left| \cos\left(\frac{2}{t}\right) \right| + C$$

$$51. \int \frac{3}{\sqrt{6x - x^2}} dx = 3 \int \frac{1}{\sqrt{9 - (x - 3)^2}} dx = 3 \arcsin\left(\frac{x - 3}{3}\right) + C$$

$$53. \int \frac{4}{4x^2 + 4x + 65} dx = \int \frac{1}{[x + (1/2)]^2 + 16} dx = \frac{1}{4} \arctan\left[\frac{x + (1/2)}{4}\right] + C = \frac{1}{4} \arctan\left(\frac{2x + 1}{8}\right) + C$$

$$55. \frac{ds}{dt} = \frac{t}{\sqrt{1 - t^4}}, \left(0, -\frac{1}{2}\right)$$

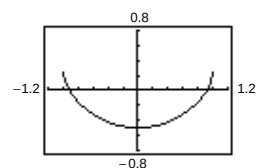
(a)

(b) $u = t^2, du = 2t dt$

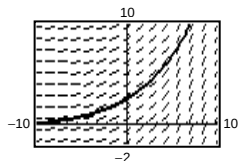
$$\int \frac{t}{\sqrt{1 - t^4}} dt = \frac{1}{2} \int \frac{2t}{\sqrt{1 - (t^2)^2}} dt = \frac{1}{2} \arcsin t^2 + C$$

$$\left(0, -\frac{1}{2}\right): -\frac{1}{2} = \frac{1}{2} \arcsin 0 + C \Rightarrow C = -\frac{1}{2}$$

$$s = \frac{1}{2} \arcsin t^2 - \frac{1}{2}$$



57.



$$y = 3e^{0.2x}$$

$$61. \frac{dy}{dx} = \frac{\sec^2 x}{4 + \tan^2 x}$$

$$\text{Let } u = \tan x, du = \sec^2 x \, dx.$$

$$y = \int \frac{\sec^2 x}{4 + \tan^2 x} dx = \frac{1}{2} \arctan\left(\frac{\tan x}{2}\right) + C$$

$$65. \text{ Let } u = -x^2, du = -2x \, dx.$$

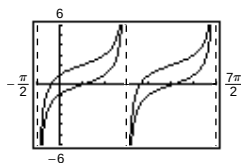
$$\begin{aligned} \int_0^1 x e^{-x^2} dx &= -\frac{1}{2} \int_0^1 e^{-x^2} (-2x) dx = \left[-\frac{1}{2} e^{-x^2} \right]_0^1 \\ &= \frac{1}{2} (1 - e^{-1}) \approx 0.316 \end{aligned}$$

$$69. \text{ Let } u = 3x, du = 3 \, dx.$$

$$\begin{aligned} \int_0^{2/\sqrt{3}} \frac{1}{4 + 9x^2} dx &= \frac{1}{3} \int_0^{2/\sqrt{3}} \frac{3}{4 + (3x)^2} dx \\ &= \left[\frac{1}{6} \arctan\left(\frac{3x}{2}\right) \right]_0^{2/\sqrt{3}} \\ &= \frac{\pi}{18} \approx 0.175 \end{aligned}$$

$$73. \int \frac{1}{1 + \sin \theta} d\theta = \tan \theta - \sec \theta + C \left(\text{or } \frac{-2}{1 + \tan(\theta/2)} \right)$$

The antiderivatives are vertical translations of each other.



$$77. \text{ Log Rule: } \int \frac{du}{u} = \ln|u| + C, u = x^2 + 1.$$

$$\begin{aligned} 59. y &= \int (1 + e^x)^2 dx = \int (e^{2x} + 2e^x + 1) dx \\ &= \frac{1}{2} e^{2x} + 2e^x + x + C \end{aligned}$$

$$63. \text{ Let } u = 2x, du = 2 \, dx.$$

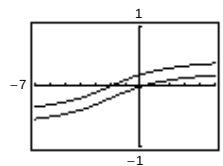
$$\begin{aligned} \int_0^{\pi/4} \cos 2x \, dx &= \frac{1}{2} \int_0^{\pi/4} \cos 2x (2) \, dx \\ &= \left[\frac{1}{2} \sin 2x \right]_0^{\pi/4} = \frac{1}{2} \end{aligned}$$

$$67. \text{ Let } u = x^2 + 9, du = 2x \, dx.$$

$$\begin{aligned} \int_0^4 \frac{2x}{\sqrt{x^2 + 9}} dx &= \int_0^4 (x^2 + 9)^{-1/2} (2x) \, dx \\ &= \left[2\sqrt{x^2 + 9} \right]_0^4 = 4 \end{aligned}$$

$$71. \int \frac{1}{x^2 + 4x + 13} dx = \frac{1}{3} \arctan\left(\frac{x+2}{3}\right) + C$$

The antiderivatives are vertical translations of each other.



$$75. \text{ Power Rule: } \int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1.$$

$$u = x^2 + 1, n = 3$$

$$79. \text{ The are equivalent because}$$

$$e^{x+C_1} = e^x \cdot e^{C_1} = C e^x, C = e^{C_1}$$

81. $\sin x + \cos x = a \sin(x + b)$

$$\sin x + \cos x = a \sin x \cos b + a \cos x \sin b$$

$$\sin x + \cos x = (a \cos b) \sin x + (a \sin b) \cos x$$

Equate coefficients of like terms to obtain the following.

$$1 = a \cos b \quad \text{and} \quad 1 = a \sin b$$

Thus, $a = 1/\cos b$. Now, substitute for a in $1 = a \sin b$.

$$1 = \left(\frac{1}{\cos b}\right) \sin b$$

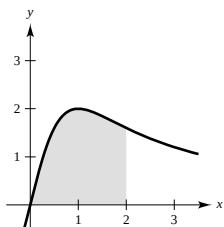
$$1 = \tan b \Rightarrow b = \frac{\pi}{4}$$

Since $b = \frac{\pi}{4}$, $a = \frac{1}{\cos(\pi/4)} = \sqrt{2}$. Thus, $\sin x + \cos x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$.

$$\int \frac{dx}{\sin x + \cos x} = \int \frac{dx}{\sqrt{2} \sin(x + (\pi/4))} = \frac{1}{\sqrt{2}} \int \csc\left(x + \frac{\pi}{4}\right) dx = -\frac{1}{\sqrt{2}} \ln \left| \csc\left(x + \frac{\pi}{4}\right) + \cot\left(x + \frac{\pi}{4}\right) \right| + C$$

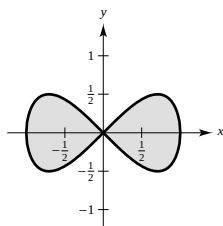
83. $\int_0^2 \frac{4x}{x^2 + 1} dx \approx 3$

Matches (a).



85. Let $u = 1 - x^2$, $du = -2x dx$.

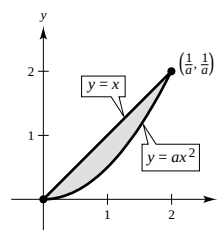
$$\begin{aligned} A &= 4 \int_0^1 x \sqrt{1 - x^2} dx \\ &= -2 \int_0^1 (1 - x^2)^{1/2} (-2x) dx \\ &= \left[-\frac{4}{3} (1 - x^2)^{3/2} \right]_0^1 = \frac{4}{3} \end{aligned}$$



87. $\int_0^{1/a} (x - ax^2) dx = \left[\frac{1}{2}x^2 - \frac{a}{3}x^3 \right]_0^{1/a}$

$$= \frac{1}{6a^2}$$

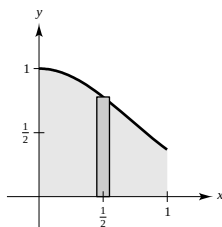
Let $\frac{1}{6a^2} = \frac{2}{3}$, $12a^2 = 3$, $a = \frac{1}{2}$.



89. (a) **Shell Method:**

Let $u = -x^2$, $du = -2x dx$.

$$\begin{aligned} V &= 2\pi \int_0^1 x e^{-x^2} dx \\ &= -\pi \int_0^1 e^{-x^2} (-2x) dx \\ &= \left[-\pi e^{-x^2} \right]_0^1 \\ &= \pi(1 - e^{-1}) \approx 1.986 \end{aligned}$$

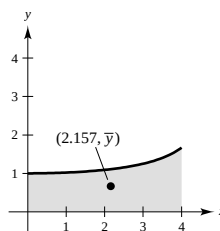


(b) **Shell Method:**

$$\begin{aligned} V &= 2\pi \int_0^b x e^{-x^2} dx \\ &= \left[-\pi e^{-x^2} \right]_0^b \\ &= \pi(1 - e^{-b^2}) = \frac{4}{3} \\ e^{-b^2} &= \frac{3\pi - 4}{3\pi} \\ b &= \sqrt{\ln\left(\frac{3\pi}{3\pi - 4}\right)} \\ &\approx 0.743 \end{aligned}$$

$$91. A = \int_0^4 \frac{5}{\sqrt{25-x^2}} dx = \left[5 \arcsin \frac{x}{5} \right]_0^4 = 5 \arcsin \frac{4}{5}$$

$$\begin{aligned} \bar{x} &= \frac{1}{A} \int_0^4 x \left(\frac{5}{\sqrt{25-x^2}} \right) dx \\ &= \frac{1}{5 \arcsin(4/5)} \left(-\frac{5}{2} \right) \int_0^4 (25-x^2)^{-1/2} (-2x) dx \\ &= \frac{1}{5 \arcsin(4/5)} (-5) \left[(25-x^2)^{1/2} \right]_0^4 \\ &= -\frac{1}{\arcsin(4/5)} [3-5] \\ &= \frac{2}{\arcsin(4/5)} \approx 2.157 \end{aligned}$$



$$93. \quad y = \tan(\pi x)$$

$$y' = \pi \sec^2(\pi x)$$

$$1 + (y')^2 = 1 + \pi^2 \sec^4(\pi x)$$

$$\begin{aligned} s &= \int_0^{1/4} \sqrt{1 + \pi^2 \sec^4(\pi x)} dx \\ &\approx 1.0320 \end{aligned}$$

Section 7.2 Integration by Parts

$$1. \frac{d}{dx} [\sin x - x \cos x] = \cos x - (-x \sin x + \cos x) = x \sin x. \text{ Matches (b)}$$

$$3. \frac{d}{dx} [x^2 e^x - 2x e^x + 2e^x] = x^2 e^x + 2x e^x - 2x e^x - 2e^x + 2e^x = x^2 e^x. \text{ Matches (c)}$$

$$5. \int x e^{2x} dx$$

$$u = x, dv = e^{2x} dx$$

$$7. \int (\ln x)^2 dx$$

$$u = (\ln x)^2, dv = dx$$

$$9. \int x \sec^2 x dx$$

$$u = x, dv = \sec^2 x dx$$

$$11. dv = e^{-2x} dx \Rightarrow v = \int e^{-2x} dx = -\frac{1}{2} e^{-2x}$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x e^{-2x} dx &= -\frac{1}{2} x e^{-2x} - \int -\frac{1}{2} e^{-2x} dx \\ &= -\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} + C = \frac{-1}{4e^{2x}} (2x + 1) + C \end{aligned}$$

13. Use integration by parts three times.

$$(1) \, dv = e^x dx \Rightarrow v = \int e^x dx = e^x \quad (2) \, dv = e^x dx \Rightarrow v = \int e^x dx = e^x \quad (3) \, dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = x^3 \Rightarrow du = 3x^2 dx \quad u = x^2 \Rightarrow du = 2x dx \quad u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x^3 e^x dx &= x^3 e^x - 3 \int x^2 e^x dx = x^3 e^x - 3x^2 e^x + 6 \int x e^x dx \\ &= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C = e^x(x^3 - 3x^2 + 6x - 6) + C \end{aligned}$$

$$15. \int x^2 e^{x^3} dx = \frac{1}{3} \int e^{x^3} (3x^2) dx = \frac{1}{3} e^{x^3} + C$$

$$17. \, dv = t \, dt \Rightarrow v = \int t \, dt = \frac{t^2}{2}$$

$$u = \ln(t+1) \Rightarrow du = \frac{1}{t+1} dt$$

$$\begin{aligned} \int t \ln(t+1) dt &= \frac{t^2}{2} \ln(t+1) - \frac{1}{2} \int \frac{t^2}{t+1} dt \\ &= \frac{t^2}{2} \ln(t+1) - \frac{1}{2} \int \left(t - 1 + \frac{1}{t+1} \right) dt \\ &= \frac{t^2}{2} \ln(t+1) - \frac{1}{2} \left[\frac{t^2}{2} - t + \ln(t+1) \right] + C \\ &= \frac{1}{4} [2(t^2 - 1) \ln|t+1| - t^2 + 2t] + C \end{aligned}$$

$$19. \text{ Let } u = \ln x, \, du = \frac{1}{x} dx.$$

$$\int \frac{(\ln x)^2}{x} dx = \int (\ln x)^2 \left(\frac{1}{x} \right) dx = \frac{(\ln x)^3}{3} + C$$

$$21. \, dv = \frac{1}{(2x+1)^2} dx \Rightarrow v = \int (2x+1)^{-2} dx = -\frac{1}{2(2x+1)}$$

$$u = xe^{2x} \Rightarrow du = (2xe^{2x} + e^{2x}) dx = e^{2x}(2x+1) dx$$

$$\begin{aligned} \int \frac{xe^{2x}}{(2x+1)^2} dx &= -\frac{xe^{2x}}{2(2x+1)} + \int \frac{e^{2x}}{2} dx \\ &= \frac{-xe^{2x}}{2(2x+1)} + \frac{e^{2x}}{4} + C \\ &= \frac{e^{2x}}{4(2x+1)} + C \end{aligned}$$

23. Use integration by parts twice.

$$(1) \, dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = x^2 \Rightarrow du = 2x dx$$

$$(2) \, dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int (x^2 - 1)e^x dx &= \int x^2 e^x dx - \int e^x dx = x^2 e^x - 2 \int x e^x dx - e^x \\ &= x^2 e^x - 2 \left[x e^x - \int e^x dx \right] - e^x = x^2 e^x - 2x e^x + e^x + C = (x-1)^2 e^x + C \end{aligned}$$

$$25. dv = \sqrt{x-1} dx \Rightarrow v = \int (x-1)^{1/2} dx = \frac{2}{3}(x-1)^{3/2}$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x\sqrt{x-1} dx &= \frac{2}{3}x(x-1)^{3/2} - \frac{2}{3} \int (x-1)^{3/2} dx \\ &= \frac{2}{3}x(x-1)^{3/2} - \frac{4}{15}(x-1)^{5/2} + C \\ &= \frac{2(x-1)^{3/2}}{15}(3x+2) + C \end{aligned}$$

$$27. dv = \cos x dx \Rightarrow v = \int \cos x dx = \sin x$$

$$u = x \Rightarrow du = dx$$

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + C$$

29. Use integration by parts three times.

$$(1) u = x^3, du = 3x^2, dv = \sin x dx, v = -\cos x$$

$$\int x^3 \sin x dx = -x^3 \cos x + 3 \int x^2 \cos x dx$$

$$(2) u = x^2, du = 2x dx, dv = \cos x dx, v = \sin x$$

$$\begin{aligned} \int x^3 \sin x dx &= -x^3 \cos x + 3 \left[x^2 \sin x - 2 \int x \sin x dx \right] \\ &= -x^3 \cos x + 3x^2 \sin x - 6 \int x \sin x dx \end{aligned}$$

$$(3) u = x, du = dx, dv = \sin x dx, v = -\cos x$$

$$\begin{aligned} \int x^3 \sin x dx &= -x^3 \cos x + 3x^2 \sin x - 6 \left[-x \cos x + \int \cos x dx \right] \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C \end{aligned}$$

$$31. u = t, du = dt, dv = \csc t \cot t dt, v = -\csc t$$

$$\begin{aligned} \int t \csc t \cot t dt &= -t \csc t + \int \csc t dt \\ &= -t \csc t - \ln |\csc t + \cot t| + C \end{aligned}$$

$$33. dv = dx \Rightarrow v = \int dx = x$$

$$u = \arctan x \Rightarrow du = \frac{1}{1+x^2} dx$$

$$\begin{aligned} \int \arctan x dx &= x \arctan x - \int \frac{x}{1+x^2} dx \\ &= x \arctan x - \frac{1}{2} \ln(1+x^2) + C \end{aligned}$$

35. Use integration by parts twice.

$$(1) dv = e^{2x} dx \Rightarrow v = \frac{1}{2} e^{2x}$$

$$u = \sin x \Rightarrow du = \cos x dx$$

$$\int e^{2x} \sin x dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \int e^{2x} \cos x dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \left(\frac{1}{2} e^{2x} \cos x + \frac{1}{2} \int e^{2x} \sin x dx \right)$$

$$\frac{5}{4} \int e^{2x} \sin x dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x$$

$$\int e^{2x} \sin x dx = \frac{1}{5} e^{2x} (2 \sin x - \cos x) + C$$

$$(2) dv = e^{2x} dx \Rightarrow v = \frac{1}{2} e^{2x}$$

$$u = \cos x \Rightarrow du = -\sin x dx$$

37. $y' = xe^{x^2}$

$$y = \int xe^{x^2} dx = \frac{1}{2}e^{x^2} + C$$

39. Use integration by parts twice.

$$(1) \quad dv = \frac{1}{\sqrt{2+3t}} dt \Rightarrow v = \int (2+3t)^{-1/2} dt = \frac{2}{3}\sqrt{2+3t}$$

$$u = t^2 \Rightarrow du = 2t dt$$

$$(2) \quad dv = \sqrt{2+3t} dt \Rightarrow v = \int (2+3t)^{1/2} dt = \frac{2}{9}(2+3t)^{3/2}$$

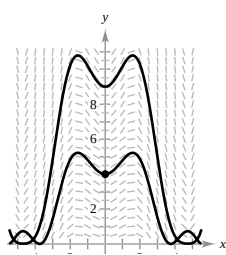
$$u = t \Rightarrow du = dt$$

$$\begin{aligned} y &= \int \frac{t^2}{\sqrt{2+3t}} dt = \frac{2t^2\sqrt{2+3t}}{3} - \frac{4}{3} \int t\sqrt{2+3t} dt \\ &= \frac{2t^2\sqrt{2+3t}}{3} - \frac{4}{3} \left[\frac{2t}{9}(2+3t)^{3/2} - \frac{2}{9} \int (2+3t)^{3/2} dt \right] \\ &= \frac{2t^2\sqrt{2+3t}}{3} - \frac{8t}{27}(2+3t)^{3/2} + \frac{16}{405}(2+3t)^{5/2} + C \\ &= \frac{2\sqrt{2+3t}}{405}(27t^2 - 24t + 32) + C \end{aligned}$$

41. $(\cos y)y' = 2x$

$$\begin{aligned} \int \cos y dy &= \int 2x dx \\ \sin y &= x^2 + C \end{aligned}$$

43. (a)



(b) $\frac{dy}{dx} = x\sqrt{y} \cos x, (0, 4)$

$$\int \frac{dy}{\sqrt{y}} = \int x \cos x dx$$

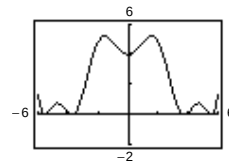
$$\int y^{-1/2} dy = \int x \cos x dx \quad (u = x, du = dx, dv = \cos x dx, v = \sin x)$$

$$2y^{1/2} = x \sin x - \int \sin x dx$$

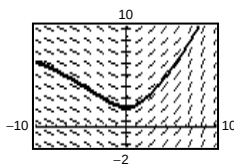
$$= x \sin x + \cos x + C$$

$$(0, 4): 2(4)^{1/2} = 0 + 1 + C \Rightarrow C = 3$$

$$2\sqrt{y} = x \sin x + \cos x + 3$$



45. $\frac{dy}{dx} = \frac{x}{y} e^{x/8}, y(0) = 2$



47. $u = x, du = dx, dv = e^{-x/2} dx, v = -2e^{-x/2}$

$$\int x e^{-x/2} dx = -2x e^{-x/2} + \int 2e^{-x/2} dx = -2x e^{-x/2} - 4e^{-x/2} + C$$

$$\begin{aligned} \text{Thus, } \int_0^4 x e^{-x/2} dx &= \left[-2x e^{-x/2} - 4e^{-x/2} \right]_0^4 \\ &= -8e^{-2} - 4e^{-2} + 4 \\ &= -12e^{-2} + 4 \approx 2.376. \end{aligned}$$

49. See Exercise 27.

$$\int_0^{\pi/2} x \cos x dx = \left[x \sin x + \cos x \right]_0^{\pi/2} = \frac{\pi}{2} - 1$$

51. $u = \arccos x, du = -\frac{1}{\sqrt{1-x^2}} dx, dv = dx, v = x$

$$\int \arccos x dx = x \arccos x + \int \frac{x}{\sqrt{1-x^2}} dx = x \arccos x - \sqrt{1-x^2} + C$$

$$\begin{aligned} \text{Thus, } \int_0^{1/2} \arccos x dx &= \left[x \arccos x - \sqrt{1-x^2} \right]_0^{1/2} \\ &= \frac{1}{2} \arccos\left(\frac{1}{2}\right) - \sqrt{\frac{3}{4}} + 1 \\ &= \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1 \approx 0.658. \end{aligned}$$

53. Use integration by parts twice.

$$(1) dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = \sin x \Rightarrow du = \cos x dx$$

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx = e^x \sin x - e^x \cos x - \int e^x \sin x dx$$

$$2 \int e^x \sin x dx = e^x (\sin x - \cos x)$$

$$\int e^x \sin x dx = \frac{e^x}{2} (\sin x - \cos x) + C$$

$$\text{Thus, } \int_0^1 e^x \sin x dx = \left[\frac{e^x}{2} (\sin x - \cos x) \right]_0^1 = \frac{e}{2} (\sin 1 - \cos 1) + \frac{1}{2} = \frac{e(\sin 1 - \cos 1) + 1}{2} \approx 0.909.$$

$$(2) dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = \cos x \Rightarrow du = -\sin x dx$$

55. $dv = x^2 dx, v = \frac{x^3}{3}, u = \ln x, du = \frac{1}{x} dx$

$$\int x^2 \ln x dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \left(\frac{1}{x} \right) dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx$$

$$\text{Hence, } \int_1^2 x^2 \ln x dx = \left[\frac{x^3}{3} \ln x - \frac{1}{9} x^3 \right]_1^2$$

$$= \frac{8}{3} \ln 2 - \frac{8}{9} + \frac{1}{9} = \frac{8}{3} \ln 2 - \frac{7}{9} \approx 1.071.$$

57. $dv = x dx$, $v = \frac{x^2}{2}$, $u = \operatorname{arcsec} x$, $du = \frac{1}{x\sqrt{x^2-1}} dx$

$$\begin{aligned}\int x \operatorname{arcsec} x dx &= \frac{x^2}{2} \operatorname{arcsec} x - \int \frac{x^2/2}{x\sqrt{x^2-1}} dx \\ &= \frac{x^2}{2} \operatorname{arcsec} x - \frac{1}{4} \int \frac{2x}{\sqrt{x^2-1}} dx \\ &= \frac{x^2}{2} \operatorname{arcsec} x - \frac{1}{2} \sqrt{x^2-1} + C\end{aligned}$$

Hence,

$$\begin{aligned}\int_2^4 x \operatorname{arcsec} x dx &= \left[\frac{x^2}{2} \operatorname{arcsec} x - \frac{1}{2} \sqrt{x^2-1} \right]_2^4 \\ &= \left(8 \operatorname{arcsec} 4 - \frac{\sqrt{15}}{2} \right) - \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) \\ &= 8 \operatorname{arcsec} 4 - \frac{\sqrt{15}}{2} + \frac{\sqrt{3}}{2} - \frac{2\pi}{3} \\ &\approx 7.380.\end{aligned}$$

59. $\int x^2 e^{2x} dx = x^2 \left(\frac{1}{2} e^{2x} \right) - (2x) \left(\frac{1}{4} e^{2x} \right) + 2 \left(\frac{1}{8} e^{2x} \right) + C$

$$\begin{aligned}&= \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} + C \\ &= \frac{1}{4} e^{2x} (2x^2 - 2x + 1) + C\end{aligned}$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^2	e^{2x}
−	$2x$	$\frac{1}{2} e^{2x}$
+	2	$\frac{1}{4} e^{2x}$
−	0	$\frac{1}{8} e^{2x}$

61. $\int x^3 \sin x dx = x^3(-\cos x) - 3x^2(-\sin x) + 6x \cos x - 6 \sin x + C$

$$\begin{aligned}&= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C \\ &= (3x^2 - 6) \sin x - (x^3 - 6x) \cos x + C\end{aligned}$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^3	$\sin x$
−	$3x^2$	$-\cos x$
+	$6x$	$-\sin x$
−	6	$\cos x$
+	0	$\sin x$

63. $\int x \sec^2 x dx = x \tan x + \ln|\cos x| + C$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x	$\sec^2 x$
−	1	$\tan x$
+	0	$-\ln \cos x $

65. Integration by parts is based on the product rule.

67. No. Substitution.

69. Yes. $u = x^2$, $dv = e^{2x} dx$

71. Yes. Let $u = x$ and $du = \frac{1}{\sqrt{x+1}} dx$.

(Substitution also works. Let $u = \sqrt{x+1}$)

73. $\int t^3 e^{-4t} dt = -\frac{e^{-4t}}{128} (32t^3 + 24t^2 + 12t + 3) + C$

75. $\int_0^{\pi/2} e^{-2x} \sin 3x dx = \left[\frac{e^{-2x}(-2 \sin 3x - 3 \cos 3x)}{13} \right]_0^{\pi/2} = \frac{1}{13} (2e^{-\pi} + 3) \approx 0.2374$

$$77. (a) \, dv = \sqrt{2x-3} \, dx \Rightarrow v = \int (2x-3)^{1/2} dx = \frac{1}{3}(2x-3)^{3/2}$$

$$u = 2x \quad \Rightarrow \quad du = 2 \, dx$$

$$\begin{aligned} \int 2x\sqrt{2x-3} \, dx &= \frac{2}{3}x(2x-3)^{3/2} - \frac{2}{3} \int (2x-3)^{3/2} dx \\ &= \frac{2}{3}x(2x-3)^{3/2} - \frac{2}{15}(2x-3)^{5/2} + C \\ &= \frac{2}{15}(2x-3)^{3/2}(3x+3) + C = \frac{2}{5}(2x-3)^{3/2}(x+1) + C \end{aligned}$$

$$(b) \, u = 2x-3 \Rightarrow x = \frac{u+3}{2} \text{ and } dx = \frac{1}{2} du$$

$$\begin{aligned} \int 2x\sqrt{2x-3} \, dx &= \int 2\left(\frac{u+3}{2}\right) u^{1/2} \left(\frac{1}{2}\right) du = \frac{1}{2} \int (u^{3/2} + 3u^{1/2}) du = \frac{1}{2} \left[\frac{2}{5} u^{5/2} + 2u^{3/2} \right] + C \\ &= \frac{1}{5} u^{3/2}(u+5) + C = \frac{1}{5} (2x-3)^{3/2} [(2x-3)+5] + C = \frac{2}{5} (2x-3)^{3/2}(x+1) + C \end{aligned}$$

$$79. (a) \, dv = \frac{x}{\sqrt{4+x^2}} dx \Rightarrow v = \int (4+x^2)^{-1/2} x \, dx = \sqrt{4+x^2}$$

$$u = x^2 \Rightarrow du = 2x \, dx$$

$$\begin{aligned} \int \frac{x^3}{\sqrt{4+x^2}} dx &= x^2 \sqrt{4+x^2} - 2 \int x \sqrt{4+x^2} dx \\ &= x^2 \sqrt{4+x^2} - \frac{2}{3} (4+x^2)^{3/2} + C = \frac{1}{3} \sqrt{4+x^2} (x^2-8) + C \end{aligned}$$

$$(b) \, u = 4+x^2 \Rightarrow x^2 = u-4 \text{ and } 2x \, dx = du \Rightarrow x \, dx = \frac{1}{2} du$$

$$\begin{aligned} \int \frac{x^3}{\sqrt{4+x^2}} dx &= \int \frac{x^2}{\sqrt{4+x^2}} x \, dx = \int \frac{u-4}{\sqrt{u}} \frac{1}{2} du \\ &= \frac{1}{2} \int (u^{1/2} - 4u^{-1/2}) du = \frac{1}{2} \left(\frac{2}{3} u^{3/2} - 8u^{1/2} \right) + C \\ &= \frac{1}{3} u^{1/2}(u-12) + C = \frac{1}{3} \sqrt{4+x^2} [(4+x^2)-12] + C = \frac{1}{3} \sqrt{4+x^2} (x^2-8) + C \end{aligned}$$

$$81. \, n=0: \int \ln x \, dx = x(\ln x - 1) + C$$

$$n=1: \int x \ln x \, dx = \frac{x^2}{4}(2 \ln x - 1) + C$$

$$n=2: \int x^2 \ln x \, dx = \frac{x^3}{9}(3 \ln x - 1) + C$$

$$n=3: \int x^3 \ln x \, dx = \frac{x^4}{16}(4 \ln x - 1) + C$$

$$n=4: \int x^4 \ln x \, dx = \frac{x^5}{25}(5 \ln x - 1) + C$$

$$\text{In general, } \int x^n \ln x \, dx = \frac{x^{n+1}}{(n+1)^2} [(n+1) \ln x - 1] + C. \text{ (See Exercise 85.)}$$

$$83. dv = \sin x \, dx \Rightarrow v = -\cos x$$

$$u = x^n \Rightarrow du = nx^{n-1} \, dx$$

$$\int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx$$

$$85. dv = x^n \, dx \Rightarrow v = \frac{x^{n+1}}{n+1}$$

$$u = \ln x \Rightarrow du = \frac{1}{x} \, dx$$

$$\begin{aligned} \int x^n \ln x \, dx &= \frac{x^{n+1}}{n+1} \ln x - \int \frac{x^n}{n+1} \, dx \\ &= \frac{x^{n+1}}{n+1} \ln x - \frac{x^{n+1}}{(n+1)^2} + C \\ &= \frac{x^{n+1}}{(n+1)^2} [(n+1) \ln x - 1] + C \end{aligned}$$

87. Use integration by parts twice.

$$(1) dv = e^{ax} \, dx \Rightarrow v = \frac{1}{a} e^{ax}$$

$$u = \sin bx \Rightarrow du = b \cos bx \, dx$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax} \sin bx}{a} - \frac{b}{a} \int e^{ax} \cos bx \, dx$$

$$= \frac{e^{ax} \sin bx}{a} - \frac{b}{a} \left[\frac{e^{ax} \cos bx}{a} + \frac{b}{a} \int e^{ax} \sin bx \, dx \right] = \frac{e^{ax} \sin bx}{a} - \frac{b^2}{a^2} \int e^{ax} \sin bx \, dx$$

$$\text{Therefore, } \left(1 + \frac{b^2}{a^2}\right) \int e^{ax} \sin bx \, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2}$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} + C.$$

$$(2) dv = e^{ax} \, dx \Rightarrow v = \frac{1}{a} e^{ax}$$

$$u = \cos bx \Rightarrow du = -b \sin bx \, dx$$

89. $n = 3$ (Use formula in Exercise 85.)

$$\int x^3 \ln x \, dx = \frac{x^4}{16} [4 \ln x - 1] + C$$

91. $a = 2, b = 3$ (Use formula in Exercise 88.)

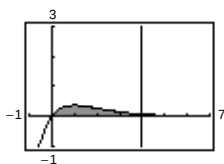
$$\int e^{2x} \cos 3x \, dx = \frac{e^{2x}(2 \cos 3x + 3 \sin 3x)}{13} + C$$

$$93. dv = e^{-x} \, dx \Rightarrow v = -e^{-x}$$

$$u = x \Rightarrow du = dx$$

$$A = \int_0^4 x e^{-x} \, dx = \left[-x e^{-x} \right]_0^4 + \int_0^4 e^{-x} \, dx = \frac{-4}{e^4} - \left[e^{-x} \right]_0^4$$

$$= 1 - \frac{5}{e^4} \approx 0.908$$

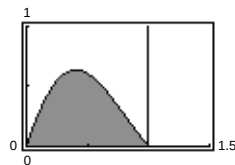


$$95. A = \int_0^1 e^{-x} \sin(\pi x) \, dx$$

$$= \left[\frac{e^{-x}(-\sin \pi x - \pi \cos \pi x)}{1 + \pi^2} \right]_0^1$$

$$= \frac{1}{1 + \pi^2} \left(\frac{\pi}{e} + \pi \right) = \frac{\pi}{1 + \pi^2} \left(\frac{1}{e} + 1 \right)$$

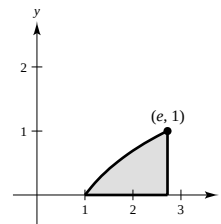
$$\approx 0.395 \text{ (See Exercise 87.)}$$



97. (a) $A = \int_1^e \ln x \, dx = \left[-x + x \ln x \right]_1^e = 1$ (See Exercise 4.)

(b) $R(x) = \ln x, r(x) = 0$

$$\begin{aligned} V &= \pi \int_1^e (\ln x)^2 \, dx \\ &= \pi \left[x(\ln x)^2 - 2x \ln x + 2x \right]_1^e \quad (\text{Use integration by parts twice, see Exercise 7.}) \\ &= \pi(e - 2) \approx 2.257 \end{aligned}$$



(c) $p(x) = x, h(x) = \ln x$

$$\begin{aligned} V &= 2\pi \int_1^e x \ln x \, dx = 2\pi \left[\frac{x^2}{4}(-1 + 2 \ln x) \right]_1^e \\ &= \frac{(e^2 + 1)\pi}{2} \approx 13.177 \quad (\text{See Exercise 85.}) \end{aligned}$$

(d) $\bar{x} = \frac{\int_1^e x \ln x \, dx}{1} = \frac{e^2 + 1}{4} \approx 2.097$

$$\bar{y} = \frac{\frac{1}{2} \int_1^e (\ln x)^2 \, dx}{1} = \frac{e - 2}{2} \approx 0.359$$

$$(\bar{x}, \bar{y}) = \left(\frac{e^2 + 1}{4}, \frac{e - 2}{2} \right) \approx (2.097, 0.359)$$

99. Average value $= \frac{1}{\pi} \int_0^\pi e^{-4t}(\cos 2t + 5 \sin 2t) \, dt$

$$\begin{aligned} &= \frac{1}{\pi} \left[e^{-4t} \left(\frac{-4 \cos 2t + 2 \sin 2t}{20} \right) + 5e^{-4t} \left(\frac{-4 \sin 2t - 2 \cos 2t}{20} \right) \right]_0^\pi \quad (\text{From Exercises 87 and 88}) \\ &= \frac{7}{10\pi} (1 - e^{-4\pi}) \approx 0.223 \end{aligned}$$

101. $c(t) = 100,000 + 4000t, r = 5\%, t_1 = 10$

$$P = \int_0^{10} (100,000 + 4000t)e^{-0.05t} \, dt = 4000 \int_0^{10} (25 + t)e^{-0.05t} \, dt$$

Let $u = 25 + t, dv = e^{-0.05t} \, dt, du = dt, v = -\frac{100}{5}e^{-0.05t}$

$$\begin{aligned} P &= 4000 \left\{ \left[(25 + t) \left(-\frac{100}{5}e^{-0.05t} \right) \right]_0^{10} + \frac{100}{5} \int_0^{10} e^{-0.05t} \, dt \right\} \\ &= 4000 \left\{ \left[(25 + t) \left(-\frac{100}{5}e^{-0.05t} \right) \right]_0^{10} - \left[\frac{10,000}{25}e^{-0.05t} \right]_0^{10} \right\} \approx \$931,265 \end{aligned}$$

103. $\int_{-\pi}^\pi x \sin nx \, dx = \left[-\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \right]_{-\pi}^\pi$

$$= -\frac{\pi}{n} \cos \pi n - \frac{\pi}{n} \cos(-\pi n)$$

$$= -\frac{2\pi}{n} \cos \pi n$$

$$= \begin{cases} -(2\pi/n), & \text{if } n \text{ is even} \\ (2\pi/n), & \text{if } n \text{ is odd} \end{cases}$$

105. Let $u = x$, $dv = \sin\left(\frac{n\pi}{2}x\right)dx$, $du = dx$, $v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$.

$$\begin{aligned} I_1 &= \int_0^1 x \sin\left(\frac{n\pi}{2}x\right) dx = \left[\frac{-2x}{n\pi} \cos\left(\frac{n\pi}{2}x\right) \right]_0^1 + \frac{2}{n\pi} \int_0^1 \cos\left(\frac{n\pi}{2}x\right) dx \\ &= -\frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right) \right]_0^1 \\ &= -\frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

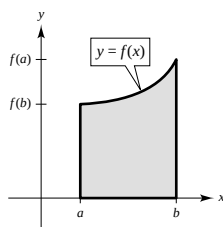
Let $u = (-x + 2)$, $dv = \sin\left(\frac{n\pi}{2}x\right)dx$, $du = -dx$, $v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$.

$$\begin{aligned} I_2 &= \int_1^2 (-x + 2) \sin\left(\frac{n\pi}{2}x\right) dx = \left[\frac{-2(-x + 2)}{n\pi} \cos\left(\frac{n\pi}{2}x\right) \right]_1^2 - \frac{2}{n\pi} \int_1^2 \cos\left(\frac{n\pi}{2}x\right) dx \\ &= \frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) - \left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right) \right]_1^2 \\ &= \frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

$$h(I_1 + I_2) = b_n = h \left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \right] = \frac{8h}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right)$$

107. Shell Method:

$$\begin{aligned} V &= 2\pi \int_a^b x f(x) dx \\ dv &= x dx \Rightarrow v = \frac{x^2}{2} \\ u &= f(x) \Rightarrow du = f'(x) dx \\ V &= 2\pi \left[\frac{x^2}{2} f(x) - \int \frac{x^2}{2} f'(x) dx \right]_a^b \\ &= \pi \left[(b^2 f(b) - a^2 f(a)) - \int_a^b x^2 f'(x) dx \right] \end{aligned}$$



Disk Method:

$$\begin{aligned} V &= \pi \int_0^{f(a)} (b^2 - a^2) dy + \pi \int_{f(a)}^{f(b)} [b^2 - [f^{-1}(y)]^2] dy \\ &= \pi(b^2 - a^2) f(a) + \pi b^2 (f(b) - f(a)) - \pi \int_{f(a)}^{f(b)} [f^{-1}(y)]^2 dy \\ &= \pi \left[(b^2 f(b) - a^2 f(a)) - \int_{f(a)}^{f(b)} [f^{-1}(y)]^2 dy \right] \end{aligned}$$

Since $x = f^{-1}(y)$, we have $f(x) = y$ and $f'(x)dx = dy$. When $y = f(a)$, $x = a$. When $y = f(b)$, $x = b$. Thus,

$$\int_{f(a)}^{f(b)} [f^{-1}(y)]^2 dy = \int_a^b x^2 f'(x) dx$$

and the volumes are the same.

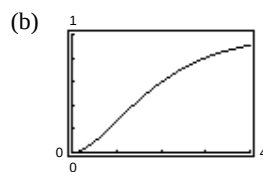
109. $f'(x) = xe^{-x}$

(a) $f(x) = \int xe^{-x} dx = -xe^{-x} - e^{-x} + C$

(Parts: $u = x, dv = e^{-x} dx$)

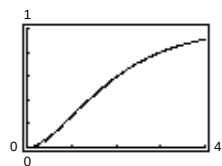
$f(0) = 0 = -1 + C \Rightarrow C = 1$

$f(x) = -xe^{-x} - e^{-x} + 1$



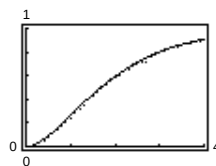
(c) You obtain the points

n	x_n	y_n
0	0	0
1	0.05	0
2	0.10	2.378×10^{-3}
3	0.15	0.0069
4	0.20	0.0134
$\vdots$	$\vdots$	$\vdots$
80	4.0	0.9064



(d) You obtain the points

n	x_n	y_n
0	0	0
1	0.1	0
2	0.2	0.0090484
3	0.3	0.025423
4	0.4	0.047648
$\vdots$	$\vdots$	$\vdots$
40	4.0	0.9039



(e) $f(4) = 0.9084$

The approximations are tangent line approximations. The results in (c) are better because Δx is smaller.

Section 7.3 Trigonometric Integrals

1. $f(x) = \sin^4 x + \cos^4 x$

$$\begin{aligned}
 \text{(a) } \sin^4 x + \cos^4 x &= \left(\frac{1 - \cos 2x}{2}\right)^2 + \left(\frac{1 + \cos 2x}{2}\right)^2 \\
 &= \frac{1}{4}[1 - 2\cos 2x + \cos^2 2x + 1 + 2\cos 2x + \cos^2 2x] \\
 &= \frac{1}{4}\left[2 + 2\frac{1 + \cos 4x}{2}\right] \\
 &= \frac{1}{4}[3 + \cos 4x]
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } \sin^4 x + \cos^4 x &= (\sin^2 x)^2 + \cos^4 x \\
 &= (1 - \cos^2 x)^2 + \cos^4 x \\
 &= 1 - 2\cos^2 x + 2\cos^4 x
 \end{aligned}$$

$$\begin{aligned}
 \text{(c) } \sin^4 x + \cos^4 x &= \sin^4 x + 2\sin^2 x \cos^2 x + \cos^4 x - 2\sin^2 x \cos^2 x \\
 &= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x \\
 &= 1 - 2\sin^2 x \cos^2 x
 \end{aligned}$$

—CONTINUED—

1. —CONTINUED—

$$\begin{aligned}
 \text{(d) } 1 - 2 \sin^2 x \cos^2 x &= 1 - (2 \sin x \cos x)(\sin x \cos x) \\
 &= 1 - (\sin 2x) \left(\frac{1}{2} \sin 2x \right) \\
 &= 1 - \frac{1}{2} \sin^2(2x)
 \end{aligned}$$

(e) Four ways. There is often more than one way to rewrite a trigonometric expression.

3. Let $u = \cos x$, $du = -\sin x \, dx$.

$$\begin{aligned}
 \int \cos^3 x \sin x \, dx &= -\int \cos^3 x (-\sin x) \, dx \\
 &= -\frac{1}{4} \cos^4 x + C
 \end{aligned}$$

5. Let $u = \sin 2x$, $du = 2 \cos 2x \, dx$.

$$\begin{aligned}
 \int \sin^5 2x \cos 2x \, dx &= \frac{1}{2} \int \sin^5 2x (2 \cos 2x) \, dx \\
 &= \frac{1}{12} \sin^6 2x + C
 \end{aligned}$$

7. Let $u = \cos x$, $du = -\sin x \, dx$.

$$\begin{aligned}
 \int \sin^5 x \cos^2 x \, dx &= \int \sin x (1 - \cos^2 x)^2 \cos^2 x \, dx \\
 &= -\int (\cos^2 x - 2 \cos^4 x + \cos^6 x)(-\sin x) \, dx = \frac{-1}{3} \cos^3 x + \frac{2}{5} \cos^5 x - \frac{1}{7} \cos^7 x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{9. } \int \cos^3 \theta \sqrt{\sin \theta} \, d\theta &= \int \cos \theta (1 - \sin^2 \theta)(\sin \theta)^{1/2} \, d\theta \\
 &= \int [(\sin \theta)^{1/2} - (\sin \theta)^{5/2}] \cos \theta \, d\theta \\
 &= \frac{2}{3} (\sin \theta)^{3/2} - \frac{2}{7} (\sin \theta)^{7/2} + C
 \end{aligned}$$

$$\begin{aligned}
 \text{11. } \int \cos^2 3x \, dx &= \int \frac{1 + \cos 6x}{2} \, dx \\
 &= \frac{1}{2} \left(x + \frac{1}{6} \sin 6x \right) + C \\
 &= \frac{1}{12} (6x + \sin 6x) + C
 \end{aligned}$$

$$\begin{aligned}
 \text{13. } \int \sin^2 \alpha \cdot \cos^2 \alpha \, d\alpha &= \int \frac{1 - \cos 2\alpha}{2} \cdot \frac{1 + \cos 2\alpha}{2} \, d\alpha \\
 &= \frac{1}{4} \int (1 - \cos^2 2\alpha) \, d\alpha \\
 &= \frac{1}{4} \int \left(1 - \frac{1 + \cos 4\alpha}{2} \right) \, d\alpha \\
 &= \frac{1}{8} \int (1 - \cos 4\alpha) \, d\alpha \\
 &= \frac{1}{8} \left[\alpha - \frac{1}{4} \sin 4\alpha \right] + C \\
 &= \frac{1}{32} [4\alpha - \sin 4\alpha] + C
 \end{aligned}$$

15. Integration by parts.

$$dv = \sin^2 x \, dx = \frac{1 - \cos 2x}{2} \Rightarrow v = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{1}{4}(2x - \sin 2x)$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x \sin^2 x \, dx &= \frac{1}{4}x(2x - \sin 2x) - \frac{1}{4} \int (2x - \sin 2x) \, dx \\ &= \frac{1}{4}x(2x - \sin 2x) - \frac{1}{4} \left(x^2 + \frac{1}{2} \cos 2x \right) + C = \frac{1}{8}(2x^2 - 2x \sin 2x - \cos 2x) + C \end{aligned}$$

17. Let $u = \sin x$, $du = \cos x \, dx$.

$$\begin{aligned} \int_0^{\pi/2} \cos^3 x \, dx &= \int_0^{\pi/2} (1 - \sin^2 x) \cos x \, dx \\ &= \left[\sin x - \frac{1}{3} \sin^3 x \right]_0^{\pi/2} = \frac{2}{3} \end{aligned}$$

19. Let $u = \sin x$, $du = \cos x \, dx$.

$$\begin{aligned} \int_0^{\pi/2} \cos^7 x \, dx &= \int_0^{\pi/2} (1 - \sin^2 x)^3 \cos x \, dx = \int_0^{\pi/2} (1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x) \cos x \, dx \\ &= \left[\sin x - \sin^3 x + \frac{3}{5} \sin^5 x - \frac{1}{7} \sin^7 x \right]_0^{\pi/2} = \frac{16}{35} \end{aligned}$$

$$21. \int \sec(3x) \, dx = \frac{1}{3} \ln |\sec 3x + \tan 3x| + C$$

$$\begin{aligned} 23. \int \sec^4 5x \, dx &= \int (1 + \tan^2 5x) \sec^2 5x \, dx \\ &= \frac{1}{5} \left(\tan 5x + \frac{\tan^3 5x}{3} \right) + C \\ &= \frac{\tan 5x}{15} (3 + \tan^2 5x) + C \end{aligned}$$

$$25. dv = \sec^2 \pi x \, dx \Rightarrow v = \frac{1}{\pi} \tan \pi x$$

$$u = \sec \pi x \Rightarrow du = \pi \sec \pi x \tan \pi x \, dx$$

$$\int \sec^3 \pi x \, dx = \frac{1}{\pi} \sec \pi x \tan \pi x - \int \sec \pi x \tan^2 \pi x \, dx = \frac{1}{\pi} \sec \pi x \tan \pi x - \int \sec \pi x (\sec^2 \pi x - 1) \, dx$$

$$2 \int \sec^3 \pi x \, dx = \frac{1}{\pi} (\sec \pi x \tan \pi x + \ln |\sec \pi x + \tan \pi x|) + C_1$$

$$\int \sec^3 \pi x \, dx = \frac{1}{2\pi} (\sec \pi x \tan \pi x + \ln |\sec \pi x + \tan \pi x|) + C$$

$$\begin{aligned} 27. \int \tan^5 \frac{x}{4} \, dx &= \int \left(\sec^2 \frac{x}{4} - 1 \right) \tan^3 \frac{x}{4} \, dx \\ &= \int \tan^3 \frac{x}{4} \sec^2 \frac{x}{4} \, dx - \int \tan^3 \frac{x}{4} \, dx \\ &= \tan^4 \frac{x}{4} - \int \left(\sec^2 \frac{x}{4} - 1 \right) \tan \frac{x}{4} \, dx \\ &= \tan^4 \frac{x}{4} - 2 \tan^2 \frac{x}{4} - 4 \ln \left| \cos \frac{x}{4} \right| + C \end{aligned}$$

$$29. u = \tan x, du = \sec^2 x \, dx$$

$$\int \sec^2 x \tan x \, dx = \frac{1}{2} \tan^2 x + C$$

$$31. \int \tan^2 x \sec^2 x \, dx = \frac{\tan^3 x}{3} + C$$

$$33. \int \sec^6 4x \tan 4x \, dx = \frac{1}{4} \int \sec^5 4x (4 \sec 4x \tan 4x) \, dx$$

$$= \frac{\sec^6 4x}{24} + C$$

$$35. \text{ Let } u = \sec x, du = \sec x \tan x \, dx.$$

$$\int \sec^3 x \tan x \, dx = \int \sec^2 x (\sec x \tan x) \, dx$$

$$= \frac{1}{3} \sec^3 x + C$$

$$37. \int \frac{\tan^2 x}{\sec x} \, dx = \int \frac{(\sec^2 x - 1)}{\sec x} \, dx$$

$$= \int (\sec x - \cos x) \, dx$$

$$= \ln|\sec x + \tan x| - \sin x + C$$

$$39. r = \int \sin^4(\pi\theta) \, d\theta = \frac{1}{4} \int [1 - \cos(2\pi\theta)]^2 \, d\theta$$

$$= \frac{1}{4} \int [1 - 2\cos(2\pi\theta) + \cos^2(2\pi\theta)] \, d\theta$$

$$= \frac{1}{4} \int \left[1 - 2\cos(2\pi\theta) + \frac{1 + \cos(4\pi\theta)}{2} \right] \, d\theta$$

$$= \frac{1}{4} \left[\theta - \frac{1}{\pi} \sin(2\pi\theta) + \frac{\theta}{2} + \frac{1}{8\pi} \sin(4\pi\theta) \right] + C$$

$$= \frac{1}{32\pi} [12\pi\theta - 8\sin(2\pi\theta) + \sin(4\pi\theta)] + C$$

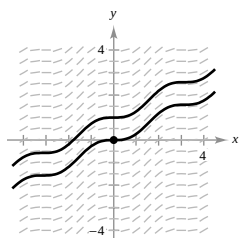
$$41. y = \int \tan^3 3x \sec 3x \, dx$$

$$= \int (\sec^2 3x - 1) \sec 3x \tan 3x \, dx$$

$$= \frac{1}{3} \int \sec^2 3x (3 \sec 3x \tan 3x) \, dx - \frac{1}{3} \int 3 \sec 3x \tan 3x \, dx$$

$$= \frac{1}{9} \sec^3 3x - \frac{1}{3} \sec 3x + C$$

43. (a)

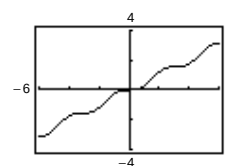


$$(b) \frac{dy}{dx} = \sin^2 x, (0, 0)$$

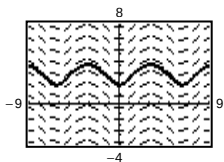
$$y = \int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx$$

$$= \frac{1}{2}x - \frac{\sin 2x}{4} + C$$

$$(0, 0): 0 = C, y = \frac{1}{2}x - \frac{\sin 2x}{4}$$



$$45. \frac{dy}{dx} = \frac{3 \sin x}{y}, y(0) = 2$$



$$47. \int \sin 3x \cos 2x \, dx = \frac{1}{2} \int (\sin 5x + \sin x) \, dx$$

$$= \frac{-1}{2} \left(\frac{1}{5} \cos 5x + \cos x \right) + C$$

$$= \frac{-1}{10} (\cos 5x + 5 \cos x) + C$$

$$49. \int \sin \theta \sin 3\theta \, d\theta = \frac{1}{2} \int (\cos 2\theta - \cos 4\theta) \, d\theta$$

$$= \frac{1}{2} \left(\frac{1}{2} \sin 2\theta - \frac{1}{4} \sin 4\theta \right) + C$$

$$= \frac{1}{8} (2 \sin 2\theta - \sin 4\theta) + C$$

$$51. \int \cot^3 2x \, dx = \int (\csc^2 2x - 1) \cot 2x \, dx$$

$$= -\frac{1}{2} \int \cot 2x (-2 \csc^2 2x) \, dx - \frac{1}{2} \int \frac{2 \cos 2x}{\sin 2x} \, dx$$

$$= -\frac{1}{4} \cot^2 2x - \frac{1}{2} \ln|\sin 2x| + C$$

$$= \frac{1}{4} (\ln|\csc^2 2x| - \cot^2 2x) + C$$

53. Let $u = \cot \theta$, $du = -\csc^2 \theta d\theta$.

$$\begin{aligned}\int \csc^4 \theta d\theta &= \int \csc^2 \theta (1 + \cot^2 \theta) d\theta \\ &= \int \csc^2 \theta d\theta + \int \csc^2 \theta \cot^2 \theta d\theta \\ &= -\cot \theta - \frac{1}{3} \cot^3 \theta + C\end{aligned}$$

$$\begin{aligned}57. \int \frac{1}{\sec x \tan x} dx &= \int \frac{\cos^2 x}{\sin x} dx = \int \frac{1 - \sin^2 x}{\sin x} dx \\ &= \int (\csc x - \sin x) dx \\ &= \ln |\csc x - \cot x| + \cos x + C\end{aligned}$$

$$\begin{aligned}59. \int (\tan^4 t - \sec^4 t) dt &= \int (\tan^2 t + \sec^2 t)(\tan^2 t - \sec^2 t) dt \quad (\tan^2 t - \sec^2 t = -1) \\ &= -\int (\tan^2 t + \sec^2 t) dt = -\int (2 \sec^2 t - 1) dt = -2 \tan t + t + C\end{aligned}$$

$$\begin{aligned}61. \int_{-\pi}^{\pi} \sin^2 x dx &= 2 \int_0^{\pi} \frac{1 - \cos 2x}{2} dx \\ &= \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi} = \pi\end{aligned}$$

$$\begin{aligned}63. \int_0^{\pi/4} \tan^3 x dx &= \int_0^{\pi/4} (\sec^2 x - 1) \tan x dx \\ &= \int_0^{\pi/4} \sec^2 x \tan x dx - \int_0^{\pi/4} \frac{\sin x}{\cos x} dx \\ &= \left[\frac{1}{2} \tan^2 x + \ln |\cos x| \right]_0^{\pi/4} \\ &= \frac{1}{2} (1 - \ln 2)\end{aligned}$$

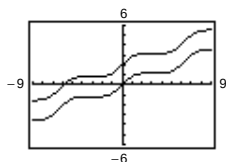
65. Let $u = 1 + \sin t$, $du = \cos t dt$.

$$\int_0^{\pi/2} \frac{\cos t}{1 + \sin t} dt = \left[\ln |1 + \sin t| \right]_0^{\pi/2} = \ln 2$$

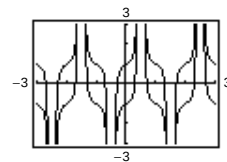
67. Let $u = \sin x$, $du = \cos x dx$.

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \cos^3 x dx &= 2 \int_0^{\pi/2} (1 - \sin^2 x) \cos x dx \\ &= 2 \left[\sin x - \frac{1}{3} \sin^3 x \right]_0^{\pi/2} = \frac{4}{3}\end{aligned}$$

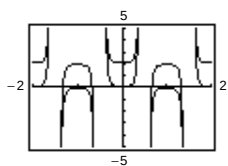
$$69. \int \cos^4 \frac{x}{2} dx = \frac{1}{16} [6x + 8 \sin x + \sin 2x] + C$$



$$71. \int \sec^5 \pi x dx = \frac{1}{4\pi} \left\{ \sec^3 \pi x \tan \pi x + \frac{3}{2} [\sec \pi x \tan \pi x + \ln |\sec \pi x + \tan \pi x|] \right\} + C$$



$$73. \int \sec^5 \pi x \tan \pi x \, dx = \frac{1}{5\pi} \sec^5 \pi x + C$$



$$75. \int_0^{\pi/4} \sin 2\theta \sin 3\theta \, d\theta = \frac{1}{2} \left[\sin \theta - \frac{1}{5} \sin 5\theta \right]_0^{\pi/4} = \frac{3\sqrt{2}}{10}$$

$$77. \int_0^{\pi/2} \sin^4 x \, dx = \frac{1}{4} \left[\frac{3x}{2} - \sin 2x + \frac{1}{8} \sin 4x \right]_0^{\pi/2} = \frac{3\pi}{16}$$

79. (a) Save one sine factor and convert the remaining sine factors to cosine. Then expand and integrate.

(b) Save one cosine factor and convert the remaining cosine factors to sine. Then expand and integrate.

(c) Make repeated use of the power reducing formula to convert the integrand to odd powers of the cosine.

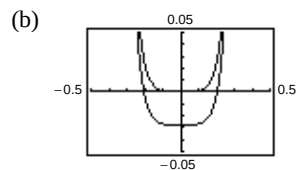
81. (a) Let $u = \tan 3x$, $du = 3 \sec^2 3x \, dx$.

$$\begin{aligned} \int \sec^4 3x \tan^3 3x \, dx &= \int \sec^2 3x \tan^3 3x \sec^2 3x \, dx \\ &= \frac{1}{3} \int (\tan^2 3x + 1) \tan^3 3x (3 \sec^2 3x) \, dx \\ &= \frac{1}{3} \int (\tan^5 3x + \tan^3 3x) (3 \sec^2 3x) \, dx \\ &= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + C_1 \end{aligned}$$

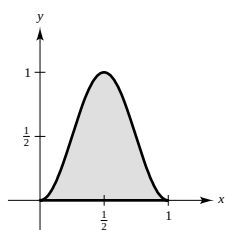
Or let $u = \sec 3x$, $du = 3 \sec 3x \tan 3x \, dx$.

$$\begin{aligned} \int \sec^4 3x \tan^3 3x \, dx &= \int \sec^3 3x \tan^2 3x \sec 3x \tan 3x \, dx \\ &= \frac{1}{3} \int \sec^3 3x (\sec^2 3x - 1) (3 \sec 3x \tan 3x) \, dx \\ &= \frac{\sec^6 3x}{18} - \frac{\sec^4 3x}{12} + C \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \frac{\sec^6 3x}{18} - \frac{\sec^4 3x}{12} + C &= \frac{(1 + \tan^2 3x)^3}{18} - \frac{(1 + \tan^2 3x)^2}{12} + C \\ &= \frac{1}{18} \tan^6 3x + \frac{1}{6} \tan^4 3x + \frac{1}{6} \tan^2 3x + \frac{1}{18} - \frac{1}{12} \tan^4 3x - \frac{1}{6} \tan^2 3x - \frac{1}{12} + C \\ &= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + \left(\frac{1}{18} - \frac{1}{12} \right) + C \\ &= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + C_2 \end{aligned}$$



$$\begin{aligned} 83. \quad A &= \int_0^1 \sin^2(\pi x) \, dx \\ &= \int_0^1 \frac{1 - \cos(2\pi x)}{2} \, dx \\ &= \left[\frac{x}{2} - \frac{1}{4\pi} \sin(2\pi x) \right]_0^1 \\ &= \frac{1}{2} \end{aligned}$$



$$85. (a) V = \pi \int_0^{\pi} \sin^2 x \, dx = \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x) \, dx = \frac{\pi}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi} = \frac{\pi^2}{2}$$

$$(b) A = \int_0^{\pi} \sin x \, dx = \left[-\cos x \right]_0^{\pi} = 1 + 1 = 2$$

Let $u = x$, $dv = \sin x \, dx$, $du = dx$, $v = -\cos x$.

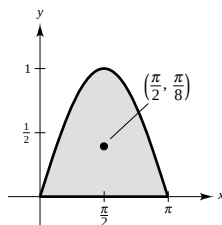
$$\bar{x} = \frac{1}{A} \int_0^{\pi} x \sin x \, dx = \frac{1}{2} \left[\left[-x \cos x \right]_0^{\pi} + \int_0^{\pi} \cos x \, dx \right] = \frac{1}{2} \left[-x \cos x + \sin x \right]_0^{\pi} = \frac{\pi}{2}$$

$$\bar{y} = \frac{1}{2A} \int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{1}{8} \int_0^{\pi} (1 - \cos 2x) \, dx$$

$$= \frac{1}{8} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi} = \frac{\pi}{8}$$

$$(\bar{x}, \bar{y}) = \left(\frac{\pi}{2}, \frac{\pi}{8} \right)$$



$$87. dv = \sin x \, dx \Rightarrow v = -\cos x$$

$$u = \sin^{n-1} x \Rightarrow du = (n-1) \sin^{n-2} x \cos x \, dx$$

$$\int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

$$\text{Therefore, } n \int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx$$

$$\int \sin^n x \, dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

$$89. \text{ Let } u = \sin^{n-1} x, du = (n-1) \sin^{n-2} x \cos x \, dx, dv = \cos^m x \sin x \, dx, v = \frac{-\cos^{m+1} x}{m+1}.$$

$$\int \cos^m x \sin^n x \, dx = \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^{m+2} x \, dx$$

$$= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^m x (1 - \sin^2 x) \, dx$$

$$= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^m x \, dx - \frac{n-1}{m+1} \int \sin^n x \cos^m x \, dx$$

$$\frac{m+n}{m+1} \int \cos^m x \sin^n x \, dx = \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^m x \, dx$$

$$\int \cos^m x \sin^n x \, dx = \frac{-\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x \, dx$$

$$\begin{aligned}
91. \int \sin^5 x \, dx &= -\frac{\sin^4 x \cos x}{5} + \frac{4}{5} \int \sin^3 x \, dx \\
&= -\frac{\sin^4 x \cos x}{5} + \frac{4}{5} \left[-\frac{\sin^2 x \cos x}{3} + \frac{2}{3} \int \sin x \, dx \right] \\
&= -\frac{1}{5} \sin^4 x \cos x - \frac{4}{15} \sin^2 x \cos x - \frac{8}{15} \cos x + C \\
&= -\frac{\cos x}{15} [3 \sin^4 x + 4 \sin^2 x + 8] + C
\end{aligned}$$

$$\begin{aligned}
93. \int \sec^4\left(\frac{2\pi x}{5}\right) dx &= \frac{5}{2\pi} \int \sec^4\left(\frac{2\pi x}{5}\right) \frac{2\pi}{5} dx \\
&= \frac{5}{2\pi} \left[\frac{1}{3} \sec^2\left(\frac{2\pi x}{5}\right) \tan\left(\frac{2\pi x}{5}\right) + \frac{2}{3} \int \sec^2\left(\frac{2\pi x}{5}\right) \frac{2\pi}{5} dx \right] \\
&= \frac{5}{6\pi} \left[\sec^2\left(\frac{2\pi x}{5}\right) \tan\left(\frac{2\pi x}{5}\right) + 2 \tan\left(\frac{2\pi x}{5}\right) \right] + C \\
&= \frac{5}{6\pi} \tan\left(\frac{2\pi x}{5}\right) \left[\sec^2\left(\frac{2\pi x}{5}\right) + 2 \right] + C
\end{aligned}$$

95. (a) $f(t) = a_0 + a_1 \cos \frac{\pi t}{6} + b_1 \sin \frac{\pi t}{6}$ where:

$$a_0 = \frac{1}{12} \int_0^{12} f(t) \, dt$$

$$a_1 = \frac{1}{6} \int_0^{12} f(t) \cos \frac{\pi t}{6} \, dt$$

$$b_1 = \frac{1}{6} \int_0^{12} f(t) \sin \frac{\pi t}{6} \, dt$$

$$\begin{aligned}
a_0 &\approx \frac{12-0}{3(12)^2} [30.9 + 4(32.2) + 2(41.1) + 4(53.7) + 2(64.6) + 4(74.0) + 2(78.2) + 4(77.0) + 2(71.0) + \\
&\quad 4(60.1) + 2(47.1) + 4(35.7) + 30.9] \approx 55.46
\end{aligned}$$

$$\begin{aligned}
a_1 &\approx \frac{12-0}{6(3)(12)} \left[30.9 \cos 0 + 4 \left(32.2 \cos \frac{\pi}{6} \right) + 2 \left(41.1 \cos \frac{\pi}{3} \right) + 4 \left(53.7 \cos \frac{\pi}{2} \right) + 2 \left(64.6 \cos \frac{2\pi}{3} \right) + \right. \\
&\quad 4 \left(74.0 \cos \frac{5\pi}{6} \right) + 2(78.2 \cos \pi) + 4 \left(77.0 \cos \frac{7\pi}{6} \right) + 2 \left(71.0 \cos \frac{4\pi}{3} \right) + \\
&\quad \left. 4 \left(60.1 \cos \frac{3\pi}{2} \right) + 2 \left(47.1 \cos \frac{5\pi}{3} \right) + 4 \left(35.7 \cos \frac{11\pi}{6} \right) + 30.9 \cos 2\pi \right] \approx -23.88
\end{aligned}$$

$$\begin{aligned}
b_1 &\approx \frac{12-0}{6(3)(12)} \left[30.9 \sin 0 + 4 \left(32.2 \sin \frac{\pi}{6} \right) + 2 \left(41.1 \sin \frac{\pi}{3} \right) + 4 \left(53.7 \sin \frac{\pi}{2} \right) + 2 \left(64.6 \sin \frac{2\pi}{3} \right) + \right. \\
&\quad 4 \left(74.0 \sin \frac{5\pi}{6} \right) + 2(78.2 \sin \pi) + 4 \left(77.0 \sin \frac{7\pi}{6} \right) + 2 \left(71.0 \sin \frac{4\pi}{3} \right) + \\
&\quad \left. 4 \left(60.1 \sin \frac{3\pi}{2} \right) + 2 \left(47.1 \sin \frac{5\pi}{3} \right) + 4 \left(35.7 \sin \frac{11\pi}{6} \right) + 30.9 \sin 2\pi \right] \approx -3.34
\end{aligned}$$

$$H(t) \approx 55.46 - 23.88 \cos \frac{\pi t}{6} - 3.34 \sin \frac{\pi t}{6}$$

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95. —CONTINUED—

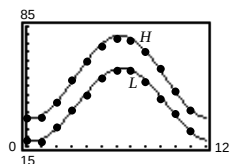
$$(b) a_0 \approx \frac{12-0}{3(12)^2} [18.0 + 4(17.7) + 2(25.8) + 4(36.1) + 2(45.4) + 4(55.2) + 2(59.9) + 4(59.4) + 2(53.1) + 4(43.2) + 2(34.3) + 4(24.2) + 18.0] \approx 39.34$$

$$a_1 \approx \frac{12-0}{6(3)(12)} \left[18.0 \cos 0 + 4 \left(17.7 \cos \frac{\pi}{6} \right) + 2 \left(25.8 \cos \frac{\pi}{3} \right) + 4 \left(36.1 \cos \frac{\pi}{2} \right) + 2 \left(45.4 \cos \frac{2\pi}{3} \right) + 4 \left(55.2 \cos \frac{5\pi}{6} \right) + 2(59.9 \cos \pi) + 4 \left(59.4 \cos \frac{7\pi}{6} \right) + 2 \left(53.1 \cos \frac{4\pi}{3} \right) + 4 \left(43.2 \cos \frac{3\pi}{2} \right) + 2 \left(34.3 \cos \frac{5\pi}{3} \right) + 4 \left(24.2 \cos \frac{11\pi}{6} \right) + 18 \cos 2\pi \right] \approx -20.78$$

$$b_1 \approx \frac{12-0}{6(3)(12)} \left[18.0 \sin 0 + 4 \left(17.7 \sin \frac{\pi}{6} \right) + 2 \left(25.8 \sin \frac{\pi}{3} \right) + 4 \left(36.1 \sin \frac{\pi}{2} \right) + 2 \left(45.4 \sin \frac{2\pi}{3} \right) + 4 \left(55.2 \sin \frac{5\pi}{6} \right) + 2(59.9 \sin \pi) + 4 \left(59.4 \sin \frac{7\pi}{6} \right) + 2 \left(53.1 \sin \frac{4\pi}{3} \right) + 4 \left(43.2 \sin \frac{3\pi}{2} \right) + 2 \left(34.3 \sin \frac{5\pi}{3} \right) + 4 \left(24.2 \sin \frac{11\pi}{6} \right) + 18 \sin 2\pi \right] \approx -4.33$$

$$L(t) \approx 39.34 - 20.78 \cos \frac{\pi t}{6} - 4.33 \sin \frac{\pi t}{6}$$

(c) The difference between the maximum and minimum temperatures is greatest in the summer.



$$97. \int_{-\pi}^{\pi} \cos(mx) \cos(nx) dx = \frac{1}{2} \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right]_{-\pi}^{\pi} = 0, \quad (m \neq n)$$

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx &= \frac{1}{2} \int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x] dx \\ &= \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_{-\pi}^{\pi} = 0, \quad (m \neq n) \end{aligned}$$

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(mx) \cos(nx) dx &= \frac{1}{2} \int_{-\pi}^{\pi} [\sin(m+n)x + \sin(m-n)x] dx \\ &= -\frac{1}{2} \left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n} \right]_{-\pi}^{\pi}, \quad (m \neq n) \\ &= -\frac{1}{2} \left[\left(\frac{\cos(m+n)\pi}{m+n} + \frac{\cos(m-n)\pi}{m-n} \right) - \left(\frac{\cos(m+n)(-\pi)}{m+n} + \frac{\cos(m-n)(-\pi)}{m-n} \right) \right] \\ &= 0, \text{ since } \cos(-\theta) = \cos\theta. \end{aligned}$$

$$\int_{-\pi}^{\pi} \sin(mx) \cos(mx) dx = \frac{1}{m} \left[\frac{\sin^2(mx)}{2} \right]_{-\pi}^{\pi} = 0$$

Section 7.4 Trigonometric Substitution

$$\begin{aligned}
1. \quad \frac{d}{dx} \left[4 \ln \left| \frac{\sqrt{x^2 + 16} - 4}{x} \right| + \sqrt{x^2 + 16} + C \right] &= \frac{d}{dx} \left[4 \ln |\sqrt{x^2 + 16} - 4| - 4 \ln |x| + \sqrt{x^2 + 16} + C \right] \\
&= 4 \left[\frac{x/\sqrt{x^2 + 16}}{\sqrt{x^2 + 16} - 4} \right] - \frac{4}{x} + \frac{x}{\sqrt{x^2 + 16}} \\
&= \frac{4x}{\sqrt{x^2 + 16}(\sqrt{x^2 + 16} - 4)} - \frac{4}{x} + \frac{x}{\sqrt{x^2 + 16}} \\
&= \frac{4x^2 - 4\sqrt{x^2 + 16}(\sqrt{x^2 + 16} - 4) + x^2(\sqrt{x^2 + 16} - 4)}{x\sqrt{x^2 + 16}(\sqrt{x^2 + 16} - 4)} \\
&= \frac{4x^2 - 4(x^2 + 16) + 16\sqrt{x^2 + 16} + x^2\sqrt{x^2 + 16} - 4x^2}{x\sqrt{x^2 + 16}(\sqrt{x^2 + 16} - 4)} \\
&= \frac{\sqrt{x^2 + 16}(x^2 + 16) - 4(x^2 + 16)}{x\sqrt{x^2 + 16}(\sqrt{x^2 + 16} - 4)} \\
&= \frac{(x^2 + 16)(\sqrt{x^2 + 16} - 4)}{x\sqrt{x^2 + 16}(\sqrt{x^2 + 16} - 4)} = \frac{\sqrt{x^2 + 16}}{x}
\end{aligned}$$

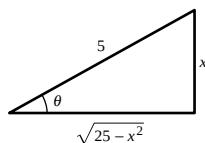
Indefinite integral: $\int \frac{\sqrt{x^2 + 16}}{x} dx$ Matches (b)

$$\begin{aligned}
3. \quad \frac{d}{dx} \left[8 \arcsin \frac{x}{4} - \frac{x\sqrt{16 - x^2}}{2} + C \right] &= 8 \frac{1/4}{\sqrt{1 - (x/4)^2}} - \frac{x(1/2)(16 - x^2)^{-1/2}(-2x) + \sqrt{16 - x^2}}{2} \\
&= \frac{8}{\sqrt{16 - x^2}} + \frac{x^2}{2\sqrt{16 - x^2}} - \frac{\sqrt{16 - x^2}}{2} \\
&= \frac{16}{2\sqrt{16 - x^2}} + \frac{x^2}{2\sqrt{16 - x^2}} - \frac{(16 - x^2)}{2\sqrt{16 - x^2}} = \frac{x^2}{\sqrt{16 - x^2}}
\end{aligned}$$

Matches (a)

5. Let $x = 5 \sin \theta$, $dx = 5 \cos \theta d\theta$, $\sqrt{25 - x^2} = 5 \cos \theta$.

$$\begin{aligned}
\int \frac{1}{(25 - x^2)^{3/2}} dx &= \int \frac{5 \cos \theta}{(5 \cos \theta)^3} d\theta \\
&= \frac{1}{25} \int \sec^2 \theta d\theta \\
&= \frac{1}{25} \tan \theta + C \\
&= \frac{x}{25\sqrt{25 - x^2}} + C
\end{aligned}$$

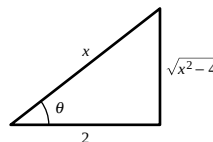


7. Same substitution as in Exercise 5

$$\begin{aligned}
\int \frac{\sqrt{25 - x^2}}{x} dx &= \int \frac{25 \cos^2 \theta d\theta}{5 \sin \theta} = 5 \int \frac{1 - \sin^2 \theta}{\sin \theta} d\theta = 5 \int (\csc \theta - \sin \theta) d\theta \\
&= 5 [\ln |\csc \theta - \cot \theta| + \cos \theta] + C = 5 \ln \left| \frac{5 - \sqrt{25 - x^2}}{x} \right| + \sqrt{25 - x^2} + C
\end{aligned}$$

9. Let $x = 2 \sec \theta$, $dx = 2 \sec \theta \tan \theta d\theta$, $\sqrt{x^2 - 4} = 2 \tan \theta$.

$$\begin{aligned}\int \frac{1}{\sqrt{x^2 - 4}} dx &= \int \frac{2 \sec \theta \tan \theta d\theta}{2 \tan \theta} = \int \sec \theta d\theta = \ln|\sec \theta + \tan \theta| + C_1 \\ &= \ln \left| \frac{x}{2} + \frac{\sqrt{x^2 - 4}}{2} \right| + C_1 \\ &= \ln|x + \sqrt{x^2 - 4}| - \ln 2 + C_1 = \ln|x + \sqrt{x^2 - 4}| + C\end{aligned}$$



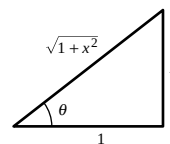
11. Same substitution as in Exercise 9

$$\begin{aligned}\int x^3 \sqrt{x^2 - 4} dx &= \int (8 \sec^3 \theta)(2 \tan \theta)(2 \sec \theta \tan \theta) d\theta = 32 \int \tan^2 \theta \sec^4 \theta d\theta \\ &= 32 \int \tan^2 \theta (1 + \tan^2 \theta) \sec^2 \theta d\theta = 32 \left(\frac{\tan^3 \theta}{3} + \frac{\tan^5 \theta}{5} \right) + C \\ &= \frac{32}{15} \tan^3 \theta [5 + 3 \tan^2 \theta] + C = \frac{32}{15} \frac{(x^2 - 4)^{3/2}}{8} \left[5 + 3 \frac{(x^2 - 4)}{4} \right] + C \\ &= \frac{1}{15} (x^2 - 4)^{3/2} [20 + 3(x^2 - 4)] + C = \frac{1}{15} (x^2 - 4)^{3/2} (3x^2 + 8) + C\end{aligned}$$

13. Let $x = \tan \theta$, $dx = \sec^2 \theta d\theta$, $\sqrt{1 + x^2} = \sec \theta$.

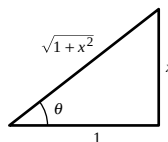
$$\int x \sqrt{1 + x^2} dx = \int \tan \theta (\sec \theta) \sec^2 \theta d\theta = \frac{\sec^3 \theta}{3} + C = \frac{1}{3} (1 + x^2)^{3/2} + C$$

Note: This integral could have been evaluated with the Power Rule.



15. Same substitution as in Exercise 13

$$\begin{aligned}\int \frac{1}{(1 + x^2)^2} dx &= \int \frac{1}{(\sec^2 \theta)^2} dx \\ &= \int \frac{\sec^2 \theta d\theta}{\sec^4 \theta} \\ &= \int \cos^2 \theta d\theta = \frac{1}{2} \int (1 + \cos 2\theta) d\theta \\ &= \frac{1}{2} \left[\theta + \frac{\sin 2\theta}{2} \right] \\ &= \frac{1}{2} [\theta + \sin \theta \cos \theta] + C \\ &= \frac{1}{2} \left[\arctan x + \left(\frac{x}{\sqrt{1 + x^2}} \right) \left(\frac{1}{\sqrt{1 + x^2}} \right) \right] + C \\ &= \frac{1}{2} \left[\arctan x + \frac{x}{1 + x^2} \right] + C\end{aligned}$$



17. Let $u = 3x$, $a = 2$, and $du = 3 dx$.

$$\begin{aligned}\int \sqrt{4 + 9x^2} dx &= \frac{1}{3} \int \sqrt{(2)^2 + (3x)^2} 3 dx \\ &= \frac{1}{3} \left(\frac{1}{2} \right) (3x \sqrt{4 + 9x^2} + 4 \ln|3x + \sqrt{4 + 9x^2}|) + C \\ &= \frac{1}{2} x \sqrt{4 + 9x^2} + \frac{2}{3} \ln|3x + \sqrt{4 + 9x^2}| + C\end{aligned}$$

$$19. \int \frac{x}{\sqrt{x^2 + 9}} dx = \frac{1}{2} \int (x^2 + 9)^{-1/2} (2x) dx$$

$$= \sqrt{x^2 + 9} + C$$

(Power Rule)

$$21. \int \frac{1}{\sqrt{16 - x^2}} dx = \arcsin\left(\frac{x}{4}\right) + C$$

23. Let $x = 2 \sin \theta$, $dx = 2 \cos \theta d\theta$, $\sqrt{4 - x^2} = 2 \cos \theta$.

$$\int \sqrt{16 - 4x^2} dx = 2 \int \sqrt{4 - x^2} dx$$

$$= 2 \int 2 \cos \theta (2 \cos \theta d\theta)$$

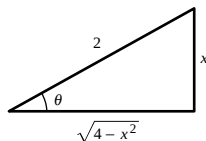
$$= 8 \int \cos^2 \theta d\theta$$

$$= 4 \int (1 + \cos 2\theta) d\theta$$

$$= 4 \left[\theta + \frac{1}{2} \sin 2\theta \right] + C$$

$$= 4\theta + 4 \sin \theta \cos \theta + C$$

$$= 4 \arcsin\left(\frac{x}{2}\right) + x\sqrt{4 - x^2} + C$$

25. Let $x = 3 \sec \theta$, $dx = 3 \sec \theta \tan \theta d\theta$,

$$\sqrt{x^2 - 9} = 3 \tan \theta.$$

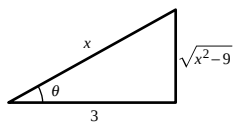
$$\int \frac{1}{\sqrt{x^2 - 9}} dx = \int \frac{3 \sec \theta \tan \theta d\theta}{3 \tan \theta}$$

$$= \int \sec \theta d\theta$$

$$= \ln|\sec \theta + \tan \theta| + C_1$$

$$= \ln\left|\frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3}\right| + C_1$$

$$= \ln|x + \sqrt{x^2 - 9}| + C$$

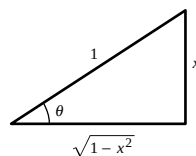
27. Let $x = \sin \theta$, $dx = \cos \theta d\theta$, $\sqrt{1 - x^2} = \cos \theta$.

$$\int \frac{\sqrt{1 - x^2}}{x^4} dx = \int \frac{\cos \theta (\cos \theta d\theta)}{\sin^4 \theta}$$

$$= \int \cot^2 \theta \csc^2 \theta d\theta$$

$$= -\frac{1}{3} \cot^3 \theta + C$$

$$= \frac{-(1 - x^2)^{3/2}}{3x^3} + C$$



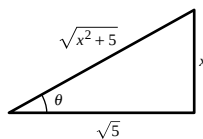
29. Same substitutions as in Exercise 28

$$\int \frac{1}{x\sqrt{4x^2 + 9}} dx = \int \frac{(3/2) \sec^2 \theta d\theta}{(3/2) \tan \theta 3 \sec \theta}$$

$$= \frac{1}{3} \int \csc \theta d\theta = -\frac{1}{3} \ln|\csc \theta + \cot \theta| + C = -\frac{1}{3} \ln\left|\frac{\sqrt{4x^2 + 9} + 3}{2x}\right| + C$$

31. Let $x = \sqrt{5} \tan \theta$, $dx = \sqrt{5} \sec^2 \theta d\theta$, $x^2 + 5 = 5 \sec^2 \theta$.

$$\begin{aligned} \int \frac{-5x}{(x^2 + 5)^{3/2}} dx &= \int \frac{-5\sqrt{5} \tan \theta}{(5 \sec^2 \theta)^{3/2}} \sqrt{5} \sec^2 \theta d\theta \\ &= -\sqrt{5} \int \frac{\tan \theta}{\sec \theta} d\theta \\ &= -\sqrt{5} \int \sin \theta d\theta \\ &= \sqrt{5} \cos \theta + C \\ &= \sqrt{5} \frac{\sqrt{5}}{\sqrt{x^2 + 5}} + C \\ &= \frac{5}{\sqrt{x^2 + 5}} + C \end{aligned}$$

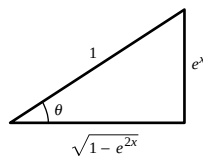


33. Let $u = 1 + e^{2x}$, $du = 2e^{2x} dx$.

$$\int e^{2x} \sqrt{1 + e^{2x}} dx = \frac{1}{2} \int (1 + e^{2x})^{1/2} (2e^{2x}) dx = \frac{1}{3} (1 + e^{2x})^{3/2} + C$$

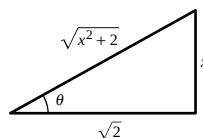
35. Let $e^x = \sin \theta$, $e^x dx = \cos \theta d\theta$, $\sqrt{1 - e^{2x}} = \cos \theta$.

$$\begin{aligned} \int e^x \sqrt{1 - e^{2x}} dx &= \int \cos^2 \theta d\theta \\ &= \frac{1}{2} \int (1 + \cos 2\theta) d\theta \\ &= \frac{1}{2} \left[\theta + \frac{\sin 2\theta}{2} \right] \\ &= \frac{1}{2} (\theta + \sin \theta \cos \theta) + C = \frac{1}{2} (\arcsin e^x + e^x \sqrt{1 - e^{2x}}) + C \end{aligned}$$



37. Let $x = \sqrt{2} \tan \theta$, $dx = \sqrt{2} \sec^2 \theta d\theta$, $x^2 + 2 = 2 \sec^2 \theta$.

$$\begin{aligned} \int \frac{1}{4 + 4x^2 + x^4} dx &= \int \frac{1}{(x^2 + 2)^2} dx \\ &= \int \frac{\sqrt{2} \sec^2 \theta d\theta}{4 \sec^4 \theta} \\ &= \frac{\sqrt{2}}{4} \int \cos^2 \theta d\theta \\ &= \frac{\sqrt{2}}{4} \left(\frac{1}{2} \right) \int (1 + \cos 2\theta) d\theta \\ &= \frac{\sqrt{2}}{8} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\ &= \frac{\sqrt{2}}{8} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{1}{4} \left[\frac{x}{x^2 + 2} + \frac{1}{\sqrt{2}} \arctan \frac{x}{\sqrt{2}} \right] + C \end{aligned}$$



39. Since $x > \frac{1}{2}$,

$$u = \operatorname{arcsec} 2x, \Rightarrow du = \frac{1}{x\sqrt{4x^2-1}} dx, dv = dx \Rightarrow v = x$$

$$\int \operatorname{arcsec} 2x dx = x \operatorname{arcsec} 2x - \int \frac{1}{\sqrt{4x^2-1}} dx$$

$$2x = \sec \theta, dx = \frac{1}{2} \sec \theta \tan \theta d\theta, \sqrt{4x^2-1} = \tan \theta$$

$$\begin{aligned} \int \operatorname{arcsec} 2x dx &= x \operatorname{arcsec} 2x - \int \frac{(1/2) \sec \theta \tan \theta d\theta}{\tan \theta} = x \operatorname{arcsec} 2x - \frac{1}{2} \int \sec \theta d\theta \\ &= x \operatorname{arcsec} 2x - \frac{1}{2} \ln |\sec \theta + \tan \theta| + C = x \operatorname{arcsec} 2x - \frac{1}{2} \ln |2x + \sqrt{4x^2-1}| + C. \end{aligned}$$

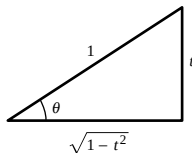
41. $\int \frac{1}{\sqrt{4x-x^2}} dx = \int \frac{1}{\sqrt{4-(x-2)^2}} dx = \arcsin\left(\frac{x-2}{2}\right) + C$

43. Let $x+2 = 2 \tan \theta$, $dx = 2 \sec^2 \theta d\theta$, $\sqrt{(x+2)^2+4} = 2 \sec \theta$.

$$\begin{aligned} \int \frac{x}{\sqrt{x^2+4x+8}} dx &= \int \frac{x}{\sqrt{(x+2)^2+4}} dx = \int \frac{(2 \tan \theta - 2)(2 \sec^2 \theta) d\theta}{2 \sec \theta} \\ &= 2 \int (\tan \theta - 1)(\sec \theta) d\theta \\ &= 2[\sec \theta - \ln |\sec \theta + \tan \theta|] + C_1 \\ &= 2 \left[\frac{\sqrt{(x+2)^2+4}}{2} - \ln \left| \frac{\sqrt{(x+2)^2+4}}{2} + \frac{x+2}{2} \right| \right] + C_1 \\ &= \sqrt{x^2+4x+8} - 2 \left[\ln \left| \sqrt{x^2+4x+8} + (x+2) \right| - \ln 2 \right] + C_1 \\ &= \sqrt{x^2+4x+8} - 2 \ln \left| \sqrt{x^2+4x+8} + (x+2) \right| + C \end{aligned}$$

45. Let $t = \sin \theta$, $dt = \cos \theta d\theta$, $1-t^2 = \cos^2 \theta$.

$$\begin{aligned} \text{(a)} \quad \int \frac{t^2}{(1-t^2)^{3/2}} dt &= \int \frac{\sin^2 \theta \cos \theta d\theta}{\cos^3 \theta} \\ &= \int \tan^2 \theta d\theta \\ &= \int (\sec^2 \theta - 1) d\theta \\ &= \tan \theta - \theta + C \\ &= \frac{t}{\sqrt{1-t^2}} - \arcsin t + C \end{aligned}$$



$$\text{Thus, } \int_0^{\sqrt{3}/2} \frac{t^2}{(1-t^2)^{3/2}} dt = \left[\frac{t}{\sqrt{1-t^2}} - \arcsin t \right]_0^{\sqrt{3}/2} = \frac{\sqrt{3}/2}{\sqrt{1/4}} - \arcsin \frac{\sqrt{3}}{2} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$$

(b) When $t = 0$, $\theta = 0$. When $t = \sqrt{3}/2$, $\theta = \pi/3$. Thus,

$$\int_0^{\sqrt{3}/2} \frac{t^2}{(1-t^2)^{3/2}} dt = \left[\tan \theta - \theta \right]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$$

47. (a) Let $x = 3 \tan \theta$, $dx = 3 \sec^2 \theta d\theta$, $\sqrt{x^2 + 9} = 3 \sec \theta$.

$$\begin{aligned} \int \frac{x^3}{\sqrt{x^2 + 9}} dx &= \int \frac{(27 \tan^3 \theta)(3 \sec^2 \theta d\theta)}{3 \sec \theta} \\ &= 27 \int (\sec^2 \theta - 1) \sec \theta \tan \theta d\theta \\ &= 27 \left[\frac{1}{3} \sec^3 \theta - \sec \theta \right] + C = 9[\sec^3 \theta - 3 \sec \theta] + C \\ &= 9 \left[\left(\frac{\sqrt{x^2 + 9}}{3} \right)^3 - 3 \left(\frac{\sqrt{x^2 + 9}}{3} \right) \right] + C = \frac{1}{3}(x^2 + 9)^{3/2} - 9\sqrt{x^2 + 9} + C \end{aligned}$$

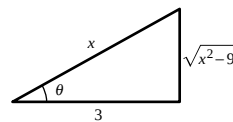
$$\begin{aligned} \text{Thus, } \int_0^3 \frac{x^3}{\sqrt{x^2 + 9}} dx &= \left[\frac{1}{3}(x^2 + 9)^{3/2} - 9\sqrt{x^2 + 9} \right]_0^3 \\ &= \left(\frac{1}{3}(54\sqrt{2}) - 27\sqrt{2} \right) - (9 - 27) \\ &= 18 - 9\sqrt{2} = 9(2 - \sqrt{2}) \approx 5.272. \end{aligned}$$

- (b) When $x = 0$, $\theta = 0$. When $x = 3$, $\theta = \pi/4$. Thus,

$$\int_0^3 \frac{x^3}{\sqrt{x^2 + 9}} dx = 9 \left[\sec^3 \theta - 3 \sec \theta \right]_0^{\pi/4} = 9(2\sqrt{2} - 3\sqrt{2}) - 9(1 - 3) = 9(2 - \sqrt{2}) \approx 5.272.$$

49. (a) Let $x = 3 \sec \theta$, $dx = 3 \sec \theta \tan \theta d\theta$, $\sqrt{x^2 - 9} = 3 \tan \theta$.

$$\begin{aligned} \int \frac{x^2}{\sqrt{x^2 - 9}} dx &= \int \frac{9 \sec^2 \theta}{3 \tan \theta} 3 \sec \theta \tan \theta d\theta \\ &= 9 \int \sec^3 \theta d\theta \\ &= 9 \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \int \sec \theta d\theta \right] \quad (7.3 \text{ Exercise 90}) \\ &= \frac{9}{2} [\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|] \\ &= \frac{9}{2} \left[\frac{x}{3} \cdot \frac{\sqrt{x^2 - 9}}{3} + \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| \right] \end{aligned}$$



Hence,

$$\begin{aligned} \int_4^6 \frac{x^2}{\sqrt{x^2 - 9}} dx &= \frac{9}{2} \left[\frac{x\sqrt{x^2 - 9}}{9} + \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| \right]_4^6 \\ &= \frac{9}{2} \left[\left(\frac{6\sqrt{27}}{9} + \ln \left| 2 + \frac{\sqrt{27}}{3} \right| \right) - \left(\frac{4\sqrt{7}}{9} + \ln \left| \frac{4}{3} + \frac{\sqrt{7}}{3} \right| \right) \right] \\ &= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2} \left(\ln \left(\frac{6 + \sqrt{27}}{3} \right) - \ln \left(\frac{4 + \sqrt{7}}{3} \right) \right) \\ &= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2} \ln \left(\frac{6 + 3\sqrt{3}}{4 + \sqrt{7}} \right) \\ &= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2} \ln \left(\frac{(4 - \sqrt{7})(2 + \sqrt{3})}{3} \right) \approx 12.644. \end{aligned}$$

—CONTINUED—

49. —CONTINUED—

(b) When $x = 4$, $\theta = \operatorname{arcsec}\left(\frac{4}{3}\right)$.

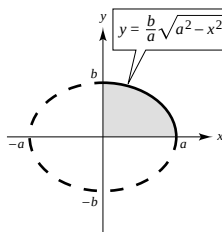
When $x = 6$, $\theta = \operatorname{arcsec}(2) = \frac{\pi}{3}$.

$$\begin{aligned}\int_4^6 \frac{x^2}{\sqrt{x^2-9}} dx &= \frac{9}{2} \left[\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right]_{\operatorname{arcsec}(4/3)}^{\pi/3} \\ &= \frac{9}{2} \left[2 \cdot \sqrt{3} + \ln |2 + \sqrt{3}| \right] - \frac{9}{2} \left[\frac{4}{3} \cdot \frac{\sqrt{7}}{3} + \ln \left| \frac{4}{3} + \frac{\sqrt{7}}{3} \right| \right] \\ &= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2} \ln \left(\frac{6 + 3\sqrt{3}}{4 + \sqrt{7}} \right) \approx 12.644\end{aligned}$$

51. $\int \frac{x^2}{\sqrt{x^2 + 10x + 9}} dx = \frac{1}{2} \sqrt{x^2 + 10x + 9} (x - 15) + 33 \ln |(x + 5) + \sqrt{x^2 + 10x + 9}| + C$

53. $\int \frac{x^2}{\sqrt{x^2 - 1}} dx = \frac{1}{2} (x\sqrt{x^2 - 1} + \ln |x + \sqrt{x^2 - 1}|) + C$ 55. (a) $u = a \sin \theta$ (b) $u = a \tan \theta$ (c) $u = a \sec \theta$

57.
$$\begin{aligned}A &= 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx \\ &= \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2} dx \\ &= \left[\frac{4b}{a} \left(\frac{1}{2} \right) \left(a^2 \arcsin \frac{x}{a} + x\sqrt{a^2 - x^2} \right) \right]_0^a \\ &= \frac{2b}{a} \left(a^2 \left(\frac{\pi}{2} \right) \right) \\ &= \pi ab\end{aligned}$$



Note: See Theorem 7.2 for $\int \sqrt{a^2 - x^2} dx$.

59. $x^2 + y^2 = a^2$

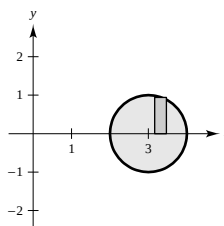
$$x = \pm \sqrt{a^2 - y^2}$$

$$\begin{aligned}A &= 2 \int_h^a \sqrt{a^2 - y^2} dy = \left[a^2 \arcsin \left(\frac{y}{a} \right) + y \sqrt{a^2 - y^2} \right]_h^a && \text{(Theorem 7.2)} \\ &= \left(a^2 \frac{\pi}{2} \right) - \left(a^2 \arcsin \left(\frac{h}{a} \right) + h \sqrt{a^2 - h^2} \right) \\ &= \frac{a^2 \pi}{2} - a^2 \arcsin \left(\frac{h}{a} \right) - h \sqrt{a^2 - h^2}\end{aligned}$$

61. Let $x - 3 = \sin \theta$, $dx = \cos \theta d\theta$, $\sqrt{1 - (x - 3)^2} = \cos \theta$.

Shell Method:

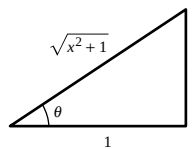
$$\begin{aligned}V &= 4\pi \int_2^4 x \sqrt{1 - (x - 3)^2} dx \\ &= 4\pi \int_{-\pi/2}^{\pi/2} (3 + \sin \theta) \cos^2 \theta d\theta \\ &= 4\pi \left[\frac{3}{2} \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) d\theta + \int_{-\pi/2}^{\pi/2} \cos^2 \theta \sin \theta d\theta \right] \\ &= 4\pi \left[\frac{3}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) - \frac{1}{3} \cos^3 \theta \right]_{-\pi/2}^{\pi/2} = 6\pi^2\end{aligned}$$



63. $y = \ln x, y' = \frac{1}{x}, 1 + (y')^2 = 1 + \frac{1}{x^2} = \frac{x^2 + 1}{x^2}$

Let $x = \tan \theta, dx = \sec^2 \theta d\theta, \sqrt{x^2 + 1} = \sec \theta$.

$$\begin{aligned} s &= \int_1^5 \sqrt{\frac{x^2 + 1}{x^2}} dx = \int_1^5 \frac{\sqrt{x^2 + 1}}{x} dx \\ &= \int_a^b \frac{\sec \theta}{\tan \theta} \sec^2 \theta d\theta = \int_a^b \frac{\sec \theta}{\tan \theta} (1 + \tan^2 \theta) d\theta \\ &= \int_a^b (\csc \theta + \sec \theta \tan \theta) d\theta \\ &= \left[-\ln|\csc \theta + \cot \theta| + \sec \theta \right]_a^b \\ &= \left[-\ln \left| \frac{\sqrt{x^2 + 1}}{x} + \frac{1}{x} \right| + \sqrt{x^2 + 1} \right]_1^5 \\ &= \left[-\ln \left(\frac{\sqrt{26} + 1}{5} \right) + \sqrt{26} \right] - \left[-\ln(\sqrt{2} + 1) + \sqrt{2} \right] \\ &= \ln \left[\frac{5(\sqrt{2} + 1)}{\sqrt{26} + 1} \right] + \sqrt{26} - \sqrt{2} \approx 4.367 \text{ or } \ln \left[\frac{\sqrt{26} - 1}{5(\sqrt{2} - 1)} \right] + \sqrt{26} - \sqrt{2} \end{aligned}$$



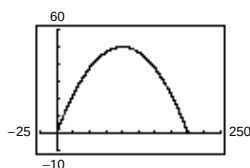
65. Length of one arch of sine curve: $y = \sin x, y' = \cos x$

$$L_1 = \int_0^\pi \sqrt{1 + \cos^2 x} dx$$

Length of one arch of cosine curve: $y = \cos x, y' = -\sin x$

$$\begin{aligned} L_2 &= \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin^2 x} dx \\ &= \int_{-\pi/2}^{\pi/2} \sqrt{1 + \cos^2 \left(x - \frac{\pi}{2} \right)} dx \quad u = x - \frac{\pi}{2}, du = dx \\ &= \int_{-\pi}^0 \sqrt{1 + \cos^2 u} du \\ &= \int_0^\pi \sqrt{1 + \cos^2 u} du = L_1 \end{aligned}$$

67. (a)



(b) $y = 0$ for $x = 200$ (range)

(c) $y = x - 0.005x^2, y' = 1 - 0.01x, 1 + (y')^2 = 1 + (1 - 0.01x)^2$

Let $u = 1 - 0.01x, du = -0.01 dx, a = 1$. (See Theorem 7.2.)

$$\begin{aligned} s &= \int_0^{200} \sqrt{1 + (1 - 0.01x)^2} dx = -100 \int_0^{200} \sqrt{(1 - 0.01x)^2 + 1} (-0.01) dx \\ &= -50 \left[(1 - 0.01x) \sqrt{(1 - 0.01x)^2 + 1} + \ln \left| (1 - 0.01x) + \sqrt{(1 - 0.01x)^2 + 1} \right| \right]_0^{200} \\ &= -50 \left[(-\sqrt{2} + \ln|-1 + \sqrt{2}|) - (\sqrt{2} + \ln|1 + \sqrt{2}|) \right] \\ &= 100\sqrt{2} + 50 \ln \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right) \approx 229.559 \end{aligned}$$

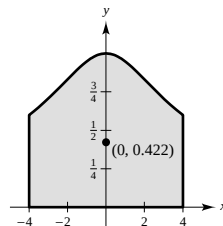
69. Let $x = 3 \tan \theta$, $dx = 3 \sec^2 \theta d\theta$, $\sqrt{x^2 + 9} = 3 \sec \theta$.

$$\begin{aligned} A &= 2 \int_0^4 \frac{3}{\sqrt{x^2 + 9}} dx = 6 \int_0^4 \frac{dx}{\sqrt{x^2 + 9}} = 6 \int_a^b \frac{3 \sec^2 \theta d\theta}{3 \sec \theta} \\ &= 6 \int_a^b \sec \theta d\theta = \left[6 \ln |\sec \theta + \tan \theta| \right]_a^b = \left[6 \ln \left| \frac{\sqrt{x^2 + 9} + x}{3} \right| \right]_0^4 = 6 \ln 3 \end{aligned}$$

$\bar{x} = 0$ (by symmetry)

$$\begin{aligned} \bar{y} &= \frac{1}{2} \left(\frac{1}{A} \right) \int_{-4}^4 \left(\frac{3}{\sqrt{x^2 + 9}} \right)^2 dx \\ &= \frac{9}{12 \ln 3} \int_{-4}^4 \frac{1}{x^2 + 9} dx \\ &= \frac{3}{4 \ln 3} \left[\frac{1}{3} \arctan \frac{x}{3} \right]_{-4}^4 \\ &= \frac{2}{4 \ln 3} \arctan \frac{4}{3} \approx 0.422 \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{1}{2 \ln 3} \arctan \frac{4}{3} \right) \approx (0, 0.422)$$



71. $y = x^2$, $y' = 2x$, $1 + (y')^2 = 1 + 4x^2$

$$2x = \tan \theta, dx = \frac{1}{2} \sec^2 \theta d\theta, \sqrt{1 + 4x^2} = \sec \theta$$

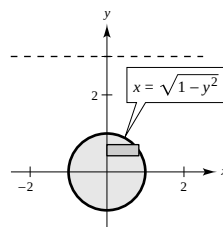
(For $\int \sec^5 \theta d\theta$ and $\int \sec^3 \theta d\theta$, see Exercise 80 in Section 7.3)

$$\begin{aligned} S &= 2\pi \int_0^{\sqrt{2}} x^2 \sqrt{1 + 4x^2} dx = 2\pi \int_a^b \left(\frac{\tan \theta}{2} \right)^2 (\sec \theta) \left(\frac{1}{2} \sec^2 \theta \right) d\theta \\ &= \frac{\pi}{4} \int_a^b \sec^3 \theta \tan^2 \theta d\theta = \frac{\pi}{4} \left[\int_a^b \sec^5 \theta d\theta - \int_a^b \sec^3 \theta d\theta \right] \\ &= \frac{\pi}{4} \left[\frac{1}{4} \left[\sec^3 \theta \tan \theta + \frac{3}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \right] - \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \right] \Big|_a^b \\ &= \frac{\pi}{4} \left[\frac{1}{4} [(1 + 4x^2)^{3/2} (2x)] - \frac{1}{8} [(1 + 4x^2)^{1/2} (2x) + \ln |\sqrt{1 + 4x^2} + 2x|] \right] \Big|_0^{\sqrt{2}} \\ &= \frac{\pi}{4} \left[\frac{54\sqrt{2}}{4} - \frac{6\sqrt{2}}{6} = \frac{1}{8} \ln(3 + 2\sqrt{2}) \right] \\ &= \frac{\pi}{4} \left(\frac{51\sqrt{2}}{4} - \frac{\ln(3 + 2\sqrt{2})}{8} \right) = \frac{\pi}{32} [102\sqrt{2} - \ln(3 + 2\sqrt{2})] \approx 13.989 \end{aligned}$$

73. (a) Area of representative rectangle: $2\sqrt{1 - y^2} \Delta y$

Pressure: $2(62.4)(3 - y)\sqrt{1 - y^2} \Delta y$

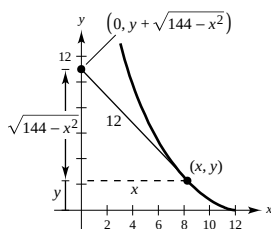
$$\begin{aligned} F &= 124.8 \int_{-1}^1 (3 - y)\sqrt{1 - y^2} dy \\ &= 124.8 \left[3 \int_{-1}^1 \sqrt{1 - y^2} dy - \int_{-1}^1 y\sqrt{1 - y^2} dy \right] \\ &= 124.8 \left[\frac{3}{2} (\arcsin y + y\sqrt{1 - y^2}) + \frac{1}{2} \left(\frac{2}{3} \right) (1 - y^2)^{3/2} \right]_{-1}^1 \\ &= (62.4)3[\arcsin 1 - \arcsin(-1)] = 187.2\pi \text{ lb} \end{aligned}$$



$$\begin{aligned} \text{(b) } F &= 124.8 \int_{-1}^1 (d - y)\sqrt{1 - y^2} dy = 124.8d \int_{-1}^1 \sqrt{1 - y^2} dy - 124.8 \int_{-1}^1 y\sqrt{1 - y^2} dy \\ &= 124.8 \left(\frac{d}{2} \right) \left[\arcsin y + y\sqrt{1 - y^2} \right]_{-1}^1 - 124.8(0) = 62.4\pi d \text{ lb} \end{aligned}$$

$$75. (a) m = \frac{dy}{dx} = \frac{y - (y + \sqrt{144 - x^2})}{x - 0}$$

$$= -\frac{\sqrt{144 - x^2}}{x}$$



$$(b) y = -\int \frac{\sqrt{144 - x^2}}{x} dx$$

Let $x = 12 \sin \theta$, $dx = 12 \cos \theta d\theta$, $\sqrt{144 - x^2} = 12 \cos \theta$.

$$y = -\int \frac{12 \cos \theta}{12 \sin \theta} 12 \cos \theta d\theta = -12 \int \frac{1 - \sin^2 \theta}{\sin \theta} d\theta$$

$$= -12 \int (\csc \theta - \sin \theta) d\theta = -12 \ln |\csc \theta - \cot \theta| - 12 \cos \theta + C$$

$$= -12 \ln \left| \frac{12}{x} - \frac{\sqrt{144 - x^2}}{x} \right| - 12 \left(\frac{\sqrt{144 - x^2}}{12} \right) + C$$

$$= -12 \ln \left| \frac{12 - \sqrt{144 - x^2}}{x} \right| - \sqrt{144 - x^2} + C$$

When $x = 12$, $y = 0 \Rightarrow C = 0$. Thus, $y = -12 \ln \left(\frac{12 - \sqrt{144 - x^2}}{x} \right) - \sqrt{144 - x^2}$.

Note: $\frac{12 - \sqrt{144 - x^2}}{x} > 0$ for $0 < x \leq 12$

(c) Vertical asymptote: $x = 0$

$$(d) y + \sqrt{144 - x^2} = 12 \Rightarrow y = 12 - \sqrt{144 - x^2}$$

Thus,

$$12 - \sqrt{144 - x^2} = -12 \ln \left(\frac{12 - \sqrt{144 - x^2}}{x} \right) - \sqrt{144 - x^2}$$

$$-1 = \ln \left(\frac{12 - \sqrt{144 - x^2}}{x} \right)$$

$$xe^{-1} = 12 - \sqrt{144 - x^2}$$

$$(xe^{-1} - 12)^2 = (-\sqrt{144 - x^2})^2$$

$$x^2 e^{-2} - 24xe^{-1} + 144 = 144 - x^2$$

$$x^2(e^{-2} + 1) - 24xe^{-1} = 0$$

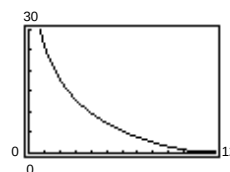
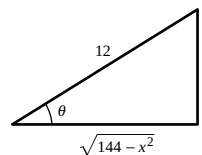
$$x[x(e^{-2} + 1) - 24e^{-1}] = 0$$

$$x = 0 \text{ or } x = \frac{24e^{-1}}{e^{-2} + 1} \approx 7.77665.$$

Therefore,

$$s = \int_{7.77665}^{12} \sqrt{1 + \left(-\frac{\sqrt{144 - x^2}}{x} \right)^2} dx = \int_{7.77665}^{12} \sqrt{\frac{x^2 + (144 - x^2)}{x^2}} dx$$

$$= \int_{7.77665}^{12} \frac{12}{x} dx = \left[12 \ln |x| \right]_{7.77665}^{12} = 12(\ln 12 - \ln 7.77665) \approx 5.2 \text{ meters.}$$



77. True

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos \theta d\theta}{\cos \theta} = \int d\theta$$

79. False

$$\int_0^{\sqrt{3}} \frac{dx}{(\sqrt{1+x^2})^3} = \int_0^{\pi/3} \frac{\sec^2 \theta d\theta}{\sec^3 \theta} = \int_0^{\pi/3} \cos \theta d\theta$$

81. Let $u = a \sin \theta$, $du = a \cos \theta d\theta$, $\sqrt{a^2 - u^2} = a \cos \theta$.

$$\begin{aligned} \int \sqrt{a^2 - u^2} du &= \int a^2 \cos^2 \theta d\theta = a^2 \int \frac{1 + \cos 2\theta}{2} d\theta \\ &= \frac{a^2}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C = \frac{a^2}{2} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{a^2}{2} \left[\arcsin \frac{u}{a} + \left(\frac{u}{a} \right) \left(\frac{\sqrt{a^2 - u^2}}{a} \right) \right] + C = \frac{1}{2} \left[a^2 \arcsin \frac{u}{a} + u \sqrt{a^2 - u^2} \right] + C \end{aligned}$$

Let $u = a \sec \theta$, $du = a \sec \theta \tan \theta d\theta$, $\sqrt{u^2 - a^2} = a \tan \theta$.

$$\begin{aligned} \int \sqrt{u^2 - a^2} du &= \int a \tan \theta (a \sec \theta \tan \theta) d\theta = a^2 \int \tan^2 \theta \sec \theta d\theta \\ &= a^2 \int (\sec^2 \theta - 1) \sec \theta d\theta = a^2 \int (\sec^3 \theta - \sec \theta) d\theta \\ &= a^2 \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \int \sec \theta d\theta \right] - a^2 \int \sec \theta d\theta = a^2 \left[\frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] \\ &= \frac{a^2}{2} \left[\frac{u}{a} \cdot \frac{\sqrt{u^2 - a^2}}{a} - \ln \left| \frac{u}{a} + \frac{\sqrt{u^2 - a^2}}{a} \right| \right] + C_1 \\ &= \frac{1}{2} [u \sqrt{u^2 - a^2} - a^2 \ln |u + \sqrt{u^2 - a^2}|] + C \end{aligned}$$

Let $u = a \tan \theta$, $du = a \sec^2 \theta d\theta$, $\sqrt{u^2 + a^2} = a \sec \theta d\theta$.

$$\begin{aligned} \int \sqrt{u^2 + a^2} du &= \int (a \sec \theta)(a \sec^2 \theta) d\theta \\ &= a^2 \int \sec^3 \theta d\theta = a^2 \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] + C_1 \\ &= \frac{a^2}{2} \left[\frac{\sqrt{u^2 + a^2}}{a} \cdot \frac{u}{a} + \ln \left| \frac{\sqrt{u^2 + a^2}}{a} + \frac{u}{a} \right| \right] + C_1 = \frac{1}{2} [u \sqrt{u^2 + a^2} + a^2 \ln |u + \sqrt{u^2 + a^2}|] + C \end{aligned}$$

Section 7.5 Partial Fractions

$$1. \frac{5}{x^2 - 10x} = \frac{5}{x(x - 10)} = \frac{A}{x} + \frac{B}{x - 10}$$

$$3. \frac{2x - 3}{x^3 + 10x} = \frac{2x - 3}{x(x^2 + 10)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 10}$$

$$5. \frac{16x}{x^3 - 10x^2} = \frac{16x}{x^2(x - 10)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 10}$$

$$7. \frac{1}{x^2 - 1} = \frac{1}{(x + 1)(x - 1)} = \frac{A}{x + 1} + \frac{B}{x - 1}$$

$$1 = A(x - 1) + B(x + 1)$$

$$\text{When } x = -1, 1 = -2A, A = -\frac{1}{2}.$$

$$\text{When } x = 1, 1 = 2B, B = \frac{1}{2}.$$

$$\begin{aligned} \int \frac{1}{x^2 - 1} dx &= -\frac{1}{2} \int \frac{1}{x + 1} dx + \frac{1}{2} \int \frac{1}{x - 1} dx \\ &= -\frac{1}{2} \ln |x + 1| + \frac{1}{2} \ln |x - 1| + C \\ &= \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| + C \end{aligned}$$

$$9. \frac{3}{x^2 + x - 2} = \frac{3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$3 = (x+2) + B(x-1)$$

When $x = 1$, $3 = 3A$, $A = 1$.

When $x = -2$, $3 = -3B$, $B = -1$.

$$\begin{aligned} \int \frac{3}{x^2 + x - 2} dx &= \int \frac{1}{x-1} dx - \int \frac{1}{x+2} dx \\ &= \ln|x-1| - \ln|x+2| + C \\ &= \ln\left|\frac{x-1}{x+2}\right| + C \end{aligned}$$

$$11. \frac{5-x}{2x^2+x-1} = \frac{5-x}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1}$$

$$5-x = A(x+1) + B(2x-1)$$

When $x = \frac{1}{2}$, $\frac{9}{2} = \frac{3}{2}A$, $A = 3$.

When $x = -1$, $6 = -3B$, $B = -2$.

$$\begin{aligned} \int \frac{5-x}{2x^2+x-1} dx &= 3 \int \frac{1}{2x-1} dx - 2 \int \frac{1}{x+1} dx \\ &= \frac{3}{2} \ln|2x-1| - 2 \ln|x+1| + C \end{aligned}$$

$$13. \frac{x^2 + 12x + 12}{x(x+2)(x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$$

$$x^2 + 12x + 12 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2)$$

When $x = 0$, $12 = -4A$, $A = -3$. When $x = -2$, $-8 = 8B$, $B = -1$. When $x = 2$, $40 = 8C$, $C = 5$.

$$\begin{aligned} \int \frac{x^2 + 12x + 12}{x^3 - 4x} dx &= 5 \int \frac{1}{x-2} dx - \int \frac{1}{x+2} dx - 3 \int \frac{1}{x} dx \\ &= 5 \ln|x-2| - \ln|x+2| - 3 \ln|x| + C \end{aligned}$$

$$15. \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8} = 2x + \frac{x+5}{(x-4)(x+2)} = 2x + \frac{A}{x-4} + \frac{B}{x+2}$$

$$x+5 = A(x+2) + B(x-4)$$

When $x = 4$, $9 = 6A$, $A = \frac{3}{2}$. When $x = -2$, $3 = -6B$, $B = -\frac{1}{2}$.

$$\begin{aligned} \int \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8} dx &= \int \left[2x + \frac{3/2}{x-4} - \frac{1/2}{x+2} \right] dx \\ &= x^2 + \frac{3}{2} \ln|x-4| - \frac{1}{2} \ln|x+2| + C \end{aligned}$$

$$17. \frac{4x^2 + 2x - 1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$4x^2 + 2x - 1 = Ax(x+1) + B(x+1) + Cx^2$$

When $x = 0$, $B = -1$. When $x = -1$, $C = 1$. When $x = 1$, $A = 3$.

$$\begin{aligned} \int \frac{4x^2 + 2x - 1}{x^3 + x^2} dx &= \int \left[\frac{3}{x} - \frac{1}{x^2} + \frac{1}{x+1} \right] dx = 3 \ln|x| + \frac{1}{x} + \ln|x+1| + C \\ &= \frac{1}{x} + \ln|x^4 + x^3| + C \end{aligned}$$

$$19. \frac{x^2 + 3x - 4}{x^3 - 4x^2 + 4x} = \frac{x^2 + 3x - 4}{x(x-2)^2} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

$$x^2 + 3x - 4 = A(x-2)^2 + Bx(x-2) + Cx$$

When $x = 0$, $-4 = -4A \Rightarrow A = -1$. When $x = 2$, $6 = 2C \Rightarrow C = 3$. When $x = 1$, $0 = -1 - B + 3 \Rightarrow B = 2$.

$$\begin{aligned} \int \frac{x^2 + 3x - 4}{x^3 - 4x^2 + 4x} dx &= \int \frac{-1}{x} dx + \int \frac{2}{(x-2)} dx + \int \frac{3}{(x-2)^2} dx \\ &= -\ln|x| + 2 \ln|x-2| - \frac{3}{(x-2)} + C \end{aligned}$$

$$21. \frac{x^2 - 1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

$$x^2 - 1 = A(x^2 + 1) + (Bx + C)x$$

When $x = 0$, $A = -1$. When $x = 1$, $0 = -2 + B + C$. When $x = -1$, $0 = -2 + B + C$. Solving these equations we have $A = -1$, $B = 2$, $C = 0$.

$$\begin{aligned} \int \frac{x^2 - 1}{x^3 + x} dx &= -\int \frac{1}{x} dx + \int \frac{2x}{x^2 + 1} dx \\ &= \ln|x^2 + 1| - \ln|x| + C \\ &= \ln\left|\frac{x^2 + 1}{x}\right| + C \end{aligned}$$

$$23. \frac{x^2}{x^4 - 2x^2 - 8} = \frac{A}{x - 2} + \frac{B}{x + 2} + \frac{Cx + D}{x^2 + 2}$$

$$x^2 = A(x + 2)(x^2 + 2) + B(x - 2)(x^2 + 2) + (Cx + D)(x + 2)(x - 2)$$

When $x = 2$, $4 = 24A$. When $x = -2$, $4 = -24B$. When $x = 0$, $0 = 4A - 4B - 4D$, and when $x = 1$, $1 = 9A - 3B - 3C - 3D$. Solving these equations we have $A = \frac{1}{6}$, $B = -\frac{1}{6}$, $C = 0$, $D = \frac{1}{3}$.

$$\begin{aligned} \int \frac{x^2}{x^4 - 2x^2 - 8} dx &= \frac{1}{6} \left[\int \frac{1}{x - 2} dx - \int \frac{1}{x + 2} dx + 2 \int \frac{1}{x^2 + 2} dx \right] \\ &= \frac{1}{6} \left[\ln\left|\frac{x - 2}{x + 2}\right| + \sqrt{2} \arctan \frac{x}{\sqrt{2}} \right] + C \end{aligned}$$

$$25. \frac{x}{(2x - 1)(2x + 1)(4x^2 + 1)} = \frac{A}{2x - 1} + \frac{B}{2x + 1} + \frac{Cx + D}{4x^2 + 1}$$

$$x = A(2x + 1)(4x^2 + 1) + B(2x - 1)(4x^2 + 1) + (Cx + D)(2x - 1)(2x + 1)$$

When $x = \frac{1}{2}$, $\frac{1}{2} = 4A$. When $x = -\frac{1}{2}$, $-\frac{1}{2} = -4B$. When $x = 0$, $0 = A - B - D$, and when $x = 1$, $1 = 15A + 5B + 3C + 3D$. Solving these equations we have $A = \frac{1}{8}$, $B = \frac{1}{8}$, $C = -\frac{1}{2}$, $D = 0$.

$$\begin{aligned} \int \frac{x}{16x^4 - 1} dx &= \frac{1}{8} \left[\int \frac{1}{2x - 1} dx + \int \frac{1}{2x + 1} dx - 4 \int \frac{x}{4x^2 + 1} dx \right] \\ &= \frac{1}{16} \ln\left|\frac{4x^2 - 1}{4x^2 + 1}\right| + C \end{aligned}$$

$$27. \frac{x^2 + 5}{(x + 1)(x^2 - 2x + 3)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 - 2x + 3}$$

$$\begin{aligned} x^2 + 5 &= A(x^2 - 2x + 3) + (Bx + C)(x + 1) \\ &= (A + B)x^2 + (-2A + B + C)x + (3A + C) \end{aligned}$$

When $x = -1$, $A = 1$. By equating coefficients of like terms, we have $A + B = 1$, $-2A + B + C = 0$, $3A + C = 5$. Solving these equations we have $A = 1$, $B = 0$, $C = 2$.

$$\begin{aligned} \int \frac{x^2 + 5}{x^3 - x^2 + x + 3} dx &= \int \frac{1}{x + 1} dx + 2 \int \frac{1}{(x - 1)^2 + 2} dx \\ &= \ln|x + 1| + \sqrt{2} \arctan\left(\frac{x - 1}{\sqrt{2}}\right) + C \end{aligned}$$

$$29. \frac{3}{(2x+1)(x+2)} = \frac{A}{2x+1} + \frac{B}{x+2}$$

$$3 = A(x+2) + B(2x+1)$$

When $x = -\frac{1}{2}$, $A = 2$. When $x = -2$, $B = -1$.

$$\begin{aligned} \int_0^1 \frac{3}{2x^2 + 5x + 2} dx &= \int_0^1 \frac{2}{2x+1} dx - \int_0^1 \frac{1}{x+2} dx \\ &= \left[\ln|2x-1| - \ln|x+2| \right]_0^1 \\ &= \ln 2 \end{aligned}$$

$$31. \frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

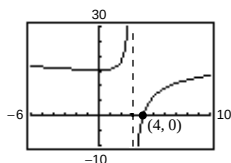
$$x+1 = A(x^2+1) + (Bx+C)x$$

When $x = 0$, $A = 1$. When $x = 1$, $2 = 2A + B + C$. When $x = -1$, $0 = 2A + B - C$. Solving these equations we have $A = 1$, $B = -1$, $C = 1$.

$$\begin{aligned} \int_1^2 \frac{x+1}{x(x^2+1)} dx &= \int_1^2 \frac{1}{x} dx - \int_1^2 \frac{x}{x^2+1} dx + \int_1^2 \frac{1}{x^2+1} dx \\ &= \left[\ln|x| - \frac{1}{2} \ln(x^2+1) + \arctan x \right]_1^2 \\ &= \frac{1}{2} \ln \frac{8}{5} - \frac{\pi}{4} + \arctan 2 \\ &\approx 0.557 \end{aligned}$$

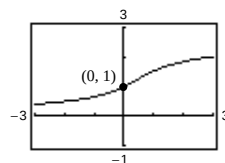
$$33. \int \frac{3x dx}{x^2 - 6x + 9} = 3 \ln|x-3| - \frac{9}{x-3} + C$$

$$(4, 0): 3 \ln|4-3| - \frac{9}{4-3} + C = 0 \Rightarrow C = 9$$



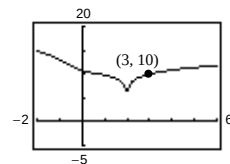
$$35. \int \frac{x^2 + x + 2}{(x^2 + 2)^2} dx = \frac{\sqrt{2}}{2} \arctan \frac{x}{\sqrt{2}} - \frac{1}{2(x^2 + 2)} + C$$

$$(0, 1): 0 - \frac{1}{4} + C = 1 \Rightarrow C = \frac{5}{4}$$



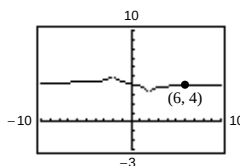
$$37. \int \frac{2x^2 - 2x + 3}{x^3 - x^2 - x - 2} dx = \ln|x-2| + \frac{1}{2} \ln|x^2 + x + 1| - \sqrt{3} \arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$(3, 10): 0 + \frac{1}{2} \ln 13 - \sqrt{3} \arctan \frac{7}{\sqrt{3}} + C = 10 \Rightarrow C = 10 - \frac{1}{2} \ln 13 + \sqrt{3} \arctan \frac{7}{\sqrt{3}}$$



$$39. \int \frac{1}{x^2 - 4} dx = \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + C$$

$$(6, 4): \frac{1}{4} \ln \left| \frac{4}{8} \right| + C = 4 \Rightarrow C = 4 - \frac{1}{4} \ln \frac{1}{2} = 4 + \frac{1}{4} \ln 2$$



41. Let
- $u = \cos x$
- ,
- $du = -\sin x \, dx$
- .

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$1 = A(u-1) + Bu$$

When $u = 0$, $A = -1$. When $u = 1$, $B = 1$, $u = \cos x$,
 $du = -\sin x \, dx$.

$$\begin{aligned} \int \frac{\sin x}{\cos x(\cos x - 1)} \, dx &= - \int \frac{1}{u(u-1)} \, du \\ &= \int \frac{1}{u} \, du - \int \frac{1}{u-1} \, du \\ &= \ln|u| - \ln|u-1| + C \\ &= \ln \left| \frac{u}{u-1} \right| + C \\ &= \ln \left| \frac{\cos x}{\cos x - 1} \right| + C \end{aligned}$$

45. Let
- $u = e^x$
- ,
- $du = e^x \, dx$
- .

$$\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}$$

$$1 = A(u+4) + B(u-1)$$

When $u = 1$, $A = \frac{1}{5}$. When $u = -4$, $B = -\frac{1}{5}$, $u = e^x$,
 $du = e^x \, dx$.

$$\begin{aligned} \int \frac{e^x}{(e^x-1)(e^x+4)} \, dx &= \int \frac{1}{(u-1)(u+4)} \, du \\ &= \frac{1}{5} \left(\int \frac{1}{u-1} \, du - \int \frac{1}{u+4} \, du \right) \\ &= \frac{1}{5} \ln \left| \frac{u-1}{u+4} \right| + C \\ &= \frac{1}{5} \ln \left| \frac{e^x-1}{e^x+4} \right| + C \end{aligned}$$

- 49.
- $\frac{x}{(a+bx)^2} = \frac{A}{a+bx} + \frac{B}{(a+bx)^2}$

$$x = A(a+bx) + B$$

When $x = -a/b$, $B = -a/b$.

When $x = 0$, $0 = aA + B \Rightarrow A = 1/b$.

$$\begin{aligned} \int \frac{x}{(a+bx)^2} \, dx &= \int \left(\frac{1/b}{a+bx} + \frac{-a/b}{(a+bx)^2} \right) \, dx \\ &= \frac{1}{b} \int \frac{1}{a+bx} \, dx - \frac{a}{b} \int \frac{1}{(a+bx)^2} \, dx \\ &= \frac{1}{b^2} \ln|a+bx| + \frac{a}{b^2} \left(\frac{1}{a+bx} \right) + C \\ &= \frac{1}{b^2} \left(\frac{a}{a+bx} + \ln|a+bx| \right) + C \end{aligned}$$

- 43.
- $\int \frac{3 \cos x}{\sin^2 x + \sin x - 2} \, dx = 3 \int \frac{1}{u^2 + u - 2} \, du$

$$= \ln \left| \frac{u-1}{u+2} \right| + C$$

$$= \ln \left| \frac{-1 + \sin x}{2 + \sin x} \right| + C$$

(From Exercise 9 with $u = \sin x$, $du = \cos x \, dx$)

- 47.
- $\frac{1}{x(a+bx)} = \frac{A}{x} + \frac{B}{a+bx}$

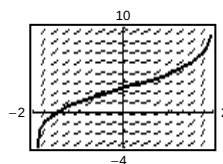
$$1 = A(a+bx) + Bx$$

When $x = 0$, $1 = aA \Rightarrow A = 1/a$.

When $x = -a/b$, $1 = -(a/b)B \Rightarrow B = -b/a$.

$$\begin{aligned} \int \frac{1}{x(a+bx)} \, dx &= \frac{1}{a} \int \left(\frac{1}{x} - \frac{b}{a+bx} \right) \, dx \\ &= \frac{1}{a} (\ln|x| - \ln|a+bx|) + C \\ &= \frac{1}{a} \ln \left| \frac{x}{a+bx} \right| + C \end{aligned}$$

- 51.
- $\frac{dy}{dx} = \frac{6}{4-x^2}$
- ,
- $y(0) = 3$



53. Dividing x^3 by $x - 5$.

55. (a) Substitution: $u = x^2 + 2x - 8$

(b) Partial fractions

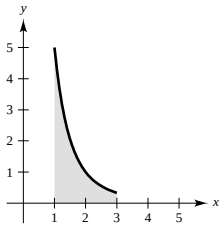
(c) Trigonometric substitution (tan) or inverse tangent rule

$$\begin{aligned}
 57. \text{ Average Cost} &= \frac{1}{80 - 75} \int_{75}^{80} \frac{124p}{(10 + p)(100 - p)} dp \\
 &= \frac{1}{5} \int_{75}^{80} \left(\frac{-124}{(10 + p)11} + \frac{1240}{(100 - p)11} \right) dp \\
 &= \frac{1}{5} \left[\frac{-124}{11} \ln(10 + p) - \frac{1240}{11} \ln(100 - p) \right]_{75}^{80} \\
 &\approx \frac{1}{5} (24.51) = 4.9
 \end{aligned}$$

Approximately \$490,000.

$$59. A = \int_1^3 \frac{10}{x(x^2 + 1)} dx \approx 3$$

Matches (c)



$$61. \quad \frac{1}{(x+1)(n-x)} = \frac{A}{x+1} + \frac{B}{n-x}, A = B = \frac{1}{n+1}$$

$$\frac{1}{n+1} \int \left(\frac{1}{x+1} + \frac{1}{n-x} \right) dx = kt + C$$

$$\frac{1}{n+1} \ln \left| \frac{x+1}{n-x} \right| = kt + C$$

$$\text{When } t = 0, x = 0, C = \frac{1}{n+1} \ln \frac{1}{n}.$$

$$\frac{1}{n+1} \ln \left| \frac{x+1}{n-x} \right| = kt + \frac{1}{n+1} \ln \frac{1}{n}$$

$$\frac{1}{n+1} \left[\ln \left| \frac{x+1}{n-x} \right| - \ln \frac{1}{n} \right] = kt$$

$$\ln \frac{nx + n}{n - x} = (n+1)kt$$

$$\frac{nx + n}{n - x} = e^{(n+1)kt}$$

$$x = \frac{n[e^{(n+1)kt} - 1]}{n + e^{(n+1)kt}} \quad \text{Note: } \lim_{t \rightarrow \infty} x = n$$

$$\begin{aligned}
 63. \quad \frac{x}{1+x^4} &= \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1} \\
 x &= (Ax+B)(x^2-\sqrt{2}x+1) + (Cx+D)(x^2+\sqrt{2}x+1) \\
 &= (A+C)x^3 + (B+D-\sqrt{2}A+\sqrt{2}C)x^2 + (A+C-\sqrt{2}B+\sqrt{2}D)x + (B+D)
 \end{aligned}$$

$$0 = A + C \Rightarrow C = -A$$

$$0 = B + D - \sqrt{2}A + \sqrt{2}C \quad \left\{ \begin{array}{l} -2\sqrt{2}A = 0 \Rightarrow A = 0 \text{ and } C = 0 \end{array} \right.$$

$$1 = A + C - \sqrt{2}B + \sqrt{2}D \quad \left\{ \begin{array}{l} -2\sqrt{2}B = 1 \Rightarrow B = -\frac{\sqrt{2}}{4} \text{ and } D = \frac{\sqrt{2}}{4} \end{array} \right.$$

$$0 = B + D \Rightarrow D = -B$$

Thus,

$$\begin{aligned}
 \int_0^1 \frac{x}{1+x^4} dx &= \int_0^1 \left[\frac{-\sqrt{2}/4}{x^2+\sqrt{2}x+1} + \frac{\sqrt{2}/4}{x^2-\sqrt{2}x+1} \right] dx \\
 &= \frac{\sqrt{2}}{4} \int_0^1 \left[\frac{-1}{\left[x + (\sqrt{2}/2)\right]^2 + (1/2)} + \frac{1}{\left[x - (\sqrt{2}/2)\right]^2 + (1/2)} \right] dx \\
 &= \frac{\sqrt{2}}{4} \cdot \frac{1}{1/\sqrt{2}} \left[-\arctan\left(\frac{x + (\sqrt{2}/2)}{1/\sqrt{2}}\right) + \arctan\left(\frac{x - (\sqrt{2}/2)}{1/\sqrt{2}}\right) \right]_0^1 \\
 &= \frac{1}{2} \left[-\arctan(\sqrt{2}x+1) + \arctan(\sqrt{2}x-1) \right]_0^1 \\
 &= \frac{1}{2} \left[(-\arctan(\sqrt{2}+1) + \arctan(\sqrt{2}-1)) - (-\arctan 1 + \arctan(-1)) \right] \\
 &= \frac{1}{2} \left[\arctan(\sqrt{2}-1) - \arctan(\sqrt{2}+1) + \frac{\pi}{4} + \frac{\pi}{4} \right].
 \end{aligned}$$

Since $\arctan x - \arctan y = \arctan[(x-y)/(1+xy)]$, we have:

$$\int_0^1 \frac{x}{1+x^4} dx = \frac{1}{2} \left[\arctan\left(\frac{(\sqrt{2}-1) - (\sqrt{2}+1)}{1 + (\sqrt{2}-1)(\sqrt{2}+1)}\right) + \frac{\pi}{2} \right] = \frac{1}{2} \left[\arctan\left(\frac{-2}{2}\right) + \frac{\pi}{2} \right] = \frac{1}{2} \left[-\frac{\pi}{4} + \frac{\pi}{2} \right] = \frac{\pi}{8}$$

Section 7.6 Integration by Tables and Other Integration Techniques

$$1. \text{ By Formula 6: } \int \frac{x^2}{1+x} dx = -\frac{x}{2}(2-x) + \ln|1+x| + C$$

$$\begin{aligned}
 3. \text{ By Formula 26: } \int e^x \sqrt{1+e^{2x}} dx &= \frac{1}{2} [e^x \sqrt{e^{2x}+1} + \ln|e^x + \sqrt{e^{2x}+1}|] + C \\
 u &= e^x, du = e^x dx
 \end{aligned}$$

$$5. \text{ By Formula 44: } \int \frac{1}{x^2 \sqrt{1-x^2}} dx = -\frac{\sqrt{1-x^2}}{x} + C$$

$$\begin{aligned}
7. \text{ By Formulas 50 and 48: } \int \sin^4(2x) dx &= \frac{1}{2} \int \sin^4(2x)(2) dx \\
&= \frac{1}{2} \left[\frac{-\sin^3(2x) \cos(2x)}{4} + \frac{3}{4} \int \sin^2(2x)(2) dx \right] \\
&= \frac{1}{2} \left[\frac{-\sin^3(2x) \cos(2x)}{4} + \frac{3}{8} (2x - \sin 2x \cos 2x) \right] + C \\
&= \frac{1}{16} (6x - 3 \sin 2x \cos 2x - 2 \sin^3 2x \cos 2x) + C
\end{aligned}$$

$$\begin{aligned}
9. \text{ By Formula 57: } \int \frac{1}{\sqrt{x}(1 - \cos \sqrt{x})} dx &= 2 \int \frac{1}{1 - \cos \sqrt{x}} \left(\frac{1}{2\sqrt{x}} \right) dx \\
&= -2(\cot \sqrt{x} + \csc \sqrt{x}) + C \\
u &= \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx
\end{aligned}$$

11. By Formula 84:

$$\int \frac{1}{1 + e^{2x}} dx = x - \frac{1}{2} \ln(1 + e^{2x}) + C$$

13. By Formula 89:

$$\int x^3 \ln x dx = \frac{x^4}{16} (4 \ln|x| - 1) + C$$

$$\begin{aligned}
15. (a) \text{ By Formulas 83 and 82: } \int x^2 e^x dx &= x^2 e^x - 2 \int x e^x dx \\
&= x^2 e^x - 2[(x - 1)e^x + C_1] \\
&= x^2 e^x - 2x e^x + 2e^x + C
\end{aligned}$$

(b) Integration by parts: $u = x^2$, $du = 2x dx$, $dv = e^x dx$, $v = e^x$

$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx$$

Parts again: $u = 2x$, $du = 2 dx$, $dv = e^x dx$, $v = e^x$

$$\int x^2 e^x dx = x^2 e^x - \left[2x e^x - \int 2e^x dx \right] = x^2 e^x - 2x e^x + 2e^x + C$$

17. (a) By Formula: 12, $a = b = 1$, $u = x$, and

$$\begin{aligned}
\int \frac{1}{x^2(x+1)} dx &= \frac{-1}{1} \left(\frac{1}{x} + \frac{1}{1} \ln \left| \frac{x}{1+x} \right| \right) + C \\
&= \frac{-1}{x} - \ln \left| \frac{x}{1+x} \right| + C \\
&= \frac{-1}{x} + \ln \left| \frac{x+1}{x} \right| + C
\end{aligned}$$

(b) Partial fractions:

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$1 = Ax(x+1) + B(x+1) + Cx^2$$

$$x = 0: 1 = B$$

$$x = -1: 1 = C$$

$$x = 1: 1 = 2A + 2 + 1 \Rightarrow A = -1$$

$$\begin{aligned}
\int \frac{1}{x^2(x+1)} dx &= \int \left[\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1} \right] dx \\
&= -\ln|x| - \frac{1}{x} + \ln|x+1| + C \\
&= -\frac{1}{x} - \ln \left| \frac{x}{x+1} \right| + C
\end{aligned}$$

19. By Formula 81: $\int x e^{x^2} = \frac{1}{2} e^{x^2} + C$

21. By Formula 79: $\int x \operatorname{arcsec}(x^2 + 1) dx = \frac{1}{2} \int \operatorname{arcsec}(x^2 + 1)(2x) dx$
 $= \frac{1}{2} [(x^2 + 1) \operatorname{arcsec}(x^2 + 1) - \ln((x^2 + 1) + \sqrt{x^4 + 2x^2})] + C$
 $u = x^2 + 1, du = 2x dx$

23. By Formula 89: $\int x^2 \ln x dx = \frac{x^3}{9}(-1 + 3 \ln|x|) + C$

25. By Formula 35: $\int \frac{1}{x^2 \sqrt{x^2 - 4}} dx = \frac{\sqrt{x^2 - 4}}{4x} + C$

27. By Formula 4: $\int \frac{2x}{(1 - 3x)^2} dx = 2 \int \frac{x}{(1 - 3x)^2} dx = \frac{2}{9} \left(\ln|1 - 3x| + \frac{1}{1 - 3x} \right) + C$

29. By Formula 76:

$$\int e^x \arccos e^x dx = e^x \arccos e^x - \sqrt{1 - e^{2x}} + C$$

$u = e^x, du = e^x dx$

31. By Formula 73:

$$\int \frac{x}{1 - \sec x^2} dx = \frac{1}{2} \int \frac{2x}{1 - \sec x^2} dx$$

$$= \frac{1}{2} (x^2 + \cot x^2 + \csc x^2) + C$$

33. By Formula 23: $\int \frac{\cos x}{1 + \sin^2 x} dx = \arctan(\sin x) + C$
 $u = \sin x, du = \cos x dx$

35. By Formula 14: $\int \frac{\cos \theta}{3 + 2 \sin \theta + \sin^2 \theta} d\theta = \frac{\sqrt{2}}{2} \arctan\left(\frac{1 + \sin \theta}{\sqrt{2}}\right) + C$
 $u = \sin \theta, du = \cos \theta d\theta$

37. By Formula 35: $\int \frac{1}{x^2 \sqrt{2 + 9x^2}} dx = 3 \int \frac{3}{(3x)^2 \sqrt{(\sqrt{2})^2 + (3x)^2}} dx$
 $= -\frac{3\sqrt{2 + 9x^2}}{6x} + C$
 $= -\frac{\sqrt{2 + 9x^2}}{2x} + C$

39. By Formulas 54 and 55:

$$\int t^3 \cos t dt = t^3 \sin t - 3 \int t^2 \sin t dt$$

$$= t^3 \sin t - 3 \left[-t^2 \cos t + 2 \int t \cos t dt \right]$$

$$= t^3 \sin t + 3t^2 \cos t - 6 \left[t \sin t - \int \sin t dt \right]$$

$$= t^3 \sin t + 3t^2 \cos t - 6t \sin t - 6 \cos t + C$$

41. By Formula 3: $\int \frac{\ln x}{x(3 + 2 \ln x)} dx = \frac{1}{4}(2 \ln|x| - 3 \ln|3 + 2 \ln|x||) + C$

$$u = \ln x, du = \frac{1}{x} dx$$

43. By Formulas 1, 25, and 33: $\int \frac{x}{(x^2 - 6x + 10)^2} dx = \frac{1}{2} \int \frac{2x - 6 + 6}{(x^2 - 6x + 10)^2} dx$

$$= \frac{1}{2} \int (x^2 - 6x + 10)^{-2} (2x - 6) dx + 3 \int \frac{1}{[(x - 3)^2 + 1]^2} dx$$

$$= -\frac{1}{2(x^2 - 6x + 10)} + \frac{3}{2} \left[\frac{x - 3}{x^2 - 6x + 10} + \arctan(x - 3) \right] + C$$

$$= \frac{3x - 10}{2(x^2 - 6x + 10)} + \frac{3}{2} \arctan(x - 3) + C$$

45. By Formula 31: $\int \frac{x}{\sqrt{x^4 - 6x^2 + 5}} dx = \frac{1}{2} \int \frac{2x}{\sqrt{(x^2 - 3)^2 - 4}} dx$

$$= \frac{1}{2} \ln|x^2 - 3 + \sqrt{x^4 - 6x^2 + 5}| + C$$

$$u = x^2 - 3, du = 2x dx$$

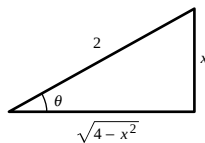
47. $\int \frac{x^3}{\sqrt{4 - x^2}} dx = \int \frac{8 \sin^3 \theta (2 \cos \theta d\theta)}{2 \cos \theta}$

$$= 8 \int (1 - \cos^2 \theta) \sin \theta d\theta$$

$$= 8 \int [\sin \theta - \cos^2 \theta (\sin \theta)] d\theta$$

$$= -8 \cos \theta + \frac{8 \cos^3 \theta}{3} + C$$

$$= \frac{-\sqrt{4 - x^2}}{3} (x^2 + 8) + C$$



$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4 - x^2} = 2 \cos \theta$$

49. By Formula 8: $\int \frac{e^{3x}}{(1 + e^x)^3} dx = \int \frac{(e^x)^2}{(1 + e^x)^3} (e^x) dx$

$$= \frac{2}{1 + e^x} - \frac{1}{2(1 + e^x)^2} + \ln|1 + e^x| + C$$

$$u = e^x, du = e^x dx$$

51. $\frac{u^2}{(a + bu)^2} = \frac{1}{b^2} - \frac{(2a/b)u + (a^2/b^2)}{(a + bu)^2} = \frac{1}{b^2} + \frac{A}{a + bu} + \frac{B}{(a + bu)^2}$

$$-\frac{2a}{b}u - \frac{a^2}{b^2} = A(a + bu) + B = (aA + B) + bAu$$

Equating the coefficients of like terms we have $aA + B = -a^2/b^2$ and $bA = -2a/b$. Solving these equations we have $A = -2a/b^2$ and $B = a^2/b^2$.

$$\int \frac{u^2}{(a + bu)^2} du = \frac{1}{b^2} \int du - \frac{2a}{b^2} \left(\frac{1}{b} \right) \int \frac{1}{a + bu} b du + \frac{a^2}{b^2} \left(\frac{1}{b} \right) \int \frac{1}{(a + bu)^2} b du = \frac{1}{b^2} u - \frac{2a}{b^3} \ln|a + bu| - \frac{a^2}{b^3} \left(\frac{1}{a + bu} \right) + C$$

$$= \frac{1}{b^3} \left(bu - \frac{a^2}{a + bu} - 2a \ln|a + bu| \right) + C$$

53. When we have $u^2 + a^2$:

$$u = a \tan \theta$$

$$du = a \sec^2 \theta d\theta$$

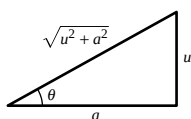
$$u^2 + a^2 = a^2 \sec^2 \theta$$

$$\int \frac{1}{(u^2 + a^2)^{3/2}} du = \int \frac{a \sec^2 \theta d\theta}{a^3 \sec^3 \theta}$$

$$= \frac{1}{a^2} \int \cos \theta d\theta$$

$$= \frac{1}{a^2} \sin \theta + C$$

$$= \frac{u}{a^2 \sqrt{u^2 + a^2}} + C$$

When we have $u^2 - a^2$:

$$u = a \sec \theta$$

$$du = a \sec \theta \tan \theta d\theta$$

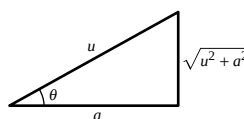
$$u^2 - a^2 = a^2 \tan^2 \theta$$

$$\int \frac{1}{(u^2 - a^2)^{3/2}} du = \int \frac{a \sec \theta \tan \theta d\theta}{a^3 \tan^3 \theta}$$

$$= \frac{1}{a^2} \int \frac{\cos \theta}{\sin^2 \theta} d\theta$$

$$= -\frac{1}{a^2} \csc \theta + C$$

$$= \frac{-u}{a^2 \sqrt{u^2 - a^2}} + C$$

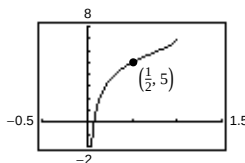


$$\begin{aligned} 55. \int (\arctan u) du &= u \arctan u - \frac{1}{2} \int \frac{2u}{1+u^2} du \\ &= u \arctan u - \frac{1}{2} \ln(1+u^2) + C \end{aligned}$$

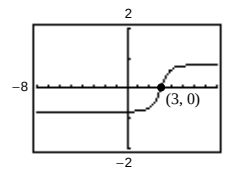
$$= u \arctan u - \ln \sqrt{1+u^2} + C$$

$$w = \arctan u, dv = du, dw = \frac{du}{1+u^2}, v = u$$

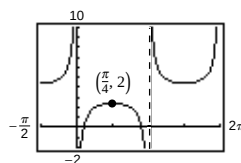
$$\begin{aligned} 57. \int \frac{1}{x^{3/2} \sqrt{1-x}} dx &= \frac{-2\sqrt{1-x}}{\sqrt{x}} + C \\ \left(\frac{1}{2}, 5\right): \frac{-2\sqrt{1/2}}{\sqrt{1/2}} + C &= 5 \Rightarrow C = 7 \\ y &= \frac{-2\sqrt{1-x}}{\sqrt{x}} + 7 \end{aligned}$$



$$\begin{aligned} 59. \int \frac{1}{(x^2 - 6x + 10)^2} dx &= \frac{1}{2} \left[\tan^{-1}(x-3) + \frac{x-3}{x^2 - 6x + 10} \right] + C \\ (3, 0): \frac{1}{2} \left[0 + \frac{0}{10} \right] + C &= 0 \Rightarrow C = 0 \\ y &= \frac{1}{2} \left[\tan^{-1}(x-3) + \frac{x-3}{x^2 - 6x + 10} \right] \end{aligned}$$



$$\begin{aligned} 61. \int \frac{1}{\sin \theta \tan \theta} d\theta &= -\csc \theta + C \\ \left(\frac{\pi}{4}, 2\right): -\frac{2}{\sqrt{2}} + C &= 2 \Rightarrow C = 2 + \sqrt{2} \\ y &= -\csc \theta + 2 + \sqrt{2} \end{aligned}$$



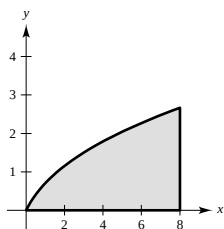
$$\begin{aligned}
 63. \int \frac{1}{2 - 3 \sin \theta} d\theta &= \int \left[\frac{\frac{2 du}{1 + u^2}}{2 - 3 \left(\frac{2u}{1 + u^2} \right)} \right] \\
 &= \int \frac{2}{(1 + u^2) - 6u} du \\
 &= \int \frac{1}{u^2 - 3u + 1} du \\
 &= \int \frac{1}{\left(u - \frac{3}{2}\right)^2 - \frac{5}{4}} du \\
 &= \frac{1}{\sqrt{5}} \ln \left| \frac{\left(u - \frac{3}{2}\right) - \frac{\sqrt{5}}{2}}{\left(u - \frac{3}{2}\right) + \frac{\sqrt{5}}{2}} \right| + C \\
 &= \frac{1}{\sqrt{5}} \ln \left| \frac{2u - 3 - \sqrt{5}}{2u - 3 + \sqrt{5}} \right| + C \\
 &= \frac{1}{\sqrt{5}} \ln \left| \frac{2 \tan\left(\frac{\theta}{2}\right) - 3 - \sqrt{5}}{2 \tan\left(\frac{\theta}{2}\right) - 3 + \sqrt{5}} \right| + C
 \end{aligned}$$

$$u = \tan \frac{\theta}{2}$$

$$\begin{aligned}
 67. \int \frac{\sin \theta}{3 - 2 \cos \theta} d\theta &= \frac{1}{2} \int \frac{2 \sin \theta}{3 - 2 \cos \theta} d\theta \\
 &= \frac{1}{2} \ln |u| + C \\
 &= \frac{1}{2} \ln(3 - 2 \cos \theta) + C
 \end{aligned}$$

$$u = 3 - 2 \cos \theta, du = 2 \sin \theta d\theta$$

$$\begin{aligned}
 71. A &= \int_0^8 \frac{x}{\sqrt{x+1}} dx \\
 &= \left[\frac{-2(2-x)}{3} \sqrt{x+1} \right]_0^8 \\
 &= 12 - \left(-\frac{4}{3}\right) \\
 &= \frac{40}{3} \approx 13.333 \text{ square units}
 \end{aligned}$$



$$\begin{aligned}
 65. \int_0^{\pi/2} \frac{1}{1 + \sin \theta + \cos \theta} d\theta &= \int_0^1 \left[\frac{\frac{2 du}{1 + u^2}}{1 + \frac{2u}{1 + u^2} + \frac{1 - u^2}{1 + u^2}} \right] \\
 &= \int_0^1 \frac{1}{1 + u} du \\
 &= \left[\ln |1 + u| \right]_0^1 \\
 &= \ln 2
 \end{aligned}$$

$$u = \tan \frac{\theta}{2}$$

$$\begin{aligned}
 69. \int \frac{\cos \sqrt{\theta}}{\sqrt{\theta}} d\theta &= 2 \int \cos \sqrt{\theta} \left(\frac{1}{2\sqrt{\theta}} \right) d\theta \\
 &= 2 \sin \sqrt{\theta} + C \\
 u &= \sqrt{\theta}, du = \frac{1}{2\sqrt{\theta}} d\theta
 \end{aligned}$$

73. Arctangent Formula, Formula 23,

$$\int \frac{1}{u^2 + 1} du, u = e^x$$

75. Substitution: $u = x^2, du = 2x dx$
Then Formula 81.

77. Cannot be integrated.

79. Answers will vary. For example,

$$\int (2x)e^{2x} dx$$

can be integrated by first letting
 $u = 2x$ and then using Formula 82.

$$\begin{aligned}
 81. \quad W &= \int_0^5 2000xe^{-x} dx \\
 &= -2000 \int_0^5 -xe^{-x} dx \\
 &= 2000 \int_0^5 (-x)e^{-x}(-1) dx \\
 &= 2000 \left[(-x)e^{-x} - e^{-x} \right]_0^5 \\
 &= 2000 \left(-\frac{6}{e^5} + 1 \right) \\
 &\approx 1919.145 \text{ ft} \cdot \text{lbs}
 \end{aligned}$$

$$\begin{aligned}
 83. \quad (a) \quad V &= 20(2) \int_0^3 \frac{2}{\sqrt{1+y^2}} dy & W &= 148(80 \ln(3 + \sqrt{10})) \\
 &= \left[80 \ln|y + \sqrt{1+y^2}| \right]_0^3 & &= 11,840 \ln(3 + \sqrt{10}) \\
 &= 80 \ln(3 + \sqrt{10}) & &\approx 21,530.4 \text{ lb} \\
 &\approx 145.5 \text{ cubic feet}
 \end{aligned}$$

(b) By symmetry, $\bar{x} = 0$.

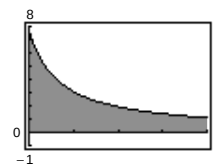
$$\begin{aligned}
 M &= \rho(2) \int_0^3 \frac{2}{\sqrt{1+y^2}} dy = \left[4\rho \ln|y + \sqrt{1+y^2}| \right]_0^3 = 4\rho \ln(3 + \sqrt{10}) \\
 M_x &= 2\rho \int_0^3 \frac{2y}{\sqrt{1+y^2}} dy = \left[4\rho \sqrt{1+y^2} \right]_0^3 = 4\rho(\sqrt{10} - 1) \\
 \bar{y} &= \frac{M_x}{M} = \frac{4\rho(\sqrt{10} - 1)}{4\rho \ln(3 + \sqrt{10})} \approx 1.19
 \end{aligned}$$

Centroid: $(\bar{x}, \bar{y}) \approx (0, 1.19)$

$$85. \quad (a) \quad \int_0^4 \frac{k}{2+3x} dx = 10$$

$$\begin{aligned}
 k &= \frac{10}{\int_0^4 \frac{1}{2+3x} dx} \approx \frac{10}{0.6486} \\
 &= 15.417 \quad \left(= \frac{30}{\ln 7} \right)
 \end{aligned}$$

$$(b) \quad \int_0^4 \frac{15.417}{2+3x} dx$$

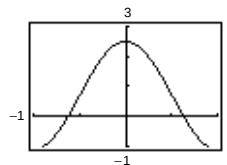


87. False. You might need to convert your integral using substitution or algebra.

Section 7.7 Indeterminate Forms and L'Hôpital's Rule

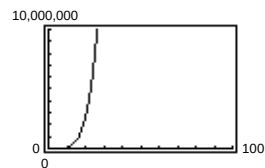
$$1. \quad \lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 2x} \approx 2.5 \left(\text{exact: } \frac{5}{2} \right)$$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	2.4132	2.4991	2.500	2.500	2.4991	2.4132



3. $\lim_{x \rightarrow \infty} x^5 e^{-x/100} \approx 0$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$	0.9901	90,484	3.7×10^9	4.5×10^{10}	0	0



5. (a) $\lim_{x \rightarrow 3} \frac{2(x-3)}{x^2-9} = \lim_{x \rightarrow 3} \frac{2(x-3)}{(x+3)(x-3)} = \lim_{x \rightarrow 3} \frac{2}{x+3} = \frac{1}{3}$

(b) $\lim_{x \rightarrow 3} \frac{2(x-3)}{x^2-9} = \lim_{x \rightarrow 3} \frac{(d/dx)[2(x-3)]}{(d/dx)[x^2-9]} = \lim_{x \rightarrow 3} \frac{2}{2x} = \frac{2}{6} = \frac{1}{3}$

7. (a) $\lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x \rightarrow 3} \frac{(x+1)-4}{(x-3)(\sqrt{x+1}+2)} = \lim_{x \rightarrow 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{4}$

(b) $\lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x \rightarrow 3} \frac{(d/dx)[\sqrt{x+1}-2]}{(d/dx)[x-3]} = \lim_{x \rightarrow 3} \frac{1/(2\sqrt{x+1})}{1} = \frac{1}{4}$

9. (a) $\lim_{x \rightarrow \infty} \frac{5x^2-3x+1}{3x^2-5} = \lim_{x \rightarrow \infty} \frac{5-(3/x)+(1/x^2)}{3-(5/x^2)} = \frac{5}{3}$

(b) $\lim_{x \rightarrow \infty} \frac{5x^2-3x+1}{3x^2-5} = \lim_{x \rightarrow \infty} \frac{(d/dx)[5x^2-3x+1]}{(d/dx)[3x^2-5]} = \lim_{x \rightarrow \infty} \frac{10x-3}{6x} = \lim_{x \rightarrow \infty} \frac{(d/dx)[10x-3]}{(d/dx)[6x]} = \lim_{x \rightarrow \infty} \frac{10}{6} = \frac{5}{3}$

11. $\lim_{x \rightarrow 2} \frac{x^2-x-2}{x-2} = \lim_{x \rightarrow 2} \frac{2x-1}{1} = 3$

13. $\lim_{x \rightarrow 0} \frac{\sqrt{4-x^2}-2}{x} = \lim_{x \rightarrow 0} \frac{-x/\sqrt{4-x^2}}{1} = 0$

15. $\lim_{x \rightarrow 0} \frac{e^x - (1-x)}{x} = \lim_{x \rightarrow 0} \frac{e^x + 1}{1} = 2$

17. Case 1: $n = 1$

$$\lim_{x \rightarrow 0^+} \frac{e^x - (1+x)}{x} = \lim_{x \rightarrow 0^+} \frac{e^x - 1}{1} = 0$$

Case 2: $n = 2$

$$\lim_{x \rightarrow 0^+} \frac{e^x - (1+x)}{x^2} = \lim_{x \rightarrow 0^+} \frac{e^x - 1}{2x} = \lim_{x \rightarrow 0^+} \frac{e^x}{2} = \frac{1}{2}$$

Case 3: $n \geq 3$

$$\lim_{x \rightarrow 0^+} \frac{e^x - (1+x)}{x^n} = \lim_{x \rightarrow 0^+} \frac{e^x - 1}{nx^{n-1}} = \lim_{x \rightarrow 0^+} \frac{e^x}{n(n-1)x^{n-2}} = \infty$$

19. $\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x} = \lim_{x \rightarrow 0} \frac{2 \cos 2x}{3 \cos 3x} = \frac{2}{3}$

21. $\lim_{x \rightarrow 0} \frac{\arcsin x}{x} = \lim_{x \rightarrow 0} \frac{1/\sqrt{1-x^2}}{1} = 1$

23. $\lim_{x \rightarrow \infty} \frac{3x^2-2x+1}{2x^2+3} = \lim_{x \rightarrow \infty} \frac{6x-2}{4x} = \lim_{x \rightarrow \infty} \frac{6}{4} = \frac{3}{2}$

25. $\lim_{x \rightarrow \infty} \frac{x^2+2x+3}{x-1} = \lim_{x \rightarrow \infty} \frac{2x+2}{1} = \infty$

27. $\lim_{x \rightarrow \infty} \frac{x^3}{e^{x/2}} = \lim_{x \rightarrow \infty} \frac{3x^2}{(1/2)e^{x/2}}$

$$= \lim_{x \rightarrow \infty} \frac{6x}{(1/4)e^{x/2}} = \lim_{x \rightarrow \infty} \frac{6}{(1/8)e^{x/2}} = 0$$

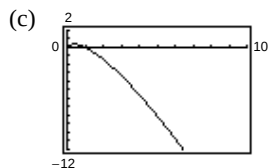
$$29. \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + (1/x^2)}} = 1$$

Note: L'Hôpital's Rule does not work on this limit.
See Exercise 79.

$$33. \lim_{x \rightarrow \infty} \frac{\ln x}{x^2} = \lim_{x \rightarrow \infty} \frac{1/x}{2x} = \lim_{x \rightarrow \infty} \frac{1}{2x^2} = 0$$

$$37. (a) \lim_{x \rightarrow 0^+} (-x \ln x) = (-0)(-\infty) = (0)(\infty)$$

$$\begin{aligned} (b) \lim_{x \rightarrow 0^+} (-x \ln x) &= \lim_{x \rightarrow 0^+} \frac{\ln x}{-1/x} \\ &= \lim_{x \rightarrow 0^+} \frac{1/x}{1/x^2} \\ &= \lim_{x \rightarrow 0^+} x = 0 \end{aligned}$$



$$41. (a) \lim_{x \rightarrow 0^+} x^{1/x} = 0^\infty = 0, \text{ not indeterminate}$$

(See Exercise 95)

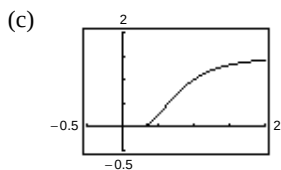
$$(b) \text{ Let } y = x^{1/x}$$

$$\ln y = \ln x^{1/x} = \frac{1}{x} \ln x.$$

Since $x \rightarrow 0^+$, $\frac{1}{x} \ln x \rightarrow (\infty)(-\infty) = -\infty$. Hence,

$$\ln y \rightarrow -\infty \Rightarrow y \rightarrow 0^+.$$

Therefore, $\lim_{x \rightarrow 0^+} x^{1/x} = 0$.



$$45. (a) \lim_{x \rightarrow 0^+} (1+x)^{1/x} = 1^\infty$$

$$(b) \text{ Let } y = \lim_{x \rightarrow 0^+} (1+x)^{1/x}.$$

$$\begin{aligned} \ln y &= \lim_{x \rightarrow 0^+} \frac{\ln(1+x)}{x} \\ &= \lim_{x \rightarrow 0^+} \left(\frac{1/(1+x)}{1} \right) = 1 \end{aligned}$$

Thus, $\ln y = 1 \Rightarrow y = e^1 = e$.

Therefore, $\lim_{x \rightarrow 0^+} (1+x)^{1/x} = e$.

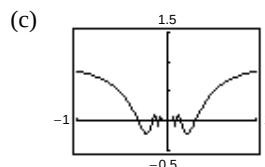
$$31. \lim_{x \rightarrow \infty} \frac{\cos x}{x} = 0 \text{ by Squeeze Theorem}$$

$$\left(\frac{\cos x}{x} \leq \frac{1}{x} \right)$$

$$35. \lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty$$

$$39. (a) \lim_{x \rightarrow \infty} \left(x \sin \frac{1}{x} \right) = (\infty)(0)$$

$$\begin{aligned} (b) \lim_{x \rightarrow \infty} x \sin \frac{1}{x} &= \lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} \\ &= \lim_{x \rightarrow \infty} \frac{(-1/x^2) \cos(1/x)}{-1/x^2} \\ &= \lim_{x \rightarrow \infty} \cos\left(\frac{1}{x}\right) = 1 \end{aligned}$$



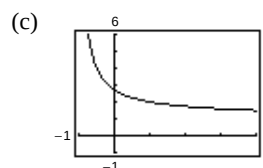
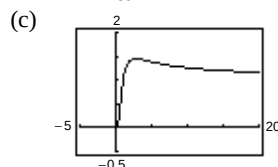
$$43. (a) \lim_{x \rightarrow \infty} x^{1/x} = \infty^0$$

$$(b) \text{ Let } y = \lim_{x \rightarrow \infty} x^{1/x}.$$

$$\ln y = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \left(\frac{1/x}{1} \right) = 0$$

Thus, $\ln y = 0 \Rightarrow y = e^0 = 1$. Therefore,

$$\lim_{x \rightarrow \infty} x^{1/x} = 1.$$



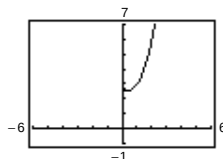
47. (a) $\lim_{x \rightarrow 0^+} [3(x)^{x/2}] = 0^0$

(b) Let $y = \lim_{x \rightarrow 0^+} 3(x)^{x/2}$.

$$\begin{aligned} \ln y &= \lim_{x \rightarrow 0^+} \left[\ln 3 + \frac{x}{2} \ln x \right] \\ &= \lim_{x \rightarrow 0^+} \left[\ln 3 + \frac{\ln x}{2/x} \right] \\ &= \lim_{x \rightarrow 0^+} \ln 3 + \lim_{x \rightarrow 0^+} \frac{1/x}{-2/x^2} \\ &= \lim_{x \rightarrow 0^+} \ln 3 - \lim_{x \rightarrow 0^+} \frac{x}{2} \\ &= \ln 3 \end{aligned}$$

Hence, $\lim_{x \rightarrow 0^+} 3(x)^{x/2} = 3$.

(c)

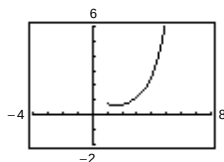


49. (a) $\lim_{x \rightarrow 1^+} (\ln x)^{x-1} = 0^0$

(b) Let $y = \lim_{x \rightarrow 1^+} (\ln x)^{x-1}$
 $= \lim_{x \rightarrow 1^+} (x-1) \ln x = 0$

Hence, $\lim_{x \rightarrow 1^+} (\ln x)^{x-1} = 1$

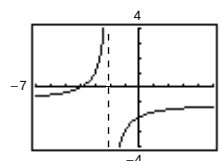
(c)



51. (a) $\lim_{x \rightarrow 2^+} \left(\frac{8}{x^2 - 4} - \frac{x}{x - 2} \right) = \infty - \infty$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow 2^+} \left(\frac{8}{x^2 - 4} - \frac{x}{x - 2} \right) &= \lim_{x \rightarrow 2^+} \frac{8 - x(x + 2)}{x^2 - 4} \\ &= \lim_{x \rightarrow 2^+} \frac{(2 - x)(4 + x)}{(x + 2)(x - 2)} \\ &= \lim_{x \rightarrow 2^+} \frac{-(x + 4)}{x + 2} = \frac{-3}{2} \end{aligned}$$

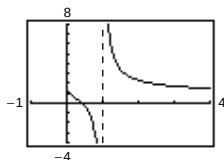
(c)



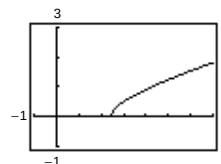
53. (a) $\lim_{x \rightarrow 1^+} \left(\frac{3}{\ln x} - \frac{2}{x - 1} \right) = \infty - \infty$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow 1^+} \left(\frac{3}{\ln x} - \frac{2}{x - 1} \right) &= \lim_{x \rightarrow 1^+} \frac{3x - 3 - 2 \ln x}{(x - 1) \ln x} \\ &= \lim_{x \rightarrow 1^+} \frac{3 - (2/x)}{[(x - 1)/x] + \ln x} = \infty \end{aligned}$$

(c)

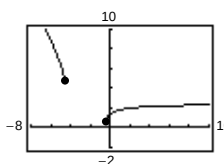


55. (a)



$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow 3} \frac{x - 3}{\ln(2x - 5)} &= \lim_{x \rightarrow 3} \frac{1}{2/(2x - 5)} \\ &= \lim_{x \rightarrow 3} \frac{2x - 5}{2} = \frac{1}{2} \end{aligned}$$

57. (a)



$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x + 2} - x) &= \lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x + 2} - x) \frac{(\sqrt{x^2 + 5x + 2} + x)}{(\sqrt{x^2 + 5x + 2} + x)} \\ &= \lim_{x \rightarrow \infty} \frac{(x^2 + 5x + 2) - x^2}{\sqrt{x^2 + 5x + 2} + x} \\ &= \lim_{x \rightarrow \infty} \frac{5x + 2}{\sqrt{x^2 + 5x + 2} + x} \\ &= \lim_{x \rightarrow \infty} \frac{5 + (2/x)}{\sqrt{1 + (5/x) + (2/x^2)} + 1} = \frac{5}{2} \end{aligned}$$

59. $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, 1^\infty, 0^0, \infty - \infty$

63. $\lim_{x \rightarrow \infty} \frac{x^2}{e^{5x}} = \lim_{x \rightarrow \infty} \frac{2x}{5e^{5x}} = \lim_{x \rightarrow \infty} \frac{2}{25e^{5x}} = 0$

65. $\lim_{x \rightarrow \infty} \frac{(\ln x)^3}{x} = \lim_{x \rightarrow \infty} \frac{3(\ln x)^2(1/x)}{1}$
 $= \lim_{x \rightarrow \infty} \frac{3(\ln x)^2}{x}$
 $= \lim_{x \rightarrow \infty} \frac{6(\ln x)(1/x)}{1}$
 $= \lim_{x \rightarrow \infty} \frac{6(\ln x)}{x} = \lim_{x \rightarrow \infty} \frac{6}{x} = 0$

69.

x	10	10^2	10^4	10^6	10^8	10^{10}
$\frac{(\ln x)^4}{x}$	2.811	4.498	0.720	0.036	0.001	0.000

71. $y = x^{1/x}, x > 0$

Horizontal asymptote: $y = 1$ (See Exercise 37)

$$\ln y = \frac{1}{x} \ln x$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} \left(\frac{1}{x} \right) + (\ln x) \left(-\frac{1}{x^2} \right)$$

$$\frac{dy}{dx} = x^{1/x} \left(\frac{1}{x^2} \right) (1 - \ln x) = x^{(1/x)-2} (1 - \ln x) = 0$$

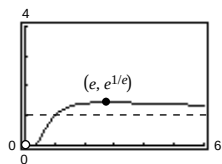
Critical number: $x = e$

Intervals: $(0, e)$ (e, ∞)

Sign of dy/dx : $+$ $-$

$y = f(x)$: Increasing Decreasing

Relative maximum: $(e, e^{1/e})$



75. $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{e^x} = \frac{0}{1} = 0$

Limit is not of the form $0/0$ or ∞/∞ .
 L'Hôpital's Rule does not apply.

61. (a) Let $f(x) = x^2 - 25$ and $g(x) = x - 5$.

(b) Let $f(x) = (x - 5)^2$ and $g(x) = x^2 - 25$.

(c) Let $f(x) = x^2 - 25$ and $g(x) = (x - 5)^3$.

67. $\lim_{x \rightarrow \infty} \frac{(\ln x)^n}{x^m} = \lim_{x \rightarrow \infty} \frac{n(\ln x)^{n-1}/x}{mx^{m-1}}$
 $= \lim_{x \rightarrow \infty} \frac{n(\ln x)^{n-1}}{mx^m}$
 $= \lim_{x \rightarrow \infty} \frac{n(n-1)(\ln x)^{n-2}}{m^2 x^m}$
 $= \dots = \lim_{x \rightarrow \infty} \frac{n!}{m^n x^m} = 0$

73. $y = 2xe^{-x}$

$$\lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0$$

Horizontal asymptote: $y = 0$

$$\frac{dy}{dx} = 2x(-e^{-x}) + 2e^{-x}$$

$$= 2e^{-x}(1 - x) = 0$$

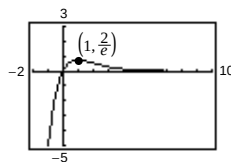
Critical number: $x = 1$

Intervals: $(-\infty, 1)$ $(1, \infty)$

Sign of dy/dx : $+$ $-$

$y = f(x)$: Increasing Decreasing

Relative maximum: $\left(1, \frac{2}{e}\right)$



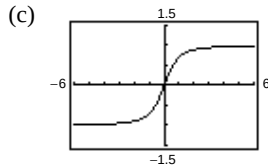
77. $\lim_{x \rightarrow \infty} x \cos \frac{1}{x} = \infty(1) = \infty$

Limit is not of the form $0/0$ or ∞/∞ .
 L'Hôpital's Rule does not apply.

$$\begin{aligned}
 79. (a) \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} &= \lim_{x \rightarrow \infty} \frac{x/x}{\sqrt{x^2 + 1/x}} \\
 &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x^2 + 1/\sqrt{x^2}}} \\
 &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + (1/x^2)}} \\
 &= \frac{1}{\sqrt{1 + 0}} = 1
 \end{aligned}$$

$$\begin{aligned}
 (b) \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} &= \lim_{x \rightarrow \infty} \frac{1}{x/\sqrt{x^2 + 1}} \\
 &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x} = \lim_{x \rightarrow \infty} \frac{x/\sqrt{x^2 + 1}}{1} \\
 &= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}}
 \end{aligned}$$

Applying L'Hôpital's rule twice results in the original limit, so L'Hôpital's rule fails.



$$\begin{aligned}
 81. \lim_{k \rightarrow 0} \frac{32 \left(1 - e^{-kt} + \frac{v_0 k e^{-kt}}{32} \right)}{k} &= \lim_{k \rightarrow 0} \frac{32(1 - e^{-kt})}{k} + \lim_{k \rightarrow 0} (v_0 e^{-kt}) \\
 &= \lim_{k \rightarrow 0} \frac{32(0 + t e^{-kt})}{1} + \lim_{k \rightarrow 0} \left(\frac{v_0}{e^{kt}} \right) = 32t + v_0
 \end{aligned}$$

83. Area of triangle: $\frac{1}{2}(2x)(1 - \cos x) = x - x \cos x$

Shaded area: Area of rectangle - Area under curve

$$\begin{aligned}
 2x(1 - \cos x) - 2 \int_0^x (1 - \cos t) dt &= 2x(1 - \cos x) - 2 \left[t - \sin t \right]_0^x \\
 &= 2x(1 - \cos x) - 2(x - \sin x) = 2 \sin x - 2x \cos x
 \end{aligned}$$

$$\begin{aligned}
 \text{Ratio: } \lim_{x \rightarrow 0} \frac{x - x \cos x}{2 \sin x - 2x \cos x} &= \lim_{x \rightarrow 0} \frac{1 + x \sin x - \cos x}{2 \cos x + 2x \sin x - 2 \cos x} \\
 &= \lim_{x \rightarrow 0} \frac{1 + x \sin x - \cos x}{2x \sin x} \\
 &= \lim_{x \rightarrow 0} \frac{x \cos x + \sin x + \sin x}{2x \cos x + 2 \sin x} \\
 &= \lim_{x \rightarrow 0} \frac{x \cos x + 2 \sin x}{2x \cos x + 2 \sin x} \cdot \frac{1/\cos x}{1/\cos x} \\
 &= \lim_{x \rightarrow 0} \frac{x + 2 \tan x}{2x + 2 \tan x} \\
 &= \lim_{x \rightarrow 0} \frac{1 + 2 \sec^2 x}{2 + 2 \sec^2 x} = \frac{3}{4}
 \end{aligned}$$

85. $f(x) = x^3, g(x) = x^2 + 1, [0, 1]$

$$\begin{aligned}
 \frac{f(b) - f(a)}{g(b) - g(a)} &= \frac{f'(c)}{g'(c)} \\
 \frac{f(1) - f(0)}{g(1) - g(0)} &= \frac{3c^2}{2c} \\
 \frac{1}{1} &= \frac{3c}{2} \\
 c &= \frac{2}{3}
 \end{aligned}$$

87. $f(x) = \sin x, g(x) = \cos x, \left[0, \frac{\pi}{2} \right]$

$$\begin{aligned}
 \frac{f(\pi/2) - f(0)}{g(\pi/2) - g(0)} &= \frac{f'(c)}{g'(c)} \\
 \frac{1}{-1} &= \frac{\cos c}{-\sin c} \\
 -1 &= -\cot c \\
 c &= \frac{\pi}{4}
 \end{aligned}$$

89. False. L'Hôpital's Rule does not apply since

$$\lim_{x \rightarrow 0} (x^2 + x + 1) \neq 0.$$

$$\lim_{x \rightarrow 0} \frac{x^2 + x + 1}{x} = \lim_{x \rightarrow 0} \left(x + 1 + \frac{1}{x} \right) = 1 + \infty = \infty$$

91. True

93. (a) $\sin \theta = BD$

$$\cos \theta = DO \Rightarrow AD = 1 - \cos \theta$$

$$\text{Area } \triangle ABD = \frac{1}{2}bh = \frac{1}{2}(1 - \cos \theta) \sin \theta = \frac{1}{2} \sin \theta - \frac{1}{2} \sin \theta \cos \theta$$

(b) Area of sector: $\frac{1}{2}\theta$

$$\text{Shaded area: } \frac{1}{2}\theta - \text{Area } \triangle OBD = \frac{1}{2}\theta - \frac{1}{2}(\cos \theta)(\sin \theta) = \frac{1}{2}\theta - \frac{1}{2} \sin \theta \cos \theta$$

$$(c) R = \frac{(1/2) \sin \theta - (1/2) \sin \theta \cos \theta}{(1/2)\theta - (1/2) \sin \theta \cos \theta} = \frac{\sin \theta - \sin \theta \cos \theta}{\theta - \sin \theta \cos \theta}$$

$$(d) \lim_{\theta \rightarrow 0} R = \lim_{\theta \rightarrow 0} \frac{\sin \theta - (1/2) \sin 2\theta}{\theta - (1/2) \sin 2\theta}$$

$$= \lim_{\theta \rightarrow 0} \frac{\cos \theta - \cos 2\theta}{1 - \cos 2\theta} = \lim_{\theta \rightarrow 0} \frac{-\sin \theta + 2 \sin 2\theta}{2 \sin 2\theta} = \lim_{\theta \rightarrow 0} \frac{-\cos \theta + 4 \cos 2\theta}{4 \cos 2\theta} = \frac{3}{4}$$

95. $\lim_{x \rightarrow a} f(x)^{g(x)}$

$$y = f(x)^{g(x)}$$

$$\ln y = g(x) \ln f(x)$$

$$\lim_{x \rightarrow a} g(x) \ln f(x) = (\infty)(-\infty) = -\infty$$

As $x \rightarrow a$, $\ln y \Rightarrow -\infty$, and hence $y = 0$. Thus,

$$\lim_{x \rightarrow a} f(x)^{g(x)} = 0.$$

$$97. f'(a)(b-a) - \int_a^b f''(t)(t-b) dt = f'(a)(b-a) - \left\{ \left[f'(t)(t-b) \right]_a^b - \int_a^b f'(t) dt \right\}$$

$$= f'(a)(b-a) + f'(a)(a-b) + \left[f(t) \right]_a^b = f(b) - f(a)$$

$$dv = f''(t)dt \Rightarrow v = f'(t)$$

$$u = t - b \Rightarrow du = dt$$

Section 7.8 Improper Integrals

1. Infinite discontinuity at $x = 0$.

$$\int_0^4 \frac{1}{\sqrt{x}} dx = \lim_{b \rightarrow 0^+} \int_b^4 \frac{1}{\sqrt{x}} dx$$

$$= \lim_{b \rightarrow 0^+} \left[2\sqrt{x} \right]_b^4$$

$$= \lim_{b \rightarrow 0^+} (4 - 2\sqrt{b}) = 4$$

Converges

3. Infinite discontinuity at $x = 1$.

$$\begin{aligned}\int_0^2 \frac{1}{(x-1)^2} dx &= \int_0^1 \frac{1}{(x-1)^2} dx + \int_1^2 \frac{1}{(x-1)^2} dx \\ &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{(x-1)^2} dx + \lim_{c \rightarrow 1^+} \int_c^2 \frac{1}{(x-1)^2} dx \\ &= \lim_{b \rightarrow 1^-} \left[-\frac{1}{x-1} \right]_0^b + \lim_{c \rightarrow 1^+} \left[-\frac{1}{x-1} \right]_c^2 = (\infty - 1) + (-1 + \infty)\end{aligned}$$

Diverges

5. Infinite limit of integration.

$$\begin{aligned}\int_0^\infty e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx \\ &= \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_0^b = 0 + 1 = 1\end{aligned}$$

Converges

$$7. \int_{-1}^1 \frac{1}{x^2} dx \neq -2$$

because the integrand is not defined at $x = 0$.

Diverges

$$\begin{aligned}9. \int_1^\infty \frac{1}{x^2} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2} dx \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_1^b = 1\end{aligned}$$

$$\begin{aligned}11. \int_1^\infty \frac{3}{\sqrt[3]{x}} dx &= \lim_{b \rightarrow \infty} \int_1^b 3x^{-1/3} dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{9}{2} x^{2/3} \right]_1^b = \infty\end{aligned}$$

Diverges

$$13. \int_{-\infty}^0 x e^{-2x} dx = \lim_{b \rightarrow -\infty} \int_b^0 x e^{-2x} dx = \lim_{b \rightarrow -\infty} \frac{1}{4} \left[(-2x - 1) e^{-2x} \right]_b^0 = \lim_{b \rightarrow -\infty} \frac{1}{4} [-1 + (2b + 1) e^{-2b}] = -\infty \quad (\text{Integration by parts})$$

Diverges

$$15. \int_0^\infty x^2 e^{-x} dx = \lim_{b \rightarrow \infty} \int_0^b x^2 e^{-x} dx = \lim_{b \rightarrow \infty} \left[-e^{-x}(x^2 + 2x + 2) \right]_0^b = \lim_{b \rightarrow \infty} \left(-\frac{b^2 + 2b + 2}{e^b} + 2 \right) = 2$$

Since $\lim_{b \rightarrow \infty} \left(-\frac{b^2 + 2b + 2}{e^b} \right) = 0$ by L'Hôpital's Rule.

$$\begin{aligned}17. \int_0^\infty e^{-x} \cos x dx &= \lim_{b \rightarrow \infty} \frac{1}{2} \left[e^{-x} (-\cos x + \sin x) \right]_0^b \\ &= \frac{1}{2} [0 - (-1)] = \frac{1}{2}\end{aligned}$$

$$\begin{aligned}19. \int_4^\infty \frac{1}{x(\ln x)^3} dx &= \lim_{b \rightarrow \infty} \int_4^b (\ln x)^{-3} \frac{1}{x} dx \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{2} (\ln x)^{-2} \right]_4^b \\ &= -\frac{1}{2} (\ln b)^{-2} + \frac{1}{2} (\ln 4)^{-2} \\ &= \frac{1}{2} \frac{1}{(\ln 2)^2} = \frac{1}{8(\ln 2)^2}\end{aligned}$$

$$\begin{aligned}21. \int_{-\infty}^\infty \frac{2}{4+x^2} dx &= \int_{-\infty}^0 \frac{2}{4+x^2} dx + \int_0^\infty \frac{2}{4+x^2} dx \\ &= \lim_{b \rightarrow -\infty} \int_b^0 \frac{2}{4+x^2} dx + \lim_{c \rightarrow \infty} \int_0^c \frac{2}{4+x^2} dx \\ &= \lim_{b \rightarrow -\infty} \left[\arctan\left(\frac{x}{2}\right) \right]_b^0 + \lim_{c \rightarrow \infty} \left[\arctan\left(\frac{x}{2}\right) \right]_0^c \\ &= \left(0 - \left(-\frac{\pi}{2} \right) \right) + \left(\frac{\pi}{2} - 0 \right) = \pi\end{aligned}$$

$$\begin{aligned}
 23. \int_0^\infty \frac{1}{e^x + e^{-x}} dx &= \lim_{b \rightarrow \infty} \int_0^b \frac{e^x}{1 + e^{2x}} dx \\
 &= \lim_{b \rightarrow \infty} \left[\arctan(e^x) \right]_0^b \\
 &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
 \end{aligned}$$

$$25. \int_0^\infty \cos \pi x dx = \lim_{b \rightarrow \infty} \left[\frac{1}{\pi} \sin \pi x \right]_0^b$$

Diverges since $\sin \pi x$ does not approach a limit as $x \rightarrow \infty$.

$$27. \int_0^1 \frac{1}{x^2} dx = \lim_{b \rightarrow 0^+} \left[-\frac{1}{x} \right]_b^1 = -1 + \infty$$

Diverges

$$29. \int_0^8 \frac{1}{\sqrt[3]{8-x}} dx = \lim_{b \rightarrow 8^-} \int_0^b \frac{1}{\sqrt[3]{8-x}} dx = \lim_{b \rightarrow 8^-} \left[\frac{-3}{2} (8-x)^{2/3} \right]_0^b = 6$$

$$31. \int_0^1 x \ln x dx = \lim_{b \rightarrow 0^+} \left[\frac{x^2}{2} \ln|x| - \frac{x^2}{4} \right]_b^1 = \lim_{b \rightarrow 0^+} \left[\frac{-1}{4} - \frac{b^2 \ln b}{2} + \frac{b^2}{4} \right] = \frac{-1}{4} \text{ since } \lim_{b \rightarrow 0^+} (b^2 \ln b) = 0 \text{ by L'Hôpital's Rule.}$$

$$33. \int_0^{\pi/2} \tan \theta d\theta = \lim_{b \rightarrow (\pi/2)^-} \left[\ln|\sec \theta| \right]_0^b = \infty$$

Diverges

$$\begin{aligned}
 35. \int_2^4 \frac{2}{x\sqrt{x^2-4}} dx &= \lim_{b \rightarrow 2^+} \int_b^4 \frac{2}{x\sqrt{x^2-4}} dx \\
 &= \lim_{b \rightarrow 2^+} \left[\operatorname{arcsec} \left| \frac{x}{2} \right| \right]_b^4 \\
 &= \lim_{b \rightarrow 2^+} \left(\operatorname{arcsec} 2 - \operatorname{arcsec} \left(\frac{b}{2} \right) \right) \\
 &= \frac{\pi}{3} - 0 = \frac{\pi}{3}
 \end{aligned}$$

$$\begin{aligned}
 37. \int_2^4 \frac{1}{\sqrt{x^2-4}} dx &= \lim_{b \rightarrow 2^+} \left[\ln|x + \sqrt{x^2-4}| \right]_b^4 \\
 &= \ln(4 + 2\sqrt{3}) - \ln 2 \\
 &= \ln(2 + \sqrt{3}) \approx 1.317
 \end{aligned}$$

$$\begin{aligned}
 39. \int_0^2 \frac{1}{\sqrt[3]{x-1}} dx &= \int_0^1 \frac{1}{\sqrt[3]{x-1}} dx + \int_1^2 \frac{1}{\sqrt[3]{x-1}} dx \\
 &= \lim_{b \rightarrow 1^-} \left[\frac{3}{2} (x-1)^{2/3} \right]_0^b + \lim_{c \rightarrow 1^+} \left[\frac{3}{2} (x-1)^{2/3} \right]_c^2 = \frac{-3}{2} + \frac{3}{2} = 0
 \end{aligned}$$

$$41. \int_0^\infty \frac{4}{\sqrt{x}(x+6)} dx = \int_0^1 \frac{4}{\sqrt{x}(x+6)} dx + \int_1^\infty \frac{4}{\sqrt{x}(x+6)} dx$$

Let $u = \sqrt{x}$, $u^2 = x$, $2u du = dx$.

$$\int \frac{4}{\sqrt{x}(x+6)} dx = \int \frac{4(2u du)}{u(u^2+6)} = 8 \int \frac{du}{u^2+6} = \frac{8}{\sqrt{6}} \arctan\left(\frac{4}{\sqrt{6}}\right) + C = \frac{8}{\sqrt{6}} \arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$

$$\begin{aligned}
 \text{Thus, } \int_0^\infty \frac{4}{\sqrt{x}(x+6)} dx &= \lim_{b \rightarrow 0^+} \left[\frac{8}{\sqrt{6}} \arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right) \right]_b^1 + \lim_{c \rightarrow \infty} \left[\frac{8}{\sqrt{6}} \arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right) \right]_1^c \\
 &= \left(\frac{8}{\sqrt{6}} \arctan\left(\frac{1}{\sqrt{6}}\right) - \frac{8}{\sqrt{6}} 0 \right) + \left(\frac{8}{\sqrt{6}} \frac{\pi}{2} - \frac{8}{\sqrt{6}} \arctan\left(\frac{1}{\sqrt{6}}\right) \right) \\
 &= \frac{8\pi}{2\sqrt{6}} = \frac{2\pi\sqrt{6}}{3}.
 \end{aligned}$$

43. If $p = 1$, $\int_1^\infty \frac{1}{x} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x} dx = \lim_{b \rightarrow \infty} \ln x \Big|_1^b$.

Diverges. For $p \neq 1$,

$$\int_1^\infty \frac{1}{x^p} dx = \lim_{b \rightarrow \infty} \left[\frac{x^{1-p}}{1-p} \right]_1^b = \lim_{b \rightarrow \infty} \left[\frac{b^{1-p}}{1-p} - \frac{1}{1-p} \right].$$

This converges to $\frac{1}{p-1}$ if $1-p < 0$ or $p > 1$.

45. For $n = 1$ we have

$$\begin{aligned} \int_0^\infty x e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b x e^{-x} dx \\ &= \lim_{b \rightarrow \infty} \left[-e^{-x} x - e^{-x} \right]_0^b \quad (\text{Parts: } u = x, dv = e^{-x} dx) \\ &= \lim_{b \rightarrow \infty} [-e^{-b} b - e^{-b} + 1] \\ &= \lim_{b \rightarrow \infty} \left[\frac{-b}{e^b} - \frac{1}{e^b} + 1 \right] = 1 \quad (\text{L'Hôpital's Rule}) \end{aligned}$$

Assume that $\int_0^\infty x^n e^{-x} dx$ converges. Then for $n+1$ we have

$$\int x^{n+1} e^{-x} dx = -x^{n+1} e^{-x} + (n+1) \int x^n e^{-x} dx$$

by parts ($u = x^{n+1}$, $du = (n+1)x^n dx$, $dv = e^{-x} dx$, $v = -e^{-x}$).

Thus,

$$\int_0^\infty x^{n+1} e^{-x} dx = \lim_{b \rightarrow \infty} \left[-x^{n+1} e^{-x} \right]_0^b + (n+1) \int_0^\infty x^n e^{-x} dx = 0 + (n+1) \int_0^\infty x^n e^{-x} dx, \text{ which converges.}$$

47. $\int_0^1 \frac{1}{x^3} dx$ diverges.

(See Exercise 44, $p = 3 \nless 1$.)

49. $\int_1^\infty \frac{1}{x^3} dx = \frac{1}{3-1} = \frac{1}{2}$ converges.

(See Exercise 43, $p = 3$.)

51. Since $\frac{1}{x^2+5} \leq \frac{1}{x^2}$ on $[1, \infty)$ and $\int_1^\infty \frac{1}{x^2} dx$ converges by Exercise 43, $\int_1^\infty \frac{1}{x^2+5} dx$ converges.

53. Since $\frac{1}{\sqrt[3]{x(x-1)}} \geq \frac{1}{\sqrt[3]{x^2}}$ on $[2, \infty)$ and $\int_2^\infty \frac{1}{\sqrt[3]{x^2}} dx$ diverges by Exercise 43, $\int_2^\infty \frac{1}{\sqrt[3]{x(x-1)}} dx$ diverges.

55. Since $e^{-x^2} \leq e^{-x}$ on $[1, \infty)$ and $\int_0^\infty e^{-x} dx$ converges (see Exercise 5), $\int_0^\infty e^{-x^2} dx$ converges.

57. Answers will vary. See pages 540, 543.

59. $\int_{-1}^1 \frac{1}{x^3} dx = \int_{-1}^0 \frac{1}{x^3} dx + \int_0^1 \frac{1}{x^3} dx$

These two integrals diverge by Exercise 44.

61. $f(t) = 1$

$$F(s) = \int_0^\infty e^{-st} dt = \lim_{b \rightarrow \infty} \left[-\frac{1}{s} e^{-st} \right]_0^b = \frac{1}{s}, s > 0$$

63. $f(t) = t^2$

$$\begin{aligned} F(s) &= \int_0^\infty t^2 e^{-st} dt = \lim_{b \rightarrow \infty} \left[\frac{1}{s^3} (-s^2 t^2 - 2st - 2) e^{-st} \right]_0^b \\ &= \frac{2}{s^3}, s > 0 \end{aligned}$$

65. $f(t) = \cos at$

$$\begin{aligned}
 F(s) &= \int_0^{\infty} e^{-st} \cos at \, dt \\
 &= \lim_{b \rightarrow \infty} \left[\frac{e^{-st}}{s^2 + a^2} (-s \cos at + a \sin at) \right]_0^b \\
 &= 0 + \frac{s}{s^2 + a^2} = \frac{s}{s^2 + a^2}, s > 0
 \end{aligned}$$

67. $f(t) = \cosh at$

$$\begin{aligned}
 F(s) &= \int_0^{\infty} e^{-st} \cosh at \, dt = \int_0^{\infty} e^{-st} \left(\frac{e^{at} + e^{-at}}{2} \right) dt = \frac{1}{2} \int_0^{\infty} \left[e^{t(-s+a)} + e^{t(-s-a)} \right] dt \\
 &= \lim_{b \rightarrow \infty} \frac{1}{2} \left[\frac{1}{(-s+a)} e^{t(-s+a)} + \frac{1}{(-s-a)} e^{t(-s-a)} \right]_0^b = 0 - \frac{1}{2} \left[\frac{1}{(-s+a)} + \frac{1}{(-s-a)} \right] \\
 &= \frac{-1}{2} \left[\frac{1}{(-s+a)} + \frac{1}{(-s-a)} \right] = \frac{s}{s^2 - a^2}, s > |a|
 \end{aligned}$$

69. (a) $A = \int_0^{\infty} e^{-x} \, dx$

$$= \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_0^b = 0 - (-1) = 1$$

(b) **Disk:**

$$\begin{aligned}
 V &= \pi \int_0^{\infty} (e^{-x})^2 \, dx \\
 &= \lim_{b \rightarrow \infty} \pi \left[-\frac{1}{2} e^{-2x} \right]_0^b = \frac{\pi}{2}
 \end{aligned}$$

(c) **Shell:**

$$\begin{aligned}
 V &= 2\pi \int_0^{\infty} x e^{-x} \, dx \\
 &= \lim_{b \rightarrow \infty} \left\{ 2\pi \left[-e^{-x}(x+1) \right]_0^b \right\} = 2\pi
 \end{aligned}$$

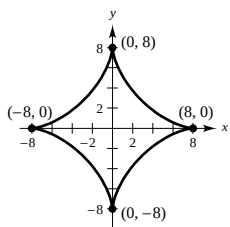
71. $x^{2/3} + y^{2/3} = 4$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = \frac{-y^{1/3}}{x^{1/3}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{y^{2/3}}{x^{2/3}}} = \sqrt{\frac{x^{2/3} + y^{2/3}}{x^{2/3}}} = \sqrt{\frac{4}{x^{2/3}}} = \frac{2}{x^{1/3}}$$

$$s = 4 \int_0^8 \frac{2}{x^{1/3}} \, dx = \lim_{b \rightarrow 0^+} \left[8 \cdot \frac{3}{2} x^{2/3} \right]_b^8 = 48$$



$$73. \Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx$$

$$(a) \Gamma(1) = \int_0^{\infty} e^{-x} dx = \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_0^b = 1$$

$$\Gamma(2) = \int_0^{\infty} x e^{-x} dx = \lim_{b \rightarrow \infty} \left[-e^{-x}(x+1) \right]_0^b = 1$$

$$\Gamma(3) = \int_0^{\infty} x^2 e^{-x} dx = \lim_{b \rightarrow \infty} \left[-x^2 e^{-x} - 2x e^{-x} - 2e^{-x} \right]_0^b = 2$$

$$(b) \Gamma(n+1) = \int_0^{\infty} x^n e^{-x} dx = \lim_{b \rightarrow \infty} \left[-x^n e^{-x} \right]_0^b + \lim_{b \rightarrow \infty} n \int_0^b x^{n-1} e^{-x} dx = 0 + n\Gamma(n) \quad (u = x^n, dv = e^{-x} dx)$$

$$(c) \Gamma(n) = (n-1)!$$

$$75. (a) \int_{-\infty}^{\infty} \frac{1}{7} e^{-t/7} dt = \int_0^{\infty} \frac{1}{7} e^{-t/7} dt = \lim_{b \rightarrow \infty} \left[-e^{-t/7} \right]_0^b = 1$$

$$(b) \int_0^4 \frac{1}{7} e^{-t/7} dt = \left[-e^{-t/7} \right]_0^4 = -e^{-4/7} + 1$$

$$\approx 0.4353 = 43.53\%$$

$$(c) \int_0^{\infty} t \left[\frac{1}{7} e^{-t/7} \right] dt = \lim_{b \rightarrow \infty} \left[-te^{-t/7} - 7e^{-t/7} \right]_0^b$$

$$= 0 + 7 = 7$$

$$77. (a) C = 650,000 + \int_0^5 25,000 e^{-0.06t} dt = 650,000 - \left[\frac{25,000}{0.06} e^{-0.06t} \right]_0^5 \approx \$757,992.41$$

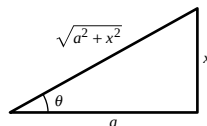
$$(b) C = 650,000 + \int_0^{10} 25,000 e^{-0.06t} dt \approx \$837,995.15$$

$$(c) C = 650,000 + \int_0^{\infty} 25,000 e^{-0.06t} dt = 650,000 - \lim_{b \rightarrow \infty} \left[\frac{25,000}{0.06} e^{-0.06t} \right]_0^b \approx \$1,066,666.67$$

$$79. \text{ Let } x = a \tan \theta, dx = a \sec^2 \theta d\theta, \sqrt{a^2 + x^2} = a \sec \theta.$$

$$\int \frac{1}{(a^2 + x^2)^{3/2}} dx = \int \frac{a \sec^2 \theta d\theta}{a^3 \sec^3 \theta} = \frac{1}{a^2} \int \cos \theta d\theta$$

$$= \frac{1}{a^2} \sin \theta = \frac{1}{a^2} \frac{x}{\sqrt{a^2 + x^2}}$$



Hence,

$$P = k \int_1^{\infty} \frac{1}{(a^2 + x^2)^{3/2}} dx = \frac{k}{a^2} \lim_{b \rightarrow \infty} \left[\frac{x}{\sqrt{a^2 + x^2}} \right]_1^b$$

$$= \frac{k}{a^2} \left[1 - \frac{1}{\sqrt{a^2 + 1}} \right] = \frac{k(\sqrt{a^2 + 1} - 1)}{a^2 \sqrt{a^2 + 1}}.$$

$$81. \frac{10}{x^2 - 2x} = \frac{10}{x(x-2)} \Rightarrow x = 0, 2.$$

You must analyze three improper integrals, and each must converge in order for the original integral to converge.

$$\int_0^3 f(x) dx = \int_0^1 f(x) dx + \int_1^2 f(x) dx + \int_2^3 f(x) dx$$

83. For $n = 1$,

$$I_1 = \int_0^\infty \frac{x}{(x^2 + 1)^4} dx = \lim_{b \rightarrow \infty} \frac{1}{2} \int_0^b (x^2 + 1)^{-4} (2x dx) = \lim_{b \rightarrow \infty} \left[-\frac{1}{6} \frac{1}{(x^2 + 1)^3} \right]_0^b = \frac{1}{6}.$$

For $n > 1$,

$$I_n = \int_0^\infty \frac{x^{2n-1}}{(x^2 + 1)^{n+3}} dx = \lim_{b \rightarrow \infty} \left[\frac{-x^{2n-2}}{2(n+2)(x^2 + 1)^n} + 2 \right]_0^b + \frac{n-1}{n+2} \int_0^\infty \frac{x^{2n-3}}{(x^2 + 1)^{n+2}} dx = 0 + \frac{n-1}{n+2} (I_{n-1})$$

$$u = x^{2n-2}, du = (2n-2)x^{2n-3} dx, dv = \frac{x}{(x^2 + 1)^{n+3}} dx, v = \frac{-1}{2(n+2)(x^2 + 1)^{n+2}}$$

$$(a) \int_0^\infty \frac{x}{(x^2 + 1)^4} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{6(x^2 + 1)^3} \right]_0^b = \frac{1}{6}$$

$$(b) \int_0^\infty \frac{x^3}{(x^2 + 1)^5} dx = \frac{1}{4} \int_0^\infty \frac{x}{(x^2 + 1)^4} dx = \frac{1}{4} \left(\frac{1}{6} \right) = \frac{1}{24}$$

$$(c) \int_0^\infty \frac{x^5}{(x^2 + 1)^6} dx = \frac{2}{5} \int_0^\infty \frac{x^3}{(x^2 + 1)^5} dx = \frac{2}{5} \left(\frac{1}{24} \right) = \frac{1}{60}$$

85. False. $f(x) = 1/(x+1)$ is continuous on $[0, \infty)$, $\lim_{x \rightarrow \infty} 1/(x+1) = 0$, but $\int_0^\infty \frac{1}{x+1} dx = \lim_{b \rightarrow \infty} \left[\ln|x+1| \right]_0^b = \infty$.

Diverges

87. True

Review Exercises for Chapter 7

$$\begin{aligned} 1. \int x\sqrt{x^2-1} dx &= \frac{1}{2} \int (x^2-1)^{1/2} (2x) dx \\ &= \frac{1}{2} \frac{(x^2-1)^{3/2}}{3/2} + C \\ &= \frac{1}{3} (x^2-1)^{3/2} + C \end{aligned}$$

$$\begin{aligned} 3. \int \frac{x}{x^2-1} dx &= \frac{1}{2} \int \frac{2x}{x^2-1} dx \\ &= \frac{1}{2} \ln|x^2-1| + C \end{aligned}$$

$$5. \int \frac{\ln(2x)}{x} dx = \frac{(\ln 2x)^2}{2} + C$$

$$7. \int \frac{16}{\sqrt{16-x^2}} dx = 16 \arcsin\left(\frac{x}{4}\right) + C$$

$$\begin{aligned} 9. \int e^{2x} \sin 3x dx &= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \int e^{2x} \cos 3x dx \\ &= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \left(\frac{1}{3} e^{2x} \sin 3x - \frac{2}{3} \int e^{2x} \sin 3x dx \right) \end{aligned}$$

$$\frac{13}{9} \int e^{2x} \sin 3x dx = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x$$

$$\int e^{2x} \sin 3x dx = \frac{e^{2x}}{13} (2 \sin 3x - 3 \cos 3x) + C$$

$$(1) dv = \sin 3x dx \Rightarrow v = -\frac{1}{3} \cos 3x$$

$$(2) dv = \cos 3x dx \Rightarrow v = \frac{1}{3} \sin 3x$$

$$u = e^{2x} \Rightarrow du = 2e^{2x} dx$$

$$u = e^{2x} \Rightarrow du = 2e^{2x} dx$$

$$11. u = x, du = dx, dv = (x - 5)^{1/2} dx, v = \frac{2}{3}(x - 5)^{3/2}$$

$$\begin{aligned}\int x\sqrt{x-5} dx &= \frac{2}{3}x(x-5)^{3/2} - \int \frac{2}{3}(x-5)^{3/2} dx \\ &= \frac{2}{3}x(x-5)^{3/2} - \frac{4}{15}(x-5)^{5/2} + C \\ &= (x-5)^{3/2} \left[\frac{2}{3}x - \frac{4}{15}(x-5) \right] + C \\ &= (x-5)^{3/2} \left[\frac{6}{15}x + \frac{4}{3} \right] + C \\ &= \frac{2}{15}(x-5)^{3/2}[3x+10] + C\end{aligned}$$

$$\begin{aligned}13. \int x^2 \sin 2x dx &= -\frac{1}{2}x^2 \cos 2x + \int x \cos 2x dx \\ &= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x - \frac{1}{2} \int \sin 2x dx \\ &= -\frac{1}{2}x^2 \cos 2x + \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x + C\end{aligned}$$

$$(1) dv = \sin 2x dx \Rightarrow v = -\frac{1}{2} \cos 2x$$

$$u = x^2 \Rightarrow du = 2x dx$$

$$(2) dv = \cos 2x dx \Rightarrow v = \frac{1}{2} \sin 2x$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned}15. \int x \arcsin 2x dx &= \frac{x^2}{2} \arcsin 2x - \int \frac{x^2}{\sqrt{1-4x^2}} dx \\ &= \frac{x^2}{2} \arcsin 2x - \frac{1}{8} \int \frac{2(2x)^2}{\sqrt{1-(2x)^2}} dx \\ &= \frac{x^2}{2} \arcsin 2x - \frac{1}{8} \left(\frac{1}{2} \right) [-(2x)\sqrt{1-4x^2} + \arcsin 2x] + C \quad (\text{by Formula 43 of Integration Tables}) \\ &= \frac{1}{16} [(8x^2 - 1)\arcsin 2x + 2x\sqrt{1-4x^2}] + C\end{aligned}$$

$$dv = x dx \Rightarrow v = \frac{x^2}{2}$$

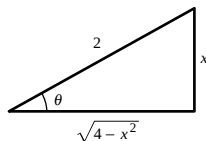
$$u = \arcsin 2x \Rightarrow du = \frac{2}{\sqrt{1-4x^2}} dx$$

$$\begin{aligned}17. \int \cos^3(\pi x - 1) dx &= \int [1 - \sin^2(\pi x - 1)] \cos(\pi x - 1) dx \\ &= \frac{1}{\pi} \left[\sin(\pi x - 1) - \frac{1}{3} \sin^3(\pi x - 1) \right] + C \\ &= \frac{1}{3\pi} \sin(\pi x - 1) [3 - \sin^2(\pi x - 1)] + C \\ &= \frac{1}{3\pi} \sin(\pi x - 1) [3 - (1 - \cos^2(\pi x - 1))] + C \\ &= \frac{1}{3\pi} \sin(\pi x - 1) [2 + \cos^2(\pi x - 1)] + C\end{aligned}$$

$$\begin{aligned}19. \int \sec^4\left(\frac{x}{2}\right) dx &= \int \left[\tan^2\left(\frac{x}{2}\right) + 1 \right] \sec^2\left(\frac{x}{2}\right) dx \\ &= \int \tan^2\left(\frac{x}{2}\right) \sec^2\left(\frac{x}{2}\right) dx + \int \sec^2\left(\frac{x}{2}\right) dx \\ &= \frac{2}{3} \tan^3\left(\frac{x}{2}\right) + 2 \tan\left(\frac{x}{2}\right) + C = \frac{2}{3} \left[\tan^3\left(\frac{x}{2}\right) + 3 \tan\left(\frac{x}{2}\right) \right] + C\end{aligned}$$

$$21. \int \frac{1}{1 - \sin \theta} d\theta = \int \frac{1 + \sin \theta}{\cos^2 \theta} d\theta = \int (\sec^2 \theta + \sec \theta \tan \theta) d\theta = \tan \theta + \sec \theta + C$$

$$\begin{aligned}
 23. \int \frac{-12}{x^2 \sqrt{4-x^2}} dx &= \int \frac{-24 \cos \theta d\theta}{(4 \sin^2 \theta)(2 \cos \theta)} \\
 &= -3 \int \csc^2 \theta d\theta \\
 &= 3 \cot \theta + C \\
 &= \frac{3\sqrt{4-x^2}}{x} + C
 \end{aligned}$$



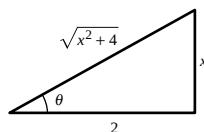
$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$$

$$25. \quad x = 2 \tan \theta$$

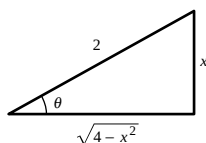
$$dx = 2 \sec^2 \theta d\theta$$

$$4 + x^2 = 4 \sec^2 \theta$$

$$\begin{aligned}
 \int \frac{x^3}{\sqrt{4+x^2}} dx &= \int \frac{8 \tan^3 \theta}{2 \sec \theta} 2 \sec^2 \theta d\theta \\
 &= 8 \int \tan^3 \theta \sec \theta d\theta \\
 &= 8 \int (\sec^2 \theta - 1) \tan \theta \sec \theta d\theta \\
 &= 8 \left[\frac{\sec^3 \theta}{3} - \sec \theta \right] + C \\
 &= 8 \left[\frac{(x^2 + 4)^{3/2}}{24} - \frac{\sqrt{x^2 + 4}}{2} \right] + C \\
 &= \sqrt{x^2 + 4} \left[\frac{1}{3}(x^2 + 4) - 4 \right] + C \\
 &= \frac{1}{3} x^2 \sqrt{x^2 + 4} - \frac{8}{3} \sqrt{x^2 + 4} + C \\
 &= \frac{1}{3} (x^2 + 4)^{1/2} (x^2 - 8) + C
 \end{aligned}$$



$$\begin{aligned}
 27. \int \sqrt{4-x^2} dx &= \int (2 \cos \theta)(2 \cos \theta) d\theta \\
 &= 2 \int (1 + \cos 2\theta) d\theta \\
 &= 2 \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\
 &= 2(\theta + \sin \theta \cos \theta) + C \\
 &= 2 \left[\arcsin\left(\frac{x}{2}\right) + \frac{x}{2} \left(\frac{\sqrt{4-x^2}}{2} \right) \right] + C \\
 &= \frac{1}{2} \left[4 \arcsin\left(\frac{x}{2}\right) + x \sqrt{4-x^2} \right] + C
 \end{aligned}$$



$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$$

$$\begin{aligned}
 29. \text{ (a) } \int \frac{x^3}{\sqrt{4+x^2}} dx &= 8 \int \frac{\sin^3 \theta}{\cos^4 \theta} d\theta \\
 &= 8 \int (\cos^{-4} \theta - \cos^{-2} \theta) \sin \theta d\theta \\
 &= \frac{8}{3} \sec \theta (\sec^2 \theta - 3) + C \\
 &= \frac{\sqrt{4+x^2}}{3} (x^2 - 8) + C
 \end{aligned}$$

$$x = 2 \tan \theta, dx = 2 \sec^2 \theta d\theta$$

$$\begin{aligned}
 \text{(b) } \int \frac{x^3}{\sqrt{4+x^2}} dx &= \int (u^2 - 4) du \\
 &= \frac{1}{3} u^3 - 4u + C \\
 &= \frac{u}{3} (u^2 - 12) + C \\
 &= \frac{\sqrt{4+x^2}}{3} (x^2 - 8) + C
 \end{aligned}$$

$$u^2 = 4 + x^2, 2u du = 2x dx$$

$$\begin{aligned}
 \text{(c) } \int \frac{x^3}{\sqrt{4+x^2}} dx &= x^2 \sqrt{4+x^2} - \int 2x \sqrt{4+x^2} dx \\
 &= x^2 \sqrt{4+x^2} - \frac{2}{3} (4+x^2)^{3/2} + C = \frac{\sqrt{4+x^2}}{3} (x^2 - 8) + C
 \end{aligned}$$

$$dv = \frac{x}{\sqrt{4+x^2}} dx \Rightarrow v = \sqrt{4+x^2}$$

$$u = x^2 \Rightarrow du = 2x dx$$

$$31. \frac{x-28}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$x-28 = A(x+2) + B(x-3)$$

$$x = -2 \Rightarrow -30 = B(-5) \Rightarrow B = 6$$

$$x = 3 \Rightarrow -25 = A(5) \Rightarrow A = -5$$

$$\int \frac{x-28}{x^2-x-6} dx = \int \left(\frac{-5}{x-3} + \frac{6}{x+2} \right) dx = -5 \ln|x-3| + 6 \ln|x+2| + C$$

$$33. \frac{x^2+2x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$x^2+2x = A(x^2+1) + (Bx+C)(x-1)$$

$$\text{Let } x = 1: 3 = 2A \Rightarrow A = \frac{3}{2}$$

$$\text{Let } x = 0: 0 = A - C \Rightarrow C = \frac{3}{2}$$

$$\text{Let } x = 2: 8 = 5A + 2B + C \Rightarrow B = -\frac{1}{2}$$

$$\begin{aligned}
 \int \frac{x^2+2x}{x^3-x^2+x-1} dx &= \frac{3}{2} \int \frac{1}{x-1} dx - \frac{1}{2} \int \frac{x-3}{x^2+1} dx \\
 &= \frac{3}{2} \int \frac{1}{x-1} dx - \frac{1}{4} \int \frac{2x}{x^2+1} dx + \frac{3}{2} \int \frac{1}{x^2+1} dx \\
 &= \frac{3}{2} \ln|x-1| - \frac{1}{4} \ln|x^2+1| + \frac{3}{2} \arctan x + C \\
 &= \frac{1}{4} [6 \ln|x-1| - \ln(x^2+1) + 6 \arctan x] + C
 \end{aligned}$$

$$35. \frac{x^2}{x^2 + 2x - 15} = 1 + \frac{15 - 2x}{x^2 + 2x - 15}$$

$$\frac{15 - 2x}{(x - 3)(x + 5)} = \frac{A}{x - 3} + \frac{B}{x + 5}$$

$$15 - 2x = A(x + 5) + B(x - 3)$$

$$\text{Let } x = 3: 9 = 8A \Rightarrow A = \frac{9}{8}$$

$$\text{Let } x = -5: 25 = -8B \Rightarrow B = -\frac{25}{8}$$

$$\begin{aligned} \int \frac{x^2}{x^2 + 2x - 15} dx &= \int dx + \frac{9}{8} \int \frac{1}{x - 3} dx - \frac{25}{8} \int \frac{1}{x + 5} dx \\ &= x + \frac{9}{8} \ln|x - 3| - \frac{25}{8} \ln|x + 5| + C \end{aligned}$$

$$37. \int \frac{x}{(2 + 3x)^2} dx = \frac{1}{9} \left[\frac{2}{2 + 3x} + \ln|2 + 3x| \right] + C$$

(Formula 4)

$$\begin{aligned} 39. \int \frac{x}{1 + \sin x^2} dx &= \frac{1}{2} \int \frac{1}{1 + \sin u} du \quad (u = x^2) \\ &= \frac{1}{2} [\tan u - \sec u] + C \quad (\text{Formula 56}) \\ &= \frac{1}{2} [\tan x^2 - \sec x^2] + C \end{aligned}$$

$$\begin{aligned} 41. \int \frac{x}{x^2 + 4x + 8} dx &= \frac{1}{2} \left[\ln|x^2 + 4x + 8| - 4 \int \frac{1}{x^2 + 4x + 8} dx \right] \quad (\text{Formula 15}) \\ &= \frac{1}{2} [\ln|x^2 + 4x + 8|] - 2 \left[\frac{2}{\sqrt{32 - 16}} \arctan \left(\frac{2x + 4}{\sqrt{32 - 16}} \right) \right] + C \quad (\text{Formula 14}) \\ &= \frac{1}{2} \ln|x^2 + 4x + 8| - \arctan \left(1 + \frac{x}{2} \right) + C \end{aligned}$$

$$\begin{aligned} 43. \int \frac{1}{\sin \pi x \cos \pi x} dx &= \frac{1}{\pi} \int \frac{1}{\sin \pi x \cos \pi x} (\pi) dx \quad (u = \pi x) \\ &= \frac{1}{\pi} \ln|\tan \pi x| + C \quad (\text{Formula 58}) \end{aligned}$$

$$\begin{aligned} 45. dv &= dx \Rightarrow v = x \\ u &= (\ln x)^n \Rightarrow du = n(\ln x)^{n-1} \frac{1}{x} dx \\ \int (\ln x)^n dx &= x(\ln x)^n - n \int (\ln x)^{n-1} dx \end{aligned}$$

$$\begin{aligned} 47. \int \theta \sin \theta \cos \theta d\theta &= \frac{1}{2} \int \theta \sin 2\theta d\theta \\ &= -\frac{1}{4} \theta \cos 2\theta + \frac{1}{4} \int \cos 2\theta d\theta = -\frac{1}{4} \theta \cos 2\theta + \frac{1}{8} \sin 2\theta + C = \frac{1}{8} (\sin 2\theta - 2\theta \cos 2\theta) + C \end{aligned}$$

$$dv = \sin 2\theta d\theta \Rightarrow v = -\frac{1}{2} \cos 2\theta$$

$$u = \theta \Rightarrow du = d\theta$$

$$\begin{aligned}
 49. \int \frac{x^{1/4}}{1+x^{1/2}} dx &= 4 \int \frac{u(u^3)}{1+u^2} du \\
 &= 4 \int \left(u^2 - 1 + \frac{1}{u^2+1} \right) du \\
 &= 4 \left(\frac{1}{3} u^3 - u + \arctan u \right) + C \\
 &= \frac{4}{3} [x^{3/4} - 3x^{1/4} + 3 \arctan(x^{1/4})] + C
 \end{aligned}$$

$$y = \sqrt[4]{x}, x = u^4, dx = 4u^3 du$$

$$\begin{aligned}
 53. \int \cos x \ln(\sin x) dx &= \sin x \ln(\sin x) - \int \cos x dx \\
 &= \sin x \ln(\sin x) - \sin x + C
 \end{aligned}$$

$$dv = \cos x dx \Rightarrow v = \sin x$$

$$u = \ln(\sin x) \Rightarrow du = \frac{\cos x}{\sin x} dx$$

$$\begin{aligned}
 57. y &= \int \ln(x^2 + x) dx = x \ln|x^2 + x| - \int \frac{2x^2 + x}{x^2 + x} dx \\
 &= x \ln|x^2 + x| - \int \frac{2x + 1}{x + 1} dx \\
 &= x \ln|x^2 + x| - \int 2 dx + \int \frac{1}{x + 1} dx \\
 &= x \ln|x^2 + x| - 2x + \ln|x + 1| + C
 \end{aligned}$$

$$dv = dx \Rightarrow v = x$$

$$u = \ln(x^2 + x) \Rightarrow du = \frac{2x + 1}{x^2 + x} dx$$

$$61. \int_1^4 \frac{\ln x}{x} dx = \left[\frac{1}{2} (\ln x)^2 \right]_1^4 = \frac{1}{2} (\ln 4)^2 = 2(\ln 2)^2 \approx 0.961$$

$$\begin{aligned}
 65. A &= \int_0^4 x \sqrt{4-x} dx = \int_2^0 (4-u^2)u(-2u) du \\
 &= \int_2^0 2(u^4 - 4u^2) du \\
 &= \left[2 \left(\frac{u^5}{5} - \frac{4u^3}{3} \right) \right]_2^0 = \frac{128}{15}
 \end{aligned}$$

$$u = \sqrt{4-x}, x = 4-u^2, dx = -2u du$$

$$69. s = \int_0^\pi \sqrt{1 + \cos^2 x} dx \approx 3.82$$

$$73. \lim_{x \rightarrow \infty} \frac{e^{2x}}{x^2} = \lim_{x \rightarrow \infty} \frac{2e^{2x}}{2x} = \lim_{x \rightarrow \infty} \frac{4e^{2x}}{2} = \infty$$

$$\begin{aligned}
 51. \int \sqrt{1 + \cos x} dx &= \int \frac{\sin x}{\sqrt{1 - \cos x}} dx \\
 &= \int (1 - \cos x)^{-1/2} (\sin x) dx \\
 &= 2\sqrt{1 - \cos x} + C
 \end{aligned}$$

$$u = 1 - \cos x, du = \sin x dx$$

$$\begin{aligned}
 55. y &= \int \frac{9}{x^2 - 9} dx = \frac{3}{2} \ln \left| \frac{x-3}{x+3} \right| + C \\
 &\text{(by Formula 24 of Integration Tables)}
 \end{aligned}$$

$$59. \int_2^{\sqrt{5}} x(x^2 - 4)^{3/2} dx = \left[\frac{1}{5} (x^2 - 4)^{5/2} \right]_2^{\sqrt{5}} = \frac{1}{5}$$

$$63. \int_0^\pi x \sin x dx = \left[-x \cos x + \sin x \right]_0^\pi = \pi$$

$$67. \text{By symmetry, } \bar{x} = 0, A = \frac{1}{2} \pi.$$

$$\begin{aligned}
 \bar{y} &= \frac{2}{\pi} \left(\frac{1}{2} \right) \int_{-1}^1 (\sqrt{1-x^2})^2 dx = \frac{1}{\pi} \left[x - \frac{1}{3} x^3 \right]_{-1}^1 = \frac{4}{3\pi} \\
 (\bar{x}, \bar{y}) &= \left(0, \frac{4}{3\pi} \right)
 \end{aligned}$$

$$71. \lim_{x \rightarrow 1} \left[\frac{(\ln x)^2}{x-1} \right] = \lim_{x \rightarrow 1} \left[\frac{2(1/x) \ln x}{1} \right] = 0$$

$$75. y = \lim_{x \rightarrow \infty} (\ln x)^{2/x}$$

$$\ln y = \lim_{x \rightarrow \infty} \frac{2 \ln(\ln x)}{x} = \lim_{x \rightarrow \infty} \left[\frac{2/(x \ln x)}{1} \right] = 0$$

$$\text{Since } \ln y = 0, y = 1.$$

C H A P T E R 7

Integration Techniques, L'Hôpital's Rule, and Improper Integrals

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CHAPTER 7

Integration Techniques, L'Hôpital's Rule, and Improper Integrals

Section 7.1 Basic Integration Rules

Solutions to Even-Numbered Exercises

$$2. (a) \frac{d}{dx}[\ln\sqrt{x^2+1} + C] = \frac{1}{2}\left(\frac{2x}{x^2+1}\right) = \frac{x}{x^2+1}$$

$$(b) \frac{d}{dx}\left[\frac{2x}{(x^2+1)^2} + C\right] = \frac{(x^2+1)^2(2) - (2x)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2(1-3x^2)}{(x^2+1)^3}$$

$$(c) \frac{d}{dx}[\arctan x + C] = \frac{1}{1+x^2}$$

$$(d) \frac{d}{dx}[\ln(x^2+1) + C] = \frac{2x}{x^2+1}$$

$$\int \frac{x}{x^2+1} dx \text{ matches (a).}$$

$$4. (a) \frac{d}{dx}[2x \sin(x^2+1) + C] = 2x[\cos(x^2+1)(2x)] + 2 \sin(x^2+1) = 2[2x^2 \cos(x^2+1) + \sin(x^2+1)]$$

$$(b) \frac{d}{dx}\left[-\frac{1}{2} \sin(x^2+1) + C\right] = -\frac{1}{2} \cos(x^2+1)(2x) = -x \cos(x^2+1)$$

$$(c) \frac{d}{dx}\left[\frac{1}{2} \sin(x^2+1) + C\right] = \frac{1}{2} \cos(x^2+1)(2x) = x \cos(x^2+1)$$

$$(d) \frac{d}{dx}[-2x \sin(x^2+1) + C] = -2x[\cos(x^2+1)(2x)] - 2 \sin(x^2+1) = -2[2x^2 \cos(x^2+1) + \sin(x^2+1)]$$

$$\int x \cos(x^2+1) dx \text{ matches (c).}$$

$$6. \int \frac{2t-1}{t^2-t+2} dt$$

$$u = t^2 - t + 2, du = (2t-1) dt$$

$$\text{Use } \int \frac{du}{u}.$$

$$8. \int \frac{2}{(2t-1)^2+4} dt$$

$$u = 2t-1, du = 2dt, a = 2$$

$$\text{Use } \int \frac{du}{u^2+a^2}.$$

$$10. \int \frac{-2x}{\sqrt{x^2-4}} dx$$

$$u = x^2 - 4, du = 2x dx, n = -\frac{1}{2}$$

$$\text{Use } \int u^n du.$$

$$12. \int \sec 3x \tan 3x dx$$

$$u = 3x, du = 3 dx$$

$$\text{Use } \int \sec u \tan u du.$$

$$14. \int \frac{1}{x\sqrt{x^2-4}} dx$$

$$u = x, du = dx, a = 2$$

$$\text{Use } \int \frac{du}{u\sqrt{u^2-a^2}}.$$

16. Let $u = x - 4$, $du = dx$.

$$\begin{aligned}\int 6(x-4)^5 dx &= 6 \int (x-4)^5 dx = 6 \frac{(x-4)^6}{6} + C \\ &= (x-4)^6 + C\end{aligned}$$

20. Let $u = 4 - 2x^2$, $du = -4x dx$.

$$\begin{aligned}\int x \sqrt{4 - 2x^2} dx &= -\frac{1}{4} \int (4 - 2x^2)^{1/2} (-4x) dx \\ &= -\frac{1}{6} (4 - 2x^2)^{3/2} + C\end{aligned}$$

24. Let $u = x^2 + 2x - 4$, $du = 2(x+1) dx$.

$$\begin{aligned}\int \frac{x+1}{\sqrt{x^2+2x-4}} dx &= \frac{1}{2} \int (x^2+2x-4)^{-1/2} (2)(x+1) dx \\ &= \sqrt{x^2+2x-4} + C\end{aligned}$$

26. $\int \frac{2x}{x-4} dx = \int 2 dx + \int \frac{8}{x-4} dx = 2x + 8 \ln|x-4| + C$

28. $\int \left(\frac{1}{3x-1} - \frac{1}{3x+1} \right) dx = \frac{1}{3} \int \frac{1}{3x-1} (3) dx - \frac{1}{3} \int \frac{1}{3x+1} (3) dx$
 $= \frac{1}{3} \ln|3x-1| - \frac{1}{3} \ln|3x+1| + C = \frac{1}{3} \ln \left| \frac{3x-1}{3x+1} \right| + C$

30. $\int x \left(1 + \frac{1}{x} \right)^3 dx = \int x \left(1 + \frac{3}{x} + \frac{3}{x^2} + \frac{1}{x^3} \right) dx = \int \left(x + 3 + \frac{3}{x} + \frac{1}{x^2} \right) dx = \frac{1}{2}x^2 + 3x + 3 \ln|x| - \frac{1}{x} + C$

32. $\int \sec 4x dx = \frac{1}{4} \int \sec(4x)(4) dx$
 $= \frac{1}{4} \ln|\sec 4x + \tan 4x| + C$

36. Let $u = \cot x$, $du = -\csc^2 x dx$.

$$\int \csc^2 x e^{\cot x} dx = -\int e^{\cot x} (-\csc^2 x) dx = -e^{\cot x} + C$$

18. Let $u = t - 9$, $du = dt$.

$$\int \frac{2}{(t-9)^2} dt = 2 \int (t-9)^{-2} dt = \frac{-2}{t-9} + C$$

22. $\int \left[x - \frac{3}{(2x+3)^2} \right] dx = \int x dx - \frac{3}{2} \int (2x+3)^{-2} (2) dx$
 $= \frac{x^2}{2} - \frac{3}{2} \frac{(2x+3)^{-1}}{-1} + C$
 $= \frac{x^2}{2} + \frac{3}{2(2x+3)} + C$

34. Let $u = \cos x$, $du = -\sin x dx$.

$$\begin{aligned}\int \frac{\sin x}{\sqrt{\cos x}} dx &= -\int (\cos x)^{-1/2} (-\sin x) dx \\ &= -2\sqrt{\cos x} + C\end{aligned}$$

38. $\int \frac{5}{3e^x - 2} dx = 5 \int \left(\frac{1}{3e^x - 2} \right) \left(\frac{e^{-x}}{e^{-x}} \right) dx$
 $= 5 \int \frac{e^{-x}}{3 - 2e^{-x}} dx$
 $= \frac{5}{2} \int \frac{1}{3 - 2e^{-x}} (2e^{-x}) dx$
 $= \frac{5}{2} \ln|3 - 2e^{-x}| + C$

40. Let $u = \ln(\cos x)$, $du = \frac{-\sin x}{\cos x} dx$
 $= -\tan x dx$

$$\int (\tan x)(\ln \cos x) dx = -\int (\ln \cos x)(-\tan x) dx$$

$$= \frac{-[\ln(\cos x)]^2}{2} + C$$

44. $\int \frac{2}{3(\sec x - 1)} dx = \frac{2}{3} \int \frac{1}{\sec x - 1} \cdot \left(\frac{\sec x + 1}{\sec x + 1} \right) dx$

$$= \frac{2}{3} \int \frac{\sec x + 1}{\tan^2 x} dx$$

$$= \frac{2}{3} \int \frac{\sec x}{\tan^2 x} dx + \frac{2}{3} \int \cot^2 x dx$$

$$= \frac{2}{3} \int \frac{\cos x}{\sin^2 x} dx + \frac{2}{3} \int (\csc^2 x - 1) dx$$

$$= \frac{2}{3} \left(-\frac{1}{\sin x} \right) - \frac{2}{3} \cot x - \frac{2}{3}x + C$$

$$= -\frac{2}{3} [\csc x + \cot x + x] + C$$

48. Let $u = \sqrt{3}x$, $du = \sqrt{3} dx$.

$$\int \frac{1}{4 + 3x^2} dx = \frac{1}{\sqrt{3}} \int \frac{\sqrt{3}}{4 + (\sqrt{3}x)^2} dx$$

$$= \frac{1}{2\sqrt{3}} \arctan\left(\frac{\sqrt{3}x}{2}\right) + C$$

42. $\int \frac{1 + \cos \alpha}{\sin \alpha} d\alpha = \int \csc \alpha d\alpha + \int \cot \alpha d\alpha$
 $= -\ln|\csc \alpha + \cot \alpha| + \ln|\sin \alpha| + C$

46. $\int \frac{3}{t^2 + 1} dt = 3 \arctan t + C$

50. Let $u = \frac{1}{t}$, $du = \frac{-1}{t^2} dt$.

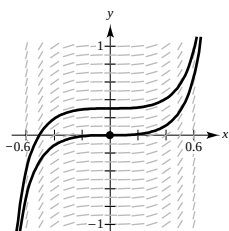
$$\int \frac{e^{1/t}}{t^2} dt = -\int e^{1/t} \left(\frac{-1}{t^2} \right) dt = -e^{1/t} + C$$

52. $\int \frac{1}{(x-1)\sqrt{4x^2 - 8x + 3}} dx = \int \frac{2}{[2(x-1)]\sqrt{[2(x-1)]^2 - 1}} dx = \operatorname{arcsec}|2(x-1)| + C$

54. $\int \frac{1}{\sqrt{1-4x-x^2}} dx = \int \frac{1}{\sqrt{5-(x^2+4x+4)}} dx = \int \frac{1}{\sqrt{5-(x+2)^2}} dx = \arcsin\left(\frac{x+2}{\sqrt{5}}\right) + C \quad (a = \sqrt{5})$

56. $\frac{dy}{dx} = \tan^2(2x)$, $(0, 0)$

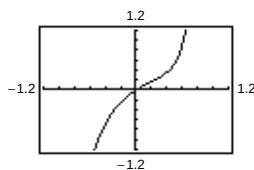
(a)



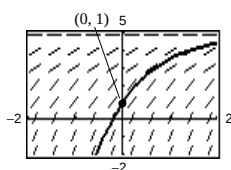
(b) $\int \tan^2(2x) dx = \int (\sec^2(2x) - 1) dx = \frac{1}{2} \tan(2x) - x + C$

$(0, 0): 0 = C$

$$y = \frac{1}{2} \tan(2x) - x$$



58.



60. $r = \int \frac{(1 + e^t)^2}{e^t} dt = \int \frac{1 + 2e^t + e^{2t}}{e^t} dt$
 $= \int (e^{-t} + 2 + e^t) dt = -e^{-t} + 2t + e^t + C$

62. Let $u = 2x$, $du = 2 dx$.

$$y = \int \frac{1}{x\sqrt{4x^2 - 1}} dx = \int \frac{2}{2x\sqrt{(2x)^2 - 1}} dx$$

$$= \operatorname{arcsec}|2x| + C$$

66. Let $u = 1 - \ln x$, $du = \frac{-1}{x} dx$.

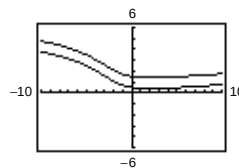
$$\int_1^e \frac{1 - \ln x}{x} dx = - \int_1^e (1 - \ln x) \left(\frac{-1}{x} \right) dx$$

$$= \left[-\frac{1}{2}(1 - \ln x)^2 \right]_1^e = \frac{1}{2}$$

70. $\int_0^4 \frac{1}{\sqrt{25 - x^2}} dx = \left[\arcsin \frac{x}{5} \right]_0^4 = \arcsin \frac{4}{5} \approx 0.927$

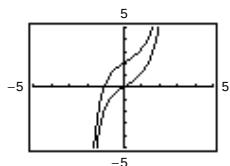
72. $\int \frac{x - 2}{x^2 + 4x + 13} dx = \frac{1}{2} \ln(x^2 + 4x + 13) - \frac{4}{3} \arctan\left(\frac{x + 2}{3}\right) + C$

The antiderivatives are vertical translations of each other.



74. $\int \left(\frac{e^x + e^{-x}}{2} \right)^3 dx = \frac{1}{24} [e^{3x} + 9e^x - 9e^{-x} - e^{-3x}] + C$

The antiderivatives are vertical translations of each other.



64. Let $u = \sin t$, $du = \cos t dt$.

$$\int_0^\pi \sin^2 t \cos t dt = \left[\frac{1}{3} \sin^3 t \right]_0^\pi = 0$$

68. $\int_1^2 \frac{x - 2}{x} dx = \int_1^2 \left(1 - \frac{2}{x} \right) dx$
- $$= \left[x - 2 \ln x \right]_1^2 = 1 - \ln 4 \approx -0.386$$

76. $\int \sec u \tan u du = \sec u + C$

78. Arctan Rule: $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C$

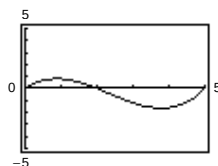
80. They differ by a constant:

$$\sec^2 x + C_1 = (\tan^2 x + 1) + C_1 = \tan^2 x + C.$$

82. $f(x) = \frac{1}{5}(x^3 - 7x^2 + 10x)$

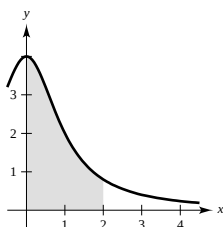
$$\int_0^5 f(x) dx < 0 \text{ because}$$

more area is below the x -axis than above.

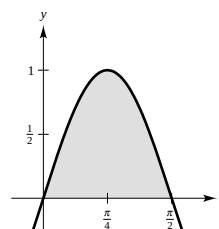


84. $\int_0^2 \frac{4}{x^2 + 1} dx \approx 4$

Matches (d).



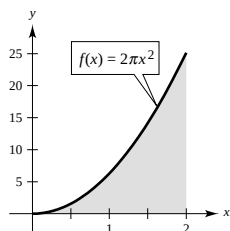
86. $A = \int_0^{\pi/2} \sin 2x dx$
- $$= \left[-\frac{1}{2} \cos 2x \right]_0^{\pi/2} = 1$$



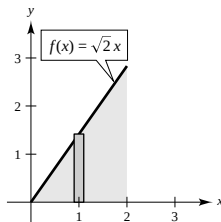
88. $\int_0^2 2\pi x^2 dx$

 (a) Let $f(x) = 2\pi x^2$ over the interval $[0, 2]$.

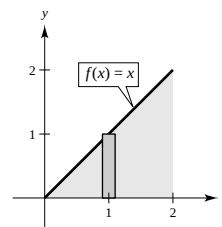
$$A = \int_0^2 (2\pi x^2) dx$$


 (b) Let $f(x) = \sqrt{2}x$ over the interval $[0, 2]$. Revolve this region about the x -axis.

$$\begin{aligned} V &= \pi \int_0^2 (\sqrt{2}x)^2 dx \\ &= \int_0^2 2\pi x^2 dx \end{aligned}$$


 (c) Let $f(x) = x$ over the interval $[0, 2]$. Revolve this region about the y -axis.

$$\begin{aligned} V &= 2\pi \int_0^2 x(x) dx \\ &= \int_0^2 2\pi x^2 dx \end{aligned}$$



90. (a) $\frac{1}{(\pi/n) - 0} \int_0^{\pi/n} \sin(nx) dx = \frac{1}{\pi} \int_0^{\pi/n} \sin(nx)(n) dx = \left[-\frac{1}{\pi} \cos(nx) \right]_0^{\pi/n} = \frac{2}{\pi}$

(b) $\frac{1}{3 - (-3)} \int_{-3}^3 \frac{1}{1 + x^2} dx = \left[\frac{1}{6} \arctan x \right]_{-3}^3 = \frac{1}{3} \arctan 3$

92. $y = 2\sqrt{x}$

$$y' = \frac{1}{\sqrt{x}}$$

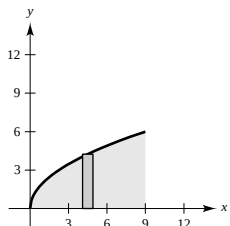
$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$S = 2\pi \int_0^9 2\sqrt{x} \sqrt{\frac{x+1}{x}} dx$$

$$= 2\pi \int_0^9 2\sqrt{x+1} dx$$

$$= \left[4\pi \left(\frac{2}{3} \right) (x+1)^{3/2} \right]_0^9$$

$$= \frac{8\pi}{3} (10\sqrt{10} - 1) \approx 256.545$$



94. $y = x^{2/3}$

$$y' = \frac{2}{3x^{1/3}}$$

$$1 + (y')^2 = 1 + \frac{4}{9x^{2/3}}$$

$$s = \int_1^8 \sqrt{1 + \frac{4}{9x^{2/3}}} dx \approx 7.6337$$

Section 7.2 Integration by Parts

2. $\frac{d}{dx} [x^2 \sin x + 2x \cos x - 2 \sin x] = x^2 \cos x + 2x \sin x - 2x \sin x + 2 \cos x - 2 \cos x = x^2 \cos x$. Matches (d)

4. $\frac{d}{dx}[-x + x \ln x] = -1 + x\left(\frac{1}{x}\right) + \ln x = \ln x$. Matches (a)

6. $\int x^2 e^{2x} dx$

$u = x^2, dv = e^{2x} dx$

8. $\int \ln 3x dx$

$u = \ln 3x, dv = dx$

10. $\int x^2 \cos x dx$

$u = x^2, dv = \cos x dx$

12. $dv = e^{-x} dx \Rightarrow v = \int e^{-x} dx = -e^{-x}$

$u = x \Rightarrow du = dx$

14. $\int \frac{e^{1/t}}{t^2} dt = -\int e^{1/t} \left(\frac{-1}{t^2}\right) dt = -e^{1/t} + C$

$$\begin{aligned} 2 \int \frac{x}{e^x} dx &= 2 \int x e^{-x} dx \\ &= 2 \left[-x e^{-x} - \int -e^{-x} dx \right] = 2[-x e^{-x} - e^{-x}] + C \\ &= -2x e^{-x} - 2e^{-x} + C \end{aligned}$$

16. $dv = x^4 dx \Rightarrow v = \frac{x^5}{5}$

$u = \ln x \Rightarrow du = \frac{1}{x} dx$

$$\begin{aligned} \int x^4 \ln x dx &= \frac{x^5}{5} \ln x - \int \frac{x^5}{5} \left(\frac{1}{x}\right) dx = \frac{x^5}{5} \ln x - \frac{1}{5} \int x^4 dx \\ &= \frac{x^5}{5} \ln x - \frac{1}{25} x^5 + C = \frac{x^5}{25} (5 \ln x - 1) + C \end{aligned}$$

18. Let $u = \ln x, du = \frac{1}{x} dx$.

$$\int \frac{1}{x(\ln x)^3} dx = \int (\ln x)^{-3} \left(\frac{1}{x}\right) dx = \frac{-1}{2(\ln x)^2} + C$$

20. $dv = \frac{1}{x^2} dx \Rightarrow v = \int \frac{1}{x^2} dx = -\frac{1}{x}$

$u = \ln x \Rightarrow du = \frac{1}{x} dx$

$$\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} + \int \frac{1}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x} + C$$

22. $dv = \frac{x}{(x^2 + 1)^2} dx \Rightarrow v = \int (x^2 + 1)^{-2} x dx = -\frac{1}{2(x^2 + 1)}$

$u = x^2 e^{x^2} \Rightarrow du = (2x^3 e^{x^2} + 2x e^{x^2}) dx = 2x e^{x^2} (x^2 + 1) dx$

$$\int \frac{x^3 e^{x^2}}{(x^2 + 1)^2} dx = -\frac{x^2 e^{x^2}}{2(x^2 + 1)} + \int x e^{x^2} dx = -\frac{x^2 e^{x^2}}{2(x^2 + 1)} + \frac{e^{x^2}}{2} + C = \frac{e^{x^2}}{2(x^2 + 1)} + C$$

24. $dv = \frac{1}{x^2} dx \Rightarrow v = \int \frac{1}{x^2} dx = -\frac{1}{x}$

$u = \ln 2x \Rightarrow du = \frac{1}{x} dx$

$$\int \frac{\ln(2x)}{x^2} dx = -\frac{\ln(2x)}{x} + \int \frac{1}{x^2} dx = -\frac{\ln(2x)}{x} - \frac{1}{x} + C = -\frac{\ln(2x) + 1}{x} + C$$

$$26. dv = \frac{1}{\sqrt{2+3x}} dx \Rightarrow v = \int (2+3x)^{-1/2} dx = \frac{2}{3} \sqrt{2+3x}$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int \frac{x}{\sqrt{2+3x}} dx &= \frac{2x\sqrt{2+3x}}{3} - \frac{2}{3} \int \sqrt{2+3x} dx \\ &= \frac{2x\sqrt{2+3x}}{3} - \frac{4}{27}(2+3x)^{3/2} + C = \frac{2\sqrt{2+3x}}{27}[9x - 2(2+3x)] + C = \frac{2\sqrt{2+3x}}{27}(3x-4) + C \end{aligned}$$

$$28. dv = \sin x dx \Rightarrow v = -\cos x$$

$$u = x \Rightarrow du = dx$$

$$\int x \sin x dx = -x \cos x - \int -\cos x dx = -x \cos x + \sin x + C$$

30. Use integration by parts twice.

$$(1) u = x^2, du = 2x dx, dv = \cos x dx, v = \sin x$$

$$\int x^2 \cos x dx = x^2 \sin x - 2 \int x \sin x dx$$

$$(2) u = x, du = dx, dv = \sin x dx, v = -\cos x$$

$$\begin{aligned} \int x^2 \cos x dx &= x^2 \sin x - 2 \left[-x \cos x + \int \cos x dx \right] \\ &= x^2 \sin x + 2x \cos x - 2 \sin x + C \end{aligned}$$

$$32. dv = \sec \theta \tan \theta d\theta \Rightarrow v = \int \sec \theta \tan \theta d\theta = \sec \theta$$

$$u = \theta \Rightarrow du = d\theta$$

$$\begin{aligned} \int \theta \sec \theta \tan \theta d\theta &= \theta \sec \theta - \int \sec \theta d\theta \\ &= \theta \sec \theta - \ln |\sec \theta + \tan \theta| + C \end{aligned}$$

$$34. dv = dx \Rightarrow v = \int dx = x$$

$$u = \arccos x \Rightarrow du = -\frac{1}{\sqrt{1-x^2}} dx$$

$$\begin{aligned} 4 \int \arccos x dx &= 4 \left[x \arccos x + \int \frac{x}{\sqrt{1-x^2}} dx \right] \\ &= 4 \left[x \arccos x - \sqrt{1-x^2} \right] + C \end{aligned}$$

36. Use integration by parts twice.

$$(1) dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = \cos 2x \Rightarrow du = -2 \sin 2x dx$$

$$\int e^x \cos 2x dx = e^x \cos 2x + 2 \int e^x \sin 2x dx = e^x \cos 2x + 2 \left(e^x \sin 2x - 2 \int e^x \cos 2x dx \right)$$

$$5 \int e^x \cos 2x dx = e^x \cos 2x + 2e^x \sin 2x$$

$$\int e^x \cos 2x dx = \frac{e^x}{5} (\cos 2x + 2 \sin 2x) + C$$

$$(2) dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = \sin 2x \Rightarrow du = 2 \cos 2x dx$$

$$38. dv = dx \Rightarrow v = x$$

$$u = \ln x \Rightarrow du = \frac{1}{x} dx$$

$$y' = \ln x$$

$$y = \int \ln x dx = x \ln x - \int x \left(\frac{1}{x} \right) dx = x \ln x - x + C = x(-1 + \ln x) + C$$

40. Use integration by parts twice.

$$(1) \quad dv = \sqrt{x-1} \, dx \Rightarrow v = \int (x-1)^{1/2} \, dx = \frac{2}{3}(x-1)^{3/2}$$

$$u = x^2 \Rightarrow du = 2x \, dx$$

$$(2) \quad dv = (x-1)^{3/2} \, dx \Rightarrow v = \int (x-1)^{3/2} \, dx = \frac{2}{5}(x-1)^{5/2}$$

$$u = x \Rightarrow du = dx$$

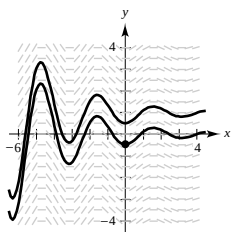
$$\begin{aligned} y &= \int x^2 \sqrt{x-1} \, dx \\ &= \frac{2}{3}x^2(x-1)^{3/2} - \frac{4}{3} \int x(x-1)^{3/2} \, dx = \frac{2}{3}x^2(x-1)^{3/2} - \frac{4}{3} \left[\frac{2}{5}x(x-1)^{5/2} - \frac{2}{5} \int (x-1)^{5/2} \, dx \right] \\ &= \frac{2}{3}x^2(x-1)^{3/2} - \frac{8}{15}x(x-1)^{5/2} + \frac{16}{105}(x-1)^{7/2} + C = \frac{2(x-1)^{3/2}}{105}(15x^2 + 12x + 8) + C \end{aligned}$$

42. $dv = dx \Rightarrow v = \int dx = x$

$$u = \arctan \frac{x}{2} \Rightarrow du = \frac{1}{1 + (x/2)^2} \left(\frac{1}{2} \right) dx = \frac{2}{4 + x^2} dx$$

$$y = \int \arctan \frac{x}{2} \, dx = x \arctan \frac{x}{2} - \int \frac{2x}{4 + x^2} \, dx = x \arctan \frac{x}{2} - \ln(4 + x^2) + C$$

44. (a)



(b) $\frac{dy}{dx} = e^{-x/3} \sin 2x, \left(0, -\frac{18}{37}\right)$

$$y = \int e^{-x/3} \sin 2x \, dx$$

Use integration by parts twice.

(1) $u = \sin 2x, \quad du = 2 \cos 2x$

$$dv = e^{-x/3} \, dx, \quad v = -3e^{-x/3}$$

$$\int e^{-x/3} \sin 2x \, dx = -3e^{-x/3} \sin 2x + \int 6e^{-x/3} \cos 2x \, dx$$

(2) $u = \cos 2x, \quad du = -2 \sin 2x$

$$dv = e^{-x/3} \, dx, \quad v = -3e^{-x/3}$$

$$\int e^{-x/3} \sin 2x \, dx = -3e^{-x/3} \sin 2x + 6 \left[-3e^{-x/3} \cos 2x - \int 6e^{-x/3} \sin 2x \, dx \right] + C$$

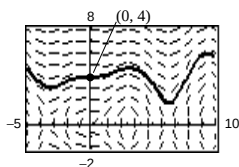
$$37 \int e^{-x/3} \sin 2x \, dx = -3e^{-x/3} \sin 2x - 18e^{-x/3} \cos 2x + C$$

$$y = \int e^{-x/3} \sin 2x \, dx = \frac{1}{37} \left[-3e^{-x/3} \sin 2x - 18e^{-x/3} \cos 2x \right] + C$$

$$\left(0, -\frac{18}{37}\right): \frac{-18}{37} = \frac{1}{37} [0 - 18] + C \Rightarrow C = 0$$

$$y = \frac{-1}{37} [3e^{-x/3} \sin 2x + 18e^{-x/3} \cos 2x]$$

46. $\frac{dy}{dx} = \frac{x}{y} \sin x, y(0) = 4$



48. See Exercise 3.

$$\int_0^1 x^2 e^x dx = \left[x^2 e^x - 2x e^x + 2e^x \right]_0^1 = e - 2 \approx 0.718$$

50. $dv = \sin 2x dx \Rightarrow v = \int \sin 2x dx = -\frac{1}{2} \cos 2x$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x \sin 2x dx &= \frac{-1}{2} x \cos 2x + \frac{1}{2} \int \cos 2x dx \\ &= \frac{-1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C \\ &= \frac{1}{4} (\sin 2x - 2x \cos 2x) + C \end{aligned}$$

$$\text{Thus, } \int_0^\pi x \sin 2x dx = \left[\frac{1}{4} (\sin 2x - 2x \cos 2x) \right]_0^\pi = -\frac{\pi}{2}.$$

52. $dv = x dx \Rightarrow v = \int x dx = \frac{x^2}{2}$

$$u = \arcsin x^2 \Rightarrow du = \frac{2x}{\sqrt{1-x^4}} dx$$

$$\begin{aligned} \int x \arcsin x^2 dx &= \frac{x^2}{2} \arcsin x^2 - \int \frac{x^3}{\sqrt{1-x^4}} dx \\ &= \frac{x^2}{2} \arcsin x^2 + \frac{1}{4} (2)(1-x^4)^{1/2} + C \\ &= \frac{1}{2} [x^2 \arcsin x^2 + \sqrt{1-x^4}] + C \end{aligned}$$

$$\begin{aligned} \text{Thus, } \int_0^1 x \arcsin x^2 dx &= \frac{1}{2} [x^2 \arcsin x^2 + \sqrt{1-x^4}]_0^1 \\ &= \frac{1}{4} (\pi - 2). \end{aligned}$$

54. Use integration by parts twice.

(1) $dv = e^{-x}, v = -e^{-x}, u = \cos x, du = -\sin x dx$

$$\int e^{-x} \cos x dx = -e^{-x} \cos x - \int e^{-x} \sin x dx$$

(2) $dv = e^{-x} dx, v = -e^{-x}, u = \sin x, du = \cos x dx$

$$\int e^{-x} \cos x dx = -e^{-x} \cos x - \left[-e^{-x} \sin x + \int e^{-x} \cos x dx \right] \Rightarrow 2 \int e^{-x} \cos x dx = e^{-x} \sin x - e^{-x} \cos x$$

Thus,

$$\begin{aligned} \int_0^2 e^{-x} \cos x dx &= \left[\frac{e^{-x} \sin x - e^{-x} \cos x}{2} \right]_0^2 \\ &= \frac{-e^{-2}}{2} [\sin 2 - \cos 2] + \frac{1}{2} \end{aligned}$$

56. $dv = dx \Rightarrow v = \int dx = x$

$$u = \ln(1+x^2) \Rightarrow du = \frac{2x}{1+x^2} dx$$

$$\begin{aligned} \int \ln(1+x^2) dx &= x \ln(1+x^2) - \int \frac{2x^2}{1+x^2} dx \\ &= x \ln(1+x^2) - 2 \int \left[1 - \frac{1}{1+x^2} \right] dx = x \ln(1+x^2) - 2x + 2 \arctan x + C \end{aligned}$$

$$\text{Thus, } \int_0^1 \ln(1+x^2) dx = \left[x \ln(1+x^2) - 2x + 2 \arctan x \right]_0^1 = \ln 2 - 2 + \frac{\pi}{2}.$$

58. $u = x, du = dx, dv = \sec^2 x dx, v = \tan x$

Hence,

$$\int x \sec^2 x dx = x \tan x - \int \tan x dx$$

$$\begin{aligned} \int_0^{\pi/4} x \sec^2 x dx &= \left[x \tan x + \ln |\cos x| \right]_0^{\pi/4} \\ &= \left(\frac{\pi}{4} + \ln \frac{\sqrt{2}}{2} \right) - 0 \\ &= \frac{\pi}{4} - \frac{1}{2} \ln 2 \end{aligned}$$

60.
$$\begin{aligned} \int x^3 e^{-2x} dx &= x^3 \left(-\frac{1}{2} e^{-2x} \right) - 3x^2 \left(\frac{1}{4} e^{-2x} \right) + 6x \left(-\frac{1}{8} e^{-2x} \right) - 6 \left(\frac{1}{16} e^{-2x} \right) + C \\ &= -\frac{1}{8} e^{-2x} (4x^3 + 6x^2 + 6x + 3) + C \end{aligned}$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^3	e^{-2x}
−	$3x^2$	$-\frac{1}{2}e^{-2x}$
+	$6x$	$\frac{1}{4}e^{-2x}$
−	6	$-\frac{1}{8}e^{-2x}$
+	0	$\frac{1}{16}e^{-2x}$

62.
$$\begin{aligned} \int x^3 \cos 2x dx &= x^3 \left(\frac{1}{2} \sin 2x \right) - 3x^2 \left(-\frac{1}{4} \cos 2x \right) + 6x \left(-\frac{1}{8} \sin 2x \right) - 6 \left(\frac{1}{16} \cos 2x \right) + C \\ &= \frac{1}{2} x^3 \sin 2x + \frac{3}{4} x^2 \cos 2x - \frac{3}{4} x \sin 2x - \frac{3}{8} \cos 2x + C \\ &= \frac{1}{8} [4x^3 \sin 2x + 6x^2 \cos 2x - 6x \sin 2x - 3 \cos 2x] + C \end{aligned}$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^3	$\cos 2x$
−	$3x^2$	$\frac{1}{2} \sin 2x$
+	$6x$	$-\frac{1}{4} \cos 2x$
−	6	$-\frac{1}{8} \sin 2x$
+	0	$\frac{1}{16} \cos 2x$

64.
$$\begin{aligned} \int x^2 (x-2)^{3/2} dx &= \frac{2}{5} x^2 (x-2)^{5/2} - \frac{8}{35} x (x-2)^{7/2} + \frac{16}{315} (x-2)^{9/2} + C \\ &= \frac{2}{315} (x-2)^{5/2} (35x^2 + 40x + 32) + C \end{aligned}$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^2	$(x-2)^{3/2}$
−	$2x$	$\frac{2}{5} (x-2)^{5/2}$
+	2	$\frac{4}{35} (x-2)^{7/2}$
−	0	$\frac{8}{315} (x-2)^{9/2}$

 66. Answers will vary.
See pages 488, 493.

 68. Yes.
 $u = \ln x, dv = x dx$

70. No. Substitution.

72. No. Substitution.

74.
$$\int \alpha^4 \sin \pi \alpha d\alpha = \frac{1}{\pi^5} [-(\alpha\pi)^4 \cos \pi\alpha + 4(\alpha\pi)^3 \sin \pi\alpha + 12(\alpha\pi)^2 \cos \pi\alpha - 24(\alpha\pi) \sin \pi\alpha - 24 \cos \pi\alpha] + C$$

76.
$$\begin{aligned} \int_0^5 x^4 (25 - x^2)^{3/2} dx &= \left[\frac{1,171,875 \arcsin(x/5)}{128} - \frac{x(2x^2 + 25)(25 - x^2)^{5/2}}{16} + \frac{625x(25 - x^2)^{3/2}}{64} + \frac{46,875x\sqrt{25 - x^2}}{128} \right]_0^5 \\ &\approx 14,381.0699 \end{aligned}$$

$$78. (a) dv = \sqrt{4+x} dx \Rightarrow v = \int (4+x)^{1/2} dx = \frac{2}{3}(4+x)^{3/2}$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x\sqrt{4+x} dx &= \frac{2}{3}x(4+x)^{3/2} - \frac{2}{3} \int (4+x)^{3/2} dx \\ &= \frac{2}{3}x(4+x)^{3/2} - \frac{4}{15}(4+x)^{5/2} + C = \frac{2}{15}(4+x)^{3/2}(3x-8) + C \end{aligned}$$

$$(b) u = 4+x \Rightarrow x = u-4 \text{ and } dx = du$$

$$\begin{aligned} \int x\sqrt{4+x} dx &= \int (u-4)u^{1/2} du = \int (u^{3/2} - 4u^{1/2}) du \\ &= \frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2} + C = \frac{2}{15}u^{3/2}(3u-20) + C \\ &= \frac{2}{15}(4+x)^{3/2}[3(4+x)-20] + C = \frac{2}{15}(4+x)^{3/2}(3x-8) + C \end{aligned}$$

$$80. (a) dv = \sqrt{4-x} dx \Rightarrow v = \int (4-x)^{1/2} dx$$

$$= -\frac{2}{3}(4-x)^{3/2}$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x\sqrt{4-x} dx &= -\frac{2}{3}x(4-x)^{3/2} + \frac{2}{3} \int (4-x)^{3/2} dx \\ &= -\frac{2}{3}x(4-x)^{3/2} - \frac{4}{15}(4-x)^{5/2} + C \\ &= -\frac{2}{15}(4-x)^{3/2}[5x+2(4-x)] + C \\ &= -\frac{2}{15}(4-x)^{3/2}(3x+8) + C \end{aligned}$$

$$(b) u = 4-x \Rightarrow x = 4-u \text{ and } dx = -du$$

$$\begin{aligned} \int x\sqrt{4-x} dx &= -\int (4-u)\sqrt{u} du \\ &= -\int (4u^{1/2} - u^{3/2}) du \\ &= -\frac{8}{3}u^{3/2} + \frac{2}{5}u^{5/2} + C \\ &= -\frac{2}{15}u^{3/2}(20-3u) + C \\ &= -\frac{2}{15}(4-x)^{3/2}[20-3(4-x)] + C \\ &= -\frac{2}{15}(4-x)^{3/2}(3x+8) + C \end{aligned}$$

$$84. dv = \cos x dx \Rightarrow v = \sin x$$

$$u = x^n \Rightarrow du = nx^{n-1} dx$$

$$\int x^n \cos x dx = x^n \sin x - n \int x^{n-1} \sin x dx$$

$$82. n = 0: \int e^x dx = e^x + C$$

$$n = 1: \int xe^x dx = xe^x - e^x + C = xe^x - \int e^x dx$$

$$\begin{aligned} n = 2: \int x^2 e^x dx &= x^2 e^x - 2xe^x + 2e^x + C \\ &= x^2 e^x - 2 \int xe^x dx \end{aligned}$$

$$\begin{aligned} n = 3: \int x^3 e^x dx &= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C \\ &= x^3 e^x - 3 \int x^2 e^x dx \end{aligned}$$

$$\begin{aligned} n = 4: \int x^4 e^x dx \\ &= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x + C \\ &= x^4 e^x - 4 \int x^3 e^x dx \end{aligned}$$

$$\text{In general, } \int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx.$$

(See Exercise 86)

$$86. dv = e^{ax} dx \Rightarrow v = \frac{1}{a}e^{ax}$$

$$u = x^n \Rightarrow du = nx^{n-1} dx$$

$$\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx$$

88. Use integration by parts twice.

$$(1) \quad dv = e^{ax} dx \Rightarrow v = \frac{1}{a}e^{ax}$$

$$u = \cos bx \Rightarrow du = -b \sin bx$$

$$\begin{aligned} \int e^{ax} \cos bx \, dx &= \frac{e^{ax} \cos bx}{a} + \frac{b}{a} \int e^{ax} \sin bx \, dx = \frac{e^{ax} \cos bx}{a} + \frac{b}{a} \left[\frac{e^{ax} \sin bx}{a} - \frac{b}{a} \int e^{ax} \cos bx \, dx \right] \\ &= \frac{e^{ax} \cos bx}{a} + \frac{be^{ax} \sin bx}{a^2} - \frac{b^2}{a^2} \int e^{ax} \cos bx \, dx \end{aligned}$$

$$\begin{aligned} \text{Therefore, } \left(1 + \frac{b^2}{a^2}\right) \int e^{ax} \cos bx \, dx &= \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2} \\ \int e^{ax} \cos bx \, dx &= \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2} + C. \end{aligned}$$

$$(2) \quad dv = e^{ax} dx \Rightarrow v = \frac{1}{a}e^{ax}$$

$$u = \sin bx \Rightarrow du = b \cos bx$$

90. $n = 2$ (Use formula in Exercise 84.)

$$\begin{aligned} \int x^2 \cos x \, dx &= x^2 \sin x - 2 \int x \sin x \, dx \quad (\text{Use formula in Exercise 83.}) \quad (n = 1) \\ &= x^2 \sin x - 2 \left[-x \cos x + \int \cos x \, dx \right] = x^2 \sin x + 2x \cos x - 2 \sin x + C \end{aligned}$$

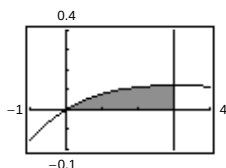
92. $n = 3, a = 2$ (Use formula in Exercise 86 three times.)

$$\begin{aligned} \int x^3 e^{2x} \, dx &= \frac{x^3 e^{2x}}{2} - \frac{3}{2} \int x^2 e^{2x} \, dx \quad (n = 3, a = 2) \\ &= \frac{x^3 e^{2x}}{2} - \frac{3}{2} \left[\frac{x^2 e^{2x}}{2} - \int x e^{2x} \, dx \right] \quad (n = 2, a = 2) \\ &= \frac{x^3 e^{2x}}{2} - \frac{3x^2 e^{2x}}{4} + \frac{3}{2} \left[\frac{x e^{2x}}{2} - \frac{1}{2} \int e^{2x} \, dx \right] = \frac{x^3 e^{2x}}{2} - \frac{3x^2 e^{2x}}{4} + \frac{3x e^{2x}}{4} - \frac{3e^{2x}}{8} + C \quad (n = 1, a = 2) \\ &= \frac{e^{2x}}{8} (4x^3 - 6x^2 + 6x - 3) + C \end{aligned}$$

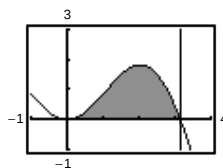
$$94. \quad dv = e^{-x/3} dx \Rightarrow v = -3e^{-x/3}$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} A &= \frac{1}{9} \int_0^3 x e^{-x/3} \, dx \\ &= \frac{1}{9} \left(\left[-3x e^{-x/3} \right]_0^3 + 3 \int_0^3 e^{-x/3} \, dx \right) \\ &= \frac{1}{9} \left(\frac{-9}{e} - \left[9e^{-x/3} \right]_0^3 \right) \\ &= -\frac{1}{e} - \frac{1}{e} + 1 = 1 - \frac{2}{e} \approx 0.264 \end{aligned}$$



$$\begin{aligned} 96. \quad A &= \int_0^\pi x \sin x \, dx = \left[-x \cos x + \sin x \right]_0^\pi \\ &= \pi \quad (\text{See Exercise 83.}) \end{aligned}$$



98. In Example 6, we showed that the centroid of an equivalent region was $(1, \pi/8)$. By symmetry, the centroid of this region is $(\pi/8, 1)$.

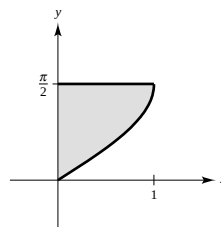
You can also solve this problem directly.

$$A = \int_0^1 \left(\frac{\pi}{2} - \arcsin x \right) dx = \left[\frac{\pi}{2}x - x \arcsin x - \sqrt{1-x^2} \right]_0^1 \quad (\text{Example 3})$$

$$= \left(\frac{\pi}{2} - \frac{\pi}{2} - 0 \right) - (-1) = 1$$

$$\bar{x} = \frac{M_y}{A} = \int_0^1 x \left[\frac{\pi}{2} - \arcsin x \right] dx = \frac{\pi}{8}$$

$$\bar{y} = \frac{M_x}{A} = \int_0^1 \frac{(\pi/2) + \arcsin x}{2} \left[\frac{\pi}{2} - \arcsin x \right] dx = 1$$



100. (a) Average $= \int_1^2 (1.6t \ln t + 1) dt = \left[0.8t^2 \ln t - 0.4t^2 + t \right]_1^2 = 3.2(\ln 2) - 0.2 \approx 2.018$

(b) Average $= \int_3^4 (1.6t \ln t + 1) dt = \left[0.8t^2 \ln t - 0.4t^2 + t \right]_3^4 = 12.8(\ln 4) - 7.2(\ln 3) - 1.8 \approx 8.035$

102. $c(t) = 30,000 + 500t$, $r = 7\%$, $t_1 = 5$

$$P \int_0^5 (30,000 + 500t)e^{-0.07t} dt = 500 \int_0^5 (60 + t)e^{-0.07t} dt$$

Let $u = 60 + t$, $dv = e^{-0.07t} dt$, $du = dt$, $v = -\frac{100}{7}e^{-0.07t}$.

$$P = 500 \left\{ \left[(60 + t) \left(-\frac{100}{7} e^{-0.07t} \right) \right]_0^5 + \frac{100}{7} \int_0^5 e^{-0.07t} dt \right\}$$

$$= 500 \left\{ \left[(60 + t) \left(-\frac{100}{7} e^{-0.07t} \right) \right]_0^5 - \left[\frac{10,000}{49} e^{-0.07t} \right]_0^5 \right\} \approx \$131,528.68$$

104. $\int_{-\pi}^{\pi} x^2 \cos nx \, dx = \left[\frac{x^2}{n} \sin nx + \frac{2x}{n^2} \cos nx - \frac{2}{n^3} \sin nx \right]_{-\pi}^{\pi}$

$$= \frac{2\pi}{n^2} \cos n\pi + \frac{2\pi}{n^2} \cos(-n\pi)$$

$$= \frac{4\pi}{n^2} \cos n\pi$$

$$= \begin{cases} (4\pi/n^2), & \text{if } n \text{ is even} \\ -(4\pi/n^2), & \text{if } n \text{ is odd} \end{cases}$$

$$= \frac{(-1)^n 4\pi}{n^2}$$

106. For any integrable function, $\int f(x) \, dx = C + \int f(x) \, dx$, but this cannot be used to imply that $C = 0$.

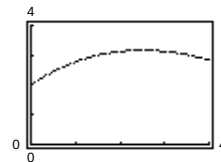
108. On $\left[0, \frac{\pi}{2}\right]$, $\sin x \leq 1 \Rightarrow x \sin x \leq x \Rightarrow \int_0^{\pi/2} x \sin x \, dx \leq \int_0^{\pi/2} x \, dx$.

110. $f'(x) = \cos \sqrt{x}, f(0) = 2$

(a) It cannot be solved by integration.

(b) You obtain the points

n	x_n	y_n
0	0	2
1	0.05	2.05
2	0.10	2.098755
3	0.15	2.146276
$\vdots$	$\vdots$	$\vdots$
80	4.0	2.8403565



Section 7.3 Trigonometric Integrals

2. (a) $y = \sec x \Rightarrow y' = \sec x \tan x = \sin x \sec^2 x$.

Matches (iii)

(b) $y = \cos x + \sec x \Rightarrow y' = -\sin x + \sec x \tan x$

$$= -\sin x + \sec^2 x \sin x$$

$$= \sin x(-1 + \sec^2 x)$$

$$= \sin x \tan^2 x \quad \text{Matches (i)}$$

(c) $y = x - \tan x + \frac{1}{3} \tan^3 x \Rightarrow y' = 1 - \sec^2 x + \tan^2 x \sec^2 x$
 $= -\tan^2 x + \tan^2 x(1 + \tan^2 x)$
 $= \tan^4 x \quad \text{Matches (iv)}$

(d) $y = 3x + 2 \sin x \cos^3 x + 3 \sin x \cos x \Rightarrow$
 $y' = 3 + 2 \cos x(\cos^3 x) + 6 \sin x \cos^2 x(-\sin x) + 3 \cos^2 x - 3 \sin^2 x$
 $= 3 + 2 \cos^4 x - 6 \cos^2 x(1 - \cos^2 x) + 3 \cos^2 x - 3(1 - \cos^2 x)$
 $= 8 \cos^4 x \quad \text{Matches (ii)}$

4. $\int \cos^3 x \sin^4 x \, dx = \int \cos x(1 - \sin^2 x) \sin^4 x \, dx$
 $= \int (\sin^4 x - \sin^6 x) \cos x \, dx$
 $= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$

6. Let $u = \cos x, du = -\sin x \, dx$.

$$\begin{aligned} \int \sin^3 x \, dx &= \int \sin x(1 - \cos^2 x) \, dx \\ &= \int \cos^2 x(-\sin x) \, dx + \int \sin x \, dx \\ &= \frac{1}{3} \cos^3 x - \cos x + C \end{aligned}$$

8. Let $u = \sin \frac{x}{3}, du = \frac{1}{3} \cos \frac{x}{3} \, dx$.

$$\begin{aligned} \int \cos^3 \frac{x}{3} \, dx &= \int \left(\cos \frac{x}{3} \right) \left(1 - \sin^2 \frac{x}{3} \right) \, dx \\ &= 3 \int \left(1 - \sin^2 \frac{x}{3} \right) \left(\frac{1}{3} \cos \frac{x}{3} \right) \, dx \\ &= 3 \left(\sin \frac{x}{3} - \frac{1}{3} \sin^3 \frac{x}{3} \right) + C \\ &= 3 \sin \frac{x}{3} - \sin^3 \frac{x}{3} + C \end{aligned}$$

$$10. \int \frac{\sin^5 t}{\sqrt{\cos t}} dt = \int \sin t (1 - \cos^2 t)^2 (\cos t)^{-1/2} dt$$

$$= \int \sin t (1 - 2 \cos^2 t + \cos^4 t) (\cos t)^{-1/2} dt$$

$$= \int [(\cos t)^{-1/2} - 2(\cos t)^{3/2} + (\cos t)^{7/2}] \sin t dt = -2(\cos t)^{1/2} + \frac{4}{5}(\cos t)^{5/2} - \frac{2}{9}(\cos t)^{9/2} + C$$

$$12. \int \sin^2 2x dx = \int \frac{1 - \cos 4x}{2} dx = \frac{1}{2} \left(x - \frac{1}{4} \sin 4x \right) + C$$

$$= \frac{1}{8} (4x - \sin 4x) + C$$

$$14. \int \sin^4 2\theta d\theta = \int \frac{1 - \cos 4\theta}{2} \cdot \frac{1 - \cos 4\theta}{2} d\theta$$

$$= \frac{1}{4} \int (1 - 2 \cos 4\theta + \cos^2 4\theta) d\theta$$

$$= \frac{1}{4} \int \left(1 - 2 \cos 4\theta + \frac{1 + \cos 8\theta}{2} \right) d\theta$$

$$= \frac{1}{4} \int \left(\frac{3}{2} - 2 \cos 4\theta + \frac{1}{2} \cos 8\theta \right) d\theta$$

$$= \frac{1}{4} \left[\frac{3}{2} \theta - \frac{1}{2} \sin 4\theta + \frac{1}{16} \sin 8\theta \right] + C$$

$$= \frac{3}{8} \theta - \frac{1}{8} \sin 4\theta + \frac{1}{64} \sin 8\theta + C$$

16. Use integration by parts twice.

$$dv = \sin^2 x dx = \frac{1 - \cos 2x}{2} \Rightarrow v = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{1}{4} (2x - \sin 2x)$$

$$u = x^2 \Rightarrow du = 2x dx$$

$$dv = \sin 2x dx \Rightarrow v = -\frac{1}{2} \cos 2x$$

$$u = x \Rightarrow du = dx$$

$$\int x^2 \sin^2 x dx = \frac{1}{4} x^2 (2x - \sin 2x) - \frac{1}{2} \int (2x^2 - x \sin 2x) dx$$

$$= \frac{1}{2} x^3 - \frac{1}{4} x^2 \sin 2x - \frac{1}{3} x^3 + \frac{1}{2} \int x \sin 2x dx$$

$$= \frac{1}{6} x^3 - \frac{1}{4} x^2 \sin 2x + \frac{1}{2} \left[-\frac{1}{2} x \cos 2x + \frac{1}{2} \int \cos 2x dx \right]$$

$$= \frac{1}{6} x^3 - \frac{1}{4} x^2 \sin 2x - \frac{1}{4} x \cos 2x + \frac{1}{8} \sin 2x + C$$

$$= \frac{1}{24} (4x^3 - 6x^2 \sin 2x - 6x \cos 2x + 3 \sin 2x) + C$$

18. Let $u = \sin x$, $du = \cos x dx$.

$$\int_0^{\pi/2} \cos^5 x dx = \int_0^{\pi/2} (1 - \sin^2 x)^2 \cos x dx$$

$$= \int_0^{\pi/2} (1 - 2 \sin^2 x + \sin^4 x) \cos x dx$$

$$= \left[\sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x \right]_0^{\pi/2}$$

$$= \frac{8}{15}$$

$$20. \int_0^{\pi/2} \sin^2 x dx = \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2x) dx$$

$$= \frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi/2} = \frac{\pi}{4}$$

$$22. \int \sec^2(2x - 1) dx = \frac{1}{2} \tan(2x - 1) + C$$

$$\begin{aligned} 24. \int \sec^6 3x dx &= \int (1 + \tan^2 3x)^2 \sec^2 3x dx \\ &= \int (1 + 2 \tan^2 3x + \tan^4 3x) \sec^2 3x dx \\ &= \frac{1}{3} \tan 3x + \frac{2}{9} \tan^3 3x + \frac{1}{15} \tan^5 3x + C \end{aligned}$$

$$26. \int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$$

$$28. \int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2} dx = \frac{1}{2\pi} \tan^4 \frac{\pi x}{2} + C$$

$$30. \text{ Let } u = \sec 2t, du = 2 \sec 2t \tan 2t$$

$$\begin{aligned} \int \tan^3 2t \cdot \sec^3 2t dt &= \int (\sec^2 2t - 1) \sec^3 2t \cdot \tan 2t dt \\ &= \int (\sec^4 2t - \sec^2 2t)(\sec 2t \tan 2t) dt \\ &= \frac{\sec^5 2t}{5} - \frac{\sec^3 2t}{3} + C \end{aligned}$$

$$32. \int \tan^5 2x \sec^2 2x dx = \frac{1}{12} \tan^6 2x + C$$

$$\begin{aligned} 34. \int \sec^2 \frac{x}{2} \tan \frac{x}{2} dx &= 2 \int \sec \frac{x}{2} \left(\frac{1}{2} \sec \frac{x}{2} \tan \frac{x}{2} \right) dx \\ &= \sec^2 \frac{x}{2} + C \\ \text{or } \int \sec^2 \frac{x}{2} \tan \frac{x}{2} dx &= 2 \int \tan \frac{x}{2} \left(\frac{1}{2} \sec^2 \frac{x}{2} \right) dx \\ &= \tan^2 \frac{x}{2} + C \end{aligned}$$

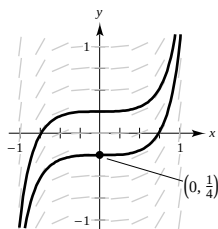
$$\begin{aligned} 36. \int \tan^3 3x dx &= \int (\sec^2 3x - 1) \tan 3x dx \\ &= \frac{1}{3} \int \tan 3x (3 \sec^2 3x) dx + \frac{1}{3} \int \frac{-3 \sin 3x}{\cos 3x} dx \\ &= \frac{1}{6} \tan^2 3x + \frac{1}{3} \ln |\cos 3x| + C \end{aligned}$$

$$\begin{aligned} 38. \int \frac{\tan^2 x}{\sec^5 x} dx &= \int \frac{\sin^2 x}{\cos^2 x} \cdot \cos^5 x dx \\ &= \int \sin^2 x \cdot \cos^3 x dx \\ &= \int \sin^2 x (1 - \sin^2 x) \cos x dx \\ &= \int (\sin^2 x - \sin^4 x) \cos x dx \\ &= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C \end{aligned}$$

$$\begin{aligned} 40. s &= \int \sin^2 \frac{\alpha}{2} \cos^2 \frac{\alpha}{2} d\alpha \\ &= \int \left(\frac{1 - \cos \alpha}{2} \right) \left(\frac{1 + \cos \alpha}{2} \right) d\alpha = \int \frac{1 - \cos^2 \alpha}{4} d\alpha \\ &= \frac{1}{4} \int \sin^2 \alpha d\alpha = \frac{1}{8} \int (1 - \cos 2\alpha) d\alpha \\ &= \frac{1}{8} \left[\theta - \frac{\sin 2\alpha}{2} \right] + C \\ &= \frac{1}{16} (2\alpha - \sin 2\alpha) + C \end{aligned}$$

$$\begin{aligned} 42. y &= \int \sqrt{\tan x} \sec^4 x dx \\ &= \int \tan^{1/2} x (\tan^2 x + 1) \sec^2 x dx \\ &= \int (\tan^{5/2} x + \tan^{1/2} x) \sec^2 x dx \\ &= \frac{2}{7} \tan^{7/2} x + \frac{2}{3} \tan^{3/2} x + C \end{aligned}$$

44. (a)



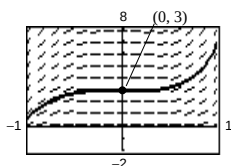
$$(b) \frac{dy}{dx} = \sec^2 x \tan^2 x, \left(0, -\frac{1}{4}\right)$$

$$y = \int \sec^2 x \tan^2 x \, dx \quad u = \tan x, \, du = \sec^2 x \, dx$$

$$y = \frac{\tan^3 x}{3} + C$$

$$\left(0, -\frac{1}{4}\right): -\frac{1}{4} = C \Rightarrow y = \frac{1}{3} \tan^3 x - \frac{1}{4}$$

$$46. \frac{dy}{dx} = 3\sqrt{y} \tan^2 x, y(0) = 3$$



$$\begin{aligned} 48. \int \cos 4\theta \cos(-3\theta) \, d\theta &= \int \cos 4\theta \cos 3\theta \, d\theta \\ &= \frac{1}{2} \int (\cos 7\theta + \cos \theta) \, d\theta \\ &= \frac{\sin 7\theta}{14} + \frac{\sin \theta}{2} + C \end{aligned}$$

$$\begin{aligned} 50. \int \sin(-4x) \cos 3x \, dx &= - \int \sin 4x \cos 3x \, dx \\ &= -\frac{1}{2} \int (\sin x + \sin 7x) \, dx \\ &= -\frac{1}{2} \left[-\cos x - \frac{1}{7} \cos 7x \right] + C \\ &= \frac{1}{14} [7 \cos x + \cos 7x] + C \end{aligned}$$

$$52. \text{ Let } u = \tan \frac{x}{2}, \, du = \frac{1}{2} \sec^2 \frac{x}{2} \, dx.$$

$$\begin{aligned} \int \tan^4 \frac{x}{2} \sec^4 \frac{x}{2} \, dx &= \int \tan^4 \frac{x}{2} \left(\tan^2 \frac{x}{2} + 1 \right) \sec^2 \frac{x}{2} \, dx \\ &= 2 \int \left(\tan^6 \frac{x}{2} + \tan^4 \frac{x}{2} \right) \left(\frac{1}{2} \sec^2 \frac{x}{2} \right) dx \\ &= \frac{2}{7} \tan^7 \frac{x}{2} + \frac{2}{5} \tan^5 \frac{x}{2} + C \end{aligned}$$

$$\begin{aligned} 54. \, u = \cot 3x, \, du = -3 \csc^2 3x \, dx \\ \int \csc^2 3x \cot 3x \, dx &= -\frac{1}{3} \int \cot 3x (-3 \csc^2 3x) \, dx \\ &= -\frac{1}{6} \cot^2 3x + C \end{aligned}$$

$$\begin{aligned} 56. \int \frac{\cot^3 t}{\csc t} \, dt &= \int \frac{\cos^3 t}{\sin^2 t} \, dt = \int \frac{(1 - \sin^2 t) \cos t}{\sin^2 t} \, dt \\ &= \int \frac{\cos t}{\sin^2 t} \, dt - \int \cos t \, dt \\ &= \frac{-1}{\sin t} - \sin t + C \\ &= -\csc t - \sin t + C \end{aligned}$$

$$58. \int \frac{\sin^2 x - \cos^2 x}{\cos x} \, dx = \int \frac{1 - 2 \cos^2 x}{\cos x} \, dx = \int (\sec x - 2 \cos x) \, dx = \ln |\sec x + \tan x| - 2 \sin x + C$$

$$\begin{aligned} 60. \int \frac{1 - \sec t}{\cos t - 1} \, dt &= \int \frac{\cos t - 1}{(\cos t - 1) \cos t} \, dt \\ &= \int \sec t \, dt = \ln |\sec t + \tan t| + C \end{aligned}$$

$$\begin{aligned} 62. \int_0^{\pi/3} \tan^2 x \, dx &= \int_0^{\pi/3} (\sec^2 x - 1) \, dx \\ &= \left[\tan x - x \right]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3} \end{aligned}$$

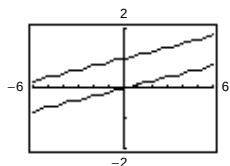
$$64. \text{ Let } u = \tan t, \, du = \sec^2 t \, dt.$$

$$\int_0^{\pi/4} \sec^2 t \sqrt{\tan t} \, dt = \left[\frac{2}{3} \tan^{3/2} t \right]_0^{\pi/4} = \frac{2}{3}$$

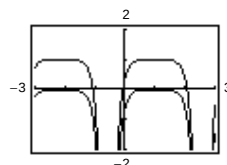
$$\begin{aligned} 66. \int_{-\pi}^{\pi} \sin 3\theta \cos \theta \, d\theta &= \frac{1}{2} \int_{-\pi}^{\pi} (\sin 4\theta + \sin 2\theta) \, d\theta \\ &= -\frac{1}{2} \left[\frac{1}{4} \cos 4\theta + \frac{1}{2} \cos 2\theta \right]_{-\pi}^{\pi} = 0 \end{aligned}$$

$$\begin{aligned}
 68. \int_{-\pi/2}^{\pi/2} (\sin^2 x + 1) dx &= \int_{-\pi/2}^{\pi/2} \left(\frac{1 - \cos 2x}{2} + 1 \right) dx \\
 &= \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} - \frac{1}{2} \cos 2x \right) dx = \left[\frac{3}{2}x - \frac{1}{4} \sin 2x \right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}
 \end{aligned}$$

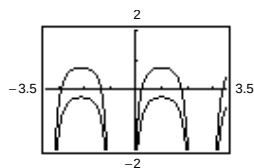
$$70. \int \sin^2 x \cos^2 x dx = \frac{1}{32} [4x - \sin 4x] + C$$



$$72. \int \tan^3(1-x) dx = -\frac{\tan^2(1-x)}{2} - \ln|\cos(1-x)| + C$$



$$74. \int \sec^4(1-x) \tan(1-x) dx = -\frac{\sec^4(1-x)}{4} + C$$



$$\begin{aligned}
 76. \int_0^{\pi/2} (1 - \cos \theta)^2 d\theta &= \left[\frac{3}{2}\theta - 2\sin \theta + \frac{1}{4}\sin 2\theta \right]_0^{\pi/2} \\
 &= \frac{3\pi}{4} - 2
 \end{aligned}$$

$$78. \int_0^{\pi/2} \sin^6 x dx = \frac{1}{8} \left[\frac{5x}{2} - 2\sin 2x + \frac{3}{8}\sin 4x + \frac{1}{6}\sin^3 2x \right]_0^{\pi/2} = \frac{5\pi}{32}$$

80. See guidelines on page 500.

82. (a) Let $u = \tan x$, $du = \sec^2 x dx$.

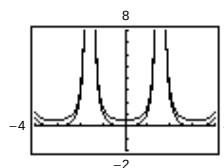
$$\int \sec^2 x \tan x dx = \frac{1}{2} \tan^2 x + C_1$$

Or let $u = \sec x$, $du = \sec x \tan x dx$.

$$\int \sec x (\sec x \tan x) dx = \frac{1}{2} \sec^2 x + C$$

$$(c) \frac{1}{2} \sec^2 x + C = \frac{1}{2} (\tan^2 x + 1) + C = \frac{1}{2} \tan^2 x + \left(\frac{1}{2} + C \right) = \frac{1}{2} \tan^2 x + C_2$$

(b)



84. Disks

$$R(x) = \tan x$$

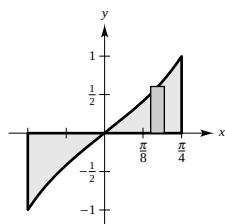
$$r(x) = 0$$

$$V = 2\pi \int_0^{\pi/4} \tan^2 x dx$$

$$= 2\pi \int_0^{\pi/4} (\sec^2 x - 1) dx$$

$$= 2\pi \left[\tan x - x \right]_0^{\pi/4}$$

$$= 2\pi \left(1 - \frac{\pi}{4} \right) \approx 1.348$$



$$86. (a) V = \pi \int_0^{\pi/2} \cos^2 x \, dx = \frac{\pi}{2} \int_0^{\pi/2} (1 + \cos 2x) \, dx = \frac{\pi}{2} \left[x + \frac{1}{2} \sin 2x \right]_0^{\pi/2} = \frac{\pi^2}{4}$$

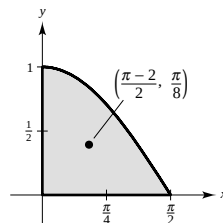
$$(b) A = \int_0^{\pi/2} \cos x \, dx = \left[\sin x \right]_0^{\pi/2} = 1$$

Let $u = x$, $dv = \cos x \, dx$, $du = dx$, $v = \sin x$.

$$\bar{x} = \int_0^{\pi/2} x \cos x \, dx = \left[x \sin x \right]_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx = \left[x \sin x + \cos x \right]_0^{\pi/2} = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$$

$$\begin{aligned} \bar{y} &= \frac{1}{2} \int_0^{\pi/2} \cos^2 x \, dx \\ &= \frac{1}{4} \int_0^{\pi/2} (1 + \cos 2x) \, dx \\ &= \frac{1}{4} \left[x + \frac{1}{2} \sin 2x \right]_0^{\pi/2} = \frac{\pi}{8} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(\frac{\pi - 2}{2}, \frac{\pi}{8} \right)$$



$$88. dv = \cos x \, dx \Rightarrow v = \sin x$$

$$u = \cos^{n-1} x \Rightarrow du = -(n-1) \cos^{n-2} x \sin x \, dx$$

$$\begin{aligned} \int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x \, dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) \, dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx \end{aligned}$$

$$\text{Therefore, } n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx$$

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

$$90. \text{ Let } u = \sec^{n-2} x, du = (n-2) \sec^{n-2} x \tan x \, dx, dv = \sec^2 x \, dx, v = \tan x.$$

$$\begin{aligned} \int \sec^n x \, dx &= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-2} x \tan^2 x \, dx \\ &= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx \\ &= \sec^{n-2} x \tan x - (n-2) \left[\int \sec^n x \, dx - \int \sec^{n-2} x \, dx \right] \end{aligned}$$

$$(n-1) \int \sec^n x \, dx = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x \, dx$$

$$\int \sec^n x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

$$92. \int \cos^4 x \, dx = \frac{\cos^3 x \sin x}{4} + \frac{3}{4} \int \cos^2 x \, dx = \frac{\cos^3 x \sin x}{4} + \frac{3}{4} \left[\frac{\cos x \sin x}{2} + \frac{1}{2} \int dx \right]$$

$$= \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} \cos x \sin x + \frac{3}{8} x + C = \frac{1}{8} [2 \cos^3 x \sin x + 3 \cos x \sin x + 3x] + C$$

$$\begin{aligned}
94. \int \sin^4 x \cos^2 x \, dx &= -\frac{\cos^3 x \sin^3 x}{6} + \frac{1}{2} \int \cos^2 x \sin^2 x \, dx \\
&= -\frac{\cos^3 x \sin^3 x}{6} + \frac{1}{2} \left[-\frac{\cos^3 x \sin x}{4} + \frac{1}{4} \int \cos^2 x \, dx \right] \\
&= -\frac{1}{6} \cos^3 x \sin^3 x - \frac{1}{8} \cos^3 x \sin x + \frac{1}{8} \left[\frac{\cos x \sin x}{2} + \frac{x}{2} \right] + C \\
&= -\frac{1}{48} [8 \cos^3 x \sin^3 x + 6 \cos^3 x \sin x - 3 \cos x \sin x - 3x] + C
\end{aligned}$$

96. (a) n is odd and $n \geq 3$.

$$\begin{aligned}
\int_0^{\pi/2} \cos^n x \, dx &= \left[\frac{\cos^{n-1} x \sin x}{n} \right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x \, dx \\
&= \frac{n-1}{n} \left[\left[\frac{\cos^{n-3} x \sin x}{n-2} \right]_0^{\pi/2} + \frac{n-3}{n-2} \int_0^{\pi/2} \cos^{n-4} x \, dx \right] \\
&= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \left[\left[\frac{\cos^{n-5} x \sin x}{n-4} \right]_0^{\pi/2} + \frac{n-5}{n-4} \int_0^{\pi/2} \cos^{n-6} x \, dx \right] \\
&= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \int_0^{\pi/2} \cos^{n-6} x \, dx \\
&= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \int_0^{\pi/2} \cos x \, dx \\
&= \left[\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots (\sin x) \right]_0^{\pi/2} \\
&= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots 1 \quad (\text{Reverse the order}) \\
&= (1) \left(\frac{2}{3} \right) \left(\frac{4}{5} \right) \left(\frac{6}{7} \right) \cdots \left(\frac{n-1}{n} \right) \\
&= \left(\frac{2}{3} \right) \left(\frac{4}{5} \right) \left(\frac{6}{7} \right) \cdots \left(\frac{n-1}{n} \right)
\end{aligned}$$

(b) n is even and $n \geq 2$.

$$\begin{aligned}
\int_0^{\pi/2} \cos^n x \, dx &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \int_0^{\pi/2} \cos^2 x \, dx \quad (\text{From part (a).}) \\
&= \left[\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \left(\frac{x}{2} + \frac{1}{4} \sin 2x \right) \right]_0^{\pi/2} \\
&= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{\pi}{4} \quad (\text{Reverse the order}) \\
&= \left(\frac{\pi}{2} \cdot \frac{1}{2} \right) \left(\frac{3}{4} \right) \left(\frac{5}{6} \right) \cdots \left(\frac{n-1}{n} \right) \\
&= \left(\frac{1}{2} \right) \left(\frac{3}{4} \right) \left(\frac{5}{6} \right) \cdots \left(\frac{n-1}{n} \right) \left(\frac{\pi}{2} \right)
\end{aligned}$$

Section 7.4 Trigonometric Substitution

$$\begin{aligned}
2. \frac{d}{dx} \left[8 \ln |\sqrt{x^2 - 16} + x| + \frac{1}{2} x \sqrt{x^2 - 16} + C \right] &= 8 \left[\frac{(x/\sqrt{x^2 - 16}) + 1}{\sqrt{x^2 - 16} + x} \right] + \frac{1}{2} x \left(\frac{x}{\sqrt{x^2 - 16}} \right) + \frac{1}{2} \sqrt{x^2 - 16} \\
&= \frac{8(x + \sqrt{x^2 - 16})}{\sqrt{x^2 - 16}(\sqrt{x^2 - 16} + x)} + \frac{x^2}{2\sqrt{x^2 - 16}} + \frac{\sqrt{x^2 - 16}}{2} \\
&= \frac{16 + x^2 + x^2 - 16}{2\sqrt{x^2 - 16}} \\
&= \frac{x^2}{\sqrt{x^2 - 16}}
\end{aligned}$$

Indefinite integral: $\int \frac{x^2}{\sqrt{x^2 - 16}} dx$ Matches (d)

$$\begin{aligned}
4. \frac{d}{dx} \left[8 \arcsin \frac{x-3}{4} + \frac{(x-3)\sqrt{7+6x-x^2}}{2} + C \right] &= 8 \left[\frac{1}{\sqrt{1 - [(x-3)/4]^2}} \cdot \frac{1}{4} \right] + \frac{1}{2} (x-3) \frac{3-x}{\sqrt{7+6x-x^2}} + \frac{1}{2} \sqrt{7+6x-x^2} \\
&= \frac{8}{\sqrt{16 - (x-3)^2}} - \frac{(x-3)^2}{2\sqrt{16 - (x-3)^2}} + \frac{\sqrt{16 - (x-3)^2}}{2} \\
&= \frac{16 - (x^2 - 6x + 9) + 16 - (x^2 - 6x + 9)}{2\sqrt{16 - (x-3)^2}} \\
&= \frac{2[16 - (x-3)^2]}{2\sqrt{16 - (x-3)^2}} \\
&= \sqrt{16 - (x-3)^2} \\
&= \sqrt{7+6x-x^2}
\end{aligned}$$

Indefinite integral: $\int \sqrt{7+6x-x^2} dx$ Matches (c)

6. Same substitution as in Exercise 5.

$$\int \frac{10}{x^2 \sqrt{25 - x^2}} dx = 10 \int \frac{5 \cos \theta d\theta}{(25 \sin^2 \theta)(5 \cos \theta)} = \frac{2}{5} \int \csc^2 \theta d\theta = -\frac{2}{5} \cot \theta + C = \frac{-2\sqrt{25 - x^2}}{5x} + C$$

8. Same substitution as in Exercise 5

$$\begin{aligned}
\int \frac{x^2}{\sqrt{25 - x^2}} dx &= \int \frac{25 \sin^2 \theta}{5 \cos \theta} (5 \cos \theta) d\theta = \frac{25}{2} \int (1 - \cos 2\theta) d\theta \\
&= \frac{25}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right) + C = \frac{25}{2} (\theta - \sin \theta \cos \theta) + C \\
&= \frac{25}{2} \left[\arcsin \left(\frac{x}{5} \right) - \left(\frac{x}{5} \right) \left(\frac{\sqrt{25 - x^2}}{5} \right) \right] + C = \frac{1}{2} \left[25 \arcsin \left(\frac{x}{5} \right) - x \sqrt{25 - x^2} \right] + C
\end{aligned}$$

10. Same substitution as in Exercise 9

$$\begin{aligned}
\int \frac{\sqrt{x^2 - 4}}{x} dx &= \int \frac{2 \tan \theta}{2 \sec \theta} (2 \sec \theta \tan \theta) d\theta = 2 \int \tan^2 \theta d\theta = 2 \int (\sec^2 \theta - 1) d\theta \\
&= 2(\tan \theta - \theta) + C = 2 \left[\frac{\sqrt{x^2 - 4}}{2} - \operatorname{arcsec} \left(\frac{x}{2} \right) \right] + C = \sqrt{x^2 - 4} - 2 \operatorname{arcsec} \left(\frac{x}{2} \right) + C
\end{aligned}$$

12. Same substitution as in Exercise 9

$$\begin{aligned}
 \int \frac{x^3}{\sqrt{x^2-4}} dx &= \int \frac{8 \sec^3 \theta}{2 \tan \theta} (2 \sec \theta \tan \theta) d\theta = 8 \int \sec^4 \theta d\theta \\
 &= 8 \int (1 + \tan^2 \theta) \sec^2 \theta d\theta = 8 \left(\tan \theta + \frac{\tan^3 \theta}{3} \right) + C = \frac{8}{3} \tan \theta (3 + \tan^2 \theta) + C \\
 &= \frac{8}{3} \left(\frac{\sqrt{x^2-4}}{2} \right) \left(3 + \frac{x^2-4}{4} \right) + C = \frac{1}{3} \sqrt{x^2-4} (12 + x^2 - 4) + C = \frac{1}{3} \sqrt{x^2-4} (x^2 + 8) + C
 \end{aligned}$$

14. Same substitution as in Exercise 13.

$$\begin{aligned}
 \int \frac{9x^3}{\sqrt{1+x^2}} dx &= 9 \int \frac{\tan^3 \theta}{\sec \theta} \sec^2 \theta d\theta = 9 \int (\sec^2 \theta - 1) \sec \theta \tan \theta d\theta = 9 \left[\frac{\sec^3 \theta}{3} - \sec \theta \right] + C \\
 &= 3 \sec \theta (\sec^2 \theta - 3) + C = 3 \sqrt{1+x^2} [(1+x^2) - 3] + C = 3 \sqrt{1+x^2} (x^2 - 2) + C
 \end{aligned}$$

16. Same substitution as in Exercise 13

$$\begin{aligned}
 \int \frac{x^2}{(1+x^2)^2} dx &= \int \frac{x^2}{(\sqrt{1+x^2})^4} dx = \int \frac{\tan^2 \theta \sec^2 \theta d\theta}{\sec^4 \theta} = \int \sin^2 \theta d\theta \\
 &= \frac{1}{2} \int (1 - \cos 2\theta) d\theta = \frac{1}{2} \left[\theta - \frac{\sin 2\theta}{2} \right] = \frac{1}{2} [\theta - \sin \theta \cos \theta] + C \\
 &= \frac{1}{2} \left[\arctan x - \left(\frac{x}{\sqrt{1+x^2}} \right) \left(\frac{1}{\sqrt{1+x^2}} \right) \right] + C = \frac{1}{2} \left[\arctan x - \frac{x}{1+x^2} \right] + C
 \end{aligned}$$

18. Let $u = x$, $a = 1$, and $du = dx$.

$$\int \sqrt{1+x^2} dx = \frac{1}{2} (x\sqrt{1+x^2} + \ln|x + \sqrt{1+x^2}|) + C$$

$$\begin{aligned}
 20. \int \frac{x}{\sqrt{9-x^2}} dx &= -\frac{1}{2} \int (9-x^2)^{-1/2} (-2x) dx \\
 &= -(9-x^2)^{1/2} + C \quad (\text{Power Rule})
 \end{aligned}$$

$$22. \int \frac{1}{\sqrt{25-x^2}} dx = \arcsin \frac{x}{5} + C$$

24. Let $u = 16 - 4x^2$, $du = -8x dx$.

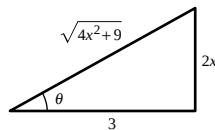
$$\int x\sqrt{16-4x^2} dx = -\frac{1}{8} \int (16-4x^2)^{1/2} (-8x) dx = \left[-\frac{1}{12} (16-4x^2)^{3/2} \right] + C = -\frac{2}{3} (4-x^2)^{3/2} + C$$

26. Let $u = 1 - t^2$, $du = -2t dt$.

$$\int \frac{t}{(1-t^2)^{3/2}} dt = -\frac{1}{2} \int (1-t^2)^{-3/2} (-2t) dt = \frac{1}{\sqrt{1-t^2}} + C$$

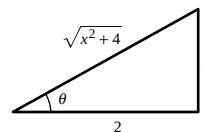
28. Let $2x = 3 \tan \theta$, $dx = \frac{3}{2} \sec^2 \theta d\theta$, $\sqrt{4x^2+9} = 3 \sec \theta$.

$$\begin{aligned}
 \int \frac{\sqrt{4x^2+9}}{x^4} dx &= \int \frac{3 \sec \theta [(3/2) \sec^2 \theta d\theta]}{(3/2)^4 \tan^4 \theta} \\
 &= \frac{8}{9} \int \frac{\cos \theta}{\sin^4 \theta} d\theta \\
 &= \frac{-8}{27 \sin^3 \theta} + C \\
 &= -\frac{8}{27} \csc^3 \theta + C \\
 &= \frac{-(4x^2+9)^{3/2}}{27x^3} + C
 \end{aligned}$$



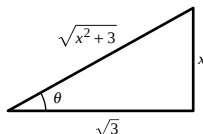
30. Let $2x = 4 \tan \theta$, $dx = 2 \sec^2 \theta d\theta$, $\sqrt{4x^2 + 16} = 4 \sec \theta$.

$$\begin{aligned} \int \frac{1}{x\sqrt{4x^2 + 16}} dx &= \int \frac{2 \sec^2 \theta d\theta}{2 \tan \theta (4 \sec \theta)} \\ &= \frac{1}{4} \int \frac{\sec \theta}{\tan \theta} d\theta = \frac{1}{4} \int \csc \theta d\theta \\ &= -\frac{1}{4} \ln |\csc \theta + \cot \theta| + C = -\frac{1}{4} \ln \left| \frac{\sqrt{x^2 + 4} + 2}{x} \right| + C \end{aligned}$$



32. Let $x = \sqrt{3} \tan \theta$, $dx = \sqrt{3} \sec^2 \theta d\theta$, $x^2 + 3 = 3 \sec^2 \theta$.

$$\begin{aligned} \int \frac{1}{(x^2 + 3)^{3/2}} dx &= \int \frac{\sqrt{3} \sec^2 \theta d\theta}{3 \sqrt{3} \sec^3 \theta} \\ &= \frac{1}{3} \int \cos \theta d\theta \\ &= \frac{1}{3} \sin \theta + C \\ &= \frac{x}{3\sqrt{x^2 + 3}} + C \end{aligned}$$

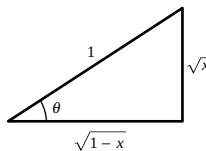


34. Let $u = x^2 + 2x + 2$, $du = (2x + 2) dx$.

$$\int (x + 1) \sqrt{x^2 + 2x + 2} dx = \frac{1}{2} \int (x^2 + 2x + 2)^{1/2} (2x + 2) dx = \frac{1}{3} (x^2 + 2x + 2)^{3/2} + C$$

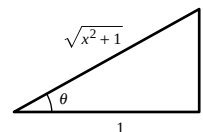
36. Let $\sqrt{x} = \sin \theta$, $x = \sin^2 \theta$, $dx = 2 \sin \theta \cos \theta d\theta$, $\sqrt{1 - x} = \cos \theta$.

$$\begin{aligned} \int \frac{\sqrt{1 - x}}{\sqrt{x}} dx &= \int \frac{\cos \theta (2 \sin \theta \cos \theta d\theta)}{\sin \theta} \\ &= 2 \int \cos^2 \theta d\theta \\ &= \int (1 + \cos 2\theta) d\theta \\ &= (\theta + \sin \theta \cos \theta) + C \\ &= \arcsin \sqrt{x} + \sqrt{x} \sqrt{1 - x} + C \end{aligned}$$



38. Let $x = \tan \theta$, $dx = \sec^2 \theta d\theta$, $x^2 + 1 = \sec^2 \theta$.

$$\begin{aligned} \int \frac{x^3 + x + 1}{x^4 + 2x^2 + 1} dx &= \frac{1}{4} \int \frac{4x^3 + 4x}{x^4 + 2x^2 + 1} dx + \int \frac{1}{(x^2 + 1)^2} dx \\ &= \frac{1}{4} \ln(x^4 + 2x^2 + 1) + \int \frac{\sec^2 \theta d\theta}{\sec^4 \theta} \\ &= \frac{1}{2} \ln(x^2 + 1) + \frac{1}{2} \int (1 + \cos 2\theta) d\theta \\ &= \frac{1}{2} \ln(x^2 + 1) + \frac{1}{2} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{1}{2} \left[\ln(x^2 + 1) + \arctan x + \frac{x}{x^2 + 1} \right] + C \end{aligned}$$



40. $u = \arcsin x, \Rightarrow du = \frac{1}{\sqrt{1-x^2}} dx, dv = x dx \Rightarrow v = \frac{x^2}{2}$

$$\int x \arcsin x dx = \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} dx$$

$$x = \sin \theta, dx = \cos \theta d\theta, \sqrt{1-x^2} = \cos \theta$$

$$\int x \arcsin x dx = \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{\sin^2 \theta}{\cos \theta} \cos \theta d\theta = \frac{x^2}{2} \arcsin x - \frac{1}{4} \int (1 - \cos 2\theta) d\theta$$

$$= \frac{x^2}{2} \arcsin x - \frac{1}{4} \left[\theta - \frac{1}{2} \sin 2\theta \right] + C = \frac{x^2}{2} \arcsin x - \frac{1}{4} [\theta - \sin \theta \cos \theta] + C$$

$$= \frac{x^2}{2} \arcsin x - \frac{1}{4} [\arcsin x - x\sqrt{1-x^2}] + C = \frac{1}{4} [(2x^2 - 1) \arcsin x + x\sqrt{1-x^2}] + C$$

42. Let $x - 1 = \sin \theta, dx = \cos \theta d\theta, \sqrt{1 - (x - 1)^2} = \sqrt{2x - x^2} = \cos \theta$.

$$\int \frac{x^2}{\sqrt{2x - x^2}} dx = \int \frac{x^2}{\sqrt{1 - (x - 1)^2}} dx$$

$$= \int \frac{(1 + \sin \theta)^2 (\cos \theta d\theta)}{\cos \theta}$$

$$= \int (1 + 2 \sin \theta + \sin^2 \theta) d\theta$$

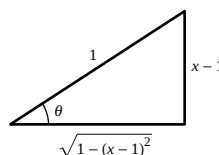
$$= \int \left(\frac{3}{2} + 2 \sin \theta - \frac{1}{2} \cos 2\theta \right) d\theta$$

$$= \frac{3}{2} \theta - 2 \cos \theta - \frac{1}{4} \sin 2\theta + C$$

$$= \frac{3}{2} \theta - 2 \cos \theta - \frac{1}{2} \sin \theta \cos \theta + C$$

$$= \frac{3}{2} \arcsin(x - 1) - 2\sqrt{2x - x^2} - \frac{1}{2}(x - 1)\sqrt{2x - x^2} + C$$

$$= \frac{3}{2} \arcsin(x - 1) - \frac{1}{2} \sqrt{2x - x^2} (x + 3) + C$$



44. Let $x - 3 = 2 \sec \theta, dx = 2 \sec \theta \tan \theta d\theta, \sqrt{(x - 3)^2 - 4} = 2 \tan \theta$.

$$\int \frac{x}{\sqrt{x^2 - 6x + 5}} dx = \int \frac{x}{\sqrt{(x - 3)^2 - 4}} dx = \int \frac{(2 \sec \theta + 3)}{2 \tan \theta} (2 \sec \theta \tan \theta) d\theta$$

$$= \int (2 \sec^2 \theta + 3 \sec \theta) d\theta$$

$$= 2 \tan \theta + 3 \ln |\sec \theta + \tan \theta| + C_1$$

$$= 2 \left(\frac{\sqrt{(x - 3)^2 - 4}}{2} \right) + 3 \ln \left| \frac{x - 3}{2} + \frac{\sqrt{(x - 3)^2 - 4}}{2} \right| + C_1$$

$$= \sqrt{x^2 - 6x + 5} + 3 \ln |(x - 3) + \sqrt{x^2 - 6x + 5}| + C$$

46. Same substitution as in Exercise 45

$$\begin{aligned}
 \text{(a)} \quad \int \frac{1}{(1-t^2)^{5/2}} dt &= \int \frac{\cos \theta d\theta}{\cos^5 \theta} = \int \sec^4 \theta d\theta = \int (\tan^2 \theta + 1) \sec^2 \theta d\theta \\
 &= \frac{1}{3} \tan^3 \theta + \tan \theta + C = \frac{1}{3} \left(\frac{t}{\sqrt{1-t^2}} \right)^3 + \frac{t}{\sqrt{1-t^2}} + C \\
 \text{Thus, } \int_0^{\sqrt{3}/2} \frac{1}{(1-t^2)^{5/2}} dt &= \left[\frac{t^3}{3(1-t^2)^{3/2}} + \frac{t}{\sqrt{1-t^2}} \right]_0^{\sqrt{3}/2} \\
 &= \frac{3\sqrt{3}/8}{3(1/4)^{3/2}} + \frac{\sqrt{3}/2}{\sqrt{1/4}} = \sqrt{3} + \sqrt{3} = 2\sqrt{3} \approx 3.464.
 \end{aligned}$$

(b) When $t = 0$, $\theta = 0$. When $t = \sqrt{3}/2$, $\theta = \pi/3$. Thus,

$$\int_0^{\sqrt{3}/2} \frac{1}{(1-t^2)^{5/2}} dt = \left[\frac{1}{3} \tan^3 \theta + \tan \theta \right]_0^{\pi/3} = \frac{1}{3} (\sqrt{3})^3 + \sqrt{3} = 2\sqrt{3} \approx 3.464.$$

48. (a) Let $5x = 3 \sin \theta$, $dx = \frac{3}{5} \cos \theta d\theta$, $\sqrt{9-25x^2} = 3 \cos \theta$.

$$\begin{aligned}
 \int \sqrt{9-25x^2} dx &= \int (3 \cos \theta) \frac{3}{5} \cos \theta d\theta \\
 &= \frac{9}{5} \int \frac{1 + \cos 2\theta}{2} d\theta \\
 &= \frac{9}{10} \left[\theta + \frac{1}{2} \sin 2\theta \right] + C \\
 &= \frac{9}{10} [\theta + \sin \theta \cos \theta] + C \\
 &= \frac{9}{10} \left[\arcsin \frac{5x}{3} + \frac{5x}{3} \cdot \frac{\sqrt{9-25x^2}}{3} \right] + C \\
 \text{Thus, } \int_0^{3/5} \sqrt{9-25x^2} dx &= \frac{9}{10} \left[\arcsin \frac{5x}{3} + \frac{5x\sqrt{9-25x^2}}{9} \right]_0^{3/5} = \frac{9}{10} \left[\frac{\pi}{2} \right] = \frac{9\pi}{20}.
 \end{aligned}$$

(b) When $x = 0$, $\theta = 0$. When $x = \frac{3}{5}$, $\theta = \frac{\pi}{2}$.

$$\text{Thus, } \int_0^{3/5} \sqrt{9-25x^2} dx = \left[\frac{9}{10} (\theta + \sin \theta \cos \theta) \right]_0^{\pi/2} = \frac{9}{10} \left(\frac{\pi}{2} \right) = \frac{9\pi}{20}.$$

50. (a) Let $x = 3 \sec \theta$, $dx = 3 \sec \theta \tan \theta d\theta$, $\sqrt{x^2-9} = 3 \tan \theta$.

$$\begin{aligned}
 \int \frac{\sqrt{x^2-9}}{x^2} dx &= \int \frac{3 \tan \theta}{9 \sec^2 \theta} 3 \sec \theta \tan \theta d\theta \\
 &= \int \frac{\tan^2 \theta}{\sec \theta} d\theta \\
 &= \int \frac{\sin^2 \theta}{\cos \theta} d\theta \\
 &= \int \frac{1 - \cos^2 \theta}{\cos \theta} d\theta \\
 &= \int (\sec \theta - \cos \theta) d\theta \\
 &= \ln |\sec \theta + \tan \theta| - \sin \theta + C \\
 &= \ln \left| \frac{x}{3} + \frac{\sqrt{x^2-9}}{3} \right| - \frac{\sqrt{x^2-9}}{x} + C
 \end{aligned}$$

—CONTINUED—

50. —CONTINUED—

$$\text{Hence, } \int_3^6 \frac{\sqrt{x^2 - 9}}{x^2} dx = \left[\ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| - \frac{\sqrt{x^2 - 9}}{x} \right]_3^6 = \ln|2 + \sqrt{3}| - \frac{\sqrt{3}}{2}.$$

(b) When $x = 3$, $\theta = 0$; when $x = 6$, $\theta = \frac{\pi}{3}$.

$$\text{Hence, } \int_3^6 \frac{\sqrt{x^2 - 9}}{x^2} dx = \left[\ln|\sec \theta + \tan \theta| - \sin \theta \right]_0^{\pi/3} = \ln|2 + \sqrt{3}| - \frac{\sqrt{3}}{2}.$$

$$52. \int (x^2 + 2x + 11)^{3/2} dx = \frac{1}{4}(x + 1)(x^2 + 2x + 26)\sqrt{x^2 + 2x + 11} + \frac{75}{2} \ln|\sqrt{x^2 + 2x + 11} + (x + 1)| + C$$

$$54. \int x^2 \sqrt{x^2 - 4} dx = \frac{1}{4}x^3 \sqrt{x^2 - 4} - \frac{1}{2}x \sqrt{x^2 - 4} - 2 \ln|x + \sqrt{x^2 - 4}| + C$$

56. (a) Substitution: $u = x^2 + 1$, $du = 2x dx$

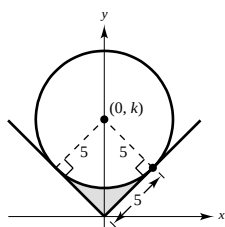
(b) Trigonometric substitution: $x = \sec \theta$

$$58. (a) x^2 + (y - k)^2 = 25$$

Radius of circle = 5

$$k^2 = 5^2 + 5^2 = 50$$

$$k = 5\sqrt{2}$$



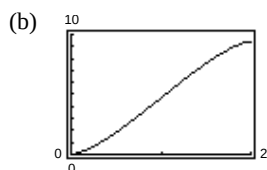
(b) Area = square $- \frac{1}{4}(\text{circle})$

$$= 25 - \frac{1}{4}\pi(5)^2 = 25\left(1 - \frac{\pi}{4}\right)$$

$$(c) \text{ Area} = r^2 - \frac{1}{4}\pi r^2 = r^2\left(1 - \frac{\pi}{4}\right)$$

60. (a) Place the center of the circle at $(0, 1)$; $x^2 + (y - 1)^2 = 1$. The depth d satisfies $0 \leq d \leq 2$. The volume is

$$\begin{aligned} V &= 3 \cdot 2 \int_0^d \sqrt{1 - (y - 1)^2} dy \\ &= 6 \cdot \frac{1}{2} \left[\arcsin(y - 1) + (y - 1)\sqrt{1 - (y - 1)^2} \right]_0^d \quad (\text{Theorem 7.2 (1)}) \\ &= 3[\arcsin(d - 1) + (d - 1)\sqrt{1 - (d - 1)^2} - \arcsin(-1)] \\ &= \frac{3\pi}{2} + 3 \arcsin(d - 1) + 3(d - 1)\sqrt{2d - d^2}. \end{aligned}$$



(c) The full tank holds $3\pi \approx 9.4248$ cubic meters. The horizontal lines

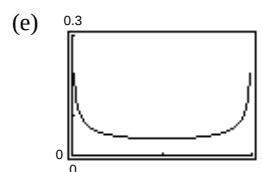
$$y = \frac{3\pi}{4}, y = \frac{3\pi}{2}, y = \frac{9\pi}{4}$$

intersect the curve at $d = 0.596, 1.0, 1.404$. The dipstick would have these markings on it.

$$(d) V = 6 \int_0^d \sqrt{1 - (y - 1)^2} dy$$

$$\frac{dV}{dt} = \frac{dV}{dd} \cdot \frac{dd}{dt} = 6\sqrt{1 - (d - 1)^2} \cdot d'(t) = \frac{1}{4}$$

$$\Rightarrow d'(t) = \frac{1}{24\sqrt{1 - (d - 1)^2}}$$

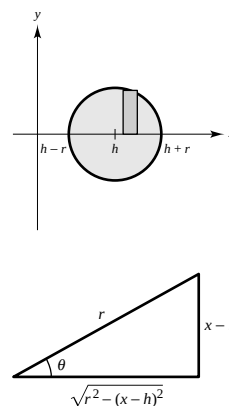


The minimum occurs at $d = 1$, which is the widest part of the tank.

62. Let $x - h = r \sin \theta$, $dx = r \cos \theta d\theta$, $\sqrt{r^2 - (x - h)^2} = r \cos \theta$.

Shell Method:

$$\begin{aligned} V &= 4\pi \int_{h-r}^{h+r} x \sqrt{r^2 - (x-h)^2} dx \\ &= 4\pi \int_{-\pi/2}^{\pi/2} (h + r \sin \theta) r \cos \theta (r \cos \theta) d\theta = 4\pi r^2 \int_{-\pi/2}^{\pi/2} (h + r \sin \theta) \cos^2 \theta d\theta \\ &= 4\pi r^2 \left[\frac{h}{2} \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) d\theta + r \int_{-\pi/2}^{\pi/2} \sin \theta \cos^2 \theta d\theta \right] \\ &= 2\pi r^2 h \left[\theta + \frac{1}{2} \sin 2\theta \right]_{-\pi/2}^{\pi/2} - \left[4\pi r^3 \left(\frac{\cos^3 \theta}{3} \right) \right]_{-\pi/2}^{\pi/2} = 2\pi^2 r^2 h \end{aligned}$$



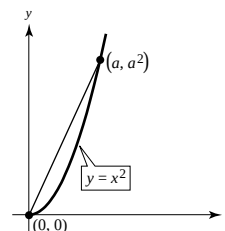
64. $y = \frac{1}{2}x^2$, $y' = x$, $1 + (y')^2 = 1 + x^2$

$$\begin{aligned} s &= \int_0^4 \sqrt{1 + x^2} dx = \left[\frac{1}{2} (x\sqrt{x^2 + 1} + \ln|x + \sqrt{x^2 + 1}|) \right]_0^4 \quad (\text{Theorem 7.2}) \\ &= \frac{1}{2} [4\sqrt{17} + \ln(4 + \sqrt{17})] \approx 9.2936 \end{aligned}$$

66. (a) Along line: $d_1 = \sqrt{a^2 + a^4} = a\sqrt{1 + a^2}$

Along parabola: $y = x^2$, $y' = 2x$

$$\begin{aligned} d_2 &= \int_0^a \sqrt{1 + 4x^2} dx \\ &= \frac{1}{4} \left[2x\sqrt{4x^2 + 1} + \ln|2x + \sqrt{4x^2 + 1}| \right]_0^a \quad (\text{Theorem 7.2}) \\ &= \frac{1}{4} [2a\sqrt{4a^2 + 1} + \ln(2a + \sqrt{4a^2 + 1})] \end{aligned}$$



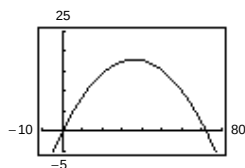
(b) For $a = 1$, $d_1 = \sqrt{2}$ and $d_2 = \frac{\sqrt{5}}{2} + \frac{1}{4} \ln(2 + \sqrt{5}) \approx 1.4789$.

For $a = 10$, $d_1 = 10\sqrt{101} \approx 100.4988$

$$d_2 \approx 101.0473.$$

(c) As a increases, $d_2 - d_1 \rightarrow 0$.

68. (a)



(b) $y = 0$ for $x = 72$

(c) $y = x - \frac{x^2}{72}$, $y' = 1 - \frac{x}{36}$, $1 + (y')^2 = 1 + \left(1 - \frac{x}{36}\right)^2$

$$\begin{aligned} s &= \int_0^{72} \sqrt{1 + \left(1 - \frac{x}{36}\right)^2} dx = -36 \int_0^{72} \sqrt{1 + \left(1 - \frac{x}{36}\right)^2} \left(-\frac{1}{36}\right) dx \\ &= -\frac{36}{2} \left[\left(1 - \frac{x}{36}\right) \sqrt{1 + \left(1 - \frac{x}{36}\right)^2} + \ln \left| \left(1 - \frac{x}{36}\right) + \sqrt{1 + \left(1 - \frac{x}{36}\right)^2} \right| \right]_0^{72} \\ &= -18 [(-\sqrt{2} + \ln|-1 + \sqrt{2}|) - (\sqrt{2} + \ln|1 + \sqrt{2}|)] = 36\sqrt{2} + 18 \ln \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right) \approx 82.641 \end{aligned}$$

70. First find where the curves intersect.

$$y^2 = 16 - (x - 4)^2 = \frac{1}{16}x^4$$

$$16^2 - 16(x - 4)^2 = x^4$$

$$16^2 - 16x^2 + 128x - 16^2 = x^4$$

$$x^4 + 16x^2 - 128x = 0$$

$$x(x - 4)(x^2 + 4x + 32) \Rightarrow x = 0, 4$$

$$A = \int_0^4 \frac{1}{4}x^2 dx + \frac{1}{4}\pi(4)^2 = \left[\frac{1}{12}x^3\right]_0^4 + 4\pi = \frac{16}{3} + 4\pi$$

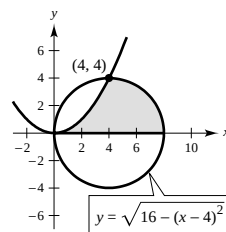
$$\begin{aligned} M_y &= \int_0^4 x \left[\frac{1}{4}x^2 \right] dx + \int_4^8 x \sqrt{16 - (x - 4)^2} dx + \int_4^8 4 \sqrt{16 - (x - 4)^2} dx \\ &= \left[\frac{x^4}{16} \right]_0^4 + \int_4^8 (x - 4) \sqrt{16 - (x - 4)^2} dx + \int_4^8 4 \sqrt{16 - (x - 4)^2} dx \\ &= 16 + \left[\frac{-1}{3} (16 - (x - 4)^2)^{3/2} \right]_4^8 + 2 \left[16 \arcsin \frac{x - 4}{4} + (x - 4) \sqrt{16 - (x - 4)^2} \right]_4^8 \\ &= 16 + \frac{1}{3} 16^{3/2} + 2 \left[16 \left(\frac{\pi}{2} \right) \right] = 16 + \frac{64}{3} + 16\pi = \frac{112}{3} + 16\pi \end{aligned}$$

$$\begin{aligned} M_x &= \int_0^4 \frac{1}{2} \left(\frac{1}{4}x^2 \right)^2 dx + \int_4^8 \frac{1}{2} (16 - (x - 4)^2) dx \\ &= \left[\frac{1}{32} \cdot \frac{x^5}{5} \right]_0^4 + \left[8x - \frac{(x - 4)^3}{6} \right]_4^8 \\ &= \frac{32}{5} + \left(64 - \frac{64}{6} \right) - 32 = \frac{416}{15} \end{aligned}$$

$$\bar{x} = \frac{M_y}{A} = \frac{112/3 + 16\pi}{16/3 + 4\pi} = \frac{112 + 48\pi}{16 + 12\pi} = \frac{28 + 12\pi}{4 + 3\pi} \approx 4.89$$

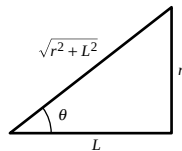
$$\bar{y} = \frac{M_x}{A} = \frac{416/15}{(16/3) + 4\pi} = \frac{104}{5(4 + 3\pi)} \approx 1.55$$

$$(\bar{x}, \bar{y}) \approx (4.89, 1.55)$$



72. Let $r = L \tan \theta$, $dr = L \sec^2 \theta d\theta$, $r^2 + L^2 = L^2 \sec^2 \theta$.

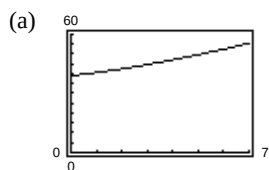
$$\begin{aligned} \frac{1}{R} \int_0^R \frac{2mL}{(r^2 + L^2)^{3/2}} dr &= \frac{2mL}{R} \int_a^b \frac{L \sec^2 \theta d\theta}{L^3 \sec^3 \theta} \\ &= \frac{2m}{RL} \int_a^b \cos \theta d\theta \\ &= \left[\frac{2m}{RL} \sin \theta \right]_a^b \\ &= \left[\frac{2m}{RL} \frac{r}{\sqrt{r^2 + L^2}} \right]_0^R \\ &= \frac{2m}{L \sqrt{R^2 + L^2}} \end{aligned}$$



$$\begin{aligned}
 74. (a) F_{\text{inside}} &= 48 \int_{-1}^{0.8} (0.8 - y)(2)\sqrt{1 - y^2} dy \\
 &= 96 \left[0.8 \int_{-1}^{0.8} \sqrt{1 - y^2} dy - \int_{-1}^{0.8} y\sqrt{1 - y^2} dy \right] \\
 &= 96 \left[\frac{0.8}{2} (\arcsin y + y\sqrt{1 - y^2}) + \frac{1}{3} (1 - y^2)^{3/2} \right]_{-1}^{0.8} \approx 96(1.263) \approx 121.3 \text{ lbs}
 \end{aligned}$$

$$\begin{aligned}
 (b) F_{\text{outside}} &= 64 \int_{-1}^{0.4} (0.4 - y)(2)\sqrt{1 - y^2} dy \\
 &= 128 \left[0.4 \int_{-1}^{0.4} \sqrt{1 - y^2} dy - \int_{-1}^{0.4} y\sqrt{1 - y^2} dy \right] \\
 &= 128 \left[\frac{0.4}{2} (\arcsin y + y\sqrt{1 - y^2}) + \frac{1}{3} (1 - y^2)^{3/2} \right]_{-1}^{0.4} \approx 92.98
 \end{aligned}$$

$$76. S = \sqrt{1520.4 + 111.2t + 15.8t^2}$$



$$(b) S'(t) = \frac{1}{2}(1520.4 + 111.2t + 15.8t^2)^{-1/2}(111.2 + 31.6t)$$

$$S'(5) \approx 2.71$$

$$(c) \text{Average value} = \frac{1}{2} \int_{10}^{12} S(t) dt \approx 68.24$$

78. False

$$\begin{aligned}
 \int \frac{\sqrt{x^2 - 1}}{x} dx &= \int \frac{\tan \theta}{\sec \theta} (\sec \theta \tan \theta d\theta) \\
 &= \int \tan^2 \theta d\theta
 \end{aligned}$$

80. True

$$\int_{-1}^1 x^2 \sqrt{1 - x^2} dx = 2 \int_0^1 x^2 \sqrt{1 - x^2} dx = 2 \int_0^{\pi/2} (\sin^2 \theta)(\cos \theta)(\cos \theta d\theta) = 2 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta$$

Section 7.5 Partial Fractions

$$2. \frac{4x^2 + 3}{(x - 5)^3} = \frac{A}{x - 5} + \frac{B}{(x - 5)^2} + \frac{C}{(x - 5)^3}$$

$$4. \frac{x - 2}{x^2 + 4x + 3} = \frac{x - 2}{(x + 1)(x + 3)} = \frac{A}{x + 1} + \frac{B}{x + 3}$$

$$6. \frac{2x - 1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

$$\begin{aligned}
 8. \frac{1}{4x^2 - 9} &= \frac{1}{(2x - 3)(2x + 3)} = \frac{A}{2x - 3} + \frac{B}{2x + 3} \\
 1 &= A(2x + 3) + B(2x - 3)
 \end{aligned}$$

$$\text{When } x = \frac{3}{2}, 1 = 6A, A = \frac{1}{6}.$$

$$\text{When } x = -\frac{3}{2}, 1 = -6B, B = -\frac{1}{6}.$$

$$\begin{aligned}
 \int \frac{1}{4x^2 - 9} dx &= \frac{1}{6} \left[\int \frac{1}{2x - 3} dx - \int \frac{1}{2x + 3} dx \right] \\
 &= \frac{1}{12} [\ln|2x - 3| - \ln|2x + 3|] + C \\
 &= \frac{1}{12} \ln \left| \frac{2x - 3}{2x + 3} \right| + C
 \end{aligned}$$

$$\begin{aligned}
 10. \int \frac{x + 1}{x^2 + 4x + 3} dx &= \int \frac{(x + 1)}{(x + 1)(x + 3)} dx \\
 &= \int \frac{1}{x + 3} dx = \ln|x + 3| + C
 \end{aligned}$$

$$12. \frac{5x^2 - 12x - 12}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

$$5x^2 - 12x - 12 = A(x^2 - 4) + Bx(x+2) + Cx(x-2)$$

When $x = 0$, $-12 = -4A \Rightarrow A = 3$. When $x = 2$, $-16 = 8B \Rightarrow B = -2$. When $x = -2$, $32 = 8C \Rightarrow C = 4$.

$$\int \frac{5x^2 - 12x - 12}{x^3 - 4x} dx = \int \frac{3}{x} dx + \int \frac{-2}{x-2} dx + \int \frac{4}{x+2} dx = 3 \ln|x| - 2 \ln|x-2| + 4 \ln|x+2| + C$$

$$14. \frac{x^3 - x + 3}{x^2 + x - 2} = x - 1 + \frac{2x + 1}{(x+2)(x-1)} = x - 1 + \frac{A}{x+2} + \frac{B}{x-1}$$

$$2x + 1 = A(x-1) + B(x+2)$$

When $x = -2$, $-3 = -3A$, $A = 1$. When $x = 1$, $3 = 3B$, $B = 1$.

$$\begin{aligned} \int \frac{x^3 - x + 3}{x^2 + x - 2} dx &= \int \left[x - 1 + \frac{1}{x+2} + \frac{1}{x-1} \right] dx \\ &= \frac{x^2}{2} - x + \ln|x+2| + \ln|x-1| + C = \frac{x^2}{2} - x + \ln|x^2 + x - 2| + C \end{aligned}$$

$$16. \frac{x+2}{x(x-4)} = \frac{A}{x-4} + \frac{B}{x}$$

$$x+2 = Ax + B(x-4)$$

When $x = 4$, $6 = 4A$, $A = \frac{3}{2}$.

When $x = 0$, $2 = -4B$, $B = -\frac{1}{2}$.

$$\begin{aligned} \int \frac{x+2}{x^2-4x} dx &= \int \left[\frac{3/2}{x-4} - \frac{1/2}{x} \right] dx \\ &= \frac{3}{2} \ln|x-4| - \frac{1}{2} \ln|x| + C \end{aligned}$$

$$18. \frac{2x-3}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$2x-3 = A(x-1) + B$$

When $x = 1$, $B = -1$. When $x = 0$, $A = 2$.

$$\begin{aligned} \int \frac{2x-3}{(x-1)^2} dx &= \int \left[\frac{2}{x-1} - \frac{1}{(x-1)^2} \right] dx \\ &= 2 \ln|x-1| + \frac{1}{x-1} + C \end{aligned}$$

$$20. \frac{4x^2}{x^3 + x^2 - x - 1} = \frac{4x^2}{x^2(x+1) - (x+1)} = \frac{4x^2}{(x^2-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$4x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

When $x = -1$, $4 = -2C \Rightarrow C = -2$. When $x = 1$, $4 = 4A \Rightarrow A = 1$. When $x = 0$, $0 = 1 - B + 2 \Rightarrow B = 3$.

$$\begin{aligned} \int \frac{4x^2}{x^3 + x^2 - x - 1} dx &= \int \frac{1}{x-1} dx + \int \frac{3}{x+1} dx - \int \frac{2}{(x+1)^2} dx \\ &= \ln|x-1| + 3 \ln|x+1| + \frac{2}{(x+1)} + C \end{aligned}$$

$$22. \frac{6x}{x^3 - 8} = \frac{6x}{(x-2)(x^2 + 2x + 4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2 + 2x + 4}$$

$$6x = A(x^2 + 2x + 4) + (Bx + C)(x-2)$$

When $x = 2$, $12 = 12A \Rightarrow A = 1$. When $x = 0$, $0 = 4 - 2C \Rightarrow C = 2$. When $x = 1$, $6 = 7 + (B+2)(-1) \Rightarrow B = -1$.

$$\begin{aligned} \int \frac{6x}{x^3 - 8} dx &= \int \frac{1}{x-2} dx + \int \frac{-x+2}{x^2 + 2x + 4} dx = \int \frac{1}{x-2} dx + \int \frac{-x-1}{x^2 + 2x + 4} dx + \int \frac{3}{(x^2 + 2x + 1) + 3} dx \\ &= \ln|x-2| - \frac{1}{2} \ln|x^2 + 2x + 4| + \frac{3}{\sqrt{3}} \arctan\left(\frac{x+1}{\sqrt{3}}\right) + C \\ &= \ln|x-2| - \frac{1}{2} \ln|x^2 + 2x + 4| + \sqrt{3} \arctan\left(\frac{\sqrt{3}(x+1)}{3}\right) + C \end{aligned}$$

$$24. \frac{x^2 - x + 9}{(x^2 + 9)^2} = \frac{Ax + B}{x^2 + 9} + \frac{Cx + D}{(x^2 + 9)^2}$$

$$\begin{aligned} x^2 - x + 9 &= (Ax + B)(x^2 + 9) + Cx + D \\ &= Ax^3 + Bx^2 + (9A + C)x + (9B + D) \end{aligned}$$

By equating coefficients of like terms, we have $A = 0$, $B = 1$, $D = 0$, and $C = -1$.

$$\begin{aligned} \int \frac{x^2 - x + 9}{(x^2 + 9)^2} dx &= \int \frac{1}{x^2 + 9} dx - \int \frac{x}{(x^2 + 9)^2} dx \\ &= \frac{1}{3} \arctan\left(\frac{x}{3}\right) + \frac{1}{2(x^2 + 9)} + C \end{aligned}$$

$$26. \frac{x^2 - 4x + 7}{(x + 1)(x^2 - 2x + 3)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 - 2x + 3}$$

$$x^2 - 4x + 7 = A(x^2 - 2x + 3) + (Bx + C)(x + 1)$$

When $x = -1$, $12 = 6A$. When $x = 0$, $7 = 3A + C$. When $x = 1$, $4 = 2A + 2B + 2C$. Solving these equations we have $A = 2$, $B = -1$, $C = 1$.

$$\begin{aligned} \int \frac{x^2 - 4x + 7}{x^3 - x^2 + x + 3} dx &= 2 \int \frac{1}{x + 1} dx + \int \frac{-x + 1}{x^2 - 2x + 3} dx \\ &= 2 \ln|x + 1| - \frac{1}{2} \ln|x^2 - 2x + 3| + C \end{aligned}$$

$$28. \frac{x^2 + x + 3}{(x^2 + 3)^2} = \frac{Ax + B}{x^2 + 3} + \frac{Cx + D}{(x^2 + 3)^2}$$

$$\begin{aligned} x^2 + x + 3 &= (Ax + B)(x^2 + 3) + Cx + D \\ &= Ax^3 + Bx^2 + (3A + C)x + (3B + D) \end{aligned}$$

By equating coefficients of like terms, we have $A = 0$, $B = 1$, $3A + C = 1$, $3B + D = 3$. Solving these equations we have $A = 0$, $B = 1$, $C = 1$, $D = 0$.

$$\begin{aligned} \int \frac{x^2 + x + 3}{x^4 + 6x^2 + 9} dx &= \int \left[\frac{1}{x^2 + 3} + \frac{x}{(x^2 + 3)^2} \right] dx \\ &= \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} - \frac{1}{2(x^2 + 3)} + C \end{aligned}$$

$$30. \frac{x - 1}{x^2(x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 1}$$

$$x - 1 = Ax(x + 1) + B(x + 1) + Cx^2$$

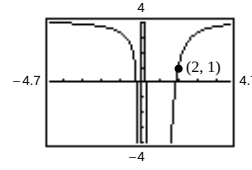
When $x = 0$, $B = -1$. When $x = -1$, $C = -2$. When $x = 1$, $0 = 2A + 2B + C$. Solving these equations we have $A = 2$, $B = -1$, $C = -2$.

$$\begin{aligned} \int_1^5 \frac{x - 1}{x^2(x + 1)} dx &= 2 \int_1^5 \frac{1}{x} dx - \int_1^5 \frac{1}{x^2} dx - 2 \int_1^5 \frac{1}{x + 1} dx \\ &= \left[2 \ln|x| + \frac{1}{x} - 2 \ln|x + 1| \right]_1^5 \\ &= \left[2 \ln \left| \frac{x}{x + 1} \right| + \frac{1}{x} \right]_1^5 \\ &= 2 \ln \frac{5}{3} - \frac{4}{5} \end{aligned}$$

$$32. \int_0^1 \frac{x^2 - x}{x^2 + x + 1} dx = \int_0^1 dx - \int_0^1 \frac{2x + 1}{x^2 + x + 1} dx = \left[x - \ln|x^2 + x + 1| \right]_0^1 = 1 - \ln 3$$

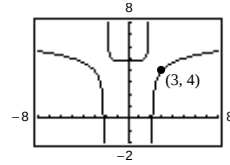
$$34. \int \frac{6x^2 + 1}{x^2(x-1)^3} dx = 3 \ln \left| \frac{x-1}{x} \right| + \frac{1}{x} + \frac{2}{x-1} - \frac{7}{2(x-1)^2} + C$$

$$(2, 1): 3 \ln \left| \frac{1}{2} \right| + \frac{1}{2} + \frac{2}{1} - \frac{7}{2} + C = 1 \Rightarrow C = 2 - 3 \ln \frac{1}{2}$$



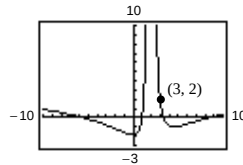
$$36. \int \frac{x^3}{(x^2 - 4)^2} dx = \frac{1}{2} \ln|x^2 - 4| - \frac{2}{x^2 - 4} + C$$

$$(3, 4): \frac{1}{2} \ln 5 - \frac{2}{5} + C = 4 \Rightarrow C = \frac{22}{5} - \frac{1}{2} \ln 5$$



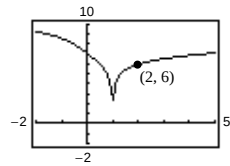
$$38. \int \frac{x(2x-9)}{x^3 - 6x^2 + 12x - 8} dx = 2 \ln|x-2| + \frac{1}{x-2} + \frac{5}{(x-2)^2} + C$$

$$(3, 2): 0 + 1 + 5 + C = 2 \Rightarrow C = -4$$



$$40. \int \frac{x^2 - x + 2}{x^3 - x^2 + x - 1} dx = -\arctan x + \ln|x-1| + C$$

$$(2, 6): -\arctan 2 + 0 + C = 6 \Rightarrow C = 6 + \arctan 2$$



$$42. \text{ Let } u = \cos x, \quad du = -\sin x \, dx.$$

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

$$1 = A(u+1) + Bu$$

When $u = 0$, $A = 1$. When $u = -1$, $B = -1$, $u = \cos x$,
 $du = -\sin x \, dx$.

$$\begin{aligned} \int \frac{\sin x}{\cos x + \cos^2 x} dx &= -\int \frac{1}{u(u+1)} du \\ &= \int \frac{1}{u+1} du - \int \frac{1}{u} du \\ &= \ln|u+1| - \ln|u| + C \\ &= \ln \left| \frac{u+1}{u} \right| + C \\ &= \ln \left| \frac{\cos x + 1}{\cos x} \right| + C \\ &= \ln|1 + \sec x| + C \end{aligned}$$

$$44. \frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}, \quad u = \tan x, \quad du = \sec^2 x \, dx$$

$$1 = A(u+1) + Bu$$

When $u = 0$, $A = 1$.

When $u = -1$, $1 = -B \Rightarrow B = -1$.

$$\begin{aligned} \int \frac{\sec^2 x \, dx}{\tan x(\tan x + 1)} &= \int \frac{1}{u(u+1)} du \\ &= \int \left(\frac{1}{u} - \frac{1}{u+1} \right) du \\ &= \ln|u| - \ln|u+1| + C \\ &= \ln \left| \frac{u}{u+1} \right| + C \\ &= \ln \left| \frac{\tan x}{\tan x + 1} \right| + C \end{aligned}$$

46. Let $u = e^x$, $du = e^x dx$.

$$\frac{1}{(u^2 + 1)(u - 1)} = \frac{A}{u - 1} + \frac{Bu + C}{u^2 + 1}$$

$$1 = A(u^2 + 1) + (Bu + C)(u - 1)$$

When $u = 1$, $A = \frac{1}{2}$.

When $u = 0$, $1 = A - C$.

When $u = -1$, $1 = 2A + 2B - 2C$. Solving these equations we have $A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = -\frac{1}{2}$, $u = e^x$, $du = e^x dx$.

$$\begin{aligned} \int \frac{e^x}{(e^{2x} + 1)(e^x - 1)} dx &= \int \frac{1}{(u^2 + 1)(u - 1)} du \\ &= \frac{1}{2} \left(\int \frac{1}{u - 1} du - \int \frac{u + 1}{u^2 + 1} du \right) \\ &= \frac{1}{2} \left(\ln|u - 1| - \frac{1}{2} \ln|u^2 + 1| - \arctan u \right) + C \\ &= \frac{1}{4} (2 \ln|e^x - 1| - \ln|e^{2x} + 1| - 2 \arctan e^x) + C \end{aligned}$$

48. $\frac{1}{a^2 - x^2} = \frac{A}{a - x} + \frac{B}{a + x}$

$$1 = A(a + x) + B(a - x)$$

When $x = a$, $A = 1/2a$.

When $x = -a$, $B = 1/2a$.

$$\begin{aligned} \int \frac{1}{a^2 - x^2} dx &= \frac{1}{2a} \int \left(\frac{1}{a - x} + \frac{1}{a + x} \right) dx \\ &= \frac{1}{2a} (-\ln|a - x| + \ln|a + x|) + C \\ &= \frac{1}{2a} \ln \left| \frac{a + x}{a - x} \right| + C \end{aligned}$$

50. $\frac{1}{x^2(a + bx)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{a + bx}$

$$1 = Ax(a + bx) + B(a + bx) + Cx^2$$

When $x = 0$, $1 = Ba \Rightarrow B = 1/a$.

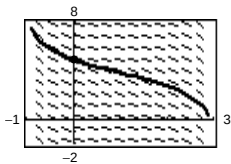
When $x = -a/b$, $1 = C(a^2/b^2) \Rightarrow C = b^2/a^2$.

When $x = 1$, $1 = (a + b)A + (a + b)B + C \Rightarrow$

$$A = -b/a^2.$$

$$\begin{aligned} \int \frac{1}{x^2(a + bx)} dx &= \int \left(\frac{-b/a^2}{x} + \frac{1/a}{x^2} + \frac{b^2/a^2}{a + bx} \right) dx \\ &= -\frac{b}{a^2} \ln|x| - \frac{1}{ax} + \frac{b}{a^2} \ln|a + bx| + C \\ &= -\frac{1}{ax} + \frac{b}{a^2} \ln \left| \frac{a + bx}{x} \right| + C \\ &= -\frac{1}{ax} - \frac{b}{a^2} \ln \left| \frac{x}{a + bx} \right| + C \end{aligned}$$

52. $\frac{dy}{dx} = \frac{4}{(x^2 - 2x - 3)}, y(0) = 5$



54. (a) $\frac{N(x)}{D(x)} = \frac{A_1}{px + q} + \frac{A_2}{(px + q)^2} + \cdots + \frac{A_m}{(px + q)^m}$

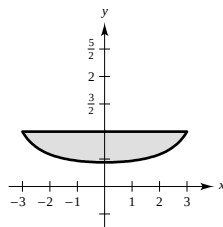
(b) $\frac{N(x)}{D(x)} = \frac{A_1 + B_1x}{(ax^2 + bx + c)} + \cdots + \frac{A_n + B_nx}{(ax^2 + bx + c)^n}$

56. $A = 2 \int_0^3 \left(1 - \frac{7}{16 - x^2} \right) dx$

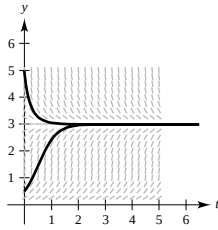
$$= 2 \int_0^3 dx - 14 \int_0^3 \frac{1}{16 - x^2} dx$$

$$= \left[2x - \frac{14}{8} \ln \left| \frac{4 + x}{4 - x} \right| \right]_0^3 \quad (\text{From Exercise 46})$$

$$= 6 - \frac{7}{4} \ln 7 \approx 2.595$$



58. (a)



(b) The slope is negative because the function is decreasing.

(c) For $y > 0$, $\lim_{t \rightarrow \infty} y(t) = 3$.

$$(d) \quad \frac{dy}{y(L-y)} = \frac{A}{y} + \frac{B}{L-y}$$

$$1 = A(L-y) + By \Rightarrow A = \frac{1}{L}, B = \frac{1}{L}$$

$$\int \frac{dy}{y(L-y)} = \int k dt$$

$$\frac{1}{L} \left[\int \frac{1}{y} dy + \int \frac{1}{L-y} dy \right] = \int k dt$$

$$\frac{1}{L} [\ln|y| - \ln|L-y|] = kt + C_1$$

$$\ln \left| \frac{y}{L-y} \right| = kLt + LC_1$$

$$C_2 e^{kLt} = \frac{y}{L-y}$$

$$\text{When } t = 0, \frac{y_0}{L-y_0} = C_2 \Rightarrow \frac{y}{L-y} = \frac{y_0}{L-y_0} e^{kLt}.$$

$$\text{Solving for } y, \text{ you obtain } y = \frac{y_0 L}{y_0 + (L-y_0)e^{-kLt}}.$$

$$60. (a) \quad V = \pi \int_0^3 \left(\frac{2x}{x^2+1} \right)^2 dx = 4\pi \int_0^3 \frac{x^2}{(x^2+1)^2} dx$$

$$= 4\pi \int_0^3 \left(\frac{1}{x^2+1} - \frac{1}{(x^2+1)^2} \right) dx \quad (\text{partial fractions})$$

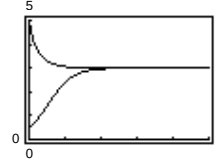
$$= 4\pi \left[\arctan x - \frac{1}{2} \left(\arctan x + \frac{x}{x^2+1} \right) \right]_0^3 \quad (\text{trigonometric substitution})$$

$$= 2\pi \left[\arctan x - \frac{x}{x^2+1} \right]_0^3 = 2\pi \left[\arctan 3 - \frac{3}{10} \right] \approx 5.963$$

(e) $k = 1, L = 3$

$$(i) y(0) = 5: y = \frac{15}{5 - 2e^{-3t}}$$

$$(ii) y(0) = \frac{1}{2}: y = \frac{3/2}{(1/2) + (5/2)e^{-3t}} = \frac{3}{1 + 5e^{-3t}}$$



$$(f) \quad \frac{dy}{dt} = ky(L-y)$$

$$\frac{d^2y}{dt^2} = k \left[y \left(\frac{-dy}{dt} \right) + (L-y) \frac{dy}{dt} \right] = 0$$

$$\Rightarrow y \frac{dy}{dt} = (L-y) \frac{dy}{dt}$$

$$\Rightarrow y = \frac{L}{2}$$

From the first derivative test, this is a maximum.

—CONTINUED—

60. —CONTINUED—

$$(b) A = \int_0^3 \frac{2x}{x^2 + 1} dx = \left[\ln(x^2 + 1) \right]_0^3 = \ln 10$$

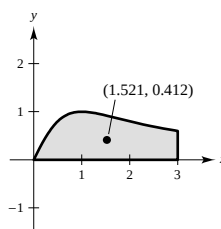
$$\begin{aligned} \bar{x} &= \frac{1}{A} \int_0^3 \frac{2x^2}{x^2 + 1} dx = \frac{1}{\ln 10} \int_0^3 \left(2 - \frac{2}{x^2 + 1} \right) dx \\ &= \frac{1}{\ln 10} \left[2x - 2 \arctan x \right]_0^3 = \frac{2}{\ln 10} [3 - \arctan 3] \approx 1.521 \end{aligned}$$

$$\begin{aligned} \bar{y} &= \frac{1}{A} \left(\frac{1}{2} \right) \int_0^3 \left(\frac{2x}{x^2 + 1} \right)^2 dx = \frac{2}{\ln 10} \int_0^3 \frac{x^2}{(x^2 + 1)^2} dx \\ &= \frac{2}{\ln 10} \int_0^3 \left(\frac{1}{x^2 + 1} - \frac{1}{(x^2 + 1)^2} \right) dx \end{aligned}$$

$$= \frac{2}{\ln 10} \left[\arctan x - \frac{1}{2} \left(\arctan x + \frac{x}{x^2 + 1} \right) \right]_0^3$$

$$= \frac{2}{\ln 10} \left[\frac{1}{2} \arctan x - \frac{x}{2(x^2 + 1)} \right]_0^3 = \frac{1}{\ln 10} \left[\arctan x - \frac{x}{x^2 + 1} \right]_0^3 = \frac{1}{\ln 10} \left[\arctan 3 - \frac{3}{10} \right] \approx 0.412$$

$$(\bar{x}, \bar{y}) \approx (1.521, 0.412)$$



(partial fractions)

(trigonometric substitution)

$$62. (a) \quad \frac{1}{(y_0 - x)(z_0 - x)} = \frac{A}{y_0 - x} + \frac{B}{z_0 - x}, A = \frac{1}{z_0 - y_0}, B = -\frac{1}{z_0 - y_0} \quad (\text{Assume } y_0 \neq z_0)$$

$$\frac{1}{z_0 - y_0} \int \left(\frac{1}{y_0 - x} - \frac{1}{z_0 - x} \right) dx = kt + C$$

$$\frac{1}{z_0 - y_0} \ln \left| \frac{z_0 - x}{y_0 - x} \right| = kt + C, \text{ when } t = 0, x = 0$$

$$C = \frac{1}{z_0 - y_0} \ln \frac{z_0}{y_0}$$

$$\frac{1}{z_0 - y_0} \left[\ln \left| \frac{z_0 - x}{y_0 - x} \right| - \ln \left(\frac{z_0}{y_0} \right) \right] = kt$$

$$\ln \left[\frac{y_0(z_0 - x)}{z_0(y_0 - x)} \right] = (z_0 - y_0)kt$$

$$\frac{y_0(z_0 - x)}{z_0(y_0 - x)} = e^{(z_0 - y_0)kt}$$

$$x = \frac{y_0 z_0 [e^{(z_0 - y_0)kt} - 1]}{z_0 e^{(z_0 - y_0)kt} - y_0}$$

$$(b) (1) \text{ If } y_0 < z_0, \lim_{t \rightarrow \infty} x = y_0.$$

$$(2) \text{ If } y_0 > z_0, \lim_{t \rightarrow \infty} x = z_0.$$

$$(c) \text{ If } y_0 = z_0, \text{ then the original equation is}$$

$$\int \frac{1}{(y_0 - x)^2} dx = \int k dt$$

$$(y_0 - x)^{-1} = kt + C_1$$

$$x = 0 \text{ when } t = 0 \Rightarrow \frac{1}{y_0} = C_1$$

$$\frac{1}{y_0 - x} = kt + \frac{1}{y_0} = \frac{kt y_0 + 1}{y_0}$$

$$y_0 - x = \frac{y_0}{kt y_0 + 1}$$

$$x = y_0 - \frac{y_0}{kt y_0 + 1}$$

$$\text{As } t \rightarrow \infty, x \rightarrow y_0 = x_0.$$

Section 7.6 Integration by Tables and Other Integration Techniques

2. By Formula 13: ($b = 2, a = -5$)

$$\begin{aligned}\frac{2}{3} \int \frac{1}{x^2(2x-5)^2} dx &= \frac{2}{3} \left(\frac{-1}{25} \right) \left[\frac{-5+4x}{x(-5+2x)} + \frac{4}{-5} \ln \left| \frac{x}{2x-5} \right| \right] + C \\ &= \frac{8}{375} \ln \left| \frac{x}{2x-5} \right| - \frac{2}{75} \frac{(4x-5)}{x(2x-5)} + C\end{aligned}$$

4. By Formula 29: ($a = 3$)

$$\frac{1}{3} \int \frac{\sqrt{x^2-9}}{x} dx = \frac{1}{3} \sqrt{x^2-9} - \operatorname{arcsec} \frac{|x|}{3} + C$$

6. By Formula 41: $\int \frac{x}{\sqrt{9-x^4}} dx = \frac{1}{2} \int \frac{2x}{\sqrt{3^2-(x^2)^2}} dx$
 $= \frac{1}{2} \arcsin \frac{x^2}{3} + C$

8. By Formulas 51 and 47: $\int \frac{\cos^3 \sqrt{x}}{\sqrt{x}} dx = 2 \int \cos^3 \sqrt{x} \left(\frac{1}{2\sqrt{x}} \right) dx$

$$= 2 \left[\frac{\cos^2 \sqrt{x} \sin \sqrt{x}}{3} + \frac{2}{3} \int \cos \sqrt{x} \left(\frac{1}{2\sqrt{x}} \right) dx \right] = \frac{2}{3} \sin \sqrt{x} (\cos^2 \sqrt{x} + 2) + C$$

$$u = \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx$$

10. By Formula 71:

$$\begin{aligned}\int \frac{1}{1-\tan 5x} dx &= \frac{1}{5} \int \frac{1}{1-\tan 5x} (5) dx \\ &= \frac{1}{5} \left(\frac{1}{2} \right) (u - \ln |\cos u - \sin u|) + C \\ &= \frac{1}{10} (5x - \ln |\cos 5x - \sin 5x|) + C\end{aligned}$$

$$u = 5x, du = 5 dx$$

12. By Formula 85: ($a = -\frac{1}{2}, b = 2$)

$$\begin{aligned}\int e^{-x/2} \sin 2x dx &= \frac{e^{-x/2}}{(1/4) + 4} \left(-\frac{1}{2} \sin 2x - 2 \cos 2x \right) + C \\ &= \frac{4}{17} e^{-x/2} \left(-\frac{1}{2} \sin 2x - 2 \cos 2x \right) + C\end{aligned}$$

14. By Formulas 90 and 91: $\int (\ln x)^3 dx = x(\ln x)^3 - 3 \int (\ln x)^2 dx$

$$= x(\ln x)^3 - 3x[2 - 2 \ln x + (\ln x)^2] + C$$

$$= x[(\ln x)^3 - 3(\ln x)^2 + 6 \ln x - 6] + C$$

16. (a) By Formula 89: $\int x^4 \ln x dx = \frac{x^5}{5^2} [-1 + (4+1) \ln x] + C = \frac{-x^5}{25} + \frac{1}{5} x^5 \ln x + C$

(b) Integration by parts: $u = \ln x, du = \frac{1}{x} dx, dv = x^4 dx, v = \frac{x^5}{5}$

$$\int x^4 \ln x dx = \frac{x^5}{5} \ln x - \int \frac{x^5}{5} \frac{1}{x} dx = \frac{x^5}{5} \ln x - \frac{x^5}{25} + C$$

18. (a) By Formula 24: $a = \sqrt{75}$, $x = u$, and

$$\begin{aligned}\int \frac{1}{x^2 - 75} dx &= \frac{1}{2\sqrt{75}} \ln \left| \frac{x - \sqrt{75}}{x + \sqrt{75}} \right| + C \\ &= \frac{\sqrt{3}}{30} \ln \left| \frac{x - \sqrt{75}}{x + \sqrt{75}} \right| + C\end{aligned}$$

(b) Partial fractions:

$$\begin{aligned}\frac{1}{x^2 - 75} &= \frac{A}{x - \sqrt{75}} + \frac{B}{x + \sqrt{75}} \\ 1 &= A(x + \sqrt{75}) + B(x - \sqrt{75})\end{aligned}$$

$$x = \sqrt{75}: 1 = 2A\sqrt{75} \Rightarrow A = \frac{1}{2\sqrt{75}} = \frac{1}{10\sqrt{3}} = \frac{\sqrt{3}}{30}$$

$$x = -\sqrt{75}: 1 = -2B\sqrt{75} \Rightarrow B = -\frac{\sqrt{3}}{30}$$

$$\begin{aligned}\int \frac{1}{x^2 - 75} dx &= \int \left[\frac{\sqrt{3}/30}{x - \sqrt{75}} - \frac{\sqrt{3}/30}{x + \sqrt{75}} \right] dx \\ &= \frac{\sqrt{3}}{30} \ln \left| \frac{x - \sqrt{75}}{x + \sqrt{75}} \right| + C\end{aligned}$$

$$20. \text{ By Formula 21: } \int \frac{x}{\sqrt{1+x}} dx = -\frac{2}{3}(2-x)\sqrt{1+x} + C$$

$$\begin{aligned}22. \text{ By Formula 79: } \int \operatorname{arcsec} 2x \, dx &= \frac{1}{2} [2x \operatorname{arcsec} 2x - \ln|2x + \sqrt{4x^2 - 1}|] + C \\ u &= 2x, du = 2 \, dx\end{aligned}$$

$$24. \text{ By Formula 52: } \int x \sin x \, dx = \sin x - x \cos x + C$$

$$26. \text{ By Formula 7: } \int \frac{x^2}{(3x-5)^2} dx = \frac{1}{27} \left(3x - \frac{25}{3x-5} + 10 \ln|3x-5| \right) + C$$

$$28. \text{ By Formula 14: } \int \frac{1}{x^2 + 2x + 2} dx = \frac{2}{\sqrt{4}} \arctan\left(\frac{2x+2}{2}\right) + C = \arctan(x+1) + C$$

30. By Formula 56:

$$\begin{aligned}\int \frac{\theta^2}{1 - \sin \theta^3} d\theta &= \frac{1}{3} \int \frac{1}{1 - \sin \theta^3} 3\theta^2 d\theta \\ &= \frac{1}{3} (\tan \theta^3 + \sec \theta^3) + C\end{aligned}$$

32. By Formula 71:

$$\begin{aligned}\int \frac{e^x}{1 - \tan e^x} dx &= \frac{1}{2} (e^x - \ln|\cos e^x - \sin e^x|) + C \\ u &= e^x, du = e^x dx\end{aligned}$$

$$\begin{aligned}34. \text{ By Formula 23: } \int \frac{1}{t[1 + (\ln t)^2]} dt &= \int \frac{1}{1 + (\ln t)^2} \left(\frac{1}{t}\right) dt = \arctan(\ln t) + C \\ u &= \ln t, du = \frac{1}{t} dt\end{aligned}$$

$$36. \text{ By Formula 26: } \int \sqrt{3+x^2} \, dx = \frac{1}{2} (x\sqrt{x^2+3} + 3 \ln|x + \sqrt{x^2+3}|) + C$$

$$\begin{aligned}38. \text{ By Formula 27: } \int x^2 \sqrt{2 + (3x)^2} \, dx &= \frac{1}{27} \int (3x)^2 \sqrt{(\sqrt{2})^2 + (3x)^2} 3 \, dx \\ &= \frac{1}{8(27)} [3x(18x^2 + 2)\sqrt{2 + 9x^2} - 4 \ln|3x + \sqrt{2 + 9x^2}|] + C\end{aligned}$$

40. By Formula 77: $\int \sqrt{x} \arctan(x^{3/2}) dx = \frac{2}{3} \int \arctan(x^{3/2}) \left(\frac{3}{2}\sqrt{x}\right) dx$

$$= \frac{2}{3} [x^{3/2} \arctan(x^{3/2}) - \ln \sqrt{1+x^3}] + C$$

42. By Formula 45: $\int \frac{e^x}{(1-e^{2x})^{3/2}} dx = \frac{e^x}{\sqrt{1-e^{2x}}} + C$

$$u = e^x, du = e^x dx$$

44. By Formula 27:

$$\begin{aligned} \int (2x-3)^2 \sqrt{(2x-3)^2+4} dx &= \frac{1}{2} \int (2x-3)^2 \sqrt{(2x-3)^2+4} (2) dx \\ &= \frac{1}{8} (2x-3) [(2x-3)^2+2] \sqrt{(2x-3)^2+4} - \ln|2x-3+\sqrt{(2x-3)^2+4}| + C \\ u &= 2x-3, du = 2 dx \end{aligned}$$

46. By Formula 31: $\int \frac{\cos x}{\sqrt{\sin^2 x + 1}} dx = \ln|\sin x + \sqrt{\sin^2 x + 1}| + C$

$$u = \sin x, du = \cos x dx$$

48. $\int \sqrt{\frac{3-x}{3+x}} dx = \int \frac{3-x}{\sqrt{9-x^2}} dx$

$$\begin{aligned} &= 3 \int \frac{1}{\sqrt{9-x^2}} dx + \int \frac{-x}{\sqrt{9-x^2}} dx \\ &= 3 \arcsin \frac{x}{3} + \sqrt{9-x^2} + C \end{aligned}$$

50. By Formula 67:

$$\begin{aligned} \int \tan^3 \theta d\theta &= \frac{\tan^2 \theta}{2} - \int \tan \theta d\theta \\ &= \frac{\tan^2 \theta}{2} + \ln|\cos \theta| + C \end{aligned}$$

52. Integration by parts: $w = u^n, dw = nu^{n-1} du, dv = \frac{du}{\sqrt{a+bu}}, v = \frac{2}{b}\sqrt{a+bu}$

$$\begin{aligned} \int \frac{u^n}{\sqrt{a+bu}} du &= \frac{2u^n}{b}\sqrt{a+bu} - \frac{2n}{b} \int u^{n-1} \sqrt{a+bu} du \\ &= \frac{2u^n}{b}\sqrt{a+bu} - \frac{2n}{b} \int u^{n-1} \sqrt{a+bu} \cdot \frac{\sqrt{a+bu}}{\sqrt{a+bu}} du \\ &= \frac{2u^n}{b}\sqrt{a+bu} - \frac{2n}{b} \int \frac{au^{n-1} + bu^n}{\sqrt{a+bu}} du \\ &= \frac{2u^n}{b}\sqrt{a+bu} - \frac{2na}{b} \int \frac{u^{n-1}}{\sqrt{a+bu}} du - 2n \int \frac{u^n}{\sqrt{a+bu}} du \end{aligned}$$

Therefore, $(2n+1) \int \frac{u^n}{\sqrt{a+bu}} du = \frac{2}{b} \left[u^n \sqrt{a+bu} - na \int \frac{u^{n-1}}{\sqrt{a+bu}} du \right]$ and

$$\int \frac{u^n}{\sqrt{a+bu}} du = \frac{2}{(2n+1)b} \left[u^n \sqrt{a+bu} - na \int \frac{u^{n-1}}{\sqrt{a+bu}} du \right].$$

54. $\int u^n (\cos u) du = u^n \sin u - n \int u^{n-1} (\sin u) du$

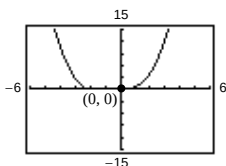
$$w = u^n, dv = \cos u du, dw = nu^{n-1} du, v = \sin u$$

$$56. \int (\ln u)^n du = u(\ln u)^n - \int n(\ln u)^{n-1} \left(\frac{1}{u}\right) u du = u(\ln u)^n - n \int (\ln u)^{n-1} du$$

$$w = (\ln u)^n, dv = du, dw = n(\ln u)^{n-1} \left(\frac{1}{u}\right) du, v = u$$

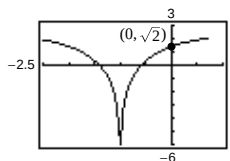
$$58. \int x\sqrt{x^2 + 2x} dx = \frac{1}{6} [2(x^2 + 2x)^{3/2} - 3(x+1)\sqrt{x^2 + 2x} + 3 \ln|x+1 + \sqrt{x^2 + 2x}|] + C$$

$$(0, 0): \frac{1}{6} [3 \ln|1|] + C = 0 \Rightarrow C = 0$$



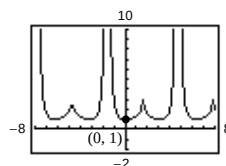
$$60. \int \frac{\sqrt{2-2x-x^2}}{x+1} dx = \sqrt{2-2x-x^2} - \sqrt{3} \ln \left| \frac{\sqrt{3} + \sqrt{2-2x-x^2}}{x+1} \right| + C$$

$$(0, \sqrt{2}): \sqrt{2} - \sqrt{3} \ln(\sqrt{3} + \sqrt{2}) + C = \sqrt{2} \Rightarrow C = \sqrt{3} \ln(\sqrt{3} + \sqrt{2})$$



$$62. \int \frac{\sin \theta}{(\cos \theta)(1 + \sin \theta)} d\theta = \frac{1}{2} \left[\frac{-\sin \theta}{1 + \sin \theta} + \ln \left| \frac{1 + \sin \theta}{\cos \theta} \right| \right] + C$$

$$(0, 1): C = 1 \Rightarrow y = \frac{1}{2} \left[\frac{-\sin \theta}{1 + \sin \theta} + \ln \left| \frac{1 + \sin \theta}{\cos \theta} \right| \right] + 1$$



$$64. \int \frac{\sin \theta}{1 + \cos^2 \theta} d\theta = - \int \frac{-\sin \theta}{1 + (\cos \theta)^2} d\theta$$

$$= -\arctan(\cos \theta) + C$$

$$66. \int_0^{\pi/2} \frac{1}{3 - 2 \cos \theta} d\theta = \int_0^1 \left[\frac{\frac{2u}{1+u^2}}{3 - \frac{2(1-u^2)}{1+u^2}} \right] du$$

$$= 2 \int_0^1 \frac{1}{5u^2 + 1} du$$

$$= \left[\frac{2}{\sqrt{5}} \arctan(\sqrt{5}u) \right]_0^1$$

$$= \frac{2}{\sqrt{5}} \arctan \sqrt{5}$$

$$u = \tan \frac{\theta}{2}$$

$$68. \int \frac{\cos \theta}{1 + \cos \theta} d\theta = \int \frac{\cos \theta(1 - \cos \theta)}{(1 + \cos \theta)(1 - \cos \theta)} d\theta$$

$$= \int \frac{\cos \theta - \cos^2 \theta}{\sin^2 \theta} d\theta$$

$$= \int (\csc \theta \cot \theta - \cot^2 \theta) d\theta$$

$$= \int (\csc \theta \cot \theta - (\csc^2 \theta - 1)) d\theta$$

$$= -\csc \theta + \cot \theta + \theta + C$$

$$70. \int \frac{1}{\sec \theta - \tan \theta} d\theta = \int \frac{1}{(1/\cos \theta) - (\sin \theta/\cos \theta)} d\theta$$

$$= - \int \frac{-\cos \theta}{1 - \sin \theta} d\theta$$

$$= -\ln|1 - \sin \theta| + C$$

$$u = 1 - \sin \theta, du = -\cos \theta d\theta$$

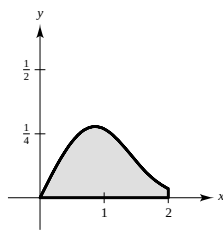
$$72. A = \int_0^2 \frac{x}{1 + e^{x^2}} dx$$

$$= \frac{1}{2} \int_0^2 \frac{2x dx}{1 + e^{x^2}}$$

$$= \frac{1}{2} \left[x^2 - \ln(1 + e^{x^2}) \right]_0^2$$

$$= \frac{1}{2} \left[4 - \ln(1 + e^4) \right] + \frac{1}{2} \ln 2$$

$$\approx 0.337 \text{ square units}$$



$$74. \text{ Log Rule: } \int \frac{1}{u} du, u = e^x + 1$$

76. Integration by parts

78. Formula 16 with $u = e^{2x}$

80. A reduction formula reduces an integral to the sum of a function and a simpler integral. For example, see Formula 50, 54.

$$82. W = \int_0^5 \frac{500x}{\sqrt{26 - x^2}} dx$$

$$= -250 \int_0^5 (26 - x^2)^{-1/2} (-2x) dx$$

$$= \left[-500 \sqrt{26 - x^2} \right]_0^5$$

$$= 500(\sqrt{26} - 1)$$

$$\approx 2049.51 \text{ ft} \cdot \text{lbs}$$

$$84. \frac{1}{2 - 0} \int_0^2 \frac{5000}{1 + e^{4.8 - 1.9t}} dt = \frac{2500}{-1.9} \int_0^2 \frac{-1.9 dt}{1 + e^{4.8 - 1.9t}}$$

$$= -\frac{2500}{1.9} \left[(4.8 - 1.9t) - \ln(1 + e^{4.8 - 1.9t}) \right]_0^2$$

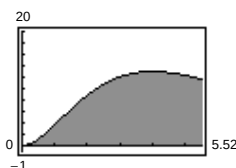
$$= -\frac{2500}{1.9} [(1 - \ln(1 + e)) - (4.8 - \ln(1 + e^{4.8}))]$$

$$= \frac{2500}{1.9} \left[3.8 + \ln\left(\frac{1 + e}{1 + e^{4.8}}\right) \right] \approx 401.4$$

$$86. (a) \int_0^k 6x^2 e^{-x/2} dx = 50$$

By trial and error, $k = 5.51897$.

$$(b) \int_0^{5.51897} 6x^2 e^{-x/2} dx$$

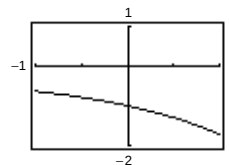


88. True

Section 7.7 Indeterminate Forms and L'Hôpital's Rule

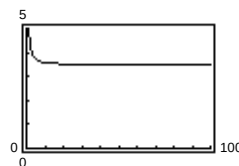
$$2. \lim_{x \rightarrow 0} \frac{1 - e^x}{x} \approx -1$$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-0.9516	-0.9950	-0.9995	-1.00005	-1.005	-1.0517



$$4. \lim_{x \rightarrow \infty} \frac{6x}{\sqrt{3x^2 - 2x}} \approx 3.4641 \left(\text{exact: } \frac{6}{\sqrt{3}} \right)$$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$	6	3.5857	3.4757	3.4653	3.4642	3.4641



$$6. (a) \lim_{x \rightarrow -1} \frac{2x^2 - x - 3}{x + 1} = \lim_{x \rightarrow -1} \frac{(2x - 3)(x + 1)}{x + 1} = \lim_{x \rightarrow -1} (2x - 3) = -5$$

$$(b) \lim_{x \rightarrow -1} \frac{2x^2 - x - 3}{x + 1} = \lim_{x \rightarrow -1} \frac{(d/dx)[2x^2 - x - 3]}{(d/dx)[x + 1]} = \lim_{x \rightarrow -1} \frac{4x - 1}{1} = -5$$

$$8. (a) \lim_{x \rightarrow 0} \frac{\sin 4x}{2x} = \lim_{x \rightarrow 0} 2 \left(\frac{\sin 4x}{4x} \right) = 2(1) = 2$$

$$(b) \lim_{x \rightarrow 0} \frac{\sin 4x}{2x} = \lim_{x \rightarrow 0} \frac{(d/dx)[\sin 4x]}{(d/dx)[2x]} = \lim_{x \rightarrow 0} \frac{4 \cos 4x}{2} = 2$$

$$10. (a) \lim_{x \rightarrow \infty} \frac{2x + 1}{4x^2 + x} = \lim_{x \rightarrow \infty} \frac{(2/x) + (1/x^2)}{4 + (1/x)} = \frac{0}{4} = 0$$

$$(b) \lim_{x \rightarrow \infty} \frac{2x + 1}{4x^2 + x} = \lim_{x \rightarrow \infty} \frac{(d/dx)[2x + 1]}{(d/dx)[4x^2 + x]} = \lim_{x \rightarrow \infty} \frac{2}{8x + 1} = 0$$

$$12. \lim_{x \rightarrow -1} \frac{x^2 - x - 2}{x + 1} = \lim_{x \rightarrow -1} \frac{2x - 1}{1} = -3$$

$$14. \lim_{x \rightarrow 2^-} \frac{\sqrt{4 - x^2}}{x - 2} = \lim_{x \rightarrow 2^-} \frac{-x/\sqrt{4 - x^2}}{1} \\ = \lim_{x \rightarrow 2^-} \frac{-x}{\sqrt{4 - x^2}} = -\infty$$

$$16. \lim_{x \rightarrow 0^+} \frac{e^x - (1 + x)}{x^3} = \lim_{x \rightarrow 0^+} \frac{e^x - 1}{3x^2} \\ = \lim_{x \rightarrow 0^+} \frac{e^x}{6x} = \infty$$

$$18. \lim_{x \rightarrow 1} \frac{\ln x^2}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{2 \ln x}{x^2 - 1} \\ = \lim_{x \rightarrow 1} \frac{2/x}{2x} \\ = \lim_{x \rightarrow 1} \frac{1}{x^2} = 1$$

$$20. \lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} = \lim_{x \rightarrow 0} \frac{a \cos ax}{b \cos bx} = \frac{a}{b}$$

$$22. \lim_{x \rightarrow 1} \frac{\arctan x - (\pi/4)}{x - 1} = \lim_{x \rightarrow 1} \frac{1/(1 + x^2)}{1} = \frac{1}{2}$$

$$24. \lim_{x \rightarrow \infty} \frac{x - 1}{x^2 + 2x + 3} = \lim_{x \rightarrow \infty} \frac{1}{2x + 2} = 0$$

$$26. \lim_{x \rightarrow \infty} \frac{x^3}{x + 1} = \lim_{x \rightarrow \infty} \frac{3x^2}{1} = \infty$$

$$28. \lim_{x \rightarrow \infty} \frac{x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0$$

$$30. \lim_{x \rightarrow \infty} \frac{x^2}{\sqrt{x^2 + 1}} = \lim_{x \rightarrow \infty} \frac{x}{\sqrt{1 + (1/x)^2}} = \infty$$

$$32. \lim_{x \rightarrow \infty} \frac{\sin x}{x - \pi} = 0$$

Note: Use the Squeeze Theorem for $x > \pi$.

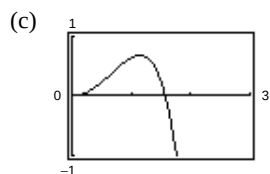
$$-\frac{1}{x - \pi} \leq \frac{\sin x}{x - \pi} \leq \frac{1}{x - \pi}$$

$$34. \lim_{x \rightarrow \infty} \frac{\ln x^4}{x^3} = \lim_{x \rightarrow \infty} \frac{4 \ln x}{x^3} = \lim_{x \rightarrow \infty} \frac{4/x}{3x^2} \\ = \lim_{x \rightarrow \infty} \frac{4}{3x^3} = 0$$

$$36. \lim_{x \rightarrow \infty} \frac{e^{x/2}}{x} = \lim_{x \rightarrow \infty} \frac{(1/2)e^{x/2}}{1} = \infty$$

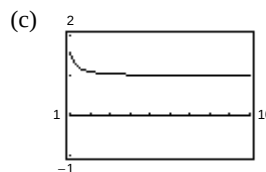
$$38. (a) \lim_{x \rightarrow 0^+} x^3 \cot x = (0)(\infty)$$

$$(b) \lim_{x \rightarrow 0^+} x^3 \cot x = \lim_{x \rightarrow 0^+} \frac{x^3}{\tan x} = \lim_{x \rightarrow 0^+} \frac{3x^2}{\sec^2 x} = 0$$



$$40. (a) \lim_{x \rightarrow \infty} \left(x \tan \frac{1}{x} \right) = (\infty)(0)$$

$$(b) \lim_{x \rightarrow \infty} x \tan \frac{1}{x} = \lim_{x \rightarrow \infty} \frac{\tan(1/x)}{1/x} \\ = \lim_{x \rightarrow \infty} \frac{-(1/x^2) \sec^2(1/x)}{-(1/x^2)} \\ = \lim_{x \rightarrow \infty} \sec^2\left(\frac{1}{x}\right) = 1$$

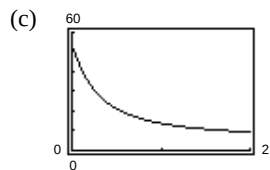


$$42. (a) \lim_{x \rightarrow 0^+} (e^x + x)^{2/x} = 1^\infty$$

$$(b) \text{ Let } y = \lim_{x \rightarrow 0^+} (e^x + x)^{2/x}.$$

$$\ln y = \lim_{x \rightarrow 0^+} \frac{2 \ln(e^x + x)}{x} \\ = \lim_{x \rightarrow 0^+} \frac{2(e^x + 1)/(e^x + x)}{1} = 4$$

Thus, $\ln y = 4 \Rightarrow y = e^4 \approx 54.598$.



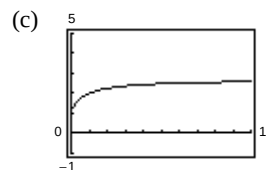
$$44. (a) \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = 1^\infty$$

$$(b) \text{ Let } y = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x.$$

$$\ln y = \lim_{x \rightarrow \infty} \left[x \ln \left(1 + \frac{1}{x} \right) \right] = \lim_{x \rightarrow \infty} \frac{\ln[1 + (1/x)]}{1/x} \\ = \lim_{x \rightarrow \infty} \frac{\left[\frac{(-1/x^2)}{1 + (1/x)} \right]}{(-1/x^2)} = \lim_{x \rightarrow \infty} \frac{1}{1 + (1/x)} = 1$$

Thus, $\ln y = 1 \Rightarrow y = e^1 = e$. Therefore,

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = e.$$



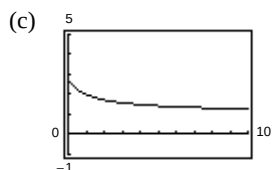
46. (a) $\lim_{x \rightarrow \infty} (1+x)^{1/x} = \infty^0$

(b) Let $y = \lim_{x \rightarrow \infty} (1+x)^{1/x}$.

$$\begin{aligned}\ln y &= \lim_{x \rightarrow \infty} \frac{\ln(1+x)}{x} \\ &= \lim_{x \rightarrow \infty} \left(\frac{1/(1+x)}{1} \right) = 0\end{aligned}$$

Thus, $\ln y = 0 \Rightarrow y = e^0 = 1$.

Therefore, $\lim_{x \rightarrow \infty} (1+x)^{1/x} = 1$.

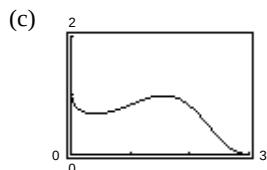


50. (a) $\lim_{x \rightarrow 0^+} \left[\cos\left(\frac{\pi}{2} - x\right) \right]^x = 0^0$

(b) Let $y = \lim_{x \rightarrow 0^+} \left[\cos\left(\frac{\pi}{2} - x\right) \right]^x$.

$$\begin{aligned}\ln y &= \lim_{x \rightarrow 0^+} x \ln \left[\cos\left(\frac{\pi}{2} - x\right) \right] \\ &= 0 \cdot 0 = 0\end{aligned}$$

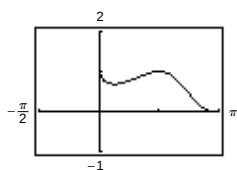
Hence, $\lim_{x \rightarrow 0^+} \left[\cos\left(\frac{\pi}{2} - x\right) \right]^x = 1$.



54. (a) $\lim_{x \rightarrow 0^+} \left(\frac{10}{x} - \frac{3}{x^2} \right) = \infty - \infty$

(b) $\lim_{x \rightarrow 0^+} \left(\frac{10}{x} - \frac{3}{x^2} \right) = \lim_{x \rightarrow 0^+} \left(\frac{10x - 3}{x^2} \right) = -\infty$

56. (a)

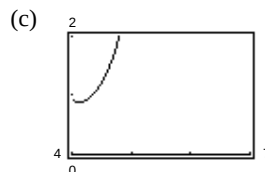


48. (a) $\lim_{x \rightarrow 4^+} [3(x-4)]^{x-4} = 0^0$

(b) Let $y = \lim_{x \rightarrow 4^+} [3(x-4)]^{x-4}$.

$$\begin{aligned}\ln y &= \lim_{x \rightarrow 4^+} (x-4) \ln[3(x-4)] \\ &= \lim_{x \rightarrow 4^+} \frac{\ln[3(x-4)]}{1/(x-4)} \\ &= \lim_{x \rightarrow 4^+} \frac{1/(x-4)}{-1/(x-4)^2} \\ &= \lim_{x \rightarrow 4^+} [-(x-4)] = 0\end{aligned}$$

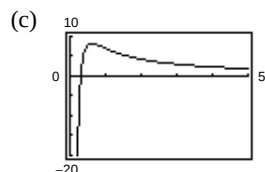
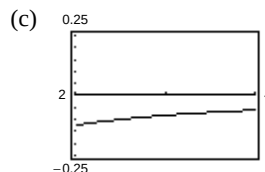
Hence, $\lim_{x \rightarrow 4^+} [3(x-4)]^{x-4} = 1$.



52. (a) $\lim_{x \rightarrow 2^+} \left(\frac{1}{x^2 - 4} - \frac{\sqrt{x-1}}{x^2 - 4} \right) = \infty - \infty$

(b) $\lim_{x \rightarrow 2^+} \left(\frac{1}{x^2 - 4} - \frac{\sqrt{x-1}}{x^2 - 4} \right) = \lim_{x \rightarrow 2^+} \frac{1 - \sqrt{x-1}}{x^2 - 4}$

$$= \lim_{x \rightarrow 2^+} \frac{-1/(2\sqrt{x-1})}{2x} = \lim_{x \rightarrow 2^+} \frac{-1}{4x\sqrt{x-1}} = -\frac{1}{8}$$

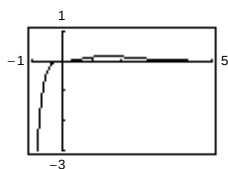


(b) Let $y = (\sin x)^x$, then $\ln y = x \ln(\sin x)$.

$$\lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{1/x} = \lim_{x \rightarrow 0^+} \frac{\cos x / \sin x}{-1/x^2} = \lim_{x \rightarrow 0^+} \frac{-x^2}{\tan x} = \lim_{x \rightarrow 0^+} \frac{-2x}{\sec^2 x} = 0$$

Therefore, since $\ln y = 0$, $y = 1$ and $\lim_{x \rightarrow 0^+} (\sin x)^x = 1$.

58. (a)



(b)
$$\lim_{x \rightarrow \infty} \frac{x^3}{e^{2x}} = \lim_{x \rightarrow \infty} \frac{3x^2}{2e^{2x}} = \lim_{x \rightarrow \infty} \frac{6x}{4e^{2x}} = \lim_{x \rightarrow \infty} \frac{6}{8e^{2x}} = 0$$

60. See Theorem 7.4.

 62. Let $f(x) = x + 25$ and $g(x) = x$.

64.
$$\lim_{x \rightarrow \infty} \frac{x^3}{e^{2x}} = \lim_{x \rightarrow \infty} \frac{3x^2}{2e^{2x}} = \lim_{x \rightarrow \infty} \frac{6x}{4e^{2x}} = \lim_{x \rightarrow \infty} \frac{6}{8e^{2x}} = 0$$

66.
$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x^3} &= \lim_{x \rightarrow \infty} \frac{(2 \ln x)/x}{3x^2} \\ &= \lim_{x \rightarrow \infty} \frac{2 \ln x}{3x^3} \\ &= \lim_{x \rightarrow \infty} \frac{2/x}{9x^2} = \lim_{x \rightarrow \infty} \frac{2}{9x^3} = 0 \end{aligned}$$

68.
$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^m}{e^{nx}} &= \lim_{x \rightarrow \infty} \frac{mx^{m-1}}{ne^{nx}} \\ &= \lim_{x \rightarrow \infty} \frac{m(m-1)x^{m-2}}{n^2e^{nx}} \\ &= \cdots = \lim_{x \rightarrow \infty} \frac{m!}{n^me^{nx}} = 0 \end{aligned}$$

70.

x	1	5	10	20	30	40	50	100
$\frac{e^x}{x^5}$	2.718	0.047	0.220	151.614	4.40×10^5	2.30×10^9	1.66×10^{13}	2.69×10^{33}

 72. $y = x^x, x > 0$

$$\lim_{x \rightarrow \infty} x^x = \infty \text{ and } \lim_{x \rightarrow 0^+} x^x = 1$$

No horizontal asymptotes

$$\ln y = x \ln x$$

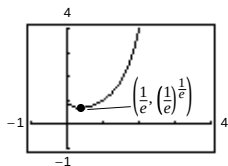
$$\frac{1}{y} \frac{dy}{dx} = x \left(\frac{1}{x} \right) + \ln x$$

$$\frac{dy}{dx} = x^x(1 + \ln x) = 0$$

 Critical number: $x = e^{-1}$

 Intervals: $(0, e^{-1})$ $(e^{-1}, 0)$

 Sign of dy/dx : $-$ $+$
 $y = f(x)$: Decreasing Increasing

 Relative minimum: $(e^{-1}, (e^{-1})^{e^{-1}}) = \left(\frac{1}{e}, \left(\frac{1}{e} \right)^{1/e} \right)$


74. $y = \frac{\ln x}{x}$

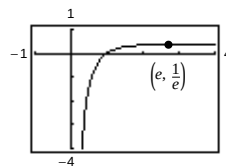
 Horizontal asymptote: $y = 0$ (See Exercise 29)

$$\frac{dy}{dx} = \frac{x(1/x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2} = 0$$

 Critical number: $x = e$

 Intervals: $(0, e)$ (e, ∞)

 Sign of dy/dx : $+$ $-$
 $y = f(x)$: Increasing Decreasing

 Relative maximum: $\left(e, \frac{1}{e} \right)$


76.
$$\lim_{x \rightarrow \infty} \frac{\sin \pi x - 1}{x} = 0 \quad (\text{Numerator is bounded})$$

 Limit is not of the form $0/0$ or ∞/∞ .

L'Hôpital's Rule does not apply.

78.
$$\lim_{x \rightarrow \infty} \frac{e^{-x}}{1 + e^{-x}} = \frac{0}{1 + 0} = 0$$

 Limit is not of the form $0/0$ or ∞/∞ .

L'Hôpital's Rule does not apply.

80. $A = P\left(1 + \frac{r}{n}\right)^{nt}$

$$\ln A = \ln P + nt \ln\left(1 + \frac{r}{n}\right) = \ln P + \frac{\ln\left(1 + \frac{r}{n}\right)}{\frac{1}{nt}}$$

$$\lim_{n \rightarrow \infty} \left[\frac{\ln\left(1 + \frac{r}{n}\right)}{\frac{1}{nt}} \right] = \lim_{n \rightarrow \infty} \left[\frac{-\frac{r}{n^2} \left(\frac{1}{1 + (r/n)} \right)}{-\left(\frac{1}{n^2 t} \right)} \right] = \lim_{n \rightarrow \infty} \left[rt \left(\frac{1}{1 + \frac{r}{n}} \right) \right] = rt$$

Since $\lim_{n \rightarrow \infty} \ln A = \ln P + rt$, we have $\lim_{n \rightarrow \infty} A = e^{(\ln P + rt)} = e^{\ln P} e^{rt} = P e^{rt}$. Alternatively,

$$\lim_{n \rightarrow \infty} A = \lim_{n \rightarrow \infty} P\left(1 + \frac{r}{n}\right)^{nt} = \lim_{n \rightarrow \infty} P\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} = P e^{rt}.$$

82. Let N be a fixed value for n . Then

$$\lim_{x \rightarrow \infty} \frac{x^{N-1}}{e^x} = \lim_{x \rightarrow \infty} \frac{(N-1)x^{N-2}}{e^x} = \lim_{x \rightarrow \infty} \frac{(N-1)(N-2)x^{N-3}}{e^x} = \dots = \lim_{x \rightarrow \infty} \left[\frac{(N-1)!}{e^x} \right] = 0. \quad (\text{See Exercise 68})$$

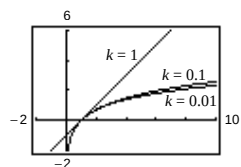
84. $f(x) = \frac{x^k - 1}{k}$

$k = 1, \quad f(x) = x - 1$

$k = 0.1, \quad f(x) = \frac{x^{0.1} - 1}{0.1} = 10(x^{0.1} - 1)$

$k = 0.01, \quad f(x) = \frac{x^{0.01} - 1}{0.01} = 100(x^{0.01} - 1)$

$$\lim_{k \rightarrow 0^+} \frac{x^k - 1}{k} = \lim_{k \rightarrow 0^+} \frac{x^k (\ln x)}{1} = \ln x$$



86. $f(x) = \frac{1}{x}, g(x) = x^2 - 4, [1, 2]$

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)}$$

$$\frac{-1/2}{3} = \frac{-1/c^2}{2c}$$

$$-\frac{1}{6} = -\frac{1}{2c^3}$$

$$2c^3 = 6$$

$$c = \sqrt[3]{3}$$

88. $f(x) = \ln x, g(x) = x^3, [1, 4]$

$$\frac{f(4) - f(1)}{g(4) - g(1)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\ln 4}{63} = \frac{1/c}{3c^2} = \frac{1}{3c^3}$$

$$3c^3 \ln 4 = 63$$

$$c^3 = \frac{21}{\ln 4}$$

$$c = \sqrt[3]{\frac{21}{\ln 4}} \approx 2.474$$

90. False. If $y = e^x/x^2$, then

$$y' = \frac{x^2 e^x - 2x e^x}{x^4} = \frac{x e^x (x - 2)}{x^4} = \frac{e^x (x - 2)}{x^3}.$$

92. False. Let $f(x) = x$ and $g(x) = x + 1$. Then

$$\lim_{x \rightarrow \infty} \frac{x}{x+1} = 1, \text{ but } \lim_{x \rightarrow \infty} [x - (x+1)] = -1.$$

$$94. g(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

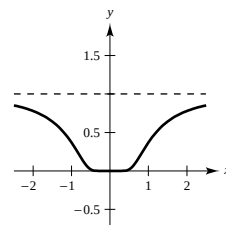
$$g'(0) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x}$$

Let $y = \frac{e^{-1/x^2}}{x}$, then $\ln y = \ln\left(\frac{e^{-1/x^2}}{x}\right) = -\frac{1}{x^2} - \ln x = \frac{-1 - x^2 \ln x}{x^2}$. Since

$$\lim_{x \rightarrow 0} x^2 \ln x = \lim_{x \rightarrow 0} \frac{\ln x}{1/x^2} = \lim_{x \rightarrow 0} \frac{1/x}{-2/x^3} = \lim_{x \rightarrow 0} \left(-\frac{x^2}{2}\right) = 0$$

we have $\lim_{x \rightarrow 0} \left(\frac{-1 - x^2 \ln x}{x^2}\right) = -\infty$. Thus, $\lim_{x \rightarrow 0} y = e^{-\infty} = 0 \Rightarrow g'(0) = 0$.

Note: The graph appears to support this conclusion—the tangent line is horizontal at $(0, 0)$.



$$96. \lim_{x \rightarrow a} f(x)^{g(x)}$$

$$y = f(x)^{g(x)}$$

$$\ln y = g(x) \ln f(x)$$

$$\lim_{x \rightarrow a} g(x) \ln f(x) = (-\infty)(-\infty) = \infty$$

As $x \rightarrow a$, $\ln y \Rightarrow \infty$, and hence $y = \infty$. Thus,

$$\lim_{x \rightarrow a} f(x)^{g(x)} = \infty.$$

$$98. \lim_{x \rightarrow 0^+} x^{\ln 2 / (1 + \ln x)}$$

Let $y = x^{\ln 2 / (1 + \ln x)}$, then:

$$\ln y = \frac{\ln 2}{1 + \ln x} \cdot \ln x = \frac{(\ln 2)(\ln x)}{1 + \ln x}$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} \ln y &= \lim_{x \rightarrow 0^+} \frac{(\ln 2)(\ln x)}{1 + \ln x} = \lim_{x \rightarrow 0^+} \frac{(\ln 2)/x}{1/x} \\ &= \lim_{x \rightarrow 0^+} (\ln 2) = \ln 2 \end{aligned}$$

Thus, $\lim_{x \rightarrow 0^+} y = e^{\ln 2} = 2$.

Section 7.8 Improper Integrals

2. Infinite discontinuity at $x = 3$.

$$\begin{aligned} \int_3^4 \frac{1}{(x-3)^{3/2}} dx &= \lim_{b \rightarrow 3^+} \int_b^4 (x-3)^{-3/2} dx \\ &= \lim_{b \rightarrow 3^+} \left[-2(x-3)^{-1/2} \right]_b^4 \\ &= \lim_{b \rightarrow 3^+} \left[-2 + \frac{2}{\sqrt{b-3}} \right] = \infty \end{aligned}$$

Diverges

4. Infinite discontinuity at $x = 1$.

$$\begin{aligned} \int_0^2 \frac{1}{(x-1)^{2/3}} dx &= \int_0^1 \frac{1}{(x-1)^{2/3}} dx + \int_1^2 \frac{1}{(x-1)^{2/3}} dx \\ &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{(x-1)^{2/3}} dx + \lim_{c \rightarrow 1^+} \int_c^2 \frac{1}{(x-1)^{2/3}} dx \\ &= \lim_{b \rightarrow 1^-} \left[3\sqrt[3]{x-1} \right]_0^b + \lim_{c \rightarrow 1^+} \left[3\sqrt[3]{x-1} \right]_c^2 = (0+3) + (3-0) = 6 \end{aligned}$$

Converges

6. Infinite limit of integration.

$$\begin{aligned}\int_{-\infty}^0 e^{2x} dx &= \lim_{b \rightarrow -\infty} \int_b^0 e^{2x} dx \\ &= \lim_{b \rightarrow -\infty} \left[\frac{1}{2} e^{2x} \right]_b^0 = \frac{1}{2} - 0 = \frac{1}{2}\end{aligned}$$

Converges

8. $\int_0^{\infty} e^{-x} dx \neq 0$. You need to evaluate the limit.

$$\begin{aligned}\lim_{b \rightarrow \infty} \int_0^b e^{-x} dx &= \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[-e^{-b} + 1 \right] = 1\end{aligned}$$

$$10. \int_1^{\infty} \frac{5}{x^3} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{5}{x^3} dx$$

$$= \lim_{b \rightarrow \infty} \left[-\frac{5}{2} x^{-2} \right]_1^b = \frac{5}{2}$$

$$12. \int_1^{\infty} \frac{4}{\sqrt[4]{x}} dx = \lim_{b \rightarrow \infty} \int_1^b 4x^{-1/4} dx$$

$$= \lim_{b \rightarrow \infty} \left[\frac{16}{3} x^{3/4} \right]_1^b = \infty \quad \text{Diverges}$$

$$14. \int_0^{\infty} x e^{-x/2} dx = \lim_{b \rightarrow \infty} \int_0^b x e^{-x/2} dx = \lim_{b \rightarrow \infty} \left[e^{-x/2} (-2x - 4) \right]_0^b = \lim_{b \rightarrow \infty} e^{-b/2} (-2b - 4) + 4 = 4$$

$$16. \int_0^{\infty} (x-1)e^{-x} dx = \lim_{b \rightarrow \infty} \int_0^b (x-1)e^{-x} dx = \lim_{b \rightarrow \infty} \left[-xe^{-x} \right]_0^b = \lim_{b \rightarrow \infty} \left(\frac{-b}{e^b} + 0 \right) = 0 \text{ by L'Hôpital's Rule.}$$

$$\begin{aligned}18. \int_0^{\infty} e^{-ax} \sin bx dx &= \lim_{c \rightarrow \infty} \left[\frac{e^{-ax} (-a \sin bx - b \cos bx)}{a^2 + b^2} \right]_0^c \\ &= 0 - \frac{-b}{a^2 + b^2} = \frac{b}{a^2 + b^2}\end{aligned}$$

$$\begin{aligned}20. \int_1^{\infty} \frac{\ln x}{x} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x} dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{(\ln x)^2}{2} \right]_1^b = \infty \quad \text{Diverges}\end{aligned}$$

$$\begin{aligned}22. \int_0^{\infty} \frac{x^3}{(x^2+1)^2} dx &= \lim_{b \rightarrow \infty} \int_0^b \frac{x}{x^2+1} dx - \lim_{b \rightarrow \infty} \int_0^b \frac{x}{(x^2+1)^2} dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} \ln(x^2+1) + \frac{1}{2(x^2+1)} \right]_0^b \\ &= \infty - \frac{1}{2}\end{aligned}$$

Diverges

$$24. \int_0^{\infty} \frac{e^x}{1+e^x} dx = \lim_{b \rightarrow \infty} \left[\ln(1+e^x) \right]_0^b = \infty - \ln 2$$

Diverges

$$26. \int_0^{\infty} \sin \frac{x}{2} dx = \lim_{b \rightarrow \infty} \left[-2 \cos \frac{x}{2} \right]_0^b$$

Diverges since $\cos \frac{x}{2}$ does not approach a limit as $x \rightarrow \infty$.

$$28. \int_0^4 \frac{8}{x} dx = \lim_{b \rightarrow 0^+} \int_b^4 \frac{8}{x} dx = \lim_{b \rightarrow 0^+} \left[8 \ln x \right]_b^4 = \infty$$

Diverges

$$\begin{aligned}30. \int_0^6 \frac{4}{\sqrt{6-x}} dx &= \lim_{b \rightarrow 6^-} \int_0^b 4(6-x)^{-1/2} dx \\ &= \lim_{b \rightarrow 6^-} \left[-8(6-x)^{1/2} \right]_0^b \\ &= -8(0) + 8\sqrt{6} \\ &= 8\sqrt{6}\end{aligned}$$

$$\begin{aligned}32. \int_0^e \ln x^2 dx &= \lim_{b \rightarrow 0^+} \int_0^b 2 \ln x dx \\ &= \lim_{b \rightarrow 0^+} \left[2x \ln x - 2x \right]_0^b \\ &= \lim_{b \rightarrow 0^+} [(2e - 2e) - (2b \ln b - 2b)] \\ &= 0\end{aligned}$$

$$34. \int_0^{\pi/2} \sec \theta d\theta = \lim_{b \rightarrow (\pi/2)} \left[\ln |\sec \theta + \tan \theta| \right]_0^b = \infty,$$

Diverges

$$36. \int_0^2 \frac{1}{\sqrt{4-x^2}} dx = \lim_{b \rightarrow 2^-} \left[\arcsin \left(\frac{x}{2} \right) \right]_0^b = \frac{\pi}{2}$$

$$38. \int_0^2 \frac{1}{4-x^2} dx = \lim_{b \rightarrow 2^-} \int_0^b \frac{1}{4} \left(\frac{1}{2+x} + \frac{1}{2-x} \right) dx = \lim_{b \rightarrow 2^-} \left[\frac{1}{4} \ln \left| \frac{2+x}{2-x} \right| \right]_0^b = \infty - 0$$

Diverges

$$\begin{aligned} 40. \int_1^3 \frac{2}{(x-2)^{8/3}} dx &= \int_1^2 2(x-2)^{-8/3} dx + \int_2^3 2(x-2)^{-8/3} dx \\ &= \lim_{b \rightarrow 2^-} \int_1^b 2(x-2)^{-8/3} dx + \lim_{c \rightarrow 2^+} \int_c^3 2(x-2)^{-8/3} dx \\ &= \lim_{b \rightarrow 2^-} \left[-\frac{6}{5} (x-2)^{-5/3} \right]_1^b + \lim_{c \rightarrow 2^+} \left[-\frac{6}{5} (x-2)^{-5/3} \right]_c^3 = \infty \end{aligned}$$

Diverges

$$42. \int \frac{1}{x \ln x} dx = \ln |\ln |x|| + C$$

Thus,

$$\begin{aligned} \int_1^\infty \frac{1}{x \ln x} dx &= \int_1^e \frac{1}{x \ln x} dx + \int_e^\infty \frac{1}{x \ln x} dx \\ &= \lim_{b \rightarrow 1^+} \left[\ln(\ln x) \right]_1^e + \lim_{c \rightarrow \infty} \left[\ln(\ln x) \right]_e^c. \end{aligned}$$

Diverges

$$44. \text{ If } p = 1, \int_0^1 \frac{1}{x} dx = \lim_{a \rightarrow 0^+} \ln x \Big|_a^1 = \lim_{a \rightarrow 0^+} -\ln a = \infty.$$

Diverges. If $p \neq 1$,

$$\int_0^1 \frac{1}{x^p} dx = \lim_{a \rightarrow 0^+} \left[\frac{x^{1-p}}{1-p} \right]_a^1 = \lim_{a \rightarrow 0^+} \left[\frac{1}{1-p} - \frac{a^{1-p}}{1-p} \right].$$

This converges to $\frac{1}{1-p}$ if $1-p > 0$ or $p < 1$.

$$46. (a) \text{ Assume } \int_a^\infty g(x) dx = L \text{ (converges).}$$

Since $0 \leq f(x) \leq g(x)$ on $[a, \infty)$, $0 \leq \int_a^\infty f(x) dx \leq \int_a^\infty g(x) dx = L$ and $\int_a^\infty f(x) dx$ converges.

$$(b) \int_a^\infty g(x) dx \text{ diverges, because otherwise, by part (a), if } \int_a^\infty g(x) dx \text{ converges, then so does } \int_a^\infty f(x) dx.$$

$$48. \int_0^1 \frac{1}{\sqrt[3]{x}} dx = \frac{1}{1 - (1/3)} = \frac{3}{2} \text{ converges.}$$

(See Exercise 44, $p = \frac{1}{3}$.)

$$50. \int_0^\infty x^4 e^{-x} dx \text{ converges.}$$

(See Exercise 45.)

$$52. \text{ Since } \frac{1}{\sqrt{x-1}} \geq \frac{1}{x} \text{ on } [2, \infty) \text{ and } \int_2^\infty \frac{1}{x} dx \text{ diverges by Exercise 43, } \int_2^\infty \frac{1}{\sqrt{x-1}} dx \text{ diverges.}$$

$$54. \text{ Since } \frac{1}{\sqrt{x}(1+x)} \leq \frac{1}{x^{3/2}} \text{ on } [1, \infty) \text{ and } \int_1^\infty \frac{1}{x^{3/2}} dx \text{ converges by Exercise 43, } \int_1^\infty \frac{1}{\sqrt{x}(1+x)} dx \text{ converges.}$$

$$56. \frac{1}{\sqrt{x} \ln x} \geq \frac{1}{x} \text{ since } \sqrt{x} \ln x < x \text{ on } [2, \infty). \text{ Since } \int_2^\infty \frac{1}{x} dx \text{ diverges by Exercise 43, } \int_2^\infty \frac{1}{\sqrt{x} \ln x} dx \text{ diverges.}$$

58. See the definitions, pages 540, 543.

60. Answers will vary.

$$(a) \int_{-\infty}^\infty \frac{e^x}{1+e^{2x}} dx$$

Converges (Example 4)

$$(b) \int_{-\infty}^\infty x dx$$

Diverges

62. $f(t) = t$

$$\begin{aligned} F(s) &= \int_0^\infty t e^{-st} dt = \lim_{b \rightarrow \infty} \left[\frac{1}{s^2} (-st - 1) e^{-st} \right]_0^b \\ &= \frac{1}{s^2}, s > 0 \end{aligned}$$

64. $f(t) = e^{at}$

$$\begin{aligned}
 F(s) &= \int_0^{\infty} e^{at} e^{-st} dt = \int_0^{\infty} e^{t(a-s)} dt \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{a-s} e^{t(a-s)} \right]_0^b \\
 &= 0 - \frac{1}{a-s} = \frac{1}{s-a}, s > a
 \end{aligned}$$

66. $f(t) = \sin at$

$$\begin{aligned}
 F(s) &= \int_0^{\infty} e^{-st} \sin at dt \\
 &= \lim_{b \rightarrow \infty} \left[\frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) \right]_0^b \\
 &= 0 + \frac{a}{s^2 + a^2} = \frac{a}{s^2 + a^2}, s > 0
 \end{aligned}$$

68. $f(t) = \sinh at$

$$\begin{aligned}
 F(s) &= \int_0^{\infty} e^{-st} \sinh at dt = \int_0^{\infty} e^{-st} \left(\frac{e^{at} - e^{-at}}{2} \right) dt = \frac{1}{2} \int_0^{\infty} [e^{t(-s+a)} - e^{t(-s-a)}] dt \\
 &= \lim_{b \rightarrow \infty} \frac{1}{2} \left[\frac{1}{(-s+a)} e^{t(-s+a)} - \frac{1}{(-s-a)} e^{t(-s-a)} \right]_0^b = 0 - \frac{1}{2} \left[\frac{1}{(-s+a)} - \frac{1}{(-s-a)} \right] \\
 &= \frac{-1}{2} \left[\frac{1}{(-s+a)} - \frac{1}{(-s-a)} \right] = \frac{a}{s^2 - a^2}, s > |a|
 \end{aligned}$$

70. (a) $A = \int_1^{\infty} \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^{\infty} = 1$

(b) **Disk:**

$$V = \pi \int_1^{\infty} \frac{1}{x^4} dx = \lim_{b \rightarrow \infty} \left[-\frac{\pi}{3x^3} \right]_1^b = \frac{\pi}{3}$$

(c) **Shell:**

$$V = 2\pi \int_1^{\infty} x \left(\frac{1}{x^2} \right) dx = \lim_{b \rightarrow \infty} \left[2\pi(\ln x) \right]_1^b = \infty$$

Diverges

72. $(x-2)^2 + y^2 = 1$

$$2(x-2) + 2yy' = 0$$

$$y' = \frac{-(x-2)}{y}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + [(x-2)^2/y^2]} = \frac{1}{y} \text{ (Assume } y > 0.)$$

$$\begin{aligned}
 S &= 4\pi \int_1^3 \frac{x}{y} dx = 4\pi \int_1^3 \frac{x}{\sqrt{1 - (x-2)^2}} dx = 4\pi \int_1^3 \left[\frac{x-2}{\sqrt{1 - (x-2)^2}} + \frac{2}{\sqrt{1 - (x-2)^2}} \right] dx \\
 &= \lim_{\substack{a \rightarrow 1^+ \\ b \rightarrow 3^-}} \left\{ 4\pi \left[-\sqrt{1 - (x-2)^2} + 2 \arcsin(x-2) \right]_a^b \right\} = 4\pi[0 + 2 \arcsin(1) - 2 \arcsin(-1)] = 8\pi^2
 \end{aligned}$$

74. (a) $F(x) = \frac{K}{x^2}, 5 = \frac{K}{(4000)^2}, K = 80,000,000$

$$W = \int_{4000}^{\infty} \frac{80,000,000}{x^2} dx = \lim_{b \rightarrow \infty} \left[\frac{-80,000,000}{x} \right]_{4000}^b = 20,000 \text{ mi-ton}$$

$$(b) \quad \frac{W}{2} = 10,000 = \left[\frac{-80,000,000}{x} \right]_{4000}^b = \frac{-80,000,000}{b} + 20,000$$

$$\frac{80,000,000}{b} = 10,000$$

$$b = 8000$$

Therefore, 4000 miles *above* the earth's surface.

$$76. (a) \int_{-\infty}^{\infty} \frac{2}{5} e^{-2t/5} dt = \int_0^{\infty} \frac{2}{5} e^{-2t/5} dt = \lim_{b \rightarrow \infty} \left[-e^{-2t/5} \right]_0^b = 1$$

$$(b) \int_0^4 \frac{2}{5} e^{-2t/5} dt = \left[-e^{-2t/5} \right]_0^4 = -e^{-8/5} + 1 \\ \approx 0.7981 = 79.81\%$$

$$(c) \int_0^{\infty} t \left[\frac{2}{5} e^{-2t/5} \right] dt = \lim_{b \rightarrow \infty} \left[-te^{2t/5} - \frac{5}{2} e^{-2t/5} \right]_0^b = \frac{5}{2}$$

$$78. (a) C = 650,000 + \int_0^5 25,000(1 + 0.08t)e^{-0.06t} dt \\ = 650,000 + 25,000 \left[-\frac{1}{0.06} e^{-0.06t} - 0.08 \left(\frac{t}{0.06} e^{-0.06t} + \frac{1}{(0.06)^2} e^{-0.06t} \right) \right]_0^5 \approx \$778,512.58$$

$$(b) C = 650,000 + \int_0^{10} 25,000(1 + 0.08t)e^{-0.06t} dt \\ = 650,000 + 25,000 \left[-\frac{1}{0.06} e^{-0.06t} - 0.08 \left(\frac{t}{0.06} e^{-0.06t} + \frac{1}{(0.06)^2} e^{-0.06t} \right) \right]_0^{10} \approx \$905,718.14$$

$$(c) C = 650,000 + \int_0^{\infty} 25,000(1 + 0.08t)e^{-0.06t} dt \\ = 650,000 + 25,000 \lim_{b \rightarrow \infty} \left[-\frac{t}{0.06} e^{-0.06t} - 0.08 \left(\frac{t}{0.06} e^{-0.06t} + \frac{1}{(0.06)^2} e^{-0.06t} \right) \right]_0^b \approx \$1,622,222.22$$

$$80. (a) \int_1^{\infty} \frac{1}{x} dx = \lim_{b \rightarrow \infty} \left[\ln|x| \right]_1^b = \infty$$

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_1^b = 1$$

$$\int_1^{\infty} \frac{1}{x^n} dx \text{ will converge if } n > 1 \text{ and will diverge if } n \leq 1.$$

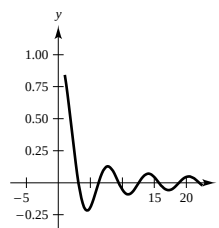
$$(c) \text{ Let } dv = \sin x \, dx \Rightarrow v = -\cos x$$

$$u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2} dx$$

$$\int_1^{\infty} \frac{\sin x}{x} dx = \lim_{b \rightarrow \infty} \left[-\frac{\cos x}{x} \right]_1^b - \int_1^{\infty} \frac{\cos x}{x^2} dx \\ = \cos 1 - \int_1^{\infty} \frac{\cos x}{x^2} dx$$

Converges

(b) It would appear to converge.



82. (a) Yes, the integral is not defined at $x = \pi/2$.

(c) As $n \rightarrow \infty$, the integral approaches $4(\pi/4) = \pi$.

$$(d) I_n = \int_0^{\pi/2} \frac{4}{1 + (\tan x)^n} dx$$

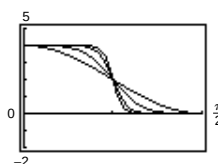
$$I_2 \approx 3.14159$$

$$I_4 \approx 3.14159$$

$$I_8 \approx 3.14159$$

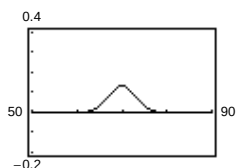
$$I_{12} \approx 3.14159$$

(b)



84. (a) $f(x) = \frac{1}{3\sqrt{2\pi}} e^{-(x-70)^2/18}$

$$\int_{50}^{90} f(x) dx \approx 1.0$$



(b) $P(72 \leq x < \infty) \approx 0.2525$

(c) $0.5 - P(70 \leq x \leq 72) \approx 0.5 - 0.2475 = 0.2525$

These are the same answers because by symmetry,

$$P(70 \leq x < \infty) = 0.5$$

and

$$0.5 = P(70 \leq x < \infty)$$

$$= P(70 \leq x \leq 72) + P(72 \leq x < \infty).$$

86. False. This is equivalent to Exercise 85.

88. True

Review Exercises for Chapter 7

$$\begin{aligned} 2. \int x e^{x^2-1} dx &= \frac{1}{2} \int e^{x^2-1} (2x) dx \\ &= \frac{1}{2} e^{x^2-1} + C \end{aligned}$$

$$\begin{aligned} 4. \int \frac{x}{\sqrt{1-x^2}} dx &= -\frac{1}{2} \int (1-x^2)^{-1/2} (-2x) dx \\ &= -\frac{1}{2} \frac{(1-x^2)^{1/2}}{1/2} + C \\ &= -\sqrt{1-x^2} + C \end{aligned}$$

$$\begin{aligned} 6. \int 2x \sqrt{2x-3} dx &= \int (u^4 + 3u^2) du = \frac{u^5}{5} + u^3 + C \\ &= \frac{2(2x-3)^{3/2}}{5} (x+1) + C \\ u = \sqrt{2x-3}, x &= \frac{u^2+3}{2}, dx = u du \end{aligned}$$

$$\begin{aligned} 8. \frac{x^4 + 2x^2 + x + 1}{x^4 + 2x^2 + 1} &= 1 + \frac{x}{(x^2+1)^2} \\ \int \frac{x^4 + 2x^2 + x + 1}{(x^2+1)^2} dx &= \int dx + \frac{1}{2} \int \frac{2x}{(x^2+1)^2} dx \\ &= x - \frac{1}{2(x^2+1)} + C \end{aligned}$$

$$10. \int (x^2 - 1)e^x dx = (x^2 - 1)e^x - 2 \int x e^x dx = (x^2 - 1)e^x - 2x e^x + 2 \int e^x dx = e^x(x^2 - 2x + 1) + 1$$

$$(1) dv = e^x dx \Rightarrow v = e^x$$

$$u = x^2 - 1 \Rightarrow du = 2x dx$$

$$(2) dv = e^x dx \Rightarrow v = e^x$$

$$u = x \Rightarrow du = dx$$

$$12. u = \arctan 2x, du = \frac{2}{1+4x^2} dx, dv = dx, v = x$$

$$\begin{aligned} \int \arctan 2x dx &= x \arctan 2x - \int \frac{2x}{1+4x^2} dx \\ &= x \arctan 2x - \frac{1}{4} \ln(1+4x^2) + C \end{aligned}$$

$$\begin{aligned} 14. \int \ln \sqrt{x^2-1} dx &= \frac{1}{2} \int \ln(x^2-1) dx \\ &= \frac{1}{2} x \ln|x^2-1| - \int \frac{x^2}{x^2-1} dx \\ &= \frac{1}{2} x \ln|x^2-1| - \int dx - \int \frac{1}{x^2-1} dx \\ &= \frac{1}{2} x \ln|x^2-1| - x - \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C \end{aligned}$$

$$dv = dx \Rightarrow v = x$$

$$u = \ln(x^2-1) \Rightarrow du = \frac{2x}{x^2-1} dx$$

$$\begin{aligned}
 16. \int e^x \arctan(e^x) dx &= e^x \arctan(e^x) - \int \frac{e^{2x}}{1 + e^{2x}} dx \\
 &= e^x \arctan(e^x) - \frac{1}{2} \ln(1 + e^{2x}) + C
 \end{aligned}$$

$$dv = e^x dx \quad \Rightarrow \quad v = e^x$$

$$u = \arctan e^x \Rightarrow du = \frac{e^x}{1 + e^{2x}} dx$$

$$18. \int \sin^2 \frac{\pi x}{2} dx = \int \frac{1}{2} (1 - \cos \pi x) dx = \frac{1}{2} \left[x - \frac{1}{\pi} \sin \pi x \right] + C = \frac{1}{2\pi} [\pi x - \sin \pi x] + C$$

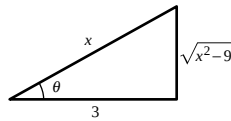
$$20. \int \tan \theta \sec^4 \theta d\theta = \int (\tan^3 \theta + \tan \theta) \sec^2 \theta d\theta = \frac{1}{4} \tan^4 \theta + \frac{1}{2} \tan^2 \theta + C_1$$

or

$$\int \tan \theta \sec^4 \theta d\theta = \int \sec^3 \theta (\sec \theta \tan \theta) d\theta = \frac{1}{4} \sec^4 \theta + C_2$$

$$\begin{aligned}
 22. \int \cos 2\theta (\sin \theta + \cos \theta)^2 d\theta &= \int (\cos^2 \theta - \sin^2 \theta) (\sin \theta + \cos \theta)^2 d\theta \\
 &= \int (\sin \theta + \cos \theta)^3 (\cos \theta - \sin \theta) d\theta = \frac{1}{4} (\sin \theta + \cos \theta)^4 + C
 \end{aligned}$$

$$\begin{aligned}
 24. \int \frac{\sqrt{x^2 - 9}}{x} dx &= \int \frac{3 \tan \theta}{3 \sec \theta} (3 \sec \theta \tan \theta d\theta) \\
 &= 3 \int \tan^2 \theta d\theta \\
 &= 3 \int (\sec^2 \theta - 1) d\theta \\
 &= 3(\tan \theta - \theta) + C \\
 &= \sqrt{x^2 - 9} - 3 \operatorname{arcsec} \left(\frac{x}{3} \right) + C
 \end{aligned}$$



$$x = 3 \sec \theta, dx = 3 \sec \theta \tan \theta d\theta, \sqrt{x^2 - 9} = 3 \tan \theta$$

$$\begin{aligned}
 26. \int \sqrt{9 - 4x^2} dx &= \frac{1}{2} \int \sqrt{9 - (2x)^2} (2) dx \\
 &= \frac{1}{2} \cdot \frac{1}{2} \left[9 \arcsin \frac{2x}{3} + 2x \sqrt{9 - 4x^2} \right] + C \\
 &= \frac{9}{4} \arcsin \frac{2x}{3} + \frac{x}{2} \sqrt{9 - 4x^2} + C
 \end{aligned}$$

$$\begin{aligned}
 28. \int \frac{\sin \theta}{1 + 2 \cos^2 \theta} d\theta &= \frac{-1}{\sqrt{2}} \int \frac{1}{1 + 2 \cos^2 \theta} (-\sqrt{2} \sin \theta) d\theta \\
 &= \frac{-1}{\sqrt{2}} \arctan(\sqrt{2} \cos \theta) + C
 \end{aligned}$$

$$u = \sqrt{2} \cos \theta, du = -\sqrt{2} \sin \theta d\theta$$

$$\begin{aligned}
 30. \quad (a) \quad \int x\sqrt{4+x} \, dx &= 64 \int \tan^3 \theta \sec^3 \theta \, d\theta \\
 &= 64 \int (\sec^4 \theta - \sec^2 \theta) \sec \theta \tan \theta \, d\theta \\
 &= \frac{64 \sec^3 \theta}{15} (3 \sec^3 \theta - 5) + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$x = 4 \tan^2 \theta, \, dx = 8 \tan \theta \sec^2 \theta \, d\theta,$$

$$\sqrt{4+x} = 2 \sec \theta$$

$$\begin{aligned}
 (c) \quad \int x\sqrt{4+x} \, dx &= \int (u^{3/2} - 4u^{1/2}) \, du \\
 &= \frac{2u^{3/2}}{15} (3u-20) + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$u = 4+x, \, du = dx$$

$$\begin{aligned}
 (b) \quad \int x\sqrt{4+x} \, dx &= 2 \int (u^4 - 4u^2) \, du \\
 &= \frac{2u^3}{15} (3u^2 - 20) + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$u^2 = 4+x, \, dx = 2u \, du$$

$$\begin{aligned}
 (d) \quad \int x\sqrt{4+x} \, dx &= \frac{2x}{3} (4+x)^{3/2} - \frac{2}{3} \int (4+x)^{3/2} \, dx \\
 &= \frac{2x}{3} (4+x)^{3/2} - \frac{4}{15} (4+x)^{5/2} + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$dv = \sqrt{4+x} \, dx \Rightarrow v = \frac{2}{3} (4+x)^{3/2}$$

$$u = x \Rightarrow du = dx$$

$$32. \quad \frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x} = 2x - 3 + \frac{4}{x} - \frac{3}{x-1}$$

$$\int \frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x} \, dx = \int \left(2x - 3 + \frac{4}{x} - \frac{3}{x-1} \right) \, dx = x^2 - 3x + 4 \ln|x| - 3 \ln|x-1| + C$$

$$34. \quad \frac{4x-2}{3(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$4x-2 = 3A(x-1) + 3B$$

$$\text{Let } x = 1: 2 = 3B \Rightarrow B = \frac{2}{3}$$

$$\text{Let } x = 2: 6 = 3A + 3B \Rightarrow A = \frac{4}{3}$$

$$\int \frac{4x-2}{3(x-1)^2} \, dx = \frac{4}{3} \int \frac{1}{x-1} \, dx + \frac{2}{3} \int \frac{1}{(x-1)^2} \, dx = \frac{4}{3} \ln|x-1| - \frac{2}{3(x-1)} + C = \frac{2}{3} \left(2 \ln|x-1| - \frac{1}{x-1} \right) + C$$

$$\begin{aligned}
 36. \quad \int \frac{\sec^2 \theta}{\tan \theta (\tan \theta - 1)} \, d\theta &= \int \frac{1}{u(u-1)} \, du = \int \frac{1}{u-1} \, du - \int \frac{1}{u} \, du \\
 &= \ln|u-1| - \ln|u| + C = \ln \left| \frac{\tan \theta - 1}{\tan \theta} \right| + C = \ln|1 - \cot \theta| + C
 \end{aligned}$$

$$u = \tan \theta, \, du = \sec^2 \theta \, d\theta$$

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$1 = A(u-1) + Bu$$

$$\text{Let } u = 0: 1 = -A \Rightarrow A = -1$$

$$\text{Let } u = 1: 1 = B$$

$$\begin{aligned}
 38. \int \frac{x}{\sqrt{2+3x}} dx &= \frac{-2(4-3x)}{27} \sqrt{2+3x} + C \quad (\text{Formula 21}) & 40. \int \frac{x}{1+e^{x^2}} dx &= \frac{1}{2} \int \frac{1}{1+e^u} du & (u = x^2) \\
 &= \frac{6x-8}{27} \sqrt{2+3x} + C & &= \frac{1}{2} [u - \ln(1+e^u)] + C & (\text{Formula 84}) \\
 & & &= \frac{1}{2} [x^2 - \ln(1+e^{x^2})] + C
 \end{aligned}$$

$$\begin{aligned}
 42. \int \frac{3}{2x\sqrt{9x^2-1}} dx &= \frac{3}{2} \int \frac{1}{3x\sqrt{(3x)^2-1}} 3 dx \quad (u = 3x) \\
 &= \frac{3}{2} \operatorname{arcsec}|3x| + C \quad (\text{Formula 33})
 \end{aligned}$$

$$\begin{aligned}
 44. \int \frac{1}{1+\tan \pi x} dx &= \frac{1}{\pi} \int \frac{1}{1+\tan \pi x} (\pi) dx & (u = \pi x) \\
 &= \frac{1}{\pi} \frac{1}{2} [\pi x + \ln|\cos \pi x + \sin \pi x|] + C \quad (\text{Formula 71})
 \end{aligned}$$

$$\begin{aligned}
 46. \int \tan^n x dx &= \int \tan^{n-2} x (\sec^2 x - 1) dx & 48. \int \frac{\csc \sqrt{2x}}{\sqrt{x}} dx &= \sqrt{2} \int \csc \sqrt{2x} \left(\frac{1}{\sqrt{2x}} \right) dx \\
 &= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx & &= -\sqrt{2} \ln|\csc \sqrt{2x} + \cot \sqrt{2x}| + C \\
 &= \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx & u = \sqrt{2x}, du &= \frac{1}{\sqrt{2x}} dx
 \end{aligned}$$

$$\begin{aligned}
 50. \int \sqrt{1+\sqrt{x}} dx &= \int u(4u^3-4u) du = \int (4u^4-4u^2) du = \frac{4u^5}{5} - \frac{4u^3}{3} + C = \frac{4}{15} (1+\sqrt{x})^{3/2} (3\sqrt{x}-2) + C \\
 u &= \sqrt{1+\sqrt{x}}, x = u^4 - 2u^2 + 1, dx = (4u^3 - 4u) du
 \end{aligned}$$

$$\begin{aligned}
 52. \frac{3x^3+4x}{(x^2+1)^2} &= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \\
 3x^3+4x &= (Ax+B)(x^2+1) + Cx+D \\
 &= Ax^3+Bx^2+(A+C)x+(B+D) \\
 A=3, B=0, A+C=4 &\Rightarrow C=1, \\
 B+D=0 &\Rightarrow D=0
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{3x^3+4x}{(x^2+1)^2} dx &= 3 \int \frac{x}{x^2+1} dx + \int \frac{x}{(x^2+1)^2} dx \\
 &= \frac{3}{2} \ln(x^2+1) - \frac{1}{2(x^2+1)} + C
 \end{aligned}$$

$$\begin{aligned}
 54. \int (\sin \theta + \cos \theta)^2 d\theta &= \int (\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta) d\theta \\
 &= \int (1 + \sin 2\theta) d\theta = \theta - \frac{1}{2} \cos 2\theta + C = \frac{1}{2} (2\theta - \cos 2\theta) + C
 \end{aligned}$$

$$\begin{aligned}
 56. \quad y &= \int \frac{\sqrt{4-x^2}}{2x} dx = \int \frac{2 \cos \theta (2 \cos \theta) d\theta}{4 \sin \theta} \\
 &= \int (\csc \theta - \sin \theta) d\theta \\
 &= [-\ln|\csc \theta + \cos \theta| + \cos \theta] + C \\
 &= -\ln \left| \frac{2 + \sqrt{4-x^2}}{x} \right| + \frac{\sqrt{4-x^2}}{2} + C
 \end{aligned}$$

$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$$

$$\begin{aligned}
 58. \quad y &= \int \sqrt{1-\cos \theta} d\theta = \int \frac{\sin \theta}{\sqrt{1+\cos \theta}} d\theta = -\int (1+\cos \theta)^{-1/2} (-\sin \theta) d\theta = -2\sqrt{1+\cos \theta} + C \\
 u &= 1 + \cos \theta, du = -\sin \theta d\theta
 \end{aligned}$$

$$\begin{aligned}
 60. \quad \int_0^1 \frac{x}{(x-2)(x-4)} dx &= \left[2 \ln|x-4| - \ln|x-2| \right]_0^1 \\
 &= 2 \ln 3 - 2 \ln 4 + \ln 2 \\
 &= \ln \frac{9}{8} \approx 0.118
 \end{aligned}$$

$$62. \quad \int_0^2 x e^{3x} dx = \left[\frac{e^{3x}}{9} (3x-1) \right]_0^2 = \frac{1}{9} (5e^6 + 1) \approx 224.238$$

$$64. \quad \int_0^3 \frac{x}{\sqrt{1+x}} dx = \left[\frac{-2(2-x)}{3} \sqrt{1+x} \right]_0^3 = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$$

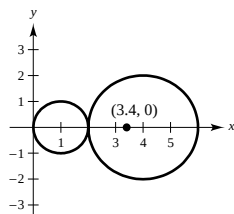
$$\begin{aligned}
 66. \quad A &= \int_0^4 \frac{1}{25-x^2} dx \\
 &= \left[-\frac{1}{10} \ln \left| \frac{x-5}{x+5} \right| \right]_0^4 = -\frac{1}{10} \ln \frac{1}{9} = \frac{1}{10} \ln 9 \approx 0.220
 \end{aligned}$$

68. By symmetry, $\bar{y} = 0$.

$$A = \pi + 4\pi = 5\pi$$

$$\begin{aligned}
 \bar{x} &= \frac{1(\pi) + 4(4\pi)}{\pi + 4\pi} \\
 &= \frac{17\pi}{5\pi} = 3.4
 \end{aligned}$$

$$(\bar{x}, \bar{y}) = (3.4, 0)$$



$$70. \quad s = \int_0^\pi \sqrt{1 + \sin^2 2x} dx \approx 3.82$$

$$72. \quad \lim_{x \rightarrow 0} \frac{\sin \pi x}{\sin 2\pi x} = \lim_{x \rightarrow 0} \frac{\pi \cos \pi x}{2\pi \cos 2\pi x} = \frac{\pi}{2\pi} = \frac{1}{2}$$

$$74. \quad \lim_{x \rightarrow \infty} x e^{-x^2} = \lim_{x \rightarrow \infty} \frac{x}{e^{x^2}} = \lim_{x \rightarrow \infty} \frac{1}{2x e^{x^2}} = 0$$

$$76. \quad y = \lim_{x \rightarrow 1^+} (x-1)^{\ln x}$$

$$\ln y = \lim_{x \rightarrow 1^+} [(\ln x) \ln(x-1)]$$

$$\begin{aligned}
 &= \lim_{x \rightarrow 1^+} \left[\frac{\ln(x-1)}{\frac{1}{\ln x}} \right] = \lim_{x \rightarrow 1^+} \left[\frac{\frac{1}{x-1}}{\left(\frac{1}{x} \right) - 1} \right] = \lim_{x \rightarrow 1^+} \left[\frac{-\ln^2 x}{\frac{x-1}{x}} \right] = \lim_{x \rightarrow 1^+} \left[\frac{-2\left(\frac{1}{x}\right)(\ln x)}{\frac{1}{x^2}} \right]
 \end{aligned}$$

$$= \lim_{x \rightarrow 1^+} 2x(\ln x) = 0$$

Since $\ln y = 0$, $y = 1$.

83. For $n = 1$,

$$I_1 = \int_0^\infty \frac{x}{(x^2 + 1)^4} dx = \lim_{b \rightarrow \infty} \frac{1}{2} \int_0^b (x^2 + 1)^{-4} (2x dx) = \lim_{b \rightarrow \infty} \left[-\frac{1}{6} \frac{1}{(x^2 + 1)^3} \right]_0^b = \frac{1}{6}.$$

For $n > 1$,

$$I_n = \int_0^\infty \frac{x^{2n-1}}{(x^2 + 1)^{n+3}} dx = \lim_{b \rightarrow \infty} \left[\frac{-x^{2n-2}}{2(n+2)(x^2 + 1)^n} + 2 \right]_0^b + \frac{n-1}{n+2} \int_0^\infty \frac{x^{2n-3}}{(x^2 + 1)^{n+2}} dx = 0 + \frac{n-1}{n+2} (I_{n-1})$$

$$u = x^{2n-2}, du = (2n-2)x^{2n-3} dx, dv = \frac{x}{(x^2 + 1)^{n+3}} dx, v = \frac{-1}{2(n+2)(x^2 + 1)^{n+2}}$$

$$(a) \int_0^\infty \frac{x}{(x^2 + 1)^4} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{6(x^2 + 1)^3} \right]_0^b = \frac{1}{6}$$

$$(b) \int_0^\infty \frac{x^3}{(x^2 + 1)^5} dx = \frac{1}{4} \int_0^\infty \frac{x}{(x^2 + 1)^4} dx = \frac{1}{4} \left(\frac{1}{6} \right) = \frac{1}{24}$$

$$(c) \int_0^\infty \frac{x^5}{(x^2 + 1)^6} dx = \frac{2}{5} \int_0^\infty \frac{x^3}{(x^2 + 1)^5} dx = \frac{2}{5} \left(\frac{1}{24} \right) = \frac{1}{60}$$

85. False. $f(x) = 1/(x+1)$ is continuous on $[0, \infty)$, $\lim_{x \rightarrow \infty} 1/(x+1) = 0$, but $\int_0^\infty \frac{1}{x+1} dx = \lim_{b \rightarrow \infty} \left[\ln|x+1| \right]_0^b = \infty$.

Diverges

87. True

Review Exercises for Chapter 7

$$\begin{aligned} 1. \int x\sqrt{x^2-1} dx &= \frac{1}{2} \int (x^2-1)^{1/2} (2x) dx \\ &= \frac{1}{2} \frac{(x^2-1)^{3/2}}{3/2} + C \\ &= \frac{1}{3} (x^2-1)^{3/2} + C \end{aligned}$$

$$\begin{aligned} 3. \int \frac{x}{x^2-1} dx &= \frac{1}{2} \int \frac{2x}{x^2-1} dx \\ &= \frac{1}{2} \ln|x^2-1| + C \end{aligned}$$

$$5. \int \frac{\ln(2x)}{x} dx = \frac{(\ln 2x)^2}{2} + C$$

$$7. \int \frac{16}{\sqrt{16-x^2}} dx = 16 \arcsin\left(\frac{x}{4}\right) + C$$

$$\begin{aligned} 9. \int e^{2x} \sin 3x dx &= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \int e^{2x} \cos 3x dx \\ &= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \left(\frac{1}{3} e^{2x} \sin 3x - \frac{2}{3} \int e^{2x} \sin 3x dx \right) \end{aligned}$$

$$\frac{13}{9} \int e^{2x} \sin 3x dx = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x$$

$$\int e^{2x} \sin 3x dx = \frac{e^{2x}}{13} (2 \sin 3x - 3 \cos 3x) + C$$

$$(1) dv = \sin 3x dx \Rightarrow v = -\frac{1}{3} \cos 3x$$

$$u = e^{2x} \Rightarrow du = 2e^{2x} dx$$

$$(2) dv = \cos 3x dx \Rightarrow v = \frac{1}{3} \sin 3x$$

$$u = e^{2x} \Rightarrow du = 2e^{2x} dx$$

$$11. u = x, du = dx, dv = (x - 5)^{1/2} dx, v = \frac{2}{3}(x - 5)^{3/2}$$

$$\begin{aligned}\int x\sqrt{x-5} dx &= \frac{2}{3}x(x-5)^{3/2} - \int \frac{2}{3}(x-5)^{3/2} dx \\ &= \frac{2}{3}x(x-5)^{3/2} - \frac{4}{15}(x-5)^{5/2} + C \\ &= (x-5)^{3/2} \left[\frac{2}{3}x - \frac{4}{15}(x-5) \right] + C \\ &= (x-5)^{3/2} \left[\frac{6}{15}x + \frac{4}{3} \right] + C \\ &= \frac{2}{15}(x-5)^{3/2}[3x+10] + C\end{aligned}$$

$$\begin{aligned}13. \int x^2 \sin 2x dx &= -\frac{1}{2}x^2 \cos 2x + \int x \cos 2x dx \\ &= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x - \frac{1}{2} \int \sin 2x dx \\ &= -\frac{1}{2}x^2 \cos 2x + \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x + C\end{aligned}$$

$$(1) dv = \sin 2x dx \Rightarrow v = -\frac{1}{2} \cos 2x$$

$$u = x^2 \Rightarrow du = 2x dx$$

$$(2) dv = \cos 2x dx \Rightarrow v = \frac{1}{2} \sin 2x$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned}15. \int x \arcsin 2x dx &= \frac{x^2}{2} \arcsin 2x - \int \frac{x^2}{\sqrt{1-4x^2}} dx \\ &= \frac{x^2}{2} \arcsin 2x - \frac{1}{8} \int \frac{2(2x)^2}{\sqrt{1-(2x)^2}} dx \\ &= \frac{x^2}{2} \arcsin 2x - \frac{1}{8} \left(\frac{1}{2} \right) [-(2x)\sqrt{1-4x^2} + \arcsin 2x] + C \quad (\text{by Formula 43 of Integration Tables}) \\ &= \frac{1}{16} [(8x^2 - 1)\arcsin 2x + 2x\sqrt{1-4x^2}] + C\end{aligned}$$

$$dv = x dx \Rightarrow v = \frac{x^2}{2}$$

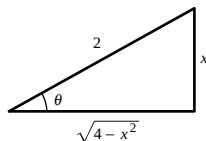
$$u = \arcsin 2x \Rightarrow du = \frac{2}{\sqrt{1-4x^2}} dx$$

$$\begin{aligned}17. \int \cos^3(\pi x - 1) dx &= \int [1 - \sin^2(\pi x - 1)] \cos(\pi x - 1) dx \\ &= \frac{1}{\pi} \left[\sin(\pi x - 1) - \frac{1}{3} \sin^3(\pi x - 1) \right] + C \\ &= \frac{1}{3\pi} \sin(\pi x - 1) [3 - \sin^2(\pi x - 1)] + C \\ &= \frac{1}{3\pi} \sin(\pi x - 1) [3 - (1 - \cos^2(\pi x - 1))] + C \\ &= \frac{1}{3\pi} \sin(\pi x - 1) [2 + \cos^2(\pi x - 1)] + C\end{aligned}$$

$$\begin{aligned}19. \int \sec^4\left(\frac{x}{2}\right) dx &= \int \left[\tan^2\left(\frac{x}{2}\right) + 1 \right] \sec^2\left(\frac{x}{2}\right) dx \\ &= \int \tan^2\left(\frac{x}{2}\right) \sec^2\left(\frac{x}{2}\right) dx + \int \sec^2\left(\frac{x}{2}\right) dx \\ &= \frac{2}{3} \tan^3\left(\frac{x}{2}\right) + 2 \tan\left(\frac{x}{2}\right) + C = \frac{2}{3} \left[\tan^3\left(\frac{x}{2}\right) + 3 \tan\left(\frac{x}{2}\right) \right] + C\end{aligned}$$

$$21. \int \frac{1}{1 - \sin \theta} d\theta = \int \frac{1 + \sin \theta}{\cos^2 \theta} d\theta = \int (\sec^2 \theta + \sec \theta \tan \theta) d\theta = \tan \theta + \sec \theta + C$$

$$\begin{aligned}
 23. \int \frac{-12}{x^2 \sqrt{4-x^2}} dx &= \int \frac{-24 \cos \theta d\theta}{(4 \sin^2 \theta)(2 \cos \theta)} \\
 &= -3 \int \csc^2 \theta d\theta \\
 &= 3 \cot \theta + C \\
 &= \frac{3\sqrt{4-x^2}}{x} + C
 \end{aligned}$$



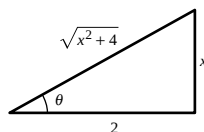
$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$$

$$25. \quad x = 2 \tan \theta$$

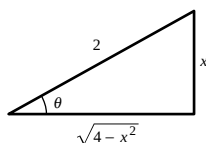
$$dx = 2 \sec^2 \theta d\theta$$

$$4 + x^2 = 4 \sec^2 \theta$$

$$\begin{aligned}
 \int \frac{x^3}{\sqrt{4+x^2}} dx &= \int \frac{8 \tan^3 \theta}{2 \sec \theta} 2 \sec^2 \theta d\theta \\
 &= 8 \int \tan^3 \theta \sec \theta d\theta \\
 &= 8 \int (\sec^2 \theta - 1) \tan \theta \sec \theta d\theta \\
 &= 8 \left[\frac{\sec^3 \theta}{3} - \sec \theta \right] + C \\
 &= 8 \left[\frac{(x^2 + 4)^{3/2}}{24} - \frac{\sqrt{x^2 + 4}}{2} \right] + C \\
 &= \sqrt{x^2 + 4} \left[\frac{1}{3}(x^2 + 4) - 4 \right] + C \\
 &= \frac{1}{3} x^2 \sqrt{x^2 + 4} - \frac{8}{3} \sqrt{x^2 + 4} + C \\
 &= \frac{1}{3} (x^2 + 4)^{1/2} (x^2 - 8) + C
 \end{aligned}$$



$$\begin{aligned}
 27. \int \sqrt{4-x^2} dx &= \int (2 \cos \theta)(2 \cos \theta) d\theta \\
 &= 2 \int (1 + \cos 2\theta) d\theta \\
 &= 2 \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\
 &= 2(\theta + \sin \theta \cos \theta) + C \\
 &= 2 \left[\arcsin\left(\frac{x}{2}\right) + \frac{x}{2} \left(\frac{\sqrt{4-x^2}}{2} \right) \right] + C \\
 &= \frac{1}{2} \left[4 \arcsin\left(\frac{x}{2}\right) + x \sqrt{4-x^2} \right] + C
 \end{aligned}$$



$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$$

$$\begin{aligned}
 29. \text{ (a) } \int \frac{x^3}{\sqrt{4+x^2}} dx &= 8 \int \frac{\sin^3 \theta}{\cos^4 \theta} d\theta \\
 &= 8 \int (\cos^{-4} \theta - \cos^{-2} \theta) \sin \theta d\theta \\
 &= \frac{8}{3} \sec \theta (\sec^2 \theta - 3) + C \\
 &= \frac{\sqrt{4+x^2}}{3} (x^2 - 8) + C
 \end{aligned}$$

$$x = 2 \tan \theta, dx = 2 \sec^2 \theta d\theta$$

$$\begin{aligned}
 \text{(b) } \int \frac{x^3}{\sqrt{4+x^2}} dx &= \int (u^2 - 4) du \\
 &= \frac{1}{3} u^3 - 4u + C \\
 &= \frac{u}{3} (u^2 - 12) + C \\
 &= \frac{\sqrt{4+x^2}}{3} (x^2 - 8) + C
 \end{aligned}$$

$$u^2 = 4 + x^2, 2u du = 2x dx$$

$$\begin{aligned}
 \text{(c) } \int \frac{x^3}{\sqrt{4+x^2}} dx &= x^2 \sqrt{4+x^2} - \int 2x \sqrt{4+x^2} dx \\
 &= x^2 \sqrt{4+x^2} - \frac{2}{3} (4+x^2)^{3/2} + C = \frac{\sqrt{4+x^2}}{3} (x^2 - 8) + C
 \end{aligned}$$

$$dv = \frac{x}{\sqrt{4+x^2}} dx \Rightarrow v = \sqrt{4+x^2}$$

$$u = x^2 \Rightarrow du = 2x dx$$

$$31. \frac{x-28}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$x-28 = A(x+2) + B(x-3)$$

$$x = -2 \Rightarrow -30 = B(-5) \Rightarrow B = 6$$

$$x = 3 \Rightarrow -25 = A(5) \Rightarrow A = -5$$

$$\int \frac{x-28}{x^2-x-6} dx = \int \left(\frac{-5}{x-3} + \frac{6}{x+2} \right) dx = -5 \ln|x-3| + 6 \ln|x+2| + C$$

$$33. \frac{x^2+2x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$x^2+2x = A(x^2+1) + (Bx+C)(x-1)$$

$$\text{Let } x = 1: 3 = 2A \Rightarrow A = \frac{3}{2}$$

$$\text{Let } x = 0: 0 = A - C \Rightarrow C = \frac{3}{2}$$

$$\text{Let } x = 2: 8 = 5A + 2B + C \Rightarrow B = -\frac{1}{2}$$

$$\begin{aligned}
 \int \frac{x^2+2x}{x^3-x^2+x-1} dx &= \frac{3}{2} \int \frac{1}{x-1} dx - \frac{1}{2} \int \frac{x-3}{x^2+1} dx \\
 &= \frac{3}{2} \int \frac{1}{x-1} dx - \frac{1}{4} \int \frac{2x}{x^2+1} dx + \frac{3}{2} \int \frac{1}{x^2+1} dx \\
 &= \frac{3}{2} \ln|x-1| - \frac{1}{4} \ln|x^2+1| + \frac{3}{2} \arctan x + C \\
 &= \frac{1}{4} [6 \ln|x-1| - \ln(x^2+1) + 6 \arctan x] + C
 \end{aligned}$$

$$35. \frac{x^2}{x^2 + 2x - 15} = 1 + \frac{15 - 2x}{x^2 + 2x - 15}$$

$$\frac{15 - 2x}{(x - 3)(x + 5)} = \frac{A}{x - 3} + \frac{B}{x + 5}$$

$$15 - 2x = A(x + 5) + B(x - 3)$$

$$\text{Let } x = 3: 9 = 8A \Rightarrow A = \frac{9}{8}$$

$$\text{Let } x = -5: 25 = -8B \Rightarrow B = -\frac{25}{8}$$

$$\begin{aligned} \int \frac{x^2}{x^2 + 2x - 15} dx &= \int dx + \frac{9}{8} \int \frac{1}{x - 3} dx - \frac{25}{8} \int \frac{1}{x + 5} dx \\ &= x + \frac{9}{8} \ln|x - 3| - \frac{25}{8} \ln|x + 5| + C \end{aligned}$$

$$37. \int \frac{x}{(2 + 3x)^2} dx = \frac{1}{9} \left[\frac{2}{2 + 3x} + \ln|2 + 3x| \right] + C$$

(Formula 4)

$$\begin{aligned} 39. \int \frac{x}{1 + \sin x^2} dx &= \frac{1}{2} \int \frac{1}{1 + \sin u} du \quad (u = x^2) \\ &= \frac{1}{2} [\tan u - \sec u] + C \quad (\text{Formula 56}) \\ &= \frac{1}{2} [\tan x^2 - \sec x^2] + C \end{aligned}$$

$$\begin{aligned} 41. \int \frac{x}{x^2 + 4x + 8} dx &= \frac{1}{2} \left[\ln|x^2 + 4x + 8| - 4 \int \frac{1}{x^2 + 4x + 8} dx \right] \quad (\text{Formula 15}) \\ &= \frac{1}{2} [\ln|x^2 + 4x + 8|] - 2 \left[\frac{2}{\sqrt{32 - 16}} \arctan \left(\frac{2x + 4}{\sqrt{32 - 16}} \right) \right] + C \quad (\text{Formula 14}) \\ &= \frac{1}{2} \ln|x^2 + 4x + 8| - \arctan \left(1 + \frac{x}{2} \right) + C \end{aligned}$$

$$\begin{aligned} 43. \int \frac{1}{\sin \pi x \cos \pi x} dx &= \frac{1}{\pi} \int \frac{1}{\sin \pi x \cos \pi x} (\pi) dx \quad (u = \pi x) \\ &= \frac{1}{\pi} \ln|\tan \pi x| + C \quad (\text{Formula 58}) \end{aligned}$$

$$\begin{aligned} 45. dv &= dx \Rightarrow v = x \\ u &= (\ln x)^n \Rightarrow du = n(\ln x)^{n-1} \frac{1}{x} dx \\ \int (\ln x)^n dx &= x(\ln x)^n - n \int (\ln x)^{n-1} dx \end{aligned}$$

$$\begin{aligned} 47. \int \theta \sin \theta \cos \theta d\theta &= \frac{1}{2} \int \theta \sin 2\theta d\theta \\ &= -\frac{1}{4} \theta \cos 2\theta + \frac{1}{4} \int \cos 2\theta d\theta = -\frac{1}{4} \theta \cos 2\theta + \frac{1}{8} \sin 2\theta + C = \frac{1}{8} (\sin 2\theta - 2\theta \cos 2\theta) + C \end{aligned}$$

$$dv = \sin 2\theta d\theta \Rightarrow v = -\frac{1}{2} \cos 2\theta$$

$$u = \theta \Rightarrow du = d\theta$$

$$\begin{aligned}
 49. \int \frac{x^{1/4}}{1+x^{1/2}} dx &= 4 \int \frac{u(u^3)}{1+u^2} du \\
 &= 4 \int \left(u^2 - 1 + \frac{1}{u^2+1} \right) du \\
 &= 4 \left(\frac{1}{3} u^3 - u + \arctan u \right) + C \\
 &= \frac{4}{3} [x^{3/4} - 3x^{1/4} + 3 \arctan(x^{1/4})] + C
 \end{aligned}$$

$$y = \sqrt[4]{x}, x = u^4, dx = 4u^3 du$$

$$\begin{aligned}
 53. \int \cos x \ln(\sin x) dx &= \sin x \ln(\sin x) - \int \cos x dx \\
 &= \sin x \ln(\sin x) - \sin x + C
 \end{aligned}$$

$$dv = \cos x dx \Rightarrow v = \sin x$$

$$u = \ln(\sin x) \Rightarrow du = \frac{\cos x}{\sin x} dx$$

$$\begin{aligned}
 57. y &= \int \ln(x^2 + x) dx = x \ln|x^2 + x| - \int \frac{2x^2 + x}{x^2 + x} dx \\
 &= x \ln|x^2 + x| - \int \frac{2x + 1}{x + 1} dx \\
 &= x \ln|x^2 + x| - \int 2 dx + \int \frac{1}{x + 1} dx \\
 &= x \ln|x^2 + x| - 2x + \ln|x + 1| + C
 \end{aligned}$$

$$dv = dx \Rightarrow v = x$$

$$u = \ln(x^2 + x) \Rightarrow du = \frac{2x + 1}{x^2 + x} dx$$

$$61. \int_1^4 \frac{\ln x}{x} dx = \left[\frac{1}{2} (\ln x)^2 \right]_1^4 = \frac{1}{2} (\ln 4)^2 = 2(\ln 2)^2 \approx 0.961$$

$$\begin{aligned}
 65. A &= \int_0^4 x \sqrt{4-x} dx = \int_2^0 (4-u^2)u(-2u) du \\
 &= \int_2^0 2(u^4 - 4u^2) du \\
 &= \left[\frac{2}{5} u^5 - \frac{4}{3} u^3 \right]_2^0 = \frac{128}{15}
 \end{aligned}$$

$$u = \sqrt{4-x}, x = 4-u^2, dx = -2u du$$

$$69. s = \int_0^\pi \sqrt{1 + \cos^2 x} dx \approx 3.82$$

$$73. \lim_{x \rightarrow \infty} \frac{e^{2x}}{x^2} = \lim_{x \rightarrow \infty} \frac{2e^{2x}}{2x} = \lim_{x \rightarrow \infty} \frac{4e^{2x}}{2} = \infty$$

$$\begin{aligned}
 51. \int \sqrt{1 + \cos x} dx &= \int \frac{\sin x}{\sqrt{1 - \cos x}} dx \\
 &= \int (1 - \cos x)^{-1/2} (\sin x) dx \\
 &= 2\sqrt{1 - \cos x} + C
 \end{aligned}$$

$$u = 1 - \cos x, du = \sin x dx$$

$$\begin{aligned}
 55. y &= \int \frac{9}{x^2 - 9} dx = \frac{3}{2} \ln \left| \frac{x-3}{x+3} \right| + C \\
 &\text{(by Formula 24 of Integration Tables)}
 \end{aligned}$$

$$59. \int_2^{\sqrt{5}} x(x^2 - 4)^{3/2} dx = \left[\frac{1}{5} (x^2 - 4)^{5/2} \right]_2^{\sqrt{5}} = \frac{1}{5}$$

$$63. \int_0^\pi x \sin x dx = \left[-x \cos x + \sin x \right]_0^\pi = \pi$$

$$67. \text{By symmetry, } \bar{x} = 0, A = \frac{1}{2} \pi.$$

$$\begin{aligned}
 \bar{y} &= \frac{2}{\pi} \left(\frac{1}{2} \right) \int_{-1}^1 (\sqrt{1-x^2})^2 dx = \frac{1}{\pi} \left[x - \frac{1}{3} x^3 \right]_{-1}^1 = \frac{4}{3\pi} \\
 (\bar{x}, \bar{y}) &= \left(0, \frac{4}{3\pi} \right)
 \end{aligned}$$

$$71. \lim_{x \rightarrow 1} \left[\frac{(\ln x)^2}{x-1} \right] = \lim_{x \rightarrow 1} \left[\frac{2(1/x) \ln x}{1} \right] = 0$$

$$75. y = \lim_{x \rightarrow \infty} (\ln x)^{2/x}$$

$$\ln y = \lim_{x \rightarrow \infty} \frac{2 \ln(\ln x)}{x} = \lim_{x \rightarrow \infty} \left[\frac{2/(x \ln x)}{1} \right] = 0$$

$$\text{Since } \ln y = 0, y = 1.$$

$$77. \lim_{n \rightarrow \infty} 1000 \left(1 + \frac{0.09}{n}\right)^n = 1000 \lim_{n \rightarrow \infty} \left(1 + \frac{0.09}{n}\right)^n$$

$$\text{Let } y = \lim_{n \rightarrow \infty} \left(1 + \frac{0.09}{n}\right)^n.$$

$$\ln y = \lim_{n \rightarrow \infty} n \ln \left(1 + \frac{0.09}{n}\right) = \lim_{n \rightarrow \infty} \frac{\ln \left(1 + \frac{0.09}{n}\right)}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(\frac{\frac{-0.09/n^2}{1 + (0.09/n)}}{-\frac{1}{n^2}} \right) = \lim_{n \rightarrow \infty} \frac{0.09}{1 + \left(\frac{0.09}{n}\right)} = 0.09$$

$$\text{Thus, } \ln y = 0.09 \Rightarrow y = e^{0.09} \text{ and } \lim_{n \rightarrow \infty} 1000 \left(1 + \frac{0.09}{n}\right)^n = 1000e^{0.09} \approx 1094.17.$$

$$79. \int_0^{16} \frac{1}{\sqrt[4]{x}} dx = \lim_{b \rightarrow 0^+} \left[\frac{4}{3} x^{3/4} \right]_b^{16} = \frac{32}{3}$$

Converges

$$81. \int_1^{\infty} x^2 \ln x dx = \lim_{b \rightarrow \infty} \left[\frac{x^3}{9} (-1 + 3 \ln x) \right]_1^b = \infty$$

Diverges

$$\begin{aligned} 83. \int_0^{t_0} 500,000 e^{-0.05t} dt &= \left[\frac{500,000}{-0.05} e^{-0.05t} \right]_0^{t_0} \\ &= \frac{-500,000}{0.05} (e^{-0.05t_0} - 1) \\ &= 10,000,000 (1 - e^{-0.05t_0}) \end{aligned}$$

$$(a) t_0 = 20: \$6,321,205.59$$

$$(b) t_0 \rightarrow \infty: \$10,000,000$$

$$85. (a) P(13 \leq x < \infty) = \frac{1}{0.95\sqrt{2\pi}} \int_{13}^{\infty} e^{-(x-12.9)^2/2(0.95)^2} dx \approx 0.4581$$

$$(b) P(15 \leq x < 20) = \frac{1}{0.95\sqrt{2\pi}} \int_{15}^{\infty} e^{-(x-12.9)^2/2(0.95)^2} dx \approx 0.0135$$

Problem Solving for Chapter 7

$$1. (a) \int_{-1}^1 (1 - x^2) dx = \left[x - \frac{x^3}{3} \right]_{-1}^1 = 2 \left(1 - \frac{1}{3} \right) = \frac{4}{3}$$

$$\int_{-1}^1 (1 - x^2)^2 dx = \int_{-1}^1 (1 - 2x^2 + x^4) dx = \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{-1}^1 = 2 \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{16}{15}$$

$$(b) \text{ Let } x = \sin u, dx = \cos u du, 1 - x^2 = 1 - \sin^2 u = \cos^2 u.$$

$$\begin{aligned} \int_{-1}^1 (1 - x^2)^n dx &= \int_{-\pi/2}^{\pi/2} (\cos^2 u)^n \cos u du \\ &= \int_{-\pi/2}^{\pi/2} \cos^{2n+1} u du \\ &= 2 \left[\frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{(2n)}{(2n+1)} \right] \quad (\text{Wallis's Formula}) \\ &= 2 \left[\frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2}{2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n)(2n+1)} \right] \\ &= \frac{2(2^n)(n!)^2}{(2n+1)!} = \frac{2^{2n+1}(n!)^2}{(2n+1)!} \end{aligned}$$

$$\begin{aligned}
 3. \quad & \lim_{x \rightarrow \infty} \left(\frac{x+c}{x-c} \right)^x = 9 \\
 & \lim_{x \rightarrow \infty} x \ln \left(\frac{x+c}{x-c} \right) = \ln 9 \\
 & \lim_{x \rightarrow \infty} \frac{\ln(x+c) - \ln(x-c)}{1/x} = \ln 9 \\
 & \lim_{x \rightarrow \infty} \frac{\frac{1}{x+c} - \frac{1}{x-c}}{-\frac{1}{x^2}} = \ln 9 \\
 & \lim_{x \rightarrow \infty} \frac{-2c}{(x+c)(x-c)} (-x^2) = \ln 9 \\
 & \lim_{x \rightarrow \infty} \left(\frac{2cx^2}{x^2 - c^2} \right) = \ln 9 \\
 & 2c = \ln 9 \\
 & 2c = 2 \ln 3 \\
 & c = \ln 3
 \end{aligned}$$

$$5. \sin \theta = \frac{PB}{OP} = PB, \cos \theta = OB$$

$$AQ = \widehat{AP} = \theta$$

$$BR = OR + OB = OR + \cos \theta$$

The triangles $\triangle AQR$ and $\triangle BPR$ are similar:

$$\frac{AR}{AQ} = \frac{BR}{BP} \Rightarrow \frac{OR+1}{\theta} = \frac{OR+\cos \theta}{\sin \theta}$$

$$\sin \theta (OR) + \sin \theta = (OR)\theta + \theta \cos \theta$$

$$OR = \frac{\theta \cos \theta - \sin \theta}{\sin \theta - \theta}$$

$$\lim_{\theta \rightarrow 0^+} OR = \lim_{\theta \rightarrow 0^+} \frac{\theta \cos \theta - \sin \theta}{\sin \theta - \theta}$$

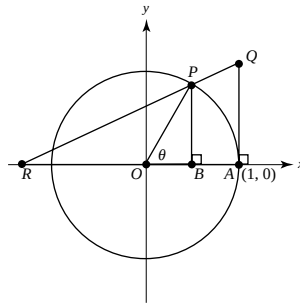
$$= \lim_{\theta \rightarrow 0^+} \frac{-\theta \sin \theta + \cos \theta - \cos \theta}{\cos \theta - 1}$$

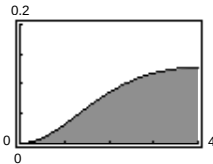
$$= \lim_{\theta \rightarrow 0^+} \frac{-\theta \sin \theta}{\cos \theta - 1}$$

$$= \lim_{\theta \rightarrow 0^+} \frac{-\sin \theta - \theta \cos \theta}{-\sin \theta}$$

$$= \lim_{\theta \rightarrow 0} \frac{\cos \theta + \cos \theta - \theta \sin \theta}{\cos \theta}$$

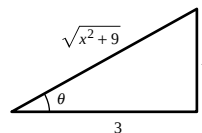
$$= 2$$



7. (a)
- 
- Area
- ≈ 0.2986

- (b) Let
- $x = 3 \tan \theta$
- ,
- $dx = 3 \sec^2 \theta d\theta$
- ,
- $x^2 + 9 = 9 \sec^2 \theta$
- .

$$\begin{aligned}
 \int \frac{x^2}{(x^2 + 9)^{3/2}} dx &= \int \frac{9 \tan^2 \theta}{(9 \sec^2 \theta)^{3/2}} (3 \sec^2 \theta d\theta) \\
 &= \int \frac{\tan^2 \theta}{\sec \theta} d\theta \\
 &= \int \frac{\sin^2 \theta}{\cos \theta} d\theta \\
 &= \int \frac{1 - \cos^2 \theta}{\cos \theta} d\theta \\
 &= \ln |\sec \theta + \tan \theta| - \sin \theta + C
 \end{aligned}$$



$$\begin{aligned}
 \text{Area} &= \int_0^4 \frac{x^2}{(x^2 + 9)^{3/2}} dx = \left[\ln |\sec \theta + \tan \theta| - \sin \theta \right]_0^{\tan^{-1}(4/3)} \\
 &= \left[\ln \left(\frac{\sqrt{x^2 + 9}}{3} + \frac{x}{3} \right) - \frac{x}{\sqrt{x^2 + 9}} \right]_0^4 \\
 &= \ln \left(\frac{5}{3} + \frac{4}{3} \right) - \frac{4}{5} = \ln 3 - \frac{4}{5}
 \end{aligned}$$

- (c)
- $x = 3 \sinh u$
- ,
- $dx = 3 \cosh u du$
- ,
- $x^2 + 9 = 9 \sinh^2 u + 9 = 9 \cosh^2 u$

$$\begin{aligned}
 A &= \int_0^4 \frac{x^2}{(x^2 + 9)^{3/2}} dx = \int_0^{\sinh^{-1}(4/3)} \frac{9 \sinh^2 u}{(9 \cosh^2 u)^{3/2}} (3 \cosh u du) \\
 &= \int_0^{\sinh^{-1}(4/3)} \tanh^2 u du \\
 &= \int_0^{\sinh^{-1}(4/3)} (1 - \operatorname{sech}^2 u) du \\
 &= \left[u - \tanh u \right]_0^{\sinh^{-1}(4/3)} \\
 &= \sinh^{-1} \left(\frac{4}{3} \right) - \tanh \left(\sinh^{-1} \left(\frac{4}{3} \right) \right) \\
 &= \ln \left(\frac{4}{3} + \sqrt{\frac{16}{9} + 1} \right) - \tanh \left[\ln \left(\frac{4}{3} + \sqrt{\frac{16}{9} + 1} \right) \right] \\
 &= \ln \left(\frac{4}{3} + \frac{5}{3} \right) - \tanh \left(\ln \left(\frac{4}{3} + \frac{5}{3} \right) \right) \\
 &= \ln 3 - \tanh(\ln 3) \\
 &= \ln 3 - \frac{3 - (1/3)}{3 + (1/3)} \\
 &= \ln 3 - \frac{4}{5}
 \end{aligned}$$

$$9. y = \ln(1 - x^2), y' = \frac{-2x}{1 - x^2}$$

$$1 + (y')^2 = 1 + \frac{4x^2}{(1 - x^2)^2} = \frac{1 - 2x^2 + x^4 + 4x^2}{(1 - x^2)^2} = \left(\frac{1 + x^2}{1 - x^2}\right)^2$$

$$\begin{aligned} \text{Arc length} &= \int_0^{1/2} \sqrt{1 + (y')^2} dx \\ &= \int_0^{1/2} \left(\frac{1 + x^2}{1 - x^2}\right) dx \\ &= \int_0^{1/2} \left(-1 + \frac{2}{1 - x^2}\right) dx \\ &= \int_0^{1/2} \left(-1 + \frac{1}{x + 1} + \frac{1}{1 - x}\right) dx \\ &= \left[-x + \ln(1 + x) - \ln(1 - x)\right]_0^{1/2} \\ &= \left(-\frac{1}{2} + \ln \frac{3}{2} - \ln \frac{1}{2}\right) \\ &= -\frac{1}{2} + \ln 3 - \ln 2 + \ln 2 \\ &= \ln 3 - \frac{1}{2} \approx 0.5986 \end{aligned}$$

$$11. \text{ Consider } \int \frac{1}{\ln x} dx.$$

$$\text{Let } u = \ln x, du = \frac{1}{x} dx, x = e^u. \text{ Then } \int \frac{1}{\ln x} dx = \int \frac{1}{u} e^u du = \int \frac{e^u}{u} du.$$

$$\text{If } \int \frac{1}{\ln x} dx \text{ were elementary, then } \int \frac{e^u}{u} du \text{ would be too, which is false.}$$

$$\text{Hence, } \int \frac{1}{\ln x} dx \text{ is not elementary.}$$

$$13. x^4 + 1 = (x^2 + ax + b)(x^2 + cx + d)$$

$$= x^4 + (a + c)x^3 + (ac + b + d)x^2 + (ad + bc)x + bd$$

$$a = -c, b = d = 1, a = \sqrt{2}$$

$$x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$$

$$\begin{aligned} \int_0^1 \frac{1}{x^4 + 1} dx &= \int_0^1 \frac{Ax + B}{x^2 + \sqrt{2}x + 1} dx + \int_0^1 \frac{Cx + D}{x^2 - \sqrt{2}x + 1} dx \\ &= \int_0^1 \frac{\frac{1}{2} + \frac{\sqrt{2}}{4}x}{x^2 + \sqrt{2}x + 1} dx - \int_0^1 \frac{-\frac{1}{2} + \frac{\sqrt{2}}{4}x}{x^2 + \sqrt{2}x + 1} dx \\ &= \frac{\sqrt{2}}{4} \left[\arctan(\sqrt{2}x + 1) + \arctan(\sqrt{2}x - 1) \right]_0^1 + \frac{\sqrt{2}}{8} \left[\ln(x^2 + \sqrt{2}x + 1) - \ln(x^2 - \sqrt{2}x + 1) \right]_0^1 \\ &= \frac{\sqrt{2}}{4} [\arctan(\sqrt{2} + 1) + \arctan(\sqrt{2} - 1)] + \frac{\sqrt{2}}{8} [\ln(2 + \sqrt{2}) - \ln(2 - \sqrt{2})] - \frac{\sqrt{2}}{4} \left[\frac{\pi}{4} - \frac{\pi}{4} \right] - \frac{\sqrt{2}}{8} [0] \\ &\approx 0.5554 + 0.3116 \\ &\approx 0.8670 \end{aligned}$$

15. Using a graphing utility,

$$(a) \lim_{x \rightarrow 0^+} \left(\cot x + \frac{1}{x} \right) = \infty$$

$$(b) \lim_{x \rightarrow 0^+} \left(\cot x - \frac{1}{x} \right) = 0$$

$$(c) \lim_{x \rightarrow 0^+} \left(\cot x + \frac{1}{x} \right) \left(\cot x - \frac{1}{x} \right) \approx -\frac{2}{3}.$$

Analytically,

$$(a) \lim_{x \rightarrow 0^+} \left(\cot x + \frac{1}{x} \right) = \infty + \infty = \infty$$

$$\begin{aligned} (b) \lim_{x \rightarrow 0^+} \left(\cot x - \frac{1}{x} \right) &= \lim_{x \rightarrow 0^+} \frac{x \cot x - 1}{x} = \lim_{x \rightarrow 0^+} \frac{x \cos x - \sin x}{x \sin x} \\ &= \lim_{x \rightarrow 0^+} \frac{\cos x - x \sin x - \cos x}{\sin x + x \cos x} = \lim_{x \rightarrow 0} \frac{-x \sin x}{\sin x + x \cos x} \\ &= \lim_{x \rightarrow 0^+} \frac{-\sin x - x \cos x}{\cos x + \cos x - x \sin x} = 0. \end{aligned}$$

$$\begin{aligned} (c) \left(\cot x + \frac{1}{x} \right) \left(\cot x - \frac{1}{x} \right) &= \cot^2 x - \frac{1}{x^2} \\ &= \frac{x^2 \cot^2 x - 1}{x^2} \\ \lim_{x \rightarrow 0^+} \frac{x^2 \cot^2 x - 1}{x^2} &= \lim_{x \rightarrow 0^+} \frac{2x \cot^2 x - 2x^2 \cot x \csc^2 x}{2x} \\ &= \lim_{x \rightarrow 0^+} \frac{\cot^2 x - x \cot x \csc^2 x}{1} \\ &= \lim_{x \rightarrow 0^+} \frac{\cos^2 x \sin x - x \cos x}{\sin^3 x} \\ &= \lim_{x \rightarrow 0^+} \frac{(1 - \sin^2 x) \sin x - x \cos x}{\sin^3 x} \\ &= \lim_{x \rightarrow 0^+} \frac{\sin x - x \cos x}{\sin^3 x} - 1 \end{aligned}$$

$$\begin{aligned} \text{Now, } \lim_{x \rightarrow 0^+} \frac{\sin x - x \cos x}{\sin^3 x} &= \lim_{x \rightarrow 0^+} \frac{\cos x - \cos x + x \sin x}{3 \sin^2 x \cos x} \\ &= \lim_{x \rightarrow 0^+} \frac{x}{3 \sin x \cdot \cos x} \\ &= \lim_{x \rightarrow 0^+} \left(\frac{x}{\sin x} \right) \frac{1}{3 \cos x} = \frac{1}{3}. \end{aligned}$$

$$\text{Thus, } \lim_{x \rightarrow 0^+} \left(\cot x + \frac{1}{x} \right) \left(\cot x - \frac{1}{x} \right) = \frac{1}{3} - 1 = -\frac{2}{3}.$$

The form $0 \cdot \infty$ is indeterminant.

$$17. \frac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x} = \frac{P_1}{x} + \frac{P_2}{x-1} + \frac{P_3}{x+4} + \frac{P_4}{x-3} \Rightarrow c_1 = 0, c_2 = 1, c_3 = -4, c_4 = 3$$

$$N(x) = x^3 - 3x^2 + 1$$

$$D'(x) = 4x^3 - 26x + 12$$

$$P_1 = \frac{N(0)}{D'(0)} = \frac{1}{12}$$

$$P_2 = \frac{N(1)}{D'(1)} = \frac{-1}{-10} = \frac{1}{10}$$

$$P_3 = \frac{N(-4)}{D'(-4)} = \frac{-111}{-140} = \frac{111}{140}$$

$$P_4 = \frac{N(3)}{D'(3)} = \frac{1}{42}$$

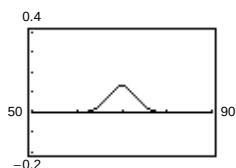
$$\text{Thus, } \frac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x} = \frac{1/12}{x} + \frac{1/10}{x-1} + \frac{111/140}{x+4} + \frac{1/42}{x-3}.$$

19. By parts,

$$\begin{aligned} \int_a^b f(x)g''(x) dx &= \left[f(x)g'(x) \right]_a^b - \int_a^b f'(x)g'(x) dx \\ &= - \int_a^b f'(x)g'(x) dx \\ &= \left[-f'(x)g(x) \right]_a^b + \int_a^b g(x)f''(x) dx \\ &= \int_a^b f''(x)g(x) dx. \end{aligned}$$

84. (a) $f(x) = \frac{1}{3\sqrt{2\pi}} e^{-(x-70)^2/18}$

$$\int_{50}^{90} f(x) dx \approx 1.0$$



(b) $P(72 \leq x < \infty) \approx 0.2525$

(c) $0.5 - P(70 \leq x \leq 72) \approx 0.5 - 0.2475 = 0.2525$

These are the same answers because by symmetry,

$$P(70 \leq x < \infty) = 0.5$$

and

$$0.5 = P(70 \leq x < \infty)$$

$$= P(70 \leq x \leq 72) + P(72 \leq x < \infty).$$

86. False. This is equivalent to Exercise 85.

88. True

Review Exercises for Chapter 7

$$\begin{aligned} 2. \int x e^{x^2-1} dx &= \frac{1}{2} \int e^{x^2-1} (2x) dx \\ &= \frac{1}{2} e^{x^2-1} + C \end{aligned}$$

$$\begin{aligned} 4. \int \frac{x}{\sqrt{1-x^2}} dx &= -\frac{1}{2} \int (1-x^2)^{-1/2} (-2x) dx \\ &= -\frac{1}{2} \frac{(1-x^2)^{1/2}}{1/2} + C \\ &= -\sqrt{1-x^2} + C \end{aligned}$$

$$\begin{aligned} 6. \int 2x \sqrt{2x-3} dx &= \int (u^4 + 3u^2) du = \frac{u^5}{5} + u^3 + C \\ &= \frac{2(2x-3)^{3/2}}{5} (x+1) + C \\ u = \sqrt{2x-3}, x &= \frac{u^2+3}{2}, dx = u du \end{aligned}$$

$$\begin{aligned} 8. \frac{x^4 + 2x^2 + x + 1}{x^4 + 2x^2 + 1} &= 1 + \frac{x}{(x^2+1)^2} \\ \int \frac{x^4 + 2x^2 + x + 1}{(x^2+1)^2} dx &= \int dx + \frac{1}{2} \int \frac{2x}{(x^2+1)^2} dx \\ &= x - \frac{1}{2(x^2+1)} + C \end{aligned}$$

$$10. \int (x^2 - 1)e^x dx = (x^2 - 1)e^x - 2 \int x e^x dx = (x^2 - 1)e^x - 2x e^x + 2 \int e^x dx = e^x(x^2 - 2x + 1) + 1$$

$$(1) dv = e^x dx \Rightarrow v = e^x$$

$$u = x^2 - 1 \Rightarrow du = 2x dx$$

$$(2) dv = e^x dx \Rightarrow v = e^x$$

$$u = x \Rightarrow du = dx$$

$$12. u = \arctan 2x, du = \frac{2}{1+4x^2} dx, dv = dx, v = x$$

$$\begin{aligned} \int \arctan 2x dx &= x \arctan 2x - \int \frac{2x}{1+4x^2} dx \\ &= x \arctan 2x - \frac{1}{4} \ln(1+4x^2) + C \end{aligned}$$

$$\begin{aligned} 14. \int \ln \sqrt{x^2-1} dx &= \frac{1}{2} \int \ln(x^2-1) dx \\ &= \frac{1}{2} x \ln|x^2-1| - \int \frac{x^2}{x^2-1} dx \\ &= \frac{1}{2} x \ln|x^2-1| - \int dx - \int \frac{1}{x^2-1} dx \\ &= \frac{1}{2} x \ln|x^2-1| - x - \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C \end{aligned}$$

$$dv = dx \Rightarrow v = x$$

$$u = \ln(x^2-1) \Rightarrow du = \frac{2x}{x^2-1} dx$$

$$\begin{aligned}
 16. \int e^x \arctan(e^x) dx &= e^x \arctan(e^x) - \int \frac{e^{2x}}{1 + e^{2x}} dx \\
 &= e^x \arctan(e^x) - \frac{1}{2} \ln(1 + e^{2x}) + C
 \end{aligned}$$

$$dv = e^x dx \quad \Rightarrow \quad v = e^x$$

$$u = \arctan e^x \Rightarrow du = \frac{e^x}{1 + e^{2x}} dx$$

$$18. \int \sin^2 \frac{\pi x}{2} dx = \int \frac{1}{2} (1 - \cos \pi x) dx = \frac{1}{2} \left[x - \frac{1}{\pi} \sin \pi x \right] + C = \frac{1}{2\pi} [\pi x - \sin \pi x] + C$$

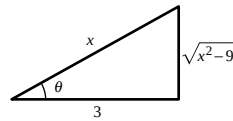
$$20. \int \tan \theta \sec^4 \theta d\theta = \int (\tan^3 \theta + \tan \theta) \sec^2 \theta d\theta = \frac{1}{4} \tan^4 \theta + \frac{1}{2} \tan^2 \theta + C_1$$

or

$$\int \tan \theta \sec^4 \theta d\theta = \int \sec^3 \theta (\sec \theta \tan \theta) d\theta = \frac{1}{4} \sec^4 \theta + C_2$$

$$\begin{aligned}
 22. \int \cos 2\theta (\sin \theta + \cos \theta)^2 d\theta &= \int (\cos^2 \theta - \sin^2 \theta) (\sin \theta + \cos \theta)^2 d\theta \\
 &= \int (\sin \theta + \cos \theta)^3 (\cos \theta - \sin \theta) d\theta = \frac{1}{4} (\sin \theta + \cos \theta)^4 + C
 \end{aligned}$$

$$\begin{aligned}
 24. \int \frac{\sqrt{x^2 - 9}}{x} dx &= \int \frac{3 \tan \theta}{3 \sec \theta} (3 \sec \theta \tan \theta d\theta) \\
 &= 3 \int \tan^2 \theta d\theta \\
 &= 3 \int (\sec^2 \theta - 1) d\theta \\
 &= 3(\tan \theta - \theta) + C \\
 &= \sqrt{x^2 - 9} - 3 \operatorname{arcsec} \left(\frac{x}{3} \right) + C
 \end{aligned}$$



$$x = 3 \sec \theta, dx = 3 \sec \theta \tan \theta d\theta, \sqrt{x^2 - 9} = 3 \tan \theta$$

$$\begin{aligned}
 26. \int \sqrt{9 - 4x^2} dx &= \frac{1}{2} \int \sqrt{9 - (2x)^2} (2) dx \\
 &= \frac{1}{2} \cdot \frac{1}{2} \left[9 \arcsin \frac{2x}{3} + 2x \sqrt{9 - 4x^2} \right] + C \\
 &= \frac{9}{4} \arcsin \frac{2x}{3} + \frac{x}{2} \sqrt{9 - 4x^2} + C
 \end{aligned}$$

$$\begin{aligned}
 28. \int \frac{\sin \theta}{1 + 2 \cos^2 \theta} d\theta &= \frac{-1}{\sqrt{2}} \int \frac{1}{1 + 2 \cos^2 \theta} (-\sqrt{2} \sin \theta) d\theta \\
 &= \frac{-1}{\sqrt{2}} \arctan(\sqrt{2} \cos \theta) + C
 \end{aligned}$$

$$u = \sqrt{2} \cos \theta, du = -\sqrt{2} \sin \theta d\theta$$

$$\begin{aligned}
 30. \text{ (a) } \int x\sqrt{4+x} \, dx &= 64 \int \tan^3 \theta \sec^3 \theta \, d\theta \\
 &= 64 \int (\sec^4 \theta - \sec^2 \theta) \sec \theta \tan \theta \, d\theta \\
 &= \frac{64 \sec^3 \theta}{15} (3 \sec^3 \theta - 5) + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$x = 4 \tan^2 \theta, \, dx = 8 \tan \theta \sec^2 \theta \, d\theta,$$

$$\sqrt{4+x} = 2 \sec \theta$$

$$\begin{aligned}
 \text{(c) } \int x\sqrt{4+x} \, dx &= \int (u^{3/2} - 4u^{1/2}) \, du \\
 &= \frac{2u^{3/2}}{15} (3u-20) + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$u = 4+x, \, du = dx$$

$$\begin{aligned}
 \text{(b) } \int x\sqrt{4+x} \, dx &= 2 \int (u^4 - 4u^2) \, du \\
 &= \frac{2u^3}{15} (3u^2 - 20) + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$u^2 = 4+x, \, dx = 2u \, du$$

$$\begin{aligned}
 \text{(d) } \int x\sqrt{4+x} \, dx &= \frac{2x}{3} (4+x)^{3/2} - \frac{2}{3} \int (4+x)^{3/2} \, dx \\
 &= \frac{2x}{3} (4+x)^{3/2} - \frac{4}{15} (4+x)^{5/2} + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$dv = \sqrt{4+x} \, dx \Rightarrow v = \frac{2}{3} (4+x)^{3/2}$$

$$u = x \Rightarrow du = dx$$

$$32. \quad \frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x} = 2x - 3 + \frac{4}{x} - \frac{3}{x-1}$$

$$\int \frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x} \, dx = \int \left(2x - 3 + \frac{4}{x} - \frac{3}{x-1} \right) \, dx = x^2 - 3x + 4 \ln|x| - 3 \ln|x-1| + C$$

$$34. \quad \frac{4x-2}{3(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$4x-2 = 3A(x-1) + 3B$$

$$\text{Let } x = 1: 2 = 3B \Rightarrow B = \frac{2}{3}$$

$$\text{Let } x = 2: 6 = 3A + 3B \Rightarrow A = \frac{4}{3}$$

$$\int \frac{4x-2}{3(x-1)^2} \, dx = \frac{4}{3} \int \frac{1}{x-1} \, dx + \frac{2}{3} \int \frac{1}{(x-1)^2} \, dx = \frac{4}{3} \ln|x-1| - \frac{2}{3(x-1)} + C = \frac{2}{3} \left(2 \ln|x-1| - \frac{1}{x-1} \right) + C$$

$$\begin{aligned}
 36. \quad \int \frac{\sec^2 \theta}{\tan \theta (\tan \theta - 1)} \, d\theta &= \int \frac{1}{u(u-1)} \, du = \int \frac{1}{u-1} \, du - \int \frac{1}{u} \, du \\
 &= \ln|u-1| - \ln|u| + C = \ln \left| \frac{\tan \theta - 1}{\tan \theta} \right| + C = \ln|1 - \cot \theta| + C
 \end{aligned}$$

$$u = \tan \theta, \, du = \sec^2 \theta \, d\theta$$

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$1 = A(u-1) + Bu$$

$$\text{Let } u = 0: 1 = -A \Rightarrow A = -1$$

$$\text{Let } u = 1: 1 = B$$

$$\begin{aligned}
 38. \int \frac{x}{\sqrt{2+3x}} dx &= \frac{-2(4-3x)}{27} \sqrt{2+3x} + C \quad (\text{Formula 21}) & 40. \int \frac{x}{1+e^{x^2}} dx &= \frac{1}{2} \int \frac{1}{1+e^u} du & (u = x^2) \\
 &= \frac{6x-8}{27} \sqrt{2+3x} + C & &= \frac{1}{2} [u - \ln(1+e^u)] + C & (\text{Formula 84}) \\
 & & &= \frac{1}{2} [x^2 - \ln(1+e^{x^2})] + C
 \end{aligned}$$

$$\begin{aligned}
 42. \int \frac{3}{2x\sqrt{9x^2-1}} dx &= \frac{3}{2} \int \frac{1}{3x\sqrt{(3x)^2-1}} 3 dx \quad (u = 3x) \\
 &= \frac{3}{2} \operatorname{arcsec}|3x| + C \quad (\text{Formula 33})
 \end{aligned}$$

$$\begin{aligned}
 44. \int \frac{1}{1+\tan \pi x} dx &= \frac{1}{\pi} \int \frac{1}{1+\tan \pi x} (\pi) dx & (u = \pi x) \\
 &= \frac{1}{\pi} \frac{1}{2} [\pi x + \ln|\cos \pi x + \sin \pi x|] + C \quad (\text{Formula 71})
 \end{aligned}$$

$$\begin{aligned}
 46. \int \tan^n x dx &= \int \tan^{n-2} x (\sec^2 x - 1) dx & 48. \int \frac{\csc \sqrt{2x}}{\sqrt{x}} dx &= \sqrt{2} \int \csc \sqrt{2x} \left(\frac{1}{\sqrt{2x}} \right) dx \\
 &= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx & &= -\sqrt{2} \ln|\csc \sqrt{2x} + \cot \sqrt{2x}| + C \\
 &= \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx & u = \sqrt{2x}, du &= \frac{1}{\sqrt{2x}} dx
 \end{aligned}$$

$$\begin{aligned}
 50. \int \sqrt{1+\sqrt{x}} dx &= \int u(4u^3-4u) du = \int (4u^4-4u^2) du = \frac{4u^5}{5} - \frac{4u^3}{3} + C = \frac{4}{15} (1+\sqrt{x})^{3/2} (3\sqrt{x}-2) + C \\
 u &= \sqrt{1+\sqrt{x}}, x = u^4 - 2u^2 + 1, dx = (4u^3 - 4u) du
 \end{aligned}$$

$$\begin{aligned}
 52. \frac{3x^3+4x}{(x^2+1)^2} &= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \\
 3x^3+4x &= (Ax+B)(x^2+1) + Cx+D \\
 &= Ax^3+Bx^2+(A+C)x+(B+D) \\
 A=3, B=0, A+C=4 &\Rightarrow C=1, \\
 B+D=0 &\Rightarrow D=0
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{3x^3+4x}{(x^2+1)^2} dx &= 3 \int \frac{x}{x^2+1} dx + \int \frac{x}{(x^2+1)^2} dx \\
 &= \frac{3}{2} \ln(x^2+1) - \frac{1}{2(x^2+1)} + C
 \end{aligned}$$

$$\begin{aligned}
 54. \int (\sin \theta + \cos \theta)^2 d\theta &= \int (\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta) d\theta \\
 &= \int (1 + \sin 2\theta) d\theta = \theta - \frac{1}{2} \cos 2\theta + C = \frac{1}{2} (2\theta - \cos 2\theta) + C
 \end{aligned}$$

$$\begin{aligned}
 56. \quad y &= \int \frac{\sqrt{4-x^2}}{2x} dx = \int \frac{2 \cos \theta (2 \cos \theta) d\theta}{4 \sin \theta} \\
 &= \int (\csc \theta - \sin \theta) d\theta \\
 &= [-\ln|\csc \theta + \cos \theta| + \cos \theta] + C \\
 &= -\ln \left| \frac{2 + \sqrt{4-x^2}}{x} \right| + \frac{\sqrt{4-x^2}}{2} + C
 \end{aligned}$$

$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$$

$$\begin{aligned}
 58. \quad y &= \int \sqrt{1-\cos \theta} d\theta = \int \frac{\sin \theta}{\sqrt{1+\cos \theta}} d\theta = -\int (1+\cos \theta)^{-1/2} (-\sin \theta) d\theta = -2\sqrt{1+\cos \theta} + C \\
 u &= 1 + \cos \theta, du = -\sin \theta d\theta
 \end{aligned}$$

$$\begin{aligned}
 60. \quad \int_0^1 \frac{x}{(x-2)(x-4)} dx &= \left[2 \ln|x-4| - \ln|x-2| \right]_0^1 \\
 &= 2 \ln 3 - 2 \ln 4 + \ln 2 \\
 &= \ln \frac{9}{8} \approx 0.118
 \end{aligned}$$

$$62. \quad \int_0^2 x e^{3x} dx = \left[\frac{e^{3x}}{9} (3x-1) \right]_0^2 = \frac{1}{9} (5e^6 + 1) \approx 224.238$$

$$64. \quad \int_0^3 \frac{x}{\sqrt{1+x}} dx = \left[\frac{-2(2-x)}{3} \sqrt{1+x} \right]_0^3 = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$$

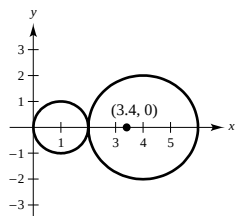
$$\begin{aligned}
 66. \quad A &= \int_0^4 \frac{1}{25-x^2} dx \\
 &= \left[-\frac{1}{10} \ln \left| \frac{x-5}{x+5} \right| \right]_0^4 = -\frac{1}{10} \ln \frac{1}{9} = \frac{1}{10} \ln 9 \approx 0.220
 \end{aligned}$$

68. By symmetry, $\bar{y} = 0$.

$$A = \pi + 4\pi = 5\pi$$

$$\begin{aligned}
 \bar{x} &= \frac{1(\pi) + 4(4\pi)}{\pi + 4\pi} \\
 &= \frac{17\pi}{5\pi} = 3.4
 \end{aligned}$$

$$(\bar{x}, \bar{y}) = (3.4, 0)$$



$$70. \quad s = \int_0^\pi \sqrt{1 + \sin^2 2x} dx \approx 3.82$$

$$72. \quad \lim_{x \rightarrow 0} \frac{\sin \pi x}{\sin 2\pi x} = \lim_{x \rightarrow 0} \frac{\pi \cos \pi x}{2\pi \cos 2\pi x} = \frac{\pi}{2\pi} = \frac{1}{2}$$

$$74. \quad \lim_{x \rightarrow \infty} x e^{-x^2} = \lim_{x \rightarrow \infty} \frac{x}{e^{x^2}} = \lim_{x \rightarrow \infty} \frac{1}{2x e^{x^2}} = 0$$

$$76. \quad y = \lim_{x \rightarrow 1^+} (x-1)^{\ln x}$$

$$\ln y = \lim_{x \rightarrow 1^+} [(\ln x) \ln(x-1)]$$

$$\begin{aligned}
 &= \lim_{x \rightarrow 1^+} \left[\frac{\ln(x-1)}{\frac{1}{\ln x}} \right] = \lim_{x \rightarrow 1^+} \left[\frac{\frac{1}{x-1}}{\left(\frac{1}{x} \right) - 1} \right] = \lim_{x \rightarrow 1^+} \left[\frac{-\ln^2 x}{\frac{x-1}{x}} \right] = \lim_{x \rightarrow 1^+} \left[\frac{-2\left(\frac{1}{x}\right)(\ln x)}{\frac{1}{x^2}} \right]
 \end{aligned}$$

$$= \lim_{x \rightarrow 1^+} 2x(\ln x) = 0$$

Since $\ln y = 0$, $y = 1$.

$$\begin{aligned}
 78. \lim_{x \rightarrow 1^+} \left(\frac{2}{\ln x} - \frac{2}{x-1} \right) &= \lim_{x \rightarrow 1^+} \left[\frac{2x - 2 - 2 \ln x}{(\ln x)(x-1)} \right] \\
 &= \lim_{x \rightarrow 1^+} \left[\frac{2 - (2/x)}{(x-1)(1/x) + \ln x} \right] \\
 &= \lim_{x \rightarrow 1^+} \frac{2x - 2}{(x-1) + x \ln x} = \lim_{x \rightarrow 1^+} \frac{2}{1 + 1 + \ln x} = 1
 \end{aligned}$$

$$80. \int_0^1 \frac{6}{x-1} dx = \lim_{b \rightarrow 1^-} \left[6 \ln |x-1| \right]_0^b = -\infty$$

Diverges

$$82. \int_0^\infty \frac{e^{-1/x}}{x^2} dx = \lim_{\substack{a \rightarrow 0^+ \\ b \rightarrow \infty}} \left[e^{-1/x} \right]_a^b = 1 - 0 = 1$$

$$\begin{aligned}
 84. V &= \pi \int_0^\infty (xe^{-x})^2 dx \\
 &= \pi \int_0^\infty x^2 e^{-2x} dx \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{\pi e^{-2x}}{4} (2x^2 + 2x + 1) \right]_0^b = \frac{\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 86. \int_2^\infty \left[\frac{1}{x^5} + \frac{1}{x^{10}} + \frac{1}{x^{15}} \right] dx &< \int_2^\infty \frac{1}{x^5 - 1} dx < \int_2^\infty \left[\frac{1}{x^5} + \frac{1}{x^{10}} + \frac{2}{x^{15}} \right] dx \\
 \lim_{b \rightarrow \infty} \left[-\frac{1}{4x^4} - \frac{1}{9x^9} - \frac{1}{14x^{14}} \right]_2^b &< \int_2^\infty \frac{1}{x^5 - 1} dx < \lim_{b \rightarrow \infty} \left[-\frac{1}{4x^4} - \frac{1}{9x^9} - \frac{1}{7x^{14}} \right]_2^b \\
 0.015846 &< \int_2^\infty \frac{1}{x^5 - 1} dx < 0.015851
 \end{aligned}$$

Problem Solving for Chapter 7

$$\begin{aligned}
 2. (a) \int_0^1 \ln x dx &= \lim_{b \rightarrow 0^+} \left[x \ln x - x \right]_b^1 \\
 &= (-1) - \lim_{b \rightarrow 0^+} (b \ln b - b) = -1
 \end{aligned}$$

$$\text{Note: } \lim_{b \rightarrow 0^+} b \ln b = \lim_{b \rightarrow 0^+} \frac{\ln b}{b^{-1}} = \lim_{b \rightarrow 0^+} \frac{1/b}{-1/b^2} = 0$$

$$\begin{aligned}
 \int_0^1 (\ln x)^2 dx &= \lim_{b \rightarrow 0^+} \left[x(\ln x)^2 - 2x \ln x + 2x \right]_b^1 \\
 &= 2 - \lim_{b \rightarrow 0^+} (b(\ln b)^2 - 2b \ln b + 2b) = 2
 \end{aligned}$$

(b) Note first that $\lim_{b \rightarrow 0^+} b(\ln b)^n = 0$ (Mathematical induction).

$$\text{Also, } \int (\ln x)^{n+1} dx = x(\ln x)^{n+1} - (n+1) \int (\ln x)^n dx.$$

$$\text{Assume } \int_0^1 (\ln x)^n dx = (-1)^n n!.$$

$$\begin{aligned}
 \text{Then, } \int_0^1 (\ln x)^{n+1} dx &= \lim_{b \rightarrow 0^+} \left[x(\ln x)^{n+1} \right]_b^1 - (n+1) \int_0^1 (\ln x)^n dx \\
 &= 0 - (n+1)(-1)^n n! = (-1)^{n+1} (n+1)!.
 \end{aligned}$$

$$4. \quad \lim_{x \rightarrow \infty} \left(\frac{x-c}{x+c} \right)^x = \frac{1}{4}$$

$$\lim_{x \rightarrow \infty} x \ln \left(\frac{x-c}{x+c} \right) = \ln \frac{1}{4}$$

$$\lim_{x \rightarrow \infty} \frac{\ln(x-c) - \ln(x+c)}{1/x} = -\ln 4$$

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x-c} - \frac{1}{x+c}}{-\frac{1}{x^2}} = -\ln 4$$

$$\lim_{x \rightarrow \infty} \frac{2c}{(x-c)(x+c)} (-x^2) = -\ln 4$$

$$\lim_{x \rightarrow \infty} \frac{2cx^2}{x^2 - c^2} = \ln 4$$

$$2c = \ln 4$$

$$2x = 2 \ln 2$$

$$c = \ln 2$$

$$6. \quad \sin \theta = BD, \cos \theta = OD$$

$$\text{Area } \triangle DAB = \frac{1}{2}(DA)(BD) = \frac{1}{2}(1 - \cos \theta) \sin \theta$$

$$\text{Shaded area} = \frac{\theta}{2} - \frac{1}{2}(1)(BD) = \frac{\theta}{2} - \frac{1}{2} \sin \theta$$

$$R = \frac{\triangle DAB}{\text{Shaded area}} = \frac{1/2(1 - \cos \theta) \sin \theta}{1/2(\theta - \sin \theta)}$$

$$\lim_{\theta \rightarrow 0^+} R = \lim_{\theta \rightarrow 0^+} \frac{(1 - \cos \theta) \sin \theta}{\theta - \sin \theta} = \lim_{\theta \rightarrow 0^+} \frac{(1 - \cos \theta) \cos \theta + \sin^2 \theta}{1 - \cos \theta}$$

$$= \lim_{\theta \rightarrow 0^+} \frac{(1 - \cos \theta)(-\sin \theta) + \cos \theta \sin \theta + 2 \sin \theta \cos \theta}{\sin \theta}$$

$$= \lim_{\theta \rightarrow 0^+} \frac{-\sin \theta - 4 \cos \theta \sin \theta}{\sin \theta} = \lim_{\theta \rightarrow 0} \frac{4 \cos \theta - 1}{1} = 3$$

$$8. \quad u = \tan \frac{x}{2}, \cos x = \frac{1-u^2}{1+u^2}, 2 + \cos x = 2 + \frac{1-u^2}{1+u^2} = \frac{3+u^2}{1+u^2}$$

$$dx = \frac{2 du}{1+u^2}$$

$$\int_0^{\pi/2} \frac{1}{2 + \cos x} dx = \int_0^1 \left(\frac{1+u^2}{3+u^2} \right) \left(\frac{2}{1+u^2} \right) du$$

$$= \int_0^1 \frac{2}{3+u^2} du$$

$$= \left[2 \frac{1}{\sqrt{3}} \arctan \left(\frac{u}{\sqrt{3}} \right) \right]_0^1$$

$$= \frac{2}{\sqrt{3}} \arctan \left(\frac{1}{\sqrt{3}} \right)$$

$$= \frac{2}{\sqrt{3}} \frac{\pi}{6} = \frac{\pi\sqrt{3}}{9} \approx 0.6046$$

10. Let $u = cx$, $du = c dx$.

$$\int_0^b e^{-c^2 x^2} dx = \int_0^{cb} e^{-u^2} \frac{du}{c} = \frac{1}{c} \int_0^{cb} e^{-u^2} du$$

$$\text{As } b \rightarrow \infty, cb \rightarrow \infty. \text{ Hence, } \int_0^\infty e^{-c^2 x^2} dx = \frac{1}{c} \int_0^\infty e^{-u^2} du.$$

$\bar{x} = 0$ by symmetry.

$$\begin{aligned} \bar{y} &= \frac{M_x}{m} = \frac{2 \int_0^\infty \frac{(e^{-c^2 x^2})}{2} dx}{2 \int_0^\infty e^{-c^2 x^2} dx} \\ &= \frac{1}{2} \frac{\int_0^\infty e^{-2c^2 x^2} dx}{\int_0^\infty e^{-c^2 x^2} dx} \\ &= \frac{1}{2} \frac{\frac{1}{\sqrt{2}c} \int_0^\infty e^{-x^2} dx}{\frac{1}{c} \int_0^\infty e^{-x^2} dx} \\ &= \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \end{aligned}$$

$$\text{Thus, } (\bar{x}, \bar{y}) = \left(0, \frac{\sqrt{2}}{4}\right).$$

12. (a) Let $y = f^{-1}(x)$, $f(y) = x$, $dx = f'(y) dy$.

$$\int f^{-1}(x) dx = \int y f'(y) dy$$

$$= y f(y) - \int f(y) dy$$

$$= x f^{-1}(x) - \int f(y) dy$$

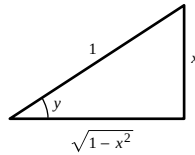
$$\left[\begin{array}{l} u = y, du = dy \\ dv = f'(y) dy, v = f(y) \end{array} \right]$$

- (b) $f^{-1}(x) = \arcsin x = y$, $f(y) = \sin y$

$$\int \arcsin x dx = x \arcsin x - \int \sin y dy$$

$$= x \arcsin x + \cos y + C$$

$$= x \arcsin x + \sqrt{1-x^2} + C$$



- (c) $f(x) = e^x$, $f^{-1}(x) = \ln x = y$ $x = 1 \Leftrightarrow y = 0$; $x = e \Leftrightarrow y = 1$

$$\int_1^e \ln x dx = \left[x \ln x \right]_1^e - \int_0^1 e^y dy$$

$$= e - \left[e^y \right]_0^1$$

$$= e - (e - 1) = 1$$

14. (a) Let $x = \frac{\pi}{2} - u$, $dx = -du$.

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x} dx = \int_{\pi/2}^0 \frac{\sin\left(\frac{\pi}{2} - u\right)}{\cos\left(\frac{\pi}{2} - u\right) + \sin\left(\frac{\pi}{2} - u\right)} (-du) \\ &= \int_0^{\pi/2} \frac{\cos u}{\sin u + \cos u} du \end{aligned}$$

Hence,

$$\begin{aligned} 2I &= \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x} dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} dx \\ &= \int_0^{\pi/2} 1 dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad I &= \int_{\pi/2}^0 \frac{\sin^n\left(\frac{\pi}{2} - u\right)}{\cos^n\left(\frac{\pi}{2} - u\right) + \sin^n\left(\frac{\pi}{2} - u\right)} (-du) \\ &= \int_0^{\pi/2} \frac{\cos^n u}{\sin^n u + \cos^n u} du \end{aligned}$$

$$\text{Thus, } 2I = \int_0^{\pi/2} 1 dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}.$$

$$16. \frac{N(x)}{D(x)} = \frac{P_1}{x - c_1} + \frac{P_2}{x - c_2} + \cdots + \frac{P_n}{x - c_n}$$

$$N(x) = P_1(x - c_2)(x - c_3) \cdots (x - c_n) + P_2(x - c_1)(x - c_3) \cdots (x - c_n) + \cdots + P_n(x - c_1)(x - c_2) \cdots (x - c_{n-1})$$

$$\text{Let } x = c_1: N(c_1) = P_1(c_1 - c_2)(c_1 - c_3) \cdots (c_1 - c_n)$$

$$P_1 = \frac{N(c_1)}{(c_1 - c_2)(c_1 - c_3) \cdots (c_1 - c_n)}$$

$$\text{Let } x = c_2: N(c_2) = P_2(c_2 - c_1)(c_2 - c_3) \cdots (c_2 - c_n)$$

$$P_2 = \frac{N(c_2)}{(c_2 - c_1)(c_2 - c_3) \cdots (c_2 - c_n)}$$

$\vdots$

$\vdots$

$$\text{Let } x = c_n: N(c_n) = P_n(c_n - c_1)(c_n - c_2) \cdots (c_n - c_{n-1})$$

$$P_n = \frac{N(c_n)}{(c_n - c_1)(c_n - c_2) \cdots (c_n - c_{n-1})}$$

If $D(x) = (x - c_1)(x - c_2)(x - c_3) \cdots (x - c_n)$, then by the Product Rule

$$D'(x) = (x - c_2)(x - c_3) \cdots (x - c_n) + (x - c_1)(x - c_3) \cdots (x - c_n) + \cdots + (x - c_1)(x - c_2)(x - c_3) \cdots (x - c_{n-1})$$

and

$$D'(c_1) = (c_1 - c_2)(c_1 - c_3) \cdots (c_1 - c_n)$$

$$D'(c_2) = (c_2 - c_1)(c_2 - c_3) \cdots (c_2 - c_n)$$

$\vdots$

$$D'(c_n) = (c_n - c_1)(c_n - c_2) \cdots (c_n - c_{n-1}).$$

Thus, $P_k = N(c_k)/D'(c_k)$ for $k = 1, 2, \dots, n$.

$$\begin{aligned}
 18. \quad s(t) &= \int \left[-32t + 12,000 \ln \frac{50,000}{50,000 - 400t} \right] dt \\
 &= -16t^2 + 12,000 \int [\ln 50,000 - \ln(50,000 - 400t)] dt \\
 &= 16t^2 + 12,000t \ln 50,000 - 12,000 \left[t \ln(50,000 - 400t) - \int \frac{-400t}{50,000 - 400t} dt \right] \\
 &= -16t^2 + 12,000t \ln \frac{50,000}{50,000 - 400t} + 12,000t \int \left[1 - \frac{50,000}{50,000 - 400t} \right] dt \\
 &= -16t^2 + 12,000t \ln \frac{50,000}{50,000 - 400t} + 12,000t + 1,500,000 \ln(50,000 - 400t) + C \\
 s(0) &= 1,500,000 \ln 50,000 + C = 0 \\
 C &= -1,500,000 \ln 50,000 \\
 s(t) &= -16t^2 + 12,000t \left[1 + \ln \frac{50,000}{50,000 - 400t} \right] + 1,500,000 \ln \frac{50,000 - 400t}{50,000}
 \end{aligned}$$

When $t = 100$, $s(100) \approx 557,168.626$ feet

$$\begin{aligned}
 20. \quad \text{Let } u &= (x - a)(x - b), \quad du = [(x - a) + (x - b)] dx, \quad dv = f'(x) dx, \quad v = f(x). \\
 \int_a^b (x - a)(x - b) dx &= \left[(x - a)(x - b) f'(x) \right]_a^b - \int_a^b [(x - a) + (x - b)] f'(x) dx \\
 &= - \int_a^b (2x - a - b) f'(x) dx \quad \left(\begin{array}{l} u = 2x - a - b \\ dv = f'(x) dx \end{array} \right) \\
 &= \left[-(2x - a - b) f(x) \right]_a^b + \int_a^b 2f(x) dx \\
 &= 2 \int_a^b f(x) dx
 \end{aligned}$$

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Infinite Series

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CHAPTER 8

Infinite Series

Section 8.1 Sequences

Solutions to Odd-Numbered Exercises

1. $a_n = 2^n$

$$a_1 = 2^1 = 2$$

$$a_2 = 2^2 = 4$$

$$a_3 = 2^3 = 8$$

$$a_4 = 2^4 = 16$$

$$a_5 = 2^5 = 32$$

3. $a_n = \left(-\frac{1}{2}\right)^n$

$$a_1 = \left(-\frac{1}{2}\right)^1 = -\frac{1}{2}$$

$$a_2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$a_3 = \left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$$

$$a_4 = \left(-\frac{1}{2}\right)^4 = \frac{1}{16}$$

$$a_5 = \left(-\frac{1}{2}\right)^5 = -\frac{1}{32}$$

5. $a_n = \sin \frac{n\pi}{2}$

$$a_1 = \sin \frac{\pi}{2} = 1$$

$$a_2 = \sin \pi = 0$$

$$a_3 = \sin \frac{3\pi}{2} = -1$$

$$a_4 = \sin 2\pi = 0$$

$$a_5 = \sin \frac{5\pi}{2} = 1$$

7. $a_n = \frac{(-1)^{n(n+1)/2}}{n^2}$

$$a_1 = \frac{(-1)^1}{1^2} = -1$$

$$a_2 = \frac{(-1)^3}{2^2} = -\frac{1}{4}$$

$$a_3 = \frac{(-1)^6}{3^2} = \frac{1}{9}$$

$$a_4 = \frac{(-1)^{10}}{4^2} = \frac{1}{16}$$

$$a_5 = \frac{(-1)^{15}}{5^2} = -\frac{1}{25}$$

9. $a_n = 5 - \frac{1}{n} + \frac{1}{n^2}$

$$a_1 = 5 - 1 + 1 = 5$$

$$a_2 = 5 - \frac{1}{2} + \frac{1}{4} = \frac{19}{4}$$

$$a_3 = 5 - \frac{1}{3} + \frac{1}{9} = \frac{43}{9}$$

$$a_4 = 5 - \frac{1}{4} + \frac{1}{16} = \frac{77}{16}$$

$$a_5 = 5 - \frac{1}{5} + \frac{1}{25} = \frac{121}{25}$$

11. $a_n = \frac{3^n}{n!}$

$$a_1 = \frac{3}{1!} = 3$$

$$a_2 = \frac{3^2}{2!} = \frac{9}{2}$$

$$a_3 = \frac{3^3}{3!} = \frac{27}{6}$$

$$a_4 = \frac{3^4}{4!} = \frac{81}{24}$$

$$a_5 = \frac{3^5}{5!} = \frac{243}{120}$$

13. $a_1 = 3, a_{k+1} = 2(a_k - 1)$

$$a_2 = 2(a_1 - 1)$$

$$= 2(3 - 1) = 4$$

$$a_3 = 2(a_2 - 1)$$

$$= 2(4 - 1) = 6$$

$$a_4 = 2(a_3 - 1)$$

$$= 2(6 - 1) = 10$$

$$a_5 = 2(a_4 - 1)$$

$$= 2(10 - 1) = 18$$

15. $a_1 = 32, a_{k+1} = \frac{1}{2}a_k$

$$a_2 = \frac{1}{2}a_1 = \frac{1}{2}(32) = 16$$

$$a_3 = \frac{1}{2}a_2 = \frac{1}{2}(16) = 8$$

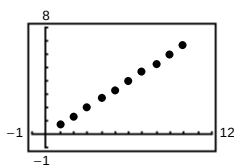
$$a_4 = \frac{1}{2}a_3 = \frac{1}{2}(8) = 4$$

$$a_5 = \frac{1}{2}a_4 = \frac{1}{2}(4) = 2$$

17. Because $a_1 = 8/(1 + 1) = 4$ and $a_2 = 8/(2 + 1) = \frac{8}{3}$, the sequence matches graph (d).

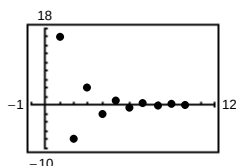
19. This sequence decreases and $a_1 = 4$, $a_2 = 4(0.5) = 2$. Matches (c).

21.



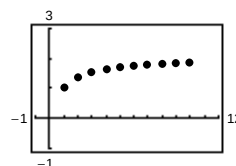
$$a_n = \frac{2}{3}n, n = 1, \dots, 10$$

23.



$$a_n = 16(-0.5)^{n-1}, n = 1, \dots, 10$$

25.



$$a_n = \frac{2n}{n+1}, n = 1, 2, \dots, 10$$

27. $a_n = 3n - 1$

$$a_5 = 3(5) - 1 = 14$$

$$a_6 = 3(6) - 1 = 17$$

Add 3 to preceeding term.

29. $a_n = \frac{3}{(-2)^{n-1}}$

$$a_n = \frac{3}{(-2)^4} = \frac{3}{16}$$

$$a_6 = \frac{3}{(-2)^5} = -\frac{3}{32}$$

Multiply the preceeding term by $-\frac{1}{2}$.

31. $\frac{10!}{8!} = \frac{8!(9)(10)}{8!}$
 $= (9)(10) = 90$

33. $\frac{(n+1)!}{n!} = \frac{n!(n+1)}{n!}$
 $= n+1$

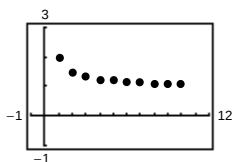
35. $\frac{(2n-1)!}{(2n+1)!} = \frac{(2n-1)!}{(2n-1)!(2n)(2n+1)}$
 $= \frac{1}{2n(2n+1)}$

37. $\lim_{n \rightarrow \infty} \frac{5n^2}{n^2 + 2} = 5$

39. $\lim_{n \rightarrow \infty} \frac{2n}{\sqrt{n^2 + 1}} = \lim_{n \rightarrow \infty} \frac{2}{\sqrt{1 + (1/n^2)}}$
 $= \frac{2}{1} = 2$

41. $\lim_{n \rightarrow \infty} \sin\left(\frac{1}{n}\right) = 0$

43.



The graph seems to indicate that the sequence converges to 1. Analytically,

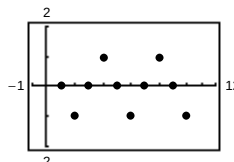
$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n+1}{n} = \lim_{x \rightarrow \infty} \frac{x+1}{x} = \lim_{x \rightarrow \infty} 1 = 1.$$

47. $\lim_{n \rightarrow \infty} (-1)^n \left(\frac{n}{n+1} \right)$

does not exist (oscillates between -1 and 1), diverges.

51. $\lim_{n \rightarrow \infty} \frac{1 + (-1)^n}{n} = 0$, converges

45.



The graph seems to indicate that the sequence diverges. Analytically, the sequence is

$$\{a_n\} = \{0, -1, 0, 1, 0, -1, \dots\}.$$

Hence, $\lim_{n \rightarrow \infty} a_n$ does not exist.

49. $\lim_{n \rightarrow \infty} \frac{3n^2 - n + 4}{2n^2 + 1} = \frac{3}{2}$, converges

53. $\lim_{n \rightarrow \infty} \frac{\ln(n^3)}{2n} = \lim_{n \rightarrow \infty} \frac{3 \ln(n)}{2n}$
 $= \lim_{n \rightarrow \infty} \frac{3 \left(\frac{1}{n} \right)}{2} = 0$, converges

(L'Hôpital's Rule)

$$55. \lim_{n \rightarrow \infty} \left(\frac{3}{4}\right)^n = 0, \text{ converges}$$

$$57. \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} = \lim_{n \rightarrow \infty} (n+1) = \infty, \text{ diverges}$$

$$59. \lim_{n \rightarrow \infty} \left(\frac{n-1}{n} - \frac{n}{n-1}\right) = \lim_{n \rightarrow \infty} \frac{(n-1)^2 - n^2}{n(n-1)} \\ = \lim_{n \rightarrow \infty} \frac{1-2n}{n^2-n} = 0, \text{ converges}$$

$$61. \lim_{n \rightarrow \infty} \frac{n^p}{e^n} = 0, \text{ converges} \\ (p > 0, n \geq 2)$$

$$63. a_n = \left(1 + \frac{k}{n}\right)^n \\ \lim_{n \rightarrow \infty} \left(1 + \frac{k}{n}\right)^n = \lim_{u \rightarrow 0} [(1+u)^{1/u}]^k = e^k$$

where $u = \frac{k}{n}$, converges

$$65. \lim_{n \rightarrow \infty} \frac{\sin n}{n} = \lim_{n \rightarrow \infty} (\sin n) \frac{1}{n} = 0, \text{ converges}$$

$$67. a_n = 3n - 2$$

$$69. a_n = n^2 - 2$$

$$71. a_n = \frac{n+1}{n+2}$$

$$73. a_n = \frac{(-1)^{n-1}}{2^{n-2}}$$

$$75. a_n = 1 + \frac{1}{n} = \frac{n+1}{n}$$

$$77. a_n = \frac{n}{(n+1)(n+2)}$$

$$79. a_n = \frac{(-1)^{n-1}}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{(-1)^{n-1} 2^n n!}{(2n)!}$$

$$81. a_n = 4 - \frac{1}{n} < 4 - \frac{1}{n+1} = a_{n+1}, \\ \text{monotonic; } |a_n| < 4 \text{ bounded.}$$

$$83. \frac{n}{2^{n+2}} \stackrel{?}{\geq} \frac{n+1}{2^{(n+1)+2}}$$

$$2^{n+3}n \stackrel{?}{\geq} 2^{n+2}(n+1)$$

$$2n \stackrel{?}{\geq} n+1$$

$$n \geq 1$$

Hence, $n \geq 1$

$$2n \geq n+1$$

$$2^{n+3}n \geq 2^{n+2}(n+1)$$

$$\frac{n}{2^{n+2}} \geq \frac{n+1}{2^{(n+1)+2}}$$

$$a_n \geq a_{n+1}$$

True; monotonic; $|a_n| \leq \frac{1}{8}$, bounded

$$85. a_n = (-1)^n \left(\frac{1}{n}\right)$$

$$a_1 = -1$$

$$a_2 = \frac{1}{2}$$

$$a_3 = -\frac{1}{3}$$

Not monotonic; $|a_n| \leq 1$, bounded

$$87. a_n = \left(\frac{2}{3}\right)^n > \left(\frac{2}{3}\right)^{n+1} = a_{n+1}$$

Monotonic; $|a_n| \leq \frac{2}{3}$, bounded

$$89. a_n = \sin\left(\frac{n\pi}{6}\right)$$

$$a_1 = 0.500$$

$$a_2 = 0.8660$$

$$a_3 = 1.000$$

$$a_4 = 0.8660$$

Not monotonic; $|a_n| \leq 1$, bounded

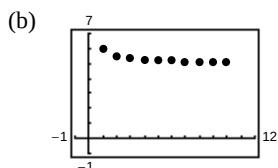
91. (a) $a_n = 5 + \frac{1}{n}$

$$\left| 5 + \frac{1}{n} \right| \leq 6 \Rightarrow \{a_n\} \text{ bounded}$$

$$a_n = 5 + \frac{1}{n} > 5 + \frac{1}{n+1}$$

$$= a_{n+1} \Rightarrow \{a_n\} \text{ monotonic}$$

Therefore, $\{a_n\}$ converges.



$$\lim_{n \rightarrow \infty} \left(5 + \frac{1}{n} \right) = 5$$

95. $A_n = P \left[1 + \frac{r}{12} \right]^n$

(a) $\lim_{n \rightarrow \infty} A_n = \infty$, divergent. The amount will grow arbitrarily large over time.

(b) $A_n = 9000 \left[1 + \frac{0.115}{12} \right]^n$

$$A_1 = \$9086.25 \quad A_6 = \$9530.06$$

$$A_2 = \$9173.33 \quad A_7 = \$9621.39$$

$$A_3 = \$9261.24 \quad A_8 = \$9713.59$$

$$A_4 = \$9349.99 \quad A_9 = \$9806.68$$

$$A_5 = \$9439.60 \quad A_{10} = \$9900.66$$

99. $a_n = 10 - \frac{1}{n}$

103. (a) $A_n = (0.8)^n (2.5)$ billion

(b) $A_1 = \$2$ billion

$$A_2 = \$1.6$$
 billion

$$A_3 = \$1.28$$
 billion

$$A_4 = \$1.024$$
 billion

(c) $\lim_{n \rightarrow \infty} (0.8)^n (2.5) = 0$

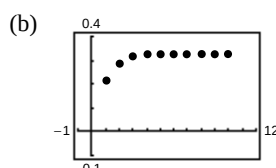
93. (a) $a_n = \frac{1}{3} \left(1 - \frac{1}{3^n} \right)$

$$\left| \frac{1}{3} \left(1 - \frac{1}{3^n} \right) \right| < \frac{1}{3} \Rightarrow \{a_n\} \text{ bounded}$$

$$a_n = \frac{1}{3} \left(1 - \frac{1}{3^n} \right) < \frac{1}{3} \left(1 - \frac{1}{3^{n+1}} \right)$$

$$= a_{n+1} \Rightarrow \{a_n\} \text{ monotonic}$$

Therefore, $\{a_n\}$ converges.



$$\lim_{n \rightarrow \infty} \left[\frac{1}{3} \left(1 - \frac{1}{3^n} \right) \right] = \frac{1}{3}$$

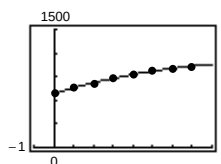
97. (a) A sequence is a function whose domain is the set of positive integers.

(b) A sequence converges if it has a limit.

(c) A bounded monotonic sequence is a sequence that has nondecreasing or nonincreasing terms, and an upper and lower bound.

101. $a_n = \frac{3n}{4n+1}$

105. (a) $a_n = -3.7262n^2 + 75.9167n + 684.25$



(b) For 2004, $n = 14$ and $a_{14} \approx 1017$, or \$1017.

107. $a_n = \frac{10^n}{n!}$

$$\begin{aligned} \text{(a) } a_9 &= a_{10} = \frac{10^9}{9!} \\ &= \frac{1,000,000,000}{362,880} \\ &= \frac{1,562,500}{567} \end{aligned}$$

(b) Decreasing

(c) Factorials increase more rapidly than exponentials.

109. $\{a_n\} = \{\sqrt[n]{n}\} = \{n^{1/n}\}$

$$a_1 = 1^{1/1} = 1$$

$$a_2 = \sqrt{2} \approx 1.4142$$

$$a_3 = \sqrt[3]{3} \approx 1.4422$$

$$a_4 = \sqrt[4]{4} \approx 1.4142$$

$$a_5 = \sqrt[5]{5} \approx 1.3797$$

$$a_6 = \sqrt[6]{6} \approx 1.3480$$

$$\text{Let } y = \lim_{n \rightarrow \infty} n^{1/n}.$$

$$\begin{aligned} \ln y &= \lim_{n \rightarrow \infty} \left(\frac{1}{n} \ln n \right) \\ &= \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{1/n}{1} = 0 \end{aligned}$$

Since $\ln y = 0$, we have $y = e^0 = 1$. Therefore,
 $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

111. $a_{n+2} = a_n + a_{n+1}$

$$\begin{array}{ll} \text{(a) } a_1 = 1 & a_7 = 8 + 5 = 13 \\ a_2 = 1 & a_8 = 13 + 8 = 21 \\ a_3 = 1 + 1 = 2 & a_9 = 21 + 13 = 34 \\ a_4 = 2 + 1 = 3 & a_{10} = 34 + 21 = 55 \\ a_5 = 3 + 2 = 5 & a_{11} = 55 + 34 = 89 \\ a_6 = 5 + 3 = 8 & a_{12} = 89 + 55 = 144 \end{array}$$

(b) $b_n = \frac{a_{n+1}}{a_n}, n \geq 1$

$$\begin{array}{ll} b_1 = \frac{1}{1} = 1 & b_6 = \frac{13}{8} \\ b_2 = \frac{2}{1} = 2 & b_7 = \frac{21}{13} \\ b_3 = \frac{3}{2} & b_8 = \frac{34}{21} \\ b_4 = \frac{5}{3} & b_9 = \frac{55}{34} \\ b_5 = \frac{8}{5} & b_{10} = \frac{89}{55} \end{array}$$

$$\begin{aligned} \text{(c) } 1 + \frac{1}{b_{n-1}} &= 1 + \frac{1}{a_n/a_{n-1}} \\ &= 1 + \frac{a_{n-1}}{a_n} \\ &= \frac{a_n + a_{n-1}}{a_n} = \frac{a_{n+1}}{a_n} = b_n \end{aligned}$$

(d) If $\lim_{n \rightarrow \infty} b_n = \rho$, then $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{b_{n-1}} \right) = \rho$.

Since $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} b_{n-1}$ we have,

$$1 + (1/\rho) = \rho.$$

$$\rho + 1 = \rho^2$$

$$0 = \rho^2 - \rho - 1$$

$$\rho = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Since a_n , and thus b_n , is positive,

$$\rho = (1 + \sqrt{5})/2 \approx 1.6180.$$

113. True

115. True

117. $a_1 = \sqrt{2} \approx 1.4142$

$$a_2 = \sqrt{2 + \sqrt{2}} \approx 1.8478$$

$$a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} \approx 1.9616$$

$$a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \approx 1.9904$$

$$a_5 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}} \approx 1.9976$$

$\{a_n\}$ is increasing and bounded by 2, and hence converges to L . Letting $\lim_{n \rightarrow \infty} a_n = L$ implies that $\sqrt{2+L} = L \Rightarrow L = 2$.
Hence, $\lim_{n \rightarrow \infty} a_n = 2$.

Section 8.2 Series and Convergence

1. $S_1 = 1$

$$S_2 = 1 + \frac{1}{4} = 1.2500$$

$$S_3 = 1 + \frac{1}{4} + \frac{1}{9} \approx 1.3611$$

$$S_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.4236$$

$$S_5 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} \approx 1.4636$$

3. $S_1 = 3$

$$S_2 = 3 - \frac{9}{2} = -1.5$$

$$S_3 = 3 - \frac{9}{2} + \frac{27}{4} = 5.25$$

$$S_4 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} = -4.875$$

$$S_5 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} = 10.3125$$

5. $S_1 = 3$

$$S_2 = 3 + \frac{3}{2} = 4.5$$

$$S_3 = 3 + \frac{3}{2} + \frac{3}{4} = 5.250$$

$$S_4 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} = 5.625$$

$$S_5 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} = 5.8125$$

7. $\sum_{n=0}^{\infty} 3\left(\frac{3}{2}\right)^n$ Geometric series

$$r = \frac{3}{2} > 1$$

Diverges by Theorem 8.6

9. $\sum_{n=0}^{\infty} 1000(1.055)^n$ Geometric series

$$r = 1.055 > 1$$

Diverges by Theorem 8.6

11. $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$$

Diverges by Theorem 8.9

13. $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^2+1} = 1 \neq 0$$

Diverges by Theorem 8.9

15. $\sum_{n=0}^{\infty} \frac{2^n+1}{2^{n+1}}$

$$\lim_{n \rightarrow \infty} \frac{2^n+1}{2^{n+1}} = \lim_{n \rightarrow \infty} \frac{1+2^{-n}}{2} = \frac{1}{2} \neq 0$$

Diverges by Theorem 8.9

17. $\sum_{n=0}^{\infty} \frac{9}{4}\left(\frac{1}{4}\right)^n = \frac{9}{4}\left[1 + \frac{1}{4} + \frac{1}{16} + \cdots\right]$

$$S_0 = \frac{9}{4}, S_1 = \frac{9}{4} \cdot \frac{5}{4} = \frac{45}{16}, S_2 = \frac{9}{4} \cdot \frac{21}{16} \approx 2.95, \dots$$

Matches graph (c).

Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \left(\frac{9}{4}\right)\left(\frac{1}{4}\right)^n = \frac{9/4}{1-1/4} = \frac{9/4}{3/4} = 3$$

19. $\sum_{n=0}^{\infty} \frac{15}{4}\left(-\frac{1}{4}\right)^n = \frac{15}{4}\left[1 - \frac{1}{4} + \frac{1}{16} - \cdots\right]$

$$S_0 = \frac{15}{4}, S_1 = \frac{45}{16}, S_2 \approx 3.05, \dots$$

Matches graph (a).

Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \frac{15}{4}\left(-\frac{1}{4}\right)^n = \frac{15/4}{1-(-1/4)} = \frac{15/4}{5/4} = 3$$

21. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1$$

23. $\sum_{n=0}^{\infty} 2\left(\frac{3}{4}\right)^n$

Geometric series with $r = \frac{3}{4} < 1$.

Converges by Theorem 8.6

25. $\sum_{n=0}^{\infty} (0.9)^n$

Geometric series with $r = 0.9 < 1$.

Converges by Theorem 8.6

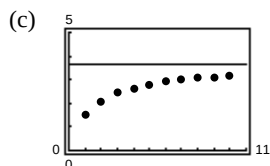
$$27. (a) \sum_{n=1}^{\infty} \frac{6}{n(n+3)} = 2 \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3} \right)$$

$$= 2 \left[\left(1 - \frac{1}{4} \right) + \left(\frac{1}{2} - \frac{1}{5} \right) + \left(\frac{1}{3} - \frac{1}{6} \right) + \left(\frac{1}{4} - \frac{1}{7} \right) + \cdots \right]$$

$$= 2 \left[1 + \frac{1}{2} + \frac{1}{3} \right] = \frac{11}{3} \approx 3.667$$

(b)

n	5	10	20	50	100
S_n	2.7976	3.1643	3.3936	3.5513	3.6078

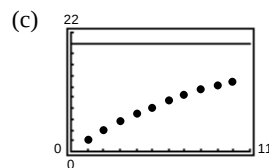


(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.

$$29. (a) \sum_{n=1}^{\infty} 2(0.9)^{n-1} = \sum_{n=0}^{\infty} 2(0.9)^n = \frac{2}{1-0.9} = 20$$

(b)

n	5	10	20	50	100
S_n	8.1902	13.0264	17.5685	19.8969	19.9995

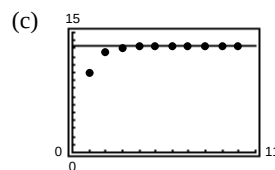


(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.

$$31. (a) \sum_{n=1}^{\infty} 10(0.25)^{n-1} = \frac{10}{1-0.25} = \frac{40}{3} \approx 13.3333$$

(b)

n	5	10	20	50	100
S_n	13.3203	13.3333	13.3333	13.3333	13.3333



(d) The terms of the series decrease in magnitude rapidly. Thus, the sequence of partial sums approaches the sum rapidly.

$$33. \sum_{n=2}^{\infty} \frac{1}{n^2-1} = \sum_{n=2}^{\infty} \left(\frac{1/2}{n-1} - \frac{1/2}{n+1} \right) = \frac{1}{2} \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$$

$$= \frac{1}{2} \left[\left(1 - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \cdots \right]$$

$$= \frac{1}{2} \left(1 + \frac{1}{2} \right) = \frac{3}{4}$$

$$35. \sum_{n=1}^{\infty} \frac{8}{(n+1)(n+2)} = 8 \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = 8 \left[\left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) + \cdots \right] = 8 \left(\frac{1}{2} \right) = 4$$

$$37. \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n = \frac{1}{1-(1/2)} = 2$$

$$39. \sum_{n=0}^{\infty} \left(-\frac{1}{2} \right)^n = \frac{1}{1-(-1/2)} = \frac{2}{3}$$

$$41. \sum_{n=0}^{\infty} \left(\frac{1}{10} \right)^n = \frac{1}{1-(1/10)} = \frac{10}{9}$$

$$43. \sum_{n=0}^{\infty} 3 \left(-\frac{1}{3} \right)^n = \frac{3}{1-(-1/3)} = \frac{9}{4}$$

$$\begin{aligned}
 45. \sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n} \right) &= \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{3} \right)^n \\
 &= \frac{1}{1 - (1/2)} - \frac{1}{1 - (1/3)} \\
 &= 2 - \frac{3}{2} = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 49. 0.075\overline{75} &= \sum_{n=0}^{\infty} \frac{3}{40} \left(\frac{1}{100} \right)^n \\
 \text{Geometric series with } a &= \frac{3}{40} \text{ and } r = \frac{1}{100} \\
 S &= \frac{a}{1 - r} = \frac{3/40}{99/100} = \frac{5}{66}
 \end{aligned}$$

$$53. \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right) = \left(1 - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \cdots = 1 + \frac{1}{2} = \frac{3}{2}, \text{ converges}$$

$$\begin{aligned}
 55. \sum_{n=1}^{\infty} \frac{3n-1}{2n+1} \\
 \lim_{n \rightarrow \infty} \frac{3n-1}{2n+1} &= \frac{3}{2} \neq 0 \\
 \text{Diverges by Theorem 8.9}
 \end{aligned}$$

$$\begin{aligned}
 61. \sum_{n=2}^{\infty} \frac{n}{\ln n} \\
 \lim_{n \rightarrow \infty} \frac{n}{\ln n} &= \lim_{n \rightarrow \infty} \frac{1}{1/n} = \infty \\
 \text{(by L'Hôpital's Rule) Diverges by Theorem 8.9}
 \end{aligned}$$

65. The series given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, a \neq 0$$

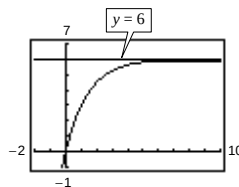
is a geometric series with ratio r . When $0 < |r| < 1$, the series converges to $\frac{a}{1-r}$. The series diverges if $|r| \geq 1$.

$$69. f(x) = 3 \left[\frac{1 - 0.5^x}{1 - 0.5} \right]$$

Horizontal asymptote: $y = 6$

$$\begin{aligned}
 \sum_{n=0}^{\infty} 3 \left(\frac{1}{2} \right)^n \\
 S &= \frac{3}{1 - (1/2)} = 6
 \end{aligned}$$

The horizontal asymptote is the sum of the series. $f(n)$ is the n^{th} partial sum.



$$\begin{aligned}
 47. 0.\overline{4} &= \sum_{n=0}^{\infty} \frac{4}{10} \left(\frac{1}{10} \right)^n \\
 \text{Geometric series with } a &= \frac{4}{10} \text{ and } r = \frac{1}{10} \\
 S &= \frac{a}{1 - r} = \frac{4/10}{1 - (1/10)} = \frac{4}{9}
 \end{aligned}$$

$$\begin{aligned}
 51. \sum_{n=1}^{\infty} \frac{n+10}{10n+1} \\
 \lim_{n \rightarrow \infty} \frac{n+10}{10n+1} &= \frac{1}{10} \neq 0 \\
 \text{Diverges by Theorem 8.9}
 \end{aligned}$$

$$\begin{aligned}
 57. \sum_{n=0}^{\infty} \frac{4}{2^n} &= 4 \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n \\
 \text{Geometric series with } r &= \frac{1}{2} \\
 \text{Converges by Theorem 8.6}
 \end{aligned}$$

$$\begin{aligned}
 59. \sum_{n=0}^{\infty} (1.075)^n \\
 \text{Geometric series with } r &= 1.075 \\
 \text{Diverges by Theorem 8.6}
 \end{aligned}$$

63. See definition, page 567.

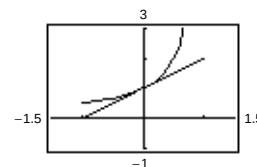
67. (a) x is the common ratio.

$$(b) 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, |x| < 1$$

Geometric series: $a = 1, r = x, |x| < 1$

$$(c) y_1 = \frac{1}{1-x}$$

$$y_2 = 1 + x$$



$$71. \frac{1}{n(n+1)} < 0.001$$

$$10,000 < n^2 + n$$

$$0 < n^2 + n - 10,000$$

$$n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-10,000)}}{2}$$

Choosing the positive value for n we have $n \approx 99.5012$. The first *term* that is less than 0.001 is $n = 100$.

$$\left(\frac{1}{8}\right)^n < 0.001$$

$$10,000 < 8^n$$

This inequality is true when $n = 5$. This series converges at a faster rate.

$$73. \sum_{i=0}^{n-1} 8000(0.9)^i = \frac{8000[1 - (0.9)^{(n-1)+1}]}{1 - 0.9}$$

$$= 80,000(1 - 0.9^n), \quad n > 0$$

$$75. \sum_{i=0}^{n-1} 100(0.75)^i = \frac{100[1 - 0.75^{(n-1)+1}]}{1 - 0.75}$$

$$= 400(1 - 0.75^n) \text{ million dollars.}$$

Sum = 400 million dollars

$$77. D_1 = 16$$

$$D_2 = \underbrace{0.81(16)}_{\text{up}} + \underbrace{0.81(16)}_{\text{down}} = 32(0.81)$$

$$D_3 = 16(0.81)^2 + 16(0.81)^2 = 32(0.81)^2$$

$\vdots$

$$D = 16 + 32(0.81) + 32(0.81)^2 + \cdots = -16 + \sum_{n=0}^{\infty} 32(0.81)^n = -16 + \frac{32}{1 - 0.81} = 152.42 \text{ ft}$$

$$79. P(n) = \frac{1}{2}\left(\frac{1}{2}\right)^n$$

$$P(2) = \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{8}$$

$$\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^n = \frac{1/2}{1 - (1/2)} = 1$$

$$81. (a) \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{1}{2}\right)^n = \frac{1}{2} \frac{1}{1 - (1/2)} = 1$$

(b) No, the series is not geometric.

$$(c) \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = 2$$

$$83. \text{ Present Value} = \sum_{n=1}^{19} 50,000 \left(\frac{1}{1.06}\right)^n$$

$$= \sum_{n=0}^{18} \frac{50,000}{1.06} \left(\frac{1}{1.06}\right)^n, \quad r = \frac{1}{1.06}$$

$$= \frac{50,000}{1.06} \left(\frac{1 - 1.06^{-19}}{1 - 1.06^{-1}}\right)$$

$$\approx \$557,905.82$$

The present value is less than \$1,000,000. After accruing interest over 20 years, it attains its full value.

$$85. w = \sum_{i=0}^{n-1} 0.01(2)^i = \frac{0.01(1 - 2^n)}{1 - 2} = 0.01(2^n - 1)$$

(a) When $n = 29$: $w = \$5,368,709.11$

(b) When $n = 30$: $w = \$10,737,418.23$

(c) When $n = 31$: $w = \$21,474,836.47$

87. $P = 50, r = 0.03, t = 20$

(a) $A = 50 \left(\frac{12}{0.03} \right) \left[\left(1 + \frac{0.03}{12} \right)^{12(20)} - 1 \right] \approx \$16,415.10$

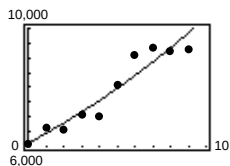
(b) $A = \frac{50 - (e^{0.03(20)} - 1)}{e^{0.03/12} - 1} \approx \$16,421.83$

89. $P = 100, r = 0.04, t = 40$

(a) $A = 100 \left(\frac{12}{0.04} \right) \left[\left(1 + \frac{0.04}{12} \right)^{12(40)} - 1 \right] \approx \$118,196.13$

(b) $A = \frac{100(e^{0.04(40)} - 1)}{e^{0.04/12} - 1} \approx \$118,393.43$

91. (a) $a_n = 6110.1832(1.0544)^x = 6110.1832e^{0.05297n}$



(b) 78,530 or \$78,530,000,000

(c) $\text{Total} = \sum_{n=0}^9 a_n \approx 78,449 \text{ or } \$78,449,000,000$

95. By letting $S_0 = 0$, we have $a_n = \sum_{k=1}^n a_k - \sum_{k=1}^{n-1} a_k = S_n - S_{n-1}$. Thus,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (S_n - S_{n-1}) = \sum_{n=1}^{\infty} (S_n - S_{n-1} + c - c) = \sum_{n=1}^{\infty} [(c - S_{n-1}) - (c - S_n)].$$

97. Let $\sum a_n = \sum_{n=0}^{\infty} 1$ and $\sum b_n = \sum_{n=0}^{\infty} (-1)$.

Both are divergent series.

$$\sum (a_n + b_n) = \sum_{n=0}^{\infty} [1 + (-1)] = \sum_{n=0}^{\infty} [1 - 1] = 0$$

99. False. $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

101. False

$$\sum_{n=1}^{\infty} ar^n = \left(\frac{a}{1-r} \right) - a$$

The formula requires that the geometric series begins with $n = 0$.103. Let H represent the half-life of the drug. If a patient receives n equal doses of P units each of this drug, administered at equal time interval of length t , the total amount of the drug in the patient's system at the time the last dose is administered is given by

$$T_n = P + Pe^{kt} + Pe^{2kt} + \cdots + Pe^{(n-1)kt}$$

where $k = -(\ln 2)/H$. One time interval *after* the last dose is administered is given by

$$T_{n+1} = Pe^{kt} + Pe^{2kt} + Pe^{3kt} + \cdots + Pe^{nkt}.$$

Two time intervals *after* the last dose is administered is given by

$$T_{n+2} = Pe^{2kt} + Pe^{3kt} + Pe^{4kt} + \cdots + Pe^{(n+1)kt}$$

and so on. Since $k < 0$, $T_{n+s} \rightarrow 0$ as $s \rightarrow \infty$, where s is an integer.

Section 8.3 The Integral Test and p -Series

1. $\sum_{n=1}^{\infty} \frac{1}{n+1}$

Let $f(x) = \frac{1}{x+1}$.

 f is positive, continuous and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x+1} dx = \left[\ln(x+1) \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

5. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Let $f(x) = \frac{1}{x^2+1}$.

 f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^2+1} dx = \left[\arctan x \right]_1^{\infty} = \frac{\pi}{4}$$

Converges by Theorem 8.10

9. $\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k + c}$

Let $f(x) = \frac{x^{k-1}}{x^k + c}$.

 f is positive, continuous, and decreasing for $x > \sqrt[k]{c(k-1)}$ since

$$f'(x) = \frac{x^{k-2}[c(k-1) - x^k]}{(x^k + c)^2} < 0$$

for $x > \sqrt[k]{c(k-1)}$.

$$\int_1^{\infty} \frac{x^{k-1}}{x^k + c} dx = \left[\frac{1}{k} \ln(x^k + c) \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

13. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/5}}$

Divergent p -series with $p = \frac{1}{5} < 1$

17. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

Convergent p -series with $p = \frac{3}{2} > 1$

3. $\sum_{n=1}^{\infty} e^{-n}$

Let $f(x) = e^{-x}$.

 f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} e^{-x} dx = \left[-e^{-x} \right]_1^{\infty} = \frac{1}{e}$$

Converges by Theorem 8.10

7. $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$

Let $f(x) = \frac{\ln(x+1)}{x+1}$.

 f is positive, continuous, and decreasing for $x \geq 2$ since

$$f'(x) = \frac{1 - \ln(x+1)}{(x+1)^2} < 0 \text{ for } x \geq 2.$$

$$\int_1^{\infty} \frac{\ln(x+1)}{x+1} dx = \left[\frac{\ln^2(x+1)}{2} \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

11. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Let $f(x) = \frac{1}{x^3}$.

 f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^3} dx = \left[-\frac{1}{2x^2} \right]_1^{\infty} = \frac{1}{2}$$

Converges by Theorem 8.10

15. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

Divergent p -series with $p = \frac{1}{2} < 1$

19. $\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

Convergent p -series with $p = 1.04 > 1$

$$21. \sum_{n=1}^{\infty} \frac{2}{\sqrt[4]{n^3}} = \frac{2}{1} + \frac{2}{2^{3/4}} + \frac{2}{3^{3/4}} + \cdots$$

$$S_1 = 2$$

$$S_2 \approx 3.189$$

$$S_3 \approx 4.067$$

Matches (a)

Diverges— p -series with $p = \frac{3}{4} < 1$

$$23. \sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}} = 2 + 2/2^{3/2} + 2/3^{3/2} + \cdots$$

$$S_1 = 2$$

$$S_2 \approx 2.707$$

$$S_3 \approx 3.092$$

Matches (b)

Converges— p -series with $p = 3/2 > 1$

25. No. Theorem 8.9 says that if the series converges, then the terms a_n tend to zero. Some of the series in Exercises 21–24 converge because the terms tend to 0 very rapidly.

$$27. \sum_{n=1}^N \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N} > M$$

(a)

M	2	4	6	8
N	4	31	227	1674

(b) No. Since the terms are decreasing (approaching zero), more and more terms are required to increase the partial sum by 2.

$$29. \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

If $p = 1$, then the series diverges by the Integral Test. If $p \neq 1$,

$$\int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \int_2^{\infty} (\ln x)^{-p} \frac{1}{x} dx = \left[\frac{(\ln x)^{-p+1}}{-p+1} \right]_2^{\infty}.$$

Converges for $-p + 1 < 0$ or $p > 1$.

31. Let f be positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$. Then,

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x) dx$$

either both converge or both diverge (Theorem 8.10).

See Example 1, page 578.

33. Your friend is not correct. The series

$$\sum_{n=10,000}^{\infty} \frac{1}{n} = \frac{1}{10,000} + \frac{1}{10,001} + \cdots$$

is the harmonic series, starting with the 10,000th term, and hence diverges.

35. Since f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, we have,

$$R_N = S - S_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^N a_n = \sum_{n=N+1}^{\infty} a_n > 0.$$

$$\text{Also, } R_N = S - S_N = \sum_{n=N+1}^{\infty} a_n \leq a_{N+1} + \int_{N+1}^{\infty} f(x) dx \leq \int_N^{\infty} f(x) dx. \text{ Thus,}$$

$$0 \leq R_N \leq \int_N^{\infty} f(x) dx.$$

$$37. S_6 = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} \approx 1.0811$$

$$R_6 \leq \int_6^{\infty} \frac{1}{x^4} dx = \left[-\frac{1}{3x^3} \right]_6^{\infty} \approx 0.0015$$

$$1.0811 \leq \sum_{n=1}^{\infty} \frac{1}{n^4} \leq 1.0811 + 0.0015 = 1.0826$$

$$39. S_{10} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \frac{1}{37} + \frac{1}{50} + \frac{1}{65} + \frac{1}{82} + \frac{1}{101} \approx 0.9818$$

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^2 + 1} dx = \left[\arctan x \right]_{10}^{\infty} = \frac{\pi}{2} - \arctan 10 \approx 0.0997$$

$$0.9818 \leq \sum_{n=1}^{\infty} \frac{1}{n^5} \leq 0.9818 + 0.0997 = 1.0815$$

$$41. S_4 = \frac{1}{e} + \frac{2}{e^4} + \frac{3}{e^9} + \frac{4}{e^{16}} \approx 0.4049$$

$$R_4 \leq \int_4^{\infty} x e^{-x^2} dx = \left[-\frac{1}{2} e^{-x^2} \right]_4^{\infty} = 5.6 \times 10^{-8}$$

$$0.4049 \leq \sum_{n=1}^{\infty} n e^{-n^2} \leq 0.4049 + 5.6 \times 10^{-8}$$

$$43. 0 \leq R_N \leq \int_N^{\infty} \frac{1}{x^4} dx = \left[-\frac{1}{3x^3} \right]_N^{\infty} = \frac{1}{3N^3} < 0.001$$

$$\frac{1}{N^3} < 0.003$$

$$N^3 > 333.33$$

$$N > 6.93$$

$$N \geq 7$$

$$45. R_N \leq \int_N^{\infty} e^{-5x} dx = \left[-\frac{1}{5} e^{-5x} \right]_N^{\infty} = \frac{e^{-5N}}{5} < 0.001$$

$$\frac{1}{e^{5N}} < 0.005$$

$$e^{5N} > 200$$

$$5N > \ln 200$$

$$N > \frac{\ln 200}{5}$$

$$N > 1.0597$$

$$N \geq 2$$

$$47. R_N \leq \int_N^{\infty} \frac{1}{x^2 + 1} dx = \left[\arctan x \right]_N^{\infty} = \frac{\pi}{2} - \arctan N < 0.001$$

$$-\arctan N < -1.5698$$

$$\arctan N > 1.5698$$

$$N > \tan 1.5698$$

$$N \geq 1004$$

49. (a) $\sum_{n=2}^{\infty} \frac{1}{n^{1.1}}$. This is a convergent p -series with $p = 1.1 > 1$.

$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is a divergent series. Use the Integral Test.

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \left[\ln |\ln x| \right]_2^{\infty} = \infty$$

(b) $\sum_{n=2}^6 \frac{1}{n^{1.1}} = \frac{1}{2^{1.1}} + \frac{1}{3^{1.1}} + \frac{1}{4^{1.1}} + \frac{1}{5^{1.1}} + \frac{1}{6^{1.1}} \approx 0.4665 + 0.2987 + 0.2176 + 0.1703 + 0.1393$

$$\sum_{n=2}^6 \frac{1}{n \ln n} = \frac{1}{2 \ln 2} + \frac{1}{3 \ln 3} + \frac{1}{4 \ln 4} + \frac{1}{5 \ln 5} + \frac{1}{6 \ln 6} \approx 0.7213 + 0.3034 + 0.1803 + 0.1243 + 0.0930$$

The terms of the convergent series **seem** to be larger than those of the divergent series!

(c) $\frac{1}{n^{1.1}} < \frac{1}{n \ln n}$

$$n \ln n < n^{1.1}$$

$$\ln n < n^{0.1}$$

This inequality holds when $n \geq 3.5 \times 10^{15}$. Or, $n > e^{40}$. Then $\ln e^{40} = 40 < (e^{40})^{0.1} = e^4 \approx 55$.

51. (a) Let $f(x) = 1/x$. f is positive, continuous, and decreasing on $[1, \infty)$.

$$S_n - 1 \leq \int_1^n \frac{1}{x} dx$$

$$S_n - 1 \leq \ln n$$

Hence, $S_n \leq 1 + \ln n$. Similarly,

$$S_n \geq \int_1^{n+1} \frac{1}{x} dx = \ln(n+1).$$

Thus, $\ln(n+1) \leq S_n \leq 1 + \ln n$.

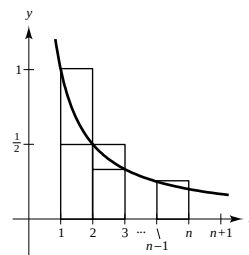
- (b) Since $\ln(n+1) \leq S_n \leq 1 + \ln n$, we have $\ln(n+1) - \ln n \leq S_n - \ln n \leq 1$. Also, since $\ln x$ is an increasing function, $\ln(n+1) - \ln n > 0$ for $n \geq 1$. Thus, $0 \leq S_n - \ln n \leq 1$ and the sequence $\{a_n\}$ is bounded.

$$(c) a_n - a_{n+1} = [S_n - \ln n] - [S_{n+1} - \ln(n+1)] = \int_n^{n+1} \frac{1}{x} dx - \frac{1}{n+1} \geq 0$$

Thus, $a_n \geq a_{n+1}$ and the sequence is decreasing.

- (d) Since the sequence is bounded and monotonic, it converges to a limit, γ .

$$(e) a_{100} = S_{100} - \ln 100 \approx 0.5822 \text{ (Actually } \gamma \approx 0.577216.)$$



$$53. \sum_{n=1}^{\infty} \frac{1}{2n-1}$$

$$\text{Let } f(x) = \frac{1}{2x-1}.$$

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{2x-1} dx = \left[\ln \sqrt{2x-1} \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

$$55. \sum_{n=1}^{\infty} \frac{1}{n^{5/4}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$$

p -series with $p = \frac{5}{4}$

Converges by Theorem 8.11

$$57. \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$$

Geometric series with $r = \frac{2}{3}$

Converges by Theorem 8.6

$$59. \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$$

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 \neq 0$$

Diverges by Theorem 8.9

$$61. \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0$$

Fails n th Term Test

Diverges by Theorem 8.9

$$63. \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$$

$$\text{Let } f(x) = \frac{1}{x(\ln x)^3}.$$

f is positive, continuous and decreasing for $x \geq 2$.

$$\int_2^{\infty} \frac{1}{x(\ln x)^3} dx = \int_2^{\infty} (\ln x)^{-3} \frac{1}{x} dx = \left[\frac{(\ln x)^{-2}}{-2} \right]_2^{\infty} = \left[-\frac{1}{2(\ln x)^2} \right]_2^{\infty} = \frac{1}{2(\ln 2)^2}$$

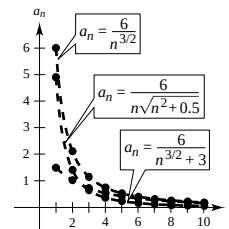
Converges by Theorem 8.10. See Exercise 13.

Section 8.4 Comparisons of Series

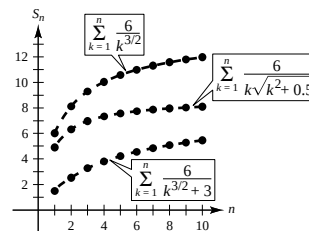
$$1. (a) \sum_{n=1}^{\infty} \frac{6}{n^{3/2}} = \frac{6}{1} + \frac{6}{2^{3/2}} + \cdots \quad S_1 = 6$$

$$\sum_{n=1}^{\infty} \frac{6}{n^{3/2} + 3} = \frac{6}{4} + \frac{6}{2^{3/2} + 3} + \cdots \quad S_1 = \frac{3}{2}$$

$$\sum_{n=1}^{\infty} \frac{6}{n\sqrt{n^2 + 0.5}} = \frac{6}{1\sqrt{1.5}} + \frac{6}{2\sqrt{4.5}} + \cdots \quad S_1 = \frac{6}{\sqrt{1.5}} \approx 4.9$$



- (b) The first series is a p -series. It converges ($p = 3/2 > 1$).
- (c) The magnitude of the terms of the other two series are less than the corresponding terms at the convergent p -series. Hence, the other two series converge.
- (d) The smaller the magnitude of the terms, the smaller the magnitude of the terms of the sequence of partial sums.



$$3. \frac{1}{n^2 + 1} < \frac{1}{n^2}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

converges by comparison with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$7. \frac{1}{3^n + 1} < \frac{1}{3^n}$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{3^n + 1}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n.$$

$$11. \text{ For } n > 3, \frac{1}{n^2} > \frac{1}{n!}.$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

converges by comparison with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$5. \frac{1}{n-1} > \frac{1}{n} \text{ for } n \geq 2$$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{n-1}$$

diverges by comparison with the divergent p -series

$$\sum_{n=2}^{\infty} \frac{1}{n}.$$

$$9. \text{ For } n \geq 3, \frac{\ln n}{n+1} > \frac{1}{n+1}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{\ln n}{n+1}$$

diverges by comparison with the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n+1}.$$

Note: $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges by the integral test.

$$13. \frac{1}{e^{n^2}} \leq \frac{1}{e^n}$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n.$$

$$15. \lim_{n \rightarrow \infty} \frac{n/(n^2 + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$19. \lim_{n \rightarrow \infty} \frac{\frac{2n^2 - 1}{3n^5 + 2n + 1}}{1/n^3} = \lim_{n \rightarrow \infty} \frac{2n^5 - n^3}{3n^5 + 2n + 1} = \frac{2}{3}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2n^2 - 1}{3n^5 + 2n + 1}$$

converges by a limit comparison with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

$$23. \lim_{n \rightarrow \infty} \frac{1/(n\sqrt{n^2 + 1})}{1/n^2} = \lim_{n \rightarrow \infty} \frac{n^2}{n\sqrt{n^2 + 1}} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2 + 1}}$$

converges by a limit comparison with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$27. \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = \lim_{n \rightarrow \infty} \frac{(-1/n^2) \cos(1/n)}{-1/n^2} \\ = \lim_{n \rightarrow \infty} \cos\left(\frac{1}{n}\right) = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$31. \sum_{n=1}^{\infty} \frac{1}{3^n + 2}$$

Converges

Direct comparison with $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

$$33. \sum_{n=1}^{\infty} \frac{n}{2n + 3}$$

Diverges; n th Term Test

$$\lim_{n \rightarrow \infty} \frac{n}{2n + 3} = \frac{1}{2} \neq 0$$

$$17. \lim_{n \rightarrow \infty} \frac{1/\sqrt{n^2 + 1}}{1/n} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + 1}} = 1$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$21. \lim_{n \rightarrow \infty} \frac{\frac{n + 3}{n(n + 2)}}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2 + 3n}{n^2 + 2n} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n + 3}{n(n + 2)}$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$25. \lim_{n \rightarrow \infty} \frac{(n^{k-1})/(n^k + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^k}{n^k + 1} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k + 1}$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$29. \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

Diverges

p -series with $p = \frac{1}{2}$

$$35. \sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}$$

Converges; integral test

37. $\lim_{n \rightarrow \infty} \frac{a_n}{1/n} = \lim_{n \rightarrow \infty} na_n$ by given conditions $\lim_{n \rightarrow \infty} na_n$ is finite and nonzero.

Therefore,

$$\sum_{n=1}^{\infty} a_n$$

diverges by a limit comparison with the p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

41. $\sum_{n=1}^{\infty} \frac{1}{n^3 + 1}$

converges since the degree of the numerator is three less than the degree of the denominator.

45. See Theorem 8.12, page 583. One example is $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ converges because

$$\frac{1}{n^2 + 1} < \frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges (p -series).

49. $\frac{1}{200} + \frac{1}{400} + \frac{1}{600} + \cdots = \sum_{n=1}^{\infty} \frac{1}{200n}$, diverges

53. Some series diverge or converge very slowly. You cannot decide convergence or divergence of a series by comparing the first few terms.

57. True

59. Since $\sum_{n=1}^{\infty} b_n$ converges, $\lim_{n \rightarrow \infty} b_n = 0$. There exists N such that $b_n < 1$ for $n > N$. Thus,

$$a_n b_n < a_n \text{ for } n > N \text{ and } \sum_{n=1}^{\infty} a_n b_n$$

converges by comparison to the convergent series $\sum_{i=1}^{\infty} a_n$.

61. $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n^3}$ both converge, and hence so does $\sum \left(\frac{1}{n^2} \right) \left(\frac{1}{n^3} \right) = \sum \frac{1}{n^5}$.

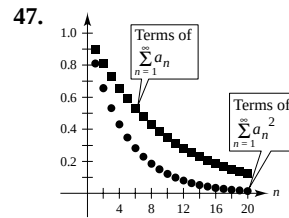
39. $\frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \frac{4}{17} + \frac{5}{26} + \cdots = \sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$,

which diverges since the degree of the numerator is only one less than the degree of the denominator.

43. $\lim_{n \rightarrow \infty} n \left(\frac{n^3}{5n^4 + 3} \right) = \lim_{n \rightarrow \infty} \frac{n^4}{5n^4 + 3} = \frac{1}{5} \neq 0$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n^3}{5n^4 + 3} \text{ diverges.}$$



For $0 < a_n < 1$, $0 < a_n^2 < a_n < 1$. Hence, the lower terms are those of $\sum a_n^2$.

51. $\frac{1}{201} + \frac{1}{204} + \frac{1}{209} + \frac{1}{216} = \sum_{n=1}^{\infty} \frac{1}{200 + n^2}$, converges

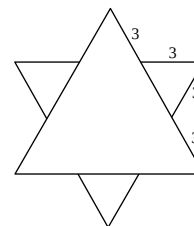
55. False. Let $a_n = 1/n^3$ and $b_n = 1/n^2$. $0 < a_n \leq b_n$ and both

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converge.

63. (a) Suppose $\sum b_n$ converges and $\sum a_n$ diverges. Then there exists N such that $0 < b_n < a_n$ for $n \geq N$. This means that $1 < a_n/b_n$ for $n \geq N$. Therefore, $\lim_{n \rightarrow \infty} a_n/b_n \neq 0$. Thus, $\sum a_n$ must also converge.
- (b) Suppose $\sum b_n$ diverges and $\sum a_n$ converges. Then there exists N such that $0 < a_n < b_n$ for $n \geq N$. This means that $0 < a_n/b_n < 1$ for $n \geq N$. Therefore, $\lim_{n \rightarrow \infty} a_n/b_n \neq \infty$. Thus, $\sum a_n$ must also diverge.
65. Start with one triangle whose sides have length 9. At the n th step, each side is replaced by four smaller line segments each having $\frac{1}{3}$ the length of the original side.

#Sides	Length of sides
3	9
$3 \cdot 4$	$9(\frac{1}{3})$
$3 \cdot 4^2$	$9(\frac{1}{3})^2$
$\vdots$	
$3 \cdot 4^n$	$9(\frac{1}{3})^n$



At the n th step there are $3 \cdot 4^n$ sides, each of length $9(\frac{1}{3})^n$. At the next step, there are $3 \cdot 4^n$ new triangles of side $9(\frac{1}{3})^{n+1}$. The area of an equilateral triangle of side x is $\frac{1}{4}\sqrt{3}x^2$. Thus, the new triangles each have area

$$9 \frac{\sqrt{3}}{4} \left(\frac{1}{3^{n+1}} \right)^2 = \frac{\sqrt{3}}{4} \frac{1}{3^{2n}}.$$

The area of the $3 \cdot 4^n$ new triangles is

$$(3 \cdot 4^n) \left(\frac{\sqrt{3}}{4} \frac{1}{3^{2n}} \right) = \frac{3\sqrt{3}}{4} \left(\frac{4}{9} \right)^n.$$

The total area is the infinite sum

$$\frac{9\sqrt{3}}{4} + \sum_{n=0}^{\infty} \frac{3\sqrt{3}}{4} \left(\frac{4}{9} \right)^n = \frac{9\sqrt{3}}{4} + \frac{3\sqrt{3}}{4} \left(\frac{1}{1 - 4/9} \right) = \frac{9\sqrt{3}}{4} + \frac{3\sqrt{3}}{4} \left(\frac{9}{5} \right) = \frac{18\sqrt{3}}{5}.$$

The perimeter is infinite, since at step n there are $3 \cdot 4^n$ sides of length $9(\frac{1}{3})^n$. Thus, the perimeter at step n is $27(\frac{4}{3})^n \rightarrow \infty$.

Section 8.5 Alternating Series

$$1. \sum_{n=1}^{\infty} \frac{6}{n^2} = \frac{6}{1} + \frac{6}{4} + \frac{6}{9} + \dots$$

$$S_1 = 6, S_2 = 7.5$$

Matches (b)

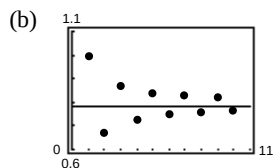
$$3. \sum_{n=1}^{\infty} \frac{10}{n2^n} = \frac{10}{2} + \frac{10}{8} + \dots$$

$$S_1 = 5, S_2 = 6.25$$

Matches (c)

$$5. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} \approx 0.7854$$

(a)	n	1	2	3	4	5	6	7	8	9	10
	S_n	1	0.6667	0.8667	0.7238	0.8349	0.7440	0.8209	0.7543	0.8131	0.7605



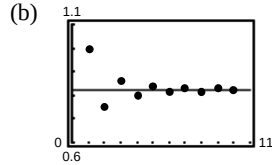
(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next term of the series.

$$7. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} \approx 0.8225$$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0.75	0.8611	0.7986	0.8386	0.8108	0.8312	0.8156	0.8280	0.8180



(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next term in the series.

$$9. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Converges by Theorem 8.14.

$$11. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

$$a_{n+1} = \frac{1}{2(n+1)-1} < \frac{1}{2n-1} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{2n-1} = 0$$

Converges by Theorem 8.14

$$13. \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2 + 1}$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = 1$$

Diverges by the n th Term Test

$$15. \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

$$a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

Converges by Theorem 8.14

$$17. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{\ln(n+1)}$$

$$\lim_{n \rightarrow \infty} \frac{n+1}{\ln(n+1)} = \lim_{n \rightarrow \infty} \frac{1}{1/(n+1)} = \lim_{n \rightarrow \infty} (n+1) = \infty$$

Diverges by the n th Term Test

$$19. \sum_{n=1}^{\infty} \sin\left[\frac{(2n-1)\pi}{2}\right] = \sum_{n=1}^{\infty} (-1)^{n+1}$$

Diverges by the n th Term Test

$$21. \sum_{n=1}^{\infty} \cos n\pi = \sum_{n=1}^{\infty} (-1)^n$$

Diverges by the n th Term Test

$$23. \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

$$a_{n+1} = \frac{1}{(n+1)!} < \frac{1}{n!} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n!} = 0$$

Converges by Theorem 8.14

$$25. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sqrt{n}}{n+2}$$

$$a_{n+1} = \frac{\sqrt{n+1}}{(n+1)+2} < \frac{\sqrt{n}}{n+2} \text{ for } n \geq 2$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+2} = 0$$

Converges by Theorem 8.14

$$27. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2)}{e^n - e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2e^n)}{e^{2n} - 1}$$

Let $f(x) = \frac{2e^x}{e^{2x} - 1}$. Then

$$f'(x) = \frac{-2e^x(e^{2x} + 1)}{(e^{2x} - 1)^2} < 0.$$

Thus, $f(x)$ is decreasing. Therefore, $a_{n+1} < a_n$, and

$$\lim_{n \rightarrow \infty} \frac{2e^n}{e^{2n} - 1} = \lim_{n \rightarrow \infty} \frac{2e^n}{2e^{2n}} = \lim_{n \rightarrow \infty} \frac{1}{e^n} = 0.$$

The series converges by Theorem 8.14.

$$29. S_6 = \sum_{n=1}^6 \frac{3(-1)^{n+1}}{n^2} = 2.4325$$

$$|R_6| = |S - S_6| \leq a_7 = \frac{3}{49} \approx 0.0612; 2.3713 \leq S \leq 2.4937$$

$$31. S_6 = \sum_{n=0}^5 \frac{2(-1)^n}{n!} \approx 0.7333$$

$$|R_6| = |S - S_6| \leq a_7 = \frac{2}{6!} = 0.002778; 0.7305 \leq S \leq 0.7361$$

$$33. \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{(N+1)!} < 0.001.$$

This inequality is valid when $N = 6$.

(b) We may approximate the series by

$$\begin{aligned} \sum_{n=0}^6 \frac{(-1)^n}{n!} &= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \\ &\approx 0.368. \end{aligned}$$

(7 terms. Note that the sum begins with $n = 0$.)

$$35. \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{[2(N+1)+1]!} < 0.001.$$

This inequality is valid when $N = 2$.

(b) We may approximate the series by

$$\sum_{n=0}^2 \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{6} + \frac{1}{120} \approx 0.842.$$

(3 terms. Note that the sum begins with $n = 0$.)

$$37. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{N+1} < 0.001.$$

This inequality is valid when $N = 1000$.

(b) We may approximate the series by

$$\begin{aligned} \sum_{n=1}^{1000} \frac{(-1)^{n+1}}{n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1000} \\ &\approx 0.693. \end{aligned}$$

(1000 terms)

$$39. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3 - 1}$$

By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{2(N+1)^3 - 1} < 0.001.$$

This inequality is valid when $N = 7$.

$$41. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \text{ converges by comparison to the } p\text{-series}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Therefore, the given series converge absolutely.

$$45. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(n+1)^2}$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = 1$$

Therefore, the series diverges by the n th Term Test.

$$49. \sum_{n=2}^{\infty} \frac{(-1)^n n}{n^3 - 1}$$

$$\sum_{n=2}^{\infty} \frac{n}{n^3 - 1}$$

converges by a limit comparison to the convergent p -series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

Therefore, the given series converges absolutely.

$$53. \sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

The given series converges by the Alternating Series Test, but

$$\sum_{n=0}^{\infty} \frac{|\cos n\pi|}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

diverges by a limit comparison to the divergent harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$\lim_{n \rightarrow \infty} \frac{|\cos n\pi|/(n+1)}{1/n} = 1, \text{ therefore the series}$$

converges conditionally.

$$43. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

The given series converges by the Alternating Series Test, but does not converge absolutely since

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is a divergent p -series. Therefore, the series converges conditionally.

$$47. \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$$

The given series converges by the Alternating Series Test, but does not converge absolutely since the series

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

diverges by comparison to the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Therefore, the series converges conditionally.

$$51. \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$$

is convergent by comparison to the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

since

$$\frac{1}{(2n+1)!} < \frac{1}{2^n} \text{ for } n > 0.$$

Therefore, the given series converges absolutely.

$$55. \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p -series. Therefore, the given series converges absolutely.

57. An alternating series is a series whose terms alternate in sign. See Theorem 8.14.
59. $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.
 $\sum a_n$ is conditionally convergent if $\sum |a_n|$ diverges, but $\sum a_n$ converges.
61. (b). The partial sums alternate above and below the horizontal line representing the sum.
63. Since $\sum_{n=1}^{\infty} |a_n|$ converges we have

$$\lim_{n \rightarrow \infty} |a_n| = 0.$$
 Thus, there must exist an $N > 0$ such that $|a_n| < 1$ for all $n > N$ and it follows that $a_n^2 \leq |a_n|$ for all $n > N$. Hence, by the Comparison Test,

$$\sum_{n=1}^{\infty} a_n^2$$
 converges. Let $a_n = 1/n$ to see that the converse is false.
67. False
 Let $a_n = \frac{(-1)^n}{n}$.
69. $\sum_{n=1}^{\infty} \frac{10}{n^{3/2}} = 10 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ convergent p -series
71. Diverges by n th Term Test. $\lim_{n \rightarrow \infty} a_n = \infty$
73. Convergent Geometric Series ($r = \frac{7}{8} < 1$)
75. Convergent Geometric Series ($r = \frac{1}{\sqrt{e}}$) or Integral Test
77. Converges (absolutely) by Alternating Series Test
79. The first term of the series is zero, not one. You cannot regroup series terms arbitrarily.

Section 8.6 The Ratio and Root Tests

1.
$$\frac{(n+1)!}{(n-2)!} = \frac{(n+1)(n)(n-1)(n-2)!}{(n-2)!}$$

$$= (n+1)(n)(n-1)$$
3. Use the Principle of Mathematical Induction. When $k = 1$, the formula is valid since $1 = \frac{(2(1))!}{2^1 \cdot 1!}$. Assume that

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{(2n)!}{2^n n!}$$

and show that

$$1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1) = \frac{(2n+2)!}{2^{n+1}(n+1)!}$$

—CONTINUED—

3. —CONTINUED—

To do this, note that:

$$\begin{aligned}
 1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1) &= [1 \cdot 3 \cdot 5 \cdots (2n-1)](2n+1) \\
 &= \frac{(2n)!}{2^n n!} \cdot (2n+1) \\
 &= \frac{(2n)!(2n+1)}{2^n n!} \cdot \frac{(2n+2)}{2(n+1)} \\
 &= \frac{(2n)!(2n+1)(2n+2)}{2^{n+1} n! (n+1)} \\
 &= \frac{(2n+2)!}{2^{n+1} (n+1)}
 \end{aligned}$$

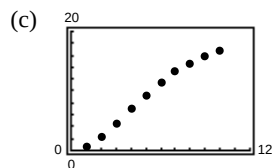
The formula is valid for all $n \geq 1$.

$$\begin{array}{lll}
 5. \sum_{n=1}^{\infty} n \left(\frac{3}{4}\right)^n = 1\left(\frac{3}{4}\right) + 2\left(\frac{9}{16}\right) + \cdots & 7. \sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n!} = 9 - \frac{3^3}{2} + \cdots & 9. \sum_{n=1}^{\infty} \left(\frac{4n}{5n-3}\right)^n = \frac{4}{2} + \left(\frac{8}{7}\right)^2 + \cdots \\
 S_1 = \frac{3}{4}, S_2 \approx 1.875 & S_1 = 9 & S_1 = 2 \\
 \text{Matches (d)} & \text{Matches (f)} & \text{Matches (a)}
 \end{array}$$

11. (a) Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^2(5/8)^{n+1}}{n^2(5/8)^n} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^2 \frac{5}{8} = \frac{5}{8} < 1$. Converges

(b)

n	5	10	15	20	25
S_n	9.2104	16.7598	18.8016	19.1878	19.2491



(d) The sum is approximately 19.26.

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of the partial sums approaches the sum of the series.

13. $\sum_{n=0}^{\infty} \frac{n!}{3^n}$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{n+1}{3} = \infty
 \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

15. $\sum_{n=1}^{\infty} n \left(\frac{3}{4}\right)^n$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(3/4)^{n+1}}{n(3/4)^n} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{3(n+1)}{4n} \right| = \frac{3}{4}
 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

17. $\sum_{n=1}^{\infty} \frac{n}{2^n}$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{n+1}{2n} = \frac{1}{2}
 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

19. $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{2n^2}{(n+1)^2} = 2
 \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

$$21. \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

$$23. \sum_{n=1}^{\infty} \frac{n!}{n3^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{(n+1)3^{n+1}} \cdot \frac{n3^n}{n!} \right| \\ &= \lim_{n \rightarrow \infty} \frac{n}{3} = \infty \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

$$25. \sum_{n=0}^{\infty} \frac{4^n}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{4^{n+1}}{(n+1)!} \cdot \frac{n!}{4^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{4}{n+1} = 0 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

$$27. \sum_{n=0}^{\infty} \frac{3^n}{(n+1)^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}}{(n+2)^{n+1}} \cdot \frac{(n+1)^n}{3^n} \right| = \lim_{n \rightarrow \infty} \frac{3(n+1)^n}{(n+2)^{n+1}} = \lim_{n \rightarrow \infty} \frac{3}{n+2} \left(\frac{n+1}{n+2} \right)^n = (0) \left(\frac{1}{e} \right) = 0$$

To find $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right)^n$, let $y = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right)^n$. Then,

$$\ln y = \lim_{n \rightarrow \infty} n \ln \left(\frac{n+1}{n+2} \right) = \lim_{n \rightarrow \infty} \frac{\ln[(n+1)/(n+2)]}{1/n} = \frac{0}{0}$$

$$\ln y = \lim_{n \rightarrow \infty} \frac{[(1)/(n+1)] - [(1)/(n+2)]}{-(1/n^2)} = -1 \text{ by L'Hôpital's Rule}$$

$$y = e^{-1} = \frac{1}{e}.$$

Therefore, by the Ratio Test, the series converges.

$$29. \sum_{n=0}^{\infty} \frac{4^n}{3^n + 1}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{4^{n+1}}{3^{n+1} + 1} \cdot \frac{3^n + 1}{4^n} \right| = \lim_{n \rightarrow \infty} \frac{4(3^n + 1)}{3^{n+1} + 1} = \lim_{n \rightarrow \infty} \frac{4(1 + 1/3^n)}{3 + 1/3^n} = \frac{4}{3}$$

Therefore, by the Ratio Test, the series diverges.

$$31. \sum_{n=0}^{\infty} \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdots (2n+1)(2n+3)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{n!} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{2n+3} = \frac{1}{2}$$

Therefore, by the Ratio Test, the series converges.

Note: The first few terms of this series are $-1 + \frac{1}{1 \cdot 3} - \frac{2!}{1 \cdot 3 \cdot 5} + \frac{3!}{1 \cdot 3 \cdot 5 \cdot 7} - \cdots$

$$33. (a) \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{1} \right| = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^{3/2} = 1$$

$$(b) \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^{1/2}} \cdot \frac{n^{1/2}}{1} \right| = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^{1/2} = 1$$

$$35. \sum_{n=1}^{\infty} \left(\frac{n}{2n+1} \right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n}{2n+1} \right)^n} \\ &= \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2} \end{aligned}$$

Therefore, by the Root Test, the series converges.

$$37. \sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{(-1)^n}{(\ln n)^n} \right|} \\ &= \lim_{n \rightarrow \infty} \frac{1}{|\ln n|} = 0 \end{aligned}$$

Therefore, by the Root Test, the series converges.

$$39. \sum_{n=1}^{\infty} (2\sqrt[n]{n} + 1)^n$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{(2\sqrt[n]{n} + 1)^n} = \lim_{n \rightarrow \infty} (2\sqrt[n]{n} + 1)$$

To find $\lim_{n \rightarrow \infty} \sqrt[n]{n}$, let $y = \lim_{n \rightarrow \infty} \sqrt[n]{x}$. Then

$$\ln y = \lim_{n \rightarrow \infty} (\ln \sqrt[n]{x}) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln x = \lim_{n \rightarrow \infty} \frac{\ln x}{x} = \lim_{n \rightarrow \infty} \frac{1/x}{1} = 0.$$

Thus, $\ln y = 0$, so $y = e^0 = 1$ and $\lim_{n \rightarrow \infty} (2\sqrt[n]{n} + 1) = 2(1) + 1 = 3$. Therefore, by the Root Test, the series diverges.

$$41. \sum_{n=3}^{\infty} \frac{1}{(\ln n)^n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0$$

Therefore, by the Root Test, the series converges.

$$43. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5}{n}$$

$$a_{n+1} = \frac{5}{n+1} < \frac{5}{n} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{5}{n} = 0$$

Therefore, by the Alternating Series Test, the series converges (conditional convergence).

$$45. \sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}} = 3 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

This is convergent p -series.

$$47. \sum_{n=1}^{\infty} \frac{2n}{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{2n}{n+1} = 2 \neq 0$$

This diverges by the n th Term Test for Divergence.

$$49. \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n 3^{-2}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{9} \left(-\frac{3}{2} \right)^n$$

Since $|r| = \frac{3}{2} > 1$, this is a divergent geometric series.

$$51. \sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$$

$$\lim_{n \rightarrow \infty} \frac{(10n+3)/n2^n}{1/2^n} = \lim_{n \rightarrow \infty} \frac{10n+3}{n} = 10$$

Therefore, the series converges by a limit comparison test with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n.$$

$$53. \sum_{n=1}^{\infty} \frac{\cos(n)}{2^n}$$

$$\left| \frac{\cos(n)}{2^n} \right| \leq \frac{1}{2^n}$$

Therefore, the series

$$\sum_{n=1}^{\infty} \left| \frac{\cos(n)}{2^n} \right|$$

converges by comparison with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n.$$

$$57. \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-1}}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^n}{(n+1)!} \cdot \frac{n!}{3^{n-1}} \right| = \lim_{n \rightarrow \infty} \frac{3}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

$$59. \sum_{n=1}^{\infty} \frac{(-3)^n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-3)^{n+1}}{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)} \cdot \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{(-3)^n} \right| = \lim_{n \rightarrow \infty} \frac{3}{2n+3} = 0$$

Therefore, by the Ratio Test, the series converges.

61. (a) and (c)

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n5^n}{n!} &= \sum_{n=0}^{\infty} \frac{(n+1)5^{n+1}}{(n+1)!} \\ &= 5 + \frac{(2)(5)^2}{2!} + \frac{(3)(5)^3}{3!} + \frac{(4)(5)^4}{4!} + \cdots \end{aligned}$$

$$55. \sum_{n=1}^{\infty} \frac{n7^n}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)7^{n+1}}{(n+1)!} \cdot \frac{n!}{n7^n} \right| = \lim_{n \rightarrow \infty} \frac{7}{n} = 0$$

Therefore, by the Ratio Test, the series converges.

63. (a) and (b) are the same.

65. Replace n with $n+1$.

$$\sum_{n=1}^{\infty} \frac{n}{4^n} = \sum_{n=0}^{\infty} \frac{n+1}{4^{n+1}}$$

67. Since

$$\frac{3^{10}}{2^{10}10!} = 1.59 \times 10^{-5},$$

use 9 terms.

$$\sum_{k=1}^9 \frac{(-3)^k}{2^k k!} \approx -0.7769$$

69. See Theorem 8.17, page 597.

71. No. Let $a_n = \frac{1}{n+10,000}$.

The series $\sum_{n=1}^{\infty} \frac{1}{n+10,000}$ diverges.

73. The series converges absolutely. See Theorem 8.17.

75. First, let

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = r < 1$$

and choose R such that $0 \leq r < R < 1$. There must exist some $N > 0$ such that $\sqrt[n]{|a_n|} < R$ for all $n > N$. Thus, for $n > N$, we $|a_n| < R^n$ and since the geometric series

$$\sum_{n=0}^{\infty} R^n$$

converges, we can apply the Comparison Test to conclude that

$$\sum_{n=1}^{\infty} |a_n|$$

converges which in turn implies that $\sum_{n=1}^{\infty} a_n$ converges.

Second, let

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = r > R > 1.$$

Then there must exist some $M > 0$ such that $\sqrt[n]{|a_n|} > R$ for all $n > M$. Thus, for $n > M$, we have $|a_n| > R^n > 1$ which implies that $\lim_{n \rightarrow \infty} a_n \neq 0$ which in turn implies that

$$\sum_{n=1}^{\infty} a_n \text{ diverges.}$$

Section 8.7 Taylor Polynomials and Approximations

1. $y = -\frac{1}{2}x^2 + 1$

Parabola

Matches (d)

3. $y = e^{-1/2}[(x+1) + 1]$

Linear

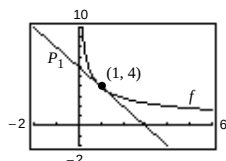
Matches (a)

5. $f(x) = \frac{4}{\sqrt{x}} = 4x^{-1/2} \quad f(1) = 4$

$$f'(x) = -2x^{-3/2} \quad f'(1) = -2$$

$$\begin{aligned} P_1(x) &= f(1) + f'(1)(x-1) \\ &= 4 + (-2)(x-1) \end{aligned}$$

$$P_1(x) = -2x + 6$$

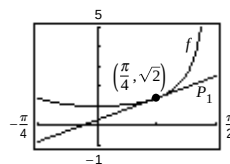


7. $f(x) = \sec x \quad f\left(\frac{\pi}{4}\right) = \sqrt{2}$

$$f'(x) = \sec x \tan x \quad f'\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$P_1(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$

$$P_1(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right)$$

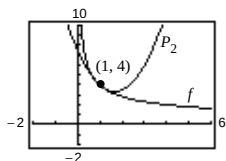


9. $f(x) = \frac{4}{\sqrt{x}} = 4x^{-1/2} \quad f(1) = 4$

$$f'(x) = -2x^{-3/2} \quad f'(1) = -2$$

$$f''(x) = 3x^{-5/2} \quad f''(1) = 3$$

$$\begin{aligned} P_2 &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 \\ &= 4 - 2(x-1) + \frac{3}{2}(x-1)^2 \end{aligned}$$



x	0	0.8	0.9	1.0	1.1	1.2	2
$f(x)$	Error	4.4721	4.2164	4.0	3.8139	3.6515	2.8284
$P_2(x)$	7.5	4.46	4.215	4.0	3.815	3.66	3.5

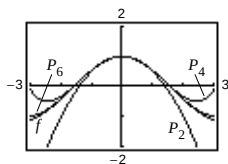
11. $f(x) = \cos x$

$$P_2(x) = 1 - \frac{1}{2}x^2$$

$$P_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

$$P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$$

(a)



13. $f(x) = e^{-x}$ $f(0) = 1$

$$f'(x) = -e^{-x} \quad f'(0) = -1$$

$$f''(x) = e^{-x} \quad f''(0) = 1$$

$$f'''(x) = -e^{-x} \quad f'''(0) = -1$$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$$

17. $f(x) = \sin x$ $f(0) = 0$

$$f'(x) = \cos x \quad f'(0) = 1$$

$$f''(x) = -\sin x \quad f''(0) = 0$$

$$f'''(x) = -\cos x \quad f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \quad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \cos x \quad f^{(5)}(0) = 1$$

$$P_5(x) = 0 + (1)x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5$$

$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

21. $f(x) = \frac{1}{x+1}$ $f(0) = 1$

$$f'(x) = -\frac{1}{(x+1)^2} \quad f'(0) = -1$$

$$f''(x) = \frac{2}{(x+1)^3} \quad f''(0) = 2$$

$$f'''(x) = \frac{-6}{(x+1)^4} \quad f'''(0) = -6$$

$$f^{(4)}(x) = \frac{24}{(x+1)^5} \quad f^{(4)}(0) = 24$$

$$P_4(x) = 1 - x + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{24}{4!}x^4$$

$$= 1 - x + x^2 - x^3 + x^4$$

(b) $f'(x) = -\sin x$ $P_2'(x) = -x$

$$f''(x) = -\cos x \quad P_2''(x) = -1$$

$$f'''(0) = P_2'''(0) = -1$$

$$f^{(4)}(x) = \sin x \quad P_4^{(4)}(x) = x$$

$$f^{(4)}(0) = 1 = P_4^{(4)}(0)$$

$$f^{(5)}(x) = \cos x \quad P_6^{(5)}(x) = -x$$

$$f^{(6)}(x) = -\sin x \quad P_6^{(6)}(x) = -1$$

$$f^{(6)}(0) = -1 = P_6^{(6)}(0)$$

(c) In general, $f^{(n)}(0) = P_n^{(n)}(0)$ for all n .

15. $f(x) = e^{2x}$ $f(0) = 1$

$$f'(x) = 2e^{2x} \quad f'(0) = 2$$

$$f''(x) = 4e^{2x} \quad f''(0) = 4$$

$$f'''(x) = 8e^{2x} \quad f'''(0) = 8$$

$$f^{(4)}(x) = 16e^{2x} \quad f^{(4)}(0) = 16$$

$$P_4(x) = 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4$$

19. $f(x) = xe^x$ $f(0) = 0$

$$f'(x) = xe^x + e^x \quad f'(0) = 1$$

$$f''(x) = xe^x + 2e^x \quad f''(0) = 2$$

$$f'''(x) = xe^x + 3e^x \quad f'''(0) = 3$$

$$f^{(4)}(x) = xe^x + 4e^x \quad f^{(4)}(0) = 4$$

$$P_4(x) = 0 + x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4$$

$$= x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$$

23. $f(x) = \sec x$ $f(0) = 1$

$$f'(x) = \sec x \tan x \quad f'(0) = 0$$

$$f''(x) = \sec^3 x + \sec x \tan^2 x \quad f''(0) = 1$$

$$P_2(x) = 1 + 0x + \frac{1}{2!}x^2 = 1 + \frac{1}{2}x^2$$

25. $f(x) = \frac{1}{x} \quad f(1) = 1$

$f'(x) = -\frac{1}{x^2} \quad f'(1) = -1$

$f''(x) = \frac{2}{x^3} \quad f''(1) = 2$

$f'''(x) = -\frac{6}{x^4} \quad f'''(1) = -6$

$f^{(4)}(x) = \frac{24}{x^5} \quad f^{(4)}(1) = 24$

$$P_4(x) = 1 - (x-1) + \frac{2}{2!}(x-1)^2 + \frac{-6}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4$$

27. $f(x) = \sqrt{x} \quad f(1) = 1$

$f'(x) = \frac{1}{2\sqrt{x}} \quad f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1}{4x\sqrt{x}} \quad f''(1) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8x^2\sqrt{x}} \quad f'''(1) = \frac{3}{8}$

$f^{(4)}(x) = -\frac{15}{16x^3\sqrt{x}} \quad f^{(4)}(1) = -\frac{15}{16}$

$$P_4(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4$$

31. $f(x) = \tan x$

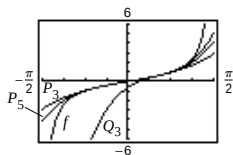
$f'(x) = \sec^2 x$

$f''(x) = 2 \sec^2 x \tan x$

$f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x$

$f^{(4)}(x) = 8 \sec^2 x \tan^3 x + 16 \sec^4 x \tan x$

$f^{(5)}(x) = 16 \sec^2 x \tan^4 x + 88 \sec^4 x \tan^2 x + 16 \sec^6 x$



29. $f(x) = \ln x \quad f(1) = 0$

$f'(x) = \frac{1}{x} \quad f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \quad f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \quad f'''(1) = 2$

$f^{(4)}(x) = -\frac{6}{x^4} \quad f^{(4)}(1) = -6$

$$P_4(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

(a) $n = 3, c = 0$

$$P_3(x) = 0 + x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 = x + \frac{1}{3}x^3$$

(b) $n = 5, c = 0$

$$P_5(x) = 0 + x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + \frac{0}{4!}x^4 + \frac{16}{5!}x^5 = x + \frac{1}{3}x^3 + \frac{2}{15}x^5$$

(c) $n = 3, c = \frac{\pi}{4}$

$$Q_3(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{16}{3!}\left(x - \frac{\pi}{4}\right)^3$$

$$= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$$

33. $f(x) = \sin x$

$P_1(x) = x$

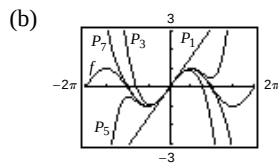
$P_3(x) = x - \frac{1}{6}x^3$

$P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

$P_7(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7$

(a)

x	0.00	0.25	0.50	0.75	1.00
$\sin x$	0.0000	0.2474	0.4794	0.6816	0.8415
$P_1(x)$	0.0000	0.2500	0.5000	0.7500	1.0000
$P_3(x)$	0.0000	0.2474	0.4792	0.6797	0.8333
$P_5(x)$	0.0000	0.2474	0.4794	0.6817	0.8417
$P_7(x)$	0.0000	0.2474	0.4794	0.6816	0.8415



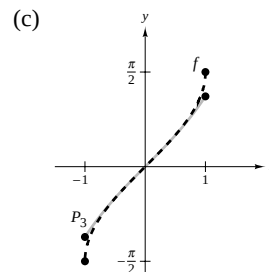
(c) As the distance increases, the accuracy decreases

35. $f(x) = \arcsin x$

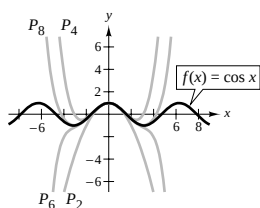
(a) $P_3(x) = x + \frac{x^3}{6}$

(b)

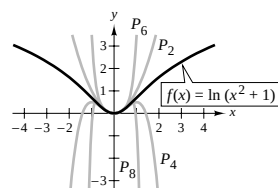
x	-0.75	-0.50	-0.25	0	0.25	0.50	0.75
$f(x)$	-0.848	-0.524	-0.253	0	0.253	0.524	0.848
$P_3(x)$	-0.820	-0.521	-0.253	0	0.253	0.521	0.820



37. $f(x) = \cos x$



39. $f(x) = \ln(x^2 + 1)$



41. $f(x) = e^{-x} \approx 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$

$f\left(\frac{1}{2}\right) \approx 0.6042$

43. $f(x) = \ln x \approx (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4$

$f(1.2) \approx 0.1823$

45. $f(x) = \cos x; f^{(5)}(x) = -\sin x \Rightarrow \text{Max on } [0, 0.3] \text{ is } 1.$

$R_4(x) \leq \frac{1}{5!}(0.3)^5 = 2.025 \times 10^{-5}$

47. $f(x) = \arcsin x$; $f^{(4)}(x) = \frac{x(6x^2 + 9)}{(1 - x^2)^{7/2}} \Rightarrow \text{Max on } [0, 0.4] \text{ is } f^{(4)}(0.4) \approx 7.3340.$

$$R_3(x) \leq \frac{7.3340}{4!}(0.4)^4 \approx 0.00782 = 7.82 \times 10^{-3}$$

49. $g(x) = \sin x$

$$g^{(n+1)}(x) \leq 1 \text{ for all } x$$

$$R_n(x) \leq \frac{1}{(n+1)!}(0.3)^{n+1} < 0.001$$

By trial and error, $n = 3$.

51. $f(x) = \ln(x + 1)$

$$f^{(n+1)}(x) = \frac{(-1)^{n+1}n!}{(x+1)^{n+1}} \Rightarrow \text{Max on } [0, 0.5] \text{ is } n!.$$

$$R_n \leq \frac{n!}{(n+1)!}(0.5)^{n+1} = \frac{(0.5)^{n+1}}{n+1} < 0.0001$$

By trial and error, $n = 9$. (See Example 9.) Using 9 terms, $\ln(1.5) \approx 0.4055$.

53. $f(x) = e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}, x < 0$

$$R_3(x) = \frac{e^z}{4!}x^4 < 0.001$$

$$e^z x^4 < 0.024$$

$$xe^{z/4} < 0.3936$$

$$x < \frac{0.3936}{e^{z/4}} < 0.3936, z < 0$$

$$-0.3936 < x < 0$$

57. See definition on page 607.

55. The graph of the approximating polynomial P and the elementary function f both pass through the point $(c, f(c))$ and the slopes of P and f agree at $(c, f(c))$. Depending on the degree of P , the n th derivatives of P and f agree at $(c, f(c))$.

59. The accuracy increases as the degree increases (for values within the interval of convergence).

61. (a) $f(x) = e^x$

$$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

$$g(x) = xe^x$$

$$Q_5(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$$

$$Q_5(x) = xP_4(x)$$

(b) $f(x) = \sin x$

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$g(x) = x \sin x$$

$$Q_6(x) = xP_5(x) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}$$

(c) $g(x) = \frac{\sin x}{x} = \frac{1}{x}P_5(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!}$

63. (a) $Q_2(x) = -1 + \frac{\pi^2(x+2)^2}{32}$

(b) $R_2(x) = -1 + \frac{\pi^2(x-6)^2}{32}$

(c) No. The polynomial will be linear.
Translations are possible at $x = -2 + 8n$.

65. Let f be an even function and P_n be the n th Maclaurin polynomial for f . Since f is even, f' is odd, f'' is even, f''' is odd, etc. (see Exercise 45). All of the odd derivatives of f are odd and thus, all of the odd powers of x will have coefficients of zero. P_n will only have terms with even powers of x .

67. As you move away from $x = c$, the Taylor Polynomial becomes less and less accurate.

Section 8.8 Power Series

1. Centered at 0

$$\begin{aligned}
 5. \quad & \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1} \\
 L &= \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n x^n} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{n+1}{n+2} \right| |x| = |x| \\
 |x| < 1 &\Rightarrow R = 1
 \end{aligned}$$

$$\begin{aligned}
 9. \quad & \sum_{n=0}^{\infty} \frac{(2x)^{2n}}{(2n)!} \\
 L &= \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{2n+2}/(2n+2)!}{(2x)^{2n}/(2n)!} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{(2x)^2}{(2n+2)(2n+1)} \right| = 0
 \end{aligned}$$

Thus, the series converges for all x . $R = \infty$.

$$\begin{aligned}
 13. \quad & \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} \\
 \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{n+1} \cdot \frac{n}{(-1)^n x^n} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{nx}{n+1} \right| = |x|
 \end{aligned}$$

Interval: $-1 < x < 1$ When $x = 1$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.When $x = -1$, the p -series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.Therefore, the interval of convergence is $-1 < x \leq 1$.

$$\begin{aligned}
 17. \quad & \sum_{n=0}^{\infty} (2n)! \left(\frac{x}{2} \right)^n \\
 \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(2n+2)! x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(2n)! x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)(2n+1)x}{2} \right| = \infty
 \end{aligned}$$

Therefore, the series converges only for $x = 0$.

$$19. \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{4^n}$$

Since the series is geometric, it converges only if $|x/4| < 1$ or $-4 < x < 4$.

3. Centered at 2

$$\begin{aligned}
 7. \quad & \sum_{n=1}^{\infty} \frac{(2x)^n}{n^2} \\
 L &= \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(2x)^n} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{2n^2 x}{(n+1)^2} \right| = 2|x| \\
 2|x| < 1 &\Rightarrow R = \frac{1}{2}
 \end{aligned}$$

$$11. \quad \sum_{n=0}^{\infty} \left(\frac{x}{2} \right)^n$$

Since the series is geometric, it converges only if $|x/2| < 1$ or $-2 < x < 2$.

$$\begin{aligned}
 15. \quad & \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
 \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right| = 0
 \end{aligned}$$

The series converges for all x . Therefore, the interval of convergence is $-\infty < x < \infty$.

$$21. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(x-5)^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n5^n}{(-1)^{n+1}(x-5)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n(x-5)}{5(n+1)} \right| = \frac{1}{5}|x-5|$$

$$R = 5$$

$$\text{Center: } x = 5$$

$$\text{Interval: } -5 < x - 5 < 5 \text{ or } 0 < x < 10$$

When $x = 0$, the p -series $\sum_{n=1}^{\infty} \frac{-1}{n}$ diverges.

When $x = 10$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

Therefore, the interval of convergence is $0 < x \leq 10$.

$$23. \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(x-1)^{n+2}}{n+2} \cdot \frac{n+1}{(-1)^{n+1}(x-1)^{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x-1)}{n+2} \right| = |x-1|$$

$$R = 1$$

$$\text{Center: } x = 1$$

$$\text{Interval: } -1 < x - 1 < 1 \text{ or } 0 < x < 2$$

When $x = 0$, the series $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges by the integral test.

When $x = 2$, the alternating series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$ converges.

Therefore, the interval of convergence is $0 < x \leq 2$.

$$25. \sum_{n=1}^{\infty} \frac{(x-c)^{n-1}}{c^{n-1}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-c)^n}{c^n} \cdot \frac{c^{n-1}}{(x-c)^{n-1}} \right| = \frac{1}{c}|x-c|$$

$$R = c$$

$$\text{Center: } x = c$$

$$\text{Interval: } -c < x - c < c \text{ or } 0 < x < 2c$$

When $x = 0$, the series $\sum_{n=1}^{\infty} (-1)^{n-1}$ diverges.

When $x = 2c$, the series $\sum_{n=1}^{\infty} 1$ diverges.

Therefore, the interval of convergence is $0 < x < 2c$.

$$27. \sum_{n=1}^{\infty} \frac{n}{n+1} (-2x)^{n-1}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(-2x)^n}{n+2} \cdot \frac{n+1}{n(-2x)^{n-1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(-2x)(n+1)^2}{n(n+2)} \right| = 2|x| \end{aligned}$$

$$R = \frac{1}{2}$$

$$\text{Interval: } -\frac{1}{2} < x < \frac{1}{2}$$

When $x = -\frac{1}{2}$, the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges by the n th Term Test.

When $x = \frac{1}{2}$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n+1}$ diverges.

Therefore, the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$.

$$29. \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x^2}{(2n+2)(2n+3)} \right| = 0 \end{aligned}$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

$$31. \sum_{n=1}^{\infty} \frac{k(k+1) \cdots (k+n-1)x^n}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{k(k+1) \cdots (k+n-1)(k+n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k+1) \cdots (k+n-1)x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(k+n)x}{n+1} \right| = |x|$$

$$R = 1$$

When $x = \pm 1$, the series diverges and the interval of convergence is $-1 < x < 1$.

$$\left[\frac{k(k+1) \cdots (k+n-1)}{1 \cdot 2 \cdots n} \geq 1 \right]$$

$$33. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3 \cdot 7 \cdot 11 \cdots (4n-1)(x-3)^n}{4^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2} \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)(4n+3)(x-3)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{(-1)^{n+1} \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)(x-3)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(4n+3)(x-3)}{4} \right| = \infty \end{aligned}$$

$$R = 0$$

Center: $x = 3$

Therefore, the series converges only for $x = 3$.

$$35. (a) f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n, -2 < x < 2 \quad (\text{Geometric})$$

$$(b) f'(x) = \sum_{n=1}^{\infty} \left(\frac{n}{2}\right) \left(\frac{x}{2}\right)^{n-1}, -2 < x < 2$$

$$(c) f''(x) = \sum_{n=2}^{\infty} \left(\frac{n}{2}\right) \left(\frac{n-1}{2}\right) \left(\frac{x}{2}\right)^{n-2}, -2 < x < 2$$

$$(d) \int f(x) dx = \sum_{n=0}^{\infty} \frac{2}{n+1} \left(\frac{x}{2}\right)^{n+1}, -2 \leq x < 2$$

$$37. (a) f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}, 0 < x \leq 2$$

$$(b) f'(x) = \sum_{n=0}^{\infty} (-1)^{n+1}(x-1)^n, 0 < x < 2$$

$$(c) f''(x) = \sum_{n=1}^{\infty} (-1)^{n+1}n(x-1)^{n-1}, 0 < x < 2$$

$$(d) \int f(x) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+2}}{(n+1)(n+2)}, 0 \leq x \leq 2$$

$$39. g(1) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = 1 + \frac{1}{3} + \frac{1}{9} + \cdots$$

$$S_1 = 1, S_2 = 1.33. \text{ Matches (c)}$$

$$41. g(3.1) = \sum_{n=0}^{\infty} \left(\frac{3.1}{3}\right)^n \text{ diverges. Matches (b)}$$

43. A series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

is called a power series centered at c .

45. A single point, a_n interval, or the entire real line.

47. (a) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, -\infty < x < \infty$ (See Exercise 29.)

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, -\infty < x < \infty$$

(b) $f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = g(x)$

(c) $g'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} = -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = -f(x)$

(d) $f(x) = \sin x$ and $g(x) = \cos x$

49. $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$

$$y' = \sum_{n=1}^{\infty} \frac{2nx^{2n-1}}{2^n n!}$$

$$y'' = \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!}$$

$$\begin{aligned} y'' - xy' - y &= \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=1}^{\infty} \frac{2nx^{2n}}{2^n n!} - \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} \\ &= \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{2^n n!} \\ &= \sum_{n=0}^{\infty} \left[\frac{(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!} - \frac{(2n+1)x^{2n}}{2^n n!} \cdot \frac{2(n+1)}{2(n+1)} \right] \\ &= \sum_{n=0}^{\infty} \frac{2(n+1)x^{2n}[(2n+1) - (2n+1)]}{2^{n+1}(n+1)!} = 0 \end{aligned}$$

51. $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} (k!)^2}$

(a) $\lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+1} x^{2k+2}}{2^{2k+2} [(k+1)!]^2} \cdot \frac{2^{2k} (k!)^2}{(-1)^k x^{2k}} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)x^2}{2^2(k+1)^2} \right| = 0$

Therefore, the interval of convergence is $-\infty < x < \infty$.

(b) $J_0 = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{4^k (k!)^2}$

$$J_0' = \sum_{k=1}^{\infty} (-1)^k \frac{2kx^{2k-1}}{4^k (k!)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(2k+2)x^{2k+1}}{4^{k+1} [(k+1)!]^2}$$

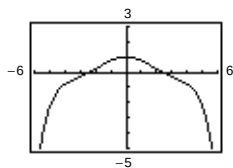
$$J_0'' = \sum_{k=1}^{\infty} (-1)^k \frac{2k(2k-1)x^{2k-2}}{4^k (k!)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(2k+2)(2k+1)x^{2k}}{4^{k+1} [(k+1)!]^2}$$

$$\begin{aligned} x^2 J_0'' + x J_0' + J_0 &= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2(2k+1)x^{2k+2}}{4^{k+1}(k+1)!k!} + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2x^{2k+2}}{4^{k+1}(k+1)!k!} + \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+2}}{4^k (k!)^2} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{4^k (k!)^2} \left[(-1) \frac{2(2k+1)}{4(k+1)} + (-1) \frac{2}{4(k+1)} + 1 \right] \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{4^k (k!)^2} \left[\frac{-4k-2}{4k+4} - \frac{2}{4k+4} + \frac{4k+4}{4k+4} \right] = 0 \end{aligned}$$

—CONTINUED—

51. —CONTINUED—

$$(c) P_6(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304}$$

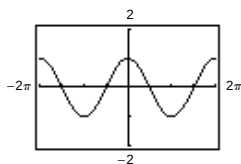


$$\begin{aligned} (d) \int_0^1 J_0 dx &= \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{4^k (k!)^2} dx \\ &= \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{4^k (k!)^2 (2k+1)} \right]_0^1 \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (k!)^2 (2k+1)} \\ &= 1 - \frac{1}{12} + \frac{1}{320} \approx 0.92 \end{aligned}$$

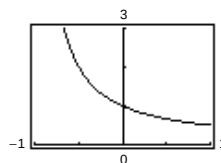
(exact integral is 0.9197304101)

$$53. f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$

(See Exercise 47.)

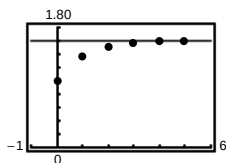


$$\begin{aligned} 55. f(x) &= \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n \\ &= \frac{1}{1 - (-x)} = \frac{1}{1+x} \text{ for } -1 < x < 1 \end{aligned}$$

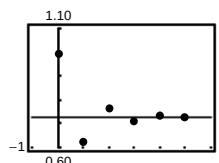


$$57. \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

$$\begin{aligned} (a) \sum_{n=0}^{\infty} \left(\frac{3/4}{2}\right)^n &= \sum_{n=0}^{\infty} \left(\frac{3}{8}\right)^n \\ &= \frac{1}{1 - (3/8)} = \frac{8}{5} = 1.6 \end{aligned}$$



$$\begin{aligned} (b) \sum_{n=0}^{\infty} \left(\frac{-3/4}{2}\right)^n &= \sum_{n=0}^{\infty} \left(-\frac{3}{8}\right)^n \\ &= \frac{1}{1 - (-3/8)} = \frac{8}{11} \approx 0.7272 \end{aligned}$$



(c) The alternating series converges more rapidly. The partial sums of the series of positive terms approach the sum from below. The partial sums of the alternating series alternate sides of the horizontal line representing the sum.

$$(d) \sum_{n=0}^N \left(\frac{3}{2}\right)^n > M$$

M	10	100	1000	10,000
N	4	9	15	21

59. False;

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n 2^n}$$

converges for $x = 2$ but diverges for $x = -2$.61. True; the radius of convergence is $R = 1$ for both series.

Section 8.9 Representation of Functions by Power Series

$$\begin{aligned}
 1. \quad (a) \quad \frac{1}{2-x} &= \frac{1/2}{1-(x/2)} = \frac{a}{1-r} \\
 &= \sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}
 \end{aligned}$$

This series converges on $(-2, 2)$.

$$\begin{array}{r}
 \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \cdots \\
 \hline
 (b) \quad 2-x \quad 1
 \end{array}$$

$$\begin{array}{r}
 1 - \frac{x}{2} \\
 \hline
 \frac{x}{2} \\
 \frac{x}{2} - \frac{x^2}{4} \\
 \hline
 \frac{x^2}{4} \\
 \frac{x^2}{4} - \frac{x^3}{8} \\
 \hline
 \frac{x^3}{8} \\
 \frac{x^3}{8} - \frac{x^4}{16} \\
 \hline
 \vdots
 \end{array}$$

$$\begin{aligned}
 3. \quad (a) \quad \frac{1}{2+x} &= \frac{1/2}{1-(-x/2)} = \frac{a}{1-r} \\
 &= \sum_{n=0}^{\infty} \frac{1}{2} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}}
 \end{aligned}$$

This series converges on $(-2, 2)$.

$$\begin{array}{r}
 \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \cdots \\
 \hline
 (b) \quad 2+x \quad 1
 \end{array}$$

$$\begin{array}{r}
 1 + \frac{x}{2} \\
 \hline
 -\frac{x}{2} \\
 -\frac{x}{2} - \frac{x^2}{4} \\
 \hline
 \frac{x^2}{4} \\
 \frac{x^2}{4} + \frac{x^3}{8} \\
 \hline
 -\frac{x^3}{8} \\
 -\frac{x^3}{8} - \frac{x^4}{16} \\
 \hline
 \vdots
 \end{array}$$

5. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{1}{2-x} = \frac{1}{-3-(x-5)} = \frac{-1/3}{1+(1/3)(x-5)}$$

which implies that $a = -1/3$ and $r = (-1/3)(x-5)$.

Therefore, the power series for $f(x)$ is given by

$$\begin{aligned}
 \frac{1}{2-x} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} -\frac{1}{3} \left[-\frac{1}{3}(x-5)\right]^n \\
 &= \sum_{n=0}^{\infty} \frac{(x-5)^n}{(-3)^{n+1}}, \quad |x-5| < 3 \text{ or } 2 < x < 8.
 \end{aligned}$$

9. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\begin{aligned}
 \frac{1}{2x-5} &= \frac{-1}{11-2(x+3)} \\
 &= \frac{-1/11}{1-(2/11)(x+3)} = \frac{a}{1-r}
 \end{aligned}$$

which implies that $a = -1/11$ and $r = (2/11)(x+3)$.

Therefore, the power series for $f(x)$ is given by

$$\begin{aligned}
 \frac{1}{2x-5} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \left(-\frac{1}{11}\right) \left[\frac{2}{11}(x+3)\right]^n \\
 &= -\sum_{n=0}^{\infty} \frac{2^n(x+3)^n}{11^{n+1}},
 \end{aligned}$$

$$|x+3| < \frac{11}{2} \text{ or } -\frac{17}{2} < x < \frac{5}{2}.$$

7. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{2x-1} = \frac{-3}{1-2x} = \frac{a}{1-r}$$

which implies that $a = -3$ and $r = 2x$.

Therefore, the power series for $f(x)$ is given by

$$\begin{aligned}
 \frac{3}{2x-1} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (-3)(2x)^n \\
 &= -3 \sum_{n=0}^{\infty} (2x)^n, \quad |2x| < 1 \text{ or } -\frac{1}{2} < x < \frac{1}{2}.
 \end{aligned}$$

11. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{x+2} = \frac{3}{2+x} = \frac{3/2}{1+(1/2)x} = \frac{a}{1-r}$$

which implies that $a = 3/2$ and $r = (-1/2)x$. Therefore, the power series for $f(x)$ is given by

$$\begin{aligned}
 \frac{3}{x+2} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{3}{2} \left(-\frac{1}{2}x\right)^n \\
 &= 3 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}} = \frac{3}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n,
 \end{aligned}$$

$$|x| < 2 \text{ or } -2 < x < 2.$$

$$13. \frac{3x}{x^2 + x - 2} = \frac{2}{x + 2} + \frac{1}{x - 1} = \frac{2}{2 + x} + \frac{1}{-1 + x} = \frac{1}{1 + (1/2)x} + \frac{-1}{1 - x}$$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{3x}{x^2 + x - 2} = \sum_{n=0}^{\infty} \left(-\frac{1}{2}x\right)^n + \sum_{n=0}^{\infty} (-1)(x)^n = \sum_{n=0}^{\infty} \left[\frac{1}{(-2)^n} - 1\right]x^n.$$

The interval of convergence is $-1 < x < 1$ since

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(1 - (-2)^{n+1})x^{n+1}}{(-2)^{n+1}} \cdot \frac{(-2)^n}{(1 - (-2)^n)x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(1 - (-2)^{n+1})x}{-2 - (-2)^{n+1}} \right| = |x|.$$

$$15. \frac{2}{1 - x^2} = \frac{1}{1 - x} + \frac{1}{1 + x}$$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{2}{1 - x^2} = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (1 + (-1)^n)x^n = \sum_{n=0}^{\infty} 2x^{2n}.$$

The interval of convergence is $|x^2| < 1$ or $-1 < x < 1$ since $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2x^{2n+2}}{2x^{2n}} \right| = |x^2|.$

$$17. \frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} (-1)^n (-x)^n = \sum_{n=0}^{\infty} (-1)^{2n} x^n = \sum_{n=0}^{\infty} x^n$$

$$h(x) = \frac{-2}{x^2 - 1} = \frac{1}{1 + x} + \frac{1}{1 - x} = \sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} [(-1)^n + 1]x^n$$

$$= 2 + 0x + 2x^2 + 0x^3 + 2x^4 + 0x^5 + 2x^6 + \cdots = \sum_{n=0}^{\infty} 2x^{2n}, \quad -1 < x < 1 \text{ (See Exercise 15.)}$$

19. By taking the first derivative, we have $\frac{d}{dx} \left[\frac{1}{x + 1} \right] = \frac{-1}{(x + 1)^2}$. Therefore,

$$\begin{aligned} \frac{-1}{(x + 1)^2} &= \frac{d}{dx} \left[\sum_{n=0}^{\infty} (-1)^n x^n \right] = \sum_{n=1}^{\infty} (-1)^n n x^{n-1} \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} (n + 1) x^n, \quad -1 < x < 1. \end{aligned}$$

21. By integrating, we have $\int \frac{1}{x + 1} dx = \ln(x + 1)$. Therefore,

$$\ln(x + 1) = \int \left[\sum_{n=0}^{\infty} (-1)^n x^n \right] dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n + 1}, \quad -1 < x \leq 1.$$

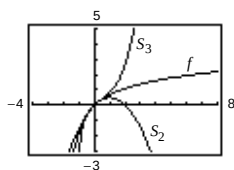
To solve for C , let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(x + 1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n + 1}, \quad -1 < x \leq 1.$$

$$23. \frac{1}{x^2 + 1} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad -1 < x < 1$$

$$25. \text{ Since, } \frac{1}{x + 1} = \sum_{n=0}^{\infty} (-1)^n x^n, \text{ we have } \frac{1}{4x^2 + 1} = \sum_{n=0}^{\infty} (-1)^n (4x^2)^n = \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n (2x)^{2n}, \quad -\frac{1}{2} < x < \frac{1}{2}.$$

$$27. x - \frac{x^2}{2} \leq \ln(x+1) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$$



x	0.0	0.2	0.4	0.6	0.8	1.0
$x - \frac{x^2}{2}$	0.000	0.180	0.320	0.420	0.480	0.500
$\ln(x+1)$	0.000	0.180	0.336	0.470	0.588	0.693
$x - \frac{x^2}{2} + \frac{x^3}{3}$	0.000	0.183	0.341	0.492	0.651	0.833

$$29. g(x) = x, \text{ line, Matches (c)}$$

$$31. g(x) = x - \frac{x^3}{3} + \frac{x^5}{5}, \text{ Matches (a)}$$

$$33. f(x) = \arctan x \text{ is an odd function (symmetric to the origin)}$$

$$\text{In Exercises 35 and 37, } \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

$$35. \arctan \frac{1}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{(1/4)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)4^{2n+1}} = \frac{1}{4} - \frac{1}{192} + \frac{1}{5120} + \dots$$

Since $\frac{1}{5120} < 0.001$, we can approximate the series by its first two terms: $\arctan \frac{1}{4} \approx \frac{1}{4} - \frac{1}{192} \approx 0.245$.

$$37. \frac{\arctan x^2}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{2n+1}$$

$$\int \frac{\arctan x^2}{x} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(4n+2)(2n+1)}$$

$$\int_0^{1/2} \frac{\arctan x^2}{x} dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(4n+2)(2n+1)2^{4n+2}} = \frac{1}{8} - \frac{1}{1152} + \dots$$

Since $\frac{1}{1152} < 0.001$, we can approximate the series by its first term: $\int_0^{1/2} \frac{\arctan x^2}{x} dx \approx 0.125$

$$\text{In Exercises 39 and 41, use } \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1.$$

$$39. (a) \frac{1}{(1-x)^2} = \frac{d}{dx} \left[\frac{1}{1-x} \right] = \frac{d}{dx} \left[\sum_{n=0}^{\infty} x^n \right] = \sum_{n=1}^{\infty} nx^{n-1}, |x| < 1$$

$$(b) \frac{x}{(1-x)^2} = x \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^n, |x| < 1$$

$$(c) \frac{1+x}{(1-x)^2} = \frac{1}{(1-x)^2} + \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n(x^{n-1} + x^n), |x| < 1$$

$$= \sum_{n=0}^{\infty} (2n+1)x^n, |x| < 1$$

$$(d) \frac{x(1+x)}{(1-x)^2} = x \sum_{n=0}^{\infty} (2n+1)x^n = \sum_{n=0}^{\infty} (2n+1)x^{n+1}, |x| < 1$$

$$41. P(n) = \left(\frac{1}{2}\right)^n$$

$$E(n) = \sum_{n=1}^{\infty} nP(n) = \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = \frac{1}{2} \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^{n-1}$$

$$= \frac{1}{2} \frac{1}{[1 - (1/2)]^2} = 2$$

Since the probability of obtaining a head on a single toss is $\frac{1}{2}$, it is expected that, on average, a head will be obtained in two tosses.

43. Replace x with $(-x)$.45. Replace x with $(-x)$ and multiply the series by 5.47. Let $\arctan x + \arctan y = \theta$. Then,

$$\tan(\arctan x + \arctan y) = \tan \theta$$

$$\frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x) \tan(\arctan y)} = \tan \theta$$

$$\frac{x + y}{1 - xy} = \tan \theta$$

$$\arctan\left(\frac{x + y}{1 - xy}\right) = \theta. \text{ Therefore, } \arctan x + \arctan y = \arctan\left(\frac{x + y}{1 - xy}\right) \text{ for } xy \neq 1.$$

$$49. (a) 2 \arctan \frac{1}{2} = \arctan \frac{1}{2} + \arctan \frac{1}{2} = \arctan \left[\frac{2(1/2)}{1 - (1/2)^2} \right] = \arctan \frac{4}{3}$$

$$2 \arctan \frac{1}{2} - \arctan \frac{1}{7} = \arctan \frac{4}{3} + \arctan \left(-\frac{1}{7} \right) = \arctan \left[\frac{(4/3) - (1/7)}{1 + (4/3)(1/7)} \right] = \arctan \frac{25}{25} = \arctan 1 = \frac{\pi}{4}$$

$$(b) \pi = 8 \arctan \frac{1}{2} - 4 \arctan \frac{1}{7} \approx 8 \left[\frac{1}{2} - \frac{(0.5)^3}{3} + \frac{(0.5)^5}{5} - \frac{(0.5)^7}{7} \right] - 4 \left[\frac{1}{7} - \frac{(1/7)^3}{3} + \frac{(1/7)^5}{5} - \frac{(1/7)^7}{7} \right] \approx 3.14$$

51. From Exercise 21, we have

$$\begin{aligned} \ln(x + 1) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}. \end{aligned}$$

$$\begin{aligned} \text{Thus, } \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2^n n} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (1/2)^n}{n} \\ &= \ln\left(\frac{1}{2} + 1\right) = \ln \frac{3}{2} \approx 0.4055 \end{aligned}$$

53. From Exercise 51, we have

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{5^n n} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2/5)^n}{n} \\ &= \ln\left(\frac{2}{5} + 1\right) = \ln \frac{7}{5} \approx 0.3365. \end{aligned}$$

55. From Exercise 54, we have

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n+1}(2n+1)} = \sum_{n=0}^{\infty} (-1)^n \frac{(1/2)^{2n+1}}{2n+1} = \arctan \frac{1}{2} \approx 0.4636.$$

57. The series in Exercise 54 converges to its sum at a slower rate because its terms approach 0 at a much slower rate.

$$59. f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \leq 2$$

$$f(0.5) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-0.5)^n}{n} = \sum_{n=1}^{\infty} -\frac{(1/2)^n}{n}$$

$$\sum_{n=1}^{\infty} -\frac{(1/2)^n}{n} = -0.6931$$

Section 8.10 Taylor and Maclaurin Series

1. For $c = 0$, we have:

$$f(x) = e^{2x}$$

$$f^{(n)}(x) = 2^n e^{2x} \Rightarrow f^{(n)}(0) = 2^n$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

3. For $c = \pi/4$, we have:

$$f(x) = \cos(x) \quad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = -\sin(x) \quad f'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f''(x) = -\cos(x) \quad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f'''(x) = \sin(x) \quad f'''\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f^{(4)}(x) = \cos(x) \quad f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

and so on. Therefore, we have:

$$\begin{aligned} \cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)[x - (\pi/4)]^n}{n!} \\ &= \frac{\sqrt{2}}{2} \left[1 - \left(x - \frac{\pi}{4}\right) - \frac{[x - (\pi/4)]^2}{2!} + \frac{[x - (\pi/4)]^3}{3!} + \frac{[x - (\pi/4)]^4}{4!} - \dots \right] \\ &= \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2} [x - (\pi/4)]^n}{n!}. \end{aligned}$$

[Note: $(-1)^{n(n+1)/2} = 1, -1, -1, 1, 1, -1, -1, 1, \dots$]

5. For $c = 1$, we have,

$$f(x) = \ln x \quad f(1) = 0$$

$$f'(x) = \frac{1}{x} \quad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \quad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \quad f'''(1) = 2$$

$$f^{(4)}(x) = -\frac{6}{x^4} \quad f^{(4)}(1) = -6$$

$$f^{(5)}(x) = \frac{24}{x^5} \quad f^{(5)}(1) = 24$$

and so on. Therefore, we have:

$$\begin{aligned} \ln x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} \\ &= 0 + (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} + \frac{24(x-1)^5}{5!} - \dots \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} \end{aligned}$$

7. For $c = 0$, we have:

$$\begin{aligned}
 f(x) &= \sin 2x & f(0) &= 0 \\
 f'(x) &= 2 \cos 2x & f'(0) &= 2 \\
 f''(x) &= -4 \sin 2x & f''(0) &= 0 \\
 f'''(x) &= -8 \cos 2x & f'''(0) &= -8 \\
 f^{(4)}(x) &= 16 \sin 2x & f^{(4)}(0) &= 0 \\
 f^{(5)}(x) &= 32 \cos 2x & f^{(5)}(0) &= 32 \\
 f^{(6)}(x) &= -64 \sin 2x & f^{(6)}(0) &= 0 \\
 f^{(7)}(x) &= -128 \cos 2x & f^{(7)}(0) &= -128
 \end{aligned}$$

and so on. Therefore, we have:

$$\begin{aligned}
 \sin 2x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \cdots \\
 &= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}
 \end{aligned}$$

9. For $c = 0$, we have:

$$\begin{aligned}
 f(x) &= \sec(x) & f(0) &= 1 \\
 f'(x) &= \sec(x)\tan(x) & f'(0) &= 0 \\
 f''(x) &= \sec^3(x) + \sec(x)\tan^2(x) & f''(0) &= 1 \\
 f'''(x) &= 5 \sec^3(x)\tan(x) + \sec(x)\tan^3(x) & f'''(0) &= 0 \\
 f^{(4)}(x) &= 5 \sec^5(x) + 18 \sec^3(x)\tan^2(x) + \sec(x)\tan^4(x) & f^{(4)}(0) &= 5 \\
 \sec(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \cdots
 \end{aligned}$$

11. The Maclaurin series for $f(x) = \cos x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

Because $f^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x$, we have $|f^{(n+1)}(z)| \leq 1$ for all z . Hence by Taylor's Theorem,

$$0 \leq |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

Since $\lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$, it follows that $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$. Hence, the Maclaurin series for $\cos x$ converges to $\cos x$ for all x .

13. Since $(1+x)^{-k} = 1 - kx + \frac{k(k+1)x^2}{2!} - \frac{k(k+1)(k+2)x^3}{3!} + \cdots$, we have

$$\begin{aligned}
 (1+x)^{-2} &= 1 - 2x + \frac{2(3)x^2}{2!} - \frac{2(3)(4)x^3}{3!} + \frac{2(3)(4)(5)x^4}{5!} - \cdots = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots \\
 &= \sum_{n=0}^{\infty} (-1)^n (n+1)x^n.
 \end{aligned}$$

15. $\frac{1}{\sqrt{4+x^2}} = \left(\frac{1}{2}\right) \left[1 + \left(\frac{x}{2}\right)^2\right]^{-1/2}$ and since $(1+x)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$, we have

$$\frac{1}{\sqrt{4+x^2}} = \frac{1}{2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)(x/2)^{2n}}{2^n n!} \right] = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^{3n+1} n!}.$$

17. Since $(1+x)^{1/2} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)x^n}{2^n n!}$ (Exercise 14)

we have $(1+x^2)^{1/2} = 1 + \frac{x^2}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)x^{2n}}{2^n n!}.$

19. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$$e^{x^2/2} = \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = 1 + \frac{x^2}{2} + \frac{x^4}{2^2 2!} + \frac{x^6}{2^3 3!} + \frac{x^8}{2^4 4!} + \cdots$$

21. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$\sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!} = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots$$

23. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$$\cos x^{3/2} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{3/2})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(2n)!} = 1 - \frac{x^3}{2!} + \frac{x^6}{4!} - \cdots$$

25. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \frac{2x^7}{7!} + \cdots$$

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

27. $\cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$

$$= \frac{1}{2} \left[1 + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots \right]$$

$$= \frac{1}{2} \left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \right]$$

29. $x \sin x = x \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right)$

$$= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}$$

31. $\frac{\sin x}{x} = \frac{x - (x^3/3!) + (x^5/5!) - \cdots}{x}$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}, x \neq 0$$

$$33. \quad e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \cdots$$

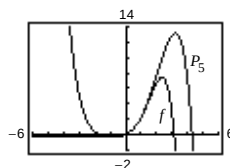
$$e^{-ix} = 1 - ix + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \cdots = 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \cdots$$

$$e^{ix} - e^{-ix} = 2ix - \frac{2ix^3}{3!} + \frac{2ix^5}{5!} - \frac{2ix^7}{7!} + \cdots$$

$$\frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin(x)$$

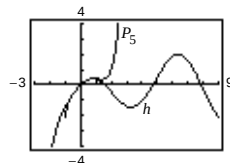
$$35. \quad f(x) = e^x \sin x$$

$$\begin{aligned} &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) \\ &= x + x^2 + \left(\frac{x^3}{2} - \frac{x^3}{6}\right) + \left(\frac{x^4}{6} - \frac{x^4}{6}\right) + \left(\frac{x^5}{120} - \frac{x^5}{12} + \frac{x^5}{24}\right) + \cdots \\ &= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots \end{aligned}$$



$$37. \quad h(x) = \cos x \ln(1+x)$$

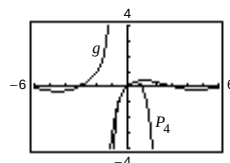
$$\begin{aligned} &= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) \\ &= x - \frac{x^2}{2} + \left(\frac{x^3}{3} - \frac{x^3}{2}\right) + \left(\frac{x^4}{4} - \frac{x^4}{4}\right) + \left(\frac{x^5}{5} - \frac{x^5}{6} + \frac{x^5}{24}\right) + \cdots \\ &= x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{3x^5}{40} + \cdots \end{aligned}$$



$$39. \quad g(x) = \frac{\sin x}{1+x}. \text{ Divide the series for } \sin x \text{ by } (1+x).$$

$$g(x) = x - x^2 + \frac{5x^3}{6} - \frac{5x^4}{6} + \cdots$$

$$\begin{array}{r} x - x^2 + \frac{5x^3}{6} - \frac{5x^4}{6} + \cdots \\ 1+x \overline{) \quad x + 0x^2 - \frac{x^3}{6} + 0x^4 + \frac{x^5}{120} + \cdots} \\ \underline{x + x^2} \\ -x^2 - \frac{x^3}{6} \\ \underline{-x^2 - x^3} \\ \frac{5x^3}{6} + 0x^4 \\ \underline{\frac{5x^3}{6} + \frac{5x^4}{6}} \\ -\frac{5x^4}{6} + \frac{x^5}{120} \\ \underline{-\frac{5x^4}{6} - \frac{5x^5}{6}} \\ \vdots \end{array}$$



$$41. \quad y = x^2 - \frac{x^4}{3!} = x\left(x - \frac{x^3}{3!}\right) \approx x \sin x.$$

Matches (a)

$$43. \quad y = x + x^2 + \frac{x^3}{2!} = x\left(1 + x + \frac{x^2}{2!}\right) \approx xe^x.$$

Matches (c)

$$\begin{aligned}
 45. \int_0^x (e^{-t^2} - 1) dt &= \int_0^x \left[\left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} \right) - 1 \right] dt \\
 &= \int_0^x \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+2}}{(n+1)!} \right] dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+3}}{(2n+3)(n+1)!} \right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)(n+1)!}
 \end{aligned}$$

$$47. \text{ Since } \ln x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

$$\text{we have } \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \approx 0.6931. \quad (10,001 \text{ terms})$$

$$49. \text{ Since } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots,$$

$$\text{we have } e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{2^n}{n!} \approx 7.3891. \quad (12 \text{ terms})$$

51. Since

$$\begin{aligned}
 \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots \\
 1 - \cos x &= \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!} \\
 \frac{1 - \cos x}{x} &= \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+2)!}
 \end{aligned}$$

$$\text{we have } \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = \lim_{x \rightarrow 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+2)!} = 0.$$

$$53. \int_0^1 \frac{\sin x}{x} dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} \right] dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} \right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}$$

Since $1/(7 \cdot 7!) < 0.0001$, we have

$$\int_0^1 \frac{\sin x}{x} dx = 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} - \dots \approx 0.9461.$$

Note: We are using $\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$.

$$55. \int_0^{\pi/2} \sqrt{x} \cos x dx = \int_0^{\pi/2} \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+1)/2}}{(2n)!} \right] dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+3)/2}}{\left(\frac{4n+3}{2}\right)(2n)!} \right]_0^{\pi/2} = \left[\sum_{n=0}^{\infty} \frac{(-1)^n 2x^{(4n+3)/2}}{(4n+3)(2n)!} \right]_0^{\pi/2}$$

Since $(\pi/2)^{19/2}/766,080 < 0.0001$, we have

$$\int_0^1 \sqrt{x} \cos x dx = 2 \left[\frac{(\pi/2)^{3/2}}{3} - \frac{(\pi/2)^{7/2}}{14} + \frac{(\pi/2)^{11/2}}{264} - \frac{(\pi/2)^{15/2}}{10,800} + \frac{(\pi/2)^{19/2}}{766,080} \right] \approx 0.7040.$$

$$57. \int_{0.1}^{0.3} \sqrt{1+x^3} dx = \int_{0.1}^{0.3} \left(1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} - \frac{5x^{12}}{128} + \dots \right) dx = \left[x + \frac{x^4}{8} - \frac{x^7}{56} + \frac{x^{10}}{160} - \frac{5x^{13}}{1664} + \dots \right]_{0.1}^{0.3}$$

Since $\frac{1}{56}(0.3^7 - 0.1^7) < 0.0001$, we have

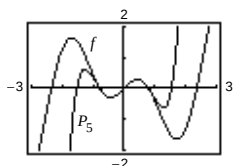
$$\int_{0.1}^{0.3} \sqrt{1+x^3} dx = \left[(0.3 - 0.1) + \frac{1}{8}(0.3^4 - 0.1^4) - \frac{1}{56}(0.3^7 - 0.1^7) \right] \approx 0.2010.$$

59. From Exercise 19, we have

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \int_0^1 e^{-x^2/2} dx &= \frac{1}{\sqrt{2\pi}} \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!} dx = \frac{1}{\sqrt{2\pi}} \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n! (2n+1)} \right]_0^1 = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n! (2n+1)} \\ &\approx \frac{1}{\sqrt{2\pi}} \left[1 - \frac{1}{2 \cdot 1 \cdot 3} + \frac{1}{2^2 \cdot 2! \cdot 5} - \frac{1}{2^3 \cdot 3! \cdot 7} \right] \approx 0.3414.\end{aligned}$$

61. $f(x) = x \cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{(2n)!}$

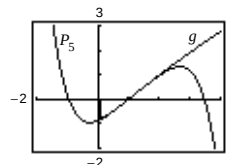
$$P_5(x) = x - 2x^3 + \frac{2x^5}{3}$$



The polynomial is a reasonable approximation on the interval $[-\frac{3}{4}, \frac{3}{4}]$.

63. $f(x) = \sqrt{x} \ln x, c = 1$

$$P_5(x) = (x-1) - \frac{(x-1)^3}{24} + \frac{(x-1)^4}{24} - \frac{71(x-1)^5}{1920}$$



The polynomial is a reasonable approximation on the interval $[\frac{1}{4}, 2]$.

65. See Guidelines, page 636.

67. (a) Replace x with $(-x)$.

(b) Replace x with $3x$.

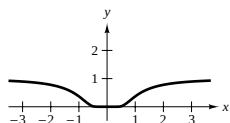
(c) Multiply series by x .

(d) Replace x with $2x$, then replace x with $-2x$, and add the two together.

$$\begin{aligned}69. y &= \left(\tan \theta - \frac{g}{kv_0 \cos \theta} \right) x - \frac{g}{k^2} \ln \left(1 - \frac{kx}{v_0 \cos \theta} \right) \\ &= (\tan \theta)x - \frac{gx}{kv_0 \cos \theta} - \frac{g}{k^2} \left[-\frac{kx}{v_0 \cos \theta} - \frac{1}{2} \left(\frac{kx}{v_0 \cos \theta} \right)^2 - \frac{1}{3} \left(\frac{kx}{v_0 \cos \theta} \right)^3 - \frac{1}{4} \left(\frac{kx}{v_0 \cos \theta} \right)^4 - \dots \right] \\ &= (\tan \theta)x - \frac{gx}{kv_0 \cos \theta} + \frac{gx}{kv_0 \cos \theta} + \frac{gx^2}{2v_0^2 \cos^2 \theta} + \frac{gkx^3}{3v_0^3 \cos^3 \theta} + \frac{gk^2x^4}{4v_0^4 \cos^4 \theta} + \dots \\ &= (\tan \theta)x + \frac{gx^2}{2v_0^2 \cos^2 \theta} + \frac{gkx^3}{3v_0^3 \cos^3 \theta} + \frac{k^2gx^4}{4v_0^4 \cos^4 \theta} + \dots\end{aligned}$$

71. $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(a)



(b) $f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{e^{-1/x^2} - 0}{x}$

Let $y = \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x}$. Then

$$\ln y = \lim_{x \rightarrow 0} \ln \left(\frac{e^{-1/x^2}}{x} \right) = \lim_{x \rightarrow 0^+} \left[-\frac{1}{x^2} - \ln x \right] = \lim_{x \rightarrow 0^+} \left[\frac{-1 - x^2 \ln x}{x^2} \right] = -\infty.$$

Thus, $y = e^{-\infty} = 0$ and we have $f'(0) = 0$.

(c) $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \dots = 0 \neq f(x)$

This series converges to f at $x = 0$ only.

73. By the Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} = 0$ which shows that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x .

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Infinite Series

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CHAPTER 8

Infinite Series

Section 8.1 Sequences

Solutions to Even-Numbered Exercises

$$2. a_n = \frac{2n}{n+3}$$

$$a_1 = \frac{2}{4} = \frac{1}{2}$$

$$a_2 = \frac{4}{5}$$

$$a_3 = \frac{6}{6} = 1$$

$$a_4 = \frac{8}{7}$$

$$a_5 = \frac{10}{8} = \frac{5}{4}$$

$$4. a_n = \left(-\frac{2}{3}\right)^n$$

$$a_1 = -\frac{2}{3}$$

$$a_2 = \frac{4}{9}$$

$$a_3 = -\frac{8}{27}$$

$$a_4 = \frac{16}{81}$$

$$a_5 = -\frac{32}{243}$$

$$6. a_n = \cos \frac{n\pi}{2}$$

$$a_1 = \cos \frac{\pi}{2} = 0$$

$$a_2 = \cos \pi = -1$$

$$a_3 = \cos \frac{3\pi}{2} = 0$$

$$a_4 = \cos 2\pi = 1$$

$$a_5 = \cos \frac{5\pi}{2} = 0$$

$$8. a_n = (-1)^{n+1} \left(\frac{2}{n}\right)$$

$$a_1 = \frac{2}{1} = 2$$

$$a_2 = -\frac{2}{2} = -1$$

$$a_3 = \frac{2}{3}$$

$$a_4 = -\frac{2}{4} = -\frac{1}{2}$$

$$a_5 = \frac{2}{5}$$

$$10. a_n = 10 + \frac{2}{n} + \frac{6}{n^2}$$

$$a_1 = 10 + 2 + 6 = 18$$

$$a_2 = 10 + 1 + \frac{3}{2} = \frac{25}{2}$$

$$a_3 = 10 + \frac{2}{3} + \frac{2}{3} = \frac{34}{3}$$

$$a_4 = 10 + \frac{1}{2} + \frac{3}{8} = \frac{87}{8}$$

$$a_5 = 10 + \frac{2}{5} + \frac{6}{25} = \frac{266}{25}$$

$$12. a_n = \frac{3n!}{(n-1)!} = 3n$$

$$a_1 = 3(1) = 3$$

$$a_2 = 3(2) = 6$$

$$a_3 = 3(3) = 9$$

$$a_4 = 3(4) = 12$$

$$a_5 = 3(5) = 15$$

$$14. a_1 = 4, a_{k+1} = \left(\frac{k+1}{2}\right)a_k$$

$$a_2 = \left(\frac{1+1}{2}\right)a_1 = 4$$

$$a_3 = \left(\frac{2+1}{2}\right)a_2 = 6$$

$$a_4 = \left(\frac{3+1}{2}\right)a_3 = 12$$

$$a_5 = \left(\frac{4+1}{2}\right)a_4 = 30$$

$$16. a_1 = 6, a_{k+1} = \frac{1}{3}a_k^2$$

$$a_2 = \frac{1}{3}a_1^2 = \frac{1}{3}(6^2) = 12$$

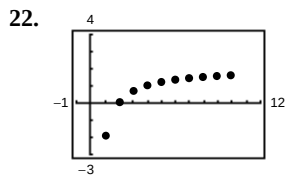
$$a_3 = \frac{1}{3}a_2^2 = \frac{1}{3}(12^2) = 48$$

$$a_4 = \frac{1}{3}a_3^2 = \frac{1}{3}(48^2) = 768$$

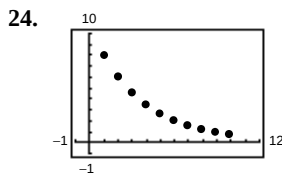
$$a_5 = \frac{1}{3}a_4^2 = \frac{1}{3}(768^2) = 196,608$$

18. Because the sequence tends to 8 as n tends to infinity, it matches (a).

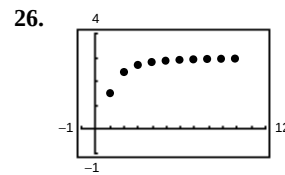
20. This sequence increases for a few terms, then decreases $a_2 = \frac{16}{2} = 8$. Matches (b).



$$a_n = 2 - \frac{4}{n}, n = 1, \dots, 10$$



$$a_n = 8(0.75)^{n-1}, n = 1, 2, \dots, 10$$



$$a_n = \frac{3n^2}{n^2 + 1}, n = 1, \dots, 10$$

28. $a_n = \frac{n+6}{2}$

$$a_5 = \frac{5+6}{2} = \frac{11}{2}$$

$$a_6 = \frac{6+6}{2} = 6$$

30. $a_{n+1} = 2a_n, a_1 = 5$

$$a_5 = 2(40) = 80$$

$$a_6 = 2(80) = 160$$

32. $\frac{25!}{23!} = \frac{23!(24)(25)}{23!}$
 $= (24)(25) = 600$

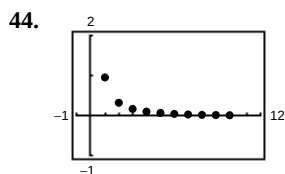
34. $\frac{(n+2)!}{n!} = \frac{n!(n+1)(n+2)}{n!}$
 $= (n+1)(n+2)$

36. $\frac{(2n+2)!}{(2n)!} = \frac{(2n)!(2n+1)(2n+2)}{(2n)!}$
 $= (2n+1)(2n+2)$

38. $\lim_{n \rightarrow \infty} \left(5 - \frac{1}{n^2}\right) = 5 - 0 = 5$

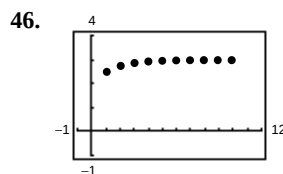
40. $\lim_{n \rightarrow \infty} \frac{5n}{\sqrt{n^2 + 4}} = \lim_{n \rightarrow \infty} \frac{5}{\sqrt{1 + (4/n^2)}}$
 $= \frac{5}{1} = 5$

42. $\lim_{n \rightarrow \infty} \cos\left(\frac{2}{n}\right) = 1$



The graph seems to indicate that the sequence converges to 0. Analytically,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n^{3/2}} = \lim_{x \rightarrow \infty} \frac{1}{x^{3/2}} = 0.$$



The graph seems to indicate that the sequence converges to 3. Analytically,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(3 - \frac{1}{2^n}\right) = 3 - 0 = 3.$$

48. $\lim_{n \rightarrow \infty} [1 + (-1)^n]$

does not exist, (alternates between 0 and 2), diverges.

50. $\lim_{n \rightarrow \infty} \frac{\sqrt[3]{n}}{\sqrt[3]{n} + 1} = 1$, converges

52. $\lim_{n \rightarrow \infty} \frac{1 + (-1)^n}{n^2} = 0$, converges

54. $\lim_{n \rightarrow \infty} \frac{\ln \sqrt{n}}{n} = \lim_{n \rightarrow \infty} \frac{1/2 \ln n}{n}$
 $= \lim_{n \rightarrow \infty} \frac{1}{2n} = 0$, converges

(L'Hôpital's Rule)

56. $\lim_{n \rightarrow \infty} (0.5)^n = 0$, converges

58. $\lim_{n \rightarrow \infty} \frac{(n-2)!}{n!} = \lim_{n \rightarrow \infty} \frac{1}{n(n-1)} = 0$, converges

$$60. \lim_{n \rightarrow \infty} \left(\frac{n^2}{2n+1} - \frac{n^2}{2n-1} \right) = \lim_{n \rightarrow \infty} \frac{-2n^2}{4n^2-1} = -\frac{1}{2}, \text{ converges}$$

$$62. a_n = n \sin \frac{1}{n}$$

$$\text{Let } f(x) = x \sin \frac{1}{x}.$$

$$\lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} = \lim_{x \rightarrow \infty} \frac{(-1/x^2) \cos(1/x)}{-1/x^2} = \lim_{x \rightarrow \infty} \cos \frac{1}{x} = \cos 0 = 1 \text{ (L'Hôpital's Rule)}$$

or,

$$\lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} = \lim_{y \rightarrow 0^+} \frac{\sin(y)}{y} = 1. \text{ Therefore } \lim_{n \rightarrow \infty} n \sin \frac{1}{n} = 1.$$

$$64. \lim_{n \rightarrow \infty} 2^{1/n} = 2^0 = 1, \text{ converges}$$

$$66. \lim_{n \rightarrow \infty} \frac{\cos \pi n}{n^2} = 0, \text{ converges}$$

$$68. a_n = 4n - 1$$

$$70. a_n = \frac{(-1)^{n-1}}{n^2}$$

$$72. a_n = \frac{n+2}{3n-1}$$

$$74. a_n = (-1)^n \frac{3^{n-2}}{2^{n-1}}$$

$$76. a_n = 1 + \frac{2^n - 1}{2^n} \\ = \frac{2^{n+1} - 1}{2^n}$$

$$78. a_n = \frac{1}{n!}$$

$$80. a_n = \frac{x^{n-1}}{(n-1)!}$$

$$82. \text{ Let } f(x) = \frac{3x}{x+2}. \text{ Then } f'(x) = \frac{6}{(x+2)^2}.$$

Thus, f is increasing which implies $\{a_n\}$ is increasing.

$$|a_n| < 3, \text{ bounded}$$

$$84. a_n = ne^{-n/2}$$

$$a_1 = 0.6065$$

$$a_2 = 0.7358$$

$$a_3 = 0.6694$$

Not monotonic; $|a_n| \leq 0.7358$, bounded

$$86. a_n = \left(-\frac{2}{3}\right)^n$$

$$a_1 = -\frac{2}{3}$$

$$a_2 = \frac{4}{9}$$

$$a_3 = -\frac{8}{27}$$

Not monotonic; $|a_n| \leq \frac{2}{3}$, bounded

$$88. a_n = \left(\frac{3}{2}\right)^n < \left(\frac{3}{2}\right)^{n+1} = a_{n+1}$$

Monotonic; $\lim_{n \rightarrow \infty} a_n = \infty$, not bounded

$$90. a_n = \frac{\cos n}{n}$$

$$a_1 = 0.5403$$

$$a_2 = -0.2081$$

$$a_3 = -0.3230$$

$$a_4 = -0.1634$$

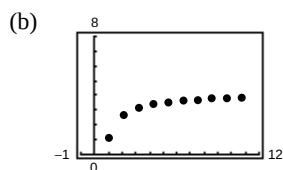
Not monotonic; $|a_n| \leq 1$, bounded

92. (a) $a_n = 4 - \frac{3}{n}$

$$\left| 4 - \frac{3}{n} \right| < 4 \Rightarrow \text{bounded}$$

$$a_n = 4 - \frac{3}{n} < 3 - \frac{4}{n+1} = a_{n+1} \Rightarrow \text{monotonic}$$

Therefore, $\{a_n\}$ converges.



$$\lim_{n \rightarrow \infty} \left(4 - \frac{3}{n} \right) = 4$$

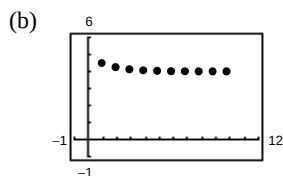
94. (a) $a_n = 4 + \frac{1}{2^n}$

$$\left| 4 + \frac{1}{2^n} \right| \leq 4.5 \Rightarrow \{a_n\} \text{ bounded}$$

$$a_n = 4 + \frac{1}{2^n} > 4 + \frac{1}{2^{n+1}}$$

$$= a_{n+1} \Rightarrow \{a_n\} \text{ monotonic}$$

Therefore, $\{a_n\}$ converges.



$$\lim_{n \rightarrow \infty} \left(4 + \frac{1}{2^n} \right) = 4$$

96. $A_n = 100(101)[(1.01)^n - 1]$

(a) $A_1 = \$101.00$

(b) $A_{60} = \$8248.64$

$A_2 = \$203.01$

(c) $A_{240} = \$99,914.79$

$A_3 = \$306.04$

$A_4 = \$410.10$

$A_5 = \$515.20$

$A_6 = \$621.35$

98. The first sequence because every other point is below the x -axis.

100. Impossible. The sequence converges by Theorem 8.5.

102. Impossible. An unbounded sequence diverges.

104. $P_n = 16,000(1.045)^n$

$P_1 = \$16,720.00$

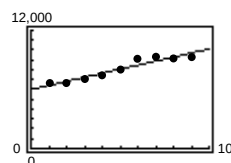
$P_2 = \$17,472.40$

$P_3 \approx \$18,258.66$

$P_4 \approx \$19,080.30$

$P_5 \approx \$19,938.91$

106. (a) $a_n = 410.9212n + 6003.8545$



(b) For 2004, $n = 14$ and $a_n = 11,757$, or \$11,757,000,000.

108. $a_n = \left(1 + \frac{1}{n} \right)^n$

$a_1 = 2.0000$

$a_2 = 2.2500$

$a_3 \approx 2.3704$

$a_4 \approx 2.4414$

$a_5 \approx 2.4883$

$a_6 \approx 2.5216$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e$$

110. Since

$$\lim_{n \rightarrow \infty} s_n = L > 0,$$

there exists for each $\epsilon > 0$, an integer N such that $|s_n - L| < \epsilon$ for every $n > N$. Let $\epsilon = L > 0$ and we have,

$$|s_n - L| < L, -L < s_n - L < L, \text{ or } 0 < s_n < 2L$$

for each $n > N$.

112. If $\{a_n\}$ is bounded, monotonic and nonincreasing, then $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$. Then

$$-a_1 \leq -a_2 \leq -a_3 \leq \cdots \leq -a_n \leq \cdots$$

is a bounded, monotonic, nondecreasing sequence which converges by the first half of the theorem. Since $\{-a_n\}$ converges, then so does $\{a_n\}$.

114. True

116. True

118. $x_0 = 1, x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, n = 1, 2, \dots$

$$x_1 = 1.5 \quad x_6 = 1.414214$$

$$x_2 = 1.41667 \quad x_7 = 1.414214$$

$$x_3 = 1.414216 \quad x_8 = 1.414114$$

$$x_4 = 1.414214 \quad x_9 = 1.414214$$

$$x_5 = 1.414214 \quad x_{10} = 1.414214$$

The limit of the sequence appears to be $\sqrt{2}$. In fact, this sequence is Newton's Method applied to $f(x) = x^2 - 2$.

Section 8.2 Series and Convergence

2. $S_1 = \frac{1}{6} \approx 0.1667$

$$S_2 = \frac{1}{6} + \frac{1}{6} \approx 0.3333$$

$$S_3 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} \approx 0.4833$$

$$S_4 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{2}{15} \approx 0.6167$$

$$S_5 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{2}{15} + \frac{5}{42} \approx 0.7357$$

4. $S_1 = 1$

$$S_2 = 1 + \frac{1}{3} \approx 1.3333$$

$$S_3 = 1 + \frac{1}{3} + \frac{1}{5} \approx 1.5333$$

$$S_4 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} \approx 1.6444$$

$$S_5 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{11} \approx 1.7354$$

6. $S_1 = 1$

$$S_2 = 1 - \frac{1}{2} = 0.5$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{6} \approx 0.6667$$

$$S_4 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} \approx 0.6250$$

$$S_5 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} \approx 0.6333$$

8. $\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n$ Geometric series

$$r = \frac{4}{3} > 1$$

Diverges by Theorem 8.6

10. $\sum_{n=0}^{\infty} 2(-1.03)^n$ Geometric series

$$|r| = 1.03 > 1$$

Diverges by Theorem 8.6

12. $\sum_{n=1}^{\infty} \frac{n}{2n+3}$

$$\lim_{n \rightarrow \infty} \frac{n}{2n+3} = \frac{1}{2} \neq 0$$

Diverges by Theorem 8.9

14. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 \neq 0$$

Diverges by Theorem 8.9

16. $\sum_{n=1}^{\infty} \frac{n!}{2^n}$

$$\lim_{n \rightarrow \infty} \frac{n!}{2^n} = \infty$$

Diverges by Theorem 8.9

$$18. \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \cdots$$

$$S_0 = 1, S_1 = \frac{5}{3}, S_2 \approx 2.11, \dots$$

Matches graph (b).

Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1 - 2/3} = \frac{1}{1/3} = 3$$

$$22. \sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1}{2n} - \frac{1}{2(n+2)} \right) = \left(\frac{1}{2} - \frac{1}{6} \right) + \left(\frac{1}{4} - \frac{1}{8} \right) + \left(\frac{1}{6} - \frac{1}{10} \right) + \left(\frac{1}{8} - \frac{1}{12} \right) + \left(\frac{1}{10} - \frac{1}{14} \right) + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[\frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} \right] = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$24. \sum_{n=0}^{\infty} 2\left(-\frac{1}{2}\right)^n$$

Geometric series with $|r| = \left| -\frac{1}{2} \right| < 1$.

Converges by Theorem 8.6

$$20. \sum_{n=0}^{\infty} \frac{17}{3} \left(-\frac{8}{9}\right)^n = \frac{17}{3} \left[1 - \frac{8}{9} + \frac{64}{81} - \cdots \right]$$

$$S_0 = \frac{17}{3}, S_1 \approx 0.63, S_3 \approx 5.1, \dots$$

Matches (d).

Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \frac{17}{3} \left(-\frac{8}{9}\right)^n = \frac{17/3}{1 - (-8/9)} = \frac{17/3}{17/9} = 3$$

$$26. \sum_{n=0}^{\infty} (-0.6)^n$$

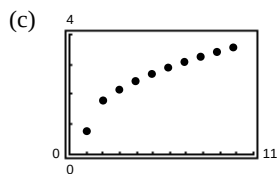
Geometric series with $|r| = |-0.6| < 1$.

Converges by Theorem 8.6

$$\begin{aligned} 28. (a) \sum_{n=1}^{\infty} \frac{4}{n(n+4)} &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+4} \right) \\ &= \left(1 - \frac{1}{5} \right) + \left(\frac{1}{2} - \frac{1}{6} \right) + \left(\frac{1}{3} - \frac{1}{7} \right) + \left(\frac{1}{4} - \frac{1}{8} \right) + \left(\frac{1}{5} - \frac{1}{9} \right) + \left(\frac{1}{6} - \frac{1}{10} \right) + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \approx 2.0833 \end{aligned}$$

(b)

n	5	10	20	50	100
S_n	1.5377	1.7607	1.9051	2.0071	2.0443

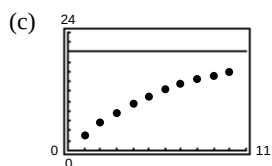


(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.

$$30. (a) \sum_{n=1}^{\infty} 3(0.85)^{n-1} = \frac{3}{1 - 0.85} = 20 \quad (\text{Geometric series})$$

(b)

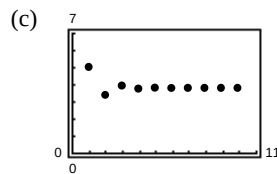
n	5	10	20	50	100
S_n	11.1259	16.0625	19.2248	19.9941	19.99998



32. (a) $\sum_{n=1}^{\infty} 5\left(-\frac{1}{3}\right)^{n-1} = \frac{5}{1 - (-1/3)} = \frac{15}{4} = 3.75$

(b)

n	5	10	20	50	100
S_n	3.7654	3.7499	3.7500	3.7500	3.7500



(d) The terms of the series decrease in magnitude rapidly. Thus, the sequence of partial sums approaches the sum rapidly.

34. $\sum_{n=1}^{\infty} \frac{4}{n(n+2)} = 2 \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right) = 2 \left[\left(1 - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \cdots \right] = 2 \left(1 + \frac{1}{2} \right) = 3$

36. $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)} = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{2n+1} - \frac{1}{2n+3} \right) = \frac{1}{2} \left[\left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{7} \right) + \left(\frac{1}{7} - \frac{1}{9} \right) + \cdots \right] = \frac{1}{2} \left(\frac{1}{3} \right) = \frac{1}{6}$

38. $\sum_{n=0}^{\infty} 6\left(\frac{4}{5}\right)^n = \frac{6}{1 - (4/5)} = 30$ (Geometric)

40. $\sum_{n=0}^{\infty} 2\left(-\frac{2}{3}\right)^n = \frac{2}{1 - (-2/3)} = \frac{6}{5}$

42. $\sum_{n=0}^{\infty} 8\left(\frac{3}{4}\right)^n = \frac{8}{1 - (3/4)} = 32$

44. $\sum_{n=0}^{\infty} 4\left(-\frac{1}{2}\right)^n = \frac{4}{1 - (-1/2)} = \frac{8}{3}$

46. $\sum_{n=1}^{\infty} [(0.7)^n + (0.9)^n] = \sum_{n=0}^{\infty} \left(\frac{7}{10}\right)^n + \sum_{n=0}^{\infty} \left(\frac{9}{10}\right)^n - 2 = \frac{1}{1 - (7/10)} + \frac{1}{1 - (9/10)} - 2 = \frac{10}{3} + 10 - 2 = \frac{34}{3}$

48. $0.81\overline{81} = \sum_{n=0}^{\infty} \frac{81}{100} \left(\frac{1}{100} \right)^n$

Geometric series with $a = \frac{81}{100}$ and $r = \frac{1}{100}$

$$S = \frac{a}{1 - r} = \frac{81/100}{1 - (1/100)} = \frac{81}{99} = \frac{9}{11}$$

50. $0.215\overline{15} = \frac{1}{5} + \sum_{n=0}^{\infty} \frac{3}{200} \left(\frac{1}{100} \right)^n$

Geometric series with $a = \frac{3}{200}$ and $r = \frac{1}{100}$

$$S = \frac{1}{5} + \frac{a}{1 - r} = \frac{1}{5} + \frac{3/200}{99/100} = \frac{71}{330}$$

52. $\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$

$$\lim_{n \rightarrow \infty} \frac{n+1}{2n-1} = \frac{1}{2} \neq 0$$

Diverges by Theorem 8.9

54. $\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3} \right)$

$$= \frac{1}{3} \left[\left(1 - \frac{1}{4} \right) + \left(\frac{1}{2} - \frac{1}{5} \right) + \left(\frac{1}{3} - \frac{1}{6} \right) + \left(\frac{1}{4} - \frac{1}{7} \right) + \left(\frac{1}{5} - \frac{1}{8} \right) + \left(\frac{1}{6} - \frac{1}{9} \right) + \cdots \right]$$

$$= \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{3} \right) = \frac{11}{18}, \text{ converges}$$

56. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

$$\lim_{n \rightarrow \infty} \frac{3^n}{n^3} = \lim_{n \rightarrow \infty} \frac{(\ln 2)3^n}{3n^2} = \lim_{n \rightarrow \infty} \frac{(\ln 2)^2 3^n}{6n} = \lim_{n \rightarrow \infty} \frac{(\ln n)^3 3^n}{6} = \infty$$

(by L'Hôpital's Rule) Diverges by Theorem 8.9

$$58. \sum_{n=0}^{\infty} \frac{1}{4^n}$$

Geometric series with $r = \frac{1}{4}$

Converges by Theorem 8.6

$$60. \sum_{n=1}^{\infty} \frac{2^n}{100}$$

Geometric series with $r = 2$

Diverges by Theorem 8.6

$$62. \sum_{n=1}^{\infty} \left(1 + \frac{k}{n}\right)^n$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{k}{n}\right)^n = e^k \neq 0$$

Diverges by Theorem 8.9

64. $\lim_{n \rightarrow \infty} a_n = 5$ means that the limit of the sequence $\{a_n\}$ is 5.

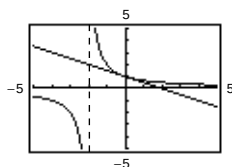
$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots = 5$ means that the limit of the partial sums is 5.

66. If $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

68. (a) $(-x/2)$ is the common ratio.

$$(c) y_1 = \frac{2}{2+x}$$

$$y_2 = 1 - \frac{x}{2}$$



$$(b) 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} = \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \frac{1}{1 - (-x/2)}$$

$$= \frac{2}{2+x}, |x| < 2$$

Geometric series:

$$a = 1, r = -\frac{x}{2}, \left|-\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$$

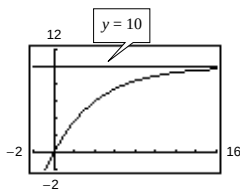
$$70. f(x) = 2 \left[\frac{1 - 0.8^x}{1 - 0.8} \right]$$

Horizontal asymptote: $y = 10$

$$\sum_{n=0}^{\infty} 2 \left(\frac{4}{5}\right)^n$$

$$S = \frac{2}{1 - (4/5)} = 10$$

The horizontal asymptote is the sum of the series. $f(n)$ is the n^{th} partial sum.



$$72. \frac{1}{2^n} < 0.0001$$

$$10,000 < 2^n$$

This inequality is true when $n = 14$.

$$(0.01)^n < 0.0001$$

$$10,000 < 10^n$$

This inequality is true when $n = 5$. This series converges at a faster rate.

$$74. V(t) = 225,000(1 - 0.3)^n = (0.7)^n(225,000)$$

$$V(5) = (0.7)^5(225,000) = \$37,815.75$$

$$76. \sum_{i=0}^{n-1} 100(0.60)^i = \frac{100[1 - 0.6^n]}{1 - 0.6}$$

$$= 250(1 - 0.6^n) \quad \text{million dollars}$$

$$\text{Sum} = 250 \text{ million dollars}$$

78. The ball in Exercise 77 takes the following times for each fall.

$$s_1 = -16t^2 + 16$$

$$s_1 = 0 \text{ if } t = 1$$

$$s_2 = -16t^2 + 16(0.81)$$

$$s_2 = 0 \text{ if } t = 0.9$$

$$s_3 = -16t^2 + 16(0.81)^2$$

$$s_3 = 0 \text{ if } t = (0.9)^2$$

$\vdots$

$\vdots$

$$s_n = -16t^2 + 16(0.81)^{n-1}$$

$$s_n = 0 \text{ if } t = (0.9)^{n-1}$$

Beginning with s_2 , the ball takes the same amount of time to bounce up as it takes to fall. The total elapsed time before the ball comes to rest is

$$t = 1 + 2 \sum_{n=1}^{\infty} (0.9)^n = -1 + 2 \sum_{n=0}^{\infty} (0.9)^n$$

$$= -1 + \frac{2}{1 - 0.9} = 19 \text{ seconds.}$$

$$80. P(n) = \frac{1}{3} \left(\frac{2}{3}\right)^n$$

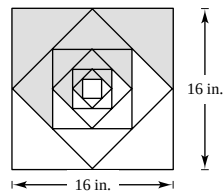
$$P(2) = \frac{1}{3} \left(\frac{2}{3}\right)^2 = \frac{4}{27}$$

$$\sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^n = \frac{1/3}{1 - (2/3)} = 1$$

82. (a) $64 + 32 + 16 + 8 + 4 + 2 = 126 \text{ in.}^2$

(b) $\sum_{n=0}^{\infty} 64\left(\frac{1}{2}\right)^n = \frac{64}{1 - (1/2)} = 128 \text{ in.}^2$

Note: This is one-half of the area of the original square!



84. Surface area $= 4\pi(1)^2 + 9\left(4\pi\left(\frac{1}{3}\right)^2\right) + 9^2 \cdot 4\pi\left(\frac{1}{9}\right)^2 + \cdots = 4[\pi + \pi + \cdots] = \infty$

86.
$$\sum_{n=0}^{12t-1} P\left(1 + \frac{r}{12}\right)^n = \frac{P\left[1 - \left(1 + \frac{r}{12}\right)^{12t}\right]}{1 - \left(1 + \frac{r}{12}\right)}$$

$$= P\left(-\frac{12}{r}\right)\left[1 - \left(1 + \frac{r}{12}\right)^{12t}\right]$$

$$= P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]$$

$$\sum_{n=0}^{12t-1} P(e^{r/12})^n = \frac{P(1 - (e^{r/12})^{12t})}{1 - e^{r/12}} = \frac{P(e^{rt} - 1)}{e^{r/12} - 1}$$

90. $P = 20, r = 0.06, t = 50$

(a) $A = 20\left(\frac{12}{0.06}\right)\left[\left(1 + \frac{0.06}{12}\right)^{12(50)} - 1\right] \approx \$75,743.82$

(b) $A = \frac{20(e^{0.06(50)} - 1)}{e^{0.06/12} - 1} \approx \$76,151.45$

88. $P = 75, r = 0.05, t = 25$

(a) $A = 75\left(\frac{12}{0.05}\right)\left[\left(1 + \frac{0.05}{12}\right)^{12(25)} - 1\right] \approx \$44,663.23$

(b) $A = \frac{75(e^{0.05(25)} - 1)}{e^{0.05/12} - 1} \approx \$44,732.85$

92. $T = 40,000 + 40,000(1.04) + \cdots + 40,000(1.04)^{39}$

$$= \sum_{n=0}^{39} 40,000(1.04)^n$$

$$= 40,000\left(\frac{1 - 1.04^{40}}{1 - 1.04}\right)$$

$$\approx \$3,801,020$$

94. $x = 0.a_1a_2a_3 \cdots \overline{a_ka_1a_2a_3 \cdots a_k}$

$$= 0.a_1a_2a_3 \cdots a_k \left[1 + \frac{1}{10^k} + \left(\frac{1}{10^k}\right)^2 + \left(\frac{1}{10^k}\right)^3 + \cdots\right]$$

$$= 0.a_1a_2a_3 \cdots a_k \sum_{n=0}^{\infty} \left(\frac{1}{10^k}\right)^n$$

$$= 0.a_1a_2a_3 \cdots a_k \left[\frac{1}{1 - (1/10^k)}\right] = \text{a rational number}$$

96. Let $\{S_n\}$ be the sequence of partial sums for the convergent series $\sum_{n=1}^{\infty} a_n = L$. Then

$$\lim_{n \rightarrow \infty} S_n = L \text{ and since } R_n = \sum_{k=n+1}^{\infty} a_k = L - S_n,$$

we have

$$\lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} (L - S_n) = \lim_{n \rightarrow \infty} L - \lim_{n \rightarrow \infty} S_n = L - L = 0.$$

98. If $\sum (a_n + b_n)$ converged, then $\sum (a_n + b_n) - \sum a_n = \sum b_n$ would converge, which is a contradiction.

Thus, $\sum (a_n + b_n)$ diverges.

100. True

102. True; $\lim_{n \rightarrow \infty} \frac{n}{1000(n+1)} = \frac{1}{1000} \neq 0$

$$104. \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \cdots = \sum_{n=0}^{\infty} \frac{1}{r} \left(\frac{1}{r}\right)^n = \frac{1/r}{1 - (1/r)} = \frac{1}{r-1} \quad \left(\text{since } \left|\frac{1}{r}\right| < 1\right)$$

This is a geometric series which converges if $\left|\frac{1}{r}\right| < 1 \Leftrightarrow |r| > 1$.

Section 8.3 The Integral Test and p -Series

$$2. \sum_{n=1}^{\infty} \frac{2}{3n+5}$$

$$\text{Let } f(x) = \frac{2}{3x+5}.$$

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{2}{3x+5} dx = \left[\frac{2}{3} \ln(3x+5) \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

$$4. \sum_{n=1}^{\infty} ne^{-n/2}$$

$$\text{Let } f(x) = xe^{-x/2}.$$

f is positive, continuous, and decreasing for $x \geq 3$.

$$\text{Since } f'(x) = \frac{2-x}{2e^{x/2}} < 0 \text{ for } x \geq 3.$$

$$\int_3^{\infty} xe^{-x/2} dx = \left[-2(x+2)e^{-x/2} \right]_3^{\infty} = 10e^{-3/2}$$

Converges by Theorem 8.10

$$6. \sum_{n=1}^{\infty} \frac{1}{2n+1}$$

$$\text{Let } f(x) = \frac{1}{2x+1}.$$

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{2x+1} dx = \left[\ln \sqrt{2x+1} \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

$$8. \sum_{n=1}^{\infty} \frac{n}{n^2+3}$$

$$\text{Let } f(x) = \frac{x}{x^2+3}.$$

$f(x)$ is positive, continuous, and decreasing for $x \geq 2$ since

$$f'(x) = \frac{3-x^2}{(x^2+3)^2} < 0 \text{ for } x \geq 2.$$

$$\int_1^{\infty} \frac{x}{x^2+3} dx = \left[\ln \sqrt{x^2+3} \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

$$10. \sum_{n=1}^{\infty} n^k e^{-n}$$

$$\text{Let } f(x) = \frac{x^k}{e^x}.$$

f is positive, continuous, and decreasing for $x > k$ since

$$f'(x) = \frac{x^{k-1}(k-x)}{e^x} < 0$$

for $x > k$. We use integration by parts.

$$\begin{aligned} \int_1^{\infty} x^k e^{-x} dx &= \left[-x^k e^{-x} \right]_1^{\infty} + k \int_1^{\infty} x^{k-1} e^{-x} dx \\ &= \frac{1}{e} + \frac{k}{e} + \frac{k(k-1)}{e} + \cdots + \frac{k!}{e} \end{aligned}$$

Converges by Theorem 8.10

$$12. \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$$

$$\text{Let } f(x) = \frac{1}{x^{1/3}}.$$

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^{1/3}} dx = \left[\frac{3}{2} x^{2/3} \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

14. $\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$

 Convergent p -series with $p = \frac{5}{3} > 1$

16. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

 Convergent p -series with $p = 2 > 1$

18. $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

 Divergent p -series with $p = \frac{2}{3} < 1$

20. $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

 Convergent p -series with $p = \pi > 1$

22. $\sum_{n=1}^{\infty} \frac{2}{n} = \frac{2}{1} + \frac{2}{2} + \frac{2}{3} + \cdots$

$S_1 = 2$

$S_2 = 3$

$S_3 \approx 3.67$

Matches (d)

Diverges—harmonic series

24. $\sum_{n=1}^{\infty} \frac{2}{n^2} = 2 + \frac{2}{2^2} + \frac{2}{3^2} + \cdots$

$S_1 = 2$

$S_2 = 2.5$

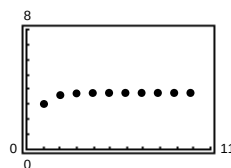
$S_3 \approx 2.722$

Matches (c)

 Converges— p -series with $p = 2 > 1$.

26. (a)

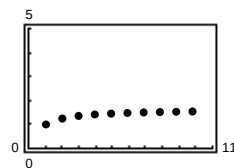
n	5	10	20	50	100
S_n	3.7488	3.75	3.75	3.75	3.75



The partial sums approach the sum 3.75 very rapidly.

(b)

n	5	10	20	50	100
S_n	1.4636	1.5498	1.5962	1.6251	1.635


 The partial sums approach the sum $\frac{\pi^2}{6} \approx 1.6449$ slower than the series in part (a).

28. $\xi(x) = \sum_{n=1}^{\infty} n^{-x} = \sum_{n=1}^{\infty} \frac{1}{n^x}$

 Converges for $x > 1$ by Theorem 8.11.

30. $\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$

 If $p = 1$, then the series diverges by the Integral Test. If $p \neq 1$,

$$\int_2^{\infty} \frac{\ln x}{x^p} dx = \int_2^{\infty} x^{-p} \ln x dx = \left[\frac{x^{-p+1}}{(-p+1)^2} [-1 + (-p+1) \ln x] \right]_2^{\infty}. \text{ (Use Integration by Parts.)}$$

 Converges for $-p + 1 < 0$ or $p > 1$.

 32. A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a p -series, $p > 0$.

 The p -series converges if $p > 1$ and diverges if $0 < p \leq 1$.

 34. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

36. From Exercise 35, we have:

$$0 \leq S - S_N \leq \int_N^\infty f(x) dx$$

$$S_N \leq S \leq S_N + \int_N^\infty f(x) dx$$

$$\sum_{n=1}^N a_n \leq S \leq \sum_{n=1}^N a_n + \int_N^\infty f(x) dx$$

40. $S_{10} = \frac{1}{2(\ln 2)^3} + \frac{1}{3(\ln 3)^3} + \frac{1}{4(\ln 4)^3} + \cdots + \frac{1}{11(\ln 11)^3} \approx 1.9821$

$$R_{10} \leq \int_{10}^\infty \frac{1}{(x+1)[\ln(x+1)]^3} dx = \left[-\frac{1}{2[\ln(x+1)]^2} \right]_{10}^\infty = \frac{1}{2(\ln 11)^3} \approx 0.0870$$

$$1.9821 \leq \sum_{n=1}^\infty \frac{1}{(n+1)[\ln(n+1)]^3} \leq 1.9821 + 0.0870 = 2.0691$$

42. $S_4 = \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \frac{1}{e^4} \approx 0.5713$

$$R_4 \leq \int_4^\infty e^{-x} dx = \left[-e^{-x} \right]_4^\infty \approx 0.0183$$

$$0.5713 \leq \sum_{n=0}^\infty e^{-n} \leq 0.5713 + 0.0183 = 0.5896$$

46. $R_N \leq \int_N^\infty e^{-x/2} dx = \left[-2e^{-x/2} \right]_N^\infty = \frac{2}{e^{N/2}} < 0.001$

$$\frac{2}{e^{N/2}} < 0.001$$

$$e^{N/2} > 2000$$

$$\frac{N}{2} > \ln 2000$$

$$N > 2 \ln 2000 \approx 15.2$$

$$N \geq 16$$

38. $S_4 = 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} \approx 1.0363$

$$R_4 \leq \int_4^\infty \frac{1}{x^5} dx = \left[-\frac{1}{4x^4} \right]_4^\infty \approx 0.0010$$

$$1.0363 \leq \sum_{n=1}^\infty \frac{1}{n^5} \leq 1.0363 + 0.0010 = 1.0373$$

44. $0 \leq R_N \leq \int_N^\infty \frac{1}{x^{3/2}} dx = \left[-\frac{2}{x^{1/2}} \right]_N^\infty = \frac{2}{\sqrt{N}} < 0.001$

$$N^{-1/2} < 0.0005$$

$$\sqrt{N} > 2000$$

$$N \geq 4,000,000$$

48. $R_n \leq \int_N^\infty \frac{2}{x^2 + 5} dx = 2 \left[\frac{1}{\sqrt{5}} \arctan\left(\frac{x}{\sqrt{5}}\right) \right]_N^\infty$

$$= \frac{2}{\sqrt{5}} \left(\frac{\pi}{2} - \arctan\left(\frac{N}{\sqrt{5}}\right) \right) < 0.001$$

$$\frac{\pi}{2} - \arctan\left(\frac{N}{\sqrt{5}}\right) < 0.001118$$

$$1.56968 < \arctan\left(\frac{N}{\sqrt{5}}\right)$$

$$\frac{N}{\sqrt{5}} > \tan 1.56968$$

$$N \geq 2004$$

50. (a) $\int_{10}^\infty \frac{1}{x^p} dx = \left[\frac{x^{-p+1}}{-p+1} \right]_{10}^\infty = \frac{1}{(p-1)10^{p-1}} \cdot p > 1$

(b) $f(x) = \frac{1}{x^p}$

$$R_{10}(p) = \sum_{n=11}^\infty \frac{1}{n^p} \leq \text{Area under the graph of } f \text{ over the interval } [10, \infty)$$

(c) The horizontal asymptote is $y = 0$. As n increases, the error decreases.

52. $\sum_{n=2}^\infty \ln\left(1 - \frac{1}{n^2}\right) = \sum_{n=2}^\infty \ln\left(\frac{n^2-1}{n^2}\right) = \sum_{n=2}^\infty \ln \frac{(n+1)(n-1)}{n^2} = \sum_{n=2}^\infty [\ln(n+1) + \ln(n-1) - 2 \ln n]$

$$= \ln 3 + \ln 1 - 2 \ln 2 + (\ln 4 + \ln 2 - 2 \ln 3) + (\ln 5 + \ln 3 - 2 \ln 4) + (\ln 6 + \ln 4 - 2 \ln 5)$$

$$+ (\ln 7 + \ln 5 - 2 \ln 6) + (\ln 8 + \ln 6 - 2 \ln 7) + (\ln 9 + \ln 7 - 2 \ln 8) + \cdots = -\ln 2$$

$$54. \sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$$

$$\text{Let } f(x) = \frac{1}{x\sqrt{x^2-1}}.$$

f is positive, continuous, and decreasing for $x \geq 2$.

$$\int_2^{\infty} \frac{1}{x\sqrt{x^2-1}} dx = \left[\operatorname{arcsec} x \right]_2^{\infty} = \frac{\pi}{2} - \frac{\pi}{3}$$

Converges by Theorem 8.10

$$56. 3 \sum_{n=1}^{\infty} \frac{1}{n^{0.95}}$$

p -series with $p = 0.95$

Diverges by Theorem 8.11

$$58. \sum_{n=0}^{\infty} (1.075)^n$$

Geometric series with $r = 1.075$

Diverges by Theorem 8.6

$$60. \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n^3} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Since these are both convergent p -series, the difference is convergent.

$$62. \sum_{n=2}^{\infty} \ln(n)$$

$$\lim_{n \rightarrow \infty} \ln(n) = \infty$$

Diverges by Theorem 8.9

$$64. \sum_{n=2}^{\infty} \frac{\ln n}{n^3}$$

$$\text{Let } f(x) = \frac{\ln x}{x^3}.$$

f is positive, continuous, and decreasing for $x \geq 2$ since $f'(x) = \frac{1 - 3 \ln x}{x^4} < 0$ for $x \geq 2$.

$$\int_2^{\infty} \frac{\ln x}{x^3} dx = \left[-\frac{\ln x}{2x^2} \right]_2^{\infty} + \frac{1}{2} \int_2^{\infty} \frac{1}{x^3} dx = \frac{\ln 2}{8} + \left[-\frac{1}{4x^2} \right]_2^{\infty} = \frac{\ln 2}{8} + \frac{1}{16} \quad (\text{Use Integration by Parts.})$$

Converges by Theorem 8.10. See Exercise 14.

Section 8.4 Comparisons of Series

$$2. (a) \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} = 2 + \frac{2}{\sqrt{2}} + \cdots \quad S_1 = 2$$

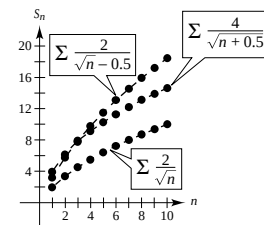
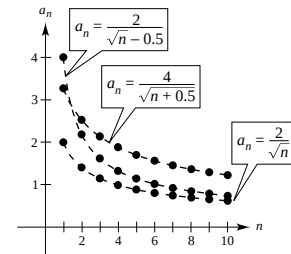
$$\sum_{n=1}^{\infty} \frac{2}{\sqrt{n} - 0.5} = \frac{2}{0.5} + \frac{2}{\sqrt{2} - 0.5} + \cdots \quad S_1 = 4$$

$$\sum_{n=1}^{\infty} \frac{4}{\sqrt{n} + 0.5} = \frac{4}{\sqrt{1.5}} + \frac{4}{\sqrt{2.5}} + \cdots \quad S_1 \approx 3.3$$

(b) The first series is a p -series. It diverges ($p = \frac{1}{2} < 1$).

(c) The magnitude of the terms of the other two series are greater than the corresponding terms of the divergent p -series. Hence, the other two series diverge.

(d) The larger the magnitude of the terms, the larger the magnitude of the terms of the sequence of partial sums.



$$4. \frac{1}{3n^2 + 2} < \frac{1}{3n^2}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{3n^2 + 2}$$

converges by comparison with the convergent p -series

$$\frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$8. \frac{3^n}{4^n + 5} < \left(\frac{3}{4}\right)^n$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{3^n}{4^n + 5}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n.$$

$$12. \frac{1}{4 \sqrt[3]{n} - 1} > \frac{1}{4 \sqrt[4]{n}}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{4 \sqrt[3]{n} - 1}$$

diverges by comparison with the divergent p -series

$$\frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}.$$

$$16. \lim_{n \rightarrow \infty} \frac{2/(3^n - 5)}{1/3^n} = \lim_{n \rightarrow \infty} \frac{2 \cdot 3^n}{3^n - 5} = 2$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2}{3^n - 5}$$

converges by a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n.$$

$$20. \lim_{n \rightarrow \infty} \frac{\frac{5n - 3}{n^2 - 2n + 5}}{1/n} = \lim_{n \rightarrow \infty} \frac{5n^2 - 3n}{n^2 - 2n + 5} = 5$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{5n - 3}{n^2 - 2n + 5}$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$6. \frac{1}{\sqrt{n} - 1} > \frac{1}{\sqrt{n}} \text{ for } n \geq 2.$$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1}$$

diverges by comparison with the divergent p -series

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}.$$

$$10. \frac{1}{\sqrt{n^3 + 1}} < \frac{1}{n^{3/2}}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 1}}$$

converges by comparison with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

$$14. \frac{4^n}{3^n - 1} > \frac{4^n}{3^n}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{4^n}{3^n - 1}$$

diverges by comparison with the divergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n.$$

$$18. \lim_{n \rightarrow \infty} \frac{3/\sqrt{n^2 - 4}}{1/n} = \lim_{n \rightarrow \infty} \frac{3n}{\sqrt{n^2 - 4}} = 3$$

Therefore,

$$\sum_{n=3}^{\infty} \frac{3}{\sqrt{n^2 - 4}}$$

diverges by a limit comparison with the divergent harmonic series

$$\sum_{n=3}^{\infty} \frac{1}{n}.$$

$$22. \lim_{n \rightarrow \infty} \frac{\frac{1}{n(n^2 + 1)}}{1/n^3} = \lim_{n \rightarrow \infty} \frac{n^3}{n^3 + n} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n(n^2 + 1)}$$

converges by a limit comparison with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

$$24. \lim_{n \rightarrow \infty} \frac{n/[(n+1)2^{n-1}]}{1/(2^{n-1})} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$$

converges by a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}.$$

$$28. \lim_{n \rightarrow \infty} \frac{\tan(1/n)}{1/n} = \lim_{n \rightarrow \infty} \frac{(-1/n^2) \sec^2(1/n)}{-1/n^2} = \lim_{n \rightarrow \infty} \sec^2\left(\frac{1}{n}\right) = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$30. \sum_{n=0}^{\infty} 5\left(-\frac{1}{5}\right)^n$$

Converges

Geometric series with $r = -\frac{1}{5}$

$$32. \sum_{n=4}^{\infty} \frac{1}{3n^2 - 2n - 15}$$

Converges

Limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$34. \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots = \frac{1}{2}$$

Converges; telescoping series

$$36. \sum_{n=1}^{\infty} \frac{3}{n(n+3)}$$

Converges; telescoping series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3}\right)$$

38. If $j < k - 1$, then $k - j > 1$. The p -series with $p = k - j$ converges and since

$$\lim_{n \rightarrow \infty} \frac{P(n)/Q(n)}{1/n^{k-j}} = L > 0,$$

the series $\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$ converges by the limit comparison test. Similarly, if $j \geq k - 1$, then $k - j \leq 1$ which implies that

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$$

diverges by the limit comparison test.

$$40. \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \cdots = \sum_{n=2}^{\infty} \frac{1}{n^2 - 1},$$

which converges since the degree of the numerator is two less than the degree of the denominator.

$$42. \sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$$

diverges since the degree of the numerator is only one less than the degree of the denominator.

44. $\lim_{n \rightarrow \infty} \frac{n}{\ln n} = \lim_{n \rightarrow \infty} \frac{1}{1/n} = \lim_{n \rightarrow \infty} n = \infty \neq 0$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} \text{ diverges.}$$

46. See Theorem 8.13, page 585.

One example is

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}} \text{ diverges because } \lim_{n \rightarrow \infty} \frac{1/\sqrt{n-1}}{1/\sqrt{n}} = 1$$

and

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges (p-series).}$$

50. $\frac{1}{200} + \frac{1}{210} + \frac{1}{220} + \cdots = \sum_{n=0}^{\infty} \frac{1}{200 + 10n},$
diverges

48. This is not correct. The beginning terms do not affect the convergence or divergence of a series.

In fact,

$$\frac{1}{1000} + \frac{1}{1001} + \cdots = \sum_{n=1000}^{\infty} \frac{1}{n} \text{ diverges (harmonic)}$$

and

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges (p-series).}$$

52. $\frac{1}{201} + \frac{1}{208} + \frac{1}{227} + \frac{1}{264} + \cdots = \sum_{n=1}^{\infty} \frac{1}{200 + n^3},$
converges

54. (a) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 4n + 1}$

converges since the degree of the numerator is two less than the degree of the denominator. (See Exercise 38.)

(b)

n	5	10	20	50	100
S_n	1.1839	1.02087	1.2212	1.2287	1.2312

(c) $\sum_{n=3}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - S_2 \approx 0.1226$

(d) $\sum_{n=10}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - S_9 \approx 0.0277$

56. True

58. False. Let $a_n = 1/n$, $b_n = 1/n$, $c_n = 1/n^2$. Then, $a_n \leq b_n + c_n$, but $\sum_{n=1}^{\infty} c_n$ converges.

60. Since $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n a_n = \sum_{n=1}^{\infty} a_n^2$ converges by Exercise 59.

62. $\sum \frac{1}{n^2}$ converge, and hence so does $\sum \left(\frac{1}{n^2}\right)^2 = \sum \frac{1}{n^4}$.

64. (a) $\sum a_n = \sum \frac{1}{n^3}$ and $\sum b_n = \sum \frac{1}{n^2}$. Since

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1/n^3}{1/n^2} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \quad \text{and} \quad \sum \frac{1}{n^2}$$

converges, so does $\sum \frac{1}{n^3}$.

(b) $\sum a_n = \sum \frac{1}{\sqrt{n}}$ and $\sum b_n = \sum \frac{1}{n}$. Since

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1/\sqrt{n}}{1/n} = \lim_{n \rightarrow \infty} \sqrt{n} = \infty \quad \text{and} \quad \sum \frac{1}{n}$$

diverges, so does $\sum \frac{1}{\sqrt{n}}$.

Section 8.5 Alternating Series

$$2. \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 6}{n^2} = \frac{6}{1} - \frac{6}{4} + \frac{6}{9} - \dots$$

$$S_1 = 6, S_2 = 4.5$$

Matches (d)

$$4. \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 10}{n2^n} = \frac{10}{2} - \frac{10}{8} + \dots$$

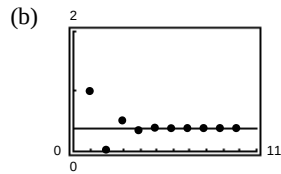
$$S_1 = 5, S_2 = 3.75$$

Matches (a)

$$6. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} = \frac{1}{e} \approx 0.3679$$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0	0.5	0.3333	0.375	0.3667	0.3681	0.3679	0.3679	0.3679



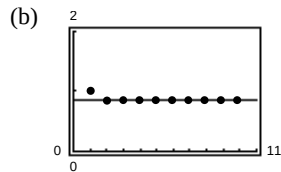
(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next series.

$$8. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} = \sin(1) \approx 0.8415$$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0.8333	0.8417	0.8415	0.8415	0.8415	0.8415	0.8415	0.8415	0.8415



(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next series.

$$10. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2n-1}$$

$$\lim_{n \rightarrow \infty} \frac{n}{2n-1} = \frac{1}{2}$$

Diverges by the n th Term Test.

$$12. \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$$

$$a_{n+1} = \frac{1}{\ln(n+2)} < \frac{1}{\ln(n+1)} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{\ln(n+1)} = 0$$

Converges by Theorem 8.14

$$14. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2 + 1}$$

$$a_{n+1} = \frac{n+1}{(n+1)^2 + 1} < \frac{n}{n^2 + 1} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{n}{n^2 + 1} = 0$$

Converges by Theorem 8.14

$$16. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2 + 5}$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 5} = 1$$

Diverges by n th Term Test

$$18. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln(n+1)}{n+1}$$

$$a_{n+1} = \frac{\ln[(n+1)+1]}{(n+1)+1} < \frac{\ln(n+1)}{n+1} \text{ for } n \geq 2$$

$$\lim_{n \rightarrow \infty} \frac{\ln(n+1)}{n+1} = \lim_{n \rightarrow \infty} \frac{1/(n+1)}{1} = 0$$

Converges by Theorem 8.14

$$22. \sum_{n=1}^{\infty} \frac{1}{n} \cos n\pi = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Converges; (see Exercise 9)

$$26. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sqrt{n}}{\sqrt[3]{n}}$$

$$\lim_{n \rightarrow \infty} \frac{n^{1/2}}{n^{1/3}} = \lim_{n \rightarrow \infty} n^{1/6} = \infty$$

Diverges by the n th Term Test

$$30. S_6 = \sum_{n=1}^6 \frac{4(-1)^{n+1}}{\ln(n+1)} \approx 2.7067$$

$$|R_6| = |S - S_6| \leq a_7 = \frac{4}{\ln 8} \approx 1.9236; 0.7831 \leq S \leq 4.6303$$

$$32. S_6 = \sum_{n=1}^6 \frac{(-1)^{n+1} n}{2^n} = 0.1875$$

$$|R_6| = |S - S_6| \leq a_7 = \frac{7}{2^7} \approx 0.05469; 0.1328 \leq S \leq 0.2422$$

$$34. \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{2^{N+1}(N+1)!} < 0.001.$$

This inequality is valid when $N = 4$.

(b) We may approximate the series by

$$\sum_{n=0}^4 \frac{(-1)^n}{2^n n!} = 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{348} \approx 0.607.$$

(5 terms. Note that the sum begins with $n = 0$.)

$$20. \sum_{n=1}^{\infty} \frac{1}{n} \sin \left[\frac{(2n-1)\pi}{2} \right] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Converges; (see Exercise 9)

$$24. \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

$$a_{n+1} = \frac{1}{(2n+3)!} < \frac{1}{(2n+1)!} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{(2n+1)!} = 0$$

Converges by Theorem 8.14

$$28. \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n + e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2e^n)}{e^{2n} + 1}$$

$$\text{Let } f(x) = \frac{2e^x}{e^{2x} + 1}. \text{ Then}$$

$$f'(x) = \frac{2e^{2x}(1 - e^{2x})}{(e^{2x} + 1)^2} < 0 \text{ for } x > 0.$$

Thus, $f(x)$ is decreasing for $x > 0$ which implies

$$a_{n+1} < a_n.$$

$$\lim_{n \rightarrow \infty} \frac{2e^n}{e^{2n} + 1} = \lim_{n \rightarrow \infty} \frac{2e^n}{2e^{2n}} = \lim_{n \rightarrow \infty} \frac{1}{e^n} = 0$$

The series converges by Theorem 8.14.

$$36. \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{(2N+2)!} < 0.001.$$

This inequality is valid when $N = 3$.

(b) We may approximate the series by

$$\sum_{n=0}^3 \frac{(-1)^n}{(2n)!} = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} \approx 0.540.$$

(4 terms. Note that the sum begins with $n = 0$.)

$$38. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^n n}$$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{4^{N+1}(N+1)} < 0.001.$$

This inequality is valid when $N = 3$.

$$40. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

By Theorem 8.15, $|R_N| \leq a_{N+1} = \frac{1}{(N+1)^4} < 0.001$.

This inequality is valid when $N = 5$.

$$44. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$$

$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a convergent p -series.

Therefore, the given series converges absolutely.

$$48. \sum_{n=0}^{\infty} \frac{(-1)^n}{e^{n^2}}$$

$$\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$$

converges by a comparison to the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n.$$

Therefore, the given series converges absolutely.

$$52. \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$$

The given series converges by the Alternating Series Test, but

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}}$$

diverges by a limit comparison to the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

Therefore, the given series converges conditionally.

(b) We may approximate the series by

$$\sum_{n=1}^3 \frac{(-1)^{n+1}}{4^n n} = \frac{1}{4} - \frac{1}{32} + \frac{1}{192} \approx 0.224.$$

(3 terms)

$$42. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

The given series converges by the Alternating Series Test, but does not converge absolutely since the series

$$\sum_{n=1}^{\infty} \frac{1}{n+1}$$

diverges by the Integral Test. Therefore, the series converge conditionally.

$$46. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n+3)}{n+10}$$

$\lim_{n \rightarrow \infty} \frac{2n+3}{n+10} = 2$ Therefore, the series diverges by the n th Term Test.

$$50. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1.5}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ is a convergent p -series.

Therefore, the given series converge absolutely.

$$54. \sum_{n=1}^{\infty} (-1)^{n+1} \arctan n$$

$\lim_{n \rightarrow \infty} \arctan n = \frac{\pi}{2} \neq 0$ Therefore, the series diverges by the n th Term Test.

$$56. \sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi/2]}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

The given series converges by the Alternating Series Test, but

$$\sum_{n=1}^{\infty} \left| \frac{\sin[(2n-1)\pi/2]}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

is a divergent p -series. Therefore, the series converges conditionally.

$$60. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \quad (\text{Alternating Harmonic Series})$$

$$62. \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$$

If $p = 0$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n^p} = 1$$

and the series diverges. If $p > 0$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0 \text{ and } \frac{1}{(n+1)^p} < \frac{1}{n^p}.$$

Therefore, the series converge by the Alternating Series Test.

$$66. (a) \sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges absolutely (by comparison) for

$$-1 < x < 1,$$

since

$$\left| \frac{x^n}{n} \right| < |x^n| \text{ and } \sum x^n$$

is a convergent geometric series for $-1 < x < 1$.

$$58. |S - S_n| = |R_n| \leq a_{n+1} \quad (\text{Theorem 8.15})$$

$$64. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \text{ converges, but } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

(b) When $x = -1$, we have the convergent alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

When $x = 1$, we have the divergent harmonic series

$$\frac{1}{n}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges conditionally for $x = -1$.

68. True, equivalent to Theorem 8.16

$$70. \sum_{n=1}^{\infty} \frac{3}{n^2 + 5}$$

converges by limit comparison to convergent p -series

$$\sum \frac{1}{n^2}.$$

72. Converges by limit comparison to convergent geometric series $\sum \frac{1}{2^n}$.

74. Diverges by n th Term Test. $\lim_{n \rightarrow \infty} a_n = \frac{3}{2}$

76. Converges (conditionally) by Alternating Series Test.

78. Diverges by comparison to Divergent Harmonic Series:

$$\frac{\ln n}{n} > \frac{1}{n} \text{ for } n \geq 3.$$

Section 8.6 The Ratio and Root Tests

$$2. \frac{(2k-2)!}{(2k)!} = \frac{(2k-2)!}{(2k)(2k-1)(2k-2)!} = \frac{1}{(2k)(2k-1)}$$

4. Use the Principle of Mathematical Induction. When $k = 3$, the formula is valid since $\frac{1}{1} = \frac{2^3 3! (3)(5)}{6!} = 1$. Assume that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)} = \frac{2^n n! (2n-3)(2n-1)}{(2n)!}$$

and show that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)(2n-3)} = \frac{2^{n+1} (n+1)! (2n-1)(2n+1)}{(2n+2)!}.$$

To do this, note that:

$$\begin{aligned} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)(2n-3)} &= \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)} \cdot \frac{1}{(2n-3)} \\ &= \frac{2^n n! (2n-3)(2n-1)}{(2n)!} \cdot \frac{1}{(2n-3)} \\ &= \frac{2^n n! (2n-1)}{(2n)!} \cdot \frac{(2n+1)(2n+2)}{(2n+1)(2n+2)} \\ &= \frac{2^n (2)(n+1)n! (2n-1)(2n+1)}{(2n)! (2n+1)(2n+2)} \\ &= \frac{2^{n+1} (n+1)! (2n-1)(2n+1)}{(2n+2)!} \end{aligned}$$

The formula is valid for all $n \geq 3$.

$$6. \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \left(\frac{1}{n!}\right) = \frac{3}{4} + \frac{9}{16} \left(\frac{1}{2}\right) + \cdots$$

$$S_1 = \frac{3}{4}, S_2 \approx 1.03$$

Matches (c)

$$8. \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 4}{(2n)!} = \frac{4}{2} - \frac{4}{24} + \cdots$$

$$S_1 = 2$$

Matches (b)

$$10. \sum_{n=0}^{\infty} 4e^{-n} = 4 + \frac{4}{e} + \cdots$$

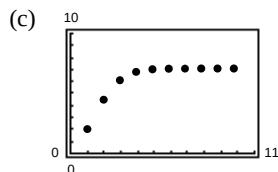
$$S_1 = 4$$

Matches (e)

$$12. (a) \text{ Ratio Test: } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^2 + 1}{(n+1)!}}{\frac{n^2 + 1}{n!}} = \lim_{n \rightarrow \infty} \left(\frac{n^2 + 2n + 2}{n^2 + 1} \right) \left(\frac{1}{n+1} \right) = 0 < 1. \text{ Converges}$$

(b)

n	5	10	15	20	25
S_n	7.0917	7.1548	7.1548	7.1548	7.1548



(d) The sum is approximately 7.15485

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of the partial sums approaches the sum of the series.

$$14. \sum_{n=0}^{\infty} \frac{3^n}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{3}{n+1} = 0 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

$$18. \sum_{n=1}^{\infty} \frac{n^3}{2^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3/2^{n+1}}{n^3/2^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3}{2n^3} \right| = \frac{1}{2} \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

$$16. \sum_{n=1}^{\infty} n \left(\frac{3}{2} \right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)3^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n3^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{3(n+1)}{2n} = \frac{3}{2} \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

$$20. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$$

$$a_{n+1} = \frac{n+3}{(n+1)(n+2)} \leq \frac{n+2}{n(n+1)} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{n+2}{n(n+1)} = 0$$

Therefore, by Theorem 8.14, the series converges.

Note: The Ratio Test is inconclusive since $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

The series converges conditionally.

$$22. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3/2)^n}{n^2}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(3/2)^{n+1}}{n^2 + 2n + 1} \cdot \frac{n^2}{(3/2)^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{3n^2}{2(n^2 + 2n + 1)} = \frac{3}{2} > 1 \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

$$24. \sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(2n+2)!}{(n+1)^5} \cdot \frac{n^5}{(2n)!} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(2n+2)(2n+1)n^5}{(n+1)^5} = \infty \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

$$26. \sum_{n=1}^{\infty} \frac{n^n}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)(n+1)^n n!}{(n+1)n!n^n} \\ &= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n = e > 1 \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

$$28. \sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{[(n+1)!]^2}{(3n+3)!} \cdot \frac{(3n)!}{(n!)^2} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} = 0 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

$$30. \sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{4n+4}}{(2n+3)!} \cdot \frac{(2n+1)!}{2^{4n}} \right| = \lim_{n \rightarrow \infty} \frac{2^4}{(2n+3)(2n+2)} = 0$$

Therefore, by the Ratio Test, the series converges.

$$32. \sum_{n=1}^{\infty} \frac{(-1)^n 2 \cdot 4 \cdot 6 \cdots 2n}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2 \cdot 4 \cdots 2n(2n+2)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{2 \cdot 4 \cdots 2n} \right| = \lim_{n \rightarrow \infty} \frac{2n+2}{3n+2} = \frac{2}{3}$$

Therefore, by the Ratio Test, the series converges.

Note: The first few terms of this series are $-\frac{2}{2} + \frac{2 \cdot 4}{2 \cdot 5} - \frac{2 \cdot 4 \cdot 6}{2 \cdot 5 \cdot 8} + \cdots$

$$34. (a) \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^4} \cdot \frac{n^4}{1} \right| = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^4 = 1$$

$$(b) \sum_{n=1}^{\infty} \frac{1}{n^p}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^p} \cdot \frac{n^p}{1} \right| = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^p = 1$$

$$36. \sum_{n=1}^{\infty} \left(\frac{2n}{n+1} \right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{2n}{n+1} \right)^n} \\ &= \lim_{n \rightarrow \infty} \frac{2n}{n+1} = 2 \end{aligned}$$

Therefore, by the Root Test, the series diverges.

$$38. \sum_{n=1}^{\infty} \left(\frac{-3n}{2n+1} \right)^{3n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left(\frac{-3n}{2n+1} \right)^{3n} \right|} \\ &= \lim_{n \rightarrow \infty} \left(\frac{3n}{2n+1} \right)^3 = \left(\frac{3}{2} \right)^3 = \frac{27}{8} \end{aligned}$$

Therefore, by the Root Test, the series diverges.

$$40. \sum_{n=0}^{\infty} e^{-n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{e^n}} = \frac{1}{e}$$

Therefore, by the Root Test, the series converges.

$$42. \sum_{n=0}^{\infty} \frac{n+1}{3^n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n+1}{3^n}} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n+1}}{3}$$

$$\text{Let } y = \lim_{n \rightarrow \infty} \sqrt[n]{x+1}$$

$$\ln y = \lim_{n \rightarrow \infty} (\ln \sqrt[n]{x+1})$$

$$= \lim_{n \rightarrow \infty} \frac{1}{x} \ln(x+1)$$

$$= \lim_{n \rightarrow \infty} \frac{\ln(x+1)}{x} = \frac{1}{x+1} = 0.$$

Since $\ln y = 0$, $y = e^0 = 1$, so

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n]{n+1}}{3} = \frac{1}{3}.$$

Therefore, by the Root Test, the series converges.

$$44. \sum_{n=1}^{\infty} \frac{5}{n} = 5 \sum_{n=1}^{\infty} \frac{1}{n}$$

This is the divergent harmonic series.

$$46. \sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^n$$

Since $\pi/4 < 1$, this is convergent geometric series.

$$48. \sum_{n=1}^{\infty} \frac{n}{2n^2 + 1}$$

$$\lim_{n \rightarrow \infty} \frac{n/(2n^2 + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{2n^2 + 1} = \frac{1}{2} > 0$$

This series diverges by limit comparison to the divergent harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$50. \sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$$

$$\lim_{n \rightarrow \infty} \frac{10/3n^{3/2}}{1/n^{3/2}} = \frac{10}{3}$$

Therefore, the series converges by a limit comparison test with the p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

$$52. \sum_{n=1}^{\infty} \frac{2^n}{4n^2 - 1}$$

$$\lim_{n \rightarrow \infty} \frac{2^n}{4n^2 - 1} = \lim_{n \rightarrow \infty} \frac{(\ln 2)2^n}{8n} = \lim_{n \rightarrow \infty} \frac{(\ln 2)^2 2^n}{8} = \infty$$

Therefore, the series diverges by the n th Term Test for Divergence.

$$54. \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)}$$

$$a_{n+1} = \frac{1}{(n+1) \ln(n+1)} \leq \frac{1}{n \ln(n)} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n \ln(n)} = 0$$

Therefore, by the Alternating Series Test, the series converges.

$$56. \sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$$

$$\frac{\ln(n)}{n^2} \leq \frac{1}{n^{3/2}}$$

Therefore, the series converges by comparison with the p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

$$58. \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n 2^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n 2^n}{3^n} \right| = \lim_{n \rightarrow \infty} \frac{3n}{2(n+1)} = \frac{3}{2}$$

Therefore, by the Ratio Test, the series diverges.

$$60. \sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{18^n (2n-1)n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)}{18^{n+1}(2n+1)(2n-1)n!} \cdot \frac{18^n (2n-1)n!}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \right| = \lim_{n \rightarrow \infty} \frac{(2n+3)(2n-1)}{18(2n+1)(2n-1)} = \frac{2}{18} = \frac{1}{9}$$

Therefore, by the Ratio Test, the series converge.

62. (b) and (c)

$$\begin{aligned} \sum_{n=0}^{\infty} (n+1) \left(\frac{3}{4}\right)^n &= \sum_{n=1}^{\infty} n \left(\frac{3}{4}\right)^{n-1} \\ &= 1 + 2\left(\frac{3}{4}\right) + 3\left(\frac{3}{4}\right)^2 + 4\left(\frac{3}{4}\right)^3 + \cdots \end{aligned}$$

64. (a) and (b) are the same.

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)2^{n-1}} &= \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \cdots \\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n 2^n} &= \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \cdots \end{aligned}$$

66. Replace n with $n + 2$.

$$\sum_{n=2}^{\infty} \frac{2^n}{(n-2)!} = \sum_{n=0}^{\infty} \frac{2^{n+2}}{n!}$$

$$\begin{aligned} 68. \sum_{k=0}^{\infty} \frac{(-3)^k}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k+1)} &= \sum_{k=0}^{\infty} \frac{(-3)^k 2^k k!}{(2k)!(2k+1)} \\ &= \sum_{k=0}^{\infty} \frac{(-6)^k k!}{(2k+1)!} \\ &\approx 0.40967 \end{aligned}$$

(See Exercise 3 and use 10 terms, $k = 9$.)

70. See Theorem 8.18.

$$72. \text{ One example is } \sum_{n=1}^{\infty} \left(-100 + \frac{1}{n} \right).$$

74. Assume that

$$\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = L > 1 \text{ or that } \lim_{n \rightarrow \infty} |a_{n+1}/a_n| = \infty.$$

Then there exists $N > 0$ such that $|a_{n+1}/a_n| > 1$ for all $n > N$. Therefore,

$$|a_{n+1}| > |a_n|, \quad n > N \Rightarrow \lim_{n \rightarrow \infty} a_n \neq 0 \Rightarrow \sum a_n \text{ diverges}$$

76. The differentiation test states that if

$$\sum_{n=1}^{\infty} U_n$$

is an infinite series with real terms and $f(x)$ is a real function such that $f(1/n) = U_n$ for all positive integers n and $d^2 f/dx^2$ exists at $x = 0$, then

$$\sum_{n=1}^{\infty} U_n$$

converges absolutely if $f(0) = f'(0) = 0$ and diverges otherwise. Below are some examples.

Convergent Series

$$\sum \frac{1}{n^3}, f(x) = x^3$$

$$\sum \left(1 - \cos \frac{1}{n} \right), f(x) = 1 - \cos x$$

Divergent Series

$$\sum \frac{1}{n}, f(x) = x$$

$$\sum \sin \frac{1}{n}, f(x) = \sin x$$

Section 8.7 Taylor Polynomials and Approximations

$$2. y = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$$

y -axis symmetry

Three relative extrema

Matches (c)

$$4. y = e^{-1/2} \left[\frac{1}{3}(x-1)^3 - (x-1) + 1 \right]$$

Cubic

Matches (b)

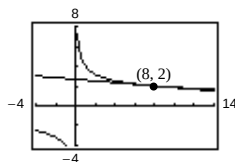
$$6. f(x) = \frac{4}{\sqrt[3]{x}} = 4x^{-1/3} \quad f(8) = 2$$

$$f'(x) = -\frac{4}{3}x^{-4/3} \quad f'(8) = -\frac{1}{12}$$

$$P_1(x) = f(8) + f'(8)(x-8)$$

$$= 2 + \left(-\frac{1}{12} \right)(x-8)$$

$$P_1(x) = -\frac{1}{12}x + \frac{8}{3}$$



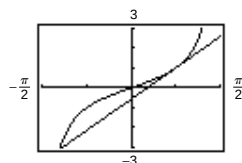
8. $f(x) = \tan x \quad f\left(\frac{\pi}{4}\right) = 1$

$f'(x) = \sec^2 x \quad f'\left(\frac{\pi}{4}\right) = 2$

$P_1 = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$

$= 1 + 2\left(x - \frac{\pi}{4}\right)$

$P_1(x) = 2x + 1 - \frac{\pi}{2}$



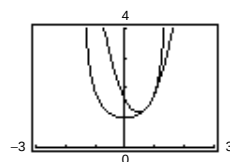
10. $f(x) = \sec x \quad f\left(\frac{\pi}{4}\right) = \sqrt{2}$

$f'(x) = \sec x \tan x \quad f'\left(\frac{\pi}{4}\right) = \sqrt{2}$

$f''(x) = \sec^3 x + \sec x \tan^2 x \quad f''\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

$P_2(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2}\left(x - \frac{\pi}{4}\right)^2$

$P_2(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3}{2}\sqrt{2}\left(x - \frac{\pi}{4}\right)^2$



x	-2.15	0.585	0.685	$\pi/4$	0.885	0.985	1.785
$f(x)$	-1.8270	1.1995	1.2913	1.4142	1.5791	1.8088	-4.7043
$P_2(x)$	15.5414	1.2160	1.2936	1.4142	1.5761	1.7810	4.9475

12. $f(x) = x^2 e^x, f(0) = 0$

(a) $f'(x) = (x^2 + 2x)e^x \quad f'(0) = 0$

$f''(x) = (x^2 + 4x + 2)e^x \quad f''(0) = 2$

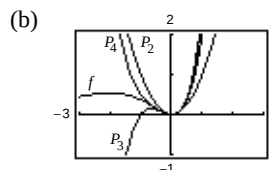
$f'''(x) = (x^2 + 6x + 6)e^x \quad f'''(0) = 6$

$f^{(4)}(x) = (x^2 + 8x + 12)e^x \quad f^{(4)}(0) = 12$

$P_2(x) = \frac{2x^2}{2!} = x^2$

$P_3(x) = x^2 + \frac{6x^3}{3!} = x^2 + x^3$

$P_4(x) = x^2 + x^3 + \frac{12x^4}{4!} = x^2 + x^3 + \frac{x^4}{2}$



(c) $f''(0) = 2 = P_2''(0)$

$f'''(0) = 6 = P_3'''(0)$

$f^{(4)}(0) = 12 = P_4^{(4)}(0)$

(d) $f^{(n)}(0) = P_n^{(n)}(0)$

14. $f(x) = e^{-x} \quad f(0) = 1$

$f'(x) = -e^{-x} \quad f'(0) = -1$

$f''(x) = e^{-x} \quad f''(0) = 1$

$f'''(x) = -e^{-x} \quad f'''(0) = -1$

$f^{(4)}(x) = e^{-x} \quad f^{(4)}(0) = 1$

$f^{(5)}(x) = -e^{-x} \quad f^{(5)}(0) = -1$

$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$

$$16. \quad f(x) = e^{3x} \quad f(0) = 1$$

$$f'(x) = 3e^{3x} \quad f'(0) = 3$$

$$f''(x) = 9e^{3x} \quad f''(0) = 9$$

$$f'''(x) = 27e^{3x} \quad f'''(0) = 27$$

$$f^{(4)}(x) = 81e^{3x} \quad f^{(4)}(0) = 81$$

$$P_4(x) = 1 + 3x + \frac{9}{2!}x^2 + \frac{27}{3!}x^3 + \frac{81}{4!}x^4 = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$$

$$18. \quad f(x) = \sin \pi x \quad f(0) = 0$$

$$f'(x) = \pi \cos \pi x \quad f'(0) = \pi$$

$$f''(x) = -\pi^2 \sin \pi x \quad f''(0) = 0$$

$$f'''(x) = -\pi^3 \cos \pi x \quad f'''(0) = -\pi^3$$

$$P_3(x) = 0 + \pi x + \frac{0}{2!}x^2 + \frac{-\pi^3}{3!}x^3 = \pi x - \frac{\pi^3}{6}x^3$$

$$20. \quad f(x) = x^2 e^{-x} \quad f(0) = 0$$

$$f'(x) = 2xe^{-x} - x^2 e^{-x} \quad f'(0) = 0$$

$$f''(x) = 2e^{-x} - 4xe^{-x} + x^2 e^{-x} \quad f''(0) = 2$$

$$f'''(x) = -6e^{-x} + 6xe^{-x} - x^2 e^{-x} \quad f'''(0) = -6$$

$$f^{(4)}(x) = 12e^{-x} - 8xe^{-x} + x^2 e^{-x} \quad f^{(4)}(0) = 12$$

$$\begin{aligned} P_4(x) &= 0 + 0x + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{12}{4!}x^4 \\ &= x^2 - x^3 + \frac{1}{2}x^4 \end{aligned}$$

$$22. \quad f(x) = \frac{x}{x+1} = \frac{x+1-1}{x+1} = 1 - (x+1)^{-1} \quad f(0) = 0$$

$$f'(x) = (x+1)^{-2} \quad f'(0) = 1$$

$$f''(x) = -2(x+1)^{-3} \quad f''(0) = -2$$

$$f'''(x) = 6(x+1)^{-4} \quad f'''(0) = 6$$

$$f^{(4)}(x) = -24(x+1)^{-5} \quad f^{(4)}(0) = -24$$

$$P_4(x) = 0 + 1(x) - \frac{2}{2}x^2 + \frac{6}{6}x^3 - \frac{24}{24}x^4 = x - x^2 + x^3 - x^4$$

$$24. \quad f(x) = \tan x \quad f(0) = 0$$

$$f'(x) = \sec^2 x \quad f'(0) = 1$$

$$f''(x) = 2 \sec^2 x \tan x \quad f''(0) = 0$$

$$f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x \quad f'''(0) = 2$$

$$P_3(x) = 0 + 1(x) + 0 + \frac{2}{6}x^3 = x + \frac{1}{3}x^3$$

$$26. \quad f(x) = 2x^{-2} \quad f(2) = \frac{1}{2}$$

$$f'(x) = -4x^{-3} \quad f'(2) = -\frac{1}{2}$$

$$f''(x) = 12x^{-4} \quad f''(2) = \frac{3}{4}$$

$$f'''(x) = -48x^{-5} \quad f'''(2) = -\frac{3}{2}$$

$$f^{(4)}(x) = 240x^{-6} \quad f^{(4)}(2) = \frac{15}{4}$$

$$P_4(x) = \frac{1}{2} - \frac{1}{2}(x-2) + \frac{3}{8}(x-2)^2 - \frac{1}{4}(x-2)^3 + \frac{5}{32}(x-2)^4$$

28. $f(x) = x^{1/3} \quad f(8) = 2$

$$f'(x) = \frac{1}{3}x^{-2/3} \quad f'(8) = \frac{1}{12}$$

$$f''(x) = -\frac{2}{9}x^{-5/3} \quad f''(8) = -\frac{1}{144}$$

$$f'''(x) = \frac{10}{27}x^{-8/3} \quad f'''(8) = \frac{10}{27} \cdot \frac{1}{2^8} = \frac{5}{3456}$$

$$P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20,736}(x-8)^3$$

32. $f(x) = \frac{1}{x^2 + 1}$

$$f'(x) = \frac{-2x}{(x^2 + 1)^2}$$

$$f''(x) = \frac{2(3x^2 - 1)}{(x^2 + 1)^3}$$

$$f'''(x) = \frac{24x(1 - x^2)}{(x^2 + 1)^4}$$

$$f^{(4)}(x) = \frac{24(5x^4 - 10x^2 + 1)}{(x^2 + 1)^5}$$

(a) $n = 2, c = 0$

$$P_2(x) = 1 + 0x + \frac{-2}{2!}x^2 = 1 - x^2$$

(c) $n = 4, c = 1$

$$Q_4(x) = \frac{1}{2} + \left(-\frac{1}{2}\right)(x-1) + \frac{1/2}{2!}(x-1)^2 + \frac{0}{3!}(x-1)^3 + \frac{-3}{4!}(x-1)^4 = \frac{1}{2} - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 - \frac{1}{8}(x-1)^4$$

34. $f(x) = \ln x$

$$P_1(x) = x - 1$$

$$P_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

(a)

x	1.00	1.25	1.50	1.75	2.00
$\ln x$	0.0000	0.2231	0.4055	0.5596	0.6931
$P_1(x)$	0.0000	0.2500	0.5000	0.7500	1.0000
$P_4(x)$	0.0000	0.2230	0.4010	0.5303	0.5833

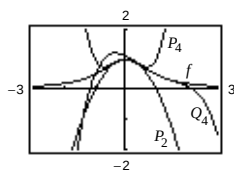
30. $f(x) = x^2 \cos x$

$$f(\pi) = -\pi^2$$

$$f'(x) = \cos x - x^2 \sin x \quad f'(\pi) = -2\pi$$

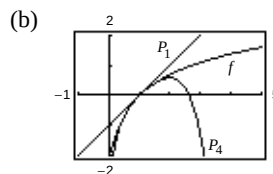
$$f''(x) = 2 \cos x - 4x \sin x - x^2 \cos x \quad f''(\pi) = -2 + \pi^2$$

$$P_2(x) = -\pi^2 - 2\pi(x - \pi) + \frac{(\pi^2 - 2)}{2}(x - \pi)^2$$



(b) $n = 4, c = 0$

$$P_4(x) = 1 + 0x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{24}{4!}x^4 = 1 - x^2 + x^4$$



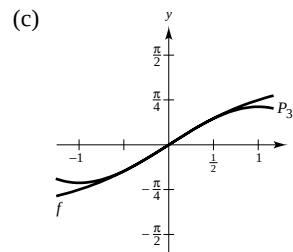
(c) As the distance increases, the accuracy decreases.

36. (a) $f(x) = \arctan x$

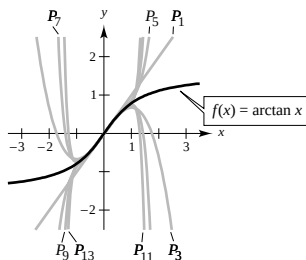
$$P_3(x) = x - \frac{x^3}{3}$$

(b)

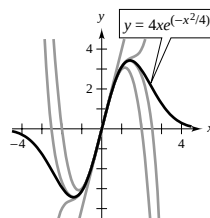
x	-0.75	-0.50	-0.25	0	0.25	0.50	0.75
$f(x)$	-0.6435	-0.4636	-0.2450	0	0.2450	0.4636	0.6435
$P_3(x)$	-0.6094	-0.4583	-0.2448	0	0.2448	0.4583	0.6094



38. $f(x) = \arctan x$



40. $f(x) = 4xe^{-x^2/4}$



42. $f(x) = x^2 e^{-x} \approx x^2 - x^3 + \frac{1}{2}x^4$

$$f\left(\frac{1}{5}\right) \approx 0.0328$$

44. $f(x) = x^2 \cos x \approx -\pi^2 - 2\pi(x - \pi) + \left(\frac{\pi^2 - 2}{2}\right)(x - \pi)^2$

$$f\left(\frac{7\pi}{8}\right) \approx -6.7954$$

46. $f(x) = e^x; f^{(6)}(x) = e^x \Rightarrow \text{Max on } [0, 1] \text{ is } e^1.$

$$R_5(x) \leq \frac{e^1}{6!} (1)^6 \approx 0.00378 = 3.78 \times 10^{-3}$$

48. $f(x) = \arctan x; f^{(4)}(x) = \frac{24x(x^2 + 1)}{(1 - x^2)^4}$
 $\Rightarrow \text{Max on } [0, 0.4] \text{ is } f^{(4)}(0.4) \approx 22.3672.$

$$R_3(x) \leq \frac{22.3672}{4!} (0.4)^4 \approx 0.0239$$

50. $f(x) = e^x$

$$f^{(n+1)}(x) = e^x$$

$$\text{Max on } [0, 0.6] \text{ is } e^{0.6} \approx 1.8221.$$

$$R_n \leq \frac{1.8221}{(n+1)!} (0.6)^{n+1} < 0.001$$

By trial and error, $n = 5$.

52. $f(x) = \cos(\pi x^2)$

$$g(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$f(x) = g(\pi x^2)$$

$$= 1 - \frac{(\pi x^2)^2}{2!} + \frac{(\pi x^2)^4}{4!} - \frac{(\pi x^2)^6}{6!} + \dots$$

$$= 1 - \frac{\pi^2 x^4}{2!} + \frac{\pi^4 x^8}{4!} - \frac{\pi^6 x^{12}}{6!} + \dots$$

$$f(0.6) = 1 - \frac{\pi^2}{2!} (0.6)^4 + \frac{\pi^4}{4!} (0.6)^8 - \frac{\pi^6}{6!} (0.6)^{12} + \dots$$

Since this is an alternating series,

$$R_n \leq a_{n+1} = \frac{\pi^{2n}}{(2n)!} (0.6)^{4n} < 0.0001.$$

By trial and error, $n = 4$. Using 4 terms $f(0.6) \approx 0.4257$.

54. $f(x) = \sin x \approx x - \frac{x^3}{3!}$

$$|R_3(x)| = \left| \frac{\sin z}{4!} x^4 \right| \leq \frac{|x^4|}{4!} < 0.001$$

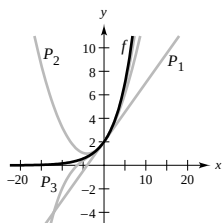
$$x^4 < 0.024$$

$$|x| < 0.3936$$

$$-0.3936 < x < 0.3936$$

56. $f(c) = P_2(c)$, $f'(c) = P_2'(c)$, and $f''(c) = P_2''(c)$

60.



58. See Theorem 8.19, page 611.

62. (a) $P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ for $f(x) = \sin x$

$$P_5'(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

This is the Maclaurin polynomial of degree 4 for $g(x) = \cos x$.

(b) $Q_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$ for $\cos x$

$$Q_6'(x) = -x + \frac{x^3}{3!} - \frac{x^5}{5!} = -P_5(x)$$

(c) $R(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$$R'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

The first four terms are the same!

64. Let f be an odd function and P_n be the n^{th} Maclaurin polynomial for f . Since f is odd, f' is even:

$$f'(-x) = \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \rightarrow 0} \frac{-f(x-h) + f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x+(-h)) - f(x)}{-h} = f'(x).$$

Similarly, f'' is odd, f''' is even, etc. Therefore, $f, f'', f^{(4)}$, etc. are all odd functions, which implies that $f(0) = f''(0) = \dots = 0$. Hence, in the formula

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \dots \text{ all the coefficients of the even power of } x \text{ are zero.}$$

66. Let $P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots + a_n(x-c)^n$ where $a_i = \frac{f^{(i)}(c)}{i!}$.

$$P_n(c) = a_0 = f(c)$$

$$\text{For } 1 \leq k \leq n, \quad P_n^{(k)}(c) = a_n k! = \left(\frac{f^{(k)}(c)}{k!} \right) k! = f^{(k)}(c).$$

Section 8.8 Power Series

2. Centered at 0

4. Centered at π

6. $\sum_{n=0}^{\infty} (2x)^n$

$$L = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{(2x)^n} \right| = 2|x|$$

$$2|x| < 1 \Rightarrow R = \frac{1}{2}$$

8. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n}$

$$L = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(-1)^n x^n} \right|$$

$$= \frac{1}{2} |x|$$

$$\frac{1}{2} |x| < 1 \Rightarrow R = 2$$

$$10. \sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{n!}$$

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)!x^{2n+2}/(n+1)!}{(2n)!x^{2n}/n!} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(2n+2)(2n+1)x^2}{(n+1)} \right| = \infty \end{aligned}$$

The series only converges at $x = 0$. $R = 0$.

$$12. \sum_{n=0}^{\infty} \left(\frac{x}{k} \right)^n$$

Since the series is geometric, it converges only if $|x/k| < 1$ or $-k < x < k$.

$$14. \sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(n+2)x^{n+1}}{(-1)^{n+1}(n+1)x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+2)x}{n+1} \right| = |x| \end{aligned}$$

Interval: $-1 < x < 1$

When $x = 1$, the series $\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)$ diverges.

When $x = -1$, the series $\sum_{n=0}^{\infty} -(n+1)$ diverges.

Therefore, the interval of convergence is $-1 < x < 1$.

$$16. \sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(3x)^{n+1}}{(2n+1)!} \cdot \frac{(2n)!}{(3x)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{3x}{(2n+2)(2n+1)} \right| = 0 \end{aligned}$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

$$18. \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)(n+2)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}x^{n+1}}{(n+2)(n+3)} \cdot \frac{(n+1)(n+2)}{(-1)^n x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)x}{n+3} \right| = |x|$$

Interval: $-1 < x < 1$

When $x = 1$, the alternating series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+2)}$ converges.

When $x = -1$, the series $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$ converges by limit comparison to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Therefore, the interval of convergence is $-1 \leq x \leq 1$.

$$20. \sum_{n=0}^{\infty} \frac{(-1)^n n!(x-4)^n}{3^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}(n+1)!(x-4)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n!(x-4)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x-4)}{3} \right| = \infty$$

$R = 0$

Center: $x = 4$

Therefore, the series converges only for $x = 4$.

$$22. \sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{(n+1)4^{n+1}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+2}}{(n+2)4^{n+2}} \cdot \frac{(n+1)4^{n+1}}{(x-2)^{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-2)(n+1)}{4(n+2)} \right| = \frac{1}{4}|x-2|$$

$$R = 4$$

$$\text{Center: } x = 2$$

$$\text{Interval: } -4 < x - 2 < 4 \text{ or } -2 < x < 6$$

When $x = -2$, the alternating series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)}$ converges.

When $x = 6$, the series $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges.

Therefore, the interval of convergence is $-2 \leq x < 6$.

$$24. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-c)^n}{nc^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(x-c)^{n+1}}{(n+1)c^{n+1}} \cdot \frac{nc^n}{(-1)^{n+1}(x-c)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n(x-c)}{c(n+1)} \right| = \frac{1}{c}|x-c|$$

$$R = c$$

$$\text{Center: } x = c$$

$$\text{Interval: } -c < x - c < c \text{ or } 0 < x < 2c$$

When $x = 0$, the p -series $\sum_{n=1}^{\infty} \frac{-1}{n}$ diverges.

When $x = 2c$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. Therefore, the interval of convergence is $0 < x \leq 2c$.

$$26. \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)} \cdot \frac{(2n+1)}{(-1)^n x^{2n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+1)}{(2n+3)} x^2 \right| = |x^2|$$

$$R = 1$$

$$\text{Interval: } -1 < x < 1$$

When $x = 1$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges.

When $x = -1$, $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ converges.

Therefore, the interval of convergence is $-1 \leq x \leq 1$.

$$28. \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(n+1)!} \cdot \frac{n!}{(-1)^n x^{2n}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x^2}{n+1} \right| = 0 \end{aligned}$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

$$30. \sum_{n=1}^{\infty} \frac{n! x^n}{(2n)!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)! x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n! x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)x}{(2n+2)(2n+1)} \right| = 0 \end{aligned}$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

$$32. \sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} (x^{2n+1})$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2 \cdot 4 \cdots (2n)(2n+2)x^{2n+3}}{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)} \cdot \frac{3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdots (2n)x^{2n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)x^2}{(2n+3)} \right| = |x^2|$$

$$R = 1$$

When $x = \pm 1$, the series diverges by comparing it to

$$\sum_{n=1}^{\infty} \frac{1}{2n+1}$$

which diverges. Therefore, the interval of convergence is $-1 < x < 1$.

$$34. \sum_{n=1}^{\infty} \frac{n!(x-c)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!(x-c)^{n+1}}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!(x-c)} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x-c)}{2n+1} \right| = \frac{1}{2} |x-c|$$

$$R = 2$$

$$\text{Interval: } -2 < x - c < 2 \text{ or } c - 2 < x < c + 2$$

The series diverges at the endpoints. Therefore, the interval of convergence is $c - 2 < x < c + 2$.

$$\left[\frac{n!(c+2-c)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{n!2^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} > 1 \right]$$

$$36. (a) f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}, 0 < x \leq 10$$

$$(b) f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^{n-1}}{5^n}, 0 < x < 10$$

$$(c) f''(x) = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(n-1)(x-5)^{n-2}}{5^n}, 0 < x < 10$$

$$(d) \int f(x) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^{n+1}}{n(n+1)5^n}, 0 \leq x \leq 10$$

$$38. (a) f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n}, 1 < x \leq 3$$

$$(b) f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1}(x-2)^{n-1}, 1 < x < 3$$

$$(c) f''(x) = \sum_{n=2}^{\infty} (-1)^{n+1}(n-1)(x-2)^{n-2}, 1 < x < 3$$

$$(d) \int f(x) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^{n+1}}{n(n+1)}, 1 \leq x \leq 3$$

$$40. g(2) = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \cdots$$

$$S_1 = 1, S_2 = 1.67. \text{ Matches (a)}$$

$$42. g(-2) = \sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n \text{ alternating. Matches (d)}$$

44. The set of all values of x for which the power series converges is the interval of convergence.

If the power series converges for all x , then the radius of convergence is $R = \infty$. If the power series converges at only c , then $R = 0$. Otherwise, according to Theorem 8.20, there exists a real number $R > 0$ (radius of convergence) such that the series converges absolutely for $|x - c| < R$ and diverges for $|x - c| > R$.

46. You differentiate and integrate the power series term by term. The radius of convergence remains the same. However, the interval of convergence might change.

$$48. (a) f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, -\infty < x < \infty \quad (\text{See Exercise 11})$$

$$(b) f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

$$(c) f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$f(0) = 1$$

$$(d) f(x) = e^x$$

$$\begin{aligned}
50. \quad y &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} \\
y' &= \sum_{n=1}^{\infty} \frac{(-1)^n 4n x^{4n-1}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} \\
y'' &= \sum_{n=1}^{\infty} \frac{(-1)^n 4n(4n-1)x^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} = -x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 4n x^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-5)} \\
y'' + x^2 y &= -x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 4n x^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-5)} + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} + x^2 \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4(n+1)x^{4n+2}}{2^{2n+2}(n+1)! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n+2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} \frac{2^2(n+1)}{2^2(n+1)} = 0
\end{aligned}$$

$$52. J_1(x) = x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+1} k! (k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k! (k+1)!}$$

$$(a) \lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+1} x^{2k+3}}{2^{2k+3} (k+1)! (k+2)!} \cdot \frac{2^{2k+1} k! (k+1)!}{(-1)^k x^{2k+1}} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)x^2}{2^2 (k+2)(k+1)} \right| = 0$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

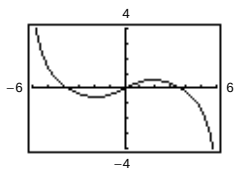
$$(b) J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k! (k+1)!}$$

$$J_1'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)x^{2k}}{2^{2k+1} k! (k+1)!}$$

$$J_1''(x) = \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k)x^{2k-1}}{2^{2k+1} k! (k+1)!}$$

$$\begin{aligned}
x^2 J_1'' + x J_1' + (x^2 - 1) J_1 &= \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k)x^{2k+1}}{2^{2k+1} k! (k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)x^{2k+1}}{2^{2k+1} k! (k+1)!} \\
&\quad + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k! (k+1)!} - \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k! (k+1)!} \\
&= \left[\sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k)x^{2k+1}}{2^{2k+1} k! (k+1)!} + \frac{x}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)x^{2k+1}}{2^{2k+1} k! (k+1)!} \right. \\
&\quad \left. - \frac{x}{2} - \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k! (k+1)!} \right] + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k! (k+1)!} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1} [(2k+1)(2k) + (2k+1) - 1]}{2^{2k+1} k! (k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k! (k+1)!} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1} 4k(k+1)}{2^{2k+1} k! (k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k! (k+1)!} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k-1} (k-1)! k!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k! (k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+3}}{2^{2k+1} k! (k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k! (k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^{k+2k+3} [(-1) + 1]}{2^{2k+1} k! (k+1)!} = 0
\end{aligned}$$

$$(c) P_7(x) = \frac{x}{2} - \frac{1}{16} x^3 + \frac{1}{384} x^5 - \frac{1}{18,432} x^7$$

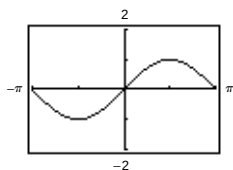


$$\begin{aligned}
(d) \quad J_0'(x) &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} 2(k+1)x^{2k+1}}{2^{2k+2} (k+1)! (k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2^{2k+1} k! (k+1)!} \\
&= -J_1(x) = -\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k! (k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2^{2k+1} k! (k+1)!}
\end{aligned}$$

Note: $J_0'(x) = -J_1(x)$

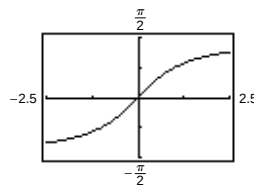
$$54. f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin x$$

(See Exercise 47.)



$$56. f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x, -1 \leq x \leq 1$$

(See Exercise 38 in Section 8.7.)



$$58. \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$$

Replace n with $n-1$.

60. True; if

$$\sum_{n=0}^{\infty} a_n x^n$$

converges for $x = 2$, then we know that it must converge on $(-2, 2]$.

62. True

$$\int_0^1 f(x) dx = \int_0^1 \left(\sum_{n=0}^{\infty} a_n x^n \right) dx = \left[\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} \right]_0^1 = \sum_{n=0}^{\infty} \frac{a_n}{n+1}$$

Section 8.9 Representation of Functions by Power Series

$$2. (a) f(x) = \frac{4}{5-x} = \frac{4/5}{1-x/5} = \frac{a}{1-r}$$

$$= \sum_{n=0}^{\infty} \frac{4}{5} \left(\frac{x}{5} \right)^n = \sum_{n=0}^{\infty} \frac{4x^n}{5^{n+1}}$$

This series converges on $(-5, 5)$.

$$\begin{array}{r} \frac{4}{5} + \frac{4}{25}x + \frac{4}{125}x^2 + \frac{4x^3}{625} + \cdots \\ (b) \quad 5-x \overline{) 4} \\ \underline{4 - \frac{4}{5}x} \\ \frac{4}{5}x \\ \underline{\frac{4}{5}x - \frac{4}{25}x^2} \\ \frac{4}{25}x^2 \\ \underline{\frac{4}{25}x^2 - \frac{4x^3}{125}} \\ \frac{4x^3}{125} \\ \underline{\frac{4x^3}{125} - \frac{4x^4}{625}} \\ \vdots \end{array}$$

$$4. (a) \frac{1}{1+x} = \frac{1}{1-(-x)} = \frac{a}{1-r}$$

$$= \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

This series converges on $(-1, 1)$.

$$(b) \quad \begin{array}{r} 1-x+x^2-x^3+\cdots \\ 1+x \overline{) 1} \\ \underline{1+x} \\ -x \\ \underline{-x-x^2} \\ x^2 \\ \underline{x^2+x^3} \\ -x^3 \\ \underline{-x^3-x^4} \\ \vdots \end{array}$$

6. Writing $f(x)$ in the form $\frac{a}{1-r}$, we have

$$\frac{4}{5-x} = \frac{4}{7-(x+2)} = \frac{4/7}{1-1/7(x+2)} = \frac{a}{1-r}.$$

Therefore, the power series for $f(x)$ is given by

$$\begin{aligned}\frac{4}{5-x} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{4}{7} \left(\frac{1}{7}(x+2) \right)^n \\ &= \sum_{n=0}^{\infty} \frac{4(x+2)^n}{7^{n+1}}.\end{aligned}$$

$$|x+2| < 7 \text{ or } -5 < x < 9$$

8. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{2x-1} = \frac{3}{3+2(x-2)} = \frac{1}{1+(2/3)(x-2)} = \frac{a}{1-r}$$

which implies that $a = 1$ and $r = (-2/3)(x-2)$.

Therefore, the power series for $f(x)$ is given by

$$\begin{aligned}\frac{3}{2x-1} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \left[-\frac{2}{3}(x-2) \right]^n \\ &= \sum_{n=0}^{\infty} \frac{(-2)^n (x-2)^n}{3^n},\end{aligned}$$

$$|x-2| < \frac{3}{2} \text{ or } \frac{1}{2} < x < \frac{7}{2}.$$

10. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{1}{2x-5} = \frac{1}{-5+2x} = \frac{-1/5}{1-(2/5)x} = \frac{a}{1-r}$$

which implies that $a = -1/5$ and $r = (2/5)x$. Therefore, the power series for $f(x)$ is given by

$$\frac{1}{2x-5} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \left(-\frac{1}{5} \right) \left(\frac{2}{5}x \right)^n = -\sum_{n=0}^{\infty} \frac{2^n x^n}{5^{n+1}},$$

$$|x| < \frac{5}{2} \text{ or } -\frac{5}{2} < x < \frac{5}{2}.$$

12. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{4}{3x+2} = \frac{4}{8+3(x-2)} = \frac{1/2}{1+(3/8)(x-2)} = \frac{a}{1-r}$$

which implies that $a = 1/2$ and $r = (-3/8)(x-2)$.

Therefore, the power series for $f(x)$ is given by

$$\begin{aligned}\frac{4}{3x+2} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{1}{2} \left[-\frac{3}{8}(x-2) \right]^n \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-3)^n (x-2)^n}{8^n},\end{aligned}$$

$$|x-2| < \frac{8}{3} \text{ or } -\frac{2}{3} < x < \frac{14}{3}.$$

$$14. \frac{4x-7}{2x^2+3x-2} = \frac{3}{x+2} - \frac{2}{2x-1} = \frac{3}{2+x} - \frac{2}{-1+2x} = \frac{3/2}{1+(1/2)x} + \frac{2}{1-2x}$$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{4x-7}{2x^2+3x-2} = \sum_{n=0}^{\infty} \left(\frac{3}{2} \right) \left(-\frac{1}{2}x \right)^n + \sum_{n=0}^{\infty} 2(2x)^n = \sum_{n=0}^{\infty} \left[\frac{3(-1)^n}{2^{n+1}} + 2^{n+1} \right] x^n, \quad |x| < \frac{1}{2} \text{ or } -\frac{1}{2} < x < \frac{1}{2}.$$

16. First finding the power series for $4/(4+x)$, we have

$$\frac{1}{1+(1/4)x} = \sum_{n=0}^{\infty} \left(-\frac{1}{4}x \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{4^n}$$

Now replace x with x^2 .

$$\frac{4}{4+x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n}.$$

The interval of convergence is $|x^2| < 4$ or $-2 < x < 2$ since

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{4^{n+1}} \cdot \frac{4^n}{(-1)^n x^{2n}} \right| = \left| -\frac{x^2}{4} \right| = \frac{|x^2|}{4}.$$

$$\begin{aligned}18. h(x) &= \frac{x}{x^2-1} = \frac{1}{2(1+x)} - \frac{1}{2(1-x)} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n x^n - \frac{1}{2} \sum_{n=0}^{\infty} x^n \\ &= \frac{1}{2} \sum_{n=0}^{\infty} [(-1)^n - 1] x^n = \frac{1}{2} [0 - 2x + 0x^2 - 2x^3 + 0x^4 - 2x^5 + \cdots] \\ &= \frac{1}{2} \sum_{n=0}^{\infty} (-2)x^{2n+1} = -\sum_{n=0}^{\infty} x^{2n+1}, \quad 1 < x < 1\end{aligned}$$

20. By taking the second derivative, we have $\frac{d^2}{dx^2} \left[\frac{1}{x+1} \right] = \frac{2}{(x+1)^3}$. Therefore,

$$\begin{aligned} \frac{2}{(x+1)^3} &= \frac{d^2}{dx^2} \left[\sum_{n=0}^{\infty} (-1)^n x^n \right] \\ &= \frac{d}{dx} \left[\sum_{n=1}^{\infty} (-1)^n n x^{n-1} \right] = \sum_{n=2}^{\infty} (-1)^n n(n-1) x^{n-2} = \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^n, \quad -1 < x < 1. \end{aligned}$$

22. By integrating, we have

$$\int \frac{1}{1+x} dx = \ln(1+x) + C_1 \text{ and } \int \frac{1}{1-x} dx = -\ln(1-x) + C_2.$$

$f(x) = \ln(1-x^2) = \ln(1+x) - [-\ln(1-x)]$. Therefore,

$$\begin{aligned} \ln(1-x^2) &= \int \frac{1}{1+x} dx - \int \frac{1}{1-x} dx \\ &= \int \left[\sum_{n=0}^{\infty} (-1)^n x^n \right] dx - \int \left[\sum_{n=0}^{\infty} x^n \right] dx = \left[C_1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \right] - \left[C_2 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \right] \\ &= C + \sum_{n=0}^{\infty} \frac{[(-1)^n - 1] x^{n+1}}{n+1} = C + \sum_{n=0}^{\infty} \frac{-2x^{2n+2}}{2n+2} = C + \sum_{n=0}^{\infty} \frac{(-1)x^{2n+2}}{n+1} \end{aligned}$$

To solve for C , let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(1-x^2) = - \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n+1}, \quad -1 < x < 1$$

24. $\frac{2x}{x^2+1} = 2x \sum_{n=0}^{\infty} (-1)^n x^{2n}$ (See Exercise 23.)

$$= \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}$$

Since $\frac{d}{dx} (\ln(x^2+1)) = \frac{2x}{x^2+1}$, we have

$$\ln(x^2+1) = \int \left[\sum_{n=0}^{\infty} (-1)^n 2x^{2n+1} \right] dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}, \quad -1 \leq x \leq 1.$$

To solve for C , let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(x^2+1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}, \quad -1 \leq x \leq 1.$$

26. Since $\int \frac{1}{4x^2+1} dx = \frac{1}{2} \arctan(2x)$, we can use the result of Exercise 25 to obtain

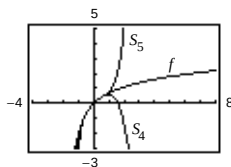
$$\arctan(2x) = 2 \int \frac{1}{4x^2+1} dx = 2 \int \left[\sum_{n=0}^{\infty} (-1)^n 4^n x^{2n} \right] dx = C + 2 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1}, \quad -\frac{1}{2} < x \leq \frac{1}{2}.$$

To solve for C , let $x = 0$ and conclude that $C = 0$. Therefore,

$$\arctan(2x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1}, \quad -\frac{1}{2} < x \leq \frac{1}{2}.$$

28. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \leq \ln(x + 1)$

$$\leq x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$



x	0.0	0.2	0.4	0.6	0.8	1.0
$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$	0.0	0.18227	0.33493	0.45960	0.54827	0.58333
$\ln(x + 1)$	0.0	0.18232	0.33647	0.47000	0.58779	0.69315
$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$	0.0	0.18233	0.33698	0.47515	0.61380	0.78333

In Exercise 35–38, $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

30. $g(x) = x - \frac{x^3}{3}$, cubic with 3 zeros.

Matches (d)

32. $g(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$,

Matches (b)

34. The approximations of degree 3, 7, 11, . . . ($4n - 1$, $n = 1, 2, \dots$) have relative extrema.

In Exercises 36 and 38, $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

36.
$$\arctan x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

$$\int \arctan x^2 dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)} + C, C = 0$$

$$\int_0^{3/4} \arctan x^2 dx = \sum_{n=0}^{\infty} (-1)^n \frac{(3/4)^{4n+3}}{(4n+3)(2n+1)}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{3^{4n+3}}{(4n+3)(2n+1)4^{4n+3}}$$

$$= \frac{27}{192} - \frac{2187}{344,064} + \frac{177,147}{230,686,720}$$

Since $177,147/230,686,720 < 0.001$, we can approximate the series by its first two terms: 0.13427

38.
$$x^2 \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{2n+1}$$

$$\int x^2 \arctan x dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{(2n+4)(2n+1)}$$

$$\int_0^{1/2} x^2 \arctan x dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+4)(2n+1)2^{2n+4}} = \frac{1}{64} - \frac{1}{1152} + \dots$$

Since $\frac{1}{1152} < 0.001$, we can approximate the series by its first term: $\int_0^{1/2} x^2 \arctan x dx \approx 0.015625$.

In Exercises 40 and 42, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$.

40. Replace n with $n+1$.

$$\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n$$

$$42. (a) \frac{1}{3} \sum_{n=1}^{\infty} n \left(\frac{2}{3}\right)^n = \frac{2}{9} \sum_{n=1}^{\infty} n \left(\frac{2}{3}\right)^{n-1} = \frac{2}{9} \frac{1}{[1 - (2/3)]^2} = 2$$

$$(b) \frac{1}{10} \sum_{n=1}^{\infty} n \left(\frac{9}{10}\right)^n = \frac{9}{100} \sum_{n=1}^{\infty} n \left(\frac{9}{10}\right)^{n-1} \\ = \frac{9}{100} \cdot \frac{1}{[1 - (9/10)]^2} = 9$$

44. Replace x with x^2 .

46. Integrate the series and multiply by (-1) .

48. (a) From Exercise 47, we have

$$\arctan \frac{120}{119} - \arctan \frac{1}{239} = \arctan \frac{120}{119} + \arctan \left(-\frac{1}{239}\right) \\ = \arctan \left[\frac{(120/119) + (-1/239)}{1 - (120/119)(-1/239)} \right] = \arctan \left(\frac{28,561}{28,561} \right) = \arctan 1 = \frac{\pi}{4}$$

$$(b) 2 \arctan \frac{1}{5} = \arctan \frac{1}{5} + \arctan \frac{1}{5} = \arctan \left[\frac{2(1/5)}{1 - (1/5)^2} \right] = \arctan \frac{10}{24} = \arctan \frac{5}{12}$$

$$4 \arctan \frac{1}{5} = 2 \arctan \frac{1}{5} + 2 \arctan \frac{1}{5} = \arctan \frac{5}{12} + \arctan \frac{5}{12} = \arctan \left[\frac{2(5/12)}{1 - (5/12)^2} \right] = \arctan \frac{120}{119}$$

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \arctan \frac{120}{119} - \arctan \frac{1}{239} = \frac{\pi}{4} \text{ (see part (a).)}$$

$$50. (a) \arctan \frac{1}{2} + \arctan \frac{1}{3} = \arctan \left[\frac{(1/2) + (1/3)}{1 - (1/2)(1/3)} \right] = \arctan \left(\frac{5/6}{5/6} \right) = \frac{\pi}{4}$$

$$(b) \pi = 4 \left[\arctan \frac{1}{2} + \arctan \frac{1}{3} \right] \\ = 4 \left[\frac{1}{2} - \frac{(1/2)^3}{3} + \frac{(1/2)^5}{5} - \frac{(1/2)^7}{7} \right] + 4 \left[\frac{1}{3} - \frac{(1/3)^3}{3} + \frac{(1/3)^5}{5} - \frac{(1/3)^7}{7} \right] \\ \approx 4(0.4635) + 4(0.3217) \approx 3.14$$

52. From Exercise 51, we have

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{3^n n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (1/3)^n}{n} \\ = \ln \left(\frac{1}{3} + 1 \right) = \ln \frac{4}{3} \approx 0.2877.$$

54. From Example 5, we have $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{(1)^{2n+1}}{2n+1} \\ = \arctan 1 = \frac{\pi}{4} \approx 0.7854$$

56. From Exercise 54, we have

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{3^{2n-1}(2n-1)} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)} \\ = \sum_{n=0}^{\infty} (-1)^n \frac{(1/3)^{2n+1}}{2n+1} \\ = \arctan \frac{1}{3} \approx 0.3218.$$

58. From Example 5, we have $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(\sqrt{3})^{2n}(2n+1)} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\ &= \sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n (1/\sqrt{3})^{2n+1}}{2n+1} \\ &= \sqrt{3} \arctan \frac{1}{\sqrt{3}} \\ &= \sqrt{3} \left(\frac{\pi}{6} \right) = \frac{\pi}{2\sqrt{3}} \end{aligned}$$

Section 8.10 Taylor and Maclaurin Series

2. For $c = 0$, we have

$$\begin{aligned} f(x) &= e^{3x} \\ f^{(n)}(x) &= 3^n e^{3x} \Rightarrow f^{(n)}(0) = 3^n \\ e^{3x} &= 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} \end{aligned}$$

4. For $c = \pi/4$, we have:

$$\begin{aligned} f(x) &= \sin x & f\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} \\ f'(x) &= \cos x & f'\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} \\ f''(x) &= -\sin x & f''\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2} \\ f'''(x) &= -\cos x & f'''\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2} \\ f^{(4)}(x) &= \sin x & f^{(4)}\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} \end{aligned}$$

and so on. Therefore we have:

$$\begin{aligned} \sin x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)[x - (\pi/4)]^n}{n!} \\ &= \frac{\sqrt{2}}{2} \left[1 + \left(x - \frac{\pi}{4}\right) - \frac{[x - (\pi/4)]^2}{2!} - \frac{[x - (\pi/4)]^3}{3!} + \frac{[x - (\pi/4)]^4}{4!} + \cdots \right] \\ &= \frac{\sqrt{2}}{2} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2} [x - (\pi/4)]^{n+1}}{(n+1)!} + 1 \right\} \end{aligned}$$

6. For $c = 1$, we have:

$$\begin{aligned} f(x) &= e^x \\ f^{(n)}(x) &= e^x \Rightarrow f^{(n)}(1) = e \\ e^x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = e \left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + \cdots \right] = e \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!} \end{aligned}$$

8. For $c = 0$, we have:

$$\begin{aligned}
 f(x) &= \ln(x^2 + 1) & f(0) &= 0 \\
 f'(x) &= \frac{2x}{x^2 + 1} & f'(0) &= 0 \\
 f''(x) &= \frac{2 - 2x^2}{(x^2 + 1)^2} & f''(0) &= 2 \\
 f'''(x) &= \frac{4x(x^2 - 3)}{(x^2 + 1)^3} & f'''(0) &= 0 \\
 f^{(4)}(x) &= \frac{12(-x^4 + 6x^2 - 1)}{(x^2 + 1)^4} & f^{(4)}(0) &= -12 \\
 f^{(5)}(x) &= \frac{48x(x^4 - 10x^2 + 5)}{(x^2 + 1)^5} & f^{(5)}(0) &= 0 \\
 f^{(6)}(x) &= \frac{-240(5x^6 - 15x^4 + 15x^2 - 1)}{(x^2 + 1)^6} & f^{(6)}(0) &= 240
 \end{aligned}$$

and so on. Therefore, we have:

$$\begin{aligned}
 \ln(x^2 + 1) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 0x + \frac{2x^2}{2!} + \frac{0x^3}{3!} - \frac{12x^4}{4!} + \frac{0x^5}{5!} + \frac{240x^6}{6!} + \cdots \\
 &= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}
 \end{aligned}$$

10. For $c = 0$, we have;

$$\begin{aligned}
 f(x) &= \tan(x) & f(0) &= 0 \\
 f'(x) &= \sec^2(x) & f'(0) &= 1 \\
 f''(x) &= 2 \sec^2(x)\tan(x) & f''(0) &= 0 \\
 f'''(x) &= 2[\sec^4(x) + 2 \sec^2(x)\tan^2(x)] & f'''(0) &= 2 \\
 f^{(4)}(x) &= 8[\sec^4(x)\tan(x) + \sec^2(x)\tan^3(x)] & f^{(4)}(0) &= 0 \\
 f^{(5)}(x) &= 8[2 \sec^6(x) + 11 \sec^4(x)\tan^2(x) + 2 \sec^2(x)\tan^4(x)] & f^{(5)}(0) &= 16 \\
 \tan(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \cdots = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots
 \end{aligned}$$

12. The Maclaurin Series for $f(x) = e^{-2x}$ is $\sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}$.

$f^{(n+1)}(x) = (-2)^{n+1}e^{-2x}$. Hence, by Taylor's Theorem,

$$0 \leq |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| = \left| \frac{(-2)^{n+1}e^{-2z}}{(n+1)!} x^{n+1} \right|.$$

Since $\lim_{n \rightarrow \infty} \left| \frac{(-2)^{n+1}x^{n+1}}{(n+1)!} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{(n+1)!} \right| = 0$, it follows that $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$.

Hence, the Maclaurin Series for e^{-2x} converges to e^{-2x} for all x .

14. Since $(1+x)^{-k} = 1 - kx + \frac{k(k+1)x^2}{2!} - \frac{k(k+1)(k+2)x^3}{3!} + \dots$, we have

$$\begin{aligned}\left[1 + (-x)\right]^{-1/2} &= 1 + \left(\frac{1}{2}\right)x + \frac{(1/2)(3/2)x^2}{2!} + \frac{(1/2)(3/2)(5/2)x^3}{3!} + \dots \\ &= 1 + \frac{x}{2} + \frac{(1)(3)x^2}{2^2 2!} + \frac{(1)(3)(5)x^3}{2^3 3!} + \dots \\ &= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)x^n}{2^n n!}\end{aligned}$$

16. Since $(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \dots$, we have

$$\begin{aligned}(1+x)^{1/3} &= 1 + \left(\frac{1}{3}\right)x + \frac{(1/3)(-2/3)x^2}{2!} + \frac{(1/3)(-2/3)(-5/3)x^3}{3!} + \dots \\ &= 1 + \frac{x}{3} - \frac{2x^2}{3^2 2!} + \frac{2 \cdot 5x^3}{3^3 3!} - \frac{2 \cdot 5 \cdot 8x^4}{3^4 4!} + \dots \\ &= 1 + \frac{x}{3} + \sum_{n=2}^{\infty} \frac{(-1)^{n+2} \cdot 5 \cdot 8 \cdot \dots \cdot (3n-4)}{3^n n!}.\end{aligned}$$

18. Since $(1+x)^{1/2} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3)x^n}{2^n n!}$ (Exercise 14)

$$\text{we have } (1+x^3)^{1/2} = 1 + \frac{x^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3)x^{3n}}{2^n n!}.$$

20. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$

$$e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^n}{n!} = 1 - 3x + \frac{9x^2}{2!} - \frac{27x^3}{3!} + \frac{81x^4}{4!} - \frac{243x^5}{5!} + \dots$$

22. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$$\begin{aligned}\cos 4x &= \sum_{n=0}^{\infty} \frac{(-1)^n (4x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n}}{(2n)!} \\ &= 1 - \frac{16x^2}{2!} + \frac{256x^4}{4!} - \dots\end{aligned}$$

24. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\begin{aligned}2 \sin x^3 &= 2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!} \\ &= 2 \left[x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \dots \right] \\ &= 2x^3 - \frac{2x^9}{3!} + \frac{2x^{15}}{5!} - \dots\end{aligned}$$

26. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

$$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \dots$$

$$2 \cosh(x) = e^x + e^{-x} = \sum_{n=0}^{\infty} 2 \frac{x^{2n}}{(2n)!}$$

28. The formula for the binomial series gives $(1+x)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$, which implies that

$$\begin{aligned}(1+x^2)^{-1/2} &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^n n!} \\ \ln(x + \sqrt{x^2 + 1}) &= \int \frac{1}{\sqrt{x^2 + 1}} dx \\ &= x + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n+1}}{2^n (2n+1)n!} \\ &= x - \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} - \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots\end{aligned}$$

$$\begin{aligned}30. \quad x \cos x &= x \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right) \\ &= x - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}\end{aligned}$$

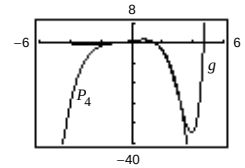
$$\begin{aligned}32. \quad \frac{\arcsin x}{x} &= \sum_{n=0}^{\infty} \frac{(2n)! x^{2n+1}}{(2^n n!)^2 (2n+1)} \cdot \frac{1}{x} \\ &= \sum_{n=0}^{\infty} \frac{(2n)! x^{2n}}{(2^n n!)^2 (2n+1)}, \quad x \neq 0\end{aligned}$$

$$34. \quad e^{ix} + e^{-ix} = 2 - \frac{2x^2}{2!} + \frac{2x^4}{4!} - \frac{2x^6}{6!} + \cdots \quad (\text{See Exercise 33.})$$

$$\frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos(x)$$

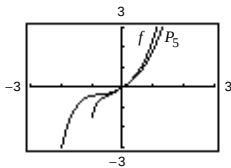
$$36. \quad g(x) = e^x \cos x$$

$$\begin{aligned}&= \left(1 + x + \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^4}{24} + \cdots \right) \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots \right) \\ &= 1 + x + \left(\frac{x^2}{2} - \frac{x^2}{2} \right) + \left(\frac{x^3}{6} - \frac{x^3}{2} \right) + \left(\frac{x^4}{24} - \frac{x^4}{4} + \frac{x^4}{24} \right) + \cdots = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots\end{aligned}$$



$$38. \quad f(x) = e^x \ln(1+x)$$

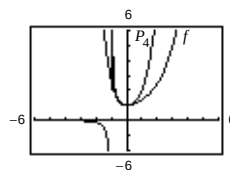
$$\begin{aligned}&= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \right) \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \right) \\ &= x + \left(x^2 - \frac{x^2}{2} \right) + \left(\frac{x^3}{3} - \frac{x^3}{2} + \frac{x^3}{2} \right) + \left(-\frac{x^4}{4} + \frac{x^4}{3} - \frac{x^4}{4} + \frac{x^4}{6} \right) + \left(\frac{x^5}{5} - \frac{x^5}{4} + \frac{x^5}{6} - \frac{x^5}{12} + \frac{x^5}{24} \right) + \cdots \\ &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + \cdots\end{aligned}$$



40. $f(x) = \frac{e^x}{1+x}$. Divide the series for e^x by $(1+x)$.

$$\begin{array}{r}
 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} + \cdots \\
 1+x \overline{) 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots} \\
 \underline{1+x} \phantom{+ \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots} \\
 0 + \frac{x^2}{2} + \frac{x^3}{6} \phantom{+ \frac{x^4}{24} + \frac{x^5}{120} + \cdots} \\
 \underline{ \frac{x^2}{2} + \frac{x^3}{6}} \phantom{+ \frac{x^4}{24} + \frac{x^5}{120} + \cdots} \\
 -\frac{x^3}{3} + \frac{x^4}{24} \phantom{+ \frac{x^5}{120} + \cdots} \\
 \underline{ -\frac{x^3}{3} - \frac{x^4}{3}} \phantom{+ \frac{x^5}{120} + \cdots} \\
 \frac{3x^4}{8} + \frac{x^5}{120} \\
 \underline{ \frac{3x^4}{8} + \frac{3x^5}{8}} \\
 \vdots
 \end{array}$$

$$f(x) = 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} - \cdots$$



42. $y = x - \frac{x^3}{2!} + \frac{x^5}{4!} = x \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \right) \approx x \cos x$.

Matches (b)

44. $y = x^2 - x^3 + x^4 = x^2(1 - x + x^2) \approx x^2 \left(\frac{1}{1+x} \right)$.

Matches (d)

$$\begin{aligned}
 46. \int_0^x \sqrt{1+t^3} dt &= \int_0^x \left[1 + \frac{t^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)t^{3n}}{2^n n!} \right] dt \\
 &= \left[t + \frac{t^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)t^{3n+1}}{(3n+1)2^n n!} \right]_0^x \\
 &= x + \frac{x^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)x^{3n+1}}{(3n+1)2^n n!}
 \end{aligned}$$

48. Since $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, we have

$$\sin(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots \approx 0.8415. \quad (4 \text{ terms})$$

50. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$, we have $e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$

$$\text{and } \frac{e-1}{e} = 1 - e^{-1} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{7!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \approx 0.6321. \quad (6 \text{ terms})$$

52. Since

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

$$\text{we have } \lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1.$$

$$54. \int_0^{1/2} \frac{\arctan x}{x} dx = \int_0^{1/2} \left(1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \cdots\right) dx = \left[x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \cdots\right]_0^{1/2}$$

Since $1/(9^2 2^9) < 0.0001$, we have

$$\int_0^{1/2} \frac{\arctan x}{x} dx \approx \left(\frac{1}{2} - \frac{1}{3^2 2^3} + \frac{1}{5^2 2^5} - \frac{1}{7^2 2^7} + \frac{1}{9^2 2^9}\right) \approx 0.4872.$$

Note: We are using $\lim_{x \rightarrow 0^+} \frac{\arctan x}{x} = 1$.

$$56. \int_{0.5}^1 \cos \sqrt{x} dx = \int_{0.5}^1 \left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots\right) dx = \left[x - \frac{x^2}{2(2!)} + \frac{x^3}{3(4!)} - \frac{x^4}{4(6!)} + \frac{x^5}{5(8!)} - \cdots\right]_{0.5}^1$$

Since $\frac{1}{210,600} (1 - 0.5^5) < 0.0001$, we have

$$\int_{0.5}^1 \cos \sqrt{x} dx \approx \left[(1 - 0.5) - \frac{1}{4}(1 - 0.5^2) + \frac{1}{72}(1 - 0.5^3) - \frac{1}{2880}(1 - 0.5^4) + \frac{1}{201,600}(1 - 0.5^5)\right] \approx 0.3243.$$

$$58. \int_0^{1/4} x \ln(x+1) dx = \int_0^{1/4} \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots\right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4 \cdot 2} + \frac{x^5}{5 \cdot 3} - \frac{x^6}{6 \cdot 4} + \cdots\right]_0^{1/4}$$

Since $\frac{(1/4)^5}{15} < 0.0001$,

$$\int_0^{1/4} x \ln(x+1) dx \approx \frac{(1/4)^3}{3} - \frac{(1/4)^4}{8} \approx 0.00472.$$

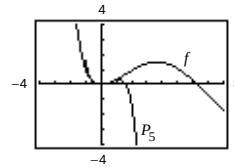
60. From Exercise 19, we have

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_1^2 e^{-x^2/2} dx &= \frac{1}{\sqrt{2\pi}} \int_1^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!} dx = \frac{1}{\sqrt{2\pi}} \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n! (2n+1)} \right]_1^2 \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n (2^{n+1} - 1)}{2^n n! (2n+1)} \\ &\approx \frac{1}{\sqrt{2\pi}} \left[1 - \frac{7}{2 \cdot 1 \cdot 3} + \frac{31}{2^2 \cdot 2! \cdot 5} - \frac{127}{2^3 \cdot 3! \cdot 7} + \frac{511}{2^4 \cdot 4! \cdot 9} - \frac{2047}{2^5 \cdot 5! \cdot 11} \right. \\ &\quad \left. + \frac{8191}{2^6 \cdot 6! \cdot 13} - \frac{32,767}{2^7 \cdot 7! \cdot 15} + \frac{131,071}{2^8 \cdot 8! \cdot 17} - \frac{524,287}{2^9 \cdot 9! \cdot 19} \right] \approx 0.1359. \end{aligned}$$

$$62. f(x) = \sin \frac{x}{2} \ln(1+x)$$

$$P_5(x) = \frac{x^2}{2} - \frac{x^3}{4} + \frac{7x^4}{48} - \frac{11x^5}{96}$$

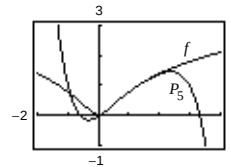
The polynomial is a reasonable approximation on the interval $(-0.60, 0.73)$.



$$64. f(x) = \sqrt[3]{x} \cdot \arctan x, c = 1$$

$$\begin{aligned} P_5(x) &\approx 0.7854 + 0.7618(x-1) - 0.3412 \left[\frac{(x-1)^2}{2!} \right] - 0.0424 \left[\frac{(x-1)^3}{3!} \right] \\ &\quad + 1.3025 \left[\frac{(x-1)^4}{4!} \right] - 5.5913 \left[\frac{(x-1)^5}{5!} \right] \end{aligned}$$

The polynomial is a reasonable approximation on the interval $(0.48, 1.75)$.



66. $a_{2n+1} = 0$ (odd coefficients are zero)

68. Answers will vary.

70. $\theta = 60^\circ, v_0 = 64, k = \frac{1}{16}, g = -32$

$$\begin{aligned}
 y &= \sqrt{3}x - \frac{32x^2}{2(64)^2(1/2)^2} - \frac{(1/16)(32)x^3}{3(64)^3(1/2)^3} - \frac{(1/16)^2(32)x^4}{4(64)^4(1/2)^4} - \cdots \\
 &= \sqrt{3}x - 32 \left[\frac{2^2x^2}{2(64)^2} + \frac{2^3x^3}{3(64)^316} + \frac{2^4x^4}{4(64)^4(16)^2} + \cdots \right] \\
 &= \sqrt{3}x - 32 \sum_{n=2}^{\infty} \frac{2^n x^n}{n(64)^n(16)^{n-2}} = \sqrt{3}x - 32 \sum_{n=2}^{\infty} \frac{x^n}{n(32)^n(16)^{n-2}}
 \end{aligned}$$

72. (a) $f(x) = \frac{\ln(x^2 + 1)}{x^2}$.

From Exercise 8, you obtain

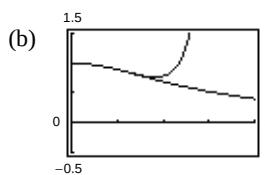
$$P = \frac{1}{x^2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n+1}$$

$$P_8 = 1 - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{4} + \frac{x^8}{5}$$

(c) $F(x) = \int_0^x \frac{\ln(t^2 + 1)}{t^2} dt$

$$G(x) = \int_0^x P_8(t) dt$$

x	0.25	0.50	0.75	1.00	1.50	2.00
$F(x)$	0.2475	0.4810	0.6920	0.8776	1.1798	1.4096
$G(x)$	0.2475	0.4810	0.6920	0.8805	5.3064	652.21

(d) The curves are nearly identical for $0 < x < 1$. Hence, the integrals nearly agree on that interval.74. Assume $e = p/q$ is rational. Let $N > q$ and form the following.

$$e - \left[1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{N!} \right] = \frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots$$

Set $a = N! \left[e - \left(1 + 1 + \cdots + \frac{1}{N!} \right) \right]$, a positive integer. But,

$$\begin{aligned}
 a &= N! \left[\frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots \right] = \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \cdots < \frac{1}{N+1} + \frac{1}{(N+1)^2} + \cdots \\
 &= \frac{1}{N+1} \left[1 + \frac{1}{N+1} + \frac{1}{(N+1)^2} + \cdots \right] = \frac{1}{N+1} \left[\frac{1}{1 - \left(\frac{1}{N+1} \right)} \right] = \frac{1}{N}, \text{ a contradiction.}
 \end{aligned}$$

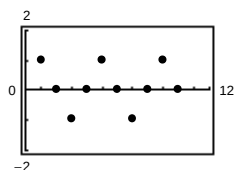
Review Exercises for Chapter 8

2. $a_n = \frac{n}{n^2 + 1}$

4. $a_n = 4 - \frac{n}{2}$: 3.5, 3, . . .
Matches (c)

6. $a_n = 6 \left(-\frac{2}{3} \right)^{n-1}$: 6, -4, . . .
Matches (b)

8. $a_n = \sin \frac{n\pi}{2}$



The sequence seems to diverge (oscillates).

$$\sin \frac{n\pi}{2}: 1, 0, -1, 0, 1, 0, \dots$$

10. $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$
Converges

12. $\lim_{n \rightarrow \infty} \frac{n}{\ln(n)} = \lim_{n \rightarrow \infty} \frac{1}{1/n} = \infty$
Diverges

14. $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2n}\right)^n = \lim_{k \rightarrow \infty} \left[\left(1 + \frac{1}{k}\right)^k\right]^{1/2} = e^{1/2}$
Converges; $k = 2n$

16. Let $y = (b^n + c^n)^{1/n}$

$$\ln y = \frac{\ln(b^n + c^n)}{n}$$

$$\lim_{n \rightarrow \infty} \ln y = \lim_{n \rightarrow \infty} \frac{1}{b^n + c^n} (b^n \ln b + c^n \ln c)$$

Assume $b \geq c$ and note that the terms

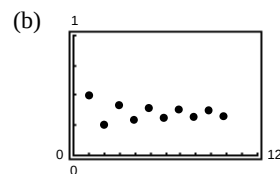
$$\frac{b^n \ln b + c^n \ln c}{b^n + c^n} = \frac{b^n \ln b}{b^n + c^n} + \frac{c^n \ln c}{b^n + c^n}$$

converge as $n \rightarrow \infty$. Hence a_n converges.

20. (a)

k	5	10	15	20	25
S_k	0.3917	0.3228	0.3627	0.3344	0.3564

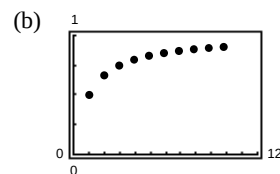
(c) The series converges by the Alternating Series Test.



22. (a)

k	5	10	15	20	25
S_k	0.8333	0.9091	0.9375	0.9524	0.9615

(c) The series converges, by the limit comparison test with $\sum \frac{1}{n^2}$.



24. Diverges. Geometric series, $r = 1.82 > 1$.

26. Diverges. n th Term Test, $\lim_{n \rightarrow \infty} a_n = \frac{2}{3}$.

28. $\sum_{n=0}^{\infty} \frac{2^{n+2}}{3^n} = 4 \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 4(3) = 12$

See Exercise 27.

30.
$$\sum_{n=0}^{\infty} \left[\left(\frac{2}{3}\right)^n - \frac{1}{(n+1)(n+2)} \right] = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$= \frac{1}{1 - (2/3)} - \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \right] = 3 - 1 = 2$$

$$32. 0.\overline{923076} = 0.923076[1 + 0.000001 + (0.000001)^2 + \cdots]$$

$$= \sum_{n=0}^{\infty} (0.923076)(0.000001)^n = \frac{0.923076}{1 - 0.000001} = \frac{923,076}{999,999} = \frac{12(76,923)}{13(76,923)} = \frac{12}{13}$$

$$34. S = \sum_{n=0}^{39} 32,000(1.055)^n = \frac{32,000(1 - 1.055^{40})}{1 - 1.055}$$

$$\approx \$4,371,379.65$$

36. See Exercise 86 in Section 8.2.

$$A = P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]$$

$$= 100\left(\frac{12}{0.065}\right)\left[\left(1 + \frac{0.065}{12}\right)^{120} - 1\right]$$

$$\approx \$16,840.32$$

$$38. \sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$$

Divergent p -series, $p = \frac{3}{4} < 1$

$$40. \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{2^n}$$

The first series is a convergent p -series and the second series is a convergent geometric series. Therefore, their difference converges.

$$42. \sum_{n=1}^{\infty} \frac{n+1}{n(n+2)}$$

$$\lim_{n \rightarrow \infty} \frac{(n+1)/n(n+2)}{1/n} = \lim_{n \rightarrow \infty} \frac{n+1}{n+2} = 1$$

By a limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$, the series diverges.

44. Since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges, $\sum_{n=1}^{\infty} \frac{1}{3^n - 5}$ converges by the Limit Comparison Test.

$$46. \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$$

$$a_{n+1} = \frac{\sqrt{n+1}}{n+2} \leq \frac{\sqrt{n}}{n+1} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+1} = 0$$

By the Alternating Series Test, the series converges.

48. Converges by the Alternating Series Test.

$$a_{n+1} = \frac{3 \ln(n+1)}{n+1} < \frac{3 \ln n}{n} = a_n, \lim_{n \rightarrow \infty} \frac{3 \ln n}{n} = 0$$

$$50. \sum_{n=1}^{\infty} \frac{n!}{e^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{n+1}{e} = \infty$$

By the Ratio Test, the series diverges.

$$52. \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$$

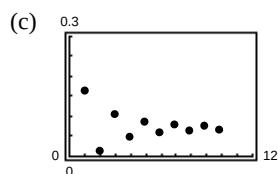
$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{1 \cdot 3 \cdots (2n-1)} \right| = \lim_{n \rightarrow \infty} \frac{2n+1}{3n+2} = \frac{2}{3}$$

By the Ratio Test, the series converges.

54. (a) The series converges by the Alternating Series Test.

(b)

x	5	10	15	20	25
S_n	0.0871	0.0669	0.0734	0.0702	0.0721



(d) The sum is approximately 0.0714.

56. No. Let $a_n = \frac{3937.5}{n^2}$, then $a_{75} = 0.7$. The series $\sum_{n=1}^{\infty} \frac{3937.5}{n^2}$ is a convergent p -series.

58. $f(x) = \tan x$ $f\left(-\frac{\pi}{4}\right) = -1$

$f'(x) = \sec^2 x$ $f'\left(-\frac{\pi}{4}\right) = 2$

$f''(x) = 2 \sec^2 x \tan x$ $f''\left(-\frac{\pi}{4}\right) = -4$

$f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x$ $f'''\left(-\frac{\pi}{4}\right) = 16$

$P_3(x) = -1 + 2\left(x + \frac{\pi}{4}\right) - 2\left(x + \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x + \frac{\pi}{4}\right)^3$

60. $\cos(0.75) \approx 1 - \frac{(0.75)^2}{2!} + \frac{(0.75)^4}{4!} - \frac{(0.75)^6}{6!} \approx 0.7317$

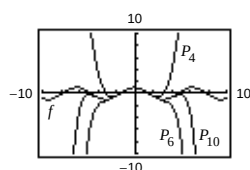
62. $e^{-0.25} \approx 1 - 0.25 + \frac{(0.25)^2}{2!} - \frac{(0.25)^3}{3!} + \frac{(0.25)^4}{4!} \approx 0.779$

64. $f(x) = \cos x$

$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

$P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

$P_{10}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$



66. $\sum_{n=0}^{\infty} (2x)^n$

Geometric series which converges only if $|2x| < 1$ or $-\frac{1}{2} < x < \frac{1}{2}$.

68. $\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}(x-2)^{n+1}}{n+1} \cdot \frac{n}{3^n(x-2)^n} \right|$$

$$= 3|x-2|$$

$R = \frac{1}{3}$

Center: 2

Since the series converges at $\frac{5}{3}$ and diverges at $\frac{7}{3}$, the interval of convergence is $\frac{5}{3} \leq x < \frac{7}{3}$.

70. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{x-2}{2}\right)^n$

Geometric series which converges only if

$$\left| \frac{x-2}{2} \right| < 1 \quad \text{or} \quad 0 < x < 4.$$

$$\begin{aligned}
72. \quad y &= \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{2^n n!} \\
y' &= \sum_{n=1}^{\infty} \frac{(-3)^n (2n) x^{2n-1}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2) x^{2n+1}}{2^{n+1} (n+1)!} \\
y'' &= \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2)(2n+1) x^{2n}}{2^{n+1} (n+1)!} \\
y'' + 3xy' + 3y &= \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2)(2n+1) x^{2n}}{2^{n+1} (n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} (2n+2) x^{2n+2}}{2^{n+1} (n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+1} (2n+2) x^{2n}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} [-(2n+1) + 1] + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} (-2n) + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^{n+1} x^{2n}}{2^n n!} (2n) + \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^{n-1} (n-1)!} \cdot \frac{2n}{2n} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} [-2n + 2n] = 0
\end{aligned}$$

$$74. \quad \frac{3}{2+x} = \frac{3/2}{1+(x/2)} = \frac{3/2}{1-(-x/2)} = \frac{a}{1-r}$$

$$\sum_{n=0}^{\infty} \frac{3}{2} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3x^n}{2^{n+1}}$$

$$76. \text{ Integral: } \sum_{n=0}^{\infty} \frac{(-1)^n 3x^{n+1}}{(n+1)2^{n+1}}$$

$$\begin{aligned}
78. \quad 8 - 2(x-3) + \frac{1}{2}(x-3)^2 - \frac{1}{8}(x-3)^3 + \cdots &= \sum_{n=0}^{\infty} 8 \left[\frac{-(x-3)}{4} \right]^n = \frac{8}{1 - [-(x-3)/4]} \\
&= \frac{32}{4 + (x-3)} = \frac{32}{1+x}, \quad -1 < x < 7
\end{aligned}$$

$$80. \quad f(x) = \cos x$$

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

$$\begin{aligned}
\cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(-\pi/4)[x + (\pi/4)]^n}{n!} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \left(x + \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!} \left(x + \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2 \cdot 3!} \left(x + \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{2 \cdot 4!} \left(x + \frac{\pi}{4}\right)^4 + \cdots \\
&= \frac{\sqrt{2}}{2} \left[1 + \left(x + \frac{\pi}{4}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{[n(n+1)]/2} [x + (\pi/4)]^{n+1}}{(n+1)!} \right]
\end{aligned}$$

$$82. \quad f(x) = \csc(x)$$

$$f'(x) = -\csc(x) \cot(x)$$

$$f''(x) = \csc^3(x) + \csc(x) \cot^2(x)$$

$$f'''(x) = -5 \csc^3(x) \cot(x) - \csc(x) \cot^3(x)$$

$$f^{(4)}(x) = 5 \csc^5(x) + 15 \csc^3(x) \cot^2(x) + \csc(x) \cot^4(x)$$

$$\csc(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)[x - (\pi/2)]^n}{n!} = 1 + \frac{1}{2!} \left(x - \frac{\pi}{2}\right)^2 + \frac{5}{4!} \left(x - \frac{\pi}{2}\right)^4 + \cdots$$

84. $f(x) = x^{1/2}$

$$f'(x) = \frac{1}{2}x^{-1/2}$$

$$f''(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-3/2}$$

$$f'''(x) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)x^{-5/2}$$

$$f^{(4)}(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)x^{-7/2}, \dots$$

$$\begin{aligned}\sqrt{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(4)(x-4)^n}{n!} \\ &= 2 + \frac{(x-4)}{2^2} - \frac{(x-4)^2}{2^5 2!} + \frac{1 \cdot 3(x-4)^3}{2^8 3!} - \frac{1 \cdot 3 \cdot 5(x-4)^4}{2^{11} 4!} + \dots \\ &= 2 + \frac{(x-4)}{2^2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)(x-4)^n}{2^{3n-1} n!}\end{aligned}$$

86. $h(x) = (1+x)^{-3}$

$$h'(x) = -3(1+x)^{-4}$$

$$h''(x) = 12(1+x)^{-5}$$

$$h'''(x) = -60(1+x)^{-6}$$

$$h^{(4)}(x) = 360(1+x)^{-7}$$

$$h^{(5)}(x) = -2520(1+x)^{-8}$$

$$\frac{1}{(1+x)^3} = 1 - 3x + \frac{12x^2}{2!} - \frac{60x^3}{3!} + \frac{360x^4}{4!} - \frac{2520x^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n(n+2)!x^n}{2n!} = \sum_{n=0}^{\infty} \frac{(-1)^n(n+2)(n+1)x}{2}$$

88. $\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \leq 2$

$$\begin{aligned}\ln\left(\frac{6}{5}\right) &= \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{(6/5)-1}{n}\right)^n \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5^n n} \approx 0.1823\end{aligned}$$

90. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$

$$e^{2/3} = \sum_{n=0}^{\infty} \frac{(2/3)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{3^n n!} \approx 1.9477$$

92. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$

$$\sin\left(\frac{1}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)!} \approx 0.3272$$

94. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots$

$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2!} + \dots$$

$$\int_0^1 xe^x dx = \left[xe^x - e^x \right]_0^1 = (e - e) - (0 - 1) = 1$$

$$\int_0^1 \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} dx = \sum_{n=0}^{\infty} \left[\frac{x^{n+2}}{(n+2)n!} \right]_0^1 = \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = 1$$

$$\begin{aligned}
96. \quad (a) \quad & f(x) = \sin 2x & f(0) &= 0 \\
& f'(x) = 2 \cos 2x & f'(0) &= 2 \\
& f''(x) = -4 \sin 2x & f''(0) &= 0 \\
& f'''(x) = -8 \cos 2x & f'''(0) &= -8 \\
& f^{(4)}(x) = 16 \sin 2x & f^{(4)}(0) &= 0 \\
& f^{(5)}(x) = 32 \cos 2x & f^{(5)}(0) &= 32 \\
& f^{(6)}(x) = -64 \sin 2x & f^{(6)}(0) &= 0 \\
& f^{(7)}(x) = -128 \cos 2x & f^{(7)}(0) &= -128
\end{aligned}$$

$$\sin 2x = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

$$(b) \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\begin{aligned}
\sin 2x &= \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots \\
&= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \frac{128x^7}{5040} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots
\end{aligned}$$

$$(c) \quad \sin 2x = 2 \sin x \cos x$$

$$\begin{aligned}
&= 2 \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \right) \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \right) \\
&= 2 \left[x + \left(-\frac{x^3}{2} - \frac{x^3}{6} \right) + \left(\frac{x^5}{24} + \frac{x^5}{12} + \frac{x^5}{120} \right) + \left(-\frac{x^7}{720} - \frac{x^7}{144} - \frac{x^7}{240} - \frac{x^7}{5040} \right) + \cdots \right] \\
&= 2 \left[x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \cdots \right] = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots
\end{aligned}$$

$$\begin{aligned}
98. \quad \cos t &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \\
\cos \frac{\sqrt{t}}{2} &= \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{2^{2n}(2n)!} \\
\int_0^x \cos \frac{\sqrt{t}}{2} dt &= \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{2^{2n}(2n)!(n+1)} \right]_0^x \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{2^{2n}(2n)!(n+1)}
\end{aligned}$$

$$\begin{aligned}
100. \quad e^t &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \\
e^t - 1 &= \sum_{n=1}^{\infty} \frac{t^n}{n!} \\
\frac{e^t - 1}{t} &= \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!} \\
\int_0^x \frac{e^t - 1}{t} dt &= \left[\sum_{n=1}^{\infty} \frac{t^n}{n \cdot n!} \right]_0^x = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}
\end{aligned}$$

$$102. \quad \arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$$

$$\frac{\arcsin x}{x} = 1 + \frac{x^2}{2 \cdot 3} + \frac{1 \cdot 3x^4}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^6}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$$

$$\lim_{x \rightarrow 0} \frac{\arcsin x}{x} = 1$$

$$\text{By L'Hôpital's Rule, } \lim_{x \rightarrow 0} \frac{\arcsin x}{x} = \lim_{x \rightarrow 0} \frac{\left(\frac{1}{\sqrt{1-x^2}} \right)}{1} = 1.$$

Review Exercises for Chapter 8

1. $a_n = \frac{1}{n!}$

3. $a_n = 4 + \frac{2}{n}$: 6, 5, 4.67, . . .

5. $a_n = 10(0.3)^{n-1}$: 10, 3, . . .

Matches (a)

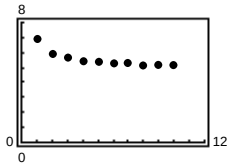
Matches (d)

7. $a_n = \frac{5n+2}{n}$

9. $\lim_{n \rightarrow \infty} \frac{n+1}{n^2} = 0$

11. $\lim_{n \rightarrow \infty} \frac{n^3}{n^2+1} = \infty$

Converges



The sequence seems to converge to 5.

$$\begin{aligned}\lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \frac{5n+2}{n} \\ &= \lim_{n \rightarrow \infty} \left(5 + \frac{2}{n} \right) = 5\end{aligned}$$

$$13. \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0 \quad \text{Converges}$$

15. $\lim_{n \rightarrow \infty} \frac{\sin(n)}{\sqrt{n}} = 0$

Converges

$$17. A_n = 5000 \left(1 + \frac{0.05}{4} \right)^n = 5000(1.0125)^n$$

$$n = 1, 2, 3$$

(a) $A_1 \approx 5062.50$ $A_5 \approx 5320.41$

$A_2 \approx 5125.78$ $A_6 \approx 5386.92$

$A_3 \approx 5189.85$ $A_7 \approx 5454.25$

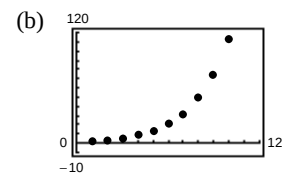
$A_4 \approx 5254.73$ $A_8 \approx 5522.43$

(b) $A_{40} \approx 8218.10$

19. (a)

k	5	10	15	20	25
S_k	13.2	113.3	873.8	6448.5	50,500.3

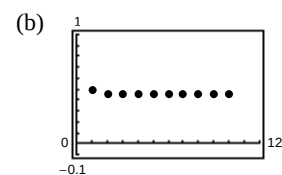
(c) The series diverges (geometric $r = \frac{3}{2} > 1$)



21. (a)

k	5	10	15	20	25
S_k	0.4597	0.4597	0.4597	0.4597	0.4597

(c) The series converges by the Alternating Series Test.



23. Converges. Geometric series, $r = 0.82$, $|r| < 1$.

25. Diverges. n th Term Test. $\lim_{n \rightarrow \infty} a_n \neq 0$.

$$27. \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$$

Geometric series with $a = 1$ and $r = \frac{2}{3}$.

$$S = \frac{a}{1-r} = \frac{1}{1-(2/3)} = \frac{1}{1/3} = 3$$

$$31. 0.\overline{09} = 0.09 + 0.0009 + 0.000009 + \cdots = 0.09(1 + 0.01 + 0.0001 + \cdots) = \sum_{n=0}^{\infty} (0.09)(0.01)^n = \frac{0.09}{1-0.01} = \frac{1}{11}$$

$$33. D_1 = 8$$

$$D_2 = 0.7(8) + 0.7(8) = 16(0.7)$$

$\vdots$

$$D = 8 + 16(0.7) + 16(0.7)^2 + \cdots + 16(0.7)^n + \cdots$$

$$= -8 + \sum_{n=0}^{\infty} 16(0.7)^n = -8 + \frac{16}{1-0.7} = 45\frac{1}{3} \text{ meters}$$

$$37. \int_1^{\infty} x^{-4} \ln(x) dx = \lim_{b \rightarrow \infty} \left[-\frac{\ln x}{3x^3} - \frac{1}{9x^3} \right]_1^b$$

$$= 0 + \frac{1}{9} = \frac{1}{9}$$

By the Integral Test, the series converges.

$$41. \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 2n}}$$

$$\lim_{n \rightarrow \infty} \frac{1/\sqrt{n^3 + 2n}}{1/(n^{3/2})} = \lim_{n \rightarrow \infty} \frac{n^{3/2}}{\sqrt{n^3 + 2n}} = 1$$

By a limit comparison test with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}, \text{ the series converges.}$$

45. Converges by the Alternating Series Test (Conditional convergence)

$$49. \sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n+1}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{e^{n^2}(n+1)}{e^{n^2+2n+1}n} \right|$$

$$= \lim_{n \rightarrow \infty} \left(\frac{1}{e^{2n+1}} \right) \left(\frac{n+1}{n} \right)$$

$$= (0)(1) = 0 < 1$$

By the Ratio Test, the series converges.

$$29. \sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n} \right) = \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{3} \right)^n$$

$$= \frac{1}{1-(1/2)} - \frac{1}{1-(1/3)} = 2 - \frac{3}{2} = \frac{1}{2}$$

35. See Exercise 86 in Section 8.2.

$$A = \frac{P(e^{rt} - 1)}{e^{r/12} - 1}$$

$$= \frac{200(e^{(0.06)(2)} - 1)}{e^{0.06/12} - 1}$$

$$\approx \$5087.14$$

$$39. \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n}$$

Since the second series is a divergent p -series while the first series is a convergent p -series, the difference diverges.

$$43. \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

$$= \left(\frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n-1}{2n-2} \right) \frac{1}{2n} > \frac{1}{2n}$$

Since $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series), so does the original series.

47. Diverges by the n th Term Test

$$51. \sum_{n=1}^{\infty} \frac{2^n}{n^3}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{2n^3}{(n+1)^3} = 2$$

Therefore, by the Ratio Test, the series diverges.

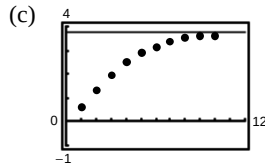
53. (a) Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)(3/5)^{n+1}}{n(3/5)^n}$

$$= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right) \left(\frac{3}{5} \right) = \frac{3}{5} < 1$$

Converges

(b)

x	5	10	15	20	25
S_n	2.8752	3.6366	3.7377	3.7488	3.7499



(d) The sum is approximately 3.75.

55. (a) $\int_N^\infty \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_N^\infty = \frac{1}{N}$

N	5	10	20	30	40
$\sum_{n=1}^N \frac{1}{n^2}$	1.4636	1.5498	1.5962	1.6122	1.6202
$\int_N^\infty \frac{1}{x^2} dx$	0.2000	0.1000	0.0500	0.0333	0.0250

(b) $\int_N^\infty \frac{1}{x^5} dx = \left[-\frac{1}{4x^4} \right]_N^\infty = \frac{1}{4N^4}$

N	5	10	20	30	40
$\sum_{n=1}^N \frac{1}{n^5}$	1.0367	1.0369	1.0369	1.0369	1.0369
$\int_N^\infty \frac{1}{x^5} dx$	0.0004	0.0000	0.0000	0.0000	0.0000

The series in part (b) converges more rapidly. The integral values represent the remainders of the partial sums.

57. $f(x) = e^{-x/2} \quad f(0) = 1$

$$f'(x) = -\frac{1}{2}e^{-x/2} \quad f'(0) = -\frac{1}{2}$$

$$f''(x) = \frac{1}{4}e^{-x/2} \quad f''(0) = \frac{1}{4}$$

$$f'''(x) = -\frac{1}{8}e^{-x/2} \quad f'''(0) = -\frac{1}{8}$$

$$P_3(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!}$$

$$= 1 - \frac{1}{2}x + \frac{1}{4}\frac{x^2}{2!} - \frac{1}{8}\frac{x^3}{3!}$$

$$= 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3$$

59. $\sin(95^\circ) = \sin\left(\frac{95\pi}{180}\right) \approx \frac{95\pi}{180} - \frac{(95\pi)^3}{180^3 3!} + \frac{(95\pi)^5}{180^5 5!} - \frac{(95\pi)^7}{180^7 7!} + \frac{(95\pi)^9}{180^9 9!} \approx 0.996$

61. $\ln(1.75) \approx (0.75) - \frac{(0.75)^2}{2} + \frac{(0.75)^3}{3} - \frac{(0.75)^4}{4} + \frac{(0.75)^5}{5} - \frac{(0.75)^6}{6} + \cdots + \frac{(0.75)^{15}}{15} \approx 0.560$

63. $f(x) = \cos x$, $c = 0$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$$

$$|f^{(n+1)}(z)| \leq 1 \Rightarrow R_n(x) \leq \frac{x^{n+1}}{(n+1)!}$$

(a) $R_n(x) \leq \frac{(0.5)^{n+1}}{(n+1)!} < 0.001$

This inequality is true for $n = 4$.

(c) $R_n(x) \leq \frac{(0.5)^{n+1}}{(n+1)!} < 0.0001$

This inequality is true for $n = 5$.

(b) $R_n(x) \leq \frac{(1)^{n+1}}{(n+1)!} < 0.001$

This inequality is true for $n = 6$.

(d) $R_n(x) \leq \frac{2^{n+1}}{(n+1)!} < 0.0001$

This inequality is true for $n = 10$.

65. $\sum_{n=0}^{\infty} \left(\frac{x}{10}\right)^n$

Geometric series which converges only if $|x/10| < 1$ or $-10 < x < 10$.

67. $\sum_{n=0}^{\infty} \frac{(-1)^n(x-2)^n}{(n+1)^2}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}(x-2)^{n+1}}{(n+2)^2} \cdot \frac{(n+1)^2}{(-1)^n(x-2)^n} \right|$$

$$= |x-2|$$

$$R = 1$$

Center: 2

Since the series converges when $x = 1$ and when $x = 3$, the interval of convergence is $1 \leq x \leq 3$.

69. $\sum_{n=0}^{\infty} n!(x-2)^n$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!(x-2)^{n+1}}{n!(x-2)^n} \right| = \infty$$

which implies that the series converges only at the center $x = 2$.

71. $y = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^n(n!)^2}$

$$y' = \sum_{n=1}^{\infty} \frac{(-1)^n(2n)x^{2n-1}}{4^n(n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)x^{2n+1}}{4^{n+1}[(n+1)!]^2}$$

$$y'' = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)(2n+1)x^{2n}}{4^{n+1}[(n+1)!]^2}$$

$$\begin{aligned} x^2 y'' + xy' + x^2 y &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)(2n+1)x^{2n+2}}{4^{n+1}[(n+1)!]^2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)x^{2n+2}}{4^{n+1}[(n+1)!]^2} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{4^n(n!)^2} \\ &= \sum_{n=0}^{\infty} \left[(-1)^{n+1} \frac{(2n+2)(2n+1)}{4^{n+1}[(n+1)!]^2} + \frac{(-1)^{n+1}(2n+2)}{4^{n+1}[(n+1)!]^2} + \frac{(-1)^n}{4^n(n!)^2} \right] x^{2n+2} \\ &= \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1}(2n+2)(2n+1+1)}{4^{n+1}[(n+1)!]^2} + (-1)^n \frac{1}{4^n(n!)^2} \right] x^{2n+2} \\ &= \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1}4(n+1)^2}{4^{n+1}[(n+1)!]^2} + (-1)^n \frac{1}{4^n(n!)^2} \right] x^{2n+2} \\ &= \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1}1}{4^n(n!)^2} + (-1)^n \frac{1}{4^n(n!)^2} \right] x^{2n+2} = 0 \end{aligned}$$

73. $\frac{2}{3-x} = \frac{2/3}{1-(x/3)} = \frac{a}{1-r}$

$$\sum_{n=0}^{\infty} \frac{2}{3} \left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{2x^n}{3^{n+1}}$$

75. Derivative: $\sum_{n=1}^{\infty} \frac{2nx^{n-1}}{3^{n+1}}$

$$77. 1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \cdots = \sum_{n=0}^{\infty} \left(\frac{2x}{3}\right)^n = \frac{1}{1 - (2x/3)} = \frac{3}{3 - 2x}, \quad -\frac{3}{2} < x < \frac{3}{2}$$

$$79. f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$

$$f''(x) = -\sin(x)$$

$$f'''(x) = -\cos(x), \dots$$

$$\begin{aligned} \sin(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x)[x - (3\pi/4)]^n}{n!} \\ &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{3\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!}\left(x - \frac{3\pi}{4}\right)^2 + \cdots = \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2}[x - (3\pi/4)]^n}{n!} \end{aligned}$$

$$81. 3^x = (e^{\ln 3})^x = e^{x \ln 3} \text{ and since } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ we have}$$

$$\begin{aligned} 3^x &= \sum_{n=0}^{\infty} \frac{(x \ln 3)^n}{n!} \\ &= 1 + x \ln 3 + \frac{x^2 \ln^2 3}{2!} + \frac{x^3 \ln^3 3}{3!} + \frac{x^4 \ln^4 3}{4!} + \cdots \end{aligned}$$

$$83. f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2}$$

$$f''(x) = \frac{2}{x^3}$$

$$f'''(x) = -\frac{6}{x^4}, \dots$$

$$\begin{aligned} \frac{1}{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)(x+1)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{-n!(x+1)^n}{n!} = -\sum_{n=0}^{\infty} (x+1)^n \end{aligned}$$

$$85. (1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots$$

$$\begin{aligned} (1+x)^{1/5} &= 1 + \frac{x}{5} + \frac{(1/5)(-4/5)x^2}{2!} + \frac{1/5(-4/5)(-9/5)x^3}{3!} + \cdots \\ &= 1 + \frac{1}{5}x - \frac{1 \cdot 4x^2}{5^2 2!} + \frac{1 \cdot 4 \cdot 9x^3}{5^3 3!} - \cdots \\ &= 1 + \frac{x}{5} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 4 \cdot 9 \cdot 14 \cdots (5n-6)x^n}{5^n n!} \\ &= 1 + \frac{x}{5} - \frac{2}{25}x^2 + \frac{6}{125}x^3 - \cdots \end{aligned}$$

$$87. \ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \leq 2$$

$$\begin{aligned} \ln\left(\frac{5}{4}\right) &= \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{5/4 - 1}{n}\right)^n \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{4^n n} \approx 0.2231 \end{aligned}$$

$$89. e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$$

$$e^{1/2} = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \approx 1.6487$$

$$91. \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty$$

$$\cos\left(\frac{2}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{3^{2n}(2n)!} \approx 0.7859$$

$$95. \quad (a) \quad f(x) = e^{2x} \quad f(0) = 1$$

$$f'(x) = 2e^{2x} \quad f'(0) = 2$$

$$f''(x) = 4e^{2x} \quad f''(0) = 4$$

$$f'''(x) = 8e^{2x} \quad f'''(0) = 8$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$$

$$(c) \quad e^{2x} = e^x \cdot e^x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right) \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)$$

$$= 1 + (x + x) + \left(x^2 + \frac{x^2}{2} + \frac{x^2}{2}\right) + \left(\frac{x^3}{6} + \frac{x^3}{6} + \frac{x^3}{2} + \frac{x^3}{2}\right) + \cdots = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$$

$$97. \quad \sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

$$\frac{\sin t}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}$$

$$\int_0^x \frac{\sin t}{t} dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)(2n+1)!} \right]_0^x$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$

$$101. \quad \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

$$\frac{\arctan x}{\sqrt{x}} = \sqrt{x} - \frac{x^{5/2}}{3} + \frac{x^{9/2}}{5} - \frac{x^{13/2}}{7} + \frac{x^{17/2}}{9} - \cdots$$

$$\lim_{x \rightarrow 0} \frac{\arctan x}{\sqrt{x}} = 0$$

By L'Hôpital's Rule, $\lim_{x \rightarrow 0} \frac{\arctan x}{\sqrt{x}} = \lim_{x \rightarrow 0} \frac{\left(\frac{1}{1+x^2}\right)}{\left(\frac{1}{2\sqrt{x}}\right)} = \lim_{x \rightarrow 0} \frac{2\sqrt{x}}{1+x^2} = 0.$

93. The series for Exercise 41 converges very slowly because the terms approach 0 at a slow rate.

$$(b) \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$$

$$99. \quad \frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n$$

$$\ln(1+t) = \int \frac{1}{1+t} dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n+1}$$

$$\frac{\ln(t+1)}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n+1}$$

$$\int_0^x \frac{\ln(t+1)}{t} dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{(n+1)^2} \right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)^2}$$

Problem Solving for Chapter 8

$$1. \quad (a) \quad 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + \cdots = \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^n = \frac{1/3}{1 - (2/3)} = 1$$

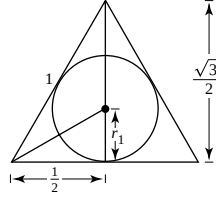
$$(b) \quad 0, \frac{1}{3}, \frac{2}{3}, 1, \text{ etc.}$$

$$(c) \quad \lim_{n \rightarrow \infty} C_n = 1 - \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^n = 1 - 1 = 0$$

3. If there are n rows, then $a_n = \frac{n(n+1)}{2}$.

For one circle,

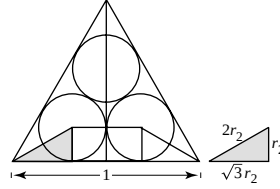
$$a_1 = 1 \text{ and } r_1 = \frac{1}{3} \left(\frac{\sqrt{3}}{2} \right) = \frac{\sqrt{3}}{6} = \frac{1}{2\sqrt{3}}$$



For three circles,

$$a_2 = 3 \text{ and } 1 = 2\sqrt{3}r_2 + 2r_2$$

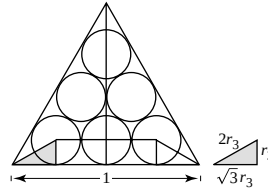
$$r_2 = \frac{1}{2 + 2\sqrt{3}}$$



For six circles,

$$a_3 = 6 \text{ and } 1 = 2\sqrt{3}r_3 + 4r_3$$

$$r_3 = \frac{1}{2\sqrt{3} + 4}$$



Continuing this pattern, $r_n = \frac{1}{2\sqrt{3} + 2(n-1)}$.

$$\text{Total Area} = (\pi r_n^2) a_n = \pi \left(\frac{1}{2\sqrt{3} + 2(n-1)} \right)^2 \frac{n(n+1)}{2}$$

$$A_n = \frac{\pi}{2} \frac{n(n+1)}{[2\sqrt{3} + 2(n+1)]^2}$$

$$\lim_{n \rightarrow \infty} A_n = \frac{\pi}{2} \cdot \frac{1}{4} = \frac{\pi}{8}$$

5. (a) $\sum a_n x^n = 1 + 2x + 3x^2 + x^3 + 2x^4 + 3x^5 + \dots$

$$= (1 + x^3 + x^6 + \dots) + 2(x + x^4 + x^7 + \dots) + 3(x^2 + x^5 + x^8 + \dots)$$

$$= (1 + x^3 + x^6 + \dots)[1 + 2x + 3x^2]$$

$$= (1 + 2x + 3x^2) \frac{1}{1 - x^3}$$

$R = 1$ because each series in the second line has $R = 1$.

- (b) $\sum a_n x^n = (a_0 + a_1 x + \dots + a_{p-1} x^{p-1}) + (a_0 x^p + a_1 x^{p+1} + \dots) + \dots$

$$= a_0(1 + x^p + \dots) + a_1 x(1 + x^p + \dots) + \dots + a_{p-1} x^{p-1}(1 + x^p + \dots)$$

$$= (a_0 + a_1 x + \dots + a_{p-1} x^{p-1})(1 + x^p + \dots)$$

$$= (a_0 + a_1 x + \dots + a_{p-1} x^{p-1}) \frac{1}{1 - x^p}$$

$R = 1$

$$7. \quad e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$xe^x = x + x^2 + \frac{x^3}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

$$\int xe^x dx = xe^x - e^x + C = \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!}$$

Letting $x = 0$, $C = 1$. Letting $x = 1$,

$$1 = \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(n+2)n!}.$$

$$\text{Thus, } \sum_{n=1}^{\infty} \frac{1}{(n+2)n!} = \frac{1}{2}.$$

$$9. \text{ Let } a_1 = \int_0^{\pi} \frac{\sin x}{x} dx, a_2 = -\int_{\pi}^{2\pi} \frac{\sin x}{x} dx, a_3 = \int_{2\pi}^{3\pi} \frac{\sin x}{x} dx, \text{ etc.}$$

Then,

$$\int_0^{\infty} \frac{\sin x}{x} dx = a_1 - a_2 + a_3 - a_4 + \cdots$$

Since $\lim_{n \rightarrow \infty} a_n = 0$ and $a_{n+1} < a_n$, this series converges.

$$11. (a) \quad a_1 = 3.0$$

$$a_2 \approx 1.73205$$

$$a_3 \approx 2.17533$$

$$a_4 \approx 2.27493$$

$$a_5 \approx 2.29672$$

$$a_6 \approx 2.30146$$

$$\lim_{n \rightarrow \infty} a_n = \frac{1 + \sqrt{13}}{2} \text{ [See part (b) for proof.]}$$

$$(b) \text{ Use mathematical induction to show the sequence is increasing. Clearly, } a_2 = \sqrt{a + a_1} = \sqrt{a\sqrt{a}} > \sqrt{a} = a_1.$$

Now assume $a_n > a_{n-1}$. Then

$$\begin{aligned} a_n + a &> a_{n-1} + a \\ \sqrt{a_n + a} &> \sqrt{a_{n-1} + a} \\ a_{n+1} &> a_n. \end{aligned}$$

Use mathematical induction to show that the sequence is bounded above by a . Clearly, $a_1 = \sqrt{a} < a$.

Now assume $a_n < a$. Then $a > a_n$ and $a - 1 > 1$ implies

$$\begin{aligned} a(a-1) &> a_n(1) \\ a^2 - a &> a_n \\ a^2 &> a_n + a \\ a &> \sqrt{a_n + a} = a_{n+1}. \end{aligned}$$

Hence, the sequence converges to some number L . To find L , assume $a_{n+1} \approx a_n \approx L$:

$$L = \sqrt{a + L} \Rightarrow L^2 = a + L \Rightarrow L^2 - L - a = 0$$

$$L = \frac{1 \pm \sqrt{1 + 4a}}{2}.$$

$$\text{Hence, } L = \frac{1 + \sqrt{1 + 4a}}{2}.$$

$$13. (a) \sum_{n=1}^{\infty} \frac{1}{2^{n+(-1)^n}} = \frac{1}{2^{1-1}} + \frac{1}{2^{2+1}} + \frac{1}{2^{3-1}} + \frac{1}{2^{4+1}} + \frac{1}{2^{5-1}} + \dots$$

$$S_1 = \frac{1}{2^0} = 1$$

$$S_2 = 1 + \frac{1}{8} = \frac{9}{8}$$

$$S_3 = \frac{9}{8} + \frac{1}{4} = \frac{11}{8}$$

$$S_4 = \frac{11}{8} + \frac{1}{32} = \frac{45}{32}$$

$$S_5 = \frac{45}{32} + \frac{1}{16} = \frac{47}{32}$$

$$(b) \frac{a_{n+1}}{a_n} = \frac{2^{n+(-1)^n}}{2^{(n+1)+(-1)^{n+1}}} = \frac{2^{(-1)^n}}{2^{1+(-1)^{n+1}}}$$

This sequence is $\frac{1}{8}, 2, \frac{1}{8}, 2, \dots$ which diverges.

$$(c) \sqrt[n]{\frac{1}{2^{n+(-1)^n}}} = \left(\frac{1}{2^n \cdot 2^{(-1)^n}} \right)^{1/n}$$

$$= \frac{1}{2 \cdot \sqrt[n]{2^{(-1)^n}}} \rightarrow \frac{1}{2} < 1 \text{ converges because } \{2^{(-1)^n}\} = \frac{1}{2}, 2, \frac{1}{2}, 2, \dots \text{ and } \sqrt[n]{1/2} \rightarrow 1 \text{ and } \sqrt[n]{2} \rightarrow 1.$$

$$15. S_6 = 130 + 70 + 40 = 240$$

$$S_7 = 240 + 130 + 70 = 440$$

$$S_8 = 440 + 240 + 130 = 810$$

$$S_9 = 810 + 440 + 240 = 1490$$

$$S_{10} = 1490 + 810 + 440 = 2740$$

66. $a_{2n+1} = 0$ (odd coefficients are zero)

68. Answers will vary.

70. $\theta = 60^\circ, v_0 = 64, k = \frac{1}{16}, g = -32$

$$\begin{aligned}
 y &= \sqrt{3}x - \frac{32x^2}{2(64)^2(1/2)^2} - \frac{(1/16)(32)x^3}{3(64)^3(1/2)^3} - \frac{(1/16)^2(32)x^4}{4(64)^4(1/2)^4} - \cdots \\
 &= \sqrt{3}x - 32 \left[\frac{2^2x^2}{2(64)^2} + \frac{2^3x^3}{3(64)^316} + \frac{2^4x^4}{4(64)^4(16)^2} + \cdots \right] \\
 &= \sqrt{3}x - 32 \sum_{n=2}^{\infty} \frac{2^n x^n}{n(64)^n(16)^{n-2}} = \sqrt{3}x - 32 \sum_{n=2}^{\infty} \frac{x^n}{n(32)^n(16)^{n-2}}
 \end{aligned}$$

72. (a) $f(x) = \frac{\ln(x^2 + 1)}{x^2}$.

From Exercise 8, you obtain

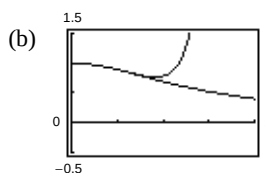
$$P = \frac{1}{x^2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n+1}$$

$$P_8 = 1 - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{4} + \frac{x^8}{5}$$

(c) $F(x) = \int_0^x \frac{\ln(t^2 + 1)}{t^2} dt$

$$G(x) = \int_0^x P_8(t) dt$$

x	0.25	0.50	0.75	1.00	1.50	2.00
$F(x)$	0.2475	0.4810	0.6920	0.8776	1.1798	1.4096
$G(x)$	0.2475	0.4810	0.6920	0.8805	5.3064	652.21

(d) The curves are nearly identical for $0 < x < 1$. Hence, the integrals nearly agree on that interval.74. Assume $e = p/q$ is rational. Let $N > q$ and form the following.

$$e - \left[1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{N!} \right] = \frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots$$

Set $a = N! \left[e - \left(1 + 1 + \cdots + \frac{1}{N!} \right) \right]$, a positive integer. But,

$$\begin{aligned}
 a &= N! \left[\frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots \right] = \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \cdots < \frac{1}{N+1} + \frac{1}{(N+1)^2} + \cdots \\
 &= \frac{1}{N+1} \left[1 + \frac{1}{N+1} + \frac{1}{(N+1)^2} + \cdots \right] = \frac{1}{N+1} \left[\frac{1}{1 - \left(\frac{1}{N+1} \right)} \right] = \frac{1}{N}, \text{ a contradiction.}
 \end{aligned}$$

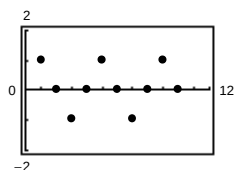
Review Exercises for Chapter 8

2. $a_n = \frac{n}{n^2 + 1}$

4. $a_n = 4 - \frac{n}{2}$: 3.5, 3, . . .
Matches (c)

6. $a_n = 6 \left(-\frac{2}{3} \right)^{n-1}$: 6, -4, . . .
Matches (b)

8. $a_n = \sin \frac{n\pi}{2}$



The sequence seems to diverge (oscillates).

$$\sin \frac{n\pi}{2}: 1, 0, -1, 0, 1, 0, \dots$$

10. $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$
Converges

12. $\lim_{n \rightarrow \infty} \frac{n}{\ln(n)} = \lim_{n \rightarrow \infty} \frac{1}{1/n} = \infty$
Diverges

14. $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2n}\right)^n = \lim_{k \rightarrow \infty} \left[\left(1 + \frac{1}{k}\right)^k\right]^{1/2} = e^{1/2}$
Converges; $k = 2n$

16. Let $y = (b^n + c^n)^{1/n}$

$$\ln y = \frac{\ln(b^n + c^n)}{n}$$

$$\lim_{n \rightarrow \infty} \ln y = \lim_{n \rightarrow \infty} \frac{1}{b^n + c^n} (b^n \ln b + c^n \ln c)$$

Assume $b \geq c$ and note that the terms

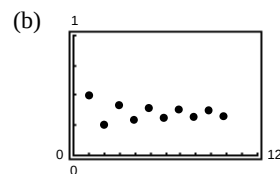
$$\frac{b^n \ln b + c^n \ln c}{b^n + c^n} = \frac{b^n \ln b}{b^n + c^n} + \frac{c^n \ln c}{b^n + c^n}$$

converge as $n \rightarrow \infty$. Hence a_n converges.

20. (a)

k	5	10	15	20	25
S_k	0.3917	0.3228	0.3627	0.3344	0.3564

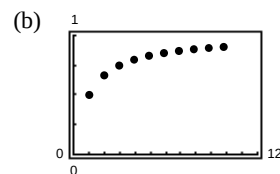
(c) The series converges by the Alternating Series Test.



22. (a)

k	5	10	15	20	25
S_k	0.8333	0.9091	0.9375	0.9524	0.9615

(c) The series converges, by the limit comparison test with $\sum \frac{1}{n^2}$.



24. Diverges. Geometric series, $r = 1.82 > 1$.

26. Diverges. n th Term Test, $\lim_{n \rightarrow \infty} a_n = \frac{2}{3}$.

28. $\sum_{n=0}^{\infty} \frac{2^{n+2}}{3^n} = 4 \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 4(3) = 12$

See Exercise 27.

30.
$$\sum_{n=0}^{\infty} \left[\left(\frac{2}{3}\right)^n - \frac{1}{(n+1)(n+2)} \right] = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$= \frac{1}{1 - (2/3)} - \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \right] = 3 - 1 = 2$$

$$32. \overline{0.923076} = 0.923076[1 + 0.000001 + (0.000001)^2 + \cdots]$$

$$= \sum_{n=0}^{\infty} (0.923076)(0.000001)^n = \frac{0.923076}{1 - 0.000001} = \frac{923,076}{999,999} = \frac{12(76,923)}{13(76,923)} = \frac{12}{13}$$

$$34. S = \sum_{n=0}^{39} 32,000(1.055)^n = \frac{32,000(1 - 1.055^{40})}{1 - 1.055}$$

$$\approx \$4,371,379.65$$

36. See Exercise 86 in Section 8.2.

$$A = P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]$$

$$= 100\left(\frac{12}{0.065}\right)\left[\left(1 + \frac{0.065}{12}\right)^{120} - 1\right]$$

$$\approx \$16,840.32$$

$$38. \sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$$

Divergent p -series, $p = \frac{3}{4} < 1$

$$40. \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{2^n}$$

The first series is a convergent p -series and the second series is a convergent geometric series. Therefore, their difference converges.

$$42. \sum_{n=1}^{\infty} \frac{n+1}{n(n+2)}$$

$$\lim_{n \rightarrow \infty} \frac{(n+1)/n(n+2)}{1/n} = \lim_{n \rightarrow \infty} \frac{n+1}{n+2} = 1$$

By a limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$, the series diverges.

44. Since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges, $\sum_{n=1}^{\infty} \frac{1}{3^n - 5}$ converges by the Limit Comparison Test.

$$46. \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$$

$$a_{n+1} = \frac{\sqrt{n+1}}{n+2} \leq \frac{\sqrt{n}}{n+1} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+1} = 0$$

By the Alternating Series Test, the series converges.

48. Converges by the Alternating Series Test.

$$a_{n+1} = \frac{3 \ln(n+1)}{n+1} < \frac{3 \ln n}{n} = a_n, \lim_{n \rightarrow \infty} \frac{3 \ln n}{n} = 0$$

$$50. \sum_{n=1}^{\infty} \frac{n!}{e^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{n+1}{e} = \infty$$

By the Ratio Test, the series diverges.

$$52. \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$$

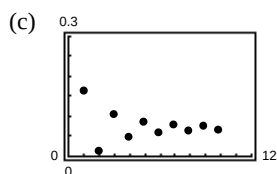
$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{1 \cdot 3 \cdots (2n-1)} \right| = \lim_{n \rightarrow \infty} \frac{2n+1}{3n+2} = \frac{2}{3}$$

By the Ratio Test, the series converges.

54. (a) The series converges by the Alternating Series Test.

(b)

x	5	10	15	20	25
S_n	0.0871	0.0669	0.0734	0.0702	0.0721



(d) The sum is approximately 0.0714.

56. No. Let $a_n = \frac{3937.5}{n^2}$, then $a_{75} = 0.7$. The series $\sum_{n=1}^{\infty} \frac{3937.5}{n^2}$ is a convergent p -series.

58. $f(x) = \tan x$ $f\left(-\frac{\pi}{4}\right) = -1$

$f'(x) = \sec^2 x$ $f'\left(-\frac{\pi}{4}\right) = 2$

$f''(x) = 2 \sec^2 x \tan x$ $f''\left(-\frac{\pi}{4}\right) = -4$

$f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x$ $f'''\left(-\frac{\pi}{4}\right) = 16$

$P_3(x) = -1 + 2\left(x + \frac{\pi}{4}\right) - 2\left(x + \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x + \frac{\pi}{4}\right)^3$

60. $\cos(0.75) \approx 1 - \frac{(0.75)^2}{2!} + \frac{(0.75)^4}{4!} - \frac{(0.75)^6}{6!} \approx 0.7317$

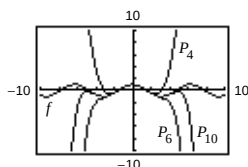
62. $e^{-0.25} \approx 1 - 0.25 + \frac{(0.25)^2}{2!} - \frac{(0.25)^3}{3!} + \frac{(0.25)^4}{4!} \approx 0.779$

64. $f(x) = \cos x$

$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

$P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

$P_{10}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$



66. $\sum_{n=0}^{\infty} (2x)^n$

Geometric series which converges only if $|2x| < 1$ or $-\frac{1}{2} < x < \frac{1}{2}$.

68. $\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}(x-2)^{n+1}}{n+1} \cdot \frac{n}{3^n(x-2)^n} \right|$$

$$= 3|x-2|$$

$R = \frac{1}{3}$

Center: 2

Since the series converges at $\frac{5}{3}$ and diverges at $\frac{7}{3}$, the interval of convergence is $\frac{5}{3} \leq x < \frac{7}{3}$.

70. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{x-2}{2}\right)^n$

Geometric series which converges only if

$$\left| \frac{x-2}{2} \right| < 1 \quad \text{or} \quad 0 < x < 4.$$

$$\begin{aligned}
72. \quad y &= \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{2^n n!} \\
y' &= \sum_{n=1}^{\infty} \frac{(-3)^n (2n) x^{2n-1}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2) x^{2n+1}}{2^{n+1} (n+1)!} \\
y'' &= \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2)(2n+1) x^{2n}}{2^{n+1} (n+1)!} \\
y'' + 3xy' + 3y &= \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2)(2n+1) x^{2n}}{2^{n+1} (n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} (2n+2) x^{2n+2}}{2^{n+1} (n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+1} (2n+2) x^{2n}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} [-(2n+1) + 1] + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} (-2n) + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^{n+1} x^{2n}}{2^n n!} (2n) + \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^{n-1} (n-1)!} \cdot \frac{2n}{2n} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} [-2n + 2n] = 0
\end{aligned}$$

$$74. \quad \frac{3}{2+x} = \frac{3/2}{1+(x/2)} = \frac{3/2}{1-(-x/2)} = \frac{a}{1-r}$$

$$\sum_{n=0}^{\infty} \frac{3}{2} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3x^n}{2^{n+1}}$$

$$76. \text{ Integral: } \sum_{n=0}^{\infty} \frac{(-1)^n 3x^{n+1}}{(n+1)2^{n+1}}$$

$$\begin{aligned}
78. \quad 8 - 2(x-3) + \frac{1}{2}(x-3)^2 - \frac{1}{8}(x-3)^3 + \cdots &= \sum_{n=0}^{\infty} 8 \left[\frac{-(x-3)}{4} \right]^n = \frac{8}{1 - [-(x-3)/4]} \\
&= \frac{32}{4 + (x-3)} = \frac{32}{1+x}, \quad -1 < x < 7
\end{aligned}$$

$$80. \quad f(x) = \cos x$$

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

$$\begin{aligned}
\cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(-\pi/4)[x + (\pi/4)]^n}{n!} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \left(x + \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!} \left(x + \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2 \cdot 3!} \left(x + \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{2 \cdot 4!} \left(x + \frac{\pi}{4}\right)^4 + \cdots \\
&= \frac{\sqrt{2}}{2} \left[1 + \left(x + \frac{\pi}{4}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{[n(n+1)]/2} [x + (\pi/4)]^{n+1}}{(n+1)!} \right]
\end{aligned}$$

$$82. \quad f(x) = \csc(x)$$

$$f'(x) = -\csc(x) \cot(x)$$

$$f''(x) = \csc^3(x) + \csc(x) \cot^2(x)$$

$$f'''(x) = -5 \csc^3(x) \cot(x) - \csc(x) \cot^3(x)$$

$$f^{(4)}(x) = 5 \csc^5(x) + 15 \csc^3(x) \cot^2(x) + \csc(x) \cot^4(x)$$

$$\csc(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)[x - (\pi/2)]^n}{n!} = 1 + \frac{1}{2!} \left(x - \frac{\pi}{2}\right)^2 + \frac{5}{4!} \left(x - \frac{\pi}{2}\right)^4 + \cdots$$

84. $f(x) = x^{1/2}$

$$f'(x) = \frac{1}{2}x^{-1/2}$$

$$f''(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-3/2}$$

$$f'''(x) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)x^{-5/2}$$

$$f^{(4)}(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)x^{-7/2}, \dots$$

$$\begin{aligned}\sqrt{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(4)(x-4)^n}{n!} \\ &= 2 + \frac{(x-4)}{2^2} - \frac{(x-4)^2}{2^5 2!} + \frac{1 \cdot 3(x-4)^3}{2^8 3!} - \frac{1 \cdot 3 \cdot 5(x-4)^4}{2^{11} 4!} + \dots \\ &= 2 + \frac{(x-4)}{2^2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)(x-4)^n}{2^{3n-1} n!}\end{aligned}$$

86. $h(x) = (1+x)^{-3}$

$$h'(x) = -3(1+x)^{-4}$$

$$h''(x) = 12(1+x)^{-5}$$

$$h'''(x) = -60(1+x)^{-6}$$

$$h^{(4)}(x) = 360(1+x)^{-7}$$

$$h^{(5)}(x) = -2520(1+x)^{-8}$$

$$\frac{1}{(1+x)^3} = 1 - 3x + \frac{12x^2}{2!} - \frac{60x^3}{3!} + \frac{360x^4}{4!} - \frac{2520x^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n(n+2)!x^n}{2n!} = \sum_{n=0}^{\infty} \frac{(-1)^n(n+2)(n+1)x}{2}$$

88. $\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \leq 2$

$$\begin{aligned}\ln\left(\frac{6}{5}\right) &= \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{(6/5)-1}{n}\right)^n \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5^n n} \approx 0.1823\end{aligned}$$

90. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$

$$e^{2/3} = \sum_{n=0}^{\infty} \frac{(2/3)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{3^n n!} \approx 1.9477$$

92. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$

$$\sin\left(\frac{1}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)!} \approx 0.3272$$

94. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots$

$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2!} + \dots$$

$$\int_0^1 xe^x dx = \left[xe^x - e^x \right]_0^1 = (e - e) - (0 - 1) = 1$$

$$\int_0^1 \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} dx = \sum_{n=0}^{\infty} \left[\frac{x^{n+2}}{(n+2)n!} \right]_0^1 = \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = 1$$

$$\begin{aligned}
96. \quad (a) \quad & f(x) = \sin 2x & f(0) &= 0 \\
& f'(x) = 2 \cos 2x & f'(0) &= 2 \\
& f''(x) = -4 \sin 2x & f''(0) &= 0 \\
& f'''(x) = -8 \cos 2x & f'''(0) &= -8 \\
& f^{(4)}(x) = 16 \sin 2x & f^{(4)}(0) &= 0 \\
& f^{(5)}(x) = 32 \cos 2x & f^{(5)}(0) &= 32 \\
& f^{(6)}(x) = -64 \sin 2x & f^{(6)}(0) &= 0 \\
& f^{(7)}(x) = -128 \cos 2x & f^{(7)}(0) &= -128
\end{aligned}$$

$$\sin 2x = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

$$(b) \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\begin{aligned}
\sin 2x &= \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots \\
&= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \frac{128x^7}{5040} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots
\end{aligned}$$

$$(c) \quad \sin 2x = 2 \sin x \cos x$$

$$\begin{aligned}
&= 2 \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \right) \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \right) \\
&= 2 \left[x + \left(-\frac{x^3}{2} - \frac{x^3}{6} \right) + \left(\frac{x^5}{24} + \frac{x^5}{12} + \frac{x^5}{120} \right) + \left(-\frac{x^7}{720} - \frac{x^7}{144} - \frac{x^7}{240} - \frac{x^7}{5040} \right) + \cdots \right] \\
&= 2 \left[x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \cdots \right] = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots
\end{aligned}$$

$$\begin{aligned}
98. \quad \cos t &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \\
\cos \frac{\sqrt{t}}{2} &= \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{2^{2n}(2n)!} \\
\int_0^x \cos \frac{\sqrt{t}}{2} dt &= \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{2^{2n}(2n)!(n+1)} \right]_0^x \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{2^{2n}(2n)!(n+1)}
\end{aligned}$$

$$\begin{aligned}
100. \quad e^t &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \\
e^t - 1 &= \sum_{n=1}^{\infty} \frac{t^n}{n!} \\
\frac{e^t - 1}{t} &= \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!} \\
\int_0^x \frac{e^t - 1}{t} dt &= \left[\sum_{n=1}^{\infty} \frac{t^n}{n \cdot n!} \right]_0^x = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}
\end{aligned}$$

$$102. \quad \arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$$

$$\frac{\arcsin x}{x} = 1 + \frac{x^2}{2 \cdot 3} + \frac{1 \cdot 3x^4}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^6}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$$

$$\lim_{x \rightarrow 0} \frac{\arcsin x}{x} = 1$$

$$\text{By L'Hôpital's Rule, } \lim_{x \rightarrow 0} \frac{\arcsin x}{x} = \lim_{x \rightarrow 0} \frac{\left(\frac{1}{\sqrt{1-x^2}} \right)}{1} = 1.$$

Problem Solving for Chapter 8

2. Let $S = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

Then $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$

$$= S + \frac{1}{2^2} + \frac{1}{4^2} + \dots$$

$$= S + \frac{1}{2^2} \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right]$$

$$= S + \frac{1}{2^2} \left(\frac{\pi^2}{6} \right).$$

Thus, $S = \frac{\pi^2}{6} - \frac{1}{4} \frac{\pi^2}{6} = \frac{\pi^2}{6} \left(\frac{3}{4} \right) = \frac{\pi^2}{8}.$

4. (a) Position the three blocks as indicated in the figure. The bottom block extends $1/6$ over the edge of the table, the middle block extends $1/4$ over the edge of the bottom block, and the top block extends $1/2$ over the edge of the middle block.

The centers of gravity are located at

bottom block: $\frac{1}{6} - \frac{1}{2} = -\frac{1}{3}$

middle block: $\frac{1}{6} + \frac{1}{4} - \frac{1}{2} = -\frac{1}{12}$

top block: $\frac{1}{6} + \frac{1}{4} + \frac{1}{2} - \frac{1}{2} = \frac{5}{12}.$

The center of gravity of the top 2 blocks is

$$\left(-\frac{1}{12} + \frac{5}{12} \right) / 2 = \frac{1}{6},$$

which lies over the bottom block. The center of gravity of the 3 blocks is

$$\left(-\frac{1}{3} - \frac{1}{12} + \frac{5}{12} \right) / 3 = 0$$

which lies over the table. Hence, the far edge of the top block lies

$$\frac{1}{6} + \frac{1}{4} + \frac{1}{2} = \frac{11}{12}$$

beyond the edge of the table.

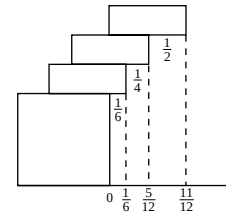
- (b) Yes. If there are n blocks, then the edge of the top block lies $\sum_{i=1}^n \frac{1}{2^i}$ from the edge of the table. Using 4 blocks,

$$\sum_{i=1}^4 \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \frac{25}{24}$$

which shows that the top block extends beyond the table.

- (c) The blocks can extend any distance beyond the table because the series diverges:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{i} = \infty.$$



$$6. a - \frac{b}{2} + \frac{a}{3} - \frac{b}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(a+b) + (a-b)}{2n}$$

$$\text{If } a = b, \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2a)}{2n} = a \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges conditionally.}$$

$$\text{If } a \neq b, \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(a+b)}{2n} + \sum_{n=1}^{\infty} \frac{a-b}{2n} \text{ diverges.}$$

No values of a and b give absolute convergence. $a = b$ implies conditional convergence.

$$8. e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \cdots + \frac{x^{12}}{6!} + \cdots$$

$$\frac{f^{(12)}(0)}{12!} = \frac{1}{6!} \Rightarrow f^{(12)}(0) = \frac{12!}{6!} = 665,280$$

$$10. (a) \text{ If } p = 1, \int_2^{\infty} \frac{1}{x \ln x} dx = \ln \ln x \Big|_2^{\infty} \text{ diverges.}$$

$$\text{If } p > 1, \int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \lim_{b \rightarrow \infty} \left[\frac{(\ln b)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} \right] \text{ converges.}$$

If $p < 1$, diverges.

$$(b) \sum_{n=4}^{\infty} \frac{1}{n \ln(n^2)} = \frac{1}{2} \sum_{n=4}^{\infty} \frac{1}{n \ln n} \text{ diverges by part (a).}$$

$$12. \text{ Let } b_n = a_n r^n.$$

$$(bn)^{1/n} = (a_n r^n)^{1/n} = a_n^{1/n} \cdot r \rightarrow Lr \text{ as } n \rightarrow \infty.$$

$$Lr < \frac{1}{r} = 1.$$

By the Root Test, $\sum b_n$ converges $\Rightarrow \sum a_n r^n$ converges.

$$14. (a) \frac{1}{0.99} = \frac{1}{1-0.01} = \sum_{n=0}^{\infty} (0.01)^n \\ = 1 + 0.01 + (0.01)^2 + \cdots \\ = 1.010101 \dots$$

$$(b) \frac{1}{0.98} = \frac{1}{1-0.02} = \sum_{n=0}^{\infty} (0.02)^n \\ = 1 + 0.02 + (0.02)^2 + \cdots \\ = 1 + 0.02 + 0.0004 + \cdots \\ = 1.0204081632 \dots$$

$$16. (a) \text{ Height} = 2 \left[1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots \right] \\ = 2 \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \infty \left(p\text{-series, } p = \frac{1}{2} < 1 \right)$$

$$(b) S = 4\pi \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots \right] 4\pi \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

$$(c) W = \frac{4}{3}\pi \left[1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \cdots \right] \\ = \frac{4}{3}\pi \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ converges.}$$

CHAPTER 9

Conics, Parametric Equations, and Polar Coordinates

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CHAPTER 9

Conics, Parametric Equations, and Polar Coordinates

Section 9.1 Conics and Calculus

Solutions to Odd-Numbered Exercises

1. $y^2 = 4x$

Vertex: $(0, 0)$

$p = 1 > 0$

Opens to the right

Matches graph (h).

5. $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Center: $(0, 0)$

Ellipse

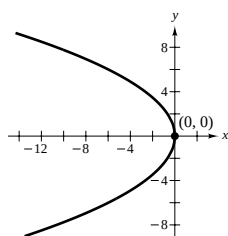
Matches (f)

9. $y^2 = -6x = 4\left(-\frac{3}{2}\right)x$

Vertex: $(0, 0)$

Focus: $\left(-\frac{3}{2}, 0\right)$

Directrix: $x = \frac{3}{2}$



3. $(x + 3)^2 = -2(y - 2)$

Vertex: $(-3, 2)$

$p = -\frac{1}{2} < 0$

Opens downward

Matches graph (e).

7. $\frac{y^2}{16} - \frac{x^2}{1} = 1$

Hyperbola

Center: $(0, 0)$

Vertical transverse axis.

Matches (c)

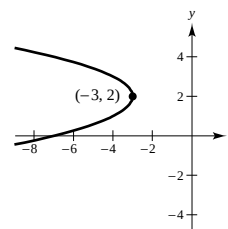
11. $(x + 3) + (y - 2)^2 = 0$

$(y - 2)^2 = 4\left(-\frac{1}{4}\right)(x + 3)$

Vertex: $(-3, 2)$

Focus: $(-3.25, 2)$

Directrix: $x = -2.75$



13. $y^2 - 4y - 4x = 0$

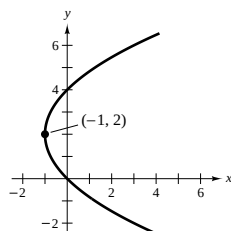
$y^2 - 4y + 4 = 4x + 4$

$(y - 2)^2 = 4(1)(x + 1)$

Vertex: $(-1, 2)$

Focus: $(0, 2)$

Directrix: $x = -2$



15. $x^2 + 4x + 4y - 4 = 0$

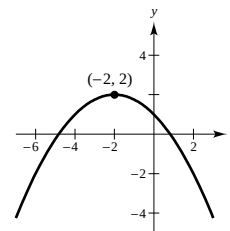
$x^2 + 4x + 4 = -4y + 4 + 4$

$(x + 2)^2 = 4(-1)(y - 2)$

Vertex: $(-2, 2)$

Focus: $(-2, 1)$

Directrix: $y = 3$



17. $y^2 + x + y = 0$

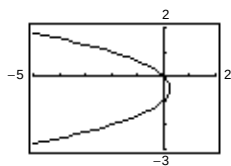
$$y^2 + y + \frac{1}{4} = -x + \frac{1}{4}$$

$$\left(y + \frac{1}{2}\right)^2 = 4\left(-\frac{1}{4}\right)\left(x - \frac{1}{4}\right)$$

Vertex: $\left(\frac{1}{4}, -\frac{1}{2}\right)$

Focus: $\left(0, -\frac{1}{2}\right)$

Directrix: $x = \frac{1}{2}$



19. $y^2 - 4x - 4 = 0$

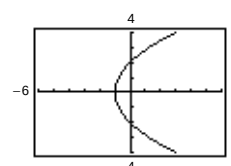
$$y^2 = 4x + 4$$

$$= 4(1)(x + 1)$$

Vertex: $(-1, 0)$

Focus: $(0, 0)$

Directrix: $x = -2$



21. $(y - 2)^2 = 4(-2)(x - 3)$

$$y^2 - 4y + 8x - 20 = 0$$

23. $(x - h)^2 = 4p(y - k)$

$$x^2 = 4(6)(y - 4)$$

$$x^2 - 24y + 96 = 0$$

25. $y = 4 - x^2$

$$x^2 + y - 4 = 0$$

27. Since the axis of the parabola is vertical, the form of the equation is $y = ax^2 + bx + c$. Now, substituting the values of the given coordinates into this equation, we obtain

$$3 = c, 4 = 9a + 3b + c, 11 = 16a + 4b + c.$$

Solving this system, we have $a = \frac{5}{3}$, $b = -\frac{14}{3}$, $c = 3$. Therefore,

$$y = \frac{5}{3}x^2 - \frac{14}{3}x + 3 \text{ or } 5x^2 - 14x - 3y + 9 = 0.$$

29. $x^2 + 4y^2 = 4$

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

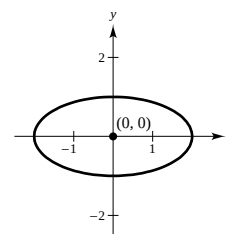
$$a^2 = 4, b^2 = 1, c^2 = 3$$

Center: $(0, 0)$

Foci: $(\pm\sqrt{3}, 0)$

Vertices: $(\pm 2, 0)$

$$e = \frac{\sqrt{3}}{2}$$



31. $\frac{(x - 1)^2}{9} + \frac{(y - 5)^2}{25} = 1$

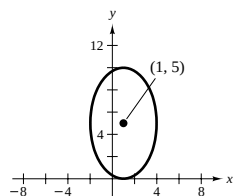
$$a^2 = 25, b^2 = 9, c^2 = 16$$

Center: $(1, 5)$

Foci: $(1, 9), (1, 1)$

Vertices: $(1, 10), (1, 0)$

$$e = \frac{4}{5}$$



33. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

$$9(x^2 + 4x + 4) + 4(y^2 - 6y + 9) = -36 + 36 + 36 = 36$$

$$\frac{(x + 2)^2}{4} + \frac{(y - 3)^2}{9} = 1$$

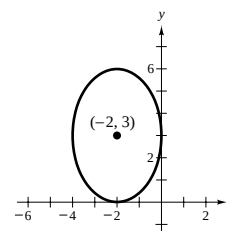
$$a^2 = 9, b^2 = 4, c^2 = 5$$

Center: $(-2, 3)$

Foci: $(-2, 3 \pm \sqrt{5})$

Vertices: $(-2, 6), (-2, 0)$

$$e = \frac{\sqrt{5}}{3}$$



35. $12x^2 + 20y^2 - 12x + 40y - 37 = 0$

$$12\left(x^2 - x + \frac{1}{4}\right) + 20(y^2 + 2y + 1) = 37 + 3 + 20$$

$$= 60$$

$$\frac{[x - (1/2)]^2}{5} + \frac{(y + 1)^2}{3} = 1$$

$$a^2 = 5, b^2 = 3, c^2 = 2$$

$$\text{Center: } \left(\frac{1}{2}, -1\right)$$

$$\text{Foci: } \left(\frac{1}{2} \pm \sqrt{2}, -1\right)$$

$$\text{Vertices: } \left(\frac{1}{2} \pm \sqrt{5}, -1\right)$$

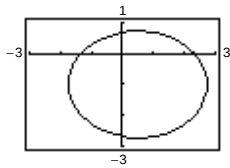
Solve for y:

$$20(y^2 + 2y + 1) = -12x^2 + 12x + 37 + 20$$

$$(y + 1)^2 = \frac{57 + 12x - 12x^2}{20}$$

$$y = -1 \pm \sqrt{\frac{57 + 12x - 12x^2}{20}}$$

(Graph each of these separately.)



39. Center: (0, 0)

Focus: (2, 0)

Vertex: (3, 0)

Horizontal major axis

$$a = 3, c = 2 \Rightarrow b = \sqrt{5}$$

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

43. Center: (0, 0)

Horizontal major axis

Points on ellipse: (3, 1), (4, 0)

Since the major axis is horizontal,

$$\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = 1.$$

Substituting the values of the coordinates of the given points into this equation, we have

$$\left(\frac{9}{a^2}\right) + \left(\frac{1}{b^2}\right) = 1, \text{ and } \frac{16}{a^2} = 1.$$

The solution to this system is $a^2 = 16, b^2 = 16/7$.

Therefore,

$$\frac{x^2}{16} + \frac{y^2}{16/7} = 1, \frac{x^2}{16} + \frac{7y^2}{16} = 1.$$

37. $x^2 + 2y^2 - 3x + 4y + 0.25 = 0$

$$\left(x^2 - 3x + \frac{9}{4}\right) + 2(y^2 + 2y + 1) = -\frac{1}{4} + \frac{9}{4} + 2 = 4$$

$$\frac{[x - (3/2)]^2}{4} + \frac{(y + 1)^2}{2} = 1$$

$$a^2 = 4, b^2 = 2, c^2 = 2$$

$$\text{Center: } \left(\frac{3}{2}, -1\right)$$

$$\text{Foci: } \left(\frac{3}{2} \pm \sqrt{2}, -1\right)$$

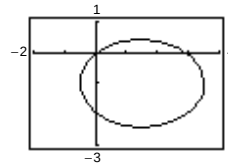
$$\text{Vertices: } \left(-\frac{1}{2}, -1\right), \left(\frac{7}{2}, -1\right)$$

$$\text{Solve for y: } 2(y^2 + 2y + 1) = -x^2 + 3x - \frac{1}{4} + 2$$

$$(y + 1)^2 = \frac{1}{2}\left(\frac{7}{4} + 3x - x^2\right)$$

$$y = -1 \pm \sqrt{\frac{7 + 12x - 4x^2}{8}}$$

(Graph each of these separately.)



41. Vertices: (3, 1), (3, 9)

Minor axis length: 6

Vertical major axis

Center: (3, 5)

$$a = 4, b = 3$$

$$\frac{(x - 3)^2}{9} + \frac{(y - 5)^2}{16} = 1$$

45. $\frac{y^2}{1} - \frac{x^2}{4} = 1$

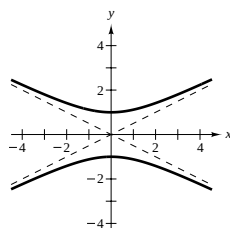
$a = 1, b = 2, c = \sqrt{5}$

Center: (0, 0)

Vertices: (0, ± 1)

Foci: (0, $\pm \sqrt{5}$)

Asymptotes: $y = \pm \frac{1}{2}x$



47. $\frac{(x-1)^2}{4} - \frac{(y+2)^2}{1} = 1$

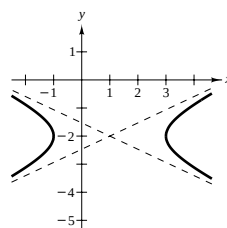
$a = 2, b = 1, c = \sqrt{5}$

Center: (1, -2)

Vertices: (-1, -2), (3, -2)

Foci: (1 $\pm \sqrt{5}$, -2)

Asymptotes: $y = -2 \pm \frac{1}{2}(x-1)$



49. $9x^2 - y^2 - 36x - 6y + 18 = 0$

$9(x^2 - 4x + 4) - (y^2 + 6y + 9) = -18 + 36 - 9$

$\frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1$

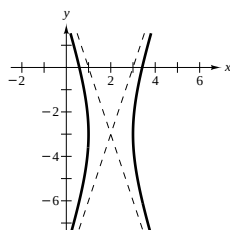
$a = 1, b = 3, c = \sqrt{10}$

Center: (2, -3)

Vertices: (1, -3), (3, -3)

Foci: (2 $\pm \sqrt{10}$, -3)

Asymptotes: $y = -3 \pm 3(x-2)$



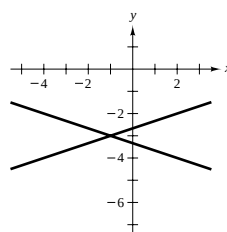
51. $x^2 - 9y^2 + 2x - 54y - 80 = 0$

$(x^2 + 2x + 1) - 9(y^2 + 6y + 9) = 80 + 1 - 81 = 0$

$(x+1)^2 - 9(y+3)^2 = 0$

$y+3 = \pm \frac{1}{3}(x+1)$

Degenerate hyperbola is two lines intersecting at (-1, -3).



53. $9y^2 - x^2 + 2x + 54y + 62 = 0$

$9(y^2 + 6y + 9) - (x^2 - 2x + 1) = -62 - 1 + 81 = 18$

$\frac{(y+3)^2}{2} - \frac{(x-1)^2}{18} = 1$

$a = \sqrt{2}, b = 3\sqrt{2}, c = 2\sqrt{5}$

Center: (1, -3)

Vertices: (1, -3 $\pm \sqrt{2}$)

Foci: (1, -3 $\pm 2\sqrt{5}$)

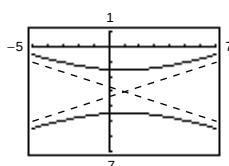
Solve for y:

$9(y^2 + 6y + 9) = x^2 - 2x - 62 + 81$

$(y+3)^2 = \frac{x^2 - 2x + 19}{9}$

$y = -3 \pm \frac{1}{3}\sqrt{x^2 - 2x + 19}$

(Graph each curve separately.)



55. $3x^2 - 2y^2 - 6x - 12y - 27 = 0$

$3(x^2 - 2x + 1) - 2(y^2 + 6y + 9) = 27 + 3 - 18 = 12$

$\frac{(x-1)^2}{4} - \frac{(y+3)^2}{6} = 1$

$a = 2, b = \sqrt{6}, c = \sqrt{10}$

Center: (1, -3)

Vertices: (-1, -3), (3, -3)

Foci: (1 $\pm \sqrt{10}$, -3)

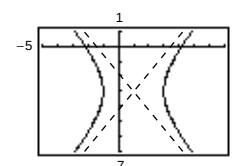
Solve for y:

$2(y^2 + 6y + 9) = 3x^2 - 6x - 27 + 18$

$(y+3)^2 = \frac{3x^2 - 6x - 9}{2}$

$y = -3 \pm \sqrt{\frac{3(x^2 - 2x - 3)}{2}}$

(Graph each curve separately.)



57. Vertices:
- $(\pm 1, 0)$

Asymptotes: $y = \pm 3x$

Horizontal transverse axis

Center: $(0, 0)$

$$a = 1, \pm \frac{b}{a} = \pm \frac{b}{1} = \pm 3 \Rightarrow b = 3$$

$$\text{Therefore, } \frac{x^2}{1} - \frac{y^2}{9} = 1.$$

61. Center:
- $(0, 0)$

Vertex: $(0, 2)$ Focus: $(0, 4)$

Vertical transverse axis

$$a = 2, c = 4, b^2 = c^2 - a^2 = 12$$

$$\text{Therefore, } \frac{y^2}{4} - \frac{x^2}{12} = 1.$$

65. (a)
- $\frac{x^2}{9} - y^2 = 1, \frac{2x}{9} - 2yy' = 0, \frac{x}{9y} = y'$

$$\text{At } x = 6: y = \pm \sqrt{3}, y' = \frac{\pm 6}{9\sqrt{3}} = \frac{\pm 2\sqrt{3}}{9}$$

$$\text{At } (6, \sqrt{3}): y - \sqrt{3} = \frac{2\sqrt{3}}{9}(x - 6)$$

$$\text{or } 2x - 3\sqrt{3}y - 3 = 0$$

$$\text{At } (6, -\sqrt{3}): y + \sqrt{3} = \frac{-2\sqrt{3}}{9}(x - 6)$$

$$\text{or } 2x + 3\sqrt{3}y - 3 = 0$$

- 67.
- $x^2 + 4y^2 - 6x + 16y + 21 = 0$

$$A = 1, C = 4$$

$$AC = 4 > 0$$

Ellipse

- 69.
- $y^2 - 4y - 4x = 0$

$$A = 0, C = 1$$

Parabola

- 71.
- $4x^2 + 4y^2 - 16y + 15 = 0$

$$A = C = 4$$

Circle

- 73.
- $9x^2 + 9y^2 - 36x + 6y + 34 = 0$

$$A = C = 9$$

Circle

59. Vertices:
- $(2, \pm 3)$

Point on graph: $(0, 5)$

Vertical transverse axis

Center: $(2, 0)$

$$a = 3$$

Therefore, the equation is of the form

$$\frac{y^2}{9} - \frac{(x - 2)^2}{b^2} = 1.$$

Substituting the coordinates of the point $(0, 5)$, we have

$$\frac{25}{9} - \frac{4}{b^2} = 1 \quad \text{or} \quad b^2 = \frac{9}{4}.$$

$$\text{Therefore, the equation is } \frac{y^2}{9} - \frac{(x - 2)^2}{9/4} = 1.$$

63. Vertices:
- $(0, 2), (6, 2)$

$$\text{Asymptotes: } y = \frac{2}{3}x, y = 4 - \frac{2}{3}x$$

Horizontal transverse axis

Center: $(3, 2)$

$$a = 3$$

$$\text{Slopes of asymptotes: } \pm \frac{b}{a} = \pm \frac{2}{3}$$

Thus, $b = 2$. Therefore,

$$\frac{(x - 3)^2}{9} - \frac{(y - 2)^2}{4} = 1.$$

- (b) From part (a) we know that the slopes of the normal lines must be
- $\mp 9/(2\sqrt{3})$
- .

$$\text{At } (6, \sqrt{3}): y - \sqrt{3} = -\frac{9}{2\sqrt{3}}(x - 6)$$

$$\text{or } 9x + 2\sqrt{3}y - 60 = 0$$

$$\text{At } (6, -\sqrt{3}): y + \sqrt{3} = \frac{9}{2\sqrt{3}}(x - 6)$$

$$\text{or } 9x - 2\sqrt{3}y - 60 = 0$$

- 75.
- $3x^2 - 6x + 3 = 6 + 2y^2 + 4y + 2$

$$3x^2 - 2y^2 - 6x - 4y - 5 = 0$$

$$A = 3, C = -2, AC < 0$$

Hyperbola

77. (a) A parabola is the set of all points (x, y) that are equidistant from a fixed line (directrix) and a fixed point (focus) not on the line.

(b) $(x - h)^2 = 4p(y - k)$ or $(y - k)^2 = 4p(x - h)$

(c) See Theorem 9.2.

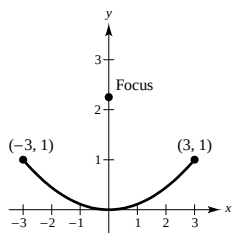
81. Assume that the vertex is at the origin.

$$x^2 = 4py$$

$$(3)^2 = 4p(1)$$

$$\frac{9}{4} = p$$

The pipe is located $\frac{9}{4}$ meters from the vertex.



79. (a) A hyperbola is the set of all points (x, y) for which the absolute value of the difference between the distances from two distance fixed points (foci) is constant.

(b) $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$ or $\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$

(c) $y = k \pm \frac{b}{a}(x - h)$ or $y = k \pm \frac{a}{b}(x - h)$

83. $y = ax^2$

$$y' = 2ax$$

The equation of the tangent line is

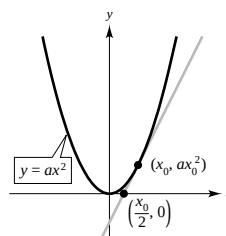
$$y - ax_0^2 = 2ax_0(x - x_0) \text{ or } y = 2ax_0x - ax_0^2.$$

Let $y = 0$. Then:

$$-ax_0^2 = 2ax_0x - 2ax_0^2$$

$$ax_0^2 = 2ax_0x$$

Therefore, $\frac{x_0}{2} = x$ is the x -intercept.



85. (a) Consider the parabola $x^2 = 4py$. Let m_0 be the slope of the one tangent line at (x_1, y_1) and therefore, $-1/m_0$ is the slope of the second at (x_2, y_2) . From the derivative given in Exercise 32 we have:

$$m_0 = \frac{1}{2p}x_1 \text{ or } x_1 = 2pm_0$$

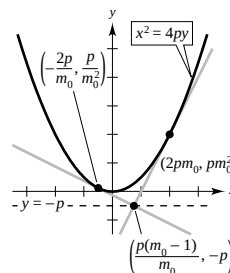
$$\frac{-1}{m_0} = \frac{1}{2p}x_2 \text{ or } x_2 = \frac{-2p}{m_0}$$

Substituting these values of x into the equation $x^2 = 4py$, we have the coordinates of the points of tangency $(2pm_0, pm_0^2)$ and $(-2p/m_0, p/m_0^2)$ and the equations of the tangent lines are

$$(y - pm_0^2) = m_0(x - 2pm_0) \text{ and } \left(y - \frac{p}{m_0^2}\right) = \frac{-1}{m_0}\left(x + \frac{2p}{m_0}\right).$$

The point of intersection of these lines is

$$\left(\frac{p(m_0^2 - 1)}{m_0}, -p\right) \text{ and is on the directrix, } y = -p.$$



—CONTINUED—

85. —CONTINUED—

$$(b) \ x^2 - 4x - 4y + 8 = 0$$

$$(x - 2)^2 = 4(y - 1). \text{ Vertex } (2, 1)$$

$$2x - 4 - 4 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{1}{2}x - 1$$

At $(-2, 5)$, $dy/dx = -2$. At $(3, \frac{5}{4})$, $dy/dx = \frac{1}{2}$.

Tangent line at $(-2, 5)$: $y - 5 = -2(x + 2) \Rightarrow 2x + y - 1 = 0$.

Tangent line at $(3, \frac{5}{4})$: $y - \frac{5}{4} = \frac{1}{2}(x - 3) \Rightarrow 2x - 4y - 1 = 0$.

Since $m_1 m_2 = (-2)(\frac{1}{2}) = -1$, the lines are perpendicular.

$$\text{Point of intersection: } -2x + 1 = \frac{1}{2}x - \frac{1}{4}$$

$$-\frac{5}{2}x = -\frac{5}{4}$$

$$x = \frac{1}{2}$$

$$y = 0$$

Directrix: $y = 0$ and the point of intersection $(\frac{1}{2}, 0)$ lies on this line.

87. $y = x - x^2$

$$\frac{dy}{dx} = 1 - 2x$$

At (x_1, y_1) on the mountain, $m = 1 - 2x_1$. Also, $m = \frac{y_1 - 1}{x_1 + 1}$.

$$\frac{y_1 - 1}{x_1 + 1} = 1 - 2x_1$$

$$(x_1 - x_1^2) - 1 = (1 - 2x_1)(x_1 + 1)$$

$$-x_1^2 + x_1 - 1 = -2x_1^2 - x_1 + 1$$

$$x_1^2 + 2x_1 - 2 = 0$$

$$x_1 = \frac{-2 \pm \sqrt{2^2 - 4(1)(-2)}}{2(1)} = \frac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}$$

Choosing the positive value for x_1 , we have $x_1 = -1 + \sqrt{3}$.

$$m = 1 - 2(-1 + \sqrt{3}) = 3 - 2\sqrt{3}$$

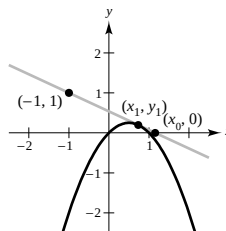
$$m = \frac{0 - 1}{x_0 + 1} = -\frac{1}{x_0 + 1}$$

$$\text{Thus, } -\frac{1}{x_0 + 1} = 3 - 2\sqrt{3}$$

$$\frac{-1}{3 - 2\sqrt{3}} = x_0 + 1$$

$$\frac{3 + 2\sqrt{3}}{3} - 1 = x_0$$

$$\frac{2\sqrt{3}}{3} = x_0$$



The closest the receiver can be to the hill is $(2\sqrt{3}/3) - 1 \approx 0.155$.

89. Parabola

Vertex: (0, 4)

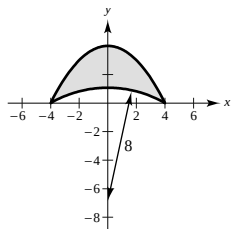
$$x^2 = 4p(y - 4)$$

$$4^2 = 4p(0 - 4)$$

$$p = -1$$

$$x^2 = -4(y - 4)$$

$$y = 4 - \frac{x^2}{4}$$



Circle

Center: (0, k)

Radius: 8

$$x^2 + (y - k)^2 = 64$$

$$4^2 + (0 - k)^2 = 64$$

$$k^2 = 48$$

$$k = -4\sqrt{3} \text{ (Center is on the negative y-axis.)}$$

$$x^2 + (y + 4\sqrt{3})^2 = 64$$

$$y = -4\sqrt{3} \pm \sqrt{64 - x^2}$$

Since the y-value is positive when $x = 0$, we have $y = -4\sqrt{3} + \sqrt{64 - x^2}$.

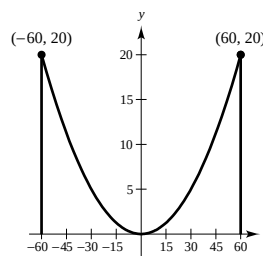
$$\begin{aligned} A &= 2 \int_0^4 \left[\left(4 - \frac{x^2}{4} \right) - \left(-4\sqrt{3} + \sqrt{64 - x^2} \right) \right] dx \\ &= 2 \left[4x - \frac{x^3}{12} + 4\sqrt{3}x - \frac{1}{2} \left(x\sqrt{64 - x^2} + 64 \arcsin \frac{x}{8} \right) \right]_0^4 \\ &= 2 \left[16 - \frac{64}{12} + 16\sqrt{3} - 2\sqrt{48} - 32 \arcsin \frac{1}{2} \right] \\ &= \frac{16(4 + 3\sqrt{3} - 2\pi)}{3} \approx 15.536 \text{ square feet} \end{aligned}$$

91. (a) Assume that $y = ax^2$.

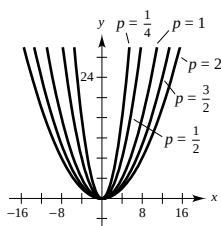
$$20 = a(60)^2 \Rightarrow a = \frac{2}{360} = \frac{1}{180} \Rightarrow y = \frac{1}{180}x^2$$

$$(b) f(x) = \frac{1}{180}x^2, f'(x) = \frac{1}{90}x$$

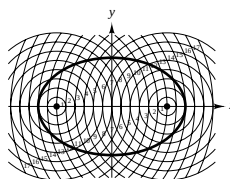
$$\begin{aligned} S &= 2 \int_0^{60} \sqrt{1 + \left(\frac{1}{90}x \right)^2} dx = \frac{2}{90} \int_0^{60} \sqrt{90^2 + x^2} dx \\ &= \frac{2}{90} \cdot \frac{1}{2} \left[x\sqrt{90^2 + x^2} + 90^2 \ln|x + \sqrt{90^2 + x^2}| \right]_0^{60} \quad (\text{formula 26}) \\ &= \frac{1}{90} [60\sqrt{11,700} + 90^2 \ln(60 + \sqrt{11,700}) - 90^2 \ln 90] \\ &= \frac{1}{90} [1800\sqrt{13} + 90^2 \ln(60 + 30\sqrt{13}) - 90^2 \ln 90] \\ &= 20\sqrt{13} + 90 \ln \left(\frac{60 + 30\sqrt{13}}{90} \right) \\ &= 10 \left[2\sqrt{13} + 9 \ln \left(\frac{2 + \sqrt{13}}{3} \right) \right] \approx 128.4 \text{ m} \end{aligned}$$

93. $x^2 = 4py$, $p = \frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}, 2$

As p increases, the graph becomes wider.



95.



$$97. a = \frac{5}{2}, b = 2, c = \sqrt{\left(\frac{5}{2}\right)^2 - (2)^2} = \frac{3}{2}$$

The tacks should be placed 1.5 feet from the center. The string should be $2a = 5$ feet long.

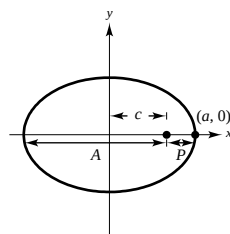
$$99. e = \frac{c}{a}$$

$$A + P = 2a$$

$$a = \frac{A + P}{2}$$

$$c = a - P = \frac{A + P}{2} - P = \frac{A - P}{2}$$

$$e = \frac{c}{a} = \frac{(A - P)/2}{(A + P)/2} = \frac{A - P}{A + P}$$



$$101. e = \frac{A - P}{A + P} = \frac{35.34au - 0.59au}{35.34au + 0.59au} \approx 0.9672$$

$$103. \frac{x^2}{10^2} + \frac{y^2}{5^2} = 1$$

$$\frac{2x}{10^2} + \frac{2yy'}{5^2} = 0$$

$$y' = \frac{-5^2x}{10^2y} = \frac{-x}{4y}$$

$$\text{At } (-8, 3): y' = \frac{8}{12} = \frac{2}{3}$$

The equation of the tangent line is $y - 3 = \frac{2}{3}(x + 8)$. It will cross the y-axis when $x = 0$ and $y = \frac{2}{3}(8) + 3 = \frac{25}{3}$.

$$105. 16x^2 + 9y^2 + 96x + 36y + 36 = 0$$

$$32x + 18yy' + 96 + 36y' = 0$$

$$y'(18y + 36) = -(32x + 96)$$

$$y' = \frac{-(32x + 96)}{18y + 36}$$

$y' = 0$ when $x = -3$. y' is undefined when $y = -2$.

At $x = -3$, $y = 2$ or -6 .

Endpoints of major axis: $(-3, 2)$, $(-3, -6)$

At $y = -2$, $x = 0$ or -6 .

Endpoints of minor axis: $(0, -2)$, $(-6, -2)$

Note: Equation of ellipse is $\frac{(x + 3)^2}{9} + \frac{(y + 2)^2}{16} = 1$

$$107. (a) A = 4 \int_0^2 \frac{1}{2} \sqrt{4 - x^2} dx = \left[x\sqrt{4 - x^2} + 4 \arcsin\left(\frac{x}{2}\right) \right]_0^2 = 2\pi \quad [\text{or, } A = \pi ab = \pi(2)(1) = 2\pi]$$

$$(b) \text{ Disk: } V = 2\pi \int_0^2 \frac{1}{4}(4 - x^2) dx = \frac{1}{2}\pi \left[4x - \frac{1}{3}x^3 \right]_0^2 = \frac{8\pi}{3}$$

$$y = \frac{1}{2}\sqrt{4 - x^2}$$

$$y' = \frac{-x}{2\sqrt{4 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{16 - 4x^2}} = \sqrt{\frac{16 - 3x^2}{4y}}$$

$$S = 2(2\pi) \int_0^2 y \left(\frac{\sqrt{16 - 3x^2}}{4y} \right) dx = \frac{\pi}{2\sqrt{3}} \left[\sqrt{3}x\sqrt{16 - 3x^2} + 16 \arcsin\left(\frac{\sqrt{3}x}{4}\right) \right]_0^2 = \frac{2\pi}{9}(9 + 4\sqrt{3}\pi) \approx 21.48$$

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107. —CONTINUED—

$$\begin{aligned}
 \text{(c) Shell: } V &= 2\pi \int_0^2 x \sqrt{4-x^2} \, dx = -\pi \int_0^2 2x(4-x^2)^{1/2} \, dx = -\frac{2\pi}{3} \left[(4-x^2)^{3/2} \right]_0^2 = \frac{16\pi}{3} \\
 x &= 2\sqrt{1-y^2} \\
 x' &= \frac{-2y}{\sqrt{1-y^2}} \\
 \sqrt{1+(x')^2} &= \sqrt{1 + \frac{4y^2}{1-y^2}} = \frac{\sqrt{1+3y^2}}{\sqrt{1-y^2}} \\
 S &= 2(2\pi) \int_0^1 2\sqrt{1-y^2} \frac{\sqrt{1+3y^2}}{\sqrt{1-y^2}} \, dy = 8\pi \int_0^1 \sqrt{1+3y^2} \, dy \\
 &= \frac{8\pi}{2\sqrt{3}} \left[\sqrt{3}y\sqrt{1+3y^2} + \ln|\sqrt{3}y + \sqrt{1+3y^2}| \right]_0^1 = \frac{4\pi}{3} \left[6 + \sqrt{3} \ln(2 + \sqrt{3}) \right] \approx 34.69
 \end{aligned}$$

109. From Example 5,

$$C = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} \, d\theta$$

For $\frac{x^2}{25} + \frac{y^2}{49} = 1$, we have

$$a = 7, b = 5, c = \sqrt{49 - 25} = 2\sqrt{6}, e = \frac{c}{a} = \frac{2\sqrt{6}}{7}.$$

$$C = 4(7) \int_0^{\pi/2} \sqrt{1 - \frac{24}{49} \sin^2 \theta} \, d\theta$$

$$\approx 28(1.3558) \approx 37.9614$$

111. Area circle = $\pi r^2 = 100\pi$ Area ellipse = $\pi ab = \pi a(10)$

$$2(100\pi) = 10\pi a \Rightarrow a = 20$$

Hence, the length of the major axis is $2a = 40$.113. The transverse axis is horizontal since $(2, 2)$ and $(10, 2)$ are the foci (see definition of hyperbola).Center: $(6, 2)$

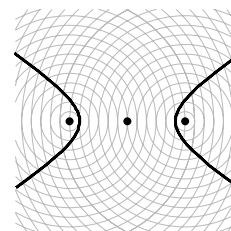
$$c = 4, 2a = 6, b^2 = c^2 - a^2 = 7$$

Therefore, the equation is

$$\frac{(x-6)^2}{9} - \frac{(y-2)^2}{7} = 1.$$

115. $2a = 10 \Rightarrow a = 5$

$$c = 6 \Rightarrow b = \sqrt{11}$$

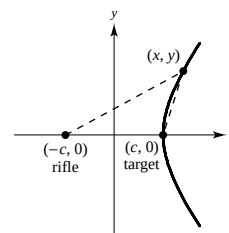
117. Time for sound of bullet hitting target to reach (x, y) : $\frac{2c}{v_m} + \frac{\sqrt{(x-c)^2 + y^2}}{v_s}$ Time for sound of rifle to reach (x, y) : $\frac{\sqrt{(x+c)^2 + y^2}}{v_s}$ Since the times are the same, we have: $\frac{2c}{v_m} + \frac{\sqrt{(x-c)^2 + y^2}}{v_s} = \frac{\sqrt{(x+c)^2 + y^2}}{v_s}$

$$\frac{4c^2}{v_m^2} + \frac{4c}{v_m v_s} \sqrt{(x-c)^2 + y^2} + \frac{(x-c)^2 + y^2}{v_s^2} = \frac{(x+c)^2 + y^2}{v_s^2}$$

$$\sqrt{(x-c)^2 + y^2} = \frac{v_m^2 x - v_s^2 c}{v_s v_m}$$

$$\left(1 - \frac{v_m^2}{v_s^2}\right)x^2 + y^2 = \left(\frac{v_s^2}{v_m^2} - 1\right)c^2$$

$$\frac{x^2}{c^2 v_s^2 / v_m^2} - \frac{y^2}{c^2 (v_m^2 - v_s^2) / v_m^2} = 1$$



119. The point (x, y) lies on the line between $(0, 10)$ and $(10, 0)$. Thus, $y = 10 - x$. The point also lies on the hyperbola $(x^2/36) - (y^2/64) = 1$. Using substitution, we have:

$$\frac{x^2}{36} - \frac{(10-x)^2}{64} = 1$$

$$16x^2 - 9(10-x)^2 = 576$$

$$7x^2 + 180x - 1476 = 0$$

$$x = \frac{-180 \pm \sqrt{180^2 - 4(7)(-1476)}}{2(7)} = \frac{-180 \pm 192\sqrt{2}}{14} = \frac{-90 \pm 96\sqrt{2}}{7}$$

Choosing the positive value for x we have:

$$x = \frac{-90 + 96\sqrt{2}}{7} \approx 6.538 \text{ and } y = \frac{160 - 96\sqrt{2}}{7} \approx 3.462$$

121.
$$\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2y^2}{b^2} = 1 - \frac{x^2}{a^2}, \quad c^2 = a^2 - b^2$$

$$\frac{x^2}{a^2 - b^2} - \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2y^2}{b^2} = \frac{x^2}{a^2 - b^2} - 1$$

$$1 - \frac{x^2}{a^2} = \frac{x^2}{a^2 - b^2} - 1 \Rightarrow 2 = x^2 \left(\frac{1}{a^2} + \frac{1}{a^2 - b^2} \right)$$

$$x^2 = \frac{2a^2(a^2 - b^2)}{2a^2 - b^2} \Rightarrow x = \pm \frac{\sqrt{2}a\sqrt{a^2 - b^2}}{\sqrt{2a^2 - b^2}} = \pm \frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}$$

$$\frac{2y^2}{b^2} = 1 - \frac{1}{a^2} \left(\frac{2a^2c^2}{2a^2 - b^2} \right) \Rightarrow \frac{2y^2}{b^2} = \frac{b^2}{2a^2 - b^2}$$

$$y^2 = \frac{b^4}{2(2a^2 - b^2)} \Rightarrow y = \pm \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}$$

There are four points of intersection: $\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \pm \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right), \left(-\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \pm \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right)$

$$\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{4yy'}{b^2} = 0 \Rightarrow y'_e = -\frac{b^2x}{2a^2y}$$

$$\frac{x^2}{a^2 - b^2} - \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2x}{c^2} - \frac{4yy'}{b^2} = 0 \Rightarrow y'_h = \frac{b^2x}{2c^2y}$$

At $\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right)$, the slopes of the tangent lines are:

$$y'_e = \frac{-b^2 \left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}} \right)}{2a^2 \left(\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right)} = -\frac{c}{a} \quad \text{and} \quad y'_h = \frac{b^2 \left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}} \right)}{2c^2 \left(\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right)} = \frac{a}{c}$$

Since the slopes are negative reciprocals, the tangent lines are perpendicular. Similarly, the curves are perpendicular at the other three points of intersection.

123. False. See the definition of a parabola.

125. True

127. False. $y^2 - x^2 + 2x + 2y = 0$ yields two intersecting lines.

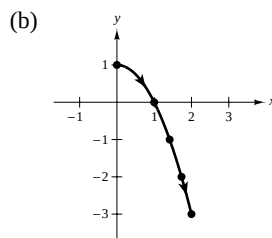
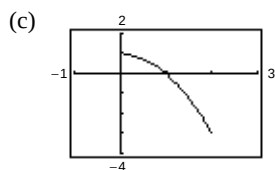
129. True

Section 9.2 Plane Curves and Parametric Equations

1. $x = \sqrt{t}$, $y = 1 - t$

(a)

t	0	1	2	3	4
x	0	1	$\sqrt{2}$	$\sqrt{3}$	2
y	1	0	-1	-2	-3



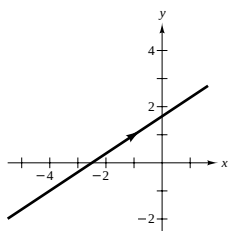
(d) $x^2 = t$
 $y = 1 - x^2, x \geq 0$

3. $x = 3t - 4$

$y = 2t + 1$

$y = 2\left(\frac{x+4}{3}\right) + 1$

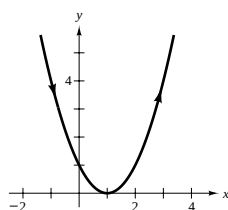
$2x - 3y + 5 = 0$



5. $x = t + 1$

$y = t^2$

$y = (x - 1)^2$

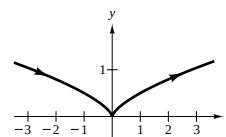


7. $x = t^3$

$y = \frac{1}{2}t^2$

$x = t^3$ implies $t = x^{1/3}$

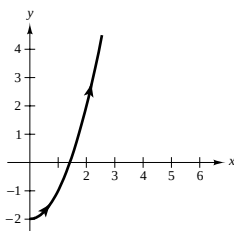
$y = \frac{1}{2}x^{2/3}$



9. $x = \sqrt{t}, t \geq 0$

$y = t - 2$

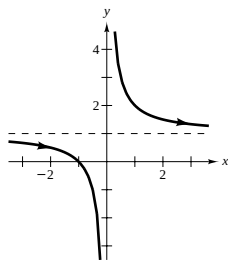
$y = x^2 - 2, x \geq 0$



11. $x = t - 1$

$y = \frac{t}{t-1}$

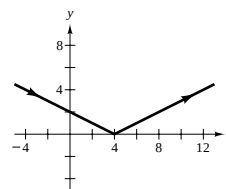
$y = \frac{x+1}{x}$



13. $x = 2t$

$y = |t - 2|$

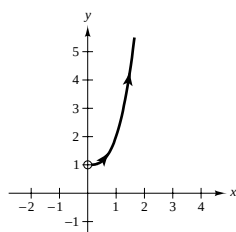
$y = \left|\frac{x}{2} - 2\right| = \frac{|x - 4|}{2}$



15. $x = e^t, x > 0$

$y = e^{3t} + 1$

$y = x^3 + 1, x > 0$



17. $x = \sec \theta$

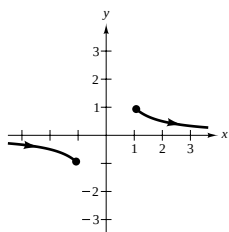
$$y = \cos \theta$$

$$0 \leq \theta < \frac{\pi}{2}, \frac{\pi}{2} < \theta \leq \pi$$

$$xy = 1$$

$$y = \frac{1}{x}$$

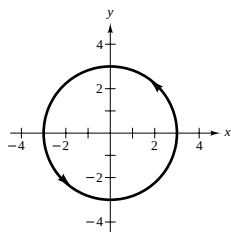
$$|x| \geq 1, |y| \leq 1$$



19. $x = 3 \cos \theta, y = 3 \sin \theta$

Squaring both equations and adding, we have

$$x^2 + y^2 = 9.$$



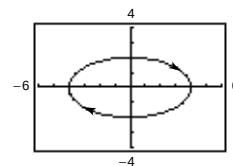
21. $x = 4 \sin 2\theta$

$$y = 2 \cos 2\theta$$

$$\frac{x^2}{16} = \sin^2 2\theta$$

$$\frac{y^2}{4} = \cos^2 2\theta$$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$



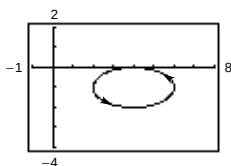
23. $x = 4 + 2 \cos \theta$

$$y = -1 + \sin \theta$$

$$\frac{(x-4)^2}{4} = \cos^2 \theta$$

$$\frac{(y+1)^2}{1} = \sin^2 \theta$$

$$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{1} = 1$$



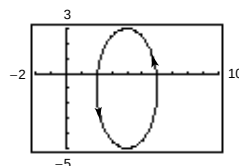
25. $x = 4 + 2 \cos \theta$

$$y = -1 + 4 \sin \theta$$

$$\frac{(x-4)^2}{4} = \cos^2 \theta$$

$$\frac{(y+1)^2}{16} = \sin^2 \theta$$

$$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{16} = 1$$



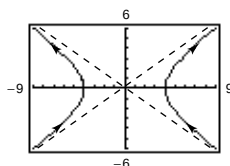
27. $x = 4 \sec \theta$

$$y = 3 \tan \theta$$

$$\frac{x^2}{16} = \sec^2 \theta$$

$$\frac{y^2}{9} = \tan^2 \theta$$

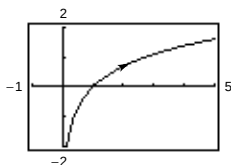
$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$



29. $x = t^3$

$$y = 3 \ln t$$

$$y = 3 \ln \sqrt[3]{x} = \ln x$$



31. $x = e^{-t}$

$$y = e^{3t}$$

$$e^t = \frac{1}{x}$$

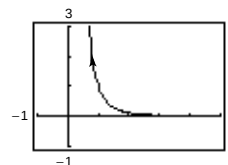
$$e^t = \sqrt[3]{y}$$

$$\sqrt[3]{y} = \frac{1}{x}$$

$$y = \frac{1}{x^3}$$

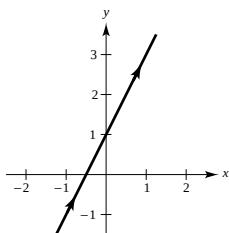
$$x > 0$$

$$y > 0$$



33. By eliminating the parameters in (a) – (d), we get $y = 2x + 1$. They differ from each other in orientation and in restricted domains. These curves are all smooth except for (b).

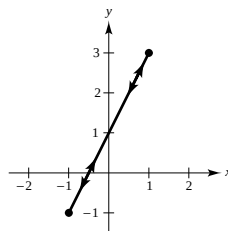
(a) $x = t, y = 2t + 1$



(b) $x = \cos \theta, y = 2 \cos \theta + 1$

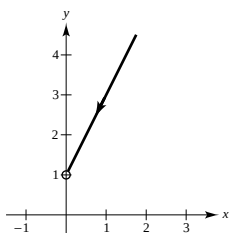
$$-1 \leq x \leq 1 \quad -1 \leq y \leq 3$$

$$\frac{dx}{d\theta} = \frac{dy}{d\theta} = 0 \text{ when } \theta = 0, \pm\pi, \pm2\pi, \dots$$



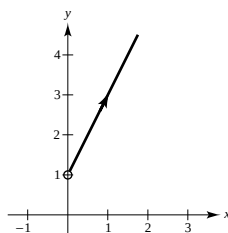
(c) $x = e^{-t}, y = 2e^{-t} + 1$

$$x > 0 \quad y > 1$$



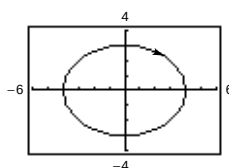
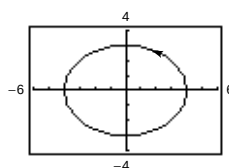
(d) $x = e^t, y = 2e^t + 1$

$$x > 0 \quad y > 1$$



35. The curves are identical on $0 < \theta < \pi$. They are both smooth. Represent $y = 2(1 - x^2)$

37. (a)



(b) The orientation of the second curve is reversed.

(c) The orientation will be reversed.

(d) Many answers possible. For example, $x = 1 + t$, $y = 1 + 2t$, and $x = 1 - t$, $y = 1 - 2t$.

39. $x = x_1 + t(x_2 - x_1)$

$$y = y_1 + t(y_2 - y_1)$$

$$\frac{x - x_1}{x_2 - x_1} = t$$

$$y = y_1 + \left(\frac{x - x_1}{x_2 - x_1}\right)(y_2 - y_1)$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - y_1 = m(x - x_1)$$

41. $x = h + a \cos \theta$

$$y = k + b \sin \theta$$

$$\frac{x - h}{a} = \cos \theta$$

$$\frac{y - k}{b} = \sin \theta$$

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

43. From Exercise 39 we have

$$x = 5t$$

$$y = -2t.$$

Solution not unique

45. From Exercise 40 we have

$$x = 2 + 4 \cos \theta$$

$$y = 1 + 4 \sin \theta.$$

Solution not unique

47. From Exercise 41 we have

$$a = 5, c = 4 \Rightarrow b = 3$$

$$x = 5 \cos \theta$$

$$y = 3 \sin \theta.$$

Center: $(0, 0)$

Solution not unique

49. From Exercise 42 we have

$$a = 4, c = 5 \Rightarrow b = 3$$

$$x = 4 \sec \theta$$

$$y = 3 \tan \theta.$$

Center: $(0, 0)$

Solution not unique

51. $y = 3x - 2$

Example

$$x = t, \quad y = 3t - 2$$

$$x = t - 3, \quad y = 3t - 11$$

53. $y = x^3$

Example

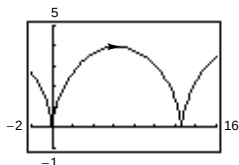
$$x = t, \quad y = t^3$$

$$x = \sqrt[3]{t}, \quad y = t$$

$$x = \tan t, \quad y = \tan^3 t$$

55. $x = 2(\theta - \sin \theta)$

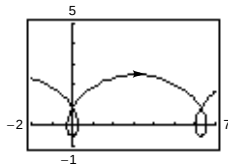
$$y = 2(1 - \cos \theta)$$



Not smooth at $\theta = 2n\pi$

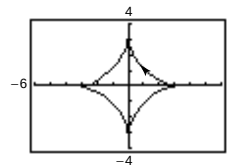
57. $x = \theta - \frac{3}{2} \sin \theta$

$$y = 1 - \frac{3}{2} \cos \theta$$



59. $x = 3 \cos^3 \theta$

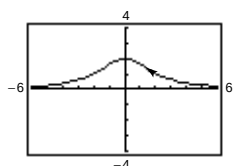
$$y = 3 \sin^3 \theta$$



Not smooth at $(x, y) = (\pm 3, 0)$ and $(0, \pm 3)$, or $\theta = \frac{1}{2}n\pi$.

61. $x = 2 \cot \theta$

$$y = 2 \sin^2 \theta$$



Smooth everywhere

63. See definition on page 665.

65. A plane curve C , represented by $x = f(t)$, $y = g(t)$, is smooth if f' and g' are continuous and not simultaneously 0. See page 670.

67. $x = 4 \cos \theta$

$$y = 2 \sin 2\theta$$

Matches (d)

69. $x = \cos \theta + \theta \sin \theta$

$$y = \sin \theta - \theta \cos \theta$$

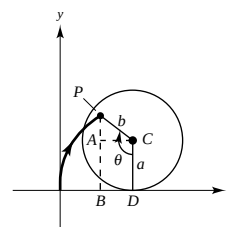
Matches (b)

71. When the circle has rolled θ radians, we know that the center is at $(a\theta, a)$.

$$\sin \theta = \sin(180^\circ - \theta) = \frac{|AC|}{b} = \frac{|BD|}{b} \quad \text{or} \quad |BD| = b \sin \theta$$

$$\cos \theta = -\cos(180^\circ - \theta) = \frac{|AP|}{-b} \quad \text{or} \quad |AP| = -b \cos \theta$$

Therefore, $x = a\theta - b \sin \theta$ and $y = a - b \cos \theta$.



73. False

$$x = t^2 \Rightarrow x \geq 0$$

$$x = t^2 \Rightarrow y \geq 0$$

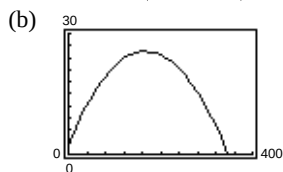
The graph of the parametric equations is only a portion of the line $y = x$.

75. (a) $100 \text{ mi/hr} = \frac{(100)(5280)}{3600} = \frac{440}{3} \text{ ft/sec}$

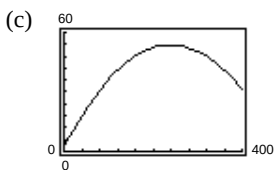
$$x = (v_0 \cos \theta)t = \left(\frac{440}{3} \cos \theta\right)t$$

$$y = h + (v_0 \sin \theta)t - 16t^2$$

$$= 3 + \left(\frac{440}{3} \sin \theta\right)t - 16t^2$$



It is not a home run—when $x = 400$, $y \leq 20$.



Yes, it's a home run when $x = 400$, $y > 10$.

(d) We need to find the angle θ (and time t) such that

$$x = \left(\frac{440}{3} \cos \theta\right)t = 400$$

$$y = 3 + \left(\frac{440}{3} \sin \theta\right)t - 16t^2 = 10.$$

From the first equation $t = 1200/440 \cos \theta$. Substituting into the second equation,

$$10 = 3 + \left(\frac{440}{3} \sin \theta\right)\left(\frac{1200}{440 \cos \theta}\right) - 16\left(\frac{1200}{440 \cos \theta}\right)^2$$

$$7 = 400 \tan \theta - 16\left(\frac{120}{44}\right)^2 \sec^2 \theta$$

$$= 400 \tan \theta - 16\left(\frac{120}{44}\right)^2 (\tan^2 \theta + 1).$$

We now solve the quadratic for $\tan \theta$:

$$16\left(\frac{120}{44}\right)^2 \tan^2 \theta - 400 \tan \theta + 7 + 16\left(\frac{120}{44}\right)^2 = 0$$

$$\tan \theta \approx 0.35185 \Rightarrow \theta \approx 19.4^\circ$$

Section 9.3 Parametric Equations and Calculus

1. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-4}{2t} = \frac{-2}{t}$

3. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2 \cos t \sin t}{2 \sin t \cos t} = -1$

[Note: $x + y = 1 \Rightarrow y = 1 - x$ and $\frac{dy}{dx} = -1$]

5. $x = 2t$, $y = 3t - 1$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{2}$$

$$\frac{d^2y}{dx^2} = 0 \text{ Line}$$

7. $x = t + 1$, $y = t^2 + 3t$

$$\frac{dy}{dx} = \frac{2t + 3}{1} = 1 \text{ when } t = -1.$$

$$\frac{d^2y}{dx^2} = 2 \text{ concave upwards}$$

9. $x = 2 \cos \theta$, $y = 2 \sin \theta$

$$\frac{dy}{dx} = \frac{2 \cos \theta}{-2 \sin \theta} = -\cot \theta = -1 \text{ when } \theta = \frac{\pi}{4}.$$

$$\frac{d^2y}{dx^2} = \frac{\csc^2 \theta}{-2 \sin \theta} = \frac{-\csc^3 \theta}{2} = -\sqrt{2} \text{ when } \theta = \frac{\pi}{4}.$$

concave downward

11. $x = 2 + \sec \theta$, $y = 1 + 2 \tan \theta$

$$\frac{dy}{dx} = \frac{2 \sec^2 \theta}{\sec \theta \tan \theta}$$

$$= \frac{2 \sec \theta}{\tan \theta} = 2 \csc \theta = 4 \text{ when } \theta = \frac{\pi}{6}.$$

$$\frac{d^2y}{dx^2} = \frac{-2 \csc \theta \cot \theta}{\sec \theta \tan \theta}$$

$$= -2 \cot^3 \theta = -6\sqrt{3} \text{ when } \theta = \frac{\pi}{6}.$$

concave downward

13. $x = \cos^3 \theta, y = \sin^3 \theta$

$$\frac{dy}{dx} = \frac{3 \sin^2 \theta \cos \theta}{-3 \cos^2 \theta \sin \theta}$$

$$= -\tan \theta = -1 \text{ when } \theta = \frac{\pi}{4}.$$

$$\frac{d^2y}{dx^2} = \frac{-\sec^2 \theta}{-3 \cos^2 \theta \sin \theta} = \frac{1}{3 \cos^4 \theta \sin \theta}$$

$$= \frac{\sec^4 \theta \csc \theta}{3} = \frac{4\sqrt{2}}{3} \text{ when } \theta = \frac{\pi}{4}.$$

concave upward

15. $x = 2 \cot \theta, y = 2 \sin^2 \theta$

$$\frac{dy}{dx} = \frac{4 \sin \theta \cos \theta}{-2 \csc^2 \theta} = -2 \sin^3 \theta \cos \theta$$

$$\text{At } \left(-\frac{2}{\sqrt{3}}, \frac{3}{2}\right), \theta = \frac{2\pi}{3}, \text{ and } \frac{dy}{dx} = \frac{3\sqrt{3}}{8}.$$

$$\text{Tangent line: } y - \frac{3}{2} = \frac{3\sqrt{3}}{8} \left(x + \frac{2}{\sqrt{3}}\right)$$

$$3\sqrt{3}x - 8y + 18 = 0$$

$$\text{At } (0, 2), \theta = \frac{\pi}{2}, \text{ and } \frac{dy}{dx} = 0.$$

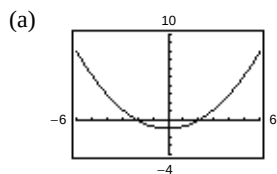
$$\text{Tangent line: } y - 2 = 0$$

$$\text{At } \left(2\sqrt{3}, \frac{1}{2}\right), \theta = \frac{\pi}{6}, \text{ and } \frac{dy}{dx} = -\frac{\sqrt{3}}{8}.$$

$$\text{Tangent line: } y - \frac{1}{2} = -\frac{\sqrt{3}}{8}(x - 2\sqrt{3})$$

$$\sqrt{3}x + 8y - 10 = 0$$

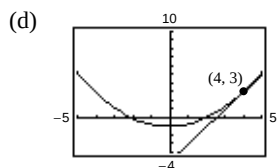
17. $x = 2t, y = t^2 - 1, t = 2$



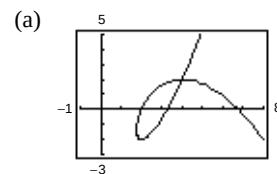
(b) At $t = 2, (x, y) = (4, 3)$, and

$$\frac{dx}{dt} = 2, \frac{dy}{dt} = 4, \frac{dy}{dx} = 2$$

(c) $\frac{dy}{dx} = 2$. At $(4, 3)$, $y - 3 = 2(x - 4)$
 $y = 2x - 5$



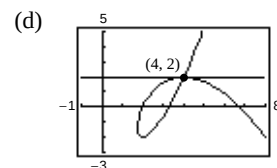
19. $x = t^2 - t + 2, y = t^3 - 3t, t = -1$



(b) At $t = -1, (x, y) = (4, 2)$, and

$$\frac{dx}{dt} = -3, \frac{dy}{dt} = 0, \frac{dy}{dx} = 0$$

(c) $\frac{dy}{dx} = 0$. At $(4, 2)$, $y - 2 = 0(x - 4)$
 $y = 2$



21. $x = 2 \sin 2t, y = 3 \sin t$ crosses itself at the origin, $(x, y) = (0, 0)$.

At this point, $t = 0$ or $t = \pi$.

$$\frac{dy}{dx} = \frac{3 \cos t}{4 \cos 2t}$$

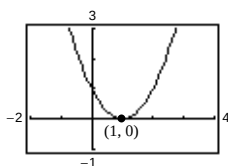
At $t = 0$: $\frac{dy}{dx} = \frac{3}{4}$ and $y = \frac{3}{4}x$. Tangent Line

At $t = \pi$, $\frac{dy}{dx} = -\frac{3}{4}$ and $y = -\frac{3}{4}x$. Tangent Line

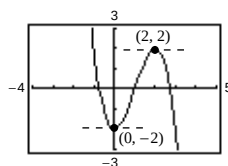
23. $x = \cos \theta + \theta \sin \theta$, $y = \sin \theta - \theta \cos \theta$

Horizontal tangents: $\frac{dy}{d\theta} = \theta \sin \theta = 0$ when $\theta = 0, \pi, 2\pi, 3\pi, \dots$ Points: $(-1, [2n-1]\pi)$, $(1, 2n\pi)$ where n is an integer.Points shown: $(1, 0)$, $(-1, \pi)$, $(1, -2\pi)$ Vertical tangents: $\frac{dx}{d\theta} = \theta \cos \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$ Points: $\left(\frac{(-1)^{n+1}(2n-1)\pi}{2}, (-1)^{n+1}\right)$ Points shown: $\left(\frac{\pi}{2}, 1\right)$, $\left(-\frac{3\pi}{2}, -1\right)$, $\left(\frac{5\pi}{2}, 1\right)$

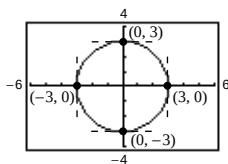
25. $x = 1 - t$, $y = t^2$

Horizontal tangents: $\frac{dy}{dt} = 2t = 0$ when $t = 0$.Point: $(1, 0)$ Vertical tangents: $\frac{dx}{dt} = -1 \neq 0$; none

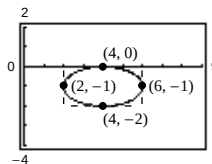
27. $x = 1 - t$, $y = t^3 - 3t$

Horizontal tangents: $\frac{dy}{dt} = 3t^2 - 3 = 0$ when $t = \pm 1$.Points: $(0, -2)$, $(2, 2)$ Vertical tangents: $\frac{dx}{dt} = -1 \neq 0$; none

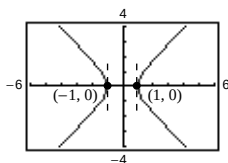
29. $x = 3 \cos \theta$, $y = 3 \sin \theta$

Horizontal tangents: $\frac{dy}{d\theta} = 3 \cos \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.Points: $(0, 3)$, $(0, -3)$ Vertical tangents: $\frac{dx}{d\theta} = -3 \sin \theta = 0$ when $\theta = 0, \pi$.Points: $(3, 0)$, $(-3, 0)$ 

31. $x = 4 + 2 \cos \theta$, $y = -1 + \sin \theta$

Horizontal tangents: $\frac{dy}{d\theta} = \cos \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.Points: $(4, 0)$, $(4, -2)$ Vertical tangents: $\frac{dx}{d\theta} = -2 \sin \theta = 0$ when $\theta = 0, \pi$.Points: $(6, -1)$, $(2, -1)$ 

33. $x = \sec \theta$, $y = \tan \theta$

Horizontal tangents: $\frac{dy}{d\theta} = \sec^2 \theta \neq 0$; noneVertical tangents: $\frac{dx}{d\theta} = \sec \theta \tan \theta = 0$ when $\theta = 0, \pi$.Points: $(1, 0)$, $(-1, 0)$ 

35. $x = t^2$, $y = 2t$, $0 \leq t \leq 2$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 2, \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 4 = 4(t^2 + 1)$$

$$\begin{aligned} s &= 2 \int_0^2 \sqrt{t^2 + 1} \, dt \\ &= \left[t\sqrt{t^2 + 1} + \ln|t + \sqrt{t^2 + 1}| \right]_0^2 \\ &= 2\sqrt{5} + \ln(2 + \sqrt{5}) \approx 5.916 \end{aligned}$$

37. $x = e^{-t} \cos t$, $y = e^{-t} \sin t$, $0 \leq t \leq \frac{\pi}{2}$

$$\frac{dx}{dt} = -e^{-t}(\sin t + \cos t), \frac{dy}{dt} = e^{-t}(\cos t - \sin t)$$

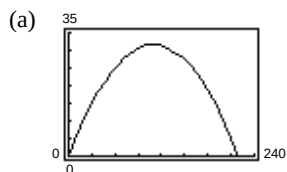
$$\begin{aligned} s &= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^{\pi/2} \sqrt{2e^{-2t}} dt = -\sqrt{2} \int_0^{\pi/2} e^{-t}(-1) dt \\ &= \left[-\sqrt{2}e^{-t}\right]_0^{\pi/2} = \sqrt{2}(1 - e^{-\pi/2}) \approx 1.12 \end{aligned}$$

41. $x = a \cos^3 \theta$, $y = a \sin^3 \theta$, $\frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta$,

$$\frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$$

$$\begin{aligned} S &= 4 \int_0^{\pi/2} \sqrt{9a^2 \cos^4 \theta \sin^2 \theta + 9a^2 \sin^4 \theta \cos^2 \theta} d\theta \\ &= 12a \int_0^{\pi/2} \sin \theta \cos \theta \sqrt{\cos^2 \theta + \sin^2 \theta} d\theta \\ &= 6a \int_0^{\pi/2} \sin 2\theta d\theta = \left[-3a \cos 2\theta\right]_0^{\pi/2} = 6a \end{aligned}$$

45. $x = (90 \cos 30^\circ)t$, $y = (90 \sin 30^\circ)t - 16t^2$



(b) Range: 219.2 ft

(c) $\frac{dx}{dt} = 90 \cos 30^\circ$, $\frac{dy}{dt} = 90 \sin 30^\circ - 32t$.

$$y = 0 \text{ for } t = \frac{45}{16}.$$

$$\begin{aligned} s &= \int_0^{45/16} \sqrt{(90 \cos 30^\circ)^2 + (90 \sin 30^\circ - 32t)^2} dt \\ &= 230.8 \text{ ft} \end{aligned}$$

39. $x = \sqrt{t}$, $y = 3t - 1$, $\frac{dx}{dt} = \frac{1}{2\sqrt{t}}$, $\frac{dy}{dt} = 3$

$$\begin{aligned} S &= \int_0^1 \sqrt{\frac{1}{4t} + 9} dt = \frac{1}{2} \int_0^1 \frac{\sqrt{1 + 36t}}{\sqrt{t}} dt \\ &= \frac{1}{6} \int_0^6 \sqrt{1 + u^2} du \\ &= \frac{1}{12} \left[\ln(\sqrt{1 + u^2} + u) + u\sqrt{1 + u^2} \right]_0^6 \\ &= \frac{1}{12} \left[\ln(\sqrt{37} + 6) + 6\sqrt{37} \right] \approx 3.249 \end{aligned}$$

$$u = 6\sqrt{t}, \quad du = \frac{3}{\sqrt{t}} dt$$

43. $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$,

$$\frac{dx}{d\theta} = a(1 - \cos \theta), \quad \frac{dy}{d\theta} = a \sin \theta$$

$$\begin{aligned} S &= 2 \int_0^\pi \sqrt{a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta} d\theta \\ &= 2\sqrt{2}a \int_0^\pi \sqrt{1 - \cos \theta} d\theta \\ &= 2\sqrt{2}a \int_0^\pi \frac{\sin \theta}{\sqrt{1 + \cos \theta}} d\theta \\ &= \left[-4\sqrt{2}a\sqrt{1 + \cos \theta}\right]_0^\pi = 8a \end{aligned}$$

(d) $y = 0 \Rightarrow (90 \sin \theta)t = 16t^2 \Rightarrow t = \frac{90}{16} \sin \theta$

$$x = (90 \cos \theta)t = \frac{90^2}{16} \cos \theta \sin \theta = \frac{90^2}{32} \sin 2\theta$$

$$x'(\theta) = \frac{90^2}{32} 2 \cos 2\theta = 0 \Rightarrow \theta = 45^\circ$$

By the First Derivative Test, $\theta = 45^\circ \left(\frac{\pi}{4}\right)$

maximizes the range.

$$\frac{dx}{dt} = 90 \cos \theta,$$

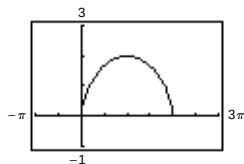
$$\frac{dy}{dt} = 90 \sin \theta - 32 = 90 \sin \theta - 32 \left(\frac{90}{16} \sin \theta\right) = -90 \sin \theta$$

$$\begin{aligned} s &= \int_0^{(90/16)\sin \theta} \sqrt{(90 \cos \theta)^2 + (-90 \sin \theta)^2} dt \\ &= \int_0^{(90/16)\sin \theta} 90 dt = 90t \Big|_0^{(90/16)\sin \theta} \\ &= \frac{90^2}{16} \sin \theta \end{aligned}$$

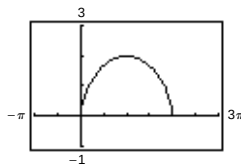
$$\frac{ds}{d\theta} = \frac{90^2}{16} \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

By the First Derivative Test, $\theta = 90^\circ$ maximizes the arc length.

47. (a) $x = t - \sin t$
 $y = 1 - \cos t$
 $0 \leq t \leq 2\pi$



$x = 2t - \sin(2t)$
 $y = 1 - \cos(2t)$
 $0 \leq t \leq \pi$



(b) The average speed of the particle on the second path is twice the average speed of a particle on the first path.

(c) $x = \frac{1}{2}t - \sin(\frac{1}{2}t)$
 $y = 1 - \cos(\frac{1}{2}t)$

The time required for the particle to traverse the same path is $t = 4\pi$.

49. $x = t$, $y = 2t$, $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 2$

(a) $S = 2\pi \int_0^4 2t\sqrt{1+4} dt = 4\sqrt{5}\pi \int_0^4 t dt$
 $= \left[2\sqrt{5}\pi t^2 \right]_0^4 = 32\pi\sqrt{5}$

(b) $S = 2\pi \int_0^4 t\sqrt{1+4} dt = 2\sqrt{5}\pi \int_0^4 t dt$
 $= \left[\sqrt{5}\pi t^2 \right]_0^4 = 16\pi\sqrt{5}$

51. $x = 4 \cos \theta$, $y = 4 \sin \theta$, $\frac{dx}{d\theta} = -4 \sin \theta$, $\frac{dy}{d\theta} = 4 \cos \theta$

$S = 2\pi \int_0^{\pi/2} 4 \cos \theta \sqrt{(-4 \sin \theta)^2 + (4 \cos \theta)^2} d\theta$
 $= 32\pi \int_0^{\pi/2} \cos \theta d\theta = \left[32\pi \sin \theta \right]_0^{\pi/2} = 32\pi$

53. $x = a \cos^3 \theta$, $y = a \sin^3 \theta$, $\frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta$, $\frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$

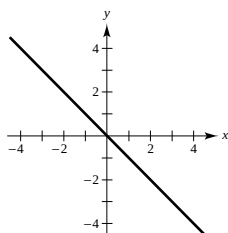
$S = 4\pi \int_0^{\pi/2} a \sin^3 \theta \sqrt{9a^2 \cos^4 \theta \sin^2 \theta + 9a^2 \sin^4 \theta \cos^2 \theta} d\theta = 12a^2\pi \int_0^{\pi/2} \sin^4 \theta \cos \theta d\theta = \frac{12\pi a^2}{5} \left[\sin^5 \theta \right]_0^{\pi/2} = \frac{12}{5}\pi a^2$

55. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

See Theorem 9.7, page 675.

57. One possible answer is the graph given by

$x = t$, $y = -t$.

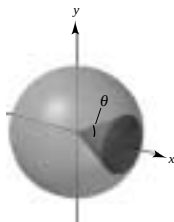


59. $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

See Theorem 9.8, page 678.

61. $x = r \cos \phi$, $y = r \sin \phi$

$S = 2\pi \int_0^\theta r \sin \phi \sqrt{r^2 \sin^2 \phi + r^2 \cos^2 \phi} d\phi$
 $= 2\pi r^2 \int_0^\theta \sin \phi d\phi$
 $= \left[-2\pi r^2 \cos \phi \right]_0^\theta$
 $= 2\pi r^2(1 - \cos \theta)$



63. $x = \sqrt{t}$, $y = 4 - t$, $0 \leq t \leq 4$

$$A = \int_0^4 (4 - t) \frac{1}{2\sqrt{t}} dt = \frac{1}{2} \int_0^4 (4t^{-1/2} - t^{1/2}) dt = \left[\frac{1}{2} \left(8\sqrt{t} - \frac{2}{3}t\sqrt{t} \right) \right]_0^4 = \frac{16}{3}$$

$$\bar{x} = \frac{3}{16} \int_0^4 (4 - t) \sqrt{t} \left(\frac{1}{2\sqrt{t}} \right) dt = \frac{3}{32} \int_0^4 (4 - t) dt = \left[\frac{3}{32} \left(4t - \frac{t^2}{2} \right) \right]_0^4 = \frac{3}{4}$$

$$\bar{y} = \frac{3}{32} \int_0^4 (4 - t)^2 \frac{1}{2\sqrt{t}} dt = \frac{3}{64} \int_0^4 [16t^{-1/2} - 8t^{1/2} + t^{3/2}] dt = \frac{3}{64} \left[32\sqrt{t} - \frac{16}{3}t\sqrt{t} + \frac{2}{5}t^2\sqrt{t} \right]_0^4 = \frac{8}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{3}{4}, \frac{8}{5} \right)$$

65. $x = 3 \cos \theta$, $y = 3 \sin \theta$, $\frac{dx}{d\theta} = -3 \sin \theta$

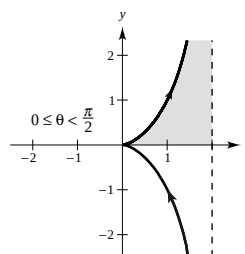
$$\begin{aligned} V &= 2\pi \int_{\pi/2}^0 (3 \sin \theta)^2 (-3 \sin \theta) d\theta \\ &= -54\pi \int_{\pi/2}^0 \sin^3 \theta d\theta \\ &= -54\pi \int_{\pi/2}^0 (1 - \cos^2 \theta) \sin \theta d\theta \\ &= -54\pi \left[-\cos \theta + \frac{\cos^3 \theta}{3} \right]_{\pi/2}^0 = 36\pi \end{aligned}$$

67. $x = 2 \sin^2 \theta$

$$y = 2 \sin^2 \theta \tan \theta$$

$$\frac{dx}{d\theta} = 4 \sin \theta \cos \theta$$

$$\begin{aligned} A &= \int_0^{\pi/2} 2 \sin^2 \theta \tan \theta (4 \sin \theta \cos \theta) d\theta = 8 \int_0^{\pi/2} \sin^4 \theta d\theta \\ &= 8 \left[\frac{-\sin^3 \theta \cos \theta}{4} - \frac{3}{8} \sin \theta \cos \theta + \frac{3}{8} \theta \right]_0^{\pi/2} = \frac{3\pi}{2} \end{aligned}$$



69. πab is area of ellipse (d).

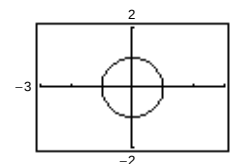
71. $6\pi a^2$ is area of cardioid (f).

73. $\frac{8}{3}ab$ is area of hourglass (a).

75. (a) $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$, $-20 \leq t \leq 20$

The graph is the circle $x^2 + y^2 = 1$, except the point $(-1, 0)$.

$$\text{Verify: } x^2 + y^2 = \left(\frac{1-t^2}{1+t^2} \right)^2 + \left(\frac{2t}{1+t^2} \right)^2 = \frac{1-2t^2+t^4+4t^2}{(1+t^2)^2} = \frac{(1+t^2)^2}{(1+t^2)^2} = 1$$



(b) As t increases from -20 to 0 , the speed increases, and as t increases from 0 to 20 , the speed decreases.

77. False

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left[\frac{g'(t)}{f'(t)} \right]}{f'(t)} = \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}$$

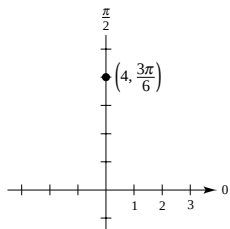
Section 9.4 Polar Coordinates and Polar Graphs

1. $\left(4, \frac{\pi}{2}\right)$

$$x = 4 \cos\left(\frac{\pi}{2}\right) = 0$$

$$y = 4 \sin\left(\frac{\pi}{2}\right) = 4$$

$$(x, y) = (0, 4)$$

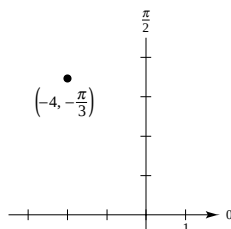


3. $\left(-4, -\frac{\pi}{3}\right)$

$$x = -4 \cos\left(-\frac{\pi}{3}\right) = -2$$

$$y = -4 \sin\left(-\frac{\pi}{3}\right) = 2\sqrt{3}$$

$$(x, y) = (-2, 2\sqrt{3})$$

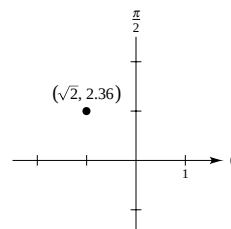


5. $(\sqrt{2}, 2.36)$

$$x = \sqrt{2} \cos(2.36) \approx -1.004$$

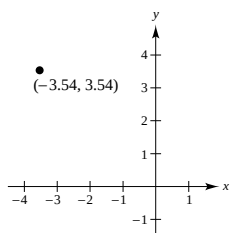
$$y = \sqrt{2} \sin(2.36) \approx 0.996$$

$$(x, y) = (-1.004, 0.996)$$



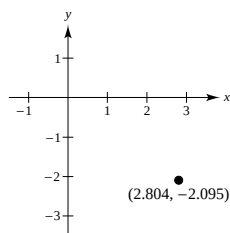
7. $(r, \theta) = \left(5, \frac{3\pi}{4}\right)$

$$(x, y) = (-3.5355, 3.5355)$$



9. $(r, \theta) = (-3.5, 2.5)$

$$(x, y) = (2.804, -2.095)$$

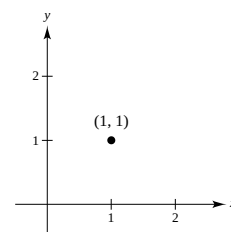


11. $(x, y) = (1, 1)$

$$r = \pm\sqrt{2}$$

$$\tan \theta = 1$$

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \left(\sqrt{2}, \frac{\pi}{4}\right), \left(-\sqrt{2}, \frac{5\pi}{4}\right)$$

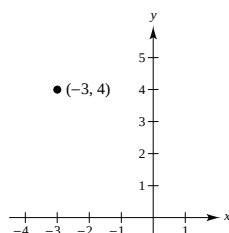


13. $(x, y) = (-3, 4)$

$$r = \pm\sqrt{9 + 16} = \pm 5$$

$$\tan \theta = -\frac{4}{3}$$

$$\theta \approx 2.214, 5.356, (5, 2.214), (-5, 5.356)$$



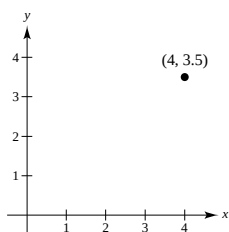
15. $(x, y) = (3, -2)$

$$(r, \theta) = (3.606, -0.588)$$

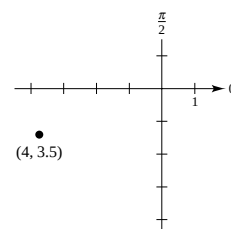
17. $(x, y) = \left(\frac{5}{2}, \frac{4}{3}\right)$

$$(r, \theta) = (2.833, 0.490)$$

19. (a) $(x, y) = (4, 3.5)$

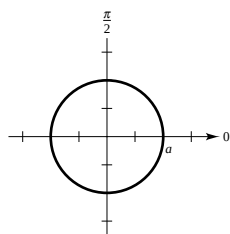


(b) $(r, \theta) = (4, 3.5)$



21. $x^2 + y^2 = a^2$

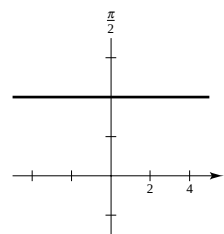
$$r = a$$



23. $y = 4$

$$r \sin \theta = 4$$

$$r = 4 \csc \theta$$

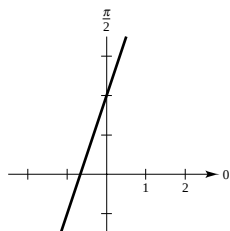


25. $3x - y + 2 = 0$

$$3r \cos \theta - r \sin \theta + 2 = 0$$

$$r(3 \cos \theta - \sin \theta) = -2$$

$$r = \frac{-2}{3 \cos \theta - \sin \theta}$$

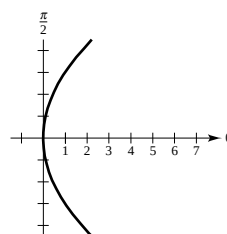


27. $y^2 = 9x$

$$r^2 \sin^2 \theta = 9r \cos \theta$$

$$r = \frac{9 \cos \theta}{\sin^2 \theta}$$

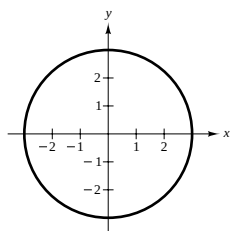
$$r = 9 \csc^2 \theta \cos \theta$$



29. $r = 3$

$$r^2 = 9$$

$$x^2 + y^2 = 9$$



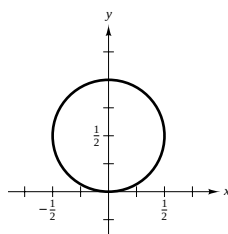
31. $r = \sin \theta$

$$r^2 = r \sin \theta$$

$$x^2 + y^2 = y$$

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

$$x^2 + y^2 - y = 0$$

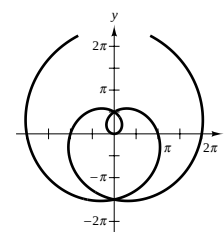


33. $r = \theta$

$$\tan r = \tan \theta$$

$$\tan \sqrt{x^2 + y^2} = \frac{y}{x}$$

$$\sqrt{x^2 + y^2} = \arctan \frac{y}{x}$$

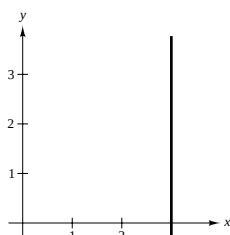


35. $r = 3 \sec \theta$

$$r \cos \theta = 3$$

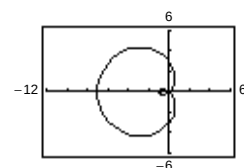
$$x = 3$$

$$x - 3 = 0$$



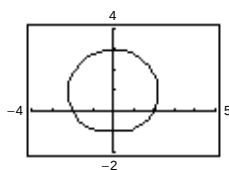
37. $r = 3 - 4 \cos \theta$

$$0 \leq \theta < 2\pi$$



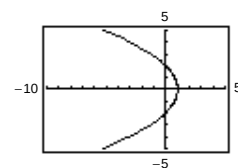
39. $r = 2 + \sin \theta$

$$0 \leq \theta < 2\pi$$



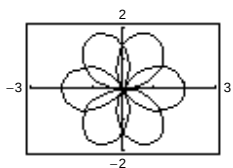
41. $r = \frac{2}{1 + \cos \theta}$

$$\text{Traced out once on } -\pi < \theta < \pi$$



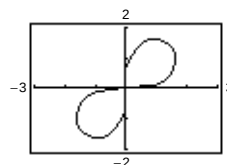
43. $r = 2 \cos\left(\frac{3\theta}{2}\right)$

$0 \leq \theta < 4\pi$



45. $r^2 = 4 \sin 2\theta$

$0 \leq \theta < \frac{\pi}{2}$



47.

$r = 2(h \cos \theta + k \sin \theta)$

Radius: $\sqrt{h^2 + k^2}$

$r^2 = 2r(h \cos \theta + k \sin \theta)$

Center: (h, k)

$r^2 = 2[h(r \cos \theta) + k(r \sin \theta)]$

$x^2 + y^2 = 2(hx + ky)$

$x^2 + y^2 - 2hx - 2ky = 0$

$(x^2 - 2hx + h^2) + (y^2 - 2ky + k^2) = 0 + h^2 + k^2$

$(x - h)^2 + (y - k)^2 = h^2 + k^2$

49. $\left(4, \frac{2\pi}{3}\right), \left(2, \frac{\pi}{6}\right)$

$$d = \sqrt{4^2 + 2^2 - 2(4)(2) \cos\left(\frac{2\pi}{3} - \frac{\pi}{6}\right)}$$

$$= \sqrt{20 - 16 \cos \frac{\pi}{2}} = 2\sqrt{5} \approx 4.5$$

51. $(2, 0.5), (7, 1.2)$

$$d = \sqrt{2^2 + 7^2 - 2(2)(7) \cos(0.5 - 1.2)}$$

$$= \sqrt{53 - 28 \cos(-0.7)} \approx 5.6$$

53. $r = 2 + 3 \sin \theta$

$$\frac{dy}{dx} = \frac{3 \cos \theta \sin \theta + \cos \theta(2 + 3 \sin \theta)}{3 \cos \theta \cos \theta - \sin \theta(2 + 3 \sin \theta)}$$

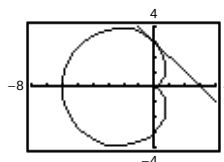
$$= \frac{2 \cos \theta(3 \sin \theta + 1)}{3 \cos 2\theta - 2 \sin \theta} = \frac{2 \cos \theta(3 \sin \theta + 1)}{6 \cos^2 \theta - 2 \sin \theta - 3}$$

At $\left(5, \frac{\pi}{2}\right), \frac{dy}{dx} = 0$.

At $(2, \pi), \frac{dy}{dx} = -\frac{2}{3}$.

At $\left(-1, \frac{3\pi}{2}\right), \frac{dy}{dx} = 0$.

55. (a), (b) $r = 3(1 - \cos \theta)$



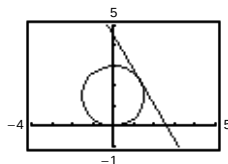
$$(r, \theta) = \left(3, \frac{\pi}{2}\right) \Rightarrow (x, y) = (0, 3)$$

Tangent line: $y - 3 = -1(x - 0)$

$$y = -x + 3$$

(c) At $\theta = \frac{\pi}{2}, \frac{dy}{dx} = -1.0$.

57. (a), (b) $r = 3 \sin \theta$



$$(r, \theta) = \left(\frac{3\sqrt{3}}{2}, \frac{\pi}{3}\right) \Rightarrow (x, y) = \left(\frac{3\sqrt{3}}{4}, \frac{9}{4}\right)$$

Tangent line: $y - \frac{9}{4} = -\sqrt{3}\left(x - \frac{3\sqrt{3}}{4}\right)$

$$y = -\sqrt{3}x + \frac{9}{2}$$

(c) At $\theta = \frac{\pi}{3}, \frac{dy}{dx} = -\sqrt{3} \approx -1.732$.

59. $r = 1 - \sin \theta$

$$\begin{aligned}\frac{dy}{d\theta} &= (1 - \sin \theta) \cos \theta - \cos \theta \sin \theta \\ &= \cos \theta (1 - 2 \sin \theta) = 0\end{aligned}$$

$$\cos \theta = 0, \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\text{Horizontal tangents: } \left(2, \frac{3\pi}{2}\right), \left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right)$$

$$\begin{aligned}\frac{dx}{d\theta} &= (-1 + \sin \theta) \sin \theta - \cos \theta \cos \theta \\ &= -\sin \theta + \sin^2 \theta + \sin^2 \theta - 1 \\ &= 2 \sin^2 \theta - \sin \theta - 1 \\ &= (2 \sin \theta + 1)(\sin \theta - 1) = 0\end{aligned}$$

$$\sin \theta = 1, \sin \theta = -\frac{1}{2} \Rightarrow \theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\text{Vertical tangents: } \left(\frac{3}{2}, \frac{7\pi}{6}\right), \left(\frac{3}{2}, \frac{11\pi}{6}\right)$$

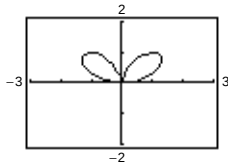
61. $r = 2 \csc \theta + 3$

$$\begin{aligned}\frac{dy}{d\theta} &= (2 \csc \theta + 3) \cos \theta + (-2 \csc \theta \cot \theta) \sin \theta \\ &= 3 \cos \theta = 0\end{aligned}$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\text{Horizontal: } \left(5, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right)$$

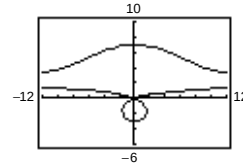
63. $r = 4 \sin \theta \cos^2 \theta$



Horizontal tangents:

$$(0, 0), (1.4142, 0.7854), (1.4142, 2.3562)$$

65. $r = 2 \csc \theta + 5$



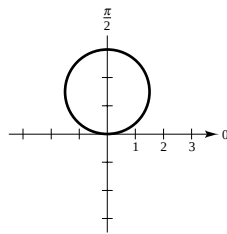
$$\text{Horizontal tangents: } \left(7, \frac{\pi}{2}\right), \left(3, \frac{3\pi}{2}\right)$$

67. $r = 3 \sin \theta$
 $r^2 = 3r \sin \theta$

$$\begin{aligned}x^2 + y^2 &= 3y \\ x^2 + \left(y - \frac{3}{2}\right)^2 &= \frac{9}{4}\end{aligned}$$

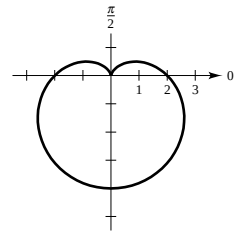
$$\text{Circle } r = \frac{3}{2}$$

$$\text{Center: } \left(0, \frac{3}{2}\right)$$

 Tangent at the pole: $\theta = 0$


69. $r = 2(1 - \sin \theta)$

Cardioid

 Symmetric to y-axis, $\theta = \frac{\pi}{2}$


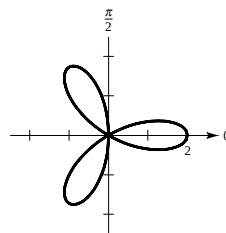
71. $r = 2 \cos(3\theta)$

Rose curve with three petals

Symmetric to the polar axis

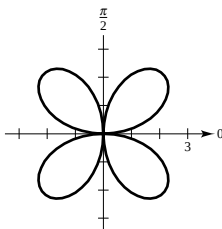
$$\text{Relative extrema: } (2, 0), \left(-2, \frac{\pi}{3}\right), \left(2, \frac{2\pi}{3}\right)$$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	2	0	$-\sqrt{2}$	-2	0	2	0	-2

 Tangents at the pole: $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$


73. $r = 3 \sin 2\theta$

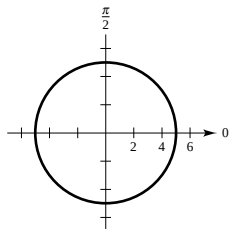
Rose curve with four petals

Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and poleRelative extrema: $\left(\pm 3, \frac{\pi}{4}\right), \left(\pm 3, \frac{5\pi}{4}\right)$ Tangents at the pole: $\theta = 0, \frac{\pi}{2}$ $(\theta = \pi, 3\pi/2 \text{ give the same tangents.})$ 

75. $r = 5$

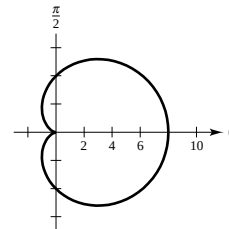
Circle radius: 5

$x^2 + y^2 = 25$



77. $r = 4(1 + \cos \theta)$

Cardioid

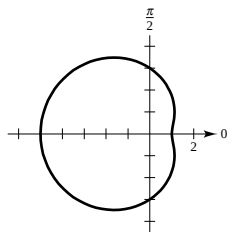


79. $r = 3 - 2 \cos \theta$

Limaçon

Symmetric to polar axis

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	1	2	3	4	5

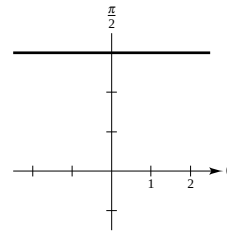


81. $r = 3 \csc \theta$

$r \sin \theta = 3$

$y = 3$

Horizontal line

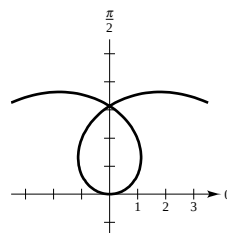


83. $r = 2\theta$

Spiral of Archimedes

Symmetric to $\theta = \frac{\pi}{2}$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π

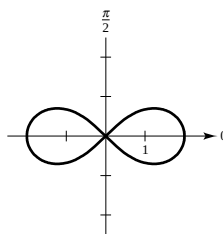
Tangent at the pole: $\theta = 0$

85. $r^2 = 4 \cos(2\theta)$

Lemniscate

Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and poleRelative extrema: $(\pm 2, 0)$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	± 2	$\pm \sqrt{2}$	0

Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$ 

87. Since

$$r = 2 - \sec \theta = 2 - \frac{1}{\cos \theta},$$

the graph has polar axis symmetry and the lengths at the pole are

$$\theta = \frac{\pi}{3}, \frac{\pi}{3}.$$

Furthermore,

$$r \Rightarrow -\infty \text{ as } \theta \Rightarrow \frac{\pi}{2}^-$$

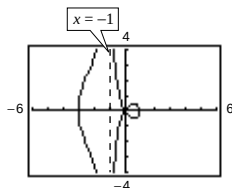
$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \frac{\pi}{2}^+.$$

$$\text{Also, } r = 2 - \frac{1}{\cos \theta} = 2 - \frac{r}{r \cos \theta} = 2 - \frac{r}{x}$$

$$rx = 2x - r$$

$$r = \frac{2x}{1+x}.$$

Thus, $r \Rightarrow \pm\infty$ as $x \Rightarrow -1$.



89. $r = \frac{2}{\theta}$

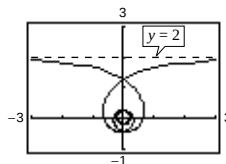
Hyperbolic spiral

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow 0$$

$$r = \frac{2}{\theta} \Rightarrow \theta = \frac{2}{r} = \frac{2 \sin \theta}{r \sin \theta} = \frac{2 \sin \theta}{y}$$

$$y = \frac{2 \sin \theta}{\theta}$$

$$\lim_{\theta \rightarrow 0} \frac{2 \sin \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{2 \cos \theta}{1} = 2$$



91. The rectangular coordinate system consists of all points of the form (x, y) where x is the directed distance from the y -axis to the point, and y is the directed distance from the x -axis to the point. Every point has a unique representation.

The polar coordinate system uses (r, θ) to designate the location of a point.

r is the directed distance to the origin and θ is the angle the point makes with the positive x -axis, measured clockwise.

Point do not have a unique polar representation.

93. $r = a$ circle

$$\theta = b \text{ line}$$

95. $r = 2 \sin \theta$ circle

Matches (c)

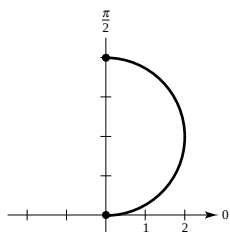
97. $r = 3(1 + \cos \theta)$

Cardioid

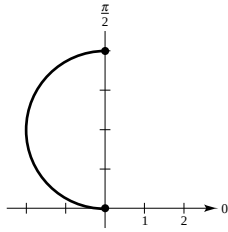
Matches (a)

99. $r = 4 \sin \theta$

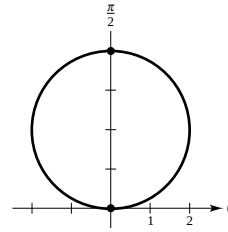
(a) $0 \leq \theta \leq \frac{\pi}{2}$



(b) $\frac{\pi}{2} \leq \theta \leq \pi$



(c) $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

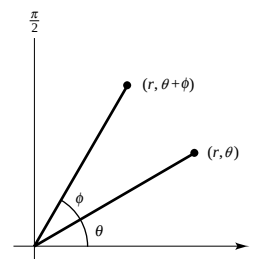


101. Let the curve $r = f(\theta)$ be rotated by ϕ to form the curve $r = g(\theta)$. If (r_1, θ_1) is a point on $r = f(\theta)$, then $(r_1, \theta_1 + \phi)$ is on $r = g(\theta)$. That is,

$$g(\theta_1 + \phi) = r_1 = f(\theta_1).$$

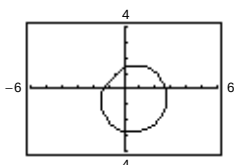
Letting $\theta = \theta_1 + \phi$, or $\theta_1 = \theta - \phi$, we see that

$$g(\theta) = g(\theta_1 + \phi) = f(\theta_1) = f(\theta - \phi).$$

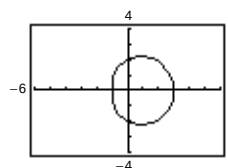


103. $r = 2 - \sin \theta$

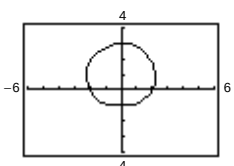
(a) $r = 2 - \sin\left(\theta - \frac{\pi}{4}\right) = 2 - \frac{\sqrt{2}}{2}(\sin \theta - \cos \theta)$



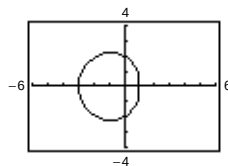
(b) $r = 2 - (-\cos \theta) = 2 + \cos \theta$



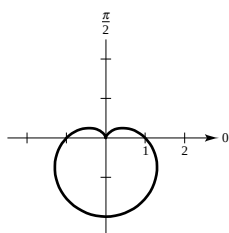
(c) $r = 2 - (-\sin \theta) = 2 + \sin \theta$



(d) $r = 2 - \cos \theta$



105. (a) $r = 1 - \sin \theta$

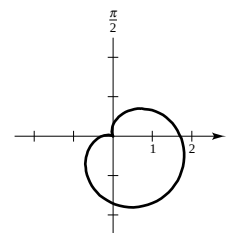


(b) $r = 1 - \sin\left(\theta - \frac{\pi}{4}\right)$

Rotate the graph of

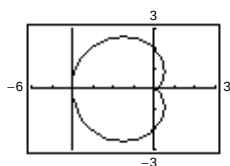
$$r = 1 - \sin \theta$$

through the angle $\pi/4$.



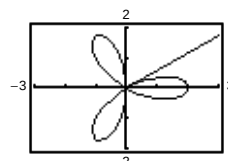
107. $\tan \psi = \frac{r}{dr/d\theta} = \frac{2(1 - \cos \theta)}{2 \sin \theta}$

At $\theta = \pi$, $\tan \psi$ is undefined $\Rightarrow \psi = \frac{\pi}{2}$.



109. $\tan \psi = \frac{r}{dr/d\theta} = \frac{2 \cos 3\theta}{-6 \sin 3\theta}$

At $\theta = \frac{\pi}{6}$, $\tan \psi = 0 \Rightarrow \psi = 0$.

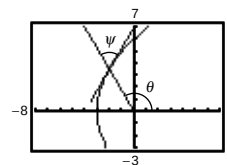


$$111. \quad r = \frac{6}{1 - \cos \theta} = 6(1 - \cos \theta)^{-1} \Rightarrow \frac{dr}{d\theta} = \frac{6 \sin \theta}{(1 - \cos \theta)^2}$$

$$\tan \psi = \frac{r}{\frac{dr}{d\theta}} = \frac{\frac{6}{(1 - \cos \theta)}}{\frac{6 \sin \theta}{(1 - \cos \theta)^2}} = \frac{1 - \cos \theta}{\sin \theta}$$

$$\text{At } \theta = \frac{2\pi}{3}, \tan \psi = \frac{1 - \left(-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} = \sqrt{3}.$$

$$\psi = \frac{\pi}{3}, (60^\circ)$$

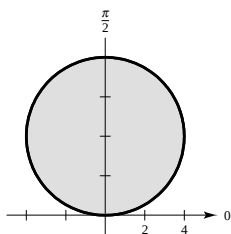


113. True

115. True

Section 9.5 Area and Arc Length in Polar Coordinates

1. (a) $r = 8 \sin \theta$



$$A = \pi(4)^2 = 16\pi$$

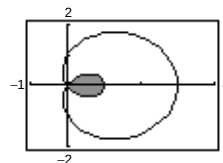
$$3. \quad A = 2 \left[\frac{1}{2} \int_0^{\pi/6} (2 \cos 3\theta)^2 d\theta \right] = 2 \left[\theta + \frac{1}{6} \sin 6\theta \right]_0^{\pi/6} = \frac{\pi}{3}$$

$$7. \quad A = 2 \left[\frac{1}{2} \int_{-\pi/2}^{\pi/2} (1 - \sin \theta)^2 d\theta \right] \\ = \left[\frac{3}{2} \theta + 2 \cos \theta - \frac{1}{4} \sin 2\theta \right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}$$

$$(b) \quad A = 2 \left(\frac{1}{2} \right) \int_0^{\pi/2} [8 \sin \theta]^2 d\theta \\ = 64 \int_0^{\pi/2} \sin^2 \theta d\theta \\ = 32 \int_0^{\pi/2} (1 - \cos 2\theta) d\theta \\ = 32 \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = 16\pi$$

$$5. \quad A = 2 \left[\frac{1}{2} \int_0^{\pi/4} (\cos 2\theta)^2 d\theta \right] \\ = \frac{1}{2} \left[\theta + \frac{1}{4} \sin 4\theta \right]_0^{\pi/4} = \frac{\pi}{8}$$

$$9. \quad A = 2 \left[\frac{1}{2} \int_{2\pi/3}^{\pi} (1 + 2 \cos \theta)^2 d\theta \right] \\ = \left[3\theta + 4 \sin \theta + \sin 2\theta \right]_{2\pi/3}^{\pi} = \frac{2\pi - 3\sqrt{3}}{2}$$

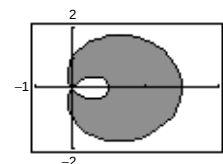


11. The area inside the outer loop is

$$2 \left[\frac{1}{2} \int_0^{2\pi/3} (1 + 2 \cos \theta)^2 d\theta \right] = \left[3\theta + 4 \sin \theta + \sin 2\theta \right]_0^{2\pi/3} = \frac{4\pi + 3\sqrt{3}}{2}.$$

From the result of Exercise 9, the area between the loops is

$$A = \left(\frac{4\pi + 3\sqrt{3}}{2} \right) - \left(\frac{2\pi - 3\sqrt{3}}{2} \right) = \pi + 3\sqrt{3}.$$



13. $r = 1 + \cos \theta$

$r = 1 - \cos \theta$

Solving simultaneously,

$$1 + \cos \theta = 1 - \cos \theta$$

$$2 \cos \theta = 0$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}.$$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-1 + \cos \theta = 1 - \cos \theta$, $\cos \theta = 1$, $\theta = 0$. Both curves pass through the pole, $(0, \pi)$, and $(0, 0)$, respectively.

Points of intersection: $\left(1, \frac{\pi}{2}\right)$, $\left(1, \frac{3\pi}{2}\right)$, $(0, 0)$

15. $r = 1 + \cos \theta$

$r = 1 - \sin \theta$

Solving simultaneously,

$$1 + \cos \theta = 1 - \sin \theta$$

$$\cos \theta = -\sin \theta$$

$$\tan \theta = -1$$

$$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}.$$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-1 + \cos \theta = 1 - \sin \theta$, $\sin \theta + \cos \theta = 2$, which has no solution. Both curves pass through the pole, $(0, \pi)$, and $(0, \pi/2)$, respectively.

Points of intersection: $\left(\frac{2 - \sqrt{2}}{2}, \frac{3\pi}{4}\right)$, $\left(\frac{2 + \sqrt{2}}{2}, \frac{7\pi}{4}\right)$, $(0, 0)$

17. $r = 4 - 5 \sin \theta$

$r = 3 \sin \theta$

Solving simultaneously,

$$4 - 5 \sin \theta = 3 \sin \theta$$

$$\sin \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

Both curves pass through the pole, $(0, \arcsin 4/5)$, and $(0, 0)$, respectively.

Points of intersection: $\left(\frac{3}{2}, \frac{\pi}{6}\right)$, $\left(\frac{3}{2}, \frac{5\pi}{6}\right)$, $(0, 0)$

19. $r = \frac{\theta}{2}$

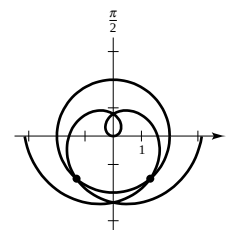
$r = 2$

Solving simultaneously, we have

$$\theta/2 = 2, \theta = 4.$$

Points of intersection:

$$(2, 4), (-2, -4)$$



21. $r = 4 \sin 2\theta$

$r = 2$

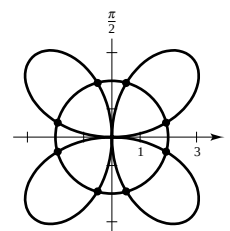
$r = 4 \sin 2\theta$ is the equation of a rose curve with four petals and is symmetric to the polar axis, $\theta = \pi/2$, and the pole. Also, $r = 2$ is the equation of a circle of radius 2 centered at the pole. Solving simultaneously,

$$4 \sin 2\theta = 2$$

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

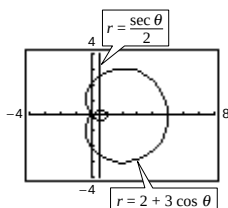
$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}.$$

Therefore, the points of intersection for one petal are $(2, \pi/12)$ and $(2, 5\pi/12)$. By symmetry, the other points of intersection are $(2, 7\pi/12)$, $(2, 11\pi/12)$, $(2, 13\pi/12)$, $(2, 17\pi/12)$, $(2, 19\pi/12)$, and $(2, 23\pi/12)$.



23. $r = 2 + 3 \cos \theta$

$$r = \frac{\sec \theta}{2}$$



The graph of $r = 2 + 3 \cos \theta$ is a limaçon with an inner loop ($b > a$) and is symmetric to the polar axis. The graph of $r = (\sec \theta)/2$ is the vertical line $x = 1/2$. Therefore, there are four points of intersection. Solving simultaneously,

$$2 + 3 \cos \theta = \frac{\sec \theta}{2}$$

$$6 \cos^2 \theta + 4 \cos \theta - 1 = 0$$

$$\cos \theta = \frac{-2 \pm \sqrt{10}}{6}$$

$$\theta = \arccos\left(\frac{-2 + \sqrt{10}}{6}\right) \approx 1.376$$

$$\theta = \arccos\left(\frac{-2 - \sqrt{10}}{6}\right) \approx 2.6068.$$

Points of intersection: $(-0.581, \pm 2.607)$, $(2.581, \pm 1.376)$

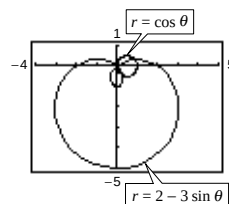
25. $r = \cos \theta$

$$r = 2 - 3 \sin \theta$$

Points of intersection:

$$(0, 0), (0.935, 0.363), (0.535, -1.006)$$

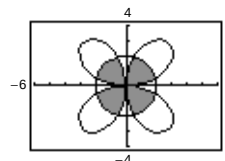
The graphs reach the pole at different times (θ values).



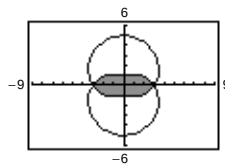
27. From Exercise 21, the points of intersection for one petal are $(2, \pi/12)$ and $(2, 5\pi/12)$. The area within one petal is

$$\begin{aligned} A &= \frac{1}{2} \int_0^{\pi/12} (4 \sin 2\theta)^2 d\theta + \frac{1}{2} \int_{\pi/12}^{5\pi/12} (2)^2 d\theta + \frac{1}{2} \int_{5\pi/12}^{\pi/2} (4 \sin 2\theta)^2 d\theta \\ &= 16 \int_0^{\pi/12} \sin^2(2\theta) d\theta + 2 \int_{\pi/12}^{5\pi/12} d\theta \quad (\text{by symmetry of the petal}) \\ &= 8 \left[\theta - \frac{1}{4} \sin 4\theta \right]_0^{\pi/12} + \left[2\theta \right]_{\pi/12}^{5\pi/12} = \frac{4\pi}{3} - \sqrt{3}. \end{aligned}$$

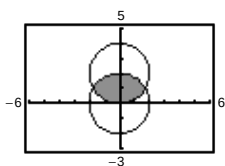
$$\text{Total area} = 4 \left(\frac{4\pi}{3} - \sqrt{3} \right) = \frac{16\pi}{3} - 4\sqrt{3} = \frac{4}{3}(4\pi - 3\sqrt{3})$$



$$\begin{aligned} 29. A &= 4 \left[\frac{1}{2} \int_0^{\pi/2} (3 - 2 \sin \theta)^2 d\theta \right] \\ &= 2 \left[11\theta + 12 \cos \theta - \sin(2\theta) \right]_0^{\pi/2} = 11\pi - 24 \end{aligned}$$

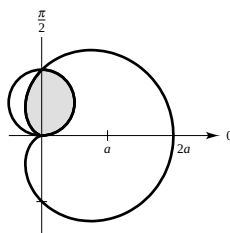


$$\begin{aligned} 31. A &= 2 \left[\frac{1}{2} \int_0^{\pi/6} (4 \sin \theta)^2 d\theta + \frac{1}{2} \int_{\pi/6}^{\pi/2} (2)^2 d\theta \right] \\ &= 16 \left[\frac{1}{2} \theta - \frac{1}{4} \sin(2\theta) \right]_0^{\pi/6} + \left[4\theta \right]_{\pi/6}^{\pi/2} \\ &= \frac{8\pi}{3} - 2\sqrt{3} = \frac{2}{3}(4\pi - 3\sqrt{3}) \end{aligned}$$

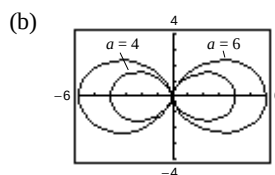


$$\begin{aligned} 33. A &= 2 \left[\frac{1}{2} \int_0^{\pi} [a(1 + \cos \theta)]^2 d\theta \right] - \frac{a^2\pi}{4} \\ &= a^2 \left[\frac{3}{2} \theta + 2 \sin \theta + \frac{\sin 2\theta}{4} \right]_0^{\pi} - \frac{a^2\pi}{4} \\ &= \frac{3a^2\pi}{2} - \frac{a^2\pi}{4} = \frac{5a^2\pi}{4} \end{aligned}$$

$$\begin{aligned}
 35. A &= \frac{\pi a^2}{8} + \frac{1}{2} \int_{\pi/2}^{\pi} [a(1 + \cos \theta)]^2 d\theta \\
 &= \frac{\pi a^2}{8} + \frac{a^2}{2} \int_{\pi/2}^{\pi} \left(\frac{3}{2} + 2 \cos \theta + \frac{\cos 2\theta}{2} \right) d\theta \\
 &= \frac{\pi a^2}{8} + \frac{a^2}{2} \left[\frac{3}{2} \theta + 2 \sin \theta + \frac{\sin 2\theta}{4} \right]_{\pi/2}^{\pi} \\
 &= \frac{\pi a^2}{8} + \frac{a^2}{2} \left[\frac{3\pi}{2} - \frac{3\pi}{4} - 2 \right] = \frac{a^2}{2} [\pi - 2]
 \end{aligned}$$



$$\begin{aligned}
 37. (a) \quad r &= a \cos^2 \theta \\
 r^3 &= ar^2 \cos^2 \theta \\
 (x^2 + y^2)^{3/2} &= ax^2
 \end{aligned}$$

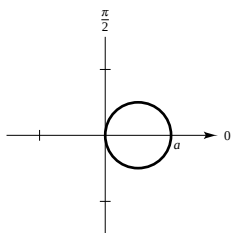


$$\begin{aligned}
 (c) \quad A &= 4 \left(\frac{1}{2} \right) \int_0^{\pi/2} [(6 \cos^2 \theta)^2 - (4 \cos^2 \theta)^2] d\theta = 40 \int_0^{\pi/2} \cos^4 \theta d\theta = 10 \int_0^{\pi/2} (1 + \cos 2\theta)^2 d\theta \\
 &= 10 \int_0^{\pi/2} \left(1 + 2 \cos 2\theta + \frac{1 - \cos 4\theta}{2} \right) d\theta = 10 \left[\frac{3}{2} \theta + \sin 2\theta + \frac{1}{8} \sin 4\theta \right]_0^{\pi/2} = \frac{15\pi}{2}
 \end{aligned}$$

$$39. r = a \cos(n\theta)$$

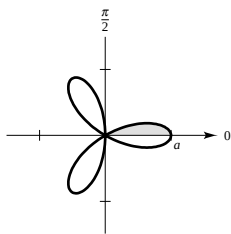
For $n = 1$:

$$\begin{aligned}
 r &= a \cos \theta \\
 A &= \pi \left(\frac{a}{2} \right)^2 = \frac{\pi a^2}{4}
 \end{aligned}$$



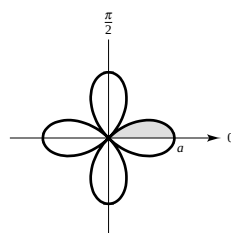
For $n = 3$:

$$\begin{aligned}
 r &= a \cos 3\theta \\
 A &= 6 \left(\frac{1}{2} \right) \int_0^{\pi/6} (a \cos 3\theta)^2 d\theta = \frac{\pi a^2}{4}
 \end{aligned}$$



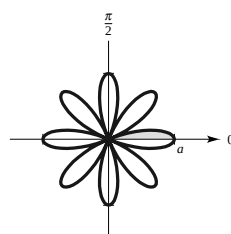
For $n = 2$:

$$\begin{aligned}
 r &= a \cos 2\theta \\
 A &= 8 \left(\frac{1}{2} \right) \int_0^{\pi/4} (a \cos 2\theta)^2 d\theta = \frac{\pi a^2}{2}
 \end{aligned}$$



For $n = 4$:

$$\begin{aligned}
 r &= a \cos 4\theta \\
 A &= 16 \left(\frac{1}{2} \right) \int_0^{\pi/8} (a \cos 4\theta)^2 d\theta = \frac{\pi a^2}{2}
 \end{aligned}$$



In general, the area of the region enclosed by $r = a \cos(n\theta)$ for $n = 1, 2, 3, \dots$ is $(\pi a^2)/4$ if n is odd and is $(\pi a^2)/2$ if n is even.

41. $r = a$

$$r' = 0$$

$$s = \int_0^{2\pi} \sqrt{a^2 + 0^2} d\theta = \left[a\theta \right]_0^{2\pi} = 2\pi a$$

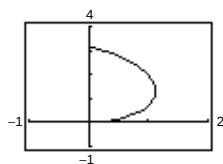
(circumference of circle of radius a)

43. $r = 1 + \sin \theta$

$$r' = \cos \theta$$

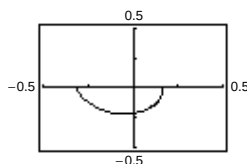
$$\begin{aligned} s &= 2 \int_{\pi/2}^{3\pi/2} \sqrt{(1 + \sin \theta)^2 + (\cos \theta)^2} d\theta \\ &= 2\sqrt{2} \int_{\pi/2}^{3\pi/2} \sqrt{1 + \sin \theta} d\theta \\ &= 2\sqrt{2} \int_{\pi/2}^{3\pi/2} \frac{-\cos \theta}{\sqrt{1 - \sin \theta}} d\theta \\ &= \left[4\sqrt{2} \sqrt{1 - \sin \theta} \right]_{\pi/2}^{3\pi/2} \\ &= 4\sqrt{2} (\sqrt{2} - 0) = 8 \end{aligned}$$

45. $r = 2\theta, 0 \leq \theta \leq \frac{\pi}{2}$



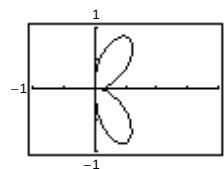
Length ≈ 4.16

47. $r = \frac{1}{\theta}, \pi \leq \theta \leq 2\pi$



Length ≈ 0.71

49. $r = \sin(3 \cos \theta), 0 \leq \theta \leq \pi$



Length ≈ 4.39

51. $r = 6 \cos \theta$

$$r' = -6 \sin \theta$$

$$\begin{aligned} S &= 2\pi \int_0^{\pi/2} 6 \cos \theta \sin \theta \sqrt{36 \cos^2 \theta + 36 \sin^2 \theta} d\theta \\ &= 72\pi \int_0^{\pi/2} \sin \theta \cos \theta d\theta \\ &= \left[36\pi \sin^2 \theta \right]_0^{\pi/2} \\ &= 36\pi \end{aligned}$$

53. $r = e^{a\theta}$

$$r' = ae^{a\theta}$$

$$\begin{aligned} S &= 2\pi \int_0^{\pi/2} e^{a\theta} \cos \theta \sqrt{(e^{a\theta})^2 + (ae^{a\theta})^2} d\theta \\ &= 2\pi \sqrt{1 + a^2} \int_0^{\pi/2} e^{2a\theta} \cos \theta d\theta \\ &= 2\pi \sqrt{1 + a^2} \left[\frac{e^{2a\theta}}{4a^2 + 1} (2a \cos \theta + \sin \theta) \right]_0^{\pi/2} \\ &= \frac{2\pi \sqrt{1 + a^2}}{4a^2 + 1} (e^{\pi a} - 2a) \end{aligned}$$

55. $r = 4 \cos 2\theta$

$$r' = -8 \sin 2\theta$$

$$\begin{aligned} S &= 2\pi \int_0^{\pi/4} 4 \cos 2\theta \sin \theta \sqrt{16 \cos^2 2\theta + 64 \sin^2 2\theta} d\theta \\ &= 32\pi \int_0^{\pi/4} \cos 2\theta \sin \theta \sqrt{\cos^2 2\theta + 4 \sin^2 2\theta} d\theta \approx 21.87 \end{aligned}$$

57. $\text{Area} = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$

$$\text{Arc length} = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} d\theta$$

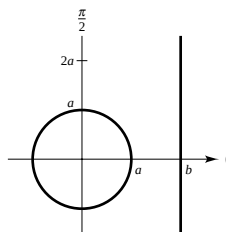
59. (a) is correct: $s \approx 33.124$.

61. Revolve $r = a$ about the line $r = b \sec \theta$ where $b > a > 0$.

$$f(\theta) = a$$

$$f'(\theta) = 0$$

$$\begin{aligned} S &= 2\pi \int_0^{2\pi} [b - a \cos \theta] \sqrt{a^2 + 0^2} d\theta \\ &= 2\pi a \left[b\theta - a \sin \theta \right]_0^{2\pi} \\ &= 2\pi a(2\pi b) = 4\pi^2 ab \end{aligned}$$



63. False. $f(\theta) = 1$ and $g(\theta) = -1$ have the same graphs.

65. In parametric form,

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Using θ instead of t , we have $x = r \cos \theta = f(\theta) \cos \theta$ and $y = r \sin \theta = f(\theta) \sin \theta$. Thus,

$$\frac{dx}{d\theta} = f'(\theta) \cos \theta - f(\theta) \sin \theta \text{ and } \frac{dy}{d\theta} = f'(\theta) \sin \theta + f(\theta) \cos \theta.$$

It follows that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2.$$

$$\text{Therefore, } s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta$$

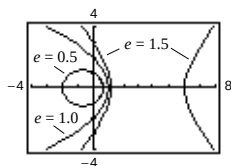
Section 9.6 Polar Equations of Conics and Kepler's Laws

1. $r = \frac{2e}{1 + e \cos \theta}$

(a) $e = 1, r = \frac{2}{1 + \cos \theta}$, parabola

(b) $e = 0.5, r = \frac{1}{1 + 0.5 \cos \theta} = \frac{2}{2 + \cos \theta}$, ellipse

(c) $e = 1.5, r = \frac{3}{1 + 1.5 \cos \theta} = \frac{6}{2 + 3 \cos \theta}$, hyperbola

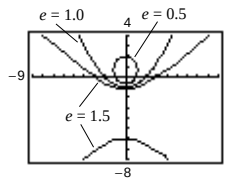


3. $r = \frac{2e}{1 - e \sin \theta}$

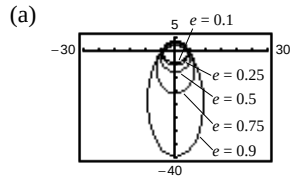
(a) $e = 1, r = \frac{2}{1 - \sin \theta}$, parabola

(b) $e = 0.5, r = \frac{1}{1 - 0.5 \sin \theta} = \frac{2}{2 - \sin \theta}$, ellipse

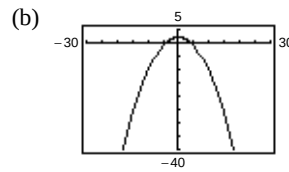
(c) $e = 1.5, r = \frac{3}{1 - 1.5 \sin \theta} = \frac{6}{2 - 3 \sin \theta}$, hyperbola



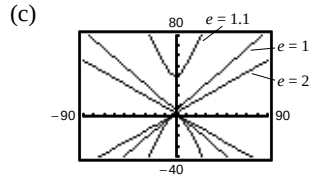
$$5. r = \frac{4}{1 + e \sin \theta}$$



The conic is an ellipse. As $e \rightarrow 1^-$, the ellipse becomes more elliptical, and as $e \rightarrow 0^+$, it becomes more circular.



The conic is a parabola.



The conic is a hyperbola. As $e \rightarrow 1^+$, the hyperbolas open more slowly, and as $e \rightarrow \infty$, they open more rapidly.

7. Parabola; Matches (c)

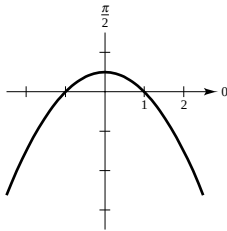
9. Hyperbola; Matches (a)

11. Ellipse; Matches (b)

$$13. r = \frac{-1}{1 - \sin \theta}$$

Parabola since $e = 1$

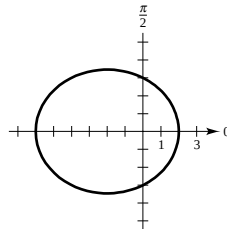
Vertex: $\left(-\frac{1}{2}, \frac{3\pi}{2}\right)$



$$15. r = \frac{6}{2 + \cos \theta} = \frac{3}{1 + (1/2) \cos \theta}$$

Ellipse since $e = \frac{1}{2} < 1$

Vertices: $(2, 0), (6, \pi)$



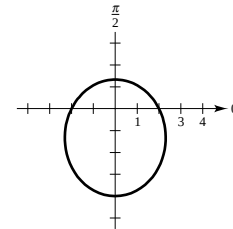
$$17. r(2 + \sin \theta) = 4$$

$$r = \frac{4}{2 + \sin \theta}$$

$$= \frac{2}{1 + (1/2) \sin \theta}$$

Ellipse since $e = \frac{1}{2} < 1$

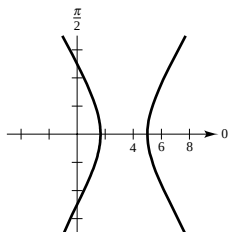
Vertices: $\left(\frac{4}{3}, \frac{\pi}{2}\right), \left(4, \frac{3\pi}{2}\right)$



$$19. r = \frac{5}{-1 + 2 \cos \theta} = \frac{-5}{1 - 2 \cos \theta}$$

Hyperbola since $e = 2 > 1$

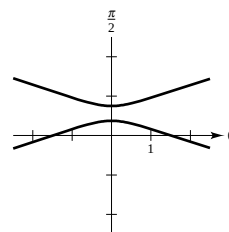
Vertices: $(5, 0), \left(-\frac{5}{3}, \pi\right)$



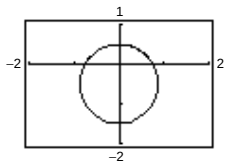
$$21. r = \frac{3}{2 + 6 \sin \theta} = \frac{3/2}{1 + 3 \sin \theta}$$

Hyperbola since $e = 3 > 1$

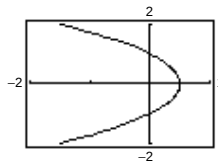
Vertices: $\left(\frac{3}{8}, \frac{\pi}{2}\right), \left(-\frac{3}{4}, \frac{3\pi}{2}\right)$



23. Ellipse



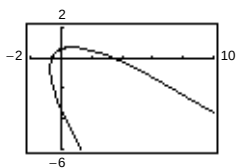
25. Parabola



$$27. r = \frac{-1}{1 - \sin\left(\theta - \frac{\pi}{4}\right)}$$

Rotate the graph of

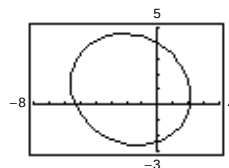
$$r = \frac{-1}{1 - \sin \theta}$$

counterclockwise through the angle $\frac{\pi}{4}$.

$$29. r = \frac{6}{2 + \cos\left(\theta + \frac{\pi}{6}\right)}$$

Rotate the graph of

$$r = \frac{6}{2 + \cos \theta}$$

clockwise through the angle $\frac{\pi}{6}$.

$$31. \text{ Change } \theta \text{ to } \theta + \frac{\pi}{4}; r = \frac{5}{5 + 3 \cos\left(\theta + \frac{\pi}{4}\right)}.$$

33. Parabola

$$e = 1, x = -1, d = 1$$

$$r = \frac{ed}{1 - e \cos \theta} = \frac{1}{1 - \cos \theta}$$

35. Ellipse

$$e = \frac{1}{2}, y = 1, d = 1$$

$$\begin{aligned} r &= \frac{ed}{1 + e \sin \theta} \\ &= \frac{1/2}{1 + (1/2) \sin \theta} \\ &= \frac{1}{2 + \sin \theta} \end{aligned}$$

37. Hyperbola

$$e = 2, x = 1, d = 1$$

$$r = \frac{ed}{1 + e \cos \theta} = \frac{2}{1 + 2 \cos \theta}$$

39. Parabola

$$\text{Vertex: } \left(1, -\frac{\pi}{2}\right)$$

$$e = 1, d = 2, r = \frac{2}{1 - \sin \theta}$$

41. Ellipse

Vertices: $(2, 0), (8, \pi)$

$$e = \frac{3}{5}, d = \frac{16}{3}$$

$$\begin{aligned} r &= \frac{ed}{1 + e \cos \theta} \\ &= \frac{16/5}{1 + (3/5) \cos \theta} \\ &= \frac{16}{5 + 3 \cos \theta} \end{aligned}$$

43. Hyperbola

$$\text{Vertices: } \left(1, \frac{3\pi}{2}\right), \left(9, \frac{3\pi}{2}\right)$$

$$e = \frac{5}{4}, d = \frac{9}{5}$$

$$\begin{aligned} r &= \frac{ed}{1 - e \sin \theta} \\ &= \frac{9/4}{1 - (5/4) \sin \theta} \\ &= \frac{9}{4 - 5 \sin \theta} \end{aligned}$$

45. Ellipse if
- $0 < e < 1$
- , parabola if
- $e = 1$
- , hyperbola if
- $e > 1$
- .

47. (a) Hyperbola ($e = 2 > 1$)

 (b) Ellipse ($e = \frac{1}{2} < 1$)

 (c) Parabola ($e = 1$)

 (d) Rotated hyperbola ($e = 3$)

49. $a = 5, c = 4, e = \frac{4}{5}, b = 3$

$$r^2 = \frac{9}{1 - (16/25) \cos^2 \theta}$$

51. $a = 3, b = 4, c = 5, e = \frac{5}{3}$

$$r^2 = \frac{-16}{1 - (25/9) \cos^2 \theta}$$

53.
$$A = 2 \left[\frac{1}{2} \int_0^\pi \left(\frac{3}{2 - \cos \theta} \right)^2 d\theta \right]$$

$$= 9 \int_0^\pi \frac{1}{(2 - \cos \theta)^2} d\theta \approx 10.88$$

 55. Vertices: (126,000, 0), (4119, π)

$$a = \frac{126,000 + 4119}{2} = 65,059.5, c = 65,059.5 - 4119 = 60,940.5, e = \frac{c}{a} = \frac{40,627}{43,373}, d = 4119 \left(\frac{84,000}{40,627} \right)$$

$$r = \frac{ed}{1 - e \cos \theta} = \frac{4119(84,000/43,373)}{1 - (40,627/43,373) \cos \theta} = \frac{345,996,000}{43,373 - 40,627 \cos \theta}$$

$$\text{When } \theta = 60^\circ, r = \frac{345,996,000}{23,059.5} \approx 15,004.49.$$

 Distance between the surface of the earth and the satellite is $r - 4000 = 11,004.49$ miles.

 57. $a = 92.957 \times 10^6$ mi, $e = 0.0167$

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta} = \frac{92,931,075.2223}{1 - 0.0167 \cos \theta}$$

 Perihelion distance: $a(1 - e) \approx 91,404,618$ mi

 Aphelion distance: $a(1 + e) \approx 94,509,382$ mi

 59. $a = 5.900 \times 10^9$ km, $e = 0.2481$

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta} \approx \frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta}$$

 Perihelion distance: $a(1 - e) = 4.436 \times 10^9$ km

 Aphelion distance: $a(1 + e) = 7.364 \times 10^9$ km

61. $r = \frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta}$

(a) $A = \frac{1}{2} \int_0^{\pi/9} \left[\frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta} \right]^2 d\theta \approx 9.341 \times 10^{18} \text{ km}^2$

$$248 \left[\frac{1}{2} \int_0^{\pi/9} \left[\frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta} \right]^2 d\theta \right] \approx 21.867 \text{ yr}$$

$$\left[\frac{1}{2} \int_0^{2\pi} \left[\frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta} \right]^2 d\theta \right]$$

(b) $\frac{1}{2} \int_\pi^{\alpha - \pi} \left[\frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta} \right]^2 d\theta = 9.341 \times 10^{18}$

$$\alpha \approx \pi + 0.8995 \text{ rad}$$

In part (a) the ray swept through a smaller angle to generate the same area since the length of the ray is longer than in part (b).

(c) $r' = \frac{(-5.537 \times 10^9)(0.2481 \sin \theta)}{(1 - 0.2481 \cos \theta)^2}$

$$s = \int_0^{\pi/9} \sqrt{\left(\frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta} \right)^2 + \left[\frac{-1.3737297 \times 10^9 \sin \theta}{(1 - 0.2481 \cos \theta)^2} \right]^2} d\theta \approx 2.559 \times 10^9 \text{ km}$$

$$\frac{2.559 \times 10^9 \text{ km}}{21.867 \text{ yr}} \approx 1.17 \times 10^8 \text{ km/yr}$$

$$s = \int_\pi^{\pi + 0.899} \sqrt{\left(\frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta} \right)^2 + \left[\frac{-1.3737297 \times 10^9 \sin \theta}{(1 - 0.2481 \cos \theta)^2} \right]^2} d\theta \approx 4.119 \times 10^9 \text{ km}$$

$$\frac{4.119 \times 10^9 \text{ km}}{21.867 \text{ yr}} \approx 1.88 \times 10^8 \text{ km/yr}$$

$$63. r_1 = \frac{ed}{1 + \sin \theta} \text{ and } r_2 = \frac{ed}{1 - \sin \theta}$$

Points of intersection: $(ed, 0), (ed, \pi)$

$$r_1: \frac{dy}{dx} = \frac{\left(\frac{ed}{1 + \sin \theta}\right)(\cos \theta) + \left(\frac{-ed \cos \theta}{(1 + \sin \theta)^2}\right)(\sin \theta)}{\left(\frac{-ed}{1 + \sin \theta}\right)(\sin \theta) + \left(\frac{-ed \cos \theta}{(1 + \sin \theta)^2}\right)(\cos \theta)}$$

At $(ed, 0), \frac{dy}{dx} = -1$. At $(ed, \pi), \frac{dy}{dx} = 1$.

$$r_2: \frac{dy}{dx} = \frac{\left(\frac{ed}{1 - \sin \theta}\right)(\cos \theta) + \left(\frac{ed \cos \theta}{(1 - \sin \theta)^2}\right)(\sin \theta)}{\left(\frac{-ed}{1 - \sin \theta}\right)(\sin \theta) + \left(\frac{ed \cos \theta}{(1 - \sin \theta)^2}\right)(\cos \theta)}$$

At $(ed, 0), \frac{dy}{dx} = 1$. At $(ed, \pi), \frac{dy}{dx} = -1$.

Therefore, at $(ed, 0)$ we have $m_1 m_2 = (-1)(1) = -1$, and at (ed, π) we have $m_1 m_2 = 1(-1) = -1$. The curves intersect at right angles.

Review Exercises for Chapter 9

1. Matches (d) - ellipse

$$5. 16x^2 + 16y^2 - 16x + 24y - 3 = 0$$

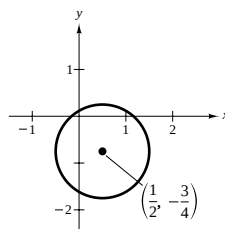
$$\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 + \frac{3}{2}y + \frac{9}{16}\right) = \frac{3}{16} + \frac{1}{4} + \frac{9}{16}$$

$$\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{4}\right)^2 = 1$$

Circle

Center: $\left(\frac{1}{2}, -\frac{3}{4}\right)$

Radius: 1



3. Matches (a) - parabola

$$7. 3x^2 - 2y^2 + 24x + 12y + 24 = 0$$

$$3(x^2 + 8x + 16) - 2(y^2 - 6y + 9) = -24 + 48 - 18$$

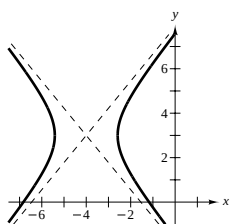
$$\frac{(x + 4)^2}{2} - \frac{(y - 3)^2}{3} = 1$$

Hyperbola

Center: $(-4, 3)$

Vertices: $(-4 \pm \sqrt{2}, 3)$

Asymptotes: $y = 3 \pm \sqrt{\frac{3}{2}}(x + 4)$



$$9. 3x^2 + 2y^2 - 12x + 12y + 29 = 0$$

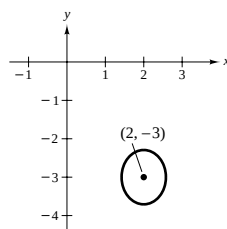
$$3(x^2 - 4x + 4) + 2(y^2 + 6y + 9) = -29 + 12 + 18$$

$$\frac{(x - 2)^2}{1/3} + \frac{(y + 3)^2}{1/2} = 1$$

Ellipse

Center: $(2, -3)$

Vertices: $\left(2, -3 \pm \frac{\sqrt{2}}{2}\right)$



11. Vertex: (0, 2)

Directrix: $x = -3$

Parabola opens to the right

 $p = 3$

$$(y - 2)^2 = 4(3)(x - 0)$$

$$y^2 - 4y - 12x + 4 = 0$$

13. Vertices: $(-3, 0), (7, 0)$ Foci: $(0, 0), (4, 0)$

Horizontal major axis

Center: $(2, 0)$

$$a = 5, c = 2, b = \sqrt{21}$$

$$\frac{(x - 2)^2}{25} + \frac{y^2}{21} = 1$$

15. Vertices: $(\pm 4, 0)$ Foci: $(\pm 6, 0)$ Center: $(0, 0)$

Horizontal transverse axis

$$a = 4, c = 6, b = \sqrt{36 - 16} = 2\sqrt{5}$$

$$\frac{x^2}{16} - \frac{y^2}{20} = 1$$

$$17. \frac{x^2}{9} + \frac{y^2}{4} = 1, a = 3, b = 2, c = \sqrt{5}, e = \frac{\sqrt{5}}{3}$$

By Example 5 of Section 9.1,

$$C = 12 \int_0^{\pi/2} \sqrt{1 - \left(\frac{5}{9}\right) \sin^2 \theta} d\theta \approx 15.87.$$

19. $y = x - 2$ has a slope of 1. The perpendicular slope is -1 .

$$y = x^2 - 2x + 2$$

$$\frac{dy}{dx} = 2x - 2 = -1 \text{ when } x = \frac{1}{2} \text{ and } y = \frac{5}{4}.$$

$$\text{Perpendicular line: } y - \frac{5}{4} = -1\left(x - \frac{1}{2}\right)$$

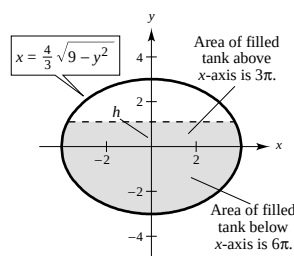
$$4x + 4y - 7 = 0$$

$$21. (a) V = (\pi ab)(\text{Length}) = 12\pi(16) = 192\pi \text{ ft}^3$$

$$\begin{aligned} (b) F &= 2(62.4) \int_{-3}^3 (3 - y) \frac{4}{3} \sqrt{9 - y^2} dy = \frac{8}{3}(62.4) \left[3 \int_{-3}^3 \sqrt{9 - y^2} dy - \int_{-3}^3 y \sqrt{9 - y^2} dy \right] \\ &= \frac{8}{3}(62.4) \left[\frac{3}{2} \left(y \sqrt{9 - y^2} + 9 \arcsin \frac{y}{3} \right) + \frac{1}{3} (9 - y^2)^{3/2} \right]_{-3}^3 \\ &= \frac{8}{3}(62.4) \left[\frac{3}{2} \left(\frac{9\pi}{2} \right) - \frac{3}{2} \left(-\frac{9\pi}{2} \right) \right] = \frac{8}{3}(62.4) \left(\frac{27\pi}{2} \right) \approx 7057.274 \end{aligned}$$

(c) You want $\frac{3}{4}$ of the total area of 12π covered. Find h so that

$$\begin{aligned} 2 \int_0^h \frac{4}{3} \sqrt{9 - y^2} dy &= 3\pi \\ \int_0^h \sqrt{9 - y^2} dy &= \frac{9\pi}{8} \\ \frac{1}{2} \left[y \sqrt{9 - y^2} + 9 \arcsin \left(\frac{y}{3} \right) \right]_0^h &= \frac{9\pi}{8} \\ h \sqrt{9 - h^2} + 9 \arcsin \left(\frac{h}{3} \right) &= \frac{9\pi}{4}. \end{aligned}$$

By Newton's Method, $h \approx 1.212$. Therefore, the total height of the water is $1.212 + 3 = 4.212$ ft.

$$(d) \text{Area of ends} = 2(12\pi) = 24\pi$$

Area of sides = (Perimeter)(Length)

$$\begin{aligned} &= 16 \int_0^{\pi/2} \left(\sqrt{1 - \left(\frac{7}{16}\right) \sin^2 \theta} \right) d\theta (16) \text{ [from Example 5 of Section 9.1]} \\ &\approx 256 \left(\frac{\pi/2}{12} \right) \left[\sqrt{1 - \left(\frac{7}{16}\right) \sin^2(0)} + 4 \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{\pi}{8}\right)} + 2 \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{\pi}{4}\right)} \right. \\ &\quad \left. + 4 \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{3\pi}{8}\right)} + \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{\pi}{2}\right)} \right] \approx 353.65 \end{aligned}$$

$$\text{Total area} = 24\pi + 353.65 \approx 429.05$$

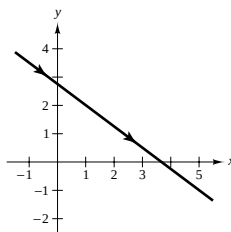
23. $x = 1 + 4t, y = 2 - 3t$

$$t = \frac{x-1}{4} \Rightarrow y = 2 - 3\left(\frac{x-1}{4}\right)$$

$$y = -\frac{3}{4}x + \frac{11}{4}$$

$$4y + 3x - 11 = 0$$

Line

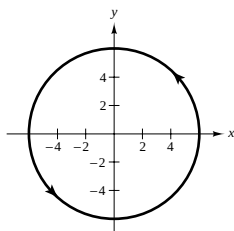


25. $x = 6 \cos \theta, y = 6 \sin \theta$

$$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$$

$$x^2 + y^2 = 36$$

Circle

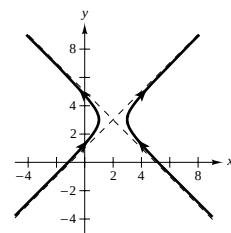


27. $x = 2 + \sec \theta, y = 3 + \tan \theta$

$$(x-2)^2 = \sec^2 \theta = 1 + \tan^2 \theta = 1 + (y-3)^2$$

$$(x-2)^2 - (y-3)^2 = 1$$

Hyperbola



29. $x = 3 + (3 - (-2))t = 3 + 5t$

$$y = 2 + (2 - 6)t = 2 - 4t$$

(other answers possible)

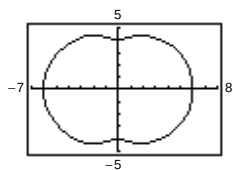
31. $\frac{(x+3)^2}{16} + \frac{(y-4)^2}{9} = 1$

$$\text{Let } \frac{(x+3)^2}{16} = \cos^2 \theta \text{ and } \frac{(y-4)^2}{9} = \sin^2 \theta.$$

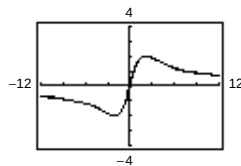
$$\text{Then } x = -3 + 4 \cos \theta \text{ and } y = 4 + 3 \sin \theta.$$

33. $x = \cos 3\theta + 5 \cos \theta$

$$y = \sin 3\theta + 5 \sin \theta$$



35. (a) $x = 2 \cot \theta, y = 4 \sin \theta \cos \theta, 0 < \theta < \pi$



(b) $(4 + x^2)y = (4 + 4 \cot^2 \theta)4 \sin \theta \cos \theta$

$$= 16 \csc^2 \theta \cdot \sin \theta \cdot \cos \theta$$

$$= 16 \frac{\cos \theta}{\sin \theta}$$

$$= 8(2 \cot \theta)$$

$$= 8x$$

37. $x = 1 + 4t$

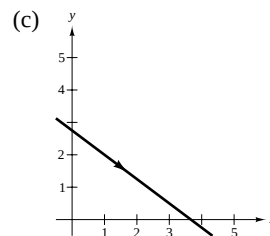
$$y = 2 - 3t$$

(a) $\frac{dy}{dx} = -\frac{3}{4}$

No horizontal tangents

(b) $t = \frac{x-1}{4}$

$$y = 2 - \frac{3}{4}(x-1) = \frac{-3x+11}{4}$$



39. $x = \frac{1}{t}$

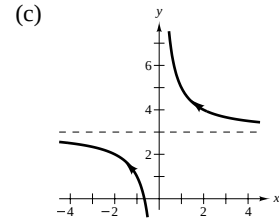
$y = 2t + 3$

(a) $\frac{dy}{dx} = \frac{2}{-1/t^2} = -2t^2$

No horizontal tangents
($t \neq 0$)

(b) $t = \frac{1}{x}$

$y = \frac{2}{x} + 3$



41. $x = \frac{1}{2t + 1}$

$y = \frac{1}{t^2 - 2t}$

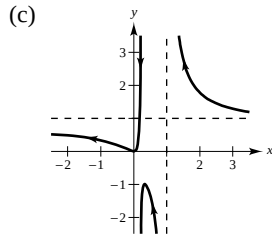
(a) $\frac{dy}{dx} = \frac{\frac{-(2t-2)}{(t^2-2t)^2}}{\frac{-2}{(2t+1)^2}} = \frac{(t-1)(2t+1)^2}{t^2(t-2)^2} = 0$ when $t = 1$.

Point of horizontal tangency: $(\frac{1}{3}, -1)$

(b) $2t + 1 = \frac{1}{x} \Rightarrow t = \frac{1}{2}\left(\frac{1}{x} - 1\right)$

$$y = \frac{1}{\frac{1}{2}\left(\frac{1-x}{x}\right)\left[\frac{1}{2}\left(\frac{1-x}{x}\right) - 2\right]}$$

$$= \frac{4x^2}{(1-x)^2 - 4x(1-x)} = \frac{4x^2}{(5x-1)(x-1)}$$



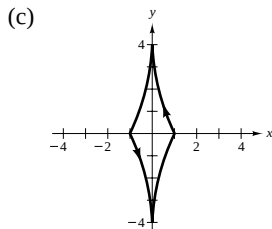
45. $x = \cos^3 \theta$

$y = 4 \sin^3 \theta$

(a) $\frac{dy}{dx} = \frac{12 \sin^2 \theta \cos \theta}{3 \cos^2 \theta (-\sin \theta)} = \frac{-4 \sin \theta}{\cos \theta} = -4 \tan \theta = 0$ when $\theta = 0, \pi$.

But, $\frac{dy}{dt} = \frac{dx}{dt} = 0$ at $\theta = 0, \pi$. Hence no points of horizontal tangency.

(b) $x^{2/3} + \left(\frac{y}{4}\right)^{2/3} = 1$



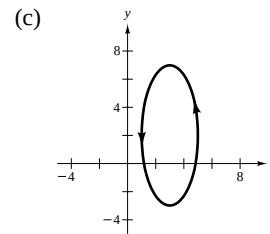
43. $x = 3 + 2 \cos \theta$

$y = 2 + 5 \sin \theta$

(a) $\frac{dy}{dx} = \frac{5 \cos \theta}{-2 \sin \theta} = -2.5 \cot \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of horizontal tangency: $(3, 7), (3, -3)$

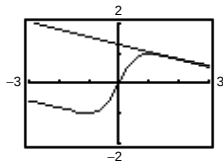
(b) $\frac{(x-3)^2}{4} + \frac{(y-2)^2}{25} = 1$



47. $x = \cot \theta$

$$y = \sin 2\theta = 2 \sin \theta \cos \theta$$

(a), (c)



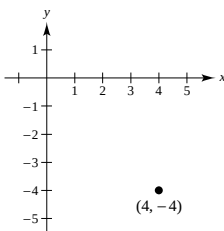
(b) At $\theta = \frac{\pi}{6}$, $\frac{dx}{d\theta} = -4$, $\frac{dy}{d\theta} = 1$, and $\frac{dy}{dx} = -\frac{1}{4}$

51. $(x, y) = (4, -4)$

$$r = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$$

$$\theta = 7\frac{\pi}{4}$$

$$(r, \theta) = \left(4\sqrt{2}, \frac{7\pi}{4}\right), \left(-4\sqrt{2}, \frac{3\pi}{4}\right)$$



53. $r = 3 \cos \theta$

$$r^2 = 3r \cos \theta$$

$$x^2 + y^2 = 3x$$

$$x^2 + y^2 - 3x = 0$$

55. $r = -2(1 + \cos \theta)$

$$r^2 = -2r(1 + \cos \theta)$$

$$x^2 + y^2 = -2(\pm\sqrt{x^2 + y^2}) - 2x$$

$$(x^2 + y^2 + 2x)^2 = 4(x^2 + y^2)$$

57. $r^2 = \cos 2\theta = \cos^2 \theta - \sin^2 \theta$

$$r^4 = r^2 \cos^2 \theta - r^2 \sin^2 \theta$$

$$(x^2 + y^2)^2 = x^2 - y^2$$

59. $r = 4 \cos 2\theta \sec \theta$

$$= 4(2 \cos^2 \theta - 1) \left(\frac{1}{\cos \theta} \right)$$

$$r \cos \theta = 8 \cos^2 \theta - 4$$

$$x = 8 \left(\frac{x^2}{x^2 + y^2} \right) - 4$$

$$x^3 + xy^2 = 4x^2 - 4y^2$$

$$y^2 = x^2 \left(\frac{4 - x}{4 + x} \right)$$

61. $(x^2 + y^2)^2 = ax^2y$

$$r^4 = a(r^2 \cos^2 \theta)(r \sin \theta)$$

$$r = a \cos^2 \theta \sin \theta$$

63. $x^2 + y^2 = a^2 \left(\arctan \frac{y}{x} \right)^2$

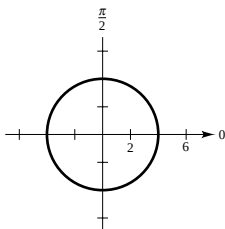
$$r^2 = a^2 \theta^2$$

65. $r = 4$

Circle of radius 4

Centered at the pole

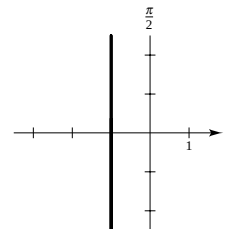
Symmetric to polar axis,

 $\theta = \pi/2$, and pole

67. $r = -\sec \theta = \frac{-1}{\cos \theta}$

$$r \cos \theta = -1, x = -1$$

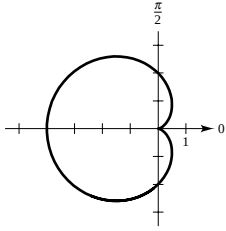
Vertical line



69. $r = -2(1 + \cos \theta)$

Cardioid

Symmetric to polar axis

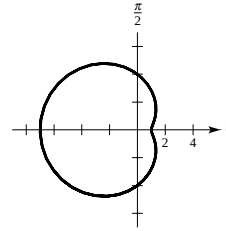


θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	-4	-3	-2	-1	0

71. $r = 4 - 3 \cos \theta$

Limaçon

Symmetric to polar axis



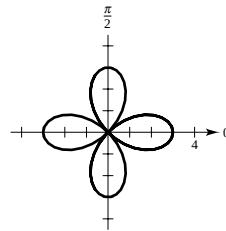
θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	1	$\frac{5}{2}$	4	$\frac{11}{2}$	7

73. $r = -3 \cos(2\theta)$

Rose curve with four petals

 Symmetric to polar axis, $\theta = \frac{\pi}{2}$, and pole

 Relative extrema: $(-3, 0)$, $(3, \frac{\pi}{2})$, $(-3, \pi)$, $(3, \frac{3\pi}{2})$

 Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$


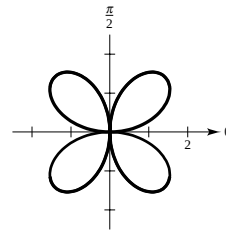
75. $r^2 = 4 \sin^2(2\theta)$

$r = \pm 2 \sin(2\theta)$

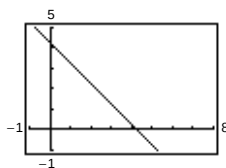
Rose curve with four petals

 Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and pole

 Relative extrema: $(\pm 2, \frac{\pi}{4})$, $(\pm 2, \frac{3\pi}{4})$

 Tangents at the pole: $\theta = 0, \frac{\pi}{2}$


77. $r = \frac{3}{\cos[\theta - (\pi/4)]}$

 Graph of $r = 3 \sec \theta$ rotated through an angle of $\pi/4$


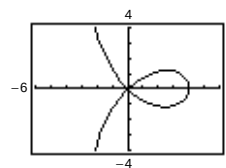
79. $r = 4 \cos 2\theta \sec \theta$

Strophoid

Symmetric to the polar axis

$$r \Rightarrow -\infty \text{ as } \theta \Rightarrow \frac{\pi^-}{2}$$

$$r \Rightarrow -\infty \text{ as } \theta \Rightarrow \frac{-\pi^+}{2}$$



81. $r = 1 - 2 \cos \theta$

(a) The graph has polar symmetry and the tangents at the pole are

$$\theta = \frac{\pi}{3}, -\frac{\pi}{3}.$$

$$(b) \frac{dy}{dx} = \frac{2 \sin^2 \theta + (1 - 2 \cos \theta) \cos \theta}{2 \sin \theta \cos \theta - (1 - 2 \cos \theta) \sin \theta}$$

$$\text{Horizontal tangents: } -4 \cos^2 \theta + \cos \theta + 2 = 0, \cos \theta = \frac{-1 \pm \sqrt{1 + 32}}{-8} = \frac{1 \pm \sqrt{33}}{8}$$

$$\text{When } \cos \theta = \frac{1 \pm \sqrt{33}}{8}, r = 1 - 2 \left(\frac{1 \pm \sqrt{33}}{8} \right) = \frac{3 \mp \sqrt{33}}{4},$$

$$\left[\frac{3 - \sqrt{33}}{4}, \arccos \left(\frac{1 + \sqrt{33}}{8} \right) \right] \approx (-0.686, 0.568)$$

$$\left[\frac{3 - \sqrt{33}}{4}, -\arccos \left(\frac{1 + \sqrt{33}}{8} \right) \right] \approx (-0.686, -0.568)$$

$$\left[\frac{3 + \sqrt{33}}{4}, \arccos \left(\frac{1 - \sqrt{33}}{8} \right) \right] \approx (2.186, 2.206)$$

$$\left[\frac{3 + \sqrt{33}}{4}, -\arccos \left(\frac{1 - \sqrt{33}}{8} \right) \right] \approx (2.186, -2.206).$$

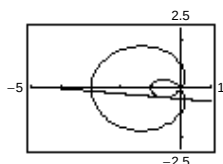
Vertical tangents:

$$\sin \theta (4 \cos \theta - 1) = 0, \sin \theta = 0, \cos \theta = \frac{1}{4},$$

$$\theta = 0, \pi, \theta = \pm \arccos \left(\frac{1}{4} \right), (-1, 0), (3, \pi)$$

$$\left(\frac{1}{2}, \pm \arccos \frac{1}{4} \right) \approx (0.5, \pm 1.318)$$

(c)



83. Circle: $r = 3 \sin \theta$

$$\frac{dy}{dx} = \frac{3 \cos \theta \sin \theta + 3 \sin \theta \cos \theta}{3 \cos \theta \cos \theta - 3 \sin \theta \sin \theta} = \frac{\sin 2\theta}{\cos^2 \theta - \sin^2 \theta} = \tan 2\theta \text{ at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \sqrt{3}$$

Limaçon: $r = 4 - 5 \sin \theta$

$$\frac{dy}{dx} = \frac{-5 \cos \theta \sin \theta + (4 - 5 \sin \theta) \cos \theta}{-5 \cos \theta \cos \theta - (4 - 5 \sin \theta) \sin \theta} \text{ at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \frac{\sqrt{3}}{9}$$

Let α be the angle between the curves:

$$\tan \alpha = \frac{\sqrt{3} - (\sqrt{3}/9)}{1 + (1/3)} = \frac{2\sqrt{3}}{3}.$$

$$\text{Therefore, } \alpha = \arctan \left(\frac{2\sqrt{3}}{3} \right) \approx 49.1^\circ.$$

85. $r = 1 + \cos \theta, r = 1 - \cos \theta$

The points $(1, \pi/2)$ and $(1, 3\pi/2)$ are the two points of intersection (other than the pole). The slope of the graph of $r = 1 + \cos \theta$ is

$$m_1 = \frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta} = \frac{-\sin^2 \theta + \cos \theta(1 + \cos \theta)}{-\sin \theta \cos \theta - \sin \theta(1 + \cos \theta)}.$$

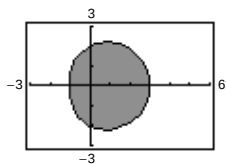
At $(1, \pi/2)$, $m_1 = -1/-1 = 1$ and at $(1, 3\pi/2)$, $m_1 = -1/1 = -1$. The slope of the graph of $r = 1 - \cos \theta$ is

$$m_2 = \frac{dy}{dx} = \frac{\sin^2 \theta + \cos \theta(1 - \cos \theta)}{\sin \theta \cos \theta - \sin \theta(1 - \cos \theta)}.$$

At $(1, \pi/2)$, $m_2 = 1/-1 = -1$ and at $(1, 3\pi/2)$, $m_2 = 1/1 = 1$. In both cases, $m_1 = -1/m_2$ and we conclude that the graphs are orthogonal at $(1, \pi/2)$ and $(1, 3\pi/2)$.

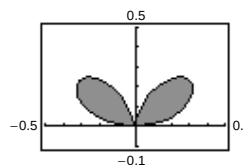
87. $r = 2 + \cos \theta$

$$A = 2 \left[\frac{1}{2} \int_0^\pi (2 + \cos \theta)^2 d\theta \right] \approx 14.14 \left(\frac{9\pi}{2} \right)$$



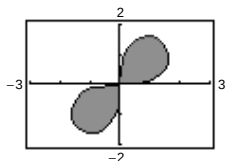
89. $r = \sin \theta \cdot \cos^2 \theta$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/2} (\sin \theta \cos^2 \theta)^2 d\theta \right] \approx 0.10 \left(\frac{\pi}{32} \right)$$



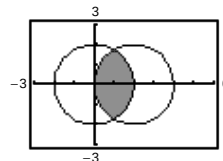
91. $r^2 = 4 \sin 2\theta$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/2} 4 \sin 2\theta d\theta \right] = 4$$



93. $r = 4 \cos \theta, r = 2$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/3} 4 d\theta + \frac{1}{2} \int_{\pi/3}^{\pi/2} (4 \cos \theta)^2 d\theta \right] \approx 4.91$$

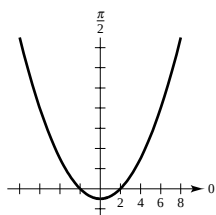


95. $s = 2 \int_0^\pi \sqrt{a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta} d\theta$

$$= 2\sqrt{2} a \int_0^\pi \sqrt{1 - \cos \theta} d\theta = 2\sqrt{2} a \int_0^\pi \frac{\sin \theta}{\sqrt{1 + \cos \theta}} d\theta = \left[-4\sqrt{2} a(1 + \cos \theta)^{1/2} \right]_0^\pi = 8a$$

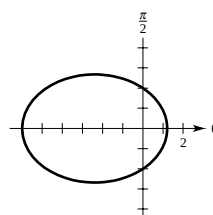
97. $r = \frac{2}{1 - \sin \theta}, e = 1$

Parabola



99. $r = \frac{6}{3 + 2 \cos \theta} = \frac{2}{1 + (2/3) \cos \theta}, e = \frac{2}{3}$

Ellipse



CHAPTER 9

Conics, Parametric Equations, and Polar Coordinates

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CHAPTER 9

Conics, Parametric Equations, and Polar Coordinates

Section 9.1 Conics and Calculus

Solutions to Even-Numbered Exercises

2. $x^2 = 8y$

Vertex: $(0, 0)$

$p = 2 > 0$

Opens upward

Matches graph (a).

6. $\frac{x^2}{9} + \frac{y^2}{9} = 1$

Circle radius 3.

Matches (g)

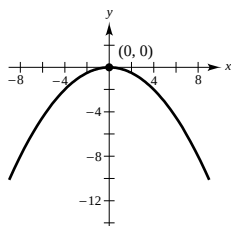
10. $x^2 + 8y = 0$

$x^2 = 4(-2)y$

Vertex: $(0, 0)$

Focus: $(0, -2)$

Directrix: $y = 2$



4. $\frac{(x-2)^2}{16} + \frac{(y+1)^2}{4} = 1$

Center: $(2, -1)$

Ellipse

Matches (b)

8. $\frac{(x-2)^2}{9} - \frac{y^2}{4} = 1$

Hyperbola

Center: $(-2, 0)$

Horizontal transverse axis.

Matches (d)

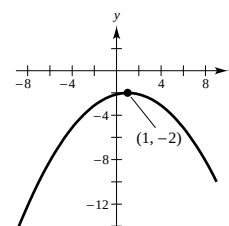
12. $(x-1)^2 + 8(y+2) = 0$

$(x-1)^2 = 4(-2)(y+2)$

Vertex: $(1, -2)$

Focus: $(1, -4)$

Directrix: $y = 0$



14. $y^2 + 6y + 8x + 25 = 0$

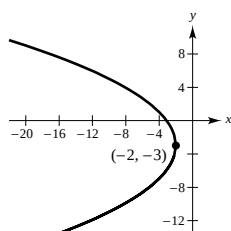
$y^2 + 6y + 9 = -8x - 25 + 9$

$(y+3)^2 = 4(-2)(x+2)$

Vertex: $(-2, -3)$

Focus: $(-4, -3)$

Directrix: $x = 0$



16. $y^2 + 4y + 8x - 12 = 0$

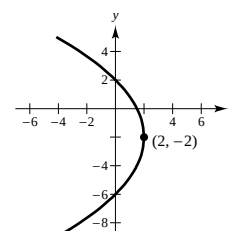
$y^2 + 4y + 4 = -8x + 12 + 4$

$(y+2)^2 = 4(-2)(x-2)$

Vertex: $(2, -2)$

Focus: $(0, -2)$

Directrix: $x = 4$



18. $y = -\frac{1}{6}(x^2 - 8x + 6) = -\frac{1}{6}(x^2 - 8x + 16 - 10)$

$$-6y = (x - 4)^2 - 10$$

$$-6y + 10 = (x - 4)^2$$

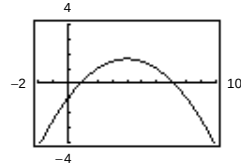
$$(x - 4)^2 = -6\left(y - \frac{5}{3}\right)$$

$$(x - 4)^2 = 4\left(-\frac{3}{2}\right)\left(y - \frac{5}{3}\right)$$

Vertex: $\left(4, \frac{5}{3}\right)$

Focus: $\left(4, \frac{1}{6}\right)$

Directrix: $y = \frac{19}{6}$



20. $x^2 - 2x + 8y + 9 = 0$

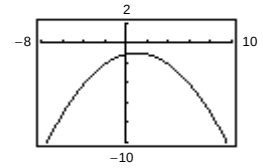
$$x^2 - 2x + 1 = -8y - 9 + 1$$

$$(x - 1)^2 = 4(-2)(y + 1)$$

Vertex: $(1, -1)$

Focus: $(1, -3)$

Directrix: $y = 1$



22. $(x + 1)^2 = 4(-2)(y - 2)$

$$x^2 + 2x + 8y - 15 = 0$$

24. Vertex: $(0, 2)$

$$(y - 2)^2 = 4(2)(x - 0)$$

$$y^2 - 8x - 4y + 4 = 0$$

26. $y = 4 - (x - 2)^2 = 4x - x^2$

$$x^2 - 4x + y = 0$$

28. From Example 2: $4p = 8$ or $p = 2$

Vertex: $(4, 0)$

$$(x - 4)^2 = 8(y - 0)$$

$$x^2 - 8x - 8y + 16 = 0$$

30. $5x^2 + 7y^2 = 70$

$$\frac{x^2}{14} + \frac{y^2}{10} = 1$$

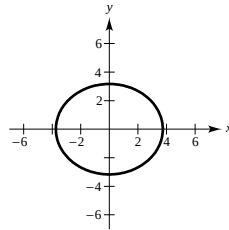
$$a^2 = 14, b^2 = 10, c^2 = 4$$

Center: $(0, 0)$

Foci: $(\pm 2, 0)$

Vertices: $(\pm \sqrt{14}, 0)$

$$e = \frac{2}{\sqrt{14}} = \frac{\sqrt{14}}{7}$$



32. $\frac{(x + 2)^2}{1} + \frac{(y + 4)^2}{1/4} = 1$

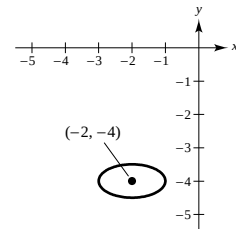
$$a^2 = 1, b^2 = \frac{1}{4}, c^2 = \frac{3}{4}$$

Center: $(-2, -4)$

Foci: $\left(-2 \pm \frac{\sqrt{3}}{2}, -4\right)$

Vertices: $(-1, -4), (-3, -4)$

$$e = \frac{\sqrt{3}}{2}$$



34. $16x^2 + 25y^2 - 64x + 150y + 279 = 0$

$$16(x^2 - 4x + 4) + 25(y^2 + 6y + 0) = -279 + 64 + 225$$

$$= 10$$

$$\frac{(x - 2)^2}{(5/8)} + \frac{(y + 3)^2}{(2/5)} = 1$$

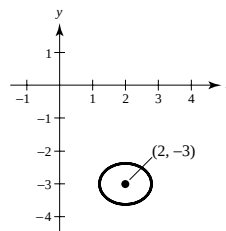
$$a^2 = \frac{5}{8}, b^2 = \frac{2}{5}, c^2 = a^2 - b^2 = \frac{9}{40}$$

Center: $(2, -3)$

Foci: $\left(2 \pm \frac{3\sqrt{10}}{20}, -3\right)$

Vertices: $\left(2 \pm \frac{\sqrt{10}}{4}, -3\right)$

$$e = \frac{c}{a} = \frac{3}{5}$$



36. $36x^2 + 9y^2 + 48x - 36y + 43 = 0$

$$36\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) + 9(y^2 - 4y + 4) = -43 + 16 + 36$$

$$= 9$$

$$\frac{[x + (2/3)]^2}{1/4} + \frac{(y - 2)^2}{1} = 1$$

$$a^2 = 1, b^2 = \frac{1}{4}, c^2 = \frac{3}{4}$$

Center: $\left(-\frac{2}{3}, 2\right)$

Foci: $\left(-\frac{2}{3}, 2 \pm \frac{\sqrt{3}}{2}\right)$

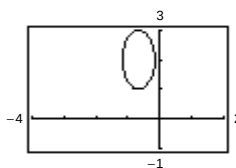
Vertices: $\left(-\frac{2}{3}, 3\right), \left(-\frac{2}{3}, 1\right)$

Solve for y:

$$9(y^2 - 4y + 4) = -36x^2 - 48x - 43 + 36$$

$$(y - 2)^2 = \frac{-(36x^2 + 48x + 7)}{9}$$

$$y = 2 \pm \frac{1}{3}\sqrt{-(36x^2 + 48x + 7)} \quad (\text{Graph each of these separately.})$$



38. $2x^2 + y^2 + 4.8x - 6.4y + 3.12 = 0$

$$50x^2 + 25y^2 + 120x - 160y + 78 = 0$$

$$50\left(x^2 + \frac{12}{5}x + \frac{36}{25}\right) + 25\left(y^2 - \frac{32}{5}y + \frac{256}{25}\right) = -78 + 72 + 256 = 250$$

$$\frac{[x + (6/5)]^2}{5} + \frac{[y - (16/5)]^2}{10} = 1$$

$$a^2 = 10, b^2 = 5, c^2 = 5$$

Center: $\left(-\frac{6}{5}, \frac{16}{5}\right)$

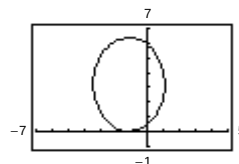
Foci: $\left(-\frac{6}{5}, \frac{16}{5} \pm \sqrt{5}\right)$

Vertices: $\left(-\frac{6}{5}, \frac{16}{5} \pm \sqrt{10}\right)$

Solve for y: $(y^2 - 6.4y + 10.24) = -2x^2 - 4.8x - 3.12 + 10.24$

$$(y - 3.2)^2 = 7.12 - 4x - 2x^2$$

$$y = 3.2 \pm \sqrt{7.12 - 4x - 2x^2} \quad (\text{Graph each of these separately.})$$



40. Vertices: $(0, 2), (4, 2)$

Eccentricity: $\frac{1}{2}$

Horizontal major axis

Center: $(2, 2)$

$$a = 2, c = 1 \Rightarrow b = \sqrt{3}$$

$$\frac{(x - 2)^2}{4} + \frac{(y - 2)^2}{3} = 1$$

42 Foci: $(0, \pm 5)$

Major axis length: 14

Vertical major axis

Center: $(0, 0)$

$$c = 5, a = 7 \Rightarrow b = \sqrt{24}$$

$$\frac{x^2}{24} + \frac{y^2}{49} = 1$$

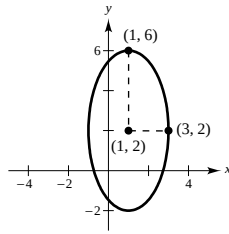
44. Center: (1, 2)

Vertical major axis

Points on ellipse: (1, 6), (3, 2)

From the sketch, we can see that
 $h = 1, k = 2, a = 4, b = 2$

$$\frac{(x-1)^2}{4} + \frac{(y-2)^2}{16} = 1.$$



48. $\frac{(y+1)^2}{12^2} - \frac{(x-4)^2}{5^2} = 1$

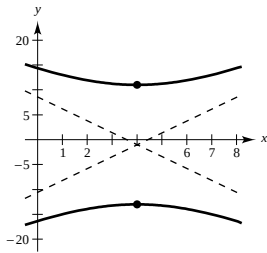
$$a = 12, b = 5, c = \sqrt{a^2 + b^2} = 13$$

Center: (4, -1)

Vertices: (4, 11), (4, -13)

Foci: (4, -14), (4, 12)

Asymptotes: $y = -1 \pm \frac{12}{5}(x - 4)$



46. $\frac{x^2}{25} - \frac{y^2}{9} = 1$

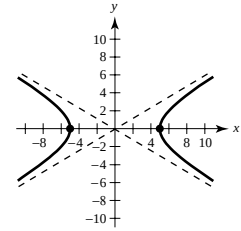
$$a = 5, b = 3, c = \sqrt{a^2 + b^2} = \sqrt{34}$$

Center: (0, 0)

Vertices: ($\pm 5, 0$)

Foci: ($\pm \sqrt{34}, 0$)

Asymptotes: $y = \pm \frac{3}{5}x$



50. $y^2 - 9x^2 + 36x - 72 = 0$

$$y^2 - 9(x^2 - 4x + 4) = 72 - 36 = 36$$

$$\frac{y^2}{36} - \frac{(x-2)^2}{4} = 1$$

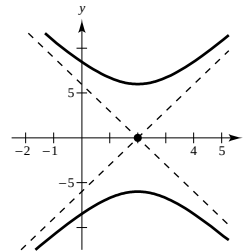
$$a = 6, b = 2, c = \sqrt{a^2 + b^2} = 2\sqrt{10}$$

Center: (2, 0)

Vertices: (2, 6), (2, -6)

Foci: $(2, 2\sqrt{10}), (2, -2\sqrt{10})$

Asymptotes: $y = \pm 3(x - 2)$



52. $9(x^2 + 6x + 9) - 4(y^2 - 2y + 1) = -78 + 81 - 4 = -1$

$$9(x+3)^2 - 4(y-1)^2 = -1$$

$$\frac{(y-1)^2}{1/4} - \frac{(x+3)^2}{1/9} = 1$$

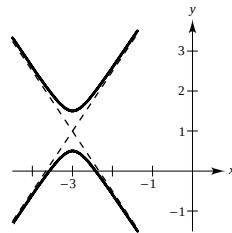
$$a = \frac{1}{2}, b = \frac{1}{3}, c = \frac{\sqrt{13}}{6}$$

Center: (-3, 1)

Vertices: $(-3, \frac{1}{2}), (-3, \frac{3}{2})$

Foci: $(-3, 1 \pm \frac{\sqrt{13}}{6})$

Asymptotes: $y = 1 \pm \frac{3}{2}(x + 3)$



54. $9x^2 - y^2 + 54x + 10y + 55 = 0$

$$9(x^2 + 6x + 9) - (y^2 - 10y + 25) = -55 + 81 - 25 = 1$$

$$\frac{(x+3)^2}{1/9} - \frac{(y-5)^2}{1} = 1$$

$$a = \frac{1}{3}, b = 1, c = \frac{\sqrt{10}}{3}$$

Center: (-3, 5)

Vertices: $(-3 \pm \frac{1}{3}, 5)$

Foci: $(-3 \pm \frac{\sqrt{10}}{3}, 5)$

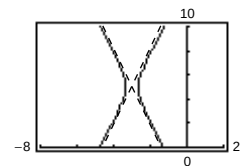
Solve for y :

$$y^2 - 10y + 25 = 9x^2 + 54x + 55 + 25$$

$$(y-5)^2 = 9x^2 + 54x + 80$$

$$y = 5 \pm \sqrt{9x^2 + 54x + 80}$$

(Graph each curve separately.)



56. $3y^2 - x^2 + 6x - 12y = 0$

$$3(y^2 - 4y + 4) - (x^2 - 6x + 9) = 0 + 12 - 9 = 3$$

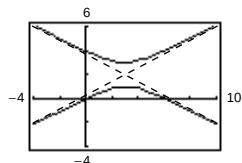
$$\frac{(y-2)^2}{1} - \frac{(x-3)^2}{3} = 1$$

$$a = 1, b = \sqrt{3}, c = 2$$

Center: (3, 2)

Vertices: (3, 1), (3, 3)

Foci: (3, 0), (3, 4)

Solve for y :

$$3(y^2 - 4y + 4) = x^2 - 6x + 12$$

$$(y-2)^2 = \frac{x^2 - 6x + 12}{3}$$

$$y = 2 \pm \sqrt{\frac{x^2 - 6x + 12}{3}}$$

(Graph each curve separately.)

60. Vertices: (2, ± 3)

Foci: (2, ± 5)

Vertical transverse axis

Center: (2, 0)

$$a = 3, c = 5, b^2 = c^2 - a^2 = 16$$

$$\text{Therefore, } \frac{y^2}{9} - \frac{(x-2)^2}{16} = 1.$$

64. Focus: (10, 0)

$$\text{Asymptotes: } y = \pm \frac{3}{4}x$$

Horizontal transverse axis

Center: (0, 0) since asymptotes intersect at the origin.

$$c = 10$$

$$\text{Slopes of asymptotes: } \pm \frac{b}{a} = \pm \frac{3}{4} \text{ and } b = \frac{3}{4}a$$

$$c^2 = a^2 + b^2 = 100$$

Solving these equations, we have $a^2 = 64$ and $b^2 = 36$.

Therefore, the equation is

$$\frac{x^2}{64} - \frac{y^2}{36} = 1.$$

68. $4x^2 - y^2 - 4x - 3 = 0$

$$A = 4, C = -1$$

$$AC < 0$$

Hyperbola

70. $25x^2 - 10x - 200y - 119 = 0$

$$A = 25, C = 0$$

Parabola

58. Vertices: (0, ± 3)

$$\text{Asymptotes: } y = \pm 3x$$

Vertical transverse axis

$$a = 3$$

$$\text{Slopes of asymptotes: } \pm \frac{a}{b} = \pm 3$$

Thus, $b = 1$. Therefore,

$$\frac{y^2}{9} - \frac{x^2}{1} = 1.$$

62. Center: (0, 0)

Vertex: (3, 0)

Focus: (5, 0)

Horizontal transverse axis

$$a = 3, c = 5, b^2 = c^2 - a^2 = 16$$

$$\text{Therefore, } \frac{x^2}{9} - \frac{y^2}{16} = 1.$$

66. (a) $\frac{y^2}{4} - \frac{x^2}{2} = 1, y^2 - 2x^2 = 4, 2yy' - 4x = 0,$

$$y' = \frac{4x}{2y} = \frac{2x}{y}$$

$$\text{At } x = 4: y = \pm 6, y' = \frac{\pm 2(4)}{6} = \pm \frac{4}{3}$$

$$\text{At } (4, 6): y - 6 = -\frac{4}{3}(x - 4) \text{ or } 4x - 3y + 2 = 0$$

$$\text{At } (4, -6): y + 6 = -\frac{4}{3}(x - 4) \text{ or } 4x + 3y + 2 = 0$$

(b) From part (a) we know that the slopes of the normal lines must be $\mp 3/4$.

$$\text{At } (4, 6): y - 6 = -\frac{3}{4}(x - 4) \text{ or } 3x + 4y - 36 = 0$$

$$\text{At } (4, -6): y + 6 = \frac{3}{4}(x - 4) \text{ or } 3x - 4y - 36 = 0$$

72. $y^2 - x - 4y - 5 = 0$

$$A = 0, C = 1$$

Parabola

$$74. \quad 2x^2 - 2xy = 3y - y^2 - 2xy$$

$$2x^2 + y^2 - 3y = 0$$

$$A = 2, C = 1, AC > 0$$

Ellipse

78. (a) An ellipse is the set of all points (x, y) , the sum of whose distance from two distinct fixed points (foci) is constant.

$$(b) \quad \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \text{ or } \frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

82. Assume that the vertex is at the origin.

$$(a) \quad x^2 = 4py$$

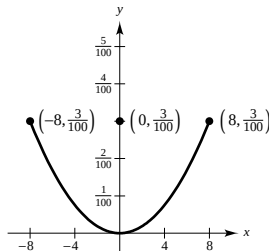
$$8^2 = 4p\left(\frac{3}{100}\right)$$

$$\frac{1600}{3} = p$$

$$x^2 = 4\left(\frac{1600}{3}\right)y = \frac{6400}{3}y$$

- (b) The deflection is 1 cm when

$$y = \frac{2}{100} \Rightarrow x = \pm \sqrt{\frac{128}{3}} \approx \pm 6.53 \text{ meters.}$$



$$76. \quad 9x^2 + 54x + 81 = 36 - 4(y^2 - 4y + 4)$$

$$9x^2 + 4y^2 + 54x - 16y + 61 = 0$$

$$A = 9, C = 4, AC > 0$$

Ellipse

$$80. \quad e = \frac{c}{a}, c = \sqrt{a^2 - b^2} \quad 0 < e < 1$$

For $e \approx 0$, the ellipse is nearly circular.

For $e \approx 1$, the ellipse is elongated.

84. (a) Without loss of generality, place the coordinate system so that the equation of the parabola is $x^2 = 4py$ and, hence,

$$y' = \left(\frac{1}{2p}\right)x.$$

Therefore, for distinct tangent lines, the slopes are unequal and the lines intersect.

$$(b) \quad x^2 - 4x - 4y = 0$$

$$2x - 4 - 4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{1}{2}x - 1$$

At $(0, 0)$, the slope is -1 : $y = -x$. At $(6, 3)$, the slope is 2 : $y = 2x - 9$. Solving for x ,

$$-x = 2x - 9$$

$$-3x = -9$$

$$x = 3$$

$$y = -3.$$

Point of intersection: $(3, -3)$

86. The focus of $x^2 = 8y = 4(2)y$ is $(0, 2)$. The distance from a point on the parabola, $(x, x^2/8)$, and the focus, $(0, 2)$, is

$$d = \sqrt{(x-0)^2 + \left(\frac{x^2}{8} - 2\right)^2}.$$

Since d is minimized when d^2 is minimized, it is sufficient to minimize the function

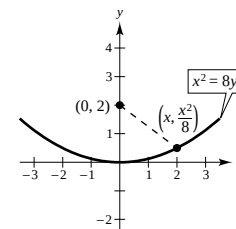
$$f(x) = x^2 + \left(\frac{x^2}{8} - 2\right)^2.$$

$$f'(x) = 2x + 2\left(\frac{x^2}{8} - 2\right)\left(\frac{x}{4}\right) = \frac{x^3}{16} + x.$$

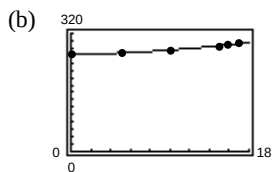
$f'(x) = 0$ implies that

$$\frac{x^3}{16} + x = x\left(\frac{x^2}{16} + 1\right) = 0 \Rightarrow x = 0.$$

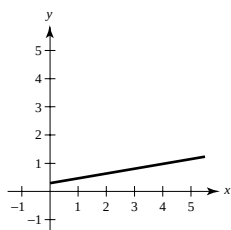
This is a minimum by the First Derivative Test. Hence, the closest point to the focus is the vertex, $(0, 0)$.



88. (a) $C = 0.0853t^2 + 0.2917t + 263.3559$



(c) $\frac{dC}{dt} = 0.1706t + 0.2971$



The consumption of fruits is increasing at a rate of 0.1706 pounds/year.

92. $x^2 = 20y$

$$y = \frac{x^2}{20}$$

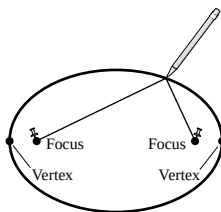
$$y' = \frac{x}{10}$$

$$S = 2\pi \int_0^r x \sqrt{1 + \left(\frac{x}{10}\right)^2} dx = 2\pi \int_0^r \frac{x\sqrt{100 + x^2}}{10} dx$$

$$= \left[\frac{\pi}{10} \cdot \frac{2}{3} (100 + x^2)^{3/2} \right]_0^r = \frac{\pi}{15} [(100 + r^2)^{3/2} - 1000]$$

96. (a) At the vertices we notice that the string is horizontal and has a length of $2a$.

- (b) The thumbtacks are located at the foci and the length of string is the constant sum of the distances from the foci.



100. $e = \frac{A - P}{A + P}$

$$= \frac{(122,000 + 4000) - (119 + 4000)}{(122,000 + 4000) + (119 + 4000)}$$

$$= \frac{121,881}{130,119} \approx 0.9367$$

90. $x = \frac{1}{4}y^2$

$$x' = \frac{1}{2}y$$

$$1 + (x')^2 = 1 + \frac{y^2}{4}$$

$$s = \int_0^4 \sqrt{1 + \left(\frac{y^2}{4}\right)} dy = \frac{1}{2} \int_0^4 \sqrt{4 + y^2} dy$$

$$= \frac{1}{4} \left[y\sqrt{4 + y^2} + 4 \ln|y + \sqrt{4 + y^2}| \right]_0^4$$

$$= \frac{1}{4} [4\sqrt{20} + 4 \ln|4 + \sqrt{20}| - 4 \ln 2]$$

$$= 2\sqrt{5} + \ln(2 + \sqrt{5}) \approx 5.916$$

94. $A = 2 \int_0^h \sqrt{4py} dy$

$$= 4\sqrt{p} \int_0^h y^{1/2} dy$$

$$= \left[4\sqrt{p} \left(\frac{2}{3} \right) y^{3/2} \right]_0^h$$

$$= \frac{8}{3} \sqrt{p} h^{3/2}$$

98. $e = \frac{c}{a}$

$$0.0167 = \frac{c}{149,570,000}$$

$$c \approx 2,497,819$$

$$\text{Least distance: } a - c = 147,072,181 \text{ km}$$

$$\text{Greatest distance: } a + c = 152,067,819 \text{ km}$$

102. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(b^2/a^2)} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(a^2 - c^2)/a^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

As $e \rightarrow 0$, $1 - e^2 \rightarrow 1$ and we have

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \text{ or the circle } x^2 + y^2 = a^2.$$

$$104. \frac{x^2}{(4.5)^2} + \frac{y^2}{(2.5)^2} = 1$$

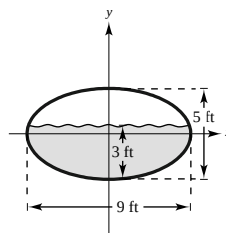
$$x^2 = (4.5)^2 \left[1 - \frac{y^2}{(2.5)^2} \right]$$

$$x = \pm \frac{9}{5} \sqrt{(2.5)^2 - y^2}$$

$$V = (\text{Area of bottom})(\text{Length}) + (\text{Area of top})(\text{Length})$$

$$V = \left[\frac{\pi(4.5)(2.5)}{2} \right](16) + 16 \int_0^{0.5} \frac{9}{5} \sqrt{(2.5)^2 - y^2} dy \quad (\text{Recall: Area of ellipse is } \pi ab.)$$

$$= 90\pi + \frac{144}{5} \cdot \frac{1}{2} \left[y \sqrt{(2.5)^2 - y^2} + (2.5)^2 \arcsin \frac{y}{2.5} \right]_0^{0.5} = 90\pi + \frac{72}{5} \left[0.5\sqrt{6} + (2.5)^2 \arcsin \frac{1}{5} \right] \approx 318.5 \text{ ft}^3$$



$$106. 9x^2 + 4y^2 + 36x - 24y + 36 = 0$$

$$18x + 8yy' + 36 - 24y' = 0$$

$$(8y - 24)y' = -(18x + 36)$$

$$y' = \frac{-(18x + 36)}{8y - 24}$$

$y' = 0$ when $x = -2$. y' undefined when $y = 3$.

At $x = -2$, $y = 0$ or 6 .

Endpoints of major axis: $(-2, 0)$, $(-2, 6)$

At $y = 3$, $x = 0$ or -4 .

Endpoints of minor axis: $(0, 3)$, $(-4, 3)$

Note: Equation of ellipse is $\frac{(x+2)^2}{4} + \frac{(y-3)^2}{9} = 1$

$$108. (a) A = 4 \int_0^4 \frac{3}{4} \sqrt{16 - x^2} dx = \frac{3}{2} \left[x \sqrt{16 - x^2} + 16 \arcsin \frac{x}{4} \right]_0^4 = 12\pi$$

$$(b) \text{ Disk: } V = 2\pi \int_0^4 \frac{9}{16} (16 - x^2) dx = \frac{9\pi}{8} \left[16x - \frac{1}{3}x^3 \right]_0^4 = 48\pi$$

$$y = \frac{3}{4} \sqrt{16 - x^2}$$

$$y' = \frac{-3x}{4\sqrt{16 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{9x^2}{16(16 - x^2)}}$$

$$S = 2(2\pi) \int_0^4 \frac{3}{4} \sqrt{16 - x^2} \sqrt{\frac{16(16 - x^2) + 9x^2}{16(16 - x^2)}} dx = 4\pi \int_0^4 \frac{3}{4} \sqrt{16 - x^2} \frac{\sqrt{256 - 7x^2}}{4\sqrt{16 - x^2}} dx = \frac{3\pi}{4} \int_0^4 \sqrt{256 - 7x^2} dx$$

$$= \frac{3\pi}{8\sqrt{7}} \left[\sqrt{7}x \sqrt{256 - 7x^2} + 256 \arcsin \frac{\sqrt{7}x}{16} \right]_0^4 = \frac{3\pi}{8\sqrt{7}} \left(48\sqrt{7} + 256 \arcsin \frac{\sqrt{7}}{4} \right) \approx 138.93$$

—CONTINUED—

108. —CONTINUED—

$$(c) \text{ Shell: } V = 4\pi \int_0^4 x \left[\frac{3}{4} \sqrt{16 - x^2} \right] dx = 3\pi \left[\left(-\frac{1}{2} \right) \left(\frac{2}{3} \right) (16 - x^2)^{3/2} \right]_0^4 = 64\pi$$

$$x = \frac{4}{3} \sqrt{9 - y^2}$$

$$x' = \frac{-4y}{3\sqrt{9 - y^2}}$$

$$\sqrt{1 + (x')^2} = \sqrt{1 + \frac{16y^2}{9(9 - y^2)}}$$

$$S = 2(2\pi) \int_0^3 \frac{4}{3} \sqrt{9 - y^2} \sqrt{\frac{9(9 - y^2) + 16y^2}{9(9 - y^2)}} dy$$

$$= 4\pi \int_0^3 \frac{4}{9} \sqrt{81 + 7y^2} dy$$

$$= \frac{16}{9} \left(\frac{\pi}{2\sqrt{7}} \right) \left[\sqrt{7}y \sqrt{81 + 7y^2} + 81 \ln \left| \sqrt{7}y + \sqrt{81 + 7y^2} \right| \right]_0^3$$

$$= \frac{8\pi}{9\sqrt{7}} 3\sqrt{7}(12) + 81 \ln(3\sqrt{7} + 12) - 81 \ln 9 \approx 168.53$$

$$110. (a) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

$$y' = -\frac{xb^2}{ya^2}$$

$$\text{At } P, y' = -\frac{b^2}{a^2} \cdot \frac{x_0}{y_0} = m.$$

$$(b) \text{ Slope of line through } (-c, 0) \text{ and } (x_0, y_0): m_1 = \frac{y_0}{x_0 + c}$$

$$\text{Slope of line through } (c, 0) \text{ and } (x_0, y_0): m_2 = \frac{y_0}{x_0 - c}$$

$$(c) \tan \alpha = \frac{m_2 - m}{1 + m_2 m} = \frac{\frac{y_0}{x_0 - c} - \left(-\frac{b^2 x_0}{a^2 y_0} \right)}{1 + \left(\frac{y_0}{x_0 - c} \right) \left(-\frac{b^2 x_0}{a^2 y_0} \right)} = \frac{a^2 y_0^2 + b^2 x_0(x_0 - c)}{a^2 y_0(x_0 - c) - b^2 x_0 y_0}$$

$$= \frac{a^2 y_0^2 + b^2 x_0^2 - b^2 x_0 c}{x_0 y_0(a^2 - b^2) - a^2 y_0 c} = \frac{a^2 b^2 - b^2 x_0 c}{x_0 y_0 c^2 - a^2 y_0 c} = \frac{b^2(a^2 - x_0 c)}{y_0 c(x_0 c - a^2)} = -\frac{b^2}{y_0 c}$$

$$\alpha = \arctan\left(-\frac{b^2}{y_0 c}\right) = -\arctan\left(\frac{b^2}{y_0 c}\right)$$

$$\tan \beta = \frac{m_1 - m}{1 + m_1 m} = \frac{\frac{y_0}{x_0 + c} - \left(-\frac{b^2 x_0}{a^2 y_0} \right)}{1 + \left(\frac{y_0}{x_0 + c} \right) \left(-\frac{b^2 x_0}{a^2 y_0} \right)} = \frac{a^2 y_0^2 + b^2 x_0(x_0 + c)}{a^2 y_0(x_0 + c) - b^2 x_0 y_0}$$

$$= \frac{a^2 y_0^2 + b^2 x_0^2 + b^2 x_0 c}{a^2 x_0 y_0 + a^2 c y_0 - b^2 x_0 y_0} = \frac{a^2 b^2 + b^2 x_0 c}{x_0 y_0(a^2 - b^2) + a^2 c y_0} = \frac{b^2(a^2 + x_0 c)}{y_0 c(x_0 c + a^2)} = \frac{b^2}{y_0 c}$$

$$\beta = \arctan\left(\frac{b^2}{y_0 c}\right)$$

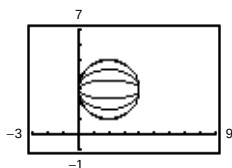
Since $|\alpha| = |\beta|$, the tangent line to an ellipse at a point P makes equal angles with the lines through P and the foci.

112. (a) $e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} \Rightarrow (ea)^2 - a^2 = b^2$. Hence,

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2(1-e^2)} = 1.$$

(b) $\frac{(x-2)^2}{4} + \frac{(y-3)^2}{4(1-e^2)} = 1$



(c) As e approaches 0, the ellipse approaches a circle.

116. Center: $(0, 0)$

Horizontal transverse axis

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

The difference of the distances from any point on the hyperbola is constant. At a vertex, this constant difference is

$$(a + c) - (c - a) = 2a.$$

Now, for any point (x, y) on the hyperbola, the difference of the distances between (x, y) and the two foci must also be $2a$.

$$\sqrt{(x-c)^2 + (y-0)^2} - \sqrt{(x+c)^2 + (y-0)^2} = 2a$$

$$\sqrt{(x-c)^2 + y^2} = 2a + \sqrt{(x+c)^2 + y^2}$$

$$(x-c)^2 + y^2 = 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

$$-4xc - 4a^2 = 4a\sqrt{(x+c)^2 + y^2}$$

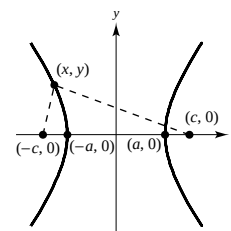
$$-(xc + a^2) = a\sqrt{(x+c)^2 + y^2}$$

$$x^2c^2 + 2a^2cx + a^4 = a^2[x^2 + 2cx + c^2 + y^2]$$

$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

Since $a^2 + b^2 = c^2$, we have $(x^2/a^2) - (y^2/b^2) = 1$.



114. The transverse axis is vertical since $(-3, 0)$ and $(-3, 3)$ are the foci.

Center: $\left(-3, \frac{3}{2}\right)$

$$c = \frac{3}{2}, 2a = 2, b^2 = c^2 - a^2 = \frac{5}{4}$$

Therefore, the equation is

$$\frac{[y - (3/2)]^2}{1} - \frac{(x + 3)^2}{5/4} = 1.$$

118. $c = 150, 2a = 0.001(186,000), a = 93,$

$$b = \sqrt{150^2 - 93^2} = \sqrt{13,851}$$

$$\frac{x^2}{93^2} - \frac{y^2}{13,851} = 1$$

When $y = 75$, we have

$$x^2 = 93^2 \left(1 + \frac{75^2}{13,851}\right)$$

$$x \approx 110.3 \text{ miles.}$$

120. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \text{ or } y' = \frac{b^2x}{a^2y}$$

$$y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0)$$

$$a^2y_0y - a^2y_0^2 = b^2x_0x - b^2x_0^2$$

$$b^2x_0^2 - a^2y_0^2 = b^2x_0x - a^2y_0y$$

$$a^2b^2 = b^2x_0x - a^2y_0y$$

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$$

- 122.
- $Ax^2 + Cy^2 + Dx + Ey + F = 0$
- (Assume
- $A \neq 0$
- and
- $C \neq 0$
- ; see (b) below)

$$A\left(x^2 + \frac{D}{A}x\right) + C\left(y^2 + \frac{E}{C}y\right) = -F$$

$$A\left(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}\right) + C\left(y^2 + \frac{E}{C}y + \frac{E^2}{4C^2}\right) = -F + \frac{D^2}{4A} + \frac{E^2}{4C} = R$$

$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{C} + \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{A} = \frac{R}{AC}$$

- (a) If
- $A = C$
- , we have

$$\left(x + \frac{D}{2A}\right)^2 + \left(y + \frac{E}{2C}\right)^2 = \frac{R}{A}$$

which is the standard equation of a circle.

- (c) If
- $AC > 0$
- , we have

$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{\left|\frac{R}{A}\right|} + \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{\left|\frac{R}{C}\right|} = 1$$

which is the equation of an ellipse.

- (b) If
- $C = 0$
- , we have

$$A\left(x + \frac{D}{2A}\right)^2 = -F - Ey + \frac{D^2}{4A}.$$

If $A = 0$, we have

$$C\left(y + \frac{E}{2C}\right)^2 = -F - Dx + \frac{E^2}{4C}.$$

These are the equations of parabolas.

- (d) If
- $AC < 0$
- , we have

$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{\left|\frac{R}{A}\right|} - \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{\left|\frac{R}{C}\right|} = \pm 1$$

which is the equation of a hyperbola.

124. True

126. False. The
- y^4
- term should be
- y^2
- .

128. True

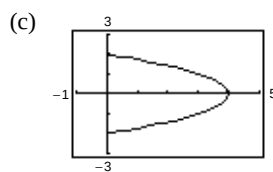
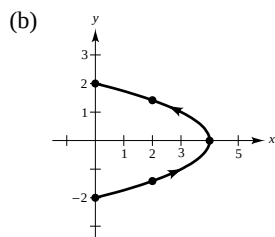
Section 9.2 Plane Curves and Parametric Equations

2. $x = 4 \cos^2 \theta$ $y = 2 \sin \theta$

$0 \leq x \leq 4$ $-2 \leq y \leq 2$

(a)

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
x	0	2	4	2	0
y	-2	$-\sqrt{2}$	0	$\sqrt{2}$	2



(d) $\frac{x}{4} = \cos^2 \theta$

$\frac{y^2}{4} = \sin^2 \theta$

$\frac{x}{4} + \frac{y^2}{4} = 1$

$x = 4 - y^2, -2 \leq y \leq 2$

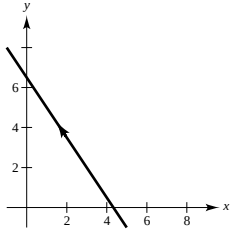
- (e) The graph would be oriented in the opposite direction.

4. $x = 3 - 2t$

$$y = 2 + 3t$$

$$y = 2 + 3\left(\frac{3-x}{2}\right)$$

$$2y + 3x - 13 = 0$$



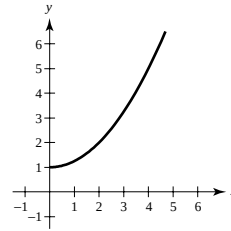
6. $x = 2t^2$

$$y = t^4 + 1$$

$$y = \left(\frac{x}{2}\right)^2 + 1 = \frac{x^2}{4} + 1, x \geq 0$$

For $t < 0$, the orientation is right to left.

For $t > 0$, the orientation is left to right.



8. $x = t^2 + t, y = t^2 - t$

Subtracting the second equation from the first, we have

$$x - y = 2t \quad \text{or} \quad t = \frac{x - y}{2}$$

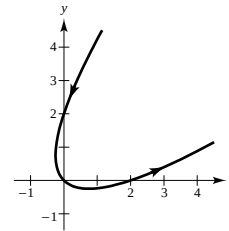
$$y = \frac{(x - y)^2}{4} - \frac{x - y}{2}$$

Since the discriminant is

$$B^2 - 4AC = (-2)^2 - 4(1)(1) = 0,$$

the graph is a rotated parabola.

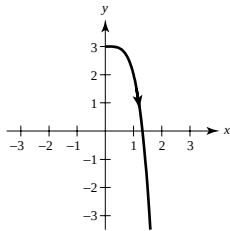
t	-2	-1	0	1	2
x	2	0	0	2	6
y	6	2	0	0	2



10. $x = \sqrt[4]{t}, t \geq 0$

$$y = 3 - t$$

$$y = 3 - x^4, x \geq 0$$

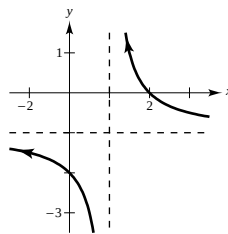


12. $x = 1 + \frac{1}{t}$

$$y = t - 1$$

$$x = 1 + \frac{1}{t} \text{ implies } t = \frac{1}{x - 1}$$

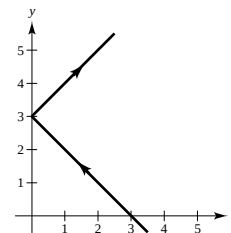
$$y = \frac{1}{x - 1} - 1$$



14. $x = |t - 1|$

$$y = t + 2$$

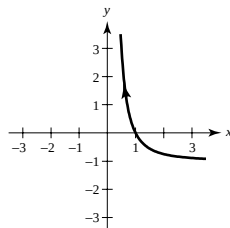
$$x = |(y - 2) - 1| = |y - 3|$$



16. $x = e^{-t}, x > 0$

$$y = e^{2t} - 1$$

$$y = x^{-2} - 1 = \frac{1}{x^2} - 1, x > 0$$



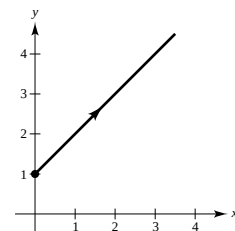
18. $x = \tan^2 \theta$

$$y = \sec^2 \theta$$

$$\sec^2 \theta = \tan^2 \theta + 1$$

$$y = x + 1$$

$$x \geq 0$$

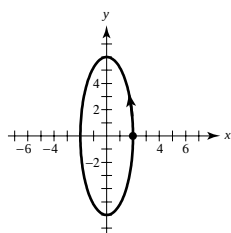


20. $x = 2 \cos \theta$

$y = 6 \sin \theta$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{6}\right)^2 = \cos^2 \theta + \sin^2 \theta = 1$$

$$\frac{x^2}{4} + \frac{y^2}{36} = 1 \text{ ellipse}$$



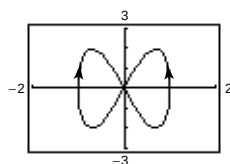
22. $x = \cos \theta$

$y = 2 \sin 2\theta$

$y = 4 \sin \theta \cos \theta$

$1 - x^2 = \sin^2 \theta$

$y = \pm 4x\sqrt{1 - x^2}$



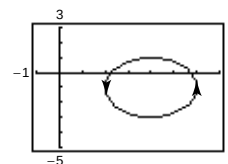
24. $x = 4 + 2 \cos \theta$

$y = -1 + 2 \sin \theta$

$(x - 4)^2 = 4 \cos^2 \theta$

$(y + 1)^2 = 4 \sin^2 \theta$

$(x - 4)^2 + (y + 1)^2 = 4$

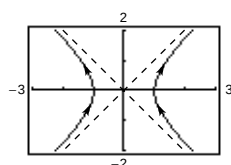


26. $x = \sec \theta$

$y = \tan \theta$

$x^2 = \sec^2 \theta$

$y^2 = \tan^2 \theta$

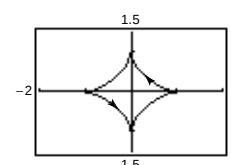


28. $x = \cos^3 \theta$

$y = \sin^3 \theta$

$x^{2/3} = \cos^2 \theta$

$y^{2/3} = \sin^2 \theta$

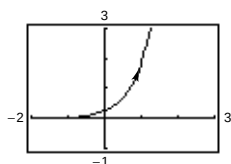


30. $x = \ln 2t$

$y = t^2$

$t = \frac{e^x}{2}$

$y = \frac{e^{2x}}{4} = \frac{1}{4}e^{2x}$



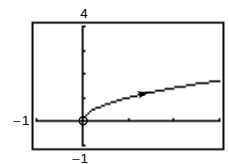
32. $x = e^{2t}$

$y = e^t$

$y^2 = x$

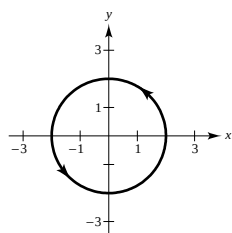
$y > 0$

$y = \sqrt{x}, x > 0$



34. By eliminating the parameters in (a) – (d), we get $x^2 + y^2 = 4$. They differ from each other in orientation and in restricted domains. These curves are all smooth.

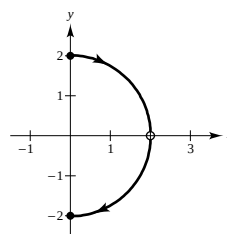
(a) $x = 2 \cos \theta, y = 2 \sin \theta$



(b) $x = \frac{\sqrt{4t^2 - 1}}{|t|} = \sqrt{4 - \frac{1}{t^2}} \quad y = \frac{1}{t}$

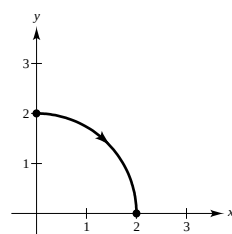
$x \geq 0, x \neq 2$

$y \neq 0$



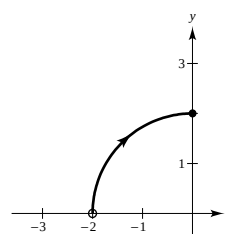
(c) $x = \sqrt{t} \quad y = \sqrt{4 - t}$

$x \geq 0 \quad y \geq 0$



(d) $x = -\sqrt{4 - e^{2t}} \quad y = e^t$

$-2 < x \leq 0 \quad y > 0$



36. The orientations are reversed. The graphs are the same. They are both smooth.

38. The set of points (x, y) corresponding to the rectangular equation of a set of parametric equations does not show the orientation of the curve nor any restriction on the domain of the original parametric equations.

40. $x = h + r \cos \theta$

$$y = k + r \sin \theta$$

$$\cos \theta = \frac{x - h}{r}$$

$$\sin \theta = \frac{y - k}{r}$$

$$\cos^2 \theta + \sin^2 \theta = \frac{(x - h)^2}{r^2} + \frac{(y - k)^2}{r^2} = 1$$

$$(x - h)^2 + (y - k)^2 = r^2$$

42.

$$x = h + a \sec \theta$$

$$y = k + b \tan \theta$$

$$\frac{x - h}{a} = \sec \theta$$

$$\frac{y - k}{b} = \tan \theta$$

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

44. From Exercise 39 we have

$$x = 1 + 4t$$

$$y = 4 - 6t.$$

Solution not unique

46. From Exercise 40 we have

$$x = -3 + 3 \cos \theta$$

$$y = 1 + 3 \sin \theta.$$

Solution not unique

48. From Exercise 41 we have

$$a = 5, c = 3 \Rightarrow b = 4$$

$$x = 4 + 5 \cos \theta$$

$$y = 2 + 4 \sin \theta.$$

Center: $(4, 2)$

Solution not unique

50. From Exercise 42 we have

$$a = 1, c = 2 \Rightarrow b = \sqrt{3}$$

$$x = \sqrt{3} \tan \theta$$

$$y = \sec \theta.$$

Center: $(0, 0)$

Solution not unique

The transverse axis is vertical, therefore, x and y are interchanged.

52. $y = \frac{2}{x - 1}$

Example

$$x = t, y = \frac{2}{t - 1}$$

$$x = -t, y = \frac{2}{-t - 1}$$

54. $y = x^2$

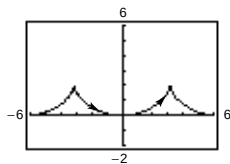
Example

$$x = t, y = t^2$$

$$x = t^3, y = t^6$$

56. $x = \theta + \sin \theta$

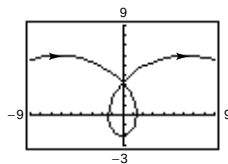
$$y = 1 - \cos \theta$$



Not smooth at $x = (2n - 1)\pi$

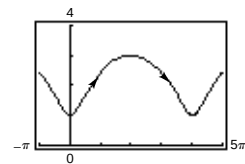
58. $x = 2\theta - 4 \sin \theta$

$$y = 2 - 4 \cos \theta$$



60. $x = 2\theta - \sin \theta$

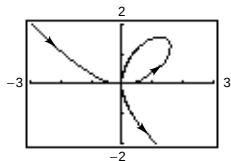
$$y = 2 - \cos \theta$$



Smooth everywhere

$$62. \quad x = \frac{3t}{1+t^3}$$

$$y = \frac{3t^2}{1+t^3}$$



Smooth everywhere

64. Each point (x, y) in the plane is determined by the plane curve $x = f(t)$, $y = g(t)$. For each t , plot (x, y) . As t increases, the curve is traced out in a specific direction called the orientation of the curve.

66. (a) Matches (ii) because $-1 \leq x \leq 0$ and $1 \leq y \leq 2$.

(b) Matches (i) because $x = (y - 2)^2 - 1$ for all y .

$$68. \quad x = \cos^3 \theta$$

$$y = 2 \sin^2 \theta$$

Matches (a)

$$70. \quad x = \cot \theta$$

$$y = 4 \sin \theta \cos \theta$$

Matches (c)

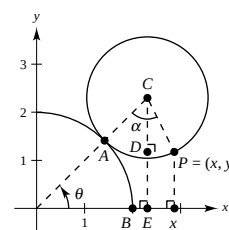
72. Let the circle of radius 1 be centered at C . A is the point of tangency on the line OC . $OA = 2$, $AC = 1$, $OC = 3$. $P = (x, y)$ is the point on the curve being traced out as the angle θ changes. $AB = AP$. $AB = 2\theta$ and $AP = \alpha \Rightarrow \alpha = 2\theta$. Form the right triangle $\triangle CDP$. The angle $OCE = (\pi/2) - \theta$ and

$$\angle DCP = \alpha - \left(\frac{\pi}{2} - \theta\right) = \alpha + \theta - \left(\frac{\pi}{2}\right) = 3\theta - \left(\frac{\pi}{2}\right).$$

$$x = OE + Ex = 3 \sin\left(\frac{\pi}{2} - \theta\right) + \sin\left(3\theta - \frac{\pi}{2}\right) = 3 \cos \theta - \cos 3\theta$$

$$y = EC - CD = 3 \sin \theta - \cos\left(3\theta - \frac{\pi}{2}\right) = 3 \sin \theta - \sin 3\theta$$

Hence, $x = 3 \cos \theta - \cos 3\theta$, $y = 3 \sin \theta - \sin 3\theta$.



74. False. Let $x = t^2$ and $y = t$. Then $x = y^2$ and y is not a function of x .

$$76. \quad (a) \quad x = (v_0 \cos \theta)t$$

$$y = h + (v_0 \sin \theta)t - 16t^2$$

$$t = \frac{x}{v_0 \cos \theta} \Rightarrow y = h + (v_0 \sin \theta) \frac{x}{v_0 \cos \theta} - 16 \left(\frac{x}{v_0 \cos \theta}\right)^2$$

$$y = h + (\tan \theta)x - \frac{16 \sec^2 \theta}{v_0^2} x^2$$

$$(b) \quad y = 5 + x - 0.005x^2 = h + (\tan \theta)x - \frac{16 \sec^2 \theta}{v_0^2} x^2$$

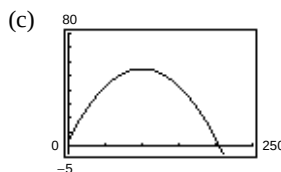
$$h = 5, \tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \text{ and}$$

$$0.005 = \frac{16 \sec^2(\pi/4)}{v_0^2} = \frac{16}{v_0^2}(2)$$

$$v_0^2 = \frac{32}{0.005} = 6400 \Rightarrow v_0 = 80.$$

$$\text{Hence, } x = (80 \cos(45^\circ))t$$

$$y = 5 + (80 \sin(45^\circ))t - 16t^2.$$



(d) Maximum height: $y = 55$ (at $x = 100$)
Range: 204.88

Section 9.3 Parametric Equations and Calculus

$$2. \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-1}{(1/3)t^{-2/3}} = -3t^{2/3}$$

$$6. x = \sqrt{t}, y = 3t - 1$$

$$\frac{dy}{dx} = \frac{3}{1/(2\sqrt{t})} = 6\sqrt{t} = 6 \text{ when } t = 1.$$

$$\frac{d^2y}{dx^2} = \frac{3/\sqrt{t}}{1/(2\sqrt{t})} = 6 \text{ concave upwards}$$

$$10. x = \cos \theta, y = 3 \sin \theta$$

$$\frac{dy}{dx} = \frac{3 \cos \theta}{-\sin \theta} = -3 \cot \theta \cdot \frac{dy}{dx} \text{ is undefined when } \theta = 0.$$

$$\frac{d^2y}{dx^2} = \frac{3 \csc^2 \theta}{-\sin \theta} = \frac{-3}{\sin^3 \theta} \cdot \frac{d^2y}{dx^2} \text{ is undefined when } \theta = 0.$$

$$12. x = \sqrt{t}, y = \sqrt{t-1}$$

$$\frac{dy}{dx} = \frac{1/(2\sqrt{t-1})}{1/(2\sqrt{t})}$$

$$= \frac{\sqrt{t}}{\sqrt{t-1}} = \sqrt{2} \text{ when } t = 2.$$

$$\frac{d^2y}{dx^2} = \frac{[\sqrt{t-1}/(2\sqrt{t}) - \sqrt{t}(1/2\sqrt{t-1})]/(t-1)}{1/(2\sqrt{t})}$$

$$= \frac{-1}{(t-1)^{3/2}} = -1 \text{ when } t = 2.$$

concave downward

$$16. x = 2 - 3 \cos \theta, y = 3 + 2 \sin \theta$$

$$\frac{dy}{dx} = \frac{2 \cos \theta}{3 \sin \theta} = \frac{2}{3} \cot \theta$$

At $(-1, 3)$, $\theta = 0$, and $\frac{dy}{dx}$ is undefined.

Tangent line: $x = -1$

At $(2, 5)$, $\theta = \frac{\pi}{2}$, and $\frac{dy}{dx} = 0$.

Tangent line: $y = 5$

At $\left(\frac{4+3\sqrt{3}}{2}, 2\right)$, $\theta = \frac{7\pi}{6}$, and $\frac{dy}{dx} = \frac{2\sqrt{3}}{3}$.

Tangent line:

$$y - 2 = \frac{2\sqrt{3}}{3} \left(x - \frac{4+3\sqrt{3}}{2} \right)$$

$$2\sqrt{3}x - 3y - 4\sqrt{3} - 3 = 0$$

$$4. \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(-1/2)e^{-\theta/2}}{2e^\theta} = -\frac{1}{4}e^{-3\theta/2} = \frac{-1}{4e^{3\theta/2}}$$

$$8. x = t^2 + 3t + 2, y = 2t$$

$$\frac{dy}{dx} = \frac{2}{2t+3} = \frac{2}{3} \text{ when } t = 0.$$

$$\frac{d^2y}{dx^2} = \frac{-2(2)/(2t+3)}{(2t+3)^2} = \frac{-4}{(2t+3)^2} = \frac{-4}{9} \text{ when } t = 0.$$

concave downward

$$14. x = \theta - \sin \theta, y = 1 - \cos \theta$$

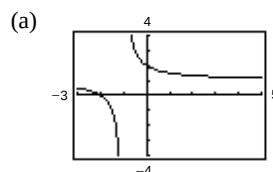
$$\frac{dy}{dx} = \frac{\sin \theta}{1 - \cos \theta} = 0 \text{ when } \theta = \pi.$$

$$\frac{d^2y}{dx^2} = \frac{[(1 - \cos \theta) \cos \theta - \sin^2 \theta]}{(1 - \cos \theta)^2}$$

$$= \frac{-1}{(1 - \cos \theta)^2} = -\frac{1}{4} \text{ when } \theta = \pi.$$

concave downward

$$18. x = t - 1, y = \frac{1}{t} + 1, t = 1$$

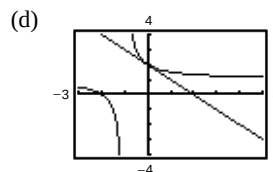


(b) At $t = 1$, $(x, y) = (0, 2)$, and

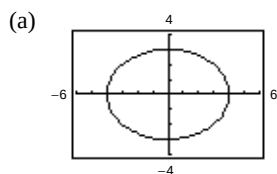
$$\frac{dx}{dt} = 1, \frac{dy}{dt} = -1, \frac{dy}{dx} = -1$$

(c) $\frac{dy}{dx} = -1$. At $(0, 2)$, $y - 2 = -1(x - 0)$

$$y = -x + 2$$



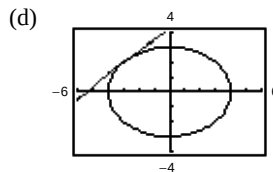
20. $x = 4 \cos \theta$, $y = 3 \sin \theta$, $\theta = \frac{3\pi}{4}$



(c) $\frac{dy}{dx} = \frac{3}{4}$. At $\left(\frac{-4}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$, $y - \frac{3}{\sqrt{2}} = \frac{3}{4}\left(x + \frac{4}{\sqrt{2}}\right)$
 $y = \frac{3}{4}x + 3\sqrt{2}$

(b) At $\theta = \frac{3\pi}{4}$, $(x, y) = \left(\frac{-4}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$, and

$$\frac{dx}{dt} = -2\sqrt{2}, \frac{dy}{dt} = -\frac{3\sqrt{2}}{2}, \frac{dy}{dx} = \frac{3}{4}$$



22. $x = t^2 - t$, $y = t^3 - 3t - 1$ crosses itself at the point $(x, y) = (2, 1)$.

At this point, $t = -1$ or $t = 2$.

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t - 1}$$

At $t = -1$, $\frac{dy}{dx} = 0$ and $y = 1$. Tangent Line

At $t = 2$, $\frac{dy}{dx} = \frac{9}{3} = 3$ and $y - 1 = 3(x - 2)$ or $y = 3x - 5$. Tangent Line

24. $x = 2\theta$, $y = 2(1 - \cos \theta)$

Horizontal tangents: $\frac{dy}{d\theta} = 2 \sin \theta = 0$ when $\theta = 0, \pm\pi, \pm2\pi, \dots$

Points: $(4n\pi, 0)$, $(2[2n - 1]\pi, 4)$ where n is an integer.

Points shown: $(0, 0)$, $(2\pi, 4)$, $(4\pi, 0)$

Vertical tangents: $\frac{dx}{d\theta} = 2 \neq 0$; none

26. $x = t + 1$, $y = t^2 + 3t$

Horizontal tangents: $\frac{dy}{dt} = 2t + 3 = 0$ when $t = -\frac{3}{2}$.

Point: $\left(-\frac{1}{2}, -\frac{9}{4}\right)$

Vertical tangents: $\frac{dx}{dt} = 1 \neq 0$; none

28. $x = t^2 - t + 2$, $y = t^3 - 3t$

Horizontal tangents: $\frac{dy}{dt} = 3t^2 - 3 = 0$ when $t = \pm 1$.

Points: $(2, -2)$, $(4, 2)$

Vertical tangents: $\frac{dx}{dt} = 2t - 1 = 0$ when $t = \frac{1}{2}$.

Point: $\left(\frac{7}{4}, -\frac{11}{8}\right)$

30. $x = \cos \theta$, $y = 2 \sin 2\theta$

Horizontal tangents: $\frac{dy}{d\theta} = 4 \cos 2\theta = 0$ when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Points: $\left(\frac{\sqrt{2}}{2}, 2\right)$, $\left(-\frac{\sqrt{2}}{2}, -2\right)$, $\left(-\frac{\sqrt{2}}{2}, 2\right)$, $\left(\frac{\sqrt{2}}{2}, -2\right)$

Vertical tangents: $\frac{dx}{d\theta} = -\sin \theta = 0$ when $\theta = 0, \pi$.

Points: $(1, 0)$, $(-1, 0)$

32. $x = 4 \cos^2 \theta$, $y = 2 \sin \theta$

Horizontal tangents: $\frac{dy}{d\theta} = 2 \cos \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Since $dx/d\theta = 0$ at $\pi/2$ and $3\pi/2$, exclude them.

Vertical tangents: $\frac{dx}{d\theta} = -8 \cos \theta \sin \theta = 0$ when

$$\theta = 0, \pi.$$

Point: $(4, 0)$

34. $x = \cos^2 \theta$, $y = \cos \theta$

Horizontal tangents: $\frac{dy}{d\theta} = -\sin \theta = 0$ when $x = 0, \pi$.

Since $dx/d\theta = 0$ at these values, exclude them.

Vertical tangents: $\frac{dx}{d\theta} = -2 \cos \theta \sin \theta = 0$ when

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}.$$

(Exclude $0, \pi$.)

Point: $(0, 0)$

36. $x = t^2 + 1$, $y = 4t^3 + 3$, $-1 \leq t \leq 0$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 12t^2, \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 144t^4$$

$$s = \int_{-1}^0 \sqrt{4t^2 + 144t^4} dt = \int_{-1}^0 -2t\sqrt{1 + 36t^2} dt$$

$$= \left[-\frac{(1 + 36t^2)^{3/2}}{54} \right]_{-1}^0 = \frac{-1}{54}(1 - 37^{3/2}) \approx 4.149$$

38. $x = \arcsin t$, $y = \ln \sqrt{1 - t^2}$, $0 \leq t \leq \frac{1}{2}$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1 - t^2}}, \frac{dy}{dt} = \frac{1}{2} \left(\frac{-2t}{1 - t^2} \right) = \frac{t}{1 - t^2}$$

$$s = \int_0^{1/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= \int_0^{1/2} \sqrt{\frac{1}{(1 - t^2)^2}} dt = \int_0^{1/2} \frac{1}{1 - t^2} dt$$

$$= \left[-\frac{1}{2} \ln \left| \frac{t - 1}{t + 1} \right| \right]_0^{1/2}$$

$$= -\frac{1}{2} \ln \left(\frac{1}{3} \right) = \frac{1}{2} \ln(3) \approx 0.549$$

40. $x = t$, $y = \frac{t^5}{10} + \frac{1}{6t^3}$, $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = \frac{t^4}{2} - \frac{1}{2t^4}$

$$S = \int_1^2 \sqrt{1 + \left(\frac{t^4}{2} - \frac{1}{2t^4}\right)^2} dt =$$

$$= \int_1^2 \sqrt{\left(\frac{t^4}{2} + \frac{1}{2t^4}\right)^2} dt$$

$$= \int_1^2 \left(\frac{t^4}{2} + \frac{1}{2t^4}\right) dt$$

$$= \left[\frac{t^5}{10} - \frac{1}{6t^3} \right]_1^2 = \frac{779}{240}$$

42. $x = a \cos \theta$, $y = a \sin \theta$, $\frac{dx}{d\theta} = -a \sin \theta$, $\frac{dy}{d\theta} = a \cos \theta$

$$S = 4 \int_0^{\pi/2} \sqrt{a^2 \sin^2 \theta + a^2 \cos^2 \theta} d\theta$$

$$= 4a \int_0^{\pi/2} d\theta = \left[4a\theta \right]_0^{\pi/2} = 2\pi a$$

44. $x = \cos \theta + \theta \sin \theta$, $y = \sin \theta - \theta \cos \theta$, $\frac{dx}{d\theta} = \theta \cos \theta$

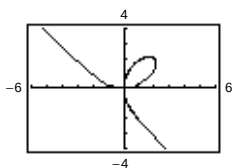
$$\frac{dy}{d\theta} = \theta \sin \theta$$

$$S = \int_0^{2\pi} \sqrt{\theta^2 \cos^2 \theta + \theta^2 \sin^2 \theta} d\theta$$

$$= \int_0^{2\pi} \theta d\theta = \left[\frac{\theta^2}{2} \right]_0^{2\pi} = 2\pi^2$$

46. $x = \frac{4t}{1+t^3}, y = \frac{4t^2}{1+t^3}$

(a) $x^3 + y^3 = 4xy$



(b) $\frac{dy}{dt} = \frac{(1+t^3)(8t) - 4t^2(3t^2)}{(1+t^3)^2}$
 $= \frac{4t(2-t^3)}{(1+t^3)^2} = 0$ when $t = 0$ or $t = \sqrt[3]{2}$.

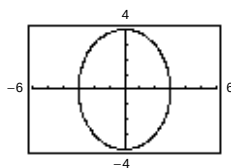
Points: $(0, 0), \left(\frac{4\sqrt[3]{2}}{3}, \frac{4\sqrt[3]{4}}{3}\right) \approx (1.6799, 2.1165)$

(c) $s = 2 \int_0^1 \sqrt{\left[\frac{4(1-2t^3)}{(1+t^3)^2}\right]^2 + \left[\frac{4t(2-t^3)}{(1+t^3)^2}\right]^2} dt = 2 \int_0^1 \sqrt{\frac{16}{(1+t^3)^4} [t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1]} dt$
 $= 8 \int_0^1 \frac{\sqrt{t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1}}{(1+t^3)^2} dt \approx 6.557$

48. $x = 3 \cos \theta, y = 4 \sin \theta$

$\frac{dx}{d\theta} = -3 \sin \theta, \frac{dy}{d\theta} = 4 \cos \theta$

$s = \int_0^{2\pi} \sqrt{9 \sin^2 \theta + 16 \cos^2 \theta} d\theta \approx 22.1$



50. $x = t, y = 4 - 2t, \frac{dx}{dt} = 1, \frac{dy}{dt} = -2$

(a) $S = 2\pi \int_0^2 (4-2t)\sqrt{1+4} dt$
 $= \left[2\sqrt{5}\pi(4t-t^2)\right]_0^2 = 8\pi\sqrt{5}$

(b) $S = 2\pi \int_0^2 t\sqrt{1+4} dt = \left[\sqrt{5}\pi t^2\right]_0^2 = 4\pi\sqrt{5}$

52. $x = \frac{1}{3}t^3, y = t + 1, 1 \leq t \leq 2, y\text{-axis}$

$\frac{dx}{dt} = t^2, \frac{dy}{dt} = 1$

$S = 2\pi \int_1^2 \frac{1}{3}t^3 \sqrt{t^4 + 1} dt = \frac{\pi}{9} \left[(x^4 + 1)^{3/2}\right]_1^2$
 $= \frac{\pi}{9}(17^{3/2} - 2^{3/2}) \approx 23.48$

54. $x = a \cos \theta, y = b \sin \theta, \frac{dx}{d\theta} = -a \sin \theta, \frac{dy}{d\theta} = b \cos \theta$

(a) $S = 4\pi \int_0^{\pi/2} b \sin \theta \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} d\theta$
 $= 4\pi \int_0^{\pi/2} ab \sin \theta \sqrt{1 - \left(\frac{a^2 - b^2}{a^2}\right) \cos^2 \theta} d\theta = \frac{-4ab\pi}{e} \int_0^{\pi/2} (-e \sin \theta) \sqrt{1 - e^2 \cos^2 \theta} d\theta$
 $= \frac{-2ab\pi}{e} \left[e \cos \theta \sqrt{1 - e^2 \cos^2 \theta} + \arcsin(e \cos \theta) \right]_0^{\pi/2} = \frac{-ab\pi}{e} [e\sqrt{1 - e^2} + \arcsin(e)]$
 $= 2\pi b^2 + \left(\frac{2\pi a^2 b}{\sqrt{a^2 - b^2}}\right) \arcsin\left(\frac{\sqrt{a^2 - b^2}}{a}\right) = 2\pi b^2 + 2\pi \left(\frac{ab}{e}\right) \arcsin(e)$
 $\left(e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{c}{a}; \text{eccentricity}\right)$

—CONTINUED—

54. —CONTINUED—

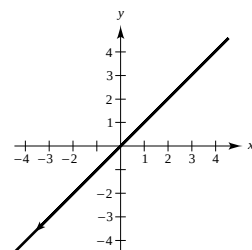
$$\begin{aligned}
 \text{(b) } S &= 4\pi \int_0^{\pi/2} a \cos \theta \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} d\theta \\
 &= 4\pi \int_0^{\pi/2} a \cos \theta \sqrt{b^2 + c^2 \sin^2 \theta} d\theta = \frac{4a\pi}{c} \int_0^{\pi/2} c \cos \theta \sqrt{b^2 + c^2 \sin^2 \theta} d\theta \\
 &= \frac{2a\pi}{c} \left[c \sin \theta \sqrt{b^2 + c^2 \sin^2 \theta} + b^2 \ln |c \sin \theta + \sqrt{b^2 + c^2 \sin^2 \theta}| \right]_0^{\pi/2} \\
 &= \frac{2a\pi}{c} [c \sqrt{b^2 + c^2} + b^2 \ln |c + \sqrt{b^2 + c^2}| - b^2 \ln b] \\
 &= 2\pi a^2 + \frac{2\pi ab^2}{\sqrt{a^2 - b^2}} \ln \left| \frac{a + \sqrt{a^2 - b^2}}{b} \right| = 2\pi a^2 + \left(\frac{\pi b^2}{e} \right) \ln \left| \frac{1+e}{1-e} \right|
 \end{aligned}$$

56. (a) 0

(b) 4

58. One possible answer is the graph given by

$$x = -t, y = -t.$$



$$\begin{aligned}
 \text{60. (a) } S &= 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
 \text{(b) } S &= 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt
 \end{aligned}$$

62. Let y be a continuous function of x on $a \leq x \leq b$. Suppose that $x = f(t)$, $y = g(t)$, and $f(t_1) = a$, $f(t_2) = b$. Then using integration by substitution, $dx = f'(t) dt$ and

$$\int_a^b y dx = \int_{t_1}^{t_2} g(t) f'(t) dt.$$

$$64. x = \sqrt{4-t}, y = \sqrt{t}, \frac{dx}{dt} = -\frac{1}{2\sqrt{4-t}}, 0 \leq t \leq 4$$

$$A = \int_4^0 \sqrt{t} \left(-\frac{1}{2\sqrt{4-t}} \right) dt = \int_0^2 \sqrt{4-u^2} du = \frac{1}{2} \left[u \sqrt{4-u^2} + 4 \arcsin \frac{u}{2} \right]_0^2 = \pi$$

Let $u = \sqrt{4-t}$, then $du = -1/(2\sqrt{4-t}) dt$ and $\sqrt{t} = \sqrt{4-u^2}$.

$$\bar{x} = \frac{1}{\pi} \int_4^0 \sqrt{4-t} \sqrt{t} \left(-\frac{1}{2\sqrt{4-t}} \right) dt = -\frac{1}{2\pi} \int_4^0 \sqrt{t} dt = \left[-\frac{1}{2\pi} \frac{2}{3} t^{3/2} \right]_4^0 = \frac{8}{3\pi}$$

$$\bar{y} = \frac{1}{2\pi} \int_4^0 (\sqrt{t})^2 \left(-\frac{1}{2\sqrt{4-t}} \right) dt = -\frac{1}{4\pi} \int_4^0 \frac{t}{\sqrt{4-t}} dt = -\frac{1}{4\pi} \left[\frac{-2(8+t)}{3} \sqrt{4-t} \right]_4^0 = \frac{8}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{3\pi}, \frac{8}{3\pi} \right)$$

$$66. x = \cos \theta, y = 3 \sin \theta, \frac{dx}{d\theta} = -\sin \theta$$

$$\begin{aligned}
 V &= 2\pi \int_{\pi/2}^0 (3 \sin \theta)^2 (-\sin \theta) d\theta \\
 &= -18\pi \int_{\pi/2}^0 \sin^3 \theta d\theta = -18\pi \left[-\cos \theta + \frac{\cos^3 \theta}{3} \right]_{\pi/2}^0 = 12\pi
 \end{aligned}$$

68. $x = 2 \cot \theta$, $y = 2 \sin^2 \theta$, $\frac{dx}{d\theta} = -2 \csc^2 \theta$

$$A = 2 \int_{\pi/2}^0 (2 \sin^2 \theta)(-2 \csc^2 \theta) d\theta = -8 \int_{\pi/2}^0 d\theta = \left[-8\theta \right]_{\pi/2}^0 = 4\pi$$

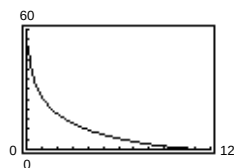
70. $\frac{3}{8}\pi a^2$ is area of asteroid (b).

72. $2\pi a^2$ is area of deltoid (c).

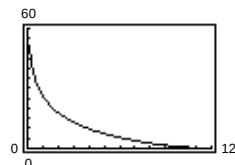
74. $2\pi ab$ is area of teardrop (e).

76. (a) $y = -12 \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$

$$0 < x \leq 12$$



(b) $x = 12 \operatorname{sech} \frac{t}{12}$, $y = t - 12 \tanh \frac{t}{12}$, $0 \leq t$



Same as the graph in (a), but has the advantage of showing the position of the object and any given time t .

(c) $\frac{dy}{dx} = \frac{1 - \operatorname{sech}^2(t/12)}{-\operatorname{sech}(t/12) \tanh(t/12)} = -\sinh \frac{t}{12}$

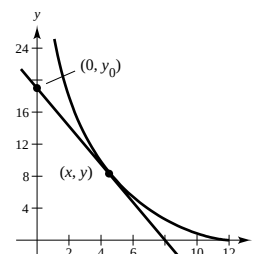
Tangent line: $y - \left(t_0 - 12 \tanh \frac{t_0}{12}\right) = -\sinh \frac{t_0}{12} \left(x - 12 \operatorname{sech} \frac{t_0}{12}\right)$

$$y = t_0 - \left(\sinh \frac{t_0}{12}\right)x$$

y -intercept: $(0, t_0)$

Distance between $(0, t_0)$ and (x, y) : $d = \sqrt{\left(12 \operatorname{sech} \frac{t_0}{12}\right)^2 + \left(-12 \tanh \frac{t_0}{12}\right)^2} = 12$

$$d = 12 \text{ for any } t \geq 0.$$



78. False. Both dx/dt and dy/dt are zero when $t = 0$. By eliminating the parameter, we have $y = x^{2/3}$ which does not have a horizontal tangent at the origin.

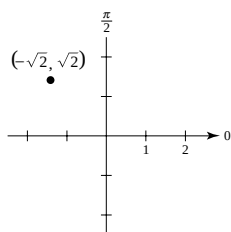
Section 9.4 Polar Coordinates and Polar Graphs

2. $\left(-2, \frac{7\pi}{4}\right)$

$$x = -2 \cos\left(\frac{7\pi}{4}\right) = -\sqrt{2}$$

$$y = -2 \sin\left(\frac{7\pi}{4}\right) = \sqrt{2}$$

$$(x, y) = (-\sqrt{2}, \sqrt{2})$$

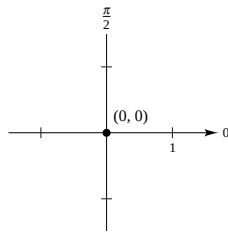


4. $\left(0, -\frac{7\pi}{6}\right)$

$$x = 0 \cos\left(-\frac{7\pi}{6}\right) = 0$$

$$y = 0 \sin\left(-\frac{7\pi}{6}\right) = 0$$

$$(x, y) = (0, 0)$$

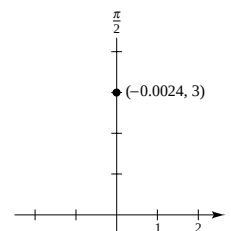


6. $(-3, -1.57)$

$$x = -3 \cos(-1.57) \approx -0.0024$$

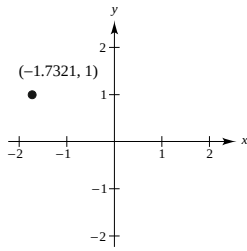
$$y = -3 \sin(-1.57) \approx 3$$

$$(x, y) = (-0.0024, 3)$$



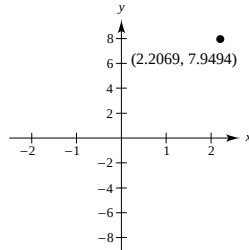
$$8. (r, \theta) = \left(-2, \frac{11\pi}{6}\right)$$

$$(x, y) = (-1.7321, 1)$$



$$10. (r, \theta) = (8.25, 1.3)$$

$$(x, y) = (2.2069, 7.9494)$$

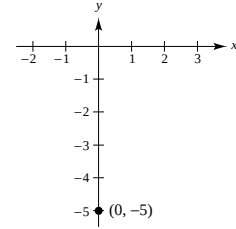


$$12. (x, y) = (0, -5)$$

$$r = \pm 5$$

$$\tan \theta \text{ undefined}$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \left(5, \frac{3\pi}{2}\right), \left(-5, \frac{\pi}{2}\right)$$



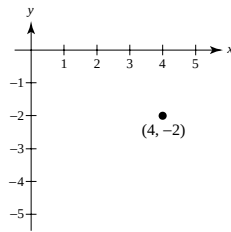
$$14. (x, y) = (4, -2)$$

$$r = \pm \sqrt{16 + 4} = \pm 2\sqrt{5}$$

$$\tan \theta = -\frac{2}{4} = -\frac{1}{2}$$

$$\theta \approx -0.464$$

$$(2\sqrt{5}, -0.464), (-2\sqrt{5}, 2.678)$$



$$16. (x, y) = (3\sqrt{2}, 3\sqrt{2})$$

$$(r, \theta) = (6, 0.785)$$

$$18. (x, y) = (0, -5)$$

$$(r, \theta) = (5, -1.571)$$

20. (a) Moving horizontally, the x -coordinate changes. Moving vertically, the y -coordinate changes.

(b) Both r and θ values change.

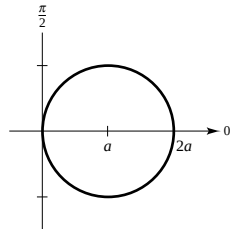
(c) In polar mode, horizontal (or vertical) changes result in changes in both r and θ .

$$22. x^2 + y^2 - 2ax = 0$$

$$r^2 - 2ar \cos \theta = 0$$

$$r(r - 2a \cos \theta) = 0$$

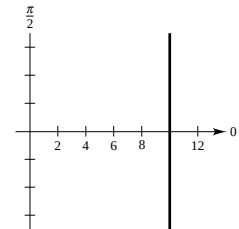
$$r = 2a \cos \theta$$



$$24. x = 10$$

$$r \cos \theta = 10$$

$$r = 10 \sec \theta$$

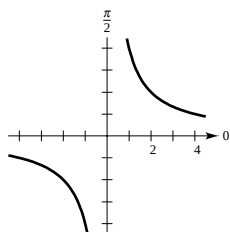


$$26. xy = 4$$

$$(r \cos \theta)(r \sin \theta) = 4$$

$$r^2 = 4 \sec \theta \csc \theta$$

$$= 8 \csc 2\theta$$

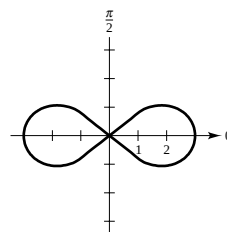


$$28. (x^2 + y^2)^2 - 9(x^2 - y^2) = 0$$

$$(r^2)^2 - 9(r^2 \cos^2 \theta - r^2 \sin^2 \theta) = 0$$

$$r^2[r^2 - 9(\cos 2\theta)] = 0$$

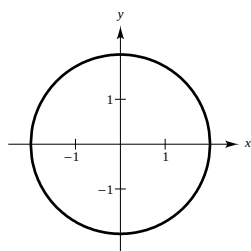
$$r^2 = 9 \cos 2\theta$$



30. $r = -2$

$$r^2 = 4$$

$$x^2 + y^2 = 4$$



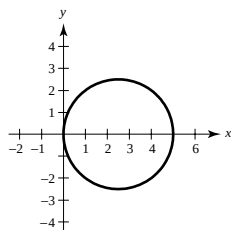
32. $r = 5 \cos \theta$

$$r^2 = 5r \cos \theta$$

$$x^2 + y^2 = 5x$$

$$x^2 - 5x + \frac{25}{4} + y^2 = \frac{25}{4}$$

$$\left(x - \frac{5}{2}\right)^2 + y^2 = \left(\frac{5}{2}\right)^2$$

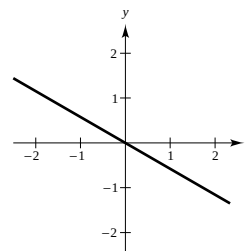


34. $\theta = \frac{5\pi}{6}$

$$\tan \theta = \tan \frac{5\pi}{6}$$

$$\frac{y}{x} = -\frac{\sqrt{3}}{3}$$

$$y = -\frac{\sqrt{3}}{3}x$$

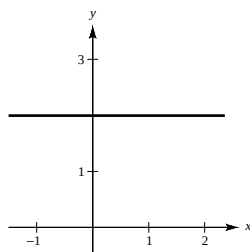


36. $r = 2 \csc \theta$

$$r \sin \theta = 2$$

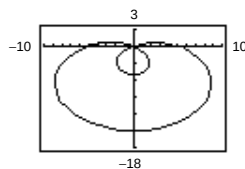
$$y = 2$$

$$y - 2 = 0$$



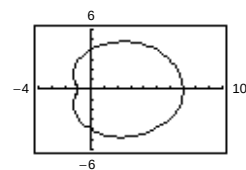
38. $r = 5(1 - 2 \sin \theta)$

$$0 \leq \theta < 2\pi$$



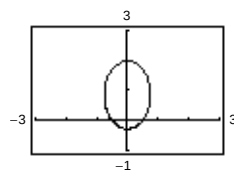
40. $r = 4 + 3 \cos \theta$

$$0 \leq \theta < 2\pi$$



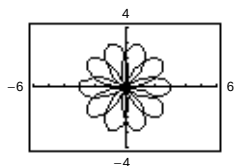
42. $r = \frac{2}{4 - 3 \sin \theta}$

Traced out once on $0 \leq \theta \leq 2\pi$



44. $r = 3 \sin\left(\frac{5\theta}{2}\right)$

$$0 \leq \theta < 4\pi$$

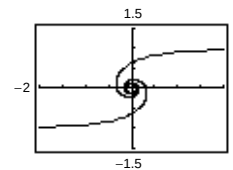


46. $r^2 = \frac{1}{\theta}$

Graph as

$$r_1 = \frac{1}{\sqrt{\theta}}, r_2 = -\frac{1}{\sqrt{\theta}}$$

It is traced out once on $[0, \infty)$.



48. (a) The rectangular coordinates of (r_1, θ_1) are $(r_1 \cos \theta_1, r_1 \sin \theta_1)$. The rectangular coordinates of (r_2, θ_2) are $(r_2 \cos \theta_2, r_2 \sin \theta_2)$.

$$\begin{aligned} d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (r_2 \cos \theta_2 - r_1 \cos \theta_1)^2 + (r_2 \sin \theta_2 - r_1 \sin \theta_1)^2 \\ &= r_2^2 \cos^2 \theta_2 - 2r_1 r_2 \cos \theta_1 \cos \theta_2 + r_1^2 \cos^2 \theta_1 + r_2^2 \sin^2 \theta_2 - 2r_1 r_2 \sin \theta_1 \sin \theta_2 + r_1^2 \sin^2 \theta_1 \\ &= r_2^2 (\cos^2 \theta_2 + \sin^2 \theta_2) + r_1^2 (\cos^2 \theta_1 + \sin^2 \theta_1) - 2r_1 r_2 (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) \\ &= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2) \\ d &= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)} \end{aligned}$$

- (b) If $\theta_1 = \theta_2$, the points lie on the same line passing through the origin. In this case,

$$\begin{aligned} d &= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(0)} \\ &= \sqrt{(r_1 - r_2)^2} = |r_1 - r_2| \end{aligned}$$

- (c) If $\theta_1 - \theta_2 = 90^\circ$, then $\cos(\theta_1 - \theta_2) = 0$ and $d = \sqrt{r_1^2 + r_2^2}$, the Pythagorean Theorem!

- (d) Many answers are possible. For example, consider the two points $(r_1, \theta_1) = (1, 0)$ and $(r_2, \theta_2) = (2, \pi/2)$.

$$d = \sqrt{1 + 2^2 - 2(1)(2) \cos\left(0 - \frac{\pi}{2}\right)} = \sqrt{5}$$

$$\text{Using } (r_1, \theta_1) = (-1, \pi) \text{ and } (r_2, \theta_2) = [2, (5\pi/2)], d = \sqrt{(-1)^2 + (2)^2 - 2(-1)(2) \cos\left(\pi - \frac{5\pi}{2}\right)} = \sqrt{5}.$$

You always obtain the same distance.

50. $\left(10, \frac{7\pi}{6}\right), (3, \pi)$

$$\begin{aligned} d &= \sqrt{10^2 + 3^2 - 2(10)(3) \cos\left(\frac{7\pi}{6} - \pi\right)} \\ &= \sqrt{109 - 60 \cos \frac{\pi}{6}} = \sqrt{109 - 30\sqrt{3}} \approx 7.6 \end{aligned}$$

52. $(4, 2.5), (12, 1)$

$$\begin{aligned} d &= \sqrt{4^2 + 12^2 - 2(4)(12) \cos(2.5 - 1)} \\ &= \sqrt{160 - 96 \cos 1.5} \approx 12.3 \end{aligned}$$

54. $r = 2(1 - \sin \theta)$

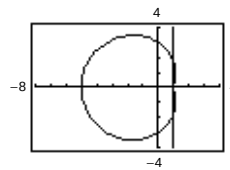
$$\frac{dy}{dx} = \frac{-2 \cos \theta \sin \theta + 2 \cos \theta (1 - \sin \theta)}{-2 \cos \theta \cos \theta - 2 \sin \theta (1 - \sin \theta)}$$

At $(2, 0)$, $\frac{dy}{dx} = -1$.

At $\left(3, \frac{7\pi}{6}\right)$, $\frac{dy}{dx}$ is undefined.

At $\left(4, \frac{3\pi}{2}\right)$, $\frac{dy}{dx} = 0$.

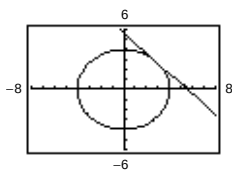
56. (a), (b) $r = 3 - 2 \cos \theta$



$$(r, \theta) = (1, 0) \Rightarrow (x, y) = (1, 0)$$

Tangent line: $x = 1$

- (c) At $\theta = 0$, $\frac{dy}{dx}$ does not exist (vertical tangent).

58. (a), (b) $r = 4$


$$\text{at } (r, \theta) = \left(4, \frac{\pi}{4}\right) \Rightarrow (x, y) = (2\sqrt{2}, 2\sqrt{2})$$

$$\begin{aligned} \text{Tangent line: } y - 2\sqrt{2} &= -1(x - 2\sqrt{2}) \\ y &= -x + 4\sqrt{2} \end{aligned}$$

 (c) At $\theta = \frac{\pi}{4}$, $\frac{dy}{dx} = -1$.

 62. $r = a \sin \theta \cos^2 \theta$

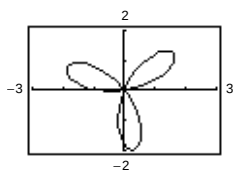
$$\frac{dy}{d\theta} = a \sin \theta \cos^3 \theta + [-2a \sin^2 \theta \cos \theta + a \cos^3 \theta] \sin \theta$$

$$= 2a[\sin \theta \cos^3 \theta - \sin^3 \theta \cos \theta]$$

$$= 2a \sin \theta \cos \theta (\cos^2 \theta - \sin^2 \theta) = 0$$

$$\theta = 0, \tan^2 \theta = 1, \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\text{Horizontal: } \left(\frac{\sqrt{2}a}{4}, \frac{\pi}{4}\right), \left(\frac{\sqrt{2}a}{4}, \frac{3\pi}{4}\right), (0, 0)$$

 66. $r = 2 \cos(3\theta - 2)$


Horizontal tangents:

$$(1.894, 0.776), (1.755, 2.594), (1.998, -1.442)$$

 70. $r = 3(1 - \cos \theta)$

Cardioid

 Symmetric to polar axis since r is a function of $\cos \theta$.

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$\frac{3}{2}$	3	$\frac{9}{2}$	6

 60. $r = a \sin \theta$

$$\frac{dy}{d\theta} = a \sin \theta \cos \theta + a \cos \theta \sin \theta$$

$$= 2a \sin \theta \cos \theta = 0$$

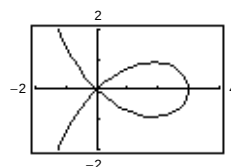
$$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

$$\frac{dx}{d\theta} = -a \sin^2 \theta + a \cos^2 \theta = a(1 - 2 \sin^2 \theta) = 0$$

$$\sin \theta = \pm \frac{1}{\sqrt{2}}, \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\text{Horizontal: } (0, 0), \left(a, \frac{\pi}{2}\right)$$

$$\text{Vertical: } \left(\frac{a\sqrt{2}}{2}, \frac{\pi}{4}\right), \left(\frac{a\sqrt{2}}{2}, \frac{3\pi}{4}\right)$$

 64. $r = 3 \cos 2\theta \sec \theta$

 Horizontal tangents: $(2.133, \pm 0.4352)$

 68. $r = 3 \cos \theta$

$$r^2 = 3r \cos \theta$$

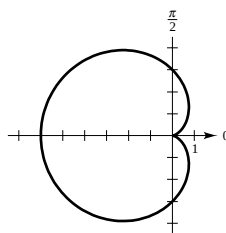
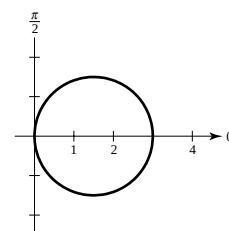
$$x^2 + y^2 = 3x$$

$$\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$$

$$\text{Circle: } r = \frac{3}{2}$$

$$\text{Center: } \left(\frac{3}{2}, 0\right)$$

$$\text{Tangent at pole: } \theta = \frac{\pi}{2}$$



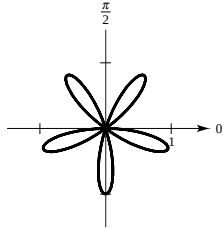
72. $r = -\sin(5\theta)$

Rose curve with five petals

 Symmetric to $\theta = \frac{\pi}{2}$

Relative extrema occur when

$$\frac{dr}{d\theta} = -5 \cos(5\theta) = 0 \text{ at } \theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}.$$

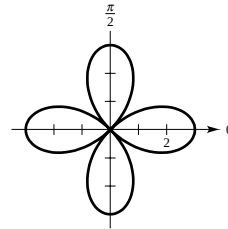
 Tangents at the pole: $\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$


74. $r = 3 \cos 2\theta$

Rose curve with four petals

 Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and pole

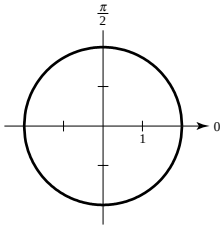
 Relative extrema: $(3, 0), (-3, \frac{\pi}{2}), (3, \pi), (-3, \frac{3\pi}{2})$

 Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$
 $\theta = \frac{5\pi}{4}$ and $\frac{7\pi}{4}$ given the same tangents.


76. $r = 2$

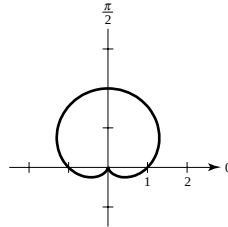
Circle radius: 2

$$x^2 + y^2 = 4$$



78. $r = 1 + \sin \theta$

Cardioid

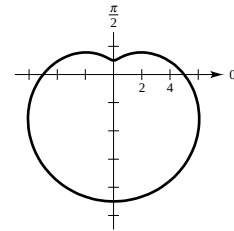


80. $r = 5 - 4 \sin \theta$

Limaçon

 Symmetric to $\theta = \frac{\pi}{2}$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$
r	9	7	5	3	1

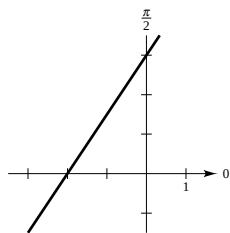


82. $r = \frac{6}{2 \sin \theta - 3 \cos \theta}$

$$2r \sin \theta - 3r \cos \theta = 6$$

$$2y - 3x = 6$$

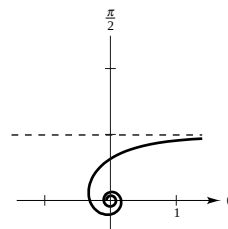
Line



84. $r = \frac{1}{\theta}$

Hyperbolic spiral

θ	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	$\frac{4}{\pi}$	$\frac{2}{\pi}$	$\frac{4}{3\pi}$	$\frac{1}{\pi}$	$\frac{4}{5\pi}$	$\frac{2}{3\pi}$

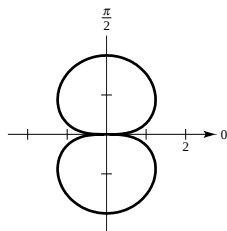


86. $r^2 = 4 \sin \theta$

Lemniscate

Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and poleRelative extrema: $(\pm 2, \frac{\pi}{2})$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π
r	0	$\pm\sqrt{2}$	± 2	$\pm\sqrt{2}$	0

Tangent at the pole: $\theta = 0$ 

88. Since

$$r = 2 + \csc \theta = 2 + \frac{1}{\sin \theta},$$

the graph has symmetry with respect to $\theta = \pi/2$.

Furthermore,

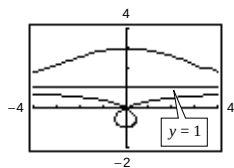
$$r \Rightarrow \infty \text{ as } \theta \Rightarrow 0^+$$

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \pi^-.$$

$$\text{Also, } r = 2 + \frac{1}{\sin \theta} = 2 + \frac{r}{\sin \theta} = 2 + \frac{r}{y}$$

$$ry = 2y + r$$

$$r = \frac{2y}{y-1}.$$

Thus, $r \Rightarrow \pm\infty$ as $y \Rightarrow 1$.

92. $x = r \cos \theta, y = r \sin \theta$

$$x^2 + y^2 = r^2, \tan \theta = \frac{y}{x}$$

96. $r = 4 \cos 2\theta$

Rose curve

Matches (b)

90. $r = 2 \cos 2\theta \sec \theta$

Strophoid

$$r \Rightarrow -\infty \text{ as } \theta \Rightarrow \frac{\pi^-}{2}$$

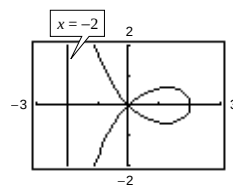
$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \frac{-\pi^+}{2}$$

$$r = 2 \cos 2\theta \sec \theta = 2(2 \cos^2 \theta - 1) \sec \theta$$

$$r \cos \theta = 4 \cos^2 \theta - 2$$

$$x = 4 \cos^2 \theta - 2$$

$$\lim_{\theta \rightarrow \pm\pi/2} (4 \cos^2 \theta - 2) = -2$$

94. Slope of tangent line to graph of $r = f(\theta)$ at (r, θ) is

$$\frac{dy}{dx} = \frac{f(\theta)\cos \theta + f'(\theta)\sin \theta}{-f(\theta)\sin \theta + f'(\theta)\cos \theta}.$$

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then $\theta = \alpha$ is tangent at the pole.

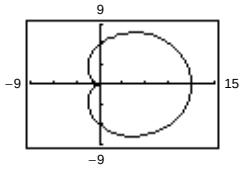
98. $r = 2 \sec \theta$

Line

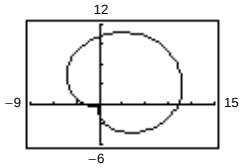
Matches (d)

100. $r = 6[1 + \cos(\theta - \phi)]$

(a) $\phi = 0, r = 6[1 + \cos \theta]$



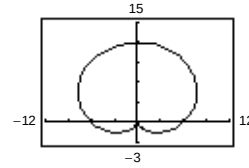
(b) $\theta = \frac{\pi}{4}, r = 6\left[1 + \cos\left(\theta - \frac{\pi}{4}\right)\right]$



The graph of $r = 6[1 + \cos \theta]$ is rotated through the angle $\pi/4$.

(c) $\theta = \frac{\pi}{2}$

$$\begin{aligned} r &= 6\left[1 + \cos\left(\theta - \frac{\pi}{2}\right)\right] \\ &= 6\left[1 + \cos \theta \cos \frac{\pi}{2} + \sin \theta \sin \frac{\pi}{2}\right] \\ &= 6[1 + \sin \theta] \end{aligned}$$



The graph of $r = 6[1 + \cos \theta]$ is rotated through the angle $\pi/2$.

102. (a) $\sin\left(\theta - \frac{\pi}{2}\right) = \sin \theta \cos\left(\frac{\pi}{2}\right) - \cos \theta \sin\left(\frac{\pi}{2}\right)$
 $= -\cos \theta$

$$\begin{aligned} r &= f\left[\sin\left(\theta - \frac{\pi}{2}\right)\right] \\ &= f(-\cos \theta) \end{aligned}$$

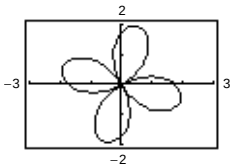
(c) $\sin\left(\theta - \frac{3\pi}{2}\right) = \sin \theta \cos\left(\frac{3\pi}{2}\right) - \cos \theta \sin\left(\frac{3\pi}{2}\right)$
 $= \cos \theta$

$$r = f\left[\sin\left(\theta - \frac{3\pi}{2}\right)\right] = f(\cos \theta)$$

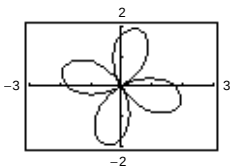
(b) $\sin(\theta - \pi) = \sin \theta \cos \pi - \cos \theta \sin \pi$
 $= -\sin \theta$
 $r = f[\sin(\theta - \pi)]$
 $= f(-\sin \theta)$

104. $r = 2 \sin 2\theta = 4 \sin \theta \cos \theta$

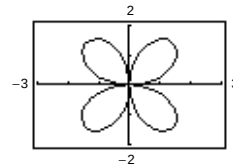
(a) $r = 4 \sin\left(\theta - \frac{\pi}{6}\right) \cos\left(\theta - \frac{\pi}{6}\right)$



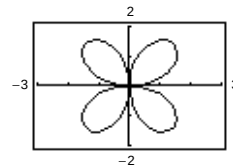
(c) $r = 4 \sin\left(\theta - \frac{2\pi}{3}\right) \cos\left(\theta - \frac{2\pi}{3}\right)$



(b) $r = 4 \sin\left(\theta - \frac{\pi}{2}\right) \cos\left(\theta - \frac{\pi}{2}\right) = -4 \sin \theta \cos \theta$



(d) $r = 4 \sin(\theta - \pi) \cos(\theta - \pi) = 4 \sin \theta \cos \theta$

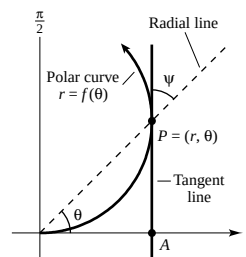


106. By Theorem 9.11, the slope of the tangent line through A and P is

$$\frac{f \cos \theta + f' \sin \theta}{-f \sin \theta + f' \cos \theta}$$

This is equal to

$$\tan(\theta + \psi) = \frac{\tan \theta + \tan \psi}{1 - \tan \theta \tan \psi} = \frac{\sin \theta + \cos \theta \tan \psi}{\cos \theta - \sin \theta \tan \psi}.$$



Equating the expressions and cross-multiplying, you obtain

$$(f \cos \theta + f' \sin \theta)(\cos \theta - \sin \theta \tan \psi) = (\sin \theta + \cos \theta \tan \psi)(-f \sin \theta + f' \cos \theta)$$

$$f \cos^2 \theta - f \cos \theta \sin \theta \tan \psi + f' \sin \theta \cos \theta - f' \sin^2 \theta \tan \psi = -f \sin^2 \theta - f \sin \theta \cos \theta \tan \psi + f' \sin \theta \cos \theta + f' \cos^2 \theta \tan \psi$$

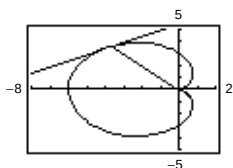
$$f(\cos^2 \theta + \sin^2 \theta) = f' \tan \psi (\cos^2 \theta + \sin^2 \theta)$$

$$\tan \psi = \frac{f}{f'} = \frac{r}{dr/d\theta}.$$

108. $\tan \psi = \frac{r}{dr/d\theta} = \frac{3(1 - \cos \theta)}{3 \sin \theta}$

At $\theta = \frac{3\pi}{4}$, $\tan \psi = \frac{1 + (\sqrt{2}/2)}{\sqrt{2}} = \frac{2 + \sqrt{2}}{\sqrt{2}}.$

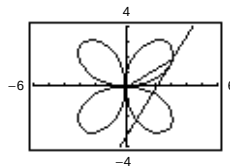
$$\psi = \arctan\left(\frac{2 + \sqrt{2}}{\sqrt{2}}\right) \approx 1.041 (\approx 59.64^\circ)$$



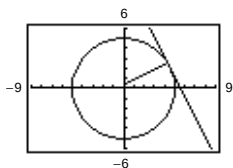
110. $\tan \psi = \frac{r}{dr/d\theta} = \frac{4 \sin 2\theta}{8 \cos 2\theta}$

At $\theta = \frac{\pi}{6}$, $\tan \psi = \frac{\sin(\pi/3)}{2 \cos(\pi/3)} = \frac{\sqrt{3}}{2}.$

$$\psi = \arctan\left(\frac{\sqrt{3}}{2}\right) \approx 0.7137 (\approx 40.89^\circ)$$



112. $\tan \psi = \frac{r}{dr/d\theta} = \frac{5}{0} \text{ undefined} \Rightarrow \psi = \frac{\pi}{2}.$

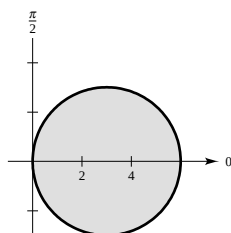


114. True

116. True

Section 9.5 Area and Arc Length in Polar Coordinates

2. (a) $r = 3 \cos \theta$



$$A = \pi \left(\frac{3}{2}\right)^2 = \frac{9\pi}{4}$$

(b) $A = 2 \left(\frac{1}{2}\right) \int_0^{\pi/2} [3 \cos \theta]^2 d\theta$

$$= 9 \int_0^{\pi/2} \cos^2 \theta d\theta$$

$$= \frac{9}{2} \int_0^{\pi/2} (1 + \cos 2\theta) d\theta$$

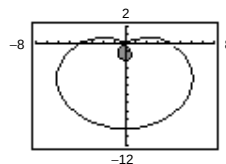
$$= \frac{9}{2} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = \frac{9\pi}{4}$$

$$\begin{aligned}
 4. \quad A &= 2 \left[\frac{1}{2} \int_0^{\pi/4} (6 \sin 2\theta)^2 d\theta \right] = 36 \int_0^{\pi/4} \sin^2 2\theta d\theta \\
 &= 36 \int_0^{\pi/4} \frac{1 - \cos 4\theta}{2} d\theta \\
 &= 18 \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/4} \\
 &= 18 \left[\frac{\pi}{4} \right] = \frac{9\pi}{2}
 \end{aligned}$$

$$\begin{aligned}
 8. \quad A &= 2 \left[\frac{1}{2} \int_0^{\pi/2} (1 - \sin \theta)^2 d\theta \right] \\
 &= \left[\frac{3}{2} \theta + 2 \cos \theta - \frac{1}{4} \sin 2\theta \right]_0^{\pi/2} = \frac{3\pi - 8}{4}
 \end{aligned}$$

$$\begin{aligned}
 6. \quad A &= 2 \left[\frac{1}{2} \int_0^{\pi/10} (\cos 5\theta)^2 d\theta \right] \\
 &= \frac{1}{2} \left[\theta + \frac{1}{10} \sin(10\theta) \right]_0^{\pi/10} = \frac{\pi}{20}
 \end{aligned}$$

$$\begin{aligned}
 10. \quad A &= 2 \left[\frac{1}{2} \int_{\arcsin(2/3)}^{\pi/2} (4 - 6 \sin \theta)^2 d\theta \right] \\
 &= \int_{\arcsin(2/3)}^{\pi/2} [16 - 48 \sin \theta + 36 \sin^2 \theta] d\theta \\
 &= \int_{\arcsin(2/3)}^{\pi/2} \left[16 - 48 \sin \theta + 36 \left(\frac{1 - \cos 2\theta}{2} \right) \right] d\theta \\
 &= \left[34\theta + 48 \cos \theta - 9 \sin 2\theta \right]_{\arcsin(2/3)}^{\pi/2} \approx 1.7635
 \end{aligned}$$



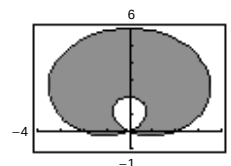
12. Four times the area in Exercise 11, $A = 4(\pi + 3\sqrt{3})$. More specifically, we see that the area inside the outer loop is

$$2 \left[\frac{1}{2} \int_{-\pi/6}^{\pi/2} (2(1 + 2 \sin \theta))^2 d\theta \right] = \int_{-\pi/6}^{\pi/2} (4 + 16 \sin \theta + 16 \sin^2 \theta) d\theta = 8\pi + 6\sqrt{3}.$$

The area inside the inner loop is

$$2 \frac{1}{2} \left[\int_{7\pi/6}^{3\pi/2} (2(1 + 2 \sin \theta))^2 d\theta \right] = 4\pi - 6\sqrt{3}.$$

Thus, the area between the loops is $(8\pi + 6\sqrt{3}) - (4\pi - 6\sqrt{3}) = 4\pi + 12\sqrt{3}$.



14. $r = 3(1 + \sin \theta)$

$r = 3(1 - \sin \theta)$

Solving simultaneously,

$$3(1 + \sin \theta) = 3(1 - \sin \theta)$$

$$2 \sin \theta = 0$$

$$\theta = 0, \pi.$$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-3(1 - \sin \theta) = 3(1 - \sin \theta)$, $\sin \theta = 1$, $\theta = \pi/2$. Both curves pass through the pole, $(0, 3\pi/2)$, and $(0, \pi/2)$, respectively.

Points of intersection: $(3, 0)$, $(3, \pi)$, $(0, 0)$

16. $r = 2 - 3 \cos \theta$

$r = \cos \theta$

Solving simultaneously,

$$2 - 3 \cos \theta = \cos \theta$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}.$$

Both curves pass through the pole, $(0, \arccos 2/3)$, and $(0, \pi/2)$, respectively.

Points of intersection: $\left(\frac{1}{2}, \frac{\pi}{3}\right)$, $\left(\frac{1}{2}, \frac{5\pi}{3}\right)$, $(0, 0)$

18. $r = 1 + \cos \theta$

$r = 3 \cos \theta$

Solving simultaneously,

$1 + \cos \theta = 3 \cos \theta$

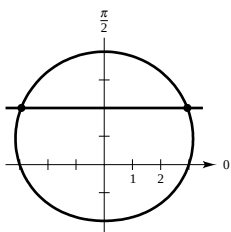
$\cos \theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

Both curves pass through the pole, $(0, \pi)$, and $(0, \pi/2)$, respectively.Points of intersection: $\left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, \frac{5\pi}{3}\right), (0, 0)$

22. $r = 3 + \sin \theta$

$r = 2 \csc \theta$



The graph of $r = 3 + \sin \theta$ is a limaçon symmetric to $\theta = \pi/2$, and the graph of $r = 2 \csc \theta$ is the horizontal line $y = 2$. Therefore, there are two points of intersection. Solving simultaneously,

$3 + \sin \theta = 2 \csc \theta$

$\sin^2 \theta + 3 \sin \theta - 2 = 0$

$\sin \theta = \frac{-3 \pm \sqrt{17}}{2}$

$\theta = \arcsin\left(\frac{\sqrt{17} - 3}{2}\right) \approx 0.596$

24. $r = 3(1 - \cos \theta)$

$r = \frac{6}{1 - \cos \theta}$

The graph of $r = 3(1 - \cos \theta)$ is a cardioid with polar axis symmetry. The graph of

$r = 6/(1 - \cos \theta)$

is a parabola with focus at the pole, vertex $(3, \pi)$, and polar axis symmetry. Therefore, there are two points of intersection. Solving simultaneously,

$3(1 - \cos \theta) = \frac{6}{1 - \cos \theta}$

$(1 - \cos \theta)^2 = 2$

$\cos \theta = 1 \pm \sqrt{2}$

$\theta = \arccos(1 - \sqrt{2})$

Points of intersection: $(3\sqrt{2}, \arccos(1 - \sqrt{2})) \approx (4.243, 1.998), (3\sqrt{2}, 2\pi - \arccos(1 - \sqrt{2})) \approx (4.243, 4.285)$

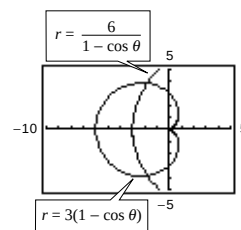
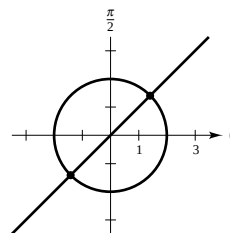
20. $\theta = \frac{\pi}{4}$

$r = 2$

Line of slope 1 passing through the pole and a circle of radius 2 centered at the pole.

Points of intersection:

$\left(2, \frac{\pi}{4}\right), \left(-2, \frac{\pi}{4}\right)$

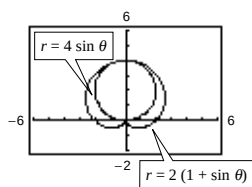


26. $r = 4 \sin \theta$

$r = 2(1 + \sin \theta)$

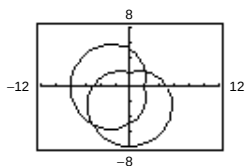
Points of intersection: $(0, 0), \left(4, \frac{\pi}{2}\right)$

The graphs reach the pole at different times (θ values).



30. $r = 5 - 3 \sin \theta$ and $r = 5 - 3 \cos \theta$ intersect at $\theta = \pi/4$ and $\theta = 5\pi/4$.

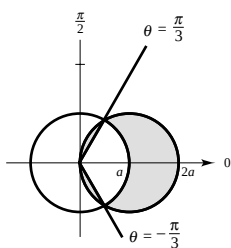
$$\begin{aligned} A &= 2 \left[\frac{1}{2} \int_{\pi/4}^{5\pi/4} (5 - 3 \sin \theta)^2 d\theta \right] \\ &= \left[\frac{59}{2} \theta + 30 \cos \theta - \frac{9}{4} \sin 2\theta \right]_{\pi/4}^{5\pi/4} \\ &= \left(\frac{59}{2} \left(\frac{5\pi}{4} \right) - 30 \frac{\sqrt{2}}{2} - \frac{9}{4} \right) - \left(\frac{59}{2} \left(\frac{\pi}{4} \right) + 30 \frac{\sqrt{2}}{2} - \frac{9}{4} \right) \\ &= \frac{59\pi}{2} - 30\sqrt{2} \approx 50.251 \end{aligned}$$



34. Area = Area of $r = 2a \cos \theta$ - Area of sector - twice area between $r = 2a \cos \theta$ and the lines

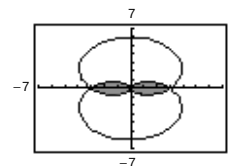
$\theta = \frac{\pi}{3}, \theta = \frac{\pi}{2}$.

$$\begin{aligned} A &= \pi a^2 - \left(\frac{\pi}{3} \right) a^2 - 2 \left[\frac{1}{2} \int_{\pi/3}^{\pi/2} (2a \cos \theta)^2 d\theta \right] \\ &= \frac{2\pi a^2}{3} - 2a^2 \int_{\pi/3}^{\pi/2} (1 + \cos 2\theta) d\theta \\ &= \frac{2\pi a^2}{3} - 2a^2 \left[\theta + \frac{\sin 2\theta}{2} \right]_{\pi/3}^{\pi/2} \\ &= \frac{2\pi a^2}{3} - 2a^2 \left[\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4} \right] = \frac{2\pi a^2 + 3\sqrt{3}a^2}{6} \end{aligned}$$

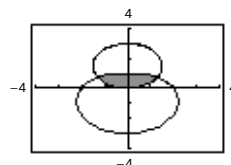


28. $A = 4 \left[\frac{1}{2} \int_0^{\pi/2} 9(1 - \sin \theta)^2 d\theta \right]$
 $= 18 \int_0^{\pi/2} (1 - \sin \theta)^2 d\theta = \frac{9}{2} (3\pi - 8)$

(from Exercise 14)



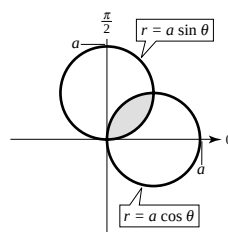
32. $A = 2 \left[\frac{1}{2} \int_{\pi/6}^{\pi/2} (3 \sin \theta)^2 d\theta - \frac{1}{2} \int_{\pi/6}^{\pi/2} (2 - \sin \theta)^2 d\theta \right]$
 $= \int_{\pi/6}^{\pi/2} (-4 \cos 2\theta + 4 \sin \theta) d\theta$
 $= \left[-2 \sin(2\theta) - 4 \cos \theta \right]_{\pi/6}^{\pi/2} = 3\sqrt{3}$



36. $r = a \cos \theta, r = a \sin \theta$

$\tan \theta = 1, \theta = \pi/4$

$$\begin{aligned} A &= 2 \left[\frac{1}{2} \int_0^{\pi/2} (a \cos \theta)^2 d\theta \right] \\ &= a^2 \int_0^{\pi/4} \frac{1 + \cos 2\theta}{2} d\theta \\ &= \frac{1}{2} a^2 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/4} \\ &= \frac{1}{2} a^2 \left[\frac{\pi}{4} + \frac{1}{2} \right] \\ &= \frac{1}{4} a^2 + \frac{1}{8} a^2 \pi \end{aligned}$$



38. By symmetry, $A_1 = A_2$ and $A_3 = A_4$.

$$\begin{aligned} A_1 &= A_2 = \frac{1}{2} \int_{-\pi/3}^{\pi/6} [(2a \cos \theta)^2 - (a)^2] d\theta + \frac{1}{2} \int_{\pi/6}^{\pi/4} [(2a \cos \theta)^2 - (2a \sin \theta)^2] d\theta \\ &= \frac{a^2}{2} \int_{-\pi/3}^{\pi/6} (4 \cos^2 \theta - 1) d\theta + 2a^2 \int_{\pi/6}^{\pi/4} \cos 2\theta d\theta \\ &= \frac{a^2}{2} \left[\theta + \sin 2\theta \right]_{-\pi/3}^{\pi/6} + a^2 \left[\sin 2\theta \right]_{\pi/6}^{\pi/4} = \frac{a^2}{2} \left(\frac{\pi}{2} + \sqrt{3} \right) + a^2 \left(1 - \frac{\sqrt{3}}{2} \right) = a^2 \left(\frac{\pi}{4} + 1 \right) \end{aligned}$$

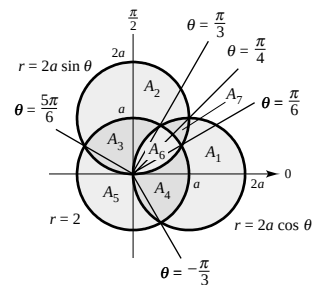
$$A_3 = A_4 = \frac{1}{2} \left(\frac{\pi}{2} \right) a^2 = \frac{\pi a^2}{4}$$

$$\begin{aligned} A_5 &= \frac{1}{2} \left(\frac{5\pi}{6} \right) a^2 - 2 \left(\frac{1}{2} \right) \int_{5\pi/6}^{\pi} (2a \sin \theta)^2 d\theta \\ &= \frac{5\pi a^2}{12} - 2a^2 \int_{5\pi/6}^{\pi} (1 - \cos 2\theta) d\theta \\ &= \frac{5\pi a^2}{12} - a^2 \left[2\theta - \sin 2\theta \right]_{5\pi/6}^{\pi} = \frac{5\pi a^2}{12} - a^2 \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right) = a^2 \left(\frac{\pi}{12} + \frac{\sqrt{3}}{2} \right) \end{aligned}$$

$$\begin{aligned} A_6 &= 2 \left(\frac{1}{2} \right) \int_0^{\pi/6} (2a \sin \theta)^2 d\theta + 2 \left(\frac{1}{2} \right) \int_{\pi/6}^{\pi/4} a^2 d\theta \\ &= 2a^2 \int_0^{\pi/6} (1 - \cos 2\theta) d\theta + \left[a^2 \theta \right]_{\pi/6}^{\pi/4} \\ &= a^2 \left[2\theta - \sin 2\theta \right]_0^{\pi/6} + \frac{\pi a^2}{12} = a^2 \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right) + \frac{\pi a^2}{12} = a^2 \left(\frac{5\pi}{12} - \frac{\sqrt{3}}{2} \right) \end{aligned}$$

$$\begin{aligned} A_7 &= 2 \left(\frac{1}{2} \right) \int_{\pi/6}^{\pi/4} [(2a \sin \theta)^2 - (a)^2] d\theta \\ &= a^2 \int_{\pi/6}^{\pi/4} (4 \sin^2 \theta - 1) d\theta = a^2 \left[\theta - \sin 2\theta \right]_{\pi/6}^{\pi/4} = a^2 \left(\frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2} \right) \end{aligned}$$

[Note: $A_1 + A_6 + A_7 + A_4 = \pi a^2 = \text{area of circle of radius } a]$



40. $r = \sec \theta - 2 \cos \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$r \cos \theta = 1 - 2 \cos^2 \theta$$

$$x = 1 - 2 \frac{r^2 \cos^2 \theta}{r^2} = 1 - 2 \left(\frac{x^2}{x^2 + y^2} \right)$$

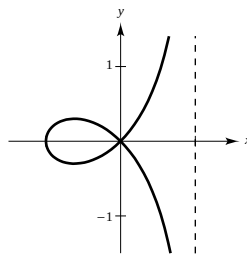
$$(x^2 + y^2)x = x^2 + y^2 - 2x^2$$

$$y^2(x - 1) = -x^2 - x^3$$

$$y^2 = \frac{x^2(1 + x)}{1 - x}$$

$$A = 2 \left(\frac{1}{2} \right) \int_0^{\pi/4} (\sec \theta - 2 \cos \theta)^2 d\theta$$

$$= \int_0^{\pi/4} (\sec^2 \theta - 4 + 4 \cos^2 \theta) d\theta = \int_0^{\pi/4} (\sec^2 \theta - 4 + 2(1 + \cos 2\theta)) d\theta = \left[\tan \theta - 2\theta + \sin 2\theta \right]_0^{\pi/4} = 2 - \frac{\pi}{2}$$



42. $r = 2a \cos \theta$

$$r' = -2a \sin \theta$$

$$s = \int_{-\pi/2}^{\pi/2} \sqrt{(2a \cos \theta)^2 + (-2a \sin \theta)^2} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 2a d\theta = \left[2\theta \right]_{-\pi/2}^{\pi/2} = 2\pi a$$

44. $r = 8(1 + \cos \theta), 0 \leq \theta \leq 2\pi$

$$r' = -8 \sin \theta$$

$$s = 2 \int_0^\pi \sqrt{[8(1 + \cos \theta)]^2 + (-8 \sin \theta)^2} d\theta$$

$$= 16 \int_0^\pi \sqrt{1 + 2 \cos \theta + \cos^2 \theta + \sin^2 \theta} d\theta$$

$$= 16 \sqrt{2} \int_0^\pi \sqrt{1 + \cos \theta} d\theta$$

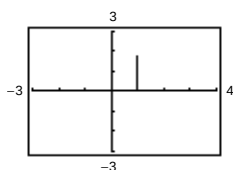
$$= 16 \sqrt{2} \int_0^\pi \sqrt{1 + \cos \theta} \cdot \left(\frac{\sqrt{1 - \cos \theta}}{\sqrt{1 - \cos \theta}} \right) d\theta$$

$$= 16 \sqrt{2} \int_0^\pi \frac{\sin \theta}{\sqrt{1 - \cos \theta}} d\theta$$

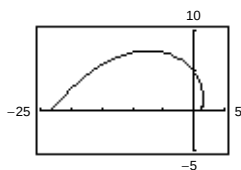
$$= \left[32 \sqrt{2} \sqrt{1 - \cos \theta} \right]_0^\pi$$

$$= 64$$

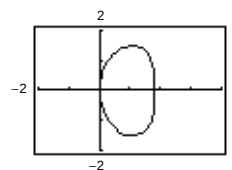
46. $r = \sec \theta, 0 \leq \theta \leq \frac{\pi}{3}$

Length ≈ 1.73 (exact $\sqrt{3}$)

48. $r = e^\theta, 0 \leq \theta \leq \pi$

Length ≈ 31.31

50. $r = 2 \sin(2 \cos \theta), 0 \leq \theta \leq \pi$

Length ≈ 7.78

52. $r = a \cos \theta$

$$r' = -a \sin \theta$$

$$S = 2\pi \int_0^{\pi/2} a \cos \theta (\cos \theta) \sqrt{a^2 \cos^2 \theta + a^2 \sin^2 \theta} d\theta$$

$$= 2\pi a^2 \int_0^{\pi/2} \cos^2 \theta d\theta = \pi a^2 \int_0^{\pi/2} (1 + \cos 2\theta) d\theta$$

$$= \left[\pi a^2 \left(\theta + \frac{\sin 2\theta}{2} \right) \right]_0^{\pi/2} = \frac{\pi^2 a^2}{2}$$

54. $r = a(1 + \cos \theta)$

$$r' = -a \sin \theta$$

$$S = 2\pi \int_0^\pi a(1 + \cos \theta) \sin \theta \sqrt{a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta} d\theta = 2\pi a^2 \int_0^\pi \sin \theta (1 + \cos \theta) \sqrt{2 + 2 \cos \theta} d\theta$$

$$= -2\sqrt{2} \pi a^2 \int_0^\pi (1 + \cos \theta)^{3/2} (-\sin \theta) d\theta = -\frac{4\sqrt{2} \pi a^2}{5} \left[(1 + \cos \theta)^{5/2} \right]_0^\pi = \frac{32 \pi a^2}{5}$$

56. $r = \theta$

$$r' = 1$$

$$S = 2\pi \int_0^\pi \theta \sin \theta \sqrt{\theta^2 + 1} d\theta \approx 42.32$$

58. The curves might intersect for different values of θ :

See page 696.

60. (a) $S = 2\pi \int_{\alpha}^{\beta} f(\theta) \sin \theta \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta$

(b) $S = 2\pi \int_{\alpha}^{\beta} f(\theta) \cos \theta \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta$

62. $r = 8 \cos \theta, 0 \leq \theta \leq \pi$

(a) $A = \frac{1}{2} \int_0^{\pi} r^2 d\theta = \frac{1}{2} \int_0^{\pi} 64 \cos^2 \theta d\theta = 32 \int_0^{\pi} \frac{1 + \cos 2\theta}{2} d\theta = 16 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi} = 16\pi$

(Area circle = $\pi r^2 = \pi 4^2 = 16\pi$)

(b)

θ	0.2	0.4	0.6	0.8	1.0	1.2	1.4
A	6.32	12.14	17.06	20.80	23.27	24.60	25.08

(c), (d) For $\frac{1}{4}$ of area ($4\pi \approx 12.57$): 0.42

For $\frac{1}{2}$ of area ($8\pi \approx 25.13$): 1.57 ($\pi/2$)

For $\frac{3}{4}$ of area ($12\pi \approx 37.70$): 2.73

(e) No, it does not depend on the radius.

64. False. $f(\theta) = 0$ and $g(\theta) = \sin 2\theta$ have only one point of intersection.

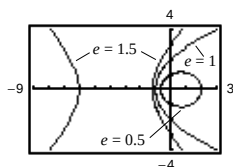
Section 9.6 Polar Equations of Conics and Kepler's Laws

2. $r = \frac{2e}{1 - e \cos \theta}$

(a) $e = 1, r = \frac{2}{1 - \cos \theta}$, parabola

(b) $e = 0.5, r = \frac{1}{1 - 0.5 \cos \theta} = \frac{2}{2 - \cos \theta}$, ellipse

(c) $e = 1.5, r = \frac{3}{1 - 1.5 \cos \theta} = \frac{6}{2 - 3 \cos \theta}$, hyperbola

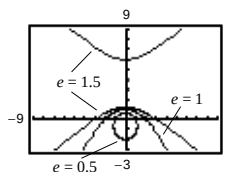


4. $r = \frac{2e}{1 + e \sin \theta}$

(a) $e = 1, r = \frac{2}{1 + \sin \theta}$, parabola

(b) $e = 0.5, r = \frac{1}{1 + 0.5 \sin \theta} = \frac{2}{2 + \sin \theta}$, ellipse

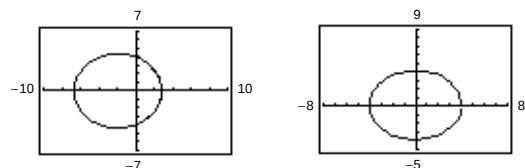
(c) $e = 1.5, r = \frac{3}{1 + 1.5 \sin \theta} = \frac{6}{2 + 3 \sin \theta}$, hyperbola



6. $r = \frac{4}{1 - 0.4 \cos \theta}$

(a) Because $e = 0.4 < 1$, the conic is an ellipse with vertical directrix to the left of the pole.

(c)



(b) $r = \frac{4}{1 + 0.4 \cos \theta}$

The ellipse is shifted to the left. The vertical directrix is to the right of the pole

$$r = \frac{4}{1 - 0.4 \sin \theta}$$

The ellipse has a horizontal directrix below the pole.

8. Ellipse; Matches (f)

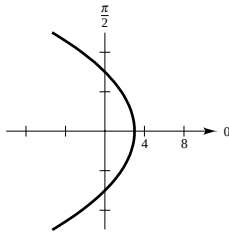
10. Parabola; Matches (e)

12. Hyperbola; Matches (d)

$$14. r = \frac{6}{1 + \cos \theta}$$

Parabola since $e = 1$

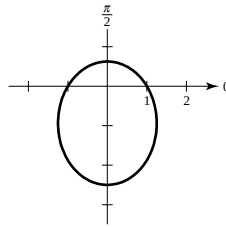
Vertex: $(3, 0)$



$$16. r = \frac{5}{5 + 3 \sin \theta} = \frac{1}{1 + (3/5) \sin \theta}$$

Ellipse since $e = \frac{3}{5} < 1$

Vertices: $(\frac{5}{8}, \frac{\pi}{2}), (\frac{5}{2}, \frac{3\pi}{2})$



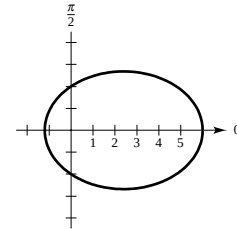
$$18. r(3 - 2 \cos \theta) = 6$$

$$r = \frac{6}{3 - 2 \cos \theta}$$

$$= \frac{2}{1 - (2/3) \cos \theta}$$

Ellipse since $e = \frac{2}{3} < 1$

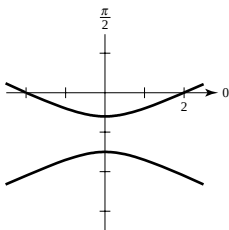
Vertices: $(6, 0), (\frac{6}{5}, \pi)$



$$20. r = \frac{-6}{3 + 7 \sin \theta} = \frac{-2}{1 + (7/3) \sin \theta}$$

Hyperbola since $e = \frac{7}{3} > 1$.

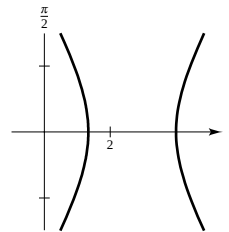
Vertices: $(-\frac{3}{5}, \frac{\pi}{2}), (\frac{3}{2}, \frac{3\pi}{2})$



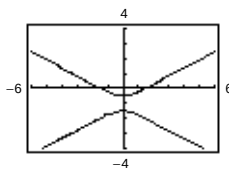
$$22. r = \frac{4}{1 + 2 \cos \theta}$$

Hyperbola since $e = 2 > 1$

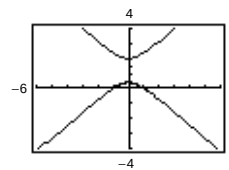
Vertices: $(\frac{4}{3}, 0), (-4, \pi)$



24. Hyperbola



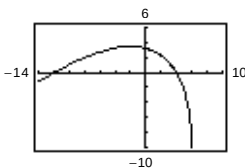
26. Hyperbola



$$28. r = \frac{6}{1 + \cos(\theta - \frac{\pi}{3})}$$

Rotate the graph of $r = \frac{6}{1 + \cos \theta}$

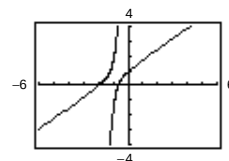
counterclockwise through the angle $\frac{\pi}{3}$.



$$30. r = \frac{-6}{3 + 7 \sin(\theta + (2\pi/3))}$$

Rotate graph of $r = \frac{-6}{3 + 7 \sin \theta}$

Clockwise through angle of $2\pi/3$.



32. Change θ to $\theta - \frac{\pi}{6}$: $r = \frac{2}{1 + \sin\left(\theta - \frac{\pi}{6}\right)}$

34. Parabola

$$e = 1, y = 1, d = 1$$

$$r = \frac{ed}{1 + e \sin \theta} = \frac{1}{1 + \sin \theta}$$

36. Ellipse

$$e = \frac{3}{4}, y = -2, d = 2$$

$$\begin{aligned} r &= \frac{ed}{1 - e \sin \theta} \\ &= \frac{2(3/4)}{1 - (3/4) \sin \theta} \\ &= \frac{6}{4 - 3 \sin \theta} \end{aligned}$$

38. Hyperbola

$$e = \frac{3}{2}, x = -1, d = 1$$

$$\begin{aligned} r &= \frac{ed}{1 - e \cos \theta} \\ &= \frac{3/2}{1 - (3/2) \cos \theta} \\ &= \frac{3}{2 - 3 \cos \theta} \end{aligned}$$

40. Parabola

Vertex: $(5, \pi)$

$$e = 1, d = 10$$

$$r = \frac{ed}{1 - e \cos \theta} = \frac{10}{1 - \cos \theta}$$

42. Ellipse

$$\text{Vertices: } \left(2, \frac{\pi}{2}\right), \left(4, \frac{3\pi}{2}\right)$$

$$e = \frac{1}{3}, d = 8$$

$$\begin{aligned} r &= \frac{ed}{1 + e \sin \theta} \\ &= \frac{8/3}{1 + (1/3) \sin \theta} \\ &= \frac{8}{3 + \sin \theta} \end{aligned}$$

44. Hyperbola

Vertices: $(2, 0), (10, 0)$

$$e = \frac{3}{2}, d = \frac{10}{3}$$

$$\begin{aligned} r &= \frac{ed}{1 + e \cos \theta} \\ &= \frac{5}{1 + (3/2) \cos \theta} \\ &= \frac{10}{2 + 3 \cos \theta} \end{aligned}$$

46. $r = \frac{4}{1 + \sin \theta}$ is a parabola with horizontal directrix above the pole.

(a) Parabola with vertical directrix to left pole.

(b) Parabola with horizontal directrix below pole.

(c) Parabola with vertical directrix to right of pole.

(d) Parabola (b) rotated counterclockwise $\pi/4$.

48. (a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$x^2 b^2 + y^2 a^2 = a^2 b^2$$

$$b^2 r^2 \cos^2 \theta + a^2 r^2 \sin^2 \theta = a^2 b^2$$

$$r^2 [b^2 \cos^2 \theta + a^2 (1 - \cos^2 \theta)] = a^2 b^2$$

$$r^2 [a^2 + \cos^2 \theta (b^2 - a^2)] = a^2 b^2$$

$$r^2 = \frac{a^2 b^2}{a^2 + (b^2 - a^2) \cos^2 \theta} = \frac{a^2 b^2}{a^2 - c^2 \cos^2 \theta}$$

$$= \frac{b^2}{1 - (c/a)^2 \cos^2 \theta} = \frac{b^2}{1 - e^2 \cos^2 \theta}$$

(b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$x^2 b^2 - y^2 a^2 = a^2 b^2$$

$$b^2 r^2 \cos^2 \theta - a^2 r^2 \sin^2 \theta = a^2 b^2$$

$$r^2 [b^2 \cos^2 \theta - a^2 (1 - \cos^2 \theta)] = a^2 b^2$$

$$r^2 [-a^2 + \cos^2 \theta (a^2 + b^2)] = a^2 b^2$$

$$r^2 = \frac{a^2 b^2}{-a^2 + c^2 \cos^2 \theta} = \frac{b^2}{-1 + (c^2/a^2) \cos^2 \theta}$$

$$= \frac{-b^2}{1 - e^2 \cos^2 \theta}$$

50. $a = 4, c = 5, b = 3, e = \frac{5}{4}$

$$r^2 = \frac{-9}{1 - (25/16) \cos^2 \theta}$$

52. $a = 2, b = 1, c = \sqrt{3}, e = \frac{\sqrt{3}}{2}$

$$r^2 = \frac{1}{1 - (3/4) \cos^2 \theta}$$

54. $A = 2 \left[\frac{1}{2} \int_{-\pi/2}^{\pi/2} \left(\frac{2}{3 - 2 \sin \theta} \right)^2 d\theta \right] = 4 \int_{-\pi/2}^{\pi/2} \frac{1}{(3 - 2 \sin \theta)^2} d\theta \approx 3.37$

56. (a) $r = \frac{ed}{1 - e \cos \theta}$

When $\theta = 0, r = c + a = ea + a = a(1 + e)$.

Therefore,

$$a(1 + e) = \frac{ed}{1 - e}$$

$$a(1 + e)(1 - e) = ed$$

$$a(1 - e^2) = ed.$$

Thus, $r = \frac{(1 - e^2)a}{1 - e \cos \theta}$.

(b) The perihelion distance is $a - c = a - ea = a(1 - e)$.

When $\theta = \pi, r = \frac{(1 - e^2)a}{1 + e} = a(1 - e)$.

The aphelion distance is $a + c = a + ea = a(1 + e)$.

When $\theta = 0, r = \frac{(1 - e^2)a}{1 - e} = a(1 + e)$.

58. $a = 1.427 \times 10^9 \text{ km}$

$$e = 0.0543$$

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta} = \frac{1.422792505 \times 10^9}{1 - 0.0543 \cos \theta}$$

Perihelion distance: $a(1 - e) = 1.3495139 \times 10^9 \text{ km}$

Aphelion distance: $a(1 + e) = 1.5044861 \times 10^9 \text{ km}$

60. $a = 36.0 \times 10^6 \text{ mi}, e = 0.206$

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta} \approx \frac{34.472 \times 10^6}{1 - 0.206 \cos \theta}$$

Perihelion distance: $a(1 - e) = 28.582 \times 10^6 \text{ mi}$

Aphelion distance: $a(1 + e) = 43.416 \times 10^6 \text{ mi}$

62. $r = a \sin \theta + b \cos \theta$

$$r^2 = ar \sin \theta + br \cos \theta$$

$$x^2 + y^2 = ay + bx$$

$x^2 + y^2 - bx - ay = 0$ represents a circle.

Review Exercises for Chapter 9

2. Matches (b) - hyperbola

4. Matches (c) - hyperbola

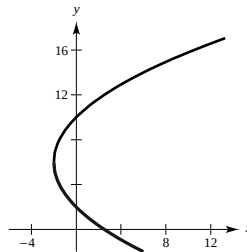
6. $y^2 - 12y - 8x + 20 = 0$

$$y^2 - 12y + 36 = 8x - 20 + 36$$

$$(y - 6)^2 = 4(2)(x + 2)$$

Parabola

Vertex: $(-2, 6)$



8. $4x^2 + y^2 - 16x + 15 = 0$

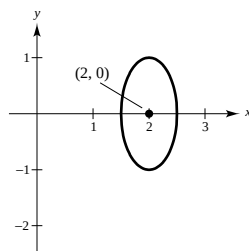
$$4(x^2 - 4x + 4) + y^2 = -15 + 16$$

$$\frac{(x-2)^2}{1/4} + \frac{y^2}{1} = 1$$

Ellipse

Center: (2, 0)

Vertices: (2, ± 1)



10. $4x^2 - 4y^2 - 4x + 8y - 11 = 0$

$$4\left(x^2 - x + \frac{1}{4}\right) - 4(y^2 - 2y + 1) = 11 + 1 - 4$$

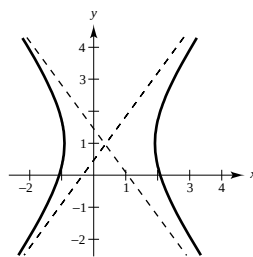
$$\frac{[x - (1/2)]^2}{2} - \frac{(y-1)^2}{2} = 1$$

Hyperbola

Center: $\left(\frac{1}{2}, 1\right)$

Vertices: $\left(\frac{1}{2} \pm \sqrt{2}, 1\right)$

Asymptotes: $y = 1 \pm \left(x - \frac{1}{2}\right)$



12. Vertex: (4, 2)

Focus: (4, 0)

Parabola opens downward

$$p = -2$$

$$(x-4)^2 = 4(-2)(y-2)$$

$$x^2 - 8x + 8y = 0$$

14. Center: (0, 0)

Solution points: (1, 2), (2, 0)

Substituting the values of the coordinates of the given points into

$$\left(\frac{x^2}{b^2}\right) + \left(\frac{y^2}{a^2}\right) = 1,$$

we obtain the system

$$\left(\frac{1}{b^2}\right) + \left(\frac{4}{a^2}\right) = 1, 4/b^2 = 1.$$

Solving the system, we have

$$a^2 = \frac{16}{3} \text{ and } b^2 = 4, \left(\frac{x^2}{4}\right) + \left(\frac{3y^2}{16}\right) = 1.$$

18. $\frac{x^2}{4} + \frac{y^2}{25} = 1, a = 5, b = 2, c = \sqrt{21}, e = \frac{\sqrt{21}}{5}$

By Example 5 of Section 9.1,

$$C = 20 \int_0^{\pi/2} \sqrt{1 - \frac{21}{25} \sin^2 \theta} d\theta \approx 23.01.$$

16. Foci: (0, ± 8)

Asymptotes: $y = \pm 4x$

Center: (0, 0)

Vertical transverse axis

$$c = 8$$

$$y = \frac{a}{b}x = 4x \text{ asymptote} \rightarrow a = 4b$$

$$b^2 = c^2 - a^2 = 64 - (4b)^2 \Rightarrow 17b^2 = 64$$

$$\Rightarrow b^2 = \frac{64}{17} \Rightarrow a^2 = \frac{1024}{17}$$

$$\frac{y^2}{1024/17} - \frac{x^2}{64/17} = 1$$

20. $y = \frac{1}{200}x^2$

(a) $x^2 = 200y$

$x^2 = 4(50)y$

Focus: (0, 50)

(b) $y = \frac{1}{200}x^2$

$y' = \frac{1}{100}x$

$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{10,000}}$

$S = 2\pi \int_0^{100} x \sqrt{1 + \frac{x^2}{10,000}} dx \approx 38,294.49$

22. (a) $A = 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx = \frac{4b}{a} \left(\frac{1}{2} \right) \left[x \sqrt{a^2 - x^2} + a^2 \arcsin\left(\frac{x}{a}\right) \right]_0^a = \pi ab$

(b) **Disk:** $V = 2\pi \int_0^b \frac{a^2}{b^2} (b^2 - y^2) dy = \frac{2\pi a^2}{b^2} \int_0^b (b^2 - y^2) dy = \frac{2\pi a^2}{b^2} \left[b^2 y - \frac{1}{3} y^3 \right]_0^b = \frac{4}{3} \pi a^2 b$

$$S = 4\pi \int_0^b \frac{a}{b} \sqrt{b^2 - y^2} \left(\frac{\sqrt{b^4 + (a^2 - b^2)y^2}}{b \sqrt{b^2 - y^2}} \right) dy$$

$$= \frac{4\pi a}{b^2} \int_0^b \sqrt{b^4 + c^2 y^2} dy = \frac{2\pi a}{b^2 c} \left[cy \sqrt{b^4 + c^2 y^2} + b^4 \ln|cy + \sqrt{b^4 + c^2 y^2}| \right]_0^b$$

$$= \frac{2\pi a}{b^2 c} [b^2 c \sqrt{b^2 + c^2} + b^4 \ln|cb + b \sqrt{b^2 + c^2}| - b^4 \ln(b^2)]$$

$$= 2\pi a^2 + \frac{\pi ab^2}{c} \ln\left(\frac{c+a}{e}\right)^2 = 2\pi a^2 + \left(\frac{\pi b^2}{e}\right) \ln\left(\frac{1+e}{1-e}\right)$$

(c) **Disk:** $V = 2\pi \int_0^a \frac{b^2}{a^2} (a^2 - x^2) dx = \frac{2\pi b^2}{a^2} \int_0^a (a^2 - x^2) dx = \frac{2\pi b^2}{a^2} \left[a^2 x - \frac{1}{3} x^3 \right]_0^a = \frac{4}{3} \pi ab^2$

$$S = 2(2\pi) \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} \left(\frac{\sqrt{a^4 - (a^2 - b^2)x^2}}{a \sqrt{a^2 - x^2}} \right) dx$$

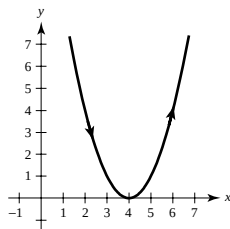
$$= \frac{4\pi b}{a^2} \int_0^a \sqrt{a^4 - c^2 x^2} dx = \frac{2\pi b}{a^2 c} \left[cx \sqrt{a^4 - c^2 x^2} + a^4 \arcsin\left(\frac{cx}{a^2}\right) \right]_0^a$$

$$= \frac{a\pi b}{a^2 c} \left[a^2 c \sqrt{a^2 - c^2} + a^4 \arcsin\left(\frac{c}{a}\right) \right] = 2\pi b^2 + 2\pi \left(\frac{ab}{e}\right) \arcsin(e)$$

24. $x = t + 4, y = t^2$

$t = x - 4 \Rightarrow y = (x - 4)^2$

Parabola

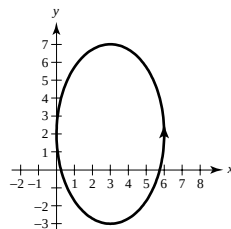


26. $x = 3 + 3 \cos \theta, y = 2 + 5 \sin \theta$

$\left(\frac{x-3}{3}\right)^2 + \left(\frac{y-2}{5}\right)^2 = 1$

$\frac{(x-3)^2}{9} + \frac{(y-2)^2}{25} = 1$

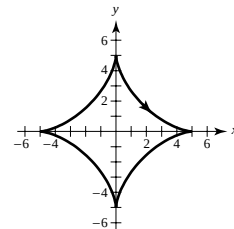
Ellipse



28. $x = 5 \sin^3 \theta, y = 5 \cos^3 \theta$

$\left(\frac{x}{5}\right)^{2/3} + \left(\frac{y}{5}\right)^{2/3} = 1$

$x^{2/3} + y^{2/3} = 5^{2/3}$



30. $(x - h)^2 + (y - k)^2 = r^2$

$$(x - 5)^2 + (y - 3)^2 = 2^2 = 4$$

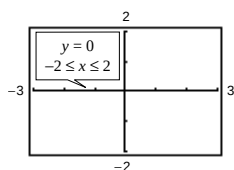
34. $x = (a - b) \cos t + b \cos \left(\frac{a - b}{b} t \right)$

$$y = (a - b) \sin t - b \sin \left(\frac{a - b}{b} t \right)$$

(a) $a = 2, b = 1$

$$x = \cos t + \cos t = 2 \cos t$$

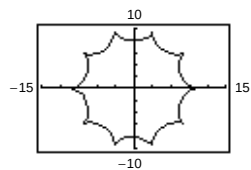
$$y = \sin t - \sin t = 0$$



(d) $a = 10, b = 1$

$$x = 9 \cos t + \cos 9t$$

$$y = 9 \sin t - \sin 9t$$

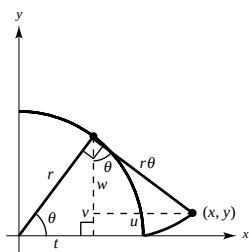


36. $x = t + u = r \cos \theta + r \theta \sin \theta$

$$= r(\cos \theta + \theta \sin \theta)$$

$$y = v - w = r \sin \theta - r \theta \cos \theta$$

$$= r(\sin \theta - \theta \cos \theta)$$



32. $a = 4, c = 5, b^2 = c^2 - a^2 = 9, \frac{y^2}{16} - \frac{x^2}{9} = 1$

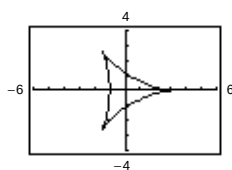
Let $\frac{y^2}{16} = \sec^2 \theta$ and $\frac{x^2}{9} = \tan^2 \theta$.

Then $x = 3 \tan \theta$ and $y = 4 \sec \theta$.

(b) $a = 3, b = 1$

$$x = 2 \cos t + \cos 2t$$

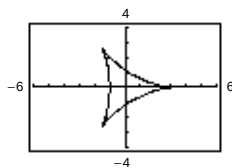
$$y = 2 \sin t - \sin 2t$$



(e) $a = 3, b = 2$

$$x = \cos t + 2 \cos \frac{t}{2}$$

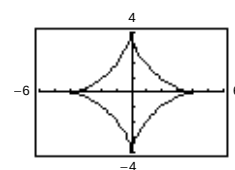
$$y = \sin t - 2 \sin \frac{t}{2}$$



(c) $a = 4, b = 1$

$$x = 3 \cos t + \cos 3t$$

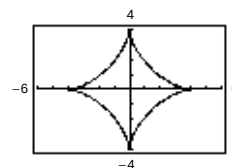
$$y = 3 \sin t - \sin 3t$$



(f) $a = 4, b = 3$

$$x = \cos t + 3 \cos \frac{t}{3}$$

$$y = \sin t - 3 \sin \frac{t}{3}$$



38. $x = t + 4$

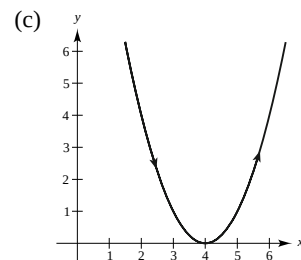
$$y = t^2$$

(a) $\frac{dy}{dx} = \frac{2t}{1} = 2t = 0$ when $t = 0$.

 Point of horizontal tangency: $(4, 0)$

(b) $t = x - 4$

$$y = (x - 4)^2$$



40. $x = \frac{1}{t}$

$y = t^2$

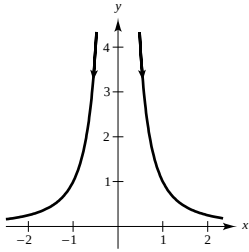
(a) $\frac{dy}{dx} = \frac{2t}{-1/t^2} = -2t^3$

No horizontal tangents ($t \neq 0$)

(b) $t = \frac{1}{x}$

$y = \frac{1}{x^2}$

(c)



44. $x = 6 \cos \theta$

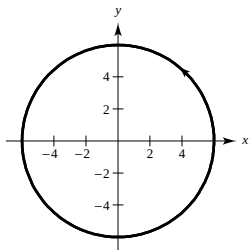
$y = 6 \sin \theta$

(a) $\frac{dy}{dx} = \frac{6 \cos \theta}{-6 \sin \theta} = -\cot \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of horizontal tangency: $(0, 6), (0, -6)$

(b) $\left(\frac{x}{6}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$

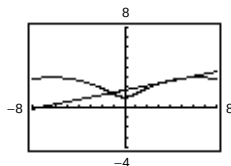
(c)



48. $x = 2\theta - \sin \theta$

$y = 2 - \cos \theta$

(a), (c)



(b) At $\theta = \frac{\pi}{6}, \frac{dx}{d\theta} \approx 1.134, \left(2 - \frac{\sqrt{3}}{2}\right),$

$\frac{dy}{dt} = 0.5,$ and $\frac{dy}{dx} \approx 0.441$

42. $x = 2t - 1$

$y = \frac{1}{t^2 - 2t}$

(a) $\frac{dy}{dx} = \frac{-(t^2 - 2t)^{-2}(2t - 2)}{2}$

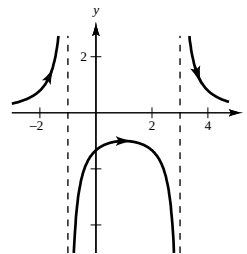
$= \frac{1 - t}{t^2(t - 2)^2} = 0$ when $t = 1$.

Point of horizontal tangency: $(1, -1)$

(b) $t = \frac{x + 1}{2}$

$y = \frac{1}{[(x + 1)/2]^2 - 2[(x + 1)/2]} = \frac{4}{(x - 3)(x + 1)}$

(c)



46. $x = e^t$

$y = e^{-t}$

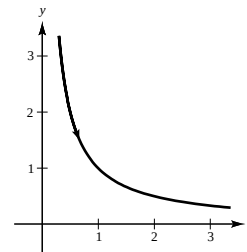
(a) $\frac{dy}{dx} = \frac{-e^{-t}}{e^t} = -\frac{1}{e^{2t}} = -\frac{1}{x^2}$

No horizontal tangents

(b) $t = \ln x$

$y = e^{-\ln x} = e^{\ln(1/x)} = \frac{1}{x}, x > 0$

(c)



50. $x = 6 \cos \theta$

$y = 6 \sin \theta$

$\frac{dx}{d\theta} = -6 \sin \theta$

$\frac{dy}{d\theta} = 6 \cos \theta$

$s = \int_0^\pi \sqrt{36 \sin^2 \theta + 36 \cos^2 \theta} d\theta = \left[6\theta\right]_0^\pi = 6\pi$

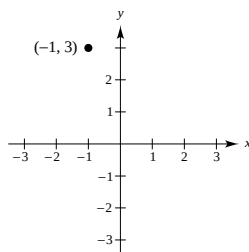
(one-half circumference of circle)

52. $(x, y) = (-1, 3)$

$$r = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\theta = \arctan(-3) \approx 1.89 \text{ (108.43}^\circ\text{)}$$

$$(r, \theta) = (\sqrt{10}, 1.89), (-\sqrt{10}, 5.03)$$



54. $r = 10$

$$r^2 = 100$$

$$x^2 + y^2 = 100$$

56.
$$r = \frac{1}{2 - \cos \theta}$$

$$2r - r \cos \theta = 1$$

$$2(\pm \sqrt{x^2 + y^2}) - x = 1$$

$$4(x^2 + y^2) = (x + 1)^2$$

$$3x^2 + 4y^2 - 2x - 1 = 0$$

58.
$$r = 4 \sec\left(\theta - \frac{\pi}{3}\right) = \frac{4}{\cos[\theta - (\pi/3)]}$$

$$= \frac{4}{(1/2) \cos \theta + (\sqrt{3}/2) \sin \theta}$$

$$r(\cos \theta + \sqrt{3} \sin \theta) = 8$$

$$x + \sqrt{3}y = 8$$

60.
$$\theta = \frac{3\pi}{4}$$

$$\tan \theta = -1$$

$$\frac{y}{x} = -1$$

$$y = -x$$

62. $x^2 + y^2 - 4x = 0$

$$r^2 - 4r \cos \theta = 0$$

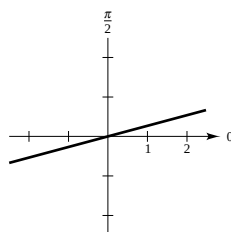
$$r = 4 \cos \theta$$

64.
$$(x^2 + y^2) \left(\arctan \frac{y}{x} \right)^2 = a^2$$

$$r^2 \theta^2 = a^2$$

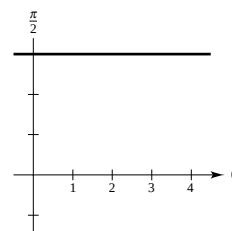
66. $\theta = \frac{\pi}{12}$

Line



68. $r = 3 \csc \theta, r \sin \theta = 3, y = 3$

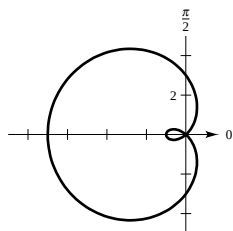
Horizontal line



70. $r = 3 - 4 \cos \theta$

Limaçon

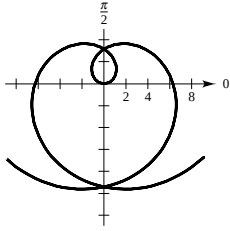
Symmetric to polar axis



θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	-1	1	3	5	7

72. $r = 2\theta$

Spiral

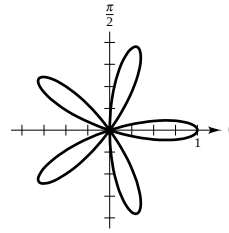
Symmetric to $\theta = \pi/2$ 

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	0	$\frac{\pi}{5}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π

74. $r = \cos(5\theta)$

Rose curve with five petals

Symmetric to polar axis

Relative extrema: $(1, 0), (-1, \frac{\pi}{5}), (1, \frac{2\pi}{5}), (-1, \frac{3\pi}{5}), (1, \frac{4\pi}{5})$ Tangents at the pole: $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$ 

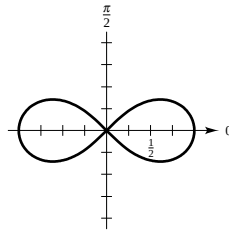
76. $r^2 = \cos(2\theta)$

Lemniscate

Symmetric to the polar axis

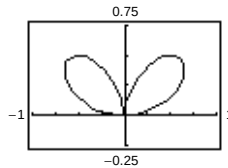
Relative extrema: $(\pm 1, 0)$ Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	± 1	$\pm \frac{\sqrt{2}}{2}$	0



78. $r = 2 \sin \theta \cos^2 \theta$

Bifolium

Symmetric to $\theta = \pi/2$ 

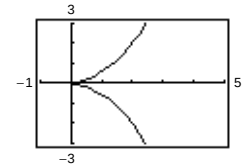
80. $r = 4(\sec \theta - \cos \theta)$

Semicubical parabola

Symmetric to the polar axis

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \frac{\pi^-}{2}$$

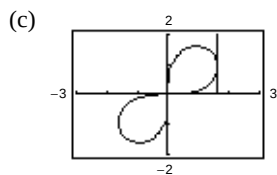
$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \frac{-\pi^+}{2}$$



82. $r^2 = 4 \sin(2\theta)$

(a) $2r \left(\frac{dr}{d\theta} \right) = 8 \cos(2\theta)$

$$\frac{dr}{d\theta} = \frac{4 \cos(2\theta)}{r}$$

Tangents at the pole: $\theta = 0, \frac{\pi}{2}$ 

$$(b) \frac{dy}{dx} = \frac{r \cos \theta + [(4 \cos 2\theta \sin \theta)/r]}{-r \sin \theta + [(4 \cos 2\theta \cos \theta)/r]}$$

$$= \frac{\cos(2\theta) \sin \theta + \sin(2\theta) \cos \theta}{\cos(2\theta) \cos \theta - \sin(2\theta) \sin \theta}$$

Horizontal tangents:

$$\frac{dy}{dx} = 0 \text{ when } \cos(2\theta) \sin \theta + \sin(2\theta) \cos \theta = 0,$$

$$\tan \theta = -\tan(2\theta), \theta = 0, \frac{\pi}{3}, (0, 0), \left(\pm \sqrt{2\sqrt{3}}, \frac{\pi}{3} \right)$$

Vertical tangents when $\cos 2\theta \cos \theta - \sin 2\theta \sin \theta = 0$:

$$\tan 2\theta \tan \theta = 1, \theta = 0, \frac{\pi}{6}, (0, 0), \left(\pm \sqrt{2\sqrt{3}}, \frac{\pi}{6} \right)$$

84. False. There are an infinite number of polar coordinate representations of a point. For example, the point $(x, y) = (1, 0)$ has polar representations $(r, \theta) = (1, 0), (1, 2\pi), (-1, \pi)$, etc.

86. $r = a \sin \theta, r = a \cos \theta$

The points of intersection are $(a/\sqrt{2}, \pi/4)$ and $(0, 0)$. For $r = a \sin \theta$,

$$m_1 = \frac{dy}{dx} = \frac{a \cos \theta \sin \theta + a \sin \theta \cos \theta}{a \cos^2 \theta - a \sin^2 \theta} = \frac{2 \sin \theta \cos \theta}{\cos 2\theta}.$$

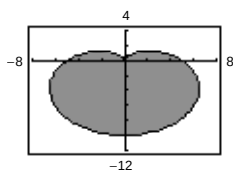
At $(a/\sqrt{2}, \pi/4)$, m_1 is undefined and at $(0, 0)$, $m_1 = 0$. For $r = a \cos \theta$,

$$m_2 = \frac{dy}{dx} = \frac{-a \sin^2 \theta + a \cos^2 \theta}{-a \sin \theta \cos \theta - a \cos \theta \sin \theta} = \frac{\cos 2\theta}{-2 \sin \theta \cos \theta}.$$

At $(a/\sqrt{2}, \pi/4)$, $m_2 = 0$ and at $(0, 0)$, m_2 is undefined. Therefore, the graphs are orthogonal at $(a/\sqrt{2}, \pi/4)$ and $(0, 0)$.

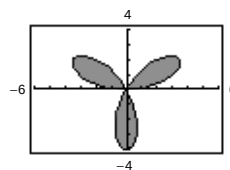
88. $r = 5(1 - \sin \theta)$

$$A = 2 \left[\frac{1}{2} \int_{\pi/2}^{3\pi/2} [5(1 - \sin \theta)]^2 d\theta \right] \approx 117.81 \left(75 \frac{\pi}{2} \right)$$



90. $r = 4 \sin 3\theta$

$$A = 3 \left[\frac{1}{2} \int_0^{\pi/3} (4 \sin 3\theta)^2 d\theta \right] \approx 12.57 (4\pi)$$



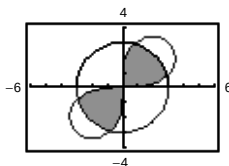
92. $r = 3, r^2 = 18 \sin 2\theta$

$$9 = r^2 = 18 \sin 2\theta$$

$$\sin 2\theta = \frac{1}{2}$$

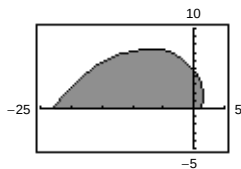
$$\theta = \frac{\pi}{12}$$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/12} 18 \sin 2\theta d\theta + \frac{1}{2} \int_{\pi/12}^{5\pi/12} 9 d\theta + \frac{1}{2} \int_{5\pi/12}^{\pi/2} 18 \sin 2\theta d\theta \right] \approx 1.2058 + 9.4248 + 1.2058 \approx 11.84$$



94. $r = e^\theta, 0 \leq \theta \leq \pi$

$$A = \frac{1}{2} \int_0^\pi (e^\theta)^2 d\theta \approx 133.62$$



96. $r = a \cos 2\theta, \frac{dr}{d\theta} = -2a \sin 2\theta$

$$\begin{aligned} s &= 8 \int_0^{\pi/4} \sqrt{a^2 \cos^2 2\theta + 4a^2 \sin^2 2\theta} d\theta \\ &= 8a \int_0^{\pi/4} \sqrt{1 + 3 \sin^2 2\theta} d\theta \quad (\text{Simpson's Rule: } n = 4) \\ &\approx \frac{\pi a}{6} [1 + 4(1.1997) + 2(1.5811) + 4(1.8870) + 2] \\ &\approx 9.69a \end{aligned}$$

$$63. r_1 = \frac{ed}{1 + \sin \theta} \text{ and } r_2 = \frac{ed}{1 - \sin \theta}$$

Points of intersection: $(ed, 0), (ed, \pi)$

$$r_1: \frac{dy}{dx} = \frac{\left(\frac{ed}{1 + \sin \theta}\right)(\cos \theta) + \left(\frac{-ed \cos \theta}{(1 + \sin \theta)^2}\right)(\sin \theta)}{\left(\frac{-ed}{1 + \sin \theta}\right)(\sin \theta) + \left(\frac{-ed \cos \theta}{(1 + \sin \theta)^2}\right)(\cos \theta)}$$

At $(ed, 0), \frac{dy}{dx} = -1$. At $(ed, \pi), \frac{dy}{dx} = 1$.

$$r_2: \frac{dy}{dx} = \frac{\left(\frac{ed}{1 - \sin \theta}\right)(\cos \theta) + \left(\frac{ed \cos \theta}{(1 - \sin \theta)^2}\right)(\sin \theta)}{\left(\frac{-ed}{1 - \sin \theta}\right)(\sin \theta) + \left(\frac{ed \cos \theta}{(1 - \sin \theta)^2}\right)(\cos \theta)}$$

At $(ed, 0), \frac{dy}{dx} = 1$. At $(ed, \pi), \frac{dy}{dx} = -1$.

Therefore, at $(ed, 0)$ we have $m_1 m_2 = (-1)(1) = -1$, and at (ed, π) we have $m_1 m_2 = 1(-1) = -1$. The curves intersect at right angles.

Review Exercises for Chapter 9

1. Matches (d) - ellipse

$$5. 16x^2 + 16y^2 - 16x + 24y - 3 = 0$$

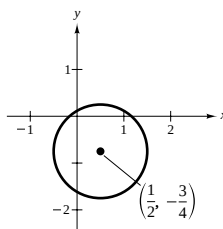
$$\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 + \frac{3}{2}y + \frac{9}{16}\right) = \frac{3}{16} + \frac{1}{4} + \frac{9}{16}$$

$$\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{4}\right)^2 = 1$$

Circle

Center: $\left(\frac{1}{2}, -\frac{3}{4}\right)$

Radius: 1



3. Matches (a) - parabola

$$7. 3x^2 - 2y^2 + 24x + 12y + 24 = 0$$

$$3(x^2 + 8x + 16) - 2(y^2 - 6y + 9) = -24 + 48 - 18$$

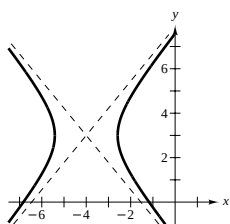
$$\frac{(x + 4)^2}{2} - \frac{(y - 3)^2}{3} = 1$$

Hyperbola

Center: $(-4, 3)$

Vertices: $(-4 \pm \sqrt{2}, 3)$

Asymptotes: $y = 3 \pm \sqrt{\frac{3}{2}}(x + 4)$



$$9. 3x^2 + 2y^2 - 12x + 12y + 29 = 0$$

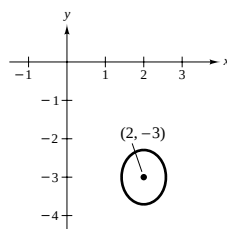
$$3(x^2 - 4x + 4) + 2(y^2 + 6y + 9) = -29 + 12 + 18$$

$$\frac{(x - 2)^2}{1/3} + \frac{(y + 3)^2}{1/2} = 1$$

Ellipse

Center: $(2, -3)$

Vertices: $\left(2, -3 \pm \frac{\sqrt{2}}{2}\right)$



11. Vertex: (0, 2)

Directrix: $x = -3$

Parabola opens to the right

 $p = 3$

$$(y - 2)^2 = 4(3)(x - 0)$$

$$y^2 - 4y - 12x + 4 = 0$$

13. Vertices: $(-3, 0), (7, 0)$ Foci: $(0, 0), (4, 0)$

Horizontal major axis

Center: $(2, 0)$

$$a = 5, c = 2, b = \sqrt{21}$$

$$\frac{(x - 2)^2}{25} + \frac{y^2}{21} = 1$$

15. Vertices: $(\pm 4, 0)$ Foci: $(\pm 6, 0)$ Center: $(0, 0)$

Horizontal transverse axis

$$a = 4, c = 6, b = \sqrt{36 - 16} = 2\sqrt{5}$$

$$\frac{x^2}{16} - \frac{y^2}{20} = 1$$

$$17. \frac{x^2}{9} + \frac{y^2}{4} = 1, a = 3, b = 2, c = \sqrt{5}, e = \frac{\sqrt{5}}{3}$$

By Example 5 of Section 9.1,

$$C = 12 \int_0^{\pi/2} \sqrt{1 - \left(\frac{5}{9}\right) \sin^2 \theta} d\theta \approx 15.87.$$

19. $y = x - 2$ has a slope of 1. The perpendicular slope is -1 .

$$y = x^2 - 2x + 2$$

$$\frac{dy}{dx} = 2x - 2 = -1 \text{ when } x = \frac{1}{2} \text{ and } y = \frac{5}{4}.$$

$$\text{Perpendicular line: } y - \frac{5}{4} = -1\left(x - \frac{1}{2}\right)$$

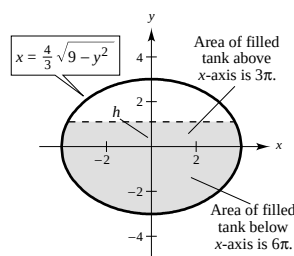
$$4x + 4y - 7 = 0$$

$$21. (a) V = (\pi ab)(\text{Length}) = 12\pi(16) = 192\pi \text{ ft}^3$$

$$\begin{aligned} (b) F &= 2(62.4) \int_{-3}^3 (3 - y) \frac{4}{3} \sqrt{9 - y^2} dy = \frac{8}{3}(62.4) \left[3 \int_{-3}^3 \sqrt{9 - y^2} dy - \int_{-3}^3 y \sqrt{9 - y^2} dy \right] \\ &= \frac{8}{3}(62.4) \left[\frac{3}{2} \left(y \sqrt{9 - y^2} + 9 \arcsin \frac{y}{3} \right) + \frac{1}{3} (9 - y^2)^{3/2} \right]_{-3}^3 \\ &= \frac{8}{3}(62.4) \left[\frac{3}{2} \left(\frac{9\pi}{2} \right) - \frac{3}{2} \left(-\frac{9\pi}{2} \right) \right] = \frac{8}{3}(62.4) \left(\frac{27\pi}{2} \right) \approx 7057.274 \end{aligned}$$

(c) You want $\frac{3}{4}$ of the total area of 12π covered. Find h so that

$$\begin{aligned} 2 \int_0^h \frac{4}{3} \sqrt{9 - y^2} dy &= 3\pi \\ \int_0^h \sqrt{9 - y^2} dy &= \frac{9\pi}{8} \\ \frac{1}{2} \left[y \sqrt{9 - y^2} + 9 \arcsin \left(\frac{y}{3} \right) \right]_0^h &= \frac{9\pi}{8} \\ h \sqrt{9 - h^2} + 9 \arcsin \left(\frac{h}{3} \right) &= \frac{9\pi}{4}. \end{aligned}$$

By Newton's Method, $h \approx 1.212$. Therefore, the total height of the water is $1.212 + 3 = 4.212$ ft.

$$(d) \text{Area of ends} = 2(12\pi) = 24\pi$$

Area of sides = (Perimeter)(Length)

$$\begin{aligned} &= 16 \int_0^{\pi/2} \left(\sqrt{1 - \left(\frac{7}{16}\right) \sin^2 \theta} \right) d\theta (16) \text{ [from Example 5 of Section 9.1]} \\ &\approx 256 \left(\frac{\pi/2}{12} \right) \left[\sqrt{1 - \left(\frac{7}{16}\right) \sin^2(0)} + 4 \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{\pi}{8}\right)} + 2 \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{\pi}{4}\right)} \right. \\ &\quad \left. + 4 \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{3\pi}{8}\right)} + \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{\pi}{2}\right)} \right] \approx 353.65 \end{aligned}$$

$$\text{Total area} = 24\pi + 353.65 \approx 429.05$$

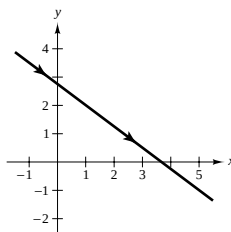
23. $x = 1 + 4t, y = 2 - 3t$

$$t = \frac{x-1}{4} \Rightarrow y = 2 - 3\left(\frac{x-1}{4}\right)$$

$$y = -\frac{3}{4}x + \frac{11}{4}$$

$$4y + 3x - 11 = 0$$

Line

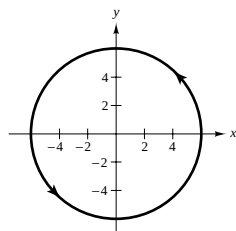


25. $x = 6 \cos \theta, y = 6 \sin \theta$

$$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$$

$$x^2 + y^2 = 36$$

Circle

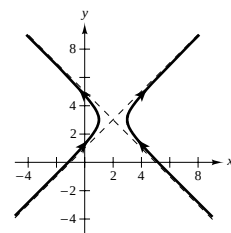


27. $x = 2 + \sec \theta, y = 3 + \tan \theta$

$$(x-2)^2 = \sec^2 \theta = 1 + \tan^2 \theta = 1 + (y-3)^2$$

$$(x-2)^2 - (y-3)^2 = 1$$

Hyperbola



29. $x = 3 + (3 - (-2))t = 3 + 5t$

$$y = 2 + (2 - 6)t = 2 - 4t$$

(other answers possible)

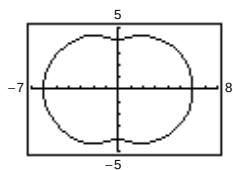
31. $\frac{(x+3)^2}{16} + \frac{(y-4)^2}{9} = 1$

$$\text{Let } \frac{(x+3)^2}{16} = \cos^2 \theta \text{ and } \frac{(y-4)^2}{9} = \sin^2 \theta.$$

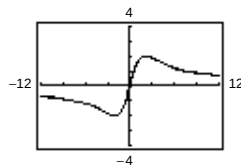
$$\text{Then } x = -3 + 4 \cos \theta \text{ and } y = 4 + 3 \sin \theta.$$

33. $x = \cos 3\theta + 5 \cos \theta$

$$y = \sin 3\theta + 5 \sin \theta$$



35. (a) $x = 2 \cot \theta, y = 4 \sin \theta \cos \theta, 0 < \theta < \pi$



(b) $(4 + x^2)y = (4 + 4 \cot^2 \theta)4 \sin \theta \cos \theta$

$$= 16 \csc^2 \theta \cdot \sin \theta \cdot \cos \theta$$

$$= 16 \frac{\cos \theta}{\sin \theta}$$

$$= 8(2 \cot \theta)$$

$$= 8x$$

37. $x = 1 + 4t$

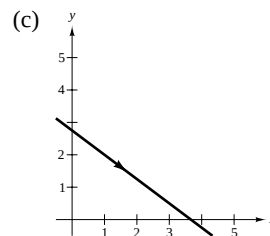
$$y = 2 - 3t$$

(a) $\frac{dy}{dx} = -\frac{3}{4}$

No horizontal tangents

(b) $t = \frac{x-1}{4}$

$$y = 2 - \frac{3}{4}(x-1) = \frac{-3x+11}{4}$$



39. $x = \frac{1}{t}$

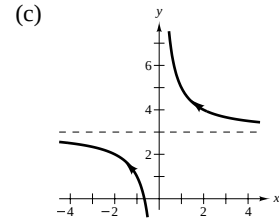
$y = 2t + 3$

(a) $\frac{dy}{dx} = \frac{2}{-1/t^2} = -2t^2$

No horizontal tangents
($t \neq 0$)

(b) $t = \frac{1}{x}$

$y = \frac{2}{x} + 3$



41. $x = \frac{1}{2t + 1}$

$y = \frac{1}{t^2 - 2t}$

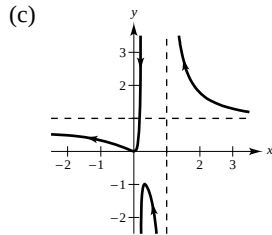
(a) $\frac{dy}{dx} = \frac{\frac{-(2t-2)}{(t^2-2t)^2}}{\frac{-2}{(2t+1)^2}} = \frac{(t-1)(2t+1)^2}{t^2(t-2)^2} = 0$ when $t = 1$.

Point of horizontal tangency: $(\frac{1}{3}, -1)$

(b) $2t + 1 = \frac{1}{x} \Rightarrow t = \frac{1}{2}\left(\frac{1}{x} - 1\right)$

$$y = \frac{1}{\frac{1}{2}\left(\frac{1}{x} - 1\right)\left[\frac{1}{2}\left(\frac{1}{x} - 1\right) - 2\right]}$$

$$= \frac{4x^2}{(1-x)^2 - 4x(1-x)} = \frac{4x^2}{(5x-1)(x-1)}$$



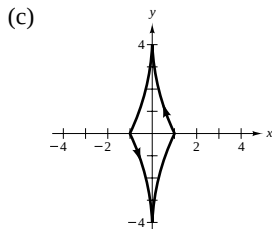
45. $x = \cos^3 \theta$

$y = 4 \sin^3 \theta$

(a) $\frac{dy}{dx} = \frac{12 \sin^2 \theta \cos \theta}{3 \cos^2 \theta (-\sin \theta)} = \frac{-4 \sin \theta}{\cos \theta} = -4 \tan \theta = 0$ when $\theta = 0, \pi$.

But, $\frac{dy}{dt} = \frac{dx}{dt} = 0$ at $\theta = 0, \pi$. Hence no points of horizontal tangency.

(b) $x^{2/3} + \left(\frac{y}{4}\right)^{2/3} = 1$



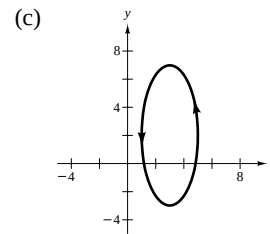
43. $x = 3 + 2 \cos \theta$

$y = 2 + 5 \sin \theta$

(a) $\frac{dy}{dx} = \frac{5 \cos \theta}{-2 \sin \theta} = -2.5 \cot \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of horizontal tangency: $(3, 7), (3, -3)$

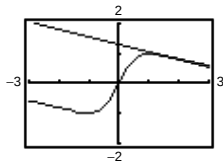
(b) $\frac{(x-3)^2}{4} + \frac{(y-2)^2}{25} = 1$



47. $x = \cot \theta$

$$y = \sin 2\theta = 2 \sin \theta \cos \theta$$

(a), (c)



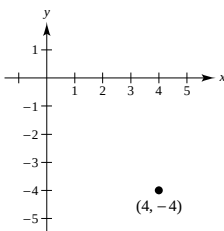
(b) At $\theta = \frac{\pi}{6}$, $\frac{dx}{d\theta} = -4$, $\frac{dy}{d\theta} = 1$, and $\frac{dy}{dx} = -\frac{1}{4}$

51. $(x, y) = (4, -4)$

$$r = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$$

$$\theta = 7\frac{\pi}{4}$$

$$(r, \theta) = \left(4\sqrt{2}, \frac{7\pi}{4}\right), \left(-4\sqrt{2}, \frac{3\pi}{4}\right)$$



53. $r = 3 \cos \theta$

$$r^2 = 3r \cos \theta$$

$$x^2 + y^2 = 3x$$

$$x^2 + y^2 - 3x = 0$$

55. $r = -2(1 + \cos \theta)$

$$r^2 = -2r(1 + \cos \theta)$$

$$x^2 + y^2 = -2(\pm\sqrt{x^2 + y^2}) - 2x$$

$$(x^2 + y^2 + 2x)^2 = 4(x^2 + y^2)$$

57. $r^2 = \cos 2\theta = \cos^2 \theta - \sin^2 \theta$

$$r^4 = r^2 \cos^2 \theta - r^2 \sin^2 \theta$$

$$(x^2 + y^2)^2 = x^2 - y^2$$

59. $r = 4 \cos 2\theta \sec \theta$

$$= 4(2 \cos^2 \theta - 1) \left(\frac{1}{\cos \theta} \right)$$

$$r \cos \theta = 8 \cos^2 \theta - 4$$

$$x = 8 \left(\frac{x^2}{x^2 + y^2} \right) - 4$$

$$x^3 + xy^2 = 4x^2 - 4y^2$$

$$y^2 = x^2 \left(\frac{4-x}{4+x} \right)$$

61. $(x^2 + y^2)^2 = ax^2y$

$$r^4 = a(r^2 \cos^2 \theta)(r \sin \theta)$$

$$r = a \cos^2 \theta \sin \theta$$

63. $x^2 + y^2 = a^2 \left(\arctan \frac{y}{x} \right)^2$

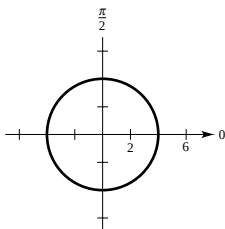
$$r^2 = a^2 \theta^2$$

65. $r = 4$

Circle of radius 4

Centered at the pole

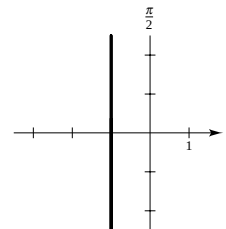
Symmetric to polar axis,

 $\theta = \pi/2$, and pole

67. $r = -\sec \theta = \frac{-1}{\cos \theta}$

$$r \cos \theta = -1, x = -1$$

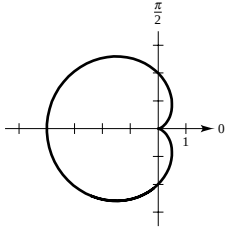
Vertical line



69. $r = -2(1 + \cos \theta)$

Cardioid

Symmetric to polar axis

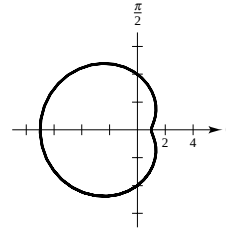


θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	-4	-3	-2	-1	0

71. $r = 4 - 3 \cos \theta$

Limaçon

Symmetric to polar axis



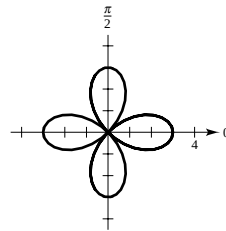
θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	1	$\frac{5}{2}$	4	$\frac{11}{2}$	7

73. $r = -3 \cos(2\theta)$

Rose curve with four petals

 Symmetric to polar axis, $\theta = \frac{\pi}{2}$, and pole

 Relative extrema: $(-3, 0)$, $(3, \frac{\pi}{2})$, $(-3, \pi)$, $(3, \frac{3\pi}{2})$

 Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$


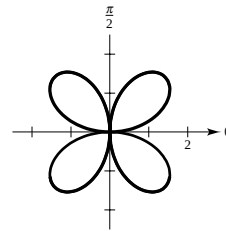
75. $r^2 = 4 \sin^2(2\theta)$

$r = \pm 2 \sin(2\theta)$

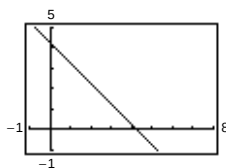
Rose curve with four petals

 Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and pole

 Relative extrema: $(\pm 2, \frac{\pi}{4})$, $(\pm 2, \frac{3\pi}{4})$

 Tangents at the pole: $\theta = 0, \frac{\pi}{2}$


77. $r = \frac{3}{\cos[\theta - (\pi/4)]}$

 Graph of $r = 3 \sec \theta$ rotated through an angle of $\pi/4$


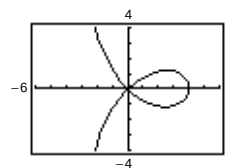
79. $r = 4 \cos 2\theta \sec \theta$

Strophoid

Symmetric to the polar axis

$$r \Rightarrow -\infty \text{ as } \theta \Rightarrow \frac{\pi^-}{2}$$

$$r \Rightarrow -\infty \text{ as } \theta \Rightarrow \frac{-\pi^+}{2}$$



81. $r = 1 - 2 \cos \theta$

(a) The graph has polar symmetry and the tangents at the pole are

$$\theta = \frac{\pi}{3}, -\frac{\pi}{3}.$$

$$(b) \frac{dy}{dx} = \frac{2 \sin^2 \theta + (1 - 2 \cos \theta) \cos \theta}{2 \sin \theta \cos \theta - (1 - 2 \cos \theta) \sin \theta}$$

$$\text{Horizontal tangents: } -4 \cos^2 \theta + \cos \theta + 2 = 0, \cos \theta = \frac{-1 \pm \sqrt{1 + 32}}{-8} = \frac{1 \pm \sqrt{33}}{8}$$

$$\text{When } \cos \theta = \frac{1 \pm \sqrt{33}}{8}, r = 1 - 2 \left(\frac{1 \pm \sqrt{33}}{8} \right) = \frac{3 \mp \sqrt{33}}{4},$$

$$\left[\frac{3 - \sqrt{33}}{4}, \arccos \left(\frac{1 + \sqrt{33}}{8} \right) \right] \approx (-0.686, 0.568)$$

$$\left[\frac{3 - \sqrt{33}}{4}, -\arccos \left(\frac{1 + \sqrt{33}}{8} \right) \right] \approx (-0.686, -0.568)$$

$$\left[\frac{3 + \sqrt{33}}{4}, \arccos \left(\frac{1 - \sqrt{33}}{8} \right) \right] \approx (2.186, 2.206)$$

$$\left[\frac{3 + \sqrt{33}}{4}, -\arccos \left(\frac{1 - \sqrt{33}}{8} \right) \right] \approx (2.186, -2.206).$$

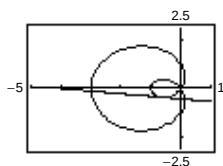
Vertical tangents:

$$\sin \theta (4 \cos \theta - 1) = 0, \sin \theta = 0, \cos \theta = \frac{1}{4},$$

$$\theta = 0, \pi, \theta = \pm \arccos \left(\frac{1}{4} \right), (-1, 0), (3, \pi)$$

$$\left(\frac{1}{2}, \pm \arccos \frac{1}{4} \right) \approx (0.5, \pm 1.318)$$

(c)



83. Circle: $r = 3 \sin \theta$

$$\frac{dy}{dx} = \frac{3 \cos \theta \sin \theta + 3 \sin \theta \cos \theta}{3 \cos \theta \cos \theta - 3 \sin \theta \sin \theta} = \frac{\sin 2\theta}{\cos^2 \theta - \sin^2 \theta} = \tan 2\theta \text{ at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \sqrt{3}$$

Limaçon: $r = 4 - 5 \sin \theta$

$$\frac{dy}{dx} = \frac{-5 \cos \theta \sin \theta + (4 - 5 \sin \theta) \cos \theta}{-5 \cos \theta \cos \theta - (4 - 5 \sin \theta) \sin \theta} \text{ at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \frac{\sqrt{3}}{9}$$

Let α be the angle between the curves:

$$\tan \alpha = \frac{\sqrt{3} - (\sqrt{3}/9)}{1 + (1/3)} = \frac{2\sqrt{3}}{3}.$$

$$\text{Therefore, } \alpha = \arctan \left(\frac{2\sqrt{3}}{3} \right) \approx 49.1^\circ.$$

85. $r = 1 + \cos \theta, r = 1 - \cos \theta$

The points $(1, \pi/2)$ and $(1, 3\pi/2)$ are the two points of intersection (other than the pole). The slope of the graph of $r = 1 + \cos \theta$ is

$$m_1 = \frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta} = \frac{-\sin^2 \theta + \cos \theta(1 + \cos \theta)}{-\sin \theta \cos \theta - \sin \theta(1 + \cos \theta)}.$$

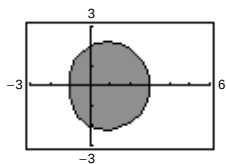
At $(1, \pi/2)$, $m_1 = -1/-1 = 1$ and at $(1, 3\pi/2)$, $m_1 = -1/1 = -1$. The slope of the graph of $r = 1 - \cos \theta$ is

$$m_2 = \frac{dy}{dx} = \frac{\sin^2 \theta + \cos \theta(1 - \cos \theta)}{\sin \theta \cos \theta - \sin \theta(1 - \cos \theta)}.$$

At $(1, \pi/2)$, $m_2 = 1/-1 = -1$ and at $(1, 3\pi/2)$, $m_2 = 1/1 = 1$. In both cases, $m_1 = -1/m_2$ and we conclude that the graphs are orthogonal at $(1, \pi/2)$ and $(1, 3\pi/2)$.

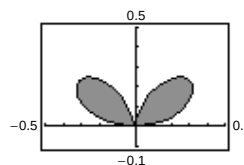
87. $r = 2 + \cos \theta$

$$A = 2 \left[\frac{1}{2} \int_0^\pi (2 + \cos \theta)^2 d\theta \right] \approx 14.14 \left(\frac{9\pi}{2} \right)$$



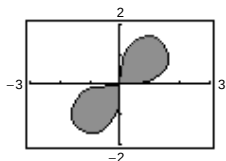
89. $r = \sin \theta \cdot \cos^2 \theta$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/2} (\sin \theta \cos^2 \theta)^2 d\theta \right] \approx 0.10 \left(\frac{\pi}{32} \right)$$



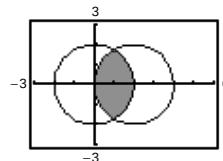
91. $r^2 = 4 \sin 2\theta$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/2} 4 \sin 2\theta d\theta \right] = 4$$



93. $r = 4 \cos \theta, r = 2$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/3} 4 d\theta + \frac{1}{2} \int_{\pi/3}^{\pi/2} (4 \cos \theta)^2 d\theta \right] \approx 4.91$$

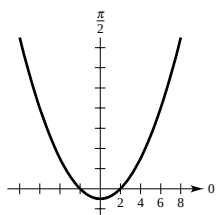


95. $s = 2 \int_0^\pi \sqrt{a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta} d\theta$

$$= 2\sqrt{2} a \int_0^\pi \sqrt{1 - \cos \theta} d\theta = 2\sqrt{2} a \int_0^\pi \frac{\sin \theta}{\sqrt{1 + \cos \theta}} d\theta = \left[-4\sqrt{2} a(1 + \cos \theta)^{1/2} \right]_0^\pi = 8a$$

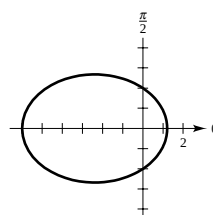
97. $r = \frac{2}{1 - \sin \theta}, e = 1$

Parabola



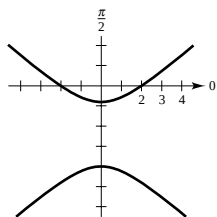
99. $r = \frac{6}{3 + 2 \cos \theta} = \frac{2}{1 + (2/3) \cos \theta}, e = \frac{2}{3}$

Ellipse



$$101. r = \frac{4}{2 - 3 \sin \theta} = \frac{2}{1 - (3/2) \sin \theta}, e = \frac{3}{2}$$

Hyperbola



105. Parabola

Vertex: $(2, \pi)$ Focus: $(0, 0)$

$$e = 1, d = 4$$

$$r = \frac{4}{1 - \cos \theta}$$

103. Circle

Center: $\left(5, \frac{\pi}{2}\right) = (0, 5)$ in rectangular coordinatesSolution point: $(0, 0)$

$$x^2 + (y - 5)^2 = 25$$

$$x^2 + y^2 - 10y = 0$$

$$r^2 - 10r \sin \theta = 0$$

$$r = 10 \sin \theta$$

107. Ellipse

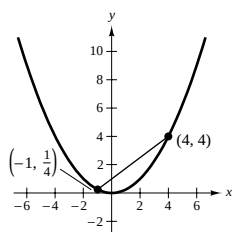
Vertices: $(5, 0), (1, \pi)$ Focus: $(0, 0)$

$$a = 3, c = 2, e = \frac{2}{3}, d = \frac{5}{2}$$

$$r = \frac{\left(\frac{2}{3}\right)\left(\frac{5}{2}\right)}{1 - \left(\frac{2}{3}\right) \cos \theta} = \frac{5}{3 - 2 \cos \theta}$$

Problem Solving for Chapter 9

1. (a)



$$(b) x^2 = 4y$$

$$2x = 4y'$$

$$y' = \frac{1}{2}x$$

$$y - 4 = 2(x - 4) \Rightarrow y = 2x - 4 \quad \text{Tangent line at } (4, 4)$$

$$y - \frac{1}{4} = -\frac{1}{2}(x + 1) \Rightarrow y = -\frac{1}{2}x - \frac{1}{4} \quad \text{Tangent line at } \left(-1, \frac{1}{4}\right)$$

Tangent lines have slopes of 2 and $-1/2 \Rightarrow$ perpendicular.

(c) Intersection:

$$2x - 4 = -\frac{1}{2}x - \frac{1}{4}$$

$$8x - 16 = -2x - 1$$

$$10x = 15$$

$$x = \frac{3}{2} \Rightarrow \left(\frac{3}{2}, -1\right)$$

Point of intersection, $(3/2, -1)$, is on directrix $y = -1$.

3. Consider $x^2 = 4py$ with focus $(0, p)$.

Let $P(a, b)$ be point on parabola.

$$zx = 4py' \Rightarrow y' = \frac{x}{2p}$$

$$y - b = \frac{a}{2p}(x - a) \quad \text{Tangent line}$$

$$\text{For } x = 0, y = b + \frac{a}{2p}(-a) = b - \frac{a^2}{2p} = b - \frac{4pb}{2p} = -b.$$

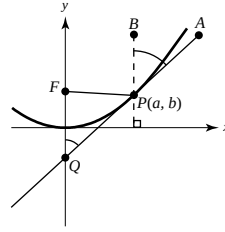
Thus, $Q = (0, -b)$.

$\triangle FQP$ is isosceles because

$$|FQ| = p + b$$

$$\begin{aligned} |FP| &= \sqrt{(a-0)^2 + (b-p)^2} = \sqrt{a^2 + b^2 - 2bp + p^2} \\ &= \sqrt{4pb + b^2 - 2bp + p^2} \\ &= \sqrt{(b+p)^2} \\ &= b + p. \end{aligned}$$

Thus, $\angle FQP = \angle BPA = \angle FPQ$.



5. (a) In $\triangle OCB$, $\cos \theta = \frac{2a}{OB} \Rightarrow OB = 2a \cdot \sec \theta$.

$$\text{In } \triangle OAC, \cos \theta = \frac{OA}{2a} \Rightarrow OA = 2a \cdot \cos \theta.$$

$$\begin{aligned} r = OP = AB = OB - OA &= 2a(\sec \theta - \cos \theta) \\ &= 2a\left(\frac{1}{\cos \theta} - \cos \theta\right) \\ &= 2a \cdot \frac{\sin^2 \theta}{\cos \theta} \\ &= 2a \cdot \tan \theta \sin \theta \end{aligned}$$

$$(b) x = r \cos \theta = (2a \tan \theta \sin \theta) \cos \theta = 2a \sin^2 \theta$$

$$y = r \sin \theta = (2a \tan \theta \sin \theta) \sin \theta = 2a \tan \theta \cdot \sin^2 \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Let $t = \tan \theta$, $-\infty < t < \infty$.

$$\text{Then } \sin^2 \theta = \frac{t^2}{1+t^2} \text{ and } x = 2a \frac{t^2}{1+t^2}, y = 2a \frac{t^3}{1+t^2}.$$

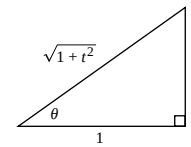
$$(c) \quad r = 2a \tan \theta \sin \theta$$

$$r \cos \theta = 2a \sin^2 \theta$$

$$r^3 \cos \theta = 2a r^2 \sin^2 \theta$$

$$(x^2 + y^2)x = 2ay^2$$

$$y^2 = \frac{x^3}{(2a-x)}$$



7. $y = a(1 - \cos \theta) \Rightarrow \cos \theta = \frac{a-y}{a}$

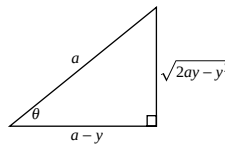
$$\theta = \arccos\left(\frac{a-y}{a}\right)$$

$$x = a(\theta - \sin \theta)$$

$$= a\left(\arccos\left(\frac{a-y}{a}\right) - \sin\left(\arccos\left(\frac{a-y}{a}\right)\right)\right)$$

$$= a\left(\arccos\left(\frac{a-y}{a}\right) - \frac{\sqrt{2ay-y^2}}{a}\right)$$

$$x = a \cdot \arccos\left(\frac{a-y}{a}\right) - \sqrt{2ay-y^2}, 0 \leq y \leq 2a$$



9. For $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \dots$

$$y = \frac{2}{\pi}, \frac{-2}{3\pi}, \frac{2}{5\pi}, \frac{-2}{7\pi}, \dots$$

Hence, the curve has length greater than

$$\begin{aligned} S &= \frac{2}{\pi} + \frac{2}{3\pi} + \frac{2}{5\pi} + \frac{2}{7\pi} + \dots \\ &= \frac{2}{\pi} \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots \right) \\ &> \frac{2}{\pi} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots \right) \\ &= \infty. \end{aligned}$$

13. If a dog is located at (r, θ) , then its neighbor is at $\left(r, \theta + \frac{\pi}{2}\right)$:

$$(x, y) = (r \cos \theta, r \sin \theta) \text{ and } (x, y) = (-r \sin \theta, r \cos \theta).$$

The slope joining these points is

$$\frac{r \cos \theta - r \sin \theta}{-r \sin \theta - r \cos \theta} = \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta} = \text{slope of tangent line at } (r, \theta).$$

$$\frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} = \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta}$$

$$\Rightarrow \frac{dr}{d\theta} = -r$$

$$\frac{dr}{r} = -d\theta$$

$$\ln r = -\theta + C_1$$

$$r = e^{-\theta + C_1}$$

$$r = Ce^{-\theta}$$

$$r\left(\frac{\pi}{4}\right) = \frac{d}{\sqrt{2}} \Rightarrow r = Ce^{-\pi/4} = \frac{d}{\sqrt{2}} \Rightarrow C = \frac{d}{\sqrt{2}} e^{\pi/4}$$

$$\text{Finally, } r = \frac{d}{\sqrt{2}} e^{((\pi/4) - \theta)}.$$

15. (a) The first plane makes an angle of 70° with the positive x -axis, and is 150 miles from P :

$$x_1 = \cos 70^\circ(150 - 375t)$$

$$y_1 = \sin 70^\circ(150 - 375t)$$

Similarly for the second plane,

$$x_2 = \cos 135^\circ(190 - 450t)$$

$$= \cos 45^\circ(-190 + 450t)$$

$$y_2 = \sin 135^\circ(190 - 450t)$$

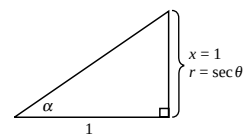
$$= \sin 45^\circ(190 - 450t)$$

(b) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= [\cos 45^\circ(-190 + 450t) - \cos 70^\circ(150 - 375t)]^2 + [\sin 45^\circ(190 - 450t) - \sin 70^\circ(150 - 375t)]^2]^{1/2}$$

11. (a) $\text{Area} = \int_0^\alpha \frac{1}{2} r^2 d\theta$

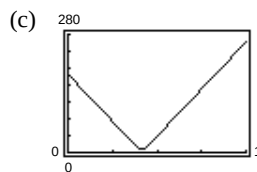
$$= \frac{1}{2} \int_0^\alpha \sec^2 \theta d\theta$$



(b) $\tan \alpha = \frac{h}{1} \Rightarrow \text{Area} = \frac{1}{2}(1)\tan \alpha$

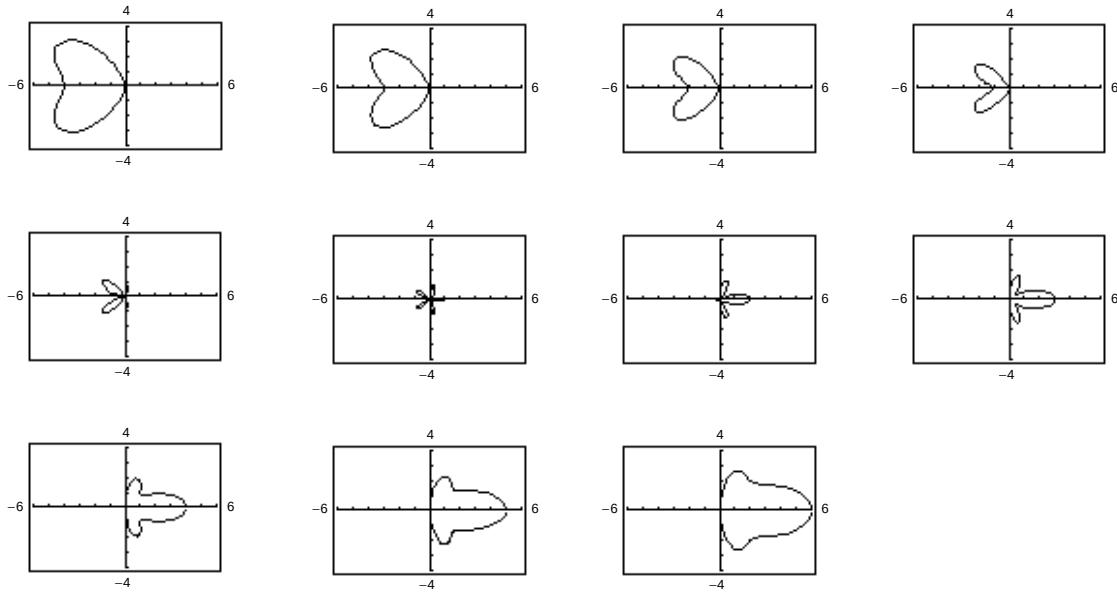
$$\Rightarrow \tan \alpha = \int_0^\alpha \sec^2 \theta d\theta$$

(c) Differentiating, $\frac{d}{d\alpha}(\tan \alpha) = \sec^2 \alpha$.



The minimum distance is 7.59 miles when $t = 0.4145$.

17.



$n = 1, 2, 3, 4, 5$ produce “bells”; $n = -1, -2, -3, -4, -5$ produce “hearts”.

50. $a = 4, c = 5, b = 3, e = \frac{5}{4}$

$$r^2 = \frac{-9}{1 - (25/16) \cos^2 \theta}$$

52. $a = 2, b = 1, c = \sqrt{3}, e = \frac{\sqrt{3}}{2}$

$$r^2 = \frac{1}{1 - (3/4) \cos^2 \theta}$$

54. $A = 2 \left[\frac{1}{2} \int_{-\pi/2}^{\pi/2} \left(\frac{2}{3 - 2 \sin \theta} \right)^2 d\theta \right] = 4 \int_{-\pi/2}^{\pi/2} \frac{1}{(3 - 2 \sin \theta)^2} d\theta \approx 3.37$

56. (a) $r = \frac{ed}{1 - e \cos \theta}$

When $\theta = 0, r = c + a = ea + a = a(1 + e)$.

Therefore,

$$a(1 + e) = \frac{ed}{1 - e}$$

$$a(1 + e)(1 - e) = ed$$

$$a(1 - e^2) = ed.$$

Thus, $r = \frac{(1 - e^2)a}{1 - e \cos \theta}$.

(b) The perihelion distance is $a - c = a - ea = a(1 - e)$.

When $\theta = \pi, r = \frac{(1 - e^2)a}{1 + e} = a(1 - e)$.

The aphelion distance is $a + c = a + ea = a(1 + e)$.

When $\theta = 0, r = \frac{(1 - e^2)a}{1 - e} = a(1 + e)$.

58. $a = 1.427 \times 10^9 \text{ km}$

$$e = 0.0543$$

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta} = \frac{1.422792505 \times 10^9}{1 - 0.0543 \cos \theta}$$

Perihelion distance: $a(1 - e) = 1.3495139 \times 10^9 \text{ km}$

Aphelion distance: $a(1 + e) = 1.5044861 \times 10^9 \text{ km}$

60. $a = 36.0 \times 10^6 \text{ mi}, e = 0.206$

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta} \approx \frac{34.472 \times 10^6}{1 - 0.206 \cos \theta}$$

Perihelion distance: $a(1 - e) = 28.582 \times 10^6 \text{ mi}$

Aphelion distance: $a(1 + e) = 43.416 \times 10^6 \text{ mi}$

62. $r = a \sin \theta + b \cos \theta$

$$r^2 = ar \sin \theta + br \cos \theta$$

$$x^2 + y^2 = ay + bx$$

$x^2 + y^2 - bx - ay = 0$ represents a circle.

Review Exercises for Chapter 9

2. Matches (b) - hyperbola

4. Matches (c) - hyperbola

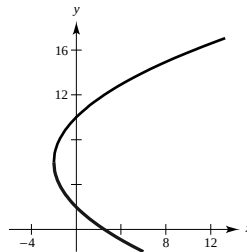
6. $y^2 - 12y - 8x + 20 = 0$

$$y^2 - 12y + 36 = 8x - 20 + 36$$

$$(y - 6)^2 = 4(2)(x + 2)$$

Parabola

Vertex: $(-2, 6)$



8. $4x^2 + y^2 - 16x + 15 = 0$

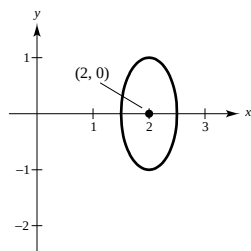
$$4(x^2 - 4x + 4) + y^2 = -15 + 16$$

$$\frac{(x - 2)^2}{1/4} + \frac{y^2}{1} = 1$$

Ellipse

Center: $(2, 0)$

Vertices: $(2, \pm 1)$



10. $4x^2 - 4y^2 - 4x + 8y - 11 = 0$

$$4\left(x^2 - x + \frac{1}{4}\right) - 4(y^2 - 2y + 1) = 11 + 1 - 4$$

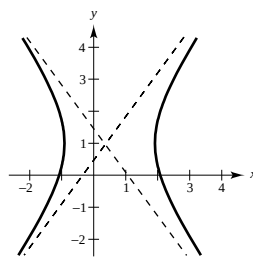
$$\frac{[x - (1/2)]^2}{2} - \frac{(y - 1)^2}{2} = 1$$

Hyperbola

Center: $\left(\frac{1}{2}, 1\right)$

Vertices: $\left(\frac{1}{2} \pm \sqrt{2}, 1\right)$

Asymptotes: $y = 1 \pm \left(x - \frac{1}{2}\right)$



12. Vertex: $(4, 2)$

Focus: $(4, 0)$

Parabola opens downward

$$p = -2$$

$$(x - 4)^2 = 4(-2)(y - 2)$$

$$x^2 - 8x + 8y = 0$$

14. Center: $(0, 0)$

Solution points: $(1, 2), (2, 0)$

Substituting the values of the coordinates of the given points into

$$\left(\frac{x^2}{b^2}\right) + \left(\frac{y^2}{a^2}\right) = 1,$$

we obtain the system

$$\left(\frac{1}{b^2}\right) + \left(\frac{4}{a^2}\right) = 1, 4/b^2 = 1.$$

Solving the system, we have

$$a^2 = \frac{16}{3} \text{ and } b^2 = 4, \left(\frac{x^2}{4}\right) + \left(\frac{3y^2}{16}\right) = 1.$$

18. $\frac{x^2}{4} + \frac{y^2}{25} = 1, a = 5, b = 2, c = \sqrt{21}, e = \frac{\sqrt{21}}{5}$

By Example 5 of Section 9.1,

$$C = 20 \int_0^{\pi/2} \sqrt{1 - \frac{21}{25} \sin^2 \theta} d\theta \approx 23.01.$$

16. Foci: $(0, \pm 8)$

Asymptotes: $y = \pm 4x$

Center: $(0, 0)$

Vertical transverse axis

$$c = 8$$

$$y = \frac{a}{b}x = 4x \text{ asymptote} \rightarrow a = 4b$$

$$b^2 = c^2 - a^2 = 64 - (4b)^2 \Rightarrow 17b^2 = 64$$

$$\Rightarrow b^2 = \frac{64}{17} \Rightarrow a^2 = \frac{1024}{17}$$

$$\frac{y^2}{1024/17} - \frac{x^2}{64/17} = 1$$

20. $y = \frac{1}{200}x^2$

(a) $x^2 = 200y$

$x^2 = 4(50)y$

Focus: (0, 50)

(b) $y = \frac{1}{200}x^2$

$y' = \frac{1}{100}x$

$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{10,000}}$

$S = 2\pi \int_0^{100} x \sqrt{1 + \frac{x^2}{10,000}} dx \approx 38,294.49$

22. (a) $A = 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx = \frac{4b}{a} \left(\frac{1}{2} \right) \left[x \sqrt{a^2 - x^2} + a^2 \arcsin\left(\frac{x}{a}\right) \right]_0^a = \pi ab$

(b) **Disk:** $V = 2\pi \int_0^b \frac{a^2}{b^2} (b^2 - y^2) dy = \frac{2\pi a^2}{b^2} \int_0^b (b^2 - y^2) dy = \frac{2\pi a^2}{b^2} \left[b^2 y - \frac{1}{3} y^3 \right]_0^b = \frac{4}{3} \pi a^2 b$

$$S = 4\pi \int_0^b \frac{a}{b} \sqrt{b^2 - y^2} \left(\frac{\sqrt{b^4 + (a^2 - b^2)y^2}}{b \sqrt{b^2 - y^2}} \right) dy$$

$$= \frac{4\pi a}{b^2} \int_0^b \sqrt{b^4 + c^2 y^2} dy = \frac{2\pi a}{b^2 c} \left[cy \sqrt{b^4 + c^2 y^2} + b^4 \ln|cy + \sqrt{b^4 + c^2 y^2}| \right]_0^b$$

$$= \frac{2\pi a}{b^2 c} [b^2 c \sqrt{b^2 + c^2} + b^4 \ln|cb + b \sqrt{b^2 + c^2}| - b^4 \ln(b^2)]$$

$$= 2\pi a^2 + \frac{\pi ab^2}{c} \ln\left(\frac{c+a}{e}\right)^2 = 2\pi a^2 + \left(\frac{\pi b^2}{e}\right) \ln\left(\frac{1+e}{1-e}\right)$$

(c) **Disk:** $V = 2\pi \int_0^a \frac{b^2}{a^2} (a^2 - x^2) dx = \frac{2\pi b^2}{a^2} \int_0^a (a^2 - x^2) dx = \frac{2\pi b^2}{a^2} \left[a^2 x - \frac{1}{3} x^3 \right]_0^a = \frac{4}{3} \pi ab^2$

$$S = 2(2\pi) \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} \left(\frac{\sqrt{a^4 - (a^2 - b^2)x^2}}{a \sqrt{a^2 - x^2}} \right) dx$$

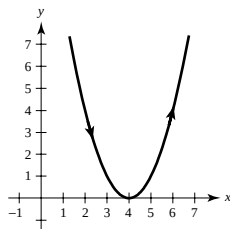
$$= \frac{4\pi b}{a^2} \int_0^a \sqrt{a^4 - c^2 x^2} dx = \frac{2\pi b}{a^2 c} \left[cx \sqrt{a^4 - c^2 x^2} + a^4 \arcsin\left(\frac{cx}{a^2}\right) \right]_0^a$$

$$= \frac{a\pi b}{a^2 c} \left[a^2 c \sqrt{a^2 - c^2} + a^4 \arcsin\left(\frac{c}{a}\right) \right] = 2\pi b^2 + 2\pi \left(\frac{ab}{e}\right) \arcsin(e)$$

24. $x = t + 4, y = t^2$

$t = x - 4 \Rightarrow y = (x - 4)^2$

Parabola

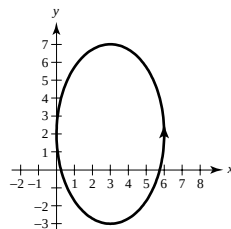


26. $x = 3 + 3 \cos \theta, y = 2 + 5 \sin \theta$

$\left(\frac{x-3}{3}\right)^2 + \left(\frac{y-2}{5}\right)^2 = 1$

$\frac{(x-3)^2}{9} + \frac{(y-2)^2}{25} = 1$

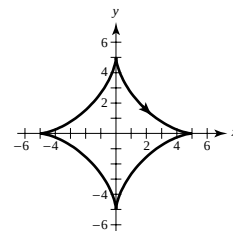
Ellipse



28. $x = 5 \sin^3 \theta, y = 5 \cos^3 \theta$

$\left(\frac{x}{5}\right)^{2/3} + \left(\frac{y}{5}\right)^{2/3} = 1$

$x^{2/3} + y^{2/3} = 5^{2/3}$



30. $(x - h)^2 + (y - k)^2 = r^2$

$$(x - 5)^2 + (y - 3)^2 = 2^2 = 4$$

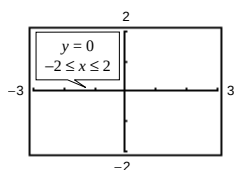
34. $x = (a - b) \cos t + b \cos \left(\frac{a - b}{b} t \right)$

$$y = (a - b) \sin t - b \sin \left(\frac{a - b}{b} t \right)$$

(a) $a = 2, b = 1$

$$x = \cos t + \cos t = 2 \cos t$$

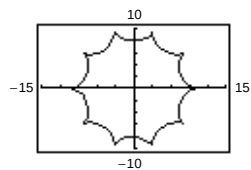
$$y = \sin t - \sin t = 0$$



(d) $a = 10, b = 1$

$$x = 9 \cos t + \cos 9t$$

$$y = 9 \sin t - \sin 9t$$

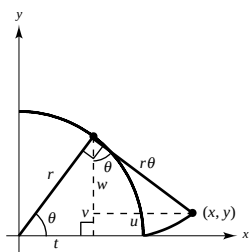


36. $x = t + u = r \cos \theta + r \theta \sin \theta$

$$= r(\cos \theta + \theta \sin \theta)$$

$$y = v - w = r \sin \theta - r \theta \cos \theta$$

$$= r(\sin \theta - \theta \cos \theta)$$



32. $a = 4, c = 5, b^2 = c^2 - a^2 = 9, \frac{y^2}{16} - \frac{x^2}{9} = 1$

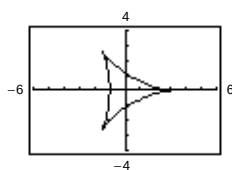
Let $\frac{y^2}{16} = \sec^2 \theta$ and $\frac{x^2}{9} = \tan^2 \theta$.

Then $x = 3 \tan \theta$ and $y = 4 \sec \theta$.

(b) $a = 3, b = 1$

$$x = 2 \cos t + \cos 2t$$

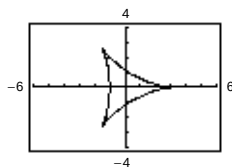
$$y = 2 \sin t - \sin 2t$$



(e) $a = 3, b = 2$

$$x = \cos t + 2 \cos \frac{t}{2}$$

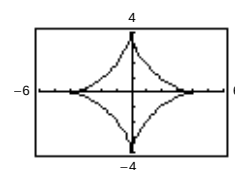
$$y = \sin t - 2 \sin \frac{t}{2}$$



(c) $a = 4, b = 1$

$$x = 3 \cos t + \cos 3t$$

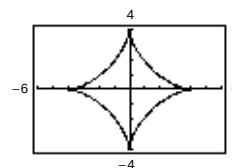
$$y = 3 \sin t - \sin 3t$$



(f) $a = 4, b = 3$

$$x = \cos t + 3 \cos \frac{t}{3}$$

$$y = \sin t - 3 \sin \frac{t}{3}$$



38. $x = t + 4$

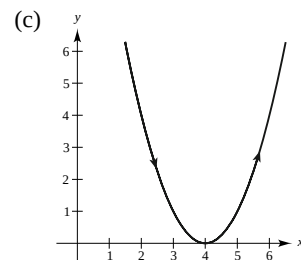
$$y = t^2$$

(a) $\frac{dy}{dx} = \frac{2t}{1} = 2t = 0$ when $t = 0$.

 Point of horizontal tangency: $(4, 0)$

(b) $t = x - 4$

$$y = (x - 4)^2$$



40. $x = \frac{1}{t}$

$y = t^2$

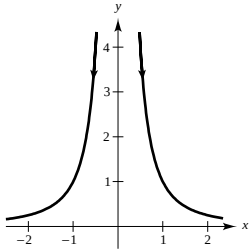
(a) $\frac{dy}{dx} = \frac{2t}{-1/t^2} = -2t^3$

No horizontal tangents ($t \neq 0$)

(b) $t = \frac{1}{x}$

$y = \frac{1}{x^2}$

(c)



44. $x = 6 \cos \theta$

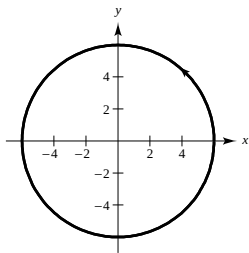
$y = 6 \sin \theta$

(a) $\frac{dy}{dx} = \frac{6 \cos \theta}{-6 \sin \theta} = -\cot \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of horizontal tangency: $(0, 6), (0, -6)$

(b) $\left(\frac{x}{6}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$

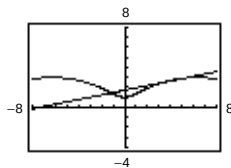
(c)



48. $x = 2\theta - \sin \theta$

$y = 2 - \cos \theta$

(a), (c)



(b) At $\theta = \frac{\pi}{6}, \frac{dx}{d\theta} \approx 1.134, \left(2 - \frac{\sqrt{3}}{2}\right),$

$\frac{dy}{dt} = 0.5,$ and $\frac{dy}{dx} \approx 0.441$

42. $x = 2t - 1$

$y = \frac{1}{t^2 - 2t}$

(a) $\frac{dy}{dx} = \frac{-(t^2 - 2t)^{-2}(2t - 2)}{2}$

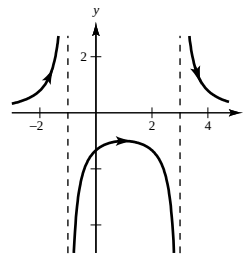
$= \frac{1 - t}{t^2(t - 2)^2} = 0$ when $t = 1$.

Point of horizontal tangency: $(1, -1)$

(b) $t = \frac{x + 1}{2}$

$y = \frac{1}{[(x + 1)/2]^2 - 2[(x + 1)/2]} = \frac{4}{(x - 3)(x + 1)}$

(c)



46. $x = e^t$

$y = e^{-t}$

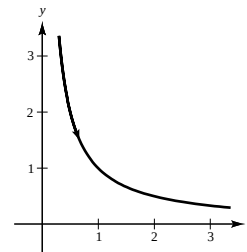
(a) $\frac{dy}{dx} = \frac{-e^{-t}}{e^t} = -\frac{1}{e^{2t}} = -\frac{1}{x^2}$

No horizontal tangents

(b) $t = \ln x$

$y = e^{-\ln x} = e^{\ln(1/x)} = \frac{1}{x}, x > 0$

(c)



50. $x = 6 \cos \theta$

$y = 6 \sin \theta$

$\frac{dx}{d\theta} = -6 \sin \theta$

$\frac{dy}{d\theta} = 6 \cos \theta$

$s = \int_0^\pi \sqrt{36 \sin^2 \theta + 36 \cos^2 \theta} d\theta = \left[6\theta\right]_0^\pi = 6\pi$

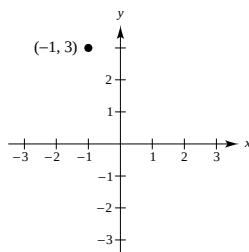
(one-half circumference of circle)

52. $(x, y) = (-1, 3)$

$$r = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\theta = \arctan(-3) \approx 1.89 \text{ (108.43}^\circ\text{)}$$

$$(r, \theta) = (\sqrt{10}, 1.89), (-\sqrt{10}, 5.03)$$



54. $r = 10$

$$r^2 = 100$$

$$x^2 + y^2 = 100$$

56.
$$r = \frac{1}{2 - \cos \theta}$$

$$2r - r \cos \theta = 1$$

$$2(\pm \sqrt{x^2 + y^2}) - x = 1$$

$$4(x^2 + y^2) = (x + 1)^2$$

$$3x^2 + 4y^2 - 2x - 1 = 0$$

58.
$$r = 4 \sec\left(\theta - \frac{\pi}{3}\right) = \frac{4}{\cos[\theta - (\pi/3)]}$$

$$= \frac{4}{(1/2) \cos \theta + (\sqrt{3}/2) \sin \theta}$$

$$r(\cos \theta + \sqrt{3} \sin \theta) = 8$$

$$x + \sqrt{3}y = 8$$

60.
$$\theta = \frac{3\pi}{4}$$

$$\tan \theta = -1$$

$$\frac{y}{x} = -1$$

$$y = -x$$

62. $x^2 + y^2 - 4x = 0$

$$r^2 - 4r \cos \theta = 0$$

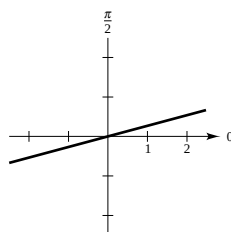
$$r = 4 \cos \theta$$

64.
$$(x^2 + y^2) \left(\arctan \frac{y}{x} \right)^2 = a^2$$

$$r^2 \theta^2 = a^2$$

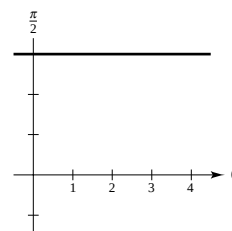
66. $\theta = \frac{\pi}{12}$

Line



68. $r = 3 \csc \theta, r \sin \theta = 3, y = 3$

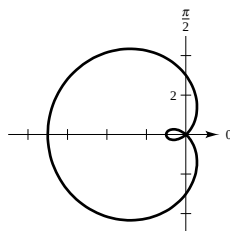
Horizontal line



70. $r = 3 - 4 \cos \theta$

Limaçon

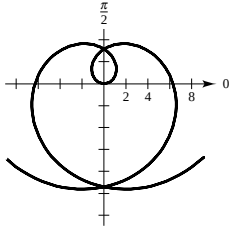
Symmetric to polar axis



θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	-1	1	3	5	7

72. $r = 2\theta$

Spiral

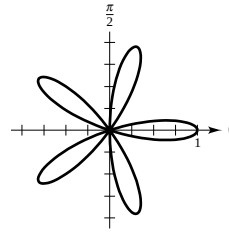
Symmetric to $\theta = \pi/2$ 

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	0	$\frac{\pi}{5}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π

74. $r = \cos(5\theta)$

Rose curve with five petals

Symmetric to polar axis

Relative extrema: $(1, 0), (-1, \frac{\pi}{5}), (1, \frac{2\pi}{5}), (-1, \frac{3\pi}{5}), (1, \frac{4\pi}{5})$ Tangents at the pole: $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$ 

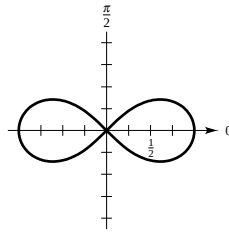
76. $r^2 = \cos(2\theta)$

Lemniscate

Symmetric to the polar axis

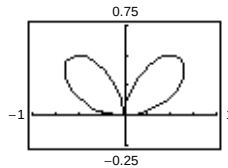
Relative extrema: $(\pm 1, 0)$ Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	± 1	$\pm \frac{\sqrt{2}}{2}$	0



78. $r = 2 \sin \theta \cos^2 \theta$

Bifolium

Symmetric to $\theta = \pi/2$ 

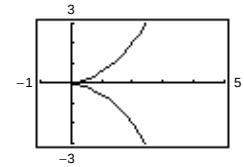
80. $r = 4(\sec \theta - \cos \theta)$

Semicubical parabola

Symmetric to the polar axis

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \frac{\pi^-}{2}$$

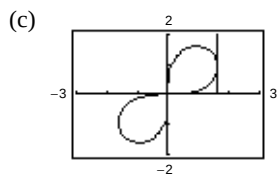
$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \frac{-\pi^+}{2}$$



82. $r^2 = 4 \sin(2\theta)$

(a) $2r \left(\frac{dr}{d\theta} \right) = 8 \cos(2\theta)$

$$\frac{dr}{d\theta} = \frac{4 \cos(2\theta)}{r}$$

Tangents at the pole: $\theta = 0, \frac{\pi}{2}$ 

$$(b) \frac{dy}{dx} = \frac{r \cos \theta + [(4 \cos 2\theta \sin \theta)/r]}{-r \sin \theta + [(4 \cos 2\theta \cos \theta)/r]}$$

$$= \frac{\cos(2\theta) \sin \theta + \sin(2\theta) \cos \theta}{\cos(2\theta) \cos \theta - \sin(2\theta) \sin \theta}$$

Horizontal tangents:

$$\frac{dy}{dx} = 0 \text{ when } \cos(2\theta) \sin \theta + \sin(2\theta) \cos \theta = 0,$$

$$\tan \theta = -\tan(2\theta), \theta = 0, \frac{\pi}{3}, (0, 0), \left(\pm \sqrt{2\sqrt{3}}, \frac{\pi}{3} \right)$$

Vertical tangents when $\cos 2\theta \cos \theta - \sin 2\theta \sin \theta = 0$:

$$\tan 2\theta \tan \theta = 1, \theta = 0, \frac{\pi}{6}, (0, 0), \left(\pm \sqrt{2\sqrt{3}}, \frac{\pi}{6} \right)$$

84. False. There are an infinite number of polar coordinate representations of a point. For example, the point $(x, y) = (1, 0)$ has polar representations $(r, \theta) = (1, 0), (1, 2\pi), (-1, \pi)$, etc.

86. $r = a \sin \theta, r = a \cos \theta$

The points of intersection are $(a/\sqrt{2}, \pi/4)$ and $(0, 0)$. For $r = a \sin \theta$,

$$m_1 = \frac{dy}{dx} = \frac{a \cos \theta \sin \theta + a \sin \theta \cos \theta}{a \cos^2 \theta - a \sin^2 \theta} = \frac{2 \sin \theta \cos \theta}{\cos 2\theta}.$$

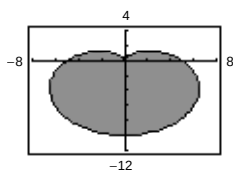
At $(a/\sqrt{2}, \pi/4)$, m_1 is undefined and at $(0, 0)$, $m_1 = 0$. For $r = a \cos \theta$,

$$m_2 = \frac{dy}{dx} = \frac{-a \sin^2 \theta + a \cos^2 \theta}{-a \sin \theta \cos \theta - a \cos \theta \sin \theta} = \frac{\cos 2\theta}{-2 \sin \theta \cos \theta}.$$

At $(a/\sqrt{2}, \pi/4)$, $m_2 = 0$ and at $(0, 0)$, m_2 is undefined. Therefore, the graphs are orthogonal at $(a/\sqrt{2}, \pi/4)$ and $(0, 0)$.

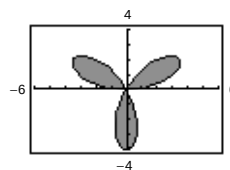
88. $r = 5(1 - \sin \theta)$

$$A = 2 \left[\frac{1}{2} \int_{\pi/2}^{3\pi/2} [5(1 - \sin \theta)]^2 d\theta \right] \approx 117.81 \left(75 \frac{\pi}{2} \right)$$



90. $r = 4 \sin 3\theta$

$$A = 3 \left[\frac{1}{2} \int_0^{\pi/3} (4 \sin 3\theta)^2 d\theta \right] \approx 12.57 (4\pi)$$



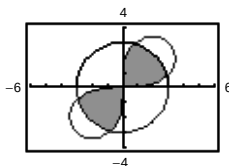
92. $r = 3, r^2 = 18 \sin 2\theta$

$$9 = r^2 = 18 \sin 2\theta$$

$$\sin 2\theta = \frac{1}{2}$$

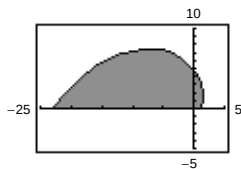
$$\theta = \frac{\pi}{12}$$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/12} 18 \sin 2\theta d\theta + \frac{1}{2} \int_{\pi/12}^{5\pi/12} 9 d\theta + \frac{1}{2} \int_{5\pi/12}^{\pi/2} 18 \sin 2\theta d\theta \right] \approx 1.2058 + 9.4248 + 1.2058 \approx 11.84$$



94. $r = e^\theta, 0 \leq \theta \leq \pi$

$$A = \frac{1}{2} \int_0^\pi (e^\theta)^2 d\theta \approx 133.62$$

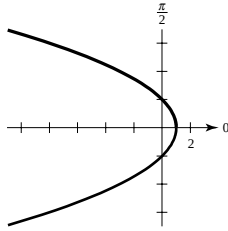


96. $r = a \cos 2\theta, \frac{dr}{d\theta} = -2a \sin 2\theta$

$$\begin{aligned} s &= 8 \int_0^{\pi/4} \sqrt{a^2 \cos^2 2\theta + 4a^2 \sin^2 2\theta} d\theta \\ &= 8a \int_0^{\pi/4} \sqrt{1 + 3 \sin^2 2\theta} d\theta \quad (\text{Simpson's Rule: } n = 4) \\ &\approx \frac{\pi a}{6} [1 + 4(1.1997) + 2(1.5811) + 4(1.8870) + 2] \\ &\approx 9.69a \end{aligned}$$

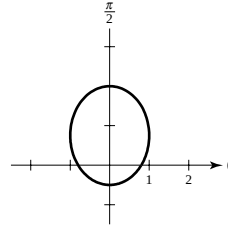
$$98. r = \frac{2}{1 + \cos \theta}, e = 1$$

Parabola



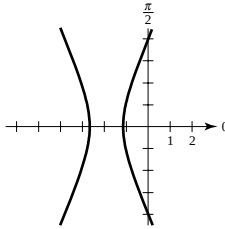
$$100. r = \frac{4}{5 - 3 \sin \theta} = \frac{4/5}{1 - (3/5) \sin \theta}, e = \frac{3}{5}$$

Ellipse



$$102. r = \frac{8}{2 - 5 \cos \theta} = \frac{4}{1 - (5/2) \cos \theta}, e = \frac{5}{2}$$

Hyperbola



104. Line

Slope: $\sqrt{3}$

Solution point: $(0, 0)$

$$y = \sqrt{3}x, r \sin \theta = \sqrt{3}r \cos \theta,$$

$$\tan \theta = \sqrt{3}, \theta = \frac{\pi}{3}$$

106. Parabola

Vertex: $\left(2, \frac{\pi}{2}\right)$

Focus: $(0, 0)$

$e = 1, d = 4$

$$r = \frac{4}{1 + \sin \theta}$$

108. Hyperbola

Vertices: $(1, 0), (7, 0)$

Focus: $(0, 0)$

$$a = 3, c = 4, e = \frac{4}{3}, d = \frac{7}{4}$$

$$r = \frac{\left(\frac{4}{3}\right)\left(\frac{7}{4}\right)}{1 + \left(\frac{4}{3}\right) \cos \theta} = \frac{7}{3 + 4 \cos \theta}$$

Problem Solving for Chapter 9

2. Assume $p > 0$.

Let $y = mx + p$ be the equation of the focal chord.

First find x -coordinates of focal chord endpoints:

$$x^2 = 4py = 4p(mx + p)$$

$$x^2 - 4pmx - 4p^2 = 0$$

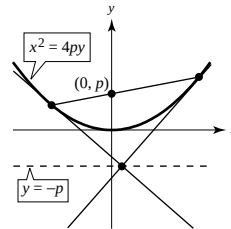
$$x = \frac{4pm \pm \sqrt{16p^2m^2 + 16p^2}}{2} = 2pm \pm 2p\sqrt{m^2 + 1}$$

$$x^2 = 4py, 2x = 4py' \Rightarrow y' = \frac{x}{2p}.$$

(a) The slopes of the tangent lines at the endpoints are perpendicular because

$$\frac{1}{2p}[2pm + 2p\sqrt{m^2 + 1}] \cdot \frac{1}{2p}[2pm - 2p\sqrt{m^2 + 1}] = \frac{1}{4p^2}[4p^2m^2 - 4p^2(m^2 + 1)] = \frac{1}{4p^2}[-4p^2] = -1$$

—CONTINUED—



2. —CONTINUED—

(b) Finally, we show that the tangent lines intersect at a point on the directrix $y = -p$.

$$\text{Let } b = 2pm + 2p\sqrt{m^2 + 1} \text{ and } c = 2pm - 2p\sqrt{m^2 + 1}.$$

$$b^2 = 8p^2m^2 + 4p^2 + 8p^2m\sqrt{m^2 + 1}$$

$$c^2 = 8p^2m^2 + 4p^2 - 8p^2m\sqrt{m^2 + 1}$$

$$\frac{b^2}{4p} = 2pm^2 + p + 2pm\sqrt{m^2 + 1}$$

$$\frac{c^2}{4p} = 2pm^2 + p - 2pm\sqrt{m^2 + 1}$$

$$\text{Tangent line at } x = b: y - \frac{b^2}{4p} = \frac{b}{2p}(x - b) \Rightarrow y = \frac{bx}{2p} - \frac{b^2}{4p}$$

$$\text{Tangent line at } x = c: y - \frac{c^2}{4p} = \frac{c}{2p}(x - c) \Rightarrow y = \frac{cx}{2p} - \frac{c^2}{4p}$$

$$\begin{aligned} \text{Intersection of tangent lines: } \quad \frac{bx}{2p} - \frac{b^2}{4p} &= \frac{cx}{2p} - \frac{c^2}{4p} \\ 2bx - b^2 &= 2cx - c^2 \\ 2x(b - c) &= b^2 - c^2 \\ 2x(4p\sqrt{m^2 + 1}) &= 16p^2m\sqrt{m^2 + 1} \\ x &= 2pm \end{aligned}$$

Finally, the corresponding y -value is $y = -p$, which shows that the intersection point lies on the directrix.

$$4. \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, a^2 + b^2 = c^2, MF_2 - MF_1 = 2a$$

$$y' = \frac{b^2x}{a^2y}$$

$$\text{Tangent line at } M(x_0, y_0): y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0)$$

$$\frac{yy_0 - y_0^2}{b^2} = \frac{x_0x - x_0^2}{a^2}$$

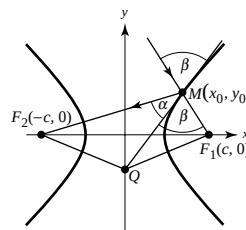
$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}$$

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$$

$$\text{At } x = 0, y = -\frac{b^2}{y_0} \Rightarrow Q = \left(0, -\frac{b^2}{y_0}\right).$$

$$QF_2 = QF_1 = \sqrt{c^2 + \frac{b^4}{y_0^2}} = d$$

$$MQ = \sqrt{x_0^2 + \left(y_0 + \frac{b^2}{y_0}\right)^2} = f$$



By the Law of Cosines,

$$(F_2Q)^2 = (MF_2)^2 + (MQ)^2 - 2(MF_2)(MQ)\cos \alpha$$

$$d^2 = (MF_2)^2 + f^2 - 2f(MF_2)\cos \alpha$$

$$(F_1Q)^2 = (MF_1)^2 + f^2 - 2f(MF_1)\cos \beta$$

$$d^2 = (MF_1)^2 + f^2 - 2f(MF_1)\cos \beta.$$

$$\cos \alpha = \frac{(MF_2)^2 f^2 - d^2}{2f(MF_2)}, \cos \beta = \frac{(MF_1)^2 f^2 - d^2}{2f(MF_1)}$$

$$MF_2 = MF_1 + 2a. \text{ Let } z = MF_1.$$

$$\text{Slopes: } MF_1: \frac{y_0}{x_0 - c}; QF_1: \frac{-b^2}{y_0 c}; QF_2: \frac{b^2}{y_0 c}$$

—CONTINUED—

4. —CONTINUED—

To show $\alpha = \beta$, consider

$$\begin{aligned}
 & [(MF_2)^2 + f^2 - d^2][2f(MF_1)] = [(MF_1)^2 + f^2 - d^2][2f(MF_2)] \\
 \Leftrightarrow & [(z + 2a)^2 + f^2 - d^2][z] = [z^2 + f^2 - d^2][z + 2a] \\
 \Leftrightarrow & z^2 + 2az = f^2 - d^2 \\
 \Leftrightarrow & (x_0 - c)^2 + y_0^2 + 2az = \left(x_0^2 + \left(y_0 + \frac{b^2}{y_0}\right)^2\right) - \left(c^2 + \frac{b^4}{y_0^2}\right) \\
 \Leftrightarrow & az - x_0c + a^2 = 0 \\
 \Leftrightarrow & a\sqrt{(x_0 - c)^2 + y_0^2} = x_0c - a^2 \\
 \Leftrightarrow & x_0^2b^2 - a^2y_0^2 = a^2b^2 \\
 \Leftrightarrow & \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1.
 \end{aligned}$$

Thus, $\alpha = \beta$ and the reflective property is verified.

$$6. (a) \quad y^2 = \frac{t^2(1 - t^2)^2}{(1 + t^2)^2}, \quad x^2 = \frac{(1 - t^2)^2}{(1 + t^2)^2}$$

$$\frac{1 - x}{1 + x} = \frac{1 - \left(\frac{1 - t^2}{1 + t^2}\right)}{1 + \left(\frac{1 - t^2}{1 + t^2}\right)} = \frac{2t^2}{2} = t^2$$

$$\text{Thus, } y^2 = x^2 \left(\frac{1 - x}{1 + x}\right).$$

$$(b) \quad r^2 \sin^2 \theta = r^2 \cos^2 \theta \left(\frac{1 - r \cos \theta}{1 + r \cos \theta}\right)$$

$$\sin^2 \theta (1 + r \cos \theta) = \cos^2 \theta (1 - r \cos \theta)$$

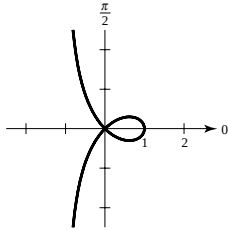
$$r \cos \theta \sin^2 \theta + \sin^2 \theta = \cos^2 \theta - r \cos^3 \theta$$

$$r \cos \theta (\sin^2 \theta + \cos^2 \theta) = \cos^2 \theta - \sin^2 \theta$$

$$r \cos \theta = \cos 2\theta$$

$$r = \cos 2\theta \cdot \sec \theta$$

(c)



$$(d) \quad r(\theta) = 0 \text{ for } \theta = \frac{\pi}{4}, \frac{3\pi}{4}.$$

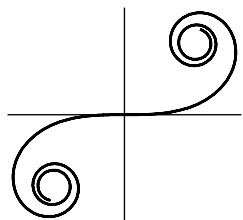
Thus, $y = x$ and $y = -x$ are tangent lines to curve at the origin.

$$(e) \quad y'(t) = \frac{(1 + t^2)(1 - 3t^2) - (t - t^3)(2t)}{(1 + t^2)^2} = \frac{1 - 4t^2 - t^4}{(1 + t^2)^2} = 0$$

$$\begin{aligned}
 t^4 + 4t^2 - 1 = 0 & \Rightarrow t^2 = -2 \pm \sqrt{5} \Rightarrow x = \frac{1 - (-2 \pm \sqrt{5})}{1 + (-2 \pm \sqrt{5})} = \frac{3 \mp \sqrt{5}}{-1 \pm \sqrt{5}} \\
 & = \frac{3 - \sqrt{5}}{-1 + \sqrt{5}} = \frac{\sqrt{5} - 1}{2}
 \end{aligned}$$

$$\left(\frac{\sqrt{5} - 1}{2}, \pm \frac{\sqrt{5} - 1}{2} \sqrt{-2 + \sqrt{5}}\right)$$

8. (a)



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$$(b) \quad (-x, -y) = \left(-\int_0^t \cos \frac{\pi u^2}{2} du, -\int_0^t \sin \frac{\pi u^2}{2} du \right) \text{ is on}$$

 the curve whenever (x, y) is on the curve.

$$(c) \quad x'(t) = \cos \frac{\pi t^2}{2}, y'(t) = \sin \frac{\pi t^2}{2}, x'(t)^2 + y'(t)^2 = 1$$

$$\text{Thus, } s = \int_0^a dt = a.$$

$$\text{On } [-\pi, \pi], s = 2\pi.$$

$$10. \quad r = \frac{ab}{a \sin \theta + b \cos \theta}, 0 \leq \theta \leq \frac{\pi}{2}$$

$$r(a \sin \theta + b \cos \theta) = ab$$

$$ay + bx = ab$$

$$\frac{y}{b} + \frac{x}{a} = 1$$

Line segment

$$\text{Area} = \frac{1}{2}ab$$

 12. Let (r, θ) be on the graph.

$$\sqrt{r^2 + 1 + 2r \cos \theta} \sqrt{r^2 + 1 - 2r \cos \theta} = 1$$

$$(r^2 + 1)^2 - 4r^2 \cos^2 \theta = 1$$

$$r^4 + 2r^2 + 1 - 4r^2 \cos^2 \theta = 1$$

$$r^2(r^2 - 4 \cos^2 \theta + 2) = 0$$

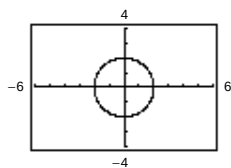
$$r^2 = 4 \cos^2 \theta - 2$$

$$r^2 = 2(2 \cos^2 \theta - 1)$$

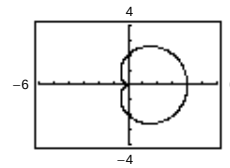
$$r^2 = 2 \cos 2\theta$$

 14. (a) $r = 2$

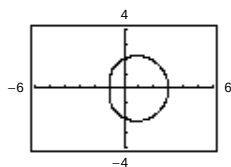
Circle radius 2

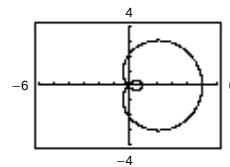

 (c) $r = 2 + 2 \cos \theta$

Cardioid


 (b) $r = 2 + \cos \theta$

Convex limaçon


 (d) $r = 2 + 3 \cos \theta$

 Limaçon with
inner loop


16. The curve is produced over the interval

$$0 \leq \theta \leq 9\pi.$$

C H A P T E R 1 0

Vectors and the Geometry of Space

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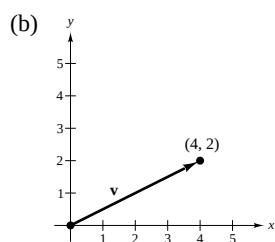
CHAPTER 10

Vectors and the Geometry of Space

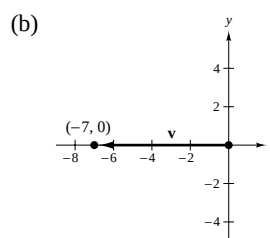
Section 10.1 Vectors in the Plane

Solutions to Odd-Numbered Exercises

1. (a) $\mathbf{v} = \langle 5 - 1, 3 - 1 \rangle = \langle 4, 2 \rangle$



3. (a) $\mathbf{v} = \langle -4 - 3, -2 - (-2) \rangle = \langle -7, 0 \rangle$



5. $\mathbf{u} = \langle 5 - 3, 6 - 2 \rangle = \langle 2, 4 \rangle$

$\mathbf{v} = \langle 1 - (-1), 8 - 4 \rangle = \langle 2, 4 \rangle$

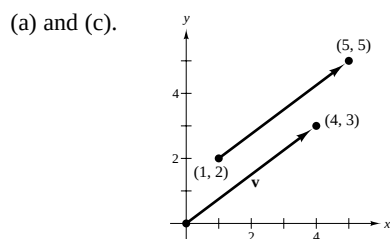
$\mathbf{u} = \mathbf{v}$

7. $\mathbf{u} = \langle 6 - 0, -2 - 3 \rangle = \langle 6, -5 \rangle$

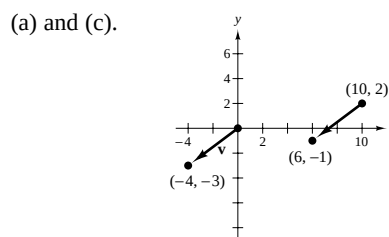
$\mathbf{v} = \langle 9 - 3, 5 - 10 \rangle = \langle 6, -5 \rangle$

$\mathbf{u} = \mathbf{v}$

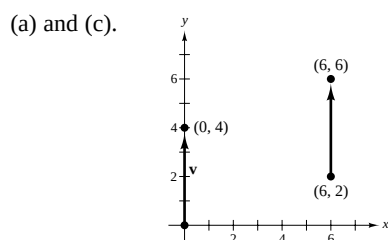
9. (b) $\mathbf{v} = \langle 5 - 1, 5 - 2 \rangle = \langle 4, 3 \rangle$



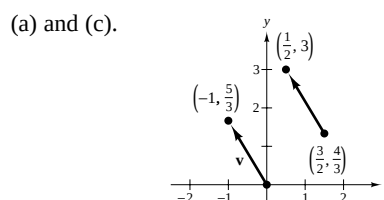
11. (b) $\mathbf{v} = \langle 6 - 10, -1 - 2 \rangle = \langle -4, -3 \rangle$



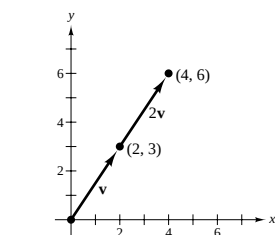
13. (b) $\mathbf{v} = \langle 6 - 6, 6 - 2 \rangle = \langle 0, 4 \rangle$



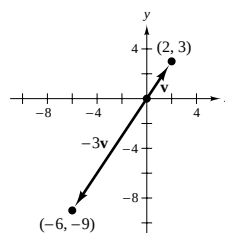
15. (b) $\mathbf{v} = \langle \frac{1}{2} - \frac{3}{2}, 3 - \frac{4}{3} \rangle = \langle -1, \frac{5}{3} \rangle$



17. (a) $2\mathbf{v} = \langle 4, 6 \rangle$



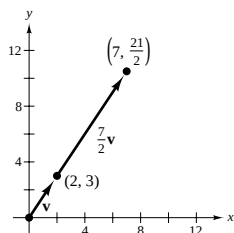
(b) $-3\mathbf{v} = \langle -6, -9 \rangle$



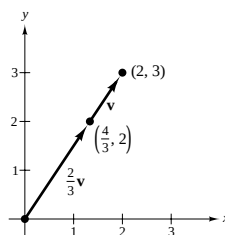
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17. —CONTINUED—

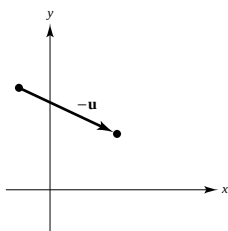
(c) $\frac{7}{2}\mathbf{v} = \left\langle 7, \frac{21}{2} \right\rangle$



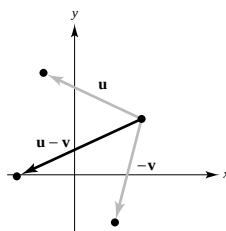
(d) $\frac{2}{3}\mathbf{v} = \left\langle \frac{4}{3}, 2 \right\rangle$



19.



21.

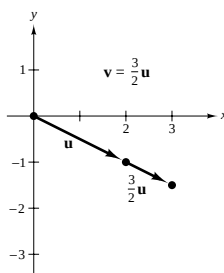


23. (a) $\frac{2}{3}\mathbf{u} = \frac{2}{3}\langle 4, 9 \rangle = \left\langle \frac{8}{3}, 6 \right\rangle$

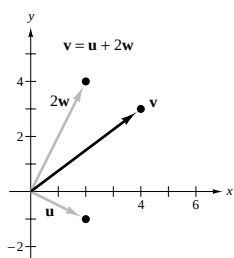
(b) $\mathbf{v} - \mathbf{u} = \langle 2, -5 \rangle - \langle 4, 9 \rangle = \langle -2, -14 \rangle$

(c) $2\mathbf{u} + 5\mathbf{v} = 2\langle 4, 9 \rangle + 5\langle 2, -5 \rangle = \langle 18, -7 \rangle$

25. $\mathbf{v} = \frac{3}{2}(2\mathbf{i} - \mathbf{j}) = 3\mathbf{i} - \frac{3}{2}\mathbf{j}$
 $= \left\langle 3, -\frac{3}{2} \right\rangle$



27. $\mathbf{v} = (2\mathbf{i} - \mathbf{j}) + 2(\mathbf{i} + 2\mathbf{j})$
 $= 4\mathbf{i} + 3\mathbf{j} = \langle 4, 3 \rangle$



29. $u_1 - 4 = -1$

$u_2 - 2 = 3$

$u_1 = 3$

$u_2 = 5$

$Q = (3, 5)$

31. $\|\mathbf{v}\| = \sqrt{16 + 9} = 5$

33. $\|\mathbf{v}\| = \sqrt{36 + 25} = \sqrt{61}$

35. $\|\mathbf{v}\| = \sqrt{0 + 16} = 4$

37. $\|\mathbf{u}\| = \sqrt{3^2 + 12^2} = \sqrt{153}$

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle 3, 12 \rangle}{\sqrt{153}} = \left\langle \frac{3}{\sqrt{153}}, \frac{12}{\sqrt{153}} \right\rangle$$

$$= \left\langle \frac{\sqrt{17}}{17}, \frac{4\sqrt{17}}{17} \right\rangle \text{ unit vector}$$

39. $\|\mathbf{u}\| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \frac{\sqrt{34}}{2}$

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle (3/2), (5/2) \rangle}{\sqrt{34}/2} = \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle$$

$$= \left\langle \frac{3\sqrt{34}}{34}, \frac{5\sqrt{34}}{34} \right\rangle \text{ unit vector}$$

$$41. \|\mathbf{u}\| = \langle 1, -1 \rangle, \mathbf{v} = \langle -1, 2 \rangle$$

$$(a) \|\mathbf{u}\| = \sqrt{1+1} = \sqrt{2}$$

$$(b) \|\mathbf{v}\| = \sqrt{1+4} = \sqrt{5}$$

$$(c) \mathbf{u} + \mathbf{v} = \langle 0, 1 \rangle$$

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{0+1} = 1$$

$$(d) \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{2}} \langle 1, -1 \rangle$$

$$\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1$$

$$(e) \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{5}} \langle -1, 2 \rangle$$

$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$$

$$(f) \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \langle 0, 1 \rangle$$

$$\left\| \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\| = 1$$

$$45. \mathbf{u} = \langle 2, 1 \rangle$$

$$\|\mathbf{u}\| = \sqrt{5} \approx 2.236$$

$$\mathbf{v} = \langle 5, 4 \rangle$$

$$\|\mathbf{v}\| = \sqrt{41} \approx 6.403$$

$$\mathbf{u} + \mathbf{v} = \langle 7, 5 \rangle$$

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{74} \approx 8.602$$

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

$$49. \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{2\sqrt{3}} \langle \sqrt{3}, 3 \rangle$$

$$2\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = \frac{1}{\sqrt{3}} \langle \sqrt{3}, 3 \rangle$$

$$\mathbf{v} = \langle 1, \sqrt{3} \rangle$$

$$53. \mathbf{v} = 2[(\cos 150^\circ)\mathbf{i} + (\sin 150^\circ)\mathbf{j}]$$

$$= -\sqrt{3}\mathbf{i} + \mathbf{j} = \langle -\sqrt{3}, 1 \rangle$$

$$43. \mathbf{u} = \left\langle 1, \frac{1}{2} \right\rangle, \mathbf{v} = \langle 2, 3 \rangle$$

$$(a) \|\mathbf{u}\| = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$$

$$(b) \|\mathbf{v}\| = \sqrt{4+9} = \sqrt{13}$$

$$(c) \mathbf{u} + \mathbf{v} = \left\langle 3, \frac{7}{2} \right\rangle$$

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{9 + \frac{49}{4}} = \frac{\sqrt{85}}{2}$$

$$(d) \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{2}{\sqrt{5}} \left\langle 1, \frac{1}{2} \right\rangle$$

$$\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1$$

$$(e) \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{13}} \langle 2, 3 \rangle$$

$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$$

$$(f) \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \frac{2}{\sqrt{85}} \left\langle 3, \frac{7}{2} \right\rangle$$

$$\left\| \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\| = 1$$

$$47. \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{2}} \langle 1, 1 \rangle$$

$$4\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = 2\sqrt{2} \langle 1, 1 \rangle$$

$$\mathbf{v} = \langle 2\sqrt{2}, 2\sqrt{2} \rangle$$

$$51. \mathbf{v} = 3[(\cos 0^\circ)\mathbf{i} + (\sin 0^\circ)\mathbf{j}] = 3\mathbf{i} = \langle 3, 0 \rangle$$

$$55. \mathbf{u} = \mathbf{i}$$

$$\mathbf{v} = \frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{u} + \mathbf{v} = \left(\frac{2+3\sqrt{2}}{2}\right)\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$$

57. $\mathbf{u} = 2(\cos 4)\mathbf{i} + 2(\sin 4)\mathbf{j}$

$$\mathbf{v} = (\cos 2)\mathbf{i} + (\sin 2)\mathbf{j}$$

$$\mathbf{u} + \mathbf{v} = (2 \cos 4 + \cos 2)\mathbf{i} + (2 \sin 4 + \sin 2)\mathbf{j}$$

61. To normalize $\mathbf{v}$, you find a unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

For Exercises 63–67, $a\mathbf{i} + b\mathbf{j} = a(\mathbf{i} + 2\mathbf{j}) + b(\mathbf{i} - \mathbf{j}) = (a + b)\mathbf{i} + (2a - b)\mathbf{j}$.63. $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$. Therefore, $a + b = 2$, $2a - b = 1$. Solving simultaneously, we have $a = 1$, $b = 1$.67. $\mathbf{v} = \mathbf{i} + \mathbf{j}$. Therefore, $a + b = 1$, $2a - b = 1$. Solving simultaneously, we have $a = \frac{2}{3}$, $b = \frac{1}{3}$.

69. $y = x^3$, $y' = 3x^2 = 3$ at $x = 1$.

(a) $m = 3$. Let $\mathbf{w} = \langle 1, 3 \rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{10}} \langle 1, 3 \rangle.$$

(b) $m = -\frac{1}{3}$. Let $\mathbf{w} = \langle 3, -1 \rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{10}} \langle 3, -1 \rangle.$$

73. $\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$

$$\mathbf{u} + \mathbf{v} = \sqrt{2}\mathbf{j}$$

$$\mathbf{v} = (\mathbf{u} + \mathbf{v}) - \mathbf{u} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

77. $\|\mathbf{F}_1\| = 2$, $\theta_{\mathbf{F}_1} = 33^\circ$

$$\|\mathbf{F}_2\| = 3$$
, $\theta_{\mathbf{F}_2} = -125^\circ$

$$\|\mathbf{F}_3\| = 2.5$$
, $\theta_{\mathbf{F}_3} = 110^\circ$

$$\|\mathbf{R}\| = \|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\| \approx 1.33$$

$$\theta_{\mathbf{R}} = \theta_{\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3} \approx 132.5^\circ$$

79. (a) $180(\cos 30\mathbf{i} + \sin 30\mathbf{j}) + 275\mathbf{i} = 430.88\mathbf{i} + 90\mathbf{j}$

$$\text{Direction: } \alpha = \arctan\left(\frac{90}{430.88}\right) = 0.206 (= 11.8^\circ)$$

$$\text{Magnitude: } \sqrt{430.88^2 + 90^2} = 440.18 \text{ newtons}$$

59. A scalar is a real number. A vector is represented by a directed line segment. A vector has both length and direction.

65. $\mathbf{v} = 3\mathbf{i}$. Therefore, $a + b = 3$, $2a - b = 0$. Solving simultaneously, we have $a = 1$, $b = 2$.

71. $f(x) = \sqrt{25 - x^2}$

$$f'(x) = \frac{-x}{\sqrt{25 - x^2}} = \frac{-3}{4} \text{ at } x = 3.$$

(a) $m = -\frac{3}{4}$. Let $\mathbf{w} = \langle -4, 3 \rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{5} \langle -4, 3 \rangle.$$

(b) $m = \frac{4}{3}$. Let $\mathbf{w} = \langle 3, 4 \rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{5} \langle 3, 4 \rangle$$

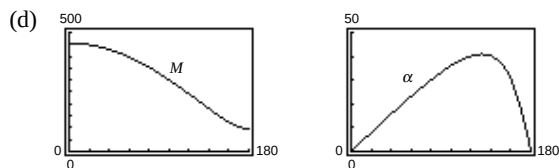
75. Programs will vary.

—CONTINUED—

79. —CONTINUED—

(c)

θ	0°	30°	60°	90°	120°	150°	180°
M	455	440.2	396.9	328.7	241.9	149.3	95
α	0°	11.8°	23.1°	33.2°	40.1°	37.1°	0



(e) M decreases because the forces change from acting in the same direction to acting in the opposite direction as θ increases from 0° to 180° .

81. $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (75 \cos 30^\circ \mathbf{i} + 75 \sin 30^\circ \mathbf{j}) + (100 \cos 45^\circ \mathbf{i} + 100 \sin 45^\circ \mathbf{j}) + (125 \cos 120^\circ \mathbf{i} + 125 \sin 120^\circ \mathbf{j})$

$$= \left(\frac{75}{2}\sqrt{3} + 50\sqrt{2} - \frac{125}{2} \right) \mathbf{i} + \left(\frac{75}{2} + 50\sqrt{2} + \frac{125}{2}\sqrt{3} \right) \mathbf{j}$$

$$\|\mathbf{R}\| = \|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\| \approx 228.5 \text{ lb}$$

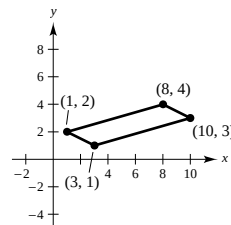
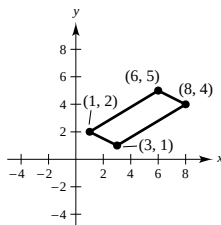
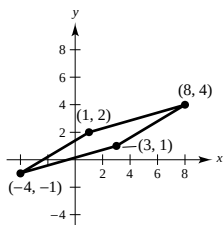
$$\theta_{\mathbf{R}} = \theta_{\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3} \approx 71.3^\circ$$

83. (a) The forces act along the same direction. $\theta = 0^\circ$.

(b) The forces cancel out each other. $\theta = 180^\circ$.

(c) No, the magnitude of the resultant can not be greater than the sum.

85. $(-4, -1), (6, 5), (10, 3)$



87. $\mathbf{u} = \overrightarrow{CB} = \|\mathbf{u}\|(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j})$

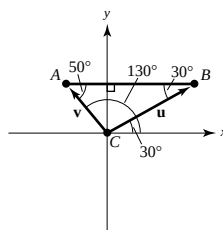
$$\mathbf{v} = \overrightarrow{CA} = \|\mathbf{v}\|(\cos 130^\circ \mathbf{i} + \sin 130^\circ \mathbf{j})$$

Vertical components: $\|\mathbf{u}\| \sin 30^\circ + \|\mathbf{v}\| \sin 130^\circ = 2000$

Horizontal components: $\|\mathbf{u}\| \cos 30^\circ + \|\mathbf{v}\| \cos 130^\circ = 0$

Solving this system, you obtain

$$\|\mathbf{u}\| \approx 1305.5 \text{ and } \|\mathbf{v}\| \approx 1758.8.$$



89. Horizontal component = $\|\mathbf{v}\| \cos \theta = 1200 \cos 6^\circ \approx 1193.43 \text{ ft/sec}$

Vertical component = $\|\mathbf{v}\| \sin \theta = 1200 \sin 6^\circ \approx 125.43 \text{ ft/sec}$

91. $\mathbf{u} = 900[\cos 148^\circ \mathbf{i} + \sin 148^\circ \mathbf{j}]$

$\mathbf{v} = 100[\cos 45^\circ \mathbf{i} + \sin 45^\circ \mathbf{j}]$

$\mathbf{u} + \mathbf{v} = [900 \cos 148^\circ + 100 \cos 45^\circ]\mathbf{i} + [900 \sin 148^\circ + 100 \sin 45^\circ]\mathbf{j}$

$\approx -692.53 \mathbf{i} + 547.64 \mathbf{j}$

$\theta \approx \arctan\left(\frac{547.64}{-692.53}\right) \approx -38.34^\circ \quad 38.34^\circ \text{ North of West.}$

$\|\mathbf{u} + \mathbf{v}\| = \sqrt{(-692.53)^2 + (547.64)^2} \approx 882.9 \text{ km/hr.}$

93. $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}$

$-3600\mathbf{j} + T_2(\cos 35^\circ \mathbf{i} - \sin 35^\circ \mathbf{j}) + T_3(\cos 92^\circ \mathbf{i} + \sin 92^\circ \mathbf{j}) = \mathbf{0}$

$T_2 \cos 35^\circ + T_3 \cos 92^\circ = 0$

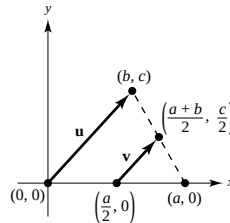
$-T_2 \sin 35^\circ + T_3 \sin 92^\circ = 3600$

$T_2 = \frac{-T_3 \cos 92^\circ}{\cos 35^\circ} \Rightarrow \frac{T_3 \cos 92^\circ}{\cos 35^\circ} \sin 35^\circ + T_3 \sin 92^\circ = 3600 \text{ and } T_3(0.97495) = 3600 \Rightarrow T_3 \approx 3692.48$

Finally, $T_2 = 157.32$

95. Let the triangle have vertices at $(0, 0)$, $(a, 0)$, and (b, c) . Let $\mathbf{u}$ be the vector joining $(0, 0)$ and (b, c) , as indicated in the figure. Then $\mathbf{v}$, the vector joining the midpoints, is

$$\begin{aligned} \mathbf{v} &= \left(\frac{a+b}{2} - \frac{a}{2}\right)\mathbf{i} + \frac{c}{2}\mathbf{j} \\ &= \frac{b}{2}\mathbf{i} + \frac{c}{2}\mathbf{j} = \frac{1}{2}(b\mathbf{i} + c\mathbf{j}) = \frac{1}{2}\mathbf{u} \end{aligned}$$



97. $\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$

$= \|\mathbf{u}\|[\|\mathbf{v}\| \cos \theta_v \mathbf{i} + \|\mathbf{v}\| \sin \theta_v \mathbf{j}] + \|\mathbf{v}\|[\|\mathbf{u}\| \cos \theta_u \mathbf{i} + \|\mathbf{u}\| \sin \theta_u \mathbf{j}] = \|\mathbf{u}\| \|\mathbf{v}\|[(\cos \theta_u + \cos \theta_v)\mathbf{i} + (\sin \theta_u + \sin \theta_v)\mathbf{j}]$

$= 2\|\mathbf{u}\| \|\mathbf{v}\| \left[\cos\left(\frac{\theta_u + \theta_v}{2}\right) \cos\left(\frac{\theta_u - \theta_v}{2}\right)\mathbf{i} + \sin\left(\frac{\theta_u + \theta_v}{2}\right) \cos\left(\frac{\theta_u - \theta_v}{2}\right)\mathbf{j} \right]$

$\tan \theta_w = \frac{\sin\left(\frac{\theta_u + \theta_v}{2}\right) \cos\left(\frac{\theta_u - \theta_v}{2}\right)}{\cos\left(\frac{\theta_u + \theta_v}{2}\right) \cos\left(\frac{\theta_u - \theta_v}{2}\right)} = \tan\left(\frac{\theta_u + \theta_v}{2}\right)$

Thus, $\theta_w = (\theta_u + \theta_v)/2$ and $\mathbf{w}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

99. True

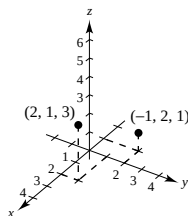
101. True

103. False

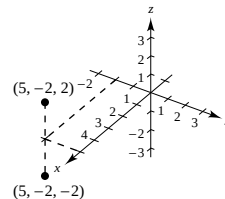
$\|a\mathbf{i} + b\mathbf{j}\| = \sqrt{2}|a|$

Section 10.2 Space Coordinates and Vectors in Space

1.



3.



5. $A(2, 3, 4)$
 $B(-1, -2, 2)$
7. $x = -3, y = 4, z = 5: (-3, 4, 5)$
9. $y = z = 0, x = 10: (10, 0, 0)$
11. The z -coordinate is 0.
13. The point is 6 units above the xy -plane.
15. The point is on the plane parallel to the yz -plane that passes through $x = 4$.
17. The point is to the left of the xz -plane.
19. The point is on or between the planes $y = 3$ and $y = -3$.
21. The point (x, y, z) is 3 units below the xy -plane, and below either quadrant I or III.
23. The point could be above the xy -plane and thus above quadrants II or IV, or below the xy -plane, and thus below quadrants I or III.
25. $d = \sqrt{(5-0)^2 + (2-0)^2 + (6-0)^2}$
 $= \sqrt{25 + 4 + 36} = \sqrt{65}$
27. $d = \sqrt{(6-1)^2 + (-2-(-2))^2 + (-2-4)^2}$
 $= \sqrt{25 + 0 + 36} = \sqrt{61}$
29. $A(0, 0, 0), B(2, 2, 1), C(2, -4, 4)$
 $|AB| = \sqrt{4 + 4 + 1} = 3$
 $|AC| = \sqrt{4 + 16 + 16} = 6$
 $|BC| = \sqrt{0 + 36 + 9} = 3\sqrt{5}$
 $|BC|^2 = |AB|^2 + |AC|^2$
 Right triangle
31. $A(1, -3, -2), B(5, -1, 2), C(-1, 1, 2)$
 $|AB| = \sqrt{16 + 4 + 16} = 6$
 $|AC| = \sqrt{4 + 16 + 16} = 6$
 $|BC| = \sqrt{36 + 4 + 0} = 2\sqrt{10}$
 Since $|AB| = |AC|$, the triangle is isosceles.
33. The z -coordinate is changed by 5 units:
 $(0, 0, 5), (2, 2, 6), (2, -4, 9)$
35. $\left(\frac{5+(-2)}{2}, \frac{-9+3}{2}, \frac{7+3}{2}\right) = \left(\frac{3}{2}, -3, 5\right)$
37. Center: $(0, 2, 5)$
 Radius: 2
 $(x-0)^2 + (y-2)^2 + (z-5)^2 = 4$
 $x^2 + y^2 + z^2 - 4y - 10z + 25 = 0$
39. Center: $\frac{(2, 0, 0) + (0, 6, 0)}{2} = (1, 3, 0)$
 Radius: $\sqrt{10}$
 $(x-1)^2 + (y-3)^2 + (z-0)^2 = 10$
 $x^2 + y^2 + z^2 - 2x - 6y = 0$
41. $x^2 + y^2 + z^2 - 2x + 6y + 8z + 1 = 0$
 $(x^2 - 2x + 1) + (y^2 + 6y + 9) + (z^2 + 8z + 16) = -1 + 1 + 9 + 16$
 $(x-1)^2 + (y+3)^2 + (z+4)^2 = 25$
 Center: $(1, -3, -4)$
 Radius: 5

43. $9x^2 + 9y^2 + 9z^2 - 6x + 18y + 1 = 0$

$$x^2 + y^2 + z^2 - \frac{2}{3}x + 2y + \frac{1}{9} = 0$$

$$\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + (y^2 + 2y + 1) + z^2 = -\frac{1}{9} + \frac{1}{9} + 1$$

$$\left(x - \frac{1}{3}\right)^2 + (y + 1)^2 + (z - 0)^2 = 1$$

Center: $\left(\frac{1}{3}, -1, 0\right)$

Radius: 1

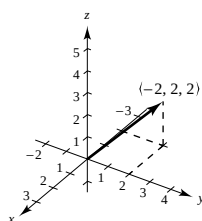
45. $x^2 + y^2 + z^2 \leq 36$

Solid ball of radius 6 centered at origin.

47. (a) $\mathbf{v} = (2 - 4)\mathbf{i} + (4 - 2)\mathbf{j} + (3 - 1)\mathbf{k}$

$$= -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} = \langle -2, 2, 2 \rangle$$

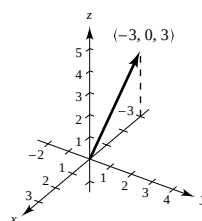
(b)



49. (a) $\mathbf{v} = (0 - 3)\mathbf{i} + (3 - 3)\mathbf{j} + (3 - 0)\mathbf{k}$

$$= -3\mathbf{i} + 3\mathbf{k} = \langle -3, 0, 3 \rangle$$

(b)



51. $\langle 4 - 3, 1 - 2, 6 - 0 \rangle = \langle 1, -1, 6 \rangle$

$$\|\langle 1, -1, 6 \rangle\| = \sqrt{1 + 1 + 36} = \sqrt{38}$$

Unit vector: $\frac{\langle 1, -1, 6 \rangle}{\sqrt{38}} = \left\langle \frac{1}{\sqrt{38}}, \frac{-1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right\rangle$

53. $\langle -5 - (-4), 3 - 3, 0 - 1 \rangle = \langle -1, 0, -1 \rangle$

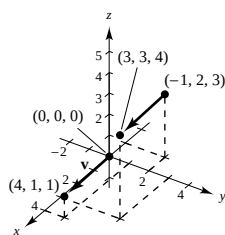
$$\|\langle -1, 0, -1 \rangle\| = \sqrt{1 + 1} = \sqrt{2}$$

Unit vector: $\left\langle \frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \right\rangle$

55. (b) $\mathbf{v} = (3 + 1)\mathbf{i} + (3 - 2)\mathbf{j} + (4 - 3)\mathbf{k}$

$$= 4\mathbf{i} + \mathbf{j} + \mathbf{k} = \langle 4, 1, 1 \rangle$$

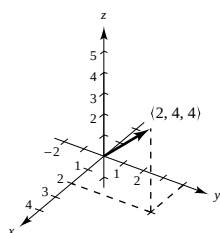
(a) and (c).



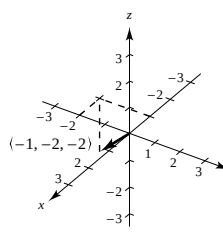
57. $(q_1, q_2, q_3) - (0, 6, 2) = (3, -5, 6)$

$$Q = (3, 1, 8)$$

59. (a) $2\mathbf{v} = \langle 2, 4, 4 \rangle$



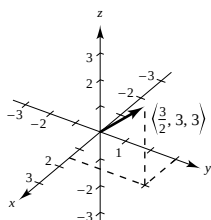
(b) $-\mathbf{v} = \langle -1, -2, -2 \rangle$



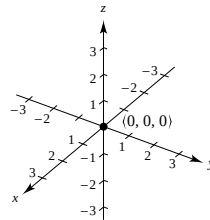
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59. —CONTINUED—

(c) $\frac{3}{2}\mathbf{v} = \left\langle \frac{3}{2}, 3, 3 \right\rangle$



(d) $0\mathbf{v} = \langle 0, 0, 0 \rangle$



61. $\mathbf{z} = \mathbf{u} - \mathbf{v} = \langle 1, 2, 3 \rangle - \langle 2, 2, -1 \rangle = \langle -1, 0, 4 \rangle$

63. $\mathbf{z} = 2\mathbf{u} + 4\mathbf{v} - \mathbf{w} = \langle 2, 4, 6 \rangle + \langle 8, 8, -4 \rangle - \langle 4, 0, -4 \rangle = \langle 6, 12, 6 \rangle$

65. $2\mathbf{z} - 3\mathbf{u} = 2\langle z_1, z_2, z_3 \rangle - 3\langle 1, 2, 3 \rangle = \langle 4, 0, -4 \rangle$

$$2z_1 - 3 = 4 \Rightarrow z_1 = \frac{7}{2}$$

$$2z_2 - 6 = 0 \Rightarrow z_2 = 3$$

$$2z_3 - 9 = -4 \Rightarrow z_3 = \frac{5}{2}$$

$$\mathbf{z} = \left\langle \frac{7}{2}, 3, \frac{5}{2} \right\rangle$$

67. (a) and (b) are parallel since $\langle -6, -4, 10 \rangle = -2\langle 3, 2, -5 \rangle$
 and $\left\langle 2, \frac{4}{3}, -\frac{10}{3} \right\rangle = \frac{2}{3}\langle 3, 2, -5 \rangle$.

69. $\mathbf{z} = -3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

(a) is parallel since $-6\mathbf{i} + 8\mathbf{j} + 4\mathbf{k} = 2\mathbf{z}$.

71. $P(0, -2, -5), Q(3, 4, 4), R(2, 2, 1)$

$$\overrightarrow{PQ} = \langle 3, 6, 9 \rangle$$

$$\overrightarrow{PR} = \langle 2, 4, 6 \rangle$$

$$\langle 3, 6, 9 \rangle = \frac{3}{2}\langle 2, 4, 6 \rangle$$

Therefore, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel. The points are collinear.

73. $P(1, 2, 4), Q(2, 5, 0), R(0, 1, 5)$

$$\overrightarrow{PQ} = \langle 1, 3, -4 \rangle$$

$$\overrightarrow{PR} = \langle -1, -1, 1 \rangle$$

Since $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are not parallel, the points are not collinear.

75. $A(2, 9, 1), B(3, 11, 4), C(0, 10, 2), D(1, 12, 5)$

$$\overrightarrow{AB} = \langle 1, 2, 3 \rangle$$

$$\overrightarrow{CD} = \langle 1, 2, 3 \rangle$$

$$\overrightarrow{AC} = \langle -2, 1, 1 \rangle$$

$$\overrightarrow{BD} = \langle -2, 1, 1 \rangle$$

Since $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$, the given points form the vertices of a parallelogram.

77. $\|\mathbf{v}\| = 0$

79. $\mathbf{v} = \langle 1, -2, -3 \rangle$

$$\|\mathbf{v}\| = \sqrt{1 + 4 + 9} = \sqrt{14}$$

81. $\mathbf{v} = \langle 0, 3, -5 \rangle$

$$\|\mathbf{v}\| = \sqrt{0 + 9 + 25} = \sqrt{34}$$

83. $\mathbf{u} = \langle 2, -1, 2 \rangle$

$$\|\mathbf{u}\| = \sqrt{4 + 1 + 4} = 3$$

(a) $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{3}\langle 2, -1, 2 \rangle$

(b) $-\frac{\mathbf{u}}{\|\mathbf{u}\|} = -\frac{1}{3}\langle 2, -1, 2 \rangle$

85. $\mathbf{u} = \langle 3, 2, -5 \rangle$

$$\|\mathbf{u}\| = \sqrt{9 + 4 + 25} = \sqrt{38}$$

(a) $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{38}}\langle 3, 2, -5 \rangle$

(b) $-\frac{\mathbf{u}}{\|\mathbf{u}\|} = -\frac{1}{\sqrt{38}}\langle 3, 2, -5 \rangle$

87. Programs will vary.

89. $c\mathbf{v} = \langle 2c, 2c, -c \rangle$

$$\|c\mathbf{v}\| = \sqrt{4c^2 + 4c^2 + c^2} = 5$$

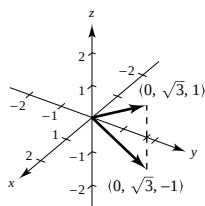
$$9c^2 = 25$$

$$c = \pm \frac{5}{3}$$

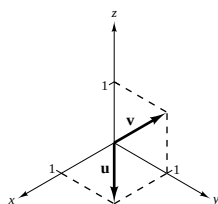
93. $\mathbf{v} = \frac{3}{2} \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{3}{2} \left\langle \frac{2}{3}, \frac{-2}{3}, \frac{1}{3} \right\rangle = \left\langle 1, -1, \frac{1}{2} \right\rangle$

95. $\mathbf{v} = 2[\cos(\pm 30^\circ)\mathbf{j} + \sin(\pm 30^\circ)\mathbf{k}]$

$$= \sqrt{3}\mathbf{j} \pm \mathbf{k} = \langle 0, \sqrt{3}, \pm 1 \rangle$$



99. (a)



(c) $a\mathbf{i} + (a+b)\mathbf{j} + b\mathbf{k} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$$a = 1, b = 1$$

$$\mathbf{w} = \mathbf{u} + \mathbf{v}$$

91. $\mathbf{v} = 10 \frac{\mathbf{u}}{\|\mathbf{u}\|} = 10 \left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

$$= \left\langle 0, \frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}} \right\rangle$$

97. $\mathbf{v} = \langle -3, -6, 3 \rangle$

$$\frac{2}{3}\mathbf{v} = \langle -2, -4, 2 \rangle$$

$$(4, 3, 0) + (-2, -4, 2) = (2, -1, 2)$$

(b) $\mathbf{w} = a\mathbf{u} + b\mathbf{v} = a\mathbf{i} + (a+b)\mathbf{j} + b\mathbf{k} = \mathbf{0}$

$$a = 0, a + b = 0, b = 0$$

 Thus, a and b are both zero.

(d) $a\mathbf{i} + (a+b)\mathbf{j} + b\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$$a = 1, a + b = 2, b = 3$$

Not possible

101. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

 105. (a) The height of the right triangle is $h = \sqrt{L^2 - 18^2}$.
The vector $\overrightarrow{PQ}$ is given by

$$\overrightarrow{PQ} = \langle 0, -18, h \rangle.$$

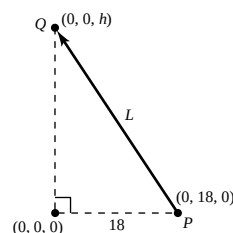
 The tension vector $\mathbf{T}$ in each wire is

$$\mathbf{T} = c\langle 0, -18, h \rangle \text{ where } ch = \frac{24}{3} = 8.$$

 Hence, $\mathbf{T} = \frac{8}{h}\langle 0, -18, h \rangle$ and

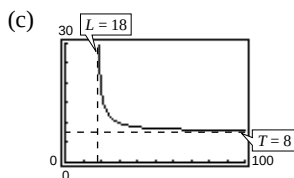
$$T = \|\mathbf{T}\| = \frac{8}{h} \sqrt{18^2 + h^2} = \frac{8}{\sqrt{L^2 - 18^2}} \sqrt{18^2 + (L^2 - 18^2)} = \frac{8L}{\sqrt{L^2 - 18^2}}$$

(b)	L	20	25	30	35	40	45	50
	T	18.4	11.5	10	9.3	9.0	8.7	8.6


 103. Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if $\mathbf{u} = c\mathbf{v}$ for some scalar c .

—CONTINUED—

105. —CONTINUED—



$x = 18$ is a vertical asymptote and $y = 8$ is a horizontal asymptote.

$$(d) \lim_{L \rightarrow 18^+} \frac{8L}{\sqrt{L^2 - 18^2}} = \infty$$

$$\lim_{L \rightarrow \infty} \frac{8L}{\sqrt{L^2 - 18^2}} = \lim_{L \rightarrow \infty} \frac{8}{\sqrt{1 - (18/L)^2}} = 8$$

(e) From the table, $T = 10$ implies $L = 30$ inches.

$$109. \overrightarrow{AB} = \langle 0, 70, 115 \rangle, \mathbf{F}_1 = C_1 \langle 0, 70, 115 \rangle$$

$$\overrightarrow{AC} = \langle -60, 0, 115 \rangle, \mathbf{F}_2 = C_2 \langle -60, 0, 115 \rangle$$

$$\overrightarrow{AD} = \langle 45, -65, 115 \rangle, \mathbf{F}_3 = C_3 \langle 45, -65, 115 \rangle$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \langle 0, 0, 500 \rangle$$

Thus:

$$-60C_2 + 45C_3 = 0$$

$$70C_1 - 65C_3 = 0$$

$$115(C_1 + C_2 + C_3) = 500$$

Solving this system yields $C_1 = \frac{104}{69}$, $C_2 = \frac{28}{23}$, and $C_3 = -\frac{112}{69}$. Thus:

$$\|\mathbf{F}_1\| \approx 202.919N$$

$$\|\mathbf{F}_2\| \approx 157.909N$$

$$\|\mathbf{F}_3\| \approx 226.521N$$

$$111. d(AP) = 2d(BP)$$

$$\sqrt{x^2 + (y+1)^2 + (z-1)^2} = 2\sqrt{(x-1)^2 + (y-2)^2 + z^2}$$

$$x^2 + y^2 + z^2 + 2y - 2z + 2 = 4(x^2 + y^2 + z^2 - 2x - 4y + 5)$$

$$0 = 3x^2 + 3y^2 + 3z^2 - 8x - 18y + 2z + 18$$

$$-6 + \frac{16}{9} + 9 + \frac{1}{9} = \left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) + (y^2 - 6y + 9) + \left(z^2 + \frac{2}{3}z + \frac{1}{9}\right)$$

$$\frac{44}{9} = \left(x - \frac{4}{3}\right)^2 + (y - 3)^2 + \left(z + \frac{1}{3}\right)^2$$

$$\text{Sphere; center: } \left(\frac{4}{3}, 3, -\frac{1}{3}\right), \text{ radius: } \frac{2\sqrt{11}}{3}$$

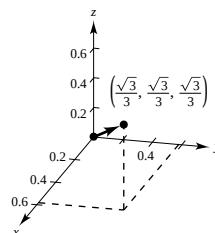
 107. Let α be the angle between $\mathbf{v}$ and the coordinate axes.

$$\mathbf{v} = (\cos \alpha)\mathbf{i} + (\cos \alpha)\mathbf{j} + (\cos \alpha)\mathbf{k}$$

$$\|\mathbf{v}\| = \sqrt{3} \cos \alpha = 1$$

$$\cos \alpha = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\mathbf{v} = \frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \frac{\sqrt{3}}{3}\langle 1, 1, 1 \rangle$$



Section 10.3 The Dot Product of Two Vectors

1. $\mathbf{u} = \langle 3, 4 \rangle, \mathbf{v} = \langle 2, -3 \rangle$

(a) $\mathbf{u} \cdot \mathbf{v} = 3(2) + 4(-3) = -6$

(b) $\mathbf{u} \cdot \mathbf{u} = 3(3) + 4(4) = 25$

(c) $\|\mathbf{u}\|^2 = 25$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = -6\langle 2, -3 \rangle = \langle -12, 18 \rangle$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(-6) = -12$

5. $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{v} = \mathbf{i} - \mathbf{k}$

(a) $\mathbf{u} \cdot \mathbf{v} = 2(1) + (-1)(0) + 1(-1) = 1$

(b) $\mathbf{u} \cdot \mathbf{u} = 2(2) + (-1)(-1) + (1)(1) = 6$

(c) $\|\mathbf{u}\|^2 = 6$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = \mathbf{v} = \mathbf{i} - \mathbf{k}$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2$

9. $\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \cos \theta$

$$\mathbf{u} \cdot \mathbf{v} = (8)(5) \cos \frac{\pi}{3} = 20$$

13. $\mathbf{u} = 3\mathbf{i} + \mathbf{j}, \mathbf{v} = -2\mathbf{i} + 4\mathbf{j}$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{-2}{\sqrt{10}\sqrt{20}} = \frac{-1}{5\sqrt{2}}$$

$$\theta = \arccos\left(-\frac{1}{5\sqrt{2}}\right) \approx 98.1^\circ$$

17. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}, \mathbf{v} = -2\mathbf{j} + 3\mathbf{k}$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{-8}{5\sqrt{13}} = \frac{-8\sqrt{13}}{65}$$

$$\theta = \arccos\left(-\frac{8\sqrt{13}}{65}\right) \approx 116.3^\circ$$

21. $\mathbf{u} = \langle 4, 3 \rangle, \mathbf{v} = \left\langle \frac{1}{2}, -\frac{2}{3} \right\rangle$

$$\mathbf{u} \neq c\mathbf{v} \Rightarrow \text{not parallel}$$

$$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow \text{orthogonal}$$

25. $\mathbf{u} = \langle 2, -3, 1 \rangle, \mathbf{v} = \langle -1, -1, -1 \rangle$

$$\mathbf{u} \neq c\mathbf{v} \Rightarrow \text{not parallel}$$

$$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow \text{orthogonal}$$

3. $\mathbf{u} = \langle 2, -3, 4 \rangle, \mathbf{v} = \langle 0, 6, 5 \rangle$

(a) $\mathbf{u} \cdot \mathbf{v} = 2(0) + (-3)(6) + (4)(5) = 2$

(b) $\mathbf{u} \cdot \mathbf{u} = 2(2) + (-3)(-3) + 4(4) = 29$

(c) $\|\mathbf{u}\|^2 = 29$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 2\langle 0, 6, 5 \rangle = \langle 0, 12, 10 \rangle$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(2) = 4$

7. $\mathbf{u} = \langle 3240, 1450, 2235 \rangle$

$$\mathbf{v} = \langle 2.22, 1.85, 3.25 \rangle$$

$$\mathbf{u} \cdot \mathbf{v} = \$17,139.05$$

This gives the total amount that the person earned on his products.

11. $\mathbf{u} = \langle 1, 1 \rangle, \mathbf{v} = \langle 2, -2 \rangle$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{0}{\sqrt{2}\sqrt{8}} = 0$$

$$\theta = \frac{\pi}{2}$$

15. $\mathbf{u} = \langle 1, 1, 1 \rangle, \mathbf{v} = \langle 2, 1, -1 \rangle$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{2}{\sqrt{3}\sqrt{6}} = \frac{\sqrt{2}}{3}$$

$$\theta = \arccos \frac{\sqrt{2}}{3} \approx 61.9^\circ$$

19. $\mathbf{u} = \langle 4, 0 \rangle, \mathbf{v} = \langle 1, 1 \rangle$

$$\mathbf{u} \neq c\mathbf{v} \Rightarrow \text{not parallel}$$

$$\mathbf{u} \cdot \mathbf{v} = 4 \neq 0 \Rightarrow \text{not orthogonal}$$

Neither

23. $\mathbf{u} = \mathbf{j} + 6\mathbf{k}, \mathbf{v} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$

$$\mathbf{u} \neq c\mathbf{v} \Rightarrow \text{not parallel}$$

$$\mathbf{u} \cdot \mathbf{v} = -8 \neq 0 \Rightarrow \text{not orthogonal}$$

Neither

27. $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\|\mathbf{u}\| = 3$

$$\cos \alpha = \frac{1}{3}$$

$$\cos \beta = \frac{2}{3}$$

$$\cos \gamma = \frac{2}{3}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1$$

31. $\mathbf{u} = \langle 3, 2, -2 \rangle$ $\|\mathbf{u}\| = \sqrt{17}$

$$\cos \alpha = \frac{3}{\sqrt{17}} \Rightarrow \alpha \approx 0.7560 \text{ or } 43.3^\circ$$

$$\cos \beta = \frac{2}{\sqrt{17}} \Rightarrow \beta \approx 1.0644 \text{ or } 61.0^\circ$$

$$\cos \gamma = \frac{-2}{\sqrt{17}} \Rightarrow \gamma \approx 2.0772 \text{ or } 119.0^\circ$$

35. $\mathbf{F}_1$: $C_1 = \frac{50}{\|\mathbf{F}_1\|} \approx 4.3193$

$\mathbf{F}_2$: $C_2 = \frac{80}{\|\mathbf{F}_2\|} \approx 5.4183$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$$

$$\approx 4.3193\langle 10, 5, 3 \rangle + 5.4183\langle 12, 7, -5 \rangle$$

$$= \langle 108.2126, 59.5246, -14.1336 \rangle$$

$$\|\mathbf{F}\| \approx 124.310 \text{ lb}$$

$$\cos \alpha \approx \frac{108.2126}{\|\mathbf{F}\|} \Rightarrow \alpha \approx 29.48^\circ$$

$$\cos \beta \approx \frac{59.5246}{\|\mathbf{F}\|} \Rightarrow \beta \approx 61.39^\circ$$

$$\cos \gamma \approx \frac{-14.1336}{\|\mathbf{F}\|} \Rightarrow \gamma \approx 96.53^\circ$$

39. $\vec{OA} = \langle 0, 10, 10 \rangle$

$$\cos \alpha = \frac{0}{\sqrt{0^2 + 10^2 + 10^2}} = 0 \Rightarrow \alpha = 90^\circ$$

$$\begin{aligned} \cos \beta = \cos \gamma &= \frac{10}{\sqrt{0^2 + 10^2 + 10^2}} \\ &= \frac{1}{\sqrt{2}} \Rightarrow \beta = \gamma = 45^\circ \end{aligned}$$

43. $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 0, 3, 3 \rangle - \langle -2, 2, 2 \rangle = \langle 2, 1, 1 \rangle$

29. $\mathbf{u} = \langle 0, 6, -4 \rangle$, $\|\mathbf{u}\| = \sqrt{52} = 2\sqrt{13}$

$$\cos \alpha = 0$$

$$\cos \beta = \frac{3}{\sqrt{13}}$$

$$\cos \gamma = -\frac{2}{\sqrt{13}}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 0 + \frac{9}{13} + \frac{4}{13} = 1$$

33. $\mathbf{u} = \langle -1, 5, 2 \rangle$ $\|\mathbf{u}\| = \sqrt{30}$

$$\cos \alpha = \frac{-1}{\sqrt{30}} \Rightarrow \alpha \approx 1.7544 \text{ or } 100.5^\circ$$

$$\cos \beta = \frac{5}{\sqrt{30}} \Rightarrow \beta \approx 0.4205 \text{ or } 24.1^\circ$$

$$\cos \gamma = \frac{2}{\sqrt{30}} \Rightarrow \gamma \approx 1.1970 \text{ or } 68.6^\circ$$

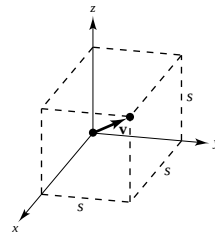
37. Let s = length of a side.

$$\mathbf{v} = \langle s, s, s \rangle$$

$$\|\mathbf{v}\| = s\sqrt{3}$$

$$\cos \alpha = \cos \beta = \cos \gamma = \frac{s}{s\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\alpha = \beta = \gamma = \arccos\left(\frac{1}{\sqrt{3}}\right) \approx 54.7^\circ$$



41. $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 6, 7 \rangle - \langle 2, 8 \rangle = \langle 4, -1 \rangle$

45. $\mathbf{u} = \langle 2, 3 \rangle$, $\mathbf{v} = \langle 5, 1 \rangle$

(a) $\mathbf{w}_1 = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \frac{13}{26} \langle 5, 1 \rangle = \left\langle \frac{5}{2}, \frac{1}{2} \right\rangle$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \left\langle -\frac{1}{2}, \frac{5}{2} \right\rangle$

47. $\mathbf{u} = \langle 2, 1, 2 \rangle, \mathbf{v} = \langle 0, 3, 4 \rangle$

$$(a) \mathbf{w}_1 = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

$$= \frac{11}{25} \langle 0, 3, 4 \rangle = \left\langle 0, \frac{33}{25}, \frac{44}{25} \right\rangle$$

$$(b) \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \left\langle 2, -\frac{8}{25}, \frac{6}{25} \right\rangle$$

49. $\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1v_1 + u_2v_2 + u_3v_3$

51. (a) Orthogonal, $\theta = \frac{\pi}{2}$

(b) Acute, $0 < \theta < \frac{\pi}{2}$

(c) Obtuse, $\frac{\pi}{2} < \theta < \pi$

53. See page 738. Direction cosines of $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are

$$\cos \alpha = \frac{v_1}{\|\mathbf{v}\|}, \cos \beta = \frac{v_2}{\|\mathbf{v}\|}, \cos \gamma = \frac{v_3}{\|\mathbf{v}\|}.$$

 α, β , and γ are the direction angles. See Figure 10.26.

55. (a) $\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \mathbf{u} \Rightarrow \mathbf{u} = c\mathbf{v} \Rightarrow \mathbf{u}$ and $\mathbf{v}$ are parallel.

$$(b) \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal.}$$

57. Programs will vary.

59. Programs will vary.

61. Because $\mathbf{u}$ appears to be perpendicular to $\mathbf{v}$, the projection of $\mathbf{u}$ onto $\mathbf{v}$ is $\mathbf{0}$. Analytically,

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} = \frac{\langle 2, -3 \rangle \cdot \langle 6, 4 \rangle}{\|\langle 6, 4 \rangle\|^2} \langle 6, 4 \rangle = 0 \langle 6, 4 \rangle = \mathbf{0}.$$

63. $\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j}$. Want $\mathbf{u} \cdot \mathbf{v} = 0$.

$\mathbf{v} = 8\mathbf{i} + 6\mathbf{j}$ and $-\mathbf{v} = -8\mathbf{i} - 6\mathbf{j}$ are orthogonal to $\mathbf{u}$.

65. $\mathbf{u} = \langle 3, 1, -2 \rangle$. Want $\mathbf{u} \cdot \mathbf{v} = 0$.

$\mathbf{v} = \langle 0, 2, 1 \rangle$ and $-\mathbf{v} = \langle 0, -2, -1 \rangle$ are orthogonal to $\mathbf{u}$.

67. (a) Gravitational Force $\mathbf{F} = -48,000 \mathbf{j}$

$\mathbf{v} = \cos 10^\circ \mathbf{i} + \sin 10^\circ \mathbf{j}$

$$\mathbf{w}_1 = \frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} = (\mathbf{F} \cdot \mathbf{v}) \mathbf{v} = (-48,000)(\sin 10^\circ) \mathbf{v}$$

$$\approx -8335.1(\cos 10^\circ \mathbf{i} + \sin 10^\circ \mathbf{j})$$

$\|\mathbf{w}_1\| \approx 8335.1 \text{ lb}$

(b) $\mathbf{w}_2 = \mathbf{F} \cdot \mathbf{w}_1 = -48,000 \mathbf{j} + 8335.1(\cos 10^\circ \mathbf{i} + \sin 10^\circ \mathbf{j})$

$= 8208.5 \mathbf{i} - 46,552.6 \mathbf{j}$

$\|\mathbf{w}_2\| \approx 47,270.8 \text{ lb}$

69. $\mathbf{F} = 85 \left(\frac{1}{2} \mathbf{i} + \frac{\sqrt{3}}{2} \mathbf{j} \right)$

$\mathbf{v} = 10\mathbf{i}$

$W = \mathbf{F} \cdot \mathbf{v} = 425 \text{ ft} \cdot \text{lb}$

71. $\overrightarrow{PQ} = \langle 4, 7, 5 \rangle$

$\mathbf{v} = \langle 1, 4, 8 \rangle$

$W = \overrightarrow{PQ} \cdot \mathbf{v} = 72$

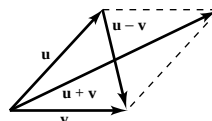
73. False. Let $\mathbf{u} = \langle 2, 4 \rangle, \mathbf{v} = \langle 1, 7 \rangle$ and $\mathbf{w} = \langle 5, 5 \rangle$. Then $\mathbf{u} \cdot \mathbf{v} = 2 + 28 = 30$ and $\mathbf{u} \cdot \mathbf{w} = 10 + 20 = 30$.75. In a rhombus, $\|\mathbf{u}\| = \|\mathbf{v}\|$. The diagonals are $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$.

$$(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = (\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} - (\mathbf{u} + \mathbf{v}) \cdot \mathbf{v}$$

$$= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{v}$$

$$= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0$$

Therefore, the diagonals are orthogonal.



77. $\mathbf{u} = \langle \cos \alpha, \sin \alpha, 0 \rangle, \mathbf{v} = \langle \cos \beta, \sin \beta, 0 \rangle$

The angle between $\mathbf{u}$ and $\mathbf{v}$ is $\alpha - \beta$. (Assuming that $\alpha > \beta$). Also,

$$\cos(\alpha - \beta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta}{(1)(1)} = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$$

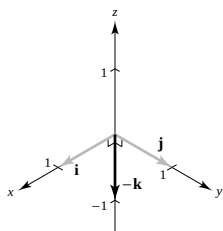
$$\begin{aligned} 79. \|\mathbf{u} - \mathbf{v}\|^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\ &= (\mathbf{u} - \mathbf{v}) \cdot \mathbf{u} - (\mathbf{u} - \mathbf{v}) \cdot \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 - \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} \end{aligned}$$

$$\begin{aligned} 81. \|\mathbf{u} + \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= (\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} + (\mathbf{u} + \mathbf{v}) \cdot \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 \\ &\leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\| \|\mathbf{v}\| + \|\mathbf{v}\|^2 \text{ from Exercise 66} \\ &\leq (\|\mathbf{u}\| + \|\mathbf{v}\|)^2 \end{aligned}$$

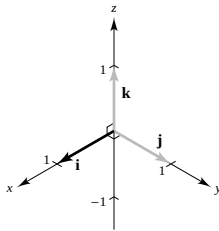
Therefore, $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

Section 10.4 The Cross Product of Two Vectors in Space

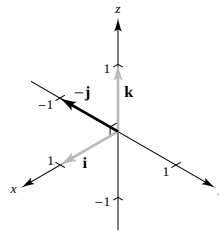
1. $\mathbf{j} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\mathbf{k}$



3. $\mathbf{j} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i}$



5. $\mathbf{i} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\mathbf{j}$



7. (a) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 4 \\ 3 & 7 & 2 \end{vmatrix} = \langle -22, 16, -23 \rangle$

(b) $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = \langle 22, -16, 23 \rangle$

(c) $\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 7 & 2 \\ 3 & 7 & 2 \end{vmatrix} = \mathbf{0}$

9. (a) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & 2 \\ 1 & -1 & 5 \end{vmatrix} = \langle 17, -33, -10 \rangle$

(b) $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = \langle -17, 33, 10 \rangle$

(c) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

11. $\mathbf{u} = \langle 2, -3, 1 \rangle, \mathbf{v} = \langle 1, -2, 1 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & -2 & 1 \end{vmatrix} = -\mathbf{i} - \mathbf{j} - \mathbf{k} = \langle -1, -1, -1 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 2(-1) + (-3)(-1) + (1)(-1) = 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 1(-1) + (-2)(-1) + (1)(-1) = 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

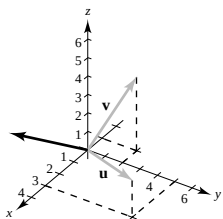
13. $\mathbf{u} = \langle 12, -3, 0 \rangle, \mathbf{v} = \langle -2, 5, 0 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & -3 & 0 \\ -2 & 5 & 0 \end{vmatrix} = 54\mathbf{k} = \langle 0, 0, 54 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 12(0) + (-3)(0) + 0(54) = 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = -2(0) + 5(0) + 0(54) = 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

17.



21. $\mathbf{u} = \langle 4, -3.5, 7 \rangle$

$\mathbf{v} = \langle -1, 8, 4 \rangle$

$$\mathbf{u} \times \mathbf{v} = \left\langle -70, -23, \frac{57}{2} \right\rangle$$

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \left\langle \frac{-140}{\sqrt{24,965}}, \frac{-46}{\sqrt{24,965}}, \frac{57}{\sqrt{24,965}} \right\rangle$$

25. Programs will vary.

27. $\mathbf{u} = \mathbf{j}$

$\mathbf{v} = \mathbf{j} + \mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i}$$

$$A = \|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{i}\| = 1$$

31. $A(1, 1, 1), B(2, 3, 4), C(6, 5, 2), D(7, 7, 5)$

$$\overrightarrow{AB} = \langle 1, 2, 3 \rangle, \overrightarrow{AC} = \langle 5, 4, 1 \rangle, \overrightarrow{CD} = \langle 1, 2, 3 \rangle, \overrightarrow{BD} = \langle 5, 4, 1 \rangle$$

Since $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$, the figure is a parallelogram. $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are adjacent sides and

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 5 & 4 & 1 \end{vmatrix} = -10\mathbf{i} + 14\mathbf{j} - 6\mathbf{k}$$

$$A = \|\overrightarrow{AB} \times \overrightarrow{AC}\| = \sqrt{332} = 2\sqrt{83}$$

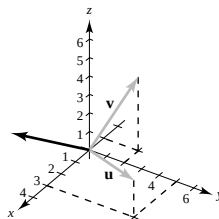
15. $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k} = \langle -2, 3, -1 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 1(-2) + 1(3) + 1(-1) = 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 2(-2) + 1(3) + (-1)(-1) = 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

19.



23. $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$

$$\mathbf{v} = \frac{1}{2}\mathbf{i} - \frac{3}{4}\mathbf{j} + \frac{1}{10}\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \left\langle -\frac{71}{20}, -\frac{11}{5}, \frac{5}{4} \right\rangle$$

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{20}{\sqrt{7602}} \left\langle -\frac{71}{20}, -\frac{11}{5}, \frac{5}{4} \right\rangle = \left\langle -\frac{71}{\sqrt{7602}}, -\frac{44}{\sqrt{7602}}, \frac{25}{\sqrt{7602}} \right\rangle$$

29. $\mathbf{u} = \langle 3, 2, -1 \rangle$

$\mathbf{v} = \langle 1, 2, 3 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix} = \langle 8, -10, 4 \rangle$$

$$A = \|\mathbf{u} \times \mathbf{v}\| = \|\langle 8, -10, 4 \rangle\| = \sqrt{180} = 6\sqrt{5}$$

33. $A(0, 0, 0), B(1, 2, 3), C(-3, 0, 0)$

$$\overrightarrow{AB} = \langle 1, 2, 3 \rangle, \overrightarrow{AC} = \langle -3, 0, 0 \rangle$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -3 & 0 & 0 \end{vmatrix} = -9\mathbf{j} + 6\mathbf{k}$$

$$A = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\sqrt{117} = \frac{3}{2}\sqrt{13}$$

35. $A(2, -7, 3), B(-1, 5, 8), C(4, 6, -1)$

$$\overrightarrow{AB} = \langle -3, 12, 5 \rangle, \overrightarrow{AC} = \langle 2, 13, -4 \rangle$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 12 & 5 \\ 2 & 13 & -4 \end{vmatrix} = \langle -113, -2, -63 \rangle$$

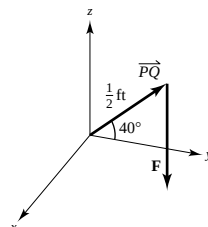
$$\text{Area} = \frac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2} \sqrt{16,742}$$

37. $\mathbf{F} = -20\mathbf{k}$

$$\overrightarrow{PQ} = \frac{1}{2}(\cos 40^\circ \mathbf{j} + \sin 40^\circ \mathbf{k})$$

$$\overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \cos 40^\circ/2 & \sin 40^\circ/2 \\ 0 & 0 & -20 \end{vmatrix} = -10 \cos 40^\circ \mathbf{i}$$

$$\|\overrightarrow{PQ} \times \mathbf{F}\| = 10 \cos 40^\circ \approx 7.66 \text{ ft} \cdot \text{lb}$$



39. (a) $\overrightarrow{OA} = \frac{3}{2}\mathbf{k}$

$$\mathbf{F} = -60(\sin \theta \mathbf{j} + \cos \theta \mathbf{k})$$

$$\overrightarrow{OA} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 3/2 \\ 0 & -60 \sin \theta & -60 \cos \theta \end{vmatrix} = 90 \sin \theta \mathbf{i}$$

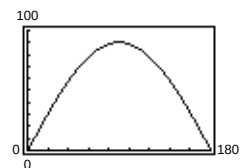
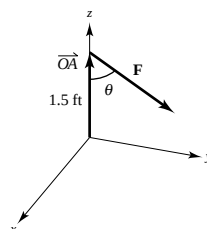
$$\|\overrightarrow{OA} \times \mathbf{F}\| = 90 \sin \theta$$

(b) When $\theta = 45^\circ$: $\|\overrightarrow{OA} \times \mathbf{F}\| = 90\left(\frac{\sqrt{2}}{2}\right) = 45\sqrt{2} \approx 63.64$.

(c) Let $T = 90 \sin \theta$.

$$\frac{dT}{d\theta} = 90 \cos \theta = 0 \text{ when } \theta = 90^\circ.$$

This is what we expected. When $\theta = 90^\circ$ the pipe wrench is horizontal.



41. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

43. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 6$

45. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2$

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 2$$

47. $\mathbf{u} = \langle 3, 0, 0 \rangle$

$$\mathbf{v} = \langle 0, 5, 1 \rangle$$

$$\mathbf{w} = \langle 2, 0, 5 \rangle$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 5 & 1 \\ 2 & 0 & 5 \end{vmatrix} = 75$$

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 75$$

49. $\mathbf{u} \times \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$

51. The magnitude of the cross product will increase by a factor of 4.

53. If the vectors are ordered pairs, then the cross product does not exist. False.

55. True

$$57. \mathbf{u} = \langle u_1, u_2, u_3 \rangle, \mathbf{v} = \langle v_1, v_2, v_3 \rangle, \mathbf{w} = \langle w_1, w_2, w_3 \rangle$$

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 + w_1 & v_2 + w_2 & v_3 + w_3 \end{vmatrix} \\ &= [u_2(v_3 + w_3) - u_3(v_2 + w_2)]\mathbf{i} - [u_1(v_3 + w_3) - u_3(v_1 + w_1)]\mathbf{j} + [u_1(v_2 + w_2) - u_2(v_1 + w_1)]\mathbf{k} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} + (u_2w_3 - u_3w_2)\mathbf{i} - \\ &\quad (u_1w_3 - u_3w_1)\mathbf{j} + (u_1w_2 - u_2w_1)\mathbf{k} \\ &= (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) \end{aligned}$$

$$59. \mathbf{u} = \langle u_1, u_2, u_3 \rangle$$

$$\mathbf{u} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = (u_2u_3 - u_3u_2)\mathbf{i} - (u_1u_3 - u_3u_1)\mathbf{j} + (u_1u_2 - u_2u_1)\mathbf{k} = \mathbf{0}$$

$$61. \mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (u_2v_3 - u_3v_2)u_1 + (u_3v_1 - u_1v_3)u_2 + (u_1v_2 - u_2v_1)u_3 = 0$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = (u_2v_3 - u_3v_2)v_1 + (u_3v_1 - u_1v_3)v_2 + (u_1v_2 - u_2v_1)v_3 = 0$$

Thus, $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}$ and $\mathbf{u} \times \mathbf{v} \perp \mathbf{v}$.

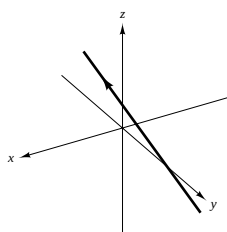
$$63. \|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$$

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, $\theta = \pi/2$ and $\sin \theta = 1$. Therefore, $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\|$.

Section 10.5 Lines and Planes in Space

$$1. x = 1 + 3t, y = 2 - t, z = 2 + 5t$$

(a)



(b) When $t = 0$ we have $P = (1, 2, 2)$. When $t = 3$ we have $Q = (10, -1, 17)$.

$$\overrightarrow{PQ} = \langle 9, -3, 15 \rangle$$

The components of the vector and the coefficients of t are proportional since the line is parallel to $\overrightarrow{PQ}$.

(c) $y = 0$ when $t = 2$. Thus, $x = 7$ and $z = 12$.

Point: $(7, 0, 12)$

$$x = 0 \text{ when } t = -\frac{1}{3}. \text{ Point: } \left(0, \frac{7}{3}, \frac{1}{3}\right)$$

$$z = 0 \text{ when } t = -\frac{2}{5}. \text{ Point: } \left(-\frac{1}{5}, \frac{12}{5}, 0\right)$$

3. Point: $(0, 0, 0)$

Direction vector: $\mathbf{v} = \langle 1, 2, 3 \rangle$

Direction numbers: 1, 2, 3

(a) Parametric: $x = t, y = 2t, z = 3t$

(b) Symmetric: $x = \frac{y}{2} = \frac{z}{3}$

5. Point: $(-2, 0, 3)$

Direction vector: $\mathbf{v} = \langle 2, 4, -2 \rangle$

Direction numbers: 2, 4, -2

(a) Parametric: $x = -2 + 2t, y = 4t, z = 3 - 2t$

(b) Symmetric: $\frac{x+2}{2} = \frac{y}{4} = \frac{z-3}{-2}$

7. Point: $(1, 0, 1)$ Direction vector: $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

Direction numbers: 3, -2, 1

(a) Parametric: $x = 1 + 3t, y = -2t, z = 1 + t$ (b) Symmetric: $\frac{x-1}{3} = \frac{y}{-2} = \frac{z-1}{1}$ 11. Points: $(2, 3, 0), (10, 8, 12)$ Direction vector: $\langle 8, 5, 12 \rangle$

Direction numbers: 8, 5, 12

(a) Parametric: $x = 2 + 8t, y = 3 + 5t, z = 12t$ (b) Symmetric: $\frac{x-2}{8} = \frac{y-3}{5} = \frac{z}{12}$ 15. Point: $(-2, 3, 1)$ Direction vector: $\mathbf{v} = 4\mathbf{i} - \mathbf{k}$

Direction numbers: 4, 0, -1

Parametric: $x = -2 + 4t, y = 3, z = 1 - t$ Symmetric: $\frac{x+2}{4} = \frac{z-1}{-1}, y = 3$

(a) On line

(b) On line

(c) Not on line ($y \neq 3$)(d) Not on line $\left(\frac{6+2}{4} \neq \frac{-2-1}{-1}\right)$ 9. Points: $(5, -3, -2), \left(\frac{-2}{3}, \frac{2}{3}, 1\right)$ Direction vector: $\mathbf{v} = \frac{17}{3}\mathbf{i} - \frac{11}{3}\mathbf{j} - 3\mathbf{k}$

Direction numbers: 17, -11, -9

(a) Parametric: $x = 5 + 17t, y = -3 - 11t, z = -2 - 9t$ (b) Symmetric: $\frac{x-5}{17} = \frac{y+3}{-11} = \frac{z+2}{-9}$ 13. Point: $(2, 3, 4)$ Direction vector: $\mathbf{v} = \mathbf{k}$

Direction numbers: 0, 0, 1

Parametric: $x = 2, y = 3, z = 4 + t$ 17. $L_1: \mathbf{v} = \langle -3, 2, 4 \rangle$ $(6, -2, 5)$ on line $L_2: \mathbf{v} = \langle 6, -4, -8 \rangle$ $(6, -2, 5)$ on line $L_3: \mathbf{v} = \langle -6, 4, 8 \rangle$ $(6, -2, 5)$ not on line $L_4: \mathbf{v} = \langle 6, 4, -6 \rangle$ not parallel to L_1, L_2 , nor L_3 Hence, L_1 and L_2 are identical. $L_1 = L_2$ and L_3 are parallel.

19. At the point of intersection, the coordinates for one line equal the corresponding coordinates for the other line. Thus,

(i) $4t + 2 = 2s + 2$, (ii) $3 = 2s + 3$, and (iii) $-t + 1 = s + 1$.From (ii), we find that $s = 0$ and consequently, from (iii), $t = 0$. Letting $s = t = 0$, we see that equation (i) is satisfied and therefore the two lines intersect. Substituting zero for s or for t , we obtain the point $(2, 3, 1)$. $\mathbf{u} = 4\mathbf{i} - \mathbf{k}$ (First line) $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ (Second line)

$$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{8 - 1}{\sqrt{17} \sqrt{9}} = \frac{7}{3\sqrt{17}} = \frac{7\sqrt{17}}{51}$$

21. Writing the equations of the lines in parametric form we have

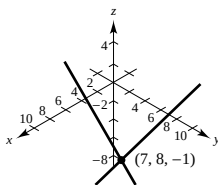
$$x = 3t \quad y = 2 - t \quad z = -1 + t$$

$$x = 1 + 4s \quad y = -2 + s \quad z = -3 - 3s.$$

For the coordinates to be equal, $3t = 1 + 4s$ and $2 - t = -2 + s$. Solving this system yields $t = \frac{17}{7}$ and $s = \frac{11}{7}$.When using these values for s and t , the z coordinates are not equal. The lines do not intersect.23. $x = 2t + 3$ $x = -2s + 7$

$$y = 5t - 2 \quad y = s + 8$$

$$z = -t + 1 \quad z = 2s - 1$$

Point of intersection: $(7, 8, -1)$ 

25. $4x - 3y - 6z = 6$

(a) $P = (0, 0, -1), Q = (0, -2, 0), R = (3, 4, -1)$

$\overrightarrow{PQ} = \langle 0, -2, 1 \rangle, \overrightarrow{PR} = \langle 3, 4, 0 \rangle$

(b) $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 1 \\ 3 & 4 & 0 \end{vmatrix} = \langle -4, 3, 6 \rangle$

The components of the cross product are proportional to the coefficients of the variables in the equation. The cross product is parallel to the normal vector.

27. Point: $(2, 1, 2)$

$\mathbf{n} = \mathbf{i} = \langle 1, 0, 0 \rangle$

$1(x - 2) + 0(y - 1) + 0(z - 2) = 0$

$x - 2 = 0$

29. Point: $(3, 2, 2)$

Normal vector: $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

$2(x - 3) + 3(y - 2) - 1(z - 2) = 0$

$2x + 3y - z = 10$

31. Point: $(0, 0, 6)$

Normal vector: $\mathbf{n} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$-1(x - 0) + 1(y - 0) - 2(z - 6) = 0$

$-x + y - 2z + 12 = 0$

$x - y + 2z = 12$

33. Let $\mathbf{u}$ be the vector from $(0, 0, 0)$ to $(1, 2, 3)$:

$\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

Let $\mathbf{v}$ be the vector from $(0, 0, 0)$ to $(-2, 3, 3)$:

$\mathbf{v} = -2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$

$$\text{Normal vector: } \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -2 & 3 & 3 \end{vmatrix}$$

$$= -3\mathbf{i} + (-9)\mathbf{j} + 7\mathbf{k}$$

$-3(x - 0) - 9(y - 0) + 7(z - 0) = 0$

$3x + 9y - 7z = 0$

35. Let $\mathbf{u}$ be the vector from $(1, 2, 3)$ to $(3, 2, 1)$: $\mathbf{u} = 2\mathbf{i} - 2\mathbf{k}$

Let $\mathbf{v}$ be the vector from $(1, 2, 3)$ to $(-1, -2, 2)$: $\mathbf{v} = -2\mathbf{i} - 4\mathbf{j} - \mathbf{k}$

$$\text{Normal vector: } \left(\frac{1}{2}\mathbf{u}\right) \times (-\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 2 & 4 & 1 \end{vmatrix} = 4\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$$

$4(x - 1) - 3(y - 2) + 4(z - 3) = 0$

$4x - 3y + 4z = 10$

37. $(1, 2, 3)$, Normal vector: $\mathbf{v} = \mathbf{k}$, $1(z - 3) = 0$, $z = 3$

39. The direction vectors for the lines are $\mathbf{u} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$,
 $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

$$\text{Normal vector: } \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ -3 & 4 & -1 \end{vmatrix} = -5(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

Point of intersection of the lines: $(-1, 5, 1)$

$(x + 1) + (y - 5) + (z - 1) = 0$

$x + y + z = 5$

41. Let $\mathbf{v}$ be the vector from $(-1, 1, -1)$ to $(2, 2, 1)$: $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

Let $\mathbf{n}$ be a vector normal to the plane $2x - 3y + z = 3$: $\mathbf{n} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

Since $\mathbf{v}$ and $\mathbf{n}$ both lie in the plane p , the normal vector to p is

$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 2 & -3 & 1 \end{vmatrix} = 7\mathbf{i} + \mathbf{j} - 11\mathbf{k}$$

$7(x - 2) + 1(y - 2) - 11(z - 1) = 0$

$7x + y - 11z = 5$

43. Let $\mathbf{u} = \mathbf{i}$ and let $\mathbf{v}$ be the vector from $(1, -2, -1)$ to $(2, 5, 6)$: $\mathbf{v} = \mathbf{i} + 7\mathbf{j} + 7\mathbf{k}$

Since $\mathbf{u}$ and $\mathbf{v}$ both lie in the plane P , the normal vector to P is:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 7 & 7 \end{vmatrix} = -7\mathbf{j} + 7\mathbf{k} = -7(\mathbf{j} - \mathbf{k})$$

$$[y - (-2)] - [z - (-1)] = 0$$

$$y - z = -1$$

47. The normal vectors to the planes are

$$\mathbf{n}_1 = \mathbf{i} - 3\mathbf{j} + 6\mathbf{k}, \quad \mathbf{n}_2 = 5\mathbf{i} + \mathbf{j} - \mathbf{k},$$

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{|5 - 3 - 6|}{\sqrt{46}\sqrt{27}} = \frac{4\sqrt{138}}{414}.$$

$$\text{Therefore, } \theta = \arccos\left(\frac{4\sqrt{138}}{414}\right) \approx 83.5^\circ.$$

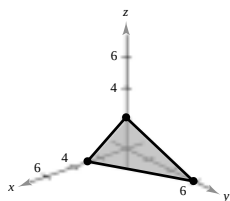
45. The normal vectors to the planes are

$$\mathbf{n}_1 = \langle 5, -3, 1 \rangle, \quad \mathbf{n}_2 = \langle 1, 4, 7 \rangle, \quad \cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = 0.$$

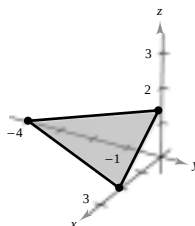
Thus, $\theta = \pi/2$ and the planes are orthogonal.

49. The normal vectors to the planes are $\mathbf{n}_1 = \langle 1, -5, -1 \rangle$ and $\mathbf{n}_2 = \langle 5, -25, -5 \rangle$. Since $\mathbf{n}_2 = 5\mathbf{n}_1$, the planes are parallel, but not equal.

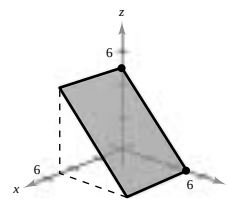
51. $4x + 2y + 6z = 12$



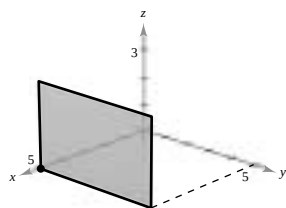
53. $2x - y + 3z = 4$



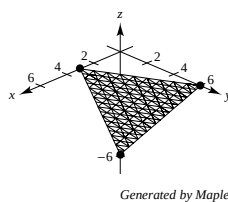
55. $y + z = 5$



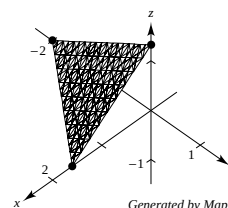
57. $x = 5$



59. $2x + y - z = 6$



61. $-5x + 4y - 6z + 8 = 0$



63. $P_1: \mathbf{n} = \langle 3, -2, 5 \rangle$ (1, -1, 1) on plane
 $P_2: \mathbf{n} = \langle -6, 4, -10 \rangle$ (1, -1, 1) not on plane
 $P_3: \mathbf{n} = \langle -3, 2, 5 \rangle$
 $P_4: \mathbf{n} = \langle 75, -50, 125 \rangle$ (1, -1, 1) on plane
 P_1 and P_4 are identical.
 $P_1 = P_4$ is parallel to P_2 .

65. Each plane passes through the points $(c, 0, 0)$, $(0, c, 0)$, and $(0, 0, c)$.

67. The normals to the planes are $\mathbf{n}_1 = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$. The direction vector for the line is

$$\mathbf{n}_2 \times \mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 2 \\ 3 & 2 & -1 \end{vmatrix} = 7(\mathbf{j} + 2\mathbf{k}).$$

Now find a point of intersection of the planes.

$$6x + 4y - 2z = 14$$

$$x - 4y + 2z = 0$$

$$7x = 14$$

$$x = 2$$

Substituting 2 for x in the second equation, we have $-4y + 2z = -2$ or $z = 2y - 1$. Letting $y = 1$, a point of intersection is $(2, 1, 1)$.

$$x = 2, y = 1 + t, z = 1 + 2t$$

71. Writing the equation of the line in parametric form and substituting into the equation of the plane we have:
- $$x = 1 + 3t, y = -1 - 2t, z = 3 + t$$
- $$2(1 + 3t) + 3(-1 - 2t) = 10, -1 = 10, \text{contradiction}$$
- Therefore, the line does not intersect the plane.

75. Point: $Q(2, 8, 4)$

$$\text{Plane: } 2x + y + z = 5$$

$$\text{Normal to plane: } \mathbf{n} = \langle 2, 1, 1 \rangle$$

$$\text{Point in plane: } P(0, 0, 5)$$

$$\text{Vector: } \overrightarrow{PQ} = \langle 2, 8, -1 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{11}{\sqrt{6}} = \frac{11\sqrt{6}}{6}$$

79. The normal vectors to the planes are $\mathbf{n}_1 = \langle -3, 6, 7 \rangle$ and $\mathbf{n}_2 = \langle 6, -12, -14 \rangle$. Since $\mathbf{n}_2 = -2\mathbf{n}_1$, the planes are parallel. Choose a point in each plane.

$$P = (0, -1, 1) \text{ is a point in } -3x + 6y + 7z = 1.$$

$$Q = \left(\frac{25}{6}, 0, 0\right) \text{ is a point in } 6x - 12y - 14z = 25.$$

$$\overrightarrow{PQ} = \left\langle \frac{25}{6}, 1, -1 \right\rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{|-27/2|}{\sqrt{94}} = \frac{27}{2\sqrt{94}} = \frac{27\sqrt{94}}{188}$$

83. The parametric equations of a line L parallel to $\mathbf{v} = \langle a, b, c \rangle$ and passing through the point $P(x_1, y_1, z_1)$ are

$$x = x_1 + at, y = y_1 + bt, z = z_1 + ct.$$

The symmetric equations are

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

69. Writing the equation of the line in parametric form and substituting into the equation of the plane we have:

$$x = \frac{1}{2} + t, y = \frac{-3}{2} - t, z = -1 + 2t$$

$$2\left(\frac{1}{2} + t\right) - 2\left(\frac{-3}{2} - t\right) + (-1 + 2t) = 12, t = \frac{3}{2}$$

Substituting $t = 3/2$ into the parametric equations for the line we have the point of intersection $(2, -3, 2)$. The line does not lie in the plane.

73. Point: $Q(0, 0, 0)$

$$\text{Plane: } 2x + 3y + z - 12 = 0$$

$$\text{Normal to plane: } \mathbf{n} = \langle 2, 3, 1 \rangle$$

$$\text{Point in plane: } P(6, 0, 0)$$

$$\text{Vector } \overrightarrow{PQ} = \langle -6, 0, 0 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-12|}{\sqrt{14}} = \frac{6\sqrt{14}}{7}$$

77. The normal vectors to the planes are $\mathbf{n}_1 = \langle 1, -3, 4 \rangle$ and $\mathbf{n}_2 = \langle 1, -3, 4 \rangle$. Since $\mathbf{n}_1 = \mathbf{n}_2$, the planes are parallel. Choose a point in each plane.

$$P = (10, 0, 0) \text{ is a point in } x - 3y + 4z = 10.$$

$$Q = (6, 0, 0) \text{ is a point in } x - 3y + 4z = 6.$$

$$\overrightarrow{PQ} = \langle -4, 0, 0 \rangle, D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{4}{\sqrt{26}} = \frac{2\sqrt{26}}{13}$$

81. $\mathbf{u} = \langle 4, 0, -1 \rangle$ is the direction vector for the line. $Q(1, 5, -2)$ is the given point, and $P(-2, 3, 1)$ is on the line. Hence, $\overrightarrow{PQ} = \langle 3, 2, -3 \rangle$ and

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -3 \\ 4 & 0 & -1 \end{vmatrix} = \langle -2, -9, -8 \rangle$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{149}}{\sqrt{17}} = \frac{\sqrt{2533}}{17}$$

85. Solve the two linear equations representing the planes to find two points of intersection. Then find the line determined by the two points.

87. (a) Sphere

$$(x - 3)^2 + (y + 2)^2 + (z - 5)^2 = 16$$

$$x^2 + y^2 + z^2 - 6x + 4y - 10z + 22 = 0$$

(b) Parallel planes

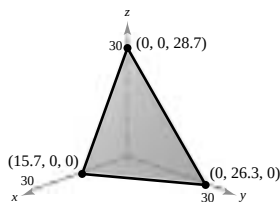
$$4x - 3y + z = 10 \pm 4\|\mathbf{n}\| = 10 \pm 4\sqrt{26}$$

 89. (a) $z = 28.7 - 1.83x - 1.09y$

Year	1980	1985	1990	1994	1995	1996	1997
z (approx.)	16.16	14.23	9.81	8.60	8.42	8.27	8.23

(b) An increase in x or y will cause a decrease in z . In fact, any increase in two variables will cause a decrease in the third.

(c)



91. True

Section 10.6 Surfaces in Space

1. Ellipsoid

Matches graph (c)

3. Hyperboloid of one sheet

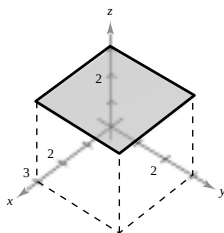
Matches graph (f)

5. Elliptic paraboloid

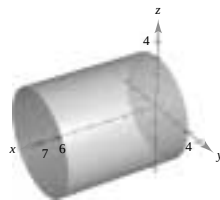
Matches graph (d)

 7. $z = 3$

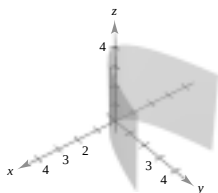
Plane parallel to the xy -coordinate plane


 9. $y^2 + z^2 = 9$

The x -coordinate is missing so we have a cylindrical surface with rulings parallel to the x -axis. The generating curve is a circle.

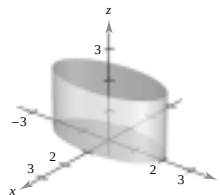

 11. $y = x^2$

The z -coordinate is missing so we have a cylindrical surface with rulings parallel to the z -axis. The generating curve is a parabola.


 13. $4x^2 + y^2 = 4$

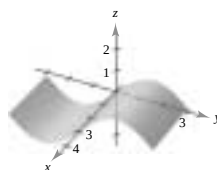
$$\frac{x^2}{1} + \frac{y^2}{4} = 1$$

The z -coordinate is missing so we have a cylindrical surface with rulings parallel to the z -axis. The generating curve is an ellipse.



15. $z = \sin y$

The x -coordinate is missing so we have a cylindrical surface with rulings parallel to the x -axis. The generating curve is the sine curve.



17. $x = x^2 + y^2$

- (a) You are viewing the paraboloid from the x -axis: $(20, 0, 0)$
 (b) You are viewing the paraboloid from above, but not on the z -axis: $(10, 10, 20)$
 (c) You are viewing the paraboloid from the z -axis: $(0, 0, 20)$
 (d) You are viewing the paraboloid from the y -axis: $(0, 20, 0)$

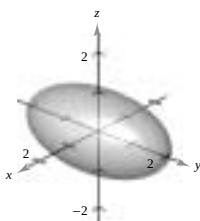
19. $\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{1} = 1$

Ellipsoid

$$xy\text{-trace: } \frac{x^2}{1} + \frac{y^2}{4} = 1 \text{ ellipse}$$

$$xz\text{-trace: } x^2 + z^2 = 1 \text{ circle}$$

$$yz\text{-trace: } \frac{y^2}{4} + \frac{z^2}{1} = 1 \text{ ellipse}$$



21. $16x^2 - y^2 + 16z^2 = 4$

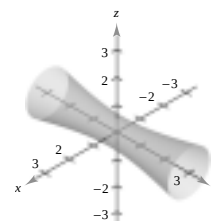
$$4x^2 - \frac{y^2}{4} + 4z^2 = 1$$

Hyperboloid on one sheet

$$xy\text{-trace: } 4x^2 - \frac{y^2}{4} = 1 \text{ hyperbola}$$

$$xz\text{-trace: } 4(x^2 + z^2) = 1 \text{ circle}$$

$$yz\text{-trace: } \frac{-y^2}{4} + 4z^2 = 1 \text{ hyperbola}$$



23. $x^2 - y + z^2 = 0$

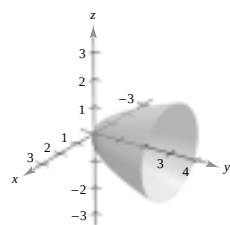
Elliptic paraboloid

$$xy\text{-trace: } y = x^2$$

$$xz\text{-trace: } x^2 + z^2 = 0, \\ \text{point } (0, 0, 0)$$

$$yz\text{-trace: } y = z^2$$

$$y = 1: x^2 + z^2 = 1$$



25. $x^2 - y^2 + z = 0$

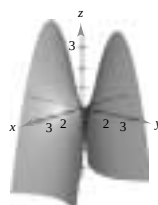
Hyperbolic paraboloid

$$xy\text{-trace: } y = \pm x$$

$$xz\text{-trace: } z = -x^2$$

$$yz\text{-trace: } z = y^2$$

$$y = \pm 1: z = 1 - x^2$$



27. $z^2 = x^2 + \frac{y^2}{4}$

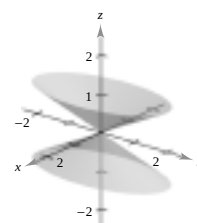
Elliptic Cone

$$xy\text{-trace: point } (0, 0, 0)$$

$$xz\text{-trace: } z = \pm x$$

$$yz\text{-trace: } z = \frac{\pm 1}{2}y$$

$$z = \pm 1: x^2 + \frac{y^2}{4} = 1$$



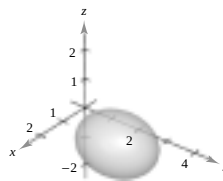
29. $16x^2 + 9y^2 + 16z^2 - 32x - 36y + 36 = 0$

$$16(x^2 - 2x + 1) + 9(y^2 - 4y + 4) + 16z^2 = -36 + 16 + 36$$

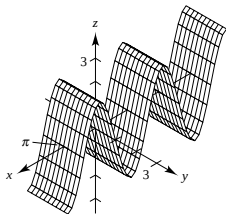
$$16(x - 1)^2 + 9(y - 2)^2 + 16z^2 = 16$$

$$\frac{(x - 1)^2}{1} + \frac{(y - 2)^2}{16/9} + \frac{z^2}{1} = 1$$

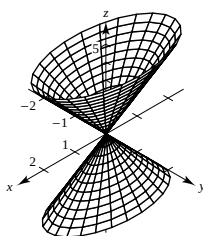
Ellipsoid with center $(1, 2, 0)$.



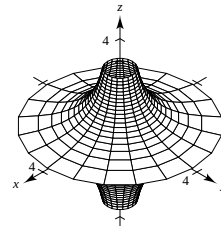
31. $z = 2 \sin x$



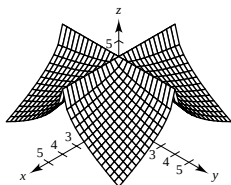
33. $z^2 = x^2 + 4y^2$
 $z = \pm \sqrt{x^2 + 4y^2}$



35. $x^2 + y^2 = \left(\frac{2}{z}\right)^2$
 $y = \pm \sqrt{\frac{4}{z^2} - x^2}$

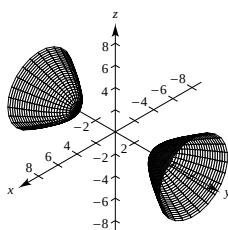


37. $z = 4 - \sqrt{|xy|}$



39. $4x^2 - y^2 + 4z^2 = -16$

$z = \pm \sqrt{\frac{y^2}{4} - x^2 - 4}$

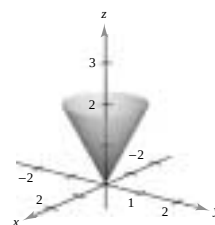


41. $z = 2\sqrt{x^2 + y^2}$

$z = 2$

$2\sqrt{x^2 + y^2} = 2$

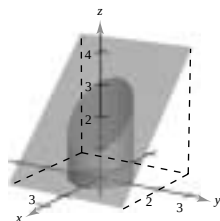
$x^2 + y^2 = 1$



43. $x^2 + y^2 = 1$

$x + z = 2$

$z = 0$



45. $x^2 + z^2 = [r(y)]^2$ and $z = r(y) = \pm 2\sqrt{y}$; therefore,

$x^2 + z^2 = 4y.$

47. $x^2 + y^2 = [r(z)]^2$ and $y = r(z) = \frac{z}{2}$; therefore,

$x^2 + y^2 = \frac{z^2}{4}, 4x^2 + 4y^2 = z^2.$

49. $y^2 + z^2 = [r(x)]^2$ and $y = r(x) = \frac{2}{x}$; therefore,

$y^2 + z^2 = \left(\frac{2}{x}\right)^2, y^2 + z^2 = \frac{4}{x^2}.$

51. $x^2 + y^2 - 2z = 0$

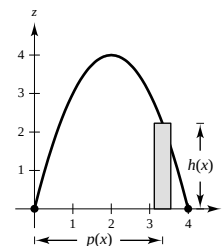
$x^2 + y^2 = (\sqrt{2z})^2$

Equation of generating curve: $y = \sqrt{2z}$ or $x = \sqrt{2z}$

 53. Let C be a curve in a plane and let L be a line not in a parallel plane. The set of all lines parallel to L and intersecting C is called a cylinder.

55. See pages 765 and 766.

57. $V = 2\pi \int_0^4 x(4x - x^2) dx$
 $= 2\pi \left[\frac{4x^3}{3} - \frac{x^4}{4} \right]_0^4 = \frac{128\pi}{3}$



59. $z = \frac{x^2}{2} + \frac{y^2}{4}$

(a) When $z = 2$ we have $2 = \frac{x^2}{2} + \frac{y^2}{4}$, or $1 = \frac{x^2}{4} + \frac{y^2}{8}$

Major axis: $2\sqrt{8} = 4\sqrt{2}$

Minor axis: $2\sqrt{4} = 4$

$c^2 = a^2 - b^2, c^2 = 4, c = 2$

Foci: $(0, \pm 2, 2)$

(b) When $z = 8$ we have $8 = \frac{x^2}{2} + \frac{y^2}{4}$, or $1 = \frac{x^2}{16} + \frac{y^2}{32}$

Major axis: $2\sqrt{32} = 8\sqrt{2}$

Minor axis: $2\sqrt{16} = 8$

$c^2 = 32 - 16 = 16, c = 4$

Foci: $(0, \pm 4, 8)$

 61. If (x, y, z) is on the surface, then

$$(y + 2)^2 = x^2 + (y - 2)^2 + z^2$$

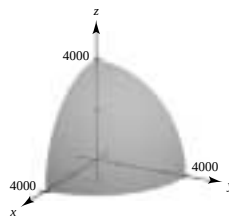
$$y^2 + 4y + 4 = x^2 + y^2 - 4y + 4 + z^2$$

$$x^2 + z^2 = 8y$$

Elliptic paraboloid

 Traces parallel to xz -plane are circles.

63. $\frac{x^2}{3963^2} + \frac{y^2}{3963^2} + \frac{z^2}{3942^2} = 1$



65. $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}, z = bx + ay$

$$bx + ay = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

$$\frac{1}{a^2} \left(x^2 + a^2 bx + \frac{a^4 b^2}{4} \right) = \frac{1}{b^2} \left(y^2 - ab^2 y + \frac{a^2 b^4}{4} \right)$$

$$\frac{\left(x + \frac{a^2 b}{2} \right)^2}{a^2} = \frac{\left(y - \frac{ab^2}{2} \right)^2}{b^2}$$

$$y = \pm \frac{b}{a} \left(x + \frac{a^2 b}{2} \right) + \frac{ab^2}{2}$$

 Letting $x = at$, you obtain the two intersecting lines

$$x = at, y = -bt, z = 0 \text{ and } x = at, y = bt + ab^2$$

$$z = 2abt + a^2 b^2.$$

 67. The Klein bottle *does not* have both an “inside” and an “outside.” It is formed by inserting the small open end through the side of the bottle and making it contiguous with the top of the bottle.

Section 10.7 Cylindrical and Spherical Coordinates

 1. $(5, 0, 2)$, cylindrical

$$x = 5 \cos 0 = 5$$

$$y = 5 \sin 0 = 0$$

$$z = 2$$

 $(5, 0, 2)$, rectangular

 3. $\left(2, \frac{\pi}{3}, 2 \right)$, cylindrical

$$x = 2 \cos \frac{\pi}{3} = 1$$

$$y = 2 \sin \frac{\pi}{3} = \sqrt{3}$$

$$z = 2$$

 $(1, \sqrt{3}, 2)$, rectangular

 5. $\left(4, \frac{7\pi}{6}, 3 \right)$, cylindrical

$$x = 4 \cos \frac{7\pi}{6} = -2\sqrt{3}$$

$$y = 4 \sin \frac{7\pi}{6} = -2$$

$$z = 3$$

 $(-2\sqrt{3}, -2, 3)$, rectangular

7. $(0, 5, 1)$, rectangular
 $r = \sqrt{(0)^2 + (5)^2} = 5$
 $\theta = \arctan \frac{5}{0} = \frac{\pi}{2}$
 $z = 1$
 $\left(5, \frac{\pi}{2}, 1\right)$, cylindrical

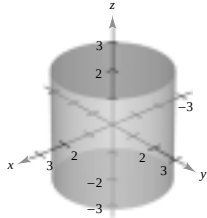
9. $(1, \sqrt{3}, 4)$, rectangular
 $r = \sqrt{1^2 + (\sqrt{3})^2} = 2$
 $\theta = \arctan \sqrt{3} = \frac{\pi}{3}$
 $z = 4$
 $\left(2, \frac{\pi}{3}, 4\right)$, cylindrical

11. $(2, -2, -4)$, rectangular
 $r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$
 $\theta = \arctan(-1) = -\frac{\pi}{4}$
 $z = -4$
 $\left(2\sqrt{2}, -\frac{\pi}{4}, -4\right)$, cylindrical

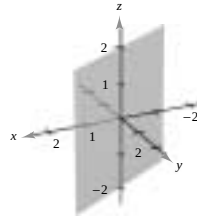
13. $x^2 + y^2 + z^2 = 10$ rectangular equation
 $r^2 + z^2 = 10$ cylindrical equation

15. $y = x^2$ rectangular equation
 $r \sin \theta = (r \cos \theta)^2$
 $\sin \theta = r \cos^2 \theta$
 $r = \sec \theta \cdot \tan \theta$ cylindrical equation

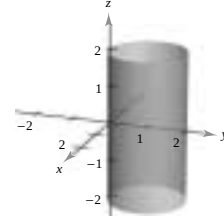
17. $r = 2$
 $\sqrt{x^2 + y^2} = 2$
 $x^2 + y^2 = 4$



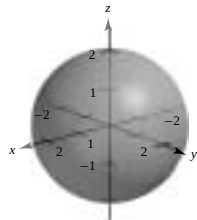
19. $\theta = \frac{\pi}{6}$
 $\tan \frac{\pi}{6} = \frac{y}{x}$
 $\frac{1}{\sqrt{3}} = \frac{y}{x}$
 $x = \sqrt{3}y$
 $x - \sqrt{3}y = 0$



21. $r = 2 \sin \theta$
 $r^2 = 2r \sin \theta$
 $x^2 + y^2 = 2y$
 $x^2 + y^2 - 2y = 0$
 $x^2 + (y - 1)^2 = 1$



23. $r^2 + z^2 = 4$
 $x^2 + y^2 + z^2 = 4$



25. $(4, 0, 0)$, rectangular
 $\rho = \sqrt{4^2 + 0^2 + 0^2} = 4$
 $\theta = \arctan 0 = 0$
 $\phi = \arccos 0 = \frac{\pi}{2}$
 $\left(4, 0, \frac{\pi}{2}\right)$, spherical

27. $(-2, 2\sqrt{3}, 4)$, rectangular
 $\rho = \sqrt{(-2)^2 + (2\sqrt{3})^2 + 4^2} = 4\sqrt{2}$
 $\theta = \arctan(-\sqrt{3}) = \frac{2\pi}{3}$
 $\phi = \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4}$
 $\left(4\sqrt{2}, \frac{2\pi}{3}, \frac{\pi}{4}\right)$, spherical

29. $(\sqrt{3}, 1, 2\sqrt{3})$, rectangular
 $\rho = \sqrt{3 + 1 + 12} = 4$
 $\theta = \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}$
 $\phi = \arccos \frac{\sqrt{3}}{2} = \frac{\pi}{6}$
 $\left(4, \frac{\pi}{6}, \frac{\pi}{6}\right)$, spherical

31. $\left(4, \frac{\pi}{6}, \frac{\pi}{4}\right)$, spherical

$$x = 4 \sin \frac{\pi}{4} \cos \frac{\pi}{6} = \sqrt{6}$$

$$y = 4 \sin \frac{\pi}{4} \sin \frac{\pi}{6} = \sqrt{2}$$

$$z = 4 \cos \frac{\pi}{4} = 2\sqrt{2}$$

$$(\sqrt{6}, \sqrt{2}, 2\sqrt{2}), \text{rectangular}$$

33. $\left(12, \frac{-\pi}{4}, 0\right)$, spherical

$$x = 12 \sin 0 \cos \left(\frac{-\pi}{4}\right) = 0$$

$$y = 12 \sin 0 \sin \left(\frac{-\pi}{4}\right) = 0$$

$$z = 12 \cos 0 = 12$$

$$(0, 0, 12), \text{rectangular}$$

35. $\left(5, \frac{\pi}{4}, \frac{3\pi}{4}\right)$, spherical

$$x = 5 \sin \frac{3\pi}{4} \cos \frac{\pi}{4} = \frac{5}{2}$$

$$y = 5 \sin \frac{3\pi}{4} \sin \frac{\pi}{4} = \frac{5}{2}$$

$$z = 5 \cos \frac{3\pi}{4} = -\frac{5\sqrt{2}}{2}$$

$$\left(\frac{5}{2}, \frac{5}{2}, -\frac{5\sqrt{2}}{2}\right), \text{rectangular}$$

37. (a) Programs will vary.

(b) $(x, y, z) = (3, -4, 2)$

$$(\rho, \theta, \phi) = (5.385, -0.927, 1.190)$$

39. $x^2 + y^2 + z^2 = 36$ rectangular equation

$$\rho^2 = 36$$

spherical equation

41. $x^2 + y^2 = 9$ rectangular equation

$$\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta = 9$$

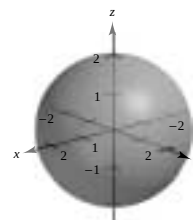
$$\rho^2 \sin^2 \phi = 9$$

$$\rho \sin \phi = 3$$

$$\rho = 3 \csc \phi \text{ spherical equation}$$

43. $\rho = 2$

$$x^2 + y^2 + z^2 = 4$$



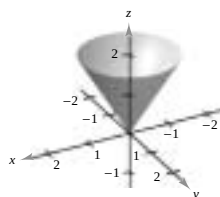
45. $\phi = \frac{\pi}{6}$

$$\cos \phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\sqrt{3}}{2} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{3}{4} = \frac{z^2}{x^2 + y^2 + z^2}$$

$$3x^2 + 3y^2 - z^2 = 0$$

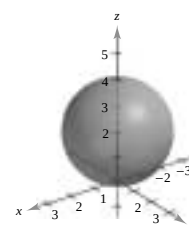


47. $\rho = 4 \cos \phi$

$$\sqrt{x^2 + y^2 + z^2} = \frac{4z}{\sqrt{x^2 + y^2 + z^2}}$$

$$x^2 + y^2 + z^2 - 4z = 0$$

$$x^2 + y^2 + (z - 2)^2 = 4$$

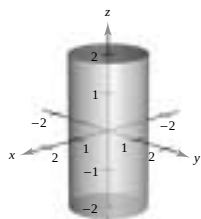


49. $\rho = \csc \phi$

$$\rho \sin \phi = 1$$

$$\sqrt{x^2 + y^2} = 1$$

$$x^2 + y^2 = 1$$



51. $\left(4, \frac{\pi}{4}, 0\right)$, cylindrical

$$\rho = \sqrt{4^2 + 0^2} = 4$$

$$\theta = \frac{\pi}{4}$$

$$\phi = \arccos 0 = \frac{\pi}{2}$$

$$\left(4, \frac{\pi}{4}, \frac{\pi}{2}\right), \text{spherical}$$

53. $\left(4, \frac{\pi}{2}, 4\right)$, cylindrical

$$\rho = \sqrt{4^2 + 4^2} = 4\sqrt{2}$$

$$\theta = \frac{\pi}{2}$$

$$\phi = \arccos \left(\frac{4}{4\sqrt{2}}\right) = \frac{\pi}{4}$$

$$\left(4\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right), \text{spherical}$$

55. $\left(4, \frac{-\pi}{6}, 6\right)$, cylindrical

$$\rho = \sqrt{4^2 + 6^2} = 2\sqrt{13}$$

$$\theta = \frac{-\pi}{6}$$

$$\phi = \arccos \frac{3}{\sqrt{13}}$$

$$\left(2\sqrt{13}, \frac{-\pi}{6}, \arccos \frac{3}{\sqrt{13}}\right),$$

spherical

57. $(12, \pi, 5)$, cylindrical

$$\rho = \sqrt{12^2 + 5^2} = 13$$

$$\theta = \pi$$

$$\phi = \arccos \frac{5}{13}$$

$$\left(13, \pi, \arccos \frac{5}{13}\right), \text{ spherical}$$

59. $\left(10, \frac{\pi}{6}, \frac{\pi}{2}\right)$, spherical

$$r = 10 \sin \frac{\pi}{2} = 10$$

$$\theta = \frac{\pi}{6}$$

$$z = 10 \cos \frac{\pi}{2} = 0$$

$$\left(10, \frac{\pi}{6}, 0\right), \text{ cylindrical}$$

61. $\left(36, \pi, \frac{\pi}{2}\right)$, spherical

$$r = \rho \sin \phi = 36 \sin \frac{\pi}{2} = 36$$

$$\theta = \pi$$

$$z = \rho \cos \phi = 36 \cos \frac{\pi}{2} = 0$$

$$(36, \pi, 0), \text{ cylindrical}$$

63. $\left(6, -\frac{\pi}{6}, \frac{\pi}{3}\right)$, spherical

$$r = 6 \sin \frac{\pi}{3} = 3\sqrt{3}$$

$$\theta = -\frac{\pi}{6}$$

$$z = 6 \cos \frac{\pi}{3} = 3$$

$$\left(3\sqrt{3}, -\frac{\pi}{6}, 3\right), \text{ cylindrical}$$

65. $\left(8, \frac{7\pi}{6}, \frac{\pi}{6}\right)$, spherical

$$r = 8 \sin \frac{\pi}{6} = 4$$

$$\theta = \frac{7\pi}{6}$$

$$z = 8 \cos \frac{\pi}{6} = \frac{8\sqrt{3}}{2}$$

$$\left(4, \frac{7\pi}{6}, 4\sqrt{3}\right), \text{ cylindrical}$$

Rectangular

67. $(4, 6, 3)$

69. $(4.698, 1.710, 8)$

71. $(-7.071, 12.247, 14.142)$

73. $(3, -2, 2)$

75. $\left(\frac{5}{2}, \frac{4}{3}, \frac{-3}{2}\right)$

77. $(-3.536, 3.536, -5)$

79. $(2.804, -2.095, 6)$

 [Note: Use the cylindrical coordinates $(3.5, 5.642, 6)$]

Cylindrical

$$(7.211, 0.983, 3)$$

$$\left(5, \frac{\pi}{9}, 8\right)$$

$$(14.142, 2.094, 14.142)$$

$$(3.606, -0.588, 2)$$

$$(2.833, 0.490, -1.5)$$

$$\left(5, \frac{3\pi}{4}, -5\right)$$

$$(-3.5, 2.5, 6)$$

Spherical

$$(7.810, 0.983, 1.177)$$

$$(9.434, 0.349, 0.559)$$

$$\left(20, \frac{2\pi}{3}, \frac{\pi}{4}\right)$$

$$(4.123, -0.588, 1.064)$$

$$(3.206, 0.490, 2.058)$$

$$(7.071, 2.356, 2.356)$$

$$(6.946, 5.642, 0.528)$$

81. $r = 5$

Cylinder

Matches graph (d)

83. $\rho = 5$

Sphere

Matches graph (c)

85. $r^2 = z, x^2 + y^2 = z$

Paraboloid

Matches graph (f)

87. Rectangular to cylindrical: $r^2 = x^2 + y^2$

$$\tan \theta = \frac{y}{x}$$

$$z = z$$

Cylindrical to rectangular: $x = r \cos \theta$

$$y = r \sin \theta$$

$$z = z$$

89. Rectangular to spherical: $\rho^2 = x^2 + y^2 + z^2$

$$\tan \theta = \frac{y}{x}$$

$$\phi = \arccos \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)$$

Spherical to rectangular: $x = \rho \sin \phi \cos \theta$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

91. $x^2 + y^2 + z^2 = 16$

(a) $r^2 + z^2 = 16$

(b) $\rho^2 = 16, \rho = 4$

95. $x^2 + y^2 = 4y$

(a) $r^2 = 4r \sin \theta, r = 4 \sin \theta$

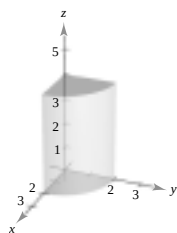
(b) $\rho^2 \sin^2 \phi = 4\rho \sin \phi \sin \theta,$
 $\rho \sin \phi (\rho \sin \phi - 4 \sin \theta) = 0,$

$\rho = \frac{4 \sin \theta}{\sin \phi}, \rho = 4 \sin \theta \csc \phi$

99. $0 \leq \theta \leq \frac{\pi}{2}$

$0 \leq r \leq 2$

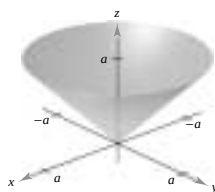
$0 \leq z \leq 4$



101. $0 \leq \theta \leq 2\pi$

$0 \leq r \leq a$

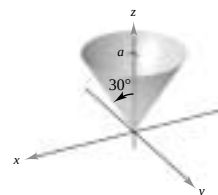
$r \leq z \leq a$



103. $0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \frac{\pi}{6}$

$0 \leq \rho \leq a \sec \phi$

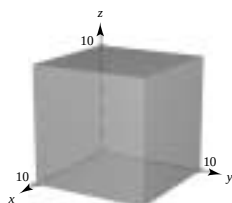


105. Rectangular

$0 \leq x \leq 10$

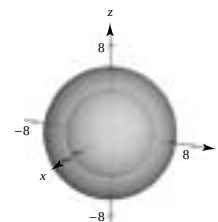
$0 \leq y \leq 10$

$0 \leq z \leq 10$



107. Spherical

$4 \leq \rho \leq 6$



109. $z = \sin \theta, r = 1$

$z = \frac{y}{r} = \frac{y}{1} = y$

The curve of intersection is the ellipse formed by the intersection of the plane $z = y$ and the cylinder $r = 1$.

Review Exercises for Chapter 10

1. $P = (1, 2), Q = (4, 1), R = (5, 4)$

(a) $\mathbf{u} = \overrightarrow{PQ} = \langle 3, -1 \rangle = 3\mathbf{i} - \mathbf{j},$

$\mathbf{v} = \overrightarrow{PR} = \langle 4, 2 \rangle = 4\mathbf{i} + 2\mathbf{j}$

(b) $\|\mathbf{v}\| = \sqrt{4^2 + 2^2} = 2\sqrt{5}$

(c) $2\mathbf{u} + \mathbf{v} = \langle 6, -2 \rangle + \langle 4, 2 \rangle = \langle 10, 0 \rangle = 10\mathbf{i}$

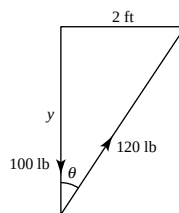
3. $\mathbf{v} = \|\mathbf{v}\| \cos \theta \mathbf{i} + \|\mathbf{v}\| \sin \theta \mathbf{j} = 8 \cos 120^\circ \mathbf{i} + 8 \sin 120^\circ \mathbf{j}$
 $= -4\mathbf{i} + 4\sqrt{3}\mathbf{j}$

5. $120 \cos \theta = 100$

$$\theta = \arccos\left(\frac{5}{6}\right)$$

$$\tan \theta = \frac{2}{y} \Rightarrow y = \frac{2}{\tan \theta}$$

$$y = \frac{2}{\tan[\arccos(5/6)]} = \frac{2}{\sqrt{11}/5} = \frac{10}{\sqrt{11}} \approx 3.015 \text{ ft}$$



7. $z = 0, y = 4, x = -5: (-5, 4, 0)$

9. Looking down from the positive x -axis towards the yz -plane, the point is either in the first quadrant ($y > 0, z > 0$) or in the third quadrant ($y < 0, z < 0$). The x -coordinate can be any number.

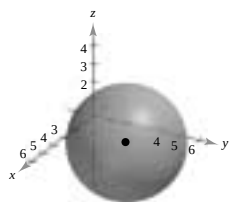
11. $(x - 3)^2 + (y + 2)^2 + (z - 6)^2 = \left(\frac{15}{2}\right)^2$

13. $(x^2 - 4x + 4) + (y^2 - 6y + 9) + z^2 = -4 + 4 + 9$

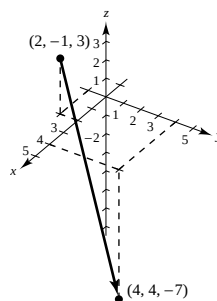
$$(x - 2)^2 + (y - 3)^2 + z^2 = 9$$

Center: $(2, 3, 0)$

Radius: 3



15. $\mathbf{v} = \langle 4 - 2, 4 + 1, -7 - 3 \rangle = \langle 2, 5, -10 \rangle$



17. $\mathbf{v} = \langle -1 - 3, 6 - 4, 9 + 1 \rangle = \langle -4, 2, 10 \rangle$

$$\mathbf{w} = \langle 5 - 3, 3 - 4, -6 + 1 \rangle = \langle 2, -1, -5 \rangle$$

Since $-2\mathbf{w} = \mathbf{v}$, the points lie in a straight line.

19. Unit vector: $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle 2, 3, 5 \rangle}{\sqrt{38}} = \left\langle \frac{2}{\sqrt{38}}, \frac{3}{\sqrt{38}}, \frac{5}{\sqrt{38}} \right\rangle$

21. $P = (5, 0, 0), Q = (4, 4, 0), R = (2, 0, 6)$

(a) $\mathbf{u} = \overrightarrow{PQ} = \langle -1, 4, 0 \rangle = -\mathbf{i} + 4\mathbf{j}$,

$$\mathbf{v} = \overrightarrow{PR} = \langle -3, 0, 6 \rangle = -3\mathbf{i} + 6\mathbf{k}$$

(b) $\mathbf{u} \cdot \mathbf{v} = (-1)(-3) + 4(0) + 0(6) = 3$

(c) $\mathbf{v} \cdot \mathbf{v} = 9 + 36 = 45$

23. $\mathbf{u} = \langle 7, -2, 3 \rangle, \mathbf{v} = \langle -1, 4, 5 \rangle$

Since $\mathbf{u} \cdot \mathbf{v} = 0$, the vectors are orthogonal.

25. $\mathbf{u} = 5\left(\cos \frac{3\pi}{4}\mathbf{i} + \sin \frac{3\pi}{4}\mathbf{j}\right) = \frac{5\sqrt{2}}{2}[-\mathbf{i} + \mathbf{j}]$

$$\mathbf{v} = 2\left(\cos \frac{2\pi}{3}\mathbf{i} + \sin \frac{2\pi}{3}\mathbf{j}\right) = -\mathbf{i} + \sqrt{3}\mathbf{j}$$

$$\mathbf{u} \cdot \mathbf{v} = \frac{5\sqrt{2}}{2}(1 + \sqrt{3})$$

$$\|\mathbf{u}\| = 5$$

$$\|\mathbf{v}\| = 2$$

$$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{(5\sqrt{2}/2)(1 + \sqrt{3})}{5(2)} = \frac{\sqrt{2} + \sqrt{6}}{4}$$

$$\theta = \arccos \frac{\sqrt{2} + \sqrt{6}}{4} = 15^\circ$$

27. $\mathbf{u} = \langle 10, -5, 15 \rangle, \mathbf{v} = \langle -2, 1, -3 \rangle$

$\mathbf{u} = -5\mathbf{v} \Rightarrow \mathbf{u}$ is parallel to $\mathbf{v}$ and in the opposite direction.

$$\theta = \pi$$

29. There are many correct answers. For example: $\mathbf{v} = \pm\langle 6, -5, 0 \rangle$.

In Exercises 31–39, $\mathbf{u} = \langle 3, -2, 1 \rangle$, $\mathbf{v} = \langle 2, -4, -3 \rangle$, $\mathbf{w} = \langle -1, 2, 2 \rangle$.

$$\begin{aligned} 31. \mathbf{u} \cdot \mathbf{u} &= 3(3) + (-2)(-2) + (1)(1) \\ &= 14 = (\sqrt{14})^2 = \|\mathbf{u}\|^2 \end{aligned}$$

$$\begin{aligned} 33. \text{proj}_{\mathbf{u}} \mathbf{w} &= \left(\frac{\mathbf{u} \cdot \mathbf{w}}{\|\mathbf{u}\|^2} \right) \mathbf{u} \\ &= -\frac{5}{14} \langle 3, -2, 1 \rangle \\ &= \left\langle -\frac{15}{14}, \frac{10}{14}, -\frac{5}{14} \right\rangle \\ &= \left\langle -\frac{15}{14}, \frac{5}{7}, -\frac{5}{14} \right\rangle \end{aligned}$$

$$35. \mathbf{n} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ -1 & 2 & 2 \end{vmatrix} = -2\mathbf{i} - \mathbf{j}$$

$$\|\mathbf{n}\| = \sqrt{5}$$

$$\frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{1}{\sqrt{5}}(-2\mathbf{i} - \mathbf{j})$$

$$\begin{aligned} 37. V &= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \\ &= |\langle 3, -2, 1 \rangle \cdot \langle -2, -1, 0 \rangle| = |-4| = 4 \end{aligned}$$

$$\begin{aligned} 39. \text{Area parallelogram} &= \|\mathbf{u} \times \mathbf{v}\| = \sqrt{10^2 + 11^2 + (-8)^2} \quad (\text{See Exercises 36, 38}) \\ &= \sqrt{285} \end{aligned}$$

$$41. \mathbf{F} = c(\cos 20^\circ \mathbf{j} + \sin 20^\circ \mathbf{k})$$

$$\overrightarrow{PQ} = 2\mathbf{k}$$

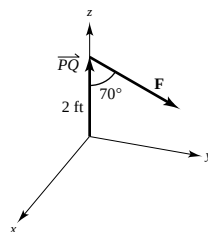
$$\overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 0 & c \cos 20^\circ & c \sin 20^\circ \end{vmatrix} = -2c \cos 20^\circ \mathbf{i}$$

$$200 = \|\overrightarrow{PQ} \times \mathbf{F}\| = 2c \cos 20^\circ$$

$$c = \frac{100}{\cos 20^\circ}$$

$$\mathbf{F} = \frac{100}{\cos 20^\circ}(\cos 20^\circ \mathbf{j} + \sin 20^\circ \mathbf{k}) = 100(\mathbf{j} + \tan 20^\circ \mathbf{k})$$

$$\|\mathbf{F}\| = 100\sqrt{1 + \tan^2 20^\circ} = 100 \sec 20^\circ \approx 106.4 \text{ lb}$$



$$43. \mathbf{v} = \mathbf{j}$$

$$(a) x = 1, y = 2 + t, z = 3$$

$$(b) \text{None}$$

$$45. 3x - 3y - 7z = -4, x - y + 2z = 3$$

Solving simultaneously, we have $z = 1$. Substituting $z = 1$ into the second equation we have $y = x - 1$. Substituting for x in this equation we obtain two points on the line of intersection, $(0, -1, 1)$, $(1, 0, 1)$. The direction vector of the line of intersection is $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

$$(a) x = t, y = -1 + t, z = 1$$

$$(b) x = y + 1, z = 1$$

47. The two lines are parallel as they have the same direction numbers, $-2, 1, 1$. Therefore, a vector parallel to the plane is $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$. A point on the first line is $(1, 0, -1)$ and a point on the second line is $(-1, 1, 2)$. The vector $\mathbf{u} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ connecting these two points is also parallel to the plane. Therefore, a normal to the plane is

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 2 & -1 & -3 \end{vmatrix} \\ = -2\mathbf{i} - 4\mathbf{j} = -2(\mathbf{i} + 2\mathbf{j}).$$

Equation of the plane: $(x - 1) + 2y = 0$

$$x + 2y = 1$$

51. $Q(3, -2, 4)$ point

$P(5, 0, 0)$ point on plane

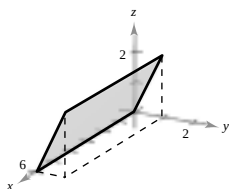
$\mathbf{n} = \langle 2, -5, 1 \rangle$ normal to plane

$$\overrightarrow{PQ} = \langle -2, -2, 4 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{10}{\sqrt{30}} = \frac{\sqrt{30}}{3}$$

55. $y = \frac{1}{2}z$

Plane with rulings parallel to the x -axis



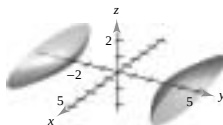
59. $\frac{x^2}{16} - \frac{y^2}{9} + z^2 = -1$

Hyperboloid of two sheets

$$xy\text{-trace: } \frac{y^2}{4} - \frac{x^2}{16} = 1$$

xz -trace: None

$$yz\text{-trace: } \frac{y^2}{9} - z^2 = 1$$



49. $Q = (1, 0, 2)$

$$2x - 3y + 6z = 6$$

A point P on the plane is $(3, 0, 0)$.

$$\overrightarrow{PQ} = \langle -2, 0, 2 \rangle$$

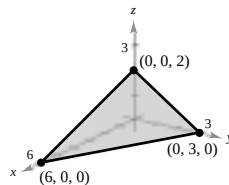
$$\mathbf{n} = \langle 2, -3, 6 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{8}{7}$$

53. $x + 2y + 3z = 6$

Plane

Intercepts: $(6, 0, 0)$, $(0, 3, 0)$, $(0, 0, 2)$



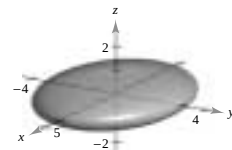
57. $\frac{x^2}{16} + \frac{y^2}{9} + z^2 = 1$

Ellipsoid

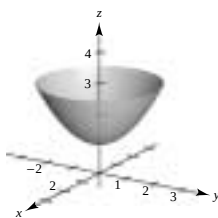
$$xy\text{-trace: } \frac{x^2}{16} + \frac{y^2}{9} = 1$$

$$xz\text{-trace: } \frac{x^2}{16} + z^2 = 1$$

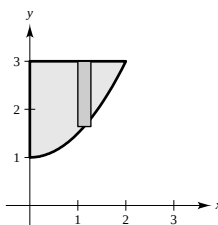
$$yz\text{-trace: } \frac{y^2}{9} + z^2 = 1$$



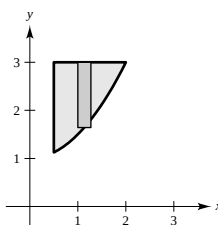
$$\begin{aligned}
 61. \quad (a) \quad x^2 + y^2 &= [r(z)]^2 \\
 &= [\sqrt{2(z-1)}]^2 \\
 x^2 + y^2 - 2z + 2 &= 0
 \end{aligned}$$



$$\begin{aligned}
 (b) \quad V &= 2\pi \int_0^2 x \left[3 - \left(\frac{1}{2}x^2 + 1 \right) \right] dx \\
 &= 2\pi \int_0^2 \left(2x - \frac{1}{2}x^3 \right) dx \\
 &= 2\pi \left[x^2 - \frac{x^4}{8} \right]_0^2 \\
 &= 4\pi \approx 12.6 \text{ cm}^3
 \end{aligned}$$



$$\begin{aligned}
 (c) \quad V &= 2\pi \int_{1/2}^2 x \left[3 - \left(\frac{1}{2}x^2 + 1 \right) \right] dx \\
 &= 2\pi \int_{1/2}^2 \left(2x - \frac{1}{2}x^3 \right) dx \\
 &= 2\pi \left[x^2 - \frac{x^4}{8} \right]_{1/2}^2 \\
 &= 4\pi - \frac{31\pi}{64} = \frac{225\pi}{64} \approx 11.04 \text{ cm}^3
 \end{aligned}$$



63. $(-2\sqrt{2}, 2\sqrt{2}, 2)$, rectangular

$$(a) \quad r = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2} = 4, \quad \theta = \arctan(-1) = \frac{3\pi}{4}, \quad z = 2, \quad \left(4, \frac{3\pi}{4}, 2 \right), \text{ cylindrical}$$

$$(b) \quad \rho = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2 + (2)^2} = 2\sqrt{5}, \quad \theta = \frac{3\pi}{4}, \quad \phi = \arccos \frac{2}{2\sqrt{5}} = \arccos \frac{1}{\sqrt{5}}, \quad \left(2\sqrt{5}, \frac{3\pi}{4}, \arccos \frac{\sqrt{5}}{5} \right), \text{ spherical}$$

65. $\left(100, -\frac{\pi}{6}, 50 \right)$, cylindrical

$$\rho = \sqrt{100^2 + 50^2} = 50\sqrt{5}$$

$$\theta = -\frac{\pi}{6}$$

$$\phi = \arccos \left(\frac{50}{50\sqrt{5}} \right) = \arccos \left(\frac{1}{\sqrt{5}} \right) \approx 63.4^\circ$$

$$\left(50\sqrt{5}, -\frac{\pi}{6}, 63.4^\circ \right), \text{ spherical}$$

67. $\left(25, -\frac{\pi}{4}, \frac{3\pi}{4} \right)$, spherical

$$r^2 = \left(25 \sin \left(\frac{3\pi}{4} \right) \right)^2 \Rightarrow r = 25 \frac{\sqrt{2}}{2}$$

$$\theta = -\frac{\pi}{4}$$

$$z = \rho \cos \phi = 25 \cos \frac{3\pi}{4} = -25 \frac{\sqrt{2}}{2}$$

$$\left(25 \frac{\sqrt{2}}{2}, -\frac{\pi}{4}, -\frac{25\sqrt{2}}{2} \right), \text{ cylindrical}$$

69. $x^2 - y^2 = 2z$

$$(a) \text{ Cylindrical: } r^2 \cos^2 \theta - r^2 \sin^2 \theta = 2z, \quad r^2 \cos 2\theta = 2z$$

$$(b) \text{ Spherical: } \rho^2 \sin^2 \phi \cos^2 \theta - \rho^2 \sin^2 \phi \sin^2 \theta = 2\rho \cos \phi, \quad \rho \sin^2 \phi \cos 2\theta - 2 \cos \phi = 0, \quad \rho = 2 \sec 2\theta \cos \phi \csc^2 \phi$$

C H A P T E R 10

Vectors and the Geometry of Space

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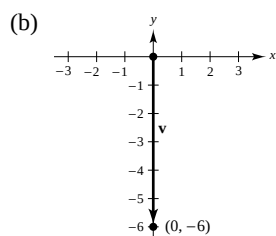
CHAPTER 10

Vectors and the Geometry of Space

Section 10.1 Vectors in the Plane

Solutions to Even-Numbered Exercises

2. (a) $\mathbf{v} = \langle 3 - 3, -2 - 4 \rangle = \langle 0, -6 \rangle$

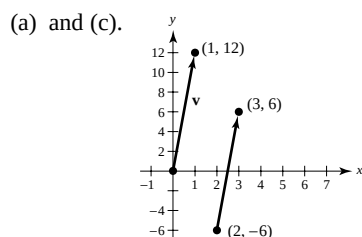


6. $\mathbf{u} = \langle 1 - (-4), 8 - 0 \rangle = \langle 5, 8 \rangle$

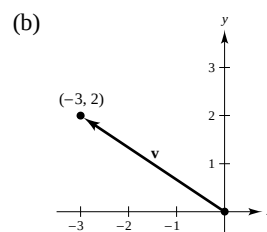
$\mathbf{v} = \langle 7 - 2, 7 - (-1) \rangle = \langle 5, 8 \rangle$

$\mathbf{u} = \mathbf{v}$

10. (b) $\mathbf{v} = \langle 3 - 2, 6 - (-6) \rangle = \langle 1, 12 \rangle$



4. (a) $\mathbf{v} = \langle -1 - 2, 3 - 1 \rangle = \langle -3, 2 \rangle$

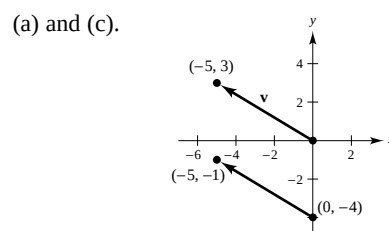


8. $\mathbf{u} = \langle 11 - (-4), -4 - (-1) \rangle = \langle 15, -3 \rangle$

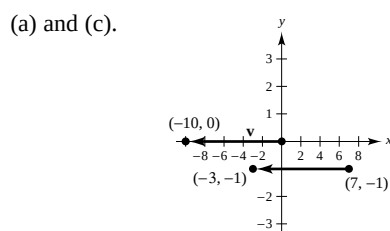
$\mathbf{v} = \langle 25 - 0, 10 - 13 \rangle = \langle 15, -3 \rangle$

$\mathbf{u} = \mathbf{v}$

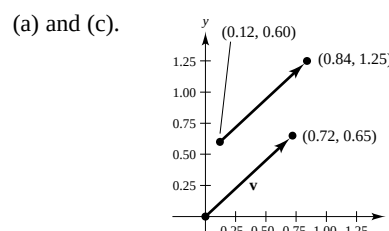
12. (b) $\mathbf{v} = \langle -5 - 0, -1 - (-4) \rangle = \langle -5, 3 \rangle$



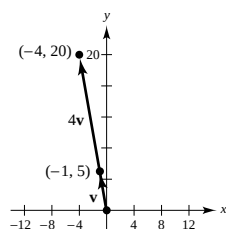
14. (b) $\mathbf{v} = \langle -3 - 7, -1 - (-1) \rangle = \langle -10, 0 \rangle$



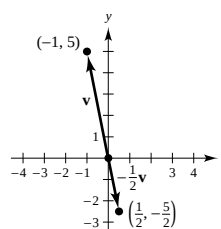
16. (b) $\mathbf{v} = \langle 0.84 - 0.12, 1.25 - 0.60 \rangle = \langle 0.72, 0.65 \rangle$



18. (a) $4\mathbf{v} = \langle -4, 20 \rangle$

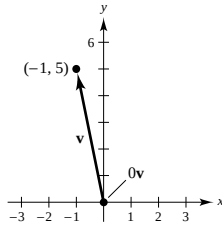


(b) $-\frac{1}{2}\mathbf{v} = \langle \frac{1}{2}, -\frac{5}{2} \rangle$

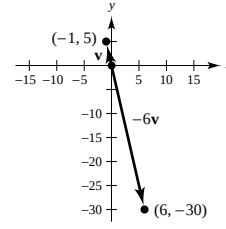
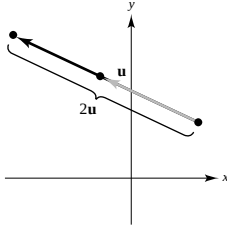


18. —CONTINUED—

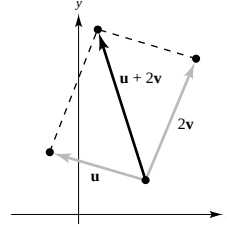
(c) $0\mathbf{v} = \langle 0, 0 \rangle$



(d) $-6\mathbf{v} = \langle 6, -30 \rangle$


 20. Twice as long as given vector $\mathbf{u}$.


22.

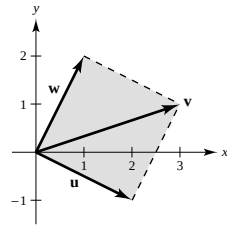


24. (a) $\frac{2}{3}\mathbf{u} = \frac{2}{3}\langle -3, -8 \rangle = \langle -2, -\frac{16}{3} \rangle$

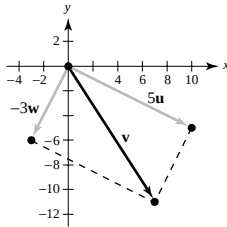
(b) $\mathbf{v} - \mathbf{u} = \langle 8, 25 \rangle - \langle -3, -8 \rangle = \langle 11, 33 \rangle$

(c) $2\mathbf{u} + 5\mathbf{v} = 2\langle -3, -8 \rangle + 5\langle 8, 25 \rangle = \langle 34, 109 \rangle$

26. $\mathbf{v} = (2\mathbf{i} - \mathbf{j}) + (\mathbf{i} + 2\mathbf{j})$
 $= 3\mathbf{i} + \mathbf{j} = \langle 3, 1 \rangle$



28. $\mathbf{v} = 5\mathbf{u} - 3\mathbf{w} = 5\langle 2, -1 \rangle - 3\langle 1, 2 \rangle = \langle 7, -11 \rangle$



30. $u_1 - 3 = 4$

$u_2 - 2 = -9$

$u_1 = 7$

$u_2 = -7$

$Q = (7, -7)$

32. $\|\mathbf{v}\| = \sqrt{144 + 25} = 13$

34. $\|\mathbf{v}\| = \sqrt{100 + 9} = \sqrt{109}$

36. $\|\mathbf{v}\| = \sqrt{1 + 1} = \sqrt{2}$

38. $\|\mathbf{u}\| = \sqrt{5^2 + 15^2} = \sqrt{250} = 5\sqrt{10}$

$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle 5, 15 \rangle}{5\sqrt{10}} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$ unit vector

40. $\|\mathbf{u}\| = \sqrt{(-6.2)^2 + (3.4)^2} = \sqrt{50} = 5\sqrt{2}$

$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle -6.2, 3.4 \rangle}{5\sqrt{2}} = \left\langle \frac{-1.24}{\sqrt{2}}, \frac{0.68}{\sqrt{2}} \right\rangle$ unit vector

42. $\mathbf{u} = \langle 0, 1 \rangle, \mathbf{v} = \langle 3, -3 \rangle$

(a) $\|\mathbf{u}\| = \sqrt{0+1} = 1$

(b) $\|\mathbf{v}\| = \sqrt{9+9} = 3\sqrt{2}$

(c) $\mathbf{u} + \mathbf{v} = \langle 3, -2 \rangle$

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{9+4} = \sqrt{13}$$

(d) $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \langle 0, 1 \rangle$

$$\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1$$

(e) $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{3\sqrt{2}} \langle 3, -3 \rangle$

$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$$

(f) $\frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \frac{1}{\sqrt{13}} \langle 3, -2 \rangle$

$$\left\| \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\| = 1$$

46. $\mathbf{u} = \langle -3, 2 \rangle$

$$\|\mathbf{u}\| = \sqrt{13} \approx 3.606$$

$$\mathbf{v} = \langle 1, -2 \rangle$$

$$\|\mathbf{v}\| = \sqrt{5} \approx 2.236$$

$$\mathbf{u} + \mathbf{v} = \langle -2, 0 \rangle$$

$$\|\mathbf{u} + \mathbf{v}\| = 2$$

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

50. $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{3} \langle 0, 3 \rangle$

$$3\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = \langle 0, 3 \rangle$$

$$\mathbf{v} = \langle 0, 3 \rangle$$

54. $\mathbf{v} = (\cos 3.5^\circ)\mathbf{i} + (\sin 3.5^\circ)\mathbf{j}$

$$\approx 0.9981\mathbf{i} + 0.0610\mathbf{j} = \langle 0.9981, 0.0610 \rangle$$

58. $\mathbf{u} = 5[\cos(-0.5)]\mathbf{i} + 5[\sin(-0.5)]\mathbf{j}$

$$= 5[\cos(0.5)]\mathbf{i} - 5[\sin(0.5)]\mathbf{j}$$

$$\mathbf{v} = 5[\cos(0.5)]\mathbf{i} + 5[\sin(0.5)]\mathbf{j}$$

$$\mathbf{u} + \mathbf{v} = 10[\cos(0.5)]\mathbf{i}$$

44. $\mathbf{u} = \langle 2, -4 \rangle, \mathbf{v} = \langle 5, 5 \rangle$

(a) $\|\mathbf{u}\| = \sqrt{4+16} = 2\sqrt{5}$

(b) $\|\mathbf{v}\| = \sqrt{25+25} = 5\sqrt{2}$

(c) $\mathbf{u} + \mathbf{v} = \langle 7, 1 \rangle$

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{49+1} = 5\sqrt{2}$$

(d) $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{2\sqrt{5}} \langle 2, -4 \rangle$

$$\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1$$

(e) $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{5\sqrt{2}} \langle 5, 5 \rangle$

$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$$

(f) $\frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \frac{1}{5\sqrt{2}} \langle 7, 1 \rangle$

$$\left\| \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\| = 1$$

48. $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{2}} \langle -1, 1 \rangle$

$$4\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = 2\sqrt{2} \langle -1, 1 \rangle$$

$$\mathbf{v} = \langle -2\sqrt{2}, 2\sqrt{2} \rangle$$

52. $\mathbf{v} = 5[(\cos 120^\circ)\mathbf{i} + (\sin 120^\circ)\mathbf{j}]$

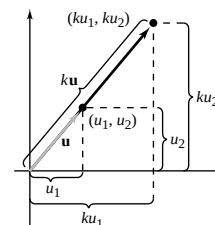
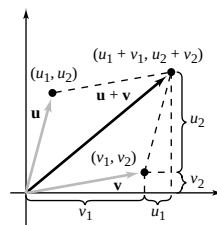
$$= -\frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j}$$

56. $\mathbf{u} = 4\mathbf{i}$

$$\mathbf{v} = \mathbf{i} + \sqrt{3}\mathbf{j}$$

$$\mathbf{u} + \mathbf{v} = 5\mathbf{i} + \sqrt{3}\mathbf{j}$$

60. See page 718:



62. See Theorem 10.1, page 719.

For Exercises 64–68, $\mathbf{a}u + \mathbf{b}w = a(\mathbf{i} + 2\mathbf{j}) + b(\mathbf{i} - \mathbf{j}) = (a + b)\mathbf{i} + (2a - b)\mathbf{j}$.

64. $\mathbf{v} = 3\mathbf{j}$. Therefore, $a + b = 0$, $2a - b = 3$. Solving simultaneously, we have $a = 1$, $b = -1$.

66. $\mathbf{v} = 3\mathbf{i} + 3\mathbf{j}$. Therefore, $a + b = 3$, $2a - b = 3$. Solving simultaneously, we have $a = 2$, $b = 1$.

68. $\mathbf{v} = -\mathbf{i} + 7\mathbf{j}$. Therefore, $a + b = -1$, $2a - b = 7$. Solving simultaneously, we have $a = 2$, $b = -3$.

70. $y = x^3$, $y' = 3x^2 = 12$ at $x = -2$.

(a) $m = 12$. Let $\mathbf{w} = \langle 1, 12 \rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{145}} \langle 1, 12 \rangle.$$

(b) $m = -\frac{1}{12}$. Let $\mathbf{w} = \langle 12, -1 \rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{145}} \langle 12, -1 \rangle.$$

72. $f(x) = \tan x$

$$f'(x) = \sec^2 x = 2 \text{ at } x = \frac{\pi}{4}.$$

(a) $m = 2$. Let $\mathbf{w} = \langle 1, 2 \rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{5}} \langle 1, 2 \rangle.$$

(b) $m = -\frac{1}{2}$. Let $\mathbf{w} = \langle -2, 1 \rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{5}} \langle -2, 1 \rangle.$$

74. $\mathbf{u} = 2\sqrt{3}\mathbf{i} + 2\mathbf{j}$

$$\mathbf{u} + \mathbf{v} = -3\mathbf{i} + 3\sqrt{3}\mathbf{j}$$

$$\mathbf{v} = (\mathbf{u} + \mathbf{v}) - \mathbf{u} = (-3 - 2\sqrt{3})\mathbf{i} + (3\sqrt{3} - 2)\mathbf{j}$$

76. magnitude ≈ 63.5

direction $\approx -8.26^\circ$

78. $\|\mathbf{F}_1\| = 2$, $\theta_{\mathbf{F}_1} = -10^\circ$

$$\|\mathbf{F}_2\| = 4$$
, $\theta_{\mathbf{F}_2} = 140^\circ$

$$\|\mathbf{F}_3\| = 3$$
, $\theta_{\mathbf{F}_3} = 200^\circ$

$$\|\mathbf{R}\| = \|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\| \approx 4.09$$

$$\theta_{\mathbf{R}} = \theta_{\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3} \approx 163.0^\circ$$

80. $\mathbf{F}_1 + \mathbf{F}_2 = (500 \cos 30^\circ \mathbf{i} + 500 \sin 30^\circ \mathbf{j}) + (200 \cos(-45^\circ) \mathbf{i} + 200 \sin(-45^\circ) \mathbf{j})$

$$= (250\sqrt{3} + 100\sqrt{2})\mathbf{i} + (250 - 100\sqrt{2})\mathbf{j}$$

$$\|\mathbf{F}_1 + \mathbf{F}_2\| = \sqrt{(250\sqrt{3} + 100\sqrt{2})^2 + (250 - 100\sqrt{2})^2} \approx 584.6 \text{ lb}$$

$$\tan \theta = \frac{250 - 100\sqrt{2}}{250\sqrt{3} + 100\sqrt{2}} \Rightarrow \theta \approx 10.7^\circ$$

82. $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = [400(\cos(-30^\circ)\mathbf{i} + \sin(-30^\circ)\mathbf{j})] + [280(\cos(45^\circ)\mathbf{i} + \sin(45^\circ)\mathbf{j})] + [350(\cos(135^\circ)\mathbf{i} + \sin(135^\circ)\mathbf{j})]$

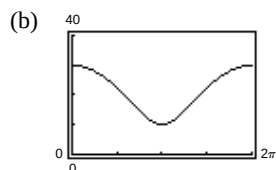
$$= [200\sqrt{3} + 140\sqrt{2} - 175\sqrt{2}]\mathbf{i} + [-200 + 140\sqrt{2} + 175\sqrt{2}]\mathbf{j}$$

$$\|\mathbf{R}\| = \sqrt{(200\sqrt{3} - 35\sqrt{2})^2 + (-200 + 315\sqrt{2})^2} \approx 385.2483 \text{ newtons}$$

$$\theta_{\mathbf{R}} = \arctan\left(\frac{-200 + 315\sqrt{2}}{200\sqrt{3} - 35\sqrt{2}}\right) \approx 0.6908 \approx 39.6^\circ$$

84. $\mathbf{F}_1 = \langle 20, 0 \rangle$, $\mathbf{F}_2 = 10\langle \cos \theta, \sin \theta \rangle$

$$\begin{aligned} \text{(a)} \quad \|\mathbf{F}_1 + \mathbf{F}_2\| &= \|\langle 20 + 10 \cos \theta, 10 \sin \theta \rangle\| \\ &= \sqrt{400 + 400 \cos \theta + 100 \cos^2 \theta + 100 \sin^2 \theta} \\ &= \sqrt{500 + 400 \cos \theta} \end{aligned}$$



(c) The range is $10 \leq \|\mathbf{F}_1 + \mathbf{F}_2\| \leq 30$.

The maximum is 30, which occur at $\theta = 0$ and $\theta = 2\pi$.

The minimum is 10 at $\theta = \pi$.

(d) The minimum of the resultant is 10.

86. $\mathbf{u} = \langle 7 - 1, 5 - 2 \rangle = \langle 6, 3 \rangle$

$$\frac{1}{3}\mathbf{u} = \langle 2, 1 \rangle$$

$$P_1 = (1, 2) + (2, 1) = (3, 3)$$

$$P_2 = (1, 2) + 2(2, 1) = (5, 4)$$

88. $\theta_1 = \arctan\left(\frac{24}{20}\right) \approx 0.8761$ or 50.2°

$$\theta_2 = \arctan\left(\frac{24}{-10}\right) + \pi \approx 1.9656 \text{ or } 112.6^\circ$$

$$\mathbf{u} = \|\mathbf{u}\|(\cos \theta_1 \mathbf{i} + \sin \theta_1 \mathbf{j})$$

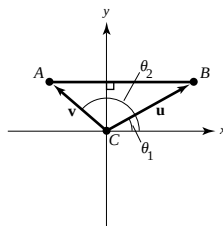
$$\mathbf{v} = \|\mathbf{v}\|(\cos \theta_2 \mathbf{i} + \sin \theta_2 \mathbf{j})$$

Vertical components: $\|\mathbf{u}\| \sin \theta_1 + \|\mathbf{v}\| \sin \theta_2 = 5000$

Horizontal components: $\|\mathbf{u}\| \cos \theta_1 + \|\mathbf{v}\| \cos \theta_2 = 0$

Solving this system, you obtain

$$\|\mathbf{u}\| \approx 2169.4 \text{ and } \|\mathbf{v}\| \approx 3611.2.$$



90. To lift the weight vertically, the sum of the vertical components of $\mathbf{u}$ and $\mathbf{v}$ must be 100 and the sum of the horizontal components must be 0.

$$\mathbf{u} = \|\mathbf{u}\|(\cos 60^\circ \mathbf{i} + \sin 60^\circ \mathbf{j})$$

$$\mathbf{v} = \|\mathbf{v}\|(\cos 110^\circ \mathbf{i} + \sin 110^\circ \mathbf{j})$$

Thus, $\|\mathbf{u}\| \sin 60^\circ + \|\mathbf{v}\| \sin 110^\circ = 100$, or

$$\|\mathbf{u}\|\left(\frac{\sqrt{3}}{2}\right) + \|\mathbf{v}\| \sin 110^\circ = 100.$$

And $\|\mathbf{u}\| \cos 60^\circ + \|\mathbf{v}\| \cos 110^\circ = 0$ or

$$\|\mathbf{u}\|\left(\frac{1}{2}\right) + \|\mathbf{v}\| \cos 110^\circ = 0$$

Multiplying the last equation by $(\sqrt{3})$ and adding to the first equation gives

$$\|\mathbf{u}\|(\sin 110^\circ - \sqrt{3} \cos 110^\circ) = 100 \Rightarrow \|\mathbf{v}\| \approx 65.27 \text{ lb.}$$

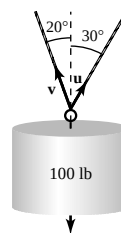
Then, $\|\mathbf{u}\|\left(\frac{1}{2}\right) + 65.27 \cos 110^\circ = 0$ gives

$$\|\mathbf{u}\| \approx 44.65 \text{ lb.}$$

(a) The tension in each rope: $\|\mathbf{u}\| = 44.65 \text{ lb}$, $\|\mathbf{v}\| = 65.27 \text{ lb}$.

(b) Vertical components: $\|\mathbf{u}\| \sin 60^\circ \approx 38.67 \text{ lb}$.

$$\|\mathbf{v}\| \sin 110^\circ \approx 61.33 \text{ lb.}$$



92. $\mathbf{u} = 400\mathbf{i}(\text{plane})$

$$\mathbf{v} = 50(\cos 135^\circ \mathbf{i} + \sin 135^\circ \mathbf{j}) = -25\sqrt{2}\mathbf{i} + 25\sqrt{2}\mathbf{j} \text{ (wind)}$$

$$\mathbf{u} + \mathbf{v} = (400 - 25\sqrt{2})\mathbf{i} + 25\sqrt{2}\mathbf{j} \approx 364.64\mathbf{i} + 35.36\mathbf{j}$$

$$\tan \theta = \frac{35.36}{364.64} \Rightarrow \theta \approx 5.54^\circ$$

Direction North of East: $\approx \text{N } 84.46^\circ \text{ E}$

Speed: $\approx 336.35 \text{ mph}$

94. $\|\mathbf{u}\| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1,$

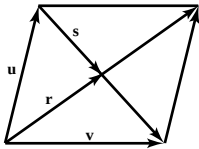
$$\|\mathbf{v}\| = \sqrt{\sin^2 \theta + \cos^2 \theta} = 1$$

96. Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors that determine the parallelogram, as indicated in the figure. The two diagonals are $\mathbf{u} + \mathbf{v}$ and $\mathbf{v} - \mathbf{u}$. Therefore, $\mathbf{r} = x(\mathbf{u} + \mathbf{v})$, $\mathbf{s} = y(\mathbf{v} - \mathbf{u})$. But,

$$\mathbf{u} = \mathbf{r} - \mathbf{s}$$

$$= x(\mathbf{u} + \mathbf{v}) - y(\mathbf{v} - \mathbf{u}) = (x + y)\mathbf{u} + (x - y)\mathbf{v}.$$

Therefore, $x + y = 1$ and $x - y = 0$. Solving we have $x = y = \frac{1}{2}$.



98. The set is a circle of radius 5, centered at the origin.

$$\|\mathbf{u}\| = \|\langle x, y \rangle\| = \sqrt{x^2 + y^2} = 5 \Rightarrow x^2 + y^2 = 25$$

100. True

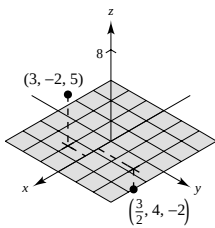
102. False

104. True

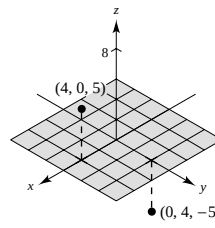
$$a = b = 0$$

Section 10.2 Space Coordinates and Vectors in Space

2.



4.



6. $A(2, -3, -1)$

$$B(-3, 1, 4)$$

8. $x = 7, y = -2, z = -1$:

$$(7, -2, -1)$$

10. $x = 0, y = 3, z = 2$: $(0, 3, 2)$

12. The x -coordinate is 0.

14. The point is 2 units in front of the xz -plane.

16. The point is on the plane $z = -3$.

18. The point is behind the yz -plane.

20. The point is in front of the plane $x = 4$.

22. The point (x, y, z) is 4 units above the xy -plane, and above either quadrant II or IV.

24. The point could be above the xy -plane, and thus above quadrants I or III, or below the xy -plane, and thus below quadrants II or IV.

$$\begin{aligned} 26. d &= \sqrt{(2 - (-2))^2 + (-5 - 3)^2 + (-2 - 2)^2} \\ &= \sqrt{16 + 64 + 16} = \sqrt{96} = 4\sqrt{6} \end{aligned}$$

$$\begin{aligned} 28. d &= \sqrt{(4 - 2)^2 + (-5 - 2)^2 + (6 - 3)^2} \\ &= \sqrt{4 + 49 + 9} = \sqrt{62} \end{aligned}$$

30. $A(5, 3, 4), B(7, 1, 3), C(3, 5, 3)$

$$|AB| = \sqrt{4 + 4 + 1} = 3$$

$$|AC| = \sqrt{4 + 4 + 1} = 3$$

$$|BC| = \sqrt{16 + 16 + 0} = 4\sqrt{2}$$

Since $|AB| = |AC|$, the triangle is isosceles.

32. $A(5, 0, 0), B(0, 2, 0), C(0, 0, -3)$

$$|AB| = \sqrt{25 + 4 + 0} = \sqrt{29}$$

$$|AC| = \sqrt{25 + 0 + 9} = \sqrt{34}$$

$$|BC| = \sqrt{0 + 4 + 9} = \sqrt{13}$$

Neither

34. The y -coordinate is changed by 3 units:

$$(5, 6, 4), (7, 4, 3), (3, 8, 3)$$

$$36. \left(\frac{4+8}{2}, \frac{0+8}{2}, \frac{-6+20}{2} \right) = (6, 4, 7)$$

38. Center: $(4, -1, 1)$

Radius: 5

$$(x - 4)^2 + (y + 1)^2 + (z - 1)^2 = 25$$

$$x^2 + y^2 + z^2 - 8x + 2y - 2z - 7 = 0$$

40. Center: $(-3, 2, 4)$

$$r = 3$$

(tangent to yz -plane)

$$(x + 3)^2 + (y - 2)^2 + (z - 4)^2 = 9$$

$$42. \quad x^2 + y^2 + z^2 + 9x - 2y + 10z + 19 = 0$$

$$\left(x^2 + 9x + \frac{81}{4} \right) + (y^2 - 2y + 1) + (z^2 + 10z + 25) = -19 + \frac{81}{4} + 1 + 25$$

$$\left(x + \frac{9}{2} \right)^2 + (y - 1)^2 + (z + 5)^2 = \frac{109}{4}$$

$$\text{Center: } \left(-\frac{9}{2}, 1, -5 \right)$$

$$\text{Radius: } \frac{\sqrt{109}}{2}$$

$$44. \quad 4x^2 + 4y^2 + 4z^2 - 4x - 32y + 8z + 33 = 0$$

$$x^2 + y^2 + z^2 - x - 8y + 2z + \frac{33}{4} = 0$$

$$\left(x^2 - x + \frac{1}{4} \right) + (y^2 - 8y + 16) + (z^2 + 2z + 1) = -\frac{33}{4} + \frac{1}{4} + 16 + 1$$

$$\left(x - \frac{1}{2} \right)^2 + (y - 4)^2 + (z + 1)^2 = 9$$

$$\text{Center: } \left(\frac{1}{2}, 4, -1 \right)$$

Radius: 3

46. $x^2 + y^2 + z^2 < 4x - 6y + 8z - 13$

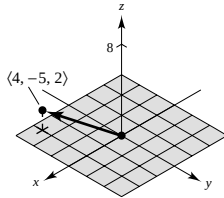
$$(x^2 - 4x + 4) + (y^2 + 6y + 9) + (z^2 - 8z + 16) < 4 + 9 + 16 - 13$$

$$(x - 2)^2 + (y + 3)^2 + (z - 4)^2 < 16$$

Interior of sphere of radius 4 centered at $(2, -3, 4)$.

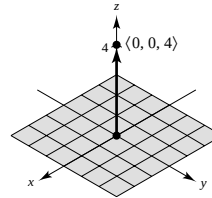
48. (a) $\mathbf{v} = (4 - 0)\mathbf{i} + (0 - 5)\mathbf{j} + (3 - 1)\mathbf{k}$
 $= 4\mathbf{i} - 5\mathbf{j} + 2\mathbf{k} = \langle 4, -5, 2 \rangle$

(b)



50. (a) $\mathbf{v} = (2 - 2)\mathbf{i} + (3 - 3)\mathbf{j} + (4 - 0)\mathbf{k}$
 $= 4\mathbf{k} = \langle 0, 0, 4 \rangle$

(b)



52. $\langle -1 - 4, 7 - (-5), -3 - 2 \rangle = \langle -5, 12, -5 \rangle$

$$\|\langle -5, 12, -5 \rangle\| = \sqrt{25 + 144 + 25} = \sqrt{194}$$

Unit vector: $\frac{\langle -5, 12, -5 \rangle}{\sqrt{194}} = \left\langle \frac{-5}{\sqrt{194}}, \frac{12}{\sqrt{194}}, \frac{-5}{\sqrt{194}} \right\rangle$

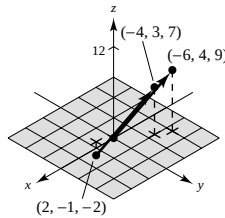
54. $\langle 2 - 1, 4 - (-2), -2 - 4 \rangle = \langle 1, 6, -6 \rangle$

$$\|\langle 1, 6, -6 \rangle\| = \sqrt{1 + 36 + 36} = \sqrt{73}$$

Unit vector: $\left\langle \frac{1}{\sqrt{73}}, \frac{6}{\sqrt{73}}, \frac{-6}{\sqrt{73}} \right\rangle$

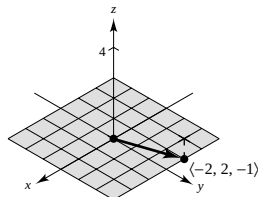
56. (b) $\mathbf{v} = (-4 - 2)\mathbf{i} + (3 + 1)\mathbf{j} + (7 + 2)\mathbf{k}$
 $= -6\mathbf{i} + 4\mathbf{j} + 9\mathbf{k} = \langle -6, 4, 9 \rangle$

(a) and (c).

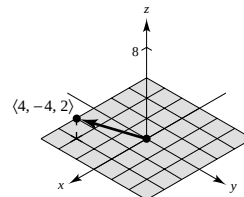


58. $(q_1, q_2, q_3) - (0, 2, \frac{5}{2}) = (1, -\frac{2}{3}, \frac{1}{2})$
 $Q = (1, -\frac{8}{3}, 3)$

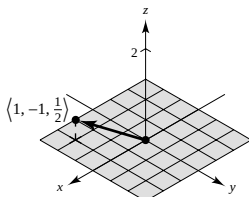
60. (a) $-\mathbf{v} = \langle -2, 2, -1 \rangle$



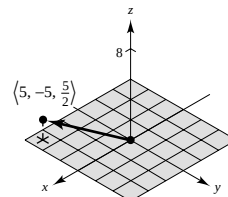
(b) $2\mathbf{v} = \langle 4, -4, 2 \rangle$



(c) $\frac{1}{2}\mathbf{v} = \langle 1, -1, \frac{1}{2} \rangle$



(d) $\frac{5}{2}\mathbf{v} = \langle 5, -5, \frac{5}{2} \rangle$



$$62. \mathbf{z} = \mathbf{u} - \mathbf{v} + 2\mathbf{w} = \langle 1, 2, 3 \rangle - \langle 2, 2, -1 \rangle + \langle 8, 0, -8 \rangle = \langle 7, 0, -4 \rangle$$

$$64. \mathbf{z} = 5\mathbf{u} - 3\mathbf{v} - \frac{1}{2}\mathbf{w} = \langle 5, 10, 15 \rangle - \langle 6, 6, -3 \rangle - \langle 2, 0, -2 \rangle = \langle -3, 4, 20 \rangle$$

$$66. 2\mathbf{u} + \mathbf{v} - \mathbf{w} + 3\mathbf{z} = 2\langle 1, 2, 3 \rangle + \langle 2, 2, -1 \rangle - \langle 4, 0, -4 \rangle + 3\langle z_1, z_2, z_3 \rangle = \langle 0, 0, 0 \rangle$$

$$\langle 0, 6, 9 \rangle + \langle 3z_1, 3z_2, 3z_3 \rangle = \langle 0, 0, 0 \rangle$$

$$0 + 3z_1 = 0 \Rightarrow z_1 = 0$$

$$6 + 3z_2 = 0 \Rightarrow z_2 = -2$$

$$9 + 3z_3 = 0 \Rightarrow z_3 = -3$$

$$\mathbf{z} = \langle 0, -2, -3 \rangle$$

$$68. (b) \text{ and } (d) \text{ are parallel since } -\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{3}{2}\mathbf{k} = -2\left(\frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{3}{4}\mathbf{k}\right) \text{ and } \frac{3}{4}\mathbf{i} - \mathbf{j} + \frac{9}{8}\mathbf{k} = \frac{3}{2}\left(\frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{3}{4}\mathbf{k}\right).$$

$$70. \mathbf{z} = \langle -7, -8, 3 \rangle$$

$$(b) \text{ is parallel since } (-z)\mathbf{z} = \langle 14, 16, -6 \rangle.$$

$$72. P(4, -2, 7), Q(-2, 0, 3), R(7, -3, 9)$$

$$\overrightarrow{PQ} = \langle -6, 2, -4 \rangle$$

$$\overrightarrow{PR} = \langle 3, -1, 2 \rangle$$

$$\langle 3, -1, 2 \rangle = -\frac{1}{2}\langle -6, 2, -4 \rangle$$

Therefore, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel.

The points are collinear.

$$74. P(0, 0, 0), Q(1, 3, -2), R(2, -6, 4)$$

$$\overrightarrow{PQ} = \langle 1, 3, -2 \rangle$$

$$\overrightarrow{PR} = \langle 2, -6, 4 \rangle$$

Since $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are not parallel, the points are not collinear.

$$76. A(1, 1, 3), B(9, -1, -2), C(11, 2, -9), D(3, 4, -4)$$

$$\overrightarrow{AB} = \langle 8, -2, -5 \rangle$$

$$\overrightarrow{DC} = \langle 8, -2, -5 \rangle$$

$$\overrightarrow{AD} = \langle 2, 3, -7 \rangle$$

$$\overrightarrow{BC} = \langle 2, 3, -7 \rangle$$

Since $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$, the given points form the vertices of a parallelogram.

$$78. \|\mathbf{v}\| = \sqrt{1 + 0 + 9} = \sqrt{10}$$

$$80. \mathbf{v} = \langle -4, 3, 7 \rangle$$

$$\|\mathbf{v}\| = \sqrt{16 + 9 + 49} = \sqrt{74}$$

$$82. \mathbf{v} = \langle 1, 3, -2 \rangle$$

$$\|\mathbf{v}\| = \sqrt{1 + 9 + 4} = \sqrt{14}$$

$$84. \mathbf{u} = \langle 6, 0, 8 \rangle$$

$$\|\mathbf{u}\| = \sqrt{36 + 0 + 64} = 10$$

$$(a) \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{10}\langle 6, 0, 8 \rangle$$

$$(b) -\frac{\mathbf{u}}{\|\mathbf{u}\|} = -\frac{1}{10}\langle 6, 0, 8 \rangle$$

$$86. \mathbf{u} = \langle 8, 0, 0 \rangle$$

$$\|\mathbf{u}\| = 8$$

$$(a) \frac{\mathbf{u}}{\|\mathbf{u}\|} = \langle 1, 0, 0 \rangle$$

$$(b) -\frac{\mathbf{u}}{\|\mathbf{u}\|} = \langle -1, 0, 0 \rangle$$

$$88. (a) \mathbf{u} + \mathbf{v} = \langle 4, 7.5, -2 \rangle$$

$$(b) \|\mathbf{u} + \mathbf{v}\| \approx 8.732$$

$$(c) \|\mathbf{u}\| \approx 5.099$$

$$(d) \|\mathbf{v}\| \approx 9.014$$

$$90. c\mathbf{u} = \langle c, 2c, 3c \rangle$$

$$\|c\mathbf{u}\| = \sqrt{c^2 + 4c^2 + 9c^2} = 3$$

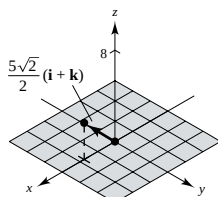
$$14c^2 = 9$$

$$c = \pm \frac{3\sqrt{14}}{14}$$

$$92. \mathbf{v} = 3 \frac{\mathbf{u}}{\|\mathbf{u}\|} = 3 \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle = \left\langle \frac{3}{\sqrt{3}}, \frac{3}{\sqrt{3}}, \frac{3}{\sqrt{3}} \right\rangle$$

$$94. \mathbf{v} = \sqrt{5} \frac{\mathbf{u}}{\|\mathbf{u}\|} = \sqrt{5} \left\langle \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right\rangle \\ = \left\langle \frac{-\sqrt{70}}{7}, \frac{3\sqrt{70}}{14}, \frac{\sqrt{70}}{14} \right\rangle$$

$$96. \mathbf{v} = 5(\cos 45^\circ \mathbf{i} + \sin 45^\circ \mathbf{k}) = \frac{5\sqrt{2}}{2}(\mathbf{i} + \mathbf{k}) \text{ or} \\ \mathbf{v} = 5(\cos 135^\circ \mathbf{i} + \sin 135^\circ \mathbf{k}) = \frac{5\sqrt{2}}{2}(-\mathbf{i} + \mathbf{k})$$



$$98. \mathbf{v} = \langle 5, 6, -3 \rangle \\ \frac{2}{3}\mathbf{v} = \left\langle \frac{10}{3}, 4, -2 \right\rangle \\ (1, 2, 5) + \left\langle \frac{10}{3}, 4, -2 \right\rangle = \left\langle \frac{13}{3}, 6, 3 \right\rangle$$

100. x_0 is directed distance to yz -plane.
 y_0 is directed distance to xz -plane.
 z_0 is directed distance to xy -plane.

$$102. (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

104. A sphere of radius 4 centered at (x_1, y_1, z_1) .
 $\|\mathbf{v}\| = \|\langle x - x_1, y - y_1, z - z_1 \rangle\| \\ = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} = 4 \\ (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = 16 \text{ sphere}$

106. As in Exercise 105(c), $x = a$ will be a vertical asymptote. Hence, $\lim_{r_0 \rightarrow a^-} T = \infty$.

$$108. 550 = \|c(75\mathbf{i} - 50\mathbf{j} - 100\mathbf{k})\| \\ 302,500 = 18,125c^2 \\ c^2 = 16.689655 \\ c \approx 4.085 \\ \mathbf{F} \approx 4.085(75\mathbf{i} - 50\mathbf{j} - 100\mathbf{k}) \\ \approx 306\mathbf{i} - 204\mathbf{j} - 409\mathbf{k}$$

110. Let A lie on the y -axis and the wall on the x -axis. Then
 $A = (0, 10, 0)$, $B = (8, 0, 6)$, $C = (-10, 0, 6)$ and

$$\overrightarrow{AB} = \langle 8, -10, 6 \rangle, \overrightarrow{AC} = \langle -10, -10, 6 \rangle. \\ \|\overrightarrow{AB}\| = 10\sqrt{2}, \|\overrightarrow{AC}\| = 2\sqrt{59}$$

$$\text{Thus, } \mathbf{F}_1 = 420 \frac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|}, \mathbf{F}_2 = 650 \frac{\overrightarrow{AC}}{\|\overrightarrow{AC}\|}$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 \approx \langle 237.6, -297.0, 178.2 \rangle \\ + \langle -423.1, -423.1, 253.9 \rangle \\ \approx \langle -185.5, -720.1, 432.1 \rangle$$

$$\|\mathbf{F}\| \approx 860.0 \text{ lb}$$

Section 10.3 The Dot Product of Two Vectors

2. $\mathbf{u} = \langle 4, 10 \rangle$, $\mathbf{v} = \langle -2, 3 \rangle$
 (a) $\mathbf{u} \cdot \mathbf{v} = 4(-2) + 10(3) = 22$
 (b) $\mathbf{u} \cdot \mathbf{u} = 4(4) + 10(10) = 116$
 (c) $\|\mathbf{u}\|^2 = 116$
 (d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 22\langle -2, 3 \rangle = \langle -44, 66 \rangle$
 (e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(22) = 44$

4. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{i}$
 (a) $\mathbf{u} \cdot \mathbf{v} = 1$
 (b) $\mathbf{u} \cdot \mathbf{u} = 1$
 (c) $\|\mathbf{u}\|^2 = 1$
 (d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = \mathbf{i}$
 (e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2$

6. $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

(a) $\mathbf{u} \cdot \mathbf{v} = 2(1) + 1(-3) + (-2)(2) = -5$

(b) $\mathbf{u} \cdot \mathbf{u} = 2(2) + 1(1) + (-2)(-2) = 9$

(c) $\|\mathbf{u}\|^2 = 9$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = -5(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) = -5\mathbf{i} + 15\mathbf{j} - 10\mathbf{k}$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(-5) = -10$

8. $\mathbf{u} = \langle 3240, 1450, 2235 \rangle$

$\mathbf{v} = \langle 2.22, 1.85, 3.25 \rangle$

Increase prices by 4%: $1.04\langle 2.22, 1.85, 3.25 \rangle$.

New total amount: $1.04(\mathbf{u} \cdot \mathbf{v}) = 1.04(17,139.05)$
 $= \$17,824.61$

10. $\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \cos \theta$

$\mathbf{u} \cdot \mathbf{v} = (40)(25) \cos \frac{5\pi}{6} = -500\sqrt{3}$

12. $\mathbf{u} = \langle 3, 1 \rangle$, $\mathbf{v} = \langle 2, -1 \rangle$

$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{5}{\sqrt{10}\sqrt{5}} = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}$

14. $\mathbf{u} = \cos\left(\frac{\pi}{6}\right)\mathbf{i} + \sin\left(\frac{\pi}{6}\right)\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$

$\mathbf{v} = \cos\left(\frac{3\pi}{4}\right)\mathbf{i} + \sin\left(\frac{3\pi}{4}\right)\mathbf{j} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$

$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$

$= \frac{\sqrt{3}}{2} \left(-\frac{\sqrt{2}}{2} \right) + \frac{1}{2} \left(\frac{\sqrt{2}}{2} \right) = \frac{\sqrt{2}}{4} (1 - \sqrt{3})$

$\theta = \arccos \left[\frac{\sqrt{2}}{4} (1 - \sqrt{3}) \right] = 105^\circ$

16. $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$

$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{3(2) + 2(-3) + 0}{\|\mathbf{u}\| \|\mathbf{v}\|} = 0$

$\theta = \frac{\pi}{2}$

18. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$

$= \frac{9}{\sqrt{14}\sqrt{6}} = \frac{9}{2\sqrt{21}} = \frac{3\sqrt{21}}{14}$

$\theta = \arccos \left(\frac{3\sqrt{21}}{14} \right) \approx 10.9^\circ$

20. $\mathbf{u} = \langle 2, 18 \rangle$, $\mathbf{v} = \left\langle \frac{3}{2}, -\frac{1}{6} \right\rangle$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$ not parallel

$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow$ orthogonal

22. $\mathbf{u} = -\frac{1}{3}(\mathbf{i} - 2\mathbf{j})$, $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$

$\mathbf{u} = -\frac{1}{6}\mathbf{v} \Rightarrow$ parallel

24. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$ not parallel

$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow$ orthogonal

26. $\mathbf{u} = \langle \cos \theta, \sin \theta, -1 \rangle$,

$\mathbf{v} = \langle \sin \theta, -\cos \theta, 0 \rangle$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$ not parallel

$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow$ orthogonal

28. $\mathbf{u} = \langle 5, 3, -1 \rangle$ $\|\mathbf{u}\| = \sqrt{35}$

$\cos \alpha = \frac{5}{\sqrt{35}}$

$\cos \beta = \frac{3}{\sqrt{35}}$

$\cos \gamma = \frac{-1}{\sqrt{35}}$

$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{25}{35} + \frac{9}{35} + \frac{1}{35} = 1$

$$30. \mathbf{u} = \langle a, b, c \rangle, \|\mathbf{u}\| = \sqrt{a^2 + b^2 + c^2}$$

$$\cos \alpha = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$\cos \beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$\cos \gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{a^2}{a^2 + b^2 + c^2} + \frac{b^2}{a^2 + b^2 + c^2} + \frac{c^2}{a^2 + b^2 + c^2} = 1$$

$$32. \mathbf{u} = \langle -4, 3, 5 \rangle \quad \|\mathbf{u}\| = \sqrt{50} = 5\sqrt{2}$$

$$\cos \alpha = \frac{-4}{5\sqrt{2}} \Rightarrow \alpha \approx 2.1721 \text{ or } 124.4^\circ$$

$$\cos \beta = \frac{3}{5\sqrt{2}} \Rightarrow \beta \approx 1.1326 \text{ or } 64.9^\circ$$

$$\cos \gamma = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \gamma \approx \frac{\pi}{4} \text{ or } 45^\circ$$

$$34. \mathbf{u} = \langle -2, 6, 1 \rangle \quad \|\mathbf{u}\| = \sqrt{41}$$

$$\cos \alpha = \frac{-2}{\sqrt{41}} \Rightarrow \alpha \approx 1.8885 \text{ or } 108.2^\circ$$

$$\cos \beta = \frac{6}{\sqrt{41}} \Rightarrow \alpha \approx 0.3567 \text{ or } 20.4^\circ$$

$$\cos \gamma = \frac{1}{\sqrt{41}} \Rightarrow \alpha \approx 1.4140 \text{ or } 81.0^\circ$$

$$36. \mathbf{F}_1: C_1 = \frac{300}{\|\mathbf{F}_1\|} \approx 13.0931$$

$$\mathbf{F}_2: C_2 = \frac{100}{\|\mathbf{F}_2\|} \approx 6.3246$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$$

$$\approx 13.0931\langle -20, -10, 5 \rangle + 6.3246\langle 5, 15, 0 \rangle$$

$$= \langle -230.239, -36.062, 65.4655 \rangle$$

$$\|\mathbf{F}\| \approx 242.067 \text{ lb}$$

$$\cos \alpha \approx \frac{-230.239}{\|\mathbf{F}\|} \Rightarrow \alpha \approx 162.02^\circ$$

$$\cos \beta \approx \frac{-36.062}{\|\mathbf{F}\|} \Rightarrow \beta \approx 98.57^\circ$$

$$\cos \gamma \approx \frac{65.4655}{\|\mathbf{F}\|} \Rightarrow \gamma \approx 74.31^\circ$$

$$38. \mathbf{v}_1 = \langle s, s, s \rangle$$

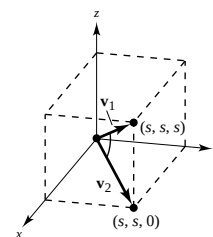
$$\|\mathbf{v}_1\| = s\sqrt{3}$$

$$\mathbf{v}_2 = \langle s, s, 0 \rangle$$

$$\|\mathbf{v}_2\| = s\sqrt{2}$$

$$\cos \theta = \frac{s\sqrt{2}}{s\sqrt{3}} = \frac{\sqrt{6}}{3}$$

$$\theta = \arccos \frac{\sqrt{6}}{3} \approx 35.26^\circ$$



$$40. \mathbf{F}_1 = C_1\langle 0, 10, 10 \rangle, \|\mathbf{F}_1\| = 200 = C_1 10\sqrt{2} \Rightarrow C_1 = 10\sqrt{2}$$

$$\text{and } \mathbf{F}_1 = \langle 0, 100\sqrt{2}, 100\sqrt{2} \rangle$$

$$\mathbf{F}_2 = C_2\langle -4, -6, 10 \rangle$$

$$\mathbf{F}_2 = C_3\langle 4, -6, 10 \rangle$$

$$\mathbf{F} = \langle 0, 0, w \rangle$$

$$\mathbf{F} + \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}$$

$$-4C_2 + 4C_3 = 0 \Rightarrow C_2 = C_3$$

$$100\sqrt{2} - 6C_2 - 6C_3 = 0 \Rightarrow C_2 = C_3 = \frac{25\sqrt{2}}{3} N$$

$$42. \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 9, 7 \rangle - \langle 3, 9 \rangle = \langle 6, -2 \rangle$$

$$44. \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 8, 2, 0 \rangle - \langle 6, 3, -3 \rangle = \langle 2, -1, 3 \rangle$$

46. $\mathbf{u} = \langle 2, -3 \rangle, \mathbf{v} = \langle 3, 2 \rangle$

(a) $\mathbf{w}_1 = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = 0\mathbf{v} = \langle 0, 0 \rangle$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 2, -3 \rangle$

48. $\mathbf{u} = \langle 1, 0, 4 \rangle, \mathbf{v} = \langle 3, 0, 2 \rangle$

(a) $\mathbf{w}_1 = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \frac{11}{13} \langle 3, 0, 2 \rangle = \left\langle \frac{33}{13}, 0, \frac{22}{13} \right\rangle$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 1, 0, 4 \rangle - \left\langle \frac{33}{13}, 0, \frac{22}{13} \right\rangle$
 $= \left\langle -\frac{20}{13}, 0, \frac{30}{13} \right\rangle$

50. The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.The angle θ between $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}.$$

54. See figure 10.29, page 739.

56. Yes, $\left\| \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} \right\| = \left\| \frac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{u}\|^2} \mathbf{u} \right\|$

$$|\mathbf{u} \cdot \mathbf{v}| \frac{\|\mathbf{v}\|}{\|\mathbf{v}\|^2} = |\mathbf{v} \cdot \mathbf{u}| \frac{\|\mathbf{u}\|}{\|\mathbf{u}\|^2}$$

$$\frac{1}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{u}\|}$$

$$\|\mathbf{u}\| = \|\mathbf{v}\|$$

58. (a) $\|\mathbf{u}\| = 5, \|\mathbf{v}\| \approx 8.602, \theta \approx 91.33^\circ$

(b) $\|\mathbf{u}\| \approx 9.165, \|\mathbf{v}\| \approx 5.745, \theta = 90^\circ$

60. (a) $\left\langle \frac{64}{17}, \frac{16}{17} \right\rangle$

(b) $\left\langle -\frac{21}{26}, \frac{63}{26}, \frac{42}{13} \right\rangle$

62. Because $\mathbf{u}$ appears to be a multiple of $\mathbf{v}$, the projection of $\mathbf{u}$ onto $\mathbf{v}$ is $\mathbf{u}$. Analytically,

$$\begin{aligned} \text{proj}_{\mathbf{v}} \mathbf{u} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} = \frac{\langle -3, -2 \rangle \cdot \langle 6, 4 \rangle}{\langle 6, 4 \rangle \cdot \langle 6, 4 \rangle} \langle 6, 4 \rangle \\ &= \frac{-26}{52} \langle 6, 4 \rangle = \langle -3, -2 \rangle = \mathbf{u}. \end{aligned}$$

64. $\mathbf{u} = -8\mathbf{i} + 3\mathbf{j}$. Want $\mathbf{u} \cdot \mathbf{v} = 0$.

$\mathbf{v} = 3\mathbf{i} + 8\mathbf{j}$ and $-\mathbf{v} = -3\mathbf{i} - 8\mathbf{j}$ are orthogonal to $\mathbf{u}$.

66. $\mathbf{u} = \langle 0, -3, 6 \rangle$. Want $\mathbf{u} \cdot \mathbf{v} = 0$.

$\mathbf{v} = \langle 0, 6, 3 \rangle$ and $-\mathbf{v} = \langle 0, -6, -3 \rangle$ are orthogonal to $\mathbf{u}$.

68. $\overrightarrow{OA} = \langle 10, 5, 20 \rangle, \mathbf{v} = \langle 0, 0, 1 \rangle$

$$\text{proj}_{\mathbf{v}} \overrightarrow{OA} = \frac{20}{1^2} \langle 0, 0, 1 \rangle = \langle 0, 0, 20 \rangle$$

$$\|\text{proj}_{\mathbf{v}} \overrightarrow{OA}\| = 20$$

70. $\mathbf{F} = 25(\cos 20^\circ \mathbf{i} + \sin 20^\circ \mathbf{j})$

$$\mathbf{v} = 50\mathbf{i}$$

$$W = \mathbf{F} \cdot \mathbf{v} = 1250 \cos 20^\circ \approx 1174.6 \text{ ft} \cdot \text{lb}$$

72. $\overrightarrow{PQ} = \langle -4, 2, 10 \rangle$

$$\vec{V} = \langle -2, 3, 6 \rangle$$

$$W = \overrightarrow{PQ} \cdot \vec{V} = 74$$

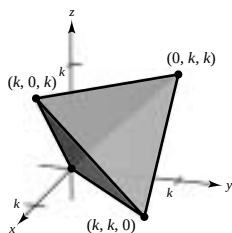
74. True

$$\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v}$$

$$= 0 + 0 = 0 \Rightarrow \mathbf{w}$$

and $\mathbf{u} + \mathbf{v}$ are orthogonal.

76. (a)



(b) Length of each edge:

$$\sqrt{k^2 + k^2 + 0^2} = k\sqrt{2}$$

$$(c) \cos \theta = \frac{k^2}{(k\sqrt{2})(k\sqrt{2})} = \frac{1}{2}$$

$$\theta = \arccos\left(\frac{1}{2}\right) = 60^\circ$$

$$(d) \vec{r}_1 = \langle k, k, 0 \rangle - \left\langle \frac{k}{2}, \frac{k}{2}, \frac{k}{2} \right\rangle = \left\langle \frac{k}{2}, \frac{k}{2}, -\frac{k}{2} \right\rangle$$

$$\vec{r}_2 = \langle 0, 0, 0 \rangle - \left\langle \frac{k}{2}, \frac{k}{2}, \frac{k}{2} \right\rangle = \left\langle -\frac{k}{2}, -\frac{k}{2}, -\frac{k}{2} \right\rangle$$

$$\cos \theta = \frac{\frac{k^2}{4}}{\left(\frac{k}{2}\right)^2 \cdot 3} = -\frac{1}{3}$$

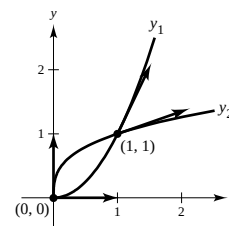
$$\theta = 109.5^\circ$$

 78. The curves $y_1 = x^2$ and $y_2 = x^{1/3}$ intersect at $(0, 0)$ and at $(1, 1)$.

At $(0, 0)$: $\langle 1, 0 \rangle$ is tangent to y_1 and $\langle 0, 1 \rangle$ is tangent to y_2 . The angle between these vectors is 90° .

At $(1, 1)$: $\langle 1/\sqrt{5}, 1/2 \rangle$ is tangent to y_1 and $\langle 3/\sqrt{10}, 1/3 \rangle = \langle 1/\sqrt{10}, 3/1 \rangle$ is tangent to y_2 . To find the angle between these vectors,

$$\cos \theta = \frac{1}{\sqrt{5}} \frac{1}{\sqrt{10}} (3 + 2) = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45^\circ.$$



$$80. \mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

$$|\mathbf{u} \cdot \mathbf{v}| = \|\mathbf{u}\| \|\mathbf{v}\| |\cos \theta|$$

$$= \|\mathbf{u}\| \|\mathbf{v}\| |\cos \theta|$$

$$\leq \|\mathbf{u}\| \|\mathbf{v}\| \text{ since } |\cos \theta| \leq 1.$$

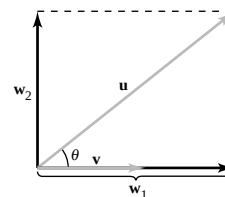
 82. Let $\mathbf{w}_1 = \text{proj}_{\mathbf{v}} \mathbf{u}$, as indicated in the figure. Because $\mathbf{w}_1$ is a scalar multiple of $\mathbf{v}$, you can write

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 = c\mathbf{v} + \mathbf{w}_2.$$

Taking the dot product of both sides with $\mathbf{v}$ produces

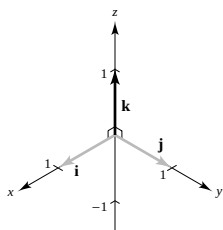
$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (c\mathbf{v} + \mathbf{w}_2) \cdot \mathbf{v} = c\mathbf{v} \cdot \mathbf{v} + \mathbf{w}_2 \cdot \mathbf{v} \\ &= c\|\mathbf{v}\|^2, \text{ since } \mathbf{w}_2 \text{ and } \mathbf{v} \text{ are orthogonal.} \end{aligned}$$

$$\text{Thus, } \mathbf{u} \cdot \mathbf{v} = c\|\mathbf{v}\|^2 \Rightarrow c = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \text{ and } \mathbf{w}_1 = \text{proj}_{\mathbf{v}} \mathbf{u} = c\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}.$$

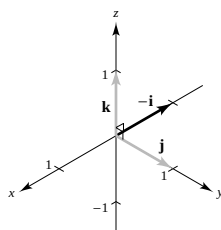


Section 10.4 The Cross Product of Two Vectors in Space

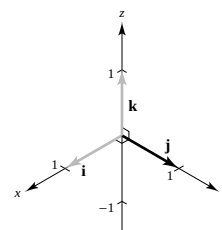
$$2. \mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k}$$



$$4. \mathbf{k} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i}$$



$$6. \mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j}$$



$$8. (a) \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 5 \\ 2 & 3 & -2 \end{vmatrix} = \langle -15, 16, 9 \rangle$$

$$(b) \mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = \langle 15, -16, -9 \rangle$$

$$(c) \mathbf{v} \times \mathbf{v} = \mathbf{0}$$

$$12. \mathbf{u} = \langle -1, 1, 2 \rangle, \mathbf{v} = \langle 0, 1, 0 \rangle$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\mathbf{i} - \mathbf{k} = \langle -2, 0, -1 \rangle$$

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= (-1)(-2) + (1)(0) + (2)(-1) \\ &= 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v} \end{aligned}$$

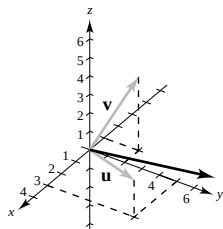
$$\begin{aligned} \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) &= (0)(-2) + (1)(0) + (0)(-1) \\ &= 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v} \end{aligned}$$

$$16. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 6 & 0 \\ -2 & 1 & 1 \end{vmatrix} = 6\mathbf{i} - \mathbf{j} + 13\mathbf{k}$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 1(6) + 6(-1) = 0 \Rightarrow \mathbf{u} \perp (\mathbf{u} \times \mathbf{v})$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = -2(6) + 1(-1) + 1(13) = 0 \Rightarrow \mathbf{v} \perp (\mathbf{u} \times \mathbf{v})$$

18.



$$10. (a) \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & -2 \\ 1 & 5 & 1 \end{vmatrix} = \langle 8, -5, 17 \rangle$$

$$(b) \mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = \langle -8, 5, -17 \rangle$$

$$(c) \mathbf{v} \times \mathbf{v} = \mathbf{0}$$

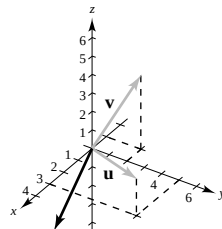
$$14. \mathbf{u} = \langle -10, 0, 6 \rangle, \mathbf{v} = \langle 7, 0, 0 \rangle$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -10 & 0 & 6 \\ 7 & 0 & 0 \end{vmatrix} = 42\mathbf{j} = \langle 0, 42, 0 \rangle$$

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= (-10)(0) + (0)(42) + 6(0) \\ &= 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v} \end{aligned}$$

$$\begin{aligned} \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) &= 7(0) + (0)(42) + (0)(0) \\ &= 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v} \end{aligned}$$

20.



$$22. \mathbf{u} = \langle -8, -6, 4 \rangle$$

$$\mathbf{v} = \langle 10, -12, -2 \rangle$$

$$\mathbf{u} \times \mathbf{v} = \langle 60, 24, 156 \rangle$$

$$\begin{aligned} \frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} &= \frac{1}{36\sqrt{22}} \langle 60, 24, 156 \rangle \\ &= \left\langle \frac{5}{3\sqrt{22}}, \frac{2}{3\sqrt{22}}, \frac{13}{3\sqrt{22}} \right\rangle \end{aligned}$$

$$26. (a) \mathbf{u} \times \mathbf{v} = \langle -18, -12, 48 \rangle$$

$$\|\mathbf{u} \times \mathbf{v}\| \approx 52.650$$

$$(b) \mathbf{u} \times \mathbf{v} = \langle -50, 40, -34 \rangle$$

$$\|\mathbf{u} \times \mathbf{v}\| \approx 72.498$$

$$24. \mathbf{u} = \frac{2}{3}\mathbf{k}$$

$$\mathbf{v} = \frac{1}{2}\mathbf{i} + 6\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \left\langle 0, \frac{1}{3}, 0 \right\rangle$$

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \langle 0, 1, 0 \rangle$$

$$28. \mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\mathbf{v} = \mathbf{j} + \mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -\mathbf{j} + \mathbf{k}$$

$$A = \|\mathbf{u} \times \mathbf{v}\| = \|-\mathbf{j} + \mathbf{k}\| = \sqrt{2}$$

$$30. \mathbf{u} = \langle 2, -1, 0 \rangle$$

$$\mathbf{v} = \langle -1, 2, 0 \rangle$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ -1 & 2 & 0 \end{vmatrix} = \langle 0, 0, 3 \rangle$$

$$A = \|\mathbf{u} \times \mathbf{v}\| = \|\langle 0, 0, 3 \rangle\| = 3$$

$$32. A(2, -3, 1), B(6, 5, -1), C(3, -6, 4), D(7, 2, 2)$$

$$\overrightarrow{AB} = \langle 4, 8, -2 \rangle, \overrightarrow{AC} = \langle 1, -3, 3 \rangle, \overrightarrow{CD} = \langle 4, 8, -2 \rangle, \overrightarrow{BD} = \langle 1, -3, 3 \rangle$$

Since $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$, the figure is a parallelogram.

$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are adjacent sides and

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 8 & -2 \\ 1 & -3 & 3 \end{vmatrix} = \langle 18, -14, -20 \rangle.$$

$$\text{Area} = \|\overrightarrow{AB} \times \overrightarrow{AC}\| = \sqrt{920} = 2\sqrt{230}$$

$$34. A(2, -3, 4), B(0, 1, 2), C(-1, 2, 0)$$

$$\overrightarrow{AB} = \langle -2, 4, -2 \rangle, \overrightarrow{AC} = \langle -3, 5, -4 \rangle$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & -2 \\ -3 & 5 & -4 \end{vmatrix} = -6\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

$$A = \frac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2} \sqrt{44} = \sqrt{11}$$

$$38. \mathbf{F} = -2000(\cos 30^\circ \mathbf{j} + \sin 30^\circ \mathbf{k}) = -1000\sqrt{3}\mathbf{j} - 1000\mathbf{k}$$

$$\overrightarrow{PQ} = 0.16\mathbf{k}$$

$$\overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.16 \\ 0 & -1000\sqrt{3} & -1000 \end{vmatrix} = 160\sqrt{3}\mathbf{i}$$

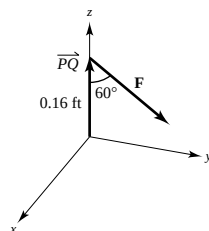
$$\|\overrightarrow{PQ} \times \mathbf{F}\| = 160\sqrt{3} \text{ ft} \cdot \text{lb}$$

$$36. A(1, 2, 0), B(-2, 1, 0), C(0, 0, 0)$$

$$\overrightarrow{AB} = \langle -3, -1, 0 \rangle, \overrightarrow{AC} = \langle -1, -2, 0 \rangle$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -1 & 0 \\ -1 & -2 & 0 \end{vmatrix} = 5\mathbf{k}$$

$$A = \frac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{5}{2}$$



$$40. (a) B \text{ is } -\frac{15}{12} = -\frac{5}{4} \text{ to the left of } A, \text{ and one foot upwards:}$$

$$\overrightarrow{AB} = -\frac{5}{4}\mathbf{j} + \mathbf{k}$$

$$\mathbf{F} = -200(\cos \theta \mathbf{j} + \sin \theta \mathbf{k})$$

$$(b) \overrightarrow{AB} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -5/4 & 1 \\ 0 & -200 \cos \theta & -200 \sin \theta \end{vmatrix}$$

$$= (250 \sin \theta + 200 \cos \theta) \mathbf{i}$$

$$\|\overrightarrow{AB} \times \mathbf{F}\| = |250 \sin \theta + 200 \cos \theta|$$

$$= 25(10 \sin \theta + 8 \cos \theta)$$

$$(c) \text{ For } \theta = 30^\circ,$$

$$\begin{aligned} \|\overrightarrow{AB} \times \mathbf{F}\| &= 25 \left(10 \left(\frac{1}{2} \right) + 8 \left(\frac{\sqrt{3}}{2} \right) \right) \\ &= 25(5 + 4\sqrt{3}) \approx 298.2. \end{aligned}$$

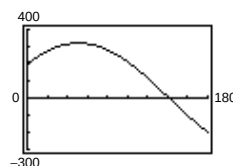
$$(d) \text{ If } T = \|\overrightarrow{AB} \times \mathbf{F}\|,$$

$$\frac{dT}{d\theta} = 25(10 \cos \theta - 8 \sin \theta) = 0 \Rightarrow \tan \theta = \frac{5}{4}$$

$$\Rightarrow \theta \approx 51.34^\circ.$$

The vectors are orthogonal.

$$(e) \text{ The zero is } \theta \approx 141.34^\circ, \text{ the angle making } \overrightarrow{AB} \text{ parallel to } \mathbf{F}.$$



$$42. \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$$

$$44. \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{vmatrix} = 0$$

$$46. \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 3 & 1 \\ 0 & 6 & 6 \\ -4 & 0 & -4 \end{vmatrix} = -72$$

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 72$$

$$48. \mathbf{u} = \langle 1, 1, 0 \rangle$$

$$\mathbf{v} = \langle 1, 0, 2 \rangle$$

$$\mathbf{w} = \langle 0, 1, 1 \rangle$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -3$$

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 3$$

50. See Theorem 10.8, page 746.

52. Form the vectors for two sides of the triangle, and compute their cross product:

$$\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \times \langle x_3 - x_1, y_3 - y_1, z_3 - z_1 \rangle$$

54. False, let $\mathbf{u} = \langle 1, 0, 0 \rangle$, $\mathbf{v} = \langle 1, 0, 0 \rangle$, $\mathbf{w} = \langle -1, 0, 0 \rangle$.

Then,

$$\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w} = \mathbf{0}, \text{ but } \mathbf{v} \neq \mathbf{w}.$$

$$56. \mathbf{u} = \langle u_1, u_2, u_3 \rangle, \mathbf{v} = \langle v_1, v_2, v_3 \rangle, \mathbf{w} = \langle w_1, w_2, w_3 \rangle$$

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$$

$$\mathbf{v} \times \mathbf{w} = (v_2w_3 - v_3w_2)\mathbf{i} - (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

58. $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, c is a scalar.

$$\begin{aligned} (c\mathbf{u}) \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ cu_1 & cu_2 & cu_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (cu_2v_3 - cu_3v_2)\mathbf{i} - (cu_1v_3 - cu_3v_1)\mathbf{j} + (cu_1v_2 - cu_2v_1)\mathbf{k} \\ &= c[(u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}] = c(\mathbf{u} \times \mathbf{v}) \end{aligned}$$

$$60. \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= w_1(u_2v_3 - v_2u_3) - w_2(u_1v_3 - v_1u_3) + w_3(u_1v_2 - v_1u_2) \\ &= u_1(v_2w_3 - w_2v_3) - u_2(v_1w_3 - w_1v_3) + u_3(v_1w_2 - w_1v_2) \\ &= \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \end{aligned}$$

62. If $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other, $\mathbf{u} = c\mathbf{v}$ for some scalar c .

$$\mathbf{u} \times \mathbf{v} = (c\mathbf{v}) \times \mathbf{v} = c(\mathbf{v} \times \mathbf{v}) = c(\mathbf{0}) = \mathbf{0}$$

If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\|\mathbf{u}\| \|\mathbf{v}\| \sin \theta = 0$. (Assume $\mathbf{u} \neq \mathbf{0}$, $\mathbf{v} \neq \mathbf{0}$.) Thus, $\sin \theta = 0$, $\theta = 0$, and $\mathbf{u}$ and $\mathbf{v}$ are parallel. Therefore,

$$\mathbf{u} = c\mathbf{v} \text{ for some scalar } c.$$

64. $\mathbf{u} = \langle a_1, b_1, c_1 \rangle$, $\mathbf{v} = \langle a_2, b_2, c_2 \rangle$, $\mathbf{w} = \langle a_3, b_3, c_3 \rangle$

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)\mathbf{i} - (a_2c_3 - a_3c_2)\mathbf{j} + (a_2b_3 - a_3b_2)\mathbf{k}$$

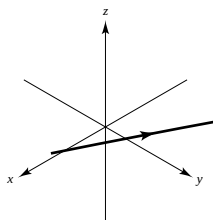
$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ (b_2c_3 - b_3c_2) & (a_3c_2 - a_2c_3) & (a_2b_3 - a_3b_2) \end{vmatrix}$$

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= [b_1(a_2b_3 - a_3b_2) - c_1(a_3c_2 - a_2c_3)]\mathbf{i} - [a_1(a_2b_3 - a_3b_2) - c_1(b_2c_3 - b_3c_2)]\mathbf{j} + \\ &\quad [a_1(a_3c_2 - a_2c_3) - b_1(b_2c_3 - b_3c_2)]\mathbf{k} \\ &= [a_2(a_1a_3 + b_1b_3 + c_1c_3) - a_3(a_1a_2 + b_1b_2 + c_1c_2)]\mathbf{i} + \\ &\quad [b_2(a_1a_3 + b_1b_3 + c_1c_3) - b_3(a_1a_2 + b_1b_2 + c_1c_2)]\mathbf{j} + \\ &\quad [c_2(a_1a_3 + b_1b_3 + c_1c_3) - c_3(a_1a_2 + b_1b_2 + c_1c_2)]\mathbf{k} \\ &= (a_1a_3 + b_1b_3 + c_1c_3)\langle a_2, b_2, c_2 \rangle - (a_1a_2 + b_1b_2 + c_1c_2)\langle a_3, b_3, c_3 \rangle \\ &= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \end{aligned}$$

Section 10.5 Lines and Planes in Space

2. $x = 2 - 3t$, $y = 2$, $z = 1 - t$

(a)



(b) When $t = 0$ we have $P = (2, 2, 1)$. When $t = 2$ we have $Q = (-4, 2, -1)$.

$$\overrightarrow{PQ} = \langle -6, 0, -2 \rangle$$

The components of the vector and the coefficients of t are proportional since the line is parallel to $\overrightarrow{PQ}$.

(c) $z = 0$ when $t = 1$. Thus, $x = -1$ and $y = 2$.

Point: $(-1, 2, 0)$

$$x = 0 \text{ when } t = \frac{2}{3}. \text{ Point: } \left(0, 2, \frac{1}{3}\right)$$

4. Point: $(0, 0, 0)$

$$\text{Direction vector: } \mathbf{v} = \left\langle -2, \frac{5}{2}, 1 \right\rangle$$

Direction numbers: $-4, 5, 2$

(a) Parametric: $x = -4t$, $y = 5t$, $z = 2t$

(b) Symmetric: $\frac{x}{-4} = \frac{y}{5} = \frac{z}{2}$

6. Point: $(-3, 0, 2)$

$$\text{Direction vector: } \mathbf{v} = \langle 0, 6, 3 \rangle$$

Direction numbers: $0, 2, 1$

(a) Parametric: $x = -3$, $y = 2t$, $z = 2 + t$

(b) Symmetric: $\frac{y}{2} = z - 2$, $x = -3$

8. Point: $(-3, 5, 4)$

 Direction numbers: $3, -2, 1$

 (a) Parametric: $x = -3 + 3t, y = 5 - 2t, z = 4 + t$

 (b) Symmetric: $\frac{x+3}{3} = \frac{y-5}{-2} = z-4$

 12. Points: $(0, 0, 25), (10, 10, 0)$

 Direction vector: $\langle 10, 10, -25 \rangle$

 Direction numbers: $2, 2, -5$

 (a) Parametric: $x = 2t, y = 2t, z = 25 - 5t$

 (b) Symmetric: $\frac{x}{2} = \frac{y}{2} = \frac{z-25}{-5}$

 16. Points: $(2, 0, -3), (4, 2, -2)$

 Direction vector: $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

 Direction numbers: $2, 2, 1$

 Parametric: $x = 2 + 2t, y = 2t, z = -3 + t$

 Symmetric: $\frac{x-2}{2} = \frac{y}{2} = \frac{z+3}{1}$

 (a) Not on line $\left(1 \neq \frac{1}{2} \neq 1\right)$

(b) On line

 (c) Not on line $\left(\frac{-3}{2} = \frac{-3}{2} \neq -1\right)$

 10. Points: $(2, 0, 2), (1, 4, -3)$

 Direction vector: $\langle 1, -4, 5 \rangle$

 Direction numbers: $1, -4, 5$

 (a) Parametric: $x = 2 + t, y = -4t, z = 2 + 5t$

 (b) Symmetric: $x - 2 = \frac{y}{-4} = \frac{z - 2}{5}$

 14. Point: $(2, 3, 4)$

 Direction vector: $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

 Direction numbers: $3, 2, -1$

 Parametric: $x = 2 + 3t, y = 3 + 2t, z = 4 - t$

 18. $L_1: \mathbf{v} = \langle 4, -2, 3 \rangle$ $(8, -5, -9)$ on line

 $L_2: \mathbf{v} = \langle 2, 1, 5 \rangle$
 $L_3: \mathbf{v} = \langle -8, 4, -6 \rangle$ $(8, -5, -9)$ on line

 $L_4: \mathbf{v} = \langle -2, 1, 1.5 \rangle$
 L_1 and L_2 are identical.

20. By equating like variables, we have

 (i) $-3t + 1 = 3s + 1$, (ii) $4t + 1 = 2s + 4$, and (iii) $2t + 4 = -s + 1$.

 From (i) we have $s = -t$, and consequently from (ii), $t = \frac{1}{2}$ and from (iii), $t = -3$. The lines do not intersect.

22. Writing the equations of the lines in parametric form we have

 $x = 2 - 3t \quad y = 2 + 6t \quad z = 3 + t$
 $x = 3 + 2s \quad y = -5 + s \quad z = -2 + 4s$

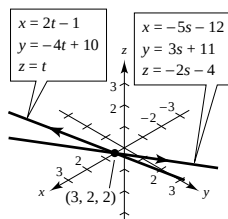
 By equating like variables, we have $2 - 3t = 3 + 2s$, $2 + 6t = -5 + s$, $3 + t = -2 + 4s$. Thus, $t = -1$, $s = 1$ and the point of intersection is $(5, -4, 2)$.

 $\mathbf{u} = \langle -3, 6, 1 \rangle$ (First line)

 $\mathbf{v} = \langle 2, 1, 4 \rangle$ (Second line)

$$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{4}{\sqrt{46}\sqrt{21}} = \frac{4}{\sqrt{966}} = \frac{2\sqrt{966}}{483}$$

 24. $x = 2t - 1 \quad x = -5s - 12$
 $y = -4t + 10 \quad y = 3s + 11$
 $z = t \quad z = -2s - 4$

 Point of intersection: $(3, 2, 2)$


26. $2x + 3y + 4z = 4$

$P = (0, 0, 1), Q = (2, 0, 0), R = (3, 2, -2)$

(a) $\overrightarrow{PQ} = \langle 2, 0, -1 \rangle, \overrightarrow{PR} = \langle 3, 2, -3 \rangle$

(b) $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 3 & 2 & -3 \end{vmatrix} = \langle 2, 3, 4 \rangle$

The components of the cross product are proportional (for this choice of P, Q , and R , they are the same) to the coefficients of the variables in the equation. The cross product is parallel to the normal vector.

30. Point: $(0, 0, 0)$

Normal vector: $\mathbf{n} = -3\mathbf{i} + 2\mathbf{k}$

$-3(x - 0) + 0(y - 0) + 2(z - 0) = 0$

$-3x + 2z = 0$

34. Let $\mathbf{u}$ be vector from $(2, 3, -2)$ to $(3, 4, 2)$: $\langle 1, 1, 4 \rangle$.

Let $\mathbf{v}$ be vector from $(2, 3, -2)$ to $(1, -1, 0)$: $\langle -1, -4, 2 \rangle$.

Normal vector: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 4 \\ -1 & -4 & 2 \end{vmatrix} = \langle 18, -6, -3 \rangle$
 $= -3\langle -6, 2, 1 \rangle$

$-6(x - 2) + 2(y - 3) + 1(z + 2) = 0$

$-6x + 2y + z = -8$

38. The plane passes through the three points $(0, 0, 0)$, $(0, 1, 0)$, $(\sqrt{3}, 0, 1)$.

The vector from $(0, 0, 0)$ to $(0, 1, 0)$: $\mathbf{u} = \mathbf{j}$

The vector from $(0, 0, 0)$ to $(\sqrt{3}, 0, 1)$: $\mathbf{v} = \sqrt{3}\mathbf{i} + \mathbf{k}$

Normal vector: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ \sqrt{3} & 0 & 1 \end{vmatrix} = \mathbf{i} - \sqrt{3}\mathbf{k}$

$x - \sqrt{3}z = 0$

42. Let $\mathbf{v}$ be the vector from $(3, 2, 1)$ to $(3, 1, -5)$:

$\mathbf{v} = -\mathbf{j} - 6\mathbf{k}$

Let $\mathbf{n}$ be the normal to the given plane: $\mathbf{n} = 6\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$

Since $\mathbf{v}$ and $\mathbf{n}$ both lie in the plane P , the normal vector to P is:

$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & -6 \\ 6 & 7 & 2 \end{vmatrix} = 40\mathbf{i} - 36\mathbf{j} + 6\mathbf{k}$
 $= 2(20\mathbf{i} - 18\mathbf{j} + 3\mathbf{k})$

$20(x - 3) - 18(y - 2) + 3(z - 1) = 0$

$20x - 18y + 3z = 27$

28. Point: $(1, 0, -3)$

$\mathbf{n} = \mathbf{k} = \langle 0, 0, 1 \rangle$

$0(x - 1) + 0(y - 0) + 1[z - (-3)] = 0$

$z + 3 = 0$

32. Point: $(3, 2, 2)$

Normal vector: $\mathbf{v} = 4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

$4(x - 3) + (y - 2) - 3(z - 2) = 0$

$4x + y - 3z = 8$

36. $(1, 2, 3)$, Normal vector: $\mathbf{v} = \mathbf{i}, 1(x - 1) = 0, x = 1$

40. The direction of the line is $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$. Choose any point on the line, $[(0, 4, 0)$, for example], and let $\mathbf{v}$ be the vector from $(0, 4, 0)$ to the given point $(2, 2, 1)$:

$\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

Normal vector: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 2 & -2 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{k}$

$(x - 2) - 2(z - 1) = 0$

$x - 2z = 0$

44. Let $\mathbf{u} = \mathbf{k}$ and let $\mathbf{v}$ be the vector from $(4, 2, 1)$ to $(-3, 5, 7)$: $\mathbf{v} = -7\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$

Since $\mathbf{u}$ and $\mathbf{v}$ both lie in the plane P , the normal vector to P is:

$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ -7 & 3 & 6 \end{vmatrix} = -3\mathbf{i} - 7\mathbf{j} = -(3\mathbf{i} + 7\mathbf{j})$

$3(x - 4) + 7(y - 2) = 0$

$3x + 7y = 26$

46. The normal vectors to the planes are $\mathbf{n}_1 = \langle 3, 1, -4 \rangle$, $\mathbf{n}_2 = \langle -9, -3, 12 \rangle$. Since $\mathbf{n}_2 = -3\mathbf{n}_1$, the planes are parallel, but not equal

48. The normal vectors to the planes are

$$\mathbf{n}_1 = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \mathbf{n}_2 = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k},$$

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{|3 - 8 - 2|}{\sqrt{14}\sqrt{21}} = \frac{1}{\sqrt{6}}.$$

$$\text{Therefore, } \theta = \arccos\left(\frac{1}{\sqrt{6}}\right) \approx 65.9^\circ.$$

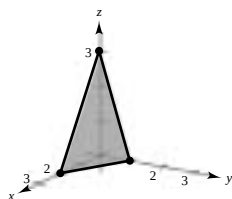
50. The normal vectors to the planes are

$$\mathbf{n}_1 = \langle 2, 0, -1 \rangle, \quad \mathbf{n}_2 = \langle 4, 1, 8 \rangle,$$

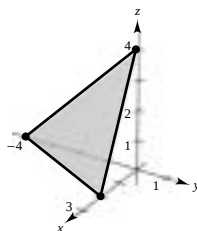
$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = 0.$$

Thus, $\theta = \frac{\pi}{2}$ and the planes are orthogonal.

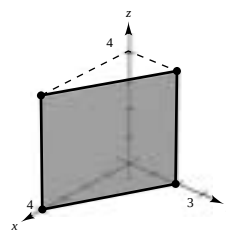
52. $3x + 6y + 2z = 6$



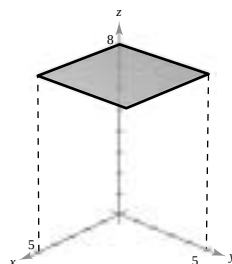
54. $2x - y + z = 4$



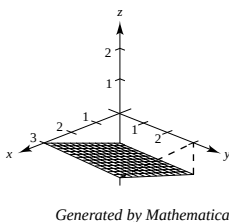
56. $x + 2y = 4$



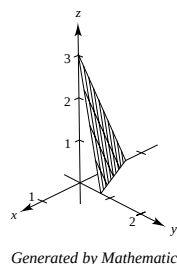
58. $z = 8$



60. $x - 3z = 3$



62. $2.1x - 4.7y - z + 3 = 0$



64. $P_1: \mathbf{n} = \langle -60, 90, 30 \rangle$ or $\langle -2, 3, 1 \rangle$ $(0, 0, \frac{9}{10})$ on plane
 $P_2: \mathbf{n} = \langle 6, -9, -3 \rangle$ or $\langle -2, 3, 1 \rangle$ $(0, 0, -\frac{2}{3})$ on plane
 $P_3: \mathbf{n} = \langle -20, 30, 10 \rangle$ or $\langle -2, 3, 1 \rangle$ $(0, 0, \frac{5}{6})$ on plane
 $P_4: \mathbf{n} = \langle 12, -18, 6 \rangle$ or $\langle -2, 3, -1 \rangle$
 $P_1, P_2,$ and P_3 are parallel.

66. If $c = 0$, $z = 0$ is xy -plane.

If $c \neq 0$, $cy + z = 0 \Rightarrow y = -\frac{1}{c}z$ is a plane parallel to x -axis and passing through the points $(0, 0, 0)$ and $(0, 1, -c)$.

68. The normals to the planes are $\mathbf{n}_1 = \langle 6, -3, 1 \rangle$ and $\mathbf{n}_2 = \langle -1, 1, 5 \rangle$.

The direction vector for the line is

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -3 & 1 \\ -1 & 1 & 5 \end{vmatrix} = \langle -16, -31, 3 \rangle.$$

Now find a point of intersection of the planes.

$$\begin{aligned} 6x - 3y + z &= 5 \Rightarrow 6x - 3y + z = 5 \\ -x + y + 5z &= 5 \Rightarrow \frac{-6x + 6y + 30z}{3y + 31z} = 30 \end{aligned}$$

Let $y = -9, z = 2 \Rightarrow x = -4 \Rightarrow (-4, -9, 2)$.

$$x = -4 - 16t, y = -9 - 31t, z = 2 + 3t$$

70. Writing the equation of the line in parametric form and substituting into the equation of the plane we have:

$$x = 1 + 4t, y = 2t, z = 3 + 6t$$

$$2(1 + 4t) + 3(2t) = -5, t = \frac{-1}{2}$$

Substituting $t = -\frac{1}{2}$ into the parametric equations for the line we have the point of intersection $(-1, -1, 0)$. The line does not lie in the plane.

74. Point: $Q(0, 0, 0)$

$$\text{Plane: } 8x - 4y + z = 8$$

$$\text{Normal to plane: } \mathbf{n} = \langle 8, -4, 1 \rangle$$

$$\text{Point in plane: } P(1, 0, 0)$$

$$\text{Vector: } \overrightarrow{PQ} = \langle -1, 0, 0 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-8|}{\sqrt{81}} = \frac{8}{9}$$

78. The normal vectors to the planes are $\mathbf{n}_1 = \langle 4, -4, 9 \rangle$ and $\mathbf{n}_2 = \langle 4, -4, 9 \rangle$. Since $\mathbf{n}_1 = \mathbf{n}_2$, the planes are parallel. Choose a point in each plane.

$$P = (-5, 0, 3) \text{ is a point in } 4x - 4y + 9z = 7.$$

$$Q = (0, 0, 2) \text{ is a point in } 4x - 4y + 9z = 18.$$

$$\overrightarrow{PQ} = \langle 5, 0, -1 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{11}{\sqrt{113}} = \frac{11\sqrt{113}}{113}$$

82. $\mathbf{u} = \langle 2, 1, 2 \rangle$ is the direction vector for the line.

$$P = \langle 0, -3, 2 \rangle \text{ is a point on the line (let } t = 0).$$

$$\overrightarrow{PQ} = \langle 1, 1, 2 \rangle$$

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{vmatrix} = \langle 0, 2, -1 \rangle$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{5}}{\sqrt{9}} = \frac{\sqrt{5}}{3}$$

86. $x = a$: plane parallel to yz -plane containing $(a, 0, 0)$

$$y = b$$
: plane parallel to xz -plane containing $(0, b, 0)$

$$z = c$$
: plane parallel to xy -plane containing $(0, 0, c)$

72. Writing the equation of the line in parametric form and substituting into the equation of the plane we have:

$$x = 4 + 2t, y = -1 - 3t, z = -2 + 5t$$

$$5(4 + 2t) + 3(-1 - 3t) = 17, t = 0$$

Substituting $t = 0$ into the parametric equations for the line we have the point of intersection $(4, -1, -2)$. The line does not lie in the plane.

76. Point: $Q(3, 2, 1)$

$$\text{Plane: } x - y + 2z = 4$$

$$\text{Normal to plane: } \mathbf{n} = \langle 1, -1, 2 \rangle$$

$$\text{Point in plane: } P(4, 0, 0)$$

$$\text{Vector: } \overrightarrow{PQ} = \langle -1, 2, 1 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-1|}{\sqrt{6}} = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

80. The normal vectors to the planes are $\mathbf{n}_1 = \langle 2, 0, -4 \rangle$ and $\mathbf{n}_2 = \langle 2, 0, -4 \rangle$. Since $\mathbf{n}_1 = \mathbf{n}_2$, the planes are parallel. Choose a point in each plane.

$$P = (2, 0, 0) \text{ is a point in } 2x - 4z = 4. Q = (5, 0, 0) \text{ is a point in } 2x - 4z = 10.$$

$$\overrightarrow{PQ} = \langle 3, 0, 0 \rangle, D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{6}{\sqrt{20}} = \frac{3\sqrt{5}}{5}$$

84. The equation of the plane containing $P(x_1, y_1, z_1)$ and having normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

You need $\mathbf{n}$ and P to find the equation.

88. (a) $t\mathbf{v}$ represents a line parallel to $\mathbf{v}$.

(b) $\mathbf{u} + t\mathbf{v}$ represents a line through the terminal point of $\mathbf{u}$ parallel to $\mathbf{v}$.

(c) $s\mathbf{u} + t\mathbf{v}$ represent the plane containing $\mathbf{u}$ and $\mathbf{v}$.

90. On one side we have the points $(0, 0, 0)$, $(6, 0, 0)$, and $(-1, -1, 8)$.

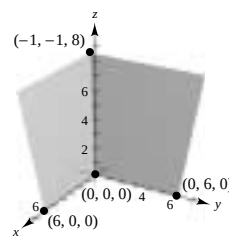
$$\mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 0 & 0 \\ -1 & -1 & 8 \end{vmatrix} = -48\mathbf{j} - 6\mathbf{k}$$

On the adjacent side we have the points $(0, 0, 0)$, $(0, 6, 0)$, and $(-1, -1, 8)$.

$$\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 6 & 0 \\ -1 & -1 & 8 \end{vmatrix} = 48\mathbf{i} + 6\mathbf{k}$$

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{36}{2340} = \frac{1}{65}$$

$$\theta = \arccos \frac{1}{65} \approx 89.1^\circ$$



92. False. They may be skew lines. (See Section Project)

Section 10.6 Surfaces in Space

2. Hyperboloid of two sheets

Matches graph (e)

4. Elliptic cone

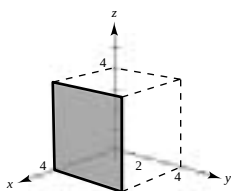
Matches graph (b)

6. Hyperbolic paraboloid

Matches graph (a)

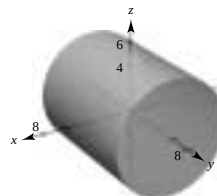
8. $x = 4$

Plane parallel to the yz -coordinate plane



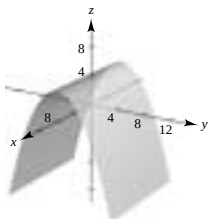
10. $x^2 + z^2 = 25$

The y -coordinate is missing so we have a cylindrical surface with rulings parallel to the y -axis. The generating curve is a circle.



12. $z = 4 - y^2$

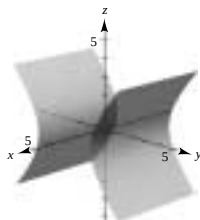
The x -coordinate is missing so we have a cylindrical surface with rulings parallel to the x -axis. The generating curve is a parabola.



14. $y^2 - z^2 = 4$

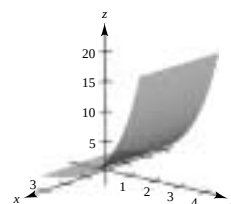
$$\frac{y^2}{4} - \frac{z^2}{4} = 1$$

The x -coordinate is missing so we have a cylindrical surface with rulings parallel to the x -axis. The generating curve is a hyperbola.



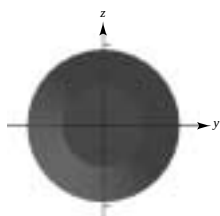
16. $z = e^y$

The x -coordinate is missing so we have a cylindrical surface with rulings parallel to the x -axis. The generating curve is the exponential curve.

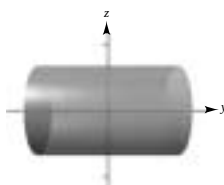


18. $y^2 + z^2 = 4$

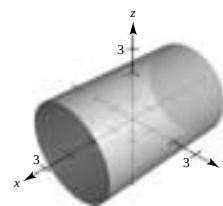
(a) From (10, 0, 0):



(b) From (0, 10, 0):



(c) From (10, 10, 10):



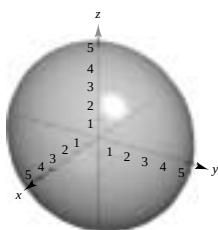
20. $\frac{x^2}{16} + \frac{y^2}{25} + \frac{z^2}{25} = 1$

Ellipsoid

$$xy\text{-trace: } \frac{x^2}{16} + \frac{y^2}{25} = 1 \text{ ellipse}$$

$$xz\text{-trace: } \frac{x^2}{16} + \frac{z^2}{25} = 1 \text{ ellipse}$$

$$yz\text{-trace: } y^2 + z^2 = 25 \text{ circle}$$

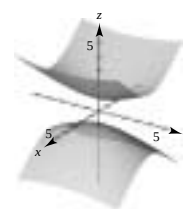


22. $z^2 - x^2 - \frac{y^2}{4} = 1$

Hyperboloid of two sheets

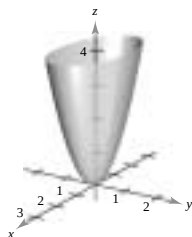
 $xy\text{-trace: none}$
 $xz\text{-trace: } z^2 - x^2 = 1 \text{ hyperbola}$
 $yz\text{-trace: } z^2 - \frac{y^2}{4} = 1 \text{ hyperbola}$

$$z = \pm\sqrt{10}: \frac{x^2}{9} + \frac{y^2}{36} = 1 \text{ ellipse}$$



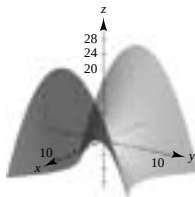
24. $z = x^2 + 4y^2$

Elliptic paraboloid

 $xy\text{-trace: point } (0, 0, 0)$
 $xz\text{-trace: } z = x^2 \text{ parabola}$
 $yz\text{-trace: } z = 4y^2 \text{ parabola}$


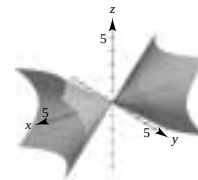
26. $3z = -y^2 + x^2$

Hyperbolic paraboloid

 $xy\text{-trace: } y = \pm x$
 $xz\text{-trace: } z = \frac{1}{3}x^2$
 $yz\text{-trace: } z = -\frac{1}{3}y^2$


28. $x^2 = 2y^2 + 2z^2$

Elliptic Cone

 $xy\text{-trace: } x = \pm\sqrt{2}y$
 $xz\text{-trace: } x = \pm\sqrt{2}z$
 $yz\text{-trace: point } (0, 0, 0)$


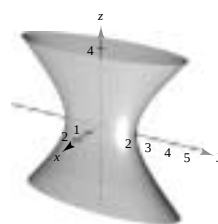
30. $9x^2 + y^2 - 9z^2 - 54x - 4y - 54z + 4 = 0$

$$9(x^2 - 6x + 9) + (y^2 - 4y + 4) - 9(z^2 + 6z + 9) = 81 + 4 - 81$$

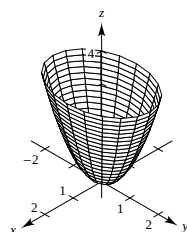
$$9(x - 3)^2 + (y - 2)^2 - 9(z + 3)^2 = 4$$

$$\frac{(x - 3)^2}{4/9} + \frac{(y - 2)^2}{4} - \frac{(z + 3)^2}{4/9} = 1$$

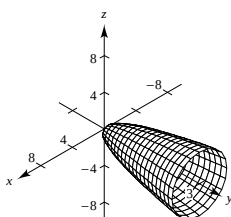
Hyperboloid of one sheet with center (3, 2, -3).



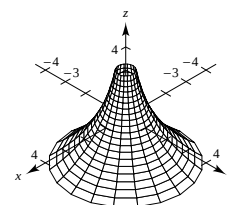
32. $z = x^2 + 0.5y^2$



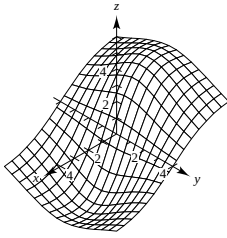
34. $z^2 = 4y - x^2$
 $z = \pm\sqrt{4y - x^2}$



36. $x^2 + y^2 = e^{-z}$
 $-\ln(x^2 + y^2) = z$

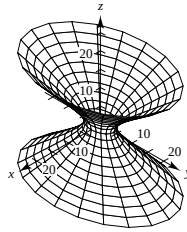


38. $z = \frac{-x}{8 + x^2 + y^2}$



40. $9x^2 + 4y^2 - 8z^2 = 72$

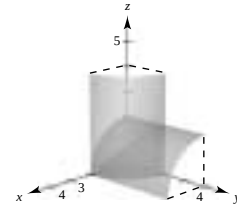
$$z = \pm \sqrt{\frac{9}{8}x^2 + \frac{1}{2}y^2 - 9}$$



42. $z = \sqrt{4 - x^2}$

$$y = \sqrt{4 - x^2}$$

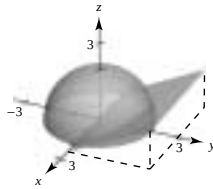
$$x = 0, y = 0, z = 0$$



44. $z = \sqrt{4 - x^2 - y^2}$

$$y = 2z$$

$$z = 0$$



46. $x^2 + z^2 = [r(y)]^2$ and $z = r(y) = 3y$; therefore,

$$x^2 + z^2 = 9y^2.$$

48. $y^2 + z^2 = [r(x)]^2$ and $z = r(x) = \frac{1}{2}\sqrt{4 - x^2}$; therefore,

$$y^2 + z^2 = \frac{1}{4}(4 - x^2), x^2 + 4y^2 + 4z^2 = 4.$$

50. $x^2 + y^2 = [r(z)]^2$ and $y = r(z) = e^z$; therefore,

$$x^2 + y^2 = e^{2z}.$$

52. $x^2 + z^2 = \cos^2 y$

Equation of generating curve:

$$x = \cos y \text{ or } z = \cos y$$

54. The trace of a surface is the intersection of the surface with a plane. You find a trace by setting one variable equal to a constant, such as $x = 0$ or $z = 2$.

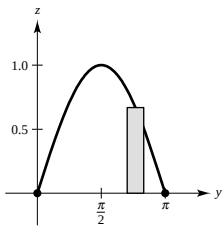
56. About x-axis: $y^2 + z^2 = [r(x)]^2$

About y-axis: $x^2 + z^2 = [r(y)]^2$

About z-axis: $x^2 + y^2 = [r(z)]^2$

58. $V = 2\pi \int_0^\pi y \sin y \, dy$

$$= 2\pi \left[\sin y - y \cos y \right]_0^\pi = 2\pi^2$$



60. $z = \frac{x^2}{2} + \frac{y^2}{4}$

(a) When $y = 4$ we have $z = \frac{x^2}{2} + 4$, $4\left(\frac{1}{2}\right)(z - 4) = x^2$.

Focus: $\left(0, 4, \frac{9}{2}\right)$

(b) When $x = 2$ we have

$$z = 2 + \frac{y^2}{4}, 4(z - 2) = y^2.$$

Focus: $(2, 0, 3)$

62. If (x, y, z) is on the surface, then

$$z^2 = x^2 + y^2 + (z - 4)^2$$

$$z^2 = x^2 + y^2 + z^2 - 8z + 16$$

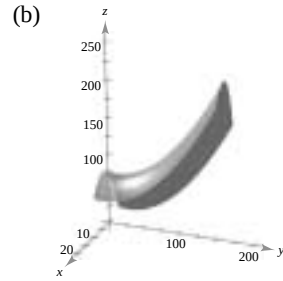
$$8z = x^2 + y^2 + 16 \Rightarrow z = \frac{x^2}{8} + \frac{y^2}{8} + 2$$

Elliptic paraboloid shifted up 2 units. Traces parallel to xy -plane are circles.

64. $z = -0.775x^2 + 0.007y^2 + 22.15x - 0.54y - 45.4$

(a)

Year	1980	1985	1990	1995	1996	1997
z	37.5	72.2	111.5	185.2	200.1	214.6
Model	37.8	72.0	112.2	185.8	204.5	214.7



(c) For y constant, the traces parallel to the xz -plane are concave downward. That is, for fixed y (public assistance), the rate of increase of z (Medicare) is decreasing with respect to x (worker's compensation).

(d) The traces parallel to the yz -plane (x constant) are concave upward. That is, for fixed x (worker's compensation), the rate of increase of z (Medicare) is increasing with respect to y (public assistance).

66. Equating twice the first equation with the second equation,

$$2x^2 + 6y^2 - 4z^2 + 4y - 8 = 2x^2 + 6y^2 - 4z^2 - 3x - 2$$

$$4y - 8 = -3x - 2$$

$$3x + 4y = 6, \text{ a plane}$$

Section 10.7 Cylindrical and Spherical Coordinates

2. $\left(4, \frac{\pi}{2}, -2\right)$, cylindrical

$$x = 4 \cos \frac{\pi}{2} = 0$$

$$y = 4 \sin \frac{\pi}{2} = 4$$

$$z = -2$$

$$(0, 4, -2), \text{ rectangular}$$

4. $\left(6, -\frac{\pi}{4}, 2\right)$, cylindrical

$$x = 6 \cos\left(-\frac{\pi}{4}\right) = 3\sqrt{2}$$

$$y = 6 \sin\left(-\frac{\pi}{4}\right) = -3\sqrt{2}$$

$$z = 2$$

$$(3\sqrt{2}, -3\sqrt{2}, 2)$$

6. $\left(1, \frac{3\pi}{2}, 1\right)$, cylindrical

$$x = \cos \frac{3\pi}{2} = 0$$

$$y = \sin \frac{3\pi}{2} = -1$$

$$z = 1$$

$$(0, -1, 1), \text{ rectangular}$$

8. $(2\sqrt{2}, -2\sqrt{2}, 4)$, rectangular

$$r = \sqrt{(2\sqrt{2})^2 + (-2\sqrt{2})^2} = 4$$

$$\theta = \arctan(-1) = -\frac{\pi}{4}$$

$$z = 4$$

$$\left(4, -\frac{\pi}{4}, 4\right), \text{ cylindrical}$$

10. $(2\sqrt{3}, -2, 6)$, rectangular

$$r = \sqrt{12 + 4} = 4$$

$$\theta = \arctan\left(-\frac{1}{\sqrt{3}}\right) = \frac{5\pi}{6}$$

$$z = 1$$

$$\left(4, -\frac{\pi}{6}, 1\right), \text{ cylindrical}$$

12. $(-3, 2, -1)$, rectangular

$$r = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$$

$$\theta = \arctan\left(\frac{-2}{3}\right) = -\arctan \frac{2}{3}$$

$$z = -1$$

$$\left(\sqrt{13}, -\arctan \frac{2}{3}, -1\right), \text{ cylindrical}$$

14. $z = x^2 + y^2 - 2$ rectangular equation

$$z = r^2 - 2 \quad \text{cylindrical equation}$$

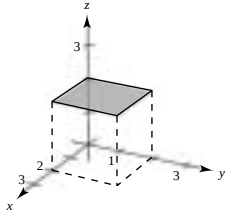
16. $x^2 + y^2 = 8x$ rectangular equation

$$r^2 = 8r \cos \theta$$

$$r = 8 \cos \theta \quad \text{cylindrical equation}$$

18. $z = 2$

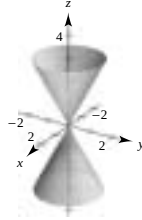
Same



20. $r = \frac{z}{2}$

$$\sqrt{x^2 + y^2} = \frac{z}{2}$$

$$x^2 + y^2 - \frac{z^2}{4} = 0$$



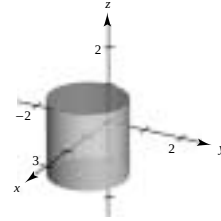
22. $r = 2 \cos \theta$

$$r^2 = 2r \cos \theta$$

$$x^2 + y^2 = 2x$$

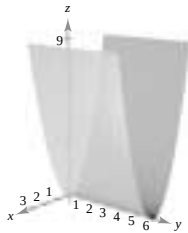
$$x^2 + y^2 - 2x = 0$$

$$(x - 1)^2 + y^2 = 1$$



24. $z = r^2 \cos^2 \theta$

$$z = x^2$$



26. $(1, 1, 1)$, rectangular

$$\rho = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\theta = \arctan 1 = \frac{\pi}{4}$$

$$\phi = \arccos \frac{1}{\sqrt{3}}$$

$$\left(\sqrt{3}, \frac{\pi}{4}, \arccos \frac{1}{\sqrt{3}} \right), \text{ spherical}$$

28. $(2, 2, 4\sqrt{2})$, rectangular

$$\rho = \sqrt{2^2 + 2^2 + (4\sqrt{2})^2} = 2\sqrt{10}$$

$$\theta = \arctan 1 = \frac{\pi}{4}$$

$$\phi = \arccos \frac{2}{\sqrt{5}}$$

$$\left(2\sqrt{10}, \frac{\pi}{4}, \arccos \frac{2}{\sqrt{5}} \right), \text{ spherical}$$

30. $(-4, 0, 0)$, rectangular

$$\rho = \sqrt{(-4)^2 + 0^2 + 0^2} = 4$$

$$\theta = \pi$$

$$\phi = \arccos 0 = \frac{\pi}{2}$$

$$\left(4, \pi, \frac{\pi}{2} \right), \text{ spherical}$$

32. $\left(12, \frac{3\pi}{4}, \frac{\pi}{9} \right)$, spherical

$$x = 12 \sin \frac{\pi}{9} \cos \frac{3\pi}{4} \approx -2.902$$

$$y = 12 \sin \frac{\pi}{9} \sin \frac{3\pi}{4} \approx 2.902$$

$$z = 12 \cos \frac{\pi}{9} \approx 11.276$$

$$(-2.902, 2.902, 11.276), \text{ rectangular}$$

34. $\left(9, \frac{\pi}{4}, \pi \right)$, spherical

$$x = 9 \sin \pi \cos \frac{\pi}{4} = 0$$

$$y = 9 \sin \pi \sin \frac{\pi}{4} = 0$$

$$z = 9 \cos \pi = -9$$

$$(0, 0, -9), \text{ rectangular}$$

36. $\left(6, \pi, \frac{\pi}{2} \right)$, spherical

$$x = 6 \sin \frac{\pi}{2} \cos \pi = -6$$

$$y = 6 \sin \frac{\pi}{2} \sin \pi = 0$$

$$z = 6 \cos \frac{\pi}{2} = 0$$

$$(-6, 0, 0), \text{ rectangular}$$

38. (a) Programs will vary.

(b) $(\rho, \theta, \phi) = (5, 1, 0.5)$

$$(x, y, z) = (1.295, 2.017, 4.388)$$

40. $x^2 + y^2 - 3z^2 = 0$ rectangular equation

$$x^2 + y^2 + z^2 = 4z^2$$

$$\rho^2 = 4\rho^2 \cos^2 \phi$$

$$1 = 4 \cos^2 \phi$$

$$\cos \phi = \frac{1}{2}$$

$$\phi = \frac{\pi}{3} \quad (\text{cone}) \text{ spherical equation}$$

42. $x = 10$ rectangular equation

$$\rho \sin \phi \cos \theta = 10$$

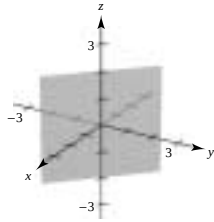
$$\rho = 10 \csc \phi \sec \theta \text{ spherical equation}$$

44. $\theta = \frac{3\pi}{4}$

$$\tan \theta = \frac{y}{x}$$

$$-1 = \frac{y}{x}$$

$$x + y = 0$$



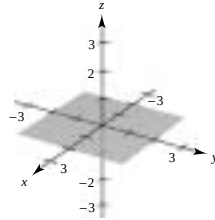
46. $\phi = \frac{\pi}{2}$

$$\cos \phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$0 = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$z = 0$$

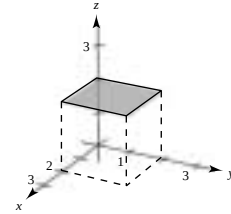
xy-plane



48. $\rho = 2 \sec \phi$

$$\rho \cos \phi = 2$$

$$z = 2$$

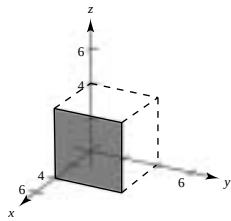


50. $\rho = 4 \csc \phi \sec \theta$

$$= \frac{4}{\sin \phi \cos \theta}$$

$$\rho \sin \phi \cos \theta = 4$$

$$x = 4$$



52. $\left(3, -\frac{\pi}{4}, 0\right)$, cylindrical

$$\rho = \sqrt{3^2 + 0^2} = 3$$

$$\theta = -\frac{\pi}{4}$$

$$\phi = \arccos\left(\frac{0}{3}\right) = \frac{\pi}{2}$$

$$\left(3, -\frac{\pi}{4}, \frac{\pi}{2}\right), \text{ spherical}$$

54. $\left(2, \frac{2\pi}{3}, -2\right)$, cylindrical

$$\rho = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$$

$$\theta = \frac{2\pi}{3}$$

$$\phi = \arccos\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$$

$$\left(2\sqrt{2}, \frac{2\pi}{3}, \frac{3\pi}{4}\right), \text{ spherical}$$

56. $\left(-4, \frac{\pi}{3}, 4\right)$, cylindrical

$$\rho = \sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$$

$$\theta = \frac{\pi}{3}$$

$$\phi = \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

$$\left(4\sqrt{2}, \frac{\pi}{3}, \frac{\pi}{4}\right), \text{ spherical}$$

58. $\left(4, \frac{\pi}{2}, 3\right)$, cylindrical

$$\rho = \sqrt{4^2 + 3^2} = 5$$

$$\theta = \frac{\pi}{2}$$

$$\phi = \arccos\left(\frac{3}{5}\right)$$

$$\left(5, \frac{\pi}{2}, \arccos\left(\frac{3}{5}\right)\right), \text{ spherical}$$

60. $\left(4, \frac{\pi}{18}, \frac{\pi}{2}\right)$, spherical

$$r = 4 \sin \frac{\pi}{2} = 4$$

$$\theta = \frac{\pi}{18}$$

$$z = 4 \cos \frac{\pi}{2} = 0$$

$$\left(4, \frac{\pi}{18}, 0\right), \text{ cylindrical}$$

62. $\left(18, \frac{\pi}{3}, \frac{\pi}{3}\right)$, spherical

$$r = \rho \sin \phi = 18 \sin \frac{\pi}{3} = 9$$

$$\theta = \frac{\pi}{3}$$

$$z = \rho \cos \phi = 18 \cos \frac{\pi}{3} = 9\sqrt{3}$$

$$\left(9, \frac{\pi}{3}, 9\sqrt{3}\right), \text{cylindrical}$$

64. $\left(5, -\frac{5\pi}{6}, \pi\right)$, spherical

$$r = 5 \sin \pi = 0$$

$$\theta = -\frac{5\pi}{6}$$

$$z = 5 \cos \pi = -5$$

$$\left(0, -\frac{5\pi}{6}, -5\right), \text{cylindrical}$$

66. $\left(7, \frac{\pi}{4}, \frac{3\pi}{4}\right)$, spherical

$$r = 7 \sin \frac{3\pi}{4} = \frac{7\sqrt{2}}{2}$$

$$\theta = \frac{\pi}{4}$$

$$z = 7 \cos \frac{3\pi}{4} = -\frac{7\sqrt{2}}{2}$$

$$\left(\frac{7\sqrt{2}}{2}, \frac{\pi}{4}, -\frac{7\sqrt{2}}{2}\right), \text{cylindrical}$$

Rectangular

68. $(6, -2, -3)$

70. $(7.317, -6.816, 6)$

72. $(6.115, 1.561, 4.052)$

74. $(3\sqrt{2}, 3\sqrt{2}, -3)$

76. $(0, -5, 4)$

78. $(-1.732, 1, 3)$

$$\left[\text{Note: Use the cylindrical coordinate } \left(2, \frac{5\pi}{6}, 3\right)\right]$$

80. $(2.207, 7.949, -4)$

Cylindrical

$$(6.325, -0.322, -3)$$

$$(10, -0.75, 6)$$

$$(6.311, 0.25, 4.052)$$

$$(6, 0.785, -3)$$

$$(5, -1.571, 4)$$

$$\left(-2, \frac{11\pi}{6}, 3\right)$$

$$(8.25, 1.3, -4)$$

Spherical

$$(7.000, -0.322, 2.014)$$

$$(11.662, -0.750, 1.030)$$

$$(7.5, 0.25, 1)$$

$$(6.708, 0.785, 2.034)$$

$$(6.403, -1.571, 0.896)$$

$$(3.606, 2.618, 0.588)$$

$$(9.169, 1.3, 2.022)$$

82. $\theta = \frac{\pi}{4}$

Plane

Matches graph (e)

84. $\phi = \frac{\pi}{4}$

Cone

Matches graph (a)

86. $\rho = 4 \sec \phi, z = \rho \cos \phi = 4$

Plane

Matches graph (b)

88. $r = a$ Cylinder with z -axis symmetry

$$\theta = b$$
 Plane perpendicular to xy -plane

$$z = c$$
 Plane parallel to xy -plane

90. $\rho = a$ Sphere

$$\theta = b$$
 Vertical half-plane

$$\phi = c$$
 Half-cone

92. $4(x^2 + y^2) = z^2$

(a) $4r^2 = z^2, 2r = z$

(b) $4(\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta) = \rho^2 \cos^2 \phi,$

$$4 \sin^2 \phi = \cos^2 \phi, \tan^2 \phi = \frac{1}{4},$$

$$\tan \phi = \frac{1}{2}, \phi = \arctan \frac{1}{2}$$

94. $x^2 + y^2 = z$

(a) $r^2 = z$

(b) $\rho^2 \sin^2 \phi = \rho \cos \phi, \rho \sin^2 \phi = \cos \phi,$

$$\rho = \frac{\cos \phi}{\sin^2 \phi}, \rho = \csc \phi \cot \phi$$

96. $x^2 + y^2 = 16$

(a) $r^2 = 16, r = 4$

(b) $\rho^2 \sin^2 \phi = 16, \rho^2 \sin^2 \phi - 16 = 0,$

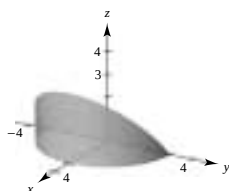
$$(\rho \sin \phi - 4)(\rho \sin \phi + 4) = 0, \rho = 4 \csc \phi$$

98. $y = 4$

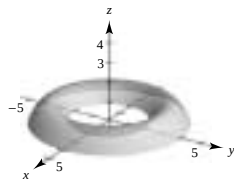
(a) $r \sin \theta = 4, r = 4 \csc \theta$

(b) $\rho \sin \phi \sin \theta = 4, \rho = 4 \csc \phi \csc \theta$

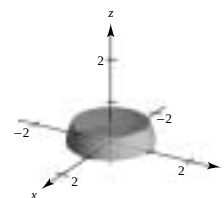
100. $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
 $0 \leq r \leq 3$
 $0 \leq z \leq r \cos \theta$



102. $0 \leq \theta \leq 2\pi$
 $2 \leq r \leq 4$
 $z^2 \leq -r^2 + 6r - 8$



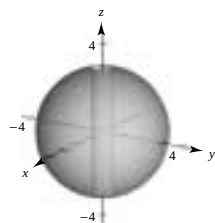
104. $0 \leq \theta \leq 2\pi$
 $\frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}$
 $0 \leq \rho \leq 1$



106. Cylindrical: $0.75 \leq r \leq 1.25, z = 8$

108. Cylindrical

$\frac{1}{2} \leq r \leq 3$
 $0 \leq \theta \leq 2\pi$
 $-\sqrt{9-r^2} \leq z \leq \sqrt{9-r^2}$



110. $\rho = 2 \sec \phi \Rightarrow \rho \cos \phi = 2 \Rightarrow z = 2$ plane
 $\rho = 4$ sphere
The intersection of the plane and the sphere is a circle.

Review Exercises for Chapter 10

2. $P = (-2, -1), Q = (5, -1), R = (2, 4)$

(a) $\mathbf{u} = \overrightarrow{PQ} = \langle 7, 0 \rangle = 7\mathbf{i}, \mathbf{v} = \overrightarrow{PR} = \langle 4, 5 \rangle = 4\mathbf{i} + 5\mathbf{j}$

(b) $\|\mathbf{v}\| = \sqrt{4^2 + 5^2} = \sqrt{41}$

(c) $2\mathbf{u} + \mathbf{v} = 14\mathbf{i} + (4\mathbf{i} + 5\mathbf{j}) = 18\mathbf{i} + 5\mathbf{j}$

4. $\mathbf{v} = \|\mathbf{v}\| \cos \theta \mathbf{i} + \|\mathbf{v}\| \sin \theta \mathbf{j} = \frac{1}{2} \cos 225^\circ \mathbf{i} + \frac{1}{2} \sin 225^\circ \mathbf{j}$
 $= -\frac{\sqrt{2}}{4} \mathbf{i} + \frac{\sqrt{2}}{4} \mathbf{j}$

6. (a) The length of cable POQ is L .

$\overrightarrow{OQ} = 9\mathbf{i} - y\mathbf{j}$

$L = 2\sqrt{9^2 + y^2} \Rightarrow \sqrt{\frac{L^2}{4} - 81} = y$

Tension: $T = c\|\overrightarrow{OQ}\| = c\sqrt{81 + y^2}$

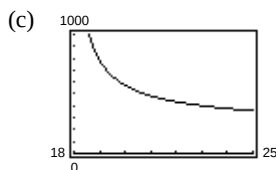
Also,

$cy = 250 \Rightarrow T = \frac{250}{y} \sqrt{81 + y^2} \Rightarrow T = \frac{250}{\sqrt{(L^2/4) - 81}} \cdot \frac{L}{2} = \frac{250L}{\sqrt{L^2 - 324}}$

Domain: $L > 18$ inches

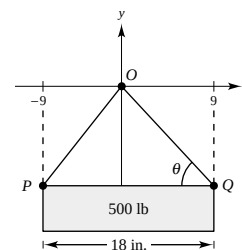
(b)

L	19	20	21	22	23	24	25
T	780.9	573.54	485.36	434.81	401.60	377.96	360.24



(d) The line $T = 400$ intersects the curve at

$L = 23.06$ inches.



(e) $\lim_{L \rightarrow \infty} T = 250$

The maximum tension is 250 pounds in each side of the cable since the total weight is 500 pounds.

8. $x = z = 0, y = -7: (0, -7, 0)$

10. Looking towards the xy -plane from the positive z -axis. The point is either in the second quadrant ($x < 0, y > 0$) or in the fourth quadrant ($x > 0, y < 0$). The z -coordinate can be any number.

12. Center: $\left(\frac{0+4}{2}, \frac{0+6}{2}, \frac{4+0}{2}\right) = (2, 3, 2)$

Radius: $\sqrt{(2-0)^2 + (3-0)^2 + (2-4)^2} = \sqrt{4+9+4} = \sqrt{17}$

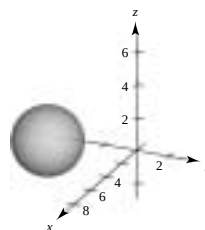
$(x-2)^2 + (y-3)^2 + (z-2)^2 = 17$

14. $(x^2 - 10x + 25) + (y^2 + 6y + 9) + (z^2 - 4z + 4) = -34 + 25 + 9 + 4$

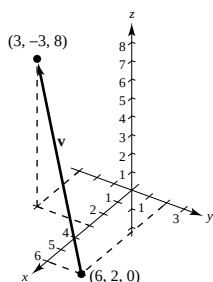
$(x-5)^2 + (y+3)^2 + (z-2)^2 = 4$

Center: $(5, -3, 2)$

Radius: 2



16. $\mathbf{v} = \langle 3-6, -3-2, 8-0 \rangle = \langle -3, -5, 8 \rangle$



18. $\mathbf{v} = \langle 8-5, -5+4, 5-7 \rangle = \langle 3, -1, -2 \rangle$

$\mathbf{w} = \langle 11-5, 6+4, 3-7 \rangle = \langle 6, 10, -4 \rangle$

Since $\mathbf{v}$ and $\mathbf{w}$ are not parallel, the points do not lie in a straight line.

20. $8 \frac{\langle 6, -3, 2 \rangle}{\sqrt{49}} = \frac{8}{7} \langle 6, -3, 2 \rangle = \left\langle \frac{48}{7}, -\frac{24}{7}, \frac{16}{7} \right\rangle$

22. $P = (2, -1, 3), Q = (0, 5, 1), R = (5, 5, 0)$

(a) $\mathbf{u} = \overrightarrow{PQ} = \langle -2, 6, -2 \rangle = -2\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$

$\mathbf{v} = \overrightarrow{PR} = \langle 3, 6, -3 \rangle = 3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$

(b) $\mathbf{u} \cdot \mathbf{v} = (-2)(3) + (6)(6) + (-2)(-3) = 36$

(c) $\mathbf{v} \cdot \mathbf{v} = 9 + 36 + 9 = 54$

24. $\mathbf{u} = \langle -4, 3, -6 \rangle, \mathbf{v} = \langle 16, -12, 24 \rangle$

Since $\mathbf{v} = -4\mathbf{u}$, the vectors are parallel.

26. $\mathbf{u} = \langle 4, -1, 5 \rangle, \mathbf{v} = \langle 3, 2, -2 \rangle$

$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow$ is orthogonal to $\mathbf{v}$.

$\theta = \frac{\pi}{2}$

28. $\mathbf{u} = \langle 1, 0, -3 \rangle$

$\mathbf{v} = \langle 2, -2, 1 \rangle$

$\mathbf{u} \cdot \mathbf{v} = -1$

$\|\mathbf{u}\| = \sqrt{10}$

$\|\mathbf{v}\| = 3$

$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{1}{3\sqrt{10}}$

$\theta \approx 83.9^\circ$

30. $W = \mathbf{F} \cdot \overrightarrow{PQ} = \|\mathbf{F}\| \|\overrightarrow{PQ}\| \cos \theta = (75)(8) \cos 30^\circ$
 $= 300\sqrt{3} \text{ ft} \cdot \text{lb}$

In Exercises 32–40, $\mathbf{u} = \langle 3, -2, 1 \rangle$, $\mathbf{v} = \langle 2, -4, -3 \rangle$, $\mathbf{w} = \langle -1, 2, 2 \rangle$.

$$32. \cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{11}{\sqrt{14}\sqrt{29}}$$

$$\theta = \arccos\left(\frac{11}{\sqrt{14}\sqrt{29}}\right) \approx 56.9^\circ$$

$$34. \text{Work} = |\mathbf{u} \cdot \mathbf{w}| = |-3 - 4 + 2| = 5$$

$$36. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 2 & -4 & -3 \end{vmatrix} = 10\mathbf{i} + 11\mathbf{j} - 8\mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ 3 & -2 & 1 \end{vmatrix} = -10\mathbf{i} - 11\mathbf{j} + 8\mathbf{k}$$

Thus, $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.

$$38. \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \langle 3, -2, 1 \rangle \times \langle 1, -2, -1 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 1 & -2 & -1 \end{vmatrix} = 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 2 & -4 & -3 \end{vmatrix} = 10\mathbf{i} + 11\mathbf{j} - 8\mathbf{k}$$

$$\mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ -1 & 2 & 2 \end{vmatrix} = -6\mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$$

$$(\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) = 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k} = \mathbf{u} \times (\mathbf{v} + \mathbf{w})$$

$$40. \text{Area triangle} = \frac{1}{2}\|\mathbf{v} \times \mathbf{w}\| = \frac{1}{2}\sqrt{(-2)^2 + (-1)^2} = \frac{\sqrt{5}}{2} \quad (\text{See Exercise 35})$$

$$42. V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 2(5) = 10$$

44. Direction numbers: 1, 1, 1

$$(a) x = 1 + t, y = 2 + t, z = 3 + t$$

$$(b) x - 1 = y - 2 = z - 3$$

$$46. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5 & 1 \\ -3 & 1 & 4 \end{vmatrix} = -21\mathbf{i} - 11\mathbf{j} - 13\mathbf{k}$$

Direction numbers: 21, 11, 13

$$(a) x = 21t, y = 1 + 11t, z = 4 + 13t$$

$$(b) \frac{x}{21} = \frac{y-1}{11} = \frac{z-4}{13}$$

48. $P = (-3, -4, 2)$, $Q = (-3, 4, 1)$, $R = (1, 1, -2)$

$$\overrightarrow{PQ} = \langle 0, 8, -1 \rangle, \overrightarrow{PR} = \langle 4, 5, -4 \rangle$$

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 8 & -1 \\ 4 & 5 & -4 \end{vmatrix} = -27\mathbf{i} - 4\mathbf{j} - 32\mathbf{k}$$

$$-27(x + 3) - 4(y + 4) - 32(z - 2) = 0$$

$$27x + 4y + 32z = -33$$

50. The normal vectors to the planes are the same,

$$\mathbf{n} = \langle 5, -3, 1 \rangle.$$

Choose a point in the first plane, $P = (0, 0, 2)$. Choose a point in the second plane, $Q = (0, 0, -3)$.

$$\overrightarrow{PQ} = \langle 0, 0, -5 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-5|}{\sqrt{35}} = \frac{5}{\sqrt{35}} = \frac{\sqrt{35}}{7}$$

- 52.
- $Q(-5, 1, 3)$
- point

$$\mathbf{u} = \langle 1, -2, -1 \rangle \text{ direction vector}$$

$$P = (1, 3, 5) \text{ point on line}$$

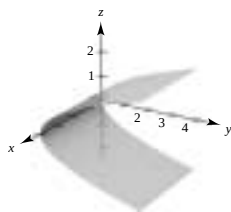
$$\overrightarrow{PQ} = \langle -6, -2, -2 \rangle$$

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & -2 & -2 \\ 1 & -2 & -1 \end{vmatrix} = \langle -2, -8, 14 \rangle$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{264}}{\sqrt{6}} = 2\sqrt{11}$$

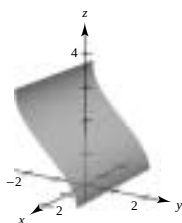
- 54.
- $y = z^2$

Since the x -coordinate is missing, we have a cylindrical surface with rulings parallel to the x -axis. The generating curve is a parabola in the yz -coordinate plane.



- 56.
- $y = \cos z$

Since the x -coordinate is missing, we have a cylindrical surface with rulings parallel to the x -axis. The generating curve is $y = \cos z$.



- 58.
- $16x^2 + 16y^2 - 9z^2 = 0$

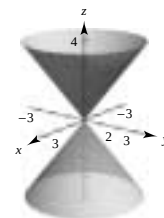
Cone

xy -trace: point $(0, 0, 0)$

$$xz\text{-trace: } z = \pm \frac{4x}{3}$$

$$yz\text{-trace: } z = \pm \frac{4y}{3}$$

$$z = 4, x^2 + y^2 = 9$$



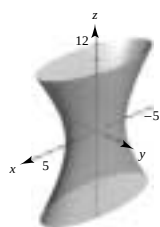
- 60.
- $\frac{x^2}{25} + \frac{y^2}{4} - \frac{z^2}{100} = 1$

Hyperboloid of one sheet

$$xy\text{-trace: } \frac{x^2}{25} + \frac{y^2}{4} = 1$$

$$xz\text{-trace: } \frac{x^2}{25} - \frac{z^2}{100} = 1$$

$$yz\text{-trace: } \frac{y^2}{4} - \frac{z^2}{100} = 1$$



62. Let
- $y = r(x) = 2\sqrt{x}$
- and revolve the curve about the
- x
- axis.

- 64.
- $\left(\frac{\sqrt{3}}{4}, \frac{3}{4}, \frac{3\sqrt{3}}{2}\right)$
- , rectangular

$$(a) r = \sqrt{\left(\frac{\sqrt{3}}{4}\right)^2 + \left(\frac{3}{4}\right)^2} = \frac{\sqrt{3}}{2}, \theta = \arctan \sqrt{3} = \frac{\pi}{3}, z = \frac{3\sqrt{3}}{2}, \left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}, \frac{3\sqrt{3}}{2}\right), \text{ cylindrical}$$

$$(b) \rho = \sqrt{\left(\frac{\sqrt{3}}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2} = \frac{\sqrt{30}}{2}, \theta = \frac{\pi}{3}, \phi = \arccos \frac{3}{\sqrt{10}}, \left(\frac{\sqrt{30}}{2}, \frac{\pi}{3}, \arccos \frac{3}{\sqrt{10}}\right), \text{ spherical}$$

91. $x^2 + y^2 + z^2 = 16$

(a) $r^2 + z^2 = 16$

(b) $\rho^2 = 16, \rho = 4$

95. $x^2 + y^2 = 4y$

(a) $r^2 = 4r \sin \theta, r = 4 \sin \theta$

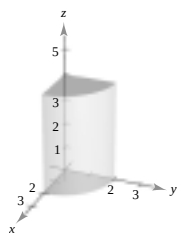
(b) $\rho^2 \sin^2 \phi = 4\rho \sin \phi \sin \theta,$
 $\rho \sin \phi (\rho \sin \phi - 4 \sin \theta) = 0,$

$\rho = \frac{4 \sin \theta}{\sin \phi}, \rho = 4 \sin \theta \csc \phi$

99. $0 \leq \theta \leq \frac{\pi}{2}$

$0 \leq r \leq 2$

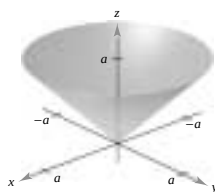
$0 \leq z \leq 4$



101. $0 \leq \theta \leq 2\pi$

$0 \leq r \leq a$

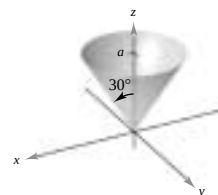
$r \leq z \leq a$



103. $0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \frac{\pi}{6}$

$0 \leq \rho \leq a \sec \phi$

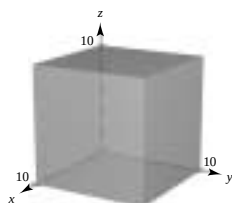


105. Rectangular

$0 \leq x \leq 10$

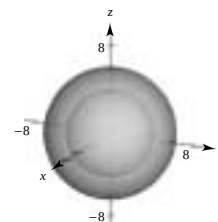
$0 \leq y \leq 10$

$0 \leq z \leq 10$



107. Spherical

$4 \leq \rho \leq 6$



109. $z = \sin \theta, r = 1$

$z = \frac{y}{r} = \frac{y}{1} = y$

The curve of intersection is the ellipse formed by the intersection of the plane $z = y$ and the cylinder $r = 1$.

Review Exercises for Chapter 10

1. $P = (1, 2), Q = (4, 1), R = (5, 4)$

(a) $\mathbf{u} = \overrightarrow{PQ} = \langle 3, -1 \rangle = 3\mathbf{i} - \mathbf{j},$

$\mathbf{v} = \overrightarrow{PR} = \langle 4, 2 \rangle = 4\mathbf{i} + 2\mathbf{j}$

(b) $\|\mathbf{v}\| = \sqrt{4^2 + 2^2} = 2\sqrt{5}$

(c) $2\mathbf{u} + \mathbf{v} = \langle 6, -2 \rangle + \langle 4, 2 \rangle = \langle 10, 0 \rangle = 10\mathbf{i}$

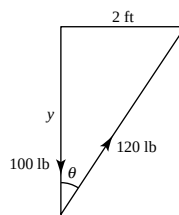
3. $\mathbf{v} = \|\mathbf{v}\| \cos \theta \mathbf{i} + \|\mathbf{v}\| \sin \theta \mathbf{j} = 8 \cos 120^\circ \mathbf{i} + 8 \sin 120^\circ \mathbf{j}$
 $= -4\mathbf{i} + 4\sqrt{3}\mathbf{j}$

5. $120 \cos \theta = 100$

$$\theta = \arccos\left(\frac{5}{6}\right)$$

$$\tan \theta = \frac{2}{y} \Rightarrow y = \frac{2}{\tan \theta}$$

$$y = \frac{2}{\tan[\arccos(5/6)]} = \frac{2}{\sqrt{11}/5} = \frac{10}{\sqrt{11}} \approx 3.015 \text{ ft}$$



7. $z = 0, y = 4, x = -5: (-5, 4, 0)$

9. Looking down from the positive x -axis towards the yz -plane, the point is either in the first quadrant ($y > 0, z > 0$) or in the third quadrant ($y < 0, z < 0$). The x -coordinate can be any number.

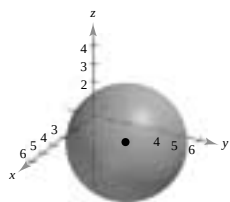
11. $(x - 3)^2 + (y + 2)^2 + (z - 6)^2 = \left(\frac{15}{2}\right)^2$

13. $(x^2 - 4x + 4) + (y^2 - 6y + 9) + z^2 = -4 + 4 + 9$

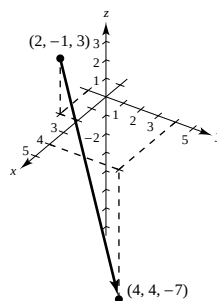
$$(x - 2)^2 + (y - 3)^2 + z^2 = 9$$

Center: $(2, 3, 0)$

Radius: 3



15. $\mathbf{v} = \langle 4 - 2, 4 + 1, -7 - 3 \rangle = \langle 2, 5, -10 \rangle$



17. $\mathbf{v} = \langle -1 - 3, 6 - 4, 9 + 1 \rangle = \langle -4, 2, 10 \rangle$

$$\mathbf{w} = \langle 5 - 3, 3 - 4, -6 + 1 \rangle = \langle 2, -1, -5 \rangle$$

Since $-2\mathbf{w} = \mathbf{v}$, the points lie in a straight line.

19. Unit vector: $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle 2, 3, 5 \rangle}{\sqrt{38}} = \left\langle \frac{2}{\sqrt{38}}, \frac{3}{\sqrt{38}}, \frac{5}{\sqrt{38}} \right\rangle$

21. $P = (5, 0, 0), Q = (4, 4, 0), R = (2, 0, 6)$

(a) $\mathbf{u} = \overrightarrow{PQ} = \langle -1, 4, 0 \rangle = -\mathbf{i} + 4\mathbf{j}$,

$$\mathbf{v} = \overrightarrow{PR} = \langle -3, 0, 6 \rangle = -3\mathbf{i} + 6\mathbf{k}$$

(b) $\mathbf{u} \cdot \mathbf{v} = (-1)(-3) + 4(0) + 0(6) = 3$

(c) $\mathbf{v} \cdot \mathbf{v} = 9 + 36 = 45$

23. $\mathbf{u} = \langle 7, -2, 3 \rangle, \mathbf{v} = \langle -1, 4, 5 \rangle$

Since $\mathbf{u} \cdot \mathbf{v} = 0$, the vectors are orthogonal.

25. $\mathbf{u} = 5\left(\cos \frac{3\pi}{4}\mathbf{i} + \sin \frac{3\pi}{4}\mathbf{j}\right) = \frac{5\sqrt{2}}{2}[-\mathbf{i} + \mathbf{j}]$

$$\mathbf{v} = 2\left(\cos \frac{2\pi}{3}\mathbf{i} + \sin \frac{2\pi}{3}\mathbf{j}\right) = -\mathbf{i} + \sqrt{3}\mathbf{j}$$

$$\mathbf{u} \cdot \mathbf{v} = \frac{5\sqrt{2}}{2}(1 + \sqrt{3})$$

$$\|\mathbf{u}\| = 5$$

$$\|\mathbf{v}\| = 2$$

$$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{(5\sqrt{2}/2)(1 + \sqrt{3})}{5(2)} = \frac{\sqrt{2} + \sqrt{6}}{4}$$

$$\theta = \arccos \frac{\sqrt{2} + \sqrt{6}}{4} = 15^\circ$$

27. $\mathbf{u} = \langle 10, -5, 15 \rangle, \mathbf{v} = \langle -2, 1, -3 \rangle$

$\mathbf{u} = -5\mathbf{v} \Rightarrow \mathbf{u}$ is parallel to $\mathbf{v}$ and in the opposite direction.

$$\theta = \pi$$

29. There are many correct answers. For example: $\mathbf{v} = \pm\langle 6, -5, 0 \rangle$.

In Exercises 31–39, $\mathbf{u} = \langle 3, -2, 1 \rangle$, $\mathbf{v} = \langle 2, -4, -3 \rangle$, $\mathbf{w} = \langle -1, 2, 2 \rangle$.

$$\begin{aligned} 31. \mathbf{u} \cdot \mathbf{u} &= 3(3) + (-2)(-2) + (1)(1) \\ &= 14 = (\sqrt{14})^2 = \|\mathbf{u}\|^2 \end{aligned}$$

$$\begin{aligned} 33. \text{proj}_{\mathbf{u}} \mathbf{w} &= \left(\frac{\mathbf{u} \cdot \mathbf{w}}{\|\mathbf{u}\|^2} \right) \mathbf{u} \\ &= -\frac{5}{14} \langle 3, -2, 1 \rangle \\ &= \left\langle -\frac{15}{14}, \frac{10}{14}, -\frac{5}{14} \right\rangle \\ &= \left\langle -\frac{15}{14}, \frac{5}{7}, -\frac{5}{14} \right\rangle \end{aligned}$$

$$35. \mathbf{n} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ -1 & 2 & 2 \end{vmatrix} = -2\mathbf{i} - \mathbf{j}$$

$$\|\mathbf{n}\| = \sqrt{5}$$

$$\frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{1}{\sqrt{5}}(-2\mathbf{i} - \mathbf{j})$$

$$\begin{aligned} 37. V &= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \\ &= |\langle 3, -2, 1 \rangle \cdot \langle -2, -1, 0 \rangle| = |-4| = 4 \end{aligned}$$

$$\begin{aligned} 39. \text{Area parallelogram} &= \|\mathbf{u} \times \mathbf{v}\| = \sqrt{10^2 + 11^2 + (-8)^2} \quad (\text{See Exercises 36, 38}) \\ &= \sqrt{285} \end{aligned}$$

$$41. \mathbf{F} = c(\cos 20^\circ \mathbf{j} + \sin 20^\circ \mathbf{k})$$

$$\overrightarrow{PQ} = 2\mathbf{k}$$

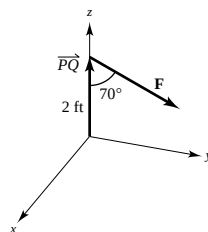
$$\overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 0 & c \cos 20^\circ & c \sin 20^\circ \end{vmatrix} = -2c \cos 20^\circ \mathbf{i}$$

$$200 = \|\overrightarrow{PQ} \times \mathbf{F}\| = 2c \cos 20^\circ$$

$$c = \frac{100}{\cos 20^\circ}$$

$$\mathbf{F} = \frac{100}{\cos 20^\circ}(\cos 20^\circ \mathbf{j} + \sin 20^\circ \mathbf{k}) = 100(\mathbf{j} + \tan 20^\circ \mathbf{k})$$

$$\|\mathbf{F}\| = 100\sqrt{1 + \tan^2 20^\circ} = 100 \sec 20^\circ \approx 106.4 \text{ lb}$$



$$43. \mathbf{v} = \mathbf{j}$$

(a) $x = 1, y = 2 + t, z = 3$

(b) None

$$45. 3x - 3y - 7z = -4, x - y + 2z = 3$$

Solving simultaneously, we have $z = 1$. Substituting $z = 1$ into the second equation we have $y = x - 1$. Substituting for x in this equation we obtain two points on the line of intersection, $(0, -1, 1)$, $(1, 0, 1)$. The direction vector of the line of intersection is $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

(a) $x = t, y = -1 + t, z = 1$

(b) $x = y + 1, z = 1$

47. The two lines are parallel as they have the same direction numbers, $-2, 1, 1$. Therefore, a vector parallel to the plane is $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$. A point on the first line is $(1, 0, -1)$ and a point on the second line is $(-1, 1, 2)$. The vector $\mathbf{u} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ connecting these two points is also parallel to the plane. Therefore, a normal to the plane is

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 2 & -1 & -3 \end{vmatrix} \\ = -2\mathbf{i} - 4\mathbf{j} = -2(\mathbf{i} + 2\mathbf{j}).$$

Equation of the plane: $(x - 1) + 2y = 0$

$$x + 2y = 1$$

51. $Q(3, -2, 4)$ point

$P(5, 0, 0)$ point on plane

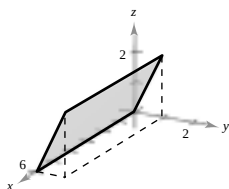
$\mathbf{n} = \langle 2, -5, 1 \rangle$ normal to plane

$$\overrightarrow{PQ} = \langle -2, -2, 4 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{10}{\sqrt{30}} = \frac{\sqrt{30}}{3}$$

55. $y = \frac{1}{2}z$

Plane with rulings parallel to the x -axis



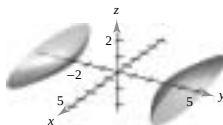
59. $\frac{x^2}{16} - \frac{y^2}{9} + z^2 = -1$

Hyperboloid of two sheets

$$xy\text{-trace: } \frac{y^2}{4} - \frac{x^2}{16} = 1$$

xz -trace: None

$$yz\text{-trace: } \frac{y^2}{9} - z^2 = 1$$



49. $Q = (1, 0, 2)$

$$2x - 3y + 6z = 6$$

A point P on the plane is $(3, 0, 0)$.

$$\overrightarrow{PQ} = \langle -2, 0, 2 \rangle$$

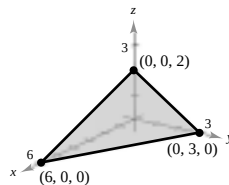
$$\mathbf{n} = \langle 2, -3, 6 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{8}{7}$$

53. $x + 2y + 3z = 6$

Plane

Intercepts: $(6, 0, 0)$, $(0, 3, 0)$, $(0, 0, 2)$



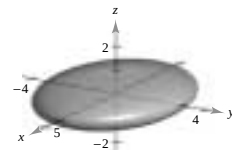
57. $\frac{x^2}{16} + \frac{y^2}{9} + z^2 = 1$

Ellipsoid

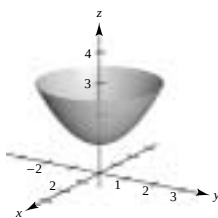
$$xy\text{-trace: } \frac{x^2}{16} + \frac{y^2}{9} = 1$$

$$xz\text{-trace: } \frac{x^2}{16} + z^2 = 1$$

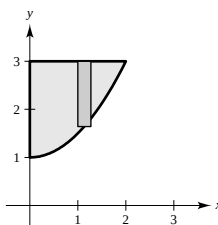
$$yz\text{-trace: } \frac{y^2}{9} + z^2 = 1$$



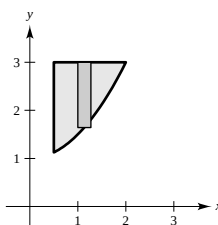
$$\begin{aligned}
 61. \quad (a) \quad x^2 + y^2 &= [r(z)]^2 \\
 &= [\sqrt{2(z-1)}]^2 \\
 x^2 + y^2 - 2z + 2 &= 0
 \end{aligned}$$



$$\begin{aligned}
 (b) \quad V &= 2\pi \int_0^2 x \left[3 - \left(\frac{1}{2}x^2 + 1 \right) \right] dx \\
 &= 2\pi \int_0^2 \left(2x - \frac{1}{2}x^3 \right) dx \\
 &= 2\pi \left[x^2 - \frac{x^4}{8} \right]_0^2 \\
 &= 4\pi \approx 12.6 \text{ cm}^3
 \end{aligned}$$



$$\begin{aligned}
 (c) \quad V &= 2\pi \int_{1/2}^2 x \left[3 - \left(\frac{1}{2}x^2 + 1 \right) \right] dx \\
 &= 2\pi \int_{1/2}^2 \left(2x - \frac{1}{2}x^3 \right) dx \\
 &= 2\pi \left[x^2 - \frac{x^4}{8} \right]_{1/2}^2 \\
 &= 4\pi - \frac{31\pi}{64} = \frac{225\pi}{64} \approx 11.04 \text{ cm}^3
 \end{aligned}$$



63. $(-2\sqrt{2}, 2\sqrt{2}, 2)$, rectangular

(a) $r = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2} = 4$, $\theta = \arctan(-1) = \frac{3\pi}{4}$, $z = 2$, $\left(4, \frac{3\pi}{4}, 2\right)$, cylindrical

(b) $\rho = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2 + (2)^2} = 2\sqrt{5}$, $\theta = \frac{3\pi}{4}$, $\phi = \arccos \frac{2}{2\sqrt{5}} = \arccos \frac{1}{\sqrt{5}}$, $\left(2\sqrt{5}, \frac{3\pi}{4}, \arccos \frac{\sqrt{5}}{5}\right)$, spherical

65. $\left(100, -\frac{\pi}{6}, 50\right)$, cylindrical

$$\rho = \sqrt{100^2 + 50^2} = 50\sqrt{5}$$

$$\theta = -\frac{\pi}{6}$$

$$\phi = \arccos\left(\frac{50}{50\sqrt{5}}\right) = \arccos\left(\frac{1}{\sqrt{5}}\right) \approx 63.4^\circ$$

$$\left(50\sqrt{5}, -\frac{\pi}{6}, 63.4^\circ\right), \text{ spherical}$$

67. $\left(25, -\frac{\pi}{4}, \frac{3\pi}{4}\right)$, spherical

$$r^2 = \left(25 \sin\left(\frac{3\pi}{4}\right)\right)^2 \Rightarrow r = 25\frac{\sqrt{2}}{2}$$

$$\theta = -\frac{\pi}{4}$$

$$z = \rho \cos \phi = 25 \cos \frac{3\pi}{4} = -25\frac{\sqrt{2}}{2}$$

$$\left(25\frac{\sqrt{2}}{2}, -\frac{\pi}{4}, -\frac{25\sqrt{2}}{2}\right), \text{ cylindrical}$$

69. $x^2 - y^2 = 2z$

(a) Cylindrical: $r^2 \cos^2 \theta - r^2 \sin^2 \theta = 2z$, $r^2 \cos 2\theta = 2z$

(b) Spherical: $\rho^2 \sin^2 \phi \cos^2 \theta - \rho^2 \sin^2 \phi \sin^2 \theta = 2\rho \cos \phi$, $\rho \sin^2 \phi \cos 2\theta - 2 \cos \phi = 0$, $\rho = 2 \sec 2\theta \cos \phi \csc^2 \phi$

Problem Solving for Chapter 10

1. $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

$\mathbf{b} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$

$(\mathbf{b} \times \mathbf{a}) + (\mathbf{b} \times \mathbf{c}) = \mathbf{0}$

$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{b} \times \mathbf{c}\|$

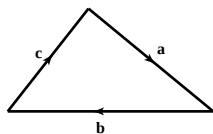
$\|\mathbf{b} \times \mathbf{c}\| = \|\mathbf{b}\| \|\mathbf{c}\| \sin A$

$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin C$

Then,

$$\begin{aligned} \frac{\sin A}{\|\mathbf{a}\|} &= \frac{\|\mathbf{b} \times \mathbf{c}\|}{\|\mathbf{a}\| \|\mathbf{b}\| \|\mathbf{c}\|} \\ &= \frac{\|\mathbf{a} \times \mathbf{b}\|}{\|\mathbf{a}\| \|\mathbf{b}\| \|\mathbf{c}\|} \\ &= \frac{\sin C}{\|\mathbf{c}\|}. \end{aligned}$$

The other case, $\frac{\sin A}{\|\mathbf{a}\|} = \frac{\sin B}{\|\mathbf{b}\|}$ is similar.



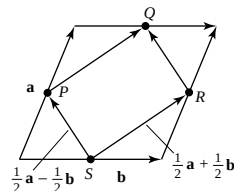
3. Label the figure as indicated.

From the figure, you see that

$$\overrightarrow{SP} = \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b} = \overrightarrow{RQ} \text{ and}$$

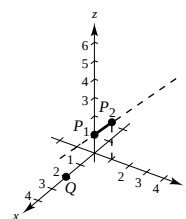
$$\overrightarrow{SR} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} = \overrightarrow{PQ}.$$

Since $\overrightarrow{SP} = \overrightarrow{RQ}$ and $\overrightarrow{SR} = \overrightarrow{PQ}$, PSRQ is a parallelogram.



5. (a) $\mathbf{u} = \langle 0, 1, 1 \rangle$ direction vector of line determined by P_1 and P_2 .

$$\begin{aligned} D &= \frac{\|\overrightarrow{P_1Q} \times \mathbf{u}\|}{\|\mathbf{u}\|} \\ &= \frac{\|\langle 2, 0, -1 \rangle \times \langle 0, 1, 1 \rangle\|}{\sqrt{2}} \\ &= \frac{\|\langle 1, -2, 2 \rangle\|}{\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2} \end{aligned}$$



(b) The shortest distance to the line **segment** is $\|P_1Q\| = \|\langle 2, 0, -1 \rangle\| = \sqrt{5}$.

7. (a) $V = \pi \int_0^1 (\sqrt{2})^2 dz = \left[\pi \frac{z^2}{2} \right]_0^1 = \frac{1}{2} \pi$

Note: $\frac{1}{2}(\text{base})(\text{altitude}) = \frac{1}{2}\pi(1) = \frac{1}{2}\pi$

(b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$: (slice at $z = c$)

$$\frac{x^2}{(\sqrt{ca})^2} + \frac{y^2}{(\sqrt{cb})^2} = 1$$

At $z = c$, figure is ellipse of area

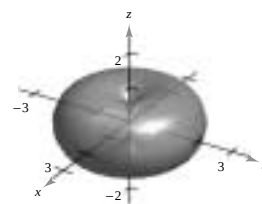
$$\pi(\sqrt{ca})(\sqrt{cb}) = \pi abc.$$

$$V = \int_0^k \pi abc \cdot dc = \left[\frac{\pi abc^2}{2} \right]_0^k = \frac{\pi abk^2}{2}$$

(c) $V = \frac{1}{2}(\pi abk)k = \frac{1}{2}(\text{base})(\text{height})$

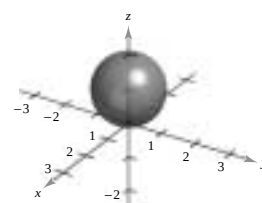
9. (a) $\rho = 2 \sin \phi$

Torus



(b) $\rho = 2 \cos \phi$

Sphere



11. From Exercise 64, Section 10.4, $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{w} \times \mathbf{z}) = [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{z}]\mathbf{w} - [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}]\mathbf{z}$.

13. (a) $\mathbf{u} = \|\mathbf{u}\|(\cos 0 \mathbf{i} + \sin 0 \mathbf{j}) = \|\mathbf{u}\|\mathbf{i}$

Downward force $\mathbf{w} = -\mathbf{j}$

$\mathbf{T} = \|\mathbf{T}\|(\cos(90^\circ + \theta)\mathbf{i} + \sin(90^\circ + \theta)\mathbf{j})$

$= \|\mathbf{T}\|(-\sin \theta \mathbf{i} + \cos \theta \mathbf{j})$

$\mathbf{0} = \mathbf{u} + \mathbf{w} + \mathbf{T} = \|\mathbf{u}\|\mathbf{i} - \mathbf{j} + \|\mathbf{T}\|(-\sin \theta \mathbf{i} + \cos \theta \mathbf{j})$

$\|\mathbf{u}\| = \sin \theta \|\mathbf{T}\|$

$1 = \cos \theta \|\mathbf{T}\|$

If $\theta = 30^\circ$, $\|\mathbf{u}\| = (1/2)\|\mathbf{T}\|$ and $1 = (\sqrt{3}/2)\|\mathbf{T}\|$

$\Rightarrow \|\mathbf{T}\| = \frac{2}{\sqrt{3}} \approx 1.1547 \text{ lb}$

and

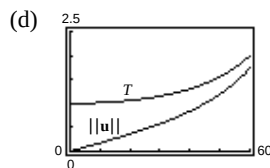
$\|\mathbf{u}\| = \frac{1}{2} \left(\frac{2}{\sqrt{3}} \right) \approx 0.5774 \text{ lb}$

(b) From part (a), $\|\mathbf{u}\| = \tan \theta$ and $\|\mathbf{T}\| = \sec \theta$.

Domain: $0 \leq \theta \leq 90^\circ$

(c)

θ	0°	10°	20°	30°	40°	50°	60°
T	1	1.0154	1.0642	1.1547	1.3054	1.5557	2
$\ \mathbf{u}\ $	0	0.1763	0.3640	0.5774	0.8391	1.1918	1.7321



(e) Both are increasing functions.

(f) $\lim_{\theta \rightarrow \pi/2^-} T = \infty$ and $\lim_{\theta \rightarrow \pi/2^-} \|\mathbf{u}\| = \infty$.

15. Let $\theta = \alpha - \beta$, the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$\sin(\alpha - \beta) = \frac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{\|\mathbf{v} \times \mathbf{u}\|}{\|\mathbf{u}\| \|\mathbf{v}\|}.$

For $\mathbf{u} = \langle \cos \alpha, \sin \alpha, 0 \rangle$ and $\mathbf{v} = \langle \cos \beta, \sin \beta, 0 \rangle$, $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$ and

$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \beta & \sin \beta & 0 \\ \cos \alpha & \sin \alpha & 0 \end{vmatrix} = (\sin \alpha \cos \beta - \cos \alpha \sin \beta)\mathbf{k}.$

Thus, $\sin(\alpha - \beta) = \|\mathbf{v} \times \mathbf{u}\| = \sin \alpha \cos \beta - \cos \alpha \sin \beta.$

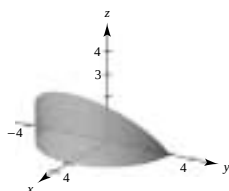
17. From Theorem 10.13 and Theorem 10.7 (6) we have

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

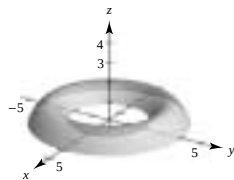
$$= \frac{|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{\|\mathbf{u} \times \mathbf{v}\|}.$$

19. a_1, b_1, c_1 , and a_2, b_2, c_2 are two sets of direction numbers for the same line. The line is parallel to both $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$. Therefore, $\mathbf{u}$ and $\mathbf{v}$ are parallel, and there exists a scalar d such that $\mathbf{u} = d\mathbf{v}$, $a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k} = d(a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k})$, $a_1 = a_2d$, $b_1 = b_2d$, $c_1 = c_2d$.

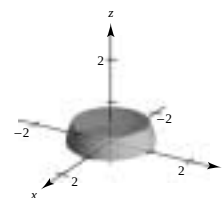
100. $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
 $0 \leq r \leq 3$
 $0 \leq z \leq r \cos \theta$



102. $0 \leq \theta \leq 2\pi$
 $2 \leq r \leq 4$
 $z^2 \leq -r^2 + 6r - 8$



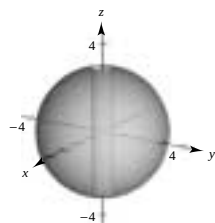
104. $0 \leq \theta \leq 2\pi$
 $\frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}$
 $0 \leq \rho \leq 1$



106. Cylindrical: $0.75 \leq r \leq 1.25, z = 8$

108. Cylindrical

$\frac{1}{2} \leq r \leq 3$
 $0 \leq \theta \leq 2\pi$
 $-\sqrt{9-r^2} \leq z \leq \sqrt{9-r^2}$



110. $\rho = 2 \sec \phi \Rightarrow \rho \cos \phi = 2 \Rightarrow z = 2$ plane

$\rho = 4$ sphere

The intersection of the plane and the sphere is a circle.

Review Exercises for Chapter 10

2. $P = (-2, -1), Q = (5, -1), R = (2, 4)$

(a) $\mathbf{u} = \overrightarrow{PQ} = \langle 7, 0 \rangle = 7\mathbf{i}, \mathbf{v} = \overrightarrow{PR} = \langle 4, 5 \rangle = 4\mathbf{i} + 5\mathbf{j}$

(b) $\|\mathbf{v}\| = \sqrt{4^2 + 5^2} = \sqrt{41}$

(c) $2\mathbf{u} + \mathbf{v} = 14\mathbf{i} + (4\mathbf{i} + 5\mathbf{j}) = 18\mathbf{i} + 5\mathbf{j}$

4. $\mathbf{v} = \|\mathbf{v}\| \cos \theta \mathbf{i} + \|\mathbf{v}\| \sin \theta \mathbf{j} = \frac{1}{2} \cos 225^\circ \mathbf{i} + \frac{1}{2} \sin 225^\circ \mathbf{j}$
 $= -\frac{\sqrt{2}}{4} \mathbf{i} + \frac{\sqrt{2}}{4} \mathbf{j}$

6. (a) The length of cable POQ is L .

$\overrightarrow{OQ} = 9\mathbf{i} - y\mathbf{j}$

$L = 2\sqrt{9^2 + y^2} \Rightarrow \sqrt{\frac{L^2}{4} - 81} = y$

Tension: $T = c\|\overrightarrow{OQ}\| = c\sqrt{81 + y^2}$

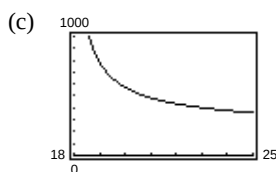
Also,

$cy = 250 \Rightarrow T = \frac{250}{y} \sqrt{81 + y^2} \Rightarrow T = \frac{250}{\sqrt{(L^2/4) - 81}} \cdot \frac{L}{2} = \frac{250L}{\sqrt{L^2 - 324}}$

Domain: $L > 18$ inches

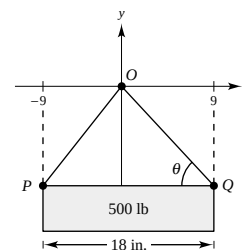
(b)

L	19	20	21	22	23	24	25
T	780.9	573.54	485.36	434.81	401.60	377.96	360.24



(d) The line $T = 400$ intersects the curve at

$L = 23.06$ inches.



(e) $\lim_{L \rightarrow \infty} T = 250$

The maximum tension is 250 pounds in each side of the cable since the total weight is 500 pounds.

8. $x = z = 0, y = -7: (0, -7, 0)$

10. Looking towards the xy -plane from the positive z -axis. The point is either in the second quadrant ($x < 0, y > 0$) or in the fourth quadrant ($x > 0, y < 0$). The z -coordinate can be any number.

12. Center: $\left(\frac{0+4}{2}, \frac{0+6}{2}, \frac{4+0}{2}\right) = (2, 3, 2)$

Radius: $\sqrt{(2-0)^2 + (3-0)^2 + (2-4)^2} = \sqrt{4+9+4} = \sqrt{17}$

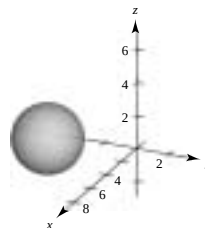
$(x-2)^2 + (y-3)^2 + (z-2)^2 = 17$

14. $(x^2 - 10x + 25) + (y^2 + 6y + 9) + (z^2 - 4z + 4) = -34 + 25 + 9 + 4$

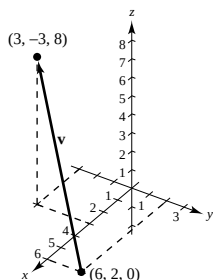
$(x-5)^2 + (y+3)^2 + (z-2)^2 = 4$

Center: $(5, -3, 2)$

Radius: 2



16. $\mathbf{v} = \langle 3-6, -3-2, 8-0 \rangle = \langle -3, -5, 8 \rangle$



18. $\mathbf{v} = \langle 8-5, -5+4, 5-7 \rangle = \langle 3, -1, -2 \rangle$

$\mathbf{w} = \langle 11-5, 6+4, 3-7 \rangle = \langle 6, 10, -4 \rangle$

Since $\mathbf{v}$ and $\mathbf{w}$ are not parallel, the points do not lie in a straight line.

20. $8 \frac{\langle 6, -3, 2 \rangle}{\sqrt{49}} = \frac{8}{7} \langle 6, -3, 2 \rangle = \left\langle \frac{48}{7}, -\frac{24}{7}, \frac{16}{7} \right\rangle$

22. $P = (2, -1, 3), Q = (0, 5, 1), R = (5, 5, 0)$

(a) $\mathbf{u} = \overrightarrow{PQ} = \langle -2, 6, -2 \rangle = -2\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$

$\mathbf{v} = \overrightarrow{PR} = \langle 3, 6, -3 \rangle = 3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$

(b) $\mathbf{u} \cdot \mathbf{v} = (-2)(3) + (6)(6) + (-2)(-3) = 36$

(c) $\mathbf{v} \cdot \mathbf{v} = 9 + 36 + 9 = 54$

24. $\mathbf{u} = \langle -4, 3, -6 \rangle, \mathbf{v} = \langle 16, -12, 24 \rangle$

Since $\mathbf{v} = -4\mathbf{u}$, the vectors are parallel.

26. $\mathbf{u} = \langle 4, -1, 5 \rangle, \mathbf{v} = \langle 3, 2, -2 \rangle$

$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow$ is orthogonal to $\mathbf{v}$.

$\theta = \frac{\pi}{2}$

28. $\mathbf{u} = \langle 1, 0, -3 \rangle$

$\mathbf{v} = \langle 2, -2, 1 \rangle$

$\mathbf{u} \cdot \mathbf{v} = -1$

$\|\mathbf{u}\| = \sqrt{10}$

$\|\mathbf{v}\| = 3$

$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{1}{3\sqrt{10}}$

$\theta \approx 83.9^\circ$

30. $W = \mathbf{F} \cdot \overrightarrow{PQ} = \|\mathbf{F}\| \|\overrightarrow{PQ}\| \cos \theta = (75)(8) \cos 30^\circ$
 $= 300\sqrt{3} \text{ ft} \cdot \text{lb}$

In Exercises 32–40, $\mathbf{u} = \langle 3, -2, 1 \rangle$, $\mathbf{v} = \langle 2, -4, -3 \rangle$, $\mathbf{w} = \langle -1, 2, 2 \rangle$.

$$32. \cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{11}{\sqrt{14}\sqrt{29}}$$

$$\theta = \arccos\left(\frac{11}{\sqrt{14}\sqrt{29}}\right) \approx 56.9^\circ$$

$$34. \text{Work} = |\mathbf{u} \cdot \mathbf{w}| = |-3 - 4 + 2| = 5$$

$$36. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 2 & -4 & -3 \end{vmatrix} = 10\mathbf{i} + 11\mathbf{j} - 8\mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ 3 & -2 & 1 \end{vmatrix} = -10\mathbf{i} - 11\mathbf{j} + 8\mathbf{k}$$

Thus, $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.

$$38. \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \langle 3, -2, 1 \rangle \times \langle 1, -2, -1 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 1 & -2 & -1 \end{vmatrix} = 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 2 & -4 & -3 \end{vmatrix} = 10\mathbf{i} + 11\mathbf{j} - 8\mathbf{k}$$

$$\mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ -1 & 2 & 2 \end{vmatrix} = -6\mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$$

$$(\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) = 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k} = \mathbf{u} \times (\mathbf{v} + \mathbf{w})$$

$$40. \text{Area triangle} = \frac{1}{2}\|\mathbf{v} \times \mathbf{w}\| = \frac{1}{2}\sqrt{(-2)^2 + (-1)^2} = \frac{\sqrt{5}}{2} \quad (\text{See Exercise 35})$$

$$42. V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 2(5) = 10$$

44. Direction numbers: 1, 1, 1

$$(a) x = 1 + t, y = 2 + t, z = 3 + t$$

$$(b) x - 1 = y - 2 = z - 3$$

$$46. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5 & 1 \\ -3 & 1 & 4 \end{vmatrix} = -21\mathbf{i} - 11\mathbf{j} - 13\mathbf{k}$$

Direction numbers: 21, 11, 13

$$(a) x = 21t, y = 1 + 11t, z = 4 + 13t$$

$$(b) \frac{x}{21} = \frac{y-1}{11} = \frac{z-4}{13}$$

48. $P = (-3, -4, 2)$, $Q = (-3, 4, 1)$, $R = (1, 1, -2)$

$$\overrightarrow{PQ} = \langle 0, 8, -1 \rangle, \overrightarrow{PR} = \langle 4, 5, -4 \rangle$$

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 8 & -1 \\ 4 & 5 & -4 \end{vmatrix} = -27\mathbf{i} - 4\mathbf{j} - 32\mathbf{k}$$

$$-27(x + 3) - 4(y + 4) - 32(z - 2) = 0$$

$$27x + 4y + 32z = -33$$

50. The normal vectors to the planes are the same,

$$\mathbf{n} = \langle 5, -3, 1 \rangle.$$

Choose a point in the first plane, $P = (0, 0, 2)$. Choose a point in the second plane, $Q = (0, 0, -3)$.

$$\overrightarrow{PQ} = \langle 0, 0, -5 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-5|}{\sqrt{35}} = \frac{5}{\sqrt{35}} = \frac{\sqrt{35}}{7}$$

- 52.
- $Q(-5, 1, 3)$
- point

$$\mathbf{u} = \langle 1, -2, -1 \rangle \text{ direction vector}$$

$$P = (1, 3, 5) \text{ point on line}$$

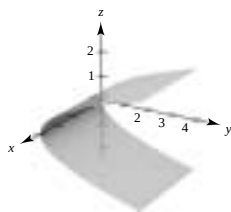
$$\overrightarrow{PQ} = \langle -6, -2, -2 \rangle$$

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & -2 & -2 \\ 1 & -2 & -1 \end{vmatrix} = \langle -2, -8, 14 \rangle$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{264}}{\sqrt{6}} = 2\sqrt{11}$$

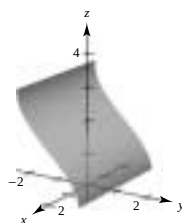
- 54.
- $y = z^2$

Since the x -coordinate is missing, we have a cylindrical surface with rulings parallel to the x -axis. The generating curve is a parabola in the yz -coordinate plane.



- 56.
- $y = \cos z$

Since the x -coordinate is missing, we have a cylindrical surface with rulings parallel to the x -axis. The generating curve is $y = \cos z$.



- 58.
- $16x^2 + 16y^2 - 9z^2 = 0$

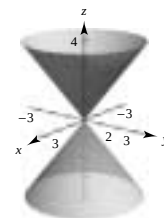
Cone

xy -trace: point $(0, 0, 0)$

$$xz\text{-trace: } z = \pm \frac{4x}{3}$$

$$yz\text{-trace: } z = \pm \frac{4y}{3}$$

$$z = 4, x^2 + y^2 = 9$$



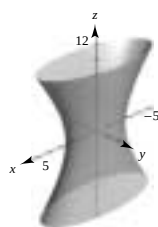
- 60.
- $\frac{x^2}{25} + \frac{y^2}{4} - \frac{z^2}{100} = 1$

Hyperboloid of one sheet

$$xy\text{-trace: } \frac{x^2}{25} + \frac{y^2}{4} = 1$$

$$xz\text{-trace: } \frac{x^2}{25} - \frac{z^2}{100} = 1$$

$$yz\text{-trace: } \frac{y^2}{4} - \frac{z^2}{100} = 1$$



62. Let
- $y = r(x) = 2\sqrt{x}$
- and revolve the curve about the
- x
- axis.

- 64.
- $\left(\frac{\sqrt{3}}{4}, \frac{3}{4}, \frac{3\sqrt{3}}{2}\right)$
- , rectangular

$$(a) r = \sqrt{\left(\frac{\sqrt{3}}{4}\right)^2 + \left(\frac{3}{4}\right)^2} = \frac{\sqrt{3}}{2}, \theta = \arctan \sqrt{3} = \frac{\pi}{3}, z = \frac{3\sqrt{3}}{2}, \left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}, \frac{3\sqrt{3}}{2}\right), \text{ cylindrical}$$

$$(b) \rho = \sqrt{\left(\frac{\sqrt{3}}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2} = \frac{\sqrt{30}}{2}, \theta = \frac{\pi}{3}, \phi = \arccos \frac{3}{\sqrt{10}}, \left(\frac{\sqrt{30}}{2}, \frac{\pi}{3}, \arccos \frac{3}{\sqrt{10}}\right), \text{ spherical}$$

66. $\left(81, -\frac{5\pi}{6}, 27\sqrt{3}\right)$, cylindrical

$$\rho = \sqrt{6561 + 2187} = 54\sqrt{3}$$

$$\theta = -\frac{5\pi}{6}$$

$$\phi = \arccos\left(\frac{27\sqrt{3}}{54\sqrt{3}}\right) = \arccos\frac{1}{2} = \frac{\pi}{3}$$

$$\left(54\sqrt{3}, -\frac{5\pi}{6}, \frac{\pi}{3}\right), \text{ spherical}$$

68. $\left(12, -\frac{\pi}{2}, \frac{2\pi}{3}\right)$, spherical

$$r^2 = \left(12 \sin\left(\frac{2\pi}{3}\right)\right)^2 \Rightarrow r = 6\sqrt{3}$$

$$\theta = -\frac{\pi}{2}$$

$$z = \rho \cos \phi = 12 \cos\left(\frac{2\pi}{3}\right) = -6$$

$$\left(6\sqrt{3}, -\frac{\pi}{2}, -6\right), \text{ cylindrical}$$

70. $x^2 + y^2 + z^2 = 16$

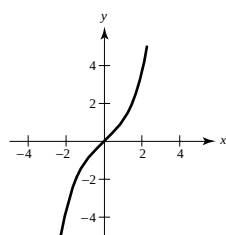
(a) Cylindrical: $r^2 + z^2 = 16$

(b) Spherical: $\rho = 4$

Problem Solving for Chapter 10

2. $f(x) = \int_0^x \sqrt{t^4 + 1} dt$

(a)



(c) $\pm \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$

(b) $f'(x) = \sqrt{x^4 + 1}$

$$f'(0) = 1 = \tan \theta$$

$$\theta = \frac{\pi}{4}$$

$$\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

(d) The line is $y = x$: $x = t, y = t$.

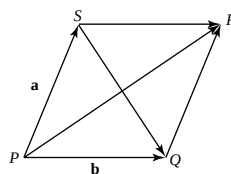
4. Label the figure as indicated.

$$\overrightarrow{PR} = \mathbf{a} + \mathbf{b}$$

$$\overrightarrow{SQ} = \mathbf{b} - \mathbf{a}$$

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} - \mathbf{a}) = \|\mathbf{b}\|^2 - \|\mathbf{a}\|^2 = 0, \text{ because}$$

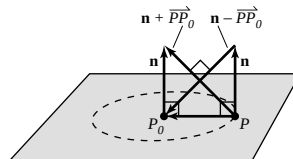
$$\|\mathbf{a}\| = \|\mathbf{b}\| \text{ in a rhombus.}$$



6. $(\mathbf{n} + \overrightarrow{PP_0}) \perp (\mathbf{n} - \overrightarrow{PP_0})$

Figure is a square.

Thus, $\|\overrightarrow{PP_0}\| = \|\mathbf{n}\|$ and the points P form a circle of radius $\|\mathbf{n}\|$ in the plane with center at P_0 .



$$8. (a) V = 2 \int_0^r \pi(r^2 - x^2) dx = 2\pi \left[r^2x - \frac{x^3}{3} \right]_0^r = \frac{4}{3} \pi r^3$$

(b) At height $z = d > 0$,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{d^2}{c^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{d^2}{c^2} = \frac{c^2 - d^2}{c^2}$$

$$\frac{\frac{x^2}{a^2}}{\frac{c^2 - d^2}{c^2}} + \frac{\frac{y^2}{b^2}}{\frac{c^2 - d^2}{c^2}} = 1.$$

$$\text{Area} = \pi \sqrt{\left(\frac{a^2(c^2 - d^2)}{c^2} \right) \left(\frac{b^2(c^2 - d^2)}{c^2} \right)} = \frac{\pi ab}{c^2} (c^2 - d^2)$$

$$V = 2 \int_0^c \frac{\pi ab}{c^2} (c^2 - d^2) dd$$

$$= \frac{2\pi ab}{c^2} \left[c^2d - \frac{d^3}{3} \right]_0^c$$

$$= \frac{4}{3} \pi abc$$

10. (a) $r = 2 \cos \theta$

Cylinder

(b) $z = r^2 \cos 2\theta$

$$z^2 = x^2 - y^2$$

Hyperbolic paraboloid

12. $x = -t + 3, y = \frac{1}{2}t + 1, z = 2t - 1; Q = (4, 3, s)$

(a) $\mathbf{u} = \langle -2, 1, 4 \rangle$ direction vector for line

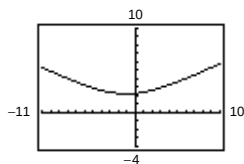
$P = (3, 1, -1)$ point on line

$$\overrightarrow{PQ} = \langle 1, 2, s + 1 \rangle$$

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & s + 1 \\ -2 & 1 & 4 \end{vmatrix} = (7 - s)\mathbf{i} + (-6 - 2s)\mathbf{j} + 5\mathbf{k}$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{(7 - s)^2 + (-6 - 2s)^2 + 25}}{\sqrt{21}}$$

(b)



The minimum is $D \approx 2.2361$ at $s = -1$.

(c) Yes, there are slant asymptotes. Using $s = x$, we have

$$D(s) = \frac{1}{\sqrt{21}} \sqrt{5x^2 + 10x + 110} = \frac{\sqrt{5}}{\sqrt{21}} \sqrt{x^2 + 2x + 22}$$

$$= \frac{\sqrt{5}}{\sqrt{21}} \sqrt{(x + 1)^2 + 21} \rightarrow \pm \sqrt{\frac{5}{21}} (x + 1)$$

$$y = \pm \frac{\sqrt{105}}{21} (s + 1) \text{ slant asymptotes.}$$

14. (a) The tension T is the same in each tow line.

$$\begin{aligned} 6000\mathbf{i} &= T(\cos 20^\circ + \cos(-20^\circ))\mathbf{i} + T(\sin 20^\circ + \sin(-20^\circ))\mathbf{j} \\ &= 2T \cos 20^\circ \mathbf{i} \end{aligned}$$

$$\Rightarrow T = \frac{6000}{2 \cos 20^\circ} \approx 3192.5 \text{ lbs}$$

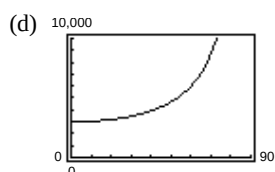
- (b) As in part (a), $6000\mathbf{i} = 2T \cos \theta$

$$\Rightarrow T = \frac{3000}{\cos \theta}$$

Domain: $0 < \theta < 90^\circ$

(c)

θ	10°	20°	30°	40°	50°	60°
T	3046.3	3192.5	3464.1	3916.2	4667.2	6000.0



- (e) As θ increases, there is less force applied in the direction of motion.

16. (a) Los Angeles: $(4000, -118.24^\circ, 55.95^\circ)$

Rio de Janeiro: $(4000, -43.22^\circ, 112.90^\circ)$

- (b) Los Angeles: $x = 4000 \sin 55.95^\circ \cos(-118.24^\circ)$

$$y = 4000 \sin 55.95^\circ \sin(-118.24^\circ)$$

$$z = 4000 \cos 55.95^\circ$$

$$(-1568.2, -2919.7, 2239.7)$$

Rio de Janeiro: $x = (4000 \sin 112.90^\circ \cos(-43.22^\circ))$

$$y = 4000 \sin 112.90^\circ \sin(-43.22^\circ)$$

$$z = 4000 \cos 112.90^\circ$$

$$(2685.2, -2523.3, -1556.5)$$

$$(c) \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{(-1568.2)(2685.2) + (-2919.7)(-2523.3) + (2239.7)(-1556.5)}{(4000)(4000)}$$

$$\theta \approx 91.18^\circ \approx 1.59 \text{ radians}$$

- (d) $s = 4000(1.59) \approx 6366 \text{ miles}$

—CONTINUED—

16. —CONTINUED—

(e) For Boston and Honolulu:

a. Boston: $(4000, -71.06^\circ, 47.64^\circ)$ Honolulu: $(4000, -157.86^\circ, 68.69^\circ)$ b. Boston: $x = 4000 \sin 47.64^\circ \cos(-71.06^\circ)$

$$y = 4000 \sin 47.64^\circ \sin(-71.06^\circ)$$

$$z = 4000 \cos 47.64^\circ$$

$$(959.4, -2795.7, 2695.1)$$

Honolulu: $x = (4000 \sin 68.69^\circ \cos(-157.86^\circ))$

$$y = 4000 \sin 68.69^\circ \sin(-157.86^\circ)$$

$$z = 4000 \cos 68.69^\circ$$

$$(-3451.7, -1404.4, 1453.7)$$

$$(f) \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{(959.4)(-3451.7) + (-2795.7)(-1404.4) + (2695.1)(1453.7)}{(4000)(4000)}$$

$$\theta \approx 73.5^\circ \approx 1.28 \text{ radians}$$

$$(g) s = 4000(1.28) \approx 5120 \text{ miles}$$

18. Assume one of a, b, c , is not zero, say a . Choose a point in the first plane such as $(-d_1/a, 0, 0)$. The distance between this point and the second plane is

$$\begin{aligned} D &= \frac{|a(-d_1/a) + b(0) + c(0) + d_2|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|-d_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}. \end{aligned}$$

20. Essay.

CHAPTER 11

Vector-Valued Functions

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CHAPTER 11

Vector-Valued Functions

Section 11.1 Vector-Valued Functions

Solutions to Odd-Numbered Exercises

$$1. \mathbf{r}(t) = 5t\mathbf{i} - 4t\mathbf{j} - \frac{1}{t}\mathbf{k}$$

Component functions: $f(t) = 5t$

$$g(t) = -4t$$

$$h(t) = -\frac{1}{t}$$

Domain: $(-\infty, 0) \cup (0, \infty)$

$$3. \mathbf{r}(t) = \ln t\mathbf{i} - e^t\mathbf{j} - t\mathbf{k}$$

Component functions: $f(t) = \ln t$

$$g(t) = -e^t$$

$$h(t) = -t$$

Domain: $(0, \infty)$

$$5. \mathbf{r}(t) = \mathbf{F}(t) + \mathbf{G}(t) = (\cos t\mathbf{i} - \sin t\mathbf{j} + \sqrt{t}\mathbf{k}) + (\cos t\mathbf{i} + \sin t\mathbf{j}) = 2\cos t\mathbf{i} + \sqrt{t}\mathbf{k}$$

Domain: $[0, \infty)$

$$7. \mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin t & \cos t & 0 \\ 0 & \sin t & \cos t \end{vmatrix} = \cos^2 t\mathbf{i} - \sin t \cos t\mathbf{j} + \sin^2 t\mathbf{k}$$

Domain: $(-\infty, \infty)$

$$9. \mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - (t-1)\mathbf{j}$$

$$(a) \mathbf{r}(1) = \frac{1}{2}\mathbf{i}$$

$$(b) \mathbf{r}(0) = \mathbf{j}$$

$$(c) \mathbf{r}(s+1) = \frac{1}{2}(s+1)^2\mathbf{i} - (s+1-1)\mathbf{j} = \frac{1}{2}(s+1)^2\mathbf{i} - s\mathbf{j}$$

$$\begin{aligned} (d) \mathbf{r}(2+\Delta t) - \mathbf{r}(2) &= \frac{1}{2}(2+\Delta t)^2\mathbf{i} - (2+\Delta t-1)\mathbf{j} - (2\mathbf{i} - \mathbf{j}) \\ &= (2+2\Delta t + \frac{1}{2}(\Delta t)^2)\mathbf{i} - (1+\Delta t)\mathbf{j} - 2\mathbf{i} + \mathbf{j} \\ &= (2\Delta t + \frac{1}{2}(\Delta t)^2)\mathbf{i} - (\Delta t)\mathbf{j} \end{aligned}$$

$$11. \mathbf{r}(t) = \ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + 3t\mathbf{k}$$

$$(a) \mathbf{r}(2) = \ln 2\mathbf{i} + \frac{1}{2}\mathbf{j} + 6\mathbf{k}$$

$$(b) \mathbf{r}(-3) \text{ is not defined. } (\ln(-3) \text{ does not exist.})$$

$$(c) \mathbf{r}(t-4) = \ln(t-4)\mathbf{i} + \frac{1}{t-4}\mathbf{j} + 3(t-4)\mathbf{k}$$

$$\begin{aligned} (d) \mathbf{r}(1+\Delta t) - \mathbf{r}(1) &= \ln(1+\Delta t)\mathbf{i} + \frac{1}{1+\Delta t}\mathbf{j} + 3(1+\Delta t)\mathbf{k} - (0\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \\ &= \ln(1+\Delta t)\mathbf{i} + \left(\frac{1}{1+\Delta t} - 1\right)\mathbf{j} + (3\Delta t)\mathbf{k} \end{aligned}$$

13. $\mathbf{r}(t) = \sin 3t\mathbf{i} + \cos 3t\mathbf{j} + t\mathbf{k}$

$$\|\mathbf{r}(t)\| = \sqrt{(\sin 3t)^2 + (\cos 3t)^2 + t^2} = \sqrt{1 + t^2}$$

17. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}, -2 \leq t \leq 2$

$$x = t, y = 2t, z = t^2$$

Thus, $z = x^2$. Matches (b)

21. (a) View from the negative x -axis: $(-20, 0, 0)$

(c) View from the z -axis: $(0, 0, 20)$

15. $\mathbf{r}(t) \cdot \mathbf{u}(t) = (3t - 1)(t^2) + \left(\frac{1}{4}t^3\right)(-8) + 4(t^3)$
$$= 3t^3 - t^2 - 2t^3 + 4t^3 = 5t^3 - t^2, \text{ a scalar.}$$

The dot product is a scalar-valued function.

19. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\mathbf{k}, -2 \leq t \leq 2$

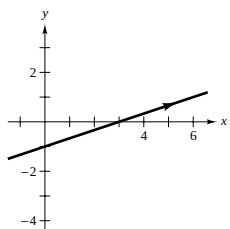
$$x = t, y = t^2, z = e^{0.75t}$$

Thus, $y = x^2$. Matches (d)

23. $x = 3t$

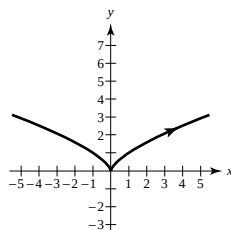
$$y = t - 1$$

$$y = \frac{x}{3} - 1$$



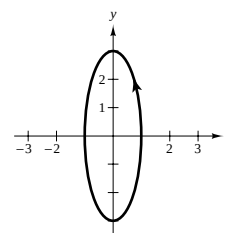
25. $x = t^3, y = t^2$

$$y = x^{2/3}$$



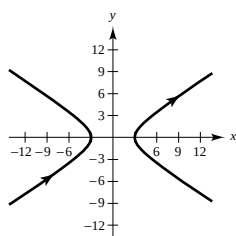
27. $x = \cos \theta, y = 3 \sin \theta$

$$x^2 + \frac{y^2}{9} = 1 \text{ Ellipse}$$



29. $x = 3 \sec \theta, y = 2 \tan \theta$

$$\frac{x^2}{9} = \frac{y^2}{4} + 1 \text{ Hyperbola}$$



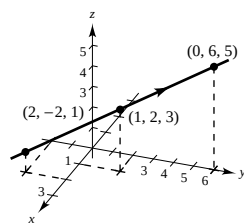
31. $x = -t + 1$

$$y = 4t + 2$$

$$z = 2t + 3$$

Line passing through the points:

$$(0, 6, 5), (1, 2, 3)$$

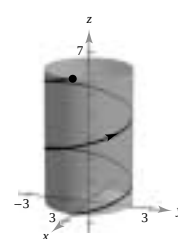


33. $x = 2 \cos t, y = 2 \sin t, z = t$

$$\frac{x^2}{4} + \frac{y^2}{4} = 1$$

$$z = t$$

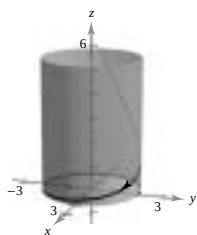
Circular helix



35. $x = 2 \sin t, y = 2 \cos t, z = e^{-t}$

$$x^2 + y^2 = 4$$

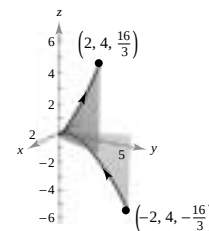
$$z = e^{-t}$$



37. $x = t, y = t^2, z = \frac{2}{3}t^3$

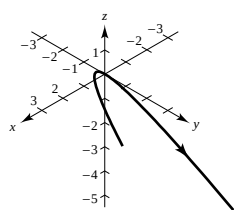
$$y = x^2, z = \frac{2}{3}x^3$$

t	-2	-1	0	1	2
x	-2	-1	0	1	2
y	4	1	0	1	4
z	$-\frac{16}{3}$	$-\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{16}{3}$



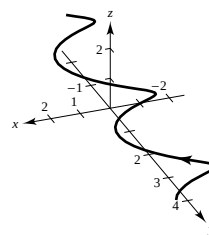
39. $\mathbf{r}(t) = -\frac{1}{2}t^2\mathbf{i} + t\mathbf{j} - \frac{\sqrt{3}}{2}t^2\mathbf{k}$

Parabola

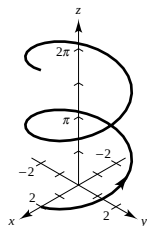


41. $\mathbf{r}(t) = \sin t\mathbf{i} + \left(\frac{\sqrt{3}}{2}\cos t - \frac{1}{2}t\right)\mathbf{j} + \left(\frac{1}{2}\cos t + \frac{\sqrt{3}}{2}\right)\mathbf{k}$

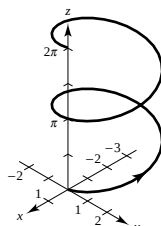
Helix



43.

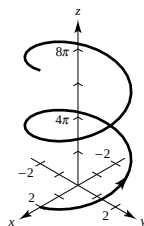


(a)



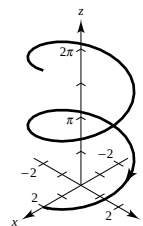
The helix is translated 2 units back on the x-axis.

(b)



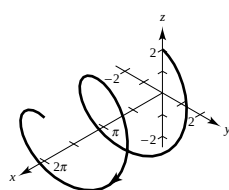
The height of the helix increases at a faster rate.

(c)



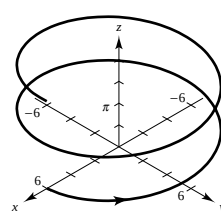
The orientation of the helix is reversed.

(d)



The axis of the helix is the x-axis.

(e)



The radius of the helix is increased from 2 to 6.

45. $y = 4 - x$

 Let $x = t$, then $y = 4 - t$.

$$\mathbf{r}(t) = t\mathbf{i} + (4 - t)\mathbf{j}$$

47. $y = (x - 2)^2$

 Let $x = t$, then $y = (t - 2)^2$.

$$\mathbf{r}(t) = t\mathbf{i} + (t - 2)^2\mathbf{j}$$

49. $x^2 + y^2 = 25$

 Let $x = 5 \cos t$, then $y = 5 \sin t$.

$$\mathbf{r}(t) = 5 \cos t\mathbf{i} + 5 \sin t\mathbf{j}$$

51. $\frac{x^2}{16} - \frac{y^2}{4} = 1$

 Let $x = 4 \sec t$, $y = 2 \tan t$.

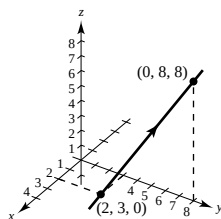
$$\mathbf{r}(t) = 4 \sec t\mathbf{i} + 2 \tan t\mathbf{j}$$

53. The parametric equations for the line are

$$x = 2 - 2t, y = 3 + 5t, z = 8t.$$

One possible answer is

$$\mathbf{r}(t) = (2 - 2t)\mathbf{i} + (3 + 5t)\mathbf{j} + 8t\mathbf{k}.$$



55. $\mathbf{r}_1(t) = t\mathbf{i}, \quad 0 \leq t \leq 4 \quad (\mathbf{r}_1(0) = \mathbf{0}, \mathbf{r}_1(4) = 4\mathbf{i})$

$$\mathbf{r}_2(t) = (4 - 4t)\mathbf{i} + 6t\mathbf{j}, \quad 0 \leq t \leq 1 \quad (\mathbf{r}_2(0) = 4\mathbf{i}, \mathbf{r}_2(1) = 6\mathbf{j})$$

$$\mathbf{r}_3(t) = (6 - t)\mathbf{j}, \quad 0 \leq t \leq 6 \quad (\mathbf{r}_3(0) = 6\mathbf{j}, \mathbf{r}_3(6) = \mathbf{0})$$

(Other answers possible)

57. $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}$, $0 \leq t \leq 2$ ($y = x^2$)

$\mathbf{r}_2(t) = (2 - t)\mathbf{i}$, $0 \leq t \leq 2$

$\mathbf{r}_3(t) = (4 - t)\mathbf{j}$, $0 \leq t \leq 4$

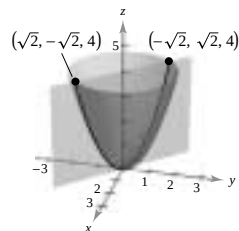
(Other answers possible)

59. $z = x^2 + y^2$, $x + y = 0$

Let $x = t$, then $y = -x = -t$ and $z = x^2 + y^2 = 2t^2$.
Therefore,

$x = t$, $y = -t$, $z = 2t^2$.

$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$



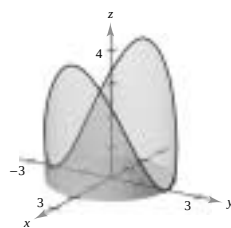
61. $x^2 + y^2 = 4$, $z = x^2$

$x = 2 \sin t$, $y = 2 \cos t$

$z = x^2 = 4 \sin^2 t$

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
x	0	1	$\sqrt{2}$	2	$\sqrt{2}$	0
y	2	$\sqrt{3}$	$\sqrt{2}$	0	$-\sqrt{2}$	-2
z	0	1	2	4	2	0

$\mathbf{r}(t) = 2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + 4 \sin^2 t \mathbf{k}$



63. $x^2 + y^2 + z^2 = 4$, $x + z = 2$

Let $x = 1 + \sin t$, then $z = 2 - x = 1 - \sin t$ and $x^2 + y^2 + z^2 = 4$.

$(1 + \sin t)^2 + y^2 + (1 - \sin t)^2 = 2 + 2 \sin^2 t + y^2 = 4$

$y^2 = 2 \cos^2 t$, $y = \pm \sqrt{2} \cos t$

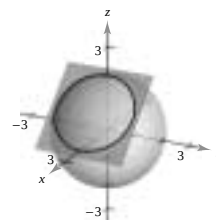
$x = 1 + \sin t$, $y = \pm \sqrt{2} \cos t$

$z = 1 - \sin t$

t	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$
x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	0	$\pm \frac{\sqrt{6}}{2}$	$\pm \sqrt{2}$	$\pm \frac{\sqrt{6}}{2}$	0
z	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0

$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} + \sqrt{2} \cos t \mathbf{j} + (1 - \sin t)\mathbf{k}$ and

$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} - \sqrt{2} \cos t \mathbf{j} + (1 - \sin t)\mathbf{k}$

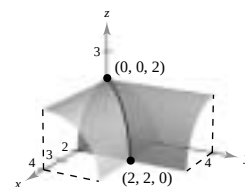


65. $x^2 + z^2 = 4$, $y^2 + z^2 = 4$

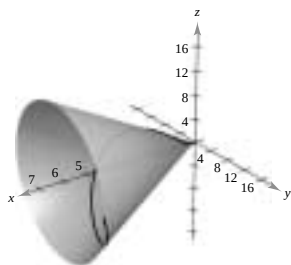
Subtracting, we have $x^2 - y^2 = 0$ or $y = \pm x$.

Therefore, in the first octant, if we let $x = t$, then $x = t$, $y = t$, $z = \sqrt{4 - t^2}$.

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\mathbf{k}$



67. $y^2 + z^2 = (2t \cos t)^2 + (2t \sin t)^2 = 4t^2 = 4x^2$



71. $\lim_{t \rightarrow 0} \left[t^2 \mathbf{i} + 3t \mathbf{j} + \frac{1 - \cos t}{t} \mathbf{k} \right] = \mathbf{0}$

since

$$\lim_{t \rightarrow 0} \frac{1 - \cos t}{t} = \lim_{t \rightarrow 0} \frac{\sin t}{1} = 0. \quad (\text{L'Hôpital's Rule})$$

75. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}$

Continuous on $(-\infty, 0), (0, \infty)$

79. $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan t \rangle$

Discontinuous at $t = \frac{\pi}{2} + n\pi$

Continuous on $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$

83. $\mathbf{r}(t) = t^2 \mathbf{i} + (t - 3)\mathbf{j} + t\mathbf{k}$

(a) $\mathbf{s}(t) = \mathbf{r}(t) + 2\mathbf{k} = t^2 \mathbf{i} + (t - 3)\mathbf{j} + (t + 3)\mathbf{k}$

(b) $\mathbf{s}(t) = \mathbf{r}(t) - 2\mathbf{i} = (t^2 - 2)\mathbf{i} + (t - 3)\mathbf{j} + t\mathbf{k}$

(c) $\mathbf{s}(t) = \mathbf{r}(t) + 5\mathbf{j} = t^2 \mathbf{i} + (t + 2)\mathbf{j} + t\mathbf{k}$

85. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$. Then:

$$\begin{aligned} \lim_{t \rightarrow c} [\mathbf{r}(t) \times \mathbf{u}(t)] &= \lim_{t \rightarrow c} \{ [y_1(t)z_2(t) - y_2(t)z_1(t)]\mathbf{i} - [x_1(t)z_2(t) - x_2(t)z_1(t)]\mathbf{j} + [x_1(t)y_2(t) - x_2(t)y_1(t)]\mathbf{k} \} \\ &= \left[\lim_{t \rightarrow c} y_1(t) \lim_{t \rightarrow c} z_2(t) - \lim_{t \rightarrow c} y_2(t) \lim_{t \rightarrow c} z_1(t) \right] \mathbf{i} - \left[\lim_{t \rightarrow c} x_1(t) \lim_{t \rightarrow c} z_2(t) - \lim_{t \rightarrow c} x_2(t) \lim_{t \rightarrow c} z_1(t) \right] \mathbf{j} \\ &\quad + \left[\lim_{t \rightarrow c} x_1(t) \lim_{t \rightarrow c} y_2(t) - \lim_{t \rightarrow c} x_2(t) \lim_{t \rightarrow c} y_1(t) \right] \mathbf{k} \\ &= \left[\lim_{t \rightarrow c} x_1(t) \mathbf{i} + \lim_{t \rightarrow c} y_1(t) \mathbf{j} + \lim_{t \rightarrow c} z_1(t) \mathbf{k} \right] \times \left[\lim_{t \rightarrow c} x_2(t) \mathbf{i} + \lim_{t \rightarrow c} y_2(t) \mathbf{j} + \lim_{t \rightarrow c} z_2(t) \mathbf{k} \right] \\ &= \lim_{t \rightarrow c} \mathbf{r}(t) \times \lim_{t \rightarrow c} \mathbf{u}(t) \end{aligned}$$

87. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Since $\mathbf{r}$ is continuous at $t = c$, then $\lim_{t \rightarrow c} \mathbf{r}(t) = \mathbf{r}(c)$.

$$\mathbf{r}(c) = x(c)\mathbf{i} + y(c)\mathbf{j} + z(c)\mathbf{k} \Rightarrow x(c), y(c), z(c)$$

are defined at c .

$$\|\mathbf{r}\| = \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2}$$

$$\lim_{t \rightarrow c} \|\mathbf{r}\| = \sqrt{(x(c))^2 + (y(c))^2 + (z(c))^2} = \|\mathbf{r}(c)\|$$

Therefore, $\|\mathbf{r}\|$ is continuous at c .

69. $\lim_{t \rightarrow 2} \left[t\mathbf{i} + \frac{t^2 - 4}{t^2 - 2t}\mathbf{j} + \frac{1}{t}\mathbf{k} \right] = 2\mathbf{i} + 2\mathbf{j} + \frac{1}{2}\mathbf{k}$

since

$$\lim_{t \rightarrow 2} \frac{t^2 - 4}{t^2 - 2t} = \lim_{t \rightarrow 2} \frac{2t}{2t - 2} = 2. \quad (\text{L'Hôpital's Rule})$$

73. $\lim_{t \rightarrow 0} \left[\frac{1}{t}\mathbf{i} + \cos t \mathbf{j} + \sin t \mathbf{k} \right]$

does not exist since $\lim_{t \rightarrow 0} \frac{1}{t}$ does not exist.

77. $\mathbf{r}(t) = t\mathbf{i} + \arcsin t \mathbf{j} + (t - 1)\mathbf{k}$

Continuous on $[-1, 1]$

81. See the definition on page 786.

89. True

Section 11.2 Differentiation and Integration of Vector-Valued Functions

1. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$, $t_0 = 2$

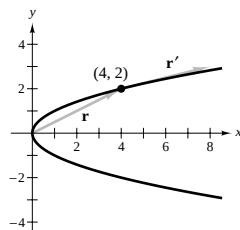
$x(t) = t^2$, $y(t) = t$

$x = y^2$

$\mathbf{r}(2) = 4\mathbf{i} + 2\mathbf{j}$

$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(2) = 4\mathbf{i} + \mathbf{j}$

 $\mathbf{r}'(t_0)$ is tangent to the curve.

3. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $t_0 = \frac{\pi}{2}$

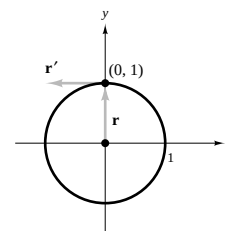
$x(t) = \cos t$, $y(t) = \sin t$

$x^2 + y^2 = 1$

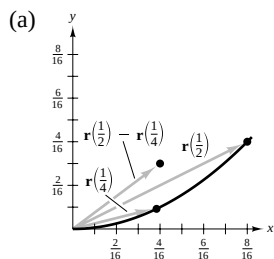
$\mathbf{r}\left(\frac{\pi}{2}\right) = \mathbf{j}$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\mathbf{r}'\left(\frac{\pi}{2}\right) = -\mathbf{i}$

 $\mathbf{r}'(t_0)$ is tangent to the curve.

5. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$



(b) $\mathbf{r}\left(\frac{1}{4}\right) = \frac{1}{4}\mathbf{i} + \frac{1}{16}\mathbf{j}$

$\mathbf{r}\left(\frac{1}{2}\right) = \frac{1}{2}\mathbf{i} + \frac{1}{4}\mathbf{j}$

$\mathbf{r}\left(\frac{1}{2}\right) - \mathbf{r}\left(\frac{1}{4}\right) = \frac{1}{4}\mathbf{i} + \frac{3}{16}\mathbf{j}$

(c) $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{r}'\left(\frac{1}{4}\right) = \mathbf{i} + \frac{1}{2}\mathbf{j}$

$$\frac{\mathbf{r}(1/2) - \mathbf{r}(1/4)}{(1/2) - (1/4)} = \frac{(1/4)\mathbf{i} + (3/16)\mathbf{j}}{1/4} = \mathbf{i} + \frac{3}{4}\mathbf{j}$$

This vector approximates $\mathbf{r}'\left(\frac{1}{4}\right)$.

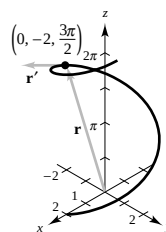
7. $\mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} + t\mathbf{k}$, $t_0 = \frac{3\pi}{2}$

$x^2 + y^2 = 4$, $z = t$

$\mathbf{r}'(t) = -2 \sin t\mathbf{i} + 2 \cos t\mathbf{j} + \mathbf{k}$

$\mathbf{r}'\left(\frac{3\pi}{2}\right) = -2\mathbf{j} + \frac{3\pi}{2}\mathbf{k}$

$\mathbf{r}'\left(\frac{3\pi}{2}\right) = 2\mathbf{i} + \mathbf{k}$



9. $\mathbf{r}(t) = 6t\mathbf{i} - 7t^2\mathbf{j} + t^3\mathbf{k}$

$\mathbf{r}'(t) = 6\mathbf{i} - 14t\mathbf{j} + 3t^2\mathbf{k}$

13. $\mathbf{r}(t) = e^{-t}\mathbf{i} + 4\mathbf{j}$

$\mathbf{r}'(t) = -e^{-t}\mathbf{i}$

17. $\mathbf{r}(t) = t^3\mathbf{i} + \frac{1}{2}t^2\mathbf{j}$

(a) $\mathbf{r}'(t) = 3t^2\mathbf{i} + t\mathbf{j}$

$\mathbf{r}''(t) = 6t\mathbf{i} + \mathbf{j}$

11. $\mathbf{r}(t) = a \cos^3 t\mathbf{i} + a \sin^3 t\mathbf{j} + \mathbf{k}$

$\mathbf{r}'(t) = -3a \cos^2 t \sin t\mathbf{i} + 3a \sin^2 t \cos t\mathbf{j}$

15. $\mathbf{r}(t) = \langle t \sin t, t \cos t, t \rangle$

$\mathbf{r}'(t) = \langle \sin t + t \cos t, \cos t - t \sin t, 1 \rangle$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 3t^2(6t) + t = 18t^3 + t$

19. $\mathbf{r}(t) = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j}$

(a) $\mathbf{r}'(t) = -4 \sin t \mathbf{i} + 4 \cos t \mathbf{j}$

$$\mathbf{r}''(t) = -4 \cos t \mathbf{i} - 4 \sin t \mathbf{j}$$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (-4 \sin t)(-4 \cos t) + 4 \cos t(-4 \sin t)$

$$= 0$$

21. $\mathbf{r}(t) = \frac{1}{2}t^2 \mathbf{i} - t \mathbf{j} + \frac{1}{6}t^3 \mathbf{k}$

(a) $\mathbf{r}'(t) = t \mathbf{i} - \mathbf{j} + \frac{1}{2}t^2 \mathbf{k}$

$$\mathbf{r}''(t) = \mathbf{i} + t \mathbf{k}$$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = t(1) - 1(0) + \frac{1}{2}t^2(t) = t + \frac{t^3}{2}$

23. $\mathbf{r}(t) = \langle \cos t + t \sin t, \sin t - t \cos t, t \rangle$

(a) $\mathbf{r}'(t) = \langle -\sin t + \sin t + t \cos t, \cos t - \cos t + t \sin t, 1 \rangle$

$$= \langle t \cos t, t \sin t, 1 \rangle$$

$$\mathbf{r}''(t) = \langle \cos t - t \sin t, \sin t + t \cos t, 0 \rangle$$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (t \cos t)(\cos t - t \sin t) + (t \sin t)(\sin t + t \cos t) = t$

25. $\mathbf{r}(t) = \cos(\pi t) \mathbf{i} + \sin(\pi t) \mathbf{j} + t^2 \mathbf{k}, t_0 = -\frac{1}{4}$

$$\mathbf{r}'(t) = -\pi \sin(\pi t) \mathbf{i} + \pi \cos(\pi t) \mathbf{j} + 2t \mathbf{k}$$

$$\mathbf{r}'\left(-\frac{1}{4}\right) = \frac{\sqrt{2}\pi}{2} \mathbf{i} + \frac{\sqrt{2}\pi}{2} \mathbf{j} - \frac{1}{2} \mathbf{k}$$

$$\|\mathbf{r}'\left(\frac{1}{4}\right)\| = \sqrt{\left(\frac{\sqrt{2}\pi}{2}\right)^2 + \left(\frac{\sqrt{2}\pi}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\pi^2 + \frac{1}{4}} = \frac{\sqrt{4\pi^2 + 1}}{2}$$

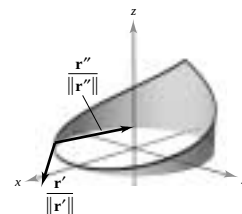
$$\frac{\mathbf{r}'(-1/4)}{\|\mathbf{r}'(-1/4)\|} = \frac{1}{\sqrt{4\pi^2 + 1}}(\sqrt{2}\pi \mathbf{i} + \sqrt{2}\pi \mathbf{j} - \mathbf{k})$$

$$\mathbf{r}''(t) = -\pi^2 \cos(\pi t) \mathbf{i} - \pi^2 \sin(\pi t) \mathbf{j} + 2 \mathbf{k}$$

$$\mathbf{r}''\left(-\frac{1}{4}\right) = -\frac{\sqrt{2}\pi^2}{2} \mathbf{i} + \frac{\sqrt{2}\pi^2}{2} \mathbf{j} + 2 \mathbf{k}$$

$$\|\mathbf{r}''\left(-\frac{1}{4}\right)\| = \sqrt{\left(-\frac{\sqrt{2}\pi^2}{2}\right)^2 + \left(\frac{\sqrt{2}\pi^2}{2}\right)^2 + (2)^2} = \sqrt{\pi^4 + 4}$$

$$\frac{\mathbf{r}''(-1/4)}{\|\mathbf{r}''(-1/4)\|} = \frac{1}{2\sqrt{\pi^4 + 4}}(-\sqrt{2}\pi^2 \mathbf{i} + \sqrt{2}\pi^2 \mathbf{j} + 4 \mathbf{k})$$



27. $\mathbf{r}(t) = t^2 \mathbf{i} + t^3 \mathbf{j}$

$$\mathbf{r}'(t) = 2t \mathbf{i} + 3t^2 \mathbf{j}$$

$$\mathbf{r}'(0) = \mathbf{0}$$

Smooth on $(-\infty, 0), (0, \infty)$

29. $\mathbf{r}(\theta) = 2 \cos^3 \theta \mathbf{i} + 3 \sin^3 \theta \mathbf{j}$

$$\mathbf{r}'(\theta) = -6 \cos^2 \theta \sin \theta \mathbf{i} + 9 \sin^2 \theta \cos \theta \mathbf{j}$$

$$\mathbf{r}'\left(\frac{n\pi}{2}\right) = \mathbf{0}$$

Smooth on $\left(\frac{n\pi}{2}, \frac{(n+1)\pi}{2}\right), n$ any integer.

31. $\mathbf{r}(\theta) = (\theta - 2 \sin \theta) \mathbf{i} + (1 - 2 \cos \theta) \mathbf{j}$

$$\mathbf{r}'(\theta) = (1 - 2 \cos \theta) \mathbf{i} + (1 + 2 \sin \theta) \mathbf{j}$$

$$\mathbf{r}'(\theta) \neq \mathbf{0} \text{ for any value of } \theta$$

Smooth on $(-\infty, \infty)$

33. $\mathbf{r}(t) = (t - 1) \mathbf{i} + \frac{1}{t} \mathbf{j} - t^2 \mathbf{k}$

$$\mathbf{r}'(t) = \mathbf{i} - \frac{1}{t^2} \mathbf{j} - 2t \mathbf{k} \neq \mathbf{0}$$

$\mathbf{r}$ is smooth for all $t \neq 0$: $(-\infty, 0) \cup (0, \infty)$

35. $\mathbf{r}(t) = t\mathbf{i} - 3t\mathbf{j} + \tan t\mathbf{k}$

$$\mathbf{r}'(t) = \mathbf{i} - 3\mathbf{j} + \sec^2 t\mathbf{k} \neq \mathbf{0}$$

$$\mathbf{r} \text{ is smooth for all } t \neq \frac{\pi}{2} + n\pi = \frac{2n+1}{2}\pi.$$

$$\text{Smooth on intervals of form } \left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$$

37. $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}$, $\mathbf{u}(t) = 4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

(a) $\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$

(c) $\mathbf{r}(t) \cdot \mathbf{u}(t) = 4t^2 + 3t^3 + t^5$

$$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 8t + 9t^2 + 5t^4$$

(e) $\mathbf{r}(t) \times \mathbf{u}(t) = 2t^4\mathbf{i} - (t^4 - 4t^3)\mathbf{j} + (t^3 - 12t^2)\mathbf{k}$

$$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = 8t^3\mathbf{i} + (12t^2 - 4t^3)\mathbf{j} + (3t^2 - 24t)\mathbf{k}$$

(b) $\mathbf{r}''(t) = 2\mathbf{k}$

(d) $3\mathbf{r}(t) - \mathbf{u}(t) = -t\mathbf{i} + (9t - t^2)\mathbf{j} + (3t^2 - t^3)\mathbf{k}$

$$D_t[3\mathbf{r}(t) - \mathbf{u}(t)] = -\mathbf{i} + (9 - 2t)\mathbf{j} + (6t - 3t^2)\mathbf{k}$$

(f) $\|\mathbf{r}(t)\| = \sqrt{10t^2 + t^4} = t\sqrt{10 + t^2}$

$$D_t[\|\mathbf{r}(t)\|] = \frac{10 + 2t^2}{\sqrt{10 + t^2}}$$

39. $\mathbf{r}(t) = 3 \sin t\mathbf{i} + 4 \cos t\mathbf{j}$

$$\mathbf{r}'(t) = 3 \cos t\mathbf{i} - 4 \sin t\mathbf{j}$$

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 9 \sin t \cos t - 16 \cos t \sin t = -7 \sin t \cos t$$

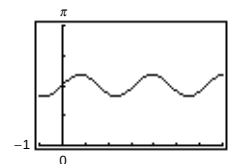
$$\cos \theta = \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\| \|\mathbf{r}'(t)\|} = \frac{-7 \sin t \cos t}{\sqrt{9 \sin^2 t + 16 \cos^2 t} \sqrt{9 \cos^2 t + 16 \sin^2 t}}$$

$$\theta = \arccos \left[\frac{-7 \sin t \cos t}{\sqrt{(9 \sin^2 t + 16 \cos^2 t)(9 \cos^2 t + 16 \sin^2 t)}} \right]$$

$$\theta = 1.855 \text{ maximum at } t = 3.927\left(\frac{5\pi}{4}\right) \text{ and } t = 0.785\left(\frac{\pi}{4}\right).$$

$$\theta = 1.287 \text{ minimum at } t = 2.356\left(\frac{3\pi}{4}\right) \text{ and } t = 5.498\left(\frac{7\pi}{4}\right).$$

$$\theta = \frac{\pi}{2}(1.571) \text{ for } t = n\frac{\pi}{2}, n = 0, 1, 2, 3, \dots$$



41. $\mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$

$$= \lim_{\Delta t \rightarrow 0} \frac{[3(t + \Delta t) + 2]\mathbf{i} + [1 - (t + \Delta t)^2]\mathbf{j} - (3t + 2)\mathbf{i} - (1 - t^2)\mathbf{j}}{\Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} \frac{(3\Delta t)\mathbf{i} - (2t(\Delta t) + (\Delta t)^2)\mathbf{j}}{\Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} 3\mathbf{i} - (2t + \Delta t)\mathbf{j} = 3\mathbf{i} - 2t\mathbf{j}$$

43. $\int (2t\mathbf{i} + \mathbf{j} + \mathbf{k}) dt = t^2\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

45. $\int \left(\frac{1}{t}\mathbf{i} + \mathbf{j} - t^{3/2}\mathbf{k}\right) dt = \ln t\mathbf{i} + t\mathbf{j} - \frac{2}{5}t^{5/2}\mathbf{k} + \mathbf{C}$

47. $\int [(2t - 1)\mathbf{i} + 4t^3\mathbf{j} + 3\sqrt{t}\mathbf{k}] dt = (t^2 - t)\mathbf{i} + t^4\mathbf{j} + 2t^{3/2}\mathbf{k} + \mathbf{C}$

49. $\int \left[\sec^2 t\mathbf{i} + \frac{1}{1 + t^2}\mathbf{j}\right] dt = \tan t\mathbf{i} + \arctan t\mathbf{j} + \mathbf{C}$

$$51. \int_0^1 (8t\mathbf{i} + t\mathbf{j} - \mathbf{k}) dt = \left[4t^2\mathbf{i} \right]_0^1 + \left[\frac{t^2}{2}\mathbf{j} \right]_0^1 - \left[t\mathbf{k} \right]_0^1 = 4\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$$

$$53. \int_0^{\pi/2} [(a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + \mathbf{k}] dt = \left[a \sin t\mathbf{i} \right]_0^{\pi/2} - \left[a \cos t\mathbf{j} \right]_0^{\pi/2} + \left[t\mathbf{k} \right]_0^{\pi/2} = a\mathbf{i} + a\mathbf{j} + \frac{\pi}{2}\mathbf{k}$$

$$55. \mathbf{r}(t) = \int (4e^{2t}\mathbf{i} + 3e^t\mathbf{j}) dt = 2e^{2t}\mathbf{i} + 3e^t\mathbf{j} + \mathbf{C}$$

$$\mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{C} = 2\mathbf{i} \Rightarrow \mathbf{C} = -3\mathbf{j}$$

$$\mathbf{r}(t) = 2e^{2t}\mathbf{i} + 3(e^t - 1)\mathbf{j}$$

$$57. \mathbf{r}'(t) = \int -32\mathbf{j} dt = -32t\mathbf{j} + \mathbf{C}_1$$

$$\mathbf{r}'(0) = \mathbf{C}_1 = 600\sqrt{3}\mathbf{i} + 600\mathbf{j}$$

$$\mathbf{r}'(t) = 600\sqrt{3}\mathbf{i} + (600 - 32t)\mathbf{j}$$

$$\mathbf{r}(t) = \int [600\sqrt{3}\mathbf{i} + (600 - 32t)\mathbf{j}] dt$$

$$= 600\sqrt{3}t\mathbf{i} + (600t - 16t^2)\mathbf{j} + \mathbf{C}$$

$$\mathbf{r}(0) = \mathbf{C} = \mathbf{0}$$

$$\mathbf{r}(t) = 600\sqrt{3}t\mathbf{i} + (600t - 16t^2)\mathbf{j}$$

$$59. \mathbf{r}(t) = \int (te^{-t^2}\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}) dt = -\frac{1}{2}e^{-t^2}\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(0) = -\frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{C} = \frac{1}{2}\mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}(t) = \left(1 - \frac{1}{2}e^{-t^2}\right)\mathbf{i} + (e^{-t} - 2)\mathbf{j} + (t + 1)\mathbf{k} = \left(\frac{2 - e^{-t^2}}{2}\right)\mathbf{i} + (e^{-t} - 2)\mathbf{j} + (t + 1)\mathbf{k}$$

61. See “Definition of the Derivative of a Vector-Valued Function” and Figure 11.8 on page 794.

63. At $t = t_0$, the graph of $\mathbf{u}(t)$ is increasing in the x , y , and z directions simultaneously.

65. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $c\mathbf{r}(t) = cx(t)\mathbf{i} + cy(t)\mathbf{j} + cz(t)\mathbf{k}$ and

$$\begin{aligned} D_t[c\mathbf{r}(t)] &= cx'(t)\mathbf{i} + cy'(t)\mathbf{j} + cz'(t)\mathbf{k} \\ &= c[x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}] = c\mathbf{r}'(t). \end{aligned}$$

67. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then $f(t)\mathbf{r}(t) = f(t)x(t)\mathbf{i} + f(t)y(t)\mathbf{j} + f(t)z(t)\mathbf{k}$.

$$\begin{aligned} D_t[f(t)\mathbf{r}(t)] &= [f(t)x'(t) + f'(t)x(t)]\mathbf{i} + [f(t)y'(t) + f'(t)y(t)]\mathbf{j} + [f(t)z'(t) + f'(t)z(t)]\mathbf{k} \\ &= f(t)[x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}] + f'(t)[x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}] \\ &= f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t) \end{aligned}$$

69. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{r}(f(t)) = x(f(t))\mathbf{i} + y(f(t))\mathbf{j} + z(f(t))\mathbf{k}$ and

$$\begin{aligned} D_t[\mathbf{r}(f(t))] &= x'(f(t))f'(t)\mathbf{i} + y'(f(t))f'(t)\mathbf{j} + z'(f(t))f'(t)\mathbf{k} \quad (\text{Chain Rule}) \\ &= f'(t)[x'(f(t))\mathbf{i} + y'(f(t))\mathbf{j} + z'(f(t))\mathbf{k}] = f'(t)\mathbf{r}'(f(t)). \end{aligned}$$

71. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$, $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$, and $\mathbf{v}(t) = x_3(t)\mathbf{i} + y_3(t)\mathbf{j} + z_3(t)\mathbf{k}$. Then:

$$\begin{aligned}\mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] &= x_1(t)[y_2(t)z_3(t) - z_2(t)y_3(t)] - y_1(t)[x_2(t)z_3(t) - z_2(t)x_3(t)] + z_1(t)[x_2(t)y_3(t) - y_2(t)x_3(t)] \\ D_t[\mathbf{r}(t) \cdot (\mathbf{u}(t) \times \mathbf{v}(t))] &= x_1'(t)y_2(t)z_3(t) + x_1(t)y_2'(t)z_3(t) + x_1(t)y_2(t)z_3'(t) - x_1(t)y_3(t)z_2'(t) - \\ &\quad x_1(t)y_3'(t)z_2(t) - x_1'(t)y_3(t)z_2(t) - y_1(t)x_2(t)z_3'(t) - y_1(t)x_2'(t)z_3(t) - y_1'(t)x_2(t)z_3(t) + \\ &\quad y_1(t)z_2(t)x_3'(t) + y_1(t)z_2'(t)x_3(t) + y_1'(t)z_2(t)x_3(t) + z_1(t)x_2(t)y_3'(t) + z_1(t)x_2'(t)y_3(t) + \\ &\quad z_1'(t)x_2(t)y_3(t) - z_1(t)y_2(t)x_3'(t) - z_1(t)y_2'(t)x_3(t) - z_1'(t)y_2(t)x_3(t) \\ &= \{x_1'(t)[y_2(t)z_3(t) - y_3(t)z_2(t)] + y_1'(t)[-x_2(t)z_3(t) + z_2(t)x_3(t)] + z_1'(t)[x_2(t)y_3(t) - y_2(t)x_3(t)]\} + \\ &\quad \{x_1(t)[y_2'(t)z_3(t) - y_3(t)z_2'(t)] + y_1(t)[-x_2'(t)z_3(t) + z_2'(t)x_3(t)] + z_1(t)[x_2'(t)y_3(t) - y_2'(t)x_3(t)]\} + \\ &\quad \{x_1(t)[y_2(t)z_3'(t) - y_3'(t)z_2(t)] + y_1(t)[-x_2(t)z_3'(t) + z_2(t)x_3'(t)] + z_1(t)[x_2(t)y_3'(t) - y_2(t)x_3'(t)]\} \\ &= \mathbf{r}'(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}'(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}'(t)]\end{aligned}$$

73. False. Let $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$.

$$\|\mathbf{r}(t)\| = \sqrt{2}$$

$$\frac{d}{dt}[\|\mathbf{r}(t)\|] = 0$$

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 1$$

Section 11.3 Velocity and Acceleration

1. $\mathbf{r}(t) = 3t\mathbf{i} + (t-1)\mathbf{j}$

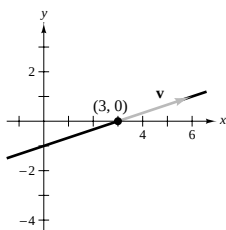
$$\mathbf{v}(t) = \mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{0}$$

$$x = 3t, y = t - 1, y = \frac{x}{3} - 1$$

$$\text{At } (3, 0), t = 1.$$

$$\mathbf{v}(1) = 3\mathbf{i} + \mathbf{j}, \mathbf{a}(1) = \mathbf{0}$$



3. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$$

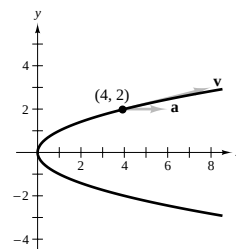
$$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i}$$

$$x = t^2, y = t, x = y^2$$

$$\text{At } (4, 2), t = 2.$$

$$\mathbf{v}(2) = 4\mathbf{i} + \mathbf{j}$$

$$\mathbf{a}(2) = 2\mathbf{i}$$



5. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$$

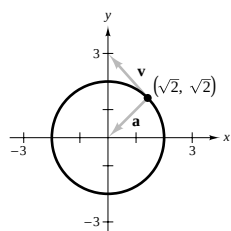
$$\mathbf{a}(t) = \mathbf{r}''(t) = -2\cos t\mathbf{i} - 2\sin t\mathbf{j}$$

$$x = 2\cos t, y = 2\sin t, x^2 + y^2 = 4$$

$$\text{At } (\sqrt{2}, \sqrt{2}), t = \frac{\pi}{4}.$$

$$\mathbf{v}\left(\frac{\pi}{4}\right) = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$$

$$\mathbf{a}\left(\frac{\pi}{4}\right) = -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j}$$



7. $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1 - \cos t, \sin t \rangle$$

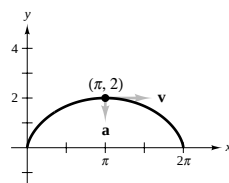
$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle \sin t, \cos t \rangle$$

$$x = t - \sin t, y = 1 - \cos t \quad (\text{cycloid})$$

$$\text{At } (\pi, 2), t = \pi.$$

$$\mathbf{v}(\pi) = \langle 2, 0 \rangle = 2\mathbf{i}$$

$$\mathbf{a}(\pi) = \langle 0, -1 \rangle = -\mathbf{j}$$



9. $\mathbf{r}(t) = t\mathbf{i} + (2t - 5)\mathbf{j} + 3t\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$s(t) = \|\mathbf{v}(t)\| = \sqrt{1 + 4 + 9} = \sqrt{14}$

$\mathbf{a}(t) = \mathbf{0}$

13. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{9 - t^2}\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - \frac{t}{\sqrt{9 - t^2}}\mathbf{k}$

$s(t) = \sqrt{1 + 1 + \frac{t^2}{9 - t^2}} = \sqrt{\frac{18 - t^2}{9 - t^2}}$

$\mathbf{a}(t) = -\frac{9}{(9 - t^2)^{3/2}}\mathbf{k}$

17. (a) $\mathbf{r}(t) = \left\langle t, -t^2, \frac{t^3}{4} \right\rangle, t_0 = 1$

$\mathbf{r}'(t) = \left\langle 1, -2t, \frac{3t^2}{4} \right\rangle$

$\mathbf{r}'(1) = \left\langle 1, -2, \frac{3}{4} \right\rangle$

$x = 1 + t, y = -1 - 2t, z = \frac{1}{4} + \frac{3}{4}t$

19. $\mathbf{a}(t) = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{v}(0) = \mathbf{0}, \mathbf{r}(0) = \mathbf{0}$

$\mathbf{v}(t) = \int (\mathbf{i} + \mathbf{j} + \mathbf{k}) dt = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

$\mathbf{v}(0) = \mathbf{C} = \mathbf{0}, \mathbf{v}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \mathbf{v}(t) = t(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\mathbf{r}(t) = \int (t\mathbf{i} + t\mathbf{j} + t\mathbf{k}) dt = \frac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = \mathbf{0}, \mathbf{r}(t) = \frac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k}),$

$\mathbf{r}(2) = 2(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

11. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$s(t) = \sqrt{1 + 4t^2 + t^2} = \sqrt{1 + 5t^2}$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

15. $\mathbf{r}(t) = \langle 4t, 3 \cos t, 3 \sin t \rangle$

$\mathbf{v}(t) = \langle 4, -3 \sin t, 3 \cos t \rangle = 4\mathbf{i} - 3 \sin t\mathbf{j} + 3 \cos t\mathbf{k}$

$s(t) = \sqrt{16 + 9 \sin^2 t + 9 \cos^2 t} = 5$

$\mathbf{a}(t) = \langle 0, -3 \cos t, -3 \sin t \rangle = -3 \cos t\mathbf{j} - 3 \sin t\mathbf{k}$

(b) $\mathbf{r}(1 + 0.1) \approx \left\langle 1 + 0.1, -1 - 2(0.1), \frac{1}{4} + \frac{3}{4}(0.1) \right\rangle$
 $= \langle 1.100, -1.200, 0.325 \rangle$

21. $\mathbf{a}(t) = t\mathbf{j} + t\mathbf{k}, \mathbf{v}(1) = 5\mathbf{j}, \mathbf{r}(1) = \mathbf{0}$

$\mathbf{v}(t) = \int (t\mathbf{j} + t\mathbf{k}) dt = \frac{t^2}{2}\mathbf{j} + \frac{t^2}{2}\mathbf{k} + \mathbf{C}$

$\mathbf{v}(1) = \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k} + \mathbf{C} = 5\mathbf{j} \Rightarrow \mathbf{C} = \frac{9}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$

$\mathbf{v}(t) = \left(\frac{t^2}{2} + \frac{9}{2}\right)\mathbf{j} + \left(\frac{t^2}{2} - \frac{1}{2}\right)\mathbf{k}$

$\mathbf{r}(t) = \int \left[\left(\frac{t^2}{2} + \frac{9}{2}\right)\mathbf{j} + \left(\frac{t^2}{2} - \frac{1}{2}\right)\mathbf{k} \right] dt$

$= \left(\frac{t^3}{6} + \frac{9}{2}t\right)\mathbf{j} + \left(\frac{t^3}{6} - \frac{1}{2}t\right)\mathbf{k} + \mathbf{C}$

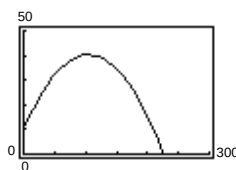
$\mathbf{r}(1) = \frac{14}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} + \mathbf{C} = \mathbf{0} \Rightarrow \mathbf{C} = -\frac{14}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$

$\mathbf{r}(t) = \left(\frac{t^3}{6} + \frac{9}{2}t - \frac{14}{3}\right)\mathbf{j} + \left(\frac{t^3}{6} - \frac{1}{2}t + \frac{1}{3}\right)\mathbf{k}$

$\mathbf{r}(2) = \frac{17}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

23. The velocity of an object involves both magnitude and direction of motion, whereas speed involves only magnitude.

25. $\mathbf{r}(t) = (88 \cos 30^\circ)t\mathbf{i} + [10 + (88 \sin 30^\circ)t - 16t^2]\mathbf{j}$
 $= 44\sqrt{3}t\mathbf{i} + (10 + 44t - 16t^2)\mathbf{j}$



$$27. \mathbf{r}(t) = (v_0 \cos \theta)t \mathbf{i} + \left[h + (v_0 \sin \theta)t - \frac{1}{2}gt^2 \right] \mathbf{j} = \frac{v_0}{\sqrt{2}}t \mathbf{i} + \left(3 + \frac{v_0}{\sqrt{2}}t - 16t^2 \right) \mathbf{j}$$

$$\frac{v_0}{\sqrt{2}}t = 300 \text{ when } 3 + \frac{v_0}{\sqrt{2}}t - 16t^2 = 3.$$

$$t = \frac{300\sqrt{2}}{v_0}, \frac{v_0}{\sqrt{2}}\left(\frac{300\sqrt{2}}{v_0}\right) - 16\left(\frac{300\sqrt{2}}{v_0}\right)^2 = 0, 300 - \frac{300^2(32)}{v_0^2} = 0$$

$$v_0^2 = 300(32), v_0 = \sqrt{9600} = 40\sqrt{6}, v_0 = 40\sqrt{6} \approx 97.98 \text{ ft/sec}$$

The maximum height is reached when the derivative of the vertical component is zero.

$$y(t) = 3 + \frac{tv_0}{\sqrt{2}} - 16t^2 = 3 + \frac{40\sqrt{6}}{\sqrt{2}}t - 16t^2 = 3 + 40\sqrt{3}t - 16t^2$$

$$y'(t) = 40\sqrt{3} - 32t = 0$$

$$t = \frac{40\sqrt{3}}{32} = \frac{5\sqrt{3}}{4}$$

$$\text{Maximum height: } y\left(\frac{5\sqrt{3}}{4}\right) = 3 + 40\sqrt{3}\left(\frac{5\sqrt{3}}{4}\right) - 16\left(\frac{5\sqrt{3}}{4}\right)^2 = 78 \text{ feet}$$

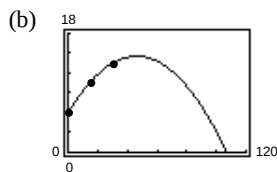
$$29. x(t) = t(v_0 \cos \theta) \text{ or } t = \frac{x}{v_0 \cos \theta}$$

$$y(t) = t(v_0 \sin \theta) - 16t^2 + h$$

$$y = \frac{x}{v_0 \cos \theta}(v_0 \sin \theta) - 16\left(\frac{x^2}{v_0^2 \cos^2 \theta}\right) + h = (\tan \theta)x - \left(\frac{16}{v_0^2} \sec^2 \theta\right)x^2 + h$$

$$31. \mathbf{r}(t) = t \mathbf{i} + (-0.004t^2 + 0.3667t + 6) \mathbf{j}, \text{ or}$$

$$(a) y = -0.004x^2 + 0.3667x + 6$$



$$(c) y' = -0.008x + 0.3667 = 0 \Rightarrow x = 45.8375 \text{ and}$$

$$y(45.8375) \approx 14.4 \text{ feet.}$$

(d) From Exercise 29,

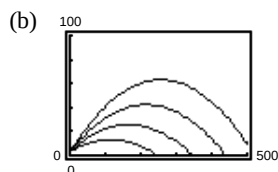
$$\tan \theta = 0.3667 \Rightarrow \theta \approx 20.14^\circ$$

$$\frac{16 \sec^2 \theta}{v_0^2} = 0.004 \Rightarrow v_0^2 = \frac{16 \sec^2 \theta}{0.004} = \frac{4000}{\cos^2 \theta}$$

$$\Rightarrow v_0 \approx 67.4 \text{ ft/sec.}$$

$$33. 100 \text{ mph} = \left(100 \frac{\text{miles}}{\text{hr}}\right)\left(5280 \frac{\text{feet}}{\text{mile}}\right)/(3600 \text{ sec/hour}) = \frac{440}{3} \text{ ft/sec}$$

$$(a) \mathbf{r}(t) = \left(\frac{440}{3} \cos \theta_0\right)t \mathbf{i} + \left[3 + \left(\frac{440}{3} \sin \theta_0\right)t - 16t^2\right] \mathbf{j}$$



Graphing these curves together with $y = 10$ shows that $\theta_0 = 20^\circ$.

—CONTINUED—

33. —CONTINUED—

(c) We want

$$x(t) = \left(\frac{440}{3} \cos \theta\right)t \geq 400 \quad \text{and} \quad y(t) = 3 + \left(\frac{440}{3} \sin \theta\right)t - 16t^2 \geq 10.$$

From $x(t)$, the minimum angle occurs when $t = 30/(11 \cos \theta)$. Substituting this for t in $y(t)$ yields:

$$3 + \left(\frac{440}{3} \sin \theta\right)\left(\frac{30}{11 \cos \theta}\right) - 16\left(\frac{30}{11 \cos \theta}\right)^2 = 10$$

$$400 \tan \theta - \frac{14,400}{121} \sec^2 \theta = 7$$

$$\frac{14,400}{121}(1 + \tan^2 \theta) - 400 \tan \theta + 7 = 0$$

$$14,400 \tan^2 \theta - 48,400 \tan \theta + 15,247 = 0$$

$$\tan \theta = \frac{48,400 \pm \sqrt{48,400^2 - 4(14,400)(15,247)}}{2(14,400)}$$

$$\theta = \tan^{-1}\left(\frac{48,400 - \sqrt{1,464,332,800}}{28,800}\right) \approx 19.38^\circ$$

35. $\mathbf{r}(t) = (v \cos \theta)t\mathbf{i} + [(v \sin \theta)t - 16t^2]\mathbf{j}$

(a) We want to find the minimum initial speed v as a function of the angle θ . Since the bale must be thrown to the position $(16, 8)$, we have

$$16 = (v \cos \theta)t$$

$$8 = (v \sin \theta)t - 16t^2.$$

 $t = 16/(v \cos \theta)$ from the first equation. Substituting into the second equation and solving for v , we obtain:

$$8 = (v \sin \theta)\left(\frac{16}{v \cos \theta}\right) - 16\left(\frac{16}{v \cos \theta}\right)^2$$

$$1 = 2 \frac{\sin \theta}{\cos \theta} - 512 \left(\frac{1}{v^2 \cos^2 \theta}\right)$$

$$512 \frac{1}{v^2 \cos^2 \theta} = 2 \frac{\sin \theta}{\cos \theta} - 1$$

$$\frac{1}{v^2} = \left(2 \frac{\sin \theta}{\cos \theta} - 1\right) \frac{\cos^2 \theta}{512} = \frac{2 \sin \theta \cos \theta - \cos^2 \theta}{512}$$

$$v^2 = \frac{512}{2 \sin \theta \cos \theta - \cos^2 \theta}$$

$$\text{We minimize } f(\theta) = \frac{512}{2 \sin \theta \cos \theta - \cos^2 \theta}.$$

$$f'(\theta) = -512 \frac{2 \cos^2 \theta - 2 \sin^2 \theta + 2 \sin \theta \cos \theta}{(2 \sin \theta \cos \theta - \cos^2 \theta)^2}$$

$$f'(\theta) = 0 \Rightarrow 2 \cos(2\theta) + \sin(2\theta) = 0$$

$$\tan(2\theta) = -2$$

$$\theta \approx 1.01722 \approx 58.28^\circ$$

Substituting into the equation for v , $v \approx 28.78$ feet per second.(b) If $\theta = 45^\circ$,

$$16 = (v \cos \theta)t = v \frac{\sqrt{2}}{2}t$$

$$8 = (v \sin \theta)t - 16t^2 = v \frac{\sqrt{2}}{2}t - 16t^2$$

$$\text{From part (a), } v^2 = \frac{512}{2(\sqrt{2}/2)(\sqrt{2}/2) - (\sqrt{2}/2)^2} = \frac{512}{1/2} = 1024 \Rightarrow v = 32 \text{ ft/sec.}$$

37. $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + [(v_0 \sin \theta)t - 16t^2]\mathbf{j}$

$$(v_0 \sin \theta)t - 16t^2 = 0 \text{ when } t = 0 \text{ and } t = \frac{v_0 \sin \theta}{16}.$$

The range is

$$x = (v_0 \cos \theta)t = (v_0 \cos \theta) \frac{v_0 \sin \theta}{16} = \frac{v_0^2}{32} \sin 2\theta.$$

Hence,

$$x = \frac{1200^2}{32} \sin(2\theta) = 3000 \Rightarrow \sin 2\theta = \frac{1}{15} \Rightarrow \theta \approx 1.91^\circ.$$

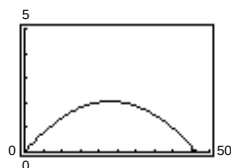
39. (a) $\theta = 10^\circ$, $v_0 = 66$ ft/sec

$$\mathbf{r}(t) = (66 \cos 10^\circ)t\mathbf{i} + [0 + (66 \sin 10^\circ)t - 16t^2]\mathbf{j}$$

$$\mathbf{r}(t) \approx (65t)\mathbf{i} + (11.46t - 16t^2)\mathbf{j}$$

Maximum height: 2.052 feet

Range: 46.557 feet



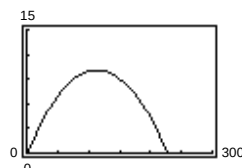
(b) $\theta = 10^\circ$, $v_0 = 146$ ft/sec

$$\mathbf{r}(t) = (146 \cos 10^\circ)t\mathbf{i} + [0 + (146 \sin 10^\circ)t - 16t^2]\mathbf{j}$$

$$\mathbf{r}(t) \approx (143.78t)\mathbf{i} + (25.35t - 16t^2)\mathbf{j}$$

Maximum height: 10.043 feet

Range: 227.828 feet



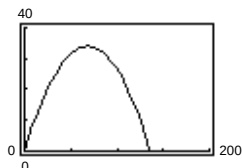
(c) $\theta = 45^\circ$, $v_0 = 66$ ft/sec

$$\mathbf{r}(t) = (66 \cos 45^\circ)t\mathbf{i} + [0 + (66 \sin 45^\circ)t - 16t^2]\mathbf{j}$$

$$\mathbf{r}(t) \approx (46.67t)\mathbf{i} + (46.67t - 16t^2)\mathbf{j}$$

Maximum height: 34.031 feet

Range: 136.125 feet



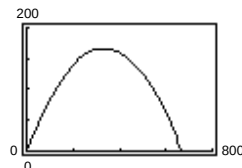
(d) $\theta = 45^\circ$, $v_0 = 146$ ft/sec

$$\mathbf{r}(t) = (146 \cos 45^\circ)t\mathbf{i} + [0 + (146 \sin 45^\circ)t - 16t^2]\mathbf{j}$$

$$\mathbf{r}(t) \approx (103.24t)\mathbf{i} + (103.24t - 16t^2)\mathbf{j}$$

Maximum height: 166.531 feet

Range: 666.125 feet



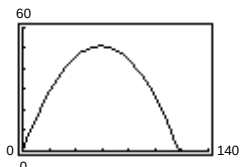
(e) $\theta = 60^\circ$, $v_0 = 66$ ft/sec

$$\mathbf{r}(t) = (66 \cos 60^\circ)t\mathbf{i} + [0 + (66 \sin 60^\circ)t - 16t^2]\mathbf{j}$$

$$\mathbf{r}(t) \approx (33t)\mathbf{i} + (57.16t - 16t^2)\mathbf{j}$$

Maximum height: 51.074 feet

Range: 117.888 feet



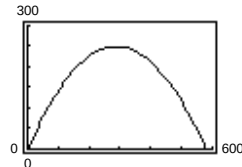
(f) $\theta = 60^\circ$, $v_0 = 146$ ft/sec

$$\mathbf{r}(t) = (146 \cos 60^\circ)t\mathbf{i} + [0 + (146 \sin 60^\circ)t - 16t^2]\mathbf{j}$$

$$\mathbf{r}(t) \approx (73t)\mathbf{i} + (126.44t - 16t^2)\mathbf{j}$$

Maximum height: 249.797 feet

Range: 576.881 feet



$$\begin{aligned} 41. \mathbf{r}(t) &= (v_0 \cos \theta)t\mathbf{i} + [h + (v_0 \sin \theta)t - 4.9t^2]\mathbf{j} \\ &= (100 \cos 30^\circ)t\mathbf{i} + [1.5 + (100 \sin 30^\circ)t - 4.9t^2]\mathbf{j} \end{aligned}$$

The projectile hits the ground when $-4.9t^2 + 100(\frac{1}{2})t + 1.5 = 0 \Rightarrow t \approx 10.234$ seconds.

The range is therefore $(100 \cos 30^\circ)(10.234) \approx 886.3$ meters.

The maximum height occurs when $dy/dt = 0$.

$$100 \sin 30 = 9.8t \Rightarrow t \approx 5.102 \text{ sec}$$

The maximum height is

$$y = 1.5 + (100 \sin 30^\circ)(5.102) - 4.9(5.102)^2 \approx 129.1 \text{ meters.}$$

$$43. \mathbf{r}(t) = b(\omega t - \sin \omega t)\mathbf{i} + b(1 - \cos \omega t)\mathbf{j}$$

$$\mathbf{v}(t) = b(\omega - \omega \cos \omega t)\mathbf{i} + b\omega \sin \omega t \mathbf{j} = b\omega(1 - \cos \omega t)\mathbf{i} + b\omega \sin \omega t \mathbf{j}$$

$$\mathbf{a}(t) = (b\omega^2 \sin \omega t)\mathbf{i} + (b\omega^2 \cos \omega t)\mathbf{j} = b\omega^2[\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}]$$

$$\|\mathbf{v}(t)\| = \sqrt{2} b\omega \sqrt{1 - \cos(\omega t)}$$

$$\|\mathbf{a}(t)\| = b\omega^2$$

$$(a) \|\mathbf{v}(t)\| = 0 \text{ when } \omega t = 0, 2\pi, 4\pi, \dots$$

$$(b) \|\mathbf{v}(t)\| \text{ is maximum when } \omega t = \pi, 3\pi, \dots, \text{ then } \|\mathbf{v}(t)\| = 2b\omega.$$

$$45. \mathbf{v}(t) = -b\omega \sin(\omega t)\mathbf{i} + b\omega \cos(\omega t)\mathbf{j}$$

$$\mathbf{r}(t) \cdot \mathbf{v}(t) = -b^2\omega \sin(\omega t) \cos(\omega t) + b^2\omega \sin(\omega t) \cos(\omega t) = 0$$

Therefore, $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are orthogonal.

$$47. \mathbf{a}(t) = -b\omega^2 \cos(\omega t)\mathbf{i} - b\omega^2 \sin(\omega t)\mathbf{j} = -b\omega^2[\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}] = -\omega^2\mathbf{r}(t)$$

$\mathbf{a}(t)$ is a negative multiple of a unit vector from $(0, 0)$ to $(\cos \omega t, \sin \omega t)$ and thus $\mathbf{a}(t)$ is directed toward the origin.

$$49. \|\mathbf{a}(t)\| = \omega^2 b$$

$$1 = m(32)$$

$$F = m(\omega^2 b) = \frac{1}{32}(2\omega^2) = 10$$

$$\omega = 4\sqrt{10} \text{ rad/sec}$$

$$\|\mathbf{v}(t)\| = b\omega = 8\sqrt{10} \text{ ft/sec}$$

$$51. \text{ To find the range, set } y(t) = h + (v_0 \sin \theta)t - \frac{1}{2}gt^2 = 0 \text{ then } 0 = \left(\frac{1}{2}g\right)t^2 - (v_0 \sin \theta)t - h.$$

By the Quadratic Formula, (discount the negative value)

$$t = \frac{v_0 \sin \theta + \sqrt{(-v_0 \sin \theta)^2 - 4[(1/2)g](-h)}}{2[(1/2)g]} = \frac{v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta + 2gh}}{g}.$$

At this time,

$$\begin{aligned} x(t) &= v_0 \cos \theta \left(\frac{v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta + 2gh}}{g} \right) = \frac{v_0 \cos \theta}{g} \left(v_0 \sin \theta + \sqrt{v_0^2 \left(\sin^2 \theta + \frac{2gh}{v_0^2} \right)} \right) \\ &= \frac{v_0^2 \cos \theta}{g} \left(\sin \theta + \sqrt{\sin^2 \theta + \frac{2gh}{v_0^2}} \right). \end{aligned}$$

53. $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ Position vector $\mathbf{v}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$ Velocity vector $\mathbf{a}(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$ Acceleration vector

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

 $= C$, C is a constant.

$$\frac{d}{dt}[x'(t)^2 + y'(t)^2 + z'(t)^2] = 0$$

$$2x'(t)x''(t) + 2y'(t)y''(t) + 2z'(t)z''(t) = 0$$

$$2[x'(t)x''(t) + y'(t)y''(t) + z'(t)z''(t)] = 0$$

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = 0$$

Orthogonal

55. $\mathbf{r}(t) = 6 \cos t \mathbf{i} + 3 \sin t \mathbf{j}$ (a) $\mathbf{v}(t) = \mathbf{r}'(t) = -6 \sin t \mathbf{i} + 3 \cos t \mathbf{j}$

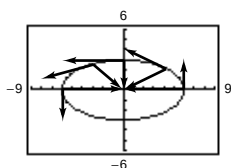
$$\|\mathbf{v}(t)\| = \sqrt{36 \sin^2 t + 9 \cos^2 t}$$

$$= 3\sqrt{4 \sin^2 t + \cos^2 t}$$

$$= 3\sqrt{3 \sin^2 t + 1}$$

 $\mathbf{a}(t) = \mathbf{v}'(t) = -6 \cos t \mathbf{i} - 3 \sin t \mathbf{j}$

(c)



t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
Speed	3	$\frac{3}{2}\sqrt{10}$	6	$\frac{3}{2}\sqrt{13}$	3

(d) The speed is increasing when the angle between $\mathbf{v}$ and $\mathbf{a}$ is in the interval

$$\left[0, \frac{\pi}{2}\right).$$

The speed is decreasing when the angle is in the interval

$$\left(\frac{\pi}{2}, \pi\right].$$

Section 11.4 Tangent Vectors and Normal Vectors

1. $\mathbf{r}(t) = t^2 \mathbf{i} + 2t \mathbf{j}$

$$\mathbf{r}'(t) = 2t \mathbf{i} + 2 \mathbf{j}, \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 4} = 2\sqrt{t^2 + 1}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{2t \mathbf{i} + 2 \mathbf{j}}{2\sqrt{t^2 + 1}} = \frac{1}{\sqrt{t^2 + 1}}(t \mathbf{i} + \mathbf{j})$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j}$$

5. $\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + t \mathbf{k}$

$$\mathbf{r}'(t) = \mathbf{i} + 2t \mathbf{j} + \mathbf{k}$$

When $t = 0$, $\mathbf{r}'(0) = \mathbf{i} + \mathbf{k}$, $[t = 0 \text{ at } (0, 0, 0)]$.

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{k})$$

Direction numbers: $a = 1$, $b = 0$, $c = 1$ Parametric equations: $x = t$, $y = 0$, $z = t$ 3. $\mathbf{r}(t) = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j}$

$$\mathbf{r}'(t) = -4 \sin t \mathbf{i} + 4 \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{16 \sin^2 t + 16 \cos^2 t} = 4$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\sin t \mathbf{i} + \cos t \mathbf{j}$$

$$\mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j}$$

7. $\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + t \mathbf{k}$

$$\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + \mathbf{k}$$

When $t = 0$, $\mathbf{r}'(0) = 2 \mathbf{j} + \mathbf{k}$, $[t = 0 \text{ at } (2, 0, 0)]$.

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\sqrt{5}}{5}(2 \mathbf{j} + \mathbf{k})$$

Direction numbers: $a = 0$, $b = 2$, $c = 1$ Parametric equations: $x = 2$, $y = 2t$, $z = t$

9. $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, 4 \rangle$

$\mathbf{r}'(t) = \langle -2 \sin t, 2 \cos t, 0 \rangle$

When $t = \frac{\pi}{4}$, $\mathbf{r}'\left(\frac{\pi}{4}\right) = \langle -\sqrt{2}, \sqrt{2}, 0 \rangle$, $\left[t = \frac{\pi}{4} \text{ at } (\sqrt{2}, \sqrt{2}, 4)\right]$.

$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{\mathbf{r}'(\pi/4)}{\|\mathbf{r}'(\pi/4)\|} = \frac{1}{2} \langle -\sqrt{2}, \sqrt{2}, 0 \rangle$

Direction numbers: $a = -\sqrt{2}$, $b = \sqrt{2}$, $c = 0$

Parametric equations: $x = -\sqrt{2}t + \sqrt{2}$, $y = \sqrt{2}t + \sqrt{2}$, $z = 4$

11. $\mathbf{r}(t) = \left\langle t, t^2, \frac{2}{3}t^3 \right\rangle$

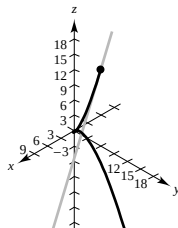
$\mathbf{r}'(t) = \langle 1, 2t, 2t^2 \rangle$

When $t = 3$, $\mathbf{r}'(3) = \langle 1, 6, 18 \rangle$, $[t = 3 \text{ at } (3, 9, 18)]$.

$\mathbf{T}(3) = \frac{\mathbf{r}'(3)}{\|\mathbf{r}'(3)\|} = \frac{1}{19} \langle 1, 6, 18 \rangle$

Direction numbers: $a = 1$, $b = 6$, $c = 18$

Parametric equations: $x = t + 3$, $y = 6t + 9$, $z = 18t + 18$



13. $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j} + \sqrt{t}\mathbf{k}$, $t_0 = 1$

$\mathbf{r}'(t) = \mathbf{i} + \frac{1}{t}\mathbf{j} + \frac{1}{2\sqrt{t}}\mathbf{k}$, $\mathbf{r}'(1) = \mathbf{i} + \mathbf{j} + \frac{1}{2}\mathbf{k}$

$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\mathbf{i} + \mathbf{j} + (1/2)\mathbf{k}}{\sqrt{1 + 1 + (1/4)}} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$

Tangent line: $x = 1 + t$, $y = t$, $z = 1 + \frac{1}{2}t$

$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(1.1) \approx 1.1\mathbf{i} + 0.1\mathbf{j} + 1.05\mathbf{k}$
 $= \langle 1.1, 0.1, 1.05 \rangle$

15. $\mathbf{r}(4) = \langle 2, 16, 2 \rangle$

$\mathbf{u}(8) = \langle 2, 16, 2 \rangle$

Hence the curves intersect.

$\mathbf{r}'(t) = \left\langle 1, 2t, \frac{1}{2} \right\rangle$, $\mathbf{r}'(4) = \left\langle 1, 8, \frac{1}{2} \right\rangle$

$\mathbf{u}'(s) = \left\langle \frac{1}{4}, 2, \frac{1}{3}s^{-2/3} \right\rangle$, $\mathbf{u}'(8) = \left\langle \frac{1}{4}, 2, \frac{1}{12} \right\rangle$

$\cos \theta = \frac{\mathbf{r}'(4) \cdot \mathbf{u}'(8)}{\|\mathbf{r}'(4)\| \|\mathbf{u}'(8)\|} \approx \frac{16.29167}{16.29513} \Rightarrow \theta \approx 1.2^\circ$

17. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j}$, $t = 2$

$\mathbf{r}'(t) = \mathbf{i} + t\mathbf{j}$

$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + t\mathbf{j}}{\sqrt{1 + t^2}}$

$\mathbf{T}'(t) = \frac{-t}{(t^2 + 1)^{3/2}}\mathbf{i} + \frac{1}{(t^2 + 1)^{3/2}}\mathbf{j}$

$\mathbf{T}'(2) = \frac{-2}{5^{3/2}}\mathbf{i} + \frac{1}{5^{3/2}}\mathbf{j}$

$\mathbf{N}(2) = \frac{\mathbf{T}'(2)}{\|\mathbf{T}'(2)\|} + \frac{1}{\sqrt{5}}(-2\mathbf{i} + \mathbf{j}) = \frac{-2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}$

19. $\mathbf{r}(t) = 6 \cos t\mathbf{i} + 6 \sin t\mathbf{j} + \mathbf{k}$, $t = \frac{3\pi}{4}$

$\mathbf{r}'(t) = -6 \sin t\mathbf{i} + 6 \cos t\mathbf{j}$

$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\mathbf{T}'(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$, $\|\mathbf{T}'(t)\| = 1$

$\mathbf{N}\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$

21. $\mathbf{r}(t) = 4t\mathbf{i}$

$\mathbf{v}(t) = 4\mathbf{i}$

$\mathbf{a}(t) = \mathbf{0}$

$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{4\mathbf{i}}{4} = \mathbf{i}$

$\mathbf{T}'(t) = \mathbf{0}$

$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ is undefined.

The path is a line and the speed is constant.

23. $\mathbf{r}(t) = 4t^2\mathbf{i}$

$\mathbf{v}(t) = 8t\mathbf{i}$

$\mathbf{a}(t) = 8\mathbf{i}$

$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{8t\mathbf{i}}{8t} = \mathbf{i}$

$\mathbf{T}'(t) = \mathbf{0}$

$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ is undefined.

The path is a line and the speed is variable.

25. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}$, $\mathbf{v}(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}$, $\mathbf{v}(1) = \mathbf{i} - \mathbf{j}$,

$\mathbf{a}(t) = \frac{2}{t^3}\mathbf{j}$, $\mathbf{a}(1) = 2\mathbf{j}$

$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{t^2}{\sqrt{t^4 + 1}}\left(\mathbf{i} - \frac{1}{t^2}\mathbf{j}\right) = \frac{1}{\sqrt{t^4 + 1}}(t^2\mathbf{i} - \mathbf{j})$

$\mathbf{T}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j}) = \frac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j})$

$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\frac{2t}{(t^4 + 1)^{3/2}}\mathbf{i} + \frac{2t^3}{(t^4 + 1)^{3/2}}\mathbf{j}}{\frac{2t}{(t^4 + 1)}}$

$= \frac{1}{\sqrt{t^4 + 1}}(\mathbf{i} + t^2\mathbf{j})$

$\mathbf{N}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$

$a_T = \mathbf{a} \cdot \mathbf{T} = -\sqrt{2}$

$a_N = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$

27. $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j}$

$\mathbf{v}(t) = e^t(\cos t - \sin t)\mathbf{i} + e^t(\cos t + \sin t)\mathbf{j}$

$\mathbf{a}(t) = e^t(-2 \sin t)\mathbf{i} + e^t(2 \cos t)\mathbf{j}$

At $t = \frac{\pi}{2}$, $\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{j})$.

Motion along $\mathbf{r}$ is counterclockwise. Therefore,

$\mathbf{N} = \frac{1}{\sqrt{2}}(-\mathbf{i} - \mathbf{j}) = -\frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$.

$a_T = \mathbf{a} \cdot \mathbf{T} = \sqrt{2}e^{\pi/2}$

$a_N = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}e^{\pi/2}$

29. $\mathbf{r}(t_0) = (\cos \omega t_0 + \omega t_0 \sin \omega t_0)\mathbf{i} + (\sin \omega t_0 - \omega t_0 \cos \omega t_0)\mathbf{j}$

$\mathbf{v}(t_0) = (\omega^2 t_0 \cos \omega t_0)\mathbf{i} + (\omega^2 t_0 \sin \omega t_0)\mathbf{j}$

$\mathbf{a}(t_0) = \omega^2[(\cos \omega t_0 - \omega t_0 \sin \omega t_0)\mathbf{i} + (\omega t_0 \cos \omega t_0 + \sin \omega t_0)\mathbf{j}]$

$\mathbf{T}(t_0) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = (\cos \omega t_0)\mathbf{i} + (\sin \omega t_0)\mathbf{j}$

Motion along $\mathbf{r}$ is counterclockwise. Therefore

$\mathbf{N}(t_0) = (-\sin \omega t_0)\mathbf{i} + (\cos \omega t_0)\mathbf{j}$.

$a_T = \mathbf{a} \cdot \mathbf{T} = \omega^2$

$a_N = \mathbf{a} \cdot \mathbf{N} = \omega^2(\omega t_0) = \omega^3 t_0$

$$31. \mathbf{r}(t) = a \cos(\omega t) \mathbf{i} + a \sin(\omega t) \mathbf{j}$$

$$\mathbf{v}(t) = -a\omega \sin(\omega t) \mathbf{i} + a\omega \cos(\omega t) \mathbf{j}$$

$$\mathbf{a}(t) = -a\omega^2 \cos(\omega t) \mathbf{i} - a\omega^2 \sin(\omega t) \mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = -\sin(\omega t) \mathbf{i} + \cos(\omega t) \mathbf{j}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\cos(\omega t) \mathbf{i} - \sin(\omega t) \mathbf{j}$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = 0$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = a\omega^2$$

$$35. \mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}, \quad t_0 = 2$$

$$x = t, \quad y = \frac{1}{t} \Rightarrow xy = 1$$

$$\mathbf{r}'(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}$$

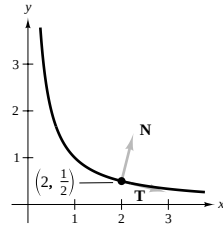
$$\mathbf{T}(t) = \frac{t^2\mathbf{i} - \mathbf{j}}{\sqrt{t^4 + 1}}$$

$$\mathbf{N}(t) = \frac{\mathbf{i} + t^2\mathbf{j}}{\sqrt{t^4 + 1}}$$

$$\mathbf{r}(2) = 2\mathbf{i} + \frac{1}{2}\mathbf{j}$$

$$\mathbf{T}(2) = \frac{\sqrt{17}}{17}(4\mathbf{i} - \mathbf{j})$$

$$\mathbf{N}(2) = \frac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{j})$$



$$39. \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}$$

$$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$$

$$\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{1 + 5t^2}}(\mathbf{i} + 2t\mathbf{j} + t\mathbf{k})$$

$$\mathbf{T}(1) = \frac{\sqrt{6}}{6}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{\frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{(1 + 5t^2)^{3/2}}}{\frac{\sqrt{5}}{1 + 5t^2}} = \frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{5}\sqrt{1 + 5t^2}}$$

$$\mathbf{N}(1) = \frac{\sqrt{30}}{30}(-5\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \frac{5\sqrt{6}}{6}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \frac{\sqrt{30}}{6}$$

$$33. \text{Speed: } \|\mathbf{v}(t)\| = a\omega$$

The speed is constant since $a_T = 0$.

$$37. \mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} - 3t\mathbf{k}$$

$$\mathbf{v}(t) = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$$

$$\mathbf{a}(t) = \mathbf{0}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = \frac{\sqrt{14}}{14}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} \text{ is undefined.}$$

a_T, a_N are not defined.

$$41. \mathbf{r}(t) = 4t\mathbf{i} + 3 \cos t\mathbf{j} + 3 \sin t\mathbf{k}$$

$$\mathbf{v}(t) = 4\mathbf{i} - 3 \sin t\mathbf{j} + 3 \cos t\mathbf{k}$$

$$\mathbf{v}\left(\frac{\pi}{2}\right) = 4\mathbf{i} - 3\mathbf{j}$$

$$\mathbf{a}(t) = -3 \cos t\mathbf{j} - 3 \sin t\mathbf{k}$$

$$\mathbf{a}\left(\frac{\pi}{2}\right) = -3\mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{5}(4\mathbf{i} - 3 \sin t\mathbf{j} + 3 \cos t\mathbf{k})$$

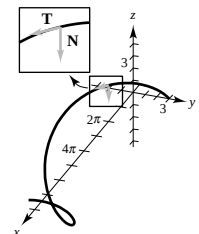
$$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = -\cos t\mathbf{j} - \sin t\mathbf{k}$$

$$\mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{k}$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = 0$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = 3$$



$$43. \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

If $\mathbf{a}(t) = a_T \mathbf{T}(t) + a_N \mathbf{N}(t)$, then a_T is the tangential component of acceleration and a_N is the normal component of acceleration.

45. If $a_N = 0$, then the motion is in a straight line.

$$47. \mathbf{r}(t) = \langle \pi t - \sin \pi t, 1 - \cos \pi t \rangle$$

The graph is a cycloid.

$$(a) \mathbf{r}(t) = \langle \pi t - \sin \pi t, 1 - \cos \pi t \rangle$$

$$\mathbf{v}(t) = \langle \pi - \pi \cos \pi t, \pi \sin \pi t \rangle$$

$$\mathbf{a}(t) = \langle \pi^2 \sin \pi t, \pi^2 \cos \pi t \rangle$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} \langle 1 - \cos \pi t, \sin \pi t \rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} \langle \sin \pi t, -1 + \cos \pi t \rangle$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} [\pi^2 \sin \pi t (1 - \cos \pi t) + \pi^2 \cos \pi t \sin \pi t] = \frac{\pi^2 \sin \pi t}{\sqrt{2(1 - \cos \pi t)}}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} [\pi^2 \sin^2 \pi t + \pi^2 \cos \pi t (-1 + \cos \pi t)] = \frac{\pi^2 (1 - \cos \pi t)}{\sqrt{2(1 - \cos \pi t)}} = \frac{\pi^2 \sqrt{2(1 - \cos \pi t)}}{2}$$

$$\text{When } t = \frac{1}{2}: a_T = \frac{\pi^2}{\sqrt{2}} = \frac{\sqrt{2}\pi^2}{2}, a_N = \frac{\sqrt{2}\pi^2}{2}$$

$$\text{When } t = 1: a_T = 0, a_N = \pi^2$$

$$\text{When } t = \frac{3}{2}: a_T = -\frac{\sqrt{2}\pi^2}{2}, a_N = \frac{\sqrt{2}\pi^2}{2}$$

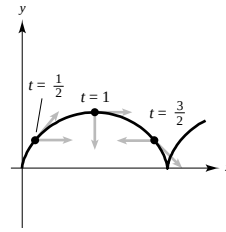
$$(b) \text{ Speed: } s = \|\mathbf{v}(t)\| = \pi \sqrt{2(1 - \cos \pi t)}$$

$$\frac{ds}{dt} = \frac{\pi^2 \sin \pi t}{\sqrt{2(1 - \cos \pi t)}} = a_T$$

$$\text{When } t = \frac{1}{2}: a_T = \frac{\sqrt{2}\pi^2}{2} > 0 \Rightarrow \text{the speed is increasing.}$$

$$\text{When } t = 1: a_T = 0 \Rightarrow \text{the height is maximum.}$$

$$\text{When } t = \frac{3}{2}: a_T = -\frac{\sqrt{2}\pi^2}{2} < 0 \Rightarrow \text{the speed is decreasing.}$$



49. $\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + \frac{t}{2} \mathbf{k}, t_0 = \frac{\pi}{2}$

$$\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + \frac{1}{2} \mathbf{k}$$

$$\mathbf{T}(t) = \frac{2\sqrt{17}}{17} \left(-2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + \frac{1}{2} \mathbf{k} \right)$$

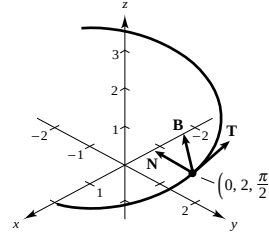
$$\mathbf{N}(t) = -\cos t \mathbf{i} - \sin t \mathbf{j}$$

$$\mathbf{r}\left(\frac{\pi}{2}\right) = 2\mathbf{j} + \frac{\pi}{4} \mathbf{k}$$

$$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{2\sqrt{17}}{17} \left(-2\mathbf{i} + \frac{1}{2} \mathbf{k} \right) = \frac{\sqrt{17}}{17} (-4\mathbf{i} + \mathbf{k})$$

$$\mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{j}$$

$$\mathbf{B}\left(\frac{\pi}{2}\right) = \mathbf{T}\left(\frac{\pi}{2}\right) \times \mathbf{N}\left(\frac{\pi}{2}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{4\sqrt{17}}{17} & 0 & \frac{\sqrt{17}}{17} \\ 0 & -1 & 0 \end{vmatrix} = \frac{\sqrt{17}}{17} \mathbf{i} + \frac{4\sqrt{17}}{17} \mathbf{k} = \frac{\sqrt{17}}{17} (\mathbf{i} + 4\mathbf{k})$$



51. From Theorem 11.3 we have:

$$\mathbf{r}(t) = (v_0 t \cos \theta) \mathbf{i} + (h + v_0 t \sin \theta - 16t^2) \mathbf{j}$$

$$\mathbf{v}(t) = v_0 \cos \theta \mathbf{i} + (v_0 \sin \theta - 32t) \mathbf{j}$$

$$\mathbf{a}(t) = -32 \mathbf{j}$$

$$\mathbf{T}(t) = \frac{(v_0 \cos \theta) \mathbf{i} + (v_0 \sin \theta - 32t) \mathbf{j}}{\sqrt{v_0^2 \cos^2 \theta + (v_0 \sin \theta - 32t)^2}}$$

$$\mathbf{N}(t) = \frac{(v_0 \sin \theta - 32t) \mathbf{i} - v_0 \cos \theta \mathbf{j}}{\sqrt{v_0^2 \cos^2 \theta + (v_0 \sin \theta - 32t)^2}} \quad (\text{Motion is clockwise.})$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \frac{-32(v_0 \sin \theta - 32t)}{\sqrt{v_0^2 \cos^2 \theta + (v_0 \sin \theta - 32t)^2}}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \frac{32v_0 \cos \theta}{\sqrt{v_0^2 \cos^2 \theta + (v_0 \sin \theta - 32t)^2}}$$

Maximum height when $v_0 \sin \theta - 32t = 0$; (vertical component of velocity)

At maximum height, $a_T = 0$ and $a_N = 32$.

53. $\mathbf{r}(t) = \langle 10 \cos 10\pi t, 10 \sin 10\pi t, 4 + 4t \rangle, 0 \leq t \leq \frac{1}{20}$

(a) $\mathbf{r}'(t) = \langle -100\pi \sin(10\pi t), 100\pi \cos(10\pi t), 4 \rangle$

$$\begin{aligned} \|\mathbf{r}'(t)\| &= \sqrt{(100\pi)^2 \sin^2(10\pi t) + (100\pi)^2 \cos^2(10\pi t) + 16} \\ &= \sqrt{(100\pi)^2 + 16} = 4\sqrt{625\pi^2 + 1} \approx 314 \text{ mi/hr} \end{aligned}$$

(b) $a_T = 0$ and $a_N = 1000\pi^2$

$a_T = 0$ because the speed is constant.

55. $\mathbf{r}(t) = (a \cos \omega t) \mathbf{i} + (a \sin \omega t) \mathbf{j}$

From Exercise 31, we know $\mathbf{a} \cdot \mathbf{T} = 0$ and $\mathbf{a} \cdot \mathbf{N} = a\omega^2$.

(a) Let $\omega_0 = 2\omega$. Then

$$\mathbf{a} \cdot \mathbf{N} = a\omega_0^2 = a(2\omega)^2 = 4a\omega^2$$

or the centripetal acceleration is increased by a factor of 4 when the velocity is doubled.

(b) Let $a_0 = a/2$. Then

$$\mathbf{a} \cdot \mathbf{N} = a_0\omega^2 = \left(\frac{a}{2}\right)\omega^2 = \left(\frac{1}{2}\right)a\omega^2$$

or the centripetal acceleration is halved when the radius is halved.

$$57. v = \sqrt{\frac{9.56 \times 10^4}{4100}} \approx 4.83 \text{ mi/sec}$$

$$59. v = \sqrt{\frac{9.56 \times 10^4}{4385}} \approx 4.67 \text{ mi/sec}$$

61. Let $\mathbf{T}(t) = \cos \phi \mathbf{i} + \sin \phi \mathbf{j}$ be the unit tangent vector. Then

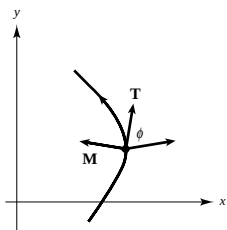
$$\mathbf{T}'(t) = \frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{d\phi} \frac{d\phi}{dt} = -(\sin \phi \mathbf{i} + \cos \phi \mathbf{j}) \frac{d\phi}{dt} = \mathbf{M} \frac{d\phi}{dt}.$$

$\mathbf{M} = -\sin \phi \mathbf{i} + \cos \phi \mathbf{j} = \cos[\phi + (\pi/2)] \mathbf{i} + \sin[\phi + (\pi/2)] \mathbf{j}$ and is rotated counterclockwise through an angle of $\pi/2$ from $\mathbf{T}$.

If $d\phi/dt > 0$, then the curve bends to the left and $\mathbf{M}$ has the same direction as $\mathbf{T}'$. Thus, $\mathbf{M}$ has the same direction as

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|},$$

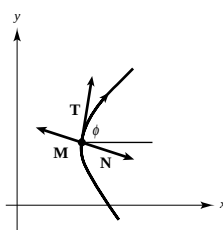
which is toward the concave side of the curve.



If $d\phi/dt < 0$, then the curve bends to the right and $\mathbf{M}$ has the opposite direction as $\mathbf{T}'$. Thus,

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|}$$

again points to the concave side of the curve.



63. Using $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$, $\mathbf{T} \times \mathbf{T} = \mathbf{O}$, and $\|\mathbf{T} \times \mathbf{N}\| = 1$, we have:

$$\begin{aligned} \mathbf{v} \times \mathbf{a} &= \|\mathbf{v}\| \mathbf{T} \times (a_T \mathbf{T} + a_N \mathbf{N}) \\ &= \|\mathbf{v}\| a_T (\mathbf{T} \times \mathbf{T}) + \|\mathbf{v}\| a_N (\mathbf{T} \times \mathbf{N}) \\ &= \|\mathbf{v}\| a_N (\mathbf{T} \times \mathbf{N}) \\ \|\mathbf{v} \times \mathbf{a}\| &= \|\mathbf{v}\| a_N \|\mathbf{T} \times \mathbf{N}\| \\ &= \|\mathbf{v}\| a_N \end{aligned}$$

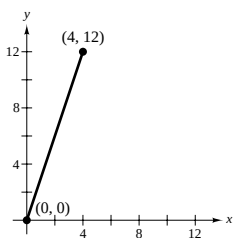
$$\text{Thus, } a_N = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}.$$

Section 11.5 Arc Length and Curvature

1. $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j}$

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 3, \quad \frac{dz}{dt} = 0$$

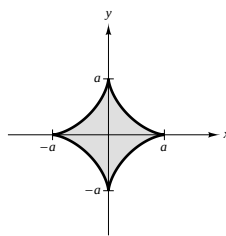
$$\begin{aligned} s &= \int_0^4 \sqrt{1 + 9} \, dt \\ &= \sqrt{10} \int_0^4 dt \\ &= \left[\sqrt{10}t \right]_0^4 = 4\sqrt{10} \end{aligned}$$



3. $\mathbf{r}(t) = a \cos^3 t \mathbf{i} + a \sin^3 t \mathbf{j}$

$$\frac{dx}{dt} = -3a \cos^2 t \sin t, \quad \frac{dy}{dt} = 3a \sin^2 t \cos t$$

$$\begin{aligned} s &= 4 \int_0^{\pi/2} \sqrt{[3a \cos^2 t (-\sin t)]^2 + [3a \sin^2 t \cos t]^2} \, dt \\ &= 12a \int_0^{\pi/2} \sin t \cos t \, dt \\ &= 3a \int_0^{\pi/2} 2 \sin 2t \, dt = \left[-3a \cos 2t \right]_0^{\pi/2} = 6a \end{aligned}$$



$$\begin{aligned}
 5. \text{ (a) } \mathbf{r}(t) &= (v_0 \cos \theta) \mathbf{i} + \left[h + (v_0 \sin \theta)t - \frac{1}{2}gt^2 \right] \mathbf{j} \\
 &= (100 \cos 45^\circ) \mathbf{i} + \left[3 + (100 \sin 45^\circ)t - \frac{1}{2}(32)t^2 \right] \mathbf{j} \\
 &= 50\sqrt{2} \mathbf{i} + [3 + 50\sqrt{2}t - 16t^2] \mathbf{j}
 \end{aligned}$$

$$(b) \mathbf{v}(t) = 50\sqrt{2} \mathbf{i} + (50\sqrt{2} - 32t) \mathbf{j}$$

$$50\sqrt{2} - 32t = 0 \Rightarrow t = \frac{25\sqrt{2}}{16}$$

$$\text{Maximum height: } 3 + 50\sqrt{2} \left(\frac{25\sqrt{2}}{16} \right) - 16 \left(\frac{25\sqrt{2}}{16} \right)^2 = 81.125 \text{ ft}$$

$$(c) 3 + 50\sqrt{2}t - 16t^2 = 0 \Rightarrow t \approx 4.4614$$

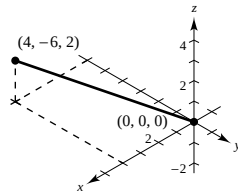
$$\text{Range: } 50\sqrt{2}(4.4614) \approx 315.5 \text{ feet}$$

$$(d) s = \int_0^{4.4614} \sqrt{(50\sqrt{2})^2 + (50\sqrt{2} - 32t)^2} dt \approx 362.9 \text{ feet}$$

$$7. \mathbf{r}(t) = 2t \mathbf{i} - 3t \mathbf{j} + t \mathbf{k}$$

$$\frac{dx}{dt} = 2, \frac{dy}{dt} = -3, \frac{dz}{dt} = 1$$

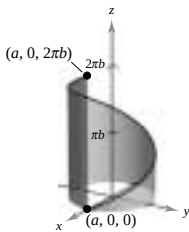
$$\begin{aligned}
 s &= \int_0^2 \sqrt{2^2 + (-3)^2 + 1^2} dt \\
 &= \int_0^2 \sqrt{14} dt = \left[\sqrt{14}t \right]_0^2 = 2\sqrt{14}
 \end{aligned}$$



$$9. \mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j} + b t \mathbf{k}$$

$$\frac{dx}{dt} = -a \sin t, \frac{dy}{dt} = a \cos t, \frac{dz}{dt} = b$$

$$\begin{aligned}
 s &= \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} dt \\
 &= \int_0^{2\pi} \sqrt{a^2 + b^2} dt = \left[\sqrt{a^2 + b^2} t \right]_0^{2\pi} = 2\pi \sqrt{a^2 + b^2}
 \end{aligned}$$



$$11. \mathbf{r}(t) = t^2 \mathbf{i} + t \mathbf{j} + \ln t \mathbf{k}$$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 1, \frac{dz}{dt} = \frac{1}{t}$$

$$\begin{aligned}
 s &= \int_1^3 \sqrt{(2t)^2 + (1)^2 + \left(\frac{1}{t}\right)^2} dt \\
 &= \int_1^3 \sqrt{\frac{4t^4 + t^2 + 1}{t^2}} dt \\
 &= \int_1^3 \frac{\sqrt{4t^4 + t^2 + 1}}{t} dt \approx 8.37
 \end{aligned}$$

$$13. \mathbf{r}(t) = t \mathbf{i} + (4 - t^2) \mathbf{j} + t^3 \mathbf{k}, \quad 0 \leq t \leq 2$$

$$(a) \mathbf{r}(0) = \langle 0, 4, 0 \rangle, \quad \mathbf{r}(2) = \langle 2, 0, 8 \rangle$$

$$\text{distance} = \sqrt{2^2 + 4^2 + 8^2} = \sqrt{84} = 2\sqrt{21} \approx 9.165$$

—CONTINUED—

13. —CONTINUED—

(b) $\mathbf{r}(0) = \langle 0, 4, 0 \rangle$

$\mathbf{r}(0.5) = \langle 0.5, 3.75, .125 \rangle$

$\mathbf{r}(1) = \langle 1, 3, 1 \rangle$

$\mathbf{r}(1.5) = \langle 1.5, 1.75, 3.375 \rangle$

$\mathbf{r}(2) = \langle 2, 0, 8 \rangle$

$$\begin{aligned} \text{distance} &\approx \sqrt{(0.5)^2 + (.25)^2 + (.125)^2} + \sqrt{(.5)^2 + (.75)^2 + (.875)^2} + \sqrt{(0.5)^2 + (1.25)^2 + (2.375)^2} + \\ &\quad \sqrt{(0.5)^2 + (1.75)^2 + (4.625)^2} \\ &\approx 0.5728 + 1.2562 + 2.7300 + 4.9702 \approx 9.529 \end{aligned}$$

(c) Increase the number of line segments.

(d) Using a graphing utility, you obtain 9.57057.

15. $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, t \rangle$

$$\begin{aligned} \text{(a) } s &= \int_0^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2} du \\ &= \int_0^t \sqrt{(-2 \sin u)^2 + (2 \cos u)^2 + (1)^2} du \\ &= \int_0^t \sqrt{5} du = \left[\sqrt{5}u \right]_0^t = \sqrt{5}t \end{aligned}$$

(c) When $s = \sqrt{5}$: $x = 2 \cos 1 \approx 1.081$

$y = 2 \sin 1 \approx 1.683$

$z = 1$

$(1.081, 1.683, 1.000)$

When $s = 4$: $x = 2 \cos \frac{4}{\sqrt{5}} \approx -0.433$

$y = 2 \sin \frac{4}{\sqrt{5}} \approx 1.953$

$z = \frac{4}{\sqrt{5}} \approx 1.789$

$(-0.433, 1.953, 1.789)$

(b) $\frac{s}{\sqrt{5}} = t$

$x = 2 \cos\left(\frac{s}{\sqrt{5}}\right), y = 2 \sin\left(\frac{s}{\sqrt{5}}\right), z = \frac{s}{\sqrt{5}}$

$\mathbf{r}(s) = 2 \cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + 2 \sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k}$

(d) $\|\mathbf{r}'(s)\| = \sqrt{\left(-\frac{2}{\sqrt{5}} \sin\left(\frac{s}{\sqrt{5}}\right)\right)^2 + \left(\frac{2}{\sqrt{5}} \cos\left(\frac{s}{\sqrt{5}}\right)\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = 1$

$$\begin{aligned} 17. \quad \mathbf{r}(s) &= \left(1 + \frac{\sqrt{2}}{2}s\right)\mathbf{i} + \left(1 - \frac{\sqrt{2}}{2}s\right)\mathbf{j} \\ \mathbf{r}'(s) &= \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} \quad \text{and} \quad \|\mathbf{r}'(s)\| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1 \end{aligned}$$

$\mathbf{T}(s) = \frac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|} = \mathbf{r}'(s)$

$\mathbf{T}'(s) = \mathbf{0} \Rightarrow K = \|\mathbf{T}'(s)\| = 0 \quad (\text{The curve is a line.})$

$$\begin{aligned} 19. \quad \mathbf{r}(s) &= 2 \cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + 2 \sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k} \\ \mathbf{T}(s) &= \mathbf{r}'(s) = -\frac{2}{\sqrt{5}} \sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + \frac{2}{\sqrt{5}} \cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k} \\ \mathbf{T}'(s) &= -\frac{2}{5} \cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} - \frac{2}{5} \sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} \\ K &= \|\mathbf{T}'(s)\| = \frac{2}{5} \end{aligned}$$

$$21. \mathbf{r}(t) = 4t\mathbf{i} - 2t\mathbf{j}$$

$$\mathbf{v}(t) = 4\mathbf{i} - 2\mathbf{j}$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{5}}(2\mathbf{i} - \mathbf{j})$$

$$\mathbf{T}'(t) = 0$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = 0 \quad (\text{The curve is a line.})$$

$$23. \mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}$$

$$\mathbf{v}(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}$$

$$\mathbf{v}(1) = \mathbf{i} - \mathbf{j}$$

$$\mathbf{a}(t) = \frac{2}{t^3}\mathbf{j}$$

$$\mathbf{a}(1) = 2\mathbf{j}$$

$$\mathbf{T}(t) = \frac{t^2\mathbf{i} - \mathbf{j}}{\sqrt{t^4 + 1}}$$

$$\mathbf{N}(t) = \frac{1}{(t^4 + 1)^{1/2}}(\mathbf{i} + t^2\mathbf{j})$$

$$\mathbf{N}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$$

$$K = \frac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2} = \frac{\sqrt{2}}{2}$$

$$25. \mathbf{r}(t) = 4 \cos(2\pi t)\mathbf{i} + 4 \sin(2\pi t)\mathbf{j}$$

$$\mathbf{r}'(t) = -8\pi \sin(2\pi t)\mathbf{i} + 8\pi \cos(2\pi t)\mathbf{j}$$

$$\mathbf{T}(t) = -\sin(2\pi t)\mathbf{i} + \cos(2\pi t)\mathbf{j}$$

$$\mathbf{T}'(t) = -2\pi \cos(2\pi t)\mathbf{i} - 2\pi \sin(2\pi t)\mathbf{j}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{2\pi}{8\pi} = \frac{1}{4}$$

$$27. \mathbf{r}(t) = a \cos(\omega t)\mathbf{i} + a \sin(\omega t)\mathbf{j}$$

$$\mathbf{r}'(t) = -a\omega \sin(\omega t)\mathbf{i} + a\omega \cos(\omega t)\mathbf{j}$$

$$\mathbf{T}(t) = -\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}$$

$$\mathbf{T}'(t) = -\omega \cos(\omega t)\mathbf{i} - \omega \sin(\omega t)\mathbf{j}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\omega}{a\omega} = \frac{1}{a}$$

$$29. \mathbf{r}(t) = e^t \cos t \mathbf{i} + e^t \sin t \mathbf{j}$$

$$\mathbf{r}'(t) = (-e^t \sin t + e^t \cos t)\mathbf{i} + (e^t \cos t + e^t \sin t)\mathbf{j}$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{2}}[(-\sin t + \cos t)\mathbf{i} + (\cos t + \sin t)\mathbf{j}]$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}[(-\cos t - \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j}]$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{2}e^t} = \frac{\sqrt{2}}{2}e^{-t}$$

$$31. \mathbf{r}(t) = \langle \cos \omega t + \omega t \sin \omega t, \sin \omega t - \omega t \cos \omega t \rangle$$

From Exercise 21, Section 11.4, we have:

$$\mathbf{a} \cdot \mathbf{N} = \omega^3 t$$

$$K = \frac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{v}\|^2} = \frac{\omega^3 t}{\omega^4 t^2} = \frac{1}{\omega t}$$

$$33. \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}}{\sqrt{1 + 5t^2}}$$

$$\mathbf{T}'(t) = \frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{(1 + 5t^2)^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

$$= \frac{\frac{\sqrt{5}}{(1 + 5t^2)}}{\sqrt{1 + 5t^2}} = \frac{\sqrt{5}}{(1 + 5t^2)^{3/2}}$$

$$35. \mathbf{r}(t) = 4t\mathbf{i} + 3 \cos t\mathbf{j} + 3 \sin t\mathbf{k}$$

$$\mathbf{r}'(t) = 4\mathbf{i} - 3 \sin t\mathbf{j} + 3 \cos t\mathbf{k}$$

$$\mathbf{T}(t) = \frac{1}{5}[4\mathbf{i} - 3 \sin t\mathbf{j} + 3 \cos t\mathbf{k}]$$

$$\mathbf{T}'(t) = \frac{1}{5}[-3 \cos t\mathbf{j} - 3 \sin t\mathbf{k}]$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{3/5}{5} = \frac{3}{25}$$

37. $y = 3x - 2$

Since $y'' = 0$, $K = 0$, and the radius of curvature is undefined.

41. $y = \sqrt{a^2 - x^2}$

$$y' = \frac{-x}{\sqrt{a^2 - x^2}}$$

$$y'' = \frac{-(2x^2 - a^2)}{(a^2 - x^2)^{3/2}}$$

At $x = 0$: $y' = 0$

$$y'' = \frac{1}{a}$$

$$K = \frac{1/a}{(1 + 0^2)^{3/2}} = \frac{1}{a}$$

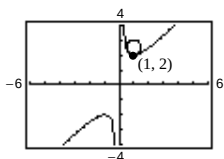
$$\frac{1}{K} = a \quad (\text{radius of curvature})$$

45. $y = x + \frac{1}{x}, y' = 1 - \frac{1}{x^2}, y'' = \frac{2}{x^3}$

$$K = \frac{2}{(1 + 0^2)^{3/2}} = 2$$

Radius of curvature $= 1/2$. Since the tangent line is horizontal at $(1, 2)$, the normal line is vertical. The center of the circle is $1/2$ unit above the point $(1, 2)$ at $(1, 5/2)$.

Circle: $(x - 1)^2 + \left(y - \frac{5}{2}\right)^2 = \frac{1}{4}$



39. $y = 2x^2 + 3$

$$y' = 4x$$

$$y'' = 4$$

$$K = \frac{4}{[1 + (-4)^2]^{3/2}} = \frac{4}{17^{3/2}} \approx 0.057$$

$$\frac{1}{K} = \frac{17^{3/2}}{4} \approx 17.523 \quad (\text{radius of curvature})$$

43. (a) Point on circle: $\left(\frac{\pi}{2}, 1\right)$

Center: $\left(\frac{\pi}{2}, 0\right)$

Equation: $\left(x - \frac{\pi}{2}\right)^2 + y^2 = 1$

(b) The circles have different radii since the curvature is different and

$$r = \frac{1}{K}.$$

47. $y = e^x, \quad x = 0$

$$y' = e^x, \quad y'' = e^x$$

$$y'(0) = 1, \quad y''(0) = 1$$

$$K = \frac{1}{(1 + 1^2)^{3/2}} = \frac{1}{2^{3/2}} = \frac{1}{2\sqrt{2}}, \quad r = \frac{1}{K} = 2\sqrt{2}$$

The slope of the tangent line at $(0, 1)$ is $y'(0) = 1$.

The slope of the normal line is -1 .

Equation of normal line: $y - 1 = -x$ or $y = -x + 1$

The center of the circle is on the normal line $2\sqrt{2}$ units away from the point $(0, 1)$.

$$\sqrt{(0 - x)^2 + (1 - y)^2} = 2\sqrt{2}$$

$$x^2 + x^2 = 8$$

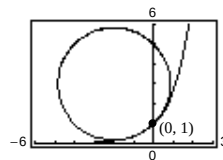
$$x^2 = 4$$

$$x = \pm 2$$

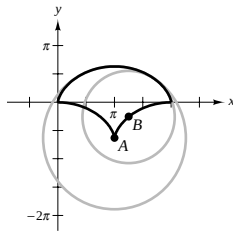
Since the circle is above the curve, $x = -2$ and $y = 3$.

Center of circle: $(-2, 3)$

Equation of circle: $(x + 2)^2 + (y - 3)^2 = 8$



49.



51. $y = (x - 1)^2 + 3$, $y' = 2(x - 1)$, $y'' = 2$

$$K = \frac{2}{(1 + [2(x - 1)]^2)^{3/2}} = \frac{2}{[1 + 4(x - 1)^2]^{3/2}}$$

 (a) K is maximum when $x = 1$ or at the vertex $(1, 3)$.

(b) $\lim_{x \rightarrow \infty} K = 0$

53. $y = x^{2/3}$, $y' = \frac{2}{3}x^{-1/3}$, $y'' = -\frac{2}{9}x^{-4/3}$

$$K = \left| \frac{(-2/9)x^{-4/3}}{[1 + (4/9)x^{-2/3}]^{3/2}} \right| = \left| \frac{6}{x^{1/3}(9x^{2/3} + 4)^{3/2}} \right|$$

 (a) $K \Rightarrow \infty$ as $x \Rightarrow 0$. No maximum

(b) $\lim_{x \rightarrow \infty} K = 0$

55. $y = (x - 1)^3 + 3$

$$y' = 3(x - 1)^2$$

$$y'' = 6(x - 1)$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|6(x - 1)|}{[1 + 9(x - 1)^4]^{3/2}} = 0 \text{ at } x = 1.$$

 Curvature is 0 at $(1, 3)$.

57. $K = \frac{|y''|}{[1 + (y')^2]^{3/2}}$

 The curvature is zero when $y'' = 0$.

59. $s = \int_a^b \|\mathbf{r}'(t)\| dt$

61. The curve is a line.

 63. Endpoints of the major axis: $(\pm 2, 0)$

 Endpoints of the minor axis: $(0, \pm 1)$

$$x^2 + 4y^2 = 4$$

$$2x + 8yy' = 0$$

$$y' = -\frac{x}{4y}$$

$$y'' = \frac{(4y)(-1) - (-x)(4y')}{16y^2} = \frac{-4y - (x^2/y)}{16y^2} = \frac{-(4y^2 + x^2)}{16y^3} = \frac{-1}{4y^3}$$

$$K = \frac{|-1/4y^3|}{[1 + (-x/4y)^2]^{3/2}} = \frac{|-16|}{(16y^2 + x^2)^{3/2}} = \frac{16}{(12y^2 + 4)^{3/2}} = \frac{16}{(16 - 3x^2)^{3/2}}$$

 Therefore, since $-2 \leq x \leq 2$, K is largest when $x = \pm 2$ and smallest when $x = 0$.

65. $f(x) = x^4 - x^2$

(a) $K = \frac{2|6x^2 - 1|}{[16x^6 - 16x^4 + 4x^2 + 1]^{3/2}}$

 (b) For $x = 0$, $K = 2$. $f(0) = 0$. At $(0, 0)$, the circle of curvature has radius $\frac{1}{2}$. Using the symmetry of the graph of f , you obtain

$$x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4}.$$

 For $x = 1$, $K = (2\sqrt{5})/5$. $f(1) = 0$. At $(1, 0)$, the circle of curvature has radius

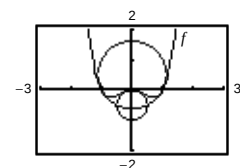
$$\frac{\sqrt{5}}{2} = \frac{1}{K}.$$

 Using the graph of f , you see that the center of curvature is $(0, \frac{1}{2})$. Thus,

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{4}.$$

To graph these circles, use

$$y = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - x^2} \quad \text{and} \quad y = \frac{1}{2} \pm \sqrt{\frac{5}{4} - x^2}.$$

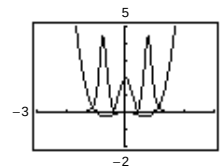


65. —CONTINUED—

- (c) The curvature tends to be greatest near the extrema of f , and K decreases as $x \rightarrow \pm\infty$. However, f and K do not have the same critical numbers.

$$\text{Critical numbers of } f: x = 0, \pm \frac{\sqrt{2}}{2} \approx \pm 0.7071$$

$$\text{Critical numbers of } K: x = 0, \pm 0.7647, \pm 0.4082$$



67. (a) Imagine dropping the circle $x^2 + (y - k)^2 = 16$ into the parabola $y = x^2$. The circle will drop to the point where the tangents to the circle and parabola are equal.

$$y = x^2 \quad \text{and} \quad x^2 + (y - k)^2 = 16 \Rightarrow x^2 + (x^2 - k)^2 = 16$$

Taking derivatives, $2x + 2(y - k)y' = 0$ and $y' = 2x$. Hence,

$$(y - k)y' = -x \Rightarrow y' = \frac{-x}{y - k}.$$

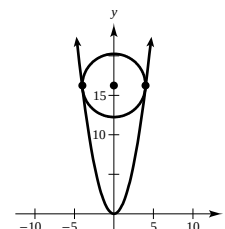
Thus,

$$\frac{-x}{y - k} = 2x \Rightarrow -x = 2x(y - k) \Rightarrow -1 = 2(x^2 - k) \Rightarrow x^2 - k = -\frac{1}{2}.$$

Thus,

$$x^2 + (x^2 - k)^2 = x^2 + \left(-\frac{1}{2}\right)^2 = 16 \Rightarrow x^2 = 15.75.$$

Finally, $k = x^2 + \frac{1}{2} = 16.25$, and the center of the circle is 16.25 units from the vertex of the parabola. Since the radius of the circle is 4, the circle is 12.25 units from the vertex.



- (b) In 2-space, the parabola $z = y^2$ (or $z = x^2$) has a curvature of $K = 2$ at $(0, 0)$. The radius of the largest sphere that will touch the vertex has radius $= 1/K = \frac{1}{2}$.

69. Given $y = f(x)$: $K = \frac{|y''|}{(1 + [y']^2)^{3/2}}$

$$R = \frac{1}{K}$$

The center of the circle is on the normal line at a distance of R from (x, y) .

$$\text{Equation of normal line: } y - y_0 = -\frac{1}{y'}(x - x_0)$$

$$\sqrt{(x - x_0)^2 + \left[-\frac{1}{y'}(x - x_0)\right]^2} = \frac{(1 + [y']^2)^{3/2}}{|y''|}$$

$$(x - x_0)^2 \left[1 + \frac{1}{(y')^2}\right] = \frac{(1 + [y']^2)^3}{(y'')^2}$$

$$(x - x_0)^2 = \frac{(y')^2(1 + [y']^2)^2}{(y'')^2}$$

$$x - x_0 = \frac{y'(1 + [y']^2)}{y''} = y'z$$

$$x_0 = x - y'z$$

$$y - y_0 = -\frac{1}{y'}(x - (x - y'z)) = -z$$

$$y_0 = y + z$$

$$\text{Thus, } (x_0, y_0) = (x - y'z, y + z).$$

$$\text{For } y = e^x, y' = e^x, y'' = e^x, z = \frac{1 + e^{2x}}{e^x} = e^{-x} + e^x.$$

$$\text{When } x = 0: x_0 = x - y'z = 0 - (1)(2) = -2$$

$$y_0 = y + z = 1 + 2 = 3$$

Center of curvature: $(-2, 3)$

(See Exercise 47)

71. $r = 1 + \sin \theta$

$$r' = \cos \theta$$

$$r'' = -\sin \theta$$

$$\begin{aligned}
 K &= \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} \\
 &= \frac{|2 \cos^2 \theta - (1 + \sin \theta)(-\sin \theta) + (1 + \sin \theta)^2|}{\sqrt{[\cos^2 \theta + (1 + \sin \theta)^2]^3}} \\
 &= \frac{3(1 + \sin \theta)}{\sqrt{8(1 + \sin \theta)^3}} = \frac{3}{2\sqrt{2}(1 + \sin \theta)}
 \end{aligned}$$

75. $r = e^{a\theta}, a > 0$

$$r' = ae^{a\theta}$$

$$r'' = a^2e^{a\theta}$$

$$\begin{aligned}
 K &= \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \frac{|2a^2e^{2a\theta} - a^2e^{2a\theta} + e^{2a\theta}|}{[a^2e^{2a\theta} + e^{2a\theta}]^{3/2}} \\
 &= \frac{1}{e^{a\theta}\sqrt{a^2 + 1}}
 \end{aligned}$$

(a) As $\theta \Rightarrow \infty$, $K \Rightarrow 0$.

(b) As $a \Rightarrow \infty$, $K \Rightarrow 0$.

79. $x = f(t)$

$$y = g(t)$$

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$y'' = \frac{\frac{d}{dt}\left[\frac{g'(t)}{f'(t)}\right]}{\frac{dx}{dt}} = \frac{\frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^2}}{f'(t)}$$

$$= \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}$$

$$\begin{aligned}
 K &= \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{\left|\frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}\right|}{\left[1 + \left(\frac{g'(t)}{f'(t)}\right)^2\right]^{3/2}} \\
 &= \frac{\left|\frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}\right|}{\sqrt{\left\{\frac{[f'(t)]^2 + [g'(t)]^2}{[f'(t)]^2}\right\}^3}} \\
 &= \frac{|f'(t)g''(t) - g'(t)f''(t)|}{([f'(t)]^2 + [g'(t)]^2)^{3/2}}
 \end{aligned}$$

83. $a_N = mK\left(\frac{ds}{dt}\right)^2 = \left(\frac{5500 \text{ lb}}{32 \text{ ft/sec}^2}\right)\left(\frac{1}{100 \text{ ft}}\right)\left(\frac{30(5280) \text{ ft}}{3600 \text{ sec}}\right)^2 = 3327.5 \text{ lb}$

73. $r = a \sin \theta$

$$r' = a \cos \theta$$

$$r'' = -a \sin \theta$$

$$\begin{aligned}
 K &= \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} \\
 &= \frac{|2a^2 \cos^2 \theta + a^2 \sin^2 \theta + a^2 \sin^2 \theta|}{\sqrt{[a^2 \cos^2 \theta + a^2 \sin^2 \theta]^3}} \\
 &= \frac{2a^2}{a^3} = \frac{2}{a}, a > 0
 \end{aligned}$$

77. $r = 4 \sin 2\theta$

$$r' = 8 \cos 2\theta$$

At the pole: $K = \frac{2}{|r'(0)|} = \frac{2}{8} = \frac{1}{4}$

81. $x(\theta) = a(\theta - \sin \theta) \quad y(\theta) = a(1 - \cos \theta)$

$$x'(\theta) = a(1 - \cos \theta) \quad y'(\theta) = a \sin \theta$$

$$x''(\theta) = a \sin \theta \quad y''(\theta) = a \cos \theta$$

$$\begin{aligned}
 K &= \frac{|x'(\theta)y''(\theta) - y'(\theta)x''(\theta)|}{[x'(\theta)^2 + y'(\theta)^2]^{3/2}} \\
 &= \frac{|a^2(1 - \cos \theta) \cos \theta - a^2 \sin^2 \theta|}{[a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta]^{3/2}} \\
 &= \frac{1}{a} \frac{|\cos \theta - 1|}{[2 - 2 \cos \theta]^{3/2}} \\
 &= \frac{1}{a} \frac{1 - \cos \theta}{2\sqrt{2}[1 - \cos \theta]^{3/2}} \quad (1 - \cos \geq 0) \\
 &= \frac{1}{2a\sqrt{2 - 2 \cos \theta}} = \frac{1}{4a} \csc\left(\frac{\theta}{2}\right)
 \end{aligned}$$

Minimum: $\frac{1}{4a} \quad (\theta = \pi)$

Maximum: none $(K \rightarrow \infty \text{ as } \theta \rightarrow 0)$

85. Let $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $r = \|\mathbf{r}\| = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}$ and $\mathbf{r}' = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$. Then,

$$\begin{aligned} r\left(\frac{d\mathbf{r}}{dt}\right) &= \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2} \left[\frac{1}{2} \{ [x(t)]^2 + [y(t)]^2 + [z(t)]^2 \}^{-1/2} \cdot (2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t)) \right] \\ &= x(t)x'(t) + y(t)y'(t) + z(t)z'(t) = \mathbf{r} \cdot \mathbf{r}'. \end{aligned}$$

87. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where x , y , and z are functions of t , and $r = \|\mathbf{r}\|$.

$$\begin{aligned} \frac{d}{dt} \left[\frac{\mathbf{r}}{r} \right] &= \frac{r\mathbf{r}' - \mathbf{r}(dr/dt)}{r^2} = \frac{r\mathbf{r}' - \mathbf{r}[(\mathbf{r} \cdot \mathbf{r}')/r]}{r^2} = \frac{r^2\mathbf{r}' - (\mathbf{r} \cdot \mathbf{r}')\mathbf{r}}{r^3} \quad (\text{using Exercise 77}) \\ &= \frac{(x^2 + y^2 + z^2)(x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}) - (xx' + yy' + zz')(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{r^3} \\ &= \frac{1}{r^3} [(x'y^2 + x'z^2 - xyy' - xzz')\mathbf{i} + (x^2y' + z^2y' - xx'y - zz'y)\mathbf{j} + (x^2z' + y^2z' - xx'z - yy'z)\mathbf{k}] \\ &= \frac{1}{r^3} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ yz' - y'z & -(xz' - x'z) & xy' - x'y \\ x & y & z \end{vmatrix} = \frac{1}{r^3} \{ [\mathbf{r} \times \mathbf{r}'] \times \mathbf{r} \} \end{aligned}$$

89. From Exercise 86, we have concluded that planetary motion is planar. Assume that the planet moves in the xy -plane with the sun at the origin. From Exercise 88, we have

$$\mathbf{r}' \times \mathbf{L} = GM \left(\frac{\mathbf{r}}{r} + \mathbf{e} \right).$$

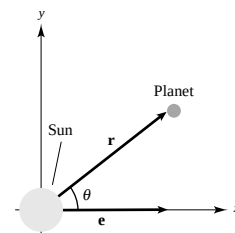
Since $\mathbf{r}' \times \mathbf{L}$ and $\mathbf{r}$ are both perpendicular to $\mathbf{L}$, so is $\mathbf{e}$. Thus, $\mathbf{e}$ lies in the xy -plane. Situate the coordinate system so that $\mathbf{e}$ lies along the positive x -axis and θ is the angle between $\mathbf{e}$ and $\mathbf{r}$. Let $e = \|\mathbf{e}\|$. Then $\mathbf{r} \cdot \mathbf{e} = \|\mathbf{r}\| \|\mathbf{e}\| \cos \theta = re \cos \theta$. Also,

$$\begin{aligned} \|\mathbf{L}\|^2 &= \mathbf{L} \cdot \mathbf{L} = (\mathbf{r} \times \mathbf{r}') \cdot \mathbf{L} \\ &= \mathbf{r} \cdot (\mathbf{r}' \times \mathbf{L}) = \mathbf{r} \cdot \left[GM \left(\mathbf{e} + \frac{\mathbf{r}}{r} \right) \right] = GM \left[\mathbf{r} \cdot \mathbf{e} + \frac{\mathbf{r} \cdot \mathbf{r}}{r} \right] = GM[re \cos \theta + r] \end{aligned}$$

Thus,

$$\frac{\|\mathbf{L}\|^2/GM}{1 + e \cos \theta} = r$$

and the planetary motion is a conic section. Since the planet returns to its initial position periodically, the conic is an ellipse.



91. $A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$

Thus,

$$\frac{dA}{dt} = \frac{dA}{d\theta} \frac{d\theta}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} \|\mathbf{L}\|$$

and $\mathbf{r}$ sweeps out area at a constant rate.

Review Exercises for Chapter 11

1. $\mathbf{r}(t) = t\mathbf{i} + \csc t\mathbf{k}$

(a) Domain: $t \neq n\pi$, n an integer

(b) Continuous except at $t = n\pi$, n an integer

3. $\mathbf{r}(t) = \ln t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$

(a) Domain: $(0, \infty)$

(b) Continuous for all $t > 0$

5. (a) $\mathbf{r}(0) = \mathbf{i}$

(b) $\mathbf{r}(-2) = -3\mathbf{i} + 4\mathbf{j} + \frac{8}{3}\mathbf{k}$

(c) $\mathbf{r}(c-1) = (2(c-1)+1)\mathbf{i} + (c-1)^2\mathbf{j} - \frac{1}{3}(c-1)^3\mathbf{k}$
 $= (2c-1)\mathbf{i} + (c-1)^2\mathbf{j} - \frac{1}{3}(c-1)^3\mathbf{k}$

(d) $\mathbf{r}(1+\Delta t) - \mathbf{r}(1) = ([2(1+\Delta t)+1]\mathbf{i} + [1+\Delta t]^2\mathbf{j} - \frac{1}{3}[1+\Delta t]^3\mathbf{k}) - (3\mathbf{i} + \mathbf{j} - \frac{1}{3}\mathbf{k})$
 $= 2\Delta t\mathbf{i} + \Delta t(\Delta t+2)\mathbf{j} - \frac{1}{3}(\Delta t^3 + 3\Delta t^2 + 3\Delta t)\mathbf{k}$

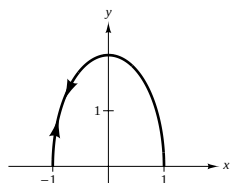
7. $\mathbf{r}(t) = \cos t\mathbf{i} + 2\sin^2 t\mathbf{j}$

$x(t) = \cos t, y(t) = 2\sin^2 t$

$x^2 + \frac{y}{2} = 1$

$y = 2(1 - x^2)$

$-1 \leq x \leq 1$

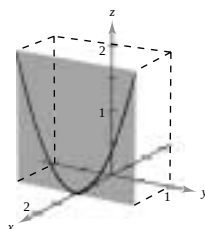


9. $\mathbf{r}(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$

$x = 1$

$y = t$

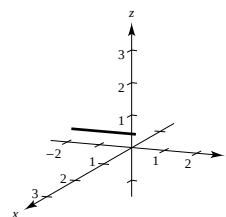
$z = t^2 \Rightarrow z = y^2$



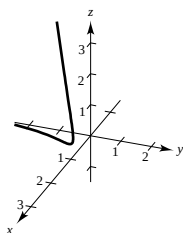
11. $\mathbf{r}(t) = \mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$

$x = 1, y = \sin t, z = 1$

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
x	1	1	1	1
y	0	1	0	-1
z	1	1	1	1



13. $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$



15. One possible answer is:

$\mathbf{r}_1(t) = 4t\mathbf{i} + 3t\mathbf{j}, \quad 0 \leq t \leq 1$

$\mathbf{r}_2(t) = 4\mathbf{i} + (3-t)\mathbf{j}, \quad 0 \leq t \leq 3$

$\mathbf{r}_3(t) = (4-t)\mathbf{i}, \quad 0 \leq t \leq 4$

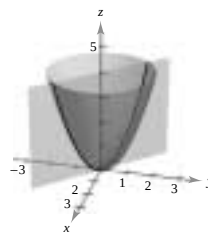
17. The vector joining the points is $\langle 7, 4, -10 \rangle$. One path is

$\mathbf{r}(t) = \langle -2 + 7t, -3 + 4t, 8 - 10t \rangle$

19. $z = x^2 + y^2, x + y = 0, t = x$

$x = t, y = -t, z = 2t^2$

$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$



21. $\lim_{t \rightarrow 2^-} (t^2\mathbf{i} + \sqrt{4-t^2}\mathbf{j} + \mathbf{k}) = 4\mathbf{i} + \mathbf{k}$

$$23. \mathbf{r}(t) = 3t\mathbf{i} + (t-1)\mathbf{j}, \mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$$

$$(a) \mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$$

$$(b) \mathbf{r}''(t) = \mathbf{0}$$

$$(c) \mathbf{r}(t) \cdot \mathbf{u}(t) = 3t^2 + t^2(t-1) = t^3 + 2t^2$$

$$(d) \mathbf{u}(t) - 2\mathbf{r}(t) = -5t\mathbf{i} + (t^2 - 2t + 2)\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$$

$$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 3t^2 + 4t$$

$$D_t[\mathbf{u}(t) - 2\mathbf{r}(t)] = -5\mathbf{i} + (2t - 2)\mathbf{j} + 2t^2\mathbf{k}$$

$$(e) \|\mathbf{r}(t)\| = \sqrt{10t^2 - 2t + 1}$$

$$(f) \mathbf{r}(t) \times \mathbf{u}(t) = \frac{2}{3}(t^4 - t^3)\mathbf{i} - 2t^4\mathbf{j} + (3t^3 - t^2 + t)\mathbf{k}$$

$$D_t[\|\mathbf{r}(t)\|] = \frac{10t - 1}{\sqrt{10t^2 - 2t + 1}}$$

$$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \left(\frac{8}{3}t^3 - 2t^2\right)\mathbf{i} - 8t^3\mathbf{j} + (9t^2 - 2t + 1)\mathbf{k}$$

$$25. x(t) \text{ and } y(t) \text{ are increasing functions at } t = t_0, \text{ and } z(t) \text{ is a decreasing function at } t = t_0.$$

$$27. \int (\cos t\mathbf{i} + t \cos t\mathbf{j}) dt = \sin t\mathbf{i} + (t \sin t + \cos t)\mathbf{j} + \mathbf{C}$$

$$29. \int \|\cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}\| dt = \int \sqrt{1 + t^2} dt = \frac{1}{2}[t\sqrt{1 + t^2} + \ln|t + \sqrt{1 + t^2}|] + \mathbf{C}$$

$$31. \mathbf{r}(t) = \int (2t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}) dt = t^2\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} + \mathbf{C}$$

$$33. \int_{-2}^2 (3t\mathbf{i} + 2t^2\mathbf{j} - t^3\mathbf{k}) dt = \left[\frac{3t^2}{2}\mathbf{i} + \frac{2t^3}{3}\mathbf{j} - \frac{t^4}{4}\mathbf{k}\right]_{-2}^2 = \frac{32}{3}\mathbf{j}$$

$$\mathbf{r}(0) = \mathbf{j} - \mathbf{k} + \mathbf{C} = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$$

$$\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (e^t + 2)\mathbf{j} - (e^{-t} + 4)\mathbf{k}$$

$$35. \int_0^2 (e^{t/2}\mathbf{i} - 3t^2\mathbf{j} - \mathbf{k}) dt = \left[2e^{t/2}\mathbf{i} - t^3\mathbf{j} - t\mathbf{k}\right]_0^2 = (2e - 2)\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$$

$$37. \mathbf{r}(t) = \langle \cos^3 t, \sin^3 t, 3t \rangle$$

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t, 3 \rangle$$

$$\|\mathbf{v}(t)\| = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t + 9}$$

$$= 3\sqrt{\cos^2 t \sin^2 t (\cos^2 t + \sin^2 t) + 1}$$

$$= 3\sqrt{\cos^2 t \sin^2 t + 1}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle -6\cos t(-\sin^2 t) + (-3\cos^2 t)\cos t, 6\sin t \cos^2 t + 3\sin^2 t(-\sin t), 0 \rangle$$

$$= \langle 3\cos t(2\sin^2 t - \cos^2 t), 3\sin t(2\cos^2 t - \sin^2 t), 0 \rangle$$

$$39. \mathbf{r}(t) = \left\langle \ln(t-3), t^2, \frac{1}{2}t \right\rangle, t_0 = 4$$

$$41. \text{Range} = x = \frac{v_0^2}{32} \sin 2\theta = \frac{(75)^2}{32} \sin 60^\circ \approx 152 \text{ feet}$$

$$\mathbf{r}'(t) = \left\langle \frac{1}{t-3}, 2t, \frac{1}{2} \right\rangle$$

$$\mathbf{r}'(4) = \left\langle 1, 8, \frac{1}{2} \right\rangle \text{ direction numbers}$$

Since $\mathbf{r}(4) = \langle 0, 16, 2 \rangle$, the parametric equations are

$$x = t, y = 16 + 8t, z = 2 + \frac{1}{2}t.$$

$$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(4.1) \approx \langle 0.1, 16.8, 2.05 \rangle$$

$$43. \text{Range} = x = \frac{v_0^2}{9.8} \sin 2\theta = 80 \Rightarrow v_0 = \sqrt{\frac{(80)(9.8)}{\sin 40^\circ}} \approx 34.9 \text{ m/sec}$$

45. $\mathbf{r}(t) = 5t\mathbf{i}$

$\mathbf{v}(t) = 5\mathbf{i}$

$\|\mathbf{v}(t)\| = 5$

$\mathbf{a}(t) = \mathbf{0}$

$\mathbf{T}(t) = \mathbf{i}$

$\mathbf{N}(t)$ does not exist

$\mathbf{a} \cdot \mathbf{T} = 0$

$\mathbf{a} \cdot \mathbf{N}$ does not exist

(The curve is a line.)

47. $\mathbf{r}(t) = t\mathbf{i} + \sqrt{t}\mathbf{j}$

$\mathbf{v}(t) = \mathbf{i} + \frac{1}{2\sqrt{t}}\mathbf{j}$

$\|\mathbf{v}(t)\| = \frac{\sqrt{4t+1}}{2\sqrt{t}}$

$\mathbf{a}(t) = -\frac{1}{4t\sqrt{t}}\mathbf{j}$

$\mathbf{T}(t) = \frac{\mathbf{i} + (1/2\sqrt{t})\mathbf{j}}{(\sqrt{4t+1})/2\sqrt{t}} = \frac{2\sqrt{t}\mathbf{i} + \mathbf{j}}{\sqrt{4t+1}}$

$\mathbf{N}(t) = \frac{\mathbf{i} - 2\sqrt{t}\mathbf{j}}{\sqrt{4t+1}}$

$\mathbf{a} \cdot \mathbf{T} = \frac{-1}{4t\sqrt{t}\sqrt{4t+1}}$

$\mathbf{a} \cdot \mathbf{N} = \frac{1}{2t\sqrt{4t+1}}$

49. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$

$\mathbf{v}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j}$

$\|\mathbf{v}(t)\| = \sqrt{e^{2t} + e^{-2t}}$

$\mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$

$\mathbf{T}(t) = \frac{e^t\mathbf{i} - e^{-t}\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}}$

$\mathbf{N}(t) = \frac{e^{-t}\mathbf{i} + e^t\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}}$

$\mathbf{a} \cdot \mathbf{T} = \frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}$

$\mathbf{a} \cdot \mathbf{N} = \frac{2}{\sqrt{e^{2t} + e^{-2t}}}$

51. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$\|\mathbf{v}\| = \sqrt{1 + 5t^2}$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

$\mathbf{T}(t) = \frac{\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}}{\sqrt{1 + 5t^2}}$

$\mathbf{N}(t) = \frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{5}\sqrt{1 + 5t^2}}$

$\mathbf{a} \cdot \mathbf{T} = \frac{5t}{\sqrt{1 + 5t^2}}$

$\mathbf{a} \cdot \mathbf{N} = \frac{5}{\sqrt{5}\sqrt{1 + 5t^2}} = \frac{\sqrt{5}}{\sqrt{1 + 5t^2}}$

53. $\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + t \mathbf{k}, x = 2 \cos t, y = 2 \sin t, z = t$

When $t = \frac{3\pi}{4}$, $x = -\sqrt{2}$, $y = \sqrt{2}$, $z = \frac{3\pi}{4}$.

$\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + \mathbf{k}$

Direction numbers when $t = \frac{3\pi}{4}$, $a = -\sqrt{2}$, $b = -\sqrt{2}$, $c = 1$

$x = -\sqrt{2}t - \sqrt{2}, y = -\sqrt{2}t + \sqrt{2}, z = t + \frac{3\pi}{4}$

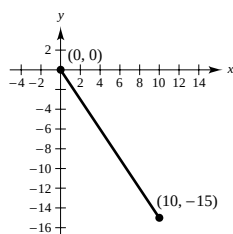
55. $v = \sqrt{\frac{9.56 \times 10^4}{4600}} \approx 4.56 \text{ mi/sec}$

57. $\mathbf{r}(t) = 2t\mathbf{i} - 3t\mathbf{j}, 0 \leq t \leq 5$

$\mathbf{r}'(t) = 2\mathbf{i} - 3\mathbf{j}$

$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^5 \sqrt{4 + 9} dt$

$= \sqrt{13}t \Big|_0^5 = 5\sqrt{13}$



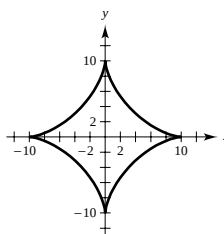
59. $\mathbf{r}(t) = 10 \cos^3 t \mathbf{i} + 10 \sin^3 t \mathbf{j}$

$$\mathbf{r}'(t) = -30 \cos^2 t \sin t \mathbf{i} + 30 \sin^2 t \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 30 \sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t}$$

$$= 30 |\cos t \sin t|$$

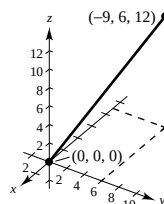
$$s = 4 \int_0^{\pi/2} 30 \cos t \cdot \sin t \, dt = \left[120 \frac{\sin^2 t}{2} \right]_0^{\pi/2} = 60$$



61. $\mathbf{r}(t) = -3t\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k}, 0 \leq t \leq 3$

$$\mathbf{r}'(t) = -3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

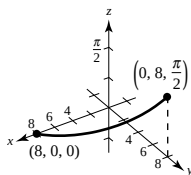
$$s = \int_a^b \|\mathbf{r}'(t)\| \, dt = \int_0^3 \sqrt{9 + 4 + 16} \, dt = \int_0^3 \sqrt{29} \, dt = 3\sqrt{29}$$



63. $\mathbf{r}(t) = \langle 8 \cos t, 8 \sin t, t \rangle, 0 \leq t \leq \frac{\pi}{2}$

$$\mathbf{r}'(t) = \langle -8 \sin t, 8 \cos t, 1 \rangle, \|\mathbf{r}'(t)\| = \sqrt{65}$$

$$s = \int_a^b \|\mathbf{r}'(t)\| \, dt = \int_0^{\pi/2} \sqrt{65} \, dt = \frac{\pi\sqrt{65}}{2}$$



65. $\mathbf{r}(t) = \frac{1}{2}t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k}, 0 \leq t \leq \pi$

$$\mathbf{r}'(t) = \frac{1}{2}\mathbf{i} + \cos t\mathbf{j} - \sin t\mathbf{k}$$

$$\begin{aligned} s &= \int_0^\pi \|\mathbf{r}'(t)\| \, dt \\ &= \int_0^\pi \sqrt{\frac{1}{4} + \cos^2 t + \sin^2 t} \, dt \\ &= \frac{\sqrt{5}}{2} \int_0^\pi dt = \left[\frac{\sqrt{5}}{2} t \right]_0^\pi = \frac{\sqrt{5}}{2} \pi \end{aligned}$$

67. $\mathbf{r}(t) = 3t\mathbf{i} + 2t\mathbf{j}$

Line

$$k = 0$$

69. $\mathbf{r}(t) = 2t\mathbf{i} + \frac{1}{2}t^2\mathbf{j} + t^2\mathbf{k}$

$$\mathbf{r}'(t) = 2\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}, \|\mathbf{r}'\| = \sqrt{5t^2 + 4}$$

$$\mathbf{r}''(t) = \mathbf{j} + 2\mathbf{k}$$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & t & 2t \\ 0 & 1 & 2 \end{vmatrix} = -4\mathbf{j} + 2\mathbf{k}, \|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{20}$$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{20}}{(5t^2 + 4)^{3/2}} = \frac{2\sqrt{5}}{(4 + 5t^2)^{3/2}}$$

71. $y = \frac{1}{2}x^2 + 2$

$$y' = x$$

$$y'' = 1$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{1}{(1 + x^2)^{3/2}}$$

$$\text{At } x = 4, K = \frac{1}{17^{3/2}} \text{ and } r = 17^{3/2} = 17\sqrt{17}.$$

73. $y = \ln x$

$$y' = \frac{1}{x}, y'' = -\frac{1}{x^2}$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{1/x^2}{[1 + (1/x)^2]^{3/2}}$$

$$\text{At } x = 1, K = \frac{1}{2^{3/2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \text{ and } r = 2\sqrt{2}.$$

75. The curvature changes abruptly from zero to a nonzero constant at the points B and C.

CHAPTER 11

Vector-Valued Functions

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CHAPTER 11

Vector-Valued Functions

Section 11.1 Vector-Valued Functions

Solutions to Even-Numbered Exercises

2. $\mathbf{r}(t) = \sqrt{4 - t^2}\mathbf{i} + t^2\mathbf{j} - 6t\mathbf{k}$

Component functions: $f(t) = \sqrt{4 - t^2}$

$$g(t) = t^2$$

$$h(t) = -6t$$

Domain: $[-2, 2]$

4. $\mathbf{r}(t) = \sin t\mathbf{i} + 4 \cos t\mathbf{j} + t\mathbf{k}$

Component functions: $f(t) = \sin t$

$$g(t) = 4 \cos t$$

$$h(t) = t$$

Domain: $(-\infty, \infty)$

$$\begin{aligned} 6. \mathbf{r}(t) &= \mathbf{F}(t) - \mathbf{G}(t) = (\ln t\mathbf{i} + 5t\mathbf{j} - 3t^2\mathbf{k}) - (\mathbf{i} + 4t\mathbf{j} - 3t^2\mathbf{k}) \\ &= (\ln t - 1)\mathbf{i} + (5t - 4t)\mathbf{j} + (-3t^2 + 3t^2)\mathbf{k} \\ &= (\ln t - 1)\mathbf{i} + t\mathbf{j} \end{aligned}$$

Domain: $(0, \infty)$

$$8. \mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^3 & -t & t \\ \sqrt[3]{t} & \frac{1}{t+1} & t+2 \end{vmatrix} = \left(-t(t+2) - \frac{t}{t+1}\right)\mathbf{i} - (t^3(t+2) - t\sqrt[3]{t})\mathbf{j} + \left(\frac{t^3}{t+1} + t\sqrt[3]{t}\right)\mathbf{k}$$

Domain: $(-\infty, -1), (-1, \infty)$

10. $\mathbf{r}(t) = \cos t\mathbf{i} + 2 \sin t\mathbf{j}$

(a) $\mathbf{r}(0) = \mathbf{i}$

(b) $\mathbf{r}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} + \sqrt{2}\mathbf{j}$

(c) $\mathbf{r}(\theta - \pi) = \cos(\theta - \pi)\mathbf{i} + 2 \sin(\theta - \pi)\mathbf{j} = -\cos \theta \mathbf{i} - 2 \sin \theta \mathbf{j}$

(d) $\mathbf{r}\left(\frac{\pi}{6} + \Delta t\right) - \mathbf{r}\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6} + \Delta t\right)\mathbf{i} + 2 \sin\left(\frac{\pi}{6} + \Delta t\right)\mathbf{j} - \left(\cos\left(\frac{\pi}{6}\right)\mathbf{i} + 2 \sin\left(\frac{\pi}{6}\right)\mathbf{j}\right)$

12. $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + t^{3/2}\mathbf{j} + e^{-t/4}\mathbf{k}$

(a) $\mathbf{r}(0) = \mathbf{k}$

(b) $\mathbf{r}(4) = 2\mathbf{i} + 8\mathbf{j} + e^{-1}\mathbf{k}$

(c) $\mathbf{r}(c+2) = \sqrt{c+2}\mathbf{i} + (c+2)^{3/2}\mathbf{j} + e^{-(c+2)/4}\mathbf{k}$

$$\begin{aligned} (d) \mathbf{r}(9 + \Delta t) - \mathbf{r}(9) &= (\sqrt{9 + \Delta t})\mathbf{i} + (9 + \Delta t)^{3/2}\mathbf{j} + e^{-(9 + \Delta t)/4}\mathbf{k} - (3\mathbf{i} + 27\mathbf{j} + e^{-9/4}\mathbf{k}) \\ &= (\sqrt{9 + \Delta t} - 3)\mathbf{i} + ((9 + \Delta t)^{3/2} - 27)\mathbf{j} + (e^{-(9 + \Delta t)/4} - e^{-9/4})\mathbf{k} \end{aligned}$$

14. $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + 3t\mathbf{j} - 4t\mathbf{k}$

$$\begin{aligned} \|\mathbf{r}(t)\| &= \sqrt{(\sqrt{t})^2 + (3t)^2 + (-4t)^2} \\ &= \sqrt{t + 9t^2 + 16t^2} = \sqrt{t(1 + 25t)} \end{aligned}$$

16. $\mathbf{r}(t) \cdot \mathbf{u}(t) = (3 \cos t)(4 \sin t) + (2 \sin t)(-6 \cos t) + (t - 2)(t^2) = t^3 - 2t^2$, a scalar.

The dot product is a scalar-valued function.

18. $\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k}$, $-1 \leq t \leq 1$

$x = \cos(\pi t)$, $y = \sin(\pi t)$, $z = t^2$

Thus, $x^2 + y^2 = 1$. Matches (c)

20. $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j} + \frac{2t}{3}\mathbf{k}$, $0.1 \leq t \leq 5$

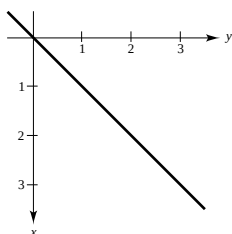
$x = t$, $y = \ln t$, $z = \frac{2t}{3}$

Thus, $z = \frac{2}{3}x$ and $y = \ln x$. Matches (a)

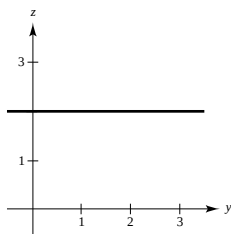
22. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + 2\mathbf{k}$

$x = t$, $y = t$, $z = 2 \Rightarrow x = y$

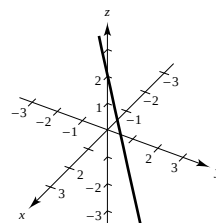
(a) (0, 0, 20)



(b) (10, 0, 0)



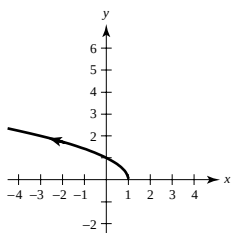
(c) (5, 5, 5)



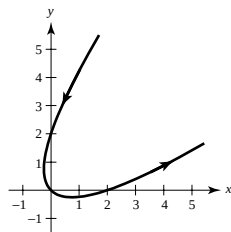
24. $x = 1 - t$, $y = \sqrt{t}$

$y = \sqrt{1 - x}$

Domain: $t \geq 0$



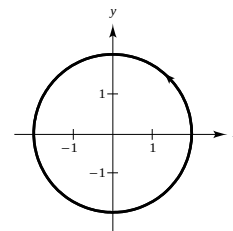
26. $x = t^2 + t$, $y = t^2 - t$



28. $x = 2 \cos t$

$y = 2 \sin t$

$x^2 + y^2 = 4$

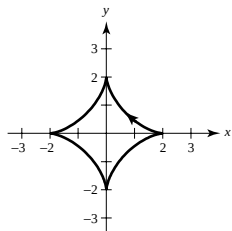


30. $x = 2 \cos^3 t$, $y = 2 \sin^3 t$

$\left(\frac{x}{2}\right)^{2/3} + \left(\frac{y}{2}\right)^{2/3} = \cos^2 t + \sin^2 t$

$= 1$

$x^{2/3} + y^{2/3} = 2^{2/3}$



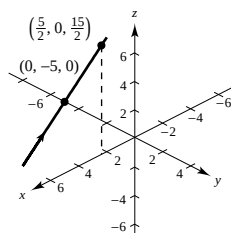
32. $x = t$

$y = 2t - 5$

$y = 3t$

Line passing through the points:

$(0, -5, 0)$, $\left(\frac{5}{2}, 0, \frac{15}{2}\right)$

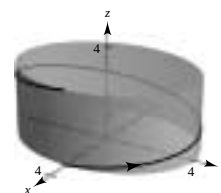


34. $x = 3 \cos t$, $y = 4 \sin t$, $z = \frac{t}{2}$

$\frac{x^2}{9} + \frac{y^2}{16} = 1$

$z = \frac{t}{2}$

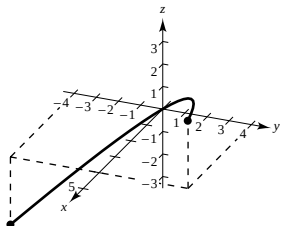
Elliptic helix



36. $x = t^2, y = 2t, z = \frac{3}{2}t$

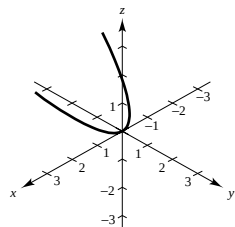
$$x = \frac{y^2}{4}, z = \frac{3}{4}y$$

t	-2	-1	0	1	2
x	4	1	0	1	4
y	-4	-2	0	2	4
z	-3	$-\frac{3}{2}$	0	$\frac{3}{2}$	3



40. $\mathbf{r}(t) = t\mathbf{i} - \frac{\sqrt{3}}{2}t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

Parabola



38. $x = \cos t + t \sin t$

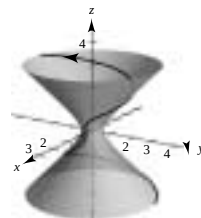
$$y = \sin t - t \cos t$$

$$z = t$$

$$x^2 + y^2 = 1 + t^2 = 1 + z^2 \text{ or } x^2 + y^2 - z^2 = 1$$

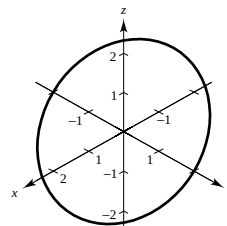
$$z = t$$

Helix along a hyperboloid of one sheet

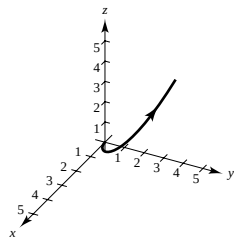


42. $\mathbf{r}(t) = -\sqrt{2} \sin t \mathbf{i} + 2 \cos t \mathbf{j} + \sqrt{2} \sin t \mathbf{k}$

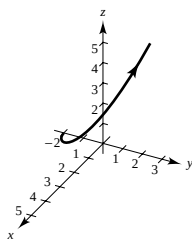
Ellipse



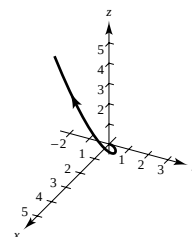
44. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$



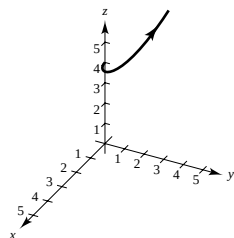
(a) $\mathbf{u}(t) = \mathbf{r}(t) - 2\mathbf{j}$ is a translation 2 units to the left along the y-axis.



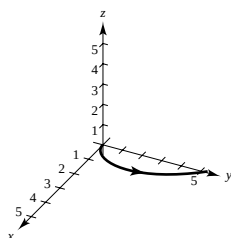
(b) $\mathbf{u}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$ has the roles of x and y interchanged. The graph is a reflection in the plane $x = y$.



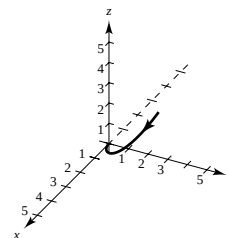
(c) $\mathbf{u}(t) = \mathbf{r}(t) + 4\mathbf{k}$ is an upward shift 4 units.



(d) $\mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{8}t^3\mathbf{k}$ shrinks the z -value by a factor of 4. The curve rises more slowly.



(e) $\mathbf{u}(t) = \mathbf{r}(-t)$ reverses the orientation.



46. $2x - 3y + 5 = 0$

Let $x = t$, then $y = \frac{1}{3}(2t + 5)$.

$$\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}(2t + 5)\mathbf{j}$$

50. $(x - 2)^2 + y^2 = 4$

Let $x - 2 = 2 \cos t$, $y = 2 \sin t$.

$$\mathbf{r}(t) = (2 + 2 \cos t)\mathbf{i} + 2 \sin t \mathbf{j}$$

54. One possible answer is

$$\mathbf{r}(t) = 1.5 \cos t \mathbf{i} + 1.5 \sin t \mathbf{j} + \frac{1}{\pi} t \mathbf{k}, 0 \leq t \leq 2\pi$$

Note that $\mathbf{r}(2\pi) = 1.5\mathbf{i} + 2\mathbf{k}$.

56. $\mathbf{r}_1(t) = t\mathbf{i}$, $0 \leq t \leq 10$ ($\mathbf{r}_1(0) = \mathbf{0}$, $\mathbf{r}_1(10) = 10\mathbf{i}$)

$$\mathbf{r}_2(t) = 10(\cos t \mathbf{i} + \sin t \mathbf{j}), 0 \leq t \leq \frac{\pi}{4} \quad \left(\mathbf{r}_2(0) = 10\mathbf{i}, \mathbf{r}_2\left(\frac{\pi}{4}\right) = 5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j} \right)$$

$$\mathbf{r}_3(t) = 5\sqrt{2}(1 - t)\mathbf{i} + 5\sqrt{2}(1 - t)\mathbf{j}, 0 \leq t \leq 1 \quad (\mathbf{r}_3(0) = 5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j}, \mathbf{r}_3(1) = \mathbf{0})$$

(Other answers possible)

58. $\mathbf{r}_1(t) = t\mathbf{i} + \sqrt{t}\mathbf{j}$, $0 \leq t \leq 1$ ($y = \sqrt{x}$)

$$\mathbf{r}_2(t) = (1 - t)\mathbf{i} + (1 - t)\mathbf{j}, 0 \leq t \leq 1 \quad (y = x)$$

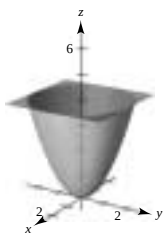
(Other answers possible)

60. $z = x^2 + y^2$, $z = 4$

Therefore, $x^2 + y^2 = 4$ or

$$x = 2 \cos t, y = 2 \sin t, z = 4.$$

$$\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + 4\mathbf{k}$$

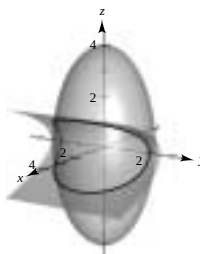


62. $4x^2 + 4y^2 + z^2 = 16$, $x = z^2$

If $z = t$, then $x = t^2$ and $y = \frac{1}{2}\sqrt{16 - 4t^4 - t^2}$.

t	-1.3	-1.2	-1	0	1	1.2
x	1.69	1.44	1	0	1	1.44
y	0.85	1.25	1.66	2	1.66	1.25
z	-1.3	-1.2	-1	0	1	1.2

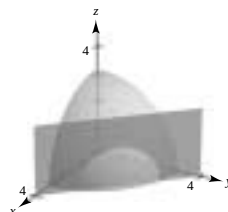
$$\mathbf{r}(t) = t^2\mathbf{i} + \frac{1}{2}\sqrt{16 - 4t^4 - t^2}\mathbf{j} + t\mathbf{k}$$



64. $x^2 + y^2 + z^2 = 10, x + y = 4$

Let $x = 2 + \sin t$, then $y = 2 - \sin t$ and $z = \sqrt{2(1 - \sin^2 t)} = \sqrt{2} \cos t$.

t	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	π
x	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	2
y	3	$\frac{5}{2}$	2	$\frac{3}{2}$	1	2
z	0	$\frac{\sqrt{6}}{2}$	$\sqrt{2}$	$\frac{\sqrt{6}}{2}$	0	$-\sqrt{2}$



$$\mathbf{r}(t) = (2 + \sin t)\mathbf{i} + (2 - \sin t)\mathbf{j} + \sqrt{2} \cos t \mathbf{k}$$

66. $x^2 + y^2 + z^2 = 16, xy = 4$ (first octant)

Let $x = t$, then

$$y = \frac{4}{t} \quad \text{and} \quad x^2 + y^2 + z^2 = t^2 + \frac{16}{t^2} + z^2 = 16.$$

$$z = \frac{1}{t} \sqrt{-t^4 + 16t^2 - 16}$$

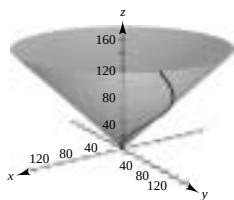
$$\left(\sqrt{8 - 4\sqrt{3}} \leq t \leq \sqrt{8 + 4\sqrt{3}} \right)$$

t	$\sqrt{8 + 4\sqrt{3}}$	1.5	2	2.5	3.0	3.5	$\sqrt{8 - 4\sqrt{3}}$
x	1.0	1.5	2	2.5	3.0	3.5	3.9
y	3.9	2.7	2	1.6	1.3	1.1	1.0
z	0	2.6	2.8	2.7	2.3	1.6	0



$$\mathbf{r}(t) = t\mathbf{i} + \frac{4}{t}\mathbf{j} + \frac{1}{t} \sqrt{-t^4 + 16t^2 - 16} \mathbf{k}$$

68. $x^2 + y^2 = (e^{-t} \cos t)^2 + (e^{-t} \sin t)^2 = e^{-2t} = z^2$



70. $\lim_{t \rightarrow 0} \left[e^t \mathbf{i} + \frac{\sin t}{t} \mathbf{j} + e^{-t} \mathbf{k} \right] = \mathbf{i} + \mathbf{j} + \mathbf{k}$

since

$$\lim_{t \rightarrow 0} \frac{\sin t}{t} = \lim_{t \rightarrow 0} \frac{\cos t}{1} = 1 \quad (\text{L'Hôpital's Rule})$$

72. $\lim_{t \rightarrow 1} \left[\sqrt{t} \mathbf{i} + \frac{\ln t}{t^2 - 1} \mathbf{j} + 2t^2 \mathbf{k} \right] = \mathbf{i} + \frac{1}{2} \mathbf{j} + 2\mathbf{k}$

since

$$\lim_{t \rightarrow 1} \frac{\ln t}{t^2 - 1} = \lim_{t \rightarrow 1} \frac{1/t}{2t} = \frac{1}{2}. \quad (\text{L'Hôpital's Rule})$$

74. $\lim_{t \rightarrow \infty} \left[e^{-t} \mathbf{i} + \frac{1}{t} \mathbf{j} + \frac{t}{t^2 + 1} \mathbf{k} \right] = \mathbf{0}$

since

$$\lim_{t \rightarrow \infty} e^{-t} = 0, \quad \lim_{t \rightarrow \infty} \frac{1}{t} = 0, \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{t}{t^2 + 1} = 0.$$

76. $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + \sqrt{t-1}\mathbf{j}$

 Continuous on $[1, \infty)$

80. $\mathbf{r}(t) = \langle 8, \sqrt{t}, \sqrt[3]{t} \rangle$

 Continuous on $[0, \infty)$

 84. A vector-valued function $\mathbf{r}$ is continuous at $t = a$ if the limit of $\mathbf{r}(t)$ exists as $t \rightarrow a$ and

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{r}(a).$$

 The function $\mathbf{r}(t) = \begin{cases} \mathbf{i} + \mathbf{j} & t \geq 0 \\ -\mathbf{i} + \mathbf{j} & t < 0 \end{cases}$ is not continuous at $t = 0$.

 86. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$. Then:

$$\begin{aligned} \lim_{t \rightarrow c} [\mathbf{r}(t) \cdot \mathbf{u}(t)] &= \lim_{t \rightarrow c} [x_1(t)x_2(t) + y_1(t)y_2(t) + z_1(t)z_2(t)] \\ &= \lim_{t \rightarrow c} x_1(t) \lim_{t \rightarrow c} x_2(t) + \lim_{t \rightarrow c} y_1(t) \lim_{t \rightarrow c} y_2(t) + \lim_{t \rightarrow c} z_1(t) \lim_{t \rightarrow c} z_2(t) \\ &= \left[\lim_{t \rightarrow c} x_1(t)\mathbf{i} + \lim_{t \rightarrow c} y_1(t)\mathbf{j} + \lim_{t \rightarrow c} z_1(t)\mathbf{k} \right] \cdot \left[\lim_{t \rightarrow c} x_2(t)\mathbf{i} + \lim_{t \rightarrow c} y_2(t)\mathbf{j} + \lim_{t \rightarrow c} z_2(t)\mathbf{k} \right] \\ &= \lim_{t \rightarrow c} \mathbf{r}(t) \cdot \lim_{t \rightarrow c} \mathbf{u}(t) \end{aligned}$$

88. Let

$$f(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ -1, & \text{if } t < 0 \end{cases}$$

 and $\mathbf{r}(t) = f(t)\mathbf{i}$. Then $\mathbf{r}$ is not continuous at $c = 0$, whereas, $\|\mathbf{r}\| = 1$ is continuous for all t .

78. $\mathbf{r}(t) = \langle 2e^{-t}, e^{-t}, \ln(t-1) \rangle$

 Continuous on $t-1 > 0$ or $t > 1$: $(1, \infty)$.

 82. No. The graph is the same because $\mathbf{r}(t) = \mathbf{u}(t+2)$. For example, if $\mathbf{r}(0)$ is on the graph of $\mathbf{r}$, then $\mathbf{u}(2)$ is the same point.

 90. False. The graph of $x = y = z = t^3$ represents a line.

Section 11.2 Differentiation and Integration of Vector-Valued Functions

2. $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j}$, $t_0 = 1$

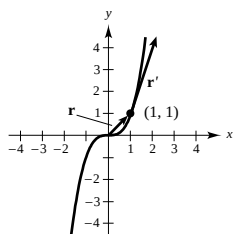
$x(t) = t, y(t) = t^3$

$y = x^3$

$\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{r}'(1) = \mathbf{i} + 3\mathbf{j}$

 $\mathbf{r}'(t_0)$ is tangent to the curve.


4. $\mathbf{r}(t) = t^2\mathbf{i} + \frac{1}{t}\mathbf{j}$, $t_0 = 2$

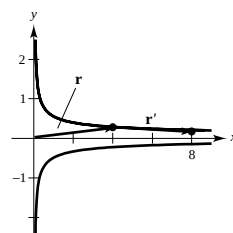
$x(t) = t^2, y(t) = \frac{1}{t}$

$x = \frac{1}{y^2}$

$\mathbf{r}(2) = 4\mathbf{i} + \frac{1}{2}\mathbf{j}$

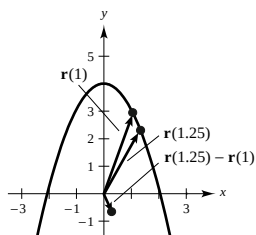
$\mathbf{r}'(t) = 2t\mathbf{i} - \frac{1}{t^2}\mathbf{j}$

$\mathbf{r}'(2) = 4\mathbf{i} - \frac{1}{4}\mathbf{j}$

 $\mathbf{r}'(t_0)$ is tangent to the curve.


6. $\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}$

(a)



(b) $\mathbf{r}(1) = \mathbf{i} + 3\mathbf{j}$

$\mathbf{r}(1.25) = 1.25\mathbf{i} + 2.4375\mathbf{j}$

$\mathbf{r}(1.25) - \mathbf{r}(1) = 0.25\mathbf{i} - 0.5625\mathbf{j}$

(c) $\mathbf{r}'(t) = \mathbf{i} - 2t\mathbf{j}$

$\mathbf{r}'(1) = \mathbf{i} - 2\mathbf{j}$

$\frac{\mathbf{r}(1.25) - \mathbf{r}(1)}{1.25 - 1} = \frac{0.25\mathbf{i} - 0.5625\mathbf{j}}{0.25} = \mathbf{i} - 2.25\mathbf{j}$

This vector approximates $\mathbf{r}'(1)$.

10. $\mathbf{r}(t) = \frac{1}{t}\mathbf{i} + 16t\mathbf{j} + \frac{t^2}{2}\mathbf{k}$

$\mathbf{r}'(t) = -\frac{1}{t^2}\mathbf{i} + 16\mathbf{j} + t\mathbf{k}$

14. $\mathbf{r}(t) = \langle \sin t - t \cos t, \cos t + t \sin t, t^2 \rangle$

$\mathbf{r}'(t) = \langle t \sin t, t \cos t, 2t \rangle$

18. $\mathbf{r}(t) = (t^2 + t)\mathbf{i} + (t^2 - t)\mathbf{j}$

(a) $\mathbf{r}'(t) = (2t + 1)\mathbf{i} + (2t - 1)\mathbf{j}$

$\mathbf{r}''(t) = 2\mathbf{i} + 2\mathbf{j}$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (2t + 1)(2) + (2t - 1)(2) = 8t$

22. $\mathbf{r}(t) = t\mathbf{i} + (2t + 3)\mathbf{j} + (3t - 5)\mathbf{k}$

(a) $\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$\mathbf{r}''(t) = \mathbf{0}$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 0$

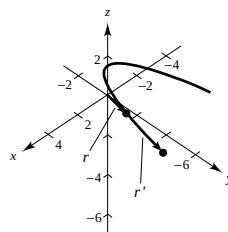
8. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{3}{2}\mathbf{k}, t_0 = 2$

$y = x^2, z = \frac{3}{2}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{r}(2) = 2\mathbf{i} + 4\mathbf{j} + \frac{3}{2}\mathbf{k}$

$\mathbf{r}'(2) = \mathbf{i} + 4\mathbf{j}$



12. $\mathbf{r}(t) = 4\sqrt{t}\mathbf{i} + t^2\sqrt{t}\mathbf{j} + \ln t^2\mathbf{k}$

$\mathbf{r}'(t) = \frac{2}{\sqrt{t}}\mathbf{i} + \left(2t\sqrt{t} + \frac{t^2}{2\sqrt{t}}\right)\mathbf{j} + \frac{2}{t}\mathbf{k}$

16. $\mathbf{r}(t) = \langle \arcsin t, \arccos t, 0 \rangle$

$\mathbf{r}'(t) = \left\langle \frac{1}{\sqrt{1-t^2}}, -\frac{1}{\sqrt{1-t^2}}, 0 \right\rangle$

20. $\mathbf{r}(t) = 8 \cos t\mathbf{i} + 3 \sin t\mathbf{j}$

(a) $\mathbf{r}'(t) = -8 \sin t\mathbf{i} + 3 \cos t\mathbf{j}$

$\mathbf{r}''(t) = -8 \cos t\mathbf{i} - 3 \sin t\mathbf{j}$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (-8 \sin t)(-8 \cos t) + 3 \cos t(-3 \sin t)$
 $= 55 \sin t \cos t$

24. $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan(t) \rangle$

(a) $\mathbf{r}'(t) = \langle -e^{-t}, 2t, \sec^2 t \rangle$

$\mathbf{r}''(t) = \langle e^{-t}, 2, 2 \sec^2 t \tan t \rangle$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = -e^{-2t} + 4t + 2 \sec^4 t \tan t$

$$26. \quad \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\mathbf{k}, \quad t_0 = \frac{1}{4}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 0.75e^{0.75t}\mathbf{k}$$

$$\mathbf{r}'\left(\frac{1}{4}\right) = \mathbf{i} + \frac{1}{2}\mathbf{j} + 0.75e^{0.1875}\mathbf{k} = \mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{3}{4}e^{3/16}\mathbf{k}$$

$$\left\|\mathbf{r}'\left(\frac{1}{4}\right)\right\| = \sqrt{1^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}e^{3/16}\right)^2} = \sqrt{\frac{5}{4} + \frac{9}{16}e^{3/8}} = \frac{\sqrt{20 + 9e^{3/8}}}{4}$$

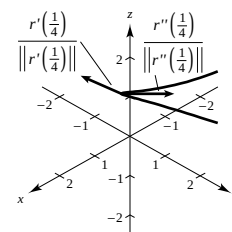
$$\frac{\mathbf{r}'(1/4)}{\|\mathbf{r}'(1/4)\|} = \frac{1}{\sqrt{20 + 9e^{3/8}}} (4\mathbf{i} + 2\mathbf{j} + 3e^{3/16}\mathbf{k})$$

$$\mathbf{r}''(t) = 2\mathbf{i} + \frac{9}{16}e^{0.75t}\mathbf{k}$$

$$\mathbf{r}''\left(\frac{1}{4}\right) = 2\mathbf{i} + \frac{9}{16}e^{3/16}\mathbf{k}$$

$$\left\|\mathbf{r}''\left(\frac{1}{4}\right)\right\| = \sqrt{2^2 + \left(\frac{9}{16}e^{3/16}\right)^2} = \sqrt{4 + \frac{81}{256}e^{3/8}} = \frac{\sqrt{1024 + 81e^{3/8}}}{16}$$

$$\frac{\mathbf{r}''(1/4)}{\|\mathbf{r}''(1/4)\|} = \frac{1}{\sqrt{1024 + 81e^{3/8}}} (32\mathbf{j} + 9e^{3/16}\mathbf{k})$$



$$28. \quad \mathbf{r}(t) = \frac{1}{t-1}\mathbf{i} + 3t\mathbf{j}$$

$$\mathbf{r}'(t) = -\frac{1}{(t-1)^2}\mathbf{i} + 3\mathbf{j}$$

Not continuous when $t = 1$

Smooth on $(-\infty, 1), (1, \infty)$

$$30. \quad \mathbf{r}(\theta) = (\theta + \sin \theta)\mathbf{i} + (1 - \cos \theta)\mathbf{j}$$

$$\mathbf{r}'(\theta) = (1 + \cos \theta)\mathbf{i} + \sin \theta\mathbf{j}$$

$$\mathbf{r}'((2n-1)\pi) = \mathbf{0}, n \text{ any integer}$$

Smooth on $((2n-1)\pi, (2n+1)\pi)$

$$32. \quad \mathbf{r}(t) = \frac{2t}{8+t^3}\mathbf{i} + \frac{2t^2}{8+t^3}\mathbf{j}$$

$$\mathbf{r}'(t) = \frac{16-4t^3}{(t^3+8)^2}\mathbf{i} + \frac{32t-2t^4}{(t^3+8)^2}\mathbf{j}$$

$\mathbf{r}'(t) \neq \mathbf{0}$ for any value of t .

$\mathbf{r}$ is not continuous when $t = -2$.

Smooth on $(-\infty, -2), (-2, \infty)$.

$$34. \quad \mathbf{r}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + 3t\mathbf{k}$$

$$\mathbf{r}'(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + 3\mathbf{k} \neq \mathbf{0}$$

$\mathbf{r}$ is smooth for all $t: (-\infty, \infty)$

$$36. \quad \mathbf{r}(t) = \sqrt{t}\mathbf{i} + (t^2-1)\mathbf{j} + \frac{1}{4}t\mathbf{k}$$

$$\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} + 2t\mathbf{j} + \frac{1}{4}\mathbf{k} \neq \mathbf{0}$$

$\mathbf{r}$ is smooth for all $t > 0: (0, \infty)$

$$38. \quad \mathbf{r}(t) = t\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$$

$$\mathbf{u}(t) = \frac{1}{t}\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$$

$$(a) \quad \mathbf{r}'(t) = \mathbf{i} + 2\cos t\mathbf{j} - 2\sin t\mathbf{k}$$

$$(c) \quad \mathbf{r}(t) \cdot \mathbf{u}(t) = 1 + 4\sin^2 t + 4\cos^2 t = 5$$

$$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 0, t \neq 0$$

$$(b) \quad \mathbf{r}''(t) = -2\sin t\mathbf{j} - 2\cos t\mathbf{k}$$

$$(d) \quad 3\mathbf{r}(t) - \mathbf{u}(t) = \left(3t - \frac{1}{t}\right)\mathbf{i} + 4\sin t\mathbf{j} + 4\cos t\mathbf{k}$$

$$D_t[3\mathbf{r}(t) - \mathbf{u}(t)] = \left(3 - \frac{1}{t^2}\right)\mathbf{i} + 4\cos t\mathbf{j} - 4\sin t\mathbf{k}$$

—CONTINUED—

38. —CONTINUED—

$$\begin{aligned}
 \text{(e) } \mathbf{r}(t) \times \mathbf{u}(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2 \sin t & 2 \cos t \\ 1/t & 2 \sin t & 2 \cos t \end{vmatrix} \\
 &= 2 \cos t \left(\frac{1}{t} - t \right) \mathbf{j} + 2 \sin t \left(t - \frac{1}{t} \right) \mathbf{k} \\
 D_t[\mathbf{r}(t) - \mathbf{u}(t)] &= \left[-2 \sin t \left(\frac{1}{t} - t \right) + 2 \cos t \left(-\frac{1}{t^2} - 1 \right) \right] \mathbf{j} \\
 &\quad + \left[2 \cos t \left(t - \frac{1}{t} \right) + 2 \sin t \left(1 + \frac{1}{t^2} \right) \right] \mathbf{k}
 \end{aligned}$$

$$\text{(f) } \|\mathbf{r}(t)\| = \sqrt{t^2 + 4}$$

$$D_t(\|\mathbf{r}(t)\|) = \frac{1}{2}(t^2 + 4)^{-1/2}(2t) = \frac{t}{\sqrt{t^2 + 4}}$$

$$40. \quad \mathbf{r}(t) = t^2 \mathbf{i} + t \mathbf{j}$$

$$\mathbf{r}'(t) = 2t \mathbf{i} + \mathbf{j}$$

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 2t^3 + t$$

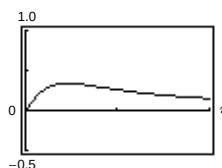
$$\|\mathbf{r}(t)\| = \sqrt{t^4 + t^2}, \quad \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 1}$$

$$\cos \theta = \frac{2t^3 + t}{\sqrt{t^4 + t^2} \sqrt{4t^2 + 1}}$$

$$\theta = \arccos \frac{2t^3 + t}{\sqrt{t^4 + t^2} \sqrt{4t^2 + 1}}$$

$$\theta = 0.340 (\approx 19.47^\circ) \text{ maximum at } t = 0.707 \left(\frac{\sqrt{2}}{2} \right).$$

$$\theta \neq \frac{\pi}{2} \text{ for any } t.$$



$$\begin{aligned}
 42. \quad \mathbf{r}'(t) &= \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \\
 &= \lim_{\Delta t \rightarrow 0} \frac{\left[\sqrt{t + \Delta t} \mathbf{i} + \frac{3}{t + \Delta t} \mathbf{j} - 2(t + \Delta t) \mathbf{k} \right] - \left[\sqrt{t} \mathbf{i} + \frac{3}{t} \mathbf{j} - 2t \mathbf{k} \right]}{\Delta t} \\
 &= \lim_{\Delta t \rightarrow 0} \left[\frac{\sqrt{t + \Delta t} - \sqrt{t}}{\Delta t} \mathbf{i} + \frac{\frac{3}{t + \Delta t} - \frac{3}{t}}{\Delta t} \mathbf{j} - 2 \mathbf{k} \right] \\
 &= \lim_{\Delta t \rightarrow 0} \left[\frac{\Delta t}{\Delta t (\sqrt{t + \Delta t} + \sqrt{t})} \mathbf{i} + \frac{-3\Delta t}{(t + \Delta t)t(\Delta t)} \mathbf{j} - 2 \mathbf{k} \right] \\
 &= \lim_{\Delta t \rightarrow 0} \left[\frac{1}{\sqrt{t + \Delta t} + \sqrt{t}} \mathbf{i} - \frac{3}{(t + \Delta t)t} \mathbf{j} - 2 \mathbf{k} \right] \\
 &= \frac{1}{2\sqrt{t}} \mathbf{i} - \frac{3}{t^2} \mathbf{j} - 2 \mathbf{k}
 \end{aligned}$$

$$44. \quad \int (4t^3 \mathbf{i} + 6t \mathbf{j} - 4\sqrt{t} \mathbf{k}) dt = t^4 \mathbf{i} + 3t^2 \mathbf{j} - \frac{8}{3} t^{3/2} \mathbf{k} + \mathbf{C}$$

$$46. \quad \int \left[\ln t \mathbf{i} + \frac{1}{t} \mathbf{j} + \mathbf{k} \right] dt = (t \ln t - t) \mathbf{i} + \ln t \mathbf{j} + t \mathbf{k} + \mathbf{C}$$

(Integration by parts)

$$48. \quad \int [e^t \mathbf{i} + \sin t \mathbf{j} + \cos t \mathbf{k}] dt = e^t \mathbf{i} - \cos t \mathbf{j} + \sin t \mathbf{k} + \mathbf{C}$$

$$50. \int [e^{-t} \sin t \mathbf{i} + e^{-t} \cos t \mathbf{j}] dt = \frac{e^{-t}}{2}(-\sin t - \cos t) \mathbf{i} + \frac{e^{-t}}{2}(-\cos t + \sin t) \mathbf{j} + \mathbf{C}$$

$$52. \int_{-1}^1 (t \mathbf{i} + t^3 \mathbf{j} + \sqrt[3]{t} \mathbf{k}) dt = \left[\frac{t^2}{2} \mathbf{i} \right]_{-1}^1 + \left[\frac{t^4}{4} \mathbf{j} \right]_{-1}^1 + \left[\frac{3}{4} t^{4/3} \mathbf{k} \right]_{-1}^1 = \mathbf{0}$$

$$54. \int_0^2 (t \mathbf{i} + e^t \mathbf{j} - t e^t \mathbf{k}) dt = \left[\frac{t^2}{2} \mathbf{i} \right]_0^2 + \left[e^t \mathbf{j} \right]_0^2 - \left[(t-1)e^t \mathbf{k} \right]_0^2$$

$$= 2 \mathbf{i} + (e^2 - 1) \mathbf{j} - (e^2 + 1) \mathbf{k}$$

$$56. \mathbf{r}(t) = \int (3t^2 \mathbf{j} + 6\sqrt{t} \mathbf{k}) dt = t^3 \mathbf{j} + 4t^{3/2} \mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(0) = \mathbf{C} = \mathbf{i} + 2 \mathbf{j}$$

$$\mathbf{r}(t) = \mathbf{i} + (2 + t^3) \mathbf{j} + 4t^{3/2} \mathbf{k}$$

$$58. \mathbf{r}''(t) = -4 \cos t \mathbf{i} - 3 \sin t \mathbf{k}$$

$$\mathbf{r}'(t) = -4 \sin t \mathbf{i} + 3 \cos t \mathbf{k} + \mathbf{C}_1$$

$$\mathbf{r}'(0) = 3 \mathbf{k} = 3 \mathbf{k} + \mathbf{C}_1 \Rightarrow \mathbf{C}_1 = \mathbf{0}$$

$$\mathbf{r}(t) = 4 \cos t \mathbf{i} + 3 \sin t \mathbf{k} + \mathbf{C}_2$$

$$\mathbf{r}(0) = 4 \mathbf{i} + \mathbf{C}_2 = 4 \mathbf{j} \Rightarrow \mathbf{C}_2 = 4 \mathbf{j} - 4 \mathbf{i}$$

$$\mathbf{r}(t) = (4 \cos t - 4) \mathbf{i} + 4 \mathbf{j} + 3 \sin t \mathbf{k}$$

$$60. \mathbf{r}(t) = \int \left[\frac{1}{1+t^2} \mathbf{i} + \frac{1}{t^2} \mathbf{j} + \frac{1}{t} \mathbf{k} \right] dt = \arctan t \mathbf{i} - \frac{1}{t} \mathbf{j} + \ln t \mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(1) = \frac{\pi}{4} \mathbf{i} - \mathbf{j} + \mathbf{C} = 2 \mathbf{i} \Rightarrow \mathbf{C} = \left(2 - \frac{\pi}{4} \right) \mathbf{i} + \mathbf{j}$$

$$\mathbf{r}(t) = \left[2 - \frac{\pi}{4} + \arctan t \right] \mathbf{i} + \left(1 - \frac{1}{t} \right) \mathbf{j} + \ln t \mathbf{k}$$

62. To find the integral of a vector-valued function, you integrate each component function separately. The constant of integration $\mathbf{C}$ is a constant vector.

64. The graph of $\mathbf{u}(t)$ does not change position relative to the xy -plane.

66. Let $\mathbf{r}(t) = x_1(t) \mathbf{i} + y_1(t) \mathbf{j} + z_1(t) \mathbf{k}$ and $\mathbf{u}(t) = x_2(t) \mathbf{i} + y_2(t) \mathbf{j} + z_2(t) \mathbf{k}$.

$$\mathbf{r}(t) \pm \mathbf{u}(t) = [x_1(t) \pm x_2(t)] \mathbf{i} + [y_1(t) \pm y_2(t)] \mathbf{j} + [z_1(t) \pm z_2(t)] \mathbf{k}$$

$$D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] = [x_1'(t) \pm x_2'(t)] \mathbf{i} + [y_1'(t) \pm y_2'(t)] \mathbf{j} + [z_1'(t) \pm z_2'(t)] \mathbf{k}$$

$$= [x_1'(t) \mathbf{i} + y_1'(t) \mathbf{j} + z_1'(t) \mathbf{k}] \pm [x_2'(t) \mathbf{i} + y_2'(t) \mathbf{j} + z_2'(t) \mathbf{k}]$$

$$= \mathbf{r}'(t) \pm \mathbf{u}'(t)$$

68. Let $\mathbf{r}(t) = x_1(t) \mathbf{i} + y_1(t) \mathbf{j} + z_1(t) \mathbf{k}$ and $\mathbf{u}(t) = x_2(t) \mathbf{i} + y_2(t) \mathbf{j} + z_2(t) \mathbf{k}$.

$$\mathbf{r}(t) \times \mathbf{u}(t) = [y_1(t)z_2(t) - z_1(t)y_2(t)] \mathbf{i} - [x_1(t)z_2(t) - z_1(t)x_2(t)] \mathbf{j} + [x_1(t)y_2(t) - y_1(t)x_2(t)] \mathbf{k}$$

$$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = [y_1(t)z_2'(t) + y_1'(t)z_2(t) - z_1(t)y_2'(t) - z_1'(t)y_2(t)] \mathbf{i} - [x_1(t)z_2'(t) + x_1'(t)z_2(t) - z_1(t)x_2'(t) - z_1'(t)x_2(t)] \mathbf{j} +$$

$$[x_1(t)y_2'(t) + x_1'(t)y_2(t) - y_1(t)x_2'(t) - y_1'(t)x_2(t)] \mathbf{k}$$

$$= \{ [y_1(t)z_2'(t) - z_1(t)y_2'(t)] \mathbf{i} - [x_1(t)z_2'(t) - z_1(t)x_2'(t)] \mathbf{j} + [x_1(t)y_2'(t) - y_1(t)x_2'(t)] \mathbf{k} \}$$

$$+ \{ [y_1'(t)z_2(t) - z_1'(t)y_2(t)] \mathbf{i} - [x_1'(t)z_2(t) - z_1'(t)x_2(t)] \mathbf{j} + [x_1'(t)y_2(t) - y_1'(t)x_2(t)] \mathbf{k} \}$$

$$= \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$$

70. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

$$\begin{aligned}\mathbf{r}(t) \times \mathbf{r}'(t) &= [y(t)z'(t) - z(t)y'(t)]\mathbf{i} - [x(t)z'(t) - z(t)x'(t)]\mathbf{j} + [x(t)y'(t) - y(t)x'(t)]\mathbf{k} \\ D_t[\mathbf{r}(t) \times \mathbf{r}'(t)] &= [y(t)z''(t) + y'(t)z'(t) - z(t)y''(t) - z'(t)y'(t)]\mathbf{i} - [x(t)z''(t) + x'(t)z'(t) - z(t)x''(t) - z'(t)x'(t)]\mathbf{j} + \\ &\quad [x(t)y''(t) + x'(t)y'(t) - y(t)x''(t) - y'(t)x'(t)]\mathbf{k} \\ &= [y(t)z''(t) - z(t)y''(t)]\mathbf{i} - [x(t)z''(t) - z(t)x''(t)]\mathbf{j} + [x(t)y''(t) - y(t)x''(t)]\mathbf{k} = \mathbf{r}(t) \times \mathbf{r}''(t)\end{aligned}$$

72. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. If $\mathbf{r}(t) \cdot \mathbf{r}(t)$ is constant, then:

$$\begin{aligned}x^2(t) + y^2(t) + z^2(t) &= C \\ D_t[x^2(t) + y^2(t) + z^2(t)] &= D_t[C] \\ 2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) &= 0 \\ 2[x(t)x'(t) + y(t)y'(t) + z(t)z'(t)] &= 0 \\ 2[\mathbf{r}(t) \cdot \mathbf{r}'(t)] &= 0\end{aligned}$$

Therefore, $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

74. False

$$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$$

(See Theorem 11.2, part 4)

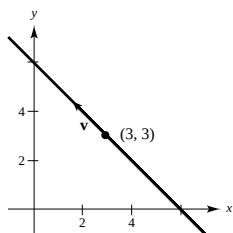
Section 11.3 Velocity and Acceleration

2. $\mathbf{r}(t) = (6 - t)\mathbf{i} + t\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = -\mathbf{i} + \mathbf{j}$

$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{0}$

$x = 6 - t, y = t, y = 6 - x$



4. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$

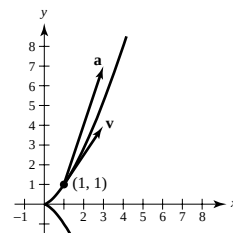
$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i} + 6t\mathbf{j}$

$x = t^2, y = t^3 \quad x = y^{2/3}$

At $(1, 1), t = 1$.

$\mathbf{v}(1) = 2\mathbf{i} + 3\mathbf{j}$

$\mathbf{a}(1) = 2\mathbf{i} + 6\mathbf{j}$



6. $\mathbf{r}(t) = 3 \cos t \mathbf{i} + 2 \sin t \mathbf{j}$

$\mathbf{v}(t) = -3 \sin t \mathbf{i} + 2 \cos t \mathbf{j}$

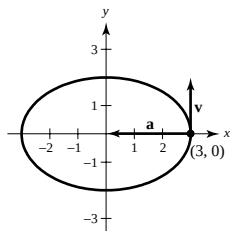
$\mathbf{a}(t) = -3 \cos t \mathbf{i} - 2 \sin t \mathbf{j}$

$x = 3 \cos t, y = 2 \sin t, \frac{x^2}{9} + \frac{y^2}{4} = 1$ Ellipse

At $(3, 0), t = 0$.

$\mathbf{v}(0) = 2\mathbf{j}$

$\mathbf{a}(0) = -3\mathbf{i}$



8. $\mathbf{r}(t) = \langle e^{-t}, e^t \rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -e^{-t}, e^t \rangle$

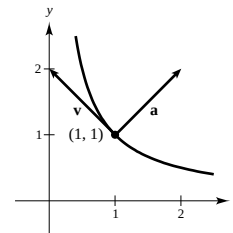
$\mathbf{a}(t) = \mathbf{r}''(t) = \langle e^{-t}, e^t \rangle$

$x = e^{-t} = \frac{1}{e^t}, y = e^t, y = \frac{1}{x}$

At $(1, 1), t = 0$.

$\mathbf{v}(0) = \langle -1, 1 \rangle = -\mathbf{i} + \mathbf{j}$

$\mathbf{a}(0) = \langle 1, 1 \rangle = \mathbf{i} + \mathbf{j}$



10. $\mathbf{r}(t) = 4t\mathbf{i} + 4t\mathbf{j} + 2t\mathbf{k}$

$\mathbf{v}(t) = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

$s(t) = \|\mathbf{v}(t)\| = \sqrt{16 + 16 + 4} = 6$

$\mathbf{a}(t) = \mathbf{0}$

$$12. \mathbf{r}(t) = 3t\mathbf{i} + t\mathbf{j} + \frac{1}{4}t^2\mathbf{k}$$

$$\mathbf{v}(t) = 3\mathbf{i} + \mathbf{j} + \frac{1}{2}t\mathbf{k}$$

$$s(t) = \sqrt{9 + 1 + \frac{1}{4}t^2} = \sqrt{10 + \frac{1}{4}t^2}$$

$$\mathbf{a}(t) = \frac{1}{2}\mathbf{k}$$

$$16. \mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$$

$$\mathbf{v}(t) = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} + e^t\mathbf{k}$$

$$s(t) = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2 + e^{2t}} \\ = e^t \sqrt{3}$$

$$\mathbf{a}(t) = -2e^t \sin t \mathbf{i} + 2e^t \cos t \mathbf{j} + e^t \mathbf{k}$$

$$14. \mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + 2t^{3/2}\mathbf{k}$$

$$\mathbf{v}(t) = 2t\mathbf{i} + \mathbf{j} + 3\sqrt{t}\mathbf{k}$$

$$s(t) = \sqrt{4t^2 + 1 + 9t} = \sqrt{4t^2 + 9t + 1}$$

$$\mathbf{a}(t) = 2\mathbf{i} + \frac{3}{2\sqrt{t}}\mathbf{k}$$

$$18. (a) \mathbf{r}(t) = \langle t, \sqrt{25 - t^2}, \sqrt{25 - t^2} \rangle, t_0 = 3$$

$$\mathbf{r}'(t) = \left\langle 1, \frac{-t}{\sqrt{25 - t^2}}, \frac{-t}{\sqrt{25 - t^2}} \right\rangle$$

$$\mathbf{r}'(3) = \left\langle 1, -\frac{3}{4}, -\frac{3}{4} \right\rangle$$

$$x = 3 + t, y = z = 4 - \frac{3}{4}t$$

$$(b) \mathbf{r}(3 + 0.1) \approx \left\langle 3 + 0.1, 4 - \frac{3}{4}(0.1), 4 - \frac{3}{4}(0.1) \right\rangle \\ = \langle 3.100, 3.925, 3.925 \rangle$$

$$20. \mathbf{a}(t) = 2\mathbf{i} + 3\mathbf{k}$$

$$\mathbf{v}(t) = \int (2\mathbf{i} + 3\mathbf{k}) dt = 2t\mathbf{i} + 3t\mathbf{k} + \mathbf{C}$$

$$\mathbf{v}(0) = \mathbf{C} = 4\mathbf{j} \Rightarrow \mathbf{v}(t) = 2t\mathbf{i} + 4\mathbf{j} + 3t\mathbf{k}$$

$$\mathbf{r}(t) = \int (2t\mathbf{i} + 4\mathbf{j} + 3t\mathbf{k}) dt = t^2\mathbf{i} + 4t\mathbf{j} + \frac{3}{2}t^2\mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(0) = \mathbf{C} = 0 \Rightarrow \mathbf{r}(t) = t^2\mathbf{i} + 4t\mathbf{j} + \frac{3}{2}t^2\mathbf{k}$$

$$\mathbf{r}(2) = 4\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$$

$$22. \mathbf{a}(t) = -\cos t \mathbf{i} - \sin t \mathbf{j}, \mathbf{v}(0) = \mathbf{j} + \mathbf{k}, \mathbf{r}(0) = \mathbf{i}$$

$$\mathbf{v}(t) = \int (-\cos t \mathbf{i} - \sin t \mathbf{j}) dt = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{C}$$

$$\mathbf{v}(0) = \mathbf{j} + \mathbf{C} = \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{C} = \mathbf{k}$$

$$\mathbf{v}(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}$$

$$\mathbf{r}(t) = \int (-\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}) dt$$

$$= \cos t \mathbf{i} + \sin t \mathbf{j} + t\mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(0) = \mathbf{i} + \mathbf{C} = \mathbf{i} \Rightarrow \mathbf{C} = \mathbf{0}$$

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t\mathbf{k}$$

$$\mathbf{r}(2) = (\cos 2)\mathbf{i} + (\sin 2)\mathbf{j} + 2\mathbf{k}$$

24. (a) The speed is increasing.

(b) The speed is decreasing.

$$26. \mathbf{r}(t) = (900 \cos 45^\circ)t\mathbf{i} + [3 + (900 \sin 45^\circ)t - 16t^2]\mathbf{j}$$

$$= 450\sqrt{2}t\mathbf{i} + (3 + 450\sqrt{2}t - 16t^2)\mathbf{j}$$

The maximum height occurs when $y'(t) = 450\sqrt{2} - 32t = 0$, which implies that $t = (225\sqrt{2})/16$.

The maximum height reached by the projectile is

$$y = 3 + 450\sqrt{2}\left(\frac{225\sqrt{2}}{16}\right) - 16\left(\frac{225\sqrt{2}}{16}\right)^2 = \frac{50,649}{8} = 6331.125 \text{ feet.}$$

The range is determined by setting $y(t) = 3 + 450\sqrt{2}t - 16t^2 = 0$ which implies that

$$t = \frac{-450\sqrt{2} \pm \sqrt{405,192}}{-32} \approx 39.779 \text{ seconds.}$$

$$\text{Range: } x = 450\sqrt{2}\left(\frac{-450\sqrt{2} - \sqrt{405,192}}{-32}\right) \approx 25,315.500 \text{ feet}$$

28. $50 \text{ mph} = \frac{220}{3} \text{ ft/sec}$

$$\mathbf{r}(t) = \left(\frac{220}{3} \cos 15^\circ \right) t \mathbf{i} + \left[5 + \left(\frac{220}{3} \sin 15^\circ \right) t - 16t^2 \right] \mathbf{j}$$

The ball is 90 feet from where it is thrown when

$$x = \frac{220}{3} \cos 15^\circ t = 90 \Rightarrow t = \frac{27}{22 \cos 15^\circ} \approx 1.2706 \text{ seconds.}$$

The height of the ball at this time is

$$y = 5 + \left(\frac{220}{3} \sin 15^\circ \right) \left(\frac{27}{22 \cos 15^\circ} \right) - 16 \left(\frac{27}{22 \cos 15^\circ} \right)^2 \approx 3.286 \text{ feet.}$$

30. $y = x - 0.005x^2$

From Exercise 34 we know that $\tan \theta$ is the coefficient of x . Therefore, $\tan \theta = 1$, $\theta = (\pi/4) \text{ rad} = 45^\circ$. Also

$$\frac{16}{v_0^2} \sec^2 \theta = \text{negative of coefficient of } x^2$$

$$\frac{16}{v_0^2}(2) = 0.005 \text{ or } v_0 = 80 \text{ ft/sec}$$

$$\mathbf{r}(t) = (40\sqrt{2}t)\mathbf{i} + (40\sqrt{2}t - 16t^2)\mathbf{j}. \text{ Position function.}$$

When $40\sqrt{2}t = 60$,

$$t = \frac{60}{40\sqrt{2}} = \frac{3\sqrt{2}}{4}$$

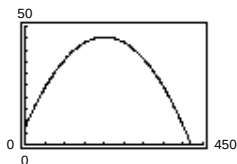
$$\mathbf{v}(t) = 40\sqrt{2}\mathbf{i} + (40\sqrt{2} - 32t)\mathbf{j}$$

$$\mathbf{v}\left(\frac{3\sqrt{2}}{4}\right) = 40\sqrt{2}\mathbf{i} + (40\sqrt{2} - 24\sqrt{2})\mathbf{j} = 8\sqrt{2}(5\mathbf{i} + 2\mathbf{j}) \text{ direction}$$

$$\text{Speed} = \left\| \mathbf{v}\left(\frac{3\sqrt{2}}{4}\right) \right\| = 8\sqrt{2}\sqrt{25+4} = 8\sqrt{58} \text{ ft/sec}$$

32. Wind: $8 \text{ mph} = \frac{176}{15} \text{ ft/sec}$

$$\mathbf{r}(t) = \left(140(\cos 22^\circ)t - \frac{176}{15} \right) \mathbf{i} + (2.5 + (140 \sin 22^\circ)t - 16t^2) \mathbf{j}$$



When $x = 375$, $t \approx 2.98$ and $y \approx 16.7$ feet.

Thus, the ball clears the 10-foot fence.

34. $h = 7$ feet, $\theta = 35^\circ$, 30 yards = 90 feet

$$\mathbf{r}(t) = (v_0 \cos 35^\circ)t\mathbf{i} + [7 + (v_0 \sin 35^\circ)t - 16t^2]\mathbf{j}$$

(a) $v_0 \cos 35^\circ t = 90$ when $7 + (v_0 \sin 35^\circ)t - 16t^2 = 4$

$$t = \frac{90}{v_0 \cos 35^\circ}$$

$$7 + (v_0 \sin 35^\circ)\left(\frac{90}{v_0 \cos 35^\circ}\right) - 16\left(\frac{90}{v_0 \cos 35^\circ}\right)^2 = 4$$

$$90 \tan 35^\circ + 3 = \frac{129,600}{v_0^2 \cos^2 35^\circ}$$

$$v_0^2 = \frac{129,600}{\cos^2 35^\circ (90 \tan 35^\circ + 3)}$$

$$v_0 \approx 54.088 \text{ feet per second}$$

- (b) The maximum height occurs when

$$y'(t) = v_0 \sin 35^\circ - 32t = 0.$$

$$t = \frac{v_0 \sin 35^\circ}{32} \approx 0.969 \text{ second}$$

At this time, the height is $y(0.969) \approx 22.0$ feet.

(c) $x(t) = 90 \Rightarrow (v_0 \cos 35^\circ)t = 90$

$$t = \frac{90}{54.088 \cos 35^\circ} \approx 2.0 \text{ seconds}$$

36. Place the origin directly below the plane. Then $\theta = 0$, $v_0 = 792$ and

$$\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + (30,000 + (v_0 \sin \theta)t - 16t^2)\mathbf{j}$$

$$= 792t\mathbf{i} + (30,000 - 16t^2)\mathbf{j}$$

$$\mathbf{v}(t) = 792\mathbf{i} - 32t\mathbf{j}.$$

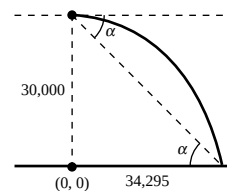
At time of impact, $30,000 - 16t^2 = 0 \Rightarrow t^2 = 1875 \Rightarrow t \approx 43.3$ seconds.

$$\mathbf{r}(43.3) = 34,294.6\mathbf{i}$$

$$\mathbf{v}(43.3) = 792\mathbf{i} - 1385.6\mathbf{j}$$

$$\|\mathbf{v}(43.3)\| = 1596 \text{ ft/sec} = 1088 \text{ mph}$$

$$\tan \alpha = \frac{30,000}{34,294.6} \approx 0.8748 \Rightarrow \alpha \approx 0.7187(41.18^\circ)$$



38. From Exercise 37, the range is

$$x = \frac{v_0^2}{32} \sin 2\theta.$$

Hence, $x = 150 = \frac{v_0^2}{32} \sin(24^\circ) \Rightarrow v_0^2 = \frac{4800}{\sin 24^\circ} \Rightarrow v_0 \approx 108.6 \text{ ft/sec}.$

40. (a) $\mathbf{r}(t) = t(v_0 \cos \theta)\mathbf{i} + (tv_0 \sin \theta - 16t^2)\mathbf{j}$

$$t(v_0 \sin \theta - 16t) = 0 \text{ when } t = \frac{v_0 \sin \theta}{16}.$$

$$\text{Range: } x = v_0 \cos \theta \left(\frac{v_0 \sin \theta}{16} \right) = \left(\frac{v_0^2}{32} \right) \sin 2\theta$$

The range will be maximum when

$$\frac{dx}{dt} = \left(\frac{v_0^2}{32} \right) 2 \cos 2\theta = 0$$

or

$$2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4} \text{ rad.}$$

- (b) $y(t) = tv_0 \sin \theta - 16t^2$

$$\frac{dy}{dt} = v_0 \sin \theta - 32t = 0 \text{ when } t = \frac{v_0 \sin \theta}{32}.$$

Maximum height:

$$y\left(\frac{v_0 \sin \theta}{32}\right) = \frac{v_0^2 \sin^2 \theta}{32} - 16 \frac{v_0^2 \sin^2 \theta}{32^2} = \frac{v_0^2 \sin^2 \theta}{64}$$

Minimum height when $\sin \theta = 1$, or $\theta = \frac{\pi}{2}.$

$$\begin{aligned} 42. \mathbf{r}(t) &= (v_0 \cos \theta)t\mathbf{i} + [h + (v_0 \sin \theta)t - 4.9t^2]\mathbf{j} \\ &= (v_0 \cos 8^\circ)t\mathbf{i} + [(v_0 \sin 8^\circ)t - 4.9t^2]\mathbf{j} \end{aligned}$$

$x = 50$ when $(v_0 \cos 8^\circ)t = 50 \Rightarrow t = \frac{50}{v_0 \cos 8^\circ}$. For this value of t , $y = 0$:

$$(v_0 \sin 8^\circ)\left(\frac{50}{v_0 \cos 8^\circ}\right) - 4.9\left(\frac{50}{v_0 \cos 8^\circ}\right)^2 = 0$$

$$\begin{aligned} 50 \tan 8^\circ &= \frac{(4.9)(2500)}{v_0^2 \cos^2 8^\circ} \Rightarrow v_0^2 = \frac{(4.9)50}{\tan 8^\circ \cos^2 8^\circ} \approx 1777.698 \\ &\Rightarrow v_0 \approx 42.2 \text{ m/sec} \end{aligned}$$

$$44. \mathbf{r}(t) = b(\omega t - \sin \omega t)\mathbf{i} + b(1 - \cos \omega t)\mathbf{j}$$

$$\mathbf{v}(t) = b\omega[(1 - \cos \omega t)\mathbf{i} + (\sin \omega t)\mathbf{j}]$$

Speed $= \|\mathbf{v}(t)\| = \sqrt{2}b\omega\sqrt{1 - \cos \omega t}$ and has a maximum value of $2b\omega$ when $\omega t = \pi, 3\pi, \dots$

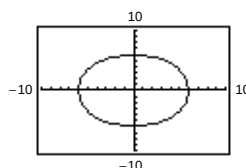
$$55 \text{ mph} = 80.67 \text{ ft/sec} = 80.67 \text{ rad/sec} = \omega \text{ since } (b = 1)$$

Therefore, the maximum speed of a point on the tire is twice the speed of the car:

$$2(80.67) \text{ ft/sec} = 110 \text{ mph}$$

$$\begin{aligned} 46. (a) \text{ Speed} &= \|\mathbf{v}\| = \sqrt{b^2\omega^2 \sin^2(\omega t) + b^2\omega^2 \cos^2(\omega t)} \\ &= \sqrt{b^2\omega^2[\sin^2(\omega t) + \cos^2(\omega t)]} = b\omega \end{aligned}$$

(b)



The graphing utility draws the circle faster for greater values of ω .

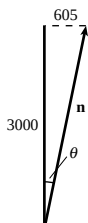
$$48. \|\mathbf{a}(t)\| = b\omega^2\|\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}\| = b\omega^2$$

$$50. \|\mathbf{v}(t)\| = 30 \text{ mph} = 44 \text{ ft/sec}$$

$$\omega = \frac{\|\mathbf{v}(t)\|}{b} = \frac{44}{300} \text{ rad/sec}$$

$$\|\mathbf{a}(t)\| = b\omega^2$$

$$F = m(b\omega^2) = \frac{3000}{32}(300)\left(\frac{44}{300}\right)^2 = 605 \text{ lb}$$



Let $\mathbf{n}$ be normal to the road.

$$\|\mathbf{n}\| \cos \theta = 3000$$

$$\|\mathbf{n}\| \sin \theta = 605$$

Dividing the second equation by the first:

$$\tan \theta = \frac{605}{3000}$$

$$\theta = \arctan\left(\frac{605}{3000}\right) \approx 11.4^\circ.$$

$$52. h = 6 \text{ feet}, v_0 = 45 \text{ feet per second}, \theta = 42.5^\circ. \text{ From Exercise 47,}$$

$$t = \frac{45 \sin 42.5^\circ + \sqrt{(45)^2 \sin^2 42.5^\circ + 2(32)(6)}}{32} \approx 2.08 \text{ seconds.}$$

At this time, $x(t) \approx 69.02$ feet.

54. $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

$y(t) = m(x(t)) + b$, m and b are constants.

$\mathbf{r}(t) = x(t)\mathbf{i} + [m(x(t)) + b]\mathbf{j}$

$\mathbf{v}(t) = x'(t)\mathbf{i} + mx'(t)\mathbf{j}$

$s(t) = \sqrt{[x'(t)]^2 + [mx'(t)]^2} = C$, C is a constant.

Thus, $x'(t) = \frac{C}{\sqrt{1 + m^2}}$

$x''(t) = 0$

$\mathbf{a}(t) = x''(t)\mathbf{i} + mx''(t)\mathbf{j} = \mathbf{0}$.

56. $\mathbf{r}_1(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

$\mathbf{r}_2(t) = \mathbf{r}_1(2t)$

Velocity: $\mathbf{r}_2'(t) = 2\mathbf{r}_1'(2t)$

Acceleration: $\mathbf{r}_2''(t) = 4\mathbf{r}_1''(2t)$

In general, if $\mathbf{r}_3(t) = \mathbf{r}_1(\omega t)$, then:

Velocity: $\mathbf{r}_3'(t) = \omega\mathbf{r}_1'(\omega t)$

Acceleration: $\mathbf{r}_3''(t) = \omega^2\mathbf{r}_1''(\omega t)$

Section 11.4 Tangent Vectors and Normal Vectors

2. $\mathbf{r}(t) = t^3\mathbf{i} + 2t^2\mathbf{j}$

$\mathbf{r}'(t) = 3t^2\mathbf{i} + 4t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{9t^4 + 16t^2}$

$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{9t^4 + 16t^2}}(3t^2\mathbf{i} + 4t\mathbf{j})$

$\mathbf{T}(1) = \frac{1}{\sqrt{9 + 16}}(3\mathbf{i} + 4\mathbf{j}) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$

4. $\mathbf{r}(t) = 6 \cos t\mathbf{i} + 2 \sin t\mathbf{j}$

$\mathbf{r}'(t) = -6 \sin t\mathbf{i} + 2 \cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{36 \sin^2 t + 4 \cos^2 t}$

$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-6 \sin t\mathbf{i} + 2 \cos t\mathbf{j}}{\sqrt{36 \sin^2 t + 4 \cos^2 t}}$

$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{-3\sqrt{3}\mathbf{i} + \mathbf{j}}{\sqrt{36(3/4) + (1/4)}} = \frac{1}{\sqrt{28}}(-3\sqrt{3}\mathbf{i} + \mathbf{j})$

6. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{4}{3}\mathbf{k}$

$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

When $t = 1$, $\mathbf{r}'(t) = \mathbf{r}'(1) = 2\mathbf{i} + \mathbf{j}$ $\left[t = 1 \text{ at } \left(1, 1, \frac{4}{3}\right)\right]$.

$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}} = \frac{\sqrt{5}}{5}(2\mathbf{i} + \mathbf{j})$

Direction numbers: $a = 2, b = 1, c = 0$

Parametric equations: $x = 2t + 1, y = t + 1, z = \frac{4}{3}$

8. $\mathbf{r}(t) = \langle t, t, \sqrt{4 - t^2} \rangle$

$\mathbf{r}'(t) = \left\langle 1, 1, -\frac{t}{\sqrt{4 - t^2}} \right\rangle$

When $t = 1$, $\mathbf{r}'(1) = \left\langle 1, 1, -\frac{1}{\sqrt{3}} \right\rangle$, $\left[t = 1 \text{ at } (1, 1, \sqrt{3})\right]$.

$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\sqrt{21}}{7} \left\langle 1, 1, -\frac{1}{\sqrt{3}} \right\rangle$

Direction numbers: $a = 1, b = 1, c = -\frac{1}{\sqrt{3}}$

Parametric equations: $x = t + 1, y = t + 1,$

$z = -\frac{1}{\sqrt{3}}t + \sqrt{3}$

10. $\mathbf{r}(t) = \langle 2 \sin t, 2 \cos t, 4 \sin^2 t \rangle$

$\mathbf{r}'(t) = \langle 2 \cos t, -2 \sin t, 8 \sin t \cos t \rangle$

When $t = \frac{\pi}{6}$, $\mathbf{r}'\left(\frac{\pi}{6}\right) = \langle \sqrt{3}, -1, 2\sqrt{3} \rangle$, $\left[t = \frac{\pi}{6} \text{ at } (1, \sqrt{3}, 1)\right]$.

$\mathbf{T}\left(\frac{\pi}{6}\right) = \frac{\mathbf{r}'(\pi/6)}{\|\mathbf{r}'(\pi/6)\|} = \frac{1}{4} \langle \sqrt{3}, -1, 2\sqrt{3} \rangle$

Direction numbers: $a = \sqrt{3}, b = -1, c = 2\sqrt{3}$

Parametric equations: $x = \sqrt{3}t + 1, y = -t + \sqrt{3}, z = 2\sqrt{3}t + 1$

12. $\mathbf{r}(t) = 3 \cos t \mathbf{i} + 4 \sin t \mathbf{j} + \frac{1}{2} \mathbf{k}$

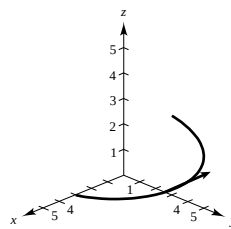
$$\mathbf{r}'(t) = -3 \sin t \mathbf{i} + 4 \cos t \mathbf{j} + \frac{1}{2} \mathbf{k}$$

When $t = \frac{\pi}{2}$, $\mathbf{r}'\left(\frac{\pi}{2}\right) = -3\mathbf{i} + \frac{1}{2}\mathbf{k}$, $\left[t = \frac{\pi}{2} \text{ at } \left(0, 4, \frac{\pi}{4}\right)\right]$.

$$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{\mathbf{r}'(\pi/2)}{\|\mathbf{r}'(\pi/2)\|} = \frac{2}{\sqrt{37}}\left(-3\mathbf{i} + \frac{1}{2}\mathbf{k}\right) = \frac{1}{\sqrt{37}}(-6\mathbf{i} + \mathbf{k})$$

Direction numbers: $a = -6$, $b = 0$, $c = 1$

Parametric equations: $x = -6t$, $y = 4$, $z = t + \frac{\pi}{4}$



14. $\mathbf{r}(t) = e^{-t} \mathbf{i} + 2 \cos t \mathbf{j} + 2 \sin t \mathbf{k}$, $t_0 = 0$

$$\mathbf{r}'(t) = -e^{-t} \mathbf{i} - 2 \sin t \mathbf{j} + 2 \cos t \mathbf{k}$$

$$\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}, \mathbf{r}'(0) = -\mathbf{i} + 2\mathbf{k}, \|\mathbf{r}'(0)\| = \sqrt{5}$$

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{-\mathbf{i} + 2\mathbf{k}}{\sqrt{5}}$$

Parametric equations: $x(s) = 1 - s$, $y(s) = 2$, $z(s) = 2s$

$$\begin{aligned} \mathbf{r}(t_0 + 0.1) &= \mathbf{r}(0 + 0.1) \approx \langle 1 - 0.1, 2, 2(0.1) \rangle \\ &= \langle 0.9, 2, 0.2 \rangle \end{aligned}$$

16. $\mathbf{r}(0) = \langle 0, 1, 0 \rangle$

$$\mathbf{u}(0) = \langle 0, 1, 0 \rangle$$

Hence the curves intersect.

$$\mathbf{r}'(t) = \langle 1, -\sin t, \cos t \rangle, \mathbf{r}'(0) = \langle 1, 0, 1 \rangle$$

$$\mathbf{u}'(s) = \left\langle -\sin s \cos s - \cos s, -\sin s \cos s - \cos s, \frac{1}{2} \cos 2s + \frac{1}{2} \right\rangle$$

$$\mathbf{u}'(0) = \langle -1, 0, 1 \rangle$$

$$\cos \theta = \frac{\mathbf{r}'(0) \cdot \mathbf{u}'(0)}{\|\mathbf{r}'(0)\| \|\mathbf{u}'(0)\|} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

18. $\mathbf{r}(t) = t\mathbf{i} + \frac{6}{t}\mathbf{j}$, $t = 3$

$$\mathbf{r}'(t) = \mathbf{i} - \frac{6}{t^2}\mathbf{j}$$

$$\begin{aligned} \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{1 + (36/t^4)}} \left(\mathbf{i} - \frac{6}{t^2}\mathbf{j} \right) \\ &= \frac{t^2}{\sqrt{t^4 + 36}} \left(\mathbf{i} - \frac{6}{t^2}\mathbf{j} \right) \end{aligned}$$

$$\mathbf{T}'(t) = \frac{72t}{(t^4 + 36)^{3/2}} \mathbf{i} + \frac{12t^3}{(t^4 + 36)^{3/2}} \mathbf{j}$$

$$\mathbf{N}(2) = \frac{\mathbf{T}'(2)}{\|\mathbf{T}'(2)\|} + \frac{1}{\sqrt{13}}(3\mathbf{i} + 2\mathbf{j})$$

20. $\mathbf{r}(t) = \cos t \mathbf{i} + 2 \sin t \mathbf{j} + \mathbf{k}$, $t = -\frac{\pi}{4}$

$$\mathbf{r}'(t) = -\sin t \mathbf{i} + 2 \cos t \mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-\sin t \mathbf{i} + 2 \cos t \mathbf{j}}{\sqrt{\sin^2 t + 4 \cos^2 t}}$$

The unit normal vector is perpendicular to this vector and points toward the z-axis:

$$\mathbf{N}(t) = \frac{-2 \cos t \mathbf{i} - \sin t \mathbf{j}}{\sqrt{\sin^2 t + 4 \cos^2 t}}$$

$$22. \mathbf{r}(t) = 4t\mathbf{i} - 2t\mathbf{j}$$

$$\mathbf{v}(t) = 4\mathbf{i} - 2\mathbf{j}$$

$$\mathbf{a}(t) = \mathbf{0}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{5}}(2\mathbf{i} - \mathbf{j})$$

$$\mathbf{T}'(t) = \mathbf{0}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \text{ is undefined.}$$

The path is a line and the speed is constant.

$$24. \mathbf{r}(t) = t^2\mathbf{j} + \mathbf{k}$$

$$\mathbf{v}(t) = 2t\mathbf{j}$$

$$\mathbf{a}(t) = 2\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{2t\mathbf{j}}{2t} = \mathbf{j}$$

$$\mathbf{T}'(t) = \mathbf{0}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \text{ is undefined.}$$

The path is a line and the speed is variable.

$$26. \mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j}, t = 1$$

$$\mathbf{v}(t) = 2t\mathbf{i} + 2\mathbf{j}, \mathbf{v}(1) = 2\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{a}(t) = 2\mathbf{i}, \mathbf{a}(1) = 2\mathbf{i}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{4t^2 + 4}}(2t\mathbf{i} + 2\mathbf{j}) = \frac{1}{\sqrt{t^2 + 1}}(t\mathbf{i} + \mathbf{j})$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\frac{1}{(t^2 + 1)^{3/2}}\mathbf{i} + \frac{-t}{(t^2 + 1)^{3/2}}\mathbf{j}}{\frac{1}{t^2 + 1}}$$

$$= \frac{1}{\sqrt{t^2 + 1}}(\mathbf{i} + t\mathbf{j})$$

$$\mathbf{N}(1) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \sqrt{2}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$$

$$30. \mathbf{r}(t_0) = (\omega t_0 - \sin \omega t_0)\mathbf{i} + (1 - \cos \omega t_0)\mathbf{j}$$

$$\mathbf{v}(t_0) = \omega[(1 - \cos \omega t_0)\mathbf{i} + (\sin \omega t_0)\mathbf{j}]$$

$$\mathbf{a}(t_0) = \omega^2[(\sin \omega t_0)\mathbf{i} + (\cos \omega t_0)\mathbf{j}]$$

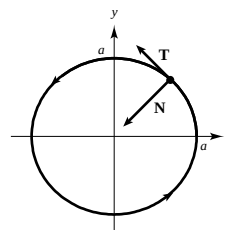
$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{(1 - \cos \omega t_0)\mathbf{i} + (\sin \omega t_0)\mathbf{j}}{\sqrt{2}\sqrt{1 - \cos \omega t_0}}$$

$$\text{Motion along } \mathbf{r} \text{ is clockwise. Therefore, } \mathbf{N} = \frac{(\sin \omega t_0)\mathbf{i} - (1 - \cos \omega t_0)\mathbf{j}}{\sqrt{2}\sqrt{1 - \cos \omega t_0}}.$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \frac{\omega^2 \sin \omega t_0}{\sqrt{2}\sqrt{1 - \cos \omega t_0}} = \frac{\omega^2}{\sqrt{2}}\sqrt{1 + \cos \omega t_0}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \frac{\omega^2}{\sqrt{2}}\sqrt{1 - \cos \omega t_0}$$

32. $\mathbf{T}(t)$ points in the direction that $\mathbf{r}$ is moving. $\mathbf{N}(t)$ points in the direction that $\mathbf{r}$ is turning, toward the concave side of the curve.



34. If the angular velocity
- ω
- is halved,

$$a_N = a \left(\frac{\omega}{2} \right)^2 = \frac{a\omega^2}{4}.$$

a_N is changed by a factor of $\frac{1}{4}$.

- 36.
- $\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j}$
- ,
- $t_0 = \frac{\pi}{4}$

$$x = 2 \cos t, y = 2 \sin t \Rightarrow x^2 + y^2 = 4$$

$$\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j}$$

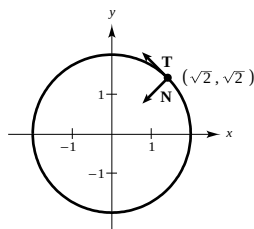
$$\mathbf{T}(t) = \frac{1}{2}(-2 \sin t \mathbf{i} + 2 \cos t \mathbf{j}) = -\sin t \mathbf{i} + \cos t \mathbf{j}$$

$$\mathbf{N}(t) = -\cos t \mathbf{i} - \sin t \mathbf{j}$$

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$$

$$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{j})$$

$$\mathbf{N}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(-\mathbf{i} - \mathbf{j})$$



- 40.
- $\mathbf{r}(t) = e^t \sin t \mathbf{i} + e^t \cos t \mathbf{j} + e^t \mathbf{k}$

$$\mathbf{v}(t) = (e^t \cos t + e^t \sin t)\mathbf{i} + (-e^t \sin t + e^t \cos t)\mathbf{j} + e^t \mathbf{k}$$

$$\mathbf{v}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\mathbf{a}(t) = 2e^t \cos t \mathbf{i} - 2e^t \sin t \mathbf{j} + e^t \mathbf{k}$$

$$\mathbf{a}(0) = 2\mathbf{i} + \mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{3}}[(\cos t + \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j} + \mathbf{k}]$$

$$\mathbf{T}(0) = \frac{1}{\sqrt{3}}[\mathbf{i} + \mathbf{j} + \mathbf{k}]$$

$$\mathbf{N}(t) = \frac{1}{\sqrt{2}}[(-\sin t + \cos t)\mathbf{i} + (-\cos t - \sin t)\mathbf{j}]$$

$$\mathbf{N}(0) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \sqrt{3}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$$

- 38.
- $\mathbf{r}(t) = 4t\mathbf{i} - 4t\mathbf{j} + 2t\mathbf{k}$

$$\mathbf{v}(t) = 4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{a}(t) = \mathbf{0}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} \text{ is undefined.}$$

a_T, a_N are not defined.

- 42.
- $\mathbf{r}(t) = t\mathbf{i} + 3t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}$

$$\mathbf{v}(t) = \mathbf{i} + 6t\mathbf{j} + t\mathbf{k}$$

$$\mathbf{v}(2) = \mathbf{i} + 12\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{a}(t) = 6\mathbf{j} + \mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{1 + 37t^2}}(\mathbf{i} + 6t\mathbf{j} + t\mathbf{k})$$

$$\mathbf{T}(2) = \frac{1}{\sqrt{149}}(\mathbf{i} + 12\mathbf{j} + 2\mathbf{k})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{1}{(1 + 37t^2)^{3/2}}[-37t\mathbf{i} + 6\mathbf{j} + \mathbf{k}]$$

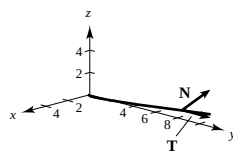
$$= \frac{1}{\sqrt{37}\sqrt{1 + 37t^2}}[-37t\mathbf{i} + 6\mathbf{j} + \mathbf{k}]$$

$$\mathbf{N}(2) = \frac{1}{\sqrt{37}\sqrt{149}}[-74\mathbf{i} + 6\mathbf{j} + \mathbf{k}]$$

$$= \frac{1}{\sqrt{5513}}(-74\mathbf{i} + 6\mathbf{j} + \mathbf{k})$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \frac{74}{\sqrt{149}}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \frac{37}{\sqrt{5513}} = \frac{\sqrt{37}}{\sqrt{149}}$$



44. The unit tangent vector points in the direction of motion.

46. If $a_T = 0$, then the speed is constant.

48. (a) $\mathbf{r}(t) = \langle \cos \pi t + \pi t \sin \pi t, \sin \pi t - \pi t \cos \pi t \rangle$

$$\mathbf{v}(t) = \langle -\pi \sin \pi t + \pi \sin \pi t + \pi^2 t \cos \pi t, \pi \cos \pi t - \pi \cos \pi t + \pi^2 t \sin \pi t \rangle = \langle \pi^2 t \cos \pi t, \pi^2 t \sin \pi t \rangle$$

$$\mathbf{a}(t) = \langle \pi^2 \cos \pi t - \pi^3 t \sin \pi t, \pi^2 \sin \pi t + \pi^3 t \cos \pi t \rangle$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \langle \cos \pi t, \sin \pi t \rangle$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \cos \pi t(\pi^2 \cos \pi t - \pi^3 t \sin \pi t) + \sin \pi t(\pi^2 \sin \pi t + \pi^3 t \cos \pi t) = \pi^2$$

$$a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2} = \sqrt{\pi^4(1 + \pi^2 t^2) - \pi^4} = \pi^3 t$$

When $t = 1$, $a_T = \pi^2$, $a_N = \pi^3$. When $t = 2$, $a_T = \pi^2$, $a_N = 2\pi^3$.

(b) Since $a_T = \pi^2 > 0$ for all values of t , the speed is increasing when $t = 1$ and $t = 2$.

50. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^3}{3}\mathbf{k}$, $t_0 = 1$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{1 + 4t^2 + t^4}}(\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k})$$

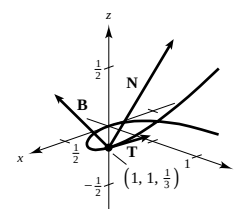
$$\mathbf{N}(t) = \frac{1}{\sqrt{1 + 4t^2 + t^4}\sqrt{1 + t^2 + t^4}}[(-2t - t^3)\mathbf{i} + (1 - t^4)\mathbf{j} + (t + 2t^3)\mathbf{k}]$$

$$\mathbf{r}(1) = \mathbf{i} + \mathbf{j} + \frac{1}{3}\mathbf{k}$$

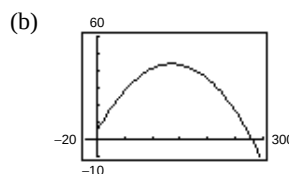
$$\mathbf{T}(1) = \frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$\mathbf{N}(1) = \frac{1}{\sqrt{6}\sqrt{3}}(-3\mathbf{i} + 3\mathbf{k}) = \frac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{k})$$

$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{vmatrix} = \frac{\sqrt{3}}{3}\mathbf{i} - \frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{3}}{3}\mathbf{k} = \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$$



52. (a) $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + \left[h + (v_0 \sin \theta)t - \frac{1}{2}gt^2 \right]\mathbf{j}$
 $= (100 \cos 30^\circ)t\mathbf{i} + [5 + (100 \sin 30^\circ)t - 16t^2]\mathbf{j}$
 $= 50\sqrt{3}t\mathbf{i} + [5 + 50t - 16t^2]\mathbf{j}$



Maximum height ≈ 44.0625

Range ≈ 279.0325

(c) $\mathbf{v}(t) = 50\sqrt{3}\mathbf{i} + (50 - 32t)\mathbf{j}$
 Speed $= \|\mathbf{v}(t)\| = \sqrt{2500(3) + (50 - 32t)^2}$
 $= 4\sqrt{64t^2 - 200t + 625}\mathbf{a}(t) = -32\mathbf{j}$

(d)

t	0.5	1.0	1.5	2.0	2.5	3.0
Speed	93.04	88.45	86.63	87.73	91.65	98.06

—CONTINUED—

52. —CONTINUED—

$$\begin{aligned}
 \text{(e)} \quad \mathbf{T}(t) &= \frac{25\sqrt{3}\mathbf{i} + (25 - 16t)\mathbf{j}}{2\sqrt{64t^2 - 200t + 625}} \\
 \mathbf{N}(t) &= \frac{(25 - 16t)\mathbf{i} - 25\sqrt{3}\mathbf{j}}{2\sqrt{64t^2 - 200t + 625}} \\
 a_{\mathbf{T}} &= \mathbf{a} \cdot \mathbf{T} = \frac{16(16t - 25)}{\sqrt{64t^2 - 200t + 625}} \\
 a_{\mathbf{N}} &= \mathbf{a} \cdot \mathbf{N} = \frac{400\sqrt{3}}{\sqrt{64t^2 - 200t + 625}} \\
 a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N} &= -32\mathbf{j}
 \end{aligned}$$

$$54. 600 \text{ mph} = 880 \text{ ft/sec}$$

$$\mathbf{r}(t) = 880t\mathbf{i} + (-16t^2 + 36,000)\mathbf{j}$$

$$\mathbf{v}(t) = 880\mathbf{i} - 32t\mathbf{j}$$

$$\mathbf{a}(t) = -32\mathbf{j}$$

$$\mathbf{T}(t) = \frac{880\mathbf{i} - 32t\mathbf{j}}{16\sqrt{4t^2 + 3025}} = \frac{55\mathbf{i} - 2t\mathbf{j}}{\sqrt{4t^2 + 3025}}$$

Motion along $\mathbf{r}$ is clockwise, therefore

$$\begin{aligned}
 \mathbf{N}(t) &= \frac{-2t\mathbf{i} - 55\mathbf{j}}{\sqrt{4t^2 + 3025}} \\
 a_{\mathbf{T}} &= \mathbf{a} \cdot \mathbf{T} = \frac{64t}{\sqrt{4t^2 + 3025}} \\
 a_{\mathbf{N}} &= \mathbf{a} \cdot \mathbf{N} = \frac{1760}{\sqrt{4t^2 + 3025}}
 \end{aligned}$$

$$58. v = \sqrt{\frac{9.56 \times 10^4}{4200}} \approx 4.77 \text{ mi/sec}$$

60. Let x = distance from the satellite to the center of the earth ($x = r + 4000$). Then:

$$\begin{aligned}
 v &= \frac{2\pi x}{t} = \frac{2\pi x}{24(3600)} = \sqrt{\frac{9.56 \times 10^4}{x}} \\
 \frac{4\pi^2 x^2}{(24)^2(3600)^2} &= \frac{9.56 \times 10^4}{x} \\
 x^3 &= \frac{(9.56 \times 10^4)(24)^2(3600)^2}{4\pi^2} \Rightarrow x \approx 26,245 \text{ mi} \\
 v &\approx \frac{2\pi(26,245)}{24(3600)} \approx 1.92 \text{ mi/sec} \approx 6871 \text{ mph}
 \end{aligned}$$

$$62. \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

$y(t) = m(x(t)) + b$, m and b are constants.

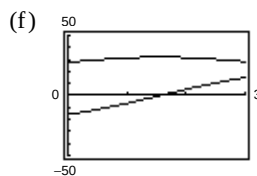
$$\mathbf{r}(t) = x(t)\mathbf{i} + [m(x(t)) + b]\mathbf{j}$$

$$\mathbf{v}(t) = x'(t)\mathbf{i} + mx'(t)\mathbf{j}$$

$$\|\mathbf{v}(t)\| = \sqrt{[x'(t)]^2 + [mx'(t)]^2} = |x'(t)|\sqrt{1 + m^2}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\pm(\mathbf{i} + m\mathbf{j})}{\sqrt{1 + m^2}}, \text{ constant}$$

Hence, $\mathbf{T}'(t) = \mathbf{0}$.



The speed is increasing when $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ have opposite signs.

$$56. \mathbf{r}(t) = (r \cos \omega t)\mathbf{i} + (r \sin \omega t)\mathbf{j}$$

$$\mathbf{v}(t) = (-r\omega \sin \omega t)\mathbf{i} + (r\omega \cos \omega t)\mathbf{j}$$

$$\|\mathbf{v}(t)\| = r\omega\sqrt{1} = r\omega = v$$

$$\mathbf{a}(t) = (-r\omega^2 \cos \omega t)\mathbf{i} - (r\omega^2 \sin \omega t)\mathbf{j}$$

$$\|\mathbf{a}(t)\| = r\omega^2$$

$$(a) F = m\|\mathbf{a}(t)\| = m(r\omega^2) = \frac{m}{r}(r^2\omega^2) = \frac{mv^2}{r}$$

(b) By Newton's Law:

$$\frac{mv^2}{r} = \frac{GMm}{r^2}, \quad v^2 = \frac{GM}{r}, \quad v = \sqrt{\frac{GM}{r}}$$

$$64. \|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}$$

$$= (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}) \cdot (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N})$$

$$= a_{\mathbf{T}}^2\|\mathbf{T}\|^2 + 2a_{\mathbf{T}}a_{\mathbf{N}}\mathbf{T} \cdot \mathbf{N} + a_{\mathbf{N}}^2\|\mathbf{N}\|^2$$

$$= a_{\mathbf{T}}^2 + a_{\mathbf{N}}^2$$

$$a_{\mathbf{N}}^2 = \|\mathbf{a}\|^2 - a_{\mathbf{T}}^2$$

Since $a_{\mathbf{N}} > 0$, we have $a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2}$.

Section 11.5 Arc Length and Curvature

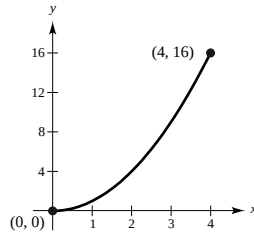
2. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{k}$

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 0, \quad \frac{dz}{dt} = 2t$$

$$s = \int_0^4 \sqrt{1 + 4t^2} dt$$

$$= \frac{1}{4} \left[2t\sqrt{1 + 4t^2} + \ln|2t + \sqrt{1 + 4t^2}| \right]_0^4$$

$$= \frac{1}{4} [8\sqrt{65} + \ln(8 + \sqrt{65})] \approx 16.819$$

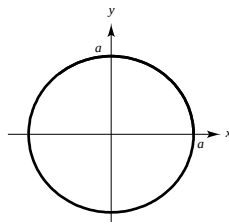


4. $\mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j}$

$$\frac{dx}{dt} = -a \sin t, \quad \frac{dy}{dt} = a \cos t$$

$$s = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} dt$$

$$= \int_0^{2\pi} a dt = \left[at \right]_0^{2\pi} = 2\pi a$$



6. (a) $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + \left[(v_0 \sin \theta)t - \frac{1}{2}gt^2 \right]\mathbf{j}$

$$y(t) = (v_0 \sin \theta)t - \frac{1}{2}gt^2$$

$$y'(t) = v_0 \sin \theta - gt = 0 \text{ when } t = \frac{v_0 \sin \theta}{g}.$$

Maximum height when $\sin \theta = 1$, or $\theta = \frac{\pi}{2}$.

(b) $y(t) = (v_0 \sin \theta)t - \frac{1}{2}gt^2 = 0 \Rightarrow t = \frac{2v_0 \sin \theta}{g}$

$$\text{Range: } x(t) = (v_0 \cos \theta) \left(\frac{2v_0 \sin \theta}{g} \right) = \frac{v_0^2}{g} \sin^2 \theta$$

The range $x(t)$ is a maximum for $\sin 2\theta = 1$, or $\theta = \frac{\pi}{4}$.

(c) $x'(t) = v_0 \cos \theta$

$$y'(t) = v_0 \sin \theta - gt$$

$$\begin{aligned} x'(t)^2 + y'(t)^2 &= v_0^2 \cos^2 \theta + (v_0 \sin \theta - gt)^2 \\ &= v_0^2 \cos^2 \theta + v_0^2 \sin^2 \theta - 2v_0^2 g \sin \theta t + g^2 t^2 \\ &= v_0^2 - 2v_0 g \sin \theta t + g^2 t^2 \end{aligned}$$

$$s(\theta) = \int_0^{2v_0 \sin \theta / g} \left[v_0^2 - 2v_0 g \sin \theta t + g^2 t^2 \right]^{1/2} dt$$

Since $v_0 = 96$ ft/sec, we have

$$s(\theta) = \int_0^{6 \sin \theta} \left[96^2 - (6144 \sin \theta)t + 1024t^2 \right]^{1/2} dt.$$

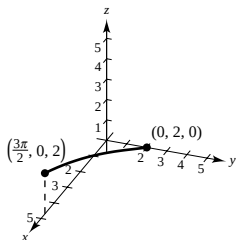
Using a computer algebra system, $s(\theta)$ is a maximum for $\theta \approx 0.9855 \approx 56.5^\circ$.

8. $\mathbf{r}(t) = \langle 3t, 2 \cos t, 2 \sin t \rangle$

$$\frac{dx}{dt} = 3, \frac{dy}{dt} = -2 \sin t, \frac{dz}{dt} = 2 \cos t$$

$$s = \int_0^{\pi/2} \sqrt{3^2 + (-2 \sin t)^2 + (2 \cos t)^2} dt$$

$$= \int_0^{\pi/2} \sqrt{13} dt = \sqrt{13}t \Big|_0^{\pi/2} = \frac{\sqrt{13}\pi}{2}$$

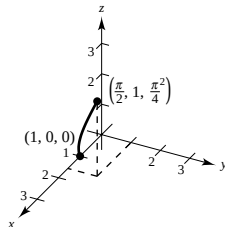


10. $\mathbf{r}(t) = \langle \cos t + t \sin t, \sin t - t \cos t, t^2 \rangle$

$$\frac{dx}{dt} = t \cos t, \frac{dy}{dt} = t \sin t, \frac{dz}{dt} = 2t$$

$$s = \int_0^{\pi/2} \sqrt{(t \cos t)^2 + (t \sin t)^2 + (2t)^2} dt$$

$$= \int_0^{\pi/2} \sqrt{5t^2} dt = \sqrt{5} \frac{t^2}{2} \Big|_0^{\pi/2} = \frac{\sqrt{5}\pi^2}{8}$$



12. $\mathbf{r}(t) = \sin \pi t \mathbf{i} + \cos \pi t \mathbf{j} + t^3 \mathbf{k}$

$$\frac{dx}{dt} = \pi \cos \pi t, \frac{dy}{dt} = -\pi \sin \pi t, \frac{dz}{dt} = 3t^2$$

$$s = \int_0^2 \sqrt{(\pi \cos \pi t)^2 + (-\pi \sin \pi t)^2 + (3t^2)^2} dt$$

$$= \int_0^2 \sqrt{\pi^2 + 9t^4} dt \approx 11.15$$

14. $\mathbf{r}(t) = 6 \cos\left(\frac{\pi t}{4}\right) \mathbf{i} + 2 \sin\left(\frac{\pi t}{4}\right) \mathbf{j} + t \mathbf{k}, 0 \leq t \leq 2$

(a) $\mathbf{r}(0) = 6\mathbf{i} = \langle 6, 0, 0 \rangle$

$$\mathbf{r}(2) = 2\mathbf{j} + 2\mathbf{k} = \langle 0, 2, 2 \rangle$$

$$\text{distance} = \sqrt{6^2 + 2^2 + 2^2} = \sqrt{44} = 2\sqrt{11} \approx 6.633$$

(b) $\mathbf{r}(0) = \langle 6, 0, 0 \rangle$

$$\mathbf{r}(0.5) = \langle 5.543, 0.765, 0.5 \rangle$$

$$\mathbf{r}(1.0) = \langle 4.243, 1.414, 1.0 \rangle$$

$$\mathbf{r}(1.5) = \langle 2.296, 1.848, 1.5 \rangle$$

$$\mathbf{r}(2.0) = \langle 0, 2, 2 \rangle$$

(c) Increase the number of line segments.

(d) Using a graphing utility, you obtain

$$s = \int_0^2 \|\mathbf{r}'(t)\| dt \approx 7.0105.$$

16. $\mathbf{r}(t) = \left\langle 4(\sin t - t \cos t), 4(\cos t + t \sin t), \frac{3}{2}t^2 \right\rangle$

(a) $s = \int_0^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2} du$

$$= \int_0^t \sqrt{(4u \sin u)^2 + (4u \cos u)^2 + (3u)^2} du = \int_0^t \sqrt{16u + 9u^2} du = \int_0^t 5u du = \frac{5}{2}t^2$$

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16. —CONTINUED—

$$(b) \quad t = \sqrt{\frac{2s}{5}}$$

$$x = 4 \left(\sin \sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}} \cos \sqrt{\frac{2s}{5}} \right)$$

$$y = 4 \left(\cos \sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}} \sin \sqrt{\frac{2s}{5}} \right)$$

$$z = \frac{3}{2} \left(\sqrt{\frac{2s}{5}} \right)^2 = \frac{3s}{5}$$

$$\mathbf{r}(s) = 4 \left(\sin \sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}} \cos \sqrt{\frac{2s}{5}} \right) \mathbf{i} + 4 \left(\cos \sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}} \sin \sqrt{\frac{2s}{5}} \right) \mathbf{j} + \frac{3s}{5} \mathbf{k}$$

$$(c) \quad \text{When } s = \sqrt{5}:$$

$$x = 4 \left(\sin \sqrt{\frac{2\sqrt{5}}{5}} - \sqrt{\frac{2\sqrt{5}}{5}} \cos \sqrt{\frac{2\sqrt{5}}{5}} \right) \approx -6.956$$

$$y = 4 \left(\cos \sqrt{\frac{2\sqrt{5}}{5}} + \sqrt{\frac{2\sqrt{5}}{5}} \sin \sqrt{\frac{2\sqrt{5}}{5}} \right) \approx 14.169$$

$$z = \frac{3\sqrt{5}}{5} \approx 1.342$$

$$(-6.956, 14.169, 1.342)$$

$$\text{When } s = 4:$$

$$x = 4 \left(\sin \sqrt{\frac{8}{5}} - \sqrt{\frac{8}{5}} \cos \sqrt{\frac{8}{5}} \right) \approx 2.291$$

$$y = 4 \left(\cos \sqrt{\frac{8}{5}} + \sqrt{\frac{8}{5}} \sin \sqrt{\frac{8}{5}} \right) \approx 6.029$$

$$z = \frac{12}{5} = 2.4$$

$$(2.291, 6.029, 2.400)$$

$$(d) \quad \|\mathbf{r}'(s)\| = \sqrt{\left(\frac{4}{5} \sin \sqrt{\frac{2s}{5}} \right)^2 + \left(\frac{4}{5} \cos \sqrt{\frac{2s}{5}} \right)^2 + \left(\frac{3}{5} \right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1$$

$$18. \quad \mathbf{r}(s) = (3 + s)\mathbf{i} + \mathbf{j}$$

$$\mathbf{r}'(s) = \mathbf{i} \quad \text{and} \quad \|\mathbf{r}'(s)\| = 1$$

$$\mathbf{T}(s) = \mathbf{r}'(s)$$

$$\mathbf{T}'(s) = \mathbf{0} \Rightarrow K = \|\mathbf{T}'(s)\| = 0 \quad (\text{The curve is a line.})$$

$$20. \quad \mathbf{r}(s) = 4 \left(\sin \sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}} \cos \sqrt{\frac{2s}{5}} \right) \mathbf{i} + 4 \left(\cos \sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}} \sin \sqrt{\frac{2s}{5}} \right) \mathbf{j} + \frac{3s}{5} \mathbf{k}$$

$$\mathbf{T}(s) = \mathbf{r}'(s) = \frac{4}{5} \sin \sqrt{\frac{2s}{5}} \mathbf{i} + \frac{4}{5} \cos \sqrt{\frac{2s}{5}} \mathbf{j} + \frac{3}{5} \mathbf{k}$$

$$\mathbf{T}'(s) = \frac{4}{25} \sqrt{\frac{5}{2s}} \cos \sqrt{\frac{2s}{5}} \mathbf{i} - \frac{4}{25} \sqrt{\frac{5}{2s}} \sin \sqrt{\frac{2s}{5}} \mathbf{j}$$

$$K = \|\mathbf{T}'(s)\| = \frac{4}{25} \sqrt{\frac{5}{2s}} = \frac{2\sqrt{10s}}{25s}$$

$$22. \quad \mathbf{r}(t) = t^2 \mathbf{j} + \mathbf{k}$$

$$\mathbf{v}(t) = 2t \mathbf{j}$$

$$\mathbf{T}(t) = \mathbf{j}$$

$$\mathbf{T}'(t) = \mathbf{0}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = 0$$

$$24. \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$$

$$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j}$$

$$\mathbf{a}(t) = 2\mathbf{j}$$

$$\mathbf{a}(1) = 2\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{i} + 2t\mathbf{j}}{\sqrt{1 + 4t^2}}$$

$$\mathbf{N}(t) = \frac{1}{\sqrt{1 + 4t^2}}(-2t\mathbf{i} + \mathbf{j})$$

$$\mathbf{N}(1) = \frac{1}{\sqrt{5}}(-2\mathbf{i} + \mathbf{j})$$

$$K = \frac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2} = \frac{2}{5\sqrt{5}}$$

$$28. \mathbf{r}(t) = a \cos(\omega t)\mathbf{i} + b \sin(\omega t)\mathbf{j}$$

$$\mathbf{r}'(t) = -a\omega \sin(\omega t)\mathbf{i} + b\omega \cos(\omega t)\mathbf{j}$$

$$\mathbf{T}(t) = \frac{-a \sin(\omega t)\mathbf{i} + b \cos(\omega t)\mathbf{j}}{\sqrt{a^2 \sin^2(\omega t) + b^2 \cos^2(\omega t)}}$$

$$\mathbf{T}'(t) = \frac{-ab^2\omega \cos(\omega t)\mathbf{i} - a^2b\omega \sin(\omega t)\mathbf{j}}{[a^2 \sin^2(\omega t) + b^2 \cos^2(\omega t)]^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\frac{ab\omega}{a^2 \sin^2(\omega t) + b^2 \cos^2(\omega t)}}{\omega \sqrt{a^2 \sin^2(\omega t) + b^2 \cos^2(\omega t)}} \\ = \frac{ab}{[a^2 \sin^2(\omega t) + b^2 \cos^2(\omega t)]^{3/2}}$$

$$32. \mathbf{r}(t) = 4t\mathbf{i} - 4t\mathbf{j} + 2t\mathbf{k}$$

$$\mathbf{r}'(t) = 4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{T}(t) = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

$$\mathbf{T}'(t) = \mathbf{0}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = 0$$

$$36. \mathbf{r}(t) = e^t \cos t \mathbf{i} + e^t \sin t \mathbf{j} + e^t \mathbf{k}$$

$$\mathbf{r}'(t) = (-e^t \sin t + e^t \cos t)\mathbf{i} + (e^t \cos t + e^t \sin t)\mathbf{j} + e^t \mathbf{k}$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{3}}[(-\sin t + \cos t)\mathbf{i} + (\cos t + \sin t)\mathbf{j} + \mathbf{k}]$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{3}}[(-\cos t - \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j}]$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{(1/\sqrt{3})\sqrt{(-\cos t - \sin t)^2 + (-\sin t + \cos t)^2}}{\sqrt{3}e^t} = \frac{\sqrt{2}}{3e^t}$$

$$26. \mathbf{r}(t) = 2 \cos \pi t \mathbf{i} + \sin \pi t \mathbf{j}$$

$$\mathbf{r}'(t) = -2\pi \sin \pi t \mathbf{i} + \pi \cos \pi t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \pi \sqrt{4 \sin^2 \pi t + \cos^2 \pi t}$$

$$\mathbf{T}(t) = \frac{-2 \sin \pi t \mathbf{i} + \cos \pi t \mathbf{j}}{\sqrt{4 \sin^2 \pi t + \cos^2 \pi t}}$$

$$\mathbf{T}'(t) = \frac{-2\pi \cos \pi t \mathbf{i} - 4\pi \sin \pi t \mathbf{j}}{(4 \sin^2 \pi t + \cos^2 \pi t)^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\frac{2\pi}{4 \sin^2 \pi t + \cos^2 \pi t}}{\pi \sqrt{4 \sin^2 \pi t + \cos^2 \pi t}} \\ = \frac{2}{(4 \sin^2 \pi t + \cos^2 \pi t)^{3/2}}$$

$$30. \mathbf{r}(t) = \langle a(\omega t - \sin \omega t), a(1 - \cos \omega t) \rangle$$

From Exercise 22, Section 11.4, we have:

$$\mathbf{a} \cdot \mathbf{N} = \frac{a\omega^2}{\sqrt{2}} \cdot \sqrt{1 - \cos \omega t}$$

$$K = \frac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{v}(t)\|^2} \\ = \frac{\left(\frac{a\omega^2}{\sqrt{2}}\right)\sqrt{1 - \cos \omega t}}{2a^2\omega^2(1 - \cos \omega t)} = \frac{\sqrt{2}}{4a\sqrt{1 - \cos \omega t}}$$

$$34. \mathbf{r}(t) = 2t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$$

$$\mathbf{r}'(t) = 4t\mathbf{i} + \mathbf{j} + t\mathbf{k}$$

$$\mathbf{T}(t) = \frac{4t\mathbf{i} + \mathbf{j} + t\mathbf{k}}{\sqrt{1 + 17t^2}}$$

$$\mathbf{T}'(t) = \frac{4\mathbf{i} - 17t\mathbf{j} + \mathbf{k}}{(1 + 17t^2)^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{289t^2 + 17}}{(1 + 17t^2)^{3/2}} / (1 + 17t^2)^{1/2} \\ = \frac{\sqrt{17}}{(1 + 17t^2)^{3/2}}$$

38. $y = mx + b$

Since $y'' = 0$, $K = 0$, and the radius of curvature is undefined.

40. $y = 2x + \frac{4}{x}, x = 1$

$$y' = 2 - \frac{4}{x^2}, y'(1) = -2$$

$$y'' = \frac{8}{x^3}, y''(1) = 8$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{8}{(1 + 4)^{3/2}} = \frac{8}{5^{3/2}}$$

$$\frac{1}{K} = \frac{5^{3/2}}{8} \quad (\text{radius of curvature})$$

44. (a) $y = \frac{4x^2}{x^2 + 3}$

$$y' = \frac{24x}{(x^2 + 3)^2}$$

$$y'' = \frac{72(1 - x^2)}{(x^2 + 3)^3}$$

At $x = 0$: $y' = 0$

$$y'' = \frac{72}{27} = \frac{8}{3}$$

$$K = \frac{8/3}{(1 + 0^2)^{3/2}} = \frac{8}{3}$$

$$r = \frac{1}{K} = \frac{3}{8}$$

Center: $\left(0, \frac{3}{8}\right)$

Equation: $x^2 + \left(y - \frac{3}{8}\right)^2 = \frac{9}{64}$

(b) The circles have different radii since the curvature is different and

$$r = \frac{1}{K}$$

42. $y = \frac{3}{4}\sqrt{16 - x^2}$

$$y' = \frac{-9x}{16y}$$

$$y'' = \frac{-[9 + (16y)^2]}{16y}$$

At $x = 0$: $y' = 0$

$$y'' = -\frac{3}{16}$$

$$K = \frac{\left|-\frac{3}{16}\right|}{(1 + 0^2)^{3/2}} = \frac{3}{16}$$

$$\frac{1}{K} = \frac{16}{3} \quad (\text{radius of curvature})$$

46. $y = \ln x, \quad x = 1$

$$y' = \frac{1}{x}, \quad y'' = -\frac{1}{x^2}$$

$y'(1) = 1, \quad y''(1) = -1$

$$K = \frac{|-1|}{(1 + (1)^2)^{3/2}} = \frac{1}{2^{3/2}}, \quad r = \frac{1}{K} = 2^{3/2} = 2\sqrt{2}$$

The slope of the tangent line at $(1, 0)$ is $y'(1) = 1$.

The slope of the normal line is -1 .

Equation of normal line: $y = -(x - 1) = -x + 1$

The center of the circle is on the normal line $2\sqrt{2}$ units away from the point $(1, 0)$.

$$\sqrt{(1 - x)^2 + (0 - y)^2} = 2\sqrt{2}$$

$$(1 - x)^2 + (x - 1)^2 = 8$$

$$2x^2 - 4x + 2 = 8$$

$$2(x^2 - 2x - 3) = 0$$

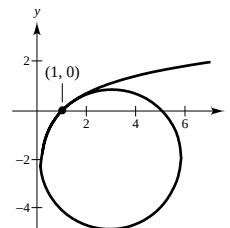
$$2(x - 3)(x + 1) = 0$$

$$x = 3 \text{ or } x = -1$$

Since the circle is below the curve, $x = 3$ and $y = -2$.

Center of circle: $(3, -2)$

Equation of circle: $(x - 3)^2 + (y + 2)^2 = 8$



48. $y = \frac{1}{3}x^3, \quad x = 1$

$$y' = x^2, \quad y''(1) = 2x$$

$$y'(1) = 1, \quad y''(1) = 2$$

$$K = \frac{2}{(1+1)^{3/2}} = \frac{1}{\sqrt{2}}, \quad r = \frac{1}{K} = \sqrt{2}$$

The slope of the tangent line at $(1, \frac{1}{3})$ is $y'(1) = 1$.

The slope of the normal line is -1 .

Equation of normal line: $y - \frac{1}{3} = -(x - 1)$ or $y = -x + \frac{4}{3}$

The center of the circle is on the normal line $\sqrt{2}$ units away from the point $(1, \frac{1}{3})$.

$$\sqrt{(1-x)^2 + (\frac{1}{3}-y)^2} = \sqrt{2}$$

$$(1-x)^2 + (x-1)^2 = 2$$

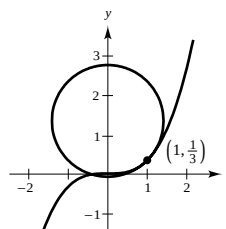
$$(x-1)^2 = 1$$

$$x = 0 \text{ or } x = 2$$

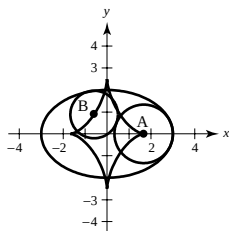
Since the circle is above the curve, $x = 0$ and $y = \frac{4}{3}$.

Center of circle: $(0, \frac{4}{3})$

Equation of circle: $x^2 + (y - \frac{4}{3})^2 = 2$



50.



52. $y = x^3, \quad y' = 3x^2, \quad y'' = 6x$

$$K = \left| \frac{6x}{(1+9x^4)^{3/2}} \right|$$

(a) K is maximum at $(\frac{1}{\sqrt[4]{45}}, \frac{1}{\sqrt[4]{45^3}}), (\frac{-1}{\sqrt[4]{45}}, \frac{-1}{\sqrt[4]{45^3}})$.

(b) $\lim_{x \rightarrow \infty} K = 0$

54. $y = \ln x, \quad y' = \frac{1}{x}, \quad y'' = -\frac{1}{x^2}$

$$K = \left| \frac{-1/x^2}{[1 + (1/x)^2]^{3/2}} \right| = \frac{x}{(x^2 + 1)^{3/2}}$$

$$\frac{dK}{dx} = \frac{-2x^2 + 1}{(x^2 + 1)^{5/2}}$$

(a) K has a maximum when $x = \frac{1}{\sqrt{2}}$.

(b) $\lim_{x \rightarrow \infty} K = 0$

56. $y = \cos x$

$$y' = -\sin x$$

$$y'' = -\cos x$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|-\cos x|}{(1 + \sin^2 x)^{3/2}} = 0 \text{ for } x = \frac{\pi}{2} + K\pi.$$

Curvature is 0 at $(\frac{\pi}{2} + K\pi, 0)$.

58. $y = \cosh x = \frac{e^x + e^{-x}}{2}$

$$y' = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$y'' = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$K = \frac{|\cosh x|}{[1 + (\sinh x)^2]^{3/2}} = \frac{\cosh x}{(\cosh^2 x)^{3/2}} = \frac{1}{\cosh^2 x} = \frac{1}{y^2}$$

60. See page 828.

$$62. K = \frac{|y''|}{[1 + (y')^2]^{3/2}}$$

At the smooth relative extremum $y' = 0$, so $K = |y''|$. Yes, for example, $y = x^4$ has a curvature of 0 at its relative minimum $(0, 0)$. The curvature is positive for any other point of the curve.

$$64. y_1 = ax(b - x), y_2 = \frac{x}{x + 2}$$

We observe that $(0, 0)$ is a solution point to both equations. Therefore, the point P is the origin.

$$y_1 = ax(b - x), \quad y_1' = a(b - 2x), \quad y_1'' = -2a$$

$$y_2 = \frac{x}{x + 2}, \quad y_2' = \frac{2}{(x + 2)^2}, \quad y_2'' = \frac{-4}{(x + 2)^3}$$

At P ,

$$y_1'(0) = ab \text{ and } y_2'(0) = \frac{2}{(0 + 2)^2} = \frac{1}{2}.$$

Since the curves have a common tangent at P , $y_1'(0) = y_2'(0)$ or $ab = \frac{1}{2}$. Therefore, $y_1'(0) = \frac{1}{2}$. Since the curves have the same curvature at P , $K_1(0) = K_2(0)$.

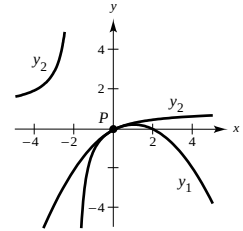
$$K_1(0) = \left| \frac{y_1''(0)}{[1 + (y_1'(0))^2]^{3/2}} \right| = \left| \frac{-2a}{[1 + (1/2)^2]^{3/2}} \right|$$

$$K_2(0) = \left| \frac{y_2''(0)}{[1 + (y_2'(0))^2]^{3/2}} \right| = \left| \frac{-1/2}{[1 + (1/2)^2]^{3/2}} \right|$$

Therefore, $2a = \pm \frac{1}{2}$ or $a = \pm \frac{1}{4}$. In order that the curves intersect at only one point, the parabola must be concave downward. Thus,

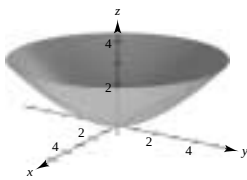
$$a = \frac{1}{4} \quad \text{and} \quad b = \frac{1}{2a} = 2.$$

$$y_1 = \frac{1}{4}x(2 - x) \quad \text{and} \quad y_2 = \frac{x}{x + 2}$$



$$66. y = \frac{1}{4}x^{8/5}, \quad 0 \leq x \leq 5$$

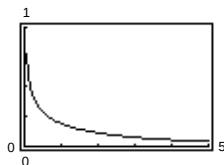
(a)



(rotated about y-axis)

$$(c) y' = \frac{2}{5}x^{3/5}, \quad y'' = \frac{6}{25}x^{-2/5} = \frac{6}{25x^{2/5}}$$

$$K = \frac{\frac{6}{25x^{2/5}}}{\left[1 + \frac{4}{25}x^{6/5}\right]^{3/2}} = \frac{6}{25x^{2/5} \left[1 + \frac{4}{25}x^{6/5}\right]^{3/2}}$$



$$(b) V = \int_0^5 2\pi x \left(\frac{1}{4}x^{8/5} \right) dx \quad (\text{shells})$$

$$= \frac{\pi}{2} \int_0^5 x^{13/5} dx = \frac{\pi}{2} \left[\frac{x^{18/5}}{18/5} \right]_0^5$$

$$= \frac{5\pi}{36} 5^{18/5} \approx 143.25 \text{ cm}^3$$

(d) No, the curvature approaches ∞ as $x \rightarrow 0^+$. Hence, any spherical object will hit the sides of the goblet before touching the bottom $(0, 0)$.

68. $s = \frac{c}{\sqrt{K}}$

$$y = \frac{1}{3}x^3$$

$$y' = x^2$$

$$y'' = 2x$$

$$K = \left| \frac{2x}{(1 + x^4)^{3/2}} \right|$$

When $x = 1$: $K = \frac{1}{\sqrt{2}}$

$$s = \frac{c}{\sqrt{1/\sqrt{2}}} = \sqrt[4]{2}c$$

$$30 = \sqrt[4]{2}c \Rightarrow c = \frac{30}{\sqrt[4]{2}}$$

At $x = \frac{3}{2}$, $K = \frac{3}{[1 + (81/16)]^{3/2}} \approx 0.201$

$$s = \left(\frac{3}{2}\right) = \frac{c}{\sqrt{K}} = \frac{30/\sqrt[4]{2}}{\sqrt{K}} \approx 56.27 \text{ mi/hr.}$$

70. $r(\theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} = f(\theta) \cos \theta \mathbf{i} + f(\theta) \sin \theta \mathbf{j}$

$$x(\theta) = f(\theta) \cos \theta$$

$$y(\theta) = f(\theta) \sin \theta$$

$$x'(\theta) = -f(\theta) \sin \theta + f'(\theta) \cos \theta$$

$$y'(\theta) = f(\theta) \cos \theta + f'(\theta) \sin \theta$$

$$x''(\theta) = -f'(\theta) \sin \theta - f(\theta) \cos \theta - f''(\theta) \sin \theta + f''(\theta) \cos \theta = -f'(\theta) \cos \theta - 2f'(\theta) \sin \theta + f''(\theta) \cos \theta$$

$$y''(\theta) = -f'(\theta) \sin \theta + f'(\theta) \cos \theta + f'(\theta) \cos \theta + f''(\theta) \sin \theta = -f'(\theta) \sin \theta + 2f'(\theta) \cos \theta + f''(\theta) \sin \theta$$

$$K = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}} = \frac{|f^2(\theta) - f(\theta)f''(\theta) + 2(f'(\theta))^2|}{[f^2(\theta) + (f'(\theta))^2]^{3/2}} = \frac{|r^2 - rr'' + 2(r')^2|}{[r^2 + (r')^2]^{3/2}}$$

72. $r = \theta$

$$r' = 1$$

$$r'' = 0$$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \frac{2 + \theta^2}{(1 + \theta^2)^{3/2}}$$

74. $r = e^\theta$

$$r' = e^\theta$$

$$r'' = e^\theta$$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \frac{2e^{2\theta}}{(2e^{2\theta})^{3/2}} = \frac{1}{\sqrt{2}e^\theta}$$

76. At the pole, $r = 0$.

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}}$$

$$= \frac{|2(r')^2|}{|r'|^3} = \frac{2}{|r'|}$$

78. $r = 6 \cos 3\theta$

$$r' = -18 \sin 3\theta$$

At the pole,

$$\theta = \frac{\pi}{6}, r'\left(\frac{\pi}{6}\right) = -18,$$

and

$$K = \frac{2}{|r'(\pi/6)|} = \frac{2}{|-18|} = \frac{1}{9}.$$

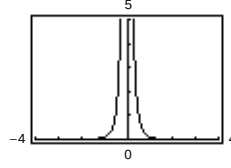
80. $x(t) = t^3$, $x'(t) = 3t^2$, $x''(t) = 6t$

$$y(t) = \frac{1}{2}t^2, \quad y'(t) = t, \quad y''(t) = 1$$

$$K = \frac{|(3t^2)(1) - (t)(6t)|}{[(3t^2)^2 + (t)^2]^{3/2}}$$

$$= \frac{3t^2}{|t^3|(9t^2 + 1)^{3/2}} = \frac{3}{|t|(9t^2 + 1)^{3/2}}$$

$$K \rightarrow 0 \text{ as } t \rightarrow \pm\infty$$



82. (a) $\mathbf{r}(t) = 3t^2\mathbf{i} + (3t - t^3)\mathbf{j}$

$$\mathbf{v}(t) = 6t\mathbf{i} + (3 - 3t^2)\mathbf{j}$$

$$\frac{ds}{dt} = \|\mathbf{v}(t)\| = 3(1 + t^2), \quad \frac{d^2s}{dt^2} = 6t$$

$$K = \frac{2}{3(1 + t^2)^2}$$

$$a_T = \frac{d^2s}{dt^2} = 6t$$

$$a_N = K\left(\frac{ds}{dt}\right)^2 = \frac{2}{3(1 + t^2)^2} \cdot 9(1 + t^2)^2 = 6$$

(b) $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$$

$$\frac{ds}{dt} = \|\mathbf{v}(t)\| = \sqrt{5t^2 + 1}$$

$$\frac{d^2s}{dt^2} = \frac{5t}{\sqrt{5t^2 + 1}}$$

$$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \mathbf{v}(t) \times \mathbf{a}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & t \\ 0 & 2 & 1 \end{vmatrix} = -\mathbf{j} + 2\mathbf{k}$$

$$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\sqrt{5}}{(5t^2 + 1)^{3/2}}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{5t}{\sqrt{5t^2 + 1}}$$

$$a_N = K\left(\frac{ds}{dt}\right)^2 = \frac{\sqrt{5}}{(5t^2 + 1)^{3/2}}(5t^2 + 1) = \frac{\sqrt{5}}{\sqrt{5t^2 + 1}}$$

84. (a) $K = \|\mathbf{T}'(s)\| = \left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \frac{d\mathbf{T}}{dt} \cdot \frac{dt}{ds} \right\|$, by the Chain Rule

$$= \left\| \frac{d\mathbf{T}/dt}{ds/dt} \right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{v}(t)\|} = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

(b) $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{r}'(t)}{ds/dt}$

$$\mathbf{r}'(t) = \frac{ds}{dt}\mathbf{T}(t)$$

$$\mathbf{r}''(t) = \left(\frac{d^2s}{dt^2}\right)\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t)$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)[\mathbf{T}(t) \times \mathbf{T}(t)] + \left(\frac{ds}{dt}\right)^2[\mathbf{T}(t) \times \mathbf{T}'(t)]$$

Since $\mathbf{T}(t) \times \mathbf{T}(t) = \mathbf{0}$ and $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$, we have:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \|\mathbf{r}'(t)\|^2[\mathbf{T}(t) \times \mathbf{T}'(t)]$$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \|\mathbf{r}'(t)\|^2\|\mathbf{T}(t) \times \mathbf{T}'(t)\| = \|\mathbf{r}'(t)\|^2\|\mathbf{T}(t)\|\|\mathbf{T}'(t)\| = \|\mathbf{r}'(t)\|^2(1)K\|\mathbf{r}'(t)\| \quad \text{from (a)}$$

Therefore, $\frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = K$.

(c) $K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^2}}{\|\mathbf{r}'(t)\|} = \frac{\frac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{\|\mathbf{v}(t)\|^2}}{\|\mathbf{v}(t)\|} = \frac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{r}'(t)\|^2}$

$$86. \mathbf{F} = m\mathbf{a} \Rightarrow m\mathbf{a} = \frac{-GmM}{r^3}\mathbf{r}$$

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r}$$

Since $\mathbf{r}$ is a constant multiple of $\mathbf{a}$, they are parallel. Since $\mathbf{a} = \mathbf{r}''$ is parallel to $\mathbf{r}$, $\mathbf{r} \times \mathbf{r}'' = \mathbf{0}$. Also,

$$\left(\frac{d}{dt}\right)(\mathbf{r} \times \mathbf{r}') = \mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'' = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Thus, $\mathbf{r} \times \mathbf{r}'$ is a constant vector which we will denote by $\mathbf{L}$.

$$\begin{aligned} 88. \frac{d}{dt}\left[\frac{\mathbf{r}'}{GM} \times \mathbf{L} - \frac{\mathbf{r}}{r}\right] &= \frac{1}{GM}[\mathbf{r}' \times \mathbf{0} + \mathbf{r}' \times \mathbf{L}] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \\ &= \frac{1}{GM}\left[\mathbf{0} + \left(\frac{-GM\mathbf{r}}{r^3}\right) \times [\mathbf{r} \times \mathbf{r}']\right] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \\ &= -\frac{\mathbf{r}}{r^3} \times [\mathbf{r} \times \mathbf{r}'] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \\ &= \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r} - [\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} = \mathbf{0} \end{aligned}$$

Thus, $\left(\frac{\mathbf{r}'}{GM}\right) \times \mathbf{L} - \left(\frac{\mathbf{r}}{r}\right)$ is a constant vector which we will denote by $\mathbf{e}$.

$$90. \|\mathbf{L}\| = \|\mathbf{r} \times \mathbf{r}'\|$$

Let: $\mathbf{r} = r(\cos \theta \mathbf{i} + \sin \theta \mathbf{j})$

$$\mathbf{r}' = r(-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) \frac{d\theta}{dt} \quad \left(\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\theta} \cdot \frac{d\theta}{dt}\right)$$

$$\begin{aligned} \text{Then: } \mathbf{r} \times \mathbf{r}' &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r \cos \theta & r \sin \theta & 0 \\ -r \sin \theta \frac{d\theta}{dt} & r \cos \theta \frac{d\theta}{dt} & 0 \end{vmatrix} \\ &= r^2 \frac{d\theta}{dt} \mathbf{k} \text{ and } \|\mathbf{L}\| = \|\mathbf{r} \times \mathbf{r}'\| = r^2 \frac{d\theta}{dt}. \end{aligned}$$

92. Let P denote the period. Then

$$A = \int_0^P \frac{dA}{dt} dt = \frac{1}{2} \|\mathbf{L}\| P.$$

Also, the area of an ellipse is πab where $2a$ and $2b$ are the lengths of the major and minor axes.

$$\pi ab = \frac{1}{2} \|\mathbf{L}\| P$$

$$P = \frac{2\pi ab}{\|\mathbf{L}\|}$$

$$P^2 = \frac{4\pi^2 a^2}{\|\mathbf{L}\|^2} (a^2 - c^2) = \frac{4\pi^2 a^2}{\|\mathbf{L}\|^2} a^2 (1 - e^2)$$

$$= \frac{4\pi^2 a^4}{\|\mathbf{L}\|^2} \left(\frac{ed}{a}\right) = \frac{4\pi^2 ed}{\|\mathbf{L}\|^2} a^3$$

$$= \frac{4\pi^2 (\|\mathbf{L}\|^2/GM)}{\|\mathbf{L}\|^2} a^3 = \frac{4\pi^2}{GM} a^3 = Ka^3$$

Review Exercises for Chapter 11

$$2. \mathbf{r}(t) = \sqrt{t} \mathbf{i} + \frac{1}{t-4} \mathbf{j} + \mathbf{k}$$

- (a) Domain: $[0, 4)$ and $(4, \infty)$
 (b) Continuous except at $t = 4$

$$4. \mathbf{r}(t) = (2t+1)\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$$

- (a) Domain: $(-\infty, \infty)$
 (b) Continuous for all t

6. (a) $\mathbf{r}(0) = 3\mathbf{i} + \mathbf{j}$

(b) $\mathbf{r}\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}\mathbf{k}$

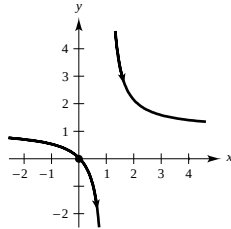
(c) $\mathbf{r}(s - \pi) = 3 \cos(s - \pi)\mathbf{i} + (1 - \sin(s - \pi))\mathbf{j} - (s - \pi)\mathbf{k}$

(d) $\mathbf{r}(\pi + \Delta t) - \mathbf{r}(\pi) = (3 \cos(\pi + \Delta t)\mathbf{i} + (1 - \sin(\pi + \Delta t))\mathbf{j} - (\pi + \Delta t)\mathbf{k}) - (-3\mathbf{i} + \mathbf{j} - \pi\mathbf{k})$
 $= (-3 \cos \Delta t + 3)\mathbf{i} + \sin \Delta t - \Delta t\mathbf{k}$

8. $\mathbf{r}(t) = t\mathbf{i} + \frac{t}{t-1}\mathbf{j}$

$x(t) = t, y(t) = \frac{t}{t-1}$

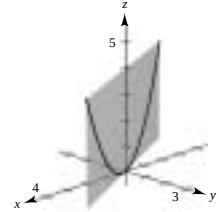
$y = \frac{x}{x-1}$



10. $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$

$x = 2t, y = t, z = t^2$
 $y = \frac{1}{2}x, z = y^2$

t	0	1	-1	2
x	0	2	-2	4
y	0	1	-1	2
z	0	1	1	4

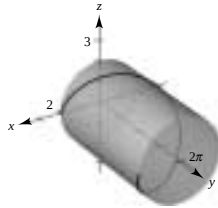


12. $\mathbf{r}(t) = 2 \cos t \mathbf{i} + t \mathbf{j} + 2 \sin t \mathbf{k}$

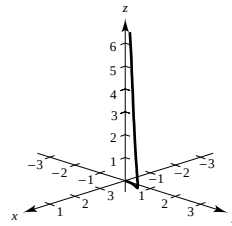
$x = 2 \cos t, y = t, z = 2 \sin t$

$x^2 + z^2 = 4$

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
x	2	0	-2	0
y	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
z	0	2	0	-2



14. $\mathbf{r}(t) = \frac{1}{2}t\mathbf{i} + \sqrt{t}\mathbf{j} + \frac{1}{4}t^3\mathbf{k}$



16. One possible answer is:

$\mathbf{r}_1(t) = 4t\mathbf{i}, \quad 0 \leq t \leq 1$

$\mathbf{r}_2(t) = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2}$

$\mathbf{r}_3(t) = (4 - t)\mathbf{j}, \quad 0 \leq t \leq 4$

18. The x - and y -components are $2 \cos t$ and $2 \sin t$. At

$t = \frac{3\pi}{2},$

the staircase has made $\frac{3}{4}$ of a revolution and is 2 meters high. Thus, one answer is

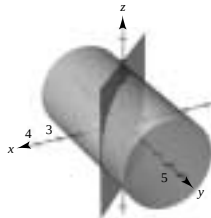
$\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + \frac{4}{3\pi}t\mathbf{k}.$

20. $x^2 + z^2 = 4, x - y = 0, t = x$

$x = t, y = t, z = \pm \sqrt{4 - t^2}$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\mathbf{k}$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} - \sqrt{4 - t^2}\mathbf{k}$



22. $\lim_{t \rightarrow 0} \left(\frac{\sin 2t}{t} \mathbf{i} + e^{-t} \mathbf{j} + e^t \mathbf{k} \right) = \left(\lim_{t \rightarrow 0} \frac{2 \cos 2t}{1} \right) \mathbf{i} + \mathbf{j} + \mathbf{k}$
 $= 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

24. $\mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + t \mathbf{k}$, $\mathbf{u}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + \frac{1}{t} \mathbf{k}$

(a) $\mathbf{r}'(t) = \cos t \mathbf{i} - \sin t \mathbf{j} + \mathbf{k}$

(b) $\mathbf{r}''(t) = -\sin t \mathbf{i} - \cos t \mathbf{j}$

(c) $\mathbf{r}(t) \cdot \mathbf{u}(t) = 2$

(d) $\mathbf{u}(t) - 2\mathbf{r}(t) = -\sin t \mathbf{i} - \cos t \mathbf{j} + \left(\frac{1}{t} - 2t\right) \mathbf{k}$

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 0$

$D_t[\mathbf{u}(t) - 2\mathbf{r}(t)] = -\cos t \mathbf{i} + \sin t \mathbf{j} + \left(-\frac{1}{t^2} - 2\right) \mathbf{k}$

(e) $\|\mathbf{r}(t)\| = \sqrt{1 + t^2}$

$D_t[\|\mathbf{r}(t)\|] = \frac{t}{\sqrt{1 + t^2}}$

(f) $\mathbf{r}(t) \times \mathbf{u}(t) = \left(\frac{1}{t} \cos t - t \cos t\right) \mathbf{i} - \left(\frac{1}{t} \sin t - t \sin t\right) \mathbf{j}$

$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \left(-\frac{1}{t} \sin t - \frac{1}{t^2} \cos t + t \sin t - \cos t\right) \mathbf{i} - \left(\frac{1}{t} \cos t - \frac{1}{t^2} \sin t - t \cos t - \sin t\right) \mathbf{j}$

26. The graph of $\mathbf{u}$ is parallel to the yz -plane.

28. $\int (\ln t \mathbf{i} + t \ln t \mathbf{j} + \mathbf{k}) dt = (t \ln t - t) \mathbf{i} + \frac{t^2}{4}(-1 + 2 \ln t) \mathbf{j} + t \mathbf{k} + \mathbf{C}$

30. $\int (t \mathbf{j} + t^2 \mathbf{k}) \times (\mathbf{i} + t \mathbf{j} + t \mathbf{k}) dt = \int [(t^2 - t^3) \mathbf{i} + t^2 \mathbf{j} - t \mathbf{k}] dt = \left(\frac{t^3}{3} - \frac{t^4}{4}\right) \mathbf{i} + \frac{t^3}{3} \mathbf{j} - \frac{t^2}{2} \mathbf{k} + \mathbf{C}$

32. $\mathbf{r}(t) = \int (\sec t \mathbf{i} + \tan t \mathbf{j} + t^2 \mathbf{k}) dt = \ln|\sec t + \tan t| \mathbf{i} - \ln|\cos t| \mathbf{j} + \frac{t^3}{3} \mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = 3\mathbf{k}$

$\mathbf{r}(t) = \ln|\sec t + \tan t| \mathbf{i} - \ln|\cos t| \mathbf{j} + \left(\frac{t^3}{3} + 3\right) \mathbf{k}$

34. $\int_0^1 (\sqrt{t} \mathbf{j} + t \sin t \mathbf{k}) dt = \left[\frac{2}{3} t^{3/2} \mathbf{j} + (\sin t - t \cos t) \mathbf{k}\right]_0^1 = \frac{2}{3} \mathbf{j} + (\sin 1 - \cos 1) \mathbf{k}$

36. $\int_{-1}^1 (t^3 \mathbf{i} - \arcsin t \mathbf{j} - t^2 \mathbf{k}) dt = \left[\frac{t^4}{4} \mathbf{i} - (t \arcsin t + \sqrt{1 - t^2}) \mathbf{j} - \frac{t^3}{3} \mathbf{k}\right]_{-1}^1$
 $= -\frac{2}{3} \mathbf{k}$

38. $\mathbf{r}(t) = \langle t, -\tan t, e^t \rangle$

$\mathbf{r}'(t) = \mathbf{v}(t) = \langle 1, -\sec^2 t, e^t \rangle$

$\|\mathbf{v}(t)\| = \sqrt{1 + \sec^4 t + e^{2t}}$

$\mathbf{r}''(t) = \mathbf{a}(t) = \langle 0, -2 \sec^2 t \cdot \tan t, e^t \rangle$

40. $\mathbf{r}(t) = \langle 3 \cosh t, \sinh t, -2t \rangle$, $t_0 = 0$

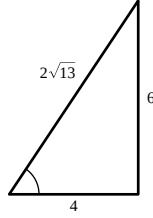
$\mathbf{r}'(t) = \langle 3 \sinh t, \cosh t, -2 \rangle$

$\mathbf{r}'(0) = \langle 0, 1, -2 \rangle$ direction numbers

Since $\mathbf{r}(0) = \langle 3, 0, 0 \rangle$, the parametric equations are $x = 3$, $y = t$, $z = -2t$.

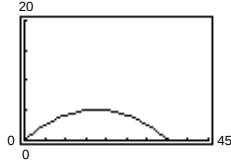
$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(0.1) \approx \langle 3, 0.1, -0.2 \rangle$

$$\begin{aligned}
 42. \text{ Range} &= 4 = \frac{v_0^2}{16} \sin \theta \cos \theta \\
 &= \frac{v_0^2}{16} \cdot \frac{6}{2\sqrt{13}} \cdot \frac{4}{2\sqrt{13}} = \frac{3v_0^2}{104} \\
 \frac{416}{3} &= v_0^2 \Rightarrow v_0 \approx 11.776 \text{ ft/sec}
 \end{aligned}$$



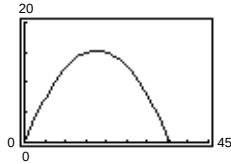
$$44. \mathbf{r}(t) = [(v_0 \cos \theta)t]\mathbf{i} + [(v_0 \sin \theta)t - \tfrac{1}{2}(9.8)t^2]\mathbf{j}$$

$$(a) \mathbf{r}(t) = [(20 \cos 30^\circ)t]\mathbf{i} + [(20 \sin 30^\circ)t - 4.9t^2]\mathbf{j}$$



Maximum height ≈ 5.1 m; Range ≈ 35.3 m

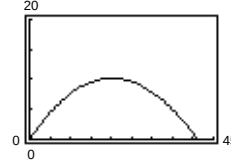
$$(c) \mathbf{r}(t) = [(20 \cos 60^\circ)t]\mathbf{i} + [(20 \sin 60^\circ)t - 4.9t^2]\mathbf{j}$$



Maximum height ≈ 15.3 m; Range ≈ 35.3 m

(Note that 45° gives the longest range)

$$(b) \mathbf{r}(t) = [(20 \cos 45^\circ)t]\mathbf{i} + [(20 \sin 45^\circ)t - 4.9t^2]\mathbf{j}$$



Maximum height ≈ 10.2 m; Range ≈ 40.8 m

$$46. \mathbf{r}(t) = (1 + 4t)\mathbf{i} + (2 - 3t)\mathbf{j}$$

$$\mathbf{v}(t) = 4\mathbf{i} - 3\mathbf{j}$$

$$\|\mathbf{v}\| = 5$$

$$\mathbf{a}(t) = \mathbf{0}$$

$$\mathbf{T}(t) = \frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$$

$\mathbf{N}(t)$ does not exist

$$\mathbf{a} \cdot \mathbf{T} = 0$$

$\mathbf{a} \cdot \mathbf{N}$ does not exist

$$48. \mathbf{r}(t) = 2(t + 1)\mathbf{i} + \frac{2}{t + 1}\mathbf{j}$$

$$\mathbf{v}(t) = 2\mathbf{i} - \frac{2}{(t + 1)^2}\mathbf{j}$$

$$\|\mathbf{v}(t)\| = \frac{2\sqrt{(t + 1)^4 + 1}}{(t + 1)^2}$$

$$\mathbf{a}(t) = \frac{4}{(t + 1)^3}\mathbf{j}$$

$$\mathbf{T}(t) = \frac{(t + 1)^2\mathbf{i} - \mathbf{j}}{\sqrt{(t + 1)^4 + 1}}$$

$$\mathbf{N}(t) = \frac{\mathbf{i} + (t + 1)^2\mathbf{j}}{\sqrt{(t + 1)^4 + 1}}$$

$$\mathbf{a} \cdot \mathbf{T} = \frac{-4}{(t + 1)^3\sqrt{(t + 1)^4 + 1}}$$

$$\mathbf{a} \cdot \mathbf{N} = \frac{4(t + 1)^2}{(t + 1)^3\sqrt{(t + 1)^4 + 1}}$$

$$= \frac{4}{(t + 1)\sqrt{(t + 1)^4 + 1}}$$

50. $\mathbf{r}(t) = t \cos t \mathbf{i} + t \sin t \mathbf{j}$

$$\mathbf{v}(t) = \mathbf{r}'(t) = (-t \sin t + \cos t) \mathbf{i} + (t \cos t + \sin t) \mathbf{j}$$

$$\|\mathbf{v}(t)\| = \text{speed} = \sqrt{(-t \sin t + \cos t)^2 + (t \cos t + \sin t)^2} = \sqrt{t^2 + 1}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = (-t \cos t - 2 \sin t) \mathbf{i} + (-t \sin t + 2 \cos t) \mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{(-t \sin t + \cos t) \mathbf{i} + (t \cos t + \sin t) \mathbf{j}}{\sqrt{t^2 + 1}}$$

$$\mathbf{N}(t) = \frac{-(t \cos t + \sin t) \mathbf{i} + (-t \sin t + \cos t) \mathbf{j}}{\sqrt{t^2 + 1}}$$

$$\mathbf{a}(t) \cdot \mathbf{T}(t) = \frac{t}{\sqrt{t^2 + 1}}$$

$$\mathbf{a}(t) \cdot \mathbf{N}(t) = \frac{t^2 + 2}{\sqrt{t^2 + 1}}$$

52. $\mathbf{r}(t) = (t - 1) \mathbf{i} + t \mathbf{j} + \frac{1}{t} \mathbf{k}$

$$\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - \frac{1}{t^2} \mathbf{k}$$

$$\|\mathbf{v}(t)\| = \frac{\sqrt{2t^4 + 1}}{t^2}$$

$$\mathbf{a}(t) = \frac{2}{t^3} \mathbf{k}$$

$$\mathbf{T}(t) = \frac{t^2 \mathbf{i} + t^2 \mathbf{j} - \mathbf{k}}{\sqrt{2t^4 + 1}}$$

$$\mathbf{N}(t) = \frac{\mathbf{i} + \mathbf{j} + 2t^2 \mathbf{k}}{\sqrt{2} \sqrt{2t^4 + 1}}$$

$$\mathbf{a} \cdot \mathbf{T} = \frac{-2}{t^3 \sqrt{2t^4 + 1}}$$

$$\mathbf{a} \cdot \mathbf{N} = \frac{4}{t \sqrt{2} \sqrt{2t^4 + 1}}$$

54. $\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + \frac{2}{3} t^3 \mathbf{k}$, $x = t$, $y = t^2$, $z = \frac{2}{3} t^3$

When $t = 2$, $x = 2$, $y = 4$, $z = \frac{16}{3}$.

$$\mathbf{r}'(t) = \mathbf{i} + 2t \mathbf{j} + 2t^2 \mathbf{k}$$

Direction numbers when $t = 2$, $a = 1$, $b = 4$, $c = 8$

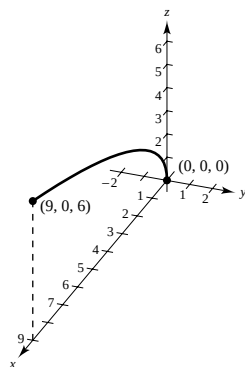
$$x = t + 2, y = 4t + 4, z = 8t + \frac{16}{3}$$

56. Factor of 4

58. $\mathbf{r}(t) = t^2 \mathbf{i} + 2t \mathbf{k}$, $0 \leq t \leq 3$

$$\mathbf{r}'(t) = 2t \mathbf{i} + 2 \mathbf{k}$$

$$\begin{aligned} s &= \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^3 \sqrt{4t^2 + 4} dt \\ &= \left[\ln \left| \sqrt{t^2 + 1} + t \right| + t \sqrt{t^2 + 1} \right]_0^3 \\ &= \ln(\sqrt{10} + 3) + 3\sqrt{10} \approx 11.3053 \end{aligned}$$

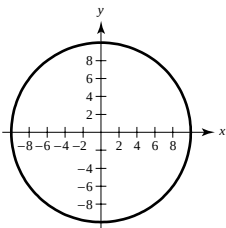


60. $\mathbf{r}(t) = 10 \cos t \mathbf{i} + 10 \sin t \mathbf{j}$

$$\mathbf{r}'(t) = -10 \sin t \mathbf{i} + 10 \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 10$$

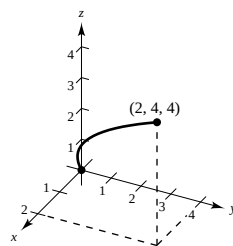
$$s = \int_0^{2\pi} 10 \, dt = 20\pi$$



62. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}, 0 \leq t \leq 2$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}, \|\mathbf{r}'(t)\| = \sqrt{5 + 4t^2}$$

$$\begin{aligned} s &= \int_a^b \|\mathbf{r}'(t)\| \, dt = \int_0^2 \sqrt{5 + 4t^2} \, dt \\ &= \sqrt{21} + \frac{5}{4} \ln 5 - \frac{5}{4} \ln(\sqrt{105} - 4\sqrt{5}) \approx 6.2638 \end{aligned}$$



64. $\mathbf{r}(t) = \langle 2(\sin t - t \cos t), 2(\cos t + t \sin t), t \rangle, 0 \leq t \leq \frac{\pi}{2}$

$$\mathbf{r}'(t) = \langle 2t \sin t, 2t \cos t, 1 \rangle, \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 1}$$

$$\begin{aligned} s &= \int_a^b \|\mathbf{r}'(t)\| \, dt = \int_0^{\pi/2} \sqrt{4t^2 + 1} \, dt \\ &= \sqrt{17} - \frac{1}{4} \ln(\sqrt{17} - 4) \approx 4.6468 \end{aligned}$$

66. $\mathbf{r}(t) = e^t \sin t \mathbf{i} + e^t \cos t \mathbf{k}, 0 \leq t \leq \pi$

$$\mathbf{r}'(t) = (e^t \cos t + e^t \sin t) \mathbf{i} + (-e^t \sin t + e^t \cos t) \mathbf{k}$$

$$\begin{aligned} \|\mathbf{r}'(t)\| &= \sqrt{(e^t \cos t + e^t \sin t)^2 + (-e^t \sin t + e^t \cos t)^2} \\ &= \sqrt{2}e^t \end{aligned}$$

$$\begin{aligned} s &= \int_0^\pi \|\mathbf{r}'(t)\| \, dt \\ &= \sqrt{2} \int_0^\pi e^t \, dt = \left[\sqrt{2}e^t \right]_0^\pi = \sqrt{2}(e^\pi - 1) \end{aligned}$$

68. $\mathbf{r}(t) = 2\sqrt{t}\mathbf{i} + 3t\mathbf{j}$

$$\mathbf{r}'(t) = \frac{1}{\sqrt{t}}\mathbf{i} + 3\mathbf{j}, \|\mathbf{r}'(t)\| = \sqrt{\frac{1}{t} + 9} = \sqrt{\frac{1 + 9t}{t}}$$

$$\mathbf{r}''(t) = -\frac{1}{2}t^{-3/2}\mathbf{i}$$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{t}} & 3 & 0 \\ -\frac{1}{2}t^{-3/2} & 0 & 0 \end{vmatrix} = \frac{3}{2}t^{-3/2}\mathbf{k}; \|\mathbf{r}' \times \mathbf{r}''\| = \frac{3}{2t^{3/2}}$$

$$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{3/2t^{3/2}}{(1 + 9t)^{3/2}/t^{3/2}} = \frac{3}{2(1 + 9t)^{3/2}}$$

70. $\mathbf{r}(t) = 2t\mathbf{i} + 5 \cos t \mathbf{j} + 5 \sin t \mathbf{k}$

$$\mathbf{r}'(t) = 2\mathbf{i} - 5 \sin t \mathbf{j} + 5 \cos t \mathbf{k}, \|\mathbf{r}'(t)\| = \sqrt{29}$$

$$\mathbf{r}''(t) = 5 \cos t \mathbf{j} - 5 \sin t \mathbf{k}$$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5 \sin t & 5 \cos t \\ 0 & -5 \cos t & -5 \sin t \end{vmatrix} = 25\mathbf{i} + 10 \sin t \mathbf{j} - 10 \cos t \mathbf{k}$$

$$\|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{725}$$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{725}}{(29)^{3/2}} = \frac{\sqrt{25 \cdot 29}}{29\sqrt{29}} = \frac{5}{29}$$

85. Let $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $r = \|\mathbf{r}\| = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}$ and $\mathbf{r}' = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$. Then,

$$\begin{aligned} r\left(\frac{d\mathbf{r}}{dt}\right) &= \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2} \left[\frac{1}{2} \{ [x(t)]^2 + [y(t)]^2 + [z(t)]^2 \}^{-1/2} \cdot (2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t)) \right] \\ &= x(t)x'(t) + y(t)y'(t) + z(t)z'(t) = \mathbf{r} \cdot \mathbf{r}'. \end{aligned}$$

87. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where x , y , and z are functions of t , and $r = \|\mathbf{r}\|$.

$$\begin{aligned} \frac{d}{dt} \left[\frac{\mathbf{r}}{r} \right] &= \frac{r\mathbf{r}' - \mathbf{r}(dr/dt)}{r^2} = \frac{r\mathbf{r}' - \mathbf{r}[(\mathbf{r} \cdot \mathbf{r}')/r]}{r^2} = \frac{r^2\mathbf{r}' - (\mathbf{r} \cdot \mathbf{r}')\mathbf{r}}{r^3} \quad (\text{using Exercise 77}) \\ &= \frac{(x^2 + y^2 + z^2)(x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}) - (xx' + yy' + zz')(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{r^3} \\ &= \frac{1}{r^3} [(x'y^2 + x'z^2 - xyy' - xzz')\mathbf{i} + (x^2y' + z^2y' - xx'y - zz'y)\mathbf{j} + (x^2z' + y^2z' - xx'z - yy'z)\mathbf{k}] \\ &= \frac{1}{r^3} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ yz' - y'z & -(xz' - x'z) & xy' - x'y \\ x & y & z \end{vmatrix} = \frac{1}{r^3} \{ [\mathbf{r} \times \mathbf{r}'] \times \mathbf{r} \} \end{aligned}$$

89. From Exercise 86, we have concluded that planetary motion is planar. Assume that the planet moves in the xy -plane with the sun at the origin. From Exercise 88, we have

$$\mathbf{r}' \times \mathbf{L} = GM \left(\frac{\mathbf{r}}{r} + \mathbf{e} \right).$$

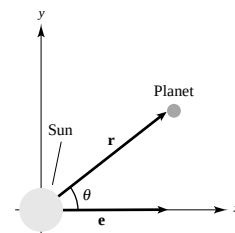
Since $\mathbf{r}' \times \mathbf{L}$ and $\mathbf{r}$ are both perpendicular to $\mathbf{L}$, so is $\mathbf{e}$. Thus, $\mathbf{e}$ lies in the xy -plane. Situate the coordinate system so that $\mathbf{e}$ lies along the positive x -axis and θ is the angle between $\mathbf{e}$ and $\mathbf{r}$. Let $e = \|\mathbf{e}\|$. Then $\mathbf{r} \cdot \mathbf{e} = \|\mathbf{r}\| \|\mathbf{e}\| \cos \theta = re \cos \theta$. Also,

$$\begin{aligned} \|\mathbf{L}\|^2 &= \mathbf{L} \cdot \mathbf{L} = (\mathbf{r} \times \mathbf{r}') \cdot \mathbf{L} \\ &= \mathbf{r} \cdot (\mathbf{r}' \times \mathbf{L}) = \mathbf{r} \cdot \left[GM \left(\mathbf{e} + \frac{\mathbf{r}}{r} \right) \right] = GM \left[\mathbf{r} \cdot \mathbf{e} + \frac{\mathbf{r} \cdot \mathbf{r}}{r} \right] = GM[re \cos \theta + r] \end{aligned}$$

Thus,

$$\frac{\|\mathbf{L}\|^2/GM}{1 + e \cos \theta} = r$$

and the planetary motion is a conic section. Since the planet returns to its initial position periodically, the conic is an ellipse.



91. $A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$

Thus,

$$\frac{dA}{dt} = \frac{dA}{d\theta} \frac{d\theta}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} \|\mathbf{L}\|$$

and $\mathbf{r}$ sweeps out area at a constant rate.

Review Exercises for Chapter 11

1. $\mathbf{r}(t) = t\mathbf{i} + \csc t\mathbf{k}$

(a) Domain: $t \neq n\pi$, n an integer

(b) Continuous except at $t = n\pi$, n an integer

3. $\mathbf{r}(t) = \ln t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$

(a) Domain: $(0, \infty)$

(b) Continuous for all $t > 0$

5. (a) $\mathbf{r}(0) = \mathbf{i}$

(b) $\mathbf{r}(-2) = -3\mathbf{i} + 4\mathbf{j} + \frac{8}{3}\mathbf{k}$

(c) $\mathbf{r}(c-1) = (2(c-1)+1)\mathbf{i} + (c-1)^2\mathbf{j} - \frac{1}{3}(c-1)^3\mathbf{k}$
 $= (2c-1)\mathbf{i} + (c-1)^2\mathbf{j} - \frac{1}{3}(c-1)^3\mathbf{k}$

(d) $\mathbf{r}(1+\Delta t) - \mathbf{r}(1) = ([2(1+\Delta t)+1]\mathbf{i} + [1+\Delta t]^2\mathbf{j} - \frac{1}{3}[1+\Delta t]^3\mathbf{k}) - (3\mathbf{i} + \mathbf{j} - \frac{1}{3}\mathbf{k})$
 $= 2\Delta t\mathbf{i} + \Delta t(\Delta t+2)\mathbf{j} - \frac{1}{3}(\Delta t^3 + 3\Delta t^2 + 3\Delta t)\mathbf{k}$

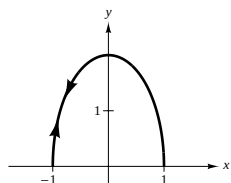
7. $\mathbf{r}(t) = \cos t\mathbf{i} + 2\sin^2 t\mathbf{j}$

$x(t) = \cos t, y(t) = 2\sin^2 t$

$x^2 + \frac{y}{2} = 1$

$y = 2(1 - x^2)$

$-1 \leq x \leq 1$

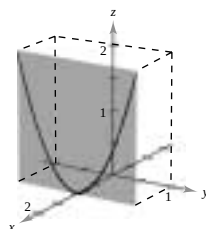


9. $\mathbf{r}(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$

$x = 1$

$y = t$

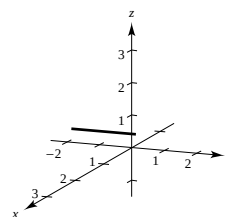
$z = t^2 \Rightarrow z = y^2$



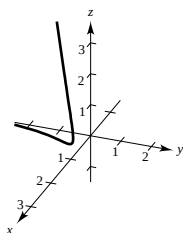
11. $\mathbf{r}(t) = \mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$

$x = 1, y = \sin t, z = 1$

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
x	1	1	1	1
y	0	1	0	-1
z	1	1	1	1



13. $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$



15. One possible answer is:

$\mathbf{r}_1(t) = 4t\mathbf{i} + 3t\mathbf{j}, \quad 0 \leq t \leq 1$

$\mathbf{r}_2(t) = 4\mathbf{i} + (3-t)\mathbf{j}, \quad 0 \leq t \leq 3$

$\mathbf{r}_3(t) = (4-t)\mathbf{i}, \quad 0 \leq t \leq 4$

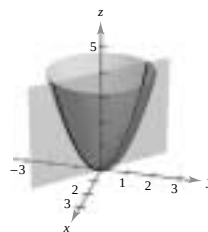
17. The vector joining the points is $\langle 7, 4, -10 \rangle$. One path is

$\mathbf{r}(t) = \langle -2 + 7t, -3 + 4t, 8 - 10t \rangle$

19. $z = x^2 + y^2, x + y = 0, t = x$

$x = t, y = -t, z = 2t^2$

$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$



21. $\lim_{t \rightarrow 2^-} (t^2\mathbf{i} + \sqrt{4-t^2}\mathbf{j} + \mathbf{k}) = 4\mathbf{i} + \mathbf{k}$

$$23. \mathbf{r}(t) = 3t\mathbf{i} + (t-1)\mathbf{j}, \mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$$

$$(a) \mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$$

$$(b) \mathbf{r}''(t) = \mathbf{0}$$

$$(c) \mathbf{r}(t) \cdot \mathbf{u}(t) = 3t^2 + t^2(t-1) = t^3 + 2t^2$$

$$(d) \mathbf{u}(t) - 2\mathbf{r}(t) = -5t\mathbf{i} + (t^2 - 2t + 2)\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$$

$$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 3t^2 + 4t$$

$$D_t[\mathbf{u}(t) - 2\mathbf{r}(t)] = -5\mathbf{i} + (2t - 2)\mathbf{j} + 2t^2\mathbf{k}$$

$$(e) \|\mathbf{r}(t)\| = \sqrt{10t^2 - 2t + 1}$$

$$(f) \mathbf{r}(t) \times \mathbf{u}(t) = \frac{2}{3}(t^4 - t^3)\mathbf{i} - 2t^4\mathbf{j} + (3t^3 - t^2 + t)\mathbf{k}$$

$$D_t[\|\mathbf{r}(t)\|] = \frac{10t - 1}{\sqrt{10t^2 - 2t + 1}}$$

$$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \left(\frac{8}{3}t^3 - 2t^2\right)\mathbf{i} - 8t^3\mathbf{j} + (9t^2 - 2t + 1)\mathbf{k}$$

$$25. x(t) \text{ and } y(t) \text{ are increasing functions at } t = t_0, \text{ and } z(t) \text{ is a decreasing function at } t = t_0.$$

$$27. \int (\cos t\mathbf{i} + t \cos t\mathbf{j}) dt = \sin t\mathbf{i} + (t \sin t + \cos t)\mathbf{j} + \mathbf{C}$$

$$29. \int \|\cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}\| dt = \int \sqrt{1 + t^2} dt = \frac{1}{2}[t\sqrt{1 + t^2} + \ln|t + \sqrt{1 + t^2}|] + \mathbf{C}$$

$$31. \mathbf{r}(t) = \int (2t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}) dt = t^2\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} + \mathbf{C}$$

$$33. \int_{-2}^2 (3t\mathbf{i} + 2t^2\mathbf{j} - t^3\mathbf{k}) dt = \left[\frac{3t^2}{2}\mathbf{i} + \frac{2t^3}{3}\mathbf{j} - \frac{t^4}{4}\mathbf{k}\right]_{-2}^2 = \frac{32}{3}\mathbf{j}$$

$$\mathbf{r}(0) = \mathbf{j} - \mathbf{k} + \mathbf{C} = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$$

$$\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (e^t + 2)\mathbf{j} - (e^{-t} + 4)\mathbf{k}$$

$$35. \int_0^2 (e^{t/2}\mathbf{i} - 3t^2\mathbf{j} - \mathbf{k}) dt = \left[2e^{t/2}\mathbf{i} - t^3\mathbf{j} - t\mathbf{k}\right]_0^2 = (2e - 2)\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$$

$$37. \mathbf{r}(t) = \langle \cos^3 t, \sin^3 t, 3t \rangle$$

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t, 3 \rangle$$

$$\|\mathbf{v}(t)\| = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t + 9}$$

$$= 3\sqrt{\cos^2 t \sin^2 t (\cos^2 t + \sin^2 t) + 1}$$

$$= 3\sqrt{\cos^2 t \sin^2 t + 1}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle -6\cos t(-\sin^2 t) + (-3\cos^2 t)\cos t, 6\sin t \cos^2 t + 3\sin^2 t(-\sin t), 0 \rangle$$

$$= \langle 3\cos t(2\sin^2 t - \cos^2 t), 3\sin t(2\cos^2 t - \sin^2 t), 0 \rangle$$

$$39. \mathbf{r}(t) = \left\langle \ln(t-3), t^2, \frac{1}{2}t \right\rangle, t_0 = 4$$

$$41. \text{Range} = x = \frac{v_0^2}{32} \sin 2\theta = \frac{(75)^2}{32} \sin 60^\circ \approx 152 \text{ feet}$$

$$\mathbf{r}'(t) = \left\langle \frac{1}{t-3}, 2t, \frac{1}{2} \right\rangle$$

$$\mathbf{r}'(4) = \left\langle 1, 8, \frac{1}{2} \right\rangle \text{ direction numbers}$$

Since $\mathbf{r}(4) = \langle 0, 16, 2 \rangle$, the parametric equations are

$$x = t, y = 16 + 8t, z = 2 + \frac{1}{2}t.$$

$$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(4.1) \approx \langle 0.1, 16.8, 2.05 \rangle$$

$$43. \text{Range} = x = \frac{v_0^2}{9.8} \sin 2\theta = 80 \Rightarrow v_0 = \sqrt{\frac{(80)(9.8)}{\sin 40^\circ}} \approx 34.9 \text{ m/sec}$$

45. $\mathbf{r}(t) = 5t\mathbf{i}$

$\mathbf{v}(t) = 5\mathbf{i}$

$\|\mathbf{v}(t)\| = 5$

$\mathbf{a}(t) = \mathbf{0}$

$\mathbf{T}(t) = \mathbf{i}$

$\mathbf{N}(t)$ does not exist

$\mathbf{a} \cdot \mathbf{T} = 0$

$\mathbf{a} \cdot \mathbf{N}$ does not exist

(The curve is a line.)

47. $\mathbf{r}(t) = t\mathbf{i} + \sqrt{t}\mathbf{j}$

$\mathbf{v}(t) = \mathbf{i} + \frac{1}{2\sqrt{t}}\mathbf{j}$

$\|\mathbf{v}(t)\| = \frac{\sqrt{4t+1}}{2\sqrt{t}}$

$\mathbf{a}(t) = -\frac{1}{4t\sqrt{t}}\mathbf{j}$

$\mathbf{T}(t) = \frac{\mathbf{i} + (1/2\sqrt{t})\mathbf{j}}{(\sqrt{4t+1})/2\sqrt{t}} = \frac{2\sqrt{t}\mathbf{i} + \mathbf{j}}{\sqrt{4t+1}}$

$\mathbf{N}(t) = \frac{\mathbf{i} - 2\sqrt{t}\mathbf{j}}{\sqrt{4t+1}}$

$\mathbf{a} \cdot \mathbf{T} = \frac{-1}{4t\sqrt{t}\sqrt{4t+1}}$

$\mathbf{a} \cdot \mathbf{N} = \frac{1}{2t\sqrt{4t+1}}$

49. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$

$\mathbf{v}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j}$

$\|\mathbf{v}(t)\| = \sqrt{e^{2t} + e^{-2t}}$

$\mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$

$\mathbf{T}(t) = \frac{e^t\mathbf{i} - e^{-t}\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}}$

$\mathbf{N}(t) = \frac{e^{-t}\mathbf{i} + e^t\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}}$

$\mathbf{a} \cdot \mathbf{T} = \frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}$

$\mathbf{a} \cdot \mathbf{N} = \frac{2}{\sqrt{e^{2t} + e^{-2t}}}$

51. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$\|\mathbf{v}\| = \sqrt{1 + 5t^2}$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

$\mathbf{T}(t) = \frac{\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}}{\sqrt{1 + 5t^2}}$

$\mathbf{N}(t) = \frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{5}\sqrt{1 + 5t^2}}$

$\mathbf{a} \cdot \mathbf{T} = \frac{5t}{\sqrt{1 + 5t^2}}$

$\mathbf{a} \cdot \mathbf{N} = \frac{5}{\sqrt{5}\sqrt{1 + 5t^2}} = \frac{\sqrt{5}}{\sqrt{1 + 5t^2}}$

53. $\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + t \mathbf{k}, x = 2 \cos t, y = 2 \sin t, z = t$

When $t = \frac{3\pi}{4}$, $x = -\sqrt{2}$, $y = \sqrt{2}$, $z = \frac{3\pi}{4}$.

$\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + \mathbf{k}$

Direction numbers when $t = \frac{3\pi}{4}$, $a = -\sqrt{2}$, $b = -\sqrt{2}$, $c = 1$

$x = -\sqrt{2}t - \sqrt{2}, y = -\sqrt{2}t + \sqrt{2}, z = t + \frac{3\pi}{4}$

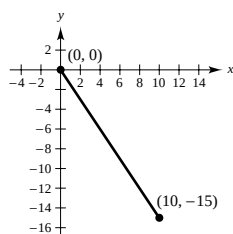
55. $v = \sqrt{\frac{9.56 \times 10^4}{4600}} \approx 4.56 \text{ mi/sec}$

57. $\mathbf{r}(t) = 2t\mathbf{i} - 3t\mathbf{j}, 0 \leq t \leq 5$

$\mathbf{r}'(t) = 2\mathbf{i} - 3\mathbf{j}$

$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^5 \sqrt{4 + 9} dt$

$= \sqrt{13}t \Big|_0^5 = 5\sqrt{13}$



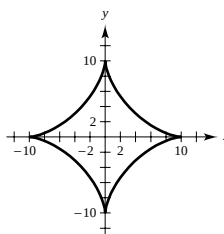
59. $\mathbf{r}(t) = 10 \cos^3 t \mathbf{i} + 10 \sin^3 t \mathbf{j}$

$$\mathbf{r}'(t) = -30 \cos^2 t \sin t \mathbf{i} + 30 \sin^2 t \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 30 \sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t}$$

$$= 30 |\cos t \sin t|$$

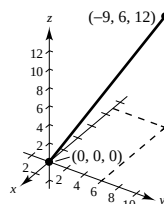
$$s = 4 \int_0^{\pi/2} 30 \cos t \cdot \sin t \, dt = \left[120 \frac{\sin^2 t}{2} \right]_0^{\pi/2} = 60$$



61. $\mathbf{r}(t) = -3t\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k}, 0 \leq t \leq 3$

$$\mathbf{r}'(t) = -3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

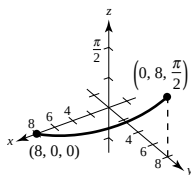
$$s = \int_a^b \|\mathbf{r}'(t)\| \, dt = \int_0^3 \sqrt{9 + 4 + 16} \, dt = \int_0^3 \sqrt{29} \, dt = 3\sqrt{29}$$



63. $\mathbf{r}(t) = \langle 8 \cos t, 8 \sin t, t \rangle, 0 \leq t \leq \frac{\pi}{2}$

$$\mathbf{r}'(t) = \langle -8 \sin t, 8 \cos t, 1 \rangle, \|\mathbf{r}'(t)\| = \sqrt{65}$$

$$s = \int_a^b \|\mathbf{r}'(t)\| \, dt = \int_0^{\pi/2} \sqrt{65} \, dt = \frac{\pi\sqrt{65}}{2}$$



65. $\mathbf{r}(t) = \frac{1}{2}t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k}, 0 \leq t \leq \pi$

$$\mathbf{r}'(t) = \frac{1}{2}\mathbf{i} + \cos t\mathbf{j} - \sin t\mathbf{k}$$

$$\begin{aligned} s &= \int_0^{\pi} \|\mathbf{r}'(t)\| \, dt \\ &= \int_0^{\pi} \sqrt{\frac{1}{4} + \cos^2 t + \sin^2 t} \, dt \\ &= \frac{\sqrt{5}}{2} \int_0^{\pi} dt = \left[\frac{\sqrt{5}}{2} t \right]_0^{\pi} = \frac{\sqrt{5}}{2} \pi \end{aligned}$$

67. $\mathbf{r}(t) = 3t\mathbf{i} + 2t\mathbf{j}$

Line

$$k = 0$$

69. $\mathbf{r}(t) = 2t\mathbf{i} + \frac{1}{2}t^2\mathbf{j} + t^2\mathbf{k}$

$$\mathbf{r}'(t) = 2\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}, \|\mathbf{r}'\| = \sqrt{5t^2 + 4}$$

$$\mathbf{r}''(t) = \mathbf{j} + 2\mathbf{k}$$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & t & 2t \\ 0 & 1 & 2 \end{vmatrix} = -4\mathbf{j} + 2\mathbf{k}, \|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{20}$$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{20}}{(5t^2 + 4)^{3/2}} = \frac{2\sqrt{5}}{(4 + 5t^2)^{3/2}}$$

71. $y = \frac{1}{2}x^2 + 2$

$$y' = x$$

$$y'' = 1$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{1}{(1 + x^2)^{3/2}}$$

$$\text{At } x = 4, K = \frac{1}{17^{3/2}} \text{ and } r = 17^{3/2} = 17\sqrt{17}.$$

73. $y = \ln x$

$$y' = \frac{1}{x}, y'' = -\frac{1}{x^2}$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{1/x^2}{[1 + (1/x)^2]^{3/2}}$$

$$\text{At } x = 1, K = \frac{1}{2^{3/2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \text{ and } r = 2\sqrt{2}.$$

75. The curvature changes abruptly from zero to a nonzero constant at the points B and C.

Problem Solving for Chapter 11

$$1. \quad x(t) = \int_0^t \cos\left(\frac{\pi u^2}{2}\right) du, y(t) = \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du$$

$$x'(t) = \cos\left(\frac{\pi t^2}{2}\right), y'(t) = \sin\left(\frac{\pi t^2}{2}\right)$$

$$(a) \quad s = \int_0^a \sqrt{x'(t)^2 + y'(t)^2} dt = \int_0^a dt = a$$

$$(b) \quad x''(t) = -\pi t \sin\left(\frac{\pi t^2}{2}\right), y''(t) = \pi t \cos\left(\frac{\pi t^2}{2}\right)$$

$$K = \frac{\left| \pi t \cos^2\left(\frac{\pi t^2}{2}\right) + \pi t \sin^2\left(\frac{\pi t^2}{2}\right) \right|}{1} = \pi t$$

$$\text{At } t = a, K = \pi a.$$

$$(c) \quad K = \pi a = \pi(\text{length})$$

$$3. \text{ Bomb: } \mathbf{r}_1(t) = \langle 5000 - 400t, 3200 - 16t^2 \rangle$$

$$\text{Projectile: } \mathbf{r}_2(t) = \langle (v_0 \cos \theta)t, (v_0 \sin \theta)t - 16t^2 \rangle$$

At 1600 feet: Bomb:

$$3200 - 16t^2 = 1600 \Rightarrow t = 10 \text{ seconds.}$$

Projectile will travel 5 seconds:

$$5(v_0 \sin \theta) - 16(25) = 1600$$

$$v_0 \sin \theta = 400.$$

Horizontal position:

$$\text{At } t = 10, \text{ bomb is at } 5000 - 400(10) = 1000.$$

$$\text{At } t = 5, \text{ projectile is at } 5v_0 \cos \theta.$$

$$\text{Thus, } v_0 \cos \theta = 200.$$

$$\text{Combining, } \frac{v_0 \sin \theta}{v_0 \cos \theta} = \frac{400}{200} \Rightarrow \tan \theta = 2 \Rightarrow \theta \approx 63.4^\circ.$$

$$v_0 = \frac{200}{\cos \theta} \approx 447.2 \text{ ft/sec}$$

$$5. \quad x'(\theta) = 1 - \cos \theta, y'(\theta) = \sin \theta, 0 \leq \theta \leq 2\pi$$

$$\sqrt{x'(\theta)^2 + y'(\theta)^2} = \sqrt{(1 - \cos \theta)^2 + \sin^2 \theta}$$

$$= \sqrt{2 - 2 \cos \theta} = \sqrt{4 \sin^2 \frac{\theta}{2}}$$

$$s(t) = \int_{\pi}^t 2 \sin \frac{\theta}{2} d\theta = \left[-4 \cos \frac{\theta}{2} \right]_{\pi}^t = -4 \cos \frac{t}{2}$$

$$x''(\theta) = \sin \theta, y''(\theta) = \cos \theta$$

$$K = \frac{1|(1 - \cos \theta)\cos \theta - \sin \theta \sin \theta|}{\left(2 \sin \frac{\theta}{2}\right)^3} = \frac{|1 \cos \theta - 1|}{8 \sin^3 \frac{\theta}{2}}$$

$$= \frac{1}{4 \sin \frac{\theta}{2}}$$

$$\text{Thus, } \rho = \frac{1}{K} = 4 \sin \frac{t}{2} \text{ and}$$

$$s^2 + \rho^2 = 16 \cos^2\left(\frac{t}{2}\right) + 16 \sin^2\left(\frac{t}{2}\right) = 16.$$

$$7. \quad \|\mathbf{r}^2(t)\| = \mathbf{r}(t) \cdot \mathbf{r}(t)$$

$$\frac{d}{dt}(\|\mathbf{r}(t)\|)^2 = 2\|\mathbf{r}(t)\| \frac{d}{dt}\|\mathbf{r}(t)\|$$

$$= \mathbf{r}(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}(t) \Rightarrow \frac{d}{dt}\|\mathbf{r}(t)\| = \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\|}$$

9. $\mathbf{r}(t) = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j} + 3t \mathbf{k}, t = \frac{\pi}{2}$

$$\mathbf{r}'(t) = -4 \sin t \mathbf{i} + 4 \cos t \mathbf{j} + 3 \mathbf{k}, \|\mathbf{r}'(t)\| = 5$$

$$\mathbf{r}''(t) = -4 \cos t \mathbf{i} - 4 \sin t \mathbf{j}$$

$$\mathbf{T} = -\frac{4}{5} \sin t \mathbf{i} + \frac{4}{5} \cos t \mathbf{j} + \frac{3}{5} \mathbf{k}$$

$$\mathbf{T}' = -\frac{4}{5} \cos t \mathbf{i} - \frac{4}{5} \sin t \mathbf{j}$$

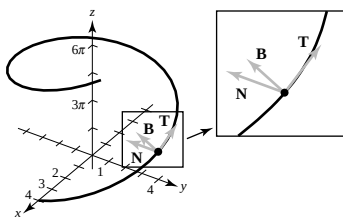
$$\mathbf{N} = -\cos t \mathbf{i} - \sin t \mathbf{j}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{3}{5} \sin t \mathbf{i} - \frac{3}{5} \cos t \mathbf{j} + \frac{4}{5} \mathbf{k}$$

$$\text{At } t = \frac{\pi}{2}, \mathbf{T}\left(\frac{\pi}{2}\right) = -\frac{4}{5} \mathbf{i} + \frac{3}{5} \mathbf{k}$$

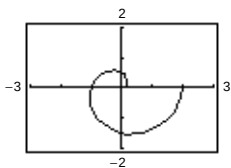
$$\mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{j}$$

$$\mathbf{B}\left(\frac{\pi}{2}\right) = \frac{3}{5} \mathbf{i} + \frac{4}{5} \mathbf{k}$$



13. $\mathbf{r}(t) = \langle t \cos \pi t, t \sin \pi t \rangle, 0 \leq t \leq 2$

(a)



(c) $K = \frac{\pi(\pi^2 t^2 + 2)}{[\pi^2 t^2 + 1]^{3/2}}$

$$K(0) = 2\pi$$

$$K(1) = \frac{\pi(\pi^2 + 2)}{(\pi^2 + 1)^{3/2}} \approx 1.04$$

$$K(2) \approx 0.51$$

(e) $\lim_{t \rightarrow \infty} K = 0$

11. (a) $\|\mathbf{B}\| = \|\mathbf{T} \times \mathbf{N}\| = 1$ constant length $\Rightarrow \frac{d\mathbf{B}}{ds} \perp \mathbf{B}$

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = (\mathbf{T} \times \mathbf{N}') + (\mathbf{T}' \times \mathbf{N})$$

$$\begin{aligned} \mathbf{T} \cdot \frac{d\mathbf{B}}{ds} &= \mathbf{T} \cdot (\mathbf{T} \times \mathbf{N}') + \mathbf{T} \cdot (\mathbf{T}' \times \mathbf{N}) \\ &= (\mathbf{T} \times \mathbf{T}) \cdot \mathbf{N}' + \mathbf{T} \cdot \left(\mathbf{T}' \times \frac{\mathbf{T}'}{\|\mathbf{T}'\|} \right) = 0 \end{aligned}$$

$$\text{Hence, } \frac{d\mathbf{B}}{ds} \perp \mathbf{B} \text{ and } \frac{d\mathbf{B}}{ds} \perp \mathbf{T} \Rightarrow \frac{d\mathbf{B}}{ds} = -\tau \mathbf{N}$$

for some scalar τ .

(b) $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Using Exercise 10.3, number 64,

$$\begin{aligned} \mathbf{B} \times \mathbf{N} &= (\mathbf{T} \times \mathbf{N}) \times \mathbf{N} = -\mathbf{N} \times (\mathbf{T} \times \mathbf{N}) \\ &= -[(\mathbf{N} \cdot \mathbf{N})\mathbf{T} - (\mathbf{N} \cdot \mathbf{T})\mathbf{N}] \\ &= -\mathbf{T} \end{aligned}$$

$$\begin{aligned} \mathbf{B} \times \mathbf{T} &= (\mathbf{T} \times \mathbf{N}) \times \mathbf{T} = -\mathbf{T} \times (\mathbf{T} \times \mathbf{N}) \\ &= -[(\mathbf{T} \cdot \mathbf{N})\mathbf{T} - (\mathbf{T} \cdot \mathbf{T})\mathbf{N}] \\ &= \mathbf{N}. \end{aligned}$$

$$\text{Now, } K\mathbf{N} = \left\| \frac{d\mathbf{T}}{ds} \right\| \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} = \mathbf{T}'(s) = \frac{d\mathbf{T}}{ds}.$$

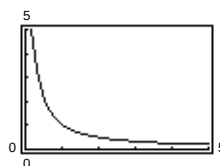
Finally,

$$\begin{aligned} \mathbf{N}'(s) &= \frac{d}{ds}(\mathbf{B} \times \mathbf{T}) = (\mathbf{B} \times \mathbf{T}') + (\mathbf{B}' \times \mathbf{T}) \\ &= (\mathbf{B} \times K\mathbf{N}) + (-\tau \mathbf{N} \times \mathbf{T}) \\ &= -K\mathbf{T} + \tau \mathbf{B}. \end{aligned}$$

(b) $\text{Length} = \int_0^2 \|\mathbf{r}'(t)\| dt$

$$= \int_0^2 \sqrt{\pi^2 t^2 + 1} dt \approx 6.766 \quad (\text{graphing utility})$$

(d)



(f) As $t \rightarrow \infty$, the graph spirals outward and the curvature decreases.

$$86. \mathbf{F} = m\mathbf{a} \Rightarrow m\mathbf{a} = \frac{-GmM}{r^3}\mathbf{r}$$

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r}$$

Since $\mathbf{r}$ is a constant multiple of $\mathbf{a}$, they are parallel. Since $\mathbf{a} = \mathbf{r}''$ is parallel to $\mathbf{r}$, $\mathbf{r} \times \mathbf{r}'' = \mathbf{0}$. Also,

$$\left(\frac{d}{dt}\right)(\mathbf{r} \times \mathbf{r}') = \mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'' = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Thus, $\mathbf{r} \times \mathbf{r}'$ is a constant vector which we will denote by $\mathbf{L}$.

$$\begin{aligned} 88. \frac{d}{dt}\left[\frac{\mathbf{r}'}{GM} \times \mathbf{L} - \frac{\mathbf{r}}{r}\right] &= \frac{1}{GM}[\mathbf{r}' \times \mathbf{0} + \mathbf{r}' \times \mathbf{L}] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \\ &= \frac{1}{GM}\left[\mathbf{0} + \left(\frac{-GM\mathbf{r}}{r^3}\right) \times [\mathbf{r} \times \mathbf{r}']\right] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \\ &= -\frac{\mathbf{r}}{r^3} \times [\mathbf{r} \times \mathbf{r}'] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} \\ &= \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r} - [\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} = \mathbf{0} \end{aligned}$$

Thus, $\left(\frac{\mathbf{r}'}{GM}\right) \times \mathbf{L} - \left(\frac{\mathbf{r}}{r}\right)$ is a constant vector which we will denote by $\mathbf{e}$.

$$90. \|\mathbf{L}\| = \|\mathbf{r} \times \mathbf{r}'\|$$

Let: $\mathbf{r} = r(\cos \theta \mathbf{i} + \sin \theta \mathbf{j})$

$$\mathbf{r}' = r(-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) \frac{d\theta}{dt} \quad \left(\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\theta} \cdot \frac{d\theta}{dt}\right)$$

$$\begin{aligned} \text{Then: } \mathbf{r} \times \mathbf{r}' &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r \cos \theta & r \sin \theta & 0 \\ -r \sin \theta \frac{d\theta}{dt} & r \cos \theta \frac{d\theta}{dt} & 0 \end{vmatrix} \\ &= r^2 \frac{d\theta}{dt} \mathbf{k} \text{ and } \|\mathbf{L}\| = \|\mathbf{r} \times \mathbf{r}'\| = r^2 \frac{d\theta}{dt}. \end{aligned}$$

92. Let P denote the period. Then

$$A = \int_0^P \frac{dA}{dt} dt = \frac{1}{2} \|\mathbf{L}\| P.$$

Also, the area of an ellipse is πab where $2a$ and $2b$ are the lengths of the major and minor axes.

$$\pi ab = \frac{1}{2} \|\mathbf{L}\| P$$

$$P = \frac{2\pi ab}{\|\mathbf{L}\|}$$

$$P^2 = \frac{4\pi^2 a^2}{\|\mathbf{L}\|^2} (a^2 - c^2) = \frac{4\pi^2 a^2}{\|\mathbf{L}\|^2} a^2 (1 - e^2)$$

$$= \frac{4\pi^2 a^4}{\|\mathbf{L}\|^2} \left(\frac{ed}{a}\right) = \frac{4\pi^2 ed}{\|\mathbf{L}\|^2} a^3$$

$$= \frac{4\pi^2 (\|\mathbf{L}\|^2/GM)}{\|\mathbf{L}\|^2} a^3 = \frac{4\pi^2}{GM} a^3 = Ka^3$$

Review Exercises for Chapter 11

$$2. \mathbf{r}(t) = \sqrt{t} \mathbf{i} + \frac{1}{t-4} \mathbf{j} + \mathbf{k}$$

- (a) Domain: $[0, 4)$ and $(4, \infty)$
 (b) Continuous except at $t = 4$

$$4. \mathbf{r}(t) = (2t+1)\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$$

- (a) Domain: $(-\infty, \infty)$
 (b) Continuous for all t

6. (a) $\mathbf{r}(0) = 3\mathbf{i} + \mathbf{j}$

(b) $\mathbf{r}\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}\mathbf{k}$

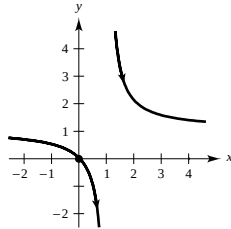
(c) $\mathbf{r}(s - \pi) = 3 \cos(s - \pi)\mathbf{i} + (1 - \sin(s - \pi))\mathbf{j} - (s - \pi)\mathbf{k}$

(d) $\mathbf{r}(\pi + \Delta t) - \mathbf{r}(\pi) = (3 \cos(\pi + \Delta t)\mathbf{i} + (1 - \sin(\pi + \Delta t))\mathbf{j} - (\pi + \Delta t)\mathbf{k}) - (-3\mathbf{i} + \mathbf{j} - \pi\mathbf{k})$
 $= (-3 \cos \Delta t + 3)\mathbf{i} + \sin \Delta t - \Delta t\mathbf{k}$

8. $\mathbf{r}(t) = t\mathbf{i} + \frac{t}{t-1}\mathbf{j}$

$x(t) = t, y(t) = \frac{t}{t-1}$

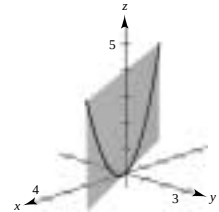
$y = \frac{x}{x-1}$



10. $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$

$x = 2t, y = t, z = t^2$
 $y = \frac{1}{2}x, z = y^2$

t	0	1	-1	2
x	0	2	-2	4
y	0	1	-1	2
z	0	1	1	4

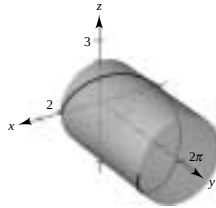


12. $\mathbf{r}(t) = 2 \cos t \mathbf{i} + t \mathbf{j} + 2 \sin t \mathbf{k}$

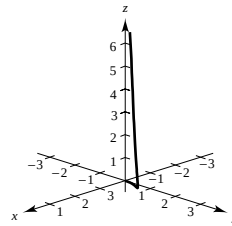
$x = 2 \cos t, y = t, z = 2 \sin t$

$x^2 + z^2 = 4$

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
x	2	0	-2	0
y	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
z	0	2	0	-2



14. $\mathbf{r}(t) = \frac{1}{2}t\mathbf{i} + \sqrt{t}\mathbf{j} + \frac{1}{4}t^3\mathbf{k}$



16. One possible answer is:

$\mathbf{r}_1(t) = 4t\mathbf{i}, \quad 0 \leq t \leq 1$

$\mathbf{r}_2(t) = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2}$

$\mathbf{r}_3(t) = (4 - t)\mathbf{j}, \quad 0 \leq t \leq 4$

18. The x - and y -components are $2 \cos t$ and $2 \sin t$. At

$t = \frac{3\pi}{2},$

the staircase has made $\frac{3}{4}$ of a revolution and is 2 meters high. Thus, one answer is

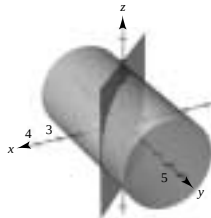
$\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + \frac{4}{3\pi}t\mathbf{k}.$

20. $x^2 + z^2 = 4, x - y = 0, t = x$

$x = t, y = t, z = \pm \sqrt{4 - t^2}$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\mathbf{k}$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} - \sqrt{4 - t^2}\mathbf{k}$



22. $\lim_{t \rightarrow 0} \left(\frac{\sin 2t}{t} \mathbf{i} + e^{-t} \mathbf{j} + e^t \mathbf{k} \right) = \left(\lim_{t \rightarrow 0} \frac{2 \cos 2t}{1} \right) \mathbf{i} + \mathbf{j} + \mathbf{k}$
 $= 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

24. $\mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + t \mathbf{k}$, $\mathbf{u}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + \frac{1}{t} \mathbf{k}$

(a) $\mathbf{r}'(t) = \cos t \mathbf{i} - \sin t \mathbf{j} + \mathbf{k}$

(b) $\mathbf{r}''(t) = -\sin t \mathbf{i} - \cos t \mathbf{j}$

(c) $\mathbf{r}(t) \cdot \mathbf{u}(t) = 2$

(d) $\mathbf{u}(t) - 2\mathbf{r}(t) = -\sin t \mathbf{i} - \cos t \mathbf{j} + \left(\frac{1}{t} - 2t\right) \mathbf{k}$

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 0$

$D_t[\mathbf{u}(t) - 2\mathbf{r}(t)] = -\cos t \mathbf{i} + \sin t \mathbf{j} + \left(-\frac{1}{t^2} - 2\right) \mathbf{k}$

(e) $\|\mathbf{r}(t)\| = \sqrt{1 + t^2}$

$D_t[\|\mathbf{r}(t)\|] = \frac{t}{\sqrt{1 + t^2}}$

(f) $\mathbf{r}(t) \times \mathbf{u}(t) = \left(\frac{1}{t} \cos t - t \cos t\right) \mathbf{i} - \left(\frac{1}{t} \sin t - t \sin t\right) \mathbf{j}$

$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \left(-\frac{1}{t} \sin t - \frac{1}{t^2} \cos t + t \sin t - \cos t\right) \mathbf{i} - \left(\frac{1}{t} \cos t - \frac{1}{t^2} \sin t - t \cos t - \sin t\right) \mathbf{j}$

26. The graph of $\mathbf{u}$ is parallel to the yz -plane.

28. $\int (\ln t \mathbf{i} + t \ln t \mathbf{j} + \mathbf{k}) dt = (t \ln t - t) \mathbf{i} + \frac{t^2}{4}(-1 + 2 \ln t) \mathbf{j} + t \mathbf{k} + \mathbf{C}$

30. $\int (t \mathbf{j} + t^2 \mathbf{k}) \times (\mathbf{i} + t \mathbf{j} + t \mathbf{k}) dt = \int [(t^2 - t^3) \mathbf{i} + t^2 \mathbf{j} - t \mathbf{k}] dt = \left(\frac{t^3}{3} - \frac{t^4}{4}\right) \mathbf{i} + \frac{t^3}{3} \mathbf{j} - \frac{t^2}{2} \mathbf{k} + \mathbf{C}$

32. $\mathbf{r}(t) = \int (\sec t \mathbf{i} + \tan t \mathbf{j} + t^2 \mathbf{k}) dt = \ln|\sec t + \tan t| \mathbf{i} - \ln|\cos t| \mathbf{j} + \frac{t^3}{3} \mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = 3\mathbf{k}$

$\mathbf{r}(t) = \ln|\sec t + \tan t| \mathbf{i} - \ln|\cos t| \mathbf{j} + \left(\frac{t^3}{3} + 3\right) \mathbf{k}$

34. $\int_0^1 (\sqrt{t} \mathbf{j} + t \sin t \mathbf{k}) dt = \left[\frac{2}{3} t^{3/2} \mathbf{j} + (\sin t - t \cos t) \mathbf{k}\right]_0^1 = \frac{2}{3} \mathbf{j} + (\sin 1 - \cos 1) \mathbf{k}$

36. $\int_{-1}^1 (t^3 \mathbf{i} - \arcsin t \mathbf{j} - t^2 \mathbf{k}) dt = \left[\frac{t^4}{4} \mathbf{i} - (t \arcsin t + \sqrt{1 - t^2}) \mathbf{j} - \frac{t^3}{3} \mathbf{k}\right]_{-1}^1$
 $= -\frac{2}{3} \mathbf{k}$

38. $\mathbf{r}(t) = \langle t, -\tan t, e^t \rangle$

$\mathbf{r}'(t) = \mathbf{v}(t) = \langle 1, -\sec^2 t, e^t \rangle$

$\|\mathbf{v}(t)\| = \sqrt{1 + \sec^4 t + e^{2t}}$

$\mathbf{r}''(t) = \mathbf{a}(t) = \langle 0, -2 \sec^2 t \cdot \tan t, e^t \rangle$

40. $\mathbf{r}(t) = \langle 3 \cosh t, \sinh t, -2t \rangle$, $t_0 = 0$

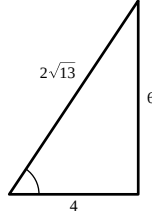
$\mathbf{r}'(t) = \langle 3 \sinh t, \cosh t, -2 \rangle$

$\mathbf{r}'(0) = \langle 0, 1, -2 \rangle$ direction numbers

Since $\mathbf{r}(0) = \langle 3, 0, 0 \rangle$, the parametric equations are $x = 3$, $y = t$, $z = -2t$.

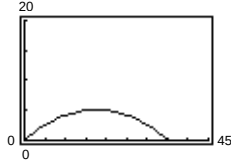
$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(0.1) \approx \langle 3, 0.1, -0.2 \rangle$

$$\begin{aligned}
 42. \text{ Range} &= 4 = \frac{v_0^2}{16} \sin \theta \cos \theta \\
 &= \frac{v_0^2}{16} \cdot \frac{6}{2\sqrt{13}} \cdot \frac{4}{2\sqrt{13}} = \frac{3v_0^2}{104} \\
 \frac{416}{3} &= v_0^2 \Rightarrow v_0 \approx 11.776 \text{ ft/sec}
 \end{aligned}$$



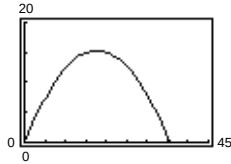
$$44. \mathbf{r}(t) = [(v_0 \cos \theta)t]\mathbf{i} + [(v_0 \sin \theta)t - \tfrac{1}{2}(9.8)t^2]\mathbf{j}$$

$$(a) \mathbf{r}(t) = [(20 \cos 30^\circ)t]\mathbf{i} + [(20 \sin 30^\circ)t - 4.9t^2]\mathbf{j}$$



Maximum height ≈ 5.1 m; Range ≈ 35.3 m

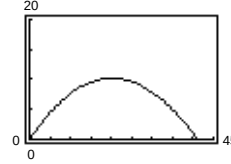
$$(c) \mathbf{r}(t) = [(20 \cos 60^\circ)t]\mathbf{i} + [(20 \sin 60^\circ)t - 4.9t^2]\mathbf{j}$$



Maximum height ≈ 15.3 m; Range ≈ 35.3 m

(Note that 45° gives the longest range)

$$(b) \mathbf{r}(t) = [(20 \cos 45^\circ)t]\mathbf{i} + [(20 \sin 45^\circ)t - 4.9t^2]\mathbf{j}$$



Maximum height ≈ 10.2 m; Range ≈ 40.8 m

$$46. \mathbf{r}(t) = (1 + 4t)\mathbf{i} + (2 - 3t)\mathbf{j}$$

$$\mathbf{v}(t) = 4\mathbf{i} - 3\mathbf{j}$$

$$\|\mathbf{v}\| = 5$$

$$\mathbf{a}(t) = \mathbf{0}$$

$$\mathbf{T}(t) = \frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$$

$\mathbf{N}(t)$ does not exist

$$\mathbf{a} \cdot \mathbf{T} = 0$$

$\mathbf{a} \cdot \mathbf{N}$ does not exist

$$48. \mathbf{r}(t) = 2(t + 1)\mathbf{i} + \frac{2}{t + 1}\mathbf{j}$$

$$\mathbf{v}(t) = 2\mathbf{i} - \frac{2}{(t + 1)^2}\mathbf{j}$$

$$\|\mathbf{v}(t)\| = \frac{2\sqrt{(t + 1)^4 + 1}}{(t + 1)^2}$$

$$\mathbf{a}(t) = \frac{4}{(t + 1)^3}\mathbf{j}$$

$$\mathbf{T}(t) = \frac{(t + 1)^2\mathbf{i} - \mathbf{j}}{\sqrt{(t + 1)^4 + 1}}$$

$$\mathbf{N}(t) = \frac{\mathbf{i} + (t + 1)^2\mathbf{j}}{\sqrt{(t + 1)^4 + 1}}$$

$$\mathbf{a} \cdot \mathbf{T} = \frac{-4}{(t + 1)^3\sqrt{(t + 1)^4 + 1}}$$

$$\mathbf{a} \cdot \mathbf{N} = \frac{4(t + 1)^2}{(t + 1)^3\sqrt{(t + 1)^4 + 1}}$$

$$= \frac{4}{(t + 1)\sqrt{(t + 1)^4 + 1}}$$

50. $\mathbf{r}(t) = t \cos t \mathbf{i} + t \sin t \mathbf{j}$

$$\mathbf{v}(t) = \mathbf{r}'(t) = (-t \sin t + \cos t) \mathbf{i} + (t \cos t + \sin t) \mathbf{j}$$

$$\|\mathbf{v}(t)\| = \text{speed} = \sqrt{(-t \sin t + \cos t)^2 + (t \cos t + \sin t)^2} = \sqrt{t^2 + 1}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = (-t \cos t - 2 \sin t) \mathbf{i} + (-t \sin t + 2 \cos t) \mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{(-t \sin t + \cos t) \mathbf{i} + (t \cos t + \sin t) \mathbf{j}}{\sqrt{t^2 + 1}}$$

$$\mathbf{N}(t) = \frac{-(t \cos t + \sin t) \mathbf{i} + (-t \sin t + \cos t) \mathbf{j}}{\sqrt{t^2 + 1}}$$

$$\mathbf{a}(t) \cdot \mathbf{T}(t) = \frac{t}{\sqrt{t^2 + 1}}$$

$$\mathbf{a}(t) \cdot \mathbf{N}(t) = \frac{t^2 + 2}{\sqrt{t^2 + 1}}$$

52. $\mathbf{r}(t) = (t - 1) \mathbf{i} + t \mathbf{j} + \frac{1}{t} \mathbf{k}$

$$\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - \frac{1}{t^2} \mathbf{k}$$

$$\|\mathbf{v}(t)\| = \frac{\sqrt{2t^4 + 1}}{t^2}$$

$$\mathbf{a}(t) = \frac{2}{t^3} \mathbf{k}$$

$$\mathbf{T}(t) = \frac{t^2 \mathbf{i} + t^2 \mathbf{j} - \mathbf{k}}{\sqrt{2t^4 + 1}}$$

$$\mathbf{N}(t) = \frac{\mathbf{i} + \mathbf{j} + 2t^2 \mathbf{k}}{\sqrt{2} \sqrt{2t^4 + 1}}$$

$$\mathbf{a} \cdot \mathbf{T} = \frac{-2}{t^3 \sqrt{2t^4 + 1}}$$

$$\mathbf{a} \cdot \mathbf{N} = \frac{4}{t \sqrt{2} \sqrt{2t^4 + 1}}$$

54. $\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + \frac{2}{3} t^3 \mathbf{k}$, $x = t$, $y = t^2$, $z = \frac{2}{3} t^3$

When $t = 2$, $x = 2$, $y = 4$, $z = \frac{16}{3}$.

$$\mathbf{r}'(t) = \mathbf{i} + 2t \mathbf{j} + 2t^2 \mathbf{k}$$

Direction numbers when $t = 2$, $a = 1$, $b = 4$, $c = 8$

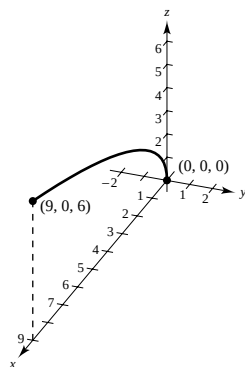
$$x = t + 2, y = 4t + 4, z = 8t + \frac{16}{3}$$

56. Factor of 4

58. $\mathbf{r}(t) = t^2 \mathbf{i} + 2t \mathbf{k}$, $0 \leq t \leq 3$

$$\mathbf{r}'(t) = 2t \mathbf{i} + 2 \mathbf{k}$$

$$\begin{aligned} s &= \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^3 \sqrt{4t^2 + 4} dt \\ &= \left[\ln \left| \sqrt{t^2 + 1} + t \right| + t \sqrt{t^2 + 1} \right]_0^3 \\ &= \ln(\sqrt{10} + 3) + 3\sqrt{10} \approx 11.3053 \end{aligned}$$

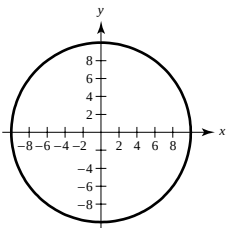


60. $\mathbf{r}(t) = 10 \cos t \mathbf{i} + 10 \sin t \mathbf{j}$

$$\mathbf{r}'(t) = -10 \sin t \mathbf{i} + 10 \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 10$$

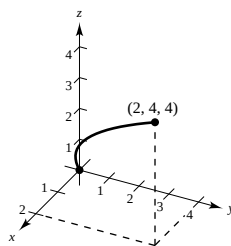
$$s = \int_0^{2\pi} 10 \, dt = 20\pi$$



62. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}, 0 \leq t \leq 2$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}, \|\mathbf{r}'(t)\| = \sqrt{5 + 4t^2}$$

$$\begin{aligned} s &= \int_a^b \|\mathbf{r}'(t)\| \, dt = \int_0^2 \sqrt{5 + 4t^2} \, dt \\ &= \sqrt{21} + \frac{5}{4} \ln 5 - \frac{5}{4} \ln(\sqrt{105} - 4\sqrt{5}) \approx 6.2638 \end{aligned}$$



64. $\mathbf{r}(t) = \langle 2(\sin t - t \cos t), 2(\cos t + t \sin t), t \rangle, 0 \leq t \leq \frac{\pi}{2}$

$$\mathbf{r}'(t) = \langle 2t \sin t, 2t \cos t, 1 \rangle, \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 1}$$

$$\begin{aligned} s &= \int_a^b \|\mathbf{r}'(t)\| \, dt = \int_0^{\pi/2} \sqrt{4t^2 + 1} \, dt \\ &= \sqrt{17} - \frac{1}{4} \ln(\sqrt{17} - 4) \approx 4.6468 \end{aligned}$$

66. $\mathbf{r}(t) = e^t \sin t \mathbf{i} + e^t \cos t \mathbf{k}, 0 \leq t \leq \pi$

$$\mathbf{r}'(t) = (e^t \cos t + e^t \sin t) \mathbf{i} + (-e^t \sin t + e^t \cos t) \mathbf{k}$$

$$\begin{aligned} \|\mathbf{r}'(t)\| &= \sqrt{(e^t \cos t + e^t \sin t)^2 + (-e^t \sin t + e^t \cos t)^2} \\ &= \sqrt{2}e^t \end{aligned}$$

$$\begin{aligned} s &= \int_0^\pi \|\mathbf{r}'(t)\| \, dt \\ &= \sqrt{2} \int_0^\pi e^t \, dt = \left[\sqrt{2}e^t \right]_0^\pi = \sqrt{2}(e^\pi - 1) \end{aligned}$$

68. $\mathbf{r}(t) = 2\sqrt{t}\mathbf{i} + 3t\mathbf{j}$

$$\mathbf{r}'(t) = \frac{1}{\sqrt{t}}\mathbf{i} + 3\mathbf{j}, \|\mathbf{r}'(t)\| = \sqrt{\frac{1}{t} + 9} = \sqrt{\frac{1 + 9t}{t}}$$

$$\mathbf{r}''(t) = -\frac{1}{2}t^{-3/2}\mathbf{i}$$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{t}} & 3 & 0 \\ -\frac{1}{2}t^{-3/2} & 0 & 0 \end{vmatrix} = \frac{3}{2}t^{-3/2}\mathbf{k}; \|\mathbf{r}' \times \mathbf{r}''\| = \frac{3}{2t^{3/2}}$$

$$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{3/2t^{3/2}}{(1 + 9t)^{3/2}/t^{3/2}} = \frac{3}{2(1 + 9t)^{3/2}}$$

70. $\mathbf{r}(t) = 2t\mathbf{i} + 5 \cos t \mathbf{j} + 5 \sin t \mathbf{k}$

$$\mathbf{r}'(t) = 2\mathbf{i} - 5 \sin t \mathbf{j} + 5 \cos t \mathbf{k}, \|\mathbf{r}'(t)\| = \sqrt{29}$$

$$\mathbf{r}''(t) = 5 \cos t \mathbf{j} - 5 \sin t \mathbf{k}$$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5 \sin t & 5 \cos t \\ 0 & -5 \cos t & -5 \sin t \end{vmatrix} = 25\mathbf{i} + 10 \sin t \mathbf{j} - 10 \cos t \mathbf{k}$$

$$\|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{725}$$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{725}}{(29)^{3/2}} = \frac{\sqrt{25 \cdot 29}}{29\sqrt{29}} = \frac{5}{29}$$

72. $y = e^{-x/2}$

$$y' = -\frac{1}{2}e^{-x/2}, y'' = \frac{1}{4}e^{-x/2}$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{\frac{1}{4}e^{-x/2}}{\left[1 + \frac{1}{4}e^{-x}\right]^{3/2}}$$

$$\text{At } x = 0, K = \frac{1/4}{(5/4)^{3/2}} = \frac{2}{5^{3/2}} = \frac{2}{5\sqrt{5}} = \frac{2\sqrt{5}}{25}, r = \frac{5\sqrt{5}}{2}.$$

74. $y = \tan x$

$$y' = \sec^2 x$$

$$y'' = 2 \sec^2 x \tan x$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|2 \sec^2 x \tan x|}{[1 + \sec^4 x]^{3/2}}$$

$$\text{At } x = \frac{\pi}{4}, K = \frac{4}{5^{3/2}} = \frac{4}{5\sqrt{5}} = \frac{4\sqrt{5}}{25} \text{ and } r = \frac{5\sqrt{5}}{4}.$$

Problem Solving for Chapter 11

2. $x^{2/3} + y^{2/3} = a^{2/3}$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = \frac{-y^{1/3}}{x^{1/3}} \text{ Slope at } P(x, y).$$

$$\mathbf{r}(t) = \cos^3 t \mathbf{i} + \sin^3 t \mathbf{j}$$

$$\mathbf{r}'(t) = -3 \cos^2 t \sin t \mathbf{i} + 3 \sin^2 t \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\mathbf{i}\| = |3 \cos t \sin t|$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\cos t \mathbf{i} + \sin t \mathbf{j}$$

$$\mathbf{T}'(t) = \sin t \mathbf{i} + \cos t \mathbf{j}$$

$$Q(0, 0, 0) \text{ origin}$$

$$P = (\cos^3 t, \sin^3 t, 0) \text{ on curve.}$$

$$\begin{aligned} \overrightarrow{PQ} \times \mathbf{T} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos^3 t & \sin^3 t & 0 \\ -\cos t & \sin t & 0 \end{vmatrix} \\ &= (\cos^3 t \sin t - \sin^3 t \cos t) \mathbf{k} \end{aligned}$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{T}\|}{\|\mathbf{T}\|} = |3 \cos t \sin t|$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{1}{|3 \cos t \sin t|}$$

Thus, the radius of curvature, $\frac{1}{K}$, is three times the distance from the origin to the tangent line.

4. Bomb: $\mathbf{r}_1(t) = \langle 5000 + 400t, 3200 - 16t^2 \rangle$

$$\text{Projectile: } \mathbf{r}_2(t) = \langle (v_0 \cos \theta)t, (v_0 \sin \theta)t - 16t^2 \rangle$$

At 1600 feet: Bomb:

$$3200 - 16t^2 = 1600 \Rightarrow t = 10$$

Projectile will travel 5 seconds:

$$5(v_0 \sin \theta) - 16(25) = 1600$$

$$v_0 \sin \theta = 400.$$

Horizontal position:

$$\text{At } t = 10, \text{ bomb is at } 5000 + 400(10) = 9000.$$

$$\text{At } t = 5, \text{ projectile is at } (v_0 \cos \theta)5.$$

Thus,

$$5v_0 \cos \theta = 9000$$

$$v_0 \cos \theta = 1800.$$

$$\text{Combining, } \frac{v_0 \sin \theta}{v_0 \cos \theta} = \frac{400}{1800} \Rightarrow \tan \theta = \frac{2}{9} \Rightarrow \theta \approx 12.5^\circ.$$

$$v_0 = \frac{1800}{\cos \theta} \approx 1843.9 \text{ ft/sec}$$

$$6. \quad r = 1 - \cos \theta$$

$$r' = \sin \theta$$

$$s(t) = \int_{\pi}^t \sqrt{(1 - \cos \theta)^2 + \sin^2 \theta} d\theta = \int_{\pi}^t \sqrt{2 - 2 \cos \theta} d\theta$$

$$= \int_{\pi}^t 2 \sin \frac{\theta}{2} d\theta = \left[-4 \cos \frac{\theta}{2} \right]_{\pi}^t = -4 \cos \frac{t}{2}$$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}}$$

$$= \frac{|2 \sin^2 \theta - (1 - \cos \theta)(\cos \theta) + (1 - \cos \theta)^2|}{8 \sin^3 \frac{\theta}{2}}$$

$$= \frac{|3 - 3 \cos \theta|}{8 \sin^3 \frac{\theta}{2}}$$

$$= \frac{3 \frac{\sin^2 \theta}{2}}{4 \frac{\sin^3 \theta}{2}} = \frac{3}{4 \sin \frac{\theta}{2}}$$

$$\rho = \frac{1}{K} = \frac{4 \sin \frac{\theta}{2}}{3}$$

$$s^2 + 9\rho^2 = 16 \cos^2 \frac{\theta}{2} + 16 \sin^2 \frac{\theta}{2} = 16$$

$$8. (a) \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} \text{ position vector}$$

$$\mathbf{r} = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j}$$

$$\frac{d\mathbf{r}}{dt} = \left[\frac{dr}{dt} \cos \theta - r \sin \theta \frac{d\theta}{dt} \right] \mathbf{i} + \left[\frac{dr}{dt} \sin \theta + r \cos \theta \frac{d\theta}{dt} \right] \mathbf{j}$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \left[\frac{d^2r}{dt^2} \cos \theta - \frac{dr}{dt} \sin \theta \frac{d\theta}{dt} - \frac{dr}{dt} \sin \theta \frac{d\theta}{dt} - r \cos \theta \left(\frac{d\theta}{dt} \right)^2 - r \sin \theta \frac{d^2\theta}{dt^2} \right] \mathbf{i}$$

$$+ \left[\frac{d^2r}{dt^2} \sin \theta + \frac{dr}{dt} \cos \theta \frac{d\theta}{dt} + \frac{dr}{dt} \cos \theta \frac{d\theta}{dt} - r \sin \theta \left(\frac{d\theta}{dt} \right)^2 + r \cos \theta \frac{d^2\theta}{dt^2} \right] \mathbf{j}$$

$$a_r = \mathbf{a} \cdot \mathbf{u}_r = \mathbf{a} \cdot (\cos \theta \mathbf{i} + \sin \theta \mathbf{j})$$

$$= \left[\frac{d^2r}{dt^2} \cos^2 \theta - 2 \frac{dr}{dt} \sin \theta \cos \theta \frac{d\theta}{dt} - r \cos^2 \theta \left(\frac{d\theta}{dt} \right)^2 - r \cos \theta \sin \theta \frac{d^2\theta}{dt^2} \right]$$

$$+ \left[\frac{d^2r}{dt^2} \sin^2 \theta + 2 \frac{dr}{dt} \sin \theta \cos \theta \frac{d\theta}{dt} - r \sin^2 \theta \left(\frac{d\theta}{dt} \right)^2 + r \cos \theta \sin \theta \frac{d^2\theta}{dt^2} \right]$$

$$= \frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2$$

$$a_{\theta} = \mathbf{a} \cdot \mathbf{u}_{\theta} = \mathbf{a} \cdot (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) = 2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2}$$

$$\mathbf{a} = (\mathbf{a} \cdot \mathbf{u}_r) \mathbf{u}_r + (\mathbf{a} \cdot \mathbf{u}_{\theta}) \mathbf{u}_{\theta}$$

$$= \left[\frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right] \mathbf{u}_r + \left[2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2} \right] \mathbf{u}_{\theta}$$

—CONTINUED—

8. —CONTINUED—

$$(b) \mathbf{r} = 42,000 \cos\left(\frac{\pi t}{12}\right) \mathbf{i} + 42,000 \sin\left(\frac{\pi t}{12}\right) \mathbf{j}$$

$$\mathbf{r} = 42,000, \frac{dr}{dt} = 0, \frac{d^2r}{dt^2} = 0$$

$$\frac{d\theta}{dt} = \frac{\pi}{12}, \frac{d^2\theta}{dt^2} = 0$$

$$\text{Therefore, } \mathbf{a} = -42000 \left(\frac{\pi}{12}\right)^2 \mathbf{u}_r = -\frac{875}{3} \pi^2 \mathbf{u}_r.$$

$$\text{Radial component: } -\frac{875}{3} \pi^2$$

$$\text{Angular component: } 0$$

$$12. y = \frac{1}{32}x^{5/2}$$

$$y' = \frac{5}{64}x^{3/2}$$

$$y'' = \frac{15}{128}x^{1/2}$$

$$K = \left| \frac{\frac{15}{128}x^{1/2}}{\left(1 + \frac{25}{4096}x^3\right)^{3/2}} \right|$$

$$\text{At the point } (4, 1), K = \frac{120}{(89)^{3/2}} \Rightarrow r = \frac{1}{K} = \frac{(89)^{3/2}}{120} \approx 7.$$

14. (a) Eliminate the parameter to see that the Ferris wheel has a radius of 15 meters and is centered at $16\mathbf{j}$.

At $t = 0$, the friend is located at $\mathbf{r}_1(0) = \mathbf{j}$, which is the low point on the Ferris wheel.

- (b) If a revolution takes Δt seconds, then

$$\frac{\pi(t + \Delta t)}{10} = \frac{\pi t}{10} + 2\pi$$

and so $\Delta t = 20$ seconds. The Ferris wheel makes three revolutions per minute.

- (c) The initial velocity is $\mathbf{r}'_2(t_0) = -8.03\mathbf{i} + 11.47\mathbf{j}$. The speed is $\sqrt{8.03^2 + 11.47^2} \approx 14$ m/sec. The angle of inclination is $\arctan(11.47/8.03) \approx 0.96$ radians or 55° .

- (d) Although you may start with other values, $t_0 = 0$ is a fine choice. The graph at the right shows two points of intersection. At $t = 3.15$ sec the friend is near the vertex of the parabola, which the object reaches when

$$t - t_0 = -\frac{11.47}{2(-4.9)} \approx 1.17 \text{ sec.}$$

Thus, after the friend reaches the low point on the Ferris wheel, wait $t_0 = 2$ sec before throwing the object in order to allow it to be within reach.

- (e) The approximate time is 3.15 seconds after starting to rise from the low point on the Ferris wheel. The friend has a constant speed of $\|\mathbf{r}'_1(t)\| = 15$ m/sec. The speed of the object at that time is

$$\|\mathbf{r}'_2(3.15)\| = \sqrt{8.03^2 + [11.47 - 9.8(3.15 - 2)]^2} \approx 8.03 \text{ m/sec.}$$

$$10. \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} - \mathbf{k}, t = \frac{\pi}{4}$$

$$\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j}, \|\mathbf{r}'(t)\| = 1$$

$$\mathbf{T} = -\sin t \mathbf{i} + \cos t \mathbf{j}$$

$$\mathbf{T}' = -\cos t \mathbf{i} - \sin t \mathbf{j}$$

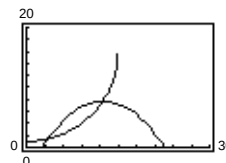
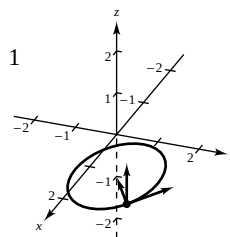
$$\mathbf{N} = -\cos t \mathbf{i} - \sin t \mathbf{j}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \mathbf{k}$$

$$\text{At } t = \frac{\pi}{4}, \mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{N}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{k}$$



C H A P T E R 12

Functions of Several Variables

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CHAPTER 12

Functions of Several Variables

Section 12.1 Introduction to Functions of Several Variables

Solutions to Odd-Numbered Exercises

1. $x^2z + yz - xy = 10$

$$z(x^2 + y) = 10 + xy$$

$$z = \frac{10 + xy}{x^2 + y}$$

Yes, z is a function of x and y .

3. $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$

No, z is not a function of x and y . For example, $(x, y) = (0, 0)$ corresponds to both $z = \pm 1$.

5. $f(x, y) = \frac{x}{y}$

(a) $f(3, 2) = \frac{3}{2}$

(b) $f(-1, 4) = -\frac{1}{4}$

(c) $f(30, 5) = \frac{30}{5} = 6$

(d) $f(5, y) = \frac{5}{y}$

(e) $f(x, 2) = \frac{x}{2}$

(f) $f(5, t) = \frac{5}{t}$

7. $f(x, y) = xe^y$

(a) $f(5, 0) = 5e^0 = 5$

(b) $f(3, 2) = 3e^2$

(c) $f(2, -1) = 2e^{-1} = \frac{2}{e}$

(d) $f(5, y) = 5e^y$

(e) $f(x, 2) = xe^2$

(f) $f(t, t) = te^t$

9. $h(x, y, z) = \frac{xy}{z}$

(a) $h(2, 3, 9) = \frac{(2)(3)}{9} = \frac{2}{3}$

(b) $h(1, 0, 1) = \frac{(1)(0)}{1} = 0$

11. $f(x, y) = x \sin y$

(a) $f\left(2, \frac{\pi}{4}\right) = 2 \sin \frac{\pi}{4} = \sqrt{2}$

(b) $f(3, 1) = 3 \sin 1$

13. $g(x, y) = \int_x^y (2t - 3) dt$

(a) $g(0, 4) = \int_0^4 (2t - 3) dt = \left[t^2 - 3t \right]_0^4 = 4$

(b) $g(1, 4) = \int_1^4 (2t - 3) dt = \left[t^2 - 3t \right]_1^4 = 6$

15. $f(x, y) = x^2 - 2y$

(a) $\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \frac{[(x + \Delta x)^2 - 2y] - (x^2 - 2y)}{\Delta x}$

$$= \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - 2y - x^2 + 2y}{\Delta x} = \frac{\Delta x(2x + \Delta x)}{\Delta x} = 2x + \Delta x, \Delta x \neq 0$$

(b) $\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \frac{[x^2 - 2(y + \Delta y)] - (x^2 - 2y)}{\Delta y} = \frac{x^2 - 2y - 2\Delta y - x^2 + 2y}{\Delta y} = \frac{-2\Delta y}{\Delta y} = -2, \Delta y \neq 0$

17. $f(x, y) = \sqrt{4 - x^2 - y^2}$

Domain: $4 - x^2 - y^2 \geq 0$

$x^2 + y^2 \leq 4$

$\{(x, y): x^2 + y^2 \leq 4\}$

Range: $0 \leq z \leq 2$

19. $f(x, y) = \arcsin(x + y)$

Domain:

$\{(x, y): -1 \leq x + y \leq 1\}$

Range: $-\frac{\pi}{2} \leq z \leq \frac{\pi}{2}$

21. $f(x, y) = \ln(4 - x - y)$

Domain: $4 - x - y > 0$

$x + y < 4$

$\{(x, y): y < -x + 4\}$

Range: all real numbers

23. $z = \frac{x + y}{xy}$

Domain: $\{(x, y): x \neq 0 \text{ and } y \neq 0\}$

Range: all real numbers

25. $f(x, y) = e^{x/y}$

Domain: $\{(x, y): y \neq 0\}$

Range: $z > 0$

27. $g(x, y) = \frac{1}{xy}$

Domain: $\{(x, y): x \neq 0 \text{ and } y \neq 0\}$

Range: all real numbers except zero

29. $f(x, y) = \frac{-4x}{x^2 + y^2 + 1}$

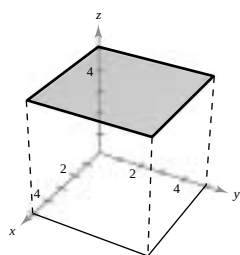
 (a) View from the positive x -axis: $(20, 0, 0)$

 (c) View from the first octant: $(20, 15, 25)$

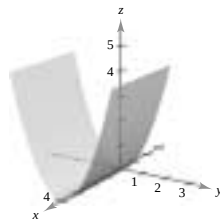
 (b) View where x is negative, y and z are positive:
 $(-15, 10, 20)$

 (d) View from the line $y = x$ in the xy -plane: $(20, 20, 0)$

31. $f(x, y) = 5$

 Plane: $z = 5$


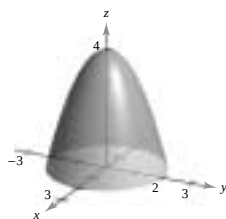
33. $f(x, y) = y^2$

 Since the variable x is missing, the surface is a cylinder with rulings parallel to the x -axis. The generating curve is $z = y^2$. The domain is the entire xy -plane and the range is $z \geq 0$.


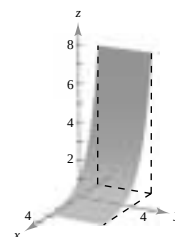
35. $z = 4 - x^2 - y^2$

Paraboloid

 Domain: entire xy -plane

 Range: $z \leq 4$


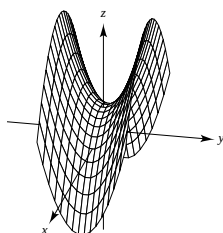
37. $f(x, y) = e^{-x}$

 Since the variable y is missing, the surface is a cylinder with rulings parallel to the y -axis. The generating curve is $z = e^{-x}$. The domain is the entire xy -plane and the range is $z > 0$.


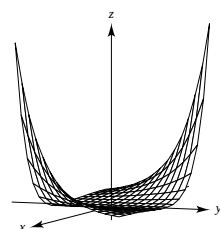
39. $z = y^2 - x^2 + 1$

Hyperbolic paraboloid

 Domain: entire xy -plane

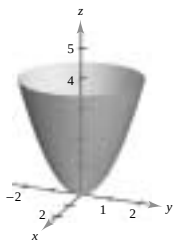
 Range: $-\infty < z < \infty$


41. $f(x, y) = x^2 e^{(-xy/2)}$

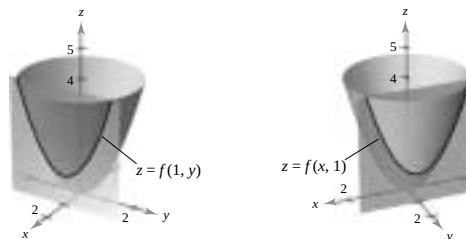


43. $f(x, y) = x^2 + y^2$

(a)

(b) g is a vertical translation of f two units upward(c) g is a horizontal translation of f two units to the right. The vertex moves from $(0, 0, 0)$ to $(0, 2, 0)$.(d) g is a reflection of f in the xy -plane followed by a vertical translation 4 units upward.

(e)



45. $z = e^{1-x^2-y^2}$

Level curves:

$$c = e^{1-x^2-y^2}$$

$$\ln c = 1 - x^2 - y^2$$

$$x^2 + y^2 = 1 - \ln c$$

Circles centered at $(0, 0)$

Matches (c)

47. $z = \ln|y - x^2|$

Level curves:

$$c = \ln|y - x^2|$$

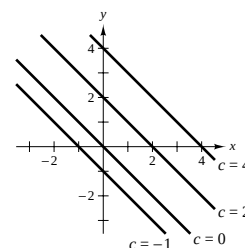
$$\pm e^c = y - x^2$$

$$y = x^2 \pm e^c$$

Parabolas

Matches (b)

49. $z = x + y$

Level curves are parallel lines of the form $x + y = c$.

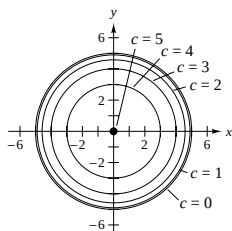
51. $f(x, y) = \sqrt{25 - x^2 - y^2}$

The level curves are of the form

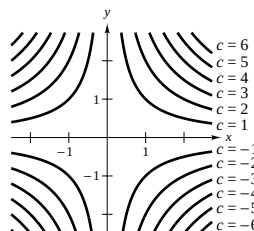
$$c = \sqrt{25 - x^2 - y^2}$$

$$x^2 + y^2 = 25 - c^2$$

Thus, the level curves are circles of radius 5 or less, centered at the origin.



53. $f(x, y) = xy$

The level curves are hyperbolas of the form $xy = c$.

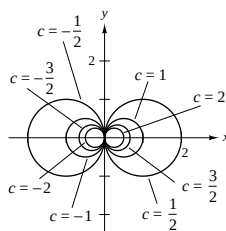
55. $f(x, y) = \frac{x}{x^2 + y^2}$

The level curves are of the form

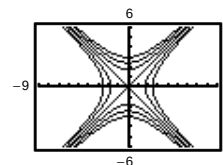
$$c = \frac{x}{x^2 + y^2}$$

$$x^2 - \frac{x}{c} + y^2 = 0$$

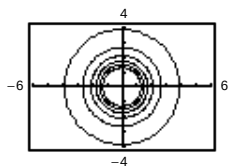
$$\left(x - \frac{1}{2c}\right)^2 + y^2 = \left(\frac{1}{2c}\right)^2$$

Thus, the level curves are circles passing through the origin and centered at $(1/2c, 0)$.

57. $f(x, y) = x^2 - y^2 + 2$



59. $g(x, y) = \frac{8}{1 + x^2 + y^2}$



61. See Definition, page 838.

63. No, The following graphs are not hemispheres.

$$z = e^{-(x^2 + y^2)}$$

$$z = x^2 + y^2$$

65. The surface is sloped like a saddle. The graph is not unique. Any vertical translation would have the same level curves.

One possible function is

$$f(x, y) = x^2 - y^2.$$

67. $V(I, R) = 1000 \left[\frac{1 + 0.10(1 - R)}{1 + I} \right]^{10}$

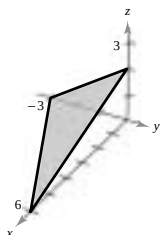
	Inflation Rate		
Tax Rate	0	0.03	0.05
0	2593.74	1929.99	1592.33
0.28	2004.23	1491.34	1230.42
0.35	1877.14	1396.77	1152.40

69. $f(x, y, z) = x - 2y + 3z$

$$c = 6$$

$$6 = x - 2y + 3z$$

Plane

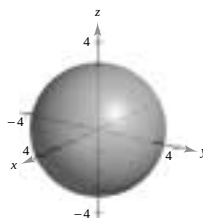


71. $f(x, y, z) = x^2 + y^2 + z^2$

$$c = 9$$

$$9 = x^2 + y^2 + z^2$$

Sphere

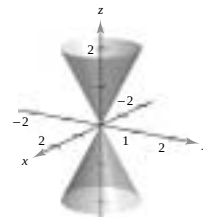


73. $f(x, y, z) = 4x^2 + 4y^2 - z^2$

$$c = 0$$

$$0 = 4x^2 + 4y^2 - z^2$$

Elliptic cone



75. $N(d, L) = \left(\frac{d - 4}{4} \right)^2 L$

(a) $N(22, 12) = \left(\frac{22 - 4}{4} \right)^2 (12) = 243$ board-feet

(b) $N(30, 12) = \left(\frac{30 - 4}{4} \right)^2 (12) = 507$ board-feet

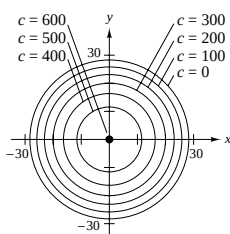
77. $T = 600 - 0.75x^2 - 0.75y^2$

The level curves are of the form

$$c = 600 - 0.75x^2 - 0.75y^2$$

$$x^2 + y^2 = \frac{600 - c}{0.75}.$$

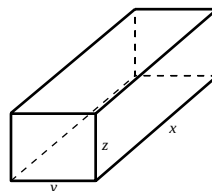
The level curves are circles centered at the origin.



79. $C = 0.75xy + 2(0.40)xz + 2(0.40)yz$

base + front & back + two ends

$$= 0.75xy + 0.80(xz + yz)$$



81. $PV = kT$, $20(2600) = k(300)$

(a) $k = \frac{20(2600)}{300} = \frac{520}{3}$

(b) $P = \frac{kT}{V} = \frac{520}{3} \left(\frac{T}{V} \right)$

The level curves are of the form: $c = \left(\frac{520}{3} \right) \left(\frac{T}{V} \right)$

$$V = \frac{520}{3c} T$$

Thus, the level curves are lines through the origin with slope $\frac{520}{3c}$.

83. (a) Highest pressure at C

(b) Lowest pressure at A

(c) Highest wind velocity at B

85. (a) The boundaries between colors represent level curves

(b) No, the colors represent intervals of different lengths, as indicated in the box

(c) You could use more colors, which means using smaller intervals

87. False. Let

$$f(x, y) = 2xy$$

$$f(1, 2) = f(2, 1), \text{ but } 1 \neq 2$$

89. False. Let

$$f(x, y) = 5.$$

$$\text{Then, } f(2x, 2y) = 5 \neq 2^2 f(x, y).$$

Section 12.2 Limits and Continuity

1. Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(x, y) - L| = |y - b| < \varepsilon$

whenever $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$. Take $\delta = \varepsilon$.

Then if $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta = \varepsilon$, we have

$$\sqrt{(y - b)^2} < \varepsilon$$

$$|y - b| < \varepsilon.$$

3. $\lim_{(x, y) \rightarrow (a, b)} [f(x, y) - g(x, y)] = \lim_{(x, y) \rightarrow (a, b)} f(x, y) - \lim_{(x, y) \rightarrow (a, b)} g(x, y) = 5 - 3 = 2$

5. $\lim_{(x, y) \rightarrow (a, b)} [f(x, y)g(x, y)] = \left[\lim_{(x, y) \rightarrow (a, b)} f(x, y) \right] \left[\lim_{(x, y) \rightarrow (a, b)} g(x, y) \right] = 5(3) = 15$

7. $\lim_{(x, y) \rightarrow (2, 1)} (x + 3y^2) = 2 + 3(1)^2 = 5$

Continuous everywhere

9. $\lim_{(x, y) \rightarrow (2, 4)} \frac{x + y}{x - y} = \frac{2 + 4}{2 - 4} = -3$

Continuous for $x \neq y$

11. $\lim_{(x, y) \rightarrow (0, 1)} \frac{\arcsin(x/y)}{1 + xy} = \arcsin 0 = 0$

Continuous for $xy \neq -1, y \neq 0, |x/y| \leq 1$

13. $\lim_{(x, y) \rightarrow (-1, 2)} e^{xy} = e^{-2} = \frac{1}{e^2}$

Continuous everywhere

15. $\lim_{(x, y, z) \rightarrow (1, 2, 5)} \sqrt{x + y + z} = \sqrt{8} = 2\sqrt{2}$

Continuous for $x + y + z \geq 0$

17. $\lim_{(x, y) \rightarrow (0, 0)} e^{xy} = 1$

Continuous everywhere

19. $\lim_{(x,y) \rightarrow (0,0)} \ln(x^2 + y^2) = \ln(0) = -\infty$

The limit does not exist.

Continuous except at $(0, 0)$

21. $f(x, y) = \frac{xy}{x^2 + y^2}$

Continuous except at $(0, 0)$

Path: $y = 0$

(x, y)	(1, 0)	(0.5, 0)	(0.1, 0)	(0.01, 0)	(0.001, 0)
$f(x, y)$	0	0	0	0	0

Path: $y = x$

(x, y)	(1, 1)	(0.5, 0.5)	(0.1, 0.1)	(0.01, 0.01)	(0.001, 0.001)
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

The limit does not exist because along the path $y = 0$ the function equals 0, whereas along the path $y = x$ the function equals $\frac{1}{2}$.

23. $f(x, y) = -\frac{xy^2}{x^2 + y^4}$

Continuous except at $(0, 0)$

Path: $x = y^2$

(x, y)	(1, 1)	(0.25, 0.5)	(0.01, 0.1)	(0.0001, 0.01)	(0.000001, 0.001)
$f(x, y)$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Path: $x = -y^2$

(x, y)	(-1, 1)	(-0.25, 0.5)	(-0.01, 0.1)	(-0.0001, 0.01)	(-0.000001, 0.001)
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

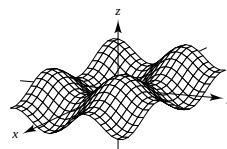
The limit does not exist because along the path $x = y^2$ the function equals $-\frac{1}{2}$, whereas along the path $x = -y^2$ the function equals $\frac{1}{2}$.

25. $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} \left(\frac{x^2 + 2xy^2 + y^2}{x^2 + y^2} \right)$
 $= \lim_{(x,y) \rightarrow (0,0)} \left(1 + \frac{2xy^2}{x^2 + y^2} \right) = 1$

(same limit for g)

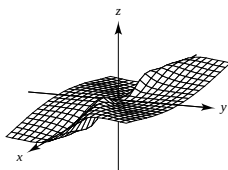
Thus, f is not continuous at $(0, 0)$, whereas g is continuous at $(0, 0)$.

27. $\lim_{(x,y) \rightarrow (0,0)} (\sin x + \sin y) = 0$



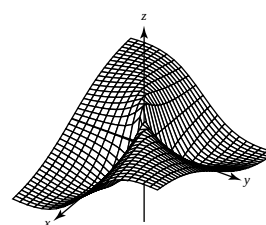
29. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + 4y^2}$

Does not exist



31. $f(x, y) = \frac{10xy}{2x^2 + 3y^2}$

The limit does not exist. Use the paths $x = 0$ and $x = y$.



$$33. \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{\sin r^2}{r^2} = \lim_{r \rightarrow 0} \frac{2r \cos r^2}{2r} = \lim_{r \rightarrow 0} \cos r^2 = 1$$

$$35. \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^3(\cos^3 \theta + \sin^3 \theta)}{r^2} = \lim_{r \rightarrow 0} r(\cos^3 \theta + \sin^3 \theta) = 0$$

$$37. f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

Continuous except at $(0, 0, 0)$

$$39. f(x, y, z) = \frac{\sin z}{e^x + e^y}$$

Continuous everywhere

$$41. f(t) = t^2$$

$$g(x, y) = 3x - 2y$$

$$f(g(x, y)) = f(3x - 2y)$$

$$= (3x - 2y)^2$$

$$= 9x^2 - 12xy + 4y^2$$

Continuous everywhere

$$43. f(t) = \frac{1}{t}$$

$$g(x, y) = 3x - 2y$$

$$f(g(x, y)) = f(3x - 2y) = \frac{1}{3x - 2y}$$

Continuous for $y \neq \frac{3x}{2}$

$$45. f(x, y) = x^2 - 4y$$

$$(a) \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 - 4y] - (x^2 - 4y)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x - (\Delta x)^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x - \Delta x) = 2x$$

$$(b) \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{[x^2 - 4(y + \Delta y)] - (x^2 - 4y)}{\Delta y}$$

$$= \lim_{\Delta y \rightarrow 0} \frac{-4\Delta y}{\Delta y} = \lim_{\Delta y \rightarrow 0} (-4) = -4$$

$$47. f(x, y) = 2x + xy - 3y$$

$$(a) \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{[2(x + \Delta x) + (x + \Delta x)y - 3y] - (2x + xy - 3y)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{2\Delta x + \Delta xy}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2 + y) = 2 + y$$

$$(b) \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{[2x + x(y + \Delta y) - 3(y + \Delta y)] - (2x + xy - 3y)}{\Delta y}$$

$$= \lim_{\Delta y \rightarrow 0} \frac{x\Delta y - 3\Delta y}{\Delta y} = \lim_{\Delta y \rightarrow 0} (x - 3) = x - 3$$

49. See the definition on page 851.

Show that the value of $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$ is not the same for two different paths to (x_0, y_0) .

51. No.

The existence of $f(2, 3)$ has no bearing on the existence of the limit as $(x, y) \rightarrow (2, 3)$.

53. Since $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L_1$, then for $\varepsilon/2 > 0$, there corresponds $\delta_1 > 0$ such that $|f(x,y) - L_1| < \varepsilon/2$ whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_1.$$

Since $\lim_{(x,y) \rightarrow (a,b)} g(x,y) = L_2$, then for $\varepsilon/2 > 0$, there corresponds $\delta_2 > 0$ such that $|g(x,y) - L_2| < \varepsilon/2$ whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_2.$$

Let δ be the smaller of δ_1 and δ_2 . By the triangle inequality, whenever $\sqrt{(x-a)^2 + (y-b)^2} < \delta$, we have

$$|f(x,y) + g(x,y) - (L_1 + L_2)| = |(f(x,y) - L_1) + (g(x,y) - L_2)| \leq |f(x,y) - L_1| + |g(x,y) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $\lim_{(x,y) \rightarrow (a,b)} [f(x,y) + g(x,y)] = L_1 + L_2$.

55. True

57. False. Let

$$f(x,y) = \begin{cases} \ln(x^2 + y^2), & (x,y) \neq (0,0) \\ 0, & x=0, y=0 \end{cases}$$

See Exercise 19.

Section 12.3 Partial Derivatives

1. $f_x(4,1) < 0$

3. $f_y(4,1) > 0$

5. $f(x,y) = 2x - 3y + 5$

$$f_x(x,y) = 2$$

$$f_y(x,y) = -3$$

7. $z = x\sqrt{y}$

$$\frac{\partial z}{\partial x} = \sqrt{y}$$

$$\frac{\partial z}{\partial y} = \frac{x}{2\sqrt{y}}$$

9. $z = x^2 - 5xy + 3y^2$

$$\frac{\partial z}{\partial x} = 2x - 5y$$

$$\frac{\partial z}{\partial y} = -5x + 6y$$

11. $z = x^2 e^{2y}$

$$\frac{\partial z}{\partial x} = 2xe^{2y}$$

$$\frac{\partial z}{\partial y} = 2x^2 e^{2y}$$

13. $z = \ln(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}$$

15. $z = \ln\left(\frac{x+y}{x-y}\right) = \ln(x+y) - \ln(x-y)$

$$\frac{\partial z}{\partial x} = \frac{1}{x+y} - \frac{1}{x-y} = -\frac{2y}{x^2 - y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x+y} + \frac{1}{x-y} = \frac{2x}{x^2 - y^2}$$

17. $z = \frac{x^2}{2y} + \frac{4y^2}{x}$

$$\frac{\partial z}{\partial x} = \frac{2x}{2y} - \frac{4y^2}{x^2} = \frac{x^3 - 4y^3}{x^2 y}$$

$$\frac{\partial z}{\partial y} = -\frac{x^2}{2y^2} + \frac{8y}{x} = \frac{-x^3 + 16y^3}{2xy^2}$$

19. $h(x,y) = e^{-(x^2+y^2)}$

$$h_x(x,y) = -2xe^{-(x^2+y^2)}$$

$$h_y(x,y) = -2ye^{-(x^2+y^2)}$$

21. $f(x,y) = \sqrt{x^2 + y^2}$

$$f_x(x,y) = \frac{1}{2}(x^2 + y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2}}$$

$$f_y(x,y) = \frac{1}{2}(x^2 + y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2 + y^2}}$$

23. $z = \tan(2x - y)$

$$\frac{\partial z}{\partial x} = 2 \sec^2(2x - y)$$

$$\frac{\partial z}{\partial y} = -\sec^2(2x - y)$$

25. $z = e^y \sin xy$

$$\frac{\partial z}{\partial x} = ye^y \cos xy$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= e^y \sin xy + xe^y \cos xy \\ &= e^y(x \cos xy + \sin xy)\end{aligned}$$

27. $f(x, y) = \int_x^y (t^2 - 1) dt$

$$= \left[\frac{t^3}{3} - t \right]_x^y = \left(\frac{y^3}{3} - y \right) - \left(\frac{x^3}{3} - x \right)$$

$$f_x(x, y) = -x^2 + 1 = 1 - x^2$$

$$f_y(x, y) = y^2 - 1$$

[You could also use the Second Fundamental Theorem of Calculus.]

29. $f(x, y) = 2x + 3y$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2(x + \Delta x) + 3y - 2x - 3y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2\Delta x}{\Delta x} = 2$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{2x + 3(y + \Delta y) - 2x - 3y}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{3\Delta y}{\Delta y} = 3$$

31. $f(x, y) = \sqrt{x + y}$

$$\begin{aligned}\frac{\partial f}{\partial x} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x + y} - \sqrt{x + y}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(\sqrt{x + \Delta x + y} - \sqrt{x + y})(\sqrt{x + \Delta x + y} + \sqrt{x + y})}{\Delta x(\sqrt{x + \Delta x + y} + \sqrt{x + y})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x + y} + \sqrt{x + y}} = \frac{1}{2\sqrt{x + y}}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\sqrt{x + y + \Delta y} - \sqrt{x + y}}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{(\sqrt{x + y + \Delta y} - \sqrt{x + y})(\sqrt{x + y + \Delta y} + \sqrt{x + y})}{\Delta y(\sqrt{x + y + \Delta y} + \sqrt{x + y})} \\ &= \lim_{\Delta y \rightarrow 0} \frac{1}{\sqrt{x + y + \Delta y} + \sqrt{x + y}} = \frac{1}{2\sqrt{x + y}}\end{aligned}$$

33. $g(x, y) = 4 - x^2 - y^2$

$$g_x(x, y) = -2x$$

$$\text{At } (1, 1): g_x(1, 1) = -2$$

$$g_y(x, y) = -2y$$

$$\text{At } (1, 1): g_y(1, 1) = -2$$

35. $z = e^{-x} \cos y$

$$\frac{\partial z}{\partial x} = -e^{-x} \cos y$$

$$\text{At } (0, 0): \frac{\partial z}{\partial x} = -1$$

$$\frac{\partial z}{\partial y} = -e^{-x} \sin y$$

$$\text{At } (0, 0): \frac{\partial z}{\partial y} = 0$$

37. $f(x, y) = \arctan \frac{y}{x}$

$$f_x(x, y) = \frac{1}{1 + (y^2/x^2)} \left(-\frac{y}{x^2} \right) = \frac{-y}{x^2 + y^2}$$

$$\text{At } (2, -2): f_x(2, -2) = \frac{1}{4}$$

$$f_y(x, y) = \frac{1}{1 + (y^2/x^2)} \left(\frac{1}{x} \right) = \frac{x}{x^2 + y^2}$$

$$\text{At } (2, -2): f_y(2, -2) = \frac{1}{4}$$

39. $f(x, y) = \frac{xy}{x - y}$

$$f_x(x, y) = \frac{y(x - y) - xy}{(x - y)^2} = \frac{-y^2}{(x - y)^2}$$

$$\text{At } (2, -2): f_x(2, -2) = -\frac{1}{4}$$

$$f_y(x, y) = \frac{x(x - y) + xy}{(x - y)^2} = \frac{x^2}{(x - y)^2}$$

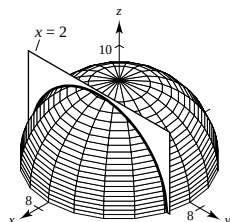
$$\text{At } (2, -2): f_y(2, -2) = \frac{1}{4}$$

41. $z = \sqrt{49 - x^2 - y^2}$, $x = 2$,
(2, 3, 6)

Intersecting curve: $z = \sqrt{45 - y^2}$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{45 - y^2}}$$

At (2, 3, 6): $\frac{\partial z}{\partial y} = \frac{-3}{\sqrt{45 - 9}} = -\frac{1}{2}$

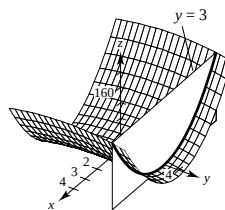


43. $z = 9x^2 - y^2$, $y = 3$, (1, 3, 0)

Intersecting curve: $z = 9x^2 - 9$

$$\frac{\partial z}{\partial x} = 18x$$

At (1, 3, 0): $\frac{\partial z}{\partial x} = 18(1) = 18$



45. $f_x(x, y) = 2x + 4y - 4$, $f_y(x, y) = 4x + 2y + 16$

$$f_x = f_y = 0: 2x + 4y = 4$$

$$4x + 2y = -16$$

Solving for x and y ,

$$x = -6 \text{ and } y = 4.$$

47. $f_x(x, y) = -\frac{1}{x^2} + y$, $f_y(x, y) = -\frac{1}{y^2} + x$

$$f_x = f_y = 0: -\frac{1}{x^2} + y = 0 \text{ and } -\frac{1}{y^2} + x = 0$$

$$y = \frac{1}{x^2} \text{ and } x = \frac{1}{y^2}$$

$$y = y^4 \Rightarrow y = 1 = x$$

Points: (1, 1)

49. (a) The graph is that of f_y .

(b) The graph is that of f_x .

51. $w = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial w}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

53. $F(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2}$
 $= \frac{1}{2} \ln(x^2 + y^2 + z^2)$

$$F_x(x, y, z) = \frac{x}{x^2 + y^2 + z^2}$$

$$F_y(x, y, z) = \frac{y}{x^2 + y^2 + z^2}$$

$$F_z(x, y, z) = \frac{z}{x^2 + y^2 + z^2}$$

55. $H(x, y, z) = \sin(x + 2y + 3z)$

$$H_x(x, y, z) = \cos(x + 2y + 3z)$$

$$H_y(x, y, z) = 2 \cos(x + 2y + 3z)$$

$$H_z(x, y, z) = 3 \cos(x + 2y + 3z)$$

57. $z = x^2 - 2xy + 3y^2$

$$\frac{\partial z}{\partial x} = 2x - 2y$$

$$\frac{\partial^2 z}{\partial x^2} = 2$$

$$\frac{\partial^2 z}{\partial y \partial x} = -2$$

$$\frac{\partial z}{\partial y} = -2x + 6y$$

$$\frac{\partial^2 z}{\partial y^2} = 6$$

$$\frac{\partial^2 z}{\partial x \partial y} = -2$$

59. $z = \sqrt{x^2 + y^2}$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{y^2}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-xy}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{x^2}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-xy}{(x^2 + y^2)^{3/2}}$$

61. $z = e^x \tan y$

$$\frac{\partial z}{\partial x} = e^x \tan y$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \tan y$$

$$\frac{\partial^2 z}{\partial y \partial x} = e^x \sec^2 y$$

$$\frac{\partial z}{\partial y} = e^x \sec^2 y$$

$$\frac{\partial^2 z}{\partial y^2} = 2e^x \sec^2 y \tan y$$

$$\frac{\partial^2 z}{\partial x \partial y} = e^x \sec^2 y$$

65. $z = x \sec y$

$$\frac{\partial z}{\partial x} = \sec y$$

$$\frac{\partial^2 z}{\partial x^2} = 0$$

$$\frac{\partial^2 z}{\partial y \partial x} = \sec y \tan y$$

$$\frac{\partial z}{\partial y} = x \sec y \tan y$$

$$\frac{\partial^2 z}{\partial y^2} = x \sec y (\sec^2 y + \tan^2 y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \sec y \tan y$$

Therefore, $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$.

There are no points for which $z_x = 0 = z_y$, because

$$\frac{\partial z}{\partial x} = \sec y \neq 0.$$

69. $f(x, y, z) = xyz$

$$f_x(x, y, z) = yz$$

$$f_y(x, y, z) = xz$$

$$f_{yy}(x, y, z) = 0$$

$$f_{xy}(x, y, z) = z$$

$$f_{yx}(x, y, z) = z$$

$$f_{yyx}(x, y, z) = 0$$

$$f_{xyy}(x, y, z) = 0$$

$$f_{yxy}(x, y, z) = 0$$

Therefore, $f_{xyy} = f_{yxy} = f_{yyx} = 0$.

71. $f(x, y, z) = e^{-x} \sin yz$

$$f_x(x, y, z) = -e^{-x} \sin yz$$

$$f_y(x, y, z) = ze^{-x} \cos yz$$

$$f_{yy}(x, y, z) = -z^2 e^{-x} \sin yz$$

$$f_{xy}(x, y, z) = -ze^{-x} \cos yz$$

$$f_{yx}(x, y, z) = -ze^{-x} \cos yz$$

$$f_{yyx}(x, y, z) = z^2 e^{-x} \sin yz$$

$$f_{xyy}(x, y, z) = z^2 e^{-x} \sin yz$$

$$f_{yxy}(x, y, z) = z^2 e^{-x} \sin yz$$

Therefore, $f_{xyy} = f_{yxy} = f_{yyx}$.

73. $z = 5xy$

$$\frac{\partial z}{\partial x} = 5y$$

$$\frac{\partial^2 z}{\partial x^2} = 0$$

$$\frac{\partial z}{\partial y} = 5x$$

$$\frac{\partial^2 z}{\partial y^2} = 0$$

Therefore, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 + 0 = 0$.

63. $z = \arctan \frac{y}{x}$

$$\frac{\partial z}{\partial x} = \frac{1}{1 + (y^2/x^2)} \left(-\frac{y}{x^2} \right) = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-(x^2 + y^2) + y(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + (y^2/x^2)} \left(\frac{1}{x} \right) = \frac{x}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

67. $z = \ln \left(\frac{x}{x^2 + y^2} \right) = \ln x - \ln(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = \frac{1}{x} - \frac{2x}{x^2 + y^2} = \frac{y^2 - x^2}{x(x^2 + y^2)}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{x^4 - 4x^2y^2 - y^4}{x^2(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{4xy}{(x^2 + y^2)^2}$$

$$\frac{\partial z}{\partial y} = -\frac{2y}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{4xy}{(x^2 + y^2)^2}$$

There are no points for which $z_x = z_y = 0$.

75. $z = e^x \sin y$

$$\frac{\partial z}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \sin y$$

$$\frac{\partial z}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 z}{\partial y^2} = -e^x \sin y$$

$$\text{Therefore, } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^x \sin y - e^x \sin y = 0.$$

77. $z = \sin(x - ct)$

$$\frac{\partial z}{\partial t} = -c \cos(x - ct)$$

$$\frac{\partial^2 z}{\partial t^2} = -c^2 \sin(x - ct)$$

$$\frac{\partial z}{\partial x} = \cos(x - ct)$$

$$\frac{\partial^2 z}{\partial x^2} = -\sin(x - ct)$$

$$\text{Therefore, } \frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}.$$

79. $z = e^{-t} \cos \frac{x}{c}$

$$\frac{\partial z}{\partial t} = -e^{-t} \cos \frac{x}{c}$$

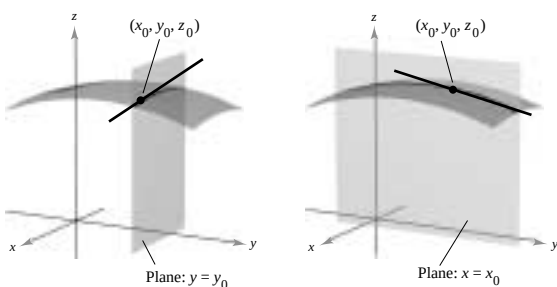
$$\frac{\partial z}{\partial x} = -\frac{1}{c} e^{-t} \sin \frac{x}{c}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{c^2} e^{-t} \cos \frac{x}{c}$$

$$\text{Therefore, } \frac{\partial z}{\partial t} = c^2 \frac{\partial^2 z}{\partial x^2}.$$

81. See the definition on page 859.

83.

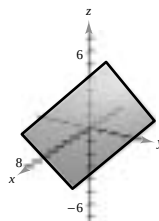


$\frac{\partial f}{\partial x}$ denotes the slope of the surface in the x -direction.

$\frac{\partial f}{\partial y}$ denotes the slope of the surface in the y -direction.

 85. The plane $z = x + y = f(x, y)$ satisfies

$$\frac{\partial f}{\partial x} > 0 \text{ and } \frac{\partial f}{\partial y} > 0.$$



87. (a) $C = 32\sqrt{xy} + 175x + 205y + 1050$

$$\frac{\partial C}{\partial x} = 16\sqrt{\frac{y}{x}} + 175$$

$$\left. \frac{\partial C}{\partial x} \right|_{(80, 20)} = 16\sqrt{\frac{1}{4}} + 175 = 183$$

$$\frac{\partial C}{\partial y} = 16\sqrt{\frac{x}{y}} + 205$$

$$\left. \frac{\partial C}{\partial y} \right|_{(80, 20)} = 16\sqrt{4} + 205 = 237$$

(b) The fireplace-insert stove results in the cost increasing at a faster rate because

$$\frac{\partial C}{\partial y} > \frac{\partial C}{\partial x}.$$

89. An increase in either price will cause a decrease in demand.

$$91. T = 500 - 0.6x^2 - 1.5y^2$$

$$\frac{\partial T}{\partial x} = -1.2x, \frac{\partial T}{\partial x}(2, 3) = -2.4^\circ/\text{m}$$

$$\frac{\partial T}{\partial y} = -3y = \frac{\partial T}{\partial y}(2, 3) = -9^\circ/\text{m}$$

$$93. PV = mRT$$

$$T = \frac{PV}{mR} \Rightarrow \frac{\partial T}{\partial P} = \frac{V}{mR}$$

$$P = \frac{mRT}{V} \Rightarrow \frac{\partial P}{\partial V} = -\frac{mRT}{V^2}$$

$$V = \frac{mRT}{P} \Rightarrow \frac{\partial V}{\partial T} = \frac{mR}{P}$$

$$\begin{aligned} \frac{\partial T}{\partial P} \cdot \frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} &= \left(\frac{V}{mR}\right) \left(-\frac{mRT}{V^2}\right) \left(\frac{mR}{P}\right) \\ &= -\frac{mRT}{VP} = -\frac{mRT}{mRT} = -1 \end{aligned}$$

$$97. f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$$(a) f_x(x, y) = \frac{(x^2 + y^2)(3x^2y - y^3) - (x^3y - xy^3)(2x)}{(x^2 + y^2)^2} = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{(x^2 + y^2)(x^3 - 3xy^2) - (x^3y - xy^3)(2y)}{(x^2 + y^2)^2} = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

$$(b) f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0/[(\Delta x)^2] - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{0/[(\Delta y)^2] - 0}{\Delta y} = 0$$

$$(c) f_{xy}(0, 0) = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \bigg|_{(0, 0)} = \lim_{\Delta y \rightarrow 0} \frac{f_x(0, \Delta y) - f_x(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\Delta y(-(\Delta y)^4)}{((\Delta y)^2)^2(\Delta y)} = \lim_{\Delta y \rightarrow 0} (-1) = -1$$

$$f_{yx}(0, 0) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \bigg|_{(0, 0)} = \lim_{\Delta x \rightarrow 0} \frac{f_y(\Delta x, 0) - f_y(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x((\Delta x)^4)}{((\Delta x)^2)^2(\Delta x)} = \lim_{\Delta x \rightarrow 0} 1 = 1$$

(d) f_{yx} or f_{xy} or both are not continuous at $(0, 0)$.

99. True

101. True

Section 12.4 Differentials

$$1. z = 3x^2y^3$$

$$dz = 6xy^3 dx + 9x^2y^2 dy$$

$$3. z = \frac{-1}{x^2 + y^2}$$

$$dz = \frac{2x}{(x^2 + y^2)^2} dx + \frac{2y}{(x^2 + y^2)^2} dy$$

$$= \frac{2}{(x^2 + y^2)^2} (x dx + y dy)$$

5. $z = x \cos y - y \cos x$

$$dz = (\cos y + y \sin x) dx + (-x \sin y - \cos x) dy = (\cos y + y \sin x) dx - (x \sin y + \cos x) dy$$

7. $z = e^x \sin y$

$$dz = (e^x \sin y) dx + (e^x \cos y) dy$$

9. $w = 2z^3 y \sin x$

$$dw = 2z^3 y \cos x dx + 2z^3 \sin x dy + 6z^2 y \sin x dz$$

11. (a) $f(1, 2) = 4$

$$f(1.05, 2.1) = 3.4875$$

$$\Delta z = f(1.05, 2.1) - f(1, 2) = -0.5125$$

(b) $dz = -2x dx - 2y dy$

$$= -2(0.05) - 4(0.1) = -0.5$$

13. (a) $f(1, 2) = \sin 2$

$$f(1.05, 2.1) = 1.05 \sin 2.1$$

$$\Delta z = f(1.05, 2.1) - f(1, 2) \approx -0.00293$$

(b) $dz = \sin y dx + x \cos y dy$

$$= (\sin 2)(0.05) + (\cos 2)(0.1) \approx 0.00385$$

15. (a) $f(1, 2) = -5$

$$f(1.05, 2.1) = -5.25$$

$$\Delta z = -0.25$$

(b) $dz = 3 dx - 4 dy$

$$= 3(0.05) - 4(0.1) \approx -0.25$$

17. Let $z = \sqrt{x^2 + y^2}$, $x = 5$, $y = 3$, $dx = 0.05$, $dy = 0.1$. Then: $dz = \frac{x}{\sqrt{x^2 + y^2}} dx + \frac{y}{\sqrt{x^2 + y^2}} dy$

$$\sqrt{(5.05)^2 + (3.1)^2} - \sqrt{5^2 + 3^2} \approx \frac{5}{\sqrt{5^2 + 3^2}}(0.05) + \frac{3}{\sqrt{5^2 + 3^2}}(0.1) = \frac{0.55}{\sqrt{34}} \approx 0.094$$

19. Let $z = (1 - x^2)/y^2$, $x = 3$, $y = 6$, $dx = 0.05$, $dy = -0.05$. Then: $dz = -\frac{2x}{y^2} dx + \frac{-2(1 - x^2)}{y^3} dy$

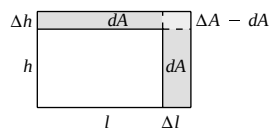
$$\frac{1 - (3.05)^2}{(5.95)^2} - \frac{1 - 3^2}{6^2} \approx -\frac{2(3)}{6^2}(0.05) - \frac{2(1 - 3^2)}{6^3}(-0.05) \approx -0.012$$

21. See the definition on page 869.

23. The tangent plane to the surface $z = f(x, y)$ at the point P is a linear approximation of z .

25. $A = lh$

$$dA = l dh + h dl$$



27. $V = \frac{\pi r^2 h}{3}$

$$r = 3$$

$$h = 6$$

$$dV = \frac{2\pi r h}{3} dr + \frac{\pi r^2}{3} dh = \frac{\pi r}{3}(2h dr + r dh)$$

Δr	Δh	dV	ΔV	$\Delta V - dV$
0.1	0.1	4.7124	4.8391	0.1267
0.1	-0.1	2.8274	2.8264	-0.0010
0.001	0.002	0.0565	0.0566	0.0001
-0.0001	0.0002	-0.0019	-0.0019	0.0000

29. (a) $dz = -1.83 dx - 1.09 dy$

$$\begin{aligned} \text{(b) } dz &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \\ &= -1.83(\pm 0.25) + (-1.09)(\pm 0.25) \\ &= \pm 0.73 \end{aligned}$$

Maximum propagated error: ± 0.73

$$\text{Relative error: } \frac{dz}{z} = \frac{\pm 0.73}{(-1.83)(7.2) - 1.09(8.5) + 28.7} = \frac{\pm 0.73}{6.259} \approx \pm 0.1166 = 11.67\%$$

31. $V = \pi r^2 h = dV = (2\pi r h) dr + (\pi r^2) dh$

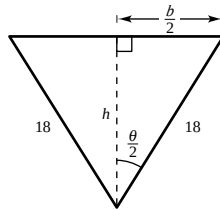
$$\begin{aligned} \frac{dV}{V} &= 2 \frac{dr}{r} + \frac{dh}{h} \\ &= 2(0.04) + (0.02) = 0.10 = 10\% \end{aligned}$$

33. $A = \frac{1}{2}ab \sin C$

$$\begin{aligned} dA &= \frac{1}{2}[(b \sin C) da + (a \sin C) db + (ab \cos C) dC] \\ &= \frac{1}{2}[4(\sin 45^\circ)(\pm \frac{1}{16}) + 3(\sin 45^\circ)(\pm \frac{1}{16}) + 12(\cos 45^\circ)(\pm 0.02)] \approx \pm 0.24 \text{ in.}^2 \end{aligned}$$

35. (a) $V = \frac{1}{2}bhl$

$$\begin{aligned} &= \left(18 \sin \frac{\theta}{2}\right) \left(18 \cos \frac{\theta}{2}\right) (16)(12) \\ &= 31,104 \sin \theta \text{ in.}^3 \\ &= 18 \sin \theta \text{ ft}^3 \end{aligned}$$



V is maximum when $\sin \theta = 1$ or $\theta = \pi/2$.

(b) $V = \frac{s^2}{2}(\sin \theta)l$

$$\begin{aligned} dV &= s(\sin \theta)l ds + \frac{s^2}{2}l(\cos \theta) d\theta + \frac{s^2}{2}(\sin \theta) dl \\ &= 18 \left(\sin \frac{\pi}{2} \right) (16)(12) \left(\frac{1}{2} \right) + \frac{18^2}{2} (16)(12) \left(\cos \frac{\pi}{2} \right) \left(\frac{\pi}{90} \right) + \frac{18^2}{2} \left(\sin \frac{\pi}{2} \right) \left(\frac{1}{2} \right) \\ &= 1809 \text{ in}^3 \approx 1.047 \text{ ft}^3 \end{aligned}$$

37. $P = \frac{E^2}{R}$

$$dP = \frac{2E}{R} dE - \frac{E^2}{R^2} dR$$

$$\frac{dP}{P} = 2 \frac{dE}{E} - \frac{dR}{R} = 2(0.02) - (-0.03) = 0.07 = 7\%$$

39. $L = 0.00021 \left(\ln \frac{2h}{r} - 0.75 \right)$

$$dL = 0.00021 \left[\frac{dh}{h} - \frac{dr}{r} \right] = 0.00021 \left[\frac{(\pm 1/100)}{100} - \frac{(\pm 1/16)}{2} \right] \approx (\pm 6.6) \times 10^{-6}$$

$$L = 0.00021(\ln 100 - 0.75) \approx 8.096 \times 10^{-4} \pm dL = 8.096 \times 10^{-4} \pm 6.6 \times 10^{-6} \text{ micro-henrys}$$

41. $z = f(x, y) = x^2 - 2x + y$

$$\begin{aligned} \Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) \\ &= (x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + y + (\Delta y)) - (x^2 - 2x + y) \\ &= 2x(\Delta x) + (\Delta x)^2 - 2(\Delta x) + (\Delta y) \\ &= (2x - 2) \Delta x + \Delta y + \Delta x(\Delta x) + 0(\Delta y) \\ &= f_x(x, y) \Delta x + f_y(x, y) \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \text{ where } \epsilon_1 = \Delta x \text{ and } \epsilon_2 = 0. \end{aligned}$$

As $(\Delta x, \Delta y) \rightarrow (0, 0)$, $\epsilon_1 \rightarrow 0$ and $\epsilon_2 \rightarrow 0$.

43. $z = f(x, y) = x^2 y$

$$\begin{aligned} \Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) \\ &= (x^2 + 2x(\Delta x) + (\Delta x)^2)(y + \Delta y) - x^2 y \\ &= 2xy(\Delta x) + y(\Delta x)^2 + x^2 \Delta y + 2x(\Delta x)(\Delta y) + (\Delta x)^2 \Delta y \\ &= 2xy(\Delta x) + x^2 \Delta y + (y \Delta x) \Delta x + [2x \Delta x + (\Delta x)^2] \Delta y \\ &= f_x(x, y) \Delta x + f_y(x, y) \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \text{ where } \epsilon_1 = y(\Delta x) \text{ and } \epsilon_2 = 2x \Delta x + (\Delta x)^2. \end{aligned}$$

As $(\Delta x, \Delta y) \rightarrow (0, 0)$, $\epsilon_1 \rightarrow 0$ and $\epsilon_2 \rightarrow 0$.

45. $f(x, y) = \begin{cases} \frac{3x^2 y}{x^4 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

$$(a) f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{0}{(\Delta x)^4} - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\frac{0}{(\Delta y)^2} - 0}{\Delta y} = 0$$

Thus, the partial derivatives exist at $(0, 0)$.

$$(b) \text{ Along the line } y = x: \lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{x \rightarrow 0} \frac{3x^3}{x^4 + x^2} = \lim_{x \rightarrow 0} \frac{3x}{x^2 + 1} = 0$$

$$\text{Along the curve } y = x^2: \lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \frac{3x^4}{2x^4} = \frac{3}{2}$$

f is not continuous at $(0, 0)$. Therefore, f is not differentiable at $(0, 0)$. (See Theorem 12.5)

47. Essay. For example, we can use the equation $F = ma$:

$$dF = \frac{\partial F}{\partial m} dm + \frac{\partial F}{\partial a} da = a dm + m da.$$

Section 12.5 Chain Rules for Functions of Several Variables

1. $w = x^2 + y^2$

$x = e^t$

$y = e^{-t}$

$$\frac{dw}{dt} = 2xe^t + 2y(-e^{-t}) = 2(e^{2t} - e^{-2t})$$

3. $w = x \sec y$

$x = e^t$

$y = \pi - t$

$$\begin{aligned}\frac{dw}{dt} &= (\sec y)(e^t) + (x \sec y \tan y)(-1) \\ &= e^t \sec(\pi - t)[1 - \tan(\pi - t)] \\ &= -e^t (\sec t + \sec t \tan t)\end{aligned}$$

5. $w = xy$, $x = 2 \sin t$, $y = \cos t$

$$\begin{aligned}\text{(a)} \quad \frac{dw}{dt} &= 2y \cos t + x(-\sin t) = 2y \cos t - x \sin t \\ &= 2(\cos^2 t - \sin^2 t) = 2 \cos 2t\end{aligned}$$

$$\text{(b)} \quad w = 2 \sin t \cos t = \sin 2t, \quad \frac{dw}{dt} = 2 \cos 2t$$

7. $w = x^2 + y^2 + z^2$

$x = e^t \cos t$

$y = e^t \sin t$

$z = e^t$

$$\text{(a)} \quad \frac{dw}{dt} = 2x(-e^t \sin t + e^t \cos t) + 2y(e^t \cos t + e^t \sin t) + 2ze^t = 4e^{2t}$$

$$\text{(b)} \quad w = 2e^{2t}, \quad \frac{dw}{dt} = 4e^{2t}$$

9. $w = xy + xz + yz$, $x = t - 1$, $y = t^2 - 1$, $z = t$

$$\begin{aligned}\text{(a)} \quad \frac{dw}{dt} &= (y + z) = (x + z)(2t) + (x + y) \\ &= (t^2 - 1 + t) + (t - 1 + 1)(2t) + (t - 1 + t^2 - 1) = 3(2t^2 - 1)\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad w &= (t - 1)(t^2 - 1) + (t - 1)t + (t^2 - 1)t \\ \frac{dw}{dt} &= 2t(t - 1) + (t^2 - 1) + 2t - 1 + 3t^2 - 1 = 3(2t^2 - 1)\end{aligned}$$

11. Distance $= f(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(10 \cos 2t - 7 \cos t)^2 + (6 \sin 2t - 4 \sin t)^2}$

$$f'(t) = \frac{1}{2}[(10 \cos 2t - 7 \cos t)^2 + (6 \sin 2t - 4 \sin t)^2]^{-1/2}$$

$$[[2(10 \cos 2t - 7 \cos t)(-20 \sin 2t + 7 \sin t)] + [2(6 \sin 2t - 4 \sin t)(12 \cos 2t - 4 \cos t)]]$$

$$f'\left(\frac{\pi}{2}\right) = \frac{1}{2}[(-10)^2 + 4^2]^{-1/2}[[2(-10)(7)] + (2(-4)(-12)]]$$

$$= \frac{1}{2}(116)^{-1/2}(-44) = \frac{22}{2\sqrt{29}} = \frac{-11\sqrt{29}}{20} \approx -2.04$$

13. $w = \arctan(2xy)$, $x = \cos t$, $y = \sin t$, $t = 0$

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\ &= \frac{2y}{1 + (4x^2y^2)}(-\sin t) + \frac{2x}{1 + (4x^2y^2)}(\cos t) \\ &= \frac{2 \sin t}{1 + 4 \cos^2 t \sin^2 t}(-\sin t) + \frac{2 \cos t}{1 + 4 \cos^2 t \sin^2 t}(\cos t) \\ &= \frac{2 \cos^2 t - 2 \sin^2 t}{1 + 4 \cos^2 t \sin^2 t} \\ \frac{d^2w}{dt^2} &= \frac{(1 + 4 \cos^2 t \sin^2 t)(-8 \cos t \sin t) - (2 \cos^2 t - 2 \sin^2 t)(8 \cos^3 t \sin t - 8 \sin^3 t \cos t)}{(1 + 4 \cos^2 t \sin^2 t)^2} \\ &= \frac{-8 \cos t \sin t(1 + 2 \sin^4 t + 2 \cos^4 t)}{(1 + 4 \cos^2 t \sin^2 t)^2}\end{aligned}$$

At $t = 0$, $\frac{d^2w}{dt^2} = 0$.

15. $w = x^2 + y^2$

$$x = s + t$$

$$y = s - t$$

$$\frac{\partial w}{\partial s} = 2x + 2y = 2(x + y) = 4s$$

$$\frac{\partial w}{\partial t} = 2x + 2y(-1) = 2(x - y) = 4t$$

When $s = 2$ and $t = -1$,

$$\frac{\partial w}{\partial s} = 8 \text{ and } \frac{\partial w}{\partial t} = -4.$$

17. $w = x^2 - y^2$

$$x = s \cos t$$

$$y = s \sin t$$

$$\begin{aligned}\frac{\partial w}{\partial s} &= 2x \cos t - 2y \sin t \\ &= 2s \cos^2 t - 2s \sin^2 t = 2s \cos 2t\end{aligned}$$

$$\frac{\partial w}{\partial t} = 2x(-s \sin t) - 2y(s \cos t) = -2s^2 \sin 2t$$

When $s = 3$ and $t = \frac{\pi}{4}$, $\frac{\partial w}{\partial s} = 0$ and $\frac{\partial w}{\partial t} = -18$.

19. $w = x^2 - 2xy + y^2$, $x = r + \theta$, $y = r - \theta$

(a) $\frac{\partial w}{\partial r} = (2x - 2y)(1) + (-2x + 2y)(1) = 0$

$$\frac{\partial w}{\partial \theta} = (2x - 2y)(1) + (-2x + 2y)(-1)$$

$$= 4x - 4y = 4(x - y)$$

$$= 4[(r + \theta) - (r - \theta)] = 8\theta$$

(b) $w = (r + \theta)^2 - 2(r + \theta)(r - \theta) + (r - \theta)^2$

$$= (r^2 + 2r\theta + \theta^2) - 2(r^2 - \theta^2) + (r^2 - 2r\theta + \theta^2)$$

$$= 4\theta^2$$

$$\frac{\partial w}{\partial r} = 0$$

$$\frac{\partial w}{\partial \theta} = 8\theta$$

21. $w = \arctan \frac{y}{x}$, $x = r \cos \theta$, $y = r \sin \theta$

(a) $\frac{\partial w}{\partial r} = \frac{-y}{x^2 + y^2} \cos \theta + \frac{x}{x^2 + y^2} \sin \theta = \frac{-r \sin \theta \cos \theta}{r^2} + \frac{r \cos \theta \sin \theta}{r^2} = 0$

$$\frac{\partial w}{\partial \theta} = \frac{-y}{x^2 + y^2} (-r \sin \theta) + \frac{x}{x^2 + y^2} (r \cos \theta) = \frac{-(r \sin \theta)(-r \sin \theta)}{r^2} + \frac{(r \cos \theta)(r \cos \theta)}{r^2} = 1$$

(b) $w = \arctan \frac{r \sin \theta}{r \cos \theta} = \arctan(\tan \theta) = \theta$

$$\frac{\partial w}{\partial r} = 0$$

$$\frac{\partial w}{\partial \theta} = 1$$

23. $w = xyz$, $x = s + t$, $y = s - t$, $z = st^2$

$$\begin{aligned} \frac{\partial w}{\partial s} &= yz(1) + xz(1) + xy(t^2) \\ &= (s - t)st^2 + (s + t)st^2 + (s + t)(s - t)t^2 \\ &= 2s^2t^2 + s^2t^2 - t^4 = 3s^2t^2 - t^4 = t^2(3s^2 - t^2) \end{aligned}$$

$$\begin{aligned} \frac{\partial w}{\partial t} &= yz(1) + xz(-1) + xy(2st) \\ &= (s - t)st^2 - (s + t)st^2 + (s + t)(s - t)(2st) \\ &= -2st^3 + 2s^3t - 2st^3 = 2s^3t - 4st^3 = 2st(s^2 - 2t^2) \end{aligned}$$

25. $w = ze^{x/y}$, $x = s - t$, $y = s + t$, $z = st$

$$\begin{aligned} \frac{\partial w}{\partial s} &= \frac{z}{y} e^{x/y}(1) + -\frac{zx}{y^2} e^{x/y}(1) + e^{x/y}(t) \\ &= e^{(s-t)/(s+t)} \left[\frac{st}{s+t} - \frac{(s-t)st}{(s+t)^2} + t \right] \\ &= e^{(s-t)/(s+t)} \left[\frac{st(s+t) - s^2t + st^2 + t(s+t)^2}{(s+t)^2} \right] \\ &= e^{(s-t)/(s+t)} \frac{t(s^2 + 4st + t^2)}{(s+t)^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{z}{y} e^{x/y}(-1) + -\frac{zx}{y^2} e^{x/y}(1) + e^{x/y}(s) \\ &= e^{(s-t)/(s+t)} \left[-\frac{st}{s+t} - \frac{st(s-t)}{(s+t)^2} + s \right] \\ &= e^{(s-t)/(s+t)} \left[\frac{-st(s+t) - st(s-t) + s(s+t)^2}{(s+t)^2} \right] \\ &= e^{(s-t)/(s+t)} \frac{s(s^2 + t^2)}{(s+t)^2} \end{aligned}$$

27. $x^2 - 3xy + y^2 - 2x + y - 5 = 0$

$$\begin{aligned} \frac{dy}{dx} &= -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{2x - 3y - 2}{-3x + 2y + 1} \\ &= \frac{3y - 2x + 2}{2y - 3x + 1} \end{aligned}$$

29. $\ln \sqrt{x^2 + y^2} + xy = 4$

$$\frac{1}{2} \ln(x^2 + y^2) + xy - 4 = 0$$

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{\frac{x}{x^2 + y^2} + y}{\frac{y}{x^2 + y^2} + x} = -\frac{x + x^2y + y^3}{y + xy^2 + x^3}$$

31. $F(x, y, z) = x^2 + y^2 + z^2 - 25$

$$F_x = 2x$$

$$F_y = 2y$$

$$F_z = 2z$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{x}{z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{y}{z}$$

33. $F(x, y, z) = \tan(x + y) + \tan(y + z) - 1$

$$F_x = \sec^2(x + y)$$

$$F_y = \sec^2(x + y) + \sec^2(y + z)$$

$$F_z = \sec^2(y + z)$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\sec^2(x + y)}{\sec^2(y + z)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\sec^2(x + y) + \sec^2(y + z)}{\sec^2(y + z)}$$

$$= -\left(\frac{\sec^2(x + y)}{\sec^2(y + z)} + 1 \right)$$

35. $x^2 + 2yz + z^2 - 1 = 0$

(i) $2x + 2y \frac{\partial z}{\partial x} + 2z \frac{\partial z}{\partial x} = 0$ implies $\frac{\partial z}{\partial x} = -\frac{x}{y+z}$.

(ii) $2y \frac{\partial z}{\partial y} + 2z + 2z \frac{\partial z}{\partial y} = 0$ implies $\frac{\partial z}{\partial y} = -\frac{z}{y+z}$.

37. $e^{xz} + xy = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = -\frac{ze^{xz} + y}{xe^{xz}}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = \frac{-x}{xe^{xz}} = \frac{-1}{e^{xz}} = -e^{-xz}$$

39. $F(x, y, z, w) = xyz + xzw - yzw + w^2 - 5$

$$F_x = yz + zw$$

$$F_y = xz - zw$$

$$F_z = xy + xw - yw$$

$$F_w = xz - yz + 2w$$

$$\frac{\partial w}{\partial x} = -\frac{F_x}{F_w} = -\frac{z(y+w)}{xz-yz+2w}$$

$$\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = -\frac{z(x-w)}{xz-yz+2w}$$

$$\frac{\partial w}{\partial z} = -\frac{F_z}{F_w} = -\frac{xy+xw-yw}{xz-yz+2w}$$

41. $F(x, y, z, w) = \cos xy + \sin yz + wz - 20$

$$\frac{\partial w}{\partial x} = \frac{-F_x}{F_w} = \frac{y \sin xy}{z}$$

$$\frac{\partial w}{\partial y} = \frac{-F_y}{F_w} = \frac{x \sin xy - z \cos yz}{z}$$

$$\frac{\partial w}{\partial z} = \frac{-F_z}{F_w} = -\frac{y \cos zy + w}{z}$$

43. $f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{(tx)(ty)}{\sqrt{(tx)^2 + (ty)^2}} = t \left(\frac{xy}{\sqrt{x^2 + y^2}} \right) = tf(x, y)$$

Degree: 1

$$\begin{aligned} xf_x(x, y) + yf_y(x, y) &= x \left(\frac{y^3}{(x^2 + y^2)^{3/2}} \right) + y \left(\frac{x^3}{(x^2 + y^2)^{3/2}} \right) \\ &= \frac{xy}{\sqrt{x^2 + y^2}} = 1f(x, y) \end{aligned}$$

45. $f(x, y) = e^{x/y}$

$$f(tx, ty) = e^{tx/ty} = e^{x/y} = f(x, y)$$

Degree: 0

$$xf_x(x, y) + yf_y(x, y) = x \left(\frac{1}{y} e^{x/y} \right) + y \left(-\frac{x}{y^2} e^{x/y} \right) = 0$$

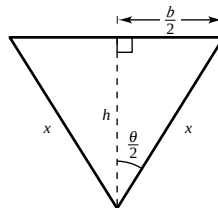
47. $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$ (Page 876)

49. $w = f(x, y)$ is the explicit form of a function of two variables, as in $z = x^2 + y^2$.
The implicit form is of the form $F(x, y, z) = 0$, as in $z - x^2 - y^2 = 0$.

51. $A = \frac{1}{2}bh = \left(x \sin \frac{\theta}{2} \right) \left(x \cos \frac{\theta}{2} \right) = \frac{x^2}{2} \sin \theta$

$$\frac{dA}{dt} = x \sin \theta \frac{dx}{dt} + \frac{x^2}{2} \cos \theta \frac{d\theta}{dt}$$

$$= 6 \left(\sin \frac{\pi}{4} \right) \left(\frac{1}{2} \right) + \frac{6^2}{2} \left(\cos \frac{\pi}{4} \right) \left(\frac{\pi}{90} \right) = \frac{3\sqrt{2}}{2} + \frac{\pi\sqrt{2}}{10} \text{ m}^2/\text{hr}$$



53. (a) $V = \frac{1}{3}\pi r^2 h$

$$\frac{dV}{dt} = \frac{1}{3}\pi \left(2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right) = \frac{1}{3}\pi [2(12)(36)(6) + (12)^2(-4)] = 1536\pi \text{ in.}^3/\text{min}$$

(b) $S = \pi r \sqrt{r^2 + h^2} + \pi r^2$ (Surface area includes base.)

$$\begin{aligned} \frac{dS}{dt} &= \pi \left[\left(\sqrt{r^2 + h^2} + \frac{r^2}{\sqrt{r^2 + h^2}} + 2r \right) \frac{dr}{dt} + \frac{rh}{\sqrt{r^2 + h^2}} \frac{dh}{dt} \right] \\ &= \pi \left[\left(\sqrt{12^2 + 36^2} + \frac{144}{\sqrt{12^2 + 36^2}} + 2(12) \right) (6) + \frac{36(12)}{\sqrt{12^2 + 36^2}} (-4) \right] \\ &= \pi \left[\left(12\sqrt{10} + \frac{12}{\sqrt{10}} \right) (6) + 144 + \frac{36}{\sqrt{10}} (-4) \right] \\ &= \frac{648\pi}{\sqrt{10}} + 144\pi \text{ in.}^2/\text{min} = \frac{36\pi}{5} (20 + 9\sqrt{10}) \text{ in.}^2/\text{min} \end{aligned}$$

55. $I = \frac{1}{2}m(r_1^2 + r_2^2)$

$$\frac{dI}{dt} = \frac{1}{2}m \left[2r_1 \frac{dr_1}{dt} + 2r_2 \frac{dr_2}{dt} \right] = m[(6)(2) + (8)(2)] = 28m \text{ cm}^2/\text{sec}$$

57. (a) $\tan \phi = \frac{2}{x}$

$$\tan(\theta + \phi) = \frac{4}{x}$$

$$\frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} = \frac{4}{x}$$

$$\frac{\tan \theta + (2/x)}{1 - (2/x)\tan \theta} = \frac{4}{x}$$

$$x \tan \theta + 2 = 4 - \frac{8}{x} \tan \theta$$

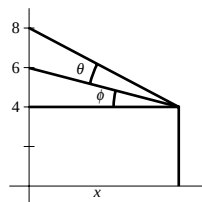
$$x^2 \tan \theta - 2x + 8 \tan \theta = 0$$

(b) $F(x, \theta) = (x^2 + 8)\tan \theta - 2x = 0$

$$\frac{d\theta}{dx} = -\frac{F_x}{F_\theta} = -\frac{2x \tan \theta - 2}{\sec^2 \theta (x^2 + 8)} = \frac{2 \cos^2 \theta - 2x \sin \theta \cos \theta}{x^2 + 8}$$

(c) $\frac{d\theta}{dx} = 0 \Rightarrow 2 \cos^2 \theta = 2x \sin \theta \cos \theta \Rightarrow \cos \theta = x \sin \theta \Rightarrow \tan \theta = \frac{1}{x}$

$$\text{Thus, } x^2 \left(\frac{1}{x} \right) - 2x + 8 \left(\frac{1}{x} \right) = 0 \Rightarrow \frac{8}{x} = x \Rightarrow x = 2\sqrt{2} \text{ ft.}$$



59. $w = f(x, y)$

$$x = u - v$$

$$y = v - u$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{dx}{du} + \frac{\partial w}{\partial y} \frac{dy}{du} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{dx}{dv} + \frac{\partial w}{\partial y} \frac{dy}{dv} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 0$$

61. $w = f(x, y)$, $x = r \cos \theta$, $y = r \sin \theta$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cos \theta + \frac{\partial w}{\partial y} \sin \theta$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} (-r \sin \theta) + \frac{\partial w}{\partial y} (r \cos \theta)$$

$$(a) \quad r \cos \theta \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} r \cos^2 \theta + \frac{\partial w}{\partial y} r \sin \theta \cos \theta$$

$$-\sin \theta \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} (r \sin^2 \theta) - \frac{\partial w}{\partial y} r \sin \theta \cos \theta$$

$$r \cos \theta \frac{\partial w}{\partial r} - \sin \theta \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} (r \cos^2 \theta + r \sin^2 \theta)$$

$$r \frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} (r \cos \theta) - \frac{\partial w}{\partial \theta} \sin \theta$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} \cos \theta - \frac{\partial w}{\partial \theta} \frac{\sin \theta}{r}$$

$$r \sin \theta \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} r \sin \theta \cos \theta + \frac{\partial w}{\partial y} r \sin^2 \theta$$

$$\cos \theta \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} (-r \sin \theta \cos \theta) + \frac{\partial w}{\partial y} (r \cos^2 \theta)$$

$$r \sin \theta \frac{\partial w}{\partial r} + \cos \theta \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial y} (r \sin^2 \theta + r \cos^2 \theta)$$

$$r \frac{\partial w}{\partial y} = \frac{\partial w}{\partial r} r \sin \theta + \frac{\partial w}{\partial \theta} \cos \theta$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r} \sin \theta + \frac{\partial w}{\partial \theta} \frac{\cos \theta}{r}$$

$$(b) \quad \left(\frac{\partial w}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial w}{\partial \theta} \right)^2 = \left(\frac{\partial w}{\partial x} \right)^2 \cos^2 \theta + 2 \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \sin \theta \cos \theta + \left(\frac{\partial w}{\partial y} \right)^2 \sin^2 \theta + \left(\frac{\partial w}{\partial x} \right)^2 \sin^2 \theta -$$

$$2 \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \sin \theta \cos \theta + \left(\frac{\partial w}{\partial y} \right)^2 \cos^2 \theta = \left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2$$

63. Given $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, $x = r \cos \theta$ and $y = r \sin \theta$.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta = \frac{\partial v}{\partial y} \cos \theta - \frac{\partial v}{\partial x} \sin \theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x} (-r \sin \theta) + \frac{\partial v}{\partial y} (r \cos \theta) = r \left[\frac{\partial v}{\partial y} \cos \theta - \frac{\partial v}{\partial x} \sin \theta \right]$$

Therefore, $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$.

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \cos \theta + \frac{\partial v}{\partial y} \sin \theta = -\frac{\partial u}{\partial y} \cos \theta + \frac{\partial u}{\partial x} \sin \theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} (-r \sin \theta) + \frac{\partial u}{\partial y} (r \cos \theta) = -r \left[-\frac{\partial u}{\partial y} \cos \theta + \frac{\partial u}{\partial x} \sin \theta \right]$$

Therefore, $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$.

Section 12.6 Directional Derivatives and Gradients

1. $f(x, y) = 3x - 4xy + 5y$

$$\mathbf{v} = \frac{1}{2}(\mathbf{i} + \sqrt{3}\mathbf{j})$$

$$\nabla f(x, y) = (3 - 4y)\mathbf{i} + (-4x + 5)\mathbf{j}$$

$$\nabla f(1, 2) = -5\mathbf{i} + \mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

$$D_{\mathbf{u}}f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = \frac{1}{2}(-5 + \sqrt{3})$$

3. $f(x, y) = xy$

$$\mathbf{v} = \mathbf{i} + \mathbf{j}$$

$$\nabla f(x, y) = y\mathbf{i} + x\mathbf{j}$$

$$\nabla f(2, 3) = 3\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$D_{\mathbf{u}}f(2, 3) = \nabla f(2, 3) \cdot \mathbf{u} = \frac{5\sqrt{2}}{2}$$

5. $g(x, y) = \sqrt{x^2 + y^2}$

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$$

$$\nabla g = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$$

$$\nabla g(3, 4) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}}g(3, 4) = \nabla g(3, 4) \cdot \mathbf{u} = -\frac{7}{25}$$

7. $h(x, y) = e^x \sin y$

$$\mathbf{v} = -\mathbf{i}$$

$$\nabla h = e^x \sin y \mathbf{i} + e^x \cos y \mathbf{j}$$

$$h\left(1, \frac{\pi}{2}\right) = e\mathbf{i}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\mathbf{i}$$

$$D_{\mathbf{u}}h\left(1, \frac{\pi}{2}\right) = \nabla h\left(1, \frac{\pi}{2}\right) \cdot \mathbf{u} = -e$$

9. $f(x, y, z) = xy + yz + xz$

$$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$$

$$\nabla f(x, y, z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$$

$$\nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{6}}{3}\mathbf{i} + \frac{\sqrt{6}}{6}\mathbf{j} - \frac{\sqrt{6}}{6}\mathbf{k}$$

$$D_{\mathbf{u}}f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \mathbf{u} = \frac{2\sqrt{6}}{3}$$

11. $h(x, y, z) = x \arctan yz$

$$\mathbf{v} = \langle 1, 2, -1 \rangle$$

$$\nabla h(x, y, z) = \arctan yz \mathbf{i} + \frac{xz}{1 + (yz)^2} \mathbf{j} + \frac{xy}{1 + (yz)^2} \mathbf{k}$$

$$\nabla h(4, 1, 1) = \frac{\pi}{4}\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\rangle$$

$$D_{\mathbf{u}}h(4, 1, 1) = \nabla h(4, 1, 1) \cdot \mathbf{u} = \frac{\pi + 8}{4\sqrt{6}} = \frac{(\pi + 8)\sqrt{6}}{24}$$

13. $f(x, y) = x^2 + y^2$

$$\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{2}{\sqrt{2}}x + \frac{2}{\sqrt{2}}y = \sqrt{2}(x + y)$$

15. $f(x, y) = \sin(2x - y)$

$$\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$$

$$\nabla f = 2 \cos(2x - y)\mathbf{i} - \cos(2x - y)\mathbf{j}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \cos(2x - y) + \frac{\sqrt{3}}{2} \cos(2x - y)$$

$$= \left(\frac{2 + \sqrt{3}}{2} \right) \cos(2x - y)$$

17. $f(x, y) = x^2 + 4y^2$

$$\mathbf{v} = -2\mathbf{i} - 2\mathbf{j}$$

$$\nabla f = 2x\mathbf{i} + 8y\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$$

$$D_{\mathbf{u}}f = -\frac{2}{\sqrt{2}}x - \frac{8}{\sqrt{2}}y = -\sqrt{2}(x + 4y)$$

At $P = (3, 1)$, $D_{\mathbf{u}}f = -7\sqrt{2}$.

21. $f(x, y) = 3x - 5y^2 + 10$

$$\nabla f(x, y) = 3\mathbf{i} - 10y\mathbf{j}$$

$$\nabla f(2, 1) = 3\mathbf{i} - 10\mathbf{j}$$

25. $w = 3x^2y - 5yz + z^2$

$$\nabla w(x, y, z) = 6xy\mathbf{i} + (3x^2 - 5z)\mathbf{j} + (2z - 5y)\mathbf{k}$$

$$\nabla w(1, 1, -2) = 6\mathbf{i} + 13\mathbf{j} - 9\mathbf{k}$$

29. $\overrightarrow{PQ} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$

$$\nabla f(x, y) = -e^{-x}\cos y\mathbf{i} - e^{-x}\sin y\mathbf{j}$$

$$\nabla f(0, 0) = -\mathbf{i}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

33. $g(x, y) = \ln \sqrt[3]{x^2 + y^2} = \frac{1}{3} \ln(x^2 + y^2)$

$$\nabla g(x, y) = \frac{1}{3} \left[\frac{2x}{x^2 + y^2} \mathbf{i} + \frac{2y}{x^2 + y^2} \mathbf{j} \right]$$

$$\nabla g(1, 2) = \frac{1}{3} \left(\frac{2}{5} \mathbf{i} + \frac{4}{5} \mathbf{j} \right) = \frac{2}{15} (\mathbf{i} + 2\mathbf{j})$$

$$\|\nabla g(1, 2)\| = \frac{2\sqrt{5}}{15}$$

37. $f(x, y, z) = xe^{yz}$

$$\nabla f(x, y, z) = e^{yz}\mathbf{i} + xze^{yz}\mathbf{j} + xye^{yz}\mathbf{k}$$

$$\nabla f(2, 0, -4) = \mathbf{i} - 8\mathbf{j}$$

$$\|\nabla f(2, 0, -4)\| = \sqrt{65}$$

19. $h(x, y, z) = \ln(x + y + z)$

$$\mathbf{v} = 3\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\nabla h = \frac{1}{x + y + z} (\mathbf{i} + \mathbf{j} + \mathbf{k})$$

At $(1, 0, 0)$, $\nabla h = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{19}} (3\mathbf{i} + 3\mathbf{j} + \mathbf{k})$$

$$D_{\mathbf{u}}h = \nabla h \cdot \mathbf{u} = \frac{7}{\sqrt{19}} = \frac{7\sqrt{19}}{19}$$

23. $z = \cos(x^2 + y^2)$

$$\nabla z(x, y) = -2x \sin(x^2 + y^2)\mathbf{i} - 2y \sin(x^2 + y^2)\mathbf{j}$$

$$\nabla z(3, -4) = -6 \sin 25\mathbf{i} + 8 \sin 25\mathbf{j} \approx 0.7941\mathbf{i} - 1.0588\mathbf{j}$$

27. $\overrightarrow{PQ} = 2\mathbf{i} + 4\mathbf{j}$, $\mathbf{u} = \frac{1}{\sqrt{5}}\mathbf{i} = \frac{2}{\sqrt{5}}\mathbf{j}$

$$\nabla g(x, y) = 2x\mathbf{i} + 2y\mathbf{j}, \nabla g(1, 2) = 2\mathbf{i} + 4\mathbf{j}$$

$$D_{\mathbf{u}}g = \nabla g \cdot \mathbf{u} = \frac{2}{\sqrt{5}} + \frac{8}{\sqrt{5}} = \frac{10}{\sqrt{5}} = 2\sqrt{5}$$

31. $h(x, y) = x \tan y$

$$\nabla h(x, y) = \tan y\mathbf{i} + x \sec^2 y\mathbf{j}$$

$$\nabla h\left(2, \frac{\pi}{4}\right) = \mathbf{i} + 4\mathbf{j}$$

$$\left\| \nabla h\left(2, \frac{\pi}{4}\right) \right\| = \sqrt{17}$$

35. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

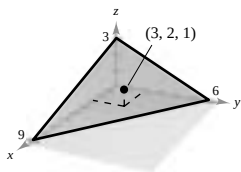
$$\nabla f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\nabla f(1, 4, 2) = \frac{1}{\sqrt{21}} (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$$

$$\|\nabla f(1, 4, 2)\| = 1$$

For Exercises 39–45, $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$ and $D_\theta f(x, y) = -\left(\frac{1}{3}\right) \cos \theta - \left(\frac{1}{2}\right) \sin \theta$.

39. $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$



41. (a) $D_{4\pi/3} f(3, 2) = -\left(\frac{1}{3}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)$
 $= \frac{2 + 3\sqrt{3}}{12}$

(b) $D_{-\pi/6} f(3, 2) = -\left(\frac{1}{3}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)$
 $= \frac{3 - 2\sqrt{3}}{12}$

43. (a) $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j}$

$$\|\mathbf{v}\| = \sqrt{9 + 16} = 5$$

$$\mathbf{u} = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{1}{5} - \frac{2}{5} = -\frac{1}{5}$$

(b) $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$

$$\|\mathbf{v}\| = \sqrt{10}$$

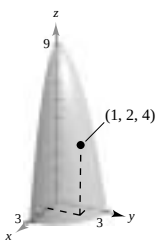
$$\mathbf{u} = \frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{-11}{6\sqrt{10}} = -\frac{11\sqrt{10}}{60}$$

45. $\|\nabla f\| = \sqrt{\frac{1}{9} + \frac{1}{4}} = \frac{1}{6}\sqrt{13}$

For Exercises 47 and 49, $f(x, y) = 9 - x^2 - y^2$ and $D_\theta f(x, y) = -2x \cos \theta - 2y \sin \theta = -2(x \cos \theta + y \sin \theta)$.

47. $f(x, y) = 9 - x^2 - y^2$



49. $\nabla f(1, 2) = -2\mathbf{i} - 4\mathbf{j}$

$$\|\nabla f(1, 2)\| = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}$$

51. (a) In the direction of the vector $-4\mathbf{i} + \mathbf{j}$.

(b) $\nabla f = \frac{1}{10}(2x - 3y)\mathbf{i} + \frac{1}{10}(-3x + 2y)\mathbf{j}$

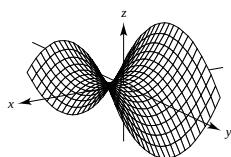
$$\nabla f(1, 2) = \frac{1}{10}(-4)\mathbf{i} + \frac{1}{10}(1)\mathbf{j} = -\frac{2}{5}\mathbf{i} + \frac{1}{10}\mathbf{j}$$

(Same direction as in part (a).)

(c) $-\nabla f = \frac{2}{5}\mathbf{i} - \frac{1}{10}\mathbf{j}$, the direction opposite that of the gradient.

53. $f(x, y) = x^2 - y^2$, $(4, -3, 7)$

(a)

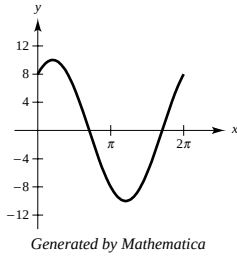


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53. —CONTINUED—

$$(b) D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = 2x \cos \theta - 2y \sin \theta$$

$$D_{\mathbf{u}} f(4, -3) = 8 \cos \theta + 6 \sin \theta$$



$$(c) \text{Zeros: } \theta \approx 2.21, 5.36$$

These are the angles θ for which $D_{\mathbf{u}} f(4, 3)$ equals zero.

$$(d) g(\theta) = D_{\mathbf{u}} f(4, -3) = 8 \cos \theta + 6 \sin \theta$$

$$g'(\theta) = -8 \sin \theta + 6 \cos \theta$$

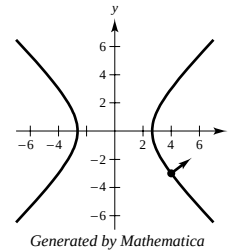
$$\text{Critical numbers: } \theta \approx 0.64, 3.79$$

These are the angles for which $D_{\mathbf{u}} f(4, -3)$ is a maximum (0.64) and minimum (3.79).

$$(e) \|\nabla f(4, -3)\| = \|2(4)\mathbf{i} - 2(3)\mathbf{j}\| = \sqrt{64 + 36} = 10, \text{ the maximum value of } D_{\mathbf{u}} f(4, -3), \text{ at } \theta = 0.64.$$

$$(f) f(x, y) = x^2 - y^2 = 7$$

$\nabla f(4, -3) = 8\mathbf{i} + 6\mathbf{j}$ is perpendicular to the level curve at $(4, -3)$.



$$55. f(x, y) = x^2 + y^2$$

$$c = 25, P = (3, 4)$$

$$\nabla f(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$$

$$x^2 + y^2 = 25$$

$$\nabla f(3, 4) = 6\mathbf{i} + 8\mathbf{j}$$

$$57. f(x, y) = \frac{x}{x^2 + y^2}$$

$$c = \frac{1}{2}, P = (1, 1)$$

$$\nabla f(x, y) = \frac{y^2 - x^2}{(x^2 + y^2)^2} \mathbf{i} - \frac{2xy}{(x^2 + y^2)^2} \mathbf{j}$$

$$\frac{x}{x^2 + y^2} = \frac{1}{2}$$

$$x^2 + y^2 - 2x = 0$$

$$\nabla f(1, 1) = -\frac{1}{2} \mathbf{j}$$

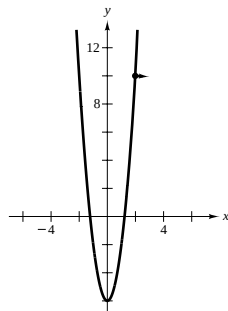
$$59. 4x^2 - y = 6$$

$$f(x, y) = 4x^2 - y$$

$$\nabla f(x, y) = 8x\mathbf{i} - \mathbf{j}$$

$$\nabla f(2, 10) = 16\mathbf{i} - \mathbf{j}$$

$$\begin{aligned} \frac{\nabla f(2, 10)}{\|\nabla f(2, 10)\|} &= \frac{1}{\sqrt{257}} (16\mathbf{i} - \mathbf{j}) \\ &= \frac{\sqrt{257}}{257} (16\mathbf{i} - \mathbf{j}) \end{aligned}$$



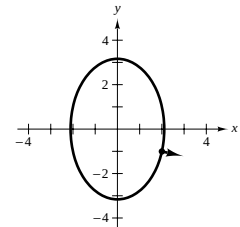
$$61. 9x^2 + 4y^2 = 40$$

$$f(x, y) = 9x^2 + 4y^2$$

$$\nabla f(x, y) = 18x\mathbf{i} + 8y\mathbf{j}$$

$$\nabla f(2, -1) = 36\mathbf{i} - 8\mathbf{j}$$

$$\begin{aligned} \frac{\nabla f(2, -1)}{\|\nabla f(2, -1)\|} &= \frac{1}{\sqrt{85}} (9\mathbf{i} - 2\mathbf{j}) \\ &= \frac{\sqrt{85}}{85} (9\mathbf{i} - 2\mathbf{j}) \end{aligned}$$



63. $T = \frac{x}{x^2 + y^2}$

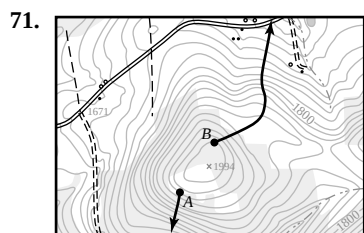
$$\nabla T = \frac{y^2 - x^2}{(x^2 + y^2)^2} \mathbf{i} - \frac{2xy}{(x^2 + y^2)^2} \mathbf{j}$$

$$\nabla T(3, 4) = \frac{7}{625} \mathbf{i} - \frac{24}{625} \mathbf{j} = \frac{1}{625} (7\mathbf{i} - 24\mathbf{j})$$

67. Let $f(x, y)$ be a function of two variables and $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ a unit vector.

(a) If $\theta = 0^\circ$, then $D_{\mathbf{u}} f = \frac{\partial f}{\partial x}$.

(b) If $\theta = 90^\circ$, then $D_{\mathbf{u}} f = \frac{\partial f}{\partial y}$.



Section 12.7 Tangent Planes and Normal Lines

1. $F(x, y, z) = 3x - 5y + 3z - 15 = 0$

$$3x - 5y + 3z = 15 \text{ Plane}$$

3. $F(x, y, z) = 4x^2 + 9y^2 - 4z^2 = 0$

$$4x^2 + 9y^2 = 4z^2 \text{ Elliptic cone}$$

5. $F(x, y, z) = x + y + z - 4$

$$\nabla F = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\begin{aligned} \mathbf{n} &= \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= \frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \end{aligned}$$

7. $F(x, y, z) = \sqrt{x^2 + y^2} - z$

$$\nabla F(x, y, z) = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j} - \mathbf{k}$$

$$\nabla F(3, 4, 5) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k}$$

$$\begin{aligned} \mathbf{n} &= \frac{\nabla F}{\|\nabla F\|} = \frac{5}{5\sqrt{2}}\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k}\right) \\ &= \frac{1}{5\sqrt{2}}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) \\ &= \frac{\sqrt{2}}{10}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) \end{aligned}$$

9. $F(x, y, z) = x^2y^4 - z$

$$\nabla F(x, y, z) = 2xy^4\mathbf{i} + 4x^2y^3\mathbf{j} - \mathbf{k}$$

$$\nabla F(1, 2, 16) = 32\mathbf{i} + 32\mathbf{j} - \mathbf{k}$$

$$\begin{aligned} \mathbf{n} &= \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{2049}}(32\mathbf{i} + 32\mathbf{j} - \mathbf{k}) \\ &= \frac{\sqrt{2049}}{2049}(32\mathbf{i} + 32\mathbf{j} - \mathbf{k}) \end{aligned}$$

11. $F(x, y, z) = \ln\left(\frac{x}{y-z}\right) = \ln x - \ln(y-z)$

$$\nabla F(x, y, z) = \frac{1}{x}\mathbf{i} - \frac{1}{y-z}\mathbf{j} + \frac{1}{y-z}\mathbf{k}$$

$$\nabla F(1, 4, 3) = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\begin{aligned} \mathbf{n} &= \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k}) \\ &= \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k}) \end{aligned}$$

13. $F(x, y, z) = -x \sin y + z - 4$

$$\nabla F(x, y, z) = -\sin y\mathbf{i} - x \cos y\mathbf{j} + \mathbf{k}$$

$$\nabla F\left(6, \frac{\pi}{6}, 7\right) = -\frac{1}{2}\mathbf{i} - 3\sqrt{3}\mathbf{j} + \mathbf{k}$$

$$\begin{aligned} \mathbf{n} &= \frac{\nabla F}{\|\nabla F\|} = \frac{2}{\sqrt{113}}\left(-\frac{1}{2}\mathbf{i} - 3\sqrt{3}\mathbf{j} + \mathbf{k}\right) \\ &= \frac{1}{\sqrt{113}}(-\mathbf{i} - 6\sqrt{3}\mathbf{j} + 2\mathbf{k}) \\ &= \frac{\sqrt{113}}{113}(-\mathbf{i} - 6\sqrt{3}\mathbf{j} + 2\mathbf{k}) \end{aligned}$$

15. $f(x, y) = 25 - x^2 - y^2, (3, 1, 15)$

$$F(x, y, z) = 25 - x^2 - y^2 - z$$

$$F_x(x, y, z) = -2x$$

$$F_y(x, y, z) = -2y$$

$$F_z(x, y, z) = -1$$

$$F_x(3, 1, 15) = -6$$

$$F_y(3, 1, 15) = -2$$

$$F_z(3, 1, 15) = -1$$

$$-6(x-3) - 2(y-1) - (z-15) = 0$$

$$0 = 6x + 2y + z - 35$$

$$6x + 2y + z = 35$$

17. $f(x, y) = \sqrt{x^2 + y^2}, (3, 4, 5)$

$$F(x, y, z) = \sqrt{x^2 + y^2} - z$$

$$F_x(x, y, z) = \frac{x}{\sqrt{x^2 + y^2}} \quad F_y(x, y, z) = \frac{y}{\sqrt{x^2 + y^2}} \quad F_z(x, y, z) = -1$$

$$F_x(3, 4, 5) = \frac{3}{5} \quad F_y(3, 4, 5) = \frac{4}{5} \quad F_z(3, 4, 5) = -1$$

$$\frac{3}{5}(x - 3) + \frac{4}{5}(y - 4) - (z - 5) = 0$$

$$3(x - 3) + 4(y - 4) - 5(z - 5) = 0$$

$$3x + 4y - 5z = 0$$

19. $g(x, y) = x^2 - y^2, (5, 4, 9)$

$$G(x, y, z) = x^2 - y^2 - z$$

$$G_x(x, y, z) = 2x \quad G_y(x, y, z) = -2y \quad G_z(x, y, z) = -1$$

$$G_x(5, 4, 9) = 10 \quad G_y(5, 4, 9) = -8 \quad G_z(5, 4, 9) = -1$$

$$10(x - 5) - 8(y - 4) - (z - 9) = 0$$

$$10x - 8y - z = 9$$

21. $z = e^x(\sin y + 1), \left(0, \frac{\pi}{2}, 2\right)$

$$F(x, y, z) = e^x(\sin y + 1) - z$$

$$F_x(x, y, z) = e^x(\sin y + 1) \quad F_y(x, y, z) = e^x \cos y \quad F_z(x, y, z) = -1$$

$$F_x\left(0, \frac{\pi}{2}, 2\right) = 2 \quad F_y\left(0, \frac{\pi}{2}, 2\right) = 0 \quad F_z\left(0, \frac{\pi}{2}, 2\right) = -1$$

$$2x - z = -2$$

23. $h(x, y) = \ln \sqrt{x^2 + y^2}, (3, 4, \ln 5)$

$$H(x, y, z) = \ln \sqrt{x^2 + y^2} - z = \frac{1}{2} \ln(x^2 + y^2) - z$$

$$H_x(x, y, z) = \frac{x}{x^2 + y^2} \quad H_y(x, y, z) = \frac{y}{x^2 + y^2} \quad H_z(x, y, z) = -1$$

$$H_x(3, 4, \ln 5) = \frac{3}{25} \quad H_y(3, 4, \ln 5) = \frac{4}{25} \quad H_z(3, 4, \ln 5) = -1$$

$$\frac{3}{25}(x - 3) + \frac{4}{25}(y - 4) - (z - \ln 5) = 0$$

$$3(x - 3) + 4(y - 4) - 25(z - \ln 5) = 0$$

$$3x + 4y - 25z = 25(1 - \ln 5)$$

25. $x^2 + 4y^2 + z^2 = 36, (2, -2, 4)$

$$F(x, y, z) = x^2 + 4y^2 + z^2 - 36$$

$$F_x(x, y, z) = 2x \quad F_y(x, y, z) = 8y \quad F_z(x, y, z) = 2z$$

$$F_x(2, -2, 4) = 4 \quad F_y(2, -2, 4) = -16 \quad F_z(2, -2, 4) = 8$$

$$4(x - 2) - 16(y + 2) + 8(z - 4) = 0$$

$$(x - 2) - 4(y + 2) + 2(z - 4) = 0$$

$$x - 4y + 2z = 18$$

27. $xy^2 + 3x - z^2 = 4$, $(2, 1, -2)$

$$F(x, y, z) = xy^2 + 3x - z^2 - 4$$

$$F_x(x, y, z) = y^2 + 3 \quad F_y(x, y, z) = 2xy \quad F_z(x, y, z) = -2z$$

$$F_x(2, 1, -2) = 4 \quad F_y(2, 1, -2) = 4 \quad F_z(2, 1, -2) = 4$$

$$4(x - 2) + 4(y - 1) + 4(z + 2) = 0$$

$$x + y + z = 1$$

29. $x^2 + y^2 + z = 9$, $(1, 2, 4)$

$$F(x, y, z) = x^2 + y^2 + z - 9$$

$$F_x(x, y, z) = 2x \quad F_y(x, y, z) = 2y \quad F_z(x, y, z) = 1$$

$$F_x(1, 2, 4) = 2 \quad F_y(1, 2, 4) = 4 \quad F_z(1, 2, 4) = 1$$

Direction numbers: 2, 4, 1

$$\text{Plane: } 2(x - 1) + 4(y - 2) + (z - 4) = 0, \quad 2x + 4y + z = 14$$

$$\text{Line: } \frac{x - 1}{2} = \frac{y - 2}{4} = \frac{z - 4}{1}$$

31. $xy - z = 0$, $(-2, -3, 6)$

$$F(x, y, z) = xy - z$$

$$F_x(x, y, z) = y \quad F_y(x, y, z) = x \quad F_z(x, y, z) = -1$$

$$F_x(-2, -3, 6) = -3 \quad F_y(-2, -3, 6) = -2 \quad F_z(-2, -3, 6) = -1$$

Direction numbers: 3, 2, 1

$$\text{Plane: } 3(x + 2) + 2(y + 3) + (z - 6) = 0, \quad 3x + 2y + z = -6$$

$$\text{Line: } \frac{x + 2}{3} = \frac{y + 3}{2} = \frac{z - 6}{1}$$

33. $z = \arctan \frac{y}{x}$, $\left(1, 1, \frac{\pi}{4}\right)$

$$F(x, y, z) = \arctan \frac{y}{x} - z$$

$$F_x(x, y, z) = \frac{-y}{x^2 + y^2} \quad F_y(x, y, z) = \frac{x}{x^2 + y^2} \quad F_z(x, y, z) = -1$$

$$F_x\left(1, 1, \frac{\pi}{4}\right) = -\frac{1}{2} \quad F_y\left(1, 1, \frac{\pi}{4}\right) = \frac{1}{2} \quad F_z\left(1, 1, \frac{\pi}{4}\right) = -1$$

Direction numbers: 1, -1, 2

$$\text{Plane: } (x - 1) - (y - 1) + 2\left(z - \frac{\pi}{4}\right) = 0, \quad x - y + 2z = \frac{\pi}{2}$$

$$\text{Line: } \frac{x - 1}{1} = \frac{y - 1}{-1} = \frac{z - (\pi/4)}{2}$$

35. $z = f(x, y) = \frac{4xy}{(x^2 + 1)(y^2 + 1)}, -2 \leq x \leq 2, 0 \leq y \leq 3$

(a) Let $F(x, y, z) = \frac{4xy}{(x^2 + 1)(y^2 + 1)} - z$

$$\begin{aligned}\nabla F(x, y, z) &= \frac{4y}{y^2 + 1} \left(\frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} \right) \mathbf{i} + \frac{4x}{x^2 + 1} \left(\frac{y^2 + 1 - 2y^2}{(y^2 + 1)^2} \right) \mathbf{j} - \mathbf{k} \\ &= \frac{4y(1 - x^2)}{(y^2 + 1)(x^2 + 1)^2} \mathbf{i} + \frac{4x(1 - y^2)}{(x^2 + 1)(y^2 + 1)^2} \mathbf{j} - \mathbf{k}\end{aligned}$$

$$\nabla F(1, 1, 1) = -\mathbf{k}.$$

Direction numbers: 0, 0, -1.

Line: $x = 1, y = 1, z = 1 - t$

Tangent plane: $0(x - 1) + 0(y - 1) - 1(z - 1) = 0 \Rightarrow z = 1$

(b) $\nabla F\left(-1, 2, -\frac{4}{5}\right) = 0\mathbf{i} + \frac{-4(-3)}{(2)(5)^2} \mathbf{j} - \mathbf{k} = \frac{6}{25} \mathbf{j} - \mathbf{k}$

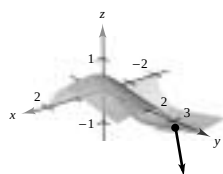
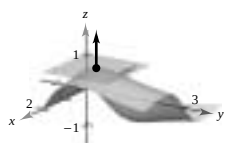
Line: $x = -1, y = 2 + \frac{6}{25}t, z = -\frac{4}{5} - t$

Plane: $0(x + 1) + \frac{6}{25}(y - 2) - 1\left(z + \frac{4}{5}\right) = 0$

$$6y - 12 - 25z - 20 = 0$$

$$6y - 25z - 32 = 0$$

(c)



(d) At $(1, 1, 1)$, the tangent plane is parallel to the xy -plane, implying that the surface is level there. At $\left(-1, 2, -\frac{4}{5}\right)$, the function does not change in the x -direction.

37. $F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$

(Theorem 12.13)

39. $F(x, y, z) = x^2 + y^2 - 5$

$G(x, y, z) = x - z$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j}$

$\nabla G(x, y, z) = \mathbf{i} - \mathbf{k}$

$\nabla F(2, 1, 2) = 4\mathbf{i} + 2\mathbf{j}$

$\nabla G(2, 1, 2) = \mathbf{i} - \mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} = -2(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

Direction numbers: $1, -2, 1, \frac{x-2}{1} = \frac{y-1}{-2} = \frac{z-2}{1}$

(b) $\cos \theta = \frac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \|\nabla G\|} = \frac{4}{\sqrt{20}\sqrt{2}} = \frac{2}{\sqrt{10}} = \frac{\sqrt{10}}{5}$; not orthogonal

41. $F(x, y, z) = x^2 + z^2 - 25$

$G(x, y, z) = y^2 + z^2 - 25$

$\nabla F = 2x\mathbf{i} + 2z\mathbf{k}$

$\nabla G = 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla F(3, 3, 4) = 6\mathbf{i} + 8\mathbf{k}$

$\nabla G(3, 3, 4) = 6\mathbf{j} + 8\mathbf{k}$

—CONTINUED—

41. —CONTINUED—

$$(a) \nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 0 & 8 \\ 0 & 6 & 8 \end{vmatrix} = -48\mathbf{i} - 48\mathbf{j} + 36\mathbf{k} = -12(4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k})$$

$$\text{Direction numbers: } 4, 4, -3, \frac{x-3}{4} = \frac{y-3}{4} = \frac{z-4}{-3}$$

$$(b) \cos \theta = \frac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \|\nabla G\|} = \frac{64}{(10)(10)} = \frac{16}{25}; \text{ not orthogonal}$$

$$43. F(x, y, z) = x^2 + y^2 + z^2 - 6 \quad G(x, y, z) = x - y - z$$

$$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \quad \nabla G(x, y, z) = \mathbf{i} - \mathbf{j} - \mathbf{k}$$

$$\nabla F(2, 1, 1) = 4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \quad \nabla G(2, 1, 1) = \mathbf{i} - \mathbf{j} - \mathbf{k}$$

$$(a) \nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 2 \\ 1 & -1 & -1 \end{vmatrix} = 6\mathbf{j} - 6\mathbf{k} = 6(\mathbf{j} - \mathbf{k}) \quad (b) \cos \theta = \frac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \|\nabla G\|} = 0; \text{ orthogonal}$$

$$\text{Direction numbers: } 0, 1, -1, x = 2, \frac{y-1}{1} = \frac{z-1}{-1}$$

$$45. f(x, y) = 6 - x^2 - \frac{y^2}{4}, \quad g(x, y) = 2x + y$$

$$(a) F(x, y, z) = z + x^2 + \frac{y^2}{4} - 6$$

$$G(x, y, z) = z - 2x - y$$

$$\nabla F(x, y, z) = 2x\mathbf{i} + \frac{1}{2}y\mathbf{j} + \mathbf{k}$$

$$\nabla G(x, y, z) = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\nabla F(1, 2, 4) = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\nabla G(1, 2, 4) = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$$

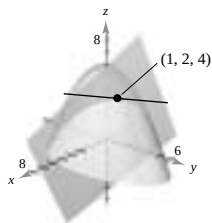
The cross product of these gradients is parallel to the curve of intersection.

$$\nabla F(1, 2, 4) \times \nabla G(1, 2, 4) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -2 & -1 & 1 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j}$$

Using direction numbers 1, -2, 0, you get $x = 1 + t$, $y = 2 - 2t$, $z = 4$.

$$\cos \theta = \frac{\nabla F \cdot \nabla G}{\|\nabla F\| \|\nabla G\|} = \frac{-4 - 1 + 1}{\sqrt{6} \sqrt{6}} = \frac{-4}{6} \Rightarrow \theta \approx 48.2^\circ$$

(b)



$$47. F(x, y, z) = 3x^2 + 2y^2 - z - 15, \quad (2, 2, 5)$$

$$\nabla F(x, y, z) = 6x\mathbf{i} + 4y\mathbf{j} - \mathbf{k}$$

$$\nabla F(2, 2, 5) = 12\mathbf{i} + 8\mathbf{j} - \mathbf{k}$$

$$\cos \theta = \frac{|\nabla F(2, 2, 5) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 5)\|} = \frac{1}{\sqrt{209}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{209}}\right) \approx 86.03^\circ$$

$$49. F(x, y, z) = x^2 - y^2 + z, \quad (1, 2, 3)$$

$$\nabla F(x, y, z) = 2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$$

$$\nabla F(1, 2, 3) = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

$$\cos \theta = \frac{|\nabla F(1, 2, 3) \cdot \mathbf{k}|}{\|\nabla F(1, 2, 3)\|} = \frac{1}{\sqrt{21}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{21}}\right) \approx 77.40^\circ$$

51. $F(x, y, z) = 3 - x^2 - y^2 + 6y - z$

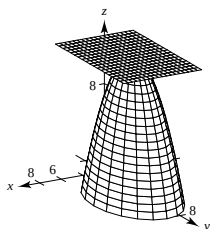
$$\nabla F(x, y, z) = -2x\mathbf{i} + (-2y + 6)\mathbf{j} - \mathbf{k}$$

$$-2x = 0, x = 0$$

$$-2y + 6 = 0, y = 3$$

$$z = 3 - 0^2 - 3^2 + 6(3) = 12$$

$$(0, 3, 12) \text{ (vertex of paraboloid)}$$



55. $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

$$F_x(x, y, z) = \frac{2x}{a^2}$$

$$F_y(x, y, z) = \frac{2y}{b^2}$$

$$F_z(x, y, z) = \frac{2z}{c^2}$$

$$\text{Plane: } \frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) + \frac{2z_0}{c^2}(z - z_0) = 0$$

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1$$

59. $f(x, y) = e^{x-y}$

$$f_x(x, y) = e^{x-y}, \quad f_y(x, y) = -e^{x-y}$$

$$f_{xx}(x, y) = e^{x-y}, \quad f_{yy}(x, y) = e^{x-y},$$

$$(a) P_1(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = 1 + x - y$$

$$(b) P_2(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2}f_{yy}(0, 0)y^2$$

$$= 1 + x - y + \frac{1}{2}x^2 - xy + \frac{1}{2}y^2$$

$$(c) \text{ If } x = 0, P_2(0, y) = 1 - y + \frac{1}{2}y^2. \text{ This is the second-degree Taylor polynomial for } e^{-y}.$$

$$\text{If } y = 0, P_2(x, 0) = 1 + x + \frac{1}{2}x^2. \text{ This is the second-degree Taylor polynomial for } e^x.$$

(d)

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1	1	1
0	0	0.9048	0.9000	0.9050
0.2	0.1	1.1052	1.1000	1.1050
0.2	0.5	0.7408	0.7000	0.7450
1	0.5	1.6487	1.5000	1.6250

53. $T(x, y, z) = 400 - 2x^2 - y^2 - 4z^2, (4, 3, 10)$

$$\frac{dx}{dt} = -4kx \quad \frac{dy}{dt} = -2ky \quad \frac{dz}{dt} = -8kz$$

$$x(t) = C_1 e^{-4kt} \quad y(t) = C_2 e^{-2kt} \quad z(t) = C_3 e^{-8kt}$$

$$x(0) = C_1 = 4 \quad y(0) = C_2 = 3 \quad z(0) = C_3 = 10$$

$$x = 4e^{-4kt} \quad y = 3e^{-2kt} \quad z = 10e^{-8kt}$$

57. $F(x, y, z) = a^2x^2 + b^2y^2 - z^2$

$$F_x(x, y, z) = 2a^2x$$

$$F_y(x, y, z) = 2b^2y$$

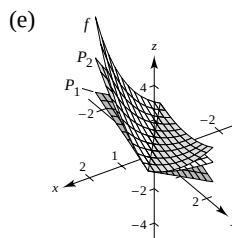
$$F_z(x, y, z) = -2z$$

$$\text{Plane: } 2a^2x_0(x - x_0) + 2b^2y_0(y - y_0) - 2z_0(z - z_0) = 0$$

$$a^2x_0x + b^2y_0y - z_0z = a^2x_0^2 + b^2y_0^2 - z_0^2 = 0$$

Hence, the plane passes through the origin.

$$f_{xy}(x, y) = -e^{x-y}$$



61. Given $w = F(x, y, z)$ where F is differentiable at

$$(x_0, y_0, z_0) \text{ and } \nabla F(x_0, y_0, z_0) \neq \mathbf{0},$$

the level surface of F at (x_0, y_0, z_0) is of the form $F(x, y, z) = C$ for some constant C . Let

$$G(x, y, z) = F(x, y, z) - C = 0.$$

Then $\nabla G(x_0, y_0, z_0) = \nabla F(x_0, y_0, z_0)$ where $\nabla G(x_0, y_0, z_0)$ is normal to $F(x_0, y_0, z_0) - C = 0$.

Therefore, $\nabla F(x_0, y_0, z_0)$ is normal to $F(x_0, y_0, z_0) = C$.

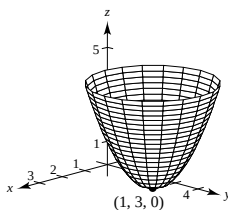
Section 12.8 Extrema of Functions of Two Variables

1. $g(x, y) = (x - 1)^2 + (y - 3)^2 \geq 0$

Relative minimum: $(1, 3, 0)$

$$g_x = 2(x - 1) = 0 \Rightarrow x = 1$$

$$g_y = 2(y - 3) = 0 \Rightarrow y = 3$$



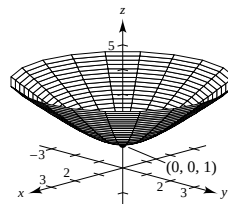
3. $f(x, y) = \sqrt{x^2 + y^2 + 1} \geq 1$

Relative minimum: $(0, 0, 1)$

Check: $f_x = \frac{x}{\sqrt{x^2 + y^2 + 1}} = 0 \Rightarrow x = 0$

$$f_y = \frac{y}{\sqrt{x^2 + y^2 + 1}} = 0 \Rightarrow y = 0$$

$$f_{xx} = \frac{y^2 + 1}{(x^2 + y^2 + 1)^{3/2}}, f_{yy} = \frac{x^2 + 1}{(x^2 + y^2 + 1)^{3/2}}, f_{xy} = \frac{-xy}{(x^2 + y^2 + 1)^{3/2}}$$

At the critical point $(0, 0)$, $f_{xx} > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(0, 0, 1)$ is a relative minimum.

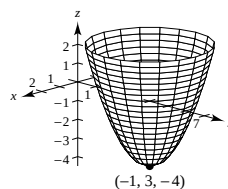
5. $f(x, y) = x^2 + y^2 + 2x - 6y + 6 = (x + 1)^2 + (y - 3)^2 - 4 \geq -4$

Relative minimum: $(-1, 3, -4)$

Check: $f_x = 2x + 2 = 0 \Rightarrow x = -1$

$$f_y = 2y - 6 = 0 \Rightarrow y = 3$$

$$f_{xx} = 2, f_{yy} = 2, f_{xy} = 0$$

At the critical point $(-1, 3)$, $f_{xx} > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(-1, 3, -4)$ is a relative minimum.

7. $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$

$$\left. \begin{aligned} f_x &= 4x + 2y + 2 = 0 \\ f_y &= 2x + 2y = 0 \end{aligned} \right\} \text{Solving simultaneously yields } x = -1 \text{ and } y = 1.$$

$$f_{xx} = 4, f_{yy} = 2, f_{xy} = 2$$

At the critical point $(-1, 1)$, $f_{xx} > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(-1, 1, -4)$ is a relative minimum.

9. $f(x, y) = -5x^2 + 4xy - y^2 + 16x + 10$

$$\left. \begin{aligned} f_x &= -10x + 4y + 16 = 0 \\ f_y &= 4x - 2y = 0 \end{aligned} \right\} \text{Solving simultaneously yields } x = 8 \text{ and } y = 16.$$

$$f_{xx} = -10, f_{yy} = -2, f_{xy} = 4$$

At the critical point $(8, 16)$, $f_{xx} < 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(8, 16, 74)$ is a relative maximum.

11. $f(x, y) = 2x^2 + 3y^2 - 4x - 12y + 13$

$$f_x = 4x - 4 = 4(x - 1) = 0 \text{ when } x = 1.$$

$$f_y = 6y - 12 = 6(y - 2) = 0 \text{ when } y = 2.$$

$$f_{xx} = 4, f_{yy} = 6, f_{xy} = 0$$

At the critical point $(1, 2)$, $f_{xx} > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(1, 2, -1)$ is a relative minimum.

13. $f(x, y) = 2\sqrt{x^2 + y^2} + 3$

$$\left. \begin{aligned} f_x &= \frac{2x}{\sqrt{x^2 + y^2}} = 0 \\ f_y &= \frac{2y}{\sqrt{x^2 + y^2}} = 0 \end{aligned} \right\} x = 0, y = 0$$

Since $f(x, y) \geq 3$ for all (x, y) , $(0, 0, 3)$ is relative minimum.

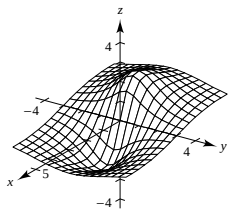
15. $g(x, y) = 4 - |x| - |y|$

$(0, 0)$ is the only critical point. Since $g(x, y) \leq 4$ for all (x, y) , $(0, 0, 4)$ is relative maximum.

17. $z = \frac{-4x}{x^2 + y^2 + 1}$

Relative minimum: $(1, 0, -2)$

Relative maximum: $(-1, 0, 2)$

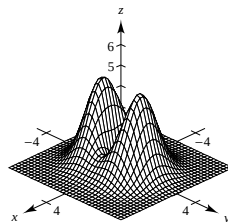


19. $z = (x^2 + 4y^2)e^{1-x^2-y^2}$

Relative minimum: $(0, 0, 0)$

Relative maxima: $(0, \pm 1, 4)$

Saddle points: $(\pm 1, 0, 1)$



21. $h(x, y) = x^2 - y^2 - 2x - 4y - 4$

$$h_x = 2x - 2 = 2(x - 1) = 0 \text{ when } x = 1.$$

$$h_y = -2y - 4 = -2(y + 2) = 0 \text{ when } y = -2.$$

$$h_{xx} = 2, h_{yy} = -2, h_{xy} = 0$$

At the critical point $(1, -2)$, $h_{xx}h_{yy} - (h_{xy})^2 < 0$. Therefore, $(1, -2, -1)$ is a saddle point.

23. $h(x, y) = x^2 - 3xy - y^2$

$$\left. \begin{array}{l} h_x = 2x - 3y = 0 \\ h_y = -3x - 2y = 0 \end{array} \right\} \text{ Solving simultaneously yields } x = 0 \text{ and } y = 0.$$

$$h_{xx} = 2, h_{yy} = -2, h_{xy} = -3$$

At the critical point $(0, 0)$, $h_{xx}h_{yy} - (h_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point.

25. $f(x, y) = x^3 - 3xy + y^3$

$$\left. \begin{array}{l} f_x = 3(x^2 - y) = 0 \\ f_y = 3(-x + y^2) = 0 \end{array} \right\} \text{ Solving by substitution yields two critical points } (0, 0) \text{ and } (1, 1).$$

$$f_{xx} = 6x, f_{yy} = 6y, f_{xy} = -3$$

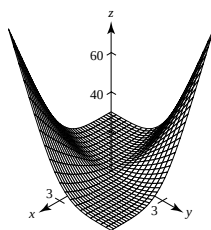
At the critical point $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point. At the critical point $(1, 1)$, $f_{xx} = 6 > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $(1, 1, -1)$ is a relative minimum.

27. $f(x, y) = e^{-x} \sin y$

$$\left. \begin{array}{l} f_x = -e^{-x} \sin y = 0 \\ f_y = e^{-x} \cos y = 0 \end{array} \right\} \text{ Since } e^{-x} > 0 \text{ for all } x \text{ and } \sin y \text{ and } \cos y \text{ are never both zero for a given value of } y, \text{ there are no critical points.}$$

29. $z = \frac{(x - y)^4}{x^2 + y^2} \geq 0$. $z = 0$ if $x = y \neq 0$.

Relative minimum at all points (x, x) , $x \neq 0$.



$$31. f_{xx} f_{yy} - (f_{xy})^2 = (9)(4) - 6^2 = 0$$

Insufficient information.

$$33. f_{xx} f_{yy} - (f_{xy})^2 = (-9)(6) - 10^2 < 0$$

f has a saddle point at (x_0, y_0) .

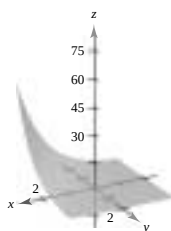
35. (a) The function f defined on a region R containing (x_0, y_0) has a relative minimum at (x_0, y_0) if $f(x, y) \geq f(x_0, y_0)$ for all (x, y) in R .

(b) The function f defined on a region R containing (x_0, y_0) has a relative maximum at (x_0, y_0) if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) in R .

(c) A saddle point is a critical point which is not a relative extremum.

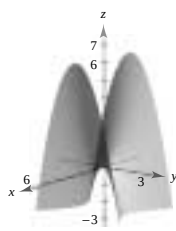
(d) See definition page 906.

37.



No extrema

39.



Saddle point

41. The point A will be a saddle point. The function could be

$$f(x, y) = x^2 - y^2.$$

$$43. d = f_{xx} f_{yy} - f_{xy}^2 = (2)(8) - f_{xy}^2 = 16 - f_{xy}^2 > 0$$

$$\Rightarrow f_{xy}^2 < 16 \Rightarrow -4 < f_{xy} < 4$$

$$45. f(x, y) = x^3 + y^3$$

$$\left. \begin{array}{l} f_x = 3x^2 = 0 \\ f_y = 3y^2 = 0 \end{array} \right\} \text{Solving yields } x = y = 0$$

$$f_{xx} = 6x, f_{yy} = 6y, f_{xy} = 0$$

At $(0, 0)$, $f_{xx} f_{yy} - (f_{xy})^2 = 0$ and the test fails. $(0, 0, 0)$ is a saddle point.

$$47. f(x, y) = (x - 1)^2(y + 4)^2 \geq 0$$

$$\left. \begin{array}{l} f_x = 2(x - 1)(y + 4)^2 = 0 \\ f_y = 2(x - 1)^2(y + 4) = 0 \end{array} \right\} \text{Solving yields the critical points } (1, a) \text{ and } (b, -4).$$

$$f_{xx} = 2(y + 4)^2, f_{yy} = 2(x - 1)^2, f_{xy} = 4(x - 1)(y + 4)$$

At both $(1, a)$ and $(b, -4)$, $f_{xx} f_{yy} - (f_{xy})^2 = 0$ and the test fails.

Absolute minima: $(1, a, 0)$ and $(b, -4, 0)$

$$49. f(x, y) = x^{2/3} + y^{2/3} \geq 0$$

$$\left. \begin{array}{l} f_x = \frac{2}{3\sqrt[3]{x}} \\ f_y = \frac{2}{3\sqrt[3]{y}} \end{array} \right\} f_x \text{ and } f_y \text{ are undefined at } x = 0, y = 0. \text{ The critical point is } (0, 0).$$

$$f_{xx} = -\frac{2}{9x\sqrt[3]{x}}, f_{yy} = -\frac{2}{9y\sqrt[3]{y}}, f_{xy} = 0$$

At $(0, 0)$, $f_{xx} f_{yy} - (f_{xy})^2$ is undefined and the test fails.

Absolute minimum: 0 at $(0, 0)$

$$51. f(x, y, z) = x^2 + (y - 3)^2 + (z + 1)^2 \geq 0$$

$$\left. \begin{array}{l} f_x = 2x = 0 \\ f_y = 2(y - 3) = 0 \\ f_z = 2(z + 1) = 0 \end{array} \right\} \text{Solving yields the critical point } (0, 3, -1).$$

Absolute minimum: 0 at $(0, 3, -1)$

53. $f(x, y) = 12 - 3x - 2y$ has no critical points. On the line $y = x + 1$, $0 \leq x \leq 1$,

$$f(x, y) = f(x) = 12 - 3x - 2(x + 1) = -5x + 10$$

and the maximum is 10, the minimum is 5. On the line $y = -2x + 4$, $1 \leq x \leq 2$,

$$f(x, y) = f(x) = 12 - 3x - 2(-2x + 4) = x + 4$$

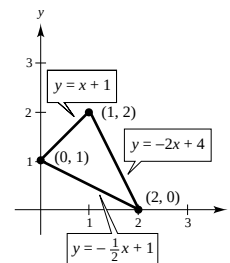
and the maximum is 6, the minimum is 5. On the line $y = -\frac{1}{2}x + 1$, $0 \leq x \leq 2$,

$$f(x, y) = f(x) = 12 - 3x - 2(-\frac{1}{2}x + 1) = -2x + 10$$

and the maximum is 10, the minimum is 6.

Absolute maximum: 10 at (0, 1)

Absolute minimum: 5 at (1, 2)



55. $f(x, y) = 3x^2 + 2y^2 - 4y$

$$\left. \begin{aligned} f_x = 6x = 0 &\Rightarrow x = 0 \\ f_y = 4y - 4 = 0 &\Rightarrow y = 1 \end{aligned} \right\} f(0, 1) = -2$$

On the line $y = 4$, $-2 \leq x \leq 2$,

$$f(x, y) = f(x) = 3x^2 + 32 - 16 = 3x^2 + 16$$

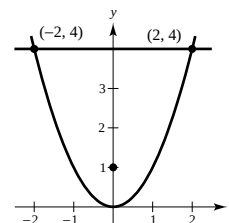
and the maximum is 28, the minimum is 16. On the curve $y = x^2$, $-2 \leq x \leq 2$,

$$f(x, y) = f(x) = 3x^2 + 2(x^2)^2 - 4x^2 = 2x^4 - x^2 = x^2(2x^2 - 1)$$

and the maximum is 28, the minimum is $-\frac{1}{8}$.

Absolute maximum: 28 at $(\pm 2, 4)$

Absolute minimum: -2 at (0, 1)



57. $f(x, y) = x^2 + xy$, $R = \{(x, y): |x| \leq 2, |y| \leq 1\}$

$$\left. \begin{aligned} f_x = 2x + y = 0 \\ f_y = x = 0 \end{aligned} \right\} x = y = 0$$

$$f(0, 0) = 0$$

Along $y = 1$, $-2 \leq x \leq 2$, $f = x^2 + x$, $f' = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$.

Thus, $f(-2, 1) = 2$, $f(-\frac{1}{2}, 1) = -\frac{1}{4}$ and $f(2, 1) = 6$.

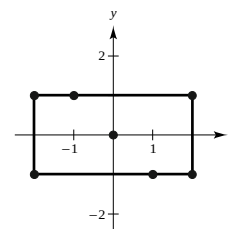
Along $y = -1$, $-2 \leq x \leq 2$, $f = x^2 - x$, $f' = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$.

Thus, $f(-2, -1) = 6$, $f(\frac{1}{2}, -1) = -\frac{1}{4}$, $f(2, -1) = 2$.

Along $x = 2$, $-1 \leq y \leq 1$, $f = 4 + 2y \Rightarrow f' = 2 \neq 0$.

Along $x = -2$, $-1 \leq y \leq 1$, $f = 4 - 2y \Rightarrow f' = -2 \neq 0$.

Thus, the maxima are $f(2, 1) = 6$ and $f(-2, -1) = 6$ and the minima are $f(-\frac{1}{2}, 1) = -\frac{1}{4}$ and $f(\frac{1}{2}, -1) = -\frac{1}{4}$.



59. $f(x, y) = x^2 + 2xy + y^2$, $R = \{(x, y): x^2 + y^2 \leq 8\}$

$$\left. \begin{aligned} f_x = 2x + 2y = 0 \\ f_y = 2x + 2y = 0 \end{aligned} \right\} y = -x$$

$$f(x, -x) = x^2 - 2x^2 + x^2 = 0$$

On the boundary $x^2 + y^2 = 8$, we have $y^2 = 8 - x^2$ and $y = \pm\sqrt{8 - x^2}$. Thus,

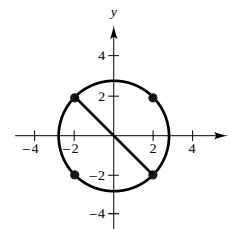
$$f = x^2 \pm 2x\sqrt{8 - x^2} + (8 - x^2) = 8 \pm 2x\sqrt{8 - x^2}$$

$$f' = \pm(8 - x^2)^{-1/2}(-2x) + 2(8 - x^2)^{1/2} = \pm \frac{16 - 4x^2}{\sqrt{8 - x^2}}$$

Then, $f' = 0$ implies $16 = 4x^2$ or $x = \pm 2$.

$$f(2, 2) = f(-2, -2) = 16 \quad \text{and} \quad f(2, -2) = f(-2, 2) = 0$$

Thus, the maxima are $f(2, 2) = 16$ and $f(-2, -2) = 16$, and the minima are $f(x, -x) = 0$, $|x| \leq 2$.



$$61. f(x, y) = \frac{4xy}{(x^2 + 1)(y^2 + 1)}, R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

$$f_x = \frac{4(1 - x^2)y}{(y^2 + 1)(x^2 + 1)} = 0 \Rightarrow x = 1 \text{ or } y = 0$$

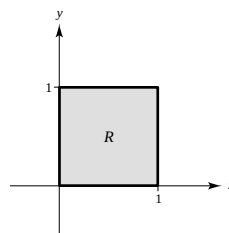
$$f_y = \frac{4(1 - y^2)x}{(x^2 + 1)(y^2 + 1)^2} \Rightarrow x = 0 \text{ or } y = 1$$

For $x = 0, y = 0$, also, and $f(0, 0) = 0$.

For $x = 1, y = 1$, $f(1, 1) = 1$.

The absolute maximum is $1 = f(1, 1)$.

The absolute minimum is $0 = f(0, 0)$. (In fact, $f(0, y) = f(x, 0) = 0$)



63. False

Let $f(x, y) = |1 - x - y|$.

$(0, 0, 1)$ is a relative maximum, but $f'_x(0, 0)$ and $f'_y(0, 0)$ do not exist.

Section 12.9 Applications of Extrema of Functions of Two Variables

1. A point on the plane is given by $(x, y, 12 - 2x - 3y)$. The square of the distance from the origin to this point is

$$S = x^2 + y^2 + (12 - 2x - 3y)^2$$

$$S_x = 2x + 2(12 - 2x - 3y)(-2)$$

$$S_y = 2y + 2(12 - 2x - 3y)(-3)$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain the system

$$5x + 6y = 24$$

$$3x + 5y = 18.$$

Solving simultaneously, we have $x = \frac{12}{7}, y = \frac{18}{7}$
 $z = 12 - \frac{24}{7} - \frac{54}{7} = \frac{6}{7}$. Therefore, the distance from the origin to $(\frac{12}{7}, \frac{18}{7}, \frac{6}{7})$ is

$$\sqrt{\left(\frac{12}{7}\right)^2 + \left(\frac{18}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \frac{6\sqrt{14}}{7}.$$

3. A point on the paraboloid is given by $(x, y, x^2 + y^2)$. The square of the distance from $(5, 5, 0)$ to a point on the paraboloid is given by

$$S = (x - 5)^2 + (y - 5)^2 + (x^2 + y^2)^2$$

$$S_x = 2(x - 5) + 4x(x^2 + y^2) = 0$$

$$S_y = 2(y - 5) + 4y(x^2 + y^2) = 0.$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain the system

$$2x^3 + 2xy^2 + x - 5 = 0$$

$$2y^3 + 2x^2y + y - 5 = 0$$

Multiply the first equation by y and the second equation by x , and subtract to obtain $x = y$. Then, we have $x = 1$, $y = 1$, $z = 2$ and the distance is

$$\sqrt{(1 - 5)^2 + (1 - 5)^2 + (2 - 0)^2} = 6.$$

5. Let x, y and z be the numbers. Since $x + y + z = 30$, $z = 30 - x - y$.

$$P = xyz = 30xy - x^2y - xy^2$$

$$P_x = 30y - 2xy - y^2 = y(30 - 2x - y) = 0 \quad \left. \begin{array}{l} 2x + y = 30 \\ P_y = 30x - x^2 - 2xy = x(30 - x - 2y) = 0 \end{array} \right\} x + 2y = 30$$

$$P_y = 30x - x^2 - 2xy = x(30 - x - 2y) = 0 \quad \left. \begin{array}{l} 2x + y = 30 \\ P_y = 30x - x^2 - 2xy = x(30 - x - 2y) = 0 \end{array} \right\} x + 2y = 30$$

Solving simultaneously yields $x = 10, y = 10$, and $z = 10$.

7. Let x, y , and z be the numbers and let $S = x^2 + y^2 + z^2$. Since $x + y + z = 30$, we have

$$S = x^2 + y^2 + (30 - x - y)^2$$

$$S_x = 2x + 2(30 - x - y)(-1) = 0 \quad \left. \begin{array}{l} 2x + y = 30 \\ S_y = 2y + 2(30 - x - y)(-1) = 0 \end{array} \right\} x + 2y = 30$$

$$S_y = 2y + 2(30 - x - y)(-1) = 0 \quad \left. \begin{array}{l} 2x + y = 30 \\ S_y = 2y + 2(30 - x - y)(-1) = 0 \end{array} \right\} x + 2y = 30$$

Solving simultaneously yields $x = 10, y = 10$, and $z = 10$.

9. Let x , y , and z be the length, width, and height, respectively. Then the sum of the length and girth is given by $x + (2y + 2z) = 108$ or $x = 108 - 2y - 2z$. The volume is given by

$$V = xyz = 108zy - 2zy^2 - 2yz^2$$

$$V_y = 108z - 4yz - 2z^2 = z(108 - 4y - 2z) = 0$$

$$V_z = 108y - 2y^2 - 4yz = y(108 - 2y - 4z) = 0.$$

Solving the system $4y + 2z = 108$ and $2y + 4z = 108$, we obtain the solution $x = 36$ inches, $y = 18$ inches, and $z = 18$ inches.

11. Let $a + b + c = k$. Then

$$V = \frac{4\pi abc}{3} = \frac{4}{3}\pi ab(k - a - b)$$

$$= \frac{4}{3}\pi(kab - a^2b - ab^2)$$

$$\left. \begin{aligned} V_a = \frac{4\pi}{3}(kb - 2ab - b^2) = 0 \\ V_b = \frac{4\pi}{3}(ka - a^2 - 2ab) = 0 \end{aligned} \right\} \begin{aligned} kb - 2ab - b^2 &= 0 \\ ka - a^2 - 2ab &= 0. \end{aligned}$$

Solving this system simultaneously yields $a = b$ and substitution yields $b = k/3$. Therefore, the solution is $a = b = c = k/3$.

13. Let x , y , and z be the length, width, and height, respectively and let V_0 be the given volume.

Then $V_0 = xyz$ and $z = V_0/xy$. The surface area is

$$S = 2xy + 2yz + 2xz = 2\left(xy + \frac{V_0}{x} + \frac{V_0}{y}\right)$$

$$S_x = 2\left(y - \frac{V_0}{x^2}\right) = 0 \quad \left\{ \begin{aligned} x^2y - V_0 &= 0 \end{aligned} \right.$$

$$S_y = 2\left(x - \frac{V_0}{y^2}\right) = 0 \quad \left\{ \begin{aligned} xy^2 - V_0 &= 0. \end{aligned} \right.$$

Solving simultaneously yields $x = \sqrt[3]{V_0}$, $y = \sqrt[3]{V_0}$, and $z = \sqrt[3]{V_0}$.

15. The distance from P to Q is $\sqrt{x^2 + 4}$. The distance from Q to R is $\sqrt{(y - x)^2 + 1}$. The distance from R to S is $10 - y$.

$$C = 3k\sqrt{x^2 + 4} + 2k\sqrt{(y - x)^2 + 1} + k(10 - y)$$

$$C_x = 3k\left(\frac{x}{\sqrt{x^2 + 4}}\right) + 2k\left(\frac{-(y - x)}{\sqrt{(y - x)^2 + 1}}\right) = 0$$

$$C_y = 2k\left(\frac{y - x}{\sqrt{(y - x)^2 + 1}}\right) - k = 0 \Rightarrow \frac{y - x}{\sqrt{(y - x)^2 + 1}} = \frac{1}{2}$$

$$3k\left(\frac{x}{\sqrt{x^2 + 4}}\right) + 2k\left(-\frac{1}{2}\right) = 0$$

$$\frac{x}{\sqrt{x^2 + 4}} = \frac{1}{3}$$

$$3x = \sqrt{x^2 + 4}$$

$$9x^2 = x^2 + 4$$

$$x^2 = \frac{1}{2}$$

$$x = \frac{\sqrt{2}}{2}$$

$$2(y - x) = \sqrt{(y - x)^2 + 1}$$

$$4(y - x)^2 = (y - x)^2 + 1$$

$$(y - x)^2 = \frac{1}{3}$$

$$y = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} = \frac{2\sqrt{3} + 3\sqrt{2}}{6}$$

Therefore, $x = \frac{\sqrt{2}}{2} \approx 0.707$ km and $y = \frac{2\sqrt{3} + 3\sqrt{2}}{6} \approx 1.284$ kms.

17. Let h be the height of the trough and r the length of the slanted sides. We observe that the area of a trapezoidal cross section is given by

$$A = h \left[\frac{(w - 2r) + [(w - 2r) + 2x]}{2} \right] = (w - 2r + x)h$$

where $x = r \cos \theta$ and $h = r \sin \theta$. Substituting these expressions for x and h , we have

$$A(r, \theta) = (w - 2r + r \cos \theta)(r \sin \theta) = wr \sin \theta - 2r^2 \sin \theta + r^2 \sin \theta \cos \theta$$

Now

$$A_r(r, \theta) = w \sin \theta - 4r \sin \theta + 2r \sin \theta \cos \theta = \sin \theta (w - 4r + 2r \cos \theta) = 0 \implies w = r(4 - 2 \cos \theta)$$

$$A_\theta(r, \theta) = wr \cos \theta - 2r^2 \cos \theta + r^2 \cos 2\theta = 0.$$

Substituting the expression for w from $A_r(r, \theta) = 0$ into the equation $A_\theta(r, \theta) = 0$, we have

$$r^2(4 - 2 \cos \theta) \cos \theta - 2r^2 \cos \theta + r^2(2 \cos^2 \theta - 1) = 0$$

$$r^2(2 \cos \theta - 1) = 0 \text{ or } \cos \theta = \frac{1}{2}.$$

Therefore, the first partial derivatives are zero when $\theta = \pi/3$ and $r = w/3$. (Ignore the solution $r = \theta = 0$.) Thus, the trapezoid of maximum area occurs when each edge of width $w/3$ is turned up 60° from the horizontal.

19. $R(x_1, x_2) = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$

$$R_{x_1} = -10x_1 - 2x_2 + 42 = 0, \quad 5x_1 + x_2 = 21$$

$$R_{x_2} = -16x_2 - 2x_1 + 102 = 0, \quad x_1 + 8x_2 = 51$$

Solving this system yields $x_1 = 3$ and $x_2 = 6$.

$$R_{x_1x_1} = -10$$

$$R_{x_1x_2} = -2$$

$$R_{x_2x_2} = -16$$

$$R_{x_1x_1} < 0 \text{ and } R_{x_1x_1}R_{x_2x_2} - (R_{x_1x_2})^2 > 0$$

Thus, revenue is maximized when $x_1 = 3$ and $x_2 = 6$.

21. $P(x_1, x_2) = 15(x_1 + x_2) - C_1 - C_2$

$$= 15x_1 + 15x_2 - (0.02x_1^2 + 4x_1 + 500) - (0.05x_2^2 + 4x_2 + 275)$$

$$= -0.02x_1^2 - 0.05x_2^2 + 11x_1 + 11x_2 - 775$$

$$P_{x_1} = -0.04x_1 + 11 = 0, \quad x_1 = 275$$

$$P_{x_2} = -0.10x_2 + 11 = 0, \quad x_2 = 110$$

$$P_{x_1x_1} = -0.04$$

$$P_{x_1x_2} = 0$$

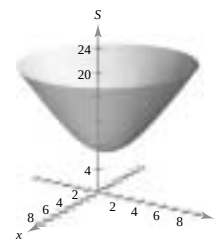
$$P_{x_2x_2} = -0.10$$

$$P_{x_1x_1} < 0 \text{ and } P_{x_1x_1}P_{x_2x_2} - (P_{x_1x_2})^2 > 0$$

Therefore, profit is maximized when $x_1 = 275$ and $x_2 = 110$.

$$\begin{aligned}
 23. (a) \quad S(x, y) &= d_1 + d_2 + d_3 \\
 &= \sqrt{(x-0)^2 + (y-0)^2} + \sqrt{(x+2)^2 + (y-2)^2} + \sqrt{(x-4)^2 + (y-2)^2} \\
 &= \sqrt{x^2 + y^2} + \sqrt{(x+2)^2 + (y-2)^2} + \sqrt{(x-4)^2 + (y-2)^2}
 \end{aligned}$$

From the graph we see that the surface has a minimum.



$$(b) \quad S_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}} + \frac{x+2}{\sqrt{(x+2)^2 + (y-2)^2}} + \frac{x-4}{\sqrt{(x-4)^2 + (y-2)^2}}$$

$$S_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}} + \frac{y-2}{\sqrt{(x+2)^2 + (y-2)^2}} + \frac{y-2}{\sqrt{(x-4)^2 + (y-2)^2}}$$

$$(c) \quad -\nabla S(1, 1) = -S_x(1, 1)\mathbf{i} - S_y(1, 1)\mathbf{j} = -\frac{1}{\sqrt{2}}\mathbf{i} - \left(\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{10}}\right)\mathbf{j}$$

$$\tan \theta = \frac{(2/\sqrt{10}) - (1/\sqrt{2})}{-1/\sqrt{2}} = 1 - \frac{2}{\sqrt{5}} \Rightarrow \theta \approx 186.027^\circ$$

$$(d) \quad (x_2, y_2) = (x_1 - S_x(x_1, y_1)t, y_1 - S_y(x_1, y_1)t) = \left(1 - \frac{1}{\sqrt{2}}t, 1 + \left(\frac{2}{\sqrt{10}} - \frac{1}{\sqrt{2}}\right)t\right)$$

$$\begin{aligned}
 S\left(1 - \frac{1}{\sqrt{2}}t, 1 + \left(\frac{2}{\sqrt{10}} - \frac{1}{\sqrt{2}}\right)t\right) &= \sqrt{2 + \left(\frac{2\sqrt{10}}{5} - 2\sqrt{2}\right)t + \left(1 - \frac{2\sqrt{5}}{5} + \frac{2}{5}\right)t^2} \\
 &\quad + \sqrt{10 - \left(\frac{2\sqrt{10}}{5} + 2\sqrt{2}\right)t + \left(1 - \frac{2\sqrt{5}}{5} + \frac{2}{5}\right)t^2} \\
 &\quad + \sqrt{10 - \left(\frac{2\sqrt{10}}{5} - 4\sqrt{2}\right)t + \left(1 - \frac{2\sqrt{5}}{5} + \frac{2}{5}\right)t^2}
 \end{aligned}$$

Using a computer algebra system, we find that the minimum occurs when $t \approx 1.344$. Thus, $(x_2, y_2) \approx (0.05, 0.90)$.

$$(e) \quad (x_3, y_3) = (x_2 - S_x(x_2, y_2)t, y_2 - S_y(x_2, y_2)t) \approx (0.05 + 0.03t, 0.90 - 0.26t)$$

$$\begin{aligned}
 S(0.05 + 0.03t, 0.90 - 0.26t) &= \sqrt{(0.05 + 0.03t)^2 + (0.90 - 0.26t)^2} + \sqrt{(2.05 + 0.03t)^2 + (-1.10 - 0.26t)^2} \\
 &\quad + \sqrt{(-3.95 + 0.03t)^2 + (-1.10 - 0.26t)^2}
 \end{aligned}$$

Using a computer algebra system, we find that the minimum occurs when $t \approx 1.78$. Thus, $(x_3, y_3) \approx (0.10, 0.44)$.

$$(x_4, y_4) = (x_3 - S_x(x_3, y_3)t, y_3 - S_y(x_3, y_3)t) \approx (0.10 - 0.09t, 0.44 - 0.01t)$$

$$\begin{aligned}
 S(0.10 - 0.09t, 0.44 - 0.01t) &= \sqrt{(0.10 - 0.09t)^2 + (0.44 - 0.01t)^2} + \sqrt{(2.10 - 0.09t)^2 + (-1.55 - 0.01t)^2} \\
 &\quad + \sqrt{(-3.90 - 0.09t)^2 + (-1.55 - 0.01t)^2}
 \end{aligned}$$

Using a computer algebra system, we find that the minimum occurs when $t \approx 0.44$. Thus, $(x_4, y_4) \approx (0.06, 0.44)$.

Note: The minimum occurs at $(x, y) = (0.0555, 0.3992)$

(f) $-\nabla S(x, y)$ points in the direction that S decreases most rapidly. You would use $\nabla S(x, y)$ for maximization problems.

25. Write the equation to be maximized or minimized as a function of two variables. Set the partial derivatives equal to zero (or undefined) to obtain the critical points. Use the Second Partial Test to test for relative extrema using the critical points. Check the boundary points, too.

27. (a)

x	y	xy	x^2
-2	0	0	4
0	1	0	0
2	3	6	4
$\sum x_i = 0$	$\sum y_i = 4$	$\sum x_i y_i = 6$	$\sum x_i^2 = 8$

$$a = \frac{3(6) - 0(4)}{3(8) - 0^2} = \frac{3}{4}, \quad b = \frac{1}{3} \left[4 - \frac{3}{4}(0) \right] = \frac{4}{3},$$

$$y = \frac{3}{4}x + \frac{4}{3}$$

$$(b) S = \left(-\frac{3}{2} + \frac{4}{3} - 0 \right)^2 + \left(\frac{4}{3} - 1 \right)^2 + \left(\frac{3}{2} + \frac{4}{3} - 3 \right)^2$$

$$= \frac{1}{6}$$

31. (0, 0), (1, 1), (3, 4), (4, 2), (5, 5)

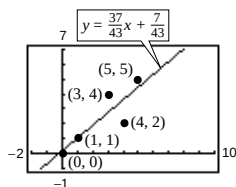
$$\sum x_i = 13, \quad \sum y_i = 12,$$

$$\sum x_i y_i = 46, \quad \sum x_i^2 = 51$$

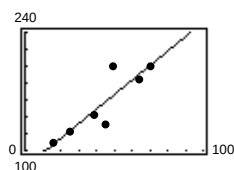
$$a = \frac{5(46) - 13(12)}{5(51) - (13)^2} = \frac{74}{86} = \frac{37}{43}$$

$$b = \frac{1}{5} \left[12 - \frac{37}{43}(13) \right] = \frac{7}{43}$$

$$y = \frac{37}{43}x + \frac{7}{43}$$


 35. (a) $y = 1.7236x + 79.7334$

(b)



(c) For each one-year increase in age, the pressure changes by 1.7236 (slope of line).

29. (a)

x	y	xy	x^2
0	4	0	0
1	3	3	1
1	1	1	1
2	0	0	4
$\sum x_i = 4$	$\sum y_i = 8$	$\sum x_i y_i = 4$	$\sum x_i^2 = 6$

$$a = \frac{4(4) - 4(8)}{4(6) - 4^2} = -2, \quad b = \frac{1}{4} [8 + 2(4)] = 4,$$

$$y = -2x + 4$$

$$(b) S = (4 - 4)^2 + (2 - 3)^2 + (2 - 1)^2 + (0 - 0)^2 = 2$$

33. (0, 6), (4, 3), (5, 0), (8, -4), (10, -5)

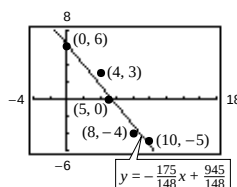
$$\sum x_i = 27, \quad \sum y_i = 0,$$

$$\sum x_i y_i = -70, \quad \sum x_i^2 = 205$$

$$a = \frac{5(-70) - (27)(0)}{5(205) - (27)^2} = \frac{-350}{296} = -\frac{175}{148}$$

$$b = \frac{1}{5} \left[0 - \left(-\frac{175}{148} \right) (27) \right] = \frac{945}{148}$$

$$y = -\frac{175}{148}x + \frac{945}{148}$$



37. (1.0, 32), (1.5, 41), (2.0, 48), (2.5, 53)

$$\sum x_i = 7, \quad \sum y_i = 174, \quad \sum x_i y_i = 322, \quad \sum x_i^2 = 13.5$$

$$a = 14, \quad b = 19, \quad y = 14x + 19$$

 When $x = 1.6$, $y = 41.4$ bushels per acre.

$$39. S(a, b, c) = \sum_{i=1}^n (y_i - ax_i^2 - bx_i - c)^2$$

$$\frac{\partial S}{\partial a} = \sum_{i=1}^n -2x_i^2(y_i - ax_i^2 - bx_i - c) = 0$$

$$\frac{\partial S}{\partial b} = \sum_{i=1}^n -2x_i(y_i - ax_i^2 - bx_i - c) = 0$$

$$\frac{\partial S}{\partial c} = -2 \sum_{i=1}^n (y_i - ax_i^2 - bx_i - c) = 0$$

$$a \sum_{i=1}^n x_i^4 + b \sum_{i=1}^n x_i^3 + c \sum_{i=1}^n x_i^2 = \sum_{i=1}^n x_i^2 y_i$$

$$a \sum_{i=1}^n x_i^3 + b \sum_{i=1}^n x_i^2 + c \sum_{i=1}^n x_i = \sum_{i=1}^n x_i y_i$$

$$a \sum_{i=1}^n x_i^2 + b \sum_{i=1}^n x_i + cn = \sum_{i=1}^n y_i$$

$$43. (0, 0), (2, 2), (3, 6), (4, 12)$$

$$\sum x_i = 9$$

$$\sum y_i = 20$$

$$\sum x_i^2 = 29$$

$$\sum x_i^3 = 99$$

$$\sum x_i^4 = 353$$

$$\sum x_i y_i = 70$$

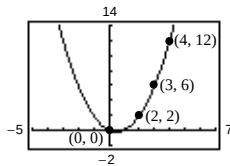
$$\sum x_i^2 y_i = 254$$

$$353a + 99b + 29c = 254$$

$$99a + 29b + 9c = 70$$

$$29a + 9b + 4c = 20$$

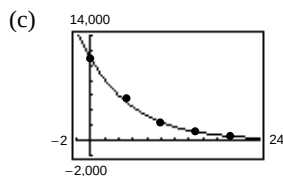
$$a = 1, b = -1, c = 0, y = x^2 - x$$



$$47. (a) \ln P = -0.1499h + 9.3018$$

$$(b) \ln P = -0.1499h + 9.3018$$

$$P = e^{-0.1499h + 9.3018} = 10,957.7e^{-0.1499h}$$



(d) Same answers.

$$41. (-2, 0), (-1, 0), (0, 1), (1, 2), (2, 5)$$

$$\sum x_i = 0$$

$$\sum y_i = 8$$

$$\sum x_i^2 = 10$$

$$\sum x_i^3 = 0$$

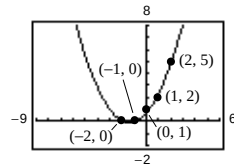
$$\sum x_i^4 = 34$$

$$\sum x_i y_i = 12$$

$$\sum x_i^2 y_i = 22$$

$$34a + 10c = 22, 10b = 12, 10a + 5c = 8$$

$$a = \frac{3}{7}, b = \frac{6}{5}, c = \frac{26}{35}, y = \frac{3}{7}x^2 + \frac{6}{5}x + \frac{26}{35}$$



$$45. (0, 0), (2, 15), (4, 30), (6, 50), (8, 65), (10, 70)$$

$$\sum x_i = 30,$$

$$\sum y_i = 230,$$

$$\sum x_i^2 = 220,$$

$$\sum x_i^3 = 1,800,$$

$$\sum x_i^4 = 15,664,$$

$$\sum x_i y_i = 1,670,$$

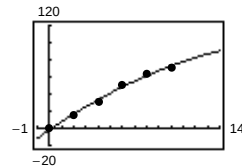
$$\sum x_i^2 y_i = 13,500$$

$$15,664a + 1,800b + 220c = 13,500$$

$$1,800a + 220b + 30c = 1,670$$

$$220a + 30b + 6c = 230$$

$$y = -\frac{25}{112}x^2 + \frac{541}{56}x - \frac{25}{14} \approx -0.22x^2 + 9.66x - 1.79$$



Section 12.10 Lagrange Multipliers

1. Maximize
- $f(x, y) = xy$
- .

Constraint: $x + y = 10$

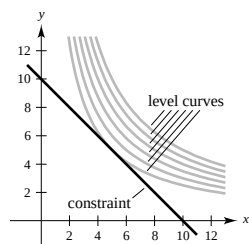
$$\nabla f = \lambda \nabla g$$

$$y\mathbf{i} + x\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j})$$

$$\left. \begin{array}{l} y = \lambda \\ x = \lambda \end{array} \right\} x = y$$

$$x + y = 10 \Rightarrow x = y = 5$$

$$f(5, 5) = 25$$



3. Minimize
- $f(x, y) = x^2 + y^2$
- .

Constraint: $x + y = 4$

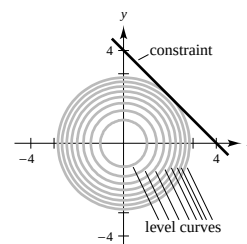
$$\nabla f = \lambda \nabla g$$

$$2x\mathbf{i} + 2y\mathbf{j} = \lambda\mathbf{i} + \lambda\mathbf{j}$$

$$\left. \begin{array}{l} 2x = \lambda \\ 2y = \lambda \end{array} \right\} x = y$$

$$x + y = 4 \Rightarrow x = y = 2$$

$$f(2, 2) = 8$$



5. Minimize
- $f(x, y) = x^2 - y^2$
- .

Constraint: $x - 2y = -6$

$$\nabla f = \lambda \nabla g$$

$$2x\mathbf{i} - 2y\mathbf{j} = \lambda\mathbf{i} - 2\lambda\mathbf{j}$$

$$2x = \lambda \Rightarrow x = \frac{\lambda}{2}$$

$$-2y = -2\lambda \Rightarrow y = \lambda$$

$$x - 2y = -6 \Rightarrow -\frac{3}{2}\lambda = -6$$

$$\lambda = 4, x = 2, y = 4$$

$$f(2, 4) = -12$$

7. Maximize
- $f(x, y) = 2x + 2xy + y$
- .

Constraint: $2x + y = 100$

$$\nabla f = \lambda \nabla g$$

$$(2 + 2y)\mathbf{i} + (2x + 1)\mathbf{j} = 2\lambda\mathbf{i} + \lambda\mathbf{j}$$

$$\left. \begin{array}{l} 2 + 2y = 2\lambda \Rightarrow y = \lambda - 1 \\ 2x + 1 = \lambda \Rightarrow x = \frac{\lambda - 1}{2} \end{array} \right\} y = 2x$$

$$2x + y = 100 \Rightarrow 4x = 100$$

$$x = 25, y = 50$$

$$f(25, 50) = 2600$$

- 9.
- Note:**
- $f(x, y) = \sqrt{6 - x^2 - y^2}$
- is maximum when
- $g(x, y)$
- is maximum.

Maximize $g(x, y) = 6 - x^2 - y^2$.Constraint: $x + y = 2$

$$\left. \begin{array}{l} -2x = \lambda \\ -2y = \lambda \end{array} \right\} x = y$$

$$x + y = 2 \Rightarrow x = y = 1$$

$$f(1, 1) = \sqrt{g(1, 1)} = 2$$

11. Maximize
- $f(x, y) = e^{xy}$
- .

Constraint: $x^2 + y^2 = 8$

$$\left. \begin{array}{l} ye^{xy} = 2x\lambda \\ xe^{xy} = 2y\lambda \end{array} \right\} x = y$$

$$x^2 + y^2 = 8 \Rightarrow 2x^2 = 8$$

$$x = y = 2$$

$$f(2, 2) = e^4$$

13. Maximize or minimize
- $f(x, y) = x^2 + 3xy + y^2$
- .

Constraint: $x^2 + y^2 \leq 1$ Case 1: On the circle $x^2 + y^2 = 1$

$$\left. \begin{array}{l} 2x + 3y = 2x\lambda \\ 3x + 2y = 2y\lambda \end{array} \right\} x^2 = y^2$$

$$x^2 + y^2 = 1 \Rightarrow x = \pm \frac{\sqrt{2}}{2}, y = \pm \frac{\sqrt{2}}{2}$$

$$\text{Maxima: } f\left(\pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}\right) = \frac{5}{2}$$

$$\text{Minima: } f\left(\pm \frac{\sqrt{2}}{2}, \mp \frac{\sqrt{2}}{2}\right) = -\frac{1}{2}$$

Case 2: Inside the circle

$$\left. \begin{array}{l} f_x = 2x + 3y = 0 \\ f_y = 3x + 2y = 0 \end{array} \right\} x = y = 0$$

$$f_{xx} = 2, f_{yy} = 2, f_{xy} = 3, f_{xx}f_{yy} - (f_{xy})^2 \leq 0$$

$$\text{Saddle point: } f(0, 0) = 0$$

By combining these two cases, we have a maximum of $\frac{5}{2}$ at

$$\left(\pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}\right)$$

and a minimum of $-\frac{1}{2}$ at

$$\left(\pm \frac{\sqrt{2}}{2}, \mp \frac{\sqrt{2}}{2}\right).$$

15. Minimize
- $f(x, y, z) = x^2 + y^2 + z^2$
- .

Constraint: $x + y + z = 6$

$$\begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \end{cases} \Rightarrow x = y = z$$

$$x + y + z = 6 \Rightarrow x = y = z = 2$$

$$f(2, 2, 2) = 12$$

19. Maximize
- $f(x, y, z) = xyz$
- .

Constraints: $x + y + z = 32$

$$x - y + z = 0$$

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

$$yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$\begin{cases} yz = \lambda + \mu \\ xz = \lambda - \mu \\ xy = \lambda + \mu \end{cases} \Rightarrow yz = xy \Rightarrow x = z$$

$$\begin{cases} x + y + z = 32 \\ x - y + z = 0 \end{cases} \Rightarrow 2x + 2z = 32 \Rightarrow x = z = 8$$

$$y = 16$$

$$f(8, 16, 8) = 1024$$

17. Minimize
- $f(x, y, z) = x^2 + y^2 + z^2$
- .

Constraint: $x + y + z = 1$

$$\begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \end{cases} \Rightarrow x = y = z$$

$$x + y + z = 1 \Rightarrow x = y = z = \frac{1}{3}$$

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$$

21. Maximize
- $f(x, y, z) = xy + yz$
- .

Constraints: $x + 2y = 6$

$$x - 3z = 0$$

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

$$y\mathbf{i} + (x + z)\mathbf{j} + y\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{j}) + \mu(\mathbf{i} - 3\mathbf{k})$$

$$\begin{cases} y = \lambda + \mu \\ x + z = 2\lambda \\ y = -3\mu \end{cases} \Rightarrow y = \frac{3}{4}\lambda \Rightarrow x + z = \frac{8}{3}y$$

$$x + 2y = 6 \Rightarrow y = 3 - \frac{x}{2}$$

$$x - 3z = 0 \Rightarrow z = \frac{x}{3}$$

$$x + \frac{x}{3} = \frac{8}{3}\left(3 - \frac{x}{2}\right)$$

$$x = 3, y = \frac{3}{2}, z = 1$$

$$f\left(3, \frac{3}{2}, 1\right) = 6$$

23. Minimize the square of the distance
- $f(x, y) = x^2 + y^2$
- subject to the constraint
- $2x + 3y = -1$
- .

$$\begin{cases} 2x = 2\lambda \\ 2y = 3\lambda \end{cases} \Rightarrow y = \frac{3x}{2}$$

$$2x + 3y = -1 \Rightarrow x = -\frac{2}{13}, y = -\frac{3}{13}$$

The point on the line is $\left(-\frac{2}{13}, -\frac{3}{13}\right)$ and the desired distance is

$$d = \sqrt{\left(-\frac{2}{13}\right)^2 + \left(-\frac{3}{13}\right)^2} = \frac{\sqrt{13}}{13}.$$

25. Minimize the square of the distance

$$f(x, y, z) = (x - 2)^2 + (y - 1)^2 + (z - 1)^2$$

subject to the constraint $x + y + z = 1$.

$$\begin{cases} 2(x - 2) = \lambda \\ 2(y - 1) = \lambda \\ 2(z - 1) = \lambda \end{cases} \Rightarrow y = z \text{ and } y = x - 1$$

$$x + y + z = 1 \Rightarrow x + 2(x - 1) = 1$$

$$x = 1, y = z = 0$$

The point on the plane is $(1, 0, 0)$ and the desired distance is

$$d = \sqrt{(1 - 2)^2 + (0 - 1)^2 + (0 - 1)^2} = \sqrt{3}.$$

27. Maximize $f(x, y, z) = z$ subject to the constraints $x^2 + y^2 + z^2 = 36$ and $2x + y - z = 2$.

$$\begin{cases} 0 = 2x\lambda + 2\mu \\ 0 = 2y\lambda + \mu \\ 1 = 2z\lambda - \mu \end{cases} \Rightarrow x = 2y$$

$$x^2 + y^2 + z^2 = 36$$

$$2x + y - z = 2 \Rightarrow z = 2x + y - 2 = 5y - 2$$

$$(2y)^2 + y^2 + (5y - 2)^2 = 36$$

$$30y^2 - 20y - 32 = 0$$

$$15y^2 - 10y - 16 = 0$$

$$y = \frac{5 \pm \sqrt{265}}{15}$$

Choosing the positive value for y we have the point

$$\left(\frac{10 + 2\sqrt{265}}{15}, \frac{5 + \sqrt{265}}{15}, \frac{-1 + \sqrt{265}}{3} \right).$$

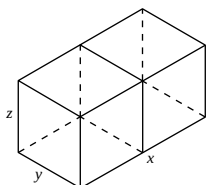
31. Maximize $V(x, y, z) = xyz$ subject to the constraint $x + 2y + 2z = 108$.

$$\begin{cases} yz = \lambda \\ xz = 2\lambda \\ xy = 2\lambda \end{cases} \Rightarrow y = z \text{ and } x = 2y$$

$$x + 2y + 2z = 108 \Rightarrow 6y = 108, y = 18$$

$$x = 36, y = z = 18$$

Volume is maximum when the dimensions are $36 \times 18 \times 18$ inches



29. Optimization problems that have restrictions or constraints on the values that can be used to produce the optimal solution are called constrained optimization problems.

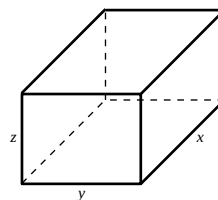
33. Minimize $C(x, y, z) = 5xy + 3(2xz + 2yz + xy)$ subject to the constraint $xyz = 480$.

$$\begin{cases} 8y + 6z = yz\lambda \\ 8x + 6z = xz\lambda \\ 6x + 6y = xy\lambda \end{cases} \Rightarrow x = y, 4y = 3z$$

$$xyz = 480 \Rightarrow \frac{4}{3}y^3 = 480$$

$$x = y = \sqrt[3]{360}, z = \frac{4}{3}\sqrt[3]{360}$$

Dimensions: $\sqrt[3]{360} \times \sqrt[3]{360} \times \frac{4}{3}\sqrt[3]{360}$ feet



35. Maximize $V(x, y, z) = (2x)(2y)(2z) = 8xyz$ subject to the constraint $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

$$\begin{cases} 8yz = \frac{2x}{a^2}\lambda \\ 8xz = \frac{2y}{b^2}\lambda \\ 8xy = \frac{2z}{c^2}\lambda \end{cases} \Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow \frac{3x^2}{a^2} = 1, \frac{3y^2}{b^2} = 1, \frac{3z^2}{c^2} = 1$$

$$x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$$

Therefore, the dimensions of the box are $\frac{2\sqrt{3}a}{3} \times \frac{2\sqrt{3}b}{3} \times \frac{2\sqrt{3}c}{3}$.

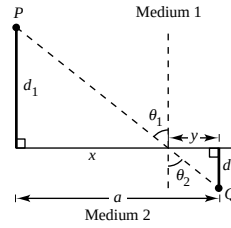
37. Using the formula $\text{Time} = \frac{\text{Distance}}{\text{Rate}}$, minimize $T(x, y) = \frac{\sqrt{d_1^2 + x^2}}{v_1} + \frac{\sqrt{d_2^2 + y^2}}{v_2}$ subject to the constraint $x + y = a$.

$$\left. \begin{aligned} \frac{x}{v_1 \sqrt{d_1^2 + x^2}} &= \lambda \\ \frac{y}{v_2 \sqrt{d_2^2 + y^2}} &= \lambda \end{aligned} \right\} \Rightarrow \frac{x}{v_1 \sqrt{d_1^2 + x^2}} = \frac{y}{v_2 \sqrt{d_2^2 + y^2}}$$

$$x + y = a$$

Since $\sin \theta_1 = \frac{x}{\sqrt{d_1^2 + x^2}}$ and $\sin \theta_2 = \frac{y}{\sqrt{d_2^2 + y^2}}$, we have

$$\frac{x/\sqrt{d_1^2 + x^2}}{v_1} = \frac{y/\sqrt{d_2^2 + y^2}}{v_2} \quad \text{or} \quad \frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.$$



39. Maximize $P(p, q, r) = 2pq + 2pr + 2qr$.

Constraint: $p + q + r = 1$

$$\nabla P = \lambda \nabla g$$

$$\left. \begin{aligned} 2q + 2r &= \lambda \\ 2p + 2r &= \lambda \\ 2p + 2q &= \lambda \end{aligned} \right\} \Rightarrow 3\lambda = 4(p + q + r) = 4(1)$$

$$\Rightarrow \lambda = \frac{4}{3}$$

$$p + q + r = 1$$

$$\left. \begin{aligned} q + r &= \frac{2}{3} \\ p + q + r &= 1 \end{aligned} \right\} \Rightarrow p = \frac{1}{3}, q = \frac{1}{3}, r = \frac{1}{3}$$

$$P\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{3}.$$

41. Maximize $P(x, y) = 100x^{0.25}y^{0.75}$

subject to the constraint $48x + 36y = 100,000$.

$$25x^{-0.75}y^{0.75} = 48\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.75} = \frac{48\lambda}{25}$$

$$75x^{0.25}y^{-0.25} = 36\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.25} = \frac{36\lambda}{75}$$

$$\left(\frac{y}{x}\right)^{0.75} \left(\frac{y}{x}\right)^{0.25} = \left(\frac{48\lambda}{25}\right) \left(\frac{75}{36\lambda}\right)$$

$$\frac{y}{x} = 4$$

$$y = 4x$$

$$48x + 36y = 100,000 \Rightarrow 192x = 100,000$$

$$x = \frac{3125}{6}, y = \frac{6250}{3}$$

Therefore, $P\left(\frac{3125}{6}, \frac{6250}{3}\right) \approx 147,314$.

43. Minimize $C(x, y) = 48x + 36y$ subject to the constraint $100x^{0.25}y^{0.75} = 20,000$.

$$48 = 25x^{-0.75}y^{0.75}\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.75} = \frac{48}{25\lambda}$$

$$36 = 75x^{0.25}y^{-0.25}\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.25} = \frac{36}{75\lambda}$$

$$\left(\frac{y}{x}\right)^{0.75} \left(\frac{y}{x}\right)^{0.25} = \left(\frac{48}{25\lambda}\right) \left(\frac{75\lambda}{36}\right)$$

$$\frac{y}{x} = 4 \Rightarrow y = 4x$$

$$100x^{0.25}y^{0.75} = 20,000 \Rightarrow x^{0.25}(4x)^{0.75} = 200$$

$$x = \frac{200}{4^{0.75}} = \frac{200}{2\sqrt{2}} = 50\sqrt{2}$$

$$y = 4x = 200\sqrt{2}$$

Therefore, $C(50\sqrt{2}, 200\sqrt{2}) \approx \$13,576.45$.

45. (a) Maximize $g(\alpha, \beta, \gamma) = \cos \alpha \cos \beta \cos \gamma$ subject to the constraint $\alpha + \beta + \gamma = \pi$.

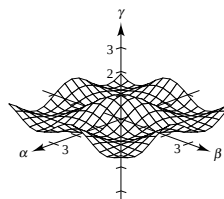
$$\left. \begin{aligned} -\sin \alpha \cos \beta \cos \gamma &= \lambda \\ -\cos \alpha \sin \beta \cos \gamma &= \lambda \\ -\cos \alpha \cos \beta \sin \gamma &= \lambda \end{aligned} \right\} \tan \alpha = \tan \beta = \tan \gamma \Rightarrow \alpha = \beta = \gamma$$

$$\alpha + \beta + \gamma = \pi \Rightarrow \alpha = \beta = \gamma = \frac{\pi}{3}$$

$$g\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{1}{8}$$

(b) $\alpha + \beta + \gamma = \pi \Rightarrow \gamma = \pi - (\alpha + \beta)$

$$\begin{aligned} g(\alpha + \beta) &= \cos \alpha \cos \beta \cos(\pi - (\alpha + \beta)) \\ &= \cos \alpha \cos \beta [\cos \pi \cos(\alpha + \beta) + \sin \pi \sin(\alpha + \beta)] \\ &= -\cos \alpha \cos \beta \cos(\alpha + \beta) \end{aligned}$$



Review Exercises for Chapter 12

1. No, it is not the graph of a function.

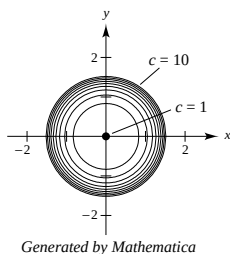
3. $f(x, y) = e^{x^2 + y^2}$

The level curves are of the form

$$c = e^{x^2 + y^2}$$

$$\ln c = x^2 + y^2.$$

The level curves are circles centered at the origin.



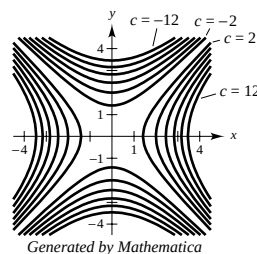
5. $f(x, y) = x^2 - y^2$

The level curves are of the form

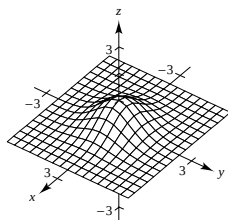
$$c = x^2 - y^2$$

$$1 = \frac{x^2}{c} - \frac{y^2}{c}.$$

The level curves are hyperbolas.



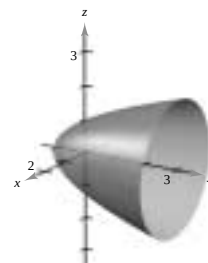
7. $f(x, y) = e^{-(x^2 + y^2)}$



9. $f(x, y, z) = x^2 - y + z^2 = 1$

$$y = x^2 + z^2 - 1$$

Elliptic paraboloid



11. $\lim_{(x,y) \rightarrow (1,1)} \frac{xy}{x^2 + y^2} = \frac{1}{2}$

Continuous except at $(0, 0)$.

13. $\lim_{(x,y) \rightarrow (0,0)} \frac{-4x^2y}{x^4 + y^2}$

$$\text{For } y = x^2, \frac{-4x^2y}{x^4 + y^2} = \frac{-4x^4}{x^4 + x^4} = -2, \text{ for } x \neq 0$$

$$\text{For } y = 0, \frac{-4x^2y}{x^4 + y^2} = 0, \text{ for } x \neq 0$$

Thus, the limit does not exist. Continuous except at $(0, 0)$.

15. $f(x, y) = e^x \cos y$

$$f_x = e^x \cos y$$

$$f_y = -e^x \sin y$$

19. $g(x, y) = \frac{xy}{x^2 + y^2}$

$$g_x = \frac{y(x^2 + y^2) - xy(2x)}{(x^2 + y^2)^2} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}$$

$$g_y = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}$$

23. $u(x, t) = ce^{-n^2 t} \sin(nx)$

$$\frac{\partial u}{\partial x} = cne^{-n^2 t} \cos(nx)$$

$$\frac{\partial u}{\partial t} = -cn^2 e^{-n^2 t} \sin(nx)$$

17. $z = xe^y + ye^x$

$$\frac{\partial z}{\partial x} = e^y + ye^x$$

$$\frac{\partial z}{\partial y} = xe^y + e^x$$

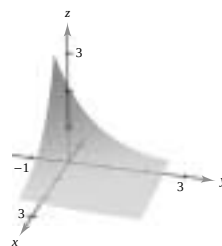
21. $f(x, y, z) = z \arctan \frac{y}{x}$

$$f_x = \frac{z}{1 + (y^2/x^2)} \left(-\frac{y}{x^2} \right) = \frac{-yz}{x^2 + y^2}$$

$$f_y = \frac{z}{1 + (y^2/x^2)} \left(\frac{1}{x} \right) = \frac{xz}{x^2 + y^2}$$

$$f_z = \arctan \frac{y}{x}$$

25.



27. $f(x, y) = 3x^2 - xy + 2y^3$

$$f_x = 6x - y$$

$$f_y = -x + 6y^2$$

$$f_{xx} = 6$$

$$f_{yy} = 12y$$

$$f_{xy} = -1$$

$$f_{yx} = -1$$

29. $h(x, y) = x \sin y + y \cos x$

$$h_x = \sin y - y \sin x$$

$$h_y = x \cos y + \cos x$$

$$h_{xx} = -y \cos x$$

$$h_{yy} = -x \sin y$$

$$h_{xy} = \cos y - \sin x$$

$$h_{yx} = \cos y - \sin x$$

31. $z = x^2 - y^2$

$$\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial^2 z}{\partial x^2} = 2$$

$$\frac{\partial z}{\partial y} = -2y$$

$$\frac{\partial^2 z}{\partial y^2} = -2$$

$$\text{Therefore, } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

33. $z = \frac{y}{x^2 + y^2}$

$$\frac{\partial z}{\partial x} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = -2y \left[\frac{-4x^2}{(x^2 + y^2)^3} + \frac{1}{(x^2 + y^2)^2} \right] = 2y \frac{3x^2 - y^2}{(x^2 + y^2)^3}$$

$$\frac{\partial z}{\partial y} = \frac{(x^2 + y^2) - 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{(x^2 + y^2)^2(-2y) - 2(x^2 - y^2)(x^2 + y^2)(2y)}{(x^2 + y^2)^4}$$

$$= -2y \frac{3x^2 - y^2}{(x^2 + y^2)^3}$$

$$\text{Therefore, } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

35. $z = x \sin \frac{y}{x}$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \left(\sin \frac{y}{x} - \frac{y}{x} \cos \frac{y}{x} \right) dx + \left(\cos \frac{y}{x} \right) dy$$

37. $z^2 = x^2 + y^2$

$$2z \, dx = 2x \, dx + 2y \, dy$$

$$dz = \frac{x}{z} \, dx + \frac{y}{z} \, dy = \frac{5}{13} \left(\frac{1}{2} \right) + \frac{12}{13} \left(\frac{1}{2} \right) = \frac{17}{26} \approx 0.654 \text{ cm}$$

Percentage error: $\frac{dz}{z} = \frac{17/26}{13} \approx 0.0503 \approx 5\%$

39. $V = \frac{1}{3}\pi r^2 h$

$$dV = \frac{2}{3}\pi r h \, dr + \frac{1}{3}\pi r^2 \, dh = \frac{2}{3}\pi(2)(5)\left(\pm\frac{1}{6}\right) + \frac{1}{3}\pi(2)^2\left(\pm\frac{1}{6}\right) \\ = \pm\frac{5}{6}\pi \pm \frac{1}{6}\pi = \pm\pi \text{ in.}^3$$

41. $w = \ln(x^2 + y^2)$, $x = 2t + 3$, $y = 4 - t$

$$\begin{aligned} \text{Chain Rule: } \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\ &= \frac{2x}{x^2 + y^2}(2) + \frac{2y}{x^2 + y^2}(-1) \\ &= \frac{2(2t + 3)2}{(2t + 3)^2 + (4 - t)^2} - \frac{2(4 - t)}{(2t + 3)^2 + (4 - t)^2} \\ &= \frac{10t + 4}{5t^2 + 4t + 25} \end{aligned}$$

Substitution: $w = \ln(x^2 + y^2) = \ln[(2t + 3)^2 + (4 - t)^2]$

$$\frac{dw}{dt} = \frac{2(2t + 3)(2) - 2(4 - t)}{(2t + 3)^2 + (4 - t)^2} = \frac{10t + 4}{5t^2 + 4t + 25}$$

43. $u = x^2 + y^2 + z^2$, $x = r \cos t$, $y = r \sin t$, $z = t$

$$\begin{aligned} \text{Chain Rule: } \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial r} \\ &= 2x \cos t + 2y \sin t + 2z(0) \\ &= 2(r \cos^2 t + r \sin^2 t) = 2r \\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t} \\ &= 2x(-r \sin t) + 2y(r \cos t) + 2z \\ &= 2(-r^2 \sin t \cos t + r^2 \sin t \cos t) + 2t \\ &= 2t \end{aligned}$$

Substitution: $u(r, t) = r^2 \cos^2 t + r^2 \sin^2 t + t^2 = r^2 + t^2$

$$\frac{\partial u}{\partial r} = 2r$$

$$\frac{\partial u}{\partial t} = 2t$$

45. $x^2 y - 2yz - xz - z^2 = 0$

$$2xy - 2y \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial x} - z - 2z \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-2xy + z}{-2y - x - 2z} = \frac{2xy - z}{x + 2y + 2z}$$

$$x^2 - 2y \frac{\partial z}{\partial y} - 2z - x \frac{\partial z}{\partial y} - 2z \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-x^2 + 2z}{-2y - x - 2z} = \frac{x^2 - 2z}{x + 2y + 2z}$$

47. $f(x, y) = x^2y$

$$\nabla f = 2xy\mathbf{i} + x^2\mathbf{j}$$

$$\nabla f(2, 1) = 4\mathbf{i} + 4\mathbf{j}$$

$$\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{v} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$$D_{\mathbf{u}}f(2, 1) = \nabla f(2, 1) \cdot \mathbf{u} = 2\sqrt{2} - 2\sqrt{2} = 0$$

51. $z = \frac{y}{x^2 + y^2}$

$$\nabla z = -\frac{2xy}{(x^2 + y^2)^2}\mathbf{i} + \frac{x^2 - y^2}{(x^2 + y^2)^2}\mathbf{j}$$

$$\nabla z(1, 1) = -\frac{1}{2}\mathbf{i} = \left\langle -\frac{1}{2}, 0 \right\rangle$$

$$\|\nabla z(1, 1)\| = \frac{1}{2}$$

55. $9x^2 - 4y^2 = 65$

$$f(x, y) = 9x^2 - 4y^2$$

$$\nabla f(x, y) = 18x\mathbf{i} + 8y\mathbf{j}$$

$$\nabla f(3, 2) = 54\mathbf{i} - 16\mathbf{j}$$

$$\text{Unit normal: } \frac{54\mathbf{i} - 16\mathbf{j}}{\|54\mathbf{i} - 16\mathbf{j}\|} = \frac{1}{\sqrt{793}}(27\mathbf{i} - 8\mathbf{j})$$

59. $F(x, y, z) = x^2 + y^2 - 4x + 6y + z + 9 = 0$

$$\nabla F = (2x - 4)\mathbf{i} + (2y + 6)\mathbf{j} + \mathbf{k}$$

$$\nabla F(2, -3, 4) = \mathbf{k}$$

Therefore, the equation of the tangent plane is

$$z - 4 = 0 \quad \text{or} \quad z = 4,$$

and the equation of the normal line is

$$x = 2, y = -3, z = 4 + t.$$

63. $f(x, y, z) = x^2 + y^2 + z^2 - 14$

$$\nabla f(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla f(2, 1, 3) = 4\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} \text{ Normal vector to plane.}$$

$$\cos \theta = \frac{|\mathbf{n} \cdot \mathbf{k}|}{\|\mathbf{n}\|} = \frac{6}{\sqrt{56}} = \frac{3\sqrt{14}}{14}$$

$$\theta = 36.7^\circ$$

49. $w = y^2 + xz$

$$\nabla w = z\mathbf{i} + 2y\mathbf{j} + x\mathbf{k}$$

$$\nabla w(1, 2, 2) = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

$$\mathbf{u} = \frac{1}{3}\mathbf{v} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

$$D_{\mathbf{u}}w(1, 2, 2) = \nabla w(1, 2, 2) \cdot \mathbf{u} = \frac{4}{3} - \frac{4}{3} + \frac{2}{3} = \frac{2}{3}$$

53. $z = e^{-x} \cos y$

$$\nabla z = -e^{-x} \cos y \mathbf{i} - e^{-x} \sin y \mathbf{j}$$

$$\nabla z\left(0, \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$$

$$\left\| \nabla z\left(0, \frac{\pi}{4}\right) \right\| = 1$$

57. $F(x, y, z) = x^2y - z = 0$

$$\nabla F = 2xy\mathbf{i} + x^2\mathbf{j} - \mathbf{k}$$

$$\nabla F(2, 1, 4) = 4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$$

Therefore, the equation of the tangent plane is

$$4(x - 2) + 4(y - 1) - (z - 4) = 0 \quad \text{or}$$

$$4x + 4y - z = 8,$$

and the equation of the normal line is

$$\frac{x - 2}{4} = \frac{y - 1}{4} = \frac{z - 4}{-1}.$$

61. $F(x, y, z) = x^2 - y^2 - z = 0$

$$G(x, y, z) = 3 - z = 0$$

$$\nabla F = 2x\mathbf{i} - 2y\mathbf{j} - \mathbf{k}$$

$$\nabla G = -\mathbf{k}$$

$$\nabla F(2, 1, 3) = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

$$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 2(\mathbf{i} + 2\mathbf{j})$$

Therefore, the equation of the tangent line is

$$\frac{x - 2}{1} = \frac{y - 1}{2}, z = 3.$$

65. $f(x, y) = x^3 - 3xy + y^2$

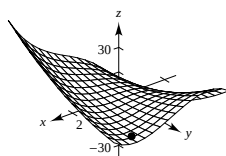
$$f_x = 3x^2 - 3y = 3(x^2 - y) = 0$$

$$f_y = -3x + 2y = 0$$

$$f_{xx} = 6x$$

$$f_{yy} = 2$$

$$f_{xy} = -3$$



From $f_x = 0$, we have $y = x^2$. Substituting this into $f_y = 0$, we have $-3x + 2x^2 = x(2x - 3) = 0$. Thus, $x = 0$ or $\frac{3}{2}$.

At the critical point $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point.

At the critical point $(\frac{3}{2}, \frac{9}{4})$, $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} > 0$. Therefore, $(\frac{3}{2}, \frac{9}{4}, -\frac{27}{16})$ is a relative minimum.

67. $f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$

$$f_x = y - \frac{1}{x^2} = 0, \quad x^2y = 1$$

$$f_y = x - \frac{1}{y^2} = 0, \quad xy^2 = 1$$

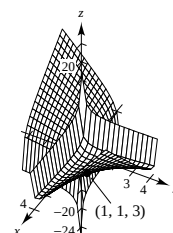
Thus, $x^2y = xy^2$ or $x = y$ and substitution yields the critical point $(1, 1)$.

$$f_{xx} = \frac{2}{x^3}$$

$$f_{xy} = 1$$

$$f_{yy} = \frac{2}{y^3}$$

At the critical point $(1, 1)$, $f_{xx} = 2 > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = 3 > 0$. Thus, $(1, 1, 3)$ is a relative minimum.



69. The level curves are hyperbolas. There is a critical point at $(0, 0)$, but there are no relative extrema. The gradient is normal to the level curve at any given point at (x_0, y_0) .

71. $P(x_1, x_2) = R - C_1 - C_2$

$$= [225 - 0.4(x_1 + x_2)](x_1 + x_2) - (0.05x_1^2 + 15x_1 + 5400) - (0.03x_2^2 + 15x_2 + 6100)$$

$$= -0.45x_1^2 - 0.43x_2^2 - 0.8x_1x_2 + 210x_1 + 210x_2 - 11,500$$

$$P_{x_1} = -0.9x_1 - 0.8x_2 + 210 = 0$$

$$0.9x_1 + 0.8x_2 = 210$$

$$P_{x_2} = -0.86x_2 - 0.8x_1 + 210 = 0$$

$$0.8x_1 + 0.86x_2 = 210$$

Solving this system yields $x_1 \approx 94$ and $x_2 \approx 157$.

$$P_{x_1x_1} = -0.9$$

$$P_{x_1x_2} = -0.8$$

$$P_{x_2x_2} = -0.86$$

$$P_{x_1x_1} < 0$$

$$P_{x_1x_1}P_{x_2x_2} - (P_{x_1x_2})^2 > 0$$

Therefore, profit is maximum when $x_1 \approx 94$ and $x_2 \approx 157$.

73. Maximize $f(x, y) = 4x + xy + 2y$ subject to the constraint $20x + 4y = 2000$.

$$\begin{cases} 4 + y = 20\lambda \\ x + 2 = 4\lambda \end{cases} \Rightarrow \begin{cases} 5x - y = -6 \end{cases}$$

$$20x + 4y = 2000 \Rightarrow 5x + y = 500$$

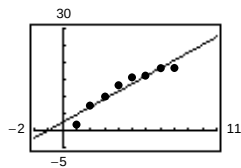
$$\begin{array}{r} 5x - y = -6 \\ 5x + y = 500 \\ \hline 10x = 494 \end{array}$$

$$x = 49.4$$

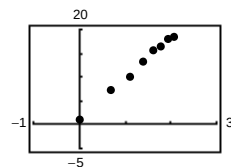
$$y = 253$$

$$f(49.4, 253) = 13,201.8$$

75. (a) $y = 2.29t + 2.34$



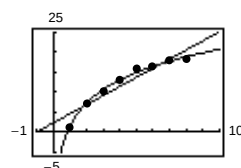
(b)



Yes, the data appears more linear.

- (c) $y = 8.37 \ln t + 1.54$

(d)



The logarithmic model is a better fit.

77. Optimize $f(x, y, z) = xy + yz + xz$ subject to the constraint $x + y + z = 1$.

$$\begin{cases} y + z = \lambda \\ x + z = \lambda \\ x + y = \lambda \end{cases} \Rightarrow x = y = z$$

$$x + y + z = 1 \Rightarrow x = y = z = \frac{1}{3}$$

$$\text{Maximum: } f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$$

79. $PQ = \sqrt{x^2 + 4}$, $QR = \sqrt{y^2 + 1}$, $RS = z$; $x + y + z = 10$

$$C = 3\sqrt{x^2 + 4} + 2\sqrt{y^2 + 1} + 2$$

$$\text{Constraint: } x + y + z = 10$$

$$\nabla C = \lambda \nabla g$$

$$\frac{3x}{\sqrt{x^2 + 4}} \mathbf{i} + \frac{2y}{\sqrt{y^2 + 1}} \mathbf{j} + \mathbf{k} = \lambda [\mathbf{i} + \mathbf{j} + \mathbf{k}]$$

$$3x = \lambda \sqrt{x^2 + 4}$$

$$2y = \lambda \sqrt{y^2 + 1}$$

$$1 = \lambda$$

$$9x^2 = x^2 + 4 \Rightarrow x^2 = \frac{1}{2}$$

$$4y^2 = y^2 + 1 \Rightarrow y^2 = \frac{1}{3}$$

$$\text{Hence, } x = \frac{\sqrt{2}}{2}, y = \frac{\sqrt{3}}{3}, z = 10 - \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{3} \approx 8.716 \text{ m.}$$

C H A P T E R 12

Functions of Several Variables

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CHAPTER 12

Functions of Several Variables

Section 12.1 Introduction to Functions of Several Variables

Solutions to Even-Numbered Exercises

2. $xz^2 + 2xy - y^2 = 4$

No, z is not a function of x and y . For example,
 $(x, y) = (1, 0)$ corresponds to both $z = \pm 2$

4. $z + x \ln y - 8 = 0$

$$z = 8 - x \ln y$$

Yes, z is a function of x and y .

6. $f(x, y) = 4 - x^2 - 4y^2$

(a) $f(0, 0) = 4$

(b) $f(0, 1) = 4 - 0 - 4 = 0$

(c) $f(2, 3) = 4 - 4 - 36 = -36$

(d) $f(1, y) = 4 - 1 - 4y^2 = 3 - 4y^2$

(e) $f(x, 0) = 4 - x^2 - 0 = 4 - x^2$

(f) $f(t, 1) = 4 - t^2 - 4 = -t^2$

8. $g(x, y) = \ln|x + y|$

(a) $g(2, 3) = \ln|2 + 3| = \ln 5$

(b) $g(5, 6) = \ln|5 + 6| = \ln 11$

(c) $g(e, 0) = \ln|e + 0| = 1$

(d) $g(0, 1) = \ln|0 + 1| = 0$

(e) $g(2, -3) = \ln|2 - 3| = \ln 1 = 0$

(f) $g(e, e) = \ln|e + e| = \ln 2e$
 $= \ln 2 + \ln e = (\ln 2) + 1$

10. $f(x, y, z) = \sqrt{x + y + z}$

(a) $f(0, 5, 4) = \sqrt{0 + 5 + 4} = 3$

(b) $f(6, 8, -3) = \sqrt{6 + 8 - 3} = \sqrt{11}$

12. $V(r, h) = \pi r^2 h$

(a) $V(3, 10) = \pi(3)^2(10) = 90\pi$

(b) $V(5, 2) = \pi(5)^2(2) = 50\pi$

14. $g(x, y) = \int_x^y \frac{1}{t} dt$

(a) $g(4, 1) = \int_4^1 \frac{1}{t} dt = \left[\ln|t| \right]_4^1 = -\ln 4$

(b) $g(6, 3) = \int_6^3 \frac{1}{t} dt = \left[\ln|t| \right]_6^3 = \ln 3 - \ln 6 = \ln\left(\frac{1}{2}\right)$

16. $f(x, y) = 3xy + y^2$

$$\begin{aligned} \text{(a)} \quad \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} &= \frac{[3(x + \Delta x)y + y^2] - (3xy + y^2)}{\Delta x} \\ &= \frac{3xy + 3(\Delta x)y + y^2 - 3xy - y^2}{\Delta x} = \frac{3(\Delta x)y}{\Delta x} = 3y, \Delta x \neq 0 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} &= \frac{[3x(y + \Delta y) + (y + \Delta y)^2] - (3xy + y^2)}{\Delta y} \\ &= \frac{3xy + 3x(\Delta y) + y^2 + 2y(\Delta y) + (\Delta y)^2 - 3xy - y^2}{\Delta y} \\ &= \frac{\Delta y(3x + 2y + \Delta y)}{\Delta y} = 3x + 2y + \Delta y, \Delta y \neq 0 \end{aligned}$$

18. $f(x, y) = \sqrt{4 - x^2 - 4y^2}$

Domain: $4 - x^2 - 4y^2 \geq 0$

$$x^2 + 4y^2 \leq 4$$

$$\frac{x^2}{4} + \frac{y^2}{1} \leq 1$$

$$\left\{ (x, y): \frac{x^2}{4} + \frac{y^2}{1} \leq 1 \right\}$$

Range: $0 \leq z \leq 2$

20. $f(x, y) = \arccos \frac{y}{x}$

Domain: $\left\{ (x, y): -1 \leq \frac{y}{x} \leq 1 \right\}$

Range: $0 \leq z \leq \pi$

22. $f(x, y) = \ln(xy - 6)$

Domain: $xy - 6 > 0$

$$xy > 6$$

$$\{(x, y): xy > 6\}$$

Range: all real numbers

24. $z = \frac{xy}{x - y}$

Domain: $\{(x, y): x \neq y\}$

Range: all real numbers

26. $f(x, y) = x^2 + y^2$

Domain: $\{(x, y): x \text{ is any real number, } y \text{ is any real number}\}$

Range: $z \geq 0$

28. $g(x, y) = x\sqrt{y}$

Domain: $\{(x, y): y \geq 0\}$

Range: all real numbers

30. (a) Domain: $\{(x, y): x \text{ is any real number, } y \text{ is any real number}\}$

Range: $-2 \leq z \leq 2$

(b) $z = 0$ when $x = 0$ which represents points on the y -axis.

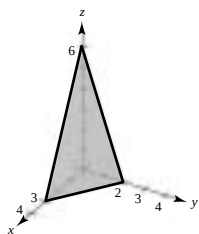
(c) No. When x is positive, z is negative. When x is negative, z is positive. The surface does not pass through the first octant, the octant where y is negative and x and z are positive, the octant where y is positive and x and z are negative, and the octant where x, y and z are all negative.

32. $f(x, y) = 6 - 2x - 3y$

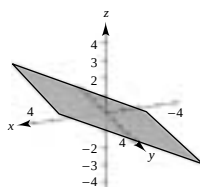
Plane

 Domain: entire xy -plane

Range: $-\infty < z < \infty$



34. $g(x, y) = \frac{1}{2}x$
Plane: $z = \frac{1}{2}x$

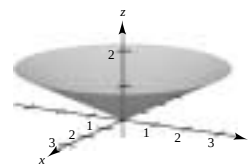


36. $z = \frac{1}{2}\sqrt{x^2 + y^2}$

Cone

 Domain of f : entire xy -plane

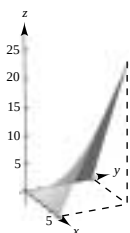
Range: $z \geq 0$



38. $f(x, y) = \begin{cases} xy, & x \geq 0, y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$

 Domain of f : entire xy -plane

Range: $z \geq 0$



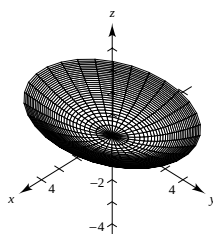
40. $f(x, y) = \frac{1}{12}\sqrt{144 - 16x^2 - 9y^2}$

Semi-ellipsoid

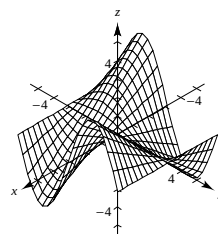
Domain: set of all points lying on or inside the ellipse

$$(x^2/9) + (y^2/16) = 1$$

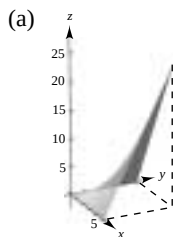
Range: $0 \leq z \leq 1$



42. $f(x, y) = x \sin y$



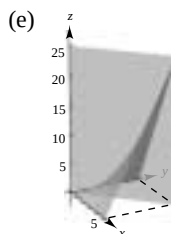
44. $f(x, y) = xy, x \geq 0, y \geq 0$



(b) g is a vertical translation of f 3 units downward

(c) g is a reflection of f in the xy -plane

(d) The graph of g is lower than the graph of f . If $z = f(x, y)$ is on the graph of f , then $\frac{1}{2}z$ is on the graph of g .



46. $z = e^{1-x^2+y^2}$

Level curves:

$$c = e^{1-x^2+y^2}$$

$$\ln c = 1 - x^2 + y^2$$

$$x^2 - y^2 = 1 - \ln c$$

Hyperbolas centered at $(0, 0)$

Matches (d)

48. $z = \cos\left(\frac{x + 2y^2}{4}\right)$

Level curves:

$$c = \cos\left(\frac{x^2 + 2y^2}{4}\right)$$

$$\cos^{-1} c = \frac{x^2 + 2y^2}{4}$$

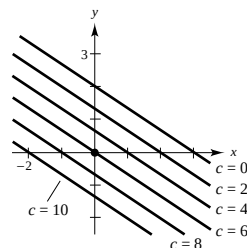
$$x^2 + 2y^2 = 4 \cos^{-1} c$$

Ellipses

Matches (a)

50. $f(x, y) = 6 - 2x - 3y$

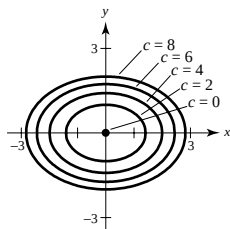
The level curves are of the form $6 - 2x - 3y = c$ or $2x + 3y = 6 - c$. Thus, the level curves are straight lines with a slope of $-\frac{2}{3}$.



52. $f(x, y) = x^2 + 2y^2$

The level curves are ellipses of the form

$$x^2 + 2y^2 = c \text{ (except } x^2 + 2y^2 = 0 \text{ is the point } (0, 0)).$$

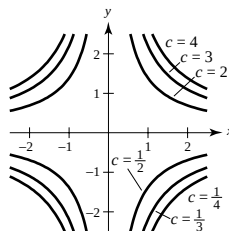


54. $f(x, y) = e^{xy/2}$

The level curves are of the form

$$e^{xy/2} = c, \text{ or } \ln c = \frac{xy}{2}.$$

Thus, the level curves are hyperbolas.



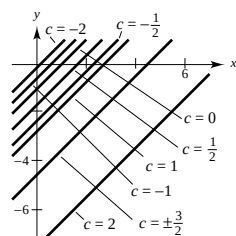
56. $f(x, y) = \ln(x - y)$

The level curves are of the form

$$c = \ln(x - y)$$

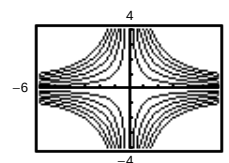
$$e^c = x - y$$

$$y = x - e^c$$

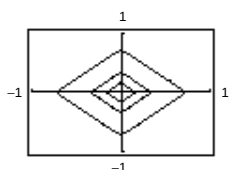


Thus, the level curves are parallel lines of slope 1 passing through the fourth quadrant.

58. $f(x, y) = |xy|$



60. $h(x, y) = 3 \sin(|x| + |y|)$



62. The graph of a function of two variables is the set of all points (x, y, z) for which $z = f(x, y)$ and (x, y) is in the domain of f . The graph can be interpreted as a surface in space. Level curves are the scalar fields $f(x, y) = c$, for c , a constant.

64. $f(x, y) = \frac{x}{y}$

The level curves are the lines

$$c = \frac{x}{y} \text{ or } y = \frac{1}{c}x$$

These lines all pass through the origin.

66. The surface could be an ellipsoid centered at $(0, 1, 0)$. One possible function is

$$f(x, y) = x^2 + \frac{(y - 1)^2}{4} = 1.$$

68. $A(r, t) = 1000e^{rt}$

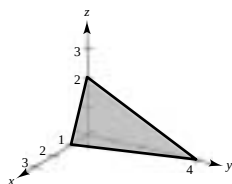
	Number of years			
Rate	5	10	15	20
0.08	\$1491.82	\$2225.54	\$3320.12	\$4953.03
0.10	\$1648.72	\$2718.28	\$4481.69	\$7389.06
0.12	\$1822.12	\$3320.12	\$6049.65	\$11,023.18
0.14	\$2013.75	\$4055.20	\$8166.17	\$16,444.65

70. $f(x, y, z) = 4x + y + 2z$

$$c = 4$$

$$4 = 4x + y + 2z$$

Plane



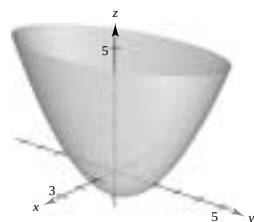
72. $f(x, y, z) = x^2 + \frac{1}{4}y^2 - z$

$$c = 1$$

$$1 = x^2 + \frac{1}{4}y^2 - z$$

Elliptic paraboloid

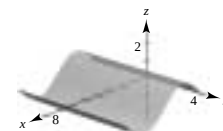
Vertex: $(0, 0, -1)$



74. $f(x, y, z) = \sin x - z$

$$c = 0$$

$$0 = \sin x - z \text{ or } z = \sin x$$



76. $W(x, y) = \frac{1}{x - y}, y < x$

(a) $W(15, 10) = \frac{1}{15 - 10} = \frac{1}{5} \text{ hr} = 12 \text{ min}$

(b) $W(12, 9) = \frac{1}{12 - 9} = \frac{1}{3} \text{ hr} = 20 \text{ min}$

(c) $W(12, 6) = \frac{1}{12 - 6} = \frac{1}{6} \text{ hr} = 10 \text{ min}$

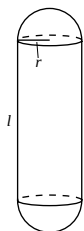
(d) $W(4, 2) = \frac{1}{4 - 2} = \frac{1}{2} \text{ hr} = 30 \text{ min}$

78. $f(x, y) = 100x^{0.6}y^{0.4}$

$f(2x, 2y) = 100(2x)^{0.6}(2y)^{0.4}$

$= 100(2)^{0.6}x^{0.6}(2)^{0.4}y^{0.4} = 100(2)^{0.6+0.4}x^{0.6}y^{0.4} = 2[100x^{0.6}y^{0.4}] = 2f(x, y)$

80. $V = \pi r^2 l + \frac{4}{3}\pi r^3 = \frac{\pi r^2}{3}(3l + 4r)$



82. (a)

Year	1995	1996	1997	1998	1999	2000
z	12.7	14.8	17.1	18.5	21.1	25.8
Model	13.09	14.79	16.45	18.47	21.38	25.78

(b) x has the greater influence because its coefficient (0.143) is larger than that of y (0.024).

(c) $f(x, 25) = 0.143x + 0.024(25) + 0.502$
 $= 0.143x + 1.102$

This function gives the shareholder's equity z in terms of net sales x and assumes constant assets of $y = 25$.

84. Southwest

86. Latitude and land versus ocean location have the greatest effect on temperature.

88. True

90. True

Section 12.2 Limits and Continuity

2. Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(x, y) - L| = |x - 4| < \varepsilon$

whenever $0 < \sqrt{(x - a)^2 + (y - b)^2} = \sqrt{(x - 4)^2 + (y + 1)^2} < \delta$. Take $\delta = \varepsilon$.

Then if $0 < \sqrt{(x - 4)^2 + (y + 1)^2} < \delta = \varepsilon$, we have

$$\sqrt{(x - 4)^2} < \varepsilon$$

$$|x - 4| < \varepsilon.$$

4.
$$\lim_{(x, y) \rightarrow (a, b)} \left[\frac{4f(x, y)}{g(x, y)} \right] = \frac{4 \left[\lim_{(x, y) \rightarrow (a, b)} f(x, y) \right]}{\lim_{(x, y) \rightarrow (a, b)} g(x, y)} = \frac{4(5)}{3} = \frac{20}{3}$$

$$6. \lim_{(x,y) \rightarrow (a,b)} \left[\frac{f(x,y) - g(x,y)}{f(x,y)} \right] = \frac{\lim_{(x,y) \rightarrow (a,b)} f(x,y) - \lim_{(x,y) \rightarrow (a,b)} g(x,y)}{\lim_{(x,y) \rightarrow (a,b)} f(x,y)} = \frac{5 - 3}{5} = \frac{2}{5}$$

$$8. \lim_{(x,y) \rightarrow (0,0)} (5x + y + 1) = 0 + 0 + 1 = 1$$

Continuous everywhere

$$10. \lim_{(x,y) \rightarrow (1,1)} \frac{x}{\sqrt{x+y}} = \frac{1}{\sqrt{1+1}} = \frac{\sqrt{2}}{2}$$

Continuous for $x + y > 0$

$$12. \lim_{(x,y) \rightarrow (\pi/4, 2)} y \cos(xy) = 2 \cos \frac{\pi}{2} = 0$$

Continuous everywhere

$$14. \lim_{(x,y) \rightarrow (1,1)} \frac{xy}{x^2 + y^2} = \frac{1}{2}$$

Continuous except at $(0, 0)$

$$16. \lim_{(x,y,z) \rightarrow (2,0,1)} xe^{yz} = 2e^0 = 2$$

Continuous everywhere

$$18. f(x,y) = \frac{x^2}{(x^2 + 1)(y^2 + 1)}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{(x^2 + 1)(y^2 + 1)} = \frac{0}{(0 + 1)(0 + 1)} = 0$$

Continuous everywhere

$$20. \lim_{(x,y) \rightarrow (0,0)} \left[1 - \frac{\cos(x^2 + y^2)}{x^2 + y^2} \right] = -\infty$$

The limit does not exist.

Continuous except at $(0, 0)$

$$22. f(x,y) = \frac{y}{x^2 + y^2}$$

Continuous except at $(0, 0)$

Path: $y = 0$

(x, y)	(1, 1)	(0.5, 0.5)	(0.1, 0.1)	(0.01, 0.01)	(0.001, 0.001)
$f(x, y)$	$\frac{1}{2}$	1	5	50	500

Path: $y = x$

(x, y)	(1, 0)	(0.5, 0)	(0.1, 0)	(0.01, 0)	(0.001, 0)
$f(x, y)$	0	0	0	0	0

The limit does not exist because along the path $y = 0$ the function equals 0, whereas along the path $y = x$ the function tends to infinity.

$$24. f(x,y) = \frac{2x - y^2}{2x^2 + y}$$

Continuous except at $(0, 0)$

Path: $y = 0$

(x, y)	(1, 0)	(0.25, 0)	(0.01, 0)	(0.001, 0)	(0.000001, 0)
$f(x, y)$	1	4	100	1000	1,000,000

Path: $y = x$

(x, y)	(1, 1)	(0.25, 0.25)	(0.01, 0.01)	(0.001, 0.001)	(0.0001, 0.0001)
$f(x, y)$	$\frac{1}{3}$	1.17	1.95	1.995	2.0

The limit does not exist because along the line $y = 0$ the function tends to infinity, whereas along the line $y = x$ the function tends to 2.

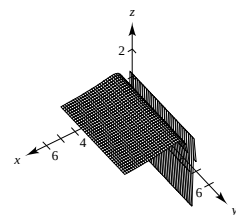
$$26. \lim_{(x,y) \rightarrow (0,0)} \frac{4x^2y^2}{(x^2 + y^2)^2} = 0$$

$$\text{Hence, } \lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} g(x,y) = 0.$$

f is continuous at $(0,0)$, whereas g is not continuous at $(0,0)$.

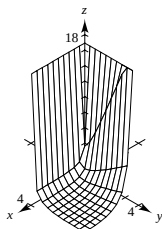
$$28. \lim_{(x,y) \rightarrow (0,0)} \left(\sin \frac{1}{x} + \cos \frac{1}{y} \right)$$

Does not exist



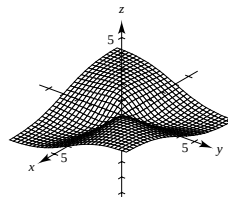
$$30. \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{x^2y}$$

Does not exist



$$32. f(x,y) = \frac{2xy}{x^2 + y^2 + 1}$$

The limit equals 0.



$$34. \lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{(r \cos \theta)(r^2 \sin^2 \theta)}{r^2} = \lim_{r \rightarrow 0} (r \cos \theta \sin^2 \theta) = 0$$

$$36. \lim_{(x,y) \rightarrow (0,0)} \frac{x^2y^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^4 \cos^2 \theta \sin^2 \theta}{r^2} = \lim_{r \rightarrow 0} r^2 \cos^2 \theta \sin^2 \theta = 0$$

$$38. f(x,y,z) = \frac{z}{x^2 + y^2 - 9}$$

Continuous for $x^2 + y^2 \neq 9$

$$40. f(x,y,z) = xy \sin z$$

Continuous everywhere

$$42. f(t) = \frac{1}{t}$$

$$g(x,y) = x^2 + y^2$$

$$f(g(x,y)) = f(x^2 + y^2)$$

$$= \frac{1}{x^2 + y^2}$$

Continuous except at $(0,0)$

$$44. f(t) = \frac{1}{4 - t}$$

$$g(x,y) = x^2 + y^2$$

$$f(g(x,y)) = f(x^2 + y^2) = \frac{1}{4 - x^2 - y^2}$$

Continuous for $x^2 + y^2 \neq 4$

$$46. f(x,y) = x^2 + y^2$$

$$\begin{aligned} \text{(a) } \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 + y^2] - (x^2 + y^2)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x \end{aligned}$$

$$\begin{aligned} \text{(b) } \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} &= \lim_{\Delta y \rightarrow 0} \frac{[x^2 + (y + \Delta y)^2] - (x^2 + y^2)}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{2y\Delta y + (\Delta y)^2}{\Delta y} = \lim_{\Delta y \rightarrow 0} (2y + \Delta y) = 2y \end{aligned}$$

48. $f(x, y) = \sqrt{y}(y + 1)$

(a) $\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{y}(y + 1) - \sqrt{y}(y + 1)}{\Delta x} = 0$

(b) $\lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{(y + \Delta y)^{3/2} + (y + \Delta y)^{1/2} - (y^{3/2} + y^{1/2})}{\Delta y}$
 $= \lim_{\Delta y \rightarrow 0} \frac{(y + \Delta y)^{3/2} - y^{3/2}}{\Delta y} + \lim_{\Delta y \rightarrow 0} \frac{(y + \Delta y)^{1/2} - y^{1/2}}{\Delta y}$
 $= \frac{3}{2}y^{1/2} + \frac{1}{2}y^{-1/2} \quad (\text{L'Hôpital's Rule})$
 $= \frac{3y + 1}{2\sqrt{y}}$

50. See the definition on page 854.

52. $\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 + y^2}{xy}$

(a) Along $y = ax$: $\lim_{(x, ax) \rightarrow (0, 0)} \frac{x^2 + (ax)^2}{x(ax)} = \lim_{x \rightarrow 0} \frac{x^2(1 + a^2)}{ax^2} = \frac{1 + a^2}{a}, a \neq 0$

If $a = 0$, then $y = 0$ and the limit does not exist.

(c) No, the limit does not exist. Different paths result in different limits.

(b) Along $y = x^2$: $\lim_{(x, x^2) \rightarrow (0, 0)} \frac{x^2 + (x^2)^2}{x(x^2)} = \lim_{x \rightarrow 0} \frac{1 + x^2}{x}$
 limit does not exist

54. Given that $f(x, y)$ is continuous, then $\lim_{(x, y) \rightarrow (a, b)} f(x, y) = f(a, b) < 0$, which means that for each $\varepsilon > 0$, there corresponds

a $\delta > 0$ such that $|f(x, y) - f(a, b)| < \varepsilon$ whenever

$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta.$$

Let $\varepsilon = |f(a, b)|/2$, then $f(x, y) < 0$ for every point in the corresponding δ neighborhood since

$$|f(x, y) - f(a, b)| < \frac{|f(a, b)|}{2} \Rightarrow -\frac{|f(a, b)|}{2} < f(x, y) - f(a, b) < \frac{|f(a, b)|}{2}$$

$$\Rightarrow \frac{3}{2}f(a, b) < f(x, y) < \frac{1}{2}f(a, b) < 0.$$

56. False. Let $f(x, y) = \frac{xy}{x^2 + y^2}$.

See Exercise 21.

58. True

Section 12.3 Partial Derivatives

2. $f_y(-1, -2) < 0$

4. $f_x(-1, -1) = 0$

6. $f(x, y) = x^2 - 3y^2 + 7$

8. $z = 2y^2\sqrt{x}$

$f_x(x, y) = 2x$

$f_y(x, y) = -6y$

$\frac{\partial z}{\partial x} = \frac{y^2}{\sqrt{x}}$

$\frac{\partial z}{\partial y} = 4y\sqrt{x}$

10. $z = y^3 - 4xy^2 - 1$

$$\frac{\partial z}{\partial x} = -4y^2$$

$$\frac{\partial z}{\partial y} = 3y^2 - 8xy$$

12. $z = xe^{x/y}$

$$\frac{\partial z}{\partial x} = \frac{x}{y}e^{x/y} + e^{x/y} = e^{x/y}\left(\frac{x}{y} + 1\right)$$

$$\frac{\partial z}{\partial y} = xe^{x/y}\left(-\frac{x}{y^2}\right) = -\frac{x^2}{y^2}e^{x/y}$$

14. $z = \ln \sqrt{xy} = \frac{1}{2} \ln(xy)$

$$\frac{\partial z}{\partial x} = \frac{1}{2} \frac{y}{xy} = \frac{1}{2x}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2} \frac{x}{xy} = \frac{1}{2y}$$

16. $z = \ln(x^2 - y^2)$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2 - y^2}(2x) = \frac{2x}{x^2 - y^2}$$

$$\frac{\partial z}{\partial y} = \frac{-2y}{x^2 - y^2}$$

18. $f(x, y) = \frac{xy}{x^2 + y^2}$

$$f_x(x, y) = \frac{(x^2 + y^2)(y) - (xy)(2x)}{(x^2 + y^2)^2} = \frac{y^3 - x^2y}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{(x^2 + y^2)(x) - (xy)(2y)}{(x^2 + y^2)^2} = \frac{x^3 - xy^2}{(x^2 + y^2)^2}$$

20. $g(x, y) = \ln \sqrt{x^2 + y^2} = \frac{1}{2} \ln(x^2 + y^2)$

$$g_x(x, y) = \frac{1}{2} \frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2}$$

$$g_y(x, y) = \frac{1}{2} \frac{2y}{x^2 + y^2} = \frac{y}{x^2 + y^2}$$

22. $f(x, y) = \sqrt{2x + y^3}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(2x + y^3)^{-1/2}(2) = \frac{1}{\sqrt{2x + y^3}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(2x + y^3)^{-1/2}(3y^2) = \frac{3y^2}{2\sqrt{2x + y^3}}$$

24. $z = \sin 3x \cos 3y$

$$\frac{\partial z}{\partial x} = 3 \cos 3x \cos 3y$$

$$\frac{\partial z}{\partial y} = -3 \sin 3x \sin 3y$$

26. $z = \cos(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = -2x \sin(x^2 + y^2)$$

$$\frac{\partial z}{\partial y} = -2y \sin(x^2 + y^2)$$

$$\begin{aligned}
 28. \quad f(x, y) &= \int_x^y (2t + 1) dt + \int_y^x (2t - 1) dt \\
 &= \int_x^y (2t + 1) dt - \int_x^y (2t - 1) dt \\
 &= \int_x^y 2 dt = \left[2t \right]_x^y = 2y - 2x
 \end{aligned}$$

$$f_x(x, y) = -2$$

$$f_y(x, y) = 2$$

30. $f(x, y) = x^2 - 2xy + y^2 = (x - y)^2$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x)y + y^2 - x^2 + 2xy - y^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x - 2y) = 2(x - y)$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$= \lim_{\Delta y \rightarrow 0} \frac{x^2 - 2x(y + \Delta y) + (y + \Delta y)^2 - x^2 + 2xy - y^2}{\Delta y} = \lim_{\Delta y \rightarrow 0} (-2x + 2y + \Delta y) = 2(y - x)$$

32. $f(x, y) = \frac{1}{x + y}$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x + \Delta x + y} - \frac{1}{x + y}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-1}{(x + \Delta x + y)(x + y)} = \frac{-1}{(x + y)^2}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\frac{1}{x + y + \Delta y} - \frac{1}{x + y}}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{-1}{(x + y + \Delta y)(x + y)} = \frac{-1}{(x + y)^2}$$

34. $h(x, y) = x^2 - y^2$

$$h_x(x, y) = 2x$$

$$\text{At } (-2, 1): h_x(-2, 1) = -4$$

$$h_y(x, y) = -2y$$

$$\text{At } (-2, 1): h_y(-2, 1) = -2$$

36. $z = \cos(2x - y)$

$$\frac{\partial z}{\partial x} = -2 \sin(2x - y)$$

$$\text{At } \left(\frac{\pi}{4}, \frac{\pi}{3}\right), \frac{\partial z}{\partial x} = -2 \sin\left(\frac{\pi}{6}\right) = -1$$

$$\frac{\partial z}{\partial y} = -\sin(2x - y)(-1) = \sin(2x - y)$$

$$\text{At } \left(\frac{\pi}{4}, \frac{\pi}{3}\right), \frac{\partial z}{\partial y} = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

38. $f(x, y) = \arccos(xy)$

$$f_x(x, y) = \frac{-y}{\sqrt{1 - x^2 y^2}}$$

$$\text{At } (1, 1), f_x \text{ is undefined.}$$

$$f_y(x, y) = \frac{-x}{\sqrt{1 - x^2 y^2}}$$

$$\text{At } (1, 1), f_y \text{ is undefined.}$$

40. $f(x, y) = \frac{6xy}{\sqrt{4x^2 + 5y^2}}$

$$f_x(x, y) = \frac{30y^3}{(4x^2 + 5y^2)^{3/2}}$$

$$\text{At } (1, 1), f_x(1, 1) = \frac{30}{27} = \frac{10}{9}$$

$$f_y(x, y) = \frac{24x^3}{(4x^2 + 5y^2)^{3/2}}$$

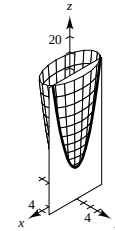
$$\text{At } (1, 1), f_y(1, 1) = \frac{8}{9}$$

42. $z = x^2 + 4y^2, y = 1, (2, 1, 8)$

$$\text{Intersecting curve: } z = x^2 + 4$$

$$\frac{\partial z}{\partial x} = 2x$$

$$\text{At } (2, 1, 8): \frac{\partial z}{\partial x} = 2(2) = 4$$

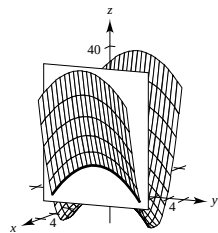


44. $z = 9x^2 - y^2, x = 1, (1, 3, 0)$

$$\text{Intersecting curve: } z = 9 - y^2$$

$$\frac{\partial z}{\partial y} = -2y$$

$$\text{At } (1, 3, 0): \frac{\partial z}{\partial y} = -2(3) = -6$$



46. $f_x(x, y) = 9x^2 - 12y, f_y(x, y) = -12x + 3y^2$

$$f_x = f_y = 0: 9x^2 - 12y = 0 \Rightarrow 3x^2 = 4y$$

$$3y^2 - 12x = 0 \Rightarrow y^2 = 4x$$

Solving for x in the second equation, $x = y^2/4$, you obtain $3(y^2/4)^2 = 4y$.

$$3y^4 = 64y \Rightarrow y = 0 \text{ or } y = \frac{4}{3^{1/3}}$$

$$\Rightarrow x = 0 \text{ or } x = \frac{1}{4} \left(\frac{16}{3^{2/3}} \right)$$

$$\text{Points: } (0, 0), \left(\frac{4}{3^{2/3}}, \frac{4}{3^{1/3}} \right)$$

$$48. f_x(x, y) = \frac{2x}{x^2 + y^2 + 1} = 0 \Rightarrow x = 0$$

$$f_y(x, y) = \frac{2y}{x^2 + y^2 + 1} = 0 \Rightarrow y = 0$$

Points: (0, 0)

$$52. w = \frac{3xz}{x + y}$$

$$\frac{\partial w}{\partial x} = \frac{(x + y)(3z) - 3xz}{(x + y)^2} = \frac{3yz}{(x + y)^2}$$

$$\frac{\partial w}{\partial y} = \frac{-3xz}{(x + y)^2}$$

$$\frac{\partial w}{\partial z} = \frac{3x}{x + y}$$

50. (a) The graph is that of f_x .

(b) The graph is that of f_y .

$$54. G(x, y, z) = \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$$

$$G_x(x, y, z) = \frac{x}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$G_y(x, y, z) = \frac{y}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$G_z(x, y, z) = \frac{z}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$56. f(x, y, z) = 3x^2y - 5xyz + 10yz^2$$

$$f_x(x, y, z) = 6xy - 5yz$$

$$f_y(x, y, z) = 3x^2 - 5xz + 10z^2$$

$$f_z(x, y, z) = -5xy + 20yz$$

$$58. z = x^4 - 3x^2y^2 + y^4$$

$$\frac{\partial z}{\partial x} = 4x^3 - 6xy^2$$

$$\frac{\partial^2 z}{\partial x^2} = 12x^2 - 6y^2$$

$$\frac{\partial^2 z}{\partial y \partial x} = -12xy$$

$$\frac{\partial z}{\partial y} = -6x^2y + 4y^3$$

$$\frac{\partial^2 z}{\partial y^2} = -6x^2 + 12y^2$$

$$\frac{\partial^2 z}{\partial x \partial y} = -12xy$$

$$60. z = \ln(x - y)$$

$$\frac{\partial z}{\partial x} = \frac{1}{x - y}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{(x - y)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{1}{(x - y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{-1}{x - y} = \frac{1}{y - x}$$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{1}{(x - y)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{(x - y)^2}$$

$$\text{Therefore, } \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}.$$

$$62. z = 2xe^y - 3ye^{-x}$$

$$\frac{\partial z}{\partial x} = 2e^y + 3ye^{-x}$$

$$\frac{\partial^2 z}{\partial x^2} = -3ye^{-x}$$

$$\frac{\partial^2 z}{\partial y \partial x} = 2e^y + 3ye^{-x}$$

$$\frac{\partial z}{\partial y} = 2xe^y - 3e^{-x}$$

$$\frac{\partial^2 z}{\partial y^2} = 2xe^y$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2e^y + 3e^{-x}$$

$$64. z = \sin(x - 2y)$$

$$\frac{\partial z}{\partial x} = \cos(x - 2y)$$

$$\frac{\partial^2 z}{\partial x^2} = -\sin(x - 2y)$$

$$\frac{\partial^2 z}{\partial y \partial x} = 2 \sin(x - 2y)$$

$$\frac{\partial z}{\partial y} = -2 \cos(x - 2y)$$

$$\frac{\partial^2 z}{\partial y^2} = -4 \sin(x - 2y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2 \sin(x - 2y)$$

$$66. z = \sqrt{9 - x^2 - y^2}$$

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{9 - x^2 - y^2}}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{y^2 - 9}{(9 - x^2 - y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-xy}{(9 - x^2 - y^2)^{3/2}}$$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{9 - x^2 - y^2}}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{x^2 - 9}{(9 - x^2 - y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-xy}{(9 - x^2 - y^2)^{3/2}}$$

$$\text{Therefore, } \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}.$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0 \text{ if } x = y = 0$$

$$\begin{aligned}
 68. \quad z &= \frac{xy}{x-y} \\
 \frac{\partial z}{\partial x} &= \frac{y(x-y) - xy}{(x-y)^2} = \frac{-y^2}{(x-y)^2} \\
 \frac{\partial^2 z}{\partial x^2} &= \frac{2y^2}{(x-y)^3} \\
 \frac{\partial^2 z}{\partial y \partial x} &= \frac{(x-y)^2(-2y) + y^2(2)(x-y)(-1)}{(x-y)^4} = \frac{-2xy}{(x-y)^3} \\
 \frac{\partial z}{\partial y} &= -\frac{x(x-y) + xy}{(x-y)^2} = \frac{-x^2}{(x-y)^2} \\
 \frac{\partial^2 z}{\partial y^2} &= \frac{2x^2}{(x-y)^3} \\
 \frac{\partial^2 z}{\partial x \partial y} &= \frac{(x-y)^2(2x) - x^2(2)(x-y)}{(x-y)^4} = \frac{-2xy}{(x-y)^3}
 \end{aligned}$$

There are no points for which $z_x = z_y = 0$.

$$\begin{aligned}
 72. \quad f(x, y, z) &= \frac{2z}{x+y} \\
 f_x(x, y, z) &= \frac{-2z}{(x+y)^2} \\
 f_y(x, y, z) &= \frac{-2z}{(x+y)^2} \\
 f_{yy}(x, y, z) &= \frac{4z}{(x+y)^3} \\
 f_{xy}(x, y, z) &= \frac{4z}{(x+y)^3} \\
 f_{yx}(x, y, z) &= \frac{4z}{(x+y)^3} \\
 f_{yyx}(x, y, z) &= \frac{-12z}{(x+y)^4} \\
 f_{xyy}(x, y, z) &= \frac{-12z}{(x+y)^4} \\
 f_{yxy}(x, y, z) &= \frac{-12z}{(x+y)^4}
 \end{aligned}$$

$$76. \quad z = \arctan \frac{y}{x}$$

From Exercise 53, we have

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2xy}{(x^2 + y^2)^2} + \frac{-2xy}{(x^2 + y^2)^2} = 0.$$

$$\begin{aligned}
 70. \quad f(x, y, z) &= x^2 - 3xy + 4yz + z^3 \\
 f_x(x, y, z) &= 2x - 3y \\
 f_y(x, y, z) &= -3x + 4z \\
 f_{yy}(x, y, z) &= 0 \\
 f_{xy}(x, y, z) &= -3 \\
 f_{yx}(x, y, z) &= -3 \\
 f_{yyx}(x, y, z) &= 0 \\
 f_{xyy}(x, y, z) &= 0 \\
 f_{yxy}(x, y, z) &= 0 \\
 \text{Therefore, } f_{xyy} &= f_{yxy} = f_{yyx} = 0.
 \end{aligned}$$

$$\begin{aligned}
 74. \quad z &= \sin x \left(\frac{e^y - e^{-y}}{2} \right) \\
 \frac{\partial z}{\partial x} &= \cos x \left(\frac{e^y - e^{-y}}{2} \right) \\
 \frac{\partial^2 z}{\partial x^2} &= -\sin x \left(\frac{e^y - e^{-y}}{2} \right) \\
 \frac{\partial z}{\partial y} &= \sin x \left(\frac{e^y + e^{-y}}{2} \right) \\
 \frac{\partial^2 z}{\partial y^2} &= \sin x \left(\frac{e^y - e^{-y}}{2} \right)
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} &= -\sin x \left(\frac{e^y - e^{-y}}{2} \right) + \sin x \left(\frac{e^y - e^{-y}}{2} \right) \\
 &= 0.
 \end{aligned}$$

$$\begin{aligned}
 78. \quad z &= \sin(wct) \sin(wx) \\
 \frac{\partial z}{\partial t} &= wc \cos(wct) \sin(wx) \\
 \frac{\partial^2 z}{\partial t^2} &= -w^2 c^2 \sin(wct) \sin(wx) \\
 \frac{\partial z}{\partial x} &= w \sin(wct) \cos(wx) \\
 \frac{\partial^2 z}{\partial x^2} &= -w^2 \sin(wct) \sin(wx) \\
 \text{Therefore, } \frac{\partial^2 z}{\partial t^2} &= c^2 \frac{\partial^2 z}{\partial x^2}.
 \end{aligned}$$

80. $z = e^{-t} \sin \frac{x}{c}$

$$\frac{\partial z}{\partial t} = -e^{-t} \sin \frac{x}{c}$$

$$\frac{\partial z}{\partial x} = \frac{1}{c} e^{-t} \cos \frac{x}{c}$$

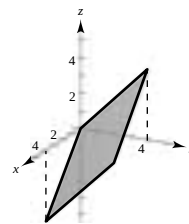
$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{c^2} e^{-t} \sin \frac{x}{c}$$

Therefore, $\frac{\partial z}{\partial t} = c^2 \frac{\partial^2 z}{\partial x^2}$.

82. If $z = f(x, y)$, then to find f_x you consider y constant and differentiate with respect to x . Similarly, to find f_y , you consider x constant and differentiate with respect to y .

84. The plane $z = -x + y = f(x, y)$ satisfies

$$\frac{\partial f}{\partial x} < 0 \text{ and } \frac{\partial f}{\partial y} > 0.$$



86. In this case, the mixed partials are equal, $f_{xy} = f_{yx}$.

See Theorem 12.3.

88. $f(x, y) = 200x^{0.7}y^{0.3}$

(a) $\frac{\partial f}{\partial x} = 140x^{-0.3}y^{0.3} = 140\left(\frac{y}{x}\right)^{0.3}$

At $(x, y) = (1000, 500)$, $\frac{\partial f}{\partial x} = 140\left(\frac{500}{1000}\right)^{0.3} = 140\left(\frac{1}{2}\right)^{0.3} \approx 113.72$

(b) $\frac{\partial f}{\partial x} = 60x^{0.7}y^{-0.7} = 60\left(\frac{x}{y}\right)^{0.7}$

At $(x, y) = (1000, 500)$, $\frac{\partial f}{\partial x} = 60\left(\frac{1000}{500}\right)^{0.7} = 60(2)^{0.7} \approx 97.47$

90. $V(I, R) = 1000\left[\frac{1 + 0.10(1 - R)}{1 + I}\right]^{10}$

$$V_I(I, R) = 10,000\left[\frac{1 + 0.10(1 - R)}{1 + I}\right]^9 \left[-\frac{1 + 0.10(1 - R)}{(1 + I)^2}\right] = -10,000 \frac{[1 + 0.10(1 - R)]^{10}}{(1 + I)^{11}}$$

$$V_I(0.03, 0.28) \approx -14,478.99$$

$$V_R(I, R) = 10,000\left[\frac{1 + 0.10(1 - R)}{1 + I}\right]^9 \left[\frac{-0.10}{1 + I}\right] = -1000 \frac{[1 + 0.10(1 - R)]^9}{(1 + I)^{10}}$$

$$V_R(0.03, 0.28) \approx -1391.17$$

The rate of inflation has the greater negative influence on the growth of the investment. (See Exercise 61 in Section 12.1.)

92. $A = 0.885t - 22.4h + 1.20th - 0.544$

(a) $\frac{\partial A}{\partial t} = 0.885 + 1.20h$

$$\frac{\partial A}{\partial t}(30^\circ, 0.80) = 0.885 + 1.20(0.80) = 1.845$$

$$\frac{\partial A}{\partial h} = -22.4 + 1.20t$$

$$\frac{\partial A}{\partial h}(30^\circ, 0.80) = -22.4 + 1.20(30^\circ) = 13.6$$

(b) The humidity has a greater effect on A since its coefficient -22.4 is larger than that of t .

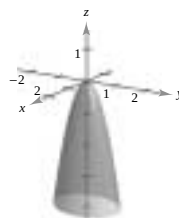
94. $U = -5x^2 + xy - 3y^2$

(a) $U_x = -10x + y$

(b) $U_y = x - 6y$

(c) $U_x(2, 3) = -17$ and $U_y(2, 3) = -16$. The person should consume one more unit of y because the rate of decrease of satisfaction is less for y .

(d)



96. (a) $\frac{\partial z}{\partial x} = -1.55x + 22.15$

$$\frac{\partial^2 z}{\partial x^2} = -1.55$$

$$\frac{\partial z}{\partial y} = 0.014y - 0.54$$

$$\frac{\partial^2 z}{\partial y^2} = 0.014$$

(b) Concave downward $\left(\frac{\partial^2 z}{\partial x^2} < 0\right)$

The rate of increase of Medicare expenses (z) is declining with respect to worker's compensation expenses (x).

(c) Concave upward $\left(\frac{\partial^2 z}{\partial y^2} > 0\right)$

The rate of increase of Medicare expenses (z) is increasing with respect to public assistance expenses (y).

98. False

$$\text{Let } z = x + y + 1.$$

100. True

102. $f(x, y) = \int_x^y \sqrt{1+t^3} dt$

By the Second Fundamental Theorem of Calculus,

$$\frac{\partial f}{\partial x} = \frac{d}{dx} \int_x^y \sqrt{1+t^3} dt = -\frac{d}{dx} \int_y^x \sqrt{1+t^3} dt = -\sqrt{1+x^3}$$

$$\frac{\partial f}{\partial y} = \frac{d}{dy} \int_x^y \sqrt{1+t^3} dt = \sqrt{1+y^3}.$$

Section 12.4 Differentials

2. $z = \frac{x^2}{y}$

$$dz = \frac{2x}{y} dx - \frac{x^2}{y^2} dy$$

4. $w = \frac{x+y}{z-2y}$

$$dw = \frac{1}{z-2y} dx + \frac{z+2x}{(z-2y)^2} dy - \frac{x+y}{(z-2y)^2} dz$$

6. $z = \left(\frac{1}{2}\right)(e^{x^2+y^2} - e^{-x^2-y^2})$

$$dz = 2x\left(\frac{e^{x^2+y^2} + e^{-x^2-y^2}}{2}\right) dx + 2y\left(\frac{e^{x^2+y^2} + e^{-x^2-y^2}}{2}\right) dy = (e^{x^2+y^2} + e^{-x^2-y^2})(x dx + y dy)$$

8. $w = e^y \cos x + z^2$

$$dw = -e^y \sin x dx + e^y \cos x dy + 2z dz$$

10. $w = x^2 y z^2 + \sin y z$

$$dw = 2xyz^2 dx + (x^2 z^2 + z \cos yz) dy + (2x^2 yz + y \cos yz) dz$$

12. (a) $f(1, 2) = \sqrt{5} \approx 2.2361$

$$f(1.05, 2.1) = \sqrt{5.5125} \approx 2.3479$$

$$\Delta z = 0.11180$$

$$\begin{aligned} \text{(b) } dz &= \frac{x}{\sqrt{x^2+y^2}} dx + \frac{y}{\sqrt{x^2+y^2}} dy \\ &= \frac{x dx + y dy}{\sqrt{x^2+y^2}} = \frac{0.05 + 2(0.1)}{\sqrt{5}} \approx 0.11180 \end{aligned}$$

14. (a) $f(1, 2) = e^2 \approx 7.3891$

$$f(1.05, 2.1) = 1.05e^{2.1} \approx 8.5745$$

$$\Delta z = 1.1854$$

$$\begin{aligned} \text{(b) } dz &= e^y dx + xe^y dy \\ &= e^2(0.05) + e^2(0.1) \approx 1.1084 \end{aligned}$$

16. (a) $f(1, 2) = \frac{1}{2} = 0.5$

$$f(1.05, 2.1) = \frac{1.05}{2.1} = 0.5$$

$$\Delta z = 0$$

(b) $dz = \frac{1}{y} dx - \frac{x}{y^2} dy$

$$= \frac{1}{2}(0.05) - \frac{1}{4}(0.1) = 0$$

18. Let $z = x^2(1 + y)^3$, $x = 2$, $y = 9$, $dx = 0.03$, $dy = -0.1$. Then: $dz = 2x(1 + y)^3 dx + 3x^2(1 + y)^2 dy$

$$(2.03)^2(1 + 8.9)^3 - 2^2(1 + 9)^3 \approx 2(2)(1 + 9)^3(0.03) + 3(2)^2(1 + 9)^2(-0.1) = 0$$

20. Let $z = \sin(x^2 + y^2)$, $x = y = 1$, $dx = 0.05$, $dy = -0.05$. Then: $dz = 2x \cos(x^2 + y^2) dx + 2y \cos(x^2 + y^2) dy$

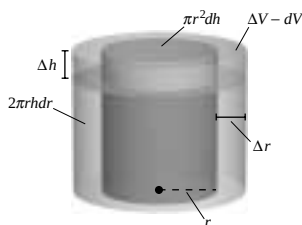
$$\sin[(1.05)^2 + (0.95)^2] - \sin 2 \approx 2(1) \cos(1^2 + 1^2)(0.05) + 2(1) \cos(1^2 + 1^2)(-0.05) = 0$$

22. In general, the accuracy worsens as Δx and Δy increase.

24. If $z = f(x, y)$, then $\Delta z \approx dz$ is the propagated error, and $\frac{\Delta z}{z} \approx \frac{dz}{z}$ is the relative error.

26. $V = \pi r^2 h$

$$dV = 2\pi r h dr + \pi r^2 dh$$



28. $S = \pi r \sqrt{r^2 + h^2}$

$$r = 8, h = 20$$

$$\frac{dS}{dr} = \pi(r^2 + h^2)^{1/2} + \pi r^2(r^2 + h^2)^{-1/2}$$

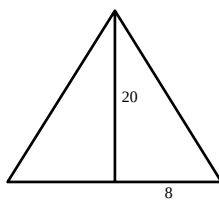
$$= \frac{\pi(r^2 + h^2) + \pi r^2}{(r^2 + h^2)^{1/2}} = \pi \frac{2r^2 + h^2}{\sqrt{r^2 + h^2}}$$

$$\frac{dS}{dh} = \pi r(r^2 + h^2)^{-1/2} h = \pi \frac{rh}{\sqrt{r^2 + h^2}}$$

$$dS = \pi \frac{2r^2 + h^2}{\sqrt{r^2 + h^2}} dr + \pi \frac{rh}{\sqrt{r^2 + h^2}} dh$$

$$= \frac{\pi}{\sqrt{r^2 + h^2}} [(2r^2 + h^2) dr + (rh) dh]$$

$$S(8, 20) = 541.3758$$



Δr	Δh	dS	ΔS	$\Delta S - dS$
0.1	0.1	10.0341	10.0768	0.0427
0.1	-0.1	5.3671	5.3596	-0.0075
0.001	0.002	0.12368	0.12368	0.683×10^{-5}
-0.0001	0.0002	-0.00303	-0.00303	-0.286×10^{-7}

$$30. \frac{\partial C}{\partial v} = 0.0817 \left[(3.71) \frac{1}{2} v^{-1/2} - 0.25 \right] (T - 91.4)$$

$$= \left[\frac{0.1516}{v^{1/2}} - 0.0204 \right] (T - 91.4)$$

$$\frac{\partial C}{\partial T} = 0.0817(3.71\sqrt{v} + 5.81 - 0.25v)$$

$$dC = C_v dv + C_T dT$$

$$= \left(\frac{0.1516}{23^{1/2}} - 0.0204 \right) (8 - 91.4)(\pm 3) + 0.0817(3.71\sqrt{23} + 5.81 - 0.25(23))(\pm 1)$$

$$= \pm 2.79 \pm 1.46 = \pm 4.25 \text{ Maximum propagated error}$$

$$\frac{dC}{C} = \frac{\pm 4.25}{-30.24} \approx \pm 0.14$$

$$32. (x, y) = (8.5, 3.2), |dx| \leq 0.05, |dy| \leq 0.05$$

$$r = \sqrt{x^2 + y^2} \Rightarrow dr = \frac{x}{\sqrt{x^2 + y^2}} dx + \frac{y}{\sqrt{x^2 + y^2}} dy$$

$$= \frac{8.5}{\sqrt{8.5^2 + 3.2^2}} dx + \frac{3.2}{\sqrt{8.5^2 + 3.2^2}} dy \approx 0.9359 dx + 0.3523 dy$$

$$|dr| \leq (1.288)(0.05) \approx 0.064$$

$$\theta = \arctan\left(\frac{y}{x}\right) \Rightarrow d\theta = \frac{-\frac{y}{x^2}}{1 + \left(\frac{y}{x}\right)^2} dx + \frac{\frac{1}{x}}{1 + \left(\frac{y}{x}\right)^2} dy$$

$$= \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy = \frac{-3.2}{8.5^2 + 3.2^2} dx + \frac{8.5}{8.5^2 + 3.2^2} dy$$

Using the worst case scenario, $dx = -0.05$ and $dy = 0.05$, you see that

$$|d\theta| \leq 0.00194 + 0.00515 = 0.0071.$$

$$34. a = \frac{v^2}{r}$$

$$da = \frac{2v}{r} dv - \frac{v^2}{r^2} dr$$

$$\frac{da}{a} = 2\frac{dv}{v} - \frac{dr}{r} = 2(0.03) - (-0.02) = 0.08 = 8\%$$

Note: The maximum error will occur when dv and dr differ in signs.

$$36. (a) \text{ Using the Law of Cosines:}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$= 330^2 + 420^2 - 2(330)(420)\cos 9^\circ$$

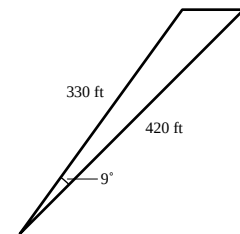
$$a \approx 107.3 \text{ ft.}$$

$$(b) a = \sqrt{b^2 + 420^2 - 2b(420)\cos \theta}$$

$$da = \frac{1}{2} \left[b^2 + 420^2 - 840b \cos \theta \right]^{-1/2} [(2b - 840 \cos \theta) db + 840b \sin \theta d\theta]$$

$$= \frac{1}{2} \left[330^2 + 420^2 - 840(330) \left(\cos \frac{\pi}{20} \right) \right]^{-1/2} \left[\left(2(330) - 840 \cos \frac{\pi}{20} \right) (6) + 840(330) \left(\sin \frac{\pi}{20} \right) \left(\frac{\pi}{180} \right) \right]$$

$$\approx \frac{1}{2} [11512.79]^{-1/2} [\pm 1774.79] \approx \pm 8.27 \text{ ft}$$



$$38. \quad \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

$$dR_1 = \Delta R_1 = 0.5$$

$$dR_2 = \Delta R_2 = -2$$

$$\Delta R \approx dR = \frac{\partial R}{\partial R_1} dR_1 + \frac{\partial R}{\partial R_2} dR_2 = \frac{R_2^2}{(R_1 + R_2)^2} \Delta R_1 + \frac{R_1^2}{(R_1 + R_2)^2} \Delta R_2$$

$$\text{When } R_1 = 10 \text{ and } R_2 = 15, \text{ we have } \Delta R \approx \frac{15^2}{(10 + 15)^2} (0.5) + \frac{10^2}{(10 + 15)^2} (-2) = -0.14 \text{ ohm.}$$

$$40. \quad T = 2\pi \sqrt{\frac{L}{g}}$$

$$dg = \Delta g = 32.24 - 32.09 = 0.15$$

$$dL = \Delta L = 2.48 - 2.5 = -0.02$$

$$\Delta T \approx dT = \frac{\partial T}{\partial g} dg + \frac{\partial T}{\partial L} dL = -\frac{\pi}{g} \sqrt{\frac{L}{g}} \Delta g + \frac{\pi}{\sqrt{Lg}} \Delta L$$

$$\text{When } g = 32.09 \text{ and } L = 2.5, \text{ we have } \Delta T \approx -\frac{\pi}{32.09} \sqrt{\frac{2.5}{32.09}} (0.15) + \frac{\pi}{\sqrt{(2.5)(32.09)}} (-0.02) \approx -0.0111 \text{ sec.}$$

$$42. \quad z = f(x, y) = x^2 + y^2$$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

$$= x^2 + 2x(\Delta x) + (\Delta x)^2 + y^2 + 2y(\Delta y) + (\Delta y)^2 - (x^2 + y^2)$$

$$= 2x(\Delta x) + 2y(\Delta y) + \Delta x(\Delta x) + \Delta y(\Delta y)$$

$$= f_x(x, y) \Delta x + f_y(x, y) \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \text{ where } \epsilon_1 = \Delta x \text{ and } \epsilon_2 = \Delta y.$$

$$\text{As } (\Delta x, \Delta y) \rightarrow (0, 0), \epsilon_1 \rightarrow 0 \text{ and } \epsilon_2 \rightarrow 0.$$

$$44. \quad z = f(x, y) = 5x - 10y + y^3$$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

$$= 5x + 5\Delta x - 10y - 10\Delta y + y^3 + 3y^2(\Delta y) + 3y(\Delta y)^2 + (\Delta y)^3 - (5x - 10y + y^3)$$

$$= 5(\Delta x) + (3y^2 - 10)(\Delta y) + 0(\Delta x) + (3y(\Delta y) + (\Delta y)^2) \Delta y$$

$$= f_x(x, y) \Delta x + f_y(x, y) \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \text{ where } \epsilon_1 = 0 \text{ and } \epsilon_2 = 3y(\Delta y) + (\Delta y)^2.$$

$$\text{As } (\Delta x, \Delta y) \rightarrow (0, 0), \epsilon_1 \rightarrow 0 \text{ and } \epsilon_2 \rightarrow 0.$$

$$46. \quad f(x, y) = \begin{cases} \frac{5x^2y}{x^3 + y^3}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$$(a) \quad f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0 - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{0 - 0}{\Delta y} = 0$$

Thus, the partial derivatives exist at $(0, 0)$.

$$(b) \quad \text{Along the line } y = x: \lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{x \rightarrow 0} \frac{5x^3}{2x^3} = \frac{5}{2}.$$

$$\text{Along the line } x = 0, \lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 0.$$

Thus, f is not continuous at $(0, 0)$. Therefore f is not differentiable at $(0, 0)$.

(See Theorem 12.5)

Section 12.5 Chain Rules for Functions of Several Variables

2. $w = \sqrt{x^2 + y^2}$

$x = \cos t, y = e^t$

$$\begin{aligned}\frac{dw}{dt} &= \frac{x}{\sqrt{x^2 + y^2}}(-\sin t) + \frac{y}{\sqrt{x^2 + y^2}}e^t \\ &= \frac{-x \sin t + ye^t}{\sqrt{x^2 + y^2}} = \frac{-\cos t \sin t + e^{2t}}{\sqrt{\cos^2 t + e^{2t}}}\end{aligned}$$

4. $w = \ln \frac{y}{x}$

$x = \cos t$

$y = \sin t$

$$\begin{aligned}\frac{dw}{dt} &= \left(\frac{-1}{x}\right)(-\sin t) + \left(\frac{1}{y}\right)(\cos t) \\ &= \tan t + \cot t = \frac{1}{\sin t \cos t}\end{aligned}$$

6. $w = \cos(x - y), x = t^2, y = 1$

$$\begin{aligned}\text{(a)} \quad \frac{dw}{dt} &= -\sin(x - y)(2t) + \sin(x - y)(0) \\ &= -2t \sin(x - y) = -2t \sin(t^2 - 1)\end{aligned}$$

$$\text{(b)} \quad w = \cos(t^2 - 1), \quad \frac{dw}{dt} = -2t \sin(t^2 - 1)$$

8. $w = xy \cos z$

$x = t$

$y = t^2$

$z = \arccos t$

$$\begin{aligned}\text{(a)} \quad \frac{dw}{dt} &= (y \cos z)(1) + (x \cos z)(2t) + (-xy \sin z)\left(-\frac{1}{\sqrt{1 - t^2}}\right) \\ &= t^2(t) + t(t)(2t) - t(t^2)\sqrt{1 - t^2}\left(\frac{-1}{\sqrt{1 - t^2}}\right) = t^3 + 2t^3 + t^3 = 4t^3\end{aligned}$$

$$\text{(b)} \quad w = t^4, \quad \frac{dw}{dt} = 4t^3$$

10. $w = xyz, x = t^2, y = 2t, z = e^{-t}$

$$\begin{aligned}\text{(a)} \quad \frac{dw}{dt} &= yz(2t) + xz(2) + (xy)(-e^{-t}) \\ &= (2t)(e^{-t})(2t) + (t^2)(e^{-t})(2) + (t^2)(2t)(-e^{-t}) \\ &= 2t^2e^{-t}(2 + 1 - t) = 2t^2e^{-t}(3 - t)\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad w &= (t^2)(2t)(e^{-t}) = 2t^3e^{-t} \\ \frac{dw}{dt} &= (2t^3)(-e^{-t}) + (e^{-t})(6t^2) = 2t^2e^{-t}(-t + 3)\end{aligned}$$

$$\begin{aligned}\text{12. Distance} = f(t) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[48 + (\sqrt{3} - \sqrt{2})]^2 + [48t(1 - \sqrt{2})]^2} \\ &= 48t\sqrt{8 - 2\sqrt{2} - 2\sqrt{6}}\end{aligned}$$

$$f'(t) = 48\sqrt{8 - 2\sqrt{2} - 2\sqrt{6}} = f'(1)$$

14. $w = \frac{x^2}{y},$

$$x = t^2,$$

$$y = t + 1,$$

$$t = 1$$

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\ &= \frac{2x}{y} (2t) + \frac{-x^2}{y^2} (1)\end{aligned}$$

$$= \frac{2t^2(2t)}{t+1} - \frac{t^4}{(t+1)^2}$$

$$= \frac{(t+1)(4t^3) - t^4}{(t+1)^2}$$

$$= \frac{3t^4 + 4t^3}{(t+1)^2}$$

$$\frac{d^2w}{dt^2} = \frac{(t+1)^2(12t^3 + 12t^2) - (3t^4 + 4t^3)2(t+1)}{(t+1)^4}$$

$$\text{At } t = 1: \frac{d^2w}{dt^2} = \frac{4(24) - (7)(4)}{16} = \frac{68}{16} = 4.25$$

16. $w = y^3 - 3x^2y$

$$x = e^s$$

$$y = e^t$$

$$\frac{\partial w}{\partial s} = -6xy(e^s) + (3y^2 - 3x^2)(0) = -6e^{2s+t}$$

$$\begin{aligned}\frac{\partial w}{\partial t} &= -6xy(0) + (3y^2 - 3x^2)(e^t) \\ &= 3e^t(e^{2t} - e^{2s})\end{aligned}$$

$$\text{When } s = 0 \text{ and } t = 1, \frac{\partial w}{\partial s} = -6e \text{ and } \frac{\partial w}{\partial t} = 3e(e^2 - 1).$$

18. $w = \sin(2x + 3y)$

$$x = s + t$$

$$y = s - t$$

$$\frac{\partial w}{\partial s} = 2 \cos(2x + 3y) + 3 \cos(2x + 3y)$$

$$= 5 \cos(2x + 3y) = 5 \cos(5s - t)$$

$$\frac{\partial w}{\partial t} = 2 \cos(2x + 3y) - 3 \cos(2x + 3y)$$

$$= -\cos(2x + 3y) = -\cos(5s - t)$$

$$\text{When } s = 0 \text{ and } t = \frac{\pi}{2}, \frac{\partial w}{\partial s} = 0 \text{ and } \frac{\partial w}{\partial t} = 0.$$

20. $w = \sqrt{25 - 5x^2 - 5y^2}, x = r \cos \theta, y = r \sin \theta$

$$\begin{aligned}\text{(a)} \quad \frac{\partial w}{\partial r} &= \frac{-5x}{\sqrt{25 - 5x^2 - 5y^2}} \cos \theta + \frac{-5y}{\sqrt{25 - 5x^2 - 5y^2}} \sin \theta \\ &= \frac{-5r \cos^2 \theta - 5r \sin^2 \theta}{\sqrt{25 - 5x^2 - 5y^2}} = \frac{-5r}{\sqrt{25 - 5r^2}}\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial \theta} &= \frac{-5x}{\sqrt{25 - 5x^2 - 5y^2}} (-r \sin \theta) + \frac{-5y}{\sqrt{25 - 5x^2 - 5y^2}} (r \cos \theta) \\ &= \frac{-5r^2 \sin^2 \theta \cos \theta - 5r^2 \sin \theta \cos \theta}{\sqrt{25 - 5x^2 - 5y^2}} = 0\end{aligned}$$

(b) $w = \sqrt{25 - 5r^2}$

$$\frac{\partial w}{\partial r} = \frac{-5r}{\sqrt{25 - 5r^2}}, \frac{\partial w}{\partial \theta} = 0$$

$$22. w = \frac{yz}{x}, x = \theta^2, y = r + \theta, z = r - \theta$$

$$\begin{aligned} (a) \quad \frac{\partial w}{\partial r} &= \frac{-yz}{x^2}(0) + \frac{z}{x}(1) + \frac{y}{x}(1) = \frac{z+y}{x} = \frac{2r}{\theta^2} \\ \frac{\partial w}{\partial \theta} &= \frac{-yz}{x^2}(2\theta) + \frac{z}{x}(1) + \frac{y}{x}(-1) \\ &= \frac{-(r+\theta)(r-\theta)}{\theta^4}(2\theta) + \frac{(r-\theta) - (r+\theta)}{\theta^2} \\ &= \frac{2(\theta^2 - r^2)}{\theta^3} - \frac{2}{\theta} = \frac{-2r^2}{\theta^3} \end{aligned}$$

$$(b) \quad w = \frac{yz}{x} = \frac{(r+\theta)(r-\theta)}{\theta^2} = \frac{r^2}{\theta^2} - 1$$

$$\frac{\partial w}{\partial r} = \frac{2r}{\theta^2}$$

$$\frac{\partial w}{\partial \theta} = \frac{-2r^2}{\theta^3}$$

$$26. w = x^2 + y^2 + z^2, x = t \sin s, y = t \cos s, z = st^2$$

$$\begin{aligned} \frac{\partial w}{\partial s} &= 2x + \cos s + 2y(-t \sin s) + 2z(t^2) \\ &= 2t^2 \sin s \cos s - 2t^2 \sin s \cos s + 2st^4 = 2st^4 \end{aligned}$$

$$\begin{aligned} \frac{\partial w}{\partial t} &= 2x \sin s + 2y \cos s + 2z(2st) \\ &= 2t \sin^2 s + 2t \cos^2 s + 4s^2 t^3 = 2t + 4s^2 t^3 \end{aligned}$$

$$30. \frac{x}{x^2 + y^2} - y^2 - 6 = 0$$

$$\begin{aligned} \frac{dy}{dx} &= -\frac{F_x(x, y)}{F_y(x, y)} \\ &= -\frac{(y^2 - x^2)/(x^2 + y^2)^2}{(-2xy)/(x^2 + y^2)^2 - 2y} \\ &= \frac{y^2 - x^2}{2xy + 2y(x^2 + y^2)^2} \\ &= \frac{y^2 - x^2}{2xy + 2yx^4 + 4x^2y^3 + 2y^5} \end{aligned}$$

$$34. F(x, y, z) = e^x \sin(y + z) - z$$

$$F_x = e^x \sin(y + z)$$

$$F_y = e^x \cos(y + z)$$

$$F_z = e^x \cos(y + z) - 1$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{e^x \sin(y + z)}{1 - e^x \cos(y + z)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{e^x \cos(y + z)}{1 - e^x \cos(y + z)}$$

$$24. w = x \cos yz, x = s^2, y = t^2, z = s - 2t$$

$$\begin{aligned} \frac{\partial w}{\partial s} &= \cos(yz)(2s) - xz \sin(yz)(0) - xy \sin(yz)(1) \\ &= \cos(st^2 - 2t^3)2s - s^2 t^2 \sin(st^2 - 2t^3) \end{aligned}$$

$$\begin{aligned} \frac{\partial w}{\partial t} &= \cos(yz)(0) - xz \sin(yz)(2t) - xy \sin(yz)(-2) \\ &= -2s^2 t(s - 2t) \sin(st^2 - 2t^3) + 2s^2 t^2 \sin(st^2 - 2t^3) \\ &= (6s^2 t^2 - 2s^3 t) \sin(st^2 - 2t^3) \end{aligned}$$

$$28. \cos x + \tan xy + 5 = 0$$

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{-\sin x + y \sec^2 xy}{x \sec^2 xy}$$

$$32. F(x, y, z) = xz + yz + xy$$

$$F_x = z + y$$

$$F_y = z + x$$

$$F_z = x + y$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y + z}{x + y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x + z}{x + y}$$

$$36. x + \sin(y + z) = 0$$

$$(i) \quad 1 + \frac{\partial z}{\partial x} \cos(y + z) = 0 \text{ implies}$$

$$\frac{\partial z}{\partial x} = -\frac{1}{\cos(y + z)} = -\sec(y + z).$$

$$(ii) \quad \left(1 + \frac{\partial z}{\partial y}\right) \cos(y + z) = 0 \text{ implies } \frac{\partial z}{\partial y} = -1.$$

38. $x \ln y + y^2 z + z^2 - 8 = 0$

(i) $\frac{\partial z}{\partial x} = \frac{-F_x(x, y, z)}{F_z(x, y, z)} = \frac{-\ln y}{y^2 + 2z}$

(ii) $\frac{\partial z}{\partial y} = \frac{-F_y(x, y, z)}{F_z(x, y, z)} = -\frac{\frac{x}{y} + 2yz}{y^2 + 2z} = -\frac{x + 2y^2 z}{y^3 + 2yz}$

42. $F(x, y, z, w) = w - \sqrt{x - y} - \sqrt{y - z} = 0$

$$\frac{\partial w}{\partial x} = \frac{-F_x}{F_w} = \frac{1}{2} \frac{(x - y)^{-1/2}}{1} = \frac{1}{2\sqrt{x - y}}$$

$$\frac{\partial w}{\partial y} = \frac{-F_y}{F_w} = \frac{-1}{2} (x - y)^{-1/2} + \frac{1}{2} (y - z)^{-1/2}$$

$$= \frac{-1}{2\sqrt{x - y}} + \frac{1}{2\sqrt{y - z}}$$

$$\frac{\partial w}{\partial z} = \frac{-F_z}{F_w} = \frac{-1}{2\sqrt{y - z}}$$

46. $f(x, y) = \frac{x^2}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{(tx)^2}{\sqrt{(tx)^2 + (ty)^2}} = t \left(\frac{x^2}{\sqrt{x^2 + y^2}} \right) = tf(x, y)$$

Degree: 1

$$\begin{aligned} xf_x(x, y) + yf_y(x, y) &= x \left[\frac{x^3 + 2xy^2}{(x^2 + y^2)^{3/2}} \right] + y \left[\frac{-x^2 y}{(x^2 + y^2)^{3/2}} \right] \\ &= \frac{x^4 + x^2 y^2}{(x^2 + y^2)^{3/2}} = \frac{x^2(x^2 + y^2)}{(x^2 + y^2)^{3/2}} \\ &= \frac{x^2}{\sqrt{x^2 + y^2}} = f(x, y) \end{aligned}$$

48. $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} \quad (\text{Page 878})$$

40. $x^2 + y^2 - z^2 - 5yw + 10w^2 - 2 = F(x, y, z, w)$

$$F_x = 2x, F_y = 2y - 5w, F_z = -2z, F_w = -5y + 20w$$

$$\frac{\partial w}{\partial x} = -\frac{F_x}{F_w} = \frac{-2x}{-5y + 20w} = \frac{2x}{5y - 20w}$$

$$\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = \frac{5w - 2y}{20w - 5y}$$

$$\frac{\partial w}{\partial z} = -\frac{F_z}{F_w} = \frac{2z}{5y - 20w}$$

44. $f(x, y) = x^3 - 3xy^2 + y^3$

$$\begin{aligned} f(tx, ty) &= (tx)^3 - 3(tx)(ty)^2 + (ty)^3 \\ &= t^3(x^3 - 3xy^2 + y^3) = t^3 f(x, y) \end{aligned}$$

Degree: 3

$$\begin{aligned} xf_x(x, y) + yf_y(x, y) &= x(3x^2 - 3y^2) + y(-6xy + 3y^2) \\ &= 3x^3 - 9xy^2 + 3y^3 = 3f(x, y) \end{aligned}$$

52. (a) $V = \pi r^2 h$

$$\frac{dV}{dt} = \pi \left(2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right) = \pi r \left(2h \frac{dr}{dt} + r \frac{dh}{dt} \right) = \pi(12)[2(36)(6) + 12(-4)] = 4608\pi \text{ in.}^3/\text{min}$$

(b) $S = 2\pi r(r + h)$

$$\frac{dS}{dt} = 2\pi \left[(2r + h) \frac{dr}{dt} + r \frac{dh}{dt} \right] = 2\pi[(24 + 36)(6) + 12(-4)] = 624\pi \text{ in.}^2/\text{min}$$

50. $\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}$

$$\frac{\partial z}{\partial x} = -\frac{f_x(x, y, z)}{f_z(x, y, z)}$$

$$\frac{\partial z}{\partial y} = -\frac{f_y(x, y, z)}{f_z(x, y, z)}$$

54. (a) $V = \frac{\pi}{3}(r^2 + rR + R^2)h$

$$\begin{aligned}\frac{dV}{dt} &= \frac{\pi}{3} \left[(2r + R)h \frac{dr}{dt} + (r + 2R)h \frac{dR}{dt} + (r^2 + rR + R^2) \frac{dh}{dt} \right] \\ &= \frac{\pi}{3} \left[[2(15) + 25](10)(4) + [15 + 2(25)](10)(4) + [(15)^2 + (15)(25) + (25)^2](12) \right] \\ &= \frac{\pi}{3}(19,500) = 6,500\pi \text{ cm}^3/\text{min}\end{aligned}$$

(b) $S = \pi(R + r)\sqrt{(R - r)^2 + h^2}$

$$\begin{aligned}\frac{dS}{dt} &= \pi \left\{ \left[\sqrt{(R - r)^2 + h^2} - (R + r) \frac{(R - r)}{\sqrt{(R - r)^2 + h^2}} \right] \frac{dr}{dt} + \left[\sqrt{(R - r)^2 + h^2} + (R + r) \frac{(R - r)}{\sqrt{(R - r)^2 + h^2}} \right] \frac{dR}{dt} + \right. \\ &\quad \left. (R + r) \frac{h}{\sqrt{(R - r)^2 + h^2}} \frac{dh}{dt} \right\} \\ &= \pi \left\{ \left[\sqrt{(25 - 15)^2 + 10^2} - (25 + 15) \frac{25 - 15}{\sqrt{(25 - 15)^2 + 10^2}} \right] (4) + \right. \\ &\quad \left[\sqrt{(25 - 15)^2 + 10^2} + (25 + 15) \frac{25 - 15}{\sqrt{(25 - 15)^2 + 10^2}} \right] (4) + (25 + 15) \frac{10}{\sqrt{(25 - 15)^2 + 10^2}} (12) \right\} \\ &= 320\sqrt{2}\pi \text{ cm}^2/\text{min}\end{aligned}$$

56. $pV = mRT$

$$T = \frac{1}{mR}(pV)$$

$$\frac{dT}{dt} = \frac{1}{mR} \left[V \frac{dp}{dt} + p \frac{dV}{dt} \right]$$

58. $g(t) = f(xt, yt) = t^n f(x, y)$

Let $u = xt$, $v = yt$, then

$$g'(t) = \frac{\partial f}{\partial u} \cdot \frac{du}{dt} + \frac{\partial f}{\partial v} \cdot \frac{dv}{dt} = \frac{\partial f}{\partial u} x + \frac{\partial f}{\partial v} y$$

and $g'(t) = nt^{n-1}f(x, y)$.

Now, let $t = 1$ and we have $u = x$, $v = y$. Thus,

$$\frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} y = nf(x, y).$$

60. $w = (x - y) \sin(y - x)$

$$\frac{\partial w}{\partial x} = -(x - y) \cos(y - x) + \sin(y - x)$$

$$\frac{\partial w}{\partial y} = (x - y) \cos(y - x) - \sin(y - x)$$

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 0$$

62. $w = \arctan \frac{y}{x}$, $x = r \cos \theta$, $y = r \sin \theta$

$$= \arctan \left(\frac{r \sin \theta}{r \cos \theta} \right) = \arctan(\tan \theta) = \theta \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\frac{\partial w}{\partial x} = \frac{-y}{x^2 + y^2}, \quad \frac{\partial w}{\partial y} = \frac{x}{x^2 + y^2}, \quad \frac{\partial w}{\partial r} = 0, \quad \frac{\partial w}{\partial \theta} = 1$$

$$\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 = \frac{y^2}{(x^2 + y^2)^2} + \frac{x^2}{(x^2 + y^2)^2} = \frac{1}{x^2 + y^2} = \frac{1}{r^2}$$

$$\left(\frac{\partial w}{\partial r} \right)^2 + \left(\frac{1}{r^2} \right) \left(\frac{\partial w}{\partial \theta} \right)^2 = 0 + \frac{1}{r^2} (1) = \frac{1}{r^2}$$

Therefore, $\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 = \left(\frac{\partial w}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial w}{\partial \theta} \right)^2$.

64. Note first that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{y}{x^2 + y^2}.$$

$$\frac{\partial u}{\partial r} = \frac{x}{x^2 + y^2} \cos \theta + \frac{y}{x^2 + y^2} \sin \theta = \frac{r \cos^2 \theta + r \sin^2 \theta}{r^2} = \frac{1}{r}$$

$$\frac{\partial v}{\partial \theta} = \frac{-y}{x^2 + y^2}(-r \sin \theta) + \frac{x}{x^2 + y^2}(r \cos \theta) = \frac{r^2 \sin^2 \theta + r^2 \cos^2 \theta}{r^2} = 1$$

$$\text{Thus, } \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}.$$

$$\frac{\partial v}{\partial r} = \frac{-y}{x^2 + y^2} \cos \theta + \frac{x}{x^2 + y^2} \sin \theta = \frac{-r \sin \theta \cos \theta + r \sin \theta \cos \theta}{r^2} = 0$$

$$\frac{\partial u}{\partial \theta} = \frac{x}{x^2 + y^2}(-r \sin \theta) + \frac{y}{x^2 + y^2}(r \cos \theta) = \frac{-r^2 \sin \theta \cos \theta + r^2 \sin \theta \cos \theta}{r^2} = 0$$

$$\text{Thus, } \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

Section 12.6 Directional Derivatives and Gradients

$$2. \quad f(x, y) = x^3 - y^3, \mathbf{v} = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$$

$$\nabla f(x, y) = 3x^2\mathbf{i} - 3y^2\mathbf{j}$$

$$\nabla f(4, 3) = 48\mathbf{i} - 27\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$D_{\mathbf{u}}f(4, 3) = \nabla f(4, 3) \cdot \mathbf{u} = 24\sqrt{2} - \frac{27}{2}\sqrt{2} = \frac{21}{2}\sqrt{2}$$

$$4. \quad f(x, y) = \frac{x}{y}$$

$$\mathbf{v} = -\mathbf{j}$$

$$\nabla f(x, y) = \frac{1}{y}\mathbf{i} - \frac{x}{y^2}\mathbf{j}$$

$$\nabla f(1, 1) = \mathbf{i} - \mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\mathbf{j}$$

$$D_{\mathbf{u}}f(1, 1) = \nabla f(1, 1) \cdot \mathbf{u} = 1$$

$$6. \quad g(x, y) = \arccos xy, \mathbf{v} = \mathbf{i} + 5\mathbf{j}$$

$$\nabla g(x, y) = \frac{-y}{\sqrt{1 - (xy)^2}}\mathbf{i} + \frac{-x}{\sqrt{1 - (xy)^2}}\mathbf{j}$$

$$\nabla g(1, 0) = -\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{26}}\mathbf{i} + \frac{5}{\sqrt{26}}\mathbf{j}$$

$$D_{\mathbf{u}}g(1, 0) = \nabla g(1, 0) \cdot \mathbf{u} = \frac{-5}{\sqrt{26}} = \frac{-5\sqrt{26}}{26}$$

$$8. \quad h(x, y) = e^{-(x^2 + y^2)}$$

$$\mathbf{v} = \mathbf{i} + \mathbf{j}$$

$$\nabla h = -2xe^{-(x^2 + y^2)}\mathbf{i} - 2ye^{-(x^2 + y^2)}\mathbf{j}$$

$$\nabla h(0, 0) = \mathbf{0}$$

$$D_{\mathbf{u}}h(0, 0) = \nabla h(0, 0) \cdot \mathbf{u} = 0$$

$$10. \quad f(x, y, z) = x^2 + y^2 + z^2$$

$$\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$$

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla f(1, 2, -1) = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{14}}\mathbf{i} - \frac{2}{\sqrt{14}}\mathbf{j} + \frac{3}{\sqrt{14}}\mathbf{k}$$

$$D_{\mathbf{u}}f(1, 2, -1) = \nabla f(1, 2, -1) \cdot \mathbf{u} = -\frac{6}{7}\sqrt{14}$$

$$12. \quad h(x, y, z) = xyz$$

$$\mathbf{v} = \langle 2, 1, 2 \rangle$$

$$\nabla h = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

$$\nabla h(2, 1, 1) = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

$$D_{\mathbf{u}}h(2, 1, 1) = \nabla h(2, 1, 1) \cdot \mathbf{u} = \frac{8}{3}$$

$$14. f(x, y) = \frac{y}{x + y}$$

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$$

$$\nabla f = -\frac{y}{(x + y)^2}\mathbf{i} + \frac{x}{(x + y)^2}\mathbf{j}$$

$$\begin{aligned} D_{\mathbf{u}}f &= \nabla f \cdot \mathbf{u} = -\frac{\sqrt{3}y}{2(x + y)^2} - \frac{x}{2(x + y)^2} \\ &= -\frac{1}{2(x + y)^2}(\sqrt{3}y + x) \end{aligned}$$

$$18. f(x, y) = \cos(x + y)$$

$$\mathbf{v} = \frac{\pi}{2}\mathbf{i} - \pi\mathbf{j}$$

$$\nabla f = -\sin(x + y)\mathbf{i} - \sin(x + y)\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$$

$$\begin{aligned} D_{\mathbf{u}}f &= -\frac{1}{\sqrt{5}}\sin(x + y) + \frac{2}{\sqrt{5}}\sin(x + y) \\ &= \frac{1}{\sqrt{5}}\sin(x + y) = \frac{\sqrt{5}}{5}\sin(x + y) \end{aligned}$$

$$\text{At } (0, \pi), D_{\mathbf{u}}f = 0.$$

$$22. g(x, y) = 2xe^{y/x}$$

$$\nabla g(x, y) = \left(-\frac{2y}{x}e^{y/x} + 2e^{y/x}\right)\mathbf{i} + 2e^{y/x}\mathbf{j}$$

$$\nabla g(2, 0) = 2\mathbf{i} + 2\mathbf{j}$$

$$26. w = x \tan(y + z)$$

$$\nabla w(x, y, z) = \tan(y + z)\mathbf{i} + x \sec^2(y + z)\mathbf{j} + x \sec^2(y + z)\mathbf{k}$$

$$\nabla w(4, 3, -1) = \tan 2\mathbf{i} + 4 \sec^2 2\mathbf{j} + 4 \sec^2 2\mathbf{k}$$

$$28. \overrightarrow{PQ} = -2\mathbf{i} + 7\mathbf{j}, \mathbf{u} = -\frac{2}{\sqrt{53}}\mathbf{i} + \frac{7}{\sqrt{53}}\mathbf{j}$$

$$\nabla f(x, y) = 6x\mathbf{i} - 2y\mathbf{j}, \nabla f(3, 1) = 18\mathbf{i} - 2\mathbf{j}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = -\frac{36}{\sqrt{53}} - \frac{14}{\sqrt{53}} = -\frac{50}{\sqrt{53}} = -\frac{50\sqrt{53}}{53}$$

$$30. \overrightarrow{PQ} = \frac{\pi}{2}\mathbf{i} + \pi\mathbf{j}, \mathbf{u} = \frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$$

$$\nabla f(x, y) = 2 \cos 2x \cos y\mathbf{i} - \sin 2x \sin y\mathbf{j}$$

$$\nabla f(0, 0) = 2\mathbf{i}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$16. g(x, y) = xe^y$$

$$\mathbf{u} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

$$\nabla g = e^y\mathbf{i} + xe^y\mathbf{j}$$

$$D_{\mathbf{u}}g = -\frac{1}{2}e^y + \frac{\sqrt{3}}{2}xe^y = \frac{e^y}{2}(\sqrt{3}x - 1)$$

$$20. g(x, y, z) = xye^z$$

$$\mathbf{v} = -2\mathbf{i} - 4\mathbf{j}$$

$$\nabla g = ye^z\mathbf{i} + xe^z\mathbf{j} + xye^z\mathbf{k}$$

$$\text{At } (2, 4, 0), \nabla g = 4\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}.$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$$

$$D_{\mathbf{u}}g = \nabla g \cdot \mathbf{u} = -\frac{4}{\sqrt{5}} - \frac{4}{\sqrt{5}} = -\frac{8}{\sqrt{5}}$$

$$24. z = \ln(x^2 - y)$$

$$\nabla z(x, y) = \frac{2x}{x^2 - y}\mathbf{i} - \frac{1}{x^2 - y}\mathbf{j}$$

$$\nabla z(2, 3) = 4\mathbf{i} - \mathbf{j}$$

32. $h(x, y) = y \cos(x - y)$

$$\nabla h(x, y) = -y \sin(x - y)\mathbf{i} + [\cos(x - y) + y \sin(x - y)]\mathbf{j}$$

$$\nabla h\left(0, \frac{\pi}{3}\right) = \frac{\sqrt{3}\pi}{6}\mathbf{i} + \left(\frac{3 - \sqrt{3}\pi}{6}\right)\mathbf{j}$$

$$\left\|\nabla h\left(0, \frac{\pi}{3}\right)\right\| = \sqrt{\frac{3\pi^2}{36} + \frac{9 - 6\sqrt{3}\pi + 3\pi^2}{36}} = \frac{\sqrt{3(2\pi^2 - 2\sqrt{3}\pi + 3)}}{6}$$

34. $g(x, y) = ye^{-x^2}$

$$\nabla g(x, y) = -2xye^{-x^2}\mathbf{i} + e^{-x^2}\mathbf{j}$$

$$\nabla g(0, 5) = \mathbf{j}$$

$$\|\nabla g(0, 5)\| = 1$$

36. $w = \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

$$\nabla w = \frac{1}{(\sqrt{1 - x^2 - y^2 - z^2})^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\nabla w(0, 0, 0) = \mathbf{0}$$

$$\|\nabla w(0, 0, 0)\| = 0$$

38. $w = xy^2z^2$

$$\nabla w = y^2z^2\mathbf{i} + 2xyz^2\mathbf{j} + 2xy^2z\mathbf{k}$$

$$\nabla w(2, 1, 1) = \mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$

$$\|\nabla w(2, 1, 1)\| = \sqrt{33}$$

For Exercises 40–46, $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$ and $D_\theta f(x, y) = -\left(\frac{1}{3}\right)\cos\theta - \left(\frac{1}{2}\right)\sin\theta$.

40. (a) $D_{\pi/4}f(3, 2) = -\left(\frac{1}{3}\right)\frac{\sqrt{2}}{2} - \left(\frac{1}{2}\right)\frac{\sqrt{2}}{2} = -\frac{5\sqrt{2}}{12}$

(b) $D_{2\pi/3}f(3, 2) = -\left(\frac{1}{3}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\frac{\sqrt{3}}{2} = \frac{2 - 3\sqrt{3}}{12}$

42. (a) $\mathbf{u} = \left(\frac{1}{\sqrt{2}}\right)(\mathbf{i} + \mathbf{j})$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$$

$$= -\left(\frac{1}{3}\right)\frac{1}{\sqrt{2}} - \left(\frac{1}{2}\right)\frac{1}{\sqrt{2}} = -\frac{5\sqrt{2}}{12}$$

(b) $\mathbf{v} = -3\mathbf{i} - 4\mathbf{j}$

$$\|\mathbf{v}\| = \sqrt{9 + 16} = 5$$

$$\mathbf{u} = -\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$$

44. $\nabla f = -\left(\frac{1}{3}\right)\mathbf{i} - \left(\frac{1}{2}\right)\mathbf{j}$

46. $\nabla f = -\frac{1}{3}\mathbf{i} - \frac{1}{2}\mathbf{j}$

$$\frac{\nabla f}{\|\nabla f\|} = \frac{1}{\sqrt{13}}(-2\mathbf{i} - 3\mathbf{j})$$

Therefore, $\mathbf{u} = (1/\sqrt{13})(3\mathbf{i} - 2\mathbf{j})$ and $D_{\mathbf{u}}f(3, 2) = \nabla f \cdot \mathbf{u} = 0$. ∇f is the direction of greatest rate of change of f . Hence, in a direction orthogonal to ∇f , the rate of change of f is 0.

For Exercises 48 and 50, $f(x, y) = 9 - x^2 - y^2$ and $D_\theta f(x, y) = -2x \cos \theta - 2y \sin \theta = -2(x \cos \theta + y \sin \theta)$.

$$48. (a) D_{-\pi/4} f(1, 2) = -2 \left(\frac{\sqrt{2}}{2} - \sqrt{2} \right) = \sqrt{2}$$

$$(b) D_{\pi/3} f(1, 2) = -2 \left(\frac{1}{2} + \sqrt{3} \right) = -(1 + 2\sqrt{3})$$

$$50. \nabla f(1, 2) = -2\mathbf{i} - 4\mathbf{j}$$

$$\frac{\nabla f(1, 2)}{\|\nabla f(1, 2)\|} = \frac{1}{\sqrt{5}}(-\mathbf{i} - 2\mathbf{j})$$

Therefore,

$$\mathbf{u} = (1/\sqrt{5})(-\mathbf{i} - 2\mathbf{j}) \text{ and}$$

$$D_{\mathbf{u}} f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = 0.$$

52. (a) In the direction of the vector $\mathbf{i} + \mathbf{j}$.

$$(b) \nabla f = \frac{1}{2}y \frac{1}{2\sqrt{x}}\mathbf{i} + \frac{1}{2}\sqrt{x}\mathbf{j} = \frac{y}{4\sqrt{x}}\mathbf{i} + \frac{1}{2}\sqrt{x}\mathbf{j}$$

$$\nabla f(1, 2) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

(Same direction as in part (a).)

(c) $-\nabla f = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$, the direction opposite that of the gradient.

$$54. (a) f(x, y) = \frac{8y}{1 + x^2 + y^2} = 2$$

$$\Rightarrow 4y = 1 + x^2 + y^2$$

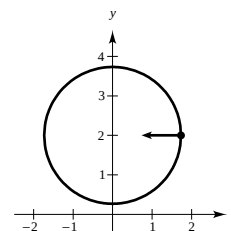
$$4 = y^2 - 4y + 4 + x^2 + 1$$

$$(y - 2)^2 + x^2 = 3$$

Circle: center: $(0, 2)$, radius: $\sqrt{3}$

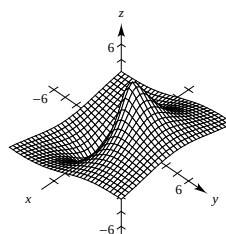
$$(b) \nabla f = \frac{-16xy}{(1 + x^2 + y^2)^2}\mathbf{i} + \frac{8 + 8x^2 - 8y^2}{(1 + x^2 + y^2)^2}\mathbf{j}$$

$$\nabla f(\sqrt{3}, 2) = \frac{-\sqrt{3}}{2}\mathbf{i}$$



(c) The directional derivative of f is 0 in the directions $\pm \mathbf{j}$.

(d)



$$56. f(x, y) = 6 - 2x - 3y$$

$$c = 6, P = (0, 0)$$

$$\nabla f(x, y) = -2\mathbf{i} - 3\mathbf{j}$$

$$6 - 2x - 3y = 6$$

$$0 = 2x + 3y$$

$$\nabla f(0, 0) = -2\mathbf{i} - 3\mathbf{j}$$

$$58. f(x, y) = xy$$

$$c = -3, P = (-1, 3)$$

$$\nabla f(x, y) = y\mathbf{i} + x\mathbf{j}$$

$$xy = -3$$

$$\nabla f(-1, 3) = 3\mathbf{i} - \mathbf{j}$$

60. $3x^2 - 2y^2 = 1$

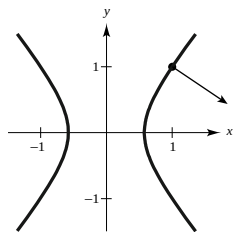
$$f(x, y) = 3x^2 - 2y^2$$

$$\nabla f(x, y) = 6x\mathbf{i} - 4y\mathbf{j}$$

$$\nabla f(1, 1) = 6\mathbf{i} - 4\mathbf{j}$$

$$\frac{\nabla f(1, 1)}{\|\nabla f(1, 1)\|} = \frac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j})$$

$$= \frac{\sqrt{13}}{13}(3\mathbf{i} - 2\mathbf{j})$$



62. $xe^y - y = 5$

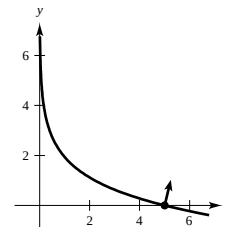
$$f(x, y) = xe^y - y$$

$$\nabla f(x, y) = e^y\mathbf{i} + (xe^y - 1)\mathbf{j}$$

$$\nabla f(5, 0) = \mathbf{i} + 4\mathbf{j}$$

$$\frac{\nabla f(5, 0)}{\|\nabla f(5, 0)\|} = \frac{1}{\sqrt{17}}(\mathbf{i} + 4\mathbf{j})$$

$$= \frac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{j})$$



64. $h(x, y) = 5000 - 0.001x^2 - 0.004y^2$

$$\nabla h = -0.002x\mathbf{i} - 0.008y\mathbf{j}$$

$$\nabla h(500, 300) = -\mathbf{i} - 2.4\mathbf{j} \text{ or}$$

$$5\nabla h = -(5\mathbf{i} + 12\mathbf{j})$$

66. The directional derivative gives the slope of a surface at a point in an arbitrary direction $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$.

68. See the definition, page 887.

70. The gradient vector is normal to the level curves.

72. The wind speed is greatest at A.

See Theorem 12.12.

74. $T(x, y) = 100 - x^2 - 2y^2$,

$$P = (4, 3)$$

$$\frac{dx}{dt} = -2x$$

$$\frac{dy}{dt} = -4y$$

$$x(t) = C_1 e^{-2t}$$

$$y(t) = C_2 e^{-4t}$$

$$4 = x(0) = C_1$$

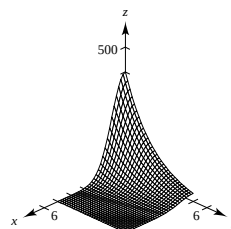
$$3 = y(0) = C_2$$

$$x(t) = 4e^{-2t}$$

$$y(t) = 3e^{-4t}$$

$$\frac{3x^2}{16} = e^{-4t} = y \Rightarrow u = \frac{3}{16}x^2$$

76. (a)



(b) $\nabla T(x, y) = 400e^{-(x^2+y)/2} \left[(-x)\mathbf{i} - \frac{1}{2}\mathbf{j} \right]$

$$\nabla T(3, 5) = 400e^{-7} \left[-3\mathbf{i} - \frac{1}{2}\mathbf{j} \right]$$

There will be no change in directions perpendicular to the gradient: $\pm(\mathbf{i} - 6\mathbf{j})$

(c) The greatest increase is in the direction of the gradient: $-3\mathbf{i} - \frac{1}{2}\mathbf{j}$

78. False

$$D_{\mathbf{u}} f(x, y) = \sqrt{2} > 1 \text{ when}$$

$$\mathbf{u} = \left(\cos \frac{\pi}{4} \right) \mathbf{i} + \left(\sin \frac{\pi}{4} \right) \mathbf{j}.$$

80. True

Section 12.7 Tangent Planes and Normal Lines

2. $F(x, y, z) = x^2 + y^2 + z^2 - 25 = 0$

$$x^2 + y^2 + z^2 = 25$$

Sphere, radius 5, centered at origin.

4. $F(x, y, z) = 16x^2 - 9y^2 + 144z = 0$

$$16x^2 - 9y^2 + 144z = 0 \text{ Hyperbolic paraboloid}$$

6. $F(x, y, z) = x^2 + y^2 + z^2 - 11$

$$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla F(3, 1, 1) = 6\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\begin{aligned}\mathbf{n} &= \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{44}}(6\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \\ &= \frac{1}{\sqrt{11}}(3\mathbf{i} + \mathbf{j} + \mathbf{k}) = \frac{\sqrt{11}}{11}(3\mathbf{i} + \mathbf{j} + \mathbf{k})\end{aligned}$$

8. $F(x, y, z) = x^3 - z$

$$\nabla F(x, y, z) = 3x^2\mathbf{i} - \mathbf{k}$$

$$\nabla F(2, 1, 8) = 12\mathbf{i} - \mathbf{k}$$

$$\begin{aligned}\mathbf{n} &= \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{145}}(12\mathbf{i} - \mathbf{k}) \\ &= \frac{\sqrt{145}}{145}(12\mathbf{i} - \mathbf{k})\end{aligned}$$

10. $F(x, y, z) = x^2 + 3y + z^3 - 9$

$$\nabla F(x, y, z) = 2x\mathbf{i} + 3\mathbf{j} + 3z^2\mathbf{k}$$

$$\nabla F(2, -1, 2) = 4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{13}(4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k})$$

12. $F(x, y, z) = ze^{x^2-y^2} - 3$

$$\nabla F(x, y, z) = 2xze^{x^2-y^2}\mathbf{i} - 2yze^{x^2-y^2}\mathbf{j} + e^{x^2-y^2}\mathbf{k}$$

$$\nabla F(2, 2, 3) = 12\mathbf{i} - 12\mathbf{j} + \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{17}(12\mathbf{i} - 12\mathbf{j} + \mathbf{k})$$

14. $F(x, y, z) = \sin(x - y) - z - 2$

$$\nabla F(x, y, z) = \cos(x - y)\mathbf{i} - \cos(x - y)\mathbf{j} - \mathbf{k}$$

$$\nabla F\left(\frac{\pi}{3}, \frac{\pi}{6}, -\frac{3}{2}\right) = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j} - \mathbf{k}$$

$$\begin{aligned}\mathbf{n} &= \frac{\nabla F}{\|\nabla F\|} = \frac{2}{\sqrt{10}}\left(\frac{\sqrt{3}}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j} - \mathbf{k}\right) \\ &= \frac{1}{\sqrt{10}}(\sqrt{3}\mathbf{i} - \sqrt{3}\mathbf{j} - 2\mathbf{k}) \\ &= \frac{\sqrt{10}}{10}(\sqrt{3}\mathbf{i} - \sqrt{3}\mathbf{j} - 2\mathbf{k})\end{aligned}$$

16. $f(x, y) = \frac{y}{x}, (1, 2, 2)$

$$F(x, y, z) = \frac{y}{x} - z$$

$$F_x(x, y, z) = -\frac{y}{x^2} \quad F_y(x, y, z) = \frac{1}{x} \quad F_z(x, y, z) = -1$$

$$F_x(1, 2, 2) = -2 \quad F_y(1, 2, 2) = 1 \quad F_z(1, 2, 2) = -1$$

$$-2(x - 1) + (y - 2) - (z - 2) = 0$$

$$-2x + y - z + 2 = 0$$

$$2x - y + z = 2$$

18. $g(x, y) = \arctan \frac{y}{x}, (1, 0, 0)$

$$G(x, y, z) = \arctan \frac{y}{x} - z$$

$$G_x(x, y, z) = \frac{-(y/x^2)}{1 + (y^2/x^2)} = \frac{-y}{x^2 + y^2} \quad G_y(x, y, z) = \frac{1/x}{1 + (y^2/x^2)} = \frac{x}{x^2 + y^2} \quad G_z(x, y, z) = -1$$

$$G_x(1, 0, 0) = 0$$

$$G_y(1, 0, 0) = 1$$

$$G_z(1, 0, 0) = -1$$

$$y - z = 0$$

20. $f(x, y) = 2 - \frac{2}{3}x - y, (3, -1, 1)$

$$F(x, y, z) = 2 - \frac{2}{3}x - y - z$$

$$F_x(x, y, z) = -\frac{2}{3}, \quad F_y(x, y, z) = -1, \quad F_z(x, y, z) = -1$$

$$-\frac{2}{3}(x - 3) - (y + 1) - (z - 1) = 0$$

$$-\frac{2}{3}x - y - z + 2 = 0$$

$$2x + 3y + 3z = 6$$

22. $z = x^2 - 2xy + y^2, (1, 2, 1)$

$$F(x, y, z) = x^2 - 2xy + y^2 - z$$

$$F_x(x, y, z) = 2x - 2y \quad F_y(x, y, z) = -2x + 2y \quad F_z(x, y, z) = -1$$

$$F_x(1, 2, 1) = -2 \quad F_y(1, 2, 1) = 2 \quad F_z(1, 2, 1) = -1$$

$$-2(x - 1) + 2(y - 2) - (z - 1) = 0$$

$$-2x + 2y - z - 1 = 0$$

$$2x - 2y + z = -1$$

24. $h(x, y) = \cos y, \left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$

$$H(x, y, z) = \cos y - z$$

$$H_x(x, y, z) = 0 \quad H_y(x, y, z) = -\sin y \quad H_z(x, y, z) = -1$$

$$H_x\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = 0 \quad H_y\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} \quad H_z\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = -1$$

$$-\frac{\sqrt{2}}{2}\left(y - \frac{\pi}{4}\right) - \left(z - \frac{\sqrt{2}}{2}\right) = 0$$

$$-\frac{\sqrt{2}}{2}y - z + \frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2} = 0$$

$$4\sqrt{2}y + 8z = \sqrt{2}(\pi + 4)$$

26. $x^2 + 2z^2 = y^2, (1, 3, -2)$

$$F(x, y, z) = x^2 - y^2 + 2z^2$$

$$F_x(x, y, z) = 2x \quad F_y(x, y, z) = -2y \quad F_z(x, y, z) = 4z$$

$$F_x(1, 3, -2) = 2 \quad F_y(1, 3, -2) = -6 \quad F_z(1, 3, -2) = -8$$

$$2(x - 1) - 6(y - 3) - 8(z + 2) = 0$$

$$(x - 1) - 3(y - 3) - 4(z + 2) = 0$$

$$x - 3y - 4z = 0$$

28. $x = y(2z - 3), (4, 4, 2)$

$$F(x, y, z) = x - 2yz + 3y$$

$$F_x(x, y, z) = 1 \quad F_y(x, y, z) = -2z + 3 \quad F_z(x, y, z) = -2y$$

$$F_x(4, 4, 2) = 1 \quad F_y(4, 4, 2) = -1 \quad F_z(4, 4, 2) = -8$$

$$(x - 4) - 1(y - 4) - 8(z - 2) = 0$$

$$x - y - 8z = -16$$

$$-x + y + 8z = 16$$

30. $x^2 + y^2 + z^2 = 9$, $(1, 2, 2)$

$$F(x, y, z) = x^2 + y^2 + z^2 - 9$$

$$F_x(x, y, z) = 2x \quad F_y(x, y, z) = 2y \quad F_z(x, y, z) = 2z$$

$$F_x(1, 2, 2) = 2 \quad F_y(1, 2, 2) = 4 \quad F_z(1, 2, 2) = 4$$

Direction numbers: 1, 2, 2

$$\text{Plane: } (x - 1) + 2(y - 2) + 2(z - 2) = 0, \quad x + 2y + 2z = 9$$

$$\text{Line: } \frac{x - 1}{1} = \frac{y - 2}{2} = \frac{z - 2}{2}$$

32. $x^2 - y^2 + z^2 = 0$, $(5, 13, -12)$

$$F(x, y, z) = x^2 - y^2 + z^2$$

$$F_x(x, y, z) = 2x \quad F_y(x, y, z) = -2y \quad F_z(x, y, z) = 2z$$

$$F_x(5, 13, -12) = 10 \quad F_y(5, 13, -12) = -26 \quad F_z(5, 13, -12) = -24$$

Direction numbers: 5, -13, -12

Plane

$$5(x - 5) - 13(y - 13) - 12(z + 12) = 0$$

$$5x - 13y - 12z = 0$$

$$\text{Line: } \frac{x - 5}{5} = \frac{y - 13}{-13} = \frac{z + 12}{-12}$$

34. $xyz = 10$, $(1, 2, 5)$

$$F(x, y, z) = xyz - 10$$

$$F_x(x, y, z) = yz \quad F_y(x, y, z) = xz \quad F_z(x, y, z) = xy$$

$$F_x(1, 2, 5) = 10 \quad F_y(1, 2, 5) = 5 \quad F_z(1, 2, 5) = 2$$

Direction numbers: 10, 5, 2

$$\text{Plane: } 10(x - 1) + 5(y - 2) + 2(z - 5) = 0, \quad 10x + 5y + 2z = 30$$

$$\text{Line: } \frac{x - 1}{10} = \frac{y - 2}{5} = \frac{z - 5}{2}$$

36. See the definition on page 897.

38. For a sphere, the common object is the center of the sphere. For a right circular cylinder, the common object is the axis of the cylinder.

40. $F(x, y, z) = x^2 + y^2 - z$ $G(x, y, z) = 4 - y - z$

$$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \quad \nabla G(x, y, z) = -\mathbf{j} - \mathbf{k}$$

$$\nabla F(2, -1, 5) = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k} \quad \nabla G(2, -1, 5) = -\mathbf{j} - \mathbf{k}$$

$$(a) \nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & -1 \\ 0 & -1 & -1 \end{vmatrix} = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$$

$$\text{Direction numbers: } 1, 4, -4, \quad \frac{x - 2}{1} = \frac{y + 1}{4} = \frac{z - 5}{-4}$$

$$(b) \cos \theta = \frac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \|\nabla G\|} = \frac{3}{\sqrt{21}\sqrt{2}} = \frac{3}{\sqrt{42}} = \frac{\sqrt{42}}{14}; \text{ not orthogonal}$$

$$42. \quad F(x, y, z) = \sqrt{x^2 + y^2} - z \qquad G(x, y, z) = 5x - 2y + 3z = 22$$

$$\nabla F(x, y, z) = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j} - \mathbf{k} \qquad \nabla G(x, y, z) = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$$

$$\nabla F(3, 4, 5) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k} \qquad \nabla G(3, 4, 5) = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$$

$$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3/5 & 4/5 & -1 \\ 5 & -2 & 3 \end{vmatrix} = \frac{2}{5}\mathbf{i} - \frac{34}{5}\mathbf{j} - \frac{26}{5}\mathbf{k}$$

Direction numbers: 1, -17, -13

$$\frac{x-3}{1} = \frac{y-4}{-17} = \frac{z-5}{-13} \text{ Tangent line}$$

$$\cos \theta = \frac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \|\nabla G\|} = \frac{-(8/5)}{\sqrt{2}\sqrt{38}} = \frac{-8}{5\sqrt{76}} \text{ Not orthogonal}$$

$$44. \quad F(x, y, z) = x^2 + y^2 - z \qquad G(x, y, z) = x + y + 6z - 33$$

$$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \qquad \nabla G(x, y, z) = \mathbf{i} + \mathbf{j} + 6\mathbf{k}$$

$$\nabla F(1, 2, 5) = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k} \qquad \nabla G(1, 2, 5) = \mathbf{i} + \mathbf{j} + 6\mathbf{k}$$

$$(a) \quad \nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -1 \\ 1 & 1 & 6 \end{vmatrix} = 25\mathbf{i} - 13\mathbf{j} - 2\mathbf{k}$$

$$\text{Direction numbers: } 25, -13, -2, \frac{x-1}{25} = \frac{y-2}{-13} = \frac{z-5}{-2}$$

$$(b) \quad \cos \theta = \frac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \|\nabla G\|} = 0; \text{ orthogonal}$$

$$46. (a) \quad f(x, y) = \sqrt{16 - x^2 - y^2 + 2x - 4y}$$

$$g(x, y) = \frac{\sqrt{2}}{2} \sqrt{1 - 3x^2 + y^2 + 6x + 4y}$$

$$(b) \quad f(x, y) = g(x, y)$$

$$16 - x^2 - y^2 + 2x - 4y = \frac{1}{2}(1 - 3x^2 + y^2 + 6x + 4y)$$

$$32 - 2x^2 - 2y^2 + 4x - 8y = 1 - 3x^2 + y^2 + 6x + 4y$$

$$x^2 - 2x + 31 = 3y^2 + 12y$$

$$(x^2 - 2x + 1) + 42 = 3(y^2 + 4y + 4)$$

$$(x - 1)^2 + 42 = 3(y + 2)^2$$

To find points of intersection, let $x = 1$. Then

$$3(y + 2)^2 = 42$$

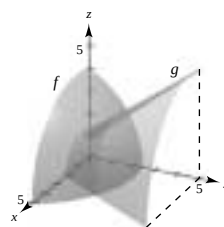
$$(y + 2)^2 = 14$$

$$y = -2 \pm \sqrt{14}$$

$\nabla f(1, -2 + \sqrt{14}) = -\sqrt{2}\mathbf{j}$, $\nabla g(1, -2 + \sqrt{14}) = (1/\sqrt{2})\mathbf{j}$. The normals to f and g at this point are $-\sqrt{2}\mathbf{j} - \mathbf{k}$ and $(1/\sqrt{2})\mathbf{j} - \mathbf{k}$, which are orthogonal.

Similarly, $\nabla f(1, -2 - \sqrt{14}) = \sqrt{2}\mathbf{j}$ and $\nabla g(1, -2 - \sqrt{14}) = (-1/\sqrt{2})\mathbf{j}$ and the normals are $\sqrt{2}\mathbf{j} - \mathbf{k}$ and $(-1/\sqrt{2})\mathbf{j} - \mathbf{k}$, which are also orthogonal.

(c) No, showing that the surfaces are orthogonal at 2 points does not imply that they are orthogonal at every point of intersection.



48. $F(x, y, z) = 2xy - z^3, (2, 2, 2)$

$$\nabla F = 2y\mathbf{i} + 2x\mathbf{j} - 3z^2\mathbf{k}$$

$$\nabla F(2, 2, 2) = 4\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$$

$$\cos \theta = \frac{|\nabla F(2, 2, 2) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 2)\|} = \frac{|-12|}{\sqrt{176}} = \frac{3\sqrt{11}}{11}$$

$$\theta = \arccos\left(\frac{3\sqrt{11}}{11}\right) \approx 25.24^\circ$$

52. $F(x, y, z) = 3x^2 + 2y^2 - 3x + 4y - z - 5$

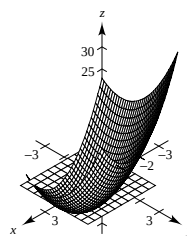
$$\nabla F(x, y, z) = (6x - 3)\mathbf{i} + (4y + 4)\mathbf{j} - \mathbf{k}$$

$$6x - 3 = 0, x = \frac{1}{2}$$

$$4y + 4 = 0, y = -1$$

$$z = 3\left(\frac{1}{2}\right)^2 + 2(-1)^2 - 3\left(\frac{1}{2}\right) + 4(-1) - 5 = -\frac{31}{4}$$

$$\left(\frac{1}{2}, -1, -\frac{31}{4}\right)$$



50. $F(x, y, z) = x^2 + y^2 - 5, (2, 1, 3)$

$$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j}$$

$$\nabla F(2, 1, 3) = 4\mathbf{i} + 2\mathbf{j}$$

$$\cos \theta = \frac{|\nabla F(2, 1, 3) \cdot \mathbf{k}|}{\|\nabla F(2, 1, 3)\|} = 0$$

$$\theta = \arccos 0 = 90^\circ$$

54. $T(x, y, z) = 100 - 3x - y - z^2, (2, 2, 5)$

$$\frac{dx}{dt} = -3$$

$$\frac{dy}{dt} = -1$$

$$\frac{dz}{dt} = -2z$$

$$x(t) = -3t + C_1$$

$$y(t) = -t + C_2$$

$$z(t) = C_3 e^{-2t}$$

$$x(0) = C_1 = 2$$

$$y(0) = C_2 = 2$$

$$z(0) = C_3 = 5$$

$$x = -3t + 2$$

$$y = -t + 2$$

$$z = 5e^{-2t}$$

56. $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1$

$$F_x(x, y, z) = \frac{2x}{a^2}$$

$$F_y(x, y, z) = \frac{2y}{b^2}$$

$$F_z(x, y, z) = \frac{-2z}{c^2}$$

$$\text{Plane: } \frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) - \frac{2z_0}{c^2}(z - z_0) = 0$$

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - \frac{z_0 z}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2} = 1$$

58. $z = xf\left(\frac{y}{x}\right)$

$$F(x, y, z) = xf\left(\frac{y}{x}\right) - z$$

$$F_x(x, y, z) = f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right)$$

$$F_y(x, y, z) = xf'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = f'\left(\frac{y}{x}\right)$$

$$F_z(x, y, z) = -1$$

Tangent plane at (x_0, y_0, z_0) :

$$\left[f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right](x - x_0) + f'\left(\frac{y_0}{x_0}\right)(y - y_0) - (z - z_0) = 0$$

$$\left[f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right]x - x_0 f\left(\frac{y_0}{x_0}\right) + y_0 f'\left(\frac{y_0}{x_0}\right) + y f'\left(\frac{y_0}{x_0}\right) - y_0 f'\left(\frac{y_0}{x_0}\right) - z + x_0 f\left(\frac{y_0}{x_0}\right) = 0$$

$$\left[f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right]x + f'\left(\frac{y_0}{x_0}\right)y - z = 0$$

Therefore, the plane passes through the origin $(x, y, z) = (0, 0, 0)$.

60. $f(x, y) = \cos(x + y)$

$f_x(x, y) = -\sin(x + y) \quad f_y(x, y) = -\sin(x + y)$

$f_{xx}(x, y) = -\cos(x + y), \quad f_{yy}(x, y) = -\cos(x + y), \quad f_{xy}(x, y) = -\cos(x + y)$

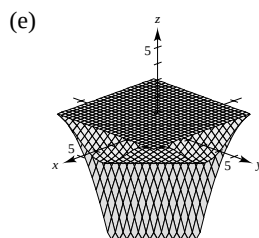
(a) $P_1(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = 1$

(b) $P_2(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2}f_{yy}(0, 0)y^2$
 $= 1 - \frac{1}{2}x^2 - xy - \frac{1}{2}y^2$

(c) If $x = 0$, $P_2(0, y) = 1 - \frac{1}{2}y^2$. This is the second-degree Taylor polynomial for $\cos y$.If $y = 0$, $P_2(x, 0) = 1 - \frac{1}{2}x^2$. This is the second-degree Taylor polynomial for $\cos x$.

(d)

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1	1	1
0	0.1	0.9950	1	0.9950
0.2	0.1	0.9553	1	0.9950
0.2	0.5	0.7648	1	0.7550
1	0.5	0.0707	1	-0.1250

62. Given $z = f(x, y)$, then:

$F(x, y, z) = f(x, y) - z = 0$

$\nabla F(x_0, y_0, z_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}$

$$\cos \theta = \frac{|\nabla F(x_0, y_0, z_0) \cdot \mathbf{k}|}{\|\nabla F(x_0, y_0, z_0)\| \|\mathbf{k}\|}$$

$$= \frac{|-1|}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + (-1)^2}}$$

$$= \frac{1}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + 1}}$$

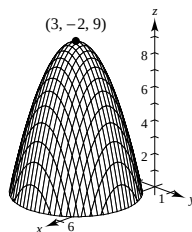
Section 12.8 Extrema of Functions of Two Variables

2. $g(x, y) = 9 - (x - 3)^2 - (y + 2)^2 \leq 9$

Relative maximum: $(3, -2, 9)$

$g_x = -2(x - 3) = 0 \Rightarrow x = 3$

$g_y = -2(y + 2) = 0 \Rightarrow y = -2$



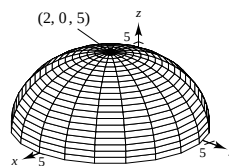
4. $f(x, y) = \sqrt{25 - (x - 2)^2 - y^2} \leq 5$

Relative maximum: $(2, 0, 5)$

Check: $f_x = -\frac{x - 2}{\sqrt{25 - (x - 2)^2 - y^2}} = 0 \Rightarrow x = 2$

$f_y = -\frac{y}{\sqrt{25 - (x - 2)^2 - y^2}} = 0 \Rightarrow y = 0$

$$f_{xx} = -\frac{25 - y^2}{[25 - (x - 2)^2 - y^2]^{3/2}}, \quad f_{yy} = -\frac{25 - (x - 2)^2}{[25 - (x - 2)^2 - y^2]^{3/2}}, \quad f_{xy} = -\frac{y(x - 2)}{[25 - (x - 2)^2 - y^2]^{3/2}}$$

At the critical point $(2, 0)$, $f_{xx} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $(2, 0, 5)$ is a relative maximum.

$$6. f(x, y) = -x^2 - y^2 + 4x + 8y - 11 = -(x - 2)^2 - (y - 4)^2 + 9 \leq 9$$

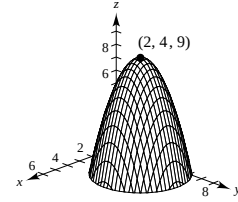
Relative maximum: $(2, 4, 9)$

Check: $f_x = -2x + 4 = 0 \Rightarrow x = 2$

$$f_y = -2y + 8 = 0 \Rightarrow y = 4$$

$$f_{xx} = -2, f_{yy} = -2, f_{xy} = 0$$

At the critical point $(2, 4)$, $f_{xx} < 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(2, 4, 9)$ is a relative maximum.



$$8. f(x, y) = -x^2 - 5y^2 + 10x - 30y - 62$$

$$\left. \begin{aligned} f_x = -2x + 10 &= 0 \\ f_y = -10y - 30 &= 0 \end{aligned} \right\} x = 5, y = -3$$

$$f_{xx} = -2, f_{yy} = -10, f_{xy} = 0$$

At the critical point $(5, -3)$, $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$.

Therefore, $(5, -3, 8)$ is a relative maximum.

$$10. f(x, y) = x^2 + 6xy + 10y^2 - 4y + 4$$

$$\left. \begin{aligned} f_x = 2x + 6y &= 0 \\ f_y = 6x + 20y - 4 &= 0 \end{aligned} \right\} \text{Solving simultaneously yields } x = -6 \text{ and } y = 2.$$

$$f_{xx} = 2, f_{yy} = 20, f_{xy} = 6$$

At the critical point $(-6, 2)$, $f_{xx} > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(-6, 2, 0)$ is a relative minimum.

$$12. f(x, y) = -3x^2 - 2y^2 + 3x - 4y + 5$$

$$f_x = -6x + 3 = 0 \text{ when } x = \frac{1}{2}.$$

$$f_y = -4y - 4 = 0 \text{ when } y = -1.$$

$$f_{xx} = -6, f_{yy} = -4, f_{xy} = 0$$

At the critical point $(\frac{1}{2}, -1)$, $f_{xx} < 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(\frac{1}{2}, -1, \frac{31}{4})$ is a relative maximum.

$$14. h(x, y) = (x^2 + y^2)^{1/3} + 2$$

$$\left. \begin{aligned} h_x = \frac{2x}{3(x^2 + y^2)^{2/3}} &= 0 \\ h_y = \frac{2y}{3(x^2 + y^2)^{2/3}} &= 0 \end{aligned} \right\} x = 0, y = 0$$

Since $h(x, y) \geq 2$ for all (x, y) , $(0, 0, 2)$ is a relative minimum.

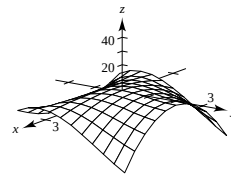
$$16. f(x, y) = |x + y| - 2$$

Since $f(x, y) \geq -2$ for all (x, y) , the relative minima of f consist of all points (x, y) satisfying $x + y = 0$.

$$18. f(x, y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$$

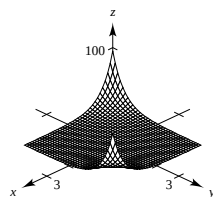
Relative maximum: $(0, 0, 1)$

Saddle points: $(0, 2, -3), (\pm\sqrt{3}, -1, -3)$



$$20. z = e^{xy}$$

Saddle point: $(0, 0, 1)$



22. $g(x, y) = 120x + 120y - xy - x^2 - y^2$

$$\left. \begin{aligned} g_x &= 120 - y - 2x = 0 \\ g_y &= 120 - x - 2y = 0 \end{aligned} \right\} \text{ Solving simultaneously yields } x = 40 \text{ and } y = 40.$$

$$g_{xx} = -2, \quad g_{yy} = -2, \quad g_{xy} = -1$$

At the critical point $(40, 40)$, $g_{xx} < 0$ and $g_{xx} g_{yy} - (g_{xy})^2 > 0$. Therefore, $(40, 40, 4800)$ is a relative maximum.

24. $g(x, y) = xy$

$$\left. \begin{aligned} g_x &= y \\ g_y &= x \end{aligned} \right\} \quad x = 0 \text{ and } y = 0$$

$$g_{xx} = 0, \quad g_{yy} = 0, \quad g_{xy} = 1$$

At the critical point $(0, 0)$, $g_{xx} g_{yy} - (g_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point.

26. $f(x, y) = 2xy - \frac{1}{2}(x^4 + y^2) + 1$

$$\left. \begin{aligned} f_x &= 2y - 2x^3 \\ f_y &= 2x - 2y^3 \end{aligned} \right\} \text{ Solving by substitution yields 3 critical points: } (0, 0), (1, 1), (-1, -1)$$

$$f_{xx} = -6x^2, \quad f_{yy} = -6y^2, \quad f_{xy} = 2$$

At $(0, 0)$, $f_{xx} f_{yy} - (f_{xy})^2 < 0 \Rightarrow (0, 0, 1)$ saddle point.

At $(1, 1)$, $f_{xx} f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} < 0 \Rightarrow (1, 1, 2)$ relative maximum.

At $(-1, -1)$, $f_{xx} f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} < 0 \Rightarrow (-1, -1, 2)$ relative maximum.

28. $f(x, y) = \left(\frac{1}{2} - x^2 + y^2\right)e^{1-x^2-y^2}$

$$\left. \begin{aligned} f_x &= (2x^3 - 2xy^2 - 3x)e^{1-x^2-y^2} = 0 \\ f_y &= (2x^2y - 2y^3 + y)e^{1-x^2-y^2} = 0 \end{aligned} \right\} \text{ Solving yields the critical points } (0, 0), \left(0, \pm \frac{\sqrt{2}}{2}\right), \left(\pm \frac{\sqrt{6}}{2}, 0\right).$$

$$f_{xx} = (-4x^4 + 4x^2y^2 + 12x^2 - 2y^2 - 3)e^{1-x^2-y^2}$$

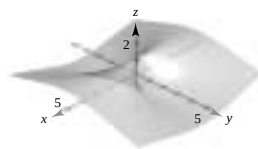
$$f_{yy} = (4y^4 - 4x^2y^2 + 2x^2 - 8y^2 + 1)e^{1-x^2-y^2}$$

$$f_{xy} = (-4x^3y + 4xy^3 + 2xy)e^{1-x^2-y^2}$$

At the critical point $(0, 0)$, $f_{xx} f_{yy} - (f_{xy})^2 < 0$. Therefore, $(0, 0, e/2)$ is a saddle point. At the critical points $(0, \pm \sqrt{2}/2)$, $f_{xx} < 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(0, \pm \sqrt{2}/2, \sqrt{e})$ are relative maxima. At the critical points $(\pm \sqrt{6}/2, 0)$, $f_{xx} > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(\pm \sqrt{6}/2, 0, -\sqrt{e}/e)$ are relative minima.

30. $z = \frac{(x^2 - y^2)^2}{x^2 + y^2} \geq 0$. $z = 0$ if $x^2 = y^2 \neq 0$.

Relative minima at all points (x, x) and $(x, -x)$, $x \neq 0$.

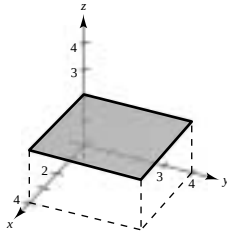


32. $f_{xx} < 0$ and $f_{xx} f_{yy} - (f_{xy})^2 = (-3)(-8) - 2^2 > 0$
 f has a relative maximum at (x_0, y_0) .

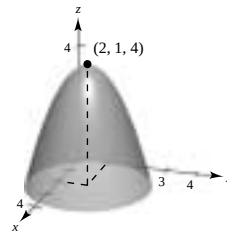
34. $f_{xx} > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 = (25)(8) - 10^2 > 0$
 f has a relative minimum at (x_0, y_0) .

36. See Theorem 12.17.

38.


 Extrema at all (x, y)

40.



Relative maximum

42. A and B are relative extrema. C and D are saddle points.

 44. $d = f_{xx}f_{yy} - f_{xy}^2 < 0$ if f_{xx} and f_{yy} have opposite signs. Hence, $(a, b, f(a, b))$ is a saddle point. For example, consider $f(x, y) = x^2 - y^2$ and $(a, b) = (0, 0)$.

 46. $f(x, y) = x^3 + y^3 - 6x^2 + 9y^2 + 12x + 27y + 19$

$$\left. \begin{aligned} f_x &= 3x^2 - 12x + 12 = 0 \\ f_y &= 3y^2 + 18y + 27 = 0 \end{aligned} \right\} \text{Solving yields } x = 2 \text{ and } y = -3.$$

$$f_{xx} = 6x - 12, f_{yy} = 6y + 18, f_{xy} = 0$$

 At $(2, -3)$, $f_{xx}f_{yy} - (f_{xy})^2 = 0$ and the test fails. $(1, -2, 0)$ is a saddle point.

 48. $f(x, y) = \sqrt{(x-1)^2 + (y+2)^2} \geq 0$

$$\left. \begin{aligned} f_x &= \frac{x-1}{\sqrt{(x-1)^2 + (y+2)^2}} = 0 \\ f_y &= \frac{y+2}{\sqrt{(x-1)^2 + (y+2)^2}} = 0 \end{aligned} \right\} \text{Solving yields } x = 1 \text{ and } y = -2.$$

$$f_{xx} = \frac{(y+2)^2}{[(x-1)^2 + (y+2)^2]^{3/2}}, f_{yy} = \frac{(x-1)^2}{[(x-1)^2 + (y+2)^2]^{3/2}}, f_{xy} = \frac{(x-1)(y+2)}{[(x-1)^2 + (y+2)^2]^{3/2}}$$

 At $(1, -2)$, $f_{xx}f_{yy} - (f_{xy})^2$ is undefined and the test fails.

 Absolute minimum: $(1, -2, 0)$

 50. $f(x, y) = (x^2 + y^2)^{2/3} \geq 0$

$$\left. \begin{aligned} f_x &= \frac{4x}{3(x^2 + y^2)^{1/3}} \\ f_y &= \frac{4y}{3(x^2 + y^2)^{1/3}} \end{aligned} \right\} f_x \text{ and } f_y \text{ are undefined at } x = 0, y = 0. \text{ The critical point is } (0, 0).$$

$$f_{xx} = \frac{4(x^2 + 3y^2)}{9(x^2 + y^2)^{4/3}}, f_{yy} = \frac{4(3x^2 + y^2)}{9(x^2 + y^2)^{4/3}}, f_{xy} = \frac{-8xy}{9(x^2 + y^2)^{4/3}}$$

 At $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2$ is undefined and the test fails.

 Absolute minimum: $(0, 0, 0)$

 52. $f(x, y, z) = 4 - [x(y-1)(z+2)]^2 \leq 4$

$$\left. \begin{aligned} f_x &= -2x(y-1)^2(z+2)^2 = 0 \\ f_y &= -2x^2(y-1)(z+2)^2 = 0 \\ f_z &= -2x(y-1)^2(z+2) = 0 \end{aligned} \right\} \text{Solving yields the critical points } (0, a, b), (c, 1, d), (e, f, -2). \text{ These points are all absolute maxima.}$$

54. $f(x, y) = (2x - y)^2$

$$f_x = 4(2x - y) = 0 \Rightarrow 2x = y$$

$$f_y = -2(2x - y) = 0 \Rightarrow 2x = y$$

On the line $y = x + 1$, $0 \leq x \leq 1$,

$$f(x, y) = f(x) = (2x - (x + 1))^2 = (x - 1)^2$$

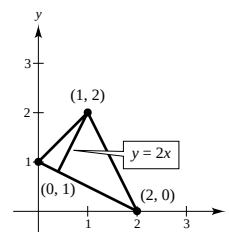
and the maximum is 1, the minimum is 0. On the line $y = -\frac{1}{2}x + 1$, $0 \leq x \leq 2$,

$$f(x, y) = f(x) = (2x - (-\frac{1}{2}x + 1))^2 = (\frac{5}{2}x - 1)^2$$

and the maximum is 16, the minimum is 0. On the line $y = -2x + 4$, $1 \leq x \leq 2$,

$$f(x, y) = f(x) = (2x - (-2x + 4))^2 = (4x - 4)^2$$

and the maximum is 16, the minimum is 0.

Absolute maximum: 16 at $(2, 0)$ Absolute minimum: 0 at $(1, 2)$ and along the line $y = 2x$.

56. $f(x, y) = 2x - 2xy + y^2$

$$f_x = 2 - 2y = 0 \Rightarrow y = 1$$

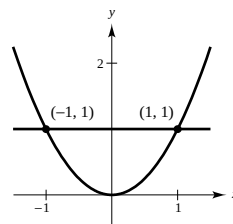
$$f_y = 2y - 2x = 0 \Rightarrow y = x \Rightarrow x = 1 \left. \vphantom{f_y = 2y - 2x = 0} \right\} f(1, 1) = 1$$

On the line $y = 1$, $-1 \leq x \leq 1$,

$$f(x, y) = f(x) = 2x - 2x + 1 = 1.$$

On the curve $y = x^2$, $-1 \leq x \leq 1$

$$f(x, y) = f(x) = 2x - 2x(x^2) + (x^2)^2 = x^4 - 2x^3 + 2x$$

and the maximum is 1, the minimum is $-\frac{11}{16}$.Absolute maximum: 1 at $(1, 1)$ and on $y = 1$ Absolute minimum: $-\frac{11}{16} = -0.6875$ at $(-\frac{1}{2}, \frac{1}{4})$ 

58. $f(x, y) = x^2 + 2xy + y^2$, $R = \{(x, y): |x| \leq 2, |y| \leq 1\}$

$$\left. \begin{aligned} f_x &= 2x + 2y = 0 \\ f_y &= 2x + 2y = 0 \end{aligned} \right\} y = -x$$

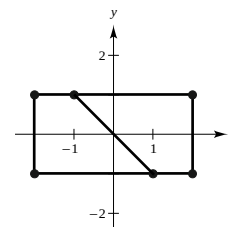
$$f(x, -x) = x^2 - 2x^2 + x^2 = 0$$

Along $y = 1$, $-2 \leq x \leq 2$,

$$f = x^2 + 2x + 1, f' = 2x + 2 = 0 \Rightarrow x = -1, f(-2, 1) = 1, f(-1, 1) = 0, f(2, 1) = 9.$$

Along $y = -1$, $-2 \leq x \leq 2$,

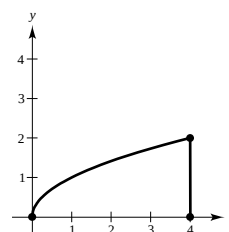
$$f = x^2 - 2x + 1, f' = 2x - 2 = 0 \Rightarrow x = 1, f(-2, -1) = 9, f(1, -1) = 0, f(2, -1) = 1.$$

Along $x = 2$, $-1 \leq y \leq 1$, $f = 4 + 4y + y^2$, $f' = 2y + 4 \neq 0$.Along $x = -2$, $-1 \leq y \leq 1$, $f = 4 - 4y + y^2$, $f' = 2y - 4 \neq 0$.Thus, the maxima are $f(-2, -1) = 9$ and $f(2, 1) = 9$, and the minima are $f(x, -x) = 0$, $-1 \leq x \leq 1$.

60. $f(x, y) = x^2 - 4xy + 5$, $R = \{(x, y): 0 \leq x \leq 4, 0 \leq y \leq \sqrt{x}\}$

$$\left. \begin{aligned} f_x &= 2x - 4y = 0 \\ f_y &= -4x = 0 \end{aligned} \right\} x = y = 0$$

$$f(0, 0) = 5$$

Along $y = 0$, $0 \leq x \leq 4$, $f = x^2 + 5$ and $f(4, 0) = 21$.Along $x = 4$, $0 \leq y \leq 2$, $f = 16 - 16y + 5$, $f' = -16 \neq 0$ and $f(4, 2) = -11$.Along $y = \sqrt{x}$, $0 \leq x \leq 4$, $f = x^2 - 4x^{3/2} + 5$, $f' = 2x - 6x^{1/2} \neq 0$ on $[0, 4]$.Thus, the maximum is $f(4, 0) = 21$ and the minimum is $f(4, 2) = -11$.

$$62. f(x, y) = \frac{4xy}{(x^2 + 1)(y^2 + 1)}, R = \{(x, y): x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$$

$$f_x = \frac{4(1 - x^2)y}{(y^2 + 1)(x^2 + 1)^2} = 0 \Rightarrow x = 1 \text{ or } y = 0$$

$$f_y = \frac{4(1 - y^2)x}{(x^2 + 1)(y^2 + 1)^2} = 0 \Rightarrow y = 1 \text{ or } x = 0$$

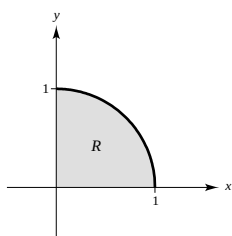
For $x = 0, y = 0$, also, and $f(0, 0) = 0$.

For $x = 1$ and $y = 1$, the point $(1, 1)$ is outside R .

For $x^2 + y^2 = 1$, $f(x, y) = f(x, \sqrt{1 - x^2}) = \frac{4x\sqrt{1 - x^2}}{2 + x^2 - x^4}$, and the maximum occurs at $x = \frac{\sqrt{2}}{2}, y = \frac{\sqrt{2}}{2}$.

Absolute maximum is $\frac{8}{9} = f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

The absolute minimum is $0 = f(0, 0)$. (In fact, $f(0, y) = f(x, 0) = 0$)



64. False

Let $f(x, y) = x^4 - 2x^2 + y^2$.

Relative minima: $(\pm 1, 0, -1)$

Saddle point: $(0, 0, 0)$

Section 12.9 Applications of Extrema of Functions of Two Variables

2. A point on the plane is given by $(x, y, 12 - 2x - 3y)$. The square of the distance from $(1, 2, 3)$ to a point on the plane is given by

$$S = (x - 1)^2 + (y - 2)^2 + (9 - 2x - 3y)^2$$

$$S_x = 2(x - 1) + 2(9 - 2x - 3y)(-2)$$

$$S_y = 2(y - 2) + 2(9 - 2x - 3y)(-3).$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain the system

$$5x + 6y = 19$$

$$6x + 10y = 29.$$

Solving simultaneously, we have $x = \frac{16}{14}, y = \frac{31}{14}, z = \frac{43}{14}$ and the distance is

$$\sqrt{\left(\frac{16}{14} - 1\right)^2 + \left(\frac{31}{14} - 2\right)^2 + \left(\frac{43}{14} - 3\right)^2} = \frac{1}{\sqrt{14}}.$$

4. A point on the paraboloid is given by $(x, y, x^2 + y^2)$. The square of the distance from $(5, 0, 0)$ to a point on the paraboloid is given by

$$S = (x - 5)^2 + y^2 + (x^2 + y^2)^2$$

$$S_x = 2(x - 5) + 4x(x^2 + y^2) = 0$$

$$S_y = 2y + 4y(x^2 + y^2) = 0.$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain the system

$$2x^3 + 2xy^2 + x - 5 = 0$$

$$2y^3 + 2x^2y + y = 0.$$

Solving as in Exercise 3, we have $x \approx 1.235, y = 0, z \approx 1.525$ and the distance is

$$\sqrt{(1.235 - 5)^2 + (1.525)^2} \approx 4.06.$$

6. Since $x + y + z = 32$, $z = 32 - x - y$. Therefore,

$$P = xy^2z = 32xy^2 - x^2y^2 - xy^3$$

$$P_x = 32y^2 - 2xy^2 - y^3 = y^2(32 - 2x - y) = 0$$

$$P_y = 64xy - 2x^2y - 3xy^2 = y(64x - 2x^2 - 3xy) = 0.$$

Ignoring the solution $y = 0$ and substituting $y = 32 - 2x$ into $P_y = 0$, we have

$$64x - 2x^2 - 3x(32 - 2x) = 0$$

$$4x(x - 8) = 0.$$

Therefore, $x = 8$, $y = 16$, and $z = 8$.

8. Let x , y , and z be the numbers and let $S = x^2 + y^2 + z^2$. Since $x + y + z = 1$, we have

$$S = x^2 + y^2 + (1 - x - y)^2$$

$$S_x = 2x - 2(1 - x - y) = 0 \quad \left. \begin{array}{l} 2x + y = 1 \\ S_y = 2y - 2(1 - x - y) = 0 \end{array} \right\} \begin{array}{l} 2x + y = 1 \\ x + 2y = 1. \end{array}$$

Solving simultaneously yields $x = \frac{1}{3}$, $y = \frac{1}{3}$, and $z = \frac{1}{3}$.

10. Let x , y , and z be the length, width, and height, respectively. Then $C_0 = 1.5xy + 2yz + 2xz$ and $z = \frac{C_0 - 1.5xy}{2(x + y)}$. The volume is given by

$$V = xyz = \frac{C_0xy - 1.5x^2y^2}{2(x + y)}$$

$$V_x = \frac{y^2(2C_0 - 3x^2 - 6xy)}{4(x + y)^2}$$

$$V_y = \frac{x^2(2C_0 - 3y^2 - 6xy)}{4(x + y)^2}.$$

In solving the system $V_x = 0$ and $V_y = 0$, we note by the symmetry of the equations that $y = x$. Substituting $y = x$ into $V_x = 0$ yields

$$\frac{x^2(2C_0 - 9x^2)}{16x^2} = 0, \quad 2C_0 = 9x^2, \quad x = \frac{1}{3}\sqrt{2C_0}, \quad y = \frac{1}{3}\sqrt{2C_0}, \quad \text{and} \quad z = \frac{1}{4}\sqrt{2C_0}.$$

12. Consider the sphere given by $x^2 + y^2 + z^2 = r^2$ and let a vertex of the rectangular box be $(x, y, \sqrt{r^2 - x^2 - y^2})$. Then the volume is given by

$$V = (2x)(2y)(2\sqrt{r^2 - x^2 - y^2}) = 8xy\sqrt{r^2 - x^2 - y^2}$$

$$V_x = 8\left(xy\frac{-x}{\sqrt{r^2 - x^2 - y^2}} + y\sqrt{r^2 - x^2 - y^2}\right) = \frac{8y}{\sqrt{r^2 - x^2 - y^2}}(r^2 - 2x^2 - y^2) = 0$$

$$V_y = 8\left(xy\frac{-y}{\sqrt{r^2 - x^2 - y^2}} + x\sqrt{r^2 - x^2 - y^2}\right) = \frac{8x}{\sqrt{r^2 - x^2 - y^2}}(r^2 - x^2 - 2y^2) = 0.$$

Solving the system

$$2x^2 + y^2 = r^2$$

$$x^2 + 2y^2 = r^2$$

yields the solution $x = y = z = r/\sqrt{3}$.

14. Let x , y , and z be the length, width, and height, respectively. Then the sum of the two perimeters of the two cross sections is given by

$$(2x + 2z) + (2y + 2z) = 108 \text{ or } x = 54 - y - 2z.$$

The volume is given by

$$V = xyz = 54yz - y^2z - 2yz^2$$

$$V_y = 54z - 2yz - 2z^2 = z(54 - 2y - 2z) = 0$$

$$V_z = 54y - y^2 - 4yz = y(54 - y - 4z) = 0.$$

Solving the system $2y + 2z = 54$ and $y + 4z = 54$, we obtain the solution

$$x = 18 \text{ inches, } y = 18 \text{ inches, and } z = 9 \text{ inches.}$$

$$16. A = \frac{1}{2}[(30 - 2x) + (30 - 2x) + 2x \cos \theta]x \sin \theta$$

$$= 30x \sin \theta - 2x^2 \sin \theta + x^2 \sin \theta \cos \theta$$

$$\frac{\partial A}{\partial x} = 30 \sin \theta - 4x \sin \theta + 2x \sin \theta \cos \theta = 0$$

$$\frac{\partial A}{\partial \theta} = 30 \cos \theta - 2x^2 \cos \theta + x^2(2 \cos^2 \theta - 1) = 0$$

$$\text{From } \frac{\partial A}{\partial x} = 0 \text{ we have } 15 - 2x + x \cos \theta = 0 \Rightarrow \cos \theta = \frac{2x - 15}{x}.$$

$$\text{From } \frac{\partial A}{\partial \theta} = 0 \text{ we obtain}$$

$$30x\left(\frac{2x - 15}{x}\right) - 2x^2\left(\frac{2x - 15}{x}\right) + x^2\left(2\left(\frac{2x - 15}{x}\right)^2 - 1\right) = 0$$

$$30(2x - 15) - 2x(2x - 15) + 2(2x - 15)^2 - x^2 = 0$$

$$3x^2 - 30x = 0$$

$$x = 10$$

$$\text{Then } \cos \theta = \frac{1}{2} \Rightarrow \theta = 60^\circ.$$

$$18. P(p, q, r) = 2pq + 2pr + 2qr.$$

$$p + q + r = 1 \text{ implies that } r = 1 - p - q.$$

$$\begin{aligned} P(p, q) &= 2pq + 2p(1 - p - q) + 2q(1 - p - q) \\ &= 2pq + 2p - 2p^2 - 2pq + 2q - 2pq - 2q^2 \\ &= -2pq + 2p + 2q - 2p^2 - 2q^2 \end{aligned}$$

$$\frac{\partial P}{\partial p} = -2q + 2 - 4p; \quad \frac{\partial P}{\partial q} = -2p + 2 - 4q$$

$$\text{Solving } \frac{\partial P}{\partial p} = \frac{\partial P}{\partial q} = 0 \text{ gives}$$

$$q + 2p = 1$$

$$p + 2q = 1$$

$$\text{and hence } p = q = \frac{1}{3} \text{ and}$$

$$\begin{aligned} P\left(\frac{1}{3}, \frac{1}{3}\right) &= -2\left(\frac{1}{9}\right) + 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right) - 2\left(\frac{1}{9}\right) - 2\left(\frac{1}{9}\right) \\ &= \frac{6}{9} = \frac{2}{3}. \end{aligned}$$

$$\begin{aligned} 22. S &= d_1 + d_2 + d_3 = \sqrt{(0 - 0)^2 + (y - 0)^2} + \sqrt{(0 - 2)^2 + (y - 2)^2} + \sqrt{(0 + 2)^2 + (y - 2)^2} \\ &= y + 2\sqrt{4 + (y - 2)^2} \end{aligned}$$

$$\frac{dS}{dy} = 1 + \frac{2(y - 2)}{\sqrt{4 + (y - 2)^2}} = 0 \text{ when } y = 2 - \frac{2\sqrt{3}}{3} = \frac{6 - 2\sqrt{3}}{3}.$$

$$\text{The sum of the distance is minimized when } y = \frac{2(3 - \sqrt{3})}{3} \approx 0.845.$$

$$20. R = 515p_1 + 805p_2 + 1.5p_1p_2 - 1.5p_1^2 - p_2^2$$

$$R_{p_1} = 515 + 1.5p_2 - 3p_1 = 0$$

$$R_{p_2} = 805 + 1.5p_1 - p_2 = 0$$

$$3p_1 - 1.5p_2 = 515$$

$$-1.5p_1 + p_2 = 805$$

$$\text{Solving this system yields } p_1 = \$2296.67, p_2 = \$4250.$$

24. (a) $S = \sqrt{(x+4)^2 + y^2} + \sqrt{(x-1)^2 + (y-6)^2} + \sqrt{(x-12)^2 + (y-2)^2}$

The surface appears to have a minimum near $(x, y) = (1, 5)$.

(b) $S_x = \frac{x+4}{\sqrt{(x+4)^2 + y^2}} + \frac{x-1}{\sqrt{(x-1)^2 + (y-6)^2}} + \frac{x-12}{\sqrt{(x-12)^2 + (y-2)^2}}$

$S_y = \frac{y}{\sqrt{(x+4)^2 + y^2}} + \frac{y-6}{\sqrt{(x-1)^2 + (y-6)^2}} + \frac{y-2}{\sqrt{(x-12)^2 + (y-2)^2}}$

(c) Let $(x_1, y_1) = (1, 5)$. Then

$$-\nabla S(1, 5) = 0.258\mathbf{i} + 0.03\mathbf{j}$$

Direction $\approx 6.6^\circ$

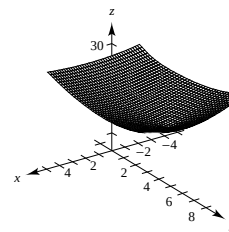
(d) $t \approx 0.94$ $x_2 \approx 1.24$ $y_2 \approx 5.03$

(e) $t \approx 3.56$, $x_3 \approx 1.24$, $y_3 \approx 5.06$,

$t \approx 1.04$, $x_4 \approx 1.23$, $y_4 \approx 5.06$

Note: Minimum occurs at $(x, y) = (1.2335, 5.0694)$

(f) $-\nabla S(x, y)$ points in the direction that S decreases most rapidly.



26. See the last paragraph on page 915 and Theorem 12.18.

28. (a)

x	y	xy	x^2
-3	0	0	9
-1	1	-1	1
1	1	1	1
3	2	6	9
$\sum x_i = 0$	$\sum y_i = 4$	$\sum x_i y_i = 6$	$\sum x_i^2 = 20$

(b) $S = \left(\frac{1}{10} - 0\right)^2 + \left(\frac{7}{10} - 1\right)^2 + \left(\frac{13}{10} - 1\right)^2 + \left(\frac{19}{10} - 2\right)^2$
 $= \frac{1}{5}$

$$a = \frac{4(6) - 0(4)}{4(20) - (0)^2} = \frac{3}{10}, \quad b = \frac{1}{4} \left[4 - \frac{3}{10}(0) \right] = 1,$$

$$y = \frac{3}{10}x + 1$$

30. (a)

x	y	xy	x^2
3	0	0	9
1	0	0	1
2	0	0	4
3	1	3	9
4	1	4	16
4	2	8	16
5	2	10	25
6	2	12	36
$\sum x_i = 28$	$\sum y_i = 8$	$\sum x_i y_i = 37$	$\sum x_i^2 = 116$

$$a = \frac{8(37) - (28)(8)}{8(116) - (28)^2} = \frac{72}{144} = \frac{1}{2}, \quad b = \frac{1}{8} \left[8 - \frac{1}{2}(28) \right] = -\frac{3}{4}, \quad y = \frac{1}{2}x - \frac{3}{4}$$

(b) $S = \left(\frac{3}{4} - 0\right)^2 + \left(-\frac{1}{4} - 0\right)^2 + \left(\frac{1}{4} - 0\right)^2 + \left(\frac{3}{4} - 1\right)^2 + \left(\frac{5}{4} - 1\right)^2 + \left(\frac{5}{4} - 2\right)^2 + \left(\frac{7}{4} - 2\right)^2 + \left(\frac{9}{4} - 2\right)^2 = \frac{3}{2}$

32. (1, 0), (3, 3), (5, 6)

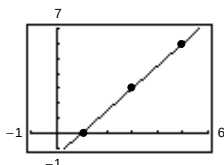
$$\sum x_i = 9, \quad \sum y_i = 9,$$

$$\sum x_i y_i = 39, \quad \sum x_i^2 = 35$$

$$a = \frac{3(39) - 9(9)}{3(35) - (9)^2} = \frac{36}{24} = \frac{3}{2}$$

$$b = \frac{1}{3} \left[9 - \frac{3}{2}(9) \right] = -\frac{9}{6} = -\frac{3}{2}$$

$$y = \frac{3}{2}x - \frac{3}{2}$$


 34. (6, 4), (1, 2), (3, 3), (8, 6), (11, 8), (13, 8); $n = 6$

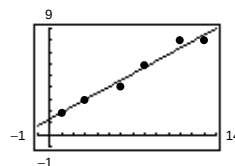
$$\sum x_i = 42, \quad \sum y_i = 31$$

$$\sum x_i y_i = 275, \quad \sum x_i^2 = 400$$

$$a = \frac{6(275) - (42)(31)}{6(400) - (42)^2} = \frac{29}{53} \approx 0.5472$$

$$b = \frac{1}{6} \left(31 - \frac{29}{53}(42) \right) = \frac{425}{318} \approx 1.3365$$

$$y = \frac{29}{53}x + \frac{425}{318}$$



36. (a) (1.00, 450), (1.25, 375), (1.50, 330)

$$\sum x_i = 3.75, \quad \sum y_i = 1,155, \quad \sum x_i^2 = 4.8125,$$

$$\sum x_i y_i = 1,413.75$$

$$a = \frac{3(1,413.75) - (3.75)(1,155)}{3(4.8125) - (3.75)^2} = -240$$

$$b = \frac{1}{3} [1,155 - (-240)(3.75)] = 685$$

$$y = -240x + 685$$

 (b) When $x = 1.40$, $y = -240(1.40) + 685 = 349$.

 38. (a) $y = 1.8311x - 47.1067$

 (b) For each 1 point increase in the percent (x), y increases by about 1.83 (slope of line).

$$40. S(a, b) = \sum_{i=1}^n (ax_i + b - y_i)^2$$

$$S_a(a, b) = 2a \sum_{i=1}^n x_i^2 + 2b \sum_{i=1}^n x_i - 2 \sum_{i=1}^n x_i y_i$$

$$S_b(a, b) = 2a \sum_{i=1}^n x_i + 2nb - 2 \sum_{i=1}^n y_i$$

$$S_{aa}(a, b) = 2 \sum_{i=1}^n x_i^2$$

$$S_{bb}(a, b) = 2n$$

$$S_{ab}(a, b) = 2 \sum_{i=1}^n x_i$$

 $S_{aa}(a, b) > 0$ as long as $x_i \neq 0$ for all i . (**Note:** If $x_i = 0$ for all i , then $x = 0$ is the least squares regression line.)

$$d = S_{aa}S_{bb} - S_{ab}^2 = 4n \sum_{i=1}^n x_i^2 - 4 \left(\sum_{i=1}^n x_i \right)^2 = 4 \left[n \sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i \right)^2 \right] \geq 0 \text{ since } n \sum_{i=1}^n x_i^2 \geq \left(\sum_{i=1}^n x_i \right)^2.$$

 As long as $d \neq 0$, the given values for a and b yield a minimum.

- 42.
- $(-4, 5), (-2, 6), (2, 6), (4, 2)$

$$\sum x_i = 0$$

$$\sum y_i = 19$$

$$\sum x_i^2 = 40$$

$$\sum x_i^3 = 0$$

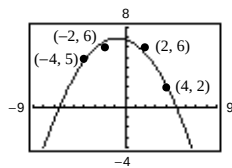
$$\sum x_i^4 = 544$$

$$\sum x_i y_i = -12$$

$$\sum x_i^2 y_i = 160$$

$$544a + 40c = 160, 40b = -12, 40a + 4c = 19$$

$$a = -\frac{5}{24}, b = -\frac{3}{10}, c = \frac{41}{6}, y = -\frac{5}{24}x^2 - \frac{3}{10}x + \frac{41}{6}$$



- 44.
- $(0, 10), (1, 9), (2, 6), (3, 0)$

$$\sum x_i = 6$$

$$\sum y_i = 25$$

$$\sum x_i^2 = 14$$

$$\sum x_i^3 = 36$$

$$\sum x_i^4 = 98$$

$$\sum x_i y_i = 21$$

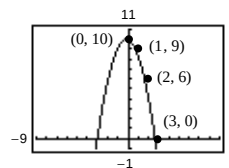
$$\sum x_i^2 y_i = 33$$

$$98a + 36b + 14c = 33$$

$$36a + 14b + 6c = 21$$

$$14a + 6b + 4c = 25$$

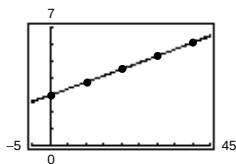
$$a = -\frac{5}{4}, b = \frac{9}{20}, c = \frac{199}{20}, y = -\frac{5}{4}x^2 + \frac{9}{20}x + \frac{199}{20}$$



46. (a)
- $y = 0.078x + 2.96$

$$(b) y = 0.0001429x^2 + 0.07229x + 2.9886$$

(c)



- (d) For the linear model,
- $x = 50$
- gives
- $y \approx 6.86$
- billion.

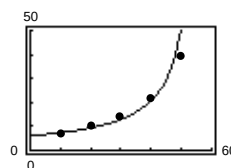
For the quadratic model, $x = 50$ gives $y \approx 6.96$ billion.

As you extrapolate into the future, the quadratic model increases more rapidly.

48. (a)
- $\frac{1}{y} = ax + b = -0.0029x + 0.1640$

$$y = \frac{1}{-0.0029x + 0.1640}$$

(b)



- (c) No. For
- $x = 60$
- ,
- $y \approx -100$
- . Note that there is a vertical asymptote at
- $x \approx 56.6$
- .

Section 12.10 Lagrange Multipliers

2. Maximize
- $f(x, y) = xy$
- .

$$\text{Constraint: } 2x + y = 4$$

$$\nabla f = \lambda \nabla g$$

$$y\mathbf{i} + x\mathbf{j} = 2\lambda\mathbf{i} + \lambda\mathbf{j}$$

$$y = 2\lambda$$

$$x = \lambda$$

$$2x + y = 4 \Rightarrow 4\lambda = 4$$

$$\lambda = 1, x = 1, y = 2$$

$$f(1, 2) = 2$$

4. Minimize
- $f(x, y) = x^2 + y^2$
- .

$$\text{Constraint: } 2x + 4y = 5$$

$$\nabla f = \lambda \nabla g$$

$$2x\mathbf{i} + 2y\mathbf{j} = 2\lambda\mathbf{i} + 4\lambda\mathbf{j}$$

$$2x = 2\lambda \Rightarrow x = \lambda$$

$$2y = 4\lambda \Rightarrow y = 2\lambda$$

$$2x + 4y = 5 \Rightarrow 10\lambda = 5$$

$$\lambda = \frac{1}{2}, x = \frac{1}{2}, y = 1$$

$$f\left(\frac{1}{2}, 1\right) = \frac{5}{4}$$

6. Maximize $f(x, y) = x^2 - y^2$.

Constraint: $2y - x^2 = 0$

$$\nabla f = \lambda \nabla g$$

$$2x\mathbf{i} - 2y\mathbf{j} = -2x\lambda\mathbf{i} + 2\lambda\mathbf{j}$$

$$2x = -2x\lambda \Rightarrow x = 0 \text{ or } \lambda = -1$$

If $x = 0$, then $y = 0$ and $f(0, 0) = 0$.

If $\lambda = -1$,

$$-2y = 2\lambda = -2 \Rightarrow y = 1 \Rightarrow x^2 = 2 \Rightarrow x = \sqrt{2}.$$

$$f(\sqrt{2}, 1) = 2 - 1 = 1 \text{ Maximum.}$$

8. Minimize $f(x, y) = 3x + y + 10$.

Constraint: $x^2y = 6$

$$\nabla f = \lambda \nabla g$$

$$3\mathbf{i} + \mathbf{j} = 2xy\lambda\mathbf{i} + x^2\lambda\mathbf{j}$$

$$\left. \begin{aligned} 3 &= 2xy\lambda \Rightarrow \lambda = \frac{3}{2xy} \\ 1 &= x^2\lambda \Rightarrow \lambda = \frac{1}{x^2} \end{aligned} \right\} 3x^2 = 2xy \Rightarrow y = \frac{3x}{2} \quad (x \neq 0)$$

$$x^2y = 6 \Rightarrow x^2\left(\frac{3x}{2}\right) = 6$$

$$x^3 = 4$$

$$x = \sqrt[3]{4}, y = \frac{3\sqrt[3]{4}}{2}$$

$$f\left(\sqrt[3]{4}, \frac{3\sqrt[3]{4}}{2}\right) = \frac{9\sqrt[3]{4} + 20}{2}$$

10. **Note:** $f(x, y) = \sqrt{x^2 + y^2}$ is minimum when $g(x, y)$ is minimum.

Minimize $g(x, y) = x^2 + y^2$.

Constraint: $2x + 4y = 15$

$$\left. \begin{aligned} 2x &= 2\lambda \\ 2y &= 4\lambda \end{aligned} \right\} y = 2x$$

$$2x + 4y = 15 \Rightarrow 10x = 15$$

$$x = \frac{3}{2}, y = 3$$

$$f\left(\frac{3}{2}, 3\right) = \sqrt{g\left(\frac{3}{2}, 3\right)} = \frac{3\sqrt{5}}{2}$$

12. Minimize $f(x, y) = 2x + y$.

Constraint: $xy = 32$

$$\left. \begin{aligned} 2 &= y\lambda \\ 1 &= x\lambda \end{aligned} \right\} y = 2x$$

$$xy = 32 \Rightarrow 2x^2 = 32$$

$$x = 4, y = 8$$

$$f(4, 8) = 16$$

14. Maximize or minimize $f(x, y) = e^{-xy/4}$.

Constraint: $x^2 + y^2 \leq 1$

Case 1: On the circle $x^2 + y^2 = 1$

$$\left. \begin{aligned} -(y/4)e^{-xy/4} &= 2x\lambda \\ -(x/4)e^{-xy/4} &= 2y\lambda \end{aligned} \right\} \Rightarrow x^2 = y^2$$

$$x^2 + y^2 = 1 \Rightarrow x = \pm \frac{\sqrt{2}}{2}$$

$$\text{Maxima: } f\left(\pm \frac{\sqrt{2}}{2}, \mp \frac{\sqrt{2}}{2}\right) = e^{1/8} \approx 1.1331$$

$$\text{Minima: } f\left(\pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}\right) = e^{-1/8} \approx 0.8825$$

Case 2: Inside the circle

$$\left. \begin{aligned} f_x &= -(y/4)e^{-xy/4} = 0 \\ f_y &= -(x/4)e^{-xy/4} = 0 \end{aligned} \right\} \Rightarrow x = y = 0$$

$$f_{xx} = \frac{y^2}{16}e^{-xy/4}, f_{yy} = \frac{x^2}{16}e^{-xy/4}, f_{xy} = e^{-xy}\left[\frac{1}{16}xy - \frac{1}{4}\right]$$

$$\text{At } (0, 0), f_{xx}f_{yy} - (f_{xy})^2 < 0.$$

$$\text{Saddle point: } f(0, 0) = 1$$

Combining the two cases, we have a maximum of $e^{1/8}$ at $\left(\pm \frac{\sqrt{2}}{2}, \mp \frac{\sqrt{2}}{2}\right)$ and a minimum of $e^{-1/8}$ at $\left(\pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}\right)$.

16. Maximize
- $f(x, y, z) = xyz$
- .

Constraint: $x + y + z = 6$

$$\left. \begin{array}{l} yz = \lambda \\ xz = \lambda \\ xy = \lambda \end{array} \right\} x = y = z$$

$$x + y + z = 6 \Rightarrow x = y = z = 2$$

$$f(2, 2, 2) = 8$$

20. Minimize
- $f(x, y, z) = x^2 + y^2 + z^2$
- .

Constraints: $x + 2z = 6$

$$x + y = 12$$

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{k}) + \mu(\mathbf{i} + \mathbf{j})$$

$$\left. \begin{array}{l} 2x = \lambda + \mu \\ 2y = \mu \\ 2z = 2\lambda \end{array} \right\} 2x = 2y + z$$

$$x + 2z = 6 \Rightarrow z = \frac{6-x}{2} = 3 - \frac{x}{2}$$

$$x + y = 12 \Rightarrow y = 12 - x$$

$$2x = 2(12 - x) + \left(3 - \frac{x}{2}\right) \Rightarrow \frac{9}{2}x = 27 \Rightarrow x = 6$$

$$x = 6, z = 0$$

$$f(6, 6, 0) = 72$$

18. Minimize
- $x^2 - 10x + y^2 - 14y + 70$

Constraint: $x + y = 10$

$$\left. \begin{array}{l} 2x - 10 = \lambda \\ 2y - 14 = \lambda \\ x + y = 8 \end{array} \right\} \begin{array}{l} x = (1/2)(\lambda + 10) \\ y = (1/2)(\lambda + 14) \end{array}$$

$$x + y = \frac{1}{2}(\lambda + 10) + \frac{1}{2}(\lambda + 14)$$

$$= \lambda + 12 = 8 \Rightarrow \lambda = -4$$

Then $x = 3, y = 5$.

$$f(3, 5) = 9 - 30 + 25 - 70 + 70 = 4$$

22. Maximize
- $f(x, y, z) = xyz$
- .

Constraints: $x^2 + z^2 = 5$

$$x - 2y = 0$$

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

$$yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(2x\mathbf{i} + 2z\mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j})$$

$$yz = 2x\lambda + \mu$$

$$xz = -2\mu \Rightarrow \mu = -\frac{xz}{2}$$

$$xy = 2z\lambda \Rightarrow \lambda = \frac{xy}{2z}$$

$$x^2 + z^2 = 5 \Rightarrow z = \sqrt{5 - x^2}$$

$$x - 2y = 0 \Rightarrow y = \frac{x}{2}$$

$$yz = 2x\left(\frac{xy}{2z}\right) - \frac{xz}{2}$$

$$\frac{x\sqrt{5-x^2}}{2} = \frac{x^3}{2\sqrt{5-x^2}} - \frac{x\sqrt{5-x^2}}{2}$$

$$x\sqrt{5-x^2} = \frac{x^3}{2\sqrt{5-x^2}}$$

$$2x(5-x^2) = x^3$$

$$0 = 3x^3 - 10x = x(3x^2 - 10)$$

$$x = 0 \text{ or } x = \sqrt{\frac{10}{3}}, y = \frac{1}{2}\sqrt{\frac{10}{3}}, z = \sqrt{\frac{5}{3}}$$

$$f\left(\sqrt{\frac{10}{3}}, \frac{1}{2}\sqrt{\frac{10}{3}}, \sqrt{\frac{5}{3}}\right) = \frac{5\sqrt{15}}{9}$$

Note: $f(0, 0, \sqrt{5}) = 0$ does not yield a maximum.

24. Minimize the square of the distance $f(x, y) = x^2 + (y - 10)^2$ subject to the constraint $(x - 4)^2 + y^2 = 4$.

$$\left. \begin{aligned} 2x &= 2(x - 4)\lambda \\ 2(y - 10) &= 2y\lambda \end{aligned} \right\} \frac{x}{x - 4} = \frac{y - 10}{y} \Rightarrow y = -\frac{5}{2}x + 10$$

$$(x - 4)^2 + y^2 = 4 \Rightarrow (x^2 - 8x + 16) + \left(\frac{25}{4}x^2 - 50x + 100\right) = 4$$

$$\frac{29}{4}x^2 - 58x + 112 = 0$$

Using a graphing utility, we obtain $x \approx 3.2572$ and $x \approx 4.7428$ or, by the Quadratic Formula,

$$x = \frac{58 \pm \sqrt{58^2 - 4(29/4)(112)}}{2(29/4)} = \frac{58 \pm 2\sqrt{29}}{29/2} = 4 \pm \frac{4\sqrt{29}}{29}.$$

Using the smaller value, we have $x = 4\left(1 - \frac{\sqrt{29}}{29}\right)$ and $y = \frac{10\sqrt{29}}{29} \approx 1.8570$.

The point on the circle is $\left[4\left(1 - \frac{\sqrt{29}}{29}\right), \frac{10\sqrt{29}}{29}\right]$

and the desired distance is $d = \sqrt{16\left(1 - \frac{\sqrt{29}}{29}\right)^2 + \left(\frac{10\sqrt{29}}{29} - 10\right)^2} \approx 8.77$.

The larger x -value does not yield a minimum.

26. Minimize the square of the distance

$$f(x, y, z) = (x - 4)^2 + y^2 + z^2$$

subject to the constraint $\sqrt{x^2 + y^2} - z = 0$.

$$\left. \begin{aligned} 2(x - 4) &= \frac{x}{\sqrt{x^2 + y^2}}\lambda = \frac{x}{z}\lambda \\ 2y &= \frac{y}{\sqrt{x^2 + y^2}}\lambda = \frac{y}{z}\lambda \\ 2z &= -\lambda \end{aligned} \right\} \begin{aligned} 2(x - 4) &= -2x \\ 2y &= -2y \end{aligned}$$

$$\sqrt{x^2 + y^2} - z = 0, \quad x = 2, \quad y = 0, \quad z = 2$$

The point on the plane is $(2, 0, 2)$ and the desired distance is

$$d = \sqrt{(2 - 4)^2 + 0^2 + 2^2} = 2\sqrt{2}.$$

28. Maximize $f(x, y, z) = z$ subject to the constraints $x^2 + y^2 - z^2 = 0$ and $x + 2z = 4$.

$$\begin{aligned} 0 &= 2x\lambda + \mu \\ 0 &= 2y\lambda \Rightarrow y = 0 \\ 1 &= -2z\lambda + 2\mu \\ x^2 + y^2 - z^2 &= 0 \\ x + 2z &= 4 \Rightarrow x = 4 - 2z \\ (4 - 2z)^2 + 0^2 - z^2 &= 0 \\ 3z^2 - 16z + 16 &= 0 \\ (3z - 4)(z - 4) &= 0 \\ z &= \frac{4}{3} \text{ or } z = 4 \end{aligned}$$

The maximum value of f occurs when $z = 4$ at the point of $(-4, 0, 4)$.

30. See explanation at the bottom of page 922.

32. Maximize $V(x, y, z) = xyz$ subject to the constraint $1.5xy + 2xz + 2yz = C$.

$$\left. \begin{aligned} yz &= (1.5y + 2z)\lambda \\ xz &= (1.5x + 2z)\lambda \\ xy &= (2x + 2y)\lambda \end{aligned} \right\} x = y \text{ and } z = \frac{3}{4}x$$

$$1.5xy + 2xz + 2yz = C \Rightarrow 1.5x^2 + \frac{3}{2}x^2 + \frac{3}{2}x^2 = C$$

$$x = \frac{\sqrt{2C}}{3}$$

Volume is maximum when

$$x = y = \frac{\sqrt{2C}}{3} \quad \text{and} \quad z = \frac{\sqrt{2C}}{4}.$$

34. Minimize $A(\pi, r) = 2\pi rh + 2\pi r^2$ subject to the constraint $\pi r^2 h = V_0$.

$$\left. \begin{aligned} 2\pi h + 4\pi r &= 2\pi rh\lambda \\ 2\pi r &= \pi r^2\lambda \end{aligned} \right\} h = 2r$$

$$\pi r^2 h = V_0 \Rightarrow 2\pi r^3 = V_0$$

$$\text{Dimensions: } r = \sqrt[3]{\frac{V_0}{2\pi}} \quad \text{and} \quad h = 2\sqrt[3]{\frac{V_0}{2\pi}}$$

36. (a) Maximize $P(x, y, z) = xyz$ subject to the constraint

$$x + y + z = S.$$

$$\begin{cases} yz = \lambda \\ xz = \lambda \\ xy = \lambda \end{cases} \Rightarrow x = y = z$$

$$x + y + z = S \Rightarrow x = y = z = \frac{S}{3}$$

Therefore,

$$xyz \leq \left(\frac{S}{3}\right)\left(\frac{S}{3}\right)\left(\frac{S}{3}\right), x, y, z > 0$$

$$xyz \leq \frac{S^3}{27}$$

$$\sqrt[3]{xyz} \leq \frac{S}{3}$$

$$\sqrt[3]{xyz} \leq \frac{x + y + z}{3}.$$

 (b) Maximize $P = x_1 x_2 x_3 \cdots x_n$ subject to the constraint

$$\sum_{i=1}^n x_i = S.$$

$$\begin{cases} x_2 x_3 \cdots x_n = \lambda \\ x_1 x_3 \cdots x_n = \lambda \\ x_1 x_2 \cdots x_{n-1} = \lambda \\ \vdots \\ x_1 x_2 x_3 \cdots x_{n-1} = \lambda \end{cases} \Rightarrow x_1 = x_2 = x_3 = \cdots = x_n$$

$$\sum_{i=1}^n x_i = S \Rightarrow x_1 = x_2 = x_3 = \cdots = x_n = \frac{S}{n}$$

Therefore,

$$x_1 x_2 x_3 \cdots x_n \leq \left(\frac{S}{n}\right)\left(\frac{S}{n}\right)\left(\frac{S}{n}\right) \cdots \left(\frac{S}{n}\right), x_i \geq 0$$

$$x_1 x_2 x_3 \cdots x_n \leq \left(\frac{S}{n}\right)^n$$

$$\sqrt[n]{x_1 x_2 x_3 \cdots x_n} \leq \frac{S}{n}$$

$$\sqrt[n]{x_1 x_2 x_3 \cdots x_n} \leq \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}.$$

 38. Case 1: Minimize $P(l, h) = 2h + l + \left(\frac{\pi l^2}{8}\right)$ subject to the constraint $lh + \left(\frac{\pi l^2}{8}\right) = A$.

$$1 + \frac{\pi}{2} = \left(h + \frac{\pi l}{4}\right)\lambda$$

$$2 = l\lambda \Rightarrow \lambda = \frac{2}{l}, 1 + \frac{\pi}{2} = \frac{2h}{l} + \frac{\pi}{2}$$

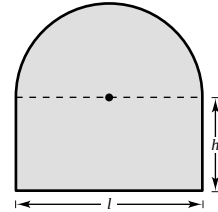
$$l = 2h$$

 Case 2: Minimize $A(l, h) = lh + \left(\frac{\pi l^2}{8}\right)$ subject to the constraint $2h + l + \left(\frac{\pi l^2}{2}\right) = P$.

$$h + \frac{\pi l}{4} = \left(1 + \frac{\pi}{2}\right)\lambda$$

$$l = 2\lambda \Rightarrow \lambda = \frac{l}{2}, h + \frac{\pi l}{4} = \frac{l}{2} + \frac{\pi l}{4}$$

$$h = \frac{l}{2} \text{ or } l = 2h$$


 40. Maximize $T(x, y, z) = 100 + x^2 + y^2$ subject to the constraints $x^2 + y^2 + z^2 = 50$ and $x - z = 0$.

$$\begin{cases} 2x = 2x\lambda + \mu \\ 2y = 2y\lambda \\ 0 = 2z\lambda - \mu \end{cases}$$

 If $y \neq 0$, then $\lambda = 1$ and $\mu = 0, z = 0$.

 Thus, $x = z = 0$ and $y = \sqrt{50}$.

$$T(0, \sqrt{50}, 0) = 100 + 50 = 150$$

 If $y = 0$, then $x^2 + z^2 = 2x^2 = 50$ and $x = z = \sqrt{50}/2$.

$$T\left(\frac{\sqrt{50}}{2}, 0, \frac{\sqrt{50}}{2}\right) = 100 + \frac{50}{4} = 112.5$$

Therefore, the maximum temperature is 150.

 42. Maximize $P(x, y) = 100x^{0.4}y^{0.6}$

 Constraint: $48x + 36y = 100,000$.

$$40x^{-0.6}y^{0.6} = 48\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.6} = \frac{48\lambda}{40}$$

$$60x^{0.4}y^{-0.4} = 36\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.4} = \frac{36\lambda}{60}$$

$$\left(\frac{y}{x}\right)^{0.6} \left(\frac{y}{x}\right)^{0.4} = \left(\frac{48\lambda}{40}\right) \left(\frac{60}{36\lambda}\right)$$

$$\frac{y}{x} = 2 \Rightarrow y = 2x$$

$$48x + 36y(2x) = 100,000 \Rightarrow x = \frac{2500}{3}, y = \frac{5000}{3}$$

$$P\left(\frac{2500}{3}, \frac{5000}{3}\right) \approx \$126,309.71.$$

44. Minimize $C(x, y) = 48x + 36y$ subject to the constraint $100x^{0.6}y^{0.4} = 20,000$.

$$48 = 60x^{-0.4}y^{0.4}\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.4} = \frac{48}{60\lambda}$$

$$36 = 40x^{0.6}y^{-0.6}\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.6} = \frac{36}{40\lambda}$$

$$\left(\frac{y}{x}\right)^{0.4} \left(\frac{y}{x}\right)^{0.6} = \left(\frac{48}{60\lambda}\right) \left(\frac{40\lambda}{36}\right)$$

$$\frac{y}{x} = \frac{8}{9} \Rightarrow y = \frac{8}{9}x$$

$$100x^{0.6}y^{0.4} = 20,000 \Rightarrow x^{0.6} \left(\frac{8}{9}x\right)^{0.4} = 200$$

$$x = \frac{200}{(8/9)^{0.4}} \approx 209.65$$

$$y = \frac{8}{9} \left[\frac{200}{(8/9)^{0.4}} \right] \approx 186.35$$

Therefore, $C(209.65, 186.35) = \$16,771.94$.

46. $f(x, y) = ax + by$, $x, y > 0$

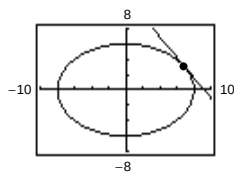
Constraint: $\frac{x^2}{64} + \frac{y^2}{36} = 1$

- (a) Level curves of $f(x, y) = 4x + 3y$ are lines of form

$$y = -\frac{4}{3}x + C.$$

Using $y = -\frac{4}{3}x + 12.3$, you obtain

$$x \approx 7, y \approx 3, \text{ and } f(7, 3) = 28 + 9 = 37.$$



Constraint is an ellipse.

- (b) Level curves of $f(x, y) = 4x + 9y$ are lines of form

$$y = -\frac{4}{9}x + C.$$

Using $y = -\frac{4}{9}x + 7$, you obtain

$$x \approx 4, y \approx 5.2, \text{ and } f(4, 5.2) = 62.8.$$

Review Exercises for Chapter 12

2. Yes, it is the graph of a function.

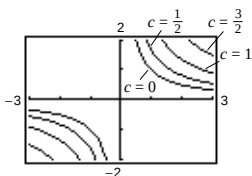
4. $f(x, y) = \ln xy$

The level curves are of the form

$$c = \ln xy$$

$$e^c = xy.$$

The level curves are hyperbolas.



6. $f(x, y) = \frac{x}{x+y}$

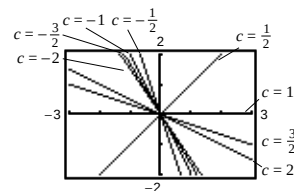
The level curves are of the form

$$c = \frac{x}{x+y}$$

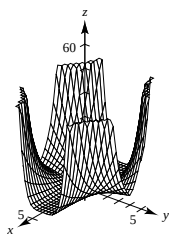
$$y = \left(\frac{1-c}{c}\right)x.$$

The level curves are passing through the origin with slope

$$\frac{1-c}{c}.$$

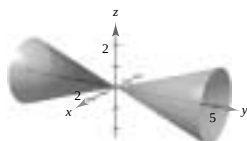


8. $g(x, y) = |y|^{1+|x|}$



10. $f(x, y, z) = 9x^2 - y^2 + 9z^2 = 0$

Elliptic cone



12. $\lim_{(x, y) \rightarrow (1, 1)} \frac{xy}{x^2 - y^2}$

Does not exist

Continuous except when $y = \pm x$.

14. $\lim_{(x, y) \rightarrow (0, 0)} \frac{y + xe^{-y^2}}{1 + x^2} = \frac{0 + 0}{1 + 0} = 0$

Continuous everywhere

16. $f(x, y) = \frac{xy}{x + y}$

$$f_x = \frac{y(x + y) - xy}{(x + y)^2} = \frac{y^2}{(x + y)^2}$$

$$f_y = \frac{x^2}{(x + y)^2}$$

18. $z = \ln(x^2 + y^2 + 1)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2 + 1}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2 + 1}$$

20. $w = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial w}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

22. $f(x, y, z) = \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

$$f_x = -\frac{1}{2}(1 - x^2 - y^2 - z^2)^{-3/2}(-2x)$$

$$= \frac{x}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$f_y = \frac{y}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$f_z = \frac{z}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

24. $u(x, t) = c(\sin akx) \cos kt$

$$\frac{\partial u}{\partial x} = akc(\cos akx) \cos kt$$

$$\frac{\partial u}{\partial t} = -kc(\sin akx) \sin kt$$

26. $z = x^2 \ln(y + 1)$

$$\frac{\partial z}{\partial x} = 2x \ln(y + 1). \text{ At } (2, 0, 0), \frac{\partial z}{\partial x} = 0.$$

Slope in x -direction.

$$\frac{\partial z}{\partial y} = \frac{x^2}{1 + y}. \text{ At } (2, 0, 0), \frac{\partial z}{\partial y} = 4.$$

Slope in y -direction.

28. $h(x, y) = \frac{x}{x + y}$

$$h_x = \frac{y}{(x + y)^2}$$

$$h_y = \frac{-x}{(x + y)^2}$$

$$h_{xx} = \frac{-2y}{(x + y)^3}$$

$$h_{yy} = \frac{2x}{(x + y)^3}$$

$$h_{xy} = \frac{(x + y)^2 - 2y(x + y)}{(x + y)^4} = \frac{x - y}{(x + y)^3}$$

$$h_{yx} = \frac{-(x + y)^2 + 2y(x + y)}{(x + y)^4} = \frac{x - y}{(x + y)^3}$$

30. $g(x, y) = \cos(x - 2y)$

$$g_x = -\sin(x - 2y)$$

$$g_y = 2 \sin(x - 2y)$$

$$g_{xx} = -\cos(x - 2y)$$

$$g_{yy} = -4 \cos(x - 2y)$$

$$g_{xy} = 2 \cos(x - 2y)$$

$$g_{yx} = 2 \cos(x - 2y)$$

32. $z = x^3 - 3xy^2$

$$\frac{\partial z}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

$$\frac{\partial z}{\partial y} = -6xy$$

$$\frac{\partial^2 z}{\partial y^2} = -6x$$

Therefore, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

34. $z = e^x \sin y$

$$\frac{\partial z}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \sin y$$

$$\frac{\partial z}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 z}{\partial y^2} = -e^x \sin y$$

Therefore, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

36. $z = \frac{xy}{\sqrt{x^2 + y^2}}$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= \left[\frac{\sqrt{x^2 + y^2}y - xy(\frac{x}{\sqrt{x^2 + y^2}})}{x^2 + y^2} \right] dx + \left[\frac{\sqrt{x^2 + y^2}x - xy(\frac{y}{\sqrt{x^2 + y^2}})}{x^2 + y^2} \right] dy = \frac{y^3}{(x^2 + y^2)^{3/2}} dx + \frac{x^3}{(x^2 + y^2)^{3/2}} dy$$

38. From the accompanying figure we observe

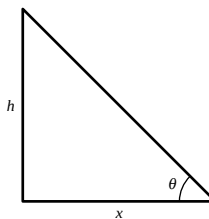
$$\tan \theta = \frac{h}{x} \text{ or } h = x \tan \theta$$

$$dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial \theta} d\theta = \tan \theta dx + x \sec^2 \theta d\theta.$$

Letting $x = 100$, $dx = \pm \frac{1}{2}$, $\theta = \frac{11\pi}{60}$, and $d\theta = \pm \frac{\pi}{180}$.

(Note that we express the measurement of the angle in radians.) The maximum error is approximately

$$dh = \tan\left(\frac{11\pi}{60}\right)\left(\pm \frac{1}{2}\right) + 100 \sec^2\left(\frac{11\pi}{60}\right)\left(\pm \frac{\pi}{180}\right) \approx \pm 0.3247 \pm 2.4814 \approx \pm 2.81 \text{ feet.}$$



40. $A = \pi r \sqrt{r^2 + h^2}$

$$dA = \left(\pi \sqrt{r^2 + h^2} + \frac{\pi r^2}{\sqrt{r^2 + h^2}} \right) dr + \frac{\pi r h}{\sqrt{r^2 + h^2}} dh$$

$$= \frac{\pi(2r^2 + h^2)}{\sqrt{r^2 + h^2}} dr + \frac{\pi r h}{\sqrt{r^2 + h^2}} dh = \frac{\pi(8 + 25)}{\sqrt{29}} \left(\pm \frac{1}{8} \right) + \frac{10\pi}{\sqrt{29}} \left(\pm \frac{1}{8} \right) = \pm \frac{43\pi}{8\sqrt{29}}$$

42. $u = y^2 - x$, $x = \cos t$, $y = \sin t$

$$\begin{aligned} \text{Chain Rule: } \frac{du}{dt} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} \\ &= -1(-\sin t) + 2y(\cos t) \\ &= \sin t + 2(\sin t) \cos t \\ &= \sin t(1 + 2 \cos t) \end{aligned}$$

Substitution: $u = \sin^2 t - \cos t$

$$\frac{du}{dt} = 2 \sin t \cos t + \sin t = \sin t(1 + 2 \cos t)$$

44. $w = \frac{xy}{z}, x = 2r + t, y = rt, z = 2r - t$

$$\begin{aligned}\text{Chain Rule: } \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r} \\ &= \frac{y}{z}(2) + \frac{x}{z}(t) - \frac{xy}{z^2}(2) \\ &= \frac{2rt}{2r-t} + \frac{(2r+t)t}{2r-t} - \frac{2(2r+t)(rt)}{(2r-t)^2} \\ &= \frac{4r^2t - 4rt^2 - t^3}{(2r-t)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t} \\ &= \frac{y}{z}(1) + \frac{x}{z}(r) - \frac{xy}{z^2}(-1) \\ &= \frac{4r^2t - rt^2 + 4r^3}{(2r-t)^2}\end{aligned}$$

Substitution: $w = \frac{xy}{z} = \frac{(2r+t)(rt)}{2r-t} = \frac{2r^2t + rt^2}{2r-t}$

$$\frac{\partial w}{\partial r} = \frac{4r^2t - 4rt^2 - t^3}{(2r-t)^2}$$

$$\frac{\partial w}{\partial t} = \frac{4r^2t - rt^2 - 4r^3}{(2r-t)^2}$$

48. $f(x, y) = \frac{1}{4}y^2 - x^2$

$$\nabla f = -2x\mathbf{i} + \frac{1}{2}y\mathbf{j}$$

$$\nabla f(1, 4) = -2\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{u} = \frac{1}{\sqrt{5}}\mathbf{v} = \frac{2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}$$

$$D_{\mathbf{u}}f(1, 4) = \nabla f(1, 4) \cdot \mathbf{u} = -\frac{4\sqrt{5}}{5} + \frac{2\sqrt{5}}{5} = -\frac{2\sqrt{5}}{5}$$

52. $z = \frac{x^2}{x-y}$

$$\nabla z = \frac{x^2 - 2xy}{(x-y)^2}\mathbf{i} + \frac{x^2}{(x-y)^2}\mathbf{j}$$

$$\nabla z(2, 1) = 4\mathbf{j}$$

$$\|\nabla z(2, 1)\| = 4$$

56. $4y \sin x - y^2 = 3$

$$f(x, y) = 4y \sin x - y^2$$

$$\nabla f(x, y) = 4y \cos x \mathbf{i} + (4 \sin x - 2y)\mathbf{j}$$

$$\nabla f\left(\frac{\pi}{2}, 1\right) = 2\mathbf{j}$$

Normal vector: $\mathbf{j}$

46. $xz^2 - y \sin z = 0$

$$2xz \frac{\partial z}{\partial x} + z^2 - y \cos z \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{z^2}{y \cos z - 2xz}$$

$$2xz \frac{\partial z}{\partial y} - y \cos z \frac{\partial z}{\partial y} - \sin z = 0$$

$$\frac{\partial z}{\partial y} = \frac{\sin z}{2xz - y \cos z}$$

50. $w = 6x^2 + 3xy - 4y^2z$

$$\nabla w = (12x + 3y)\mathbf{i} + (3x - 8yz)\mathbf{j} + (-4y^2)\mathbf{k}$$

$$\nabla w(1, 0, 1) = 12\mathbf{i} + 3\mathbf{j}$$

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{v} = \frac{\sqrt{3}}{3}\mathbf{i} + \frac{\sqrt{3}}{3}\mathbf{j} - \frac{\sqrt{3}}{3}\mathbf{k}$$

$$\begin{aligned}D_{\mathbf{u}}w(1, 0, 1) &= \nabla w(1, 0, 1) \cdot \mathbf{u} \\ &= 4\sqrt{3} + \sqrt{3} + 0 = 5\sqrt{3}\end{aligned}$$

54. $z = x^2y$

$$\nabla z = 2xy\mathbf{i} + x^2\mathbf{j}$$

$$\nabla z(2, 1) = 4\mathbf{i} + 4\mathbf{j}$$

$$\|\nabla z(2, 1)\| = 4\sqrt{2}$$

58. $F(x, y, z) = y^2 + z^2 - 25 = 0$

$$\nabla F = 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla F(2, 3, 4) = 6\mathbf{j} + 8\mathbf{k} = 2(3\mathbf{j} + 4\mathbf{k})$$

Therefore, the equation of the tangent plane is

$$3(y - 3) + 4(z - 4) = 0 \quad \text{or} \quad 3y + 4z = 25,$$

and the equation of the normal line is

$$x = 2, \frac{y-3}{3} = \frac{z-4}{4}.$$

60. $F(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$

$$\nabla F = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla F(1, 2, 2) = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} = 2(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$$

Therefore, the equation of the tangent plane is

$$(x - 1) + 2(y - 2) + 2(z - 2) = 0 \quad \text{or}$$

$$x + 2y + 2z = 9,$$

and the equation of the normal line is

$$\frac{x - 1}{1} = \frac{y - 2}{2} = \frac{z - 2}{2}.$$

62. $F(x, y, z) = y^2 + z - 25 = 0$

$$G(x, y, z) = x - y = 0$$

$$\nabla F = 2y\mathbf{i} + \mathbf{k}$$

$$\nabla G = \mathbf{i} - \mathbf{j}$$

$$\nabla F(4, 4, 9) = 8\mathbf{i} + \mathbf{k}$$

$$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \mathbf{i} + \mathbf{j} - 8\mathbf{k}$$

Therefore, the equation of the tangent line is

$$\frac{x - 4}{1} = \frac{y - 4}{1} = \frac{z - 9}{-8}.$$

64. (a) $f(x, y) = \cos x + \sin y, \quad f(0, 0) = 1$

$$f_x = -\sin x, \quad f_x(0, 0) = 0$$

$$f_y = \cos y, \quad f_y(0, 0) = 1$$

$$P_1(x, y) = 1 + y$$

(c) If $y = 0$, you obtain the 2nd degree Taylor polynomial for $\cos x$.

(b) $f_{xx} = -\cos x, \quad f_{xx}(0, 0) = -1$

$$f_{yy} = -\sin y, \quad f_{yy}(0, 0) = 0$$

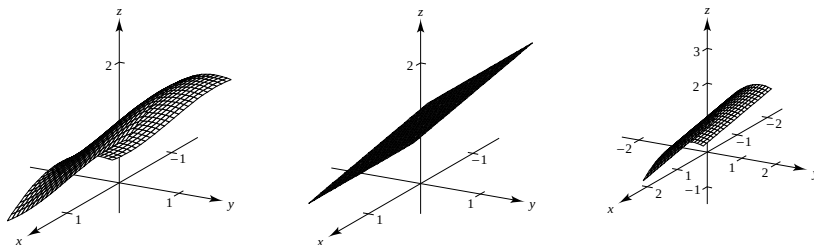
$$f_{xy} = 0, \quad f_{xy}(0, 0) = 0$$

$$P_2(x, y) = 1 + y - \frac{1}{2}x^2$$

(d)

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1.0	1.0	1.0
0	0.1	1.0998	1.1	1.1
0.2	0.1	1.0799	1.1	1.095
0.5	0.3	1.1731	1.3	1.175
1	0.5	1.0197	1.5	1.0

(e)



The accuracy lessens as the distance from $(0, 0)$ increases.

66. $f(x, y) = 2x^2 + 6xy + 9y^2 + 8x + 14$

$$f_x = 4x + 6y + 8 = 0$$

$$f_y = 6x + 18y = 0, \quad x = -3y$$

$$4(-3y) + 6y = -8 \Rightarrow y = \frac{4}{3}, \quad x = -4$$

$$f_{xx} = 4$$

$$f_{yy} = 18$$

$$f_{xy} = 6$$

$$f_{xx}f_{yy} - (f_{xy})^2 = 4(18) - (6)^2 = 36 > 0 \quad \text{Therefore, } \left(-4, \frac{4}{3}, -2\right) \text{ is a relative minimum.}$$

68. $z = 50(x + y) - (0.1x^3 + 20x + 150) - (0.05y^3 + 20.6y + 125)$

$$z_x = 50 - 0.3x^2 - 20 = 0, \quad x = \pm 10$$

$$z_y = 50 - 0.15y^2 - 20.6 = 0, \quad y = \pm 14$$

Critical Points: $(10, 14)$, $(10, -14)$, $(-10, 14)$, $(-10, -14)$

$$z_{xx} = -0.6x, \quad z_{yy} = -0.3y, \quad z_{xy} = 0$$

At $(10, 14)$, $z_{xx}z_{yy} - (z_{xy})^2 = (-6)(-4.2) - 0^2 > 0$, $z_{xx} < 0$.

$(10, 14, 199.4)$ is a relative maximum.

At $(10, -14)$, $z_{xx}z_{yy} - (z_{xy})^2 = (-6)(4.2) - 0^2 < 0$.

$(10, -14, -349.4)$ is a saddle point.

At $(-10, 14)$, $z_{xx}z_{yy} - (z_{xy})^2 = (6)(-4.2) - 0^2 < 0$.

$(-10, 14, -200.6)$ is a saddle point.

At $(-10, -14)$, $z_{xx}z_{yy} - (z_{xy})^2 = (6)(4.2) - 0^2 > 0$, $z_{xx} < 0$.

$(-10, -14, -749.4)$ is a relative minimum.

70. The level curves indicate that there is a relative extremum at A , the center of the ellipse in the second quadrant, and that there is a saddle point at B , the origin.

72. Minimize $C(x_1, x_2) = 0.25x_1^2 + 10x_1 + 0.15x_2^2 + 12x_2$ subject to the constraint $x_1 + x_2 = 1000$.

$$\begin{cases} 0.50x_1 + 10 = \lambda \\ 0.30x_2 + 12 = \lambda \end{cases} \quad 5x_1 - 3x_2 = 20$$

$$x_1 + x_2 = 1000 \Rightarrow 3x_1 + 3x_2 = 3000$$

$$\frac{5x_1 - 3x_2 = 20}{8x_1} = 3020$$

$$8x_1 = 3020$$

$$x_1 = 377.5$$

$$x_2 = 622.5$$

$$C(377.5, 622.5) = 104,997.50$$

74. Minimize the square of the distance:

$$f(x, y, z) = (x - 2)^2 + (y - 2)^2 + (x^2 + y^2 - 0)^2.$$

$$f_x = 2(x - 2) + 2(x^2 + y^2)2x = 0 \quad \left\{ \begin{array}{l} x - 2 + 2x^3 + 2xy^2 = 0 \\ y - 2 + 2y^3 + 2x^2y = 0 \end{array} \right.$$

$$f_y = 2(y - 2) + 2(x^2 + y^2)2y = 0$$

Clearly $x = y$ and hence: $4x^3 + x - 2 = 0$. Using a computer algebra system, $x \approx 0.6894$.

Thus, $(\text{distance})^2 = (0.6894 - 2)^2 + (0.6894 - 2)^2 + [2(0.6894)^2]^2 \approx 4.3389$.

distance ≈ 2.08

76. (a) $(25, 28)$, $(50, 38)$, $(75, 54)$, $(100, 75)$, $(125, 102)$

$$\sum x_i = 375, \quad \sum y_i = 297, \quad \sum x_i^2 = 34,375, \quad \sum x_i^3 = 3,515,625$$

$$\sum x_i^4 = 382,421,875, \quad \sum x_i y_i = 26,900, \quad \sum x_i^2 y_i = 2,760,000$$

$$382,421,875a + 3,515,625b + 34,375c = 2,760,000$$

$$3,515,625a + 34,375b + 375c = 26,900$$

$$34,375a + 375b + 5c = 297$$

$$a \approx 0.0045, \quad b \approx 0.0717, \quad c \approx 23.2914, \quad y \approx 0.0045x^2 + 0.0717x + 23.2914$$

- (b) When $x = 80$ km/hr, $y \approx 57.8$ km.

45. (a) Maximize $g(\alpha, \beta, \gamma) = \cos \alpha \cos \beta \cos \gamma$ subject to the constraint $\alpha + \beta + \gamma = \pi$.

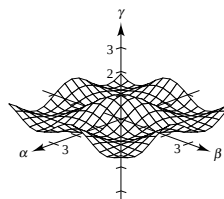
$$\left. \begin{aligned} -\sin \alpha \cos \beta \cos \gamma &= \lambda \\ -\cos \alpha \sin \beta \cos \gamma &= \lambda \\ -\cos \alpha \cos \beta \sin \gamma &= \lambda \end{aligned} \right\} \tan \alpha = \tan \beta = \tan \gamma \Rightarrow \alpha = \beta = \gamma$$

$$\alpha + \beta + \gamma = \pi \Rightarrow \alpha = \beta = \gamma = \frac{\pi}{3}$$

$$g\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{1}{8}$$

(b) $\alpha + \beta + \gamma = \pi \Rightarrow \gamma = \pi - (\alpha + \beta)$

$$\begin{aligned} g(\alpha + \beta) &= \cos \alpha \cos \beta \cos(\pi - (\alpha + \beta)) \\ &= \cos \alpha \cos \beta [\cos \pi \cos(\alpha + \beta) + \sin \pi \sin(\alpha + \beta)] \\ &= -\cos \alpha \cos \beta \cos(\alpha + \beta) \end{aligned}$$



Review Exercises for Chapter 12

1. No, it is not the graph of a function.

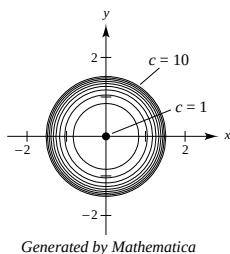
3. $f(x, y) = e^{x^2+y^2}$

The level curves are of the form

$$c = e^{x^2+y^2}$$

$$\ln c = x^2 + y^2.$$

The level curves are circles centered at the origin.



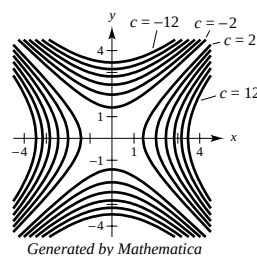
5. $f(x, y) = x^2 - y^2$

The level curves are of the form

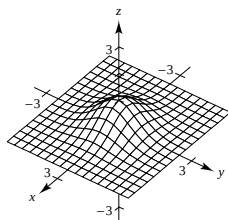
$$c = x^2 - y^2$$

$$1 = \frac{x^2}{c} - \frac{y^2}{c}.$$

The level curves are hyperbolas.



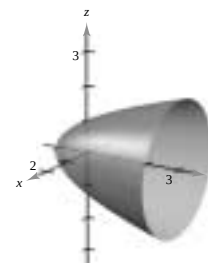
7. $f(x, y) = e^{-(x^2+y^2)}$



9. $f(x, y, z) = x^2 - y + z^2 = 1$

$$y = x^2 + z^2 - 1$$

Elliptic paraboloid



11. $\lim_{(x,y) \rightarrow (1,1)} \frac{xy}{x^2 + y^2} = \frac{1}{2}$

Continuous except at $(0, 0)$.

13. $\lim_{(x,y) \rightarrow (0,0)} \frac{-4x^2y}{x^4 + y^2}$

$$\text{For } y = x^2, \frac{-4x^2y}{x^4 + y^2} = \frac{-4x^4}{x^4 + x^4} = -2, \text{ for } x \neq 0$$

$$\text{For } y = 0, \frac{-4x^2y}{x^4 + y^2} = 0, \text{ for } x \neq 0$$

Thus, the limit does not exist. Continuous except at $(0, 0)$.

15. $f(x, y) = e^x \cos y$

$$f_x = e^x \cos y$$

$$f_y = -e^x \sin y$$

19. $g(x, y) = \frac{xy}{x^2 + y^2}$

$$g_x = \frac{y(x^2 + y^2) - xy(2x)}{(x^2 + y^2)^2} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}$$

$$g_y = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}$$

23. $u(x, t) = ce^{-n^2 t} \sin(nx)$

$$\frac{\partial u}{\partial x} = cne^{-n^2 t} \cos(nx)$$

$$\frac{\partial u}{\partial t} = -cn^2 e^{-n^2 t} \sin(nx)$$

17. $z = xe^y + ye^x$

$$\frac{\partial z}{\partial x} = e^y + ye^x$$

$$\frac{\partial z}{\partial y} = xe^y + e^x$$

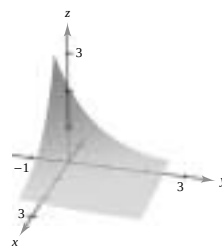
21. $f(x, y, z) = z \arctan \frac{y}{x}$

$$f_x = \frac{z}{1 + (y^2/x^2)} \left(-\frac{y}{x^2} \right) = \frac{-yz}{x^2 + y^2}$$

$$f_y = \frac{z}{1 + (y^2/x^2)} \left(\frac{1}{x} \right) = \frac{xz}{x^2 + y^2}$$

$$f_z = \arctan \frac{y}{x}$$

25.



27. $f(x, y) = 3x^2 - xy + 2y^3$

$$f_x = 6x - y$$

$$f_y = -x + 6y^2$$

$$f_{xx} = 6$$

$$f_{yy} = 12y$$

$$f_{xy} = -1$$

$$f_{yx} = -1$$

29. $h(x, y) = x \sin y + y \cos x$

$$h_x = \sin y - y \sin x$$

$$h_y = x \cos y + \cos x$$

$$h_{xx} = -y \cos x$$

$$h_{yy} = -x \sin y$$

$$h_{xy} = \cos y - \sin x$$

$$h_{yx} = \cos y - \sin x$$

31. $z = x^2 - y^2$

$$\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial^2 z}{\partial x^2} = 2$$

$$\frac{\partial z}{\partial y} = -2y$$

$$\frac{\partial^2 z}{\partial y^2} = -2$$

$$\text{Therefore, } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

33. $z = \frac{y}{x^2 + y^2}$

$$\frac{\partial z}{\partial x} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = -2y \left[\frac{-4x^2}{(x^2 + y^2)^3} + \frac{1}{(x^2 + y^2)^2} \right] = 2y \frac{3x^2 - y^2}{(x^2 + y^2)^3}$$

$$\frac{\partial z}{\partial y} = \frac{(x^2 + y^2) - 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{(x^2 + y^2)^2(-2y) - 2(x^2 - y^2)(x^2 + y^2)(2y)}{(x^2 + y^2)^4}$$

$$= -2y \frac{3x^2 - y^2}{(x^2 + y^2)^3}$$

$$\text{Therefore, } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

35. $z = x \sin \frac{y}{x}$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \left(\sin \frac{y}{x} - \frac{y}{x} \cos \frac{y}{x} \right) dx + \left(\cos \frac{y}{x} \right) dy$$

37. $z^2 = x^2 + y^2$

$$2z \, dx = 2x \, dx + 2y \, dy$$

$$dz = \frac{x}{z} \, dx + \frac{y}{z} \, dy = \frac{5}{13} \left(\frac{1}{2} \right) + \frac{12}{13} \left(\frac{1}{2} \right) = \frac{17}{26} \approx 0.654 \text{ cm}$$

Percentage error: $\frac{dz}{z} = \frac{17/26}{13} \approx 0.0503 \approx 5\%$

39. $V = \frac{1}{3}\pi r^2 h$

$$dV = \frac{2}{3}\pi r h \, dr + \frac{1}{3}\pi r^2 \, dh = \frac{2}{3}\pi(2)(5)\left(\pm\frac{1}{6}\right) + \frac{1}{3}\pi(2)^2\left(\pm\frac{1}{6}\right) \\ = \pm\frac{5}{6}\pi \pm \frac{1}{6}\pi = \pm\pi \text{ in.}^3$$

41. $w = \ln(x^2 + y^2)$, $x = 2t + 3$, $y = 4 - t$

$$\begin{aligned} \text{Chain Rule: } \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\ &= \frac{2x}{x^2 + y^2}(2) + \frac{2y}{x^2 + y^2}(-1) \\ &= \frac{2(2t + 3)2}{(2t + 3)^2 + (4 - t)^2} - \frac{2(4 - t)}{(2t + 3)^2 + (4 - t)^2} \\ &= \frac{10t + 4}{5t^2 + 4t + 25} \end{aligned}$$

Substitution: $w = \ln(x^2 + y^2) = \ln[(2t + 3)^2 + (4 - t)^2]$

$$\frac{dw}{dt} = \frac{2(2t + 3)(2) - 2(4 - t)}{(2t + 3)^2 + (4 - t)^2} = \frac{10t + 4}{5t^2 + 4t + 25}$$

43. $u = x^2 + y^2 + z^2$, $x = r \cos t$, $y = r \sin t$, $z = t$

$$\begin{aligned} \text{Chain Rule: } \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial r} \\ &= 2x \cos t + 2y \sin t + 2z(0) \\ &= 2(r \cos^2 t + r \sin^2 t) = 2r \\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t} \\ &= 2x(-r \sin t) + 2y(r \cos t) + 2z \\ &= 2(-r^2 \sin t \cos t + r^2 \sin t \cos t) + 2t \\ &= 2t \end{aligned}$$

Substitution: $u(r, t) = r^2 \cos^2 t + r^2 \sin^2 t + t^2 = r^2 + t^2$

$$\frac{\partial u}{\partial r} = 2r$$

$$\frac{\partial u}{\partial t} = 2t$$

45. $x^2 y - 2yz - xz - z^2 = 0$

$$2xy - 2y \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial x} - z - 2z \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-2xy + z}{-2y - x - 2z} = \frac{2xy - z}{x + 2y + 2z}$$

$$x^2 - 2y \frac{\partial z}{\partial y} - 2z - x \frac{\partial z}{\partial y} - 2z \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-x^2 + 2z}{-2y - x - 2z} = \frac{x^2 - 2z}{x + 2y + 2z}$$

47. $f(x, y) = x^2y$

$$\nabla f = 2xy\mathbf{i} + x^2\mathbf{j}$$

$$\nabla f(2, 1) = 4\mathbf{i} + 4\mathbf{j}$$

$$\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{v} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$$D_{\mathbf{u}}f(2, 1) = \nabla f(2, 1) \cdot \mathbf{u} = 2\sqrt{2} - 2\sqrt{2} = 0$$

51. $z = \frac{y}{x^2 + y^2}$

$$\nabla z = -\frac{2xy}{(x^2 + y^2)^2}\mathbf{i} + \frac{x^2 - y^2}{(x^2 + y^2)^2}\mathbf{j}$$

$$\nabla z(1, 1) = -\frac{1}{2}\mathbf{i} = \left\langle -\frac{1}{2}, 0 \right\rangle$$

$$\|\nabla z(1, 1)\| = \frac{1}{2}$$

55. $9x^2 - 4y^2 = 65$

$$f(x, y) = 9x^2 - 4y^2$$

$$\nabla f(x, y) = 18x\mathbf{i} + 8y\mathbf{j}$$

$$\nabla f(3, 2) = 54\mathbf{i} - 16\mathbf{j}$$

$$\text{Unit normal: } \frac{54\mathbf{i} - 16\mathbf{j}}{\|54\mathbf{i} - 16\mathbf{j}\|} = \frac{1}{\sqrt{793}}(27\mathbf{i} - 8\mathbf{j})$$

59. $F(x, y, z) = x^2 + y^2 - 4x + 6y + z + 9 = 0$

$$\nabla F = (2x - 4)\mathbf{i} + (2y + 6)\mathbf{j} + \mathbf{k}$$

$$\nabla F(2, -3, 4) = \mathbf{k}$$

Therefore, the equation of the tangent plane is

$$z - 4 = 0 \quad \text{or} \quad z = 4,$$

and the equation of the normal line is

$$x = 2, y = -3, z = 4 + t.$$

63. $f(x, y, z) = x^2 + y^2 + z^2 - 14$

$$\nabla f(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla f(2, 1, 3) = 4\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} \text{ Normal vector to plane.}$$

$$\cos \theta = \frac{|\mathbf{n} \cdot \mathbf{k}|}{\|\mathbf{n}\|} = \frac{6}{\sqrt{56}} = \frac{3\sqrt{14}}{14}$$

$$\theta = 36.7^\circ$$

49. $w = y^2 + xz$

$$\nabla w = z\mathbf{i} + 2y\mathbf{j} + x\mathbf{k}$$

$$\nabla w(1, 2, 2) = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

$$\mathbf{u} = \frac{1}{3}\mathbf{v} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

$$D_{\mathbf{u}}w(1, 2, 2) = \nabla w(1, 2, 2) \cdot \mathbf{u} = \frac{4}{3} - \frac{4}{3} + \frac{2}{3} = \frac{2}{3}$$

53. $z = e^{-x} \cos y$

$$\nabla z = -e^{-x} \cos y \mathbf{i} - e^{-x} \sin y \mathbf{j}$$

$$\nabla z\left(0, \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$$

$$\left\| \nabla z\left(0, \frac{\pi}{4}\right) \right\| = 1$$

57. $F(x, y, z) = x^2y - z = 0$

$$\nabla F = 2xy\mathbf{i} + x^2\mathbf{j} - \mathbf{k}$$

$$\nabla F(2, 1, 4) = 4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$$

Therefore, the equation of the tangent plane is

$$4(x - 2) + 4(y - 1) - (z - 4) = 0 \quad \text{or}$$

$$4x + 4y - z = 8,$$

and the equation of the normal line is

$$\frac{x - 2}{4} = \frac{y - 1}{4} = \frac{z - 4}{-1}.$$

61. $F(x, y, z) = x^2 - y^2 - z = 0$

$$G(x, y, z) = 3 - z = 0$$

$$\nabla F = 2x\mathbf{i} - 2y\mathbf{j} - \mathbf{k}$$

$$\nabla G = -\mathbf{k}$$

$$\nabla F(2, 1, 3) = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

$$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 2(\mathbf{i} + 2\mathbf{j})$$

Therefore, the equation of the tangent line is

$$\frac{x - 2}{1} = \frac{y - 1}{2}, z = 3.$$

65. $f(x, y) = x^3 - 3xy + y^2$

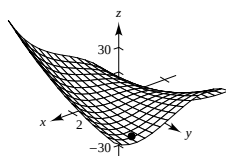
$$f_x = 3x^2 - 3y = 3(x^2 - y) = 0$$

$$f_y = -3x + 2y = 0$$

$$f_{xx} = 6x$$

$$f_{yy} = 2$$

$$f_{xy} = -3$$



From $f_x = 0$, we have $y = x^2$. Substituting this into $f_y = 0$, we have $-3x + 2x^2 = x(2x - 3) = 0$. Thus, $x = 0$ or $\frac{3}{2}$.

At the critical point $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point.

At the critical point $(\frac{3}{2}, \frac{9}{4})$, $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} > 0$. Therefore, $(\frac{3}{2}, \frac{9}{4}, -\frac{27}{16})$ is a relative minimum.

67. $f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$

$$f_x = y - \frac{1}{x^2} = 0, \quad x^2y = 1$$

$$f_y = x - \frac{1}{y^2} = 0, \quad xy^2 = 1$$

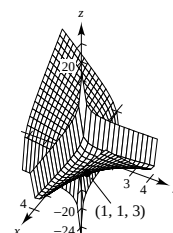
Thus, $x^2y = xy^2$ or $x = y$ and substitution yields the critical point $(1, 1)$.

$$f_{xx} = \frac{2}{x^3}$$

$$f_{xy} = 1$$

$$f_{yy} = \frac{2}{y^3}$$

At the critical point $(1, 1)$, $f_{xx} = 2 > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = 3 > 0$. Thus, $(1, 1, 3)$ is a relative minimum.



69. The level curves are hyperbolas. There is a critical point at $(0, 0)$, but there are no relative extrema. The gradient is normal to the level curve at any given point at (x_0, y_0) .

71. $P(x_1, x_2) = R - C_1 - C_2$

$$= [225 - 0.4(x_1 + x_2)](x_1 + x_2) - (0.05x_1^2 + 15x_1 + 5400) - (0.03x_2^2 + 15x_2 + 6100)$$

$$= -0.45x_1^2 - 0.43x_2^2 - 0.8x_1x_2 + 210x_1 + 210x_2 - 11,500$$

$$P_{x_1} = -0.9x_1 - 0.8x_2 + 210 = 0$$

$$0.9x_1 + 0.8x_2 = 210$$

$$P_{x_2} = -0.86x_2 - 0.8x_1 + 210 = 0$$

$$0.8x_1 + 0.86x_2 = 210$$

Solving this system yields $x_1 \approx 94$ and $x_2 \approx 157$.

$$P_{x_1x_1} = -0.9$$

$$P_{x_1x_2} = -0.8$$

$$P_{x_2x_2} = -0.86$$

$$P_{x_1x_1} < 0$$

$$P_{x_1x_1}P_{x_2x_2} - (P_{x_1x_2})^2 > 0$$

Therefore, profit is maximum when $x_1 \approx 94$ and $x_2 \approx 157$.

73. Maximize $f(x, y) = 4x + xy + 2y$ subject to the constraint $20x + 4y = 2000$.

$$\begin{cases} 4 + y = 20\lambda \\ x + 2 = 4\lambda \end{cases} \Rightarrow \begin{cases} 5x - y = -6 \end{cases}$$

$$20x + 4y = 2000 \Rightarrow 5x + y = 500$$

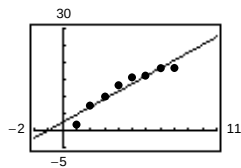
$$\begin{array}{r} 5x - y = -6 \\ 5x + y = 500 \\ \hline 10x = 494 \end{array}$$

$$x = 49.4$$

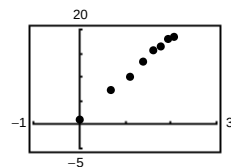
$$y = 253$$

$$f(49.4, 253) = 13,201.8$$

75. (a) $y = 2.29t + 2.34$



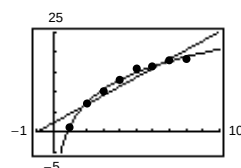
(b)



Yes, the data appears more linear.

- (c) $y = 8.37 \ln t + 1.54$

(d)



The logarithmic model is a better fit.

77. Optimize $f(x, y, z) = xy + yz + xz$ subject to the constraint $x + y + z = 1$.

$$\begin{cases} y + z = \lambda \\ x + z = \lambda \\ x + y = \lambda \end{cases} \Rightarrow x = y = z$$

$$x + y + z = 1 \Rightarrow x = y = z = \frac{1}{3}$$

$$\text{Maximum: } f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$$

79. $PQ = \sqrt{x^2 + 4}$, $QR = \sqrt{y^2 + 1}$, $RS = z$; $x + y + z = 10$

$$C = 3\sqrt{x^2 + 4} + 2\sqrt{y^2 + 1} + 2$$

$$\text{Constraint: } x + y + z = 10$$

$$\nabla C = \lambda \nabla g$$

$$\frac{3x}{\sqrt{x^2 + 4}} \mathbf{i} + \frac{2y}{\sqrt{y^2 + 1}} \mathbf{j} + \mathbf{k} = \lambda [\mathbf{i} + \mathbf{j} + \mathbf{k}]$$

$$3x = \lambda \sqrt{x^2 + 4}$$

$$2y = \lambda \sqrt{y^2 + 1}$$

$$1 = \lambda$$

$$9x^2 = x^2 + 4 \Rightarrow x^2 = \frac{1}{2}$$

$$4y^2 = y^2 + 1 \Rightarrow y^2 = \frac{1}{3}$$

$$\text{Hence, } x = \frac{\sqrt{2}}{2}, y = \frac{\sqrt{3}}{3}, z = 10 - \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{3} \approx 8.716 \text{ m.}$$

Problem Solving for Chapter 12

1. (a) The three sides have lengths 5, 6, and 5.

$$\text{Thus, } s = \frac{16}{2} = 8 \text{ and } A = \sqrt{8(3)(2)(3)} = 12$$

- (b) Let $f(a, b, c) = (\text{area})^2 = s(s-a)(s-b)(s-c)$, subject to the constraint $a + b + c = \text{constant}$ (perimeter).

Using Lagrange multipliers,

$$-s(s-b)(s-c) = \lambda$$

$$-s(s-a)(s-c) = \lambda$$

$$-s(s-b)(s-b) = \lambda$$

From the first 2 equations $s-b = s-a \Rightarrow a = b$.

Similarly, $b = c$ and hence $a = b = c$ which is an equilateral triangle.

- (c) Let $f(a, b, c) = a + b + c$, subject to $(\text{Area})^2 = s(s-a)(s-b)(s-c)$ constant.

Using Lagrange multipliers,

$$1 = -\lambda s(s-b)(s-c)$$

$$1 = -\lambda s(s-a)(s-c)$$

$$1 = -\lambda s(s-a)(s-b)$$

Hence, $s-a = s-b \Rightarrow a = b$ and $a = b = c$.

3. (a) $F(x, y, z) = xyz - 1 = 0$

$$F_x = yz, F_y = xz, F_z = xy$$

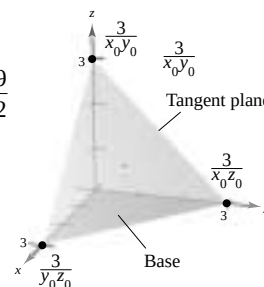
Tangent plane:

$$y_0 z_0 (x - x_0) + x_0 z_0 (y - y_0) + x_0 y_0 (z - z_0) = 0$$

$$y_0 z_0 x + x_0 z_0 y + x_0 y_0 z = 3x_0 y_0 z_0 = 3$$

$$(b) V = \frac{1}{3}(\text{base})(\text{height})$$

$$= \frac{1}{3} \left(\frac{1}{2} \frac{3}{y_0 z_0} \frac{3}{x_0 z_0} \right) \left(\frac{3}{x_0 y_0} \right) = \frac{9}{2}$$



5. We cannot use Theorem 12.9 since f is not a differentiable function of x and y . Hence, we use the definition of directional derivatives.

$$D_{\mathbf{u}} f(x, y) = \lim_{t \rightarrow 0} \frac{f(x + t \cos \theta, y + t \sin \theta) - f(x, y)}{t}$$

$$D_{\mathbf{u}} f(0, 0) = \lim_{t \rightarrow 0} \frac{f\left[0 + \left(\frac{t}{\sqrt{2}}\right), 0 + \left(\frac{t}{\sqrt{2}}\right)\right] - f(0, 0)}{t}$$

$$= \lim_{t \rightarrow 0} \frac{1}{t} \left[\frac{4 \left(\frac{t}{\sqrt{2}} \right) \left(\frac{t}{\sqrt{2}} \right)}{\left(\frac{t^2}{2} \right) + \left(\frac{t^2}{2} \right)} \right] = \lim_{t \rightarrow 0} \frac{1}{t} \left[\frac{2t^2}{t^2} \right] = \lim_{t \rightarrow 0} \frac{2}{t} \text{ which does not exist.}$$

If $f(0, 0) = 2$, then

$$D_{\mathbf{u}} f(0, 0) = \lim_{t \rightarrow 0} \frac{f\left(0 + \frac{t}{\sqrt{2}}, 0 + \frac{t}{\sqrt{2}}\right) - 2}{t} = \lim_{t \rightarrow 0} \frac{1}{t} \left[\frac{2t^2}{t^2} - 2 \right] = 0$$

which implies that the directional derivative exists.

7. $H = k(5xy + 6xz + 6yz)$

$$z = \frac{1000}{xy} \Rightarrow H = k\left(5xy + \frac{6000}{y} + \frac{6000}{x}\right).$$

$$H_x = 5y - \frac{6000}{x^2} = 0 \Rightarrow 5yx^2 = 6000$$

By symmetry, $x = y \Rightarrow x^3 = y^3 = 1200$.

Thus, $x = y = 2\sqrt[3]{150}$ and $z = \frac{5}{3}\sqrt[3]{150}$.

9. (a) $\frac{\partial f}{\partial x} = Cax^{a-1}y^{1-a}, \frac{\partial f}{\partial y} = C(1-a)x^ay^{-a}$

$$\begin{aligned} x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} &= Cax^ay^{1-a} + C(1-a)x^ay^{1-a} \\ &= [Ca + C(1-a)]x^ay^{1-a} \\ &= Cx^ay^{1-a} = f \end{aligned}$$

(b) $f(tx, ty) = C(tx)^a(ty)^{1-a} = Ct^ax^at^{1-a}y^{1-a}$
 $= Cx^ay^{1-a}(t) = tf(x, y)$

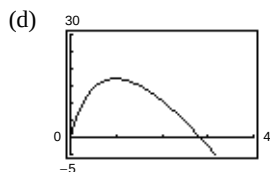
11. (a) $x = 64(\cos 45^\circ)t = 32\sqrt{2}t$

$$y = 64(\sin 45^\circ)t - 16t^2 = 32\sqrt{2}t - 16t^2$$

(b) $\tan \alpha = \frac{y}{x + 50}$

$$\alpha = \arctan\left(\frac{y}{x + 50}\right) = \arctan\left(\frac{32\sqrt{2}t - 16t^2}{32\sqrt{2}t + 50}\right)$$

(c) $\frac{d\alpha}{dt} = \frac{1}{1 + \left(\frac{32\sqrt{2}t - 16t^2}{32\sqrt{2}t + 50}\right)^2} \cdot \frac{-64(8\sqrt{2}t^2 + 25t - 25\sqrt{2})}{(32\sqrt{2}t + 50)^2} = \frac{-16(8\sqrt{2}t^2 + 25t - 25\sqrt{2})}{64t^4 - 256\sqrt{2}t^3 + 1024t^2 + 800\sqrt{2}t + 625}$



No. The rate of change of α is greatest when the projectile is closest to the camera.

(e) $\frac{d\alpha}{dt} = 0$ when

$$8\sqrt{2}t^2 + 25t - 25\sqrt{2} = 0$$

$$t = \frac{-25 + \sqrt{25^2 - 4(8\sqrt{2})(-25\sqrt{2})}}{2(8\sqrt{2})} \approx 0.98 \text{ second.}$$

No, the projectile is at its maximum height when $dy/dt = 32\sqrt{2} - 32t = 0$ or $t = \sqrt{2} \approx 1.41$ seconds.

13. (a) There is a minimum at $(0, 0, 0)$, maxima at $(0, \pm 1, 2/e)$ and saddle point at $(\pm 1, 0, 1/e)$:

$$f_x = (x^2 + 2y^2)e^{-(x^2+y^2)}(-2x) + (2x)e^{-(x^2+y^2)}$$

$$= e^{-(x^2+y^2)}[(x^2 + 2y^2)(-2x) + 2x]$$

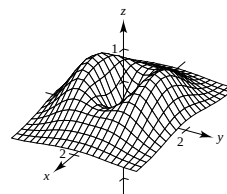
$$= e^{-(x^2+y^2)}[-2x^3 + 4xy^2 + 2x] = 0 \Rightarrow x^3 + 2xy^2 - x = 0$$

$$f_y = (x^2 + 2y^2)e^{-(x^2+y^2)}(-2y) + (4y)e^{-(x^2+y^2)}$$

$$= e^{-(x^2+y^2)}[(x^2 + 2y^2)(-2y) + 4y]$$

$$= e^{-(x^2+y^2)}[-4y^3 - 2x^2y + 4y] = 0 \Rightarrow 2y^3 + x^2y - 2y = 0$$

Solving the two equations $x^3 + 2xy^2 - x = 0$ and $2y^3 + x^2y - 2y = 0$, you obtain the following critical points: $(0, \pm 1)$, $(\pm 1, 0)$, $(0, 0)$. Using the second derivative test, you obtain the results above.



—CONTINUED—

13. —CONTINUED—

(b) As in part (a), you obtain

$$f_x = e^{-(x^2+y^2)}[2x(x^2 - 1 - 2y^2)]$$

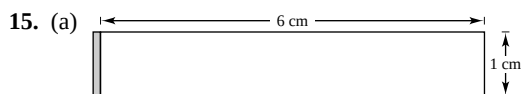
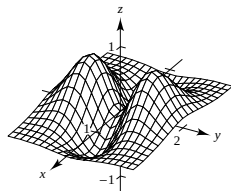
$$f_y = e^{-(x^2+y^2)}[2y(2 + x^2 - 2y^2)]$$

 The critical numbers are $(0, 0)$, $(0, \pm 1)$, $(\pm 1, 0)$.

These yield

 $(\pm 1, 0, -1/e)$ minima

 $(0, \pm 1, 2/e)$ maxima

 $(0, 0, 0)$ saddle


(c) The height has more effect since the shaded region in (b) is larger than the shaded region in (a).

 (c) In general, for $\alpha > 0$ you obtain

 $(0, 0, 0)$ minimum

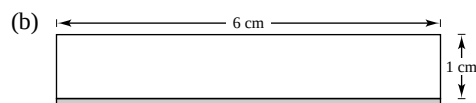
 $(0, \pm 1, \beta/e)$ maxima

 $(\pm 1, 0, \alpha/e)$ saddle

 For $\alpha < 0$, you obtain

 $(\pm 1, 0, \alpha/e)$ minima

 $(0, \pm 1, \beta/e)$ maxima

 $(0, 0, 0)$ saddle

 (d) $A = hl \Rightarrow dA = l dh + h dl$

 If $dl = 0.01$ and $dh = 0$, then $dA = 1(0.01) = 0.01$.

 If $dh = 0.01$ and $dl = 0$, then $dA = 6(0.01) = 0.06$.

17. Essay

19. $u(x, t) = \frac{1}{2}[f(x - ct) + f(x + ct)]$

 Let $r = x - ct$ and $s = x + ct$. Then $u(r, s) = \frac{1}{2}[f(r) + f(s)]$.

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial t} = \frac{1}{2} \frac{df}{dr}(-c) + \frac{1}{2} \frac{df}{ds}(c)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2} \frac{d^2 f}{dr^2}(-c)^2 + \frac{1}{2} \frac{d^2 f}{ds^2}(c)^2 = \frac{c^2}{2} \left[\frac{d^2 f}{dr^2} + \frac{d^2 f}{ds^2} \right]$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} = \frac{1}{2} \frac{df}{dr}(1) + \frac{1}{2} \frac{df}{ds}(1)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2} \frac{d^2 f}{dr^2}(1)^2 + \frac{1}{2} \frac{d^2 f}{ds^2}(1)^2 = \frac{1}{2} \left[\frac{d^2 f}{dr^2} + \frac{d^2 f}{ds^2} \right]$$

 Thus, $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$.

44. Minimize $C(x, y) = 48x + 36y$ subject to the constraint $100x^{0.6}y^{0.4} = 20,000$.

$$48 = 60x^{-0.4}y^{0.4}\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.4} = \frac{48}{60\lambda}$$

$$36 = 40x^{0.6}y^{-0.6}\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.6} = \frac{36}{40\lambda}$$

$$\left(\frac{y}{x}\right)^{0.4} \left(\frac{y}{x}\right)^{0.6} = \left(\frac{48}{60\lambda}\right) \left(\frac{40\lambda}{36}\right)$$

$$\frac{y}{x} = \frac{8}{9} \Rightarrow y = \frac{8}{9}x$$

$$100x^{0.6}y^{0.4} = 20,000 \Rightarrow x^{0.6} \left(\frac{8}{9}x\right)^{0.4} = 200$$

$$x = \frac{200}{(8/9)^{0.4}} \approx 209.65$$

$$y = \frac{8}{9} \left[\frac{200}{(8/9)^{0.4}} \right] \approx 186.35$$

Therefore, $C(209.65, 186.35) = \$16,771.94$.

46. $f(x, y) = ax + by$, $x, y > 0$

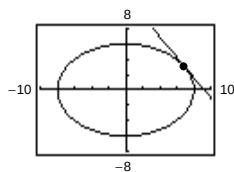
Constraint: $\frac{x^2}{64} + \frac{y^2}{36} = 1$

- (a) Level curves of $f(x, y) = 4x + 3y$ are lines of form

$$y = -\frac{4}{3}x + C.$$

Using $y = -\frac{4}{3}x + 12.3$, you obtain

$$x \approx 7, y \approx 3, \text{ and } f(7, 3) = 28 + 9 = 37.$$



Constraint is an ellipse.

- (b) Level curves of $f(x, y) = 4x + 9y$ are lines of form

$$y = -\frac{4}{9}x + C.$$

Using $y = -\frac{4}{9}x + 7$, you obtain

$$x \approx 4, y \approx 5.2, \text{ and } f(4, 5.2) = 62.8.$$

Review Exercises for Chapter 12

2. Yes, it is the graph of a function.

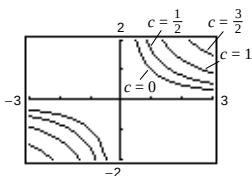
4. $f(x, y) = \ln xy$

The level curves are of the form

$$c = \ln xy$$

$$e^c = xy.$$

The level curves are hyperbolas.



6. $f(x, y) = \frac{x}{x+y}$

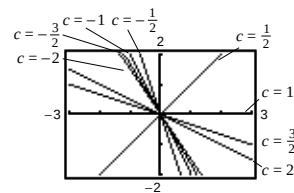
The level curves are of the form

$$c = \frac{x}{x+y}$$

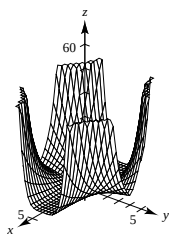
$$y = \left(\frac{1-c}{c}\right)x.$$

The level curves are passing through the origin with slope

$$\frac{1-c}{c}.$$

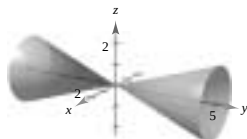


8. $g(x, y) = |y|^{1+|x|}$



10. $f(x, y, z) = 9x^2 - y^2 + 9z^2 = 0$

Elliptic cone



12. $\lim_{(x,y) \rightarrow (1,1)} \frac{xy}{x^2 - y^2}$

Does not exist

Continuous except when $y = \pm x$.

14. $\lim_{(x,y) \rightarrow (0,0)} \frac{y + xe^{-y^2}}{1 + x^2} = \frac{0+0}{1+0} = 0$

Continuous everywhere

16. $f(x, y) = \frac{xy}{x + y}$

$$f_x = \frac{y(x+y) - xy}{(x+y)^2} = \frac{y^2}{(x+y)^2}$$

$$f_y = \frac{x^2}{(x+y)^2}$$

18. $z = \ln(x^2 + y^2 + 1)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2 + 1}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2 + 1}$$

20. $w = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial w}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

22. $f(x, y, z) = \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

$$f_x = -\frac{1}{2}(1 - x^2 - y^2 - z^2)^{-3/2}(-2x)$$

$$= \frac{x}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$f_y = \frac{y}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$f_z = \frac{z}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

24. $u(x, t) = c(\sin akx) \cos kt$

$$\frac{\partial u}{\partial x} = akc(\cos akx) \cos kt$$

$$\frac{\partial u}{\partial t} = -kc(\sin akx) \sin kt$$

26. $z = x^2 \ln(y + 1)$

$$\frac{\partial z}{\partial x} = 2x \ln(y + 1). \text{ At } (2, 0, 0), \frac{\partial z}{\partial x} = 0.$$

Slope in x -direction.

$$\frac{\partial z}{\partial y} = \frac{x^2}{1 + y}. \text{ At } (2, 0, 0), \frac{\partial z}{\partial y} = 4.$$

Slope in y -direction.

28. $h(x, y) = \frac{x}{x + y}$

$$h_x = \frac{y}{(x + y)^2}$$

$$h_y = \frac{-x}{(x + y)^2}$$

$$h_{xx} = \frac{-2y}{(x + y)^3}$$

$$h_{yy} = \frac{2x}{(x + y)^3}$$

$$h_{xy} = \frac{(x + y)^2 - 2y(x + y)}{(x + y)^4} = \frac{x - y}{(x + y)^3}$$

$$h_{yx} = \frac{-(x + y)^2 + 2y(x + y)}{(x + y)^4} = \frac{x - y}{(x + y)^3}$$

30. $g(x, y) = \cos(x - 2y)$

$$g_x = -\sin(x - 2y)$$

$$g_y = 2 \sin(x - 2y)$$

$$g_{xx} = -\cos(x - 2y)$$

$$g_{yy} = -4 \cos(x - 2y)$$

$$g_{xy} = 2 \cos(x - 2y)$$

$$g_{yx} = 2 \cos(x - 2y)$$

32. $z = x^3 - 3xy^2$

$$\frac{\partial z}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

$$\frac{\partial z}{\partial y} = -6xy$$

$$\frac{\partial^2 z}{\partial y^2} = -6x$$

Therefore, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

34. $z = e^x \sin y$

$$\frac{\partial z}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \sin y$$

$$\frac{\partial z}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 z}{\partial y^2} = -e^x \sin y$$

Therefore, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

36. $z = \frac{xy}{\sqrt{x^2 + y^2}}$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= \left[\frac{\sqrt{x^2 + y^2}y - xy(x/\sqrt{x^2 + y^2})}{x^2 + y^2} \right] dx + \left[\frac{\sqrt{x^2 + y^2}x - xy(y/\sqrt{x^2 + y^2})}{x^2 + y^2} \right] dy = \frac{y^3}{(x^2 + y^2)^{3/2}} dx + \frac{x^3}{(x^2 + y^2)^{3/2}} dy$$

38. From the accompanying figure we observe

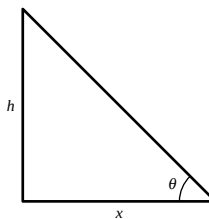
$$\tan \theta = \frac{h}{x} \text{ or } h = x \tan \theta$$

$$dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial \theta} d\theta = \tan \theta dx + x \sec^2 \theta d\theta.$$

Letting $x = 100$, $dx = \pm \frac{1}{2}$, $\theta = \frac{11\pi}{60}$, and $d\theta = \pm \frac{\pi}{180}$.

(Note that we express the measurement of the angle in radians.) The maximum error is approximately

$$dh = \tan\left(\frac{11\pi}{60}\right)\left(\pm \frac{1}{2}\right) + 100 \sec^2\left(\frac{11\pi}{60}\right)\left(\pm \frac{\pi}{180}\right) \approx \pm 0.3247 \pm 2.4814 \approx \pm 2.81 \text{ feet.}$$



40. $A = \pi r \sqrt{r^2 + h^2}$

$$dA = \left(\pi \sqrt{r^2 + h^2} + \frac{\pi r^2}{\sqrt{r^2 + h^2}} \right) dr + \frac{\pi r h}{\sqrt{r^2 + h^2}} dh$$

$$= \frac{\pi(2r^2 + h^2)}{\sqrt{r^2 + h^2}} dr + \frac{\pi r h}{\sqrt{r^2 + h^2}} dh = \frac{\pi(8 + 25)}{\sqrt{29}} \left(\pm \frac{1}{8} \right) + \frac{10\pi}{\sqrt{29}} \left(\pm \frac{1}{8} \right) = \pm \frac{43\pi}{8\sqrt{29}}$$

42. $u = y^2 - x$, $x = \cos t$, $y = \sin t$

$$\begin{aligned} \text{Chain Rule: } \frac{du}{dt} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} \\ &= -1(-\sin t) + 2y(\cos t) \\ &= \sin t + 2(\sin t) \cos t \\ &= \sin t(1 + 2 \cos t) \end{aligned}$$

Substitution: $u = \sin^2 t - \cos t$

$$\frac{du}{dt} = 2 \sin t \cos t + \sin t = \sin t(1 + 2 \cos t)$$

44. $w = \frac{xy}{z}, x = 2r + t, y = rt, z = 2r - t$

$$\begin{aligned}\text{Chain Rule: } \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r} \\ &= \frac{y}{z}(2) + \frac{x}{z}(t) - \frac{xy}{z^2}(2) \\ &= \frac{2rt}{2r-t} + \frac{(2r+t)t}{2r-t} - \frac{2(2r+t)(rt)}{(2r-t)^2} \\ &= \frac{4r^2t - 4rt^2 - t^3}{(2r-t)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t} \\ &= \frac{y}{z}(1) + \frac{x}{z}(r) - \frac{xy}{z^2}(-1) \\ &= \frac{4r^2t - rt^2 + 4r^3}{(2r-t)^2}\end{aligned}$$

Substitution: $w = \frac{xy}{z} = \frac{(2r+t)(rt)}{2r-t} = \frac{2r^2t + rt^2}{2r-t}$

$$\frac{\partial w}{\partial r} = \frac{4r^2t - 4rt^2 - t^3}{(2r-t)^2}$$

$$\frac{\partial w}{\partial t} = \frac{4r^2t - rt^2 - 4r^3}{(2r-t)^2}$$

48. $f(x, y) = \frac{1}{4}y^2 - x^2$

$$\nabla f = -2x\mathbf{i} + \frac{1}{2}y\mathbf{j}$$

$$\nabla f(1, 4) = -2\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{u} = \frac{1}{\sqrt{5}}\mathbf{v} = \frac{2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}$$

$$D_{\mathbf{u}}f(1, 4) = \nabla f(1, 4) \cdot \mathbf{u} = -\frac{4\sqrt{5}}{5} + \frac{2\sqrt{5}}{5} = -\frac{2\sqrt{5}}{5}$$

52. $z = \frac{x^2}{x-y}$

$$\nabla z = \frac{x^2 - 2xy}{(x-y)^2}\mathbf{i} + \frac{x^2}{(x-y)^2}\mathbf{j}$$

$$\nabla z(2, 1) = 4\mathbf{j}$$

$$\|\nabla z(2, 1)\| = 4$$

56. $4y \sin x - y^2 = 3$

$$f(x, y) = 4y \sin x - y^2$$

$$\nabla f(x, y) = 4y \cos x \mathbf{i} + (4 \sin x - 2y)\mathbf{j}$$

$$\nabla f\left(\frac{\pi}{2}, 1\right) = 2\mathbf{j}$$

Normal vector: $\mathbf{j}$

46. $xz^2 - y \sin z = 0$

$$2xz \frac{\partial z}{\partial x} + z^2 - y \cos z \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{z^2}{y \cos z - 2xz}$$

$$2xz \frac{\partial z}{\partial y} - y \cos z \frac{\partial z}{\partial y} - \sin z = 0$$

$$\frac{\partial z}{\partial y} = \frac{\sin z}{2xz - y \cos z}$$

50. $w = 6x^2 + 3xy - 4y^2z$

$$\nabla w = (12x + 3y)\mathbf{i} + (3x - 8yz)\mathbf{j} + (-4y^2)\mathbf{k}$$

$$\nabla w(1, 0, 1) = 12\mathbf{i} + 3\mathbf{j}$$

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{v} = \frac{\sqrt{3}}{3}\mathbf{i} + \frac{\sqrt{3}}{3}\mathbf{j} - \frac{\sqrt{3}}{3}\mathbf{k}$$

$$\begin{aligned}D_{\mathbf{u}}w(1, 0, 1) &= \nabla w(1, 0, 1) \cdot \mathbf{u} \\ &= 4\sqrt{3} + \sqrt{3} + 0 = 5\sqrt{3}\end{aligned}$$

54. $z = x^2y$

$$\nabla z = 2xy\mathbf{i} + x^2\mathbf{j}$$

$$\nabla z(2, 1) = 4\mathbf{i} + 4\mathbf{j}$$

$$\|\nabla z(2, 1)\| = 4\sqrt{2}$$

58. $F(x, y, z) = y^2 + z^2 - 25 = 0$

$$\nabla F = 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla F(2, 3, 4) = 6\mathbf{j} + 8\mathbf{k} = 2(3\mathbf{j} + 4\mathbf{k})$$

Therefore, the equation of the tangent plane is

$$3(y - 3) + 4(z - 4) = 0 \quad \text{or} \quad 3y + 4z = 25,$$

and the equation of the normal line is

$$x = 2, \frac{y-3}{3} = \frac{z-4}{4}.$$

60. $F(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$

$$\nabla F = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla F(1, 2, 2) = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} = 2(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$$

Therefore, the equation of the tangent plane is

$$(x - 1) + 2(y - 2) + 2(z - 2) = 0 \quad \text{or}$$

$$x + 2y + 2z = 9,$$

and the equation of the normal line is

$$\frac{x - 1}{1} = \frac{y - 2}{2} = \frac{z - 2}{2}.$$

62. $F(x, y, z) = y^2 + z - 25 = 0$

$$G(x, y, z) = x - y = 0$$

$$\nabla F = 2y\mathbf{i} + \mathbf{k}$$

$$\nabla G = \mathbf{i} - \mathbf{j}$$

$$\nabla F(4, 4, 9) = 8\mathbf{i} + \mathbf{k}$$

$$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \mathbf{i} + \mathbf{j} - 8\mathbf{k}$$

Therefore, the equation of the tangent line is

$$\frac{x - 4}{1} = \frac{y - 4}{1} = \frac{z - 9}{-8}.$$

64. (a) $f(x, y) = \cos x + \sin y, \quad f(0, 0) = 1$

$$f_x = -\sin x, \quad f_x(0, 0) = 0$$

$$f_y = \cos y, \quad f_y(0, 0) = 1$$

$$P_1(x, y) = 1 + y$$

(c) If $y = 0$, you obtain the 2nd degree Taylor polynomial for $\cos x$.

(b) $f_{xx} = -\cos x, \quad f_{xx}(0, 0) = -1$

$$f_{yy} = -\sin y, \quad f_{yy}(0, 0) = 0$$

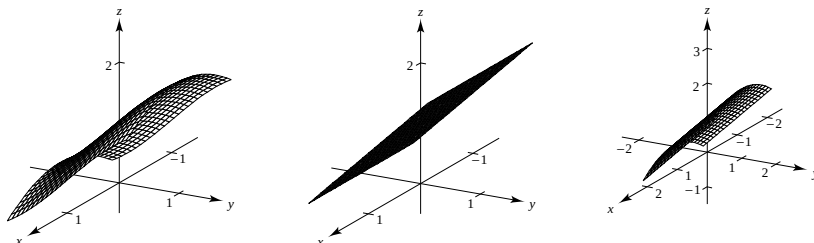
$$f_{xy} = 0, \quad f_{xy}(0, 0) = 0$$

$$P_2(x, y) = 1 + y - \frac{1}{2}x^2$$

(d)

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1.0	1.0	1.0
0	0.1	1.0998	1.1	1.1
0.2	0.1	1.0799	1.1	1.095
0.5	0.3	1.1731	1.3	1.175
1	0.5	1.0197	1.5	1.0

(e)



The accuracy lessens as the distance from $(0, 0)$ increases.

66. $f(x, y) = 2x^2 + 6xy + 9y^2 + 8x + 14$

$$f_x = 4x + 6y + 8 = 0$$

$$f_y = 6x + 18y = 0, \quad x = -3y$$

$$4(-3y) + 6y = -8 \Rightarrow y = \frac{4}{3}, \quad x = -4$$

$$f_{xx} = 4$$

$$f_{yy} = 18$$

$$f_{xy} = 6$$

$$f_{xx}f_{yy} - (f_{xy})^2 = 4(18) - (6)^2 = 36 > 0 \quad \text{Therefore, } \left(-4, \frac{4}{3}, -2\right) \text{ is a relative minimum.}$$

68. $z = 50(x + y) - (0.1x^3 + 20x + 150) - (0.05y^3 + 20.6y + 125)$

$$z_x = 50 - 0.3x^2 - 20 = 0, \quad x = \pm 10$$

$$z_y = 50 - 0.15y^2 - 20.6 = 0, \quad y = \pm 14$$

Critical Points: $(10, 14)$, $(10, -14)$, $(-10, 14)$, $(-10, -14)$

$$z_{xx} = -0.6x, \quad z_{yy} = -0.3y, \quad z_{xy} = 0$$

At $(10, 14)$, $z_{xx}z_{yy} - (z_{xy})^2 = (-6)(-4.2) - 0^2 > 0$, $z_{xx} < 0$.

$(10, 14, 199.4)$ is a relative maximum.

At $(10, -14)$, $z_{xx}z_{yy} - (z_{xy})^2 = (-6)(4.2) - 0^2 < 0$.

$(10, -14, -349.4)$ is a saddle point.

At $(-10, 14)$, $z_{xx}z_{yy} - (z_{xy})^2 = (6)(-4.2) - 0^2 < 0$.

$(-10, 14, -200.6)$ is a saddle point.

At $(-10, -14)$, $z_{xx}z_{yy} - (z_{xy})^2 = (6)(4.2) - 0^2 > 0$, $z_{xx} < 0$.

$(-10, -14, -749.4)$ is a relative minimum.

70. The level curves indicate that there is a relative extremum at A , the center of the ellipse in the second quadrant, and that there is a saddle point at B , the origin.

72. Minimize $C(x_1, x_2) = 0.25x_1^2 + 10x_1 + 0.15x_2^2 + 12x_2$ subject to the constraint $x_1 + x_2 = 1000$.

$$\begin{cases} 0.50x_1 + 10 = \lambda \\ 0.30x_2 + 12 = \lambda \end{cases} \quad \begin{cases} 5x_1 - 3x_2 = 20 \end{cases}$$

$$x_1 + x_2 = 1000 \Rightarrow 3x_1 + 3x_2 = 3000$$

$$\begin{array}{r} 5x_1 - 3x_2 = 20 \\ \hline 8x_1 = 3020 \end{array}$$

$$8x_1 = 3020$$

$$x_1 = 377.5$$

$$x_2 = 622.5$$

$$C(377.5, 622.5) = 104,997.50$$

74. Minimize the square of the distance:

$$f(x, y, z) = (x - 2)^2 + (y - 2)^2 + (x^2 + y^2 - 0)^2.$$

$$f_x = 2(x - 2) + 2(x^2 + y^2)2x = 0 \quad \left\{ \begin{array}{l} x - 2 + 2x^3 + 2xy^2 = 0 \end{array} \right.$$

$$f_y = 2(y - 2) + 2(x^2 + y^2)2y = 0 \quad \left\{ \begin{array}{l} y - 2 + 2y^3 + 2x^2y = 0 \end{array} \right.$$

Clearly $x = y$ and hence: $4x^3 + x - 2 = 0$. Using a computer algebra system, $x \approx 0.6894$.

Thus, $(\text{distance})^2 = (0.6894 - 2)^2 + (0.6894 - 2)^2 + [2(0.6894)^2]^2 \approx 4.3389$.

distance ≈ 2.08

76. (a) $(25, 28)$, $(50, 38)$, $(75, 54)$, $(100, 75)$, $(125, 102)$

$$\sum x_i = 375, \quad \sum y_i = 297, \quad \sum x_i^2 = 34,375, \quad \sum x_i^3 = 3,515,625$$

$$\sum x_i^4 = 382,421,875, \quad \sum x_i y_i = 26,900, \quad \sum x_i^2 y_i = 2,760,000$$

$$382,421,875a + 3,515,625b + 34,375c = 2,760,000$$

$$3,515,625a + 34,375b + 375c = 26,900$$

$$34,375a + 375b + 5c = 297$$

$$a \approx 0.0045, \quad b \approx 0.0717, \quad c \approx 23.2914, \quad y \approx 0.0045x^2 + 0.0717x + 23.2914$$

(b) When $x = 80$ km/hr, $y \approx 57.8$ km.

78. Optimize $f(x, y) = x^2y$ subject to the constraint $x + 2y = 2$.

$$\left. \begin{array}{l} 2xy = \lambda \\ x^2 = 2\lambda \end{array} \right\} x^2 = 4xy \Rightarrow x = 0 \text{ or } x = 4y$$

$$x + 2y = 2$$

If $x = 0$, $y = 1$. If $x = 4y$, then $y = \frac{1}{3}$, $x = \frac{4}{3}$.

$$\text{Maximum: } f\left(\frac{4}{3}, \frac{1}{3}\right) = \frac{16}{27}$$

$$\text{Minimum: } f(0, 1) = 0$$

Problem Solving for Chapter 12

$$2. V = \frac{4}{3}\pi r^3 + \pi r^2 h$$

$$\text{Material} = M = 4\pi r^2 + 2\pi r h$$

$$V = 1000 \Rightarrow h = \frac{1000 - (4/3)\pi r^3}{\pi r^2}$$

Hence,

$$M = 4\pi r^2 + 2\pi r \left(\frac{1000 - (4/3)\pi r^3}{\pi r^2} \right)$$

$$= 4\pi r^2 + \frac{2000}{r} - \frac{8}{3}\pi r^2$$

$$\frac{dM}{dr} = 8\pi r - \frac{2000}{r^2} - \frac{16}{3}\pi r = 0$$

$$8\pi r - \frac{16}{3}\pi r = \frac{2000}{r^2}$$

$$r^3 \left(\frac{8}{3}\pi \right) = 2000$$

$$r^3 = \frac{750}{\pi} \Rightarrow r = 5 \left(\frac{6}{\pi} \right)^{1/3}$$

$$\text{Then, } h = \frac{1000 - (4/3)\pi(750/\pi)}{\pi r^2} = 0.$$

$$\text{The tank is a sphere of radius } r = 5 \left(\frac{6}{\pi} \right)^{1/3}.$$

4. (a) As $x \rightarrow \pm\infty$, $f(x) = (x^3 - 1)^{1/3} \rightarrow x$ and hence

$$\lim_{x \rightarrow \infty} [f(x) - g(x)] = \lim_{x \rightarrow -\infty} [f(x) - g(x)] = 0.$$

(b) Let $(x_0, (x_0^3 - 1)^{1/3})$ be a point on the graph of f .

The line through this point perpendicular to g is

$$y = -x + x_0 + \sqrt[3]{x_0^3 - 1}.$$

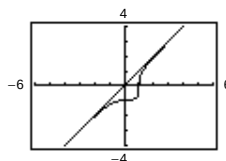
This line intersects g at the point

$$\left(\frac{1}{2}[x_0 + \sqrt[3]{x_0^3 - 1}], \frac{1}{2}[x_0 + \sqrt[3]{x_0^3 - 1}] \right).$$

The square of the distance between these two points is

$$h(x_0) = \frac{1}{2}(x_0 - \sqrt[3]{x_0^3 - 1})^2.$$

h is a maximum for $x_0 = \frac{1}{\sqrt[3]{2}}$. Hence, the point on f farthest from g is $\left(\frac{1}{\sqrt[3]{2}}, -\frac{1}{\sqrt[3]{2}} \right)$.



$$\begin{aligned} 6. \text{ Heat Loss } = H &= k(5xy + xy + 3xz + 3xz + 3yz + 3yz) \\ &= k(6xy + 6xz + 6yz) \end{aligned}$$

$$V = xyz = 1000 \Rightarrow z = \frac{1000}{xy}.$$

$$\text{Then } H = 6k\left(xy + \frac{1000}{y} + \frac{1000}{x}\right).$$

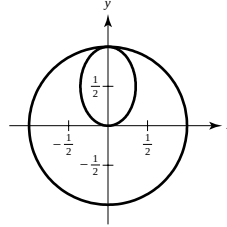
Setting $H_x = H_y = 0$, you obtain $x = y = z = 10$.

$$8. (a) T(x, y) = 2x^2 + y^2 - y + 10 = 10$$

$$2x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4}$$

$$2x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\frac{x^2}{1/8} + \frac{(y - (1/2))^2}{1/4} = 1 \quad \text{ellipse}$$



$$(b) \text{ On } x^2 + y^2 = 1, T(x, y) = T(y) = 2(1 - y^2) + y^2 - y + 10 = 12 - y^2 - y$$

$$T'(y) = -2y - 1 = 0 \Rightarrow y = -\frac{1}{2}, x = \pm\frac{\sqrt{3}}{2}.$$

$$\text{Inside: } T_x = 4x - 0, T_y = 2y - 1 = 0 \Rightarrow \left(0, \frac{1}{2}\right)$$

$$T\left(0, \frac{1}{2}\right) = \frac{39}{4} \quad \text{minimum}$$

$$T\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{49}{4} \quad \text{maximum}$$

$$10. x = r \cos \theta, y = r \sin \theta, z = z$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}(-r \sin \theta) + \frac{\partial u}{\partial y} r \cos \theta \quad \text{Similarly,}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta.$$

$$\frac{\partial^2 u}{\partial \theta^2} = (-r \sin \theta) \left[\frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial \theta} + \frac{\partial^2 u}{\partial x \partial z} \frac{\partial z}{\partial \theta} \right] - r \frac{\partial u}{\partial x} \cos \theta$$

$$+ (r \cos \theta) \left[\frac{\partial^2 u}{\partial y \partial x} \frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial \theta} + \frac{\partial^2 u}{\partial y \partial z} \frac{\partial z}{\partial \theta} \right] - r \frac{\partial u}{\partial y} \sin \theta$$

$$= \frac{\partial^2 u}{\partial x^2} r^2 \sin^2 \theta + \frac{\partial^2 u}{\partial y^2} r^2 \cos^2 \theta - 2 \frac{\partial^2 u}{\partial x \partial y} r^2 \sin \theta \cos \theta - \frac{\partial u}{\partial x} r \cos \theta - \frac{\partial u}{\partial y} r \sin \theta$$

$$\text{Similarly, } \frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 u}{\partial x^2} \cos^2 \theta + \frac{\partial^2 u}{\partial y^2} \sin^2 \theta + 2 \frac{\partial^2 u}{\partial x \partial y} \cos \theta \sin \theta.$$

Now observe that

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} &= \left[\frac{\partial^2 u}{\partial x^2} \cos^2 \theta + \frac{\partial^2 u}{\partial y^2} \sin^2 \theta + 2 \frac{\partial^2 u}{\partial x \partial y} \cos \theta \sin \theta \right] + \frac{1}{r} \left[\frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta \right] \\ &\quad + \left[\frac{\partial^2 u}{\partial x^2} \sin^2 \theta + \frac{\partial^2 u}{\partial y^2} \cos^2 \theta - 2 \frac{\partial^2 u}{\partial x \partial y} \sin \theta \cos \theta - \frac{1}{r} \frac{\partial u}{\partial x} \cos \theta - \frac{1}{r} \frac{\partial u}{\partial y} \sin \theta \right] + \frac{\partial^2 u}{\partial z^2} \\ &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}. \end{aligned}$$

Thus, Laplace's equation in cylindrical coordinates, is $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

$$\begin{aligned}
 12. (a) \quad d &= \sqrt{x^2 + y^2} = \sqrt{(32\sqrt{2}t)^2 + (32\sqrt{2}t - 16t^2)^2} \\
 &= \sqrt{4096t^2 - 1024\sqrt{2}t^3 + 256t^4} \\
 &= 16t\sqrt{t^2 - 4\sqrt{2}t + 16}
 \end{aligned}$$

$$(b) \quad \frac{dd}{dt} = \frac{32(t^2 - 3\sqrt{2}t + 8)}{\sqrt{t^2 - 4\sqrt{2}t + 16}}$$

(c) When $t = 2$:

$$\frac{dd}{dt} = \frac{32(12 - 6\sqrt{2})}{\sqrt{20 - 8\sqrt{2}}} \approx 38.16 \text{ ft/sec}$$

$$(d) \quad \frac{d^2d}{dt^2} = \frac{32(t^3 - 6\sqrt{2}t^2 + 36t - 32\sqrt{12})}{(t^2 - 4\sqrt{2}t + 16)^{3/2}} = 0$$

when $t \approx 1.943$ seconds. No. The projectile is at its maximum height when $t = \sqrt{2}$.

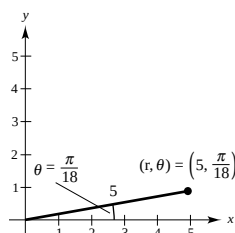
14. Given that f is a differentiable function such that $\nabla f(x_0, y_0) = \mathbf{0}$, then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Therefore, the tangent plane is $-(z - z_0) = 0$ or $z = z_0 = f(x_0, y_0)$ which is horizontal.

$$16. (r, \theta) = \left(5, \frac{\pi}{18}\right)$$

$$dr = \pm 0.05, d\theta = \pm 0.05$$

$$x = r \cos \theta = 5 \cos \frac{\pi}{18} \approx 4.924$$

$$y = r \sin \theta = 5 \sin \frac{\pi}{18} \approx 0.868$$



(a) dx should be more effected by changes in r .

$$\begin{aligned}
 dx &= (\cos \theta)dr + (-r \sin \theta)d\theta \\
 &\approx (0.985)dr - 0.868 d\theta
 \end{aligned}$$

dx is more effected by changes in r because $0.985 > 0.868$.

(b) dy should be more effected by changes in θ .

$$\begin{aligned}
 dy &= \sin \theta dr + r \cos \theta d\theta \\
 &\approx 0.174 dr + 4.924 d\theta
 \end{aligned}$$

dy is more effected by θ because $4.924 > 0.174$.

$$18. \quad \frac{\partial u}{\partial t} = \frac{1}{2}[-\cos(x - t) + \cos(x + t)]$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}[-\sin(x - t) - \sin(x + t)]$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}[\cos(x - t) + \cos(x + t)]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}[-\sin(x - t) - \sin(x + t)]$$

$$\text{Then, } \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

CHAPTER 13

Multiple Integration

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CHAPTER 13

Multiple Integration

Section 13.1 Iterated Integrals and Area in the Plane

Solutions to Odd-Numbered Exercises

$$1. \int_0^x (2x - y) dy = \left[2xy - \frac{1}{2}y^2 \right]_0^x = \frac{3}{2}x^2$$

$$3. \int_1^{2y} \frac{y}{x} dx = \left[y \ln x \right]_1^{2y} = y \ln 2y - 0 = y \ln 2y$$

$$5. \int_0^{\sqrt{4-x^2}} x^2 y dy = \left[\frac{1}{2} x^2 y^2 \right]_0^{\sqrt{4-x^2}} = \frac{4x^2 - x^4}{2}$$

$$7. \int_{e^y}^y \frac{y \ln x}{x} dx = \left[\frac{1}{2} y \ln^2 x \right]_{e^y}^y = \frac{1}{2} y [\ln^2 y - \ln^2 e^y] = \frac{y}{2} [(\ln y)^2 - y^2]$$

$$9. \int_0^{x^3} y e^{-y/x} dy = \left[-xy e^{-y/x} \right]_0^{x^3} + x \int_0^{x^3} e^{-y/x} dy = -x^4 e^{-x^2} - \left[x^2 e^{-y/x} \right]_0^{x^3} = x^2(1 - e^{-x^2} - x^2 e^{-x^2})$$

$u = y, du = dy, dv = e^{-y/x} dy, v = -x e^{-y/x}$

$$11. \int_0^1 \int_0^2 (x + y) dy dx = \int_0^1 \left[xy + \frac{1}{2} y^2 \right]_0^2 dx = \int_0^1 (2x + 2) dx = \left[x^2 + 2x \right]_0^1 = 3$$

$$13. \int_0^1 \int_0^x \sqrt{1-x^2} dy dx = \int_0^1 \left[y \sqrt{1-x^2} \right]_0^x dx = \int_0^1 x \sqrt{1-x^2} dx = \left[-\frac{1}{2} \left(\frac{2}{3} \right) (1-x^2)^{3/2} \right]_0^1 = \frac{1}{3}$$

$$15. \int_1^2 \int_0^4 (x^2 - 2y^2 + 1) dx dy = \int_1^2 \left[\frac{1}{3} x^3 - 2xy^2 + x \right]_0^4 dy$$
$$= \int_1^2 \left(\frac{64}{3} - 8y^2 + 4 \right) dy = \frac{4}{3} \int_1^2 (19 - 6y^2) dy = \left[\frac{4}{3} (19y - 2y^3) \right]_1^2 = \frac{20}{3}$$

$$17. \int_0^1 \int_0^{\sqrt{1-y^2}} (x + y) dx dy = \int_0^1 \left[\frac{1}{2} x^2 + xy \right]_0^{\sqrt{1-y^2}} dy$$
$$= \int_0^1 \left[\frac{1}{2} (1-y^2) + y \sqrt{1-y^2} \right] dy = \left[\frac{1}{2} y - \frac{1}{6} y^3 - \frac{1}{2} \left(\frac{2}{3} \right) (1-y^2)^{3/2} \right]_0^1 = \frac{2}{3}$$

$$19. \int_0^2 \int_0^{\sqrt{4-y^2}} \frac{2}{\sqrt{4-y^2}} dx dy = \int_0^2 \left[\frac{2x}{\sqrt{4-y^2}} \right]_0^{\sqrt{4-y^2}} dy = \int_0^2 2 dy = \left[2y \right]_0^2 = 4$$

$$21. \int_0^{\pi/2} \int_0^{\sin \theta} \theta r dr d\theta = \int_0^{\pi/2} \left[\theta \frac{r^2}{2} \right]_0^{\sin \theta} d\theta = \int_0^{\pi/2} \frac{1}{2} \theta \sin^2 \theta d\theta$$
$$= \frac{1}{4} \int_0^{\pi/2} (\theta - \theta \cos 2\theta) d\theta = \frac{1}{4} \left[\frac{\theta^2}{2} - \left(\frac{1}{4} \cos 2\theta + \frac{\theta}{2} \sin 2\theta \right) \right]_0^{\pi/2} = \frac{\pi^2}{32} + \frac{1}{8}$$

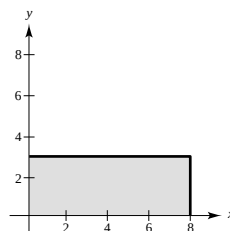
$$23. \int_1^\infty \int_0^{1/x} y \, dy \, dx = \int_1^\infty \left[\frac{y^2}{2} \right]_0^{1/x} dx = \frac{1}{2} \int_1^\infty \frac{1}{x^2} dx = \left[-\frac{1}{2x} \right]_1^\infty = 0 + \frac{1}{2} = \frac{1}{2}$$

$$25. \int_1^\infty \int_1^\infty \frac{1}{xy} dx \, dy = \int_1^\infty \left[\frac{1}{y} \ln x \right]_1^\infty dy = \int_1^\infty \left[\frac{1}{y}(\infty) - \frac{1}{y}(0) \right] dy$$

Diverges

$$27. A = \int_0^8 \int_0^3 dy \, dx = \int_0^8 \left[y \right]_0^3 dx = \int_0^8 3 \, dx = \left[3x \right]_0^8 = 24$$

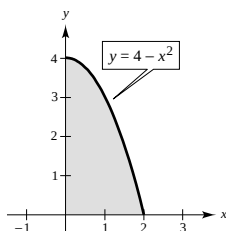
$$A = \int_0^3 \int_0^8 dx \, dy = \int_0^3 \left[x \right]_0^8 dy = \int_0^3 8 \, dy = \left[8y \right]_0^3 = 24$$



$$29. A = \int_0^2 \int_0^{4-x^2} dy \, dx = \int_0^2 \left[y \right]_0^{4-x^2} dx$$

$$= \int_0^2 (4 - x^2) dx$$

$$= \left[4x - \frac{x^3}{3} \right]_0^2 = \frac{16}{3}$$



$$A = \int_0^4 \int_0^{\sqrt{4-y}} dx \, dy$$

$$= \int_0^4 \left[x \right]_0^{\sqrt{4-y}} dy = \int_0^4 \sqrt{4-y} \, dy = -\int_0^4 (4-y)^{1/2} (-1) dy = \left[-\frac{2}{3}(4-y)^{3/2} \right]_0^4 = \frac{2}{3}(8) = \frac{16}{3}$$

$$31. A = \int_{-2}^1 \int_{x+2}^{4-x^2} dy \, dx$$

$$= \int_{-2}^1 \left[y \right]_{x+2}^{4-x^2} dx$$

$$= \int_{-2}^1 (4 - x^2 - x - 2) dx$$

$$= \int_{-2}^1 (2 - x - x^2) dx$$

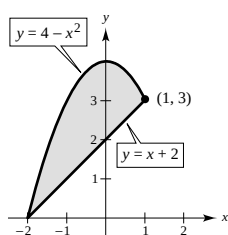
$$= \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_{-2}^1 = \frac{9}{2}$$

$$A = \int_0^3 \int_{-\sqrt{4-y}}^{y-2} dx \, dy + 2 \int_3^4 \int_0^{\sqrt{4-y}} dx \, dy$$

$$= \int_0^3 \left[x \right]_{-\sqrt{4-y}}^{y-2} dy + 2 \int_3^4 \left[x \right]_0^{\sqrt{4-y}} dy$$

$$= \int_0^3 (y - 2 + \sqrt{4-y}) dy + 2 \int_3^4 \sqrt{4-y} dy$$

$$= \left[\frac{1}{2}y^2 - 2y - \frac{2}{3}(4-y)^{3/2} \right]_0^3 - \left[\frac{4}{3}(4-y)^{3/2} \right]_3^4 = \frac{9}{2}$$



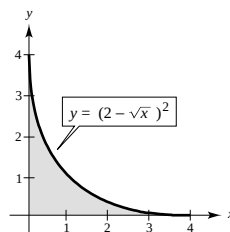
$$33. \int_0^4 \int_0^{(2-\sqrt{x})^2} dy \, dx = \int_0^4 \left[y \right]_0^{(2-\sqrt{x})^2} dx$$

$$= \int_0^4 (4 - 4\sqrt{x} + x) dx$$

$$= \left[4x - \frac{8}{3}x\sqrt{x} + \frac{x^2}{2} \right]_0^4 = \frac{8}{3}$$

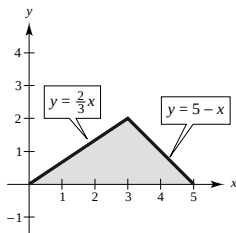
$$\int_0^4 \int_0^{(2-\sqrt{y})^2} dx \, dy = \frac{8}{3}$$

Integration steps are similar to those above.

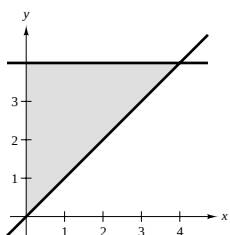


$$\begin{aligned}
 35. \quad A &= \int_0^3 \int_0^{2x/3} dy \, dx + \int_3^5 \int_0^{5-x} dy \, dx \\
 &= \int_0^3 \left[y \right]_0^{2x/3} dx + \int_3^5 \left[y \right]_0^{5-x} dx \\
 &= \int_0^3 \frac{2x}{3} dx + \int_3^5 (5-x) dx \\
 &= \left[\frac{1}{3} x^2 \right]_0^3 + \left[5x - \frac{1}{2} x^2 \right]_3^5 = 5
 \end{aligned}$$

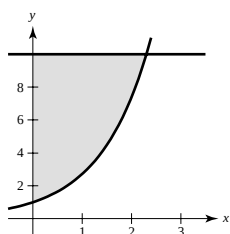
$$\begin{aligned}
 A &= \int_0^2 \int_{3y/2}^{5-y} dx \, dy \\
 &= \int_0^2 \left[x \right]_{3y/2}^{5-y} dy \\
 &= \int_0^2 \left(5 - y - \frac{3y}{2} \right) dy \\
 &= \int_0^2 \left(5 - \frac{5y}{2} \right) dy = \left[5y - \frac{5}{4} y^2 \right]_0^2 = 5
 \end{aligned}$$



$$\begin{aligned}
 39. \quad \int_0^4 \int_0^y f(x, y) \, dx \, dy, \quad 0 \leq x \leq y, \quad 0 \leq y \leq 4 \\
 = \int_0^4 \int_x^4 f(x, y) \, dy \, dx
 \end{aligned}$$



$$\begin{aligned}
 43. \quad \int_1^{10} \int_0^{\ln y} f(x, y) \, dx \, dy, \quad 0 \leq x \leq \ln y, \quad 1 \leq y \leq 10 \\
 = \int_0^{\ln 10} \int_{e^x}^{10} f(x, y) \, dy \, dx
 \end{aligned}$$

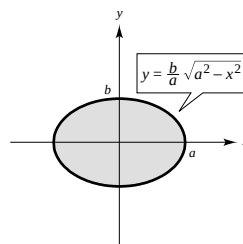


$$\begin{aligned}
 37. \quad \frac{A}{4} &= \int_0^a \int_0^{(b/a)\sqrt{a^2-x^2}} dy \, dx = \int_0^a \left[y \right]_0^{(b/a)\sqrt{a^2-x^2}} dx \\
 &= \frac{b}{a} \int_0^a \sqrt{a^2-x^2} \, dx = ab \int_0^{\pi/2} \cos^2 \theta \, d\theta \\
 (x &= a \sin \theta, \, dx = a \cos \theta \, d\theta) \\
 &= \frac{ab}{2} \int_0^{\pi/2} (1 + \cos 2\theta) \, d\theta = \left[\frac{ab}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) \right]_0^{\pi/2} \\
 &= \frac{\pi ab}{4}
 \end{aligned}$$

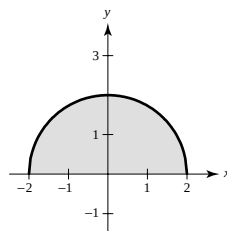
Therefore, $A = \pi ab$.

$$\frac{A}{4} = \int_0^b \int_0^{(a/b)\sqrt{b^2-y^2}} dx \, dy = \frac{\pi ab}{4}$$

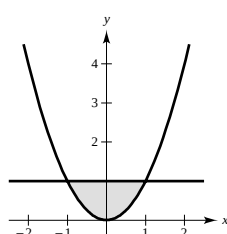
Therefore, $A = \pi ab$. Integration steps are similar to those above.



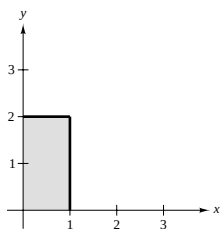
$$\begin{aligned}
 41. \quad \int_{-2}^2 \int_0^{\sqrt{4-x^2}} f(x, y) \, dy \, dx, \quad 0 \leq y \leq \sqrt{4-x^2}, \quad -2 \leq x \leq 2 \\
 = \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx \, dy
 \end{aligned}$$



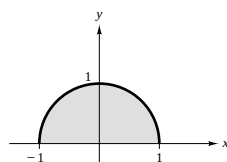
$$\begin{aligned}
 45. \quad \int_{-1}^1 \int_{x^2}^1 f(x, y) \, dy \, dx, \quad x^2 \leq y \leq 1, \quad -1 \leq x \leq 1 \\
 = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y) \, dx \, dy
 \end{aligned}$$



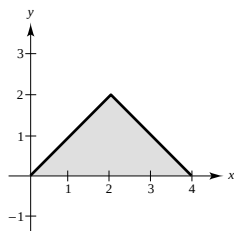
$$47. \int_0^1 \int_0^2 dy \, dx = \int_0^2 \int_0^1 dx \, dy = 2$$



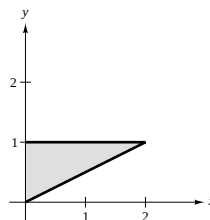
$$49. \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx \, dy = \int_{-1}^1 \int_0^{\sqrt{1-x^2}} dy \, dx = \frac{\pi}{2}$$



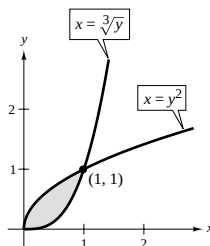
$$51. \int_0^2 \int_0^x dy \, dx + \int_2^4 \int_0^{4-x} dy \, dx = \int_0^2 \int_y^{4-y} dx \, dy = 4$$



$$53. \int_0^2 \int_{x/2}^1 dy \, dx = \int_0^1 \int_0^{2y} dx \, dy = 1$$



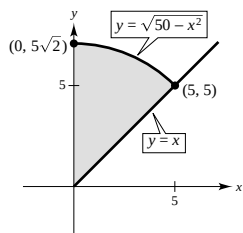
$$55. \int_0^1 \int_{y^2}^{\sqrt[3]{y}} dx \, dy = \int_0^1 \int_{x^3}^{\sqrt{x}} dy \, dx = \frac{5}{12}$$



57. The first integral arises using vertical representative rectangles. The second two integrals arise using horizontal representative rectangles.

$$\begin{aligned} \int_0^5 \int_x^{\sqrt{50-x^2}} x^2 y^2 \, dy \, dx &= \int_0^5 \left[\frac{1}{3} x^2 (50-x^2)^{3/2} - \frac{1}{3} x^5 \right] dx \\ &= \frac{15625}{24} \pi \end{aligned}$$

$$\begin{aligned} \int_0^5 \int_0^x x^2 y^2 \, dx \, dy + \int_5^{5\sqrt{2}} \int_0^{\sqrt{50-y^2}} x^2 y^2 \, dx \, dy &= \int_0^5 \frac{1}{3} y^5 \, dy + \int_5^{5\sqrt{2}} \frac{1}{3} (50-y^2)^{3/2} y^2 \, dy = \frac{15625}{18} + \left(\frac{15625}{18} \pi - \frac{15625}{18} \right) \\ &= \frac{15625}{24} \pi \end{aligned}$$



$$\begin{aligned}
 59. \int_0^2 \int_x^2 x \sqrt{1+y^3} \, dy \, dx &= \int_0^2 \int_0^y x \sqrt{1+y^3} \, dx \, dy = \int_0^2 \left[\sqrt{1+y^3} \cdot \frac{x^2}{2} \right]_0^y dy \\
 &= \frac{1}{2} \int_0^2 \sqrt{1+y^3} y^2 \, dy = \left[\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{3} (1+y^3)^{3/2} \right]_0^2 = \frac{1}{9} (27) - \frac{1}{9} (1) = \frac{26}{9}
 \end{aligned}$$

$$\begin{aligned}
 61. \int_0^1 \int_y^1 \sin(x^2) \, dx \, dy &= \int_0^1 \int_0^x \sin(x^2) \, dy \, dx = \int_0^1 \left[y \sin(x^2) \right]_0^x dx \\
 &= \int_0^1 x \sin(x^2) \, dx = \left[-\frac{1}{2} \cos(x^2) \right]_0^1 = -\frac{1}{2} \cos 1 + \frac{1}{2} (1) = \frac{1}{2} (1 - \cos 1) \approx 0.2298
 \end{aligned}$$

$$63. \int_0^2 \int_{x^2}^{2x} (x^3 + 3y^2) \, dy \, dx = \frac{1664}{105} \approx 15.848$$

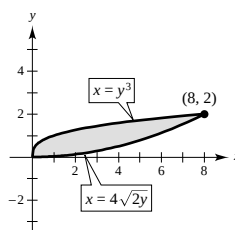
$$65. \int_0^4 \int_0^y \frac{2}{(x+1)(y+1)} \, dx \, dy = (\ln 5)^2 \approx 2.590$$

$$67. (a) x = y^3 \Leftrightarrow y = x^{1/3}$$

$$x = 4\sqrt{2y} \Leftrightarrow x^2 = 32y \Leftrightarrow y = \frac{x^2}{32}$$

$$(b) \int_0^8 \int_{x^2/32}^{x^{1/3}} (x^2 y - xy^2) \, dy \, dx$$

$$(c) \text{ Both integrals equal } 67520/693 \approx 97.43$$



$$69. \int_0^2 \int_0^{4-x^2} e^{xy} \, dy \, dx \approx 20.5648$$

$$71. \int_0^{2\pi} \int_0^{1+\cos \theta} 6r^2 \cos \theta \, dr \, d\theta = \frac{15\pi}{2}$$

73. An iterated integral is a double integral of a function of two variables. First integrate with respect to one variable while holding the other variable constant. Then integrate with respect to the second variable.

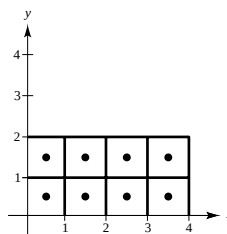
75. The region is a rectangle.

77. True

Section 13.2 Double Integrals and Volume

For Exercise 1–3, $\Delta x_i = \Delta y_i = 1$ and the midpoints of the squares are

$$\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{2}, \frac{1}{2}\right), \left(\frac{5}{2}, \frac{1}{2}\right), \left(\frac{7}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{3}{2}\right), \left(\frac{5}{2}, \frac{3}{2}\right), \left(\frac{7}{2}, \frac{3}{2}\right).$$



$$1. f(x, y) = x + y$$

$$\sum_{i=1}^8 f(x_i, y_i) \Delta x_i \Delta y_i = 1 + 2 + 3 + 4 + 2 + 3 + 4 + 5 = 24$$

$$\int_0^4 \int_0^2 (x + y) \, dy \, dx = \int_0^4 \left[xy + \frac{y^2}{2} \right]_0^2 dx = \int_0^4 (2x + 2) \, dx = \left[x^2 + 2x \right]_0^4 = 24$$

3. $f(x, y) = x^2 + y^2$

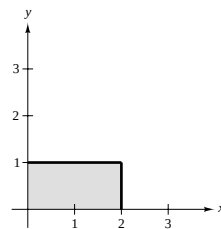
$$\sum_{i=1}^8 f(x_i, y_i) \Delta x_i \Delta y_i = \frac{2}{4} + \frac{10}{4} + \frac{26}{4} + \frac{50}{4} + \frac{10}{4} + \frac{18}{4} + \frac{34}{4} + \frac{58}{4} = 52$$

$$\int_0^4 \int_0^2 (x^2 + y^2) dy dx = \int_0^4 \left[x^2 y + \frac{y^3}{3} \right]_0^2 dx = \int_0^4 \left(2x^2 + \frac{8}{3} \right) dx = \left[\frac{2x^3}{3} + \frac{8x}{3} \right]_0^4 = \frac{160}{3}$$

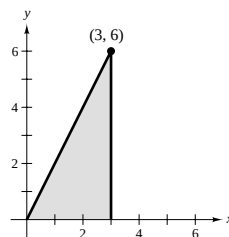
5. $\int_0^4 \int_0^4 f(x, y) dy dx \approx (32 + 31 + 28 + 23) + (31 + 30 + 27 + 22) + (28 + 27 + 24 + 19) + (23 + 22 + 19 + 14)$
 $= 400$

Using the corner of the i th square furthest from the origin, you obtain 272.

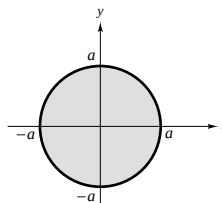
7. $\int_0^2 \int_0^1 (1 + 2x + 2y) dy dx = \int_0^2 \left[y + 2xy + y^2 \right]_0^1 dx$
 $= \int_0^2 (2 + 2x) dx$
 $= \left[2x + x^2 \right]_0^2$
 $= 8$



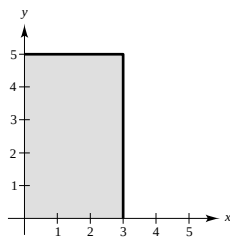
9. $\int_0^6 \int_{y/2}^3 (x + y) dx dy = \int_0^6 \left[\frac{1}{2}x^2 + xy \right]_{y/2}^3 dy$
 $= \int_0^6 \left(\frac{9}{2} + 3y - \frac{5}{8}y^2 \right) dy$
 $= \left[\frac{9}{2}y + \frac{3}{2}y^2 - \frac{5}{24}y^3 \right]_0^6$
 $= 36$



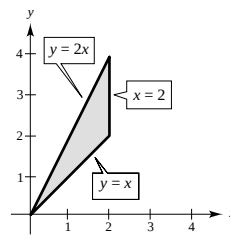
11. $\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x + y) dy dx = \int_{-a}^a \left[xy + \frac{1}{2}y^2 \right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$
 $= \int_{-a}^a 2x\sqrt{a^2-x^2} dx$
 $= \left[-\frac{2}{3}(a^2-x^2)^{3/2} \right]_{-a}^a = 0$



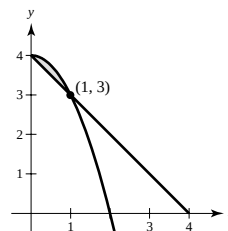
13. $\int_0^5 \int_0^3 xy dx dy = \int_0^5 \int_0^3 xy dy dx$
 $= \int_0^5 \left[\frac{1}{2}xy^2 \right]_0^3 dx$
 $= \frac{25}{2} \int_0^3 x dx$
 $= \left[\frac{25}{4}x^2 \right]_0^3 = \frac{225}{4}$



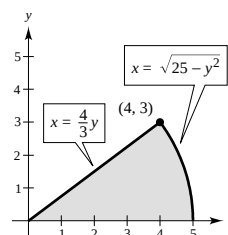
$$\begin{aligned}
 15. \quad \int_0^2 \int_{y/2}^y \frac{y}{x^2 + y^2} dx dy + \int_2^4 \int_{y/2}^2 \frac{y}{x^2 + y^2} dx dy &= \int_0^2 \int_x^{2x} \frac{y}{x^2 + y^2} dy dx \\
 &= \frac{1}{2} \int_0^2 \left[\ln(x^2 + y^2) \right]_x^{2x} dx \\
 &= \frac{1}{2} \int_0^2 (\ln 5x^2 - \ln 2x^2) dx \\
 &= \frac{1}{2} \ln \frac{5}{2} \int_0^2 dx \\
 &= \left[\frac{1}{2} \left(\ln \frac{5}{2} \right) x \right]_0^2 = \ln \frac{5}{2}
 \end{aligned}$$



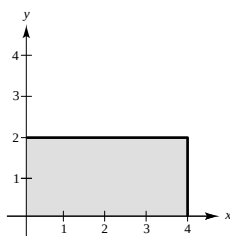
$$\begin{aligned}
 17. \quad \int_3^4 \int_{4-y}^{\sqrt{4-y}} -2y \ln x dx dy &= \int_0^1 \int_{4-x}^{4-x^2} -2y \ln x dy dx \\
 &= - \int_0^1 \left[\ln x \cdot y^2 \right]_{4-x}^{4-x^2} dx \\
 &= - \int_0^1 [\ln x (4 - x^2)^2 - (4 - x)^2] dx \\
 &= \frac{26}{25}
 \end{aligned}$$



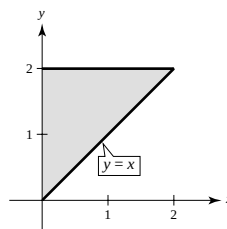
$$\begin{aligned}
 19. \quad \int_0^4 \int_0^{3x/4} x dy dx + \int_4^5 \int_0^{\sqrt{25-x^2}} x dy dx &= \int_0^3 \int_{4y/3}^{\sqrt{25-y^2}} x dx dy \\
 &= \int_0^3 \left[\frac{1}{2} x^2 \right]_{4y/3}^{\sqrt{25-y^2}} dy \\
 &= \frac{25}{18} \int_0^3 (9 - y^2) dy \\
 &= \left[\frac{25}{18} \left(9y - \frac{1}{3} y^3 \right) \right]_0^3 = 25
 \end{aligned}$$



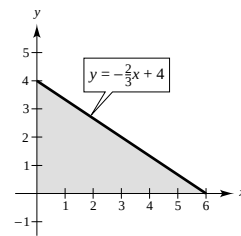
$$\begin{aligned}
 21. \quad \int_0^4 \int_0^2 \frac{y}{2} dy dx &= \int_0^4 \left[\frac{y^2}{4} \right]_0^2 dx \\
 &= \int_0^4 dx = 4
 \end{aligned}$$



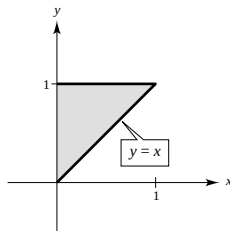
$$\begin{aligned}
 23. \quad \int_0^2 \int_0^y (4 - x - y) dx dy &= \int_0^2 \left[4x - \frac{x^2}{2} - xy \right]_0^y dy \\
 &= \int_0^2 \left(4y - \frac{y^2}{2} - y^2 \right) dy \\
 &= \left[2y^2 - \frac{y^3}{6} - \frac{y^3}{3} \right]_0^2 \\
 &= 8 - \frac{8}{6} - \frac{8}{3} = 4
 \end{aligned}$$



$$\begin{aligned}
 25. \int_0^6 \int_0^{(-2/3)x+4} \left(\frac{12-2x-3y}{4} \right) dy dx &= \int_0^6 \left[\frac{1}{4} \left(12y - 2xy - \frac{3}{2}y^2 \right) \right]_0^{(-2/3)x+4} dx \\
 &= \int_0^6 \left(\frac{1}{6}x^2 - 2x + 6 \right) dx \\
 &= \left[\frac{1}{18}x^3 - x^2 + 6x \right]_0^6 \\
 &= 12
 \end{aligned}$$



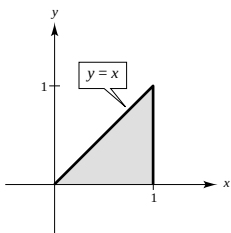
$$\begin{aligned}
 27. \int_0^1 \int_0^y (1-xy) dx dy &= \int_0^1 \left[x - \frac{x^2y}{2} \right]_0^y dy \\
 &= \int_0^1 \left(y - \frac{y^3}{2} \right) dy \\
 &= \left[\frac{y^2}{2} - \frac{y^4}{8} \right]_0^1 \\
 &= \frac{3}{8}
 \end{aligned}$$



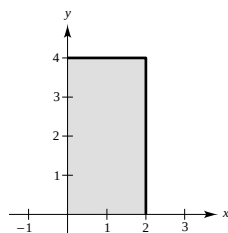
$$29. \int_0^\infty \int_0^\infty \frac{1}{(x+1)^2(y+1)^2} dy dx = \int_0^\infty \left[-\frac{1}{(x+1)^2(y+1)} \right]_0^\infty dx = \int_0^\infty \frac{1}{(x+1)^2} dx = \left[-\frac{1}{x+1} \right]_0^\infty = 1$$

$$31. 4 \int_0^2 \int_0^{\sqrt{4-x^2}} (4-x^2-y^2) dy dx = 8\pi$$

$$\begin{aligned}
 33. V &= \int_0^1 \int_0^x xy dy dx \\
 &= \int_0^1 \left[\frac{1}{2}xy^2 \right]_0^x dx = \frac{1}{2} \int_0^1 x^3 dx \\
 &= \left[\frac{1}{8}x^4 \right]_0^1 = \frac{1}{8}
 \end{aligned}$$

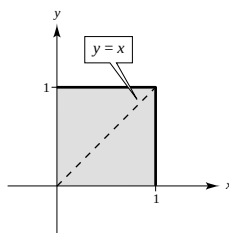


$$\begin{aligned}
 35. V &= \int_0^2 \int_0^4 x^2 dy dx \\
 &= \int_0^2 \left[x^2y \right]_0^4 dx = \int_0^2 4x^2 dx \\
 &= \left[\frac{4x^3}{3} \right]_0^2 = \frac{32}{3}
 \end{aligned}$$

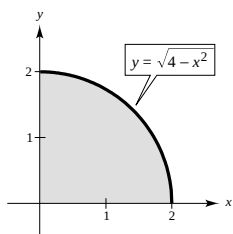


37. Divide the solid into two equal parts.

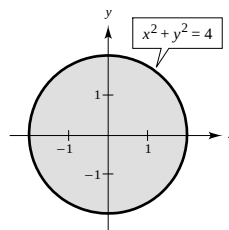
$$\begin{aligned}
 V &= 2 \int_0^1 \int_0^x \sqrt{1-x^2} dy dx \\
 &= 2 \int_0^1 \left[y\sqrt{1-x^2} \right]_0^x dx \\
 &= 2 \int_0^1 x\sqrt{1-x^2} dx \\
 &= \left[-\frac{2}{3}(1-x^2)^{3/2} \right]_0^1 = \frac{2}{3}
 \end{aligned}$$



$$\begin{aligned}
 39. V &= \int_0^2 \int_0^{\sqrt{4-x^2}} (x+y) dy dx \\
 &= \int_0^2 \left[xy + \frac{1}{2}y^2 \right]_0^{\sqrt{4-x^2}} dx \\
 &= \int_0^2 \left(x\sqrt{4-x^2} + 2 - \frac{1}{2}x^2 \right) dx \\
 &= \left[-\frac{1}{3}(4-x^2)^{3/2} + 2x - \frac{1}{6}x^3 \right]_0^2 = \frac{16}{3}
 \end{aligned}$$



$$\begin{aligned}
 41. V &= 4 \int_0^2 \int_0^{\sqrt{4-x^2}} (x^2 + y^2) dy dx \\
 &= 4 \int_0^2 \left[x^2\sqrt{4-x^2} + \frac{1}{3}(4-x^2)^{3/2} \right] dx, \quad x = 2 \sin \theta \\
 &= 4 \int_0^{\pi/2} \left(16 \cos^2 \theta - \frac{32}{3} \cos^4 \theta \right) d\theta \\
 &= 4 \left[16 \left(\frac{\pi}{4} \right) - \frac{32}{3} \left(\frac{3\pi}{16} \right) \right] \\
 &= 8\pi
 \end{aligned}$$



$$43. V = 4 \int_0^2 \int_0^{\sqrt{4-x^2}} (4-x^2-y^2) dy dx = 8\pi$$

$$45. V = \int_0^2 \int_0^{-0.5x+1} \frac{2}{1+x^2+y^2} dy dx \approx 1.2315$$

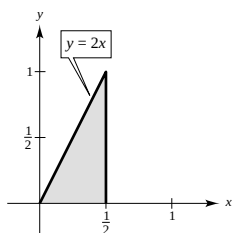
47. f is a continuous function such that $0 \leq f(x, y) \leq 1$ over a region R of area 1. Let $f(m, n)$ = the minimum value of f over R and $f(M, N)$ = the maximum value of f over R . Then

$$f(m, n) \iint_R dA \leq \iint_R f(x, y) dA \leq f(M, N) \iint_R dA.$$

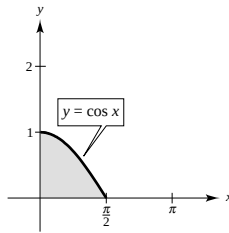
Since $\iint_R dA = 1$ and $0 \leq f(m, n) \leq f(M, N) \leq 1$, we have $0 \leq f(m, n)(1) \leq \iint_R f(x, y) dA \leq f(M, N)(1) \leq 1$.

Therefore, $0 \leq \iint_R f(x, y) dA \leq 1$.

$$\begin{aligned}
 49. \int_0^1 \int_{y/2}^{1/2} e^{-x^2} dx dy &= \int_0^{1/2} \int_{y/2}^{2x} e^{-x^2} dy dx \\
 &= \int_0^{1/2} 2xe^{-x^2} dx \\
 &= \left[-e^{-x^2} \right]_0^{1/2} \\
 &= -e^{-1/4} + 1 \\
 &= 1 - e^{-1/4} \approx 0.221
 \end{aligned}$$



$$\begin{aligned}
 51. \int_0^1 \int_0^{\arccos y} \sin x \sqrt{1 + \sin^2 x} dx dy &= \int_0^{\pi/2} \int_0^{\cos x} \sin x \sqrt{1 + \sin^2 x} dy dx \\
 &= \int_0^{\pi/2} (1 + \sin^2 x)^{1/2} \sin x \cos x dx \\
 &= \left[\frac{1}{2} \cdot \frac{2}{3} (1 + \sin^2 x)^{3/2} \right]_0^{\pi/2} = \frac{1}{3} [2\sqrt{2} - 1]
 \end{aligned}$$



$$53. \text{Average} = \frac{1}{8} \int_0^4 \int_0^2 x \, dy \, dx = \frac{1}{8} \int_0^4 2x \, dx = \left[\frac{x^2}{8} \right]_0^4 = 2$$

$$\begin{aligned} 55. \text{Average} &= \frac{1}{4} \int_0^2 \int_0^2 (x^2 + y^2) \, dx \, dy \\ &= \frac{1}{4} \int_0^2 \left[\frac{x^3}{3} + xy^2 \right]_0^2 dy = \frac{1}{4} \int_0^2 \left(\frac{8}{3} + 2y^2 \right) dy \\ &= \left[\frac{1}{4} \left(\frac{8}{3}y + \frac{2}{3}y^3 \right) \right]_0^2 = \frac{8}{3} \end{aligned}$$

57. See the definition on page 946.

59. The value of $\iint_R f(x, y) \, dA$ would be kB .

$$\begin{aligned} 61. \text{Average} &= \frac{1}{1250} \int_{300}^{325} \int_{200}^{250} 100x^{0.6}y^{0.4} \, dx \, dy \\ &= \frac{1}{1250} \int_{300}^{325} \left[(100y^{0.4}) \frac{x^{1.6}}{1.6} \right]_{200}^{250} dy = \frac{128,844.1}{1250} \int_{300}^{325} y^{0.4} \, dy = 103.0753 \left[\frac{y^{1.4}}{1.4} \right]_{300}^{325} \approx 25,645.24 \end{aligned}$$

63. $f(x, y) \geq 0$ for all (x, y) and

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dA &= \int_0^5 \int_0^2 \frac{1}{10} \, dy \, dx = \int_0^5 \frac{1}{5} \, dx = 1 \\ P(0 \leq x \leq 2, 1 \leq y \leq 2) &= \int_0^2 \int_1^2 \frac{1}{10} \, dy \, dx = \int_0^2 \frac{1}{10} \, dx = \frac{1}{5}. \end{aligned}$$

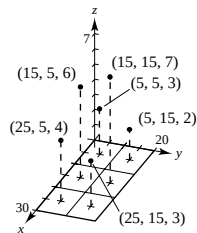
65. $f(x, y) \geq 0$ for all (x, y) and

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dA &= \int_0^3 \int_3^6 \frac{1}{27} (9 - x - y) \, dy \, dx \\ &= \int_0^3 \frac{1}{27} \left[9y - xy - \frac{y^2}{2} \right]_3^6 dx = \int_0^3 \left(\frac{1}{2} - \frac{1}{9}x \right) dx = \left[\frac{x}{2} - \frac{x^2}{18} \right]_0^3 = 1 \\ P(0 \leq x \leq 1, 4 \leq y \leq 6) &= \int_0^1 \int_4^6 \frac{1}{27} (9 - x - y) \, dy \, dx = \int_0^1 \frac{2}{27} (4 - x) \, dx = \frac{7}{27}. \end{aligned}$$

67. Divide the base into six squares, and assume the height at the center of each square is the height of the entire square.

Thus,

$$V \approx (4 + 3 + 6 + 7 + 3 + 2)(100) = 2500m^3.$$



$$69. \int_0^1 \int_0^2 \sin \sqrt{x+y} \, dy \, dx \quad m=4, n=8$$

(a) 1.78435

(b) 1.7879

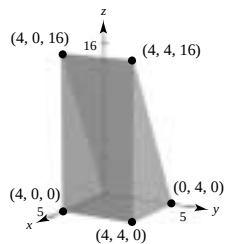
$$71. \int_4^6 \int_0^2 y \cos \sqrt{x} \, dx \, dy \quad m=4, n=8$$

(a) 11.0571

(b) 11.0414

73. $V \approx 125$

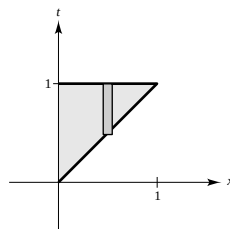
Matches d.



75. False

$$V = 8 \int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2} \, dx \, dy$$

$$\begin{aligned} 77. \text{Average} &= \int_0^1 f(x) \, dx = \int_0^1 \int_1^x e^{t^2} \, dt \, dx = - \int_0^1 \int_x^1 e^{t^2} \, dt \, dx \\ &= - \int_0^1 \int_0^t e^{t^2} \, dx \, dt = - \int_0^1 t e^{t^2} \, dt \\ &= \left[-\frac{1}{2} e^{t^2} \right]_0^1 = -\frac{1}{2}(e - 1) = \frac{1}{2}(1 - e) \end{aligned}$$



Section 13.3 Change of Variables: Polar Coordinates

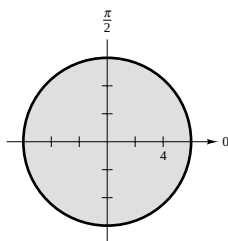
1. Rectangular coordinates

3. Polar coordinates

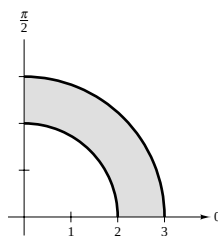
5. $R = \{(r, \theta): 0 \leq r \leq 8, 0 \leq \theta \leq \pi\}$

7. $R = \{(r, \theta): 0 \leq r \leq 3 + 3 \sin \theta, 0 \leq \theta \leq 2\pi\}$ Cardioid

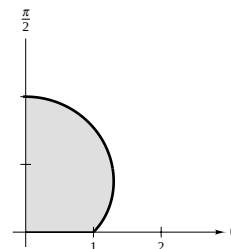
$$\begin{aligned} 9. \int_0^{2\pi} \int_0^6 3r^2 \sin \theta \, dr \, d\theta &= \int_0^{2\pi} \left[r^3 \sin \theta \right]_0^6 d\theta \\ &= \int_0^{2\pi} 216 \sin \theta \, d\theta \\ &= \left[-216 \cos \theta \right]_0^{2\pi} = 0 \end{aligned}$$



$$\begin{aligned} 11. \int_0^{\pi/2} \int_2^3 \sqrt{9-r^2} \, r \, dr \, d\theta &= \int_0^{\pi/2} \left[-\frac{1}{3}(9-r^2)^{3/2} \right]_2^3 d\theta \\ &= \left[\frac{5\sqrt{5}}{3} \theta \right]_0^{\pi/2} \\ &= \frac{5\sqrt{5}\pi}{6} \end{aligned}$$



$$\begin{aligned} 13. \int_0^{\pi/2} \int_0^{1+\sin \theta} \theta r \, dr \, d\theta &= \int_0^{\pi/2} \left[\frac{\theta r^2}{2} \right]_0^{1+\sin \theta} d\theta \\ &= \int_0^{\pi/2} \frac{1}{2} \theta (1 + \sin \theta)^2 d\theta \\ &= \left[\frac{1}{8} \theta^2 + \sin \theta - \theta \cos \theta + \frac{1}{2} \theta \left(-\frac{1}{2} \cos \theta \cdot \sin \theta + \frac{1}{2} \theta \right) + \frac{1}{8} \sin^2 \theta \right]_0^{\pi/2} \\ &= \frac{3}{32} \pi^2 + \frac{9}{8} \end{aligned}$$

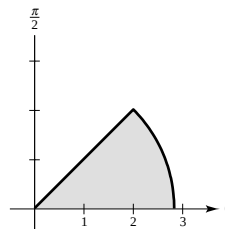


$$15. \int_0^a \int_0^{\sqrt{a^2-y^2}} y \, dx \, dy = \int_0^{\pi/2} \int_0^a r^2 \sin \theta \, dr \, d\theta = \frac{a^3}{3} \int_0^{\pi/2} \sin \theta \, d\theta = \left[\frac{a^3}{3} (-\cos \theta) \right]_0^{\pi/2} = \frac{a^3}{3}$$

$$17. \int_0^3 \int_0^{\sqrt{9-x^2}} (x^2 + y^2)^{3/2} \, dy \, dx = \int_0^{\pi/2} \int_0^3 r^4 \, dr \, d\theta = \frac{243}{5} \int_0^{\pi/2} d\theta = \frac{243\pi}{10}$$

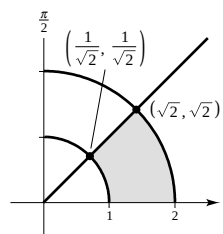
$$19. \int_0^2 \int_0^{\sqrt{2x-x^2}} xy \, dy \, dx = \int_0^{\pi/2} \int_0^{2 \cos \theta} r^3 \cos \theta \sin \theta \, dr \, d\theta = 4 \int_0^{\pi/2} \cos^5 \theta \sin \theta \, d\theta = \left[-\frac{4 \cos^6 \theta}{6} \right]_0^{\pi/2} = \frac{2}{3}$$

$$\begin{aligned} 21. \int_0^2 \int_0^x \sqrt{x^2 + y^2} \, dy \, dx + \int_2^{\sqrt{2}} \int_0^{\sqrt{8-x^2}} \sqrt{x^2 + y^2} \, dy \, dx &= \int_0^{\pi/4} \int_0^{2\sqrt{2}} r^2 \, dr \, d\theta \\ &= \int_0^{\pi/4} \frac{16\sqrt{2}}{3} d\theta \\ &= \frac{4\sqrt{2}\pi}{3} \end{aligned}$$



$$\begin{aligned} 23. \int_0^2 \int_0^{\sqrt{4-x^2}} (x+y) \, dy \, dx &= \int_0^{\pi/2} \int_0^2 (r \cos \theta + r \sin \theta) r \, dr \, d\theta = \int_0^{\pi/2} \int_0^2 (\cos \theta + \sin \theta) r^2 \, dr \, d\theta \\ &= \frac{8}{3} \int_0^{\pi/2} (\cos \theta + \sin \theta) \, d\theta = \left[\frac{8}{3} (\sin \theta - \cos \theta) \right]_0^{\pi/2} = \frac{16}{3} \end{aligned}$$

$$\begin{aligned} 25. \int_0^{1/\sqrt{2}} \int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} \arctan \frac{y}{x} \, dx \, dy + \int_{1/\sqrt{2}}^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \arctan \frac{y}{x} \, dx \, dy \\ &= \int_0^{\pi/4} \int_1^2 \theta r \, dr \, d\theta \\ &= \int_0^{\pi/4} \frac{3}{2} \theta \, d\theta = \left[\frac{3\theta^2}{4} \right]_0^{\pi/4} = \frac{3\pi^2}{64} \end{aligned}$$



$$\begin{aligned} 27. V &= \int_0^{\pi/2} \int_0^1 (r \cos \theta)(r \sin \theta) r \, dr \, d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} \int_0^1 r^3 \sin 2\theta \, dr \, d\theta = \frac{1}{8} \int_0^{\pi/2} \sin 2\theta \, d\theta = \left[-\frac{1}{16} \cos 2\theta \right]_0^{\pi/2} = \frac{1}{8} \end{aligned}$$

$$29. V = \int_0^{2\pi} \int_0^5 r^2 \, dr \, d\theta = \frac{250\pi}{3}$$

$$\begin{aligned} 31. V &= 2 \int_0^{\pi/2} \int_0^{4 \cos \theta} \sqrt{16-r^2} \, r \, dr \, d\theta = 2 \int_0^{\pi/2} \left[-\frac{1}{3} (\sqrt{16-r^2})^3 \right]_0^{4 \cos \theta} d\theta = -\frac{2}{3} \int_0^{\pi/2} (64 \sin^3 \theta - 64) \, d\theta \\ &= \frac{128}{3} \int_0^{\pi/2} [1 - \sin \theta (1 - \cos^2 \theta)] \, d\theta = \frac{128}{3} \left[\theta + \cos \theta - \frac{\cos^3 \theta}{3} \right]_0^{\pi/2} = \frac{64}{9} (3\pi - 4) \end{aligned}$$

$$33. V = \int_0^{2\pi} \int_a^4 \sqrt{16 - r^2} r \, dr \, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(\sqrt{16 - r^2})^3 \right]_a^4 d\theta = \frac{1}{3}(\sqrt{16 - a^2})^3(2\pi)$$

One-half the volume of the hemisphere is $(64\pi)/3$.

$$\frac{2\pi}{3}(16 - a^2)^{3/2} = \frac{64\pi}{3}$$

$$(16 - a^2)^{3/2} = 32$$

$$16 - a^2 = 32^{2/3}$$

$$a^2 = 16 - 32^{2/3} = 16 - 8\sqrt[3]{2}$$

$$a = \sqrt{4(4 - 2\sqrt[3]{2})} = 2\sqrt{4 - 2\sqrt[3]{2}} \approx 2.4332$$

$$\begin{aligned} 35. \text{ Total Volume} = V &= \int_0^{2\pi} \int_0^4 25e^{-r^2/4} r \, dr \, d\theta \\ &= \int_0^{2\pi} \left[-50e^{-r^2/4} \right]_0^4 d\theta \\ &= \int_0^{2\pi} -50(e^{-4} - 1) d\theta \\ &= (1 - e^{-4}) 100\pi \approx 308.40524 \end{aligned}$$

Let c be the radius of the hole that is removed.

$$\begin{aligned} \frac{1}{10} V &= \int_0^{2\pi} \int_0^c 25e^{-r^2/4} r \, dr \, d\theta = \int_0^{2\pi} \left[-50e^{-r^2/4} \right]_0^c d\theta \\ &= \int_0^{2\pi} -50(e^{-c^2/4} - 1) d\theta \Rightarrow 30.84052 = 100\pi(1 - e^{-c^2/4}) \end{aligned}$$

$$\Rightarrow e^{-c^2/4} = 0.90183$$

$$-\frac{c^2}{4} = -0.10333$$

$$c^2 = 0.41331$$

$$c = 0.6429$$

$$\Rightarrow \text{diameter} = 2c = 1.2858$$

$$37. A = \int_0^\pi \int_0^{6 \cos \theta} r \, dr \, d\theta = \int_0^\pi 18 \cos^2 \theta \, d\theta = 9 \int_0^\pi (1 + \cos 2\theta) \, d\theta = \left[9\left(\theta + \frac{1}{2} \sin 2\theta\right) \right]_0^\pi = 9\pi$$

$$\begin{aligned} 39. \int_0^{2\pi} \int_0^{1 + \cos \theta} r \, dr \, d\theta &= \frac{1}{2} \int_0^{2\pi} (1 + 2 \cos \theta + \cos^2 \theta) \, d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left(1 + 2 \cos \theta + \frac{1 + \cos 2\theta}{2} \right) d\theta = \frac{1}{2} \left[\theta + 2 \sin \theta + \frac{1}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) \right]_0^{2\pi} = \frac{3\pi}{2} \end{aligned}$$

$$41. 3 \int_0^{\pi/3} \int_0^{2 \sin 3\theta} r \, dr \, d\theta = \frac{3}{2} \int_0^{\pi/3} 4 \sin^2 3\theta \, d\theta = 3 \int_0^{\pi/3} (1 - \cos 6\theta) \, d\theta = 3 \left[\theta - \frac{1}{6} \sin 6\theta \right]_0^{\pi/3} = \pi$$

43. Let R be a region bounded by the graphs of $r = g_1(\theta)$ and $r = g_2(\theta)$, and the lines $\theta = a$ and $\theta = b$.

When using polar coordinates to evaluate a double integral over R , R can be partitioned into small polar sectors.

45. r -simple regions have fixed bounds for θ .

θ -simple regions have fixed bounds for r .

47. You would need to insert a factor of r because of the $r \, dr \, d\theta$ nature of polar coordinate integrals. The plane regions would be sectors of circles.

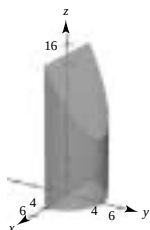
$$49. \int_{\pi/4}^{\pi/2} \int_0^5 r \sqrt{1+r^3} \sin \sqrt{\theta} \, dr \, d\theta \approx 56.051$$

$$\left[\text{Note: This integral equals } \left(\int_{\pi/4}^{\pi/2} \sin \sqrt{\theta} \, d\theta \right) \left(\int_0^5 r \sqrt{1+r^3} \, dr \right) \right]$$

51. Volume = base $\times$ height

$$\approx 8\pi \times 12 \approx 300$$

Answer (c)



53. False

Let $f(r, \theta) = r - 1$ where R is the circular sector $0 \leq r \leq 6$ and $0 \leq \theta \leq \pi$. Then,

$$\int_R \int (r - 1) \, dA > 0 \quad \text{but} \quad r - 1 \not\geq 0 \text{ for all } r.$$

$$55. (a) I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} \, dA = 4 \int_0^{\pi/2} \int_0^{\infty} e^{-r^2/2} r \, dr \, d\theta = 4 \int_0^{\pi/2} \left[-e^{-r^2/2} \right]_0^{\infty} d\theta = 4 \int_0^{\pi/2} d\theta = 2\pi$$

(b) Therefore, $I = \sqrt{2\pi}$.

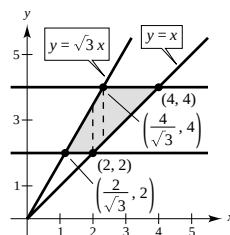
$$57. \int_{-7}^7 \int_{-\sqrt{49-x^2}}^{\sqrt{49-x^2}} 4000e^{-0.01(x^2+y^2)} \, dy \, dx = \int_0^{2\pi} \int_0^7 4000e^{-0.01r^2} r \, dr \, d\theta = \int_0^{2\pi} \left[-200,000e^{-0.01r^2} \right]_0^7 d\theta$$

$$= 2\pi(-200,000)(e^{-0.49} - 1) = 400,000\pi(1 - e^{-0.49}) \approx 486,788$$

$$59. (a) \int_2^4 \int_{y/\sqrt{3}}^y f \, dx \, dy$$

$$(b) \int_{2/\sqrt{3}}^2 \int_2^{\sqrt{3}x} f \, dy \, dx + \int_2^{4/\sqrt{3}} \int_x^{\sqrt{3}x} f \, dy \, dx + \int_{4/\sqrt{3}}^4 \int_x^4 f \, dy \, dx$$

$$(c) \int_{\pi/4}^{\pi/3} \int_{2 \csc \theta}^{4 \csc \theta} f r \, dr \, d\theta$$



$$61. A = \frac{\Delta\theta r_2^2}{2} - \frac{\Delta\theta r_1^2}{2} = \Delta\theta \left(\frac{r_1 + r_2}{2} \right) (r_2 - r_1) = r \Delta r \Delta \theta$$

Section 13.4 Center of Mass and Moments of Inertia

$$1. m = \int_0^4 \int_0^3 xy \, dy \, dx = \int_0^4 \left[\frac{xy^2}{2} \right]_0^3 dx = \int_0^4 \frac{9}{2} x \, dx = \left[\frac{9x^2}{4} \right]_0^4 = 36$$

$$3. m = \int_0^{\pi/2} \int_0^2 (r \cos \theta)(r \sin \theta) r \, dr \, d\theta = \int_0^{\pi/2} \int_0^2 \cos \theta \sin \theta \cdot r^3 \, dr \, d\theta$$

$$= \int_0^{\pi/2} 4 \cos \theta \sin \theta \, d\theta$$

$$= \left[4 \frac{\sin^2 \theta}{2} \right]_0^{\pi/2} = 2$$

$$5. (a) \quad m = \int_0^a \int_0^b k \, dy \, dx = kab$$

$$M_x = \int_0^a \int_0^b ky \, dy \, dx = \frac{kab^2}{2}$$

$$M_y = \int_0^a \int_0^b kx \, dy \, dx = \frac{ka^2b}{2}$$

$$\bar{x} = \frac{M_y}{m} = \frac{ka^2b/2}{kab} = \frac{a}{2}$$

$$\bar{y} = \frac{M_x}{m} = \frac{kab^2/2}{kab} = \frac{b}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{a}{2}, \frac{b}{2} \right)$$

$$(b) \quad m = \int_0^a \int_0^b ky \, dy \, dx = \frac{kab^2}{2}$$

$$M_x = \int_0^a \int_0^b ky^2 \, dy \, dx = \frac{kab^3}{3}$$

$$M_y = \int_0^a \int_0^b kxy \, dy \, dx = \frac{ka^2b^2}{4}$$

$$\bar{x} = \frac{M_y}{m} = \frac{ka^2b^2/4}{kab^2/2} = \frac{a}{2}$$

$$\bar{y} = \frac{M_x}{m} = \frac{kab^3/3}{kab^2/2} = \frac{2}{3}b$$

$$(\bar{x}, \bar{y}) = \left(\frac{a}{2}, \frac{2}{3}b \right)$$

$$(c) \quad m = \int_0^a \int_0^b kx \, dy \, dx = k \int_0^a xb \, dx = \frac{1}{2}ka^2b$$

$$M_x = \int_0^a \int_0^b kxy \, dy \, dx = \frac{ka^2b^2}{4}$$

$$M_y = \int_0^a \int_0^b kx^2 \, dy \, dx = \frac{ka^3b}{3}$$

$$\bar{x} = \frac{M_y}{m} = \frac{ka^3b/3}{ka^2b/2} = \frac{2}{3}a$$

$$\bar{y} = \frac{M_x}{m} = \frac{ka^2b^2/4}{ka^2b/2} = \frac{b}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{2}{3}a, \frac{b}{2} \right)$$

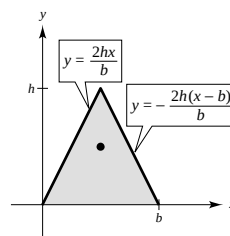
$$7. (a) \quad m = \frac{k}{2}bh$$

$$\bar{x} = \frac{b}{2} \text{ by symmetry}$$

$$\begin{aligned} M_x &= \int_0^{b/2} \int_0^{2hx/b} ky \, dy \, dx + \int_{b/2}^b \int_0^{-2h(x-b)/b} ky \, dy \, dx \\ &= \frac{kbh^2}{12} + \frac{kbh^2}{12} = \frac{kbh^2}{6} \end{aligned}$$

$$\bar{y} = \frac{M_x}{m} = \frac{kbh^2/6}{kbh/2} = \frac{h}{3}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{2}, \frac{h}{3} \right)$$



—CONTINUED—

7. —CONTINUED—

$$(b) \quad m = \int_0^{b/2} \int_0^{2hx/b} ky \, dy \, dx + \int_{b/2}^b \int_0^{-2h(x-b)/b} ky \, dy \, dx = \frac{kbh^2}{6}$$

$$M_x = \int_0^{b/2} \int_0^{2hx/b} ky^2 \, dy \, dx + \int_{b/2}^b \int_0^{-2h(x-b)/b} ky^2 \, dy \, dx = \frac{kbh^3}{12}$$

$$M_y = \int_0^{b/2} \int_0^{2hx/b} kxy \, dy \, dx + \int_{b/2}^b \int_0^{-2h(x-b)/b} kxy \, dy \, dx = \frac{kb^2h^2}{12}$$

$$\bar{x} = \frac{M_y}{m} = \frac{kb^2h^2/12}{kbh^2/6} = \frac{b}{2}$$

$$\bar{y} = \frac{M_x}{m} = \frac{kbh^3/12}{kbh^2/6} = \frac{h}{2}$$

$$(c) \quad m = \int_0^{b/2} \int_0^{2hx/b} kx \, dy \, dx + \int_{b/2}^b \int_0^{-2h(x-b)/b} kx \, dy \, dx$$

$$= \frac{1}{12}kb^2h + \frac{1}{6}kb^2h = \frac{1}{4}kb^2h$$

$$M_x = \int_0^{b/2} \int_0^{2hx/b} kxy \, dy \, dx + \int_{b/2}^b \int_0^{-2h(x-b)/b} kxy \, dy \, dx$$

$$= \frac{1}{32}kh^2b^2 + \frac{5}{96}kh^2b^2 = \frac{1}{12}kh^2b^2$$

$$M_y = \int_0^{b/2} \int_0^{2hx/b} kx^2 \, dy \, dx + \int_{b/2}^b \int_0^{-2h(x-b)/b} kx^2 \, dy \, dx$$

$$= \frac{1}{32}kb^3h + \frac{11}{96}kb^3h = \frac{7}{48}kb^3h$$

$$\bar{x} = \frac{M_y}{m} = \frac{7kb^3h/48}{kb^2h/4} = \frac{7}{12}b$$

$$\bar{y} = \frac{M_x}{m} = \frac{kh^2b^2/12}{kb^2h/4} = \frac{h}{3}$$

$$9. (a) \quad \text{The } x\text{-coordinate changes by 5: } (\bar{x}, \bar{y}) = \left(\frac{a}{2} + 5, \frac{b}{2}\right)$$

$$(b) \quad \text{The } x\text{-coordinate changes by 5: } (\bar{x}, \bar{y}) = \left(\frac{a}{2} + 5, \frac{2b}{3}\right)$$

$$(c) \quad m = \int_5^{a+5} \int_0^b kx \, dy \, dx = \frac{1}{2}k(a+5)^2b - \frac{25}{2}kb$$

$$M_x = \int_5^{a+5} \int_0^b kxy \, dy \, dx = \frac{1}{4}k(a+5)^2b^2 - \frac{25}{4}kb^2$$

$$M_y = \int_5^{a+5} \int_0^b kx^2 \, dy \, dx = \frac{1}{3}k(a+5)^3b - \frac{125}{3}kb$$

$$\bar{x} = \frac{M_y}{m} = \frac{2(a^2 + 15a + 75)}{3(a+10)}$$

$$\bar{y} = \frac{M_x}{m} = \frac{b}{2}$$

$$11. (a) \quad \bar{x} = 0 \text{ by symmetry}$$

$$m = \frac{\pi a^2 k}{2}$$

$$M_x = \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} yk \, dy \, dx = \frac{2a^3k}{3}$$

$$\bar{y} = \frac{M_x}{m} = \frac{2a^3k}{3} \cdot \frac{2}{\pi a^2k} = \frac{4a}{3\pi}$$

$$(b) \quad m = \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} k(a-y)y \, dy \, dx = \frac{a^4k}{24}(16-3\pi)$$

$$M_x = \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} k(a-y)y^2 \, dy \, dx = \frac{a^5k}{120}(15\pi-32)$$

$$M_y = \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} kx(a-y)y \, dy \, dx = 0$$

$$\bar{x} = \frac{M_y}{m} = 0$$

$$\bar{y} = \frac{M_x}{m} = \frac{a}{5} \left[\frac{15\pi-32}{16-3\pi} \right]$$

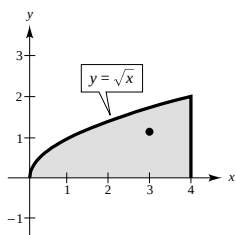
$$13. \quad m = \int_0^4 \int_0^{\sqrt{x}} kxy \, dy \, dx = \frac{32k}{3}$$

$$M_x = \int_0^4 \int_0^{\sqrt{x}} kxy^2 \, dy \, dx = \frac{256k}{21}$$

$$M_y = \int_0^4 \int_0^{\sqrt{x}} kx^2y \, dy \, dx = 32k$$

$$\bar{x} = \frac{M_y}{m} = \frac{32k}{1} \cdot \frac{3}{32k} = 3$$

$$\bar{y} = \frac{M_x}{m} = \frac{256k}{21} \cdot \frac{3}{32k} = \frac{8}{7}$$

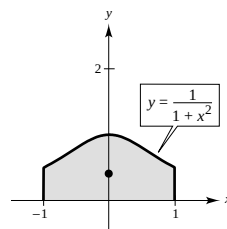


$$15. \quad \bar{x} = 0 \text{ by symmetry}$$

$$m = \int_{-1}^1 \int_0^{1/(1+x^2)} k \, dy \, dx = \frac{k\pi}{2}$$

$$M_x = \int_{-1}^1 \int_0^{1/(1+x^2)} ky \, dy \, dx = \frac{k}{8}(2 + \pi)$$

$$\bar{y} = \frac{M_x}{m} = \frac{k}{8}(2 + \pi) \cdot \frac{2}{k\pi} = \frac{2 + \pi}{4\pi}$$

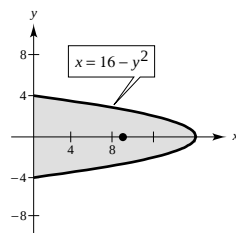


$$17. \quad \bar{y} = 0 \text{ by symmetry}$$

$$m = \int_{-4}^4 \int_0^{16-y^2} kx \, dx \, dy = \frac{8192k}{15}$$

$$M_y = \int_{-4}^4 \int_0^{16-y^2} kx^2 \, dx \, dy = \frac{524,288k}{105}$$

$$\bar{x} = \frac{M_y}{m} = \frac{524,288k}{105} \cdot \frac{15}{8192k} = \frac{64}{7}$$

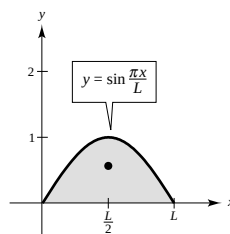


$$19. \quad \bar{x} = \frac{L}{2} \text{ by symmetry}$$

$$m = \int_0^L \int_0^{\sin \pi x/L} ky \, dy \, dx = \frac{kL}{4}$$

$$M_x = \int_0^L \int_0^{\sin \pi x/L} ky^2 \, dy \, dx = \frac{4kL}{9\pi}$$

$$\bar{y} = \frac{M_x}{m} = \frac{4kL}{9\pi} \cdot \frac{4}{kL} = \frac{16}{9\pi}$$



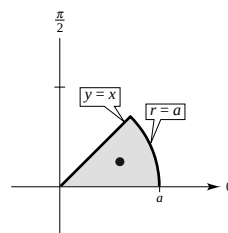
$$21. \quad m = \frac{\pi a^2 k}{8}$$

$$M_x = \int_R \int ky \, dA = \int_0^{\pi/4} \int_0^a kr^2 \sin \theta \, dr \, d\theta = \frac{ka^3(2 - \sqrt{2})}{6}$$

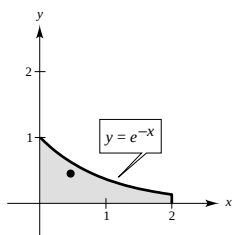
$$M_y = \int_R \int kx \, dA = \int_0^{\pi/4} \int_0^a kr^2 \cos \theta \, dr \, d\theta = \frac{ka^3\sqrt{2}}{6}$$

$$\bar{x} = \frac{M_y}{m} = \frac{ka^3\sqrt{2}}{6} \cdot \frac{8}{\pi a^2 k} = \frac{4a\sqrt{2}}{3\pi}$$

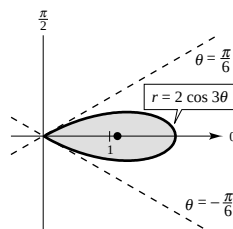
$$\bar{y} = \frac{M_x}{m} = \frac{ka^3(2 - \sqrt{2})}{6} \cdot \frac{8}{\pi a^2 k} = \frac{4a(2 - \sqrt{2})}{3\pi}$$



$$\begin{aligned}
 23. \quad m &= \int_0^2 \int_0^{e^{-x}} ky \, dy \, dx = \frac{k}{4}(1 - e^{-4}) \\
 M_x &= \int_0^2 \int_0^{e^{-x}} ky^2 \, dy \, dx = \frac{k}{9}(1 - e^{-6}) \\
 M_y &= \int_0^2 \int_0^{e^{-x}} kxy \, dy \, dx = \frac{k(1 - 5e^{-4})}{8} \\
 \bar{x} &= \frac{M_y}{m} = \frac{k(e^4 - 5)}{8e^4} \cdot \frac{4e^4}{k(e^4 - 1)} = \frac{e^4 - 5}{2(e^4 - 1)} \approx 0.46 \\
 \bar{y} &= \frac{M_x}{m} = \frac{k(e^6 - 1)}{9e^6} \cdot \frac{4e^4}{k(e^4 - 1)} = \frac{4}{9} \left[\frac{e^6 - 1}{e^6 - e^2} \right] \approx 0.45
 \end{aligned}$$



$$\begin{aligned}
 25. \quad \bar{y} &= 0 \text{ by symmetry} \\
 m &= \int_R \int k \, dA = \int_{-\pi/6}^{\pi/6} \int_0^{2 \cos 3\theta} kr \, dr \, d\theta = \frac{k\pi}{3} \\
 M_y &= \int_R \int kx \, dA \\
 &= \int_{-\pi/6}^{\pi/6} \int_0^{2 \cos 3\theta} kr^2 \cos \theta \, dr \, d\theta \approx 1.17k \\
 \bar{x} &= \frac{M_y}{m} \approx 1.17k \left(\frac{3}{\pi k} \right) \approx 1.12
 \end{aligned}$$



$$\begin{aligned}
 27. \quad m &= bh \\
 I_x &= \int_0^b \int_0^h y^2 \, dy \, dx = \frac{bh^3}{3} \\
 I_y &= \int_0^b \int_0^h x^2 \, dy \, dx = \frac{b^3h}{3} \\
 \bar{x} &= \sqrt{\frac{I_y}{m}} = \sqrt{\frac{b^3h}{3} \cdot \frac{1}{bh}} = \sqrt{\frac{b^2}{3}} = \frac{b}{\sqrt{3}} = \frac{\sqrt{3}}{3}b \\
 \bar{y} &= \sqrt{\frac{I_x}{m}} = \sqrt{\frac{bh^3}{3} \cdot \frac{1}{bh}} = \sqrt{\frac{h^2}{3}} = \frac{h}{\sqrt{3}} = \frac{\sqrt{3}}{3}h
 \end{aligned}$$

$$\begin{aligned}
 29. \quad m &= \pi a^2 \\
 I_x &= \int_R \int y^2 \, dA = \int_0^{2\pi} \int_0^a r^3 \sin^2 \theta \, dr \, d\theta = \frac{a^4\pi}{4} \\
 I_y &= \int_R \int x^2 \, dA = \int_0^{2\pi} \int_0^a r^3 \cos^2 \theta \, dr \, d\theta = \frac{a^4\pi}{4} \\
 I_0 &= I_x + I_y = \frac{a^4\pi}{4} + \frac{a^4\pi}{4} = \frac{a^4\pi}{2} \\
 \bar{x} &= \bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{a^4\pi/4}{\pi a^2}} = \frac{a}{2}
 \end{aligned}$$

$$\begin{aligned}
 31. \quad m &= \frac{\pi a^2}{4} \\
 I_x &= \int_R \int y^2 \, dA = \int_0^{\pi/2} \int_0^a r^3 \sin^2 \theta \, dr \, d\theta = \frac{\pi a^4}{16} \\
 I_y &= \int_R \int x^2 \, dA = \int_0^{\pi/2} \int_0^a r^3 \cos^2 \theta \, dr \, d\theta = \frac{\pi a^4}{16} \\
 I_0 &= I_x + I_y = \frac{\pi a^4}{16} + \frac{\pi a^4}{16} = \frac{\pi a^4}{8} \\
 \bar{x} &= \bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{\pi a^4/16}{\pi a^2/4}} = \frac{a}{2}
 \end{aligned}$$

$$\begin{aligned}
 33. \quad \rho &= ky \\
 m &= k \int_0^a \int_0^b y \, dy \, dx = \frac{kab^2}{2} \\
 I_x &= k \int_0^a \int_0^b y^3 \, dy \, dx = \frac{kab^4}{4} \\
 I_y &= k \int_0^a \int_0^b x^2 y \, dy \, dx = \frac{ka^3b^2}{6} \\
 I_0 &= I_x + I_y = \frac{3kab^4 + 2kb^2a^3}{12} \\
 \bar{x} &= \sqrt{\frac{I_y}{m}} = \sqrt{\frac{ka^3b^2/6}{kab^2/2}} = \sqrt{\frac{a^2}{3}} = \frac{a}{\sqrt{3}} = \frac{\sqrt{3}}{3}a \\
 \bar{y} &= \sqrt{\frac{I_x}{m}} = \sqrt{\frac{kab^4/4}{kab^2/2}} = \sqrt{\frac{b^2}{2}} = \frac{b}{\sqrt{2}} = \frac{\sqrt{2}}{2}b
 \end{aligned}$$

35. $\rho = kx$

$$m = k \int_0^2 \int_0^{4-x^2} x \, dy \, dx = 4k$$

$$I_x = k \int_0^2 \int_0^{4-x^2} xy^2 \, dy \, dx = \frac{32k}{3}$$

$$I_y = k \int_0^2 \int_0^{4-x^2} x^3 \, dy \, dx = \frac{16k}{3}$$

$$I_0 = I_x + I_y = 16k$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{16k/3}{4k}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{32k/3}{4k}} = \sqrt{\frac{8}{3}} = \frac{4}{\sqrt{6}} = \frac{2\sqrt{6}}{3}$$

37. $\rho = kxy$

$$m = \int_0^4 \int_0^{\sqrt{x}} kxy \, dy \, dx = \frac{32k}{3}$$

$$I_x = \int_0^4 \int_0^{\sqrt{x}} kxy^3 \, dy \, dx = 16k$$

$$I_y = \int_0^4 \int_0^{\sqrt{x}} kx^3 y \, dy \, dx = \frac{512k}{5}$$

$$I_0 = I_x + I_y = \frac{592k}{5}$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{512k}{5} \cdot \frac{3}{32k}} = \sqrt{\frac{48}{5}} = \frac{4\sqrt{15}}{5}$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{16k}{1} \cdot \frac{3}{32k}} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$$

39. $\rho = kx$

$$m = \int_0^1 \int_{x^2}^{\sqrt{x}} kx \, dy \, dx = \frac{3k}{20}$$

$$I_x = \int_0^1 \int_{x^2}^{\sqrt{x}} kxy^2 \, dy \, dx = \frac{3k}{56}$$

$$I_y = \int_0^1 \int_{x^2}^{\sqrt{x}} kx^3 \, dy \, dx = \frac{k}{18}$$

$$I_0 = I_x + I_y = \frac{55k}{504}$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{k}{18} \cdot \frac{20}{3k}} = \frac{\sqrt{30}}{9}$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{3k}{56} \cdot \frac{20}{3k}} = \frac{\sqrt{70}}{14}$$

41. $I = 2k \int_{-b}^b \int_0^{\sqrt{b^2-x^2}} (x-a)^2 \, dy \, dx = 2k \int_{-b}^b (x-a)^2 \sqrt{b^2-x^2} \, dx$

$$= 2k \left[\int_{-b}^b x^2 \sqrt{b^2-x^2} \, dx - 2a \int_{-b}^b x \sqrt{b^2-x^2} \, dx + a^2 \int_{-b}^b \sqrt{b^2-x^2} \, dx \right]$$

$$= 2k \left[\frac{\pi b^4}{8} + 0 + \frac{\pi a^2 b^2}{2} \right] = \frac{k\pi b^2}{4} (b^2 + 4a^2)$$

43. $I = \int_0^4 \int_0^{\sqrt{x}} kx(x-6)^2 \, dy \, dx = \int_0^4 kx\sqrt{x}(x^2-12x+36) \, dx = k \left[\frac{2}{9}x^{9/2} - \frac{24}{7}x^{7/2} + \frac{72}{5}x^{5/2} \right]_0^4 = \frac{42,752k}{315}$

$$\begin{aligned}
45. \quad I &= \int_0^a \int_0^{\sqrt{a^2-x^2}} k(a-y)(y-a)^2 dy dx = \int_0^a \int_0^{\sqrt{a^2-x^2}} k(a-y)^3 dy dx = \int_0^a \left[-\frac{k}{4}(a-y)^4 \right]_0^{\sqrt{a^2-x^2}} dx \\
&= -\frac{k}{4} \int_0^a \left[a^4 - 4a^3y + 6a^2y^2 - 4ay^3 + y^4 \right]_0^{\sqrt{a^2-x^2}} dx \\
&= -\frac{k}{4} \int_0^a \left[a^4 - 4a^3\sqrt{a^2-x^2} + 6a^2(a^2-x^2) - 4a(a^2-x^2)\sqrt{a^2-x^2} + (a^4 - 2a^2x^2 + x^4) - a^4 \right] dx \\
&= -\frac{k}{4} \int_0^a \left[7a^4 - 8a^2x^2 + x^4 - 8a^3\sqrt{a^2-x^2} + 4ax^2\sqrt{a^2-x^2} \right] dx \\
&= -\frac{k}{4} \left[7a^4x - \frac{8a^2}{3}x^3 + \frac{x^5}{5} - 4a^3 \left(x\sqrt{a^2-x^2} + a^2 \arcsin \frac{x}{a} \right) + \frac{a}{2} \left(x(2x^2 - a^2)\sqrt{a^2-x^2} + a^4 \arcsin \frac{x}{a} \right) \right]_0^a \\
&= -\frac{k}{4} \left(7a^5 - \frac{8}{3}a^5 + \frac{1}{5}a^5 - 2a^5\pi + \frac{1}{4}a^5\pi \right) = a^5k \left(\frac{7\pi}{16} - \frac{17}{15} \right)
\end{aligned}$$

47. $\rho(x, y) = ky$. $\bar{y}$ will increase

49. $\rho(x, y) = kxy$.

Both $\bar{x}$ and $\bar{y}$ will increase

51. Let $\rho(x, y)$ be a continuous density function on the planar lamina R .

The movements of mass with respect to the x - and y -axes are

$$M_x = \iint_R y \rho(x, y) dA \text{ and } M_y = \iint_R x \rho(x, y) dA.$$

If m is the mass of the lamina, then the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right).$$

53. See the definition on page 968

55. $\bar{y} = \frac{L}{2}$, $A = bL$, $h = \frac{L}{2}$

$$\begin{aligned}
I_{\bar{y}} &= \int_0^b \int_0^L \left(y - \frac{L}{2} \right)^2 dy dx \\
&= \int_0^b \left[\frac{\left(y - \frac{L}{2} \right)^3}{3} \right]_0^L dx = \frac{L^3b}{12} \\
y_a &= \bar{y} - \frac{I_{\bar{y}}}{hA} = \frac{L}{2} - \frac{L^3b/12}{(L/2)(bL)} = \frac{L}{3}
\end{aligned}$$

57. $\bar{y} = \frac{2L}{3}$, $A = \frac{bL}{2}$, $h = \frac{L}{3}$

$$\begin{aligned}
I_{\bar{y}} &= 2 \int_0^{b/2} \int_{2Lx/b}^L \left(y - \frac{2L}{3} \right)^2 dy dx \\
&= \frac{2}{3} \int_0^{b/2} \left[\left(y - \frac{2L}{3} \right)^3 \right]_{2Lx/b}^L dx \\
&= \frac{2}{3} \int_0^{b/2} \left[\frac{L}{27} - \left(\frac{2Lx}{b} - \frac{2L}{3} \right)^3 \right] dx \\
&= \frac{2}{3} \left[\frac{L^3x}{27} - \frac{b}{8L} \left(\frac{2Lx}{b} - \frac{2L}{3} \right)^4 \right]_0^{b/2} = \frac{L^3b}{36} \\
y_a &= \frac{2L}{3} - \frac{L^3b/36}{L^2b/6} = \frac{L}{2}
\end{aligned}$$

Section 13.5 Surface Area

1. $f(x, y) = 2x + 2y$

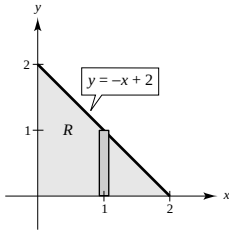
 R = triangle with vertices $(0, 0)$, $(2, 0)$, $(0, 2)$

$f_x = 2, f_y = 2$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = 3$

$$S = \int_0^2 \int_0^{2-x} 3 \, dy \, dx = 3 \int_0^2 (2-x) \, dx$$

$$= \left[3 \left(2x - \frac{x^2}{2} \right) \right]_0^2 = 6$$



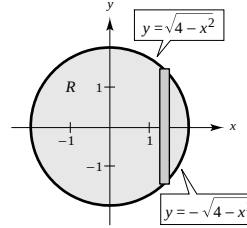
3. $f(x, y) = 8 + 2x + 2y$

 $R = \{(x, y): x^2 + y^2 \leq 4\}$

$f_x = 2, f_y = 2$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = 3$

$$S = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 3 \, dy \, dx = \int_0^{2\pi} \int_0^2 3r \, dr \, d\theta = 12\pi$$



5. $f(x, y) = 9 - x^2$

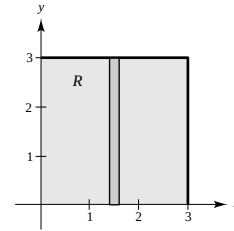
 R = square with vertices $(0, 0)$, $(3, 0)$, $(0, 3)$, $(3, 3)$

$f_x = -2x, f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2}$

$$S = \int_0^3 \int_0^3 \sqrt{1 + 4x^2} \, dy \, dx = \int_0^3 3\sqrt{1 + 4x^2} \, dx$$

$$= \left[\frac{3}{4} (2x\sqrt{1 + 4x^2} + \ln|2x + \sqrt{1 + 4x^2}|) \right]_0^3 = \frac{3}{4} (6\sqrt{37} + \ln|6 + \sqrt{37}|)$$



7. $f(x, y) = 2 + x^{3/2}$

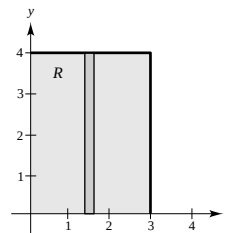
 R = rectangle with vertices $(0, 0)$, $(0, 4)$, $(3, 4)$, $(3, 0)$

$f_x = \frac{3}{2}x^{1/2}, f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \left(\frac{9}{4}\right)x} = \frac{\sqrt{4 + 9x}}{2}$

$$S = \int_0^3 \int_0^4 \frac{\sqrt{4 + 9x}}{2} \, dy \, dx = \int_0^3 4 \left(\frac{\sqrt{4 + 9x}}{2} \right) \, dx$$

$$= \left[\frac{4}{27} (4 + 9x)^{3/2} \right]_0^3 = \frac{4}{27} (31\sqrt{31} - 8)$$



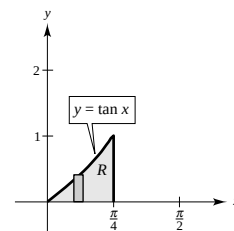
9. $f(x, y) = \ln|\sec x|$

 $R = \{(x, y): 0 \leq x \leq \frac{\pi}{4}, 0 \leq y \leq \tan x\}$

$f_x = \tan x, f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \tan^2 x} = \sec x$

$$S = \int_0^{\pi/4} \int_0^{\tan x} \sec x \, dy \, dx = \int_0^{\pi/4} \sec x \tan x \, dx = \left[\sec x \right]_0^{\pi/4} = \sqrt{2} - 1$$



11. $f(x, y) = \sqrt{x^2 + y^2}$

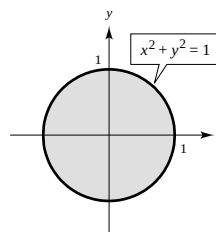
$$R = \{(x, y): 0 \leq f(x, y) \leq 1\}$$

$$0 \leq \sqrt{x^2 + y^2} \leq 1, x^2 + y^2 \leq 1$$

$$f_x = \frac{x}{\sqrt{x^2 + y^2}}, f_y = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} = \sqrt{2}$$

$$S = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{2} \, dy \, dx = \int_0^{2\pi} \int_0^1 \sqrt{2} \, r \, dr \, d\theta = \sqrt{2}\pi$$



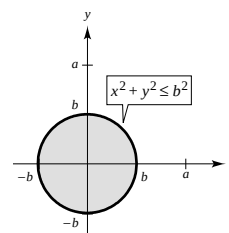
13. $f(x, y) = \sqrt{a^2 - x^2 - y^2}$

$$R = \{(x, y): x^2 + y^2 \leq b^2, b < a\}$$

$$f_x = \frac{-x}{\sqrt{a^2 - x^2 - y^2}}, f_y = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}$$

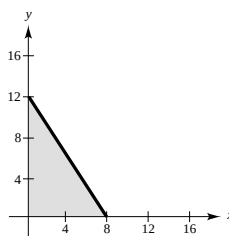
$$S = \int_{-b}^b \int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}} \, dy \, dx = \int_0^{2\pi} \int_0^b \frac{a}{\sqrt{a^2 - r^2}} \, r \, dr \, d\theta = 2\pi a(a - \sqrt{a^2 - b^2})$$



15. $z = 24 - 3x - 2y$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{14}$$

$$S = \int_0^8 \int_0^{-(3/2)x+12} \sqrt{14} \, dy \, dx = 48\sqrt{14}$$

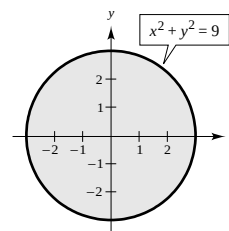


17. $z = \sqrt{25 - x^2 - y^2}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{x^2}{25 - x^2 - y^2} + \frac{y^2}{25 - x^2 - y^2}} = \frac{5}{\sqrt{25 - x^2 - y^2}}$$

$$S = 2 \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \frac{5}{\sqrt{25 - (x^2 + y^2)}} \, dy \, dx$$

$$= 2 \int_0^{2\pi} \int_0^3 \frac{5}{\sqrt{25 - r^2}} \, r \, dr \, d\theta = 20\pi$$

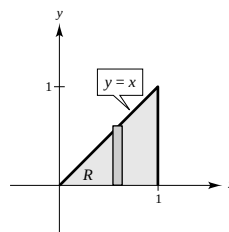


19. $f(x, y) = 2y + x^2$

$$R = \text{triangle with vertices } (0, 0), (1, 0), (1, 1)$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{5 + 4x^2}$$

$$S = \int_0^1 \int_0^x \sqrt{5 + 4x^2} \, dy \, dx = \frac{1}{12}(27 - 5\sqrt{5})$$



21. $f(x, y) = 4 - x^2 - y^2$

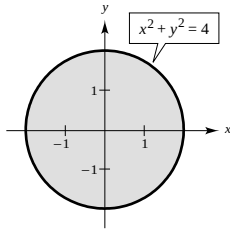
$$R = \{(x, y): 0 \leq f(x, y)\}$$

$$0 \leq 4 - x^2 - y^2, \quad x^2 + y^2 \leq 4$$

$$f_x = -2x, \quad f_y = -2y$$

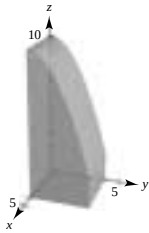
$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

$$\begin{aligned} S &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1 + 4x^2 + 4y^2} \, dy \, dx \\ &= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} \, r \, dr \, d\theta = \frac{(17\sqrt{17} - 1)\pi}{6} \end{aligned}$$



25. Surface area $> (4) \cdot (6) = 24$.

Matches (e)



29. $f(x, y) = x^3 - 3xy + y^3$

$$R = \text{square with vertices } (1, 1), (-1, 1), (-1, -1), (1, -1)$$

$$f_x = 3x^2 - 3y = 3(x^2 - y), \quad f_y = -3x + 3y^2 = 3(y^2 - x)$$

$$S = \int_{-1}^1 \int_{-1}^1 \sqrt{1 + 9(x^2 - y)^2 + 9(y^2 - x)^2} \, dy \, dx$$

33. $f(x, y) = e^{xy}$

$$R = \{(x, y): 0 \leq x \leq 4, 0 \leq y \leq 10\}$$

$$f_x = ye^{xy}, \quad f_y = xe^{xy}$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^2 e^{2xy} + x^2 e^{2xy}} = \sqrt{1 + e^{2xy}(x^2 + y^2)}$$

$$S = \int_0^4 \int_0^{10} \sqrt{1 + e^{2xy}(x^2 + y^2)} \, dy \, dx$$

23. $f(x, y) = 4 - x^2 - y^2$

$$R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

$$f_x = -2x, \quad f_y = -2y$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

$$S = \int_0^1 \int_0^1 \sqrt{1 + 4x^2 + 4y^2} \, dy \, dx \approx 1.8616$$

27. $f(x, y) = e^x$

$$R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

$$f_x = e^x, \quad f_y = 0$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + e^{2x}}$$

$$S = \int_0^1 \int_0^1 \sqrt{1 + e^{2x}} \, dy \, dx$$

$$= \int_0^1 \sqrt{1 + e^{2x}} \, dx \approx 2.0035$$

31. $f(x, y) = e^{-x} \sin y$

$$f_x = -e^{-x} \sin y, \quad f_y = e^{-x} \cos y$$

$$\begin{aligned} \sqrt{1 + f_x^2 + f_y^2} &= \sqrt{1 + e^{-2x} \sin^2 y + e^{-2x} \cos^2 y} \\ &= \sqrt{1 + e^{-2x}} \end{aligned}$$

$$S = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1 + e^{-2x}} \, dy \, dx$$

35. See the definition on page 972.

$$37. f(x, y) = \sqrt{1 - x^2}; f_x = \frac{-x}{\sqrt{1 - x^2}}, f_y = 0$$

$$\begin{aligned} S &= \iint_R \sqrt{1 + f_x^2 + f_y^2} \, dA \\ &= 16 \int_0^1 \int_0^x \frac{1}{\sqrt{1 - x^2}} \, dy \, dx \\ &= 16 \int_0^1 \frac{x}{\sqrt{1 - x^2}} \, dx = \left[-16(1 - x^2)^{1/2} \right]_0^1 = 16 \end{aligned}$$

$$39. (a) V = \int_0^{50} \int_0^{\sqrt{50^2 - x^2}} \left(20 + \frac{xy}{100} - \frac{x+y}{5} \right) dy \, dx$$

$$\begin{aligned} &= \int_0^{50} \left[20\sqrt{50^2 - x^2} + \frac{x}{200}(50^2 - x^2) - \frac{x}{5}\sqrt{50^2 - x^2} - \frac{50^2 - x^2}{10} \right] dy \\ &= \left[10 \left(x\sqrt{50^2 - x^2} + 50^2 \arcsin \frac{x}{50} \right) + \frac{25}{4}x^2 - \frac{x^4}{800} + \frac{1}{15}(50^2 - x^2)^{3/2} - 250x + \frac{x^3}{30} \right]_0^{50} \\ &\approx 30,415.74 \text{ ft}^3 \end{aligned}$$

$$(b) z = 20 + \frac{xy}{100}$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{y^2}{100^2} + \frac{x^2}{100^2}} = \frac{\sqrt{100^2 + x^2 + y^2}}{100}$$

$$\begin{aligned} S &= \frac{1}{100} \int_0^{50} \int_0^{\sqrt{50^2 - x^2}} \sqrt{100^2 + x^2 + y^2} \, dy \, dx \\ &= \frac{1}{100} \int_0^{\pi/2} \int_0^{50} \sqrt{100^2 + r^2} \, r \, dr \, d\theta \approx 2081.53 \text{ ft}^2 \end{aligned}$$

$$41. (a) V = \iint_R f(x, y) \, dA$$

$$= 8 \iint_R \sqrt{625 - x^2 - y^2} \, dA \quad \text{where } R \text{ is the region in the first quadrant}$$

$$= 8 \int_0^{\pi/2} \int_4^{25} \sqrt{625 - r^2} \, r \, dr \, d\theta$$

$$= -4 \int_0^{\pi/2} \left[\frac{2}{3} (625 - r^2)^{3/2} \right]_4^{25} d\theta$$

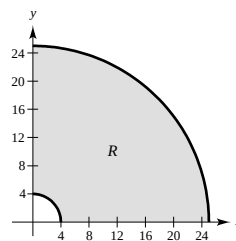
$$= -\frac{8}{3} [0 - 609\sqrt{609}] \cdot \frac{\pi}{2}$$

$$= 812\pi\sqrt{609} \text{ cm}^3$$

$$(b) A = \iint_R \sqrt{1 + (f_x)^2 + (f_y)^2} \, dA = 8 \iint_R \sqrt{1 + \frac{x^2}{625 - x^2 - y^2} + \frac{y^2}{625 - x^2 - y^2}} \, dA$$

$$= 8 \iint_R \frac{25}{\sqrt{625 - x^2 - y^2}} \, dA = 8 \int_0^{\pi/2} \int_4^{25} \frac{25}{\sqrt{625 - r^2}} \, r \, dr \, d\theta$$

$$= \lim_{b \rightarrow 25^-} \left[-200\sqrt{625 - r^2} \right]_4^b \cdot \frac{\pi}{2} = 100\pi\sqrt{609} \text{ cm}^2$$

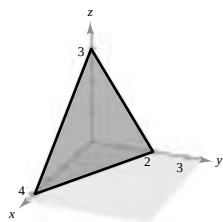


Section 13.6 Triple Integrals and Applications

1.
$$\begin{aligned}\int_0^3 \int_0^2 \int_0^1 (x + y + z) \, dx \, dy \, dz &= \int_0^3 \int_0^2 \left[\frac{1}{2}x^2 + xy + xz \right]_0^1 dy \, dz \\ &= \int_0^3 \int_0^2 \left(\frac{1}{2} + y + z \right) dy \, dz = \int_0^3 \left[\frac{1}{2}y + \frac{1}{2}y^2 + yz \right]_0^2 dz = \left[3z + z^2 \right]_0^3 = 18\end{aligned}$$
3.
$$\begin{aligned}\int_0^1 \int_0^x \int_0^{xy} x \, dz \, dy \, dx &= \int_0^1 \int_0^x \left[xz \right]_0^{xy} dy \, dx \\ &= \int_0^1 \int_0^x x^2 y \, dy \, dx = \int_0^1 \left[\frac{x^2 y^2}{2} \right]_0^x dx = \int_0^1 \frac{x^4}{2} dx = \left[\frac{x^5}{10} \right]_0^1 = \frac{1}{10}\end{aligned}$$
5.
$$\begin{aligned}\int_1^4 \int_0^1 \int_0^x 2ze^{-x^2} \, dy \, dx \, dz &= \int_1^4 \int_0^1 \left[(2ze^{-x^2})y \right]_0^x dx \, dz = \int_1^4 \int_0^1 2zxe^{-x^2} \, dx \, dz \\ &= \int_1^4 \left[-ze^{-x^2} \right]_0^1 dz = \int_1^4 z(1 - e^{-1}) \, dz = \left[(1 - e^{-1}) \frac{z^2}{2} \right]_1^4 = \frac{15}{2} \left(1 - \frac{1}{e} \right)\end{aligned}$$
7.
$$\begin{aligned}\int_0^4 \int_0^{\pi/2} \int_0^{1-x} x \cos y \, dz \, dy \, dx &= \int_0^4 \int_0^{\pi/2} \left[(x \cos y)z \right]_0^{1-x} dy \, dx = \int_0^4 \int_0^{\pi/2} x(1-x) \cos y \, dy \, dx \\ &= \int_0^4 \left[x(1-x) \sin y \right]_0^{\pi/2} dx = \int_0^4 x(1-x) \, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^4 = 8 - \frac{64}{3} = \frac{-40}{3}\end{aligned}$$
9.
$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{x^2} x \, dz \, dy \, dx = \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x^3 \, dy \, dx = \frac{128}{15}$$
11.
$$\begin{aligned}\int_0^2 \int_0^{\sqrt{4-x^2}} \int_1^4 \frac{x^2 \sin y}{z} \, dz \, dy \, dx &= \int_0^2 \int_0^{\sqrt{4-x^2}} \left[x^2 \sin y \ln |z| \right]_1^4 dy \, dx \\ &= \int_0^2 \left[x^2 \ln 4 (-\cos y) \right]_0^{\sqrt{4-x^2}} dx = \int_0^2 x^2 \ln 4 [1 - \cos \sqrt{4-x^2}] \, dx \approx 2.44167\end{aligned}$$
13.
$$\int_0^4 \int_0^{4-x} \int_0^{4-x-y} dz \, dy \, dx$$
15.
$$\int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} dz \, dy \, dx$$
17.
$$\begin{aligned}\int_{-2}^2 \int_0^{4-y^2} \int_0^x dz \, dx \, dy &= \int_{-2}^2 \int_0^{4-y^2} x \, dx \, dy \\ &= \frac{1}{2} \int_{-2}^2 (4-y^2)^2 \, dy = \int_0^2 (16-8y^2+y^4) \, dy = \left[16y - \frac{8}{3}y^3 + \frac{1}{5}y^5 \right]_0^2 = \frac{256}{15}\end{aligned}$$
19.
$$\begin{aligned}8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz \, dy \, dx &= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2} \, dy \, dx \\ &= 4 \int_0^a \left[y \sqrt{a^2-x^2-y^2} + (a^2-x^2) \arcsin \left(\frac{y}{\sqrt{a^2-x^2}} \right) \right]_0^{\sqrt{a^2-x^2}} dx \\ &= 4 \left(\frac{\pi}{2} \right) \int_0^a (a^2-x^2) \, dx = \left[2\pi \left(a^2x - \frac{1}{3}x^3 \right) \right]_0^a = \frac{4}{3} \pi a^3\end{aligned}$$

$$21. \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2} dz \, dy \, dx = \int_0^2 (4-x^2)^2 dx = \int_0^2 (16 - 8x^2 + x^4) dx = \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_0^2 = \frac{256}{15}$$

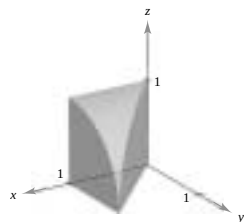
23. Plane: $3x + 6y + 4z = 12$



$$\int_0^3 \int_0^{(12-4z)/3} \int_0^{(12-4z-3x)/6} dy \, dx \, dz$$

25. Top cylinder: $y^2 + z^2 = 1$

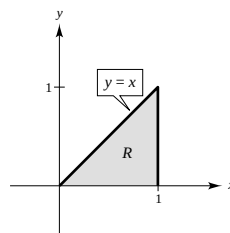
Side plane: $x = y$



$$\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz \, dy \, dx$$

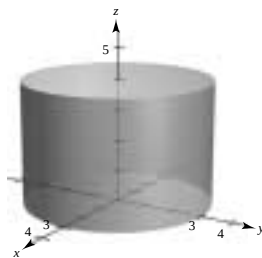
27. $Q = \{(x, y, z): 0 \leq x \leq 1, 0 \leq y \leq x, 0 \leq z \leq 3\}$

$$\begin{aligned} \iiint_Q xyz \, dV &= \int_0^3 \int_0^1 \int_y^1 xyz \, dx \, dy \, dz = \int_0^3 \int_0^1 \int_0^x xyz \, dy \, dx \, dz \\ &= \int_0^1 \int_0^3 \int_y^1 xyz \, dx \, dz \, dy \\ &= \int_0^1 \int_0^3 \int_0^x xyz \, dy \, dz \, dx \\ &= \int_0^1 \int_y^1 \int_0^3 xyz \, dz \, dx \, dy \\ &= \int_0^1 \int_0^x \int_0^3 xyz \, dz \, dy \, dx \left(= \frac{9}{16} \right) \end{aligned}$$



29. $Q = \{(x, y, z): x^2 + y^2 \leq 9, 0 \leq z \leq 4\}$

$$\begin{aligned} \iiint_Q xyz \, dV &= \int_0^4 \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xyz \, dy \, dx \, dz \\ &= \int_0^4 \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xyz \, dx \, dy \, dz \\ &= \int_{-3}^3 \int_0^4 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xyz \, dx \, dz \, dy \\ &= \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^4 xyz \, dz \, dx \, dy \\ &= \int_{-3}^3 \int_0^4 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xyz \, dy \, dz \, dx \\ &= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^4 xyz \, dz \, dy \, dx \quad (= 0) \end{aligned}$$



$$31. \quad m = k \int_0^6 \int_0^{4-(2x/3)} \int_0^{2-(y/2)-(x/3)} dz \, dy \, dx$$

$$= 8k$$

$$M_{yz} = k \int_0^6 \int_0^{4-(2x/3)} \int_0^{2-(y/2)-(x/3)} x \, dz \, dy \, dx$$

$$= 12k$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{12k}{8k} = \frac{3}{2}$$

$$33. \quad m = k \int_0^4 \int_0^4 \int_0^{4-x} x \, dz \, dy \, dx = k \int_0^4 \int_0^4 x(4-x) \, dy \, dx$$

$$= 4k \int_0^4 (4x - x^2) \, dx = \frac{128k}{3}$$

$$M_{xy} = k \int_0^4 \int_0^4 \int_0^{4-x} xz \, dz \, dy \, dx = k \int_0^4 \int_0^4 x \frac{(4-x)^2}{2} \, dy \, dx$$

$$= 2k \int_0^4 (16x - 8x^2 + x^3) \, dx = \frac{128k}{3}$$

$$\bar{z} = \frac{M_{xy}}{m} = 1$$

$$35. \quad m = k \int_0^b \int_0^b \int_0^b xy \, dz \, dy \, dx = \frac{kb^5}{4}$$

$$M_{yz} = k \int_0^b \int_0^b \int_0^b x^2 y \, dz \, dy \, dx = \frac{kb^6}{6}$$

$$M_{xz} = k \int_0^b \int_0^b \int_0^b xy^2 \, dz \, dy \, dx = \frac{kb^6}{6}$$

$$M_{xy} = k \int_0^b \int_0^b \int_0^b xyz \, dz \, dy \, dx = \frac{kb^6}{8}$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{kb^6/6}{kb^5/4} = \frac{2b}{3}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{kb^6/6}{kb^5/4} = \frac{2b}{3}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{kb^6/8}{kb^5/4} = \frac{b}{2}$$

37. $\bar{x}$ will be greater than 2, whereas $\bar{y}$ and $\bar{z}$ will be unchanged.

39. $\bar{y}$ will be greater than 0, whereas $\bar{x}$ and $\bar{z}$ will be unchanged.

$$41. \quad m = \frac{1}{3}k\pi r^2 h$$

$$\bar{x} = \bar{y} = 0 \text{ by symmetry}$$

$$M_{xy} = 4k \int_0^r \int_0^{\sqrt{r^2-x^2}} \int_{h\sqrt{x^2+y^2}/r}^h z \, dz \, dy \, dx$$

$$= \frac{3kh^2}{r^2} \int_0^r \int_0^{\sqrt{r^2-x^2}} (r^2 - x^2 - y^2) \, dy \, dx$$

$$= \frac{4kh^2}{3r^2} \int_0^r (r^2 - x^2)^{3/2} \, dx$$

$$= \frac{k\pi r^2 h^2}{4}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi r^2 h^2/4}{k\pi r^2 h/3} = \frac{3h}{4}$$

43. $m = \frac{128k\pi}{3}$

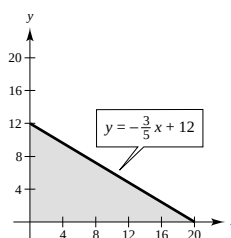
$\bar{x} = \bar{y} = 0$ by symmetry

$$z = \sqrt{4^2 - x^2 - y^2}$$

$$\begin{aligned} M_{xy} &= 4k \int_0^4 \int_0^{\sqrt{4^2 - x^2}} \int_0^{\sqrt{4^2 - x^2 - y^2}} z \, dz \, dy \, dx \\ &= 2k \int_0^4 \int_0^{\sqrt{4^2 - x^2}} (4^2 - x^2 - y^2) \, dy \, dx = 2k \int_0^4 \left[16y - x^2y - \frac{1}{3}y^3 \right]_0^{\sqrt{4^2 - x^2}} dx = \frac{4k}{3} \int_0^4 (4^2 - x^2)^{3/2} dx \\ &= \frac{1024k}{3} \int_0^{\pi/2} \cos^4 \theta \, d\theta \quad (\text{let } x = 4 \sin \theta) \\ &= 64\pi k \quad \text{by Wallis's Formula} \\ \bar{z} &= \frac{M_{xy}}{m} = \frac{64k\pi}{1} \cdot \frac{3}{128k\pi} = \frac{3}{2} \end{aligned}$$

45. $f(x, y) = \frac{5}{12}y$

$$\begin{aligned} m &= k \int_0^{20} \int_0^{-(3/5)x+12} \int_0^{(5/12)y} dz \, dy \, dx = 200k \\ M_{yz} &= k \int_0^{20} \int_0^{-(3/5)x+12} \int_0^{(5/12)y} x \, dz \, dy \, dx = 1000k \\ M_{xz} &= k \int_0^{20} \int_0^{-(3/5)x+12} \int_0^{(5/12)y} y \, dz \, dy \, dx = 1200k \\ M_{xy} &= k \int_0^{20} \int_0^{-(3/5)x+12} \int_0^{(5/12)y} z \, dz \, dy \, dx = 250k \\ \bar{x} &= \frac{M_{yz}}{m} = \frac{1000k}{200k} = 5 \\ \bar{y} &= \frac{M_{xz}}{m} = \frac{1200k}{200k} = 6 \\ \bar{z} &= \frac{M_{xy}}{m} = \frac{250k}{200k} = \frac{5}{4} \end{aligned}$$



47. (a) $I_x = k \int_0^a \int_0^a \int_0^a (y^2 + z^2) \, dx \, dy \, dz = ka \int_0^a \int_0^a (y^2 + z^2) \, dy \, dz$

$$= ka \int_0^a \left[\frac{1}{3}y^3 + z^2y \right]_0^a dz = ka \int_0^a \left(\frac{1}{3}a^3 + az^2 \right) dz = \left[ka \left(\frac{1}{3}a^3z + \frac{1}{3}az^3 \right) \right]_0^a = \frac{2ka^5}{3}$$

$I_x = I_y = I_z = \frac{2ka^5}{3}$ by symmetry

(b) $I_x = k \int_0^a \int_0^a \int_0^a (y^2 + z^2)xyz \, dx \, dy \, dz = \frac{ka^2}{2} \int_0^a \int_0^a (y^3z + yz^3) \, dy \, dz$

$$= \frac{ka^2}{2} \int_0^a \left[\frac{y^4z}{4} + \frac{y^2z^3}{2} \right]_0^a dz = \frac{ka^4}{8} \int_0^a (a^2z + 2z^3) \, dz = \left[\frac{ka^4}{8} \left(\frac{a^2z^2}{2} + \frac{2z^4}{4} \right) \right]_0^a = \frac{ka^8}{8}$$

$I_x = I_y = I_z = \frac{ka^8}{8}$ by symmetry

$$\begin{aligned}
 49. \quad (a) \quad I_x &= k \int_0^4 \int_0^4 \int_0^{4-x} (y^2 + z^2) dz dy dx = k \int_0^4 \int_0^4 \left[y^2(4-x) + \frac{1}{3}(4-x)^3 \right] dy dx \\
 &= k \int_0^4 \left[\frac{y^3}{3}(4-x) + \frac{y}{3}(4-x)^3 \right]_0^4 dx = k \int_0^4 \left[\frac{64}{3}(4-x) + \frac{4}{3}(4-x)^3 \right] dx \\
 &= k \left[-\frac{32}{3}(4-x)^2 - \frac{1}{3}(4-x)^4 \right]_0^4 = 256k
 \end{aligned}$$

$$\begin{aligned}
 I_y &= k \int_0^4 \int_0^4 \int_0^{4-x} (x^2 + z^2) dz dy dx = k \int_0^4 \int_0^4 \left[x^2(4-x) + \frac{1}{3}(4-x)^3 \right] dy dx \\
 &= 4k \int_0^4 \left[4x^2 - x^3 + \frac{1}{3}(4-x)^3 \right] dx = 4k \left[\frac{4}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{12}(4-x)^4 \right]_0^4 = \frac{512k}{3}
 \end{aligned}$$

$$\begin{aligned}
 I_z &= k \int_0^4 \int_0^4 \int_0^{4-x} (x^2 + y^2) dz dy dx = k \int_0^4 \int_0^4 (x^2 + y^2)(4-x) dy dx \\
 &= k \int_0^4 \left[\left(x^2 y + \frac{y^3}{3} \right) (4-x) \right]_0^4 dx = k \int_0^4 \left(4x^2 + \frac{64}{3} \right) (4-x) dx = 256k
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad I_x &= k \int_0^4 \int_0^4 \int_0^{4-x} y(y^2 + z^2) dz dy dx = k \int_0^4 \int_0^4 \left[y^3(4-x) + \frac{1}{3}y(4-x)^3 \right] dy dx \\
 &= k \int_0^4 \left[\frac{y^4}{4}(4-x) + \frac{y^2}{6}(4-x)^3 \right]_0^4 dx = k \int_0^4 \left[64(4-x) + \frac{8}{3}(4-x)^3 \right] dx \\
 &= k \left[-32(4-x)^2 - \frac{2}{3}(4-x)^4 \right]_0^4 = \frac{2048k}{3}
 \end{aligned}$$

$$\begin{aligned}
 I_y &= k \int_0^4 \int_0^4 \int_0^{4-x} y(x^2 + z^2) dz dy dx = k \int_0^4 \int_0^4 \left[x^2 y(4-x) + \frac{1}{3}y(4-x)^3 \right] dy dx \\
 &= 8k \int_0^4 \left[4x^2 - x^3 + \frac{1}{3}(4-x)^3 \right] dx = 8k \left[\frac{4}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{12}(4-x)^4 \right]_0^4 = \frac{1024k}{3}
 \end{aligned}$$

$$\begin{aligned}
 I_z &= k \int_0^4 \int_0^4 \int_0^{4-x} y(x^2 + y^2) dz dy dx = k \int_0^4 \int_0^4 (x^2 y + y^3)(4-x) dx \\
 &= k \int_0^4 \left[\left(\frac{x^2 y^2}{2} + \frac{y^4}{4} \right) (4-x) \right]_0^4 dx = k \int_0^4 (8x^2 + 64)(4-x) dx \\
 &= 8k \int_0^4 (32 - 8x + 4x^2 - x^3) dx = \left[8k \left(32x - 4x^2 + \frac{4}{3}x^3 - \frac{1}{4}x^4 \right) \right]_0^4 = \frac{2048k}{3}
 \end{aligned}$$

$$\begin{aligned}
 51. \quad I_{xy} &= k \int_{-L/2}^{L/2} \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} z^2 dz dx dy = k \int_{-L/2}^{L/2} \int_{-a}^a \frac{2}{3}(a^2 - x^2)\sqrt{a^2 - x^2} dx dy \\
 &= \frac{2}{3} \int_{-L/2}^{L/2} k \left[\frac{a^2}{2} \left(x\sqrt{a^2 - x^2} + a^2 \arcsin \frac{x}{a} \right) - \frac{1}{8} \left(x(2x^2 - a^2)\sqrt{x^2 - a^2} + a^4 \arcsin \frac{x}{a} \right) \right]_{-a}^a dy \\
 &= \frac{2k}{3} \int_{-L/2}^{L/2} 2 \left(\frac{a^4 \pi}{4} - \frac{a^4 \pi}{16} \right) dy = \frac{a^4 \pi L k}{4}
 \end{aligned}$$

Since $m = \pi a^2 L k$, $I_{xy} = ma^2/4$.

—CONTINUED—

51. —CONTINUED—

$$\begin{aligned}
 I_{xz} &= k \int_{-L/2}^{L/2} \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2 dz dx dy = 2k \int_{-L/2}^{L/2} \int_{-a}^a y^2 \sqrt{a^2-x^2} dx dy \\
 &= 2k \int_{-L/2}^{L/2} \left[\frac{y^2}{2} \left(x\sqrt{a^2-x^2} + a^2 \arcsin \frac{x}{a} \right) \right]_{-a}^a dy = k\pi a^2 \int_{-L/2}^{L/2} y^2 dy = \frac{2k\pi a^2}{3} \left(\frac{L^3}{8} \right) = \frac{1}{12} mL^2
 \end{aligned}$$

$$\begin{aligned}
 I_{yz} &= k \int_{-L/2}^{L/2} \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2 dz dx dy = 2k \int_{-L/2}^{L/2} \int_{-a}^a x^2 \sqrt{a^2-x^2} dx dy \\
 &= 2k \int_{-L/2}^{L/2} \frac{1}{8} \left[x(2x^2-a^2)\sqrt{a^2-x^2} + a^4 \arcsin \frac{x}{a} \right]_{-a}^a dy = \frac{ka^4\pi}{4} \int_{-L/2}^{L/2} dy = \frac{ka^4\pi L}{4} = \frac{ma^2}{4}
 \end{aligned}$$

$$I_x = I_{xy} + I_{xz} = \frac{ma^2}{4} + \frac{mL^2}{12} = \frac{m}{12}(3a^2 + L^2)$$

$$I_y = I_{xy} + I_{yz} = \frac{ma^2}{4} + \frac{ma^2}{4} = \frac{ma^2}{2}$$

$$I_z = I_{xz} + I_{yz} = \frac{mL^2}{12} + \frac{ma^2}{4} = \frac{m}{12}(3a^2 + L^2)$$

$$53. \int_{-1}^1 \int_{-1}^1 \int_0^{1-x} (x^2 + y^2) \sqrt{x^2 + y^2 + z^2} dz dy dx$$

55. See the definition, page 978.

See Theorem 13.4, page 979.

57. (a) The annular solid on the right has the greater density.

(b) The annular solid on the right has the greater movement of inertia.

(c) The solid on the left will reach the bottom first. The solid on the right has a greater resistance to rotational motion.

Section 13.7 Triple Integrals in Cylindrical and Spherical Coordinates

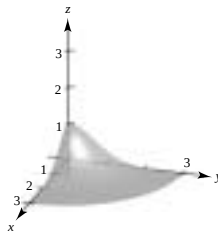
$$\begin{aligned}
 1. \int_0^4 \int_0^{\pi/2} \int_0^2 r \cos \theta dr d\theta dz &= \int_0^4 \int_0^{\pi/2} \left[\frac{r^2}{2} \cos \theta \right]_0^2 d\theta dz \\
 &= \int_0^4 \int_0^{\pi/2} 2 \cos \theta d\theta dz = \int_0^4 \left[2 \sin \theta \right]_0^{\pi/2} dz = \int_0^4 2 dz = 8
 \end{aligned}$$

$$\begin{aligned}
 3. \int_0^{\pi/2} \int_0^{2 \cos^2 \theta} \int_0^{4-r^2} r \sin \theta dz dr d\theta &= \int_0^{\pi/2} \int_0^{2 \cos^2 \theta} r(4-r^2) \sin \theta dr d\theta = \int_0^{\pi/2} \left[\left(2r^2 - \frac{r^4}{4} \right) \sin \theta \right]_0^{2 \cos^2 \theta} d\theta \\
 &= \int_0^{\pi/2} [8 \cos^4 \theta - 4 \cos^8 \theta] \sin \theta d\theta = \left[-\frac{8 \cos^5 \theta}{5} + \frac{4 \cos^9 \theta}{9} \right]_0^{\pi/2} = \frac{52}{45}
 \end{aligned}$$

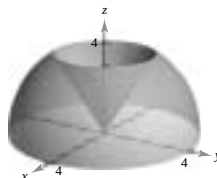
$$5. \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos \phi} \rho^2 \sin \phi d\rho d\phi d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/4} \cos^3 \phi \sin \phi d\phi d\theta = -\frac{1}{12} \int_0^{2\pi} \left[\cos^4 \phi \right]_0^{\pi/4} d\theta = \frac{\pi}{8}$$

$$7. \int_0^4 \int_0^z \int_0^{\pi/2} r e^r d\theta dr dz = \pi(e^4 + 3)$$

$$\begin{aligned}
 9. \int_0^{\pi/2} \int_0^3 \int_0^{e^{-r^2}} r \, dz \, dr \, d\theta &= \int_0^{\pi/2} \int_0^3 r e^{-r^2} \, dr \, d\theta \\
 &= \int_0^{\pi/2} \left[-\frac{1}{2} e^{-r^2} \right]_0^3 d\theta \\
 &= \int_0^{\pi/2} \frac{1}{2} (1 - e^{-9}) d\theta \\
 &= \frac{\pi}{4} (1 - e^{-9})
 \end{aligned}$$



$$\begin{aligned}
 11. \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^4 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta &= \frac{64}{3} \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \sin \phi \, d\phi \, d\theta \\
 &= \frac{64}{3} \int_0^{2\pi} \left[-\cos \phi \right]_{\pi/6}^{\pi/2} d\theta \\
 &= \frac{32\sqrt{3}}{3} \int_0^{2\pi} d\theta \\
 &= \frac{64\sqrt{3}\pi}{3}
 \end{aligned}$$



$$\begin{aligned}
 13. (a) \int_0^{2\pi} \int_0^2 \int_{r^2}^2 r^2 \cos \theta \, dz \, dr \, d\theta &= 0 \\
 (b) \int_0^{2\pi} \int_0^{\arctan(1/2)} \int_0^{4 \sec \phi} \rho^3 \sin^2 \phi \cos \theta \, d\rho \, d\phi \, d\theta &+ \int_0^{2\pi} \int_{\arctan(1/2)}^{\pi/2} \int_0^{\cot \phi \csc \phi} \rho^3 \sin^2 \phi \cos \theta \, d\rho \, d\phi \, d\theta = 0
 \end{aligned}$$

$$\begin{aligned}
 15. (a) \int_0^{2\pi} \int_0^a \int_a^{a+\sqrt{a^2-r^2}} r^2 \cos \theta \, dz \, dr \, d\theta &= 0 \\
 (b) \int_0^{\pi/4} \int_0^{2\pi} \int_{a \sec \phi}^{2a \cos \phi} \rho^3 \sin^2 \phi \cos \theta \, d\rho \, d\theta \, d\phi &= 0
 \end{aligned}$$

$$\begin{aligned}
 17. V &= 4 \int_0^{\pi/2} \int_0^{a \cos \theta} \int_0^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^{a \cos \theta} r \sqrt{a^2-r^2} \, dr \, d\theta \\
 &= \frac{4}{3} a^3 \int_0^{\pi/2} (1 - \sin^3 \theta) \, d\theta = \frac{4}{3} a^3 \left[\theta + \frac{1}{3} \cos \theta (\sin^2 \theta + 2) \right]_0^{\pi/2} = \frac{4}{3} a^3 \left(\frac{\pi}{2} - \frac{2}{3} \right) = \frac{2a^3}{9} (3\pi - 4)
 \end{aligned}$$

$$\begin{aligned}
 19. V &= 2 \int_0^{\pi} \int_0^{a \cos \theta} \int_0^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta \\
 &= 2 \int_0^{\pi} \int_0^{a \cos \theta} r \sqrt{a^2-r^2} \, dr \, d\theta \\
 &= 2 \int_0^{\pi} \left[-\frac{1}{3} (a^2-r^2)^{3/2} \right]_0^{a \cos \theta} d\theta \\
 &= \frac{2a^3}{3} \int_0^{\pi} (1 - \sin^3 \theta) \, d\theta \\
 &= \frac{2a^3}{3} \left[\theta + \cos \theta - \frac{\cos^3 \theta}{3} \right]_0^{\pi} \\
 &= \frac{2a^3}{9} (3\pi - 4)
 \end{aligned}$$

$$\begin{aligned}
 21. m &= \int_0^{2\pi} \int_0^2 \int_0^{9-r \cos \theta - 2r \sin \theta} (kr) \, dz \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^2 kr^2 (9 - r \cos \theta - 2r \sin \theta) \, dr \, d\theta \\
 &= \int_0^{2\pi} k \left[3r^3 - \frac{r^4}{4} \cos \theta - \frac{r^4}{2} \sin \theta \right]_0^2 d\theta \\
 &= \int_0^{2\pi} k [24 - 4 \cos \theta - 8 \sin \theta] \, d\theta \\
 &= k \left[24\theta - 4 \sin \theta + 8 \cos \theta \right]_0^{2\pi} \\
 &= k [48\pi + 8 - 8] = 48k\pi
 \end{aligned}$$

$$23. \quad z = h - \frac{h}{r_0} \sqrt{x^2 + y^2} = \frac{h}{r_0} (r_0 - r)$$

$$\begin{aligned} V &= 4 \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} r \, dz \, dr \, d\theta \\ &= \frac{4h}{r_0} \int_0^{\pi/2} \int_0^{r_0} (r_0 r - r^2) \, dr \, d\theta \\ &= \frac{4h}{r_0} \int_0^{\pi/2} \frac{r_0^3}{6} \, d\theta \\ &= \frac{4h}{r_0} \left(\frac{r_0^3}{6} \right) \left(\frac{\pi}{2} \right) = \frac{1}{3} \pi r_0^2 h \end{aligned}$$

$$\begin{aligned} 27. \quad I_z &= 4k \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} r^3 \, dz \, dr \, d\theta \\ &= \frac{4kh}{r_0} \int_0^{\pi/2} \int_0^{r_0} (r_0 r^3 - r^4) \, dr \, d\theta \\ &= \frac{4kh}{r_0} \left(\frac{r_0^5}{20} \right) \left(\frac{\pi}{2} \right) \\ &= \frac{1}{10} k \pi r_0^4 h \end{aligned}$$

Since the mass of the core is $m = kV = k(\frac{1}{3}\pi r_0^2 h)$ from Exercise 23, we have $k = 3m/\pi r_0^2 h$. Thus,

$$\begin{aligned} I_z &= \frac{1}{10} k \pi r_0^4 h \\ &= \frac{1}{10} \left(\frac{3m}{\pi r_0^2 h} \right) \pi r_0^4 h \\ &= \frac{3}{10} m r_0^2 \end{aligned}$$

$$31. \quad V = \int_0^{2\pi} \int_0^\pi \int_0^{4 \sin \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 16\pi^2$$

$$25. \quad \rho = k\sqrt{x^2 + y^2} = kr$$

$\bar{x} = \bar{y} = 0$ by symmetry

$$\begin{aligned} m &= 4k \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} r^2 \, dz \, dr \, d\theta \\ &= \frac{1}{6} k \pi r_0^3 h \\ M_{xy} &= 4k \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} r^2 z \, dz \, dr \, d\theta \\ &= \frac{1}{30} k \pi r_0^3 h^2 \\ \bar{z} &= \frac{M_{xy}}{m} = \frac{k \pi r_0^3 h^2 / 30}{k \pi r_0^3 h / 6} = \frac{h}{5} \end{aligned}$$

$$29. \quad m = k(\pi b^2 h - \pi a^2 h) = k\pi h(b^2 - a^2)$$

$$\begin{aligned} I_z &= 4k \int_0^{\pi/2} \int_a^b \int_0^h r^3 \, dz \, dr \, d\theta \\ &= 4kh \int_0^{\pi/2} \int_a^b r^3 \, dr \, d\theta \\ &= kh \int_0^{\pi/2} (b^4 - a^4) \, d\theta \\ &= \frac{k\pi(b^4 - a^4)h}{2} \\ &= \frac{k\pi(b^2 - a^2)(b^2 + a^2)h}{2} \\ &= \frac{1}{2} m(a^2 + b^2) \end{aligned}$$

$$\begin{aligned} 33. \quad m &= 8k \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \rho^3 \sin \phi \, d\rho \, d\theta \, d\phi \\ &= 2ka^4 \int_0^{\pi/2} \int_0^{\pi/2} \sin \phi \, d\theta \, d\phi \\ &= k\pi a^4 \int_0^{\pi/2} \sin \phi \, d\phi \\ &= \left[k\pi a^4 (-\cos \phi) \right]_0^{\pi/2} \\ &= k\pi a^4 \end{aligned}$$

$$35. \quad m = \frac{2}{3}k\pi r^3$$

$\bar{x} = \bar{y} = 0$ by symmetry

$$\begin{aligned} M_{xy} &= 4k \int_0^{\pi/2} \int_0^{\pi/2} \int_0^r \rho^3 \cos \phi \sin \phi \, d\rho \, d\theta \, d\phi \\ &= \frac{1}{2}kr^4 \int_0^{\pi/2} \int_0^{\pi/2} \sin 2\phi \, d\theta \, d\phi \\ &= \frac{kr^4\pi}{4} \int_0^{\pi/2} \sin 2\phi \, d\phi \\ &= \left[-\frac{1}{8}k\pi r^4 \cos 2\phi \right]_0^{\pi/2} = \frac{1}{4}k\pi r^4 \\ \bar{z} &= \frac{M_{xy}}{m} = \frac{k\pi r^4/4}{2k\pi r^3/3} = \frac{3r}{8} \end{aligned}$$

$$39. \quad x = r \cos \theta \quad x^2 + y^2 = r^2$$

$$y = r \sin \theta \quad \tan \theta = \frac{y}{x}$$

$$z = z \quad z = z$$

43. (a) $r = r_0$: right circular cylinder about z-axis

$\theta = \theta_0$: plane parallel to z-axis

$z = z_0$: plane parallel to xy-plane

$$\begin{aligned} 37. \quad I_z &= 4k \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \int_0^{\cos \phi} \rho^4 \sin^3 \phi \, d\rho \, d\theta \, d\phi \\ &= \frac{4}{5}k \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \cos^5 \phi \sin^3 \phi \, d\theta \, d\phi \\ &= \frac{2}{5}k\pi \int_{\pi/4}^{\pi/2} \cos^5 \phi (1 - \cos^2 \phi) \sin \phi \, d\phi \\ &= \left[\frac{2}{5}k\pi \left(-\frac{1}{6} \cos^6 \phi + \frac{1}{8} \cos^8 \phi \right) \right]_{\pi/4}^{\pi/2} \\ &= \frac{k\pi}{192} \end{aligned}$$

$$41. \quad \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r \cos \theta, r \sin \theta)}^{h_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) r \, dz \, dr \, d\theta$$

(b) $\rho = \rho_0$: sphere of radius ρ_0

$\theta = \theta_0$: plane parallel to z-axis

$\phi = \phi_0$: cone

$$\begin{aligned} 45. \quad 16 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dw \, dz \, dy \, dx \\ &= 16 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \sqrt{a^2-x^2-y^2-z^2} \, dz \, dy \, dx \\ &= 16 \int_0^{\pi/2} \int_0^a \int_0^{\sqrt{a^2-r^2}} \sqrt{(a^2-r^2)-z^2} \, dz (r \, dr \, d\theta) \\ &= 16 \int_0^{\pi/2} \int_0^a \frac{1}{2} \left[z \sqrt{(a^2-r^2)-z^2} + (a^2-r^2) \arcsin \frac{z}{\sqrt{a^2-r^2}} \right]_0^{\sqrt{a^2-r^2}} r \, dr \, d\theta \\ &= 8 \int_0^{\pi/2} \int_0^a \frac{\pi}{2} (a^2-r^2) r \, dr \, d\theta \\ &= 4\pi \int_0^{\pi/2} \left[\frac{a^2 r^2}{2} - \frac{r^4}{4} \right]_0^a d\theta \\ &= a^4 \pi \int_0^{\pi/2} d\theta = \frac{a^4 \pi^2}{2} \end{aligned}$$

Section 13.8 Change of Variables: Jacobians

1. $x = -\frac{1}{2}(u - v)$

$$y = \frac{1}{2}(u + v)$$

$$\begin{aligned}\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} &= \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \\ &= -\frac{1}{2}\end{aligned}$$

5. $x = u \cos \theta - v \sin \theta$

$$y = u \sin \theta + v \cos \theta$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \cos^2 \theta + \sin^2 \theta = 1$$

7. $x = e^u \sin v$

$$y = e^u \cos v$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = (e^u \sin v)(-e^u \sin v) - (e^u \cos v)(e^u \cos v) = -e^{2u}$$

9. $x = 3u + 2v$

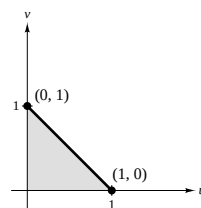
$$y = 3v$$

$$v = \frac{y}{3}$$

$$u = \frac{x - 2v}{3} = \frac{x - 2(y/3)}{3}$$

$$= \frac{x}{3} - \frac{2y}{9}$$

(x, y)	(u, v)
$(0, 0)$	$(0, 0)$
$(3, 0)$	$(1, 0)$
$(2, 3)$	$(0, 1)$



11. $x = \frac{1}{2}(u + v)$

$$y = \frac{1}{2}(u - v)$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}$$

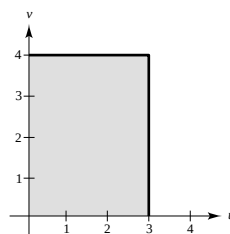
$$\begin{aligned}\iint_R 4(x^2 + y^2) dA &= \int_{-1}^1 \int_{-1}^1 4 \left[\frac{1}{4}(u + v)^2 + \frac{1}{4}(u - v)^2 \right] \left(\frac{1}{2} \right) dv du \\ &= \int_{-1}^1 \int_{-1}^1 (u^2 + v^2) dv du = \int_{-1}^1 2 \left(u^2 + \frac{1}{3} \right) du = \left[2 \left(\frac{u^3}{3} + \frac{u}{3} \right) \right]_{-1}^1 = \frac{8}{3}\end{aligned}$$

13. $x = u + v$

$$y = u$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = (1)(0) - (1)(1) = -1$$

$$\iint_R y(x - y) dA = \int_0^3 \int_0^4 uv(1) dv du = \int_0^3 8u du = 36$$

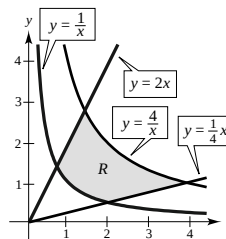


$$15. \iint_R e^{-xy/2} dA$$

$$R: y = \frac{x}{4}, y = 2x, y = \frac{1}{x}, y = \frac{4}{x}$$

$$x = \sqrt{v/u}, y = \sqrt{uv} \Rightarrow u = \frac{y}{x}, v = xy$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} \frac{v^{1/2}}{u^{3/2}} & \frac{1}{2} \frac{1}{u^{1/2} v^{1/2}} \\ \frac{1}{2} \frac{v^{1/2}}{u^{1/2}} & \frac{1}{2} \frac{u^{1/2}}{v^{1/2}} \end{vmatrix} = -\frac{1}{4} \left(\frac{1}{u} + \frac{1}{u} \right) = -\frac{1}{2u}$$



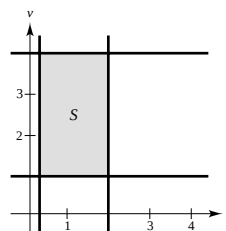
Transformed Region:

$$y = \frac{1}{x} \Rightarrow yx = 1 \Rightarrow v = 1$$

$$y = \frac{4}{x} \Rightarrow yx = 4 \Rightarrow v = 4$$

$$y = 2x \Rightarrow \frac{y}{x} = 2 \Rightarrow u = 2$$

$$y = \frac{x}{4} \Rightarrow \frac{y}{x} = \frac{1}{4} \Rightarrow u = \frac{1}{4}$$



$$\begin{aligned} \iint_R e^{-xy/2} dA &= \int_{1/4}^2 \int_1^4 e^{-v/2} \left(\frac{1}{2u} \right) dv du = - \int_{1/4}^2 \left[\frac{e^{-v/2}}{u} \right]_1^4 du = - \int_{1/4}^2 (e^{-2} - e^{-1/2}) \frac{1}{u} du \\ &= - \left[(e^{-2} - e^{-1/2}) \ln u \right]_{1/4}^2 = -(e^{-2} - e^{-1/2}) \left(\ln 2 - \ln \frac{1}{4} \right) = (e^{-1/2} - e^{-2}) \ln 8 \approx 0.9798 \end{aligned}$$

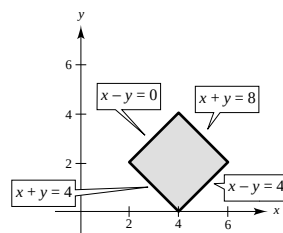
$$17. u = x + y = 4, \quad v = x - y = 0$$

$$u = x + y = 8, \quad v = x - y = 4$$

$$x = \frac{1}{2}(u + v) \quad y = \frac{1}{2}(u - v)$$

$$\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{2}$$

$$\begin{aligned} \iint_R (x + y) e^{x-y} dA &= \int_4^8 \int_0^4 u e^v \left(\frac{1}{2} \right) dv du \\ &= \frac{1}{2} \int_4^8 u (e^4 - 1) du = \left[\frac{1}{4} u^2 (e^4 - 1) \right]_4^8 = 12(e^4 - 1) \end{aligned}$$



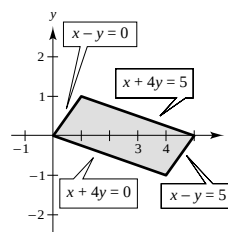
$$19. u = x + 4y = 0, \quad v = x - y = 0$$

$$u = x + 4y = 5, \quad v = x - y = 5$$

$$x = \frac{1}{5}(u + 4v), \quad y = \frac{1}{5}(u - v)$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \left(\frac{1}{5} \right) \left(-\frac{1}{5} \right) - \left(\frac{1}{5} \right) \left(\frac{4}{5} \right) = -\frac{1}{5}$$

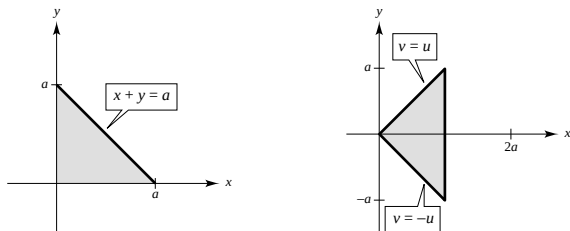
$$\begin{aligned} \iint_R \sqrt{(x-y)(x+4y)} dA &= \int_0^5 \int_0^5 \sqrt{uv} \left(\frac{1}{5} \right) du dv \\ &= \int_0^5 \left[\frac{1}{5} \left(\frac{2}{3} \right) u^{3/2} \sqrt{v} \right]_0^5 dv = \left[\frac{2\sqrt{5}}{3} \left(\frac{2}{3} \right) v^{3/2} \right]_0^5 = \frac{100}{9} \end{aligned}$$



21. $u = x + y, v = x - y, x = \frac{1}{2}(u + v), y = \frac{1}{2}(u - v)$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = -\frac{1}{2}$$

$$\iint_R \sqrt{x+y} dA = \int_0^a \int_{-u}^u \sqrt{u} \left(\frac{1}{2}\right) dv du = \int_0^a u \sqrt{u} du = \left[\frac{2}{5} u^{5/2}\right]_0^a = \frac{2}{5} a^{5/2}$$



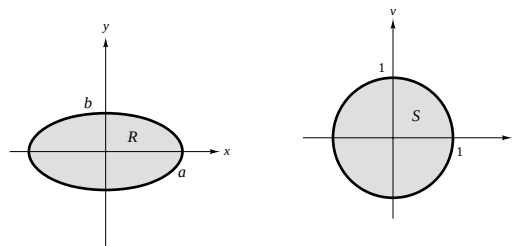
23. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, x = au, y = bv$

$$\frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} = 1$$

$$u^2 + v^2 = 1$$

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$u^2 + v^2 = 1$$



(b) $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$
 $= (a)(b) - (0)(0) = ab$

(c) $A = \iint_S ab dS$
 $= ab(\pi(1)^2) = \pi ab$

25. Jacobian $= \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$

27. $x = u(1 - v), y = uv(1 - w), z = uvw$

$$\begin{aligned} \frac{\partial(x, y, z)}{\partial(u, v, w)} &= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix} = (1-v)[u^2v(1-w) + u^2vw] + u[uv^2(1-w) + uv^2w] \\ &= (1-v)(u^2v) + u(uv^2) \\ &= u^2v \end{aligned}$$

29. $x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi$

$$\begin{aligned} \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} &= \begin{vmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \\ \cos \phi & 0 & -\rho \sin \phi \end{vmatrix} \\ &= \cos \phi [-\rho^2 \sin \phi \cos \phi \sin^2 \theta - \rho^2 \sin \phi \cos \phi \cos^2 \theta] - \rho \sin \phi [\rho \sin^2 \phi \cos^2 \theta + \rho \sin^2 \phi \sin^2 \theta] \\ &= \cos \phi [-\rho^2 \sin \phi \cos \phi (\sin^2 \theta + \cos^2 \theta)] - \rho \sin \phi [\rho \sin^2 \phi (\cos^2 \theta + \sin^2 \theta)] \\ &= -\rho^2 \sin \phi \cos^2 \phi - \rho^2 \sin^3 \phi \\ &= -\rho^2 \sin \phi (\cos^2 \phi + \sin^2 \phi) \\ &= -\rho^2 \sin \phi \end{aligned}$$

CHAPTER 13

Multiple Integration

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CHAPTER 13

Multiple Integration

Section 13.1 Iterated Integrals and Area in the Plane

Solutions to Even-Numbered Exercises

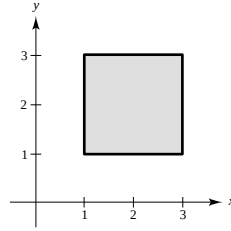
2. $\int_x^{x^2} \frac{y}{x} dy = \left[\frac{1}{2} \frac{y^2}{x} \right]_x^{x^2} = \frac{1}{2} \left(\frac{x^4}{x} - \frac{x^2}{x} \right) = \frac{x}{2} (x^2 - 1)$
4. $\int_0^{\cos y} y dx = \left[yx \right]_0^{\cos y} = y \cos y$
6. $\int_{x^3}^{\sqrt{x}} (x^2 + 3y^2) dy = \left[x^2 y + y^3 \right]_{x^3}^{\sqrt{x}} = (x^2 \sqrt{x} + (\sqrt{x})^3) - (x^2 x^3 + (x^3)^3) = x^{5/2} + x^{3/2} - x^5 - x^9$
8. $\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2 + y^2) dx = \left[\frac{1}{3} x^3 + y^2 x \right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} = 2 \left[\frac{1}{3} (1 - y^2)^{3/2} + y^2 (1 - y^2)^{1/2} \right] = \frac{2\sqrt{1-y^2}}{3} (1 + 2y^2)$
10. $\int_y^{\pi/2} \sin^3 x \cos y dx = \int_y^{\pi/2} (1 - \cos^2 x) \sin x \cos y dx$
 $= \left[\left(-\cos x + \frac{1}{3} \cos^3 x \right) \cos y \right]_y^{\pi/2} = \left(\cos y - \frac{1}{3} \cos^3 y \right) \cos y$
12. $\int_{-1}^1 \int_{-2}^2 (x^2 - y^2) dy dx = \int_{-1}^1 \left[x^2 y - \frac{y^3}{3} \right]_{-2}^2 dx = \int_{-1}^1 \left[2x^2 - \frac{8}{3} + 2x^2 - \frac{8}{3} \right] dx$
 $= \int_{-1}^1 \left(4x^2 - \frac{16}{3} \right) dx = \left[\frac{4x^3}{3} - \frac{16}{3} x \right]_{-1}^1 = \left(\frac{4}{3} - \frac{16}{3} \right) - \left(-\frac{4}{3} + \frac{16}{3} \right) = -8$
14. $\int_{-4}^4 \int_0^{x^2} \sqrt{64 - x^3} dy dx = \int_{-4}^4 \left[y \sqrt{64 - x^3} \right]_0^{x^2} dx$
 $= \int_{-4}^4 \sqrt{64 - x^3} x^2 dx = \left[-\frac{2}{9} (64 - x^3)^{3/2} \right]_{-4}^4 = 0 + \frac{2}{9} (128)^{3/2} = \frac{2048}{9} \sqrt{2}$
16. $\int_0^2 \int_y^{2y} (10 + 2x^2 + 2y^2) dx dy = \int_0^2 \left[10x + \frac{2x^3}{3} + 2y^2 x \right]_y^{2y} dy = \int_0^2 \left[\left(20y + \frac{16}{3} y^3 + 4y^3 \right) - \left(10y + \frac{2}{3} y^3 + 2y^3 \right) \right] dy$
 $= \int_0^2 \left[10y + \frac{14}{3} y^3 + 2y^3 \right] dy = \left[5y^2 + \frac{7y^4}{6} + \frac{y^4}{2} \right]_0^2 = 20 + \frac{56}{3} + 8 = \frac{140}{3}$
18. $\int_0^2 \int_{3y^2-6y}^{2y-y^2} 3y dx dy = \int_0^2 \left[3xy \right]_{3y^2-6y}^{2y-y^2} dy = 3 \int_0^2 (8y^2 - 4y^3) dy = \left[3 \left(\frac{8}{3} y^3 - y^4 \right) \right]_0^2 = 16$
20. $\int_0^{\pi/2} \int_0^{2 \cos \theta} r dr d\theta = \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_0^{2 \cos \theta} d\theta = \int_0^{\pi/2} 2 \cos^2 \theta d\theta = \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} = \frac{\pi}{2}$
22. $\int_0^{\pi/4} \int_0^{\cos \theta} 3r^2 \sin \theta dr d\theta = \int_0^{\pi/4} \left[r^3 \sin \theta \right]_0^{\cos \theta} d\theta$
 $= \int_0^{\pi/4} \cos^3 \sin \theta d\theta = \left[-\frac{\cos^4 \theta}{4} \right]_0^{\pi/4} = -\frac{1}{4} \left[\left(\frac{1}{\sqrt{2}} \right)^4 - 1 \right] = \frac{3}{16}$

$$24. \int_0^3 \int_0^\infty \frac{x^2}{1+y^2} dy dx = \int_0^3 \left[x^2 \arctan y \right]_0^\infty dx = \int_0^3 x^2 \left(\frac{\pi}{2} \right) dx = \left[\frac{\pi}{2} \cdot \frac{x^3}{3} \right]_0^3 = \frac{9\pi}{2}$$

$$26. \int_0^\infty \int_0^\infty xye^{-(x^2+y^2)} dx dy = \int_0^\infty \left[-\frac{1}{2} ye^{-(x^2+y^2)} \right]_0^\infty dy = \int_0^\infty \frac{1}{2} ye^{-y^2} dy = \left[-\frac{1}{4} e^{-y^2} \right]_0^\infty = \frac{1}{4}$$

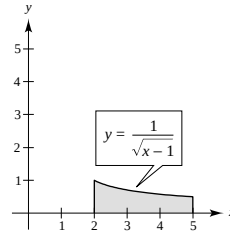
$$28. A = \int_1^3 \int_1^3 dy dx = \int_1^3 \left[y \right]_1^3 dx = \int_1^3 2 dx = \left[2x \right]_1^3 = 4$$

$$A = \int_1^3 \int_1^3 dx dy = \int_1^3 \left[x \right]_1^3 dy = \int_1^3 2 dy = \left[2y \right]_1^3 = 4$$



$$30. A = \int_2^5 \int_0^{1/\sqrt{x-1}} dy dx = \int_2^5 \left[y \right]_0^{1/\sqrt{x-1}} dx = \int_2^5 \frac{1}{\sqrt{x-1}} dx = \left[2\sqrt{x-1} \right]_2^5 = 2$$

$$\begin{aligned} A &= \int_0^{1/2} \int_2^5 dx dy + \int_{1/2}^1 \int_2^{1+(1/y^2)} dx dy \\ &= \int_0^{1/2} \left[x \right]_2^5 dy + \int_{1/2}^1 \left[x \right]_2^{1+(1/y^2)} dy \\ &= \int_0^{1/2} 3 dy + \int_{1/2}^1 \left(\frac{1}{y^2} - 1 \right) dy \\ &= \left[3y \right]_0^{1/2} + \left[-\frac{1}{y} - y \right]_{1/2}^1 = 2 \end{aligned}$$



$$32. A = \int_0^2 \int_0^{\sqrt{4-x^2}} dy dx$$

$$= \int_0^2 \sqrt{4-x^2} dx$$

$$= 4 \int_0^{\pi/2} \cos^2 \theta d\theta$$

$$= 2 \int_0^{\pi/2} (1 + \cos 2\theta) d\theta$$

$$= \left[2\left(\theta + \frac{1}{2} \sin 2\theta \right) \right]_0^{\pi/2} = \pi$$

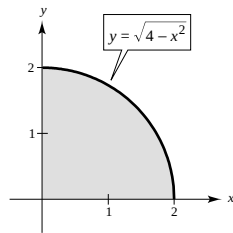
$$(x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta)$$

$$A = \int_0^2 \int_0^{\sqrt{4-y^2}} dx dy = \int_0^2 \sqrt{4-y^2} dy$$

$$= 4 \int_0^{\pi/2} \cos^2 \theta d\theta = 2 \int_0^{\pi/2} (1 + \cos 2\theta) d\theta$$

$$= \left[2\left(\theta + \frac{1}{2} \sin 2\theta \right) \right]_0^{\pi/2} = \pi$$

$$(y = 2 \sin \theta, dy = 2 \cos \theta d\theta, \sqrt{4-y^2} = 2 \cos \theta)$$



$$34. A = \int_0^4 \int_{x^{3/2}}^{2x} dy dx$$

$$= \int_0^4 \left[y \right]_{x^{3/2}}^{2x} dx$$

$$= \int_0^4 (2x - x^{3/2}) dx$$

$$= \left[x^2 - \frac{2}{5} x^{5/2} \right]_0^4$$

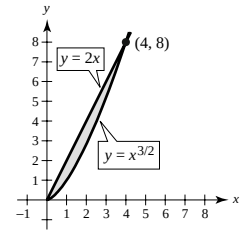
$$= 16 - \frac{2}{5}(32) = \frac{16}{5}$$

$$A = \int_0^8 \int_{y/2}^{y^{2/3}} dx dy$$

$$= \int_0^8 \left(y^{2/3} - \frac{y}{2} \right) dy$$

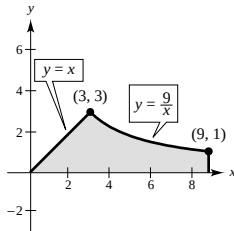
$$= \left[\frac{3}{5} y^{5/3} - \frac{y^2}{4} \right]_0^8$$

$$= \frac{3}{5}(32) - 16 = \frac{16}{5}$$

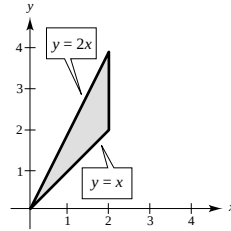


$$\begin{aligned}
 36. \quad A &= \int_0^3 \int_0^x dy \, dx + \int_3^9 \int_0^{9/x} dy \, dx \\
 &= \int_0^3 \left[y \right]_0^x dx + \int_3^9 \left[y \right]_0^{9/x} dx = \int_0^3 x \, dx + \int_3^9 \frac{9}{x} dx \\
 &= \left[\frac{1}{2} x^2 \right]_0^3 + \left[9 \ln x \right]_3^9 = \frac{9}{2} + 9(\ln 9 - \ln 3) \\
 &= \frac{9}{2}(1 + \ln 9)
 \end{aligned}$$

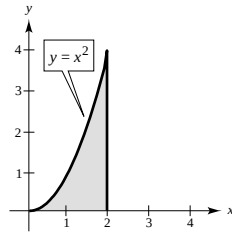
$$\begin{aligned}
 A &= \int_0^1 \int_y^9 dx \, dy + \int_1^3 \int_y^{9/y} dx \, dy \\
 &= \int_0^1 \left[x \right]_y^9 dy + \int_1^3 \left[x \right]_y^{9/y} dy \\
 &= \int_0^1 (9 - y) dy + \int_1^3 \left(\frac{9}{y} - y \right) dy \\
 &= \left[9y - \frac{1}{2} y^2 \right]_0^1 + \left[9 \ln y - \frac{1}{2} y^2 \right]_1^3 = \frac{9}{2}(1 + \ln 9)
 \end{aligned}$$



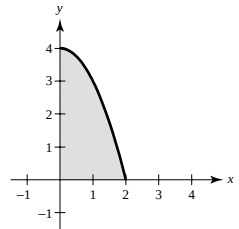
$$\begin{aligned}
 38. \quad A &= \int_0^2 \int_{y/2}^y dx \, dy + \int_2^4 \int_{y/2}^2 dx \, dy \\
 &= \int_0^2 \frac{y}{2} dy + \int_2^4 \left(2 - \frac{y}{2} \right) dy \\
 &= \left[\frac{y^2}{4} \right]_0^2 + \left[2y - \frac{y^2}{4} \right]_2^4 \\
 &= 1 + (4 - 3) = 2 \\
 A &= \int_0^2 \int_x^{2x} dy \, dx = \int_0^2 (2x - x) dx = \left[\frac{x^2}{2} \right]_0^2 = 2
 \end{aligned}$$



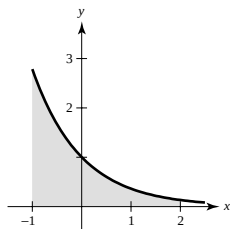
$$\begin{aligned}
 40. \quad \int_0^4 \int_{\sqrt{y}}^2 f(x, y) dx \, dy, \quad \sqrt{y} \leq x \leq 2, \quad 0 \leq y \leq 4 \\
 = \int_0^2 \int_0^{x^2} f(x, y) dy \, dx
 \end{aligned}$$



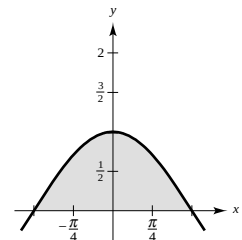
$$\begin{aligned}
 42. \quad \int_0^2 \int_0^{4-x^2} f(x, y) dy \, dx, \quad 0 \leq y \leq 4 - x^2, \quad 0 \leq x \leq 2 \\
 = \int_0^4 \int_0^{\sqrt{4-y}} f(x, y) dx \, dy
 \end{aligned}$$



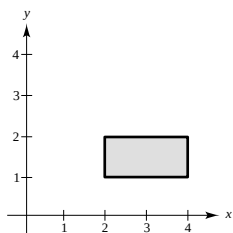
$$\begin{aligned}
 44. \quad \int_{-1}^2 \int_0^{e^{-x}} f(x, y) dy \, dx, \quad 0 \leq y \leq e^{-x}, \quad -1 \leq x \leq 2 \\
 = \int_0^{e^{-2}} \int_{-1}^2 f(x, y) dx \, dy + \int_{e^{-2}}^1 \int_{-\ln y}^2 f(x, y) dx \, dy
 \end{aligned}$$



$$\begin{aligned}
 46. \quad \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} f(x, y) dy \, dx, \quad 0 \leq y \leq \cos x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\
 = \int_0^1 \int_{-\arccos y}^{\arccos y} f(x, y) dx \, dy
 \end{aligned}$$

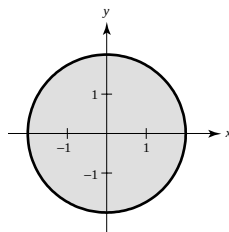


$$48. \int_1^2 \int_2^4 dx \, dy = \int_2^4 \int_1^2 dy \, dx = 2$$



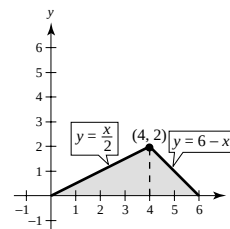
$$50. \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy \, dx = \int_{-2}^2 (\sqrt{4-x^2} + \sqrt{4-x^2}) \, dx = 4\pi$$

$$\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx \, dy = 4\pi$$



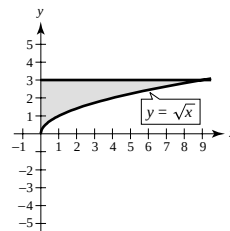
$$52. \int_0^4 \int_0^{x/2} dy \, dx + \int_4^6 \int_0^{6-x} dy \, dx = \int_0^4 \frac{x}{2} \, dx + \int_4^6 (6-x) \, dx = 4 + 2 = 6$$

$$\int_0^2 \int_{2y}^{6-y} dx \, dy = \int_0^2 (6-3y) \, dy = \left[6y - \frac{3y^2}{2} \right]_0^2 = 6$$

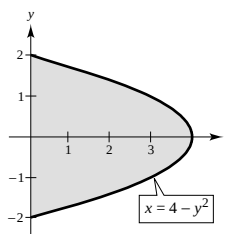


$$54. \int_0^9 \int_{\sqrt{x}}^3 dy \, dx = \int_0^9 (3 - \sqrt{x}) \, dx = \left[3x - \frac{2}{3}x^{3/2} \right]_0^9 = 27 - 18 = 9$$

$$\int_0^3 \int_0^{y^2} dx \, dy = \int_0^3 y^2 \, dy = \left[\frac{y^3}{3} \right]_0^3 = 9$$



$$56. \int_{-2}^2 \int_0^{4-y^2} dx \, dy = \int_0^4 \int_{-\sqrt{4-x}}^{\sqrt{4-x}} dy \, dx = \frac{32}{3}$$



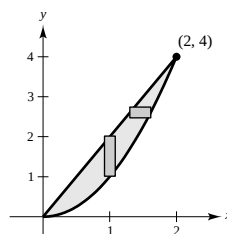
58. The first integral arises using vertical representative rectangles. The second integral arises using horizontal representative rectangles.

$$\int_0^2 \int_{x^2}^{2x} x \sin y \, dy \, dx = \int_0^2 (-x \cos(2x) + x \cos(x^2)) \, dx$$

$$= -\frac{1}{4} \cos(4) - \frac{1}{2} \sin(4) + \frac{1}{4}$$

$$\int_0^4 \int_{y/2}^{\sqrt{y}} x \sin y \, dx \, dy = \int_0^4 \left(\frac{1}{2} y \sin(y) - \frac{1}{8} y^2 \sin(y) \right) dy$$

$$= -\frac{1}{4} \cos(4) - \frac{1}{2} \sin(4) + \frac{1}{4}$$



$$\begin{aligned}
 60. \int_0^2 \int_x^2 e^{-y^2} dy dx &= \int_0^2 \int_0^y e^{-y^2} dx dy \\
 &= \int_0^2 \left[xe^{-y^2} \right]_0^y dy = \int_0^2 ye^{-y^2} dy = \left[-\frac{1}{2}e^{-y^2} \right]_0^2 = -\frac{1}{2}(e^{-4}) + \frac{1}{2}e^0 = \frac{1}{2} \left(1 - \frac{1}{e^4} \right) \approx 0.4908
 \end{aligned}$$

$$\begin{aligned}
 62. \int_0^2 \int_{y^2}^4 \sqrt{x} \sin x dx dy &= \int_0^4 \int_0^{\sqrt{x}} \sqrt{x} \sin x dy dx \\
 &= \int_0^4 \left[y\sqrt{x} \sin x \right]_0^{\sqrt{x}} dx = \int_0^4 x \sin x dx = \left[\sin x - x \cos x \right]_0^4 = \sin 4 - 4 \cos 4 \approx 1.858
 \end{aligned}$$

$$64. \int_0^1 \int_y^{2y} \sin(x+y) dx dy = \frac{\sin 2}{2} - \frac{\sin 3}{3} \approx 0.408$$

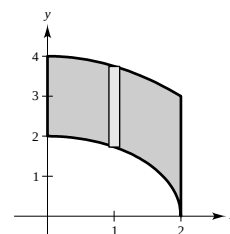
$$66. \int_0^a \int_0^{a-x} (x^2 + y^2) dy dx = \frac{a^4}{6}$$

$$68. (a) y = \sqrt{4-x^2} \Leftrightarrow x = \sqrt{4-y^2}$$

$$y = 4 - \frac{x^2}{4} \Leftrightarrow x = \sqrt{16-4y}$$

$$(b) \int_0^2 \int_{\sqrt{4-y^2}}^2 \frac{xy}{x^2+y^2+1} dx dy + \int_2^3 \int_0^2 \frac{xy}{x^2+y^2+1} dx dy + \int_3^4 \int_0^{\sqrt{16-4y}} \frac{xy}{x^2+y^2+1} dx dy$$

(c) Both orders of integration yield 1.11899.



$$70. \int_0^2 \int_x^2 \sqrt{16-x^3-y^3} dy dx \approx 6.8520$$

$$72. \int_0^{\pi/2} \int_0^{1+\sin \theta} 15\theta r dr d\theta = \frac{45\pi^2}{32} + \frac{135}{8} \approx 30.7541$$

74. A region is vertically simple if it is bounded on the left and right by vertical lines, and bounded on the top and bottom by functions of x . A region is horizontally simple if it is bounded on the top and bottom by horizontal lines, and bounded on the left and right by functions of y .

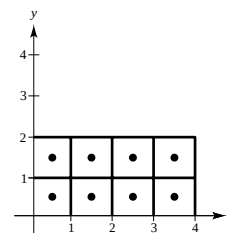
76. The integrations might be easier. See Exercise 59-62.

78. False, let $f(x, y) = x$.

Section 13.2 Double Integrals and Volume

For Exercises 2 and 4, $\Delta x_i = \Delta y_i = 1$ and the midpoints of the squares are

$$\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{2}, \frac{1}{2}\right), \left(\frac{5}{2}, \frac{1}{2}\right), \left(\frac{7}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{3}{2}\right), \left(\frac{5}{2}, \frac{3}{2}\right), \left(\frac{7}{2}, \frac{3}{2}\right).$$



$$2. f(x, y) = \frac{1}{2}x^2y$$

$$\sum_{i=1}^{\infty} f(x_i, y_i) \Delta x_i \Delta y_i = \frac{1}{16} + \frac{9}{16} + \frac{25}{16} + \frac{49}{16} + \frac{3}{16} + \frac{27}{16} + \frac{75}{16} + \frac{147}{16} = 21$$

$$\int_0^4 \int_0^2 \frac{1}{2}x^2y dy dx = \int_0^4 \left[\frac{x^2y^2}{4} \right]_0^2 dx = \int_0^4 x^2 dx = \left[\frac{x^3}{3} \right]_0^4 = \frac{64}{3} \approx 21.3$$

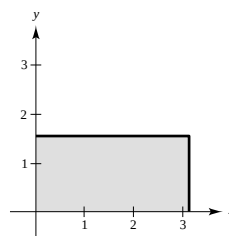
$$4. f(x, y) = \frac{1}{(x+1)(y+1)}$$

$$\sum_{i=1}^8 f(x_i, y_i) \Delta x_i \Delta y_i = \frac{4}{9} + \frac{4}{15} + \frac{4}{21} + \frac{4}{27} + \frac{4}{15} + \frac{4}{25} + \frac{4}{35} + \frac{4}{45} = \frac{7936}{4725} \approx 1.680$$

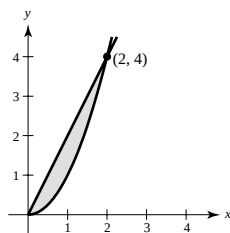
$$\begin{aligned} \int_0^4 \int_0^2 \frac{1}{(x+1)(y+1)} dy dx &= \int_0^4 \left[\frac{1}{x+1} \ln(y+1) \right]_0^2 dx \\ &= \int_0^4 \frac{\ln 3}{x+1} dx = \left[\ln 3 \cdot \ln(x+1) \right]_0^4 = (\ln 3)(\ln 5) \approx 1.768 \end{aligned}$$

$$6. \int_0^2 \int_0^2 f(x, y) dy dx \approx 4 + 2 + 8 + 6 = 20$$

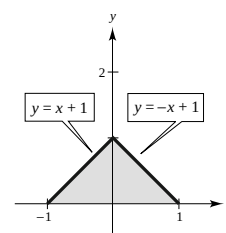
$$\begin{aligned} 8. \int_0^\pi \int_0^{\pi/2} \sin^2 x \cos^2 y dy dx &= \int_0^\pi \left[\frac{1}{2} \sin^2 x \left(y + \frac{1}{2} \sin 2y \right) \right]_0^{\pi/2} dx \\ &= \int_0^\pi \frac{1}{2} \sin^2 x \left(\frac{\pi}{2} \right) dx \\ &= \frac{\pi}{8} \int_0^\pi (1 - \cos 2x) dx \\ &= \left[\frac{\pi}{8} \left(x - \frac{1}{2} \sin 2x \right) \right]_0^\pi \\ &= \frac{\pi^2}{8} \end{aligned}$$



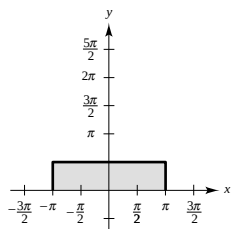
$$\begin{aligned} 10. \int_0^4 \int_{(1/2)y}^{\sqrt{y}} x^2 y^2 dx dy &= \int_0^4 \left[\frac{x^3 y^2}{3} \right]_{(1/2)y}^{\sqrt{y}} dy \\ &= \int_0^4 \left(\frac{y^{7/2}}{3} - \frac{y^5}{24} \right) dy \\ &= \left[\frac{2y^{9/2}}{27} - \frac{y^6}{144} \right]_0^4 \\ &= \frac{1024}{27} - \frac{256}{9} = \frac{256}{27} \end{aligned}$$



$$\begin{aligned} 12. \int_0^1 \int_{y-1}^0 e^{x+y} dx dy + \int_0^1 \int_0^{1-y} e^{x+y} dx dy &= \int_0^1 \left[e^{x+y} \right]_{y-1}^0 dy + \int_0^1 \left[e^{x+y} \right]_0^{1-y} dy \\ &= \int_0^1 (e - e^{2y-1}) dy \\ &= \left[ey - \frac{1}{2} e^{2y-1} \right]_0^1 \\ &= \frac{1}{2}(e + e^{-1}) \end{aligned}$$



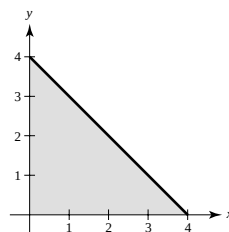
$$\begin{aligned}
 14. \int_0^{\pi/2} \int_{-\pi}^{\pi} \sin x \sin y \, dx \, dy &= \int_{-\pi}^{\pi} \int_0^{\pi/2} \sin x \sin y \, dy \, dx \\
 &= \int_{-\pi}^{\pi} \left[-\sin x \cos y \right]_0^{\pi/2} dx \\
 &= \int_{-\pi}^{\pi} \sin x \, dx \\
 &= 0
 \end{aligned}$$



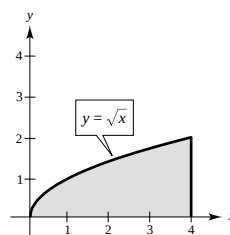
$$16. \int_0^4 \int_0^{4-x} x e^y \, dy \, dx = \int_0^4 \int_0^{4-y} x e^y \, dx \, dy$$

For the first integral, we obtain:

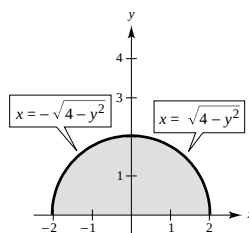
$$\begin{aligned}
 \int_0^4 \left[x e^y \right]_0^{4-x} - x \, dx &= \int_0^{4-x} (x e^4 - x) dx \\
 &= \left[-e^{4-x}(1+x) - \frac{x^2}{2} \right]_0^4 \\
 &= (-5-8) + e^4 = e^4 - 13.
 \end{aligned}$$



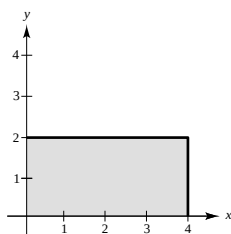
$$\begin{aligned}
 18. \int_0^2 \int_{y^2}^4 \frac{y}{1+x^2} \, dx \, dy &= \int_0^4 \int_0^{\sqrt{x}} \frac{y}{1+x^2} \, dy \, dx \\
 &= \frac{1}{2} \int_0^4 \left[\frac{y^2}{1+x^2} \right]_0^{\sqrt{x}} dx \\
 &= \frac{1}{2} \int_0^4 \frac{x}{1+x^2} dx \\
 &= \left[\frac{1}{4} \ln(1+x^2) \right]_0^4 = \frac{1}{4} \ln(17)
 \end{aligned}$$



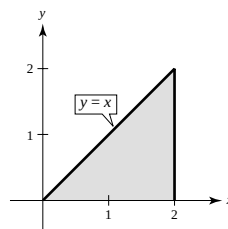
$$\begin{aligned}
 20. \int_0^2 \int_{\sqrt{4-y^2}}^{\sqrt{4-y^2}} (x^2 + y^2) \, dx \, dy &= \int_{-2}^2 \int_0^{\sqrt{4-x^2}} (x^2 + y^2) \, dy \, dx \\
 &= \int_{-2}^2 \left[x^2 y + \frac{1}{3} y^3 \right]_0^{\sqrt{4-x^2}} dx \\
 &= \int_{-2}^2 \left[x^2 \sqrt{4-x^2} + \frac{1}{3} (4-x^2)^{3/2} \right] dx \\
 &= \left[-\frac{x}{4} (4-x^2)^{3/2} + \frac{1}{2} \left(x \sqrt{4-x^2} + 4 \arcsin \frac{x}{2} \right) + \frac{1}{12} \left[x(4-x^2)^{3/2} + 6x \sqrt{4-x^2} + 24 \arctan \frac{x}{2} \right] \right]_{-2}^2 = 4\pi
 \end{aligned}$$



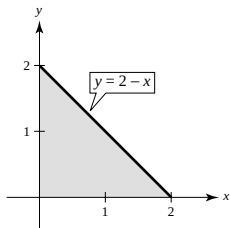
$$\begin{aligned}
 22. \int_0^4 \int_0^2 (6-2y) \, dy \, dx &= \int_0^4 \left[6y - y^2 \right]_0^2 dx \\
 &= \int_0^4 8 \, dx = 32
 \end{aligned}$$



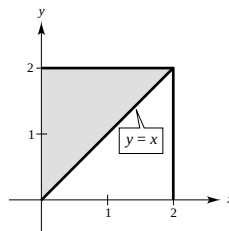
$$24. \int_0^2 \int_0^x 4 \, dy \, dx = \int_0^2 4x \, dx = 2x^2 \Big|_0^2 = 8$$



$$\begin{aligned}
 26. \int_0^2 \int_0^{2-x} (2-x-y) dy dx &= \int_0^2 \left[2y - xy - \frac{y^2}{2} \right]_0^{2-x} dx \\
 &= \int_0^2 \frac{1}{2} (2-x)^2 dx \\
 &= -\frac{1}{6} (x-2)^3 \Big|_0^2 = \frac{4}{3}
 \end{aligned}$$



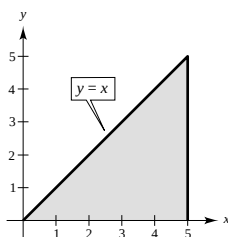
$$\begin{aligned}
 28. \int_0^2 \int_0^y (4-y^2) dx dy &= \int_0^2 (4y - y^3) dy \\
 &= \left[2y^2 - \frac{y^4}{4} \right]_0^2 \\
 &= 4
 \end{aligned}$$



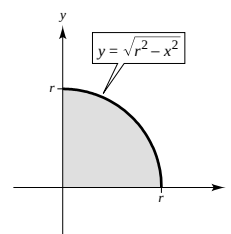
$$30. \int_0^\infty \int_0^\infty e^{-(x+y)/2} dy dx = \int_0^\infty \left[-2e^{-(x+y)/2} \right]_0^\infty dx = \int_0^\infty 2e^{-x/2} dx = \left[-4e^{-x/2} \right]_0^\infty = 4$$

$$32. \int_0^1 \int_0^x \sqrt{1-x^2} dy dx = \frac{1}{3}$$

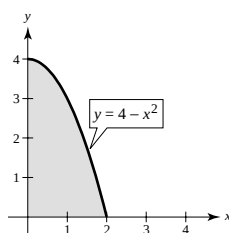
$$\begin{aligned}
 34. V &= \int_0^5 \int_0^x x dy dx \\
 &= \int_0^5 \left[xy \right]_0^x dx = \int_0^5 x^2 dx \\
 &= \left[\frac{1}{3} x^3 \right]_0^5 = \frac{125}{3}
 \end{aligned}$$



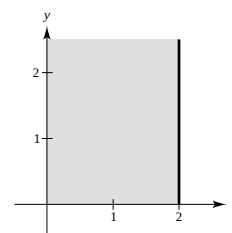
$$\begin{aligned}
 36. V &= 8 \int_0^r \int_0^{\sqrt{r^2-x^2}} \sqrt{r^2-x^2-y^2} dy dx \\
 &= 4 \int_0^r \left[\left[y \sqrt{r^2-x^2-y^2} + (r^2-x^2) \arcsin \frac{y}{\sqrt{r^2-x^2}} \right]_0^{\sqrt{r^2-x^2}} \right] dx \\
 &= 4 \left(\frac{\pi}{2} \right) \int_0^r (r^2-x^2) dx \\
 &= \left[2\pi \left(r^2x - \frac{1}{3}x^3 \right) \right]_0^r \\
 &= \frac{4\pi r^3}{3}
 \end{aligned}$$



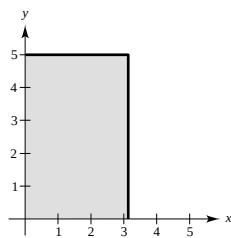
$$\begin{aligned}
 38. V &= \int_0^2 \int_0^{4-x^2} (4-x^2) dy dx \\
 &= \int_0^2 (4-x^2)(4-x^2) dx \\
 &= \int_0^2 (16-8x^2+x^4) dx \\
 &= \left[16x - 8\frac{x^3}{3} + \frac{x^5}{5} \right]_0^2 \\
 &= 32 - \frac{64}{3} + \frac{32}{5} = \frac{256}{15}
 \end{aligned}$$



$$\begin{aligned}
 40. V &= \int_0^2 \int_0^\infty \frac{1}{1+y^2} dy dx \\
 &= \int_0^2 \left[\arctan y \right]_0^\infty dx \\
 &= \int_0^2 \frac{\pi}{2} dx \\
 &= \left[\frac{\pi x}{2} \right]_0^2 = \pi
 \end{aligned}$$



$$\begin{aligned}
 42. \quad V &= \int_0^5 \int_0^\pi \sin^2 x \, dx \, dy \\
 &= \int_0^5 \frac{\pi}{2} \, dy \\
 &= \left[\frac{\pi y}{2} \right]_0^5 \\
 &= \frac{5\pi}{2}
 \end{aligned}$$



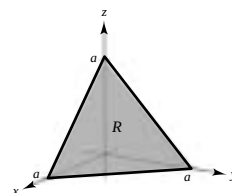
$$44. \quad V = \int_0^9 \int_0^{\sqrt{9-y}} \sqrt{9-y} \, dx \, dy = \frac{81}{2}$$

$$46. \quad V = \int_0^{16} \int_0^{4-\sqrt{y}} \ln(1+x+y) \, dx \, dy \approx 38.25$$

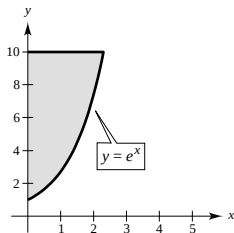
$$48. \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$z = c \left(1 - \frac{x}{a} - \frac{y}{b} \right)$$

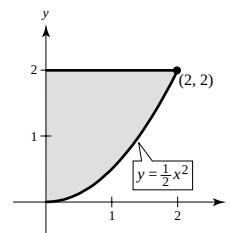
$$\begin{aligned}
 V &= \iint_R f(x, y) \, dA = \int_0^a \int_0^{b[1-(x/a)]} c \left(1 - \frac{x}{a} - \frac{y}{b} \right) \, dy \, dx \\
 &= c \int_0^a \left[y - \frac{xy}{a} - \frac{y^2}{2b} \right]_0^{b[1-(x/a)]} \, dx \\
 &= c \int_0^a \left[b \left(1 - \frac{x}{a} \right) - \frac{xb}{a} \left(1 - \frac{x}{a} \right) - \frac{b^2}{2b} \left(1 - \frac{x}{a} \right)^2 \right] \, dx \\
 &= c \left[-\frac{ab}{2} \left(1 - \frac{x}{a} \right)^2 - \frac{x^2 b}{2a} + \frac{x^3 b}{3a^2} + \frac{ab}{6} \left(1 - \frac{x}{a} \right)^3 \right]_0^a \\
 &= c \left[\left(-\frac{ab}{2} + \frac{ab}{3} \right) - \left(-\frac{ab}{2} + \frac{ab}{6} \right) \right] = \frac{abc}{6}
 \end{aligned}$$



$$\begin{aligned}
 50. \quad \int_0^{\ln 10} \int_{e^x}^{10} \frac{1}{\ln y} \, dy \, dx &= \int_1^{10} \int_0^{\ln y} \frac{1}{\ln y} \, dx \, dy \\
 &= \int_1^{10} \left[\frac{x}{\ln y} \right]_0^{\ln y} \, dy \\
 &= \int_1^{10} dy = [y]_1^{10} = 9
 \end{aligned}$$



$$\begin{aligned}
 52. \quad \int_0^2 \int_{(1/2)x^2}^2 \sqrt{y} \cos y \, dy \, dx &= \int_0^2 \int_0^{\sqrt{2y}} \sqrt{y} \cos y \, dx \, dy \\
 &= \int_0^2 \sqrt{y} \cos y \sqrt{2y} \, dy \\
 &= \sqrt{2} \int_0^2 y \cos y \, dy \\
 &= \sqrt{2} \left[\cos y + y \sin y \right]_0^2 \\
 &= \sqrt{2} [\cos 2 + 2 \sin 2 - 1]
 \end{aligned}$$



$$54. \quad \text{Average} = \frac{1}{8} \int_0^4 \int_0^2 xy \, dy \, dx = \frac{1}{8} \int_0^4 2x \, dx = \left[\frac{x^2}{8} \right]_0^4 = 2$$

$$\begin{aligned}
 56. \quad \text{Average} &= \frac{1}{1/2} \int_0^1 \int_x^1 e^{x+y} \, dy \, dx = 2 \int_0^1 e^{x+1} - e^{2x} \, dx \\
 &= 2 \left[e^{x+1} - \frac{1}{2} e^{2x} \right]_0^1 = 2 \left[e^2 - \frac{1}{2} e^2 - e + \frac{1}{2} \right] \\
 &= e^2 - 2e + 1 \\
 &= (e - 1)^2
 \end{aligned}$$

58. The second is integrable. The first contains $\int \sin y^2 dy$ which does not have an elementary antiderivation.

60. (a) The total snowfall in the county R .
(b) The average snowfall in R .

62. Average = $\frac{1}{150} \int_{45}^{60} \int_{40}^{50} [192x + 576y - x^2 - 5y^2 - 2xy - 5000] dx dy \approx 13,246.67$

64. $f(x, y) \geq 0$ for all (x, y) and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dA = \int_0^2 \int_0^2 \frac{1}{4} xy dy dx = \int_0^2 \frac{x}{2} dx = 1$$

$$P(0 \leq x \leq 1, 1 \leq y \leq 2) = \int_0^1 \int_1^2 \frac{1}{4} xy dy dx = \int_0^1 \frac{3x}{8} dx = \frac{3}{16}.$$

66. $f(x, y) \geq 0$ for all (x, y) and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dA = \int_0^{\infty} \int_0^{\infty} e^{-x-y} dy dx$$

$$= \int_0^{\infty} \lim_{b \rightarrow \infty} \left[-e^{-x-y} \right]_0^b dx = \int_0^{\infty} e^{-x} dx = \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_0^b = 1$$

$$P(0 \leq x \leq 1, x \leq y \leq 1) = \int_0^1 \int_x^1 e^{-x-y} dy dx = \int_0^1 \left[e^{-x-y} \right]_x^1 dx = \int_0^1 (e^{-2x} - e^{-x-1}) dx$$

$$= \left[-\frac{1}{2} e^{-2x} + e^{-x-1} \right]_0^1 = \frac{1}{2} e^{-2} - e^{-1} + \frac{1}{2} = \frac{1}{2} (e^{-1} - 1)^2 \approx 0.1998.$$

68. Sample Program for TI-82:

Program: DOUBLE

: Input A

: Input B

: Input M

: Input C

: Input D

: Input N

: 0 $\rightarrow$ V

: (B - A)/M $\rightarrow$ G

: (D - C)/N $\rightarrow$ H

: For (I, 1, M, 1)

: For (J, 1, N, 1)

: A + 0.5G(2I - 1) $\rightarrow$ x

: C + 0.5H(2J - 1) $\rightarrow$ y

: V + $\sin(\sqrt{x+y}) \times G \times H \rightarrow V$

: End

: End

: Disp V

70. $\int_0^2 \int_0^4 20e^{-x^3/8} dy dx \quad m = 10, n = 20$

(a) 129.2018

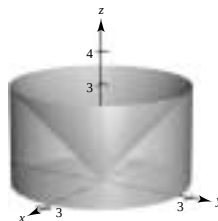
(b) 129.2756

$$72. \int_1^4 \int_1^2 \sqrt{x^3 + y^3} \, dx \, dy \quad m = 6, n = 4$$

- (a) 13.956
(b) 13.9022

$$74. V \approx 50$$

Matches a.



76. True

$$78. \int_1^2 e^{-xy} \, dy = \left[-\frac{1}{x} e^{-xy} \right]_1^2 = \frac{e^{-x} - e^{-2x}}{x}$$

Thus,

$$\begin{aligned} \int_0^\infty \frac{e^{-x} - e^{-2x}}{x} \, dx &= \int_0^\infty \int_1^2 e^{-xy} \, dx \, dy \\ &= \int_1^2 \int_0^\infty e^{-xy} \, dx \, dy \\ &= \int_1^2 \left[-\frac{e^{-xy}}{y} \right]_0^\infty \, dy \\ &= \int_1^2 \frac{1}{y} \, dy = \left[\ln y \right]_1^2 = \ln 2. \end{aligned}$$

Section 13.3 Change of Variables: Polar Coordinates

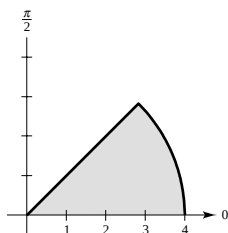
2. Polar coordinates

4. Rectangular coordinates

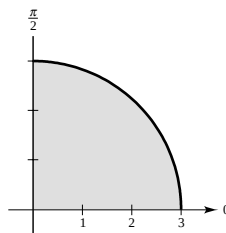
$$6. R = \{(r, \theta): 0 \leq r \leq 4 \sin \theta, 0 \leq \theta \leq \pi\}$$

$$8. R = \{(r, \theta): 0 \leq r \leq r \cos 3\theta, 0 \leq \theta \leq \pi\}$$

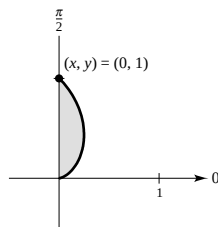
$$\begin{aligned} 10. \int_0^{\pi/4} \int_0^4 r^2 \sin \theta \cos \theta \, dr \, d\theta &= \int_0^{\pi/4} \left[\frac{r^3}{3} \sin \theta \cos \theta \right]_0^4 \, d\theta \\ &= \left[\left(\frac{64}{3} \right) \frac{\sin^2 \theta}{2} \right]_0^{\pi/4} \\ &= \frac{16}{3} \end{aligned}$$



$$\begin{aligned} 12. \int_0^{\pi/2} \int_0^3 r e^{-r^2} \, dr \, d\theta &= \int_0^{\pi/2} \left[-\frac{1}{2} e^{-r^2} \right]_0^3 \, d\theta \\ &= \left[-\frac{1}{2} (e^{-9} - 1) \theta \right]_0^{\pi/2} \\ &= \frac{\pi}{4} \left(1 - \frac{1}{e^9} \right) \end{aligned}$$



$$\begin{aligned}
 14. \int_0^{\pi/2} \int_0^{1-\cos \theta} (\sin \theta) r \, dr \, d\theta &= \int_0^{\pi/2} \left[(\sin \theta) \frac{r^2}{2} \right]_0^{1-\cos \theta} d\theta \\
 &= \int_0^{\pi/2} \frac{\sin \theta}{2} (1 - \cos \theta)^2 d\theta \\
 &= \left[\frac{1}{6} (1 - \cos(\theta))^3 \right]_0^{\pi/2} = \frac{1}{6}
 \end{aligned}$$



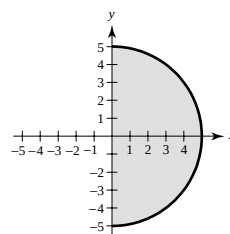
$$16. \int_0^a \int_0^{\sqrt{a^2-x^2}} x \, dy \, dx = \int_0^{\pi/2} \int_0^a r^2 \cos \theta \, dr \, d\theta = \frac{a^3}{3} \int_0^{\pi/2} \cos \theta \, d\theta = \left[\frac{a^3}{3} \sin \theta \right]_0^{\pi/2} = \frac{a^3}{3}$$

$$\begin{aligned}
 18. \int_0^2 \int_y^{\sqrt{8-y^2}} \sqrt{x^2 + y^2} \, dx \, dy &= \int_0^{\pi/4} \int_0^{2\sqrt{2}} r^2 \, dr \, d\theta \\
 &= \int_0^{\pi/4} \frac{(2\sqrt{2})^3}{3} d\theta = \left[\frac{(2\sqrt{2})^3}{3} \theta \right]_0^{\pi/4} = \frac{(2\sqrt{2})^3}{3} \cdot \frac{\pi}{4} = \frac{4\sqrt{2}\pi}{3}
 \end{aligned}$$

$$\begin{aligned}
 20. \int_0^4 \int_0^{\sqrt{4y-y^2}} x^2 \, dx \, dy &= \int_0^{\pi/2} \int_0^{4 \sin \theta} r^3 \cos^2 \theta \, dr \, d\theta = \int_0^{\pi/2} 64 \sin^4 \theta \cos^2 \theta \, d\theta \\
 &= 64 \int_0^{\pi/2} (\sin^4 \theta - \sin^6 \theta) \, d\theta = \frac{64}{6} \left[\sin^5 \theta \cos \theta - \frac{\sin^3 \theta \cos \theta}{4} + \frac{3}{8} (\theta - \sin \theta \cos \theta) \right]_0^{\pi/2} = 2\pi
 \end{aligned}$$

$$\begin{aligned}
 22. \int_0^{(5\sqrt{2})/2} \int_0^x xy \, dy \, dx + \int_{(5\sqrt{2})/2}^5 \int_0^{\sqrt{25-x^2}} xy \, dy \, dx &= \int_0^{\pi/4} \int_0^5 r^3 \sin \theta \cos \theta \, dr \, d\theta \\
 &= \int_0^{\pi/4} \frac{625}{4} \sin \theta \cos \theta \, d\theta \\
 &= \left[\frac{625}{8} \sin^2 \theta \right]_0^{\pi/4} \\
 &= \frac{625}{16}
 \end{aligned}$$

$$\begin{aligned}
 24. \int_{-\pi/2}^{\pi/2} \int_0^5 e^{-r^2/2} r \, dr \, d\theta &= \int_{-\pi/2}^{\pi/2} \left[-e^{-r^2/2} \right]_0^5 d\theta \\
 &= \int_{-\pi/2}^{\pi/2} (1 - e^{-25/2}) d\theta \\
 &= \left[(1 - e^{-25/2}) \theta \right]_{-\pi/2}^{\pi/2} = \pi(1 - e^{-25/2})
 \end{aligned}$$



$$\begin{aligned}
 26. \int_0^3 \int_0^{\sqrt{9-x^2}} (9 - x^2 - y^2) \, dy \, dx &= \int_0^{\pi/2} \int_0^3 (9 - r^2) r \, dr \, d\theta \\
 &= \int_0^{\pi/2} \int_0^3 (9r - r^3) \, dr \, d\theta = \int_0^{\pi/2} \left[\frac{9}{2} r^2 - \frac{1}{4} r^4 \right]_0^3 d\theta = \frac{81}{4} \int_0^{\pi/2} d\theta = \frac{81\pi}{8}
 \end{aligned}$$

$$\begin{aligned}
 28. \quad V &= 4 \int_0^{\pi/2} \int_0^1 (r^2 + 3)r \, dr \, d\theta = 4 \int_0^{\pi/2} \left(\frac{r^4}{4} + \frac{3r^2}{2} \right) \Big|_0^1 d\theta \\
 &= 4 \int_0^{\pi/2} \frac{7}{4} d\theta \\
 &= 7 \left(\frac{\pi}{4} \right) = \frac{7\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 30. \quad \iint_R \ln(x^2 + y^2) \, dA &= \int_0^{2\pi} \int_1^2 (\ln r^2)r \, dr \, d\theta \\
 &= 2 \int_0^{2\pi} \int_1^2 r \ln r \, dr \, d\theta \\
 &= 2 \int_0^{2\pi} \left[\frac{r^2}{4}(-1 + 2 \ln r) \right]_1^2 d\theta \\
 &= 2 \int_0^{2\pi} \left(\ln 4 - \frac{3}{4} \right) d\theta \\
 &= 4\pi \left(\ln 4 - \frac{3}{4} \right)
 \end{aligned}$$

$$32. \quad V = \int_0^{2\pi} \int_1^4 \sqrt{16 - r^2} \, r \, dr \, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(\sqrt{16 - r^2})^3 \right]_1^4 d\theta = \int_0^{2\pi} 5\sqrt{15} \, d\theta = 10\sqrt{15}\pi$$

$$\begin{aligned}
 34. \quad x^2 + y^2 + z^2 &= a^2 \Rightarrow z = \sqrt{a^2 - (x^2 + y^2)} = \sqrt{a^2 - r^2} \\
 V &= 8 \int_0^{\pi/2} \int_0^a \sqrt{a^2 - r^2} \, r \, dr \, d\theta \quad (8 \text{ times the volume in the first octant}) \\
 &= 8 \int_0^{\pi/2} \left[-\frac{1}{2} \cdot \frac{2}{3}(a^2 - r^2)^{3/2} \right]_0^a d\theta \\
 &= 8 \int_0^{\pi/2} \frac{a^3}{3} d\theta = \left[\frac{8a^3}{3} \theta \right]_0^{\pi/2} = \frac{4\pi a^3}{3}
 \end{aligned}$$

$$36. \quad \frac{-9}{4(x^2 + y^2 + 9)} \leq z \leq \frac{9}{4(x^2 + y^2 + 9)}; \quad \frac{1}{4} \leq r \leq \frac{1}{2}(1 + \cos^2 \theta)$$

$$(a) \quad \frac{-9}{4r^2 + 36} \leq z \leq \frac{9}{4r^2 + 36}$$

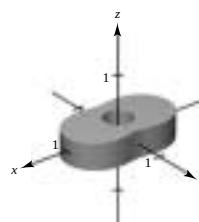
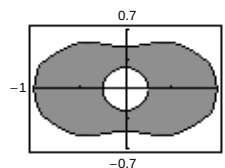
$$(b) \quad \text{Perimeter} = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} d\theta.$$

$$r = \frac{1}{2}(1 + \cos^2 \theta) = \frac{1}{2} + \frac{1}{2} \cos^2 \theta$$

$$\frac{dr}{d\theta} = -\cos \theta \sin \theta$$

$$\text{Perimeter} = 2 \int_0^{\pi} \sqrt{\frac{1}{4}(1 + \cos^2 \theta)^2 + \cos^2 \theta \sin^2 \theta} d\theta \approx 5.21$$

$$(c) \quad V = 2 \int_0^{2\pi} \int_{1/4}^{1/2(1 + \cos^2 \theta)} \frac{9}{4r^2 + 36} r \, dr \, d\theta \approx 0.8000$$



$$38. \quad A = \int_0^{2\pi} \int_2^4 r \, dr \, d\theta = \int_0^{2\pi} 6 \, d\theta = 12\pi$$

$$\begin{aligned}
 40. \quad \int_0^{2\pi} \int_0^{2 + \sin \theta} r \, dr \, d\theta &= \frac{1}{2} \int_0^{2\pi} (2 + \sin \theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (4 + 4 \sin \theta + \sin^2 \theta) d\theta = \frac{1}{2} \int_0^{2\pi} \left(4 + 4 \sin \theta + \frac{1 - \cos 2\theta}{2} \right) d\theta \\
 &= \frac{1}{2} \left[4\theta - 4 \cos \theta + \frac{1}{2}\theta - \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = \frac{1}{2} [8\pi - 4 + \pi + 4] = \frac{9\pi}{2}
 \end{aligned}$$

$$42. 8 \int_0^{\pi/4} \int_0^{3 \cos 2\theta} r \, dr \, d\theta = 4 \int_0^{\pi/4} 9 \cos^2 2\theta \, d\theta = 18 \int_0^{\pi/4} (1 + \cos 4\theta) \, d\theta = 18 \left[\theta + \frac{1}{4} \sin 4\theta \right]_0^{\pi/4} = \frac{9\pi}{2}$$

44. See Theorem 13.3.

46. (a) Horizontal or polar representative elements

(b) Polar representative element

(c) Vertical or polar

48. (a) The volume of the subregion determined by the point $(5, \pi/16, 7)$ is $\text{base} \times \text{height} = (5 \cdot 10 \cdot \pi/8)(7)$. Adding up the 20 volumes, ending with $(45 \cdot 10 \cdot \pi/8)(12)$, you obtain

$$\begin{aligned} V &\approx 10 \cdot \frac{\pi}{8} [5(7 + 9 + 9 + 5) + 15(8 + 10 + 11 + 8) + 25(10 + 14 + 15 + 11) \\ &\quad + 35(12 + 15 + 18 + 16) + 45(9 + 10 + 14 + 12)] \\ &= \frac{5\pi}{4} [150 + 555 + 1250 + 2135 + 2025] \approx \frac{5\pi}{4} [6115] \approx 24013.5 \text{ ft}^3 \end{aligned}$$

(b) $(56)(24013.5) = 1,344,759$ pounds

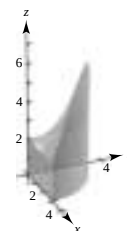
(c) $(7.48)(24013.5) \approx 179,621$ gallons

$$50. \int_0^{\pi/4} \int_0^4 5e^{\sqrt{r}\theta} r \, dr \, d\theta \approx 87.130$$

52. Volume = base $\times$ height

$$\approx \frac{9}{4}\pi \times 3 \approx 21$$

Answer (a)



54. True

$$56. (a) \text{ Let } u = \sqrt{2}x, \text{ then } \int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-u^2/2} \frac{1}{\sqrt{2}} du = \frac{1}{\sqrt{2}} (\sqrt{2\pi}) = \sqrt{\pi}.$$

$$(b) \text{ Let } u = 2x, \text{ then } \int_{-\infty}^{\infty} e^{-4x^2} dx = \int_{-\infty}^{\infty} e^{-u^2} \frac{1}{2} du = \frac{1}{2} \sqrt{\pi}.$$

$$58. \int_0^{\infty} \int_0^{\infty} k e^{-(x^2+y^2)} dy dx = \int_0^{\pi/2} \int_0^{\infty} k e^{-r^2} r \, dr \, d\theta = \int_0^{\pi/2} \left[-\frac{k}{2} e^{-r^2} \right]_0^{\infty} d\theta = \int_0^{\pi/2} \frac{k}{2} d\theta = \frac{k\pi}{4}$$

For $f(x, y)$ to be a probability density function,

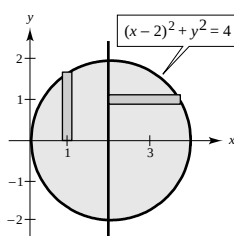
$$\frac{k\pi}{4} = 1$$

$$k = \frac{4}{\pi}.$$

$$60. (a) 4 \int_0^2 \int_2^{2+\sqrt{4-y^2}} f \, dx \, dy$$

$$(b) 4 \int_0^2 \int_0^{\sqrt{4-(x-2)^2}} f \, dy \, dx$$

$$(c) 2 \int_0^{\pi/2} \int_0^{4 \cos \theta} f r \, dr \, d\theta$$



Section 13.4 Center of Mass and Moments of Inertia

$$\begin{aligned}
 2. \quad m &= \int_0^3 \int_0^{9-x^2} xy \, dy \, dx = \int_0^3 \left[\frac{xy^2}{2} \right]_0^{9-x^2} dx \\
 &= \int_0^3 \frac{x(9-x^2)^2}{2} dx \\
 &= \left[-\frac{1}{4} \frac{(9-x^2)^3}{3} \right]_0^3 \\
 &= -\frac{1}{12}(0-9^3) = \frac{243}{4}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad m &= \int_0^3 \int_3^{3+\sqrt{9-x^2}} xy \, dy \, dx = \int_0^3 \left[x \frac{y^2}{2} \right]_3^{3+\sqrt{9-x^2}} dx \\
 &= \int_0^3 \frac{x}{2} ((3+\sqrt{9-x^2})^2 - 9) dx \\
 &= \frac{1}{2} \int_0^3 [6x\sqrt{9-x^2} + 9x - x^3] dx \\
 &= \frac{1}{2} \left[-2(9-x^2)^{3/2} + \frac{9x^2}{2} - \frac{x^4}{4} \right]_0^3 \\
 &= \frac{1}{2} \left[\frac{81}{2} - \frac{81}{4} + 54 \right] = \frac{297}{8}
 \end{aligned}$$

$$6. \quad (a) \quad m = \int_0^a \int_0^b kxy \, dy \, dx = \frac{ka^2b^2}{4}$$

$$M_x = \int_0^a \int_0^b kxy^2 \, dy \, dx = \frac{ka^2b^3}{6}$$

$$M_y = \int_0^a \int_0^b kx^2y \, dy \, dx = \frac{ka^3b^2}{6}$$

$$\bar{x} = \frac{M_y}{m} = \frac{ka^3b^2/6}{ka^2b^2/4} = \frac{2}{3}a$$

$$\bar{y} = \frac{M_x}{m} = \frac{ka^2b^3/6}{ka^2b^2/4} = \frac{2}{3}b$$

$$(b) \quad m = \int_0^a \int_0^b k(x^2 + y^2) \, dy \, dx = \frac{kab}{3}(a^2 + b^2)$$

$$M_x = \int_0^a \int_0^b k(x^2y + y^3) \, dy \, dx = \frac{kab^2}{12}(2a^2 + 3b^2)$$

$$M_y = \int_0^a \int_0^b k(x^3 + xy^2) \, dy \, dx = \frac{ka^2b}{12}(3a^2 + 2b^2)$$

$$\bar{x} = \frac{M_y}{m} = \frac{(ka^2b/12)(3a^2 + 2b^2)}{(kab/3)(a^2 + b^2)} = \frac{a}{4} \left(\frac{3a^2 + 2b^2}{a^2 + b^2} \right)$$

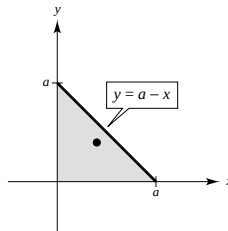
$$\bar{y} = \frac{M_x}{m} = \frac{(kab^2/12)(2a^2 + 3b^2)}{(kab/3)(a^2 + b^2)} = \frac{b}{4} \left(\frac{2a^2 + 3b^2}{a^2 + b^2} \right)$$

$$8. \quad (a) \quad m = \frac{a^2k}{2}$$

$$M_x = \int_0^a \int_0^{a-x} ky \, dy \, dx = \frac{ka^3}{6}$$

$$M_y = M_x \text{ by symmetry}$$

$$\bar{x} = \bar{y} = \frac{M_x}{m} = \frac{ka^3/6}{ka^2/2} = \frac{a}{3}$$



—CONTINUED—

8. —CONTINUED—

$$\begin{aligned}
 \text{(b)} \quad m &= \int_0^a \int_0^{a-x} (x^2 + y^2) dy dx \\
 &= \int_0^a \left[x^2 y + \frac{y^3}{3} \right]_0^{a-x} dx = \int_0^a \left[ax^2 - x^3 + \frac{1}{3}(a-x)^3 \right] dx = \frac{a^4}{6} \\
 M_y &= \int_0^a \int_0^{a-x} (x^3 + xy^2) dy dx \\
 &= \int_0^a \left(ax^3 - x^4 + \frac{1}{3}a^3x - a^2x^2 + ax^3 - \frac{1}{3}x^4 \right) dx = \frac{1}{3} \int_0^a (a^3x - 3a^2x^2 + 6ax^3 - 4x^4) dx = \frac{a^5}{15} \\
 \bar{x} &= \frac{M_y}{m} = \frac{a^5/15}{a^4/6} = \frac{2a}{5} \\
 \bar{y} &= \frac{2a}{5} \text{ by symmetry}
 \end{aligned}$$

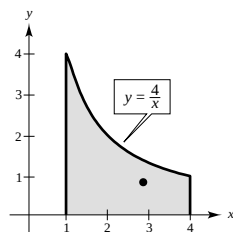
10. The x -coordinate changes by h units horizontally and k units vertically. This is not true for variable densities.

$$\begin{aligned}
 \text{12. (a)} \quad m &= \int_0^a \int_0^{\sqrt{a^2-x^2}} k dy dx = \frac{k\pi a^2}{4} \\
 M_y &= \int_0^a \int_0^{\sqrt{a^2-x^2}} kx dy dx \\
 &= k \int_0^a x \sqrt{a^2-x^2} dx \\
 &= \left[-\frac{k}{3}(a^2-x^2)^{3/2} \right]_0^a = \frac{ka^3}{3} \\
 \bar{x} &= \frac{M_y}{m} = \frac{ka^3/3}{k\pi a^2/4} = \frac{4a}{3\pi} \\
 \bar{y} &= \frac{4a}{3\pi} \text{ by symmetry}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad m &= \int_0^a \int_0^{\sqrt{a^2-x^2}} k(x^2 + y^2) dy dx \\
 &= \int_0^{\pi/2} \int_0^a kr^3 dr d\theta = \frac{ka^4\pi}{8} \\
 M_x &= \int_0^a \int_0^{\sqrt{a^2-x^2}} k(x^2 + y^2)y dy dx \\
 &= \int_0^{\pi/2} \int_0^a kr^4 \sin \theta dr d\theta = \frac{ka^5}{5} \\
 M_y &= M_x \text{ by symmetry} \\
 \bar{x} = \bar{y} &= \frac{M_y}{m} = \frac{ka^5}{5} \cdot \frac{8}{ka^4\pi} = \frac{8a}{5\pi}
 \end{aligned}$$

$$\begin{aligned}
 \text{14.} \quad m &= \int_0^2 \int_0^{x^3} kx dy dx = \int_0^2 kx^4 dx = \frac{32k}{5} \\
 M_x &= \int_0^2 \int_0^{x^3} kxy dy dx = 16k \\
 M_y &= \int_0^2 \int_0^{x^3} kx^2 dy dx = \frac{32k}{3} \\
 \bar{x} &= \frac{M_y}{m} = \frac{32k}{3} \cdot \frac{5}{32k} = \frac{5}{3} \\
 \bar{y} &= \frac{M_x}{m} = \frac{16k}{32k} \cdot \frac{5}{2} = \frac{5}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{16.} \quad m &= \int_1^4 \int_0^{4/x} kx^2 dy dx = 30k \\
 M_x &= \int_1^4 \int_0^{4/x} kx^2 y dy dx = 24k \\
 M_y &= \int_1^4 \int_0^{4/x} kx^3 dy dx = 84k \\
 \bar{x} &= \frac{M_y}{m} = \frac{84k}{30k} = \frac{14}{5} \\
 \bar{y} &= \frac{M_x}{m} = \frac{24k}{30k} = \frac{4}{5}
 \end{aligned}$$

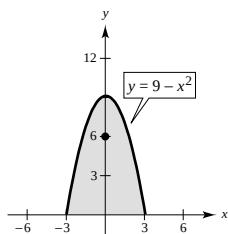


18. $\bar{x} = 0$ by symmetry

$$m = \int_{-3}^3 \int_0^{9-x^2} ky^2 dy dx = \frac{23,328k}{35}$$

$$M_x = \int_{-3}^3 \int_0^{9-x^2} ky^3 dy dx = \frac{139,968k}{35}$$

$$\bar{y} = \frac{M_x}{m} = \frac{139,968k}{35} \cdot \frac{35}{23,328k} = 6$$



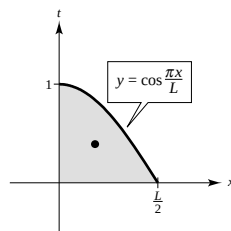
20. $m = \int_0^{L/2} \int_0^{\cos \pi x/L} k dy dx = \frac{kL}{\pi}$

$$M_x = \int_0^{L/2} \int_0^{\cos \pi x/L} ky dy dx = \frac{kL}{8}$$

$$M_y = \int_0^{L/2} \int_0^{\cos \pi x/L} kx dy dx = \frac{L^2(\pi - 2)k}{2\pi^2}$$

$$\bar{x} = \frac{M_y}{m} = \frac{L^2(\pi - 2)k}{2\pi^2} \cdot \frac{\pi}{kL} = \frac{L(\pi - 2)}{2\pi}$$

$$\bar{y} = \frac{M_x}{m} = \frac{kL}{8} \cdot \frac{\pi}{kL} = \frac{\pi}{8}$$



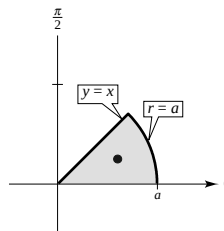
22. $m = \int_R \int k\sqrt{x^2 + y^2} dA = \int_0^{\pi/4} \int_0^a kr^2 dr d\theta = \frac{ka^3\pi}{12}$

$$M_x = \int_R \int k\sqrt{x^2 + y^2} y dA = \int_0^{\pi/4} \int_0^a kr^3 \sin \theta d\theta = \frac{ka^4(2 - \sqrt{2})}{8}$$

$$M_y = \int_R \int k\sqrt{x^2 + y^2} x dA = \int_0^{\pi/4} \int_0^a kr^3 \cos \theta d\theta = \frac{ka^4\sqrt{2}}{8}$$

$$\bar{x} = \frac{M_y}{m} = \frac{ka^4\sqrt{2}}{8} \cdot \frac{12}{ka^3\pi} = \frac{3\sqrt{2}a}{2\pi}$$

$$\bar{y} = \frac{M_x}{m} = \frac{ka^4(2 - \sqrt{2})}{8} \cdot \frac{12}{ka^3\pi} = \frac{3(2 - \sqrt{2})a}{2\pi}$$



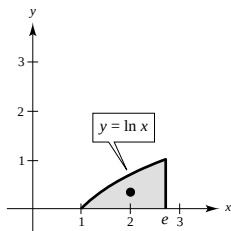
24. $m = \int_1^e \int_0^{\ln x} \frac{k}{x} dy dx = \frac{k}{2}$

$$M_x = \int_1^e \int_0^{\ln x} \frac{k}{x} y dy dx = \frac{k}{6}$$

$$M_y = \int_1^e \int_0^{\ln x} \frac{k}{x} x dy dx = k$$

$$\bar{x} = \frac{M_y}{m} = \frac{k}{1/2} \cdot \frac{2}{k} = 2$$

$$\bar{y} = \frac{M_x}{m} = \frac{k}{6} \cdot \frac{2}{k} = \frac{1}{3}$$

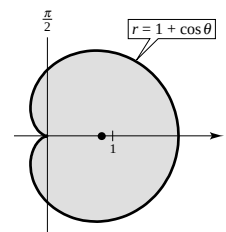


26. $\bar{y} = 0$ by symmetry

$$m = \iint_R k \, dA = \int_0^{2\pi} \int_0^{1+\cos\theta} kr \, dr \, d\theta = \frac{3\pi k}{2}$$

$$\begin{aligned} M_y &= \iint_R kx \, dA = \int_0^{2\pi} \int_0^{1+\cos\theta} kr^2 \cos\theta \, dr \, d\theta \\ &= \frac{k}{3} \int_0^{2\pi} \cos\theta (1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta) \, d\theta \\ &= \frac{k}{3} \int_0^{2\pi} \left[\cos\theta + \frac{3}{2}(1 + \cos^2\theta) + 3\cos\theta(1 - \sin^2\theta) + \frac{1}{4}(1 + \cos 2\theta)^2 \right] d\theta \\ &= \frac{5k\pi}{4} \end{aligned}$$

$$\bar{x} = \frac{M_y}{m} = \frac{5k\pi}{4} \cdot \frac{2}{3k\pi} = \frac{5}{6}$$



$$28. m = \int_0^b \int_0^{h-(hx/b)} dy \, dx = \frac{bh}{2}$$

$$I_x = \int_0^b \int_0^{h-(hx/b)} y^2 \, dy \, dx = \frac{bh^3}{12}$$

$$I_y = \int_0^b \int_0^{h-(hx/b)} x^2 \, dy \, dx = \frac{b^3h}{12}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{b^3h/12}{bh/2}} = \frac{b}{\sqrt{6}} = \frac{\sqrt{6}}{6}b$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{bh^3/12}{bh/2}} = \frac{h}{\sqrt{6}} = \frac{\sqrt{6}}{6}h$$

$$30. m = \frac{\pi a^2}{2}$$

$$I_x = \iint_R y^2 \, dA = \int_0^\pi \int_0^a r^3 \sin^2\theta \, dr \, d\theta = \frac{a^4\pi}{8}$$

$$I_y = \iint_R x^2 \, dA = \int_0^\pi \int_0^a r^3 \cos^2\theta \, dr \, d\theta = \frac{a^4\pi}{8}$$

$$I_0 = I_x + I_y = \frac{a^4\pi}{8} + \frac{a^4\pi}{8} = \frac{a^4\pi}{4}$$

$$\bar{\bar{x}} = \bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{a^4\pi}{8} \cdot \frac{2}{\pi a^2}} = \frac{a}{2}$$

32. $m = \pi ab$

$$I_x = 4 \int_0^a \int_0^{(b/a)\sqrt{a^2-x^2}} y^2 \, dy \, dx$$

$$= 4 \int_0^a \frac{b^3}{3a^3} (a^2 - x^2)^{3/2} \, dx = \frac{4b^3}{3a^3} \int_0^a [a^2\sqrt{a^2-x^2} - x^2\sqrt{a^2-x^2}] \, dx$$

$$= \frac{4b^3}{3a^3} \left[\frac{a^2}{2} \left(x\sqrt{a^2-x^2} + a^2 \arcsin \frac{x}{a} \right) - \frac{1}{8} \left[x(2x^2 - a^2)\sqrt{a^2-x^2} + a^4 \arcsin \frac{x}{a} \right] \right]_0^a = \frac{ab^3\pi}{4}$$

$$I_y = 4 \int_0^b \int_0^{(a/b)\sqrt{b^2-y^2}} x^2 \, dx \, dy = \frac{a^3b\pi}{4}$$

$$I_0 = I_y + I_x = \frac{a^3b\pi}{4} + \frac{ab^3\pi}{4} = \frac{ab\pi}{4}(a^2 + b^2)$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{a^3b\pi}{4} \cdot \frac{1}{\pi ab}} = \frac{a}{2}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{ab^3\pi}{4} \cdot \frac{1}{\pi ab}} = \frac{b}{2}$$

34. $\rho = ky$

$$\begin{aligned}
 m &= 2k \int_0^a \int_0^{\sqrt{a^2-x^2}} y \, dy \, dx \\
 &= k \int_0^a (a^2 - x^2) \, dx = \frac{2ka^3}{3} \\
 I_x &= k \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} y^3 \, dy \, dx = \frac{4ka^5}{15} \\
 I_y &= k \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} x^2 y \, dy \, dx = \frac{2ka^5}{15} \\
 I_0 &= I_x + I_y = \frac{2ka^5}{5} \\
 \bar{x} &= \sqrt{\frac{I_y}{m}} = \sqrt{\frac{2ka^5/15}{2ka^3/3}} = \sqrt{\frac{a^2}{5}} = \frac{a}{\sqrt{5}} \\
 \bar{y} &= \sqrt{\frac{I_x}{m}} = \sqrt{\frac{4ka^5/15}{2ka^3/3}} = \sqrt{\frac{2a^2}{5}} = \frac{2a}{\sqrt{10}}
 \end{aligned}$$

38. $\rho = x^2 + y^2$

$$\begin{aligned}
 m &= \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2) \, dy \, dx = \frac{6}{35} \\
 I_x &= \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2) y^2 \, dy \, dx = \frac{158}{2079} \\
 I_y &= \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2) x^2 \, dy \, dx = \frac{158}{2079} \\
 I_0 &= I_x + I_y = \frac{316}{2079} \\
 \bar{x} &= \sqrt{\frac{I_y}{m}} = \sqrt{\frac{158}{2079} \cdot \frac{35}{6}} = \sqrt{\frac{395}{891}} \\
 \bar{y} &= \sqrt{\frac{I_x}{m}} = \bar{x} = \sqrt{\frac{395}{891}}
 \end{aligned}$$

$$42. I = \int_0^4 \int_0^2 k(x-6)^2 \, dy \, dx = \int_0^4 2k(x-6)^2 \, dx = \left[\frac{2k}{3}(x-6)^3 \right]_0^4 = \frac{416k}{3}$$

$$\begin{aligned}
 44. I &= \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} ky(y-a)^2 \, dy \, dx \\
 &= \int_{-a}^a k \left[\frac{y^4}{4} - \frac{2ay^3}{3} + \frac{a^2y^2}{2} \right]_0^{\sqrt{a^2-x^2}} dx \\
 &= \int_{-a}^a k \left[\frac{1}{4}(a^4 - 2a^2x^2 + x^4) - \frac{2a}{3}(a^2\sqrt{a^2-x^2} - x^2\sqrt{a^2-x^2}) + \frac{a^2}{2}(a^2 - x^2) \right] dx \\
 &= k \left[\frac{1}{4} \left(a^4x - \frac{2a^2x^3}{3} + \frac{x^5}{5} \right) - \frac{2a}{3} \left[\frac{a^2}{2} \left(x\sqrt{a^2-x^2} + a^2 \arcsin \frac{x}{a} \right) \right. \right. \\
 &\quad \left. \left. - \frac{1}{8} \left(x(2x^2 - a^2)\sqrt{a^2-x^2} + a^4 \arcsin \frac{x}{a} \right) \right] + \frac{a^2}{2} \left(a^2x - \frac{x^3}{3} \right) \right]_{-a}^a \\
 &= 2k \left[\frac{1}{4} \left(a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5 \right) - \frac{2a}{3} \left(\frac{a^4\pi}{4} - \frac{a^4\pi}{16} \right) + \frac{a^2}{2} \left(a^3 - \frac{a^3}{3} \right) \right] = 2k \left(\frac{7a^5}{15} - \frac{a^5\pi}{8} \right) = ka^5 \left(\frac{56 - 15\pi}{60} \right)
 \end{aligned}$$

36. $\rho = kxy$

$$\begin{aligned}
 m &= k \int_0^1 \int_{x^2}^x xy \, dy \, dx = \frac{k}{2} \int_0^1 (x^3 - x^5) \, dx = \frac{k}{24} \\
 I_x &= k \int_0^1 \int_{x^2}^x xy^3 \, dy \, dx = \frac{k}{4} \int_0^1 (x^5 - x^9) \, dx = \frac{k}{60} \\
 I_y &= k \int_0^1 \int_{x^2}^x x^3 y \, dy \, dx = \frac{k}{2} \int_0^1 (x^5 - x^7) \, dx = \frac{k}{48} \\
 I_0 &= I_x + I_y = \frac{9k}{240} = \frac{3k}{80} \\
 \bar{x} &= \sqrt{\frac{I_y}{m}} = \sqrt{\frac{k/48}{k/24}} = \frac{1}{\sqrt{2}} \\
 \bar{y} &= \sqrt{\frac{I_x}{m}} = \sqrt{\frac{k/60}{k/24}} = \sqrt{\frac{2}{5}}
 \end{aligned}$$

40. $\rho = ky$

$$\begin{aligned}
 m &= 2 \int_0^2 \int_{x^3}^{4x} ky \, dy \, dx = \frac{512k}{21} \\
 I_x &= 2 \int_0^2 \int_{x^3}^{4x} ky^3 \, dy \, dx = \frac{32,768k}{65} \\
 I_y &= 2 \int_0^2 \int_{x^3}^{4x} kx^2 y \, dy \, dx = \frac{2048k}{45} \\
 I_0 &= I_x + I_y = \frac{321,536k}{585} \\
 \bar{x} &= \sqrt{\frac{I_y}{m}} = \sqrt{\frac{2048k}{45} \cdot \frac{21}{512k}} = \sqrt{\frac{28}{15}} = \frac{2\sqrt{105}}{15} \\
 \bar{y} &= \sqrt{\frac{I_x}{m}} = \sqrt{\frac{32,768k}{65} \cdot \frac{21}{512k}} = \frac{8\sqrt{1365}}{65}
 \end{aligned}$$

$$\begin{aligned}
 46. I &= \int_{-2}^2 \int_0^{4-x^2} k(y-2)^2 dy dx = \int_{-2}^2 \left[\frac{k}{3}(y-1)^3 \right]_0^{4-x^2} dx = \int_{-2}^2 \frac{k}{3}[(2-x^2)+8] dx \\
 &= \frac{k}{3} \int_{-2}^2 (16-12x^2+6x^4-x^6) dx = \left[\frac{k}{3} \left(16x-4x^3+\frac{6}{5}x^5-\frac{1}{7}x^7 \right) \right]_{-2}^2 \\
 &= \frac{2k}{3} \left(32-32+\frac{192}{5}-\frac{128}{7} \right) = \frac{1408k}{105}
 \end{aligned}$$

$$48. \rho(x, y) = k|2-x|.$$

$(\bar{x}, \bar{y})$ will be the same.

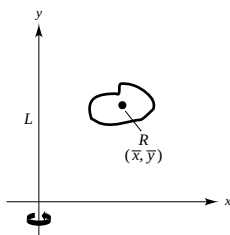
$$50. \rho(x, y) = k(4-x)(4-y). \text{ Both } \bar{x} \text{ and } \bar{y} \text{ will decrease}$$

$$52. I_x = \iint_R y^2 \rho(x, y) dA \text{ Moment of inertia about } x\text{-axis.}$$

$$I_y = \iint_R x^2 \rho(x, y) dA \text{ Moment of inertia about } y\text{-axis.}$$

54. Orient the xy -coordinate system so that L is along the y -axis and R is in the first quadrant. Then the volume of the solid is

$$\begin{aligned}
 V &= \iint_R 2\pi x dA \\
 &= 2\pi \iint_R x dA \\
 &= 2\pi \left(\frac{\iint_R x dA}{\iint_R dA} \right) \iint_R dA \\
 &= 2\pi \bar{x} A.
 \end{aligned}$$



By our positioning, $\bar{x} = r$. Therefore, $V = 2\pi rA$.

$$56. \bar{y} = \frac{a}{2}, A = ab, h = L - \frac{a}{2}$$

$$I_{\bar{y}} = \int_0^b \int_0^a \left(y - \frac{a}{2} \right)^2 dy dx = \frac{a^3 b}{12}$$

$$y_a = \frac{a}{2} - \frac{a^3 b/12}{[L - (a/2)]ab} = \frac{a(3L - 2a)}{3(2L - a)}$$

$$58. \bar{y} = 0, A = \pi a^2, h = L$$

$$I_{\bar{y}} = \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2 dy dx$$

$$= \int_0^{2\pi} \int_0^a r^3 \sin^2 \theta dr d\theta$$

$$= \int_0^{2\pi} \frac{a^4}{4} \sin^2 \theta d\theta$$

$$= \frac{a^4 \pi}{4}$$

$$y_a = -\frac{(a^4 \pi/4)}{L \pi a^2} = -\frac{a^2}{4L}$$

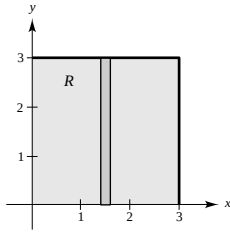
Section 13.5 Surface Area

2. $f(x, y) = 15 + 2x - 3y$

$f_x = 2, f_y = -3$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{14}$

$$S = \int_0^3 \int_0^3 \sqrt{14} \, dy \, dx = \int_0^3 3\sqrt{14} \, dx = 9\sqrt{14}$$



4. $f(x, y) = 10 + 2x - 3y$

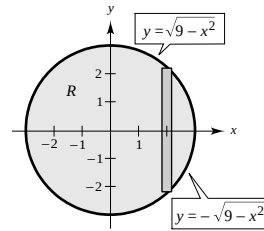
$R = \{(x, y): x^2 + y^2 \leq 9\}$

$f_x = 2, f_y = -3$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{14}$

$$S = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{14} \, dy \, dx$$

$$= \int_0^{2\pi} \int_0^3 \sqrt{14} \, r \, dr \, d\theta = 9\sqrt{14}\pi$$



6. $f(x, y) = y^2$

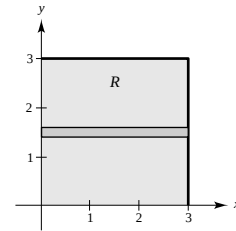
$R = \text{square with vertices } (0, 0), (3, 0), (0, 3), (3, 3)$

$f_x = 0, f_y = 2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4y^2}$

$$S = \int_0^3 \int_0^3 \sqrt{1 + 4y^2} \, dx \, dy = \int_0^3 3\sqrt{1 + 4y^2} \, dy$$

$$= \left[\frac{3}{4} (2y\sqrt{1 + 4y^2} + \ln|2y + \sqrt{1 + 4y^2}|) \right]_0^3 = \frac{3}{4} (6\sqrt{37} + \ln|6 + \sqrt{37}|)$$



8. $f(x, y) = 2 + \frac{2}{3}y^{3/2}$

$f_x = 0, f_y = y^{1/2}$

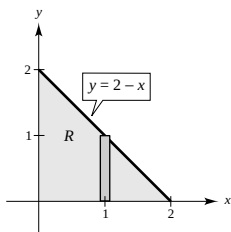
$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + y}$

$$S = \int_0^2 \int_0^{2-y} \sqrt{1 + y} \, dx \, dy = \int_0^2 \sqrt{1 + y} (2 - y) \, dy$$

$$= \left[2(1 + y)^{3/2} - \frac{2}{5}(1 + y)^{5/2} \right]_0^2$$

$$= 2 \cdot 3^{3/2} - \frac{2}{5} \cdot 3^{5/2} - 2 + \frac{2}{5}$$

$$= \frac{12}{5}\sqrt{3} - \frac{8}{5}$$



10. $f(x, y) = 9 + x^2 - y^2$

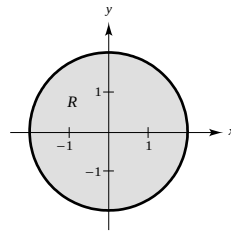
$f_x = 2x, f_y = -2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$

$$S = \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{12} (1 + 4r^2)^{3/2} \right]_0^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{12} (17^{3/2} - 1) \, d\theta = \frac{\pi}{6} (17\sqrt{17} - 1)$$



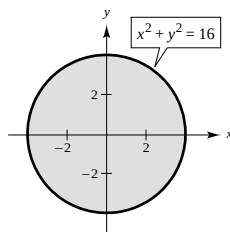
12. $f(x, y) = xy$

$$R = \{(x, y): x^2 + y^2 \leq 16\}$$

$$f_x = y, f_y = x$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^2 + x^2}$$

$$\begin{aligned} S &= \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sqrt{1 + y^2 + x^2} dy dx \\ &= \int_0^{2\pi} \int_0^4 \sqrt{1 + r^2} r dr d\theta = \frac{2\pi}{3} (17\sqrt{17} - 1) \end{aligned}$$



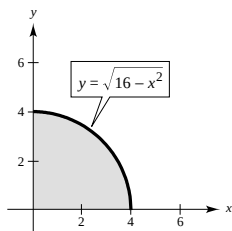
14. See Exercise 13.

$$S = \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{a}{\sqrt{a^2-x^2-y^2}} dy dx = \int_0^{2\pi} \int_0^a \frac{a}{\sqrt{a^2-r^2}} r dr d\theta = 2\pi a^2$$

16. $z = 16 - x^2 - y^2$

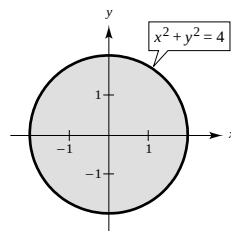
$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + 4x^2 + 4y^2}$$

$$\begin{aligned} S &= \int_0^4 \int_0^{\sqrt{16-x^2}} \sqrt{1 + 4(x^2 + y^2)} dy dx \\ &= \int_0^{2\pi} \int_0^4 \sqrt{1 + 4r^2} r dr d\theta = \frac{\pi}{24} (65\sqrt{65} - 1) \end{aligned}$$

18. $z = 2\sqrt{x^2 + y^2}$

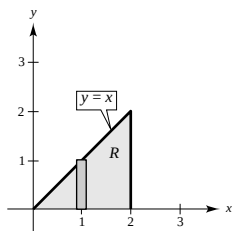
$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \frac{4x^2}{x^2 + y^2} + \frac{4y^2}{x^2 + y^2}} = \sqrt{5}$$

$$S = \int_0^{2\pi} \int_0^2 \sqrt{5} r dr d\theta = 4\pi\sqrt{5}$$

20. $f(x, y) = 2x + y^2$ R = triangle with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{5 + 4y^2}$$

$$S = \int_0^2 \int_0^x \sqrt{5 + 4y^2} dy dx = \frac{1}{12} (21\sqrt{21} - 5\sqrt{5})$$

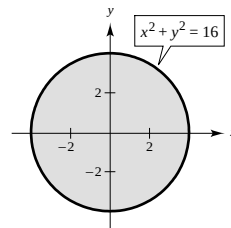
22. $f(x, y) = x^2 + y^2$ $R = \{(x, y): 0 \leq f(x, y) \leq 16\}$

$$0 \leq x^2 + y^2 \leq 16$$

$$f_x = 2x, f_y = 2y$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

$$\begin{aligned} S &= \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sqrt{1 + 4x^2 + 4y^2} dy dx \\ &= \int_0^{2\pi} \int_0^4 \sqrt{1 + 4r^2} r dr d\theta = \frac{(65\sqrt{65} - 1)\pi}{6} \end{aligned}$$



24. $f(x, y) = \frac{2}{3}x^{3/2} + \cos x$

$$R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

$$f_x = x^{1/2} - \sin x, \quad f_y = 0$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + (\sqrt{x} - \sin x)^2}$$

$$S = \int_0^1 \int_0^1 \sqrt{1 + (\sqrt{x} - \sin x)^2} dy dx \approx 1.02185$$

28. $f(x, y) = \frac{2}{5}y^{5/2}$

$$R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

$$f_x = 0, \quad f_y = y^{3/2}$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^3}$$

$$\begin{aligned} S &= \int_0^1 \int_0^1 \sqrt{1 + y^3} dx dy \\ &= \int_0^1 \sqrt{1 + y^3} dy \approx 1.1114 \end{aligned}$$

32. $f(x, y) = \cos(x^2 + y^2)$

$$R = \left\{ (x, y): x^2 + y^2 \leq \frac{\pi}{2} \right\}$$

$$f_x = -2x \sin(x^2 + y^2), \quad f_y = -2y \sin(x^2 + y^2)$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 \sin^2(x^2 + y^2) + 4y^2 \sin^2(x^2 + y^2)} = \sqrt{1 + 4[\sin^2(x^2 + y^2)](x^2 + y^2)}$$

$$S = \int_{-\sqrt{\pi/2}}^{\sqrt{\pi/2}} \int_{-\sqrt{(\pi/2)-x^2}}^{\sqrt{(\pi/2)-x^2}} \sqrt{1 + 4(x^2 + y^2) \sin^2(x^2 + y^2)} dy dx$$

34. $f(x, y) = e^{-x} \sin y$

$$R = \{(x, y): 0 \leq x \leq 4, 0 \leq y \leq x\}$$

$$f_x = -e^{-x} \sin y, \quad f_y = e^{-x} \cos y$$

$$\begin{aligned} \sqrt{1 + (f_x)^2 + (f_y)^2} &= \sqrt{1 + e^{-2x} \sin^2 y + e^{-2x} \cos^2 y} \\ &= \sqrt{1 + e^{-2x}} \end{aligned}$$

$$S = \int_0^4 \int_0^x \sqrt{1 + e^{-2x}} dy dx$$

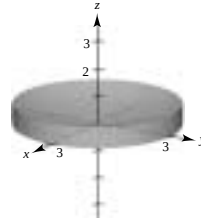
38. $f(x, y) = k\sqrt{x^2 + y^2}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{k^2 x^2}{x^2 + y^2} + \frac{k^2 y^2}{x^2 + y^2}} = \sqrt{k^2 + 1}$$

$$S = \iint_R \sqrt{1 + (f_x)^2 + (f_y)^2} dA = \iint_R \sqrt{k^2 + 1} dA = \sqrt{k^2 + 1} \iint_R dA = A\sqrt{k^2 + 1} = \pi r^2 \sqrt{k^2 + 1}$$

26. Surface area $\approx (9\pi)$

Matches (c)



30. $f(x, y) = x^2 - 3xy - y^2$

$$R = \{(x, y): 0 \leq x \leq 4, 0 \leq y \leq x\}$$

$$f_x = 2x - 3y, \quad f_y = -3x - 2y = -(3x + 2y)$$

$$\begin{aligned} \sqrt{1 + (f_x)^2 + (f_y)^2} &= \sqrt{1 + (2x - 3y)^2 + (3x + 2y)^2} \\ &= \sqrt{1 + 13(x^2 + y^2)} \end{aligned}$$

$$S = \int_0^4 \int_0^x \sqrt{1 + 13(x^2 + y^2)} dy dx$$

36. (a) Yes. For example, let R be the square given by

$$0 \leq x \leq 1, 0 \leq y \leq 1,$$

and S the square parallel to R given by

$$0 \leq x \leq 1, 0 \leq y \leq 1, z = 1.$$

(b) Yes. Let R be the region in part (a) and S the surface given by $f(x, y) = xy$.

(c) No.

40. (a) $z = -\frac{1}{75}y^3 + \frac{4}{25}y^2 - \frac{16}{15}y + 25$

(c) $f(x, y) = -\frac{1}{75}y^3 + \frac{4}{25}y^2 - \frac{16}{15}y + 25$

$$f_x = 0, f_y = -\frac{1}{25}y^2 + \frac{8}{25}y - \frac{16}{15}$$

$$S = 2 \int_0^{50} \int_0^{15} \sqrt{1 + f_y^2 + f_x^2} dy dx \approx 3087.58 \text{ sq ft}$$

(b) $V \approx 2(50) \int_0^{15} \left(-\frac{1}{75}y^3 + \frac{4}{25}y^2 - \frac{16}{15}y + 25 \right) dy$

$$= 100(266.25) = 26,625 \text{ cubic feet}$$

(d) Arc length ≈ 30.8758

$$\text{Surface area of roof} \approx 2(50)(30.8758) = 3087.58 \text{ sq ft}$$

42. False. The surface area will remain the same for any vertical translation.

Section 13.6 Triple Integrals and Applications

$$\begin{aligned} 2. \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 x^2 y^2 z^2 dx dy dz &= \frac{1}{3} \int_{-1}^1 \int_{-1}^1 \left[x^3 y^2 z^2 \right]_{-1}^1 dy dz \\ &= \frac{2}{3} \int_{-1}^1 \int_{-1}^1 y^2 z^2 dy dz = \frac{2}{9} \int_{-1}^1 \left[y^3 z^2 \right]_{-1}^1 dz = \frac{4}{9} \int_{-1}^1 z^2 dz = \left[\frac{4}{27} z^3 \right]_{-1}^1 = \frac{8}{27} \end{aligned}$$

$$\begin{aligned} 4. \int_0^9 \int_0^{y/3} \int_0^{\sqrt{y^2 - 9x^2}} z dz dx dy &= \frac{1}{2} \int_0^9 \int_0^{y/3} (y^2 - 9x^2) dx dy \\ &= \frac{1}{2} \int_0^9 \left[xy^2 - 3x^3 \right]_0^{y/3} dy = \frac{2}{18} \int_0^9 y^3 dy = \left[\frac{1}{36} y^4 \right]_0^9 = \frac{729}{4} \end{aligned}$$

$$\begin{aligned} 6. \int_1^4 \int_1^{e^2} \int_0^{1/xz} \ln z dy dz dx &= \int_1^4 \int_1^{e^2} \left[(\ln z)y \right]_0^{1/xz} dz dx = \int_1^4 \int_1^{e^2} \frac{\ln z}{xz} dz dx \\ &= \int_1^4 \left[\frac{1}{x} \frac{(\ln z)^2}{2} \right]_1^{e^2} dx = \int_1^4 \frac{2}{x} dx = \left[2 \ln |x| \right]_1^4 = 2 \ln 4 \end{aligned}$$

$$8. \int_0^{\pi/2} \int_0^{y/2} \int_0^{1/y} \sin y dz dx dy = \int_0^{\pi/2} \int_0^{y/2} \frac{\sin y}{y} dx dy = \frac{1}{2} \int_0^{\pi/2} \sin y dy = \left[-\frac{1}{2} \cos y \right]_0^{\pi/2} = \frac{1}{2}$$

$$10. \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{2x^2+y^2}^{4-y^2} y dz dy dx = \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} (4y - 2x^2y - 2y^3) dy dx = \frac{16\sqrt{2}}{15}$$

$$\begin{aligned} 12. \int_0^3 \int_0^{2-(2y/3)} \int_0^{6-2y-3z} ze^{-x^2y^2} dx dz dy &= \int_0^6 \int_0^{(6-x)/2} \int_0^{(6-x-2y)/3} ze^{-x^2y^2} dz dy dx \\ &= \int_0^6 \int_0^{3-(x/2)} \frac{1}{2} \left(\frac{6-x-2y}{3} \right)^2 e^{-x^2y^2} dy dx \approx 2.118 \end{aligned}$$

$$14. \int_0^3 \int_0^{2x} \int_0^{9-x^2} dz dy dx$$

16. $z = \frac{1}{2}(x^2 + y^2) \Rightarrow 2z = x^2 + y^2$

$$x^2 + y^2 + z^2 = 2z + z^2 = 80 \Rightarrow z^2 + 2z - 80 = 0 \Rightarrow (z - 8)(z + 10) = 0 \Rightarrow z = 8 \Rightarrow x^2 + y^2 = 2z = 16$$

$$\int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{1/2(x^2+y^2)}^{\sqrt{80-x^2-y^2}} dz \, dy \, dx$$

18. $\int_0^1 \int_0^1 \int_0^{xy} dz \, dy \, dx = \int_0^1 \int_0^1 xy \, dy \, dx = \int_0^1 \frac{x}{2} dx = \left[\frac{x^2}{4} \right]_0^1 = \frac{1}{4}$

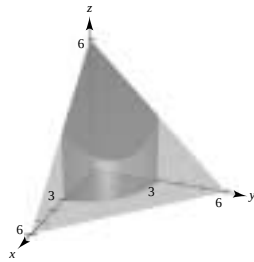
20. $4 \int_0^6 \int_0^{\sqrt{36-x^2}} \int_0^{36-x^2-y^2} dz \, dy \, dx = 4 \int_0^6 \int_0^{\sqrt{36-x^2}} (36 - x^2 - y^2) dy \, dx = 4 \int_0^6 \left[36y - x^2y - \frac{y^3}{3} \right]_0^{\sqrt{36-x^2}} dx$
 $= 4 \int_0^6 \left[36\sqrt{36-x^2} - x^2\sqrt{36-x^2} - \frac{1}{3}(36-x^2)^{3/2} \right] dx$
 $= 4 \left[9x\sqrt{36-x^2} + 324 \arcsin\left(\frac{x}{6}\right) + \frac{1}{6}x(36-x^2)^{3/2} \right]_0^6 = 4(162\pi) = 648\pi$

22. $\int_0^2 \int_0^{2-x} \int_0^{9-x^2} dz \, dy \, dx = \int_0^2 \int_0^{2-x} (9 - x^2) dy \, dx = \int_0^2 (9 - x^2)(2 - x) dx$
 $= \int_0^2 (18 - 9x - 2x^2 + x^3) dx = \left[18x - \frac{9}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 \right]_0^2 = \frac{50}{3}$

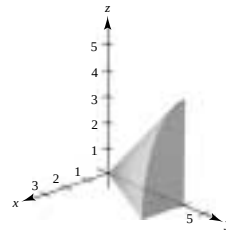
24. Top plane: $x + y + z = 6$

Side cylinder: $x^2 + y^2 = 9$

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{6-x-y} dz \, dx \, dy$$



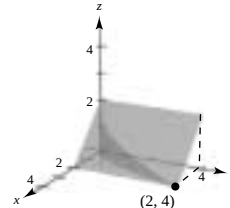
26. Elliptic cone: $4x^2 + z^2 = y^2$



$$\int_0^4 \int_z^4 \int_0^{\sqrt{y^2-z^2}/2} dx \, dy \, dz$$

28. $Q = \{(x, y, z) : 0 \leq x \leq 2, x^2 \leq y \leq 4, 0 \leq z \leq 2 - x\}$

$$\begin{aligned} \iiint_Q xyz \, dV &= \int_0^2 \int_{x^2}^4 \int_0^{2-x} xyz \, dz \, dy \, dx \\ &= \int_0^2 \int_0^{\sqrt{y}} \int_0^{2-x} xyz \, dz \, dx \, dy \\ &= \int_0^2 \int_0^{2-x} \int_{x^2}^4 xyz \, dy \, dz \, dx \\ &= \int_0^2 \int_0^{2-z} \int_{x^2}^4 xyz \, dy \, dx \, dz \\ &= \int_0^2 \int_0^{(2-z)^2} \int_0^{\sqrt{y}} xyz \, dx \, dy \, dz + \int_0^2 \int_{(2-z)^2}^4 \int_0^{2-z} xyz \, dx \, dy \, dz \\ &= \int_0^4 \int_0^{2-\sqrt{y}} \int_0^{\sqrt{y}} xyz \, dx \, dz \, dy + \int_0^4 \int_{2-\sqrt{y}}^2 \int_0^{2-z} dx \, dz \, dy \left(= \frac{104}{21} \right) \end{aligned}$$



30. $Q = \{(x, y, z): 0 \leq x \leq 1, y \leq 1 - x^2, 0 \leq z \leq 6\}$

$$\begin{aligned}
 \iiint_Q xyz \, dV &= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^6 xyz \, dz \, dy \, dx \\
 &= \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^6 xyz \, dz \, dx \, dy \\
 &= \int_0^1 \int_0^6 \int_0^{\sqrt{1-y^2}} xyz \, dx \, dz \, dy \\
 &= \int_0^6 \int_0^1 \int_0^{\sqrt{1-y^2}} xyz \, dx \, dy \, dz \\
 &= \int_0^1 \int_0^6 \int_0^{\sqrt{1-x^2}} xyz \, dy \, dz \, dx \\
 &= \int_0^6 \int_0^1 \int_0^{\sqrt{1-x^2}} xyz \, dy \, dx \, dz
 \end{aligned}$$



32. $m = k \int_0^5 \int_0^{5-x} \int_0^{1/5(15-3x-3y)} y \, dz \, dy \, dx = \frac{125}{8}k$

$M_{xz} = k \int_0^5 \int_0^{5-x} \int_0^{1/5(15-3x-3y)} y^2 \, dz \, dy \, dx = \frac{125}{4}k$

$\bar{y} = \frac{M_{xz}}{m} = 2$

34. $m = k \int_0^b \int_0^{a[1-(y/b)]} \int_0^{c[1-(y/b)-(x/a)]} dz \, dx \, dy = \frac{kabc}{6}$

$M_{xz} = k \int_0^b \int_0^{a[1-(y/b)]} \int_0^{c[1-(y/b)-(x/a)]} y \, dz \, dx \, dy = \frac{kab^2c}{24}$

$\bar{y} = \frac{M_{xz}}{m} = \frac{kab^2c/24}{kabc/6} = \frac{b}{4}$

36. $m = k \int_0^a \int_0^b \int_0^c z \, dz \, dy \, dx = \frac{kabc^2}{2}$

$M_{xy} = k \int_0^a \int_0^b \int_0^c z^2 \, dz \, dy \, dx = \frac{kabc^3}{3}$

$M_{yz} = k \int_0^a \int_0^b \int_0^c xz \, dz \, dy \, dx = \frac{ka^2bc^2}{4}$

$M_{xz} = k \int_0^a \int_0^b \int_0^c yz \, dz \, dy \, dx = \frac{kab^2c^2}{4}$

$\bar{x} = \frac{M_{yz}}{m} = \frac{ka^2bc^2/4}{kabc^2/2} = \frac{a}{2}$

$\bar{y} = \frac{M_{xz}}{m} = \frac{kab^2c^2/4}{kabc^2/2} = \frac{b}{2}$

$\bar{z} = \frac{M_{xy}}{m} = \frac{kabc^3/3}{kabc^2/2} = \frac{2c}{3}$

38. $\bar{z}$ will be greater than $8/5$, whereas $\bar{x}$ and $\bar{y}$ will be unchanged.

40. $\bar{x}, \bar{y}$ and $\bar{z}$ will all be greater than their original values.

$$42. \quad m = 2k \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^y dz \, dy \, dx$$

$$= k \int_0^2 (4 - x^2) \, dx = \frac{16k}{3}$$

$$M_{yz} = k \int_{-2}^2 \int_0^{\sqrt{4-x^2}} \int_0^y x \, dz \, dy \, dx = 0$$

$$M_{xz} = 2k \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^y y \, dz \, dy \, dx = 2k\pi$$

$$M_{xy} = 2k \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^y z \, dz \, dy \, dx = k\pi$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{0}{16k/3} = 0$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{2k\pi}{16k/3} = \frac{3\pi}{8}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi}{16k/3} = \frac{3\pi}{16}$$

$$44. \quad \bar{x} = 0$$

$$m = 2k \int_0^2 \int_0^1 \int_0^{1/(y^2+1)} dz \, dy \, dx = 2k \int_0^2 \int_0^1 \frac{1}{y^2+1} \, dy \, dx = 2k \left(\frac{\pi}{4} \right) \int_0^2 dx = k\pi$$

$$M_{xz} = 2k \int_0^2 \int_0^1 \int_0^{1/(y^2+1)} y \, dz \, dy \, dx = 2k \int_0^2 \int_0^1 \frac{y}{y^2+1} \, dy \, dx = k \int_0^2 (\ln 2) \, dx = k \ln 4$$

$$M_{xy} = 2k \int_0^2 \int_0^1 \int_0^{1/(y^2+1)} z \, dz \, dy \, dx$$

$$= k \int_0^2 \int_0^1 \frac{1}{(y^2+1)^2} \, dy \, dx = k \int_0^2 \left[\frac{y}{2(y^2+1)} + \frac{1}{2} \arctan y \right]_0^1 dx = k \left(\frac{1}{4} + \frac{\pi}{8} \right) \int_0^2 dx = k \left(\frac{1}{2} + \frac{\pi}{4} \right)$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{k \ln 4}{k\pi} = \frac{\ln 4}{\pi}$$

$$\bar{z} = \frac{M_{xy}}{m} = k \left(\frac{1}{2} + \frac{\pi}{4} \right) / k\pi = \frac{2 + \pi}{4\pi}$$

$$46. \quad f(x, y) = \frac{1}{15} (60 - 12x - 20y)$$

$$m = k \int_0^5 \int_0^{-(3/5)x+3} \int_0^{(1/15)(60-12x-20y)} dz \, dy \, dx = 10k$$

$$M_{yz} = k \int_0^5 \int_0^{-(3/5)x+3} \int_0^{(1/15)(60-12x-20y)} x \, dz \, dy \, dx = \frac{25k}{2}$$

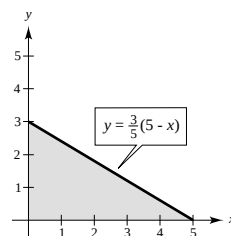
$$M_{xz} = k \int_0^5 \int_0^{-(3/5)x+3} \int_0^{(1/15)(60-12x-20y)} y \, dz \, dy \, dx = \frac{15k}{2}$$

$$M_{xy} = k \int_0^5 \int_0^{-(3/5)x+3} \int_0^{(1/15)(60-12x-20y)} z \, dz \, dy \, dx = 10k$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{25k/2}{10k} = \frac{5}{4}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{15k/2}{10k} = \frac{3}{4}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{10k}{10k} = 1$$



$$48. (a) I_{xy} = k \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} z^2 dz dy dx = \frac{ka^5}{12}$$

$$I_{xz} = I_{yz} = \frac{ka^5}{12} \text{ by symmetry}$$

$$I_x = I_y = I_z = \frac{ka^5}{12} + \frac{ka^5}{12} = \frac{ka^5}{6}$$

$$(b) I_{xy} = k \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} z^2(x^2 + y^2) dz dy dx = \frac{a^3k}{12} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} (x^2 + y^2) dy dx = \frac{a^7k}{72}$$

$$I_{xz} = k \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} y^2(x^2 + y^2) dz dy dx = ka \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} (x^2y^2 + y^4) dy dx = \frac{7ka^7}{360}$$

$$I_{yz} = I_{xz} \text{ by symmetry}$$

$$I_x = I_{xy} + I_{xz} = \frac{a^7k}{30}$$

$$I_y = I_{xy} + I_{yz} = \frac{a^7k}{30}$$

$$I_z = I_{yz} + I_{xz} = \frac{7ka^7}{180}$$

$$50. (a) I_{xy} = k \int_0^4 \int_0^2 \int_0^{4-y^2} z^3 dz dy dx = k \int_0^4 \int_0^2 \frac{1}{4} (4 - y^2)^4 dy dx$$

$$= \frac{k}{4} \int_0^4 \int_0^2 (256 - 256y^2 + 96y^4 - 16y^6 + y^8) dy dx$$

$$= \frac{k}{4} \int_0^4 \left[256y - \frac{256y^3}{3} + \frac{96y^5}{5} - \frac{16y^7}{7} + \frac{y^9}{9} \right]_0^2 dx = k \int_0^4 \frac{16,384}{945} dx = \frac{65,536k}{315}$$

$$I_{xz} = k \int_0^4 \int_0^2 \int_0^{4-y^2} y^2z dz dy dx = k \int_0^4 \int_0^2 \frac{1}{2} y^2 (4 - y^2)^2 dy dx$$

$$= k \int_0^4 \int_0^2 \frac{1}{2} (16y^2 - 8y^4 + y^6) dy dx = \frac{k}{2} \int_0^4 \left[\frac{16y^3}{3} - \frac{8y^5}{5} + \frac{y^7}{7} \right]_0^2 dx = \frac{k}{2} \int_0^4 \frac{1024}{105} dx = \frac{2048k}{105}$$

$$I_{yz} = k \int_0^4 \int_0^2 \int_0^{4-y^2} x^2z dz dy dx = k \int_0^4 \int_0^2 \frac{1}{2} x^2 (4 - y^2)^2 dy dx$$

$$= k \int_0^4 \int_0^2 \frac{1}{2} x^2 (16 - 8y^2 + y^4) dy dx = \frac{k}{2} \int_0^4 \left[x^2 \left(16y - \frac{8y^3}{3} + \frac{y^5}{5} \right) \right]_0^2 dx = \frac{k}{2} \int_0^4 \frac{256}{15} x^2 dx = \frac{8192k}{45}$$

$$I_x = I_{xz} + I_{xy} = \frac{2048k}{9}, I_y = I_{yz} + I_{xy} = \frac{8192k}{21}, I_z = I_{yz} + I_{xz} = \frac{63,488k}{315}$$

—CONTINUED—

50. —CONTINUED—

$$\begin{aligned}
 \text{(b) } I_{xy} &= \int_0^4 \int_0^2 \int_0^{4-y^2} z^2(4-z) \, dz \, dy \, dx \\
 &= k \int_0^4 \int_0^2 \int_0^{4-y^2} 4z^2 \, dz \, dy \, dx - k \int_0^4 \int_0^2 \int_0^{4-y^2} z^3 \, dz \, dy \, dx = \frac{32,768k}{105} - \frac{65,536k}{315} = \frac{32,768k}{315} \\
 I_{xz} &= \int_0^4 \int_0^2 \int_0^{4-y^2} y^2(4-z) \, dz \, dy \, dx \\
 &= k \int_0^4 \int_0^2 \int_0^{4-y^2} 4y^2 \, dz \, dy \, dx - k \int_0^4 \int_0^2 \int_0^{4-y^2} y^2 z \, dz \, dy \, dx = \frac{1024k}{15} - \frac{2048k}{105} = \frac{1024k}{21} \\
 I_{yz} &= k \int_0^4 \int_0^2 \int_0^{4-y^2} x^2(4-z) \, dz \, dy \, dx \\
 &= k \int_0^4 \int_0^2 \int_0^{4-y^2} 4x^2 \, dz \, dy \, dx - k \int_0^4 \int_0^2 \int_0^{4-y^2} x^2 z \, dz \, dy \, dx = \frac{4096k}{9} - \frac{8192k}{45} = \frac{4096k}{15} \\
 I_x &= I_{xz} + I_{xy} = \frac{48,128k}{315}, \quad I_y = I_{yz} + I_{xy} = \frac{118,784k}{315}, \quad I_z = I_{xz} + I_{yz} = \frac{11,264k}{35}
 \end{aligned}$$

$$\begin{aligned}
 52. \quad I_{xy} &= \int_{-c/2}^{c/2} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} z^2 \, dz \, dy \, dx = \frac{b^3}{12} \int_{-c/2}^{c/2} \int_{-a/2}^{a/2} dy \, dx = \frac{1}{12} b^2(abc) = \frac{1}{12} mb^2 \\
 I_{xz} &= \int_{-c/2}^{c/2} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} y^2 \, dz \, dy \, dx = b \int_{-c/2}^{c/2} \int_{-a/2}^{a/2} y^2 \, dy \, dx = \frac{ba^3}{12} \int_{-c/2}^{c/2} dx = \frac{ba^3c}{12} = \frac{1}{12} a^2(abc) = \frac{1}{12} ma^2 \\
 I_{yz} &= \int_{-c/2}^{c/2} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} x^2 \, dz \, dy \, dx = ab \int_{-c/2}^{c/2} x^2 \, dx = \frac{abc^3}{12} = \frac{1}{12} c^2(abc) = \frac{1}{12} mc^2 \\
 I_x &= I_{xy} + I_{xz} = \frac{1}{12} m(a^2 + b^2) \\
 I_y &= I_{xy} + I_{yz} = \frac{1}{12} m(b^2 + c^2) \\
 I_z &= I_{xz} + I_{yz} = \frac{1}{12} m(a^2 + c^2)
 \end{aligned}$$

$$54. \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{4-x^2-y^2} kx^2(x^2 + y^2) \, dz \, dy \, dx \qquad 56. \quad 6$$

58. Because the density increases as you move away from the axis of symmetry, the moment of inertia will increase.

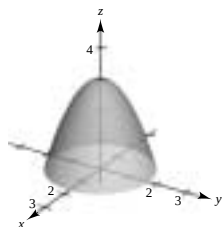
Section 13.7 Triple Integrals in Cylindrical and Spherical Coordinates

$$\begin{aligned}
 2. \quad \int_0^{\pi/4} \int_0^2 \int_0^{2-r} rz \, dz \, dr \, d\theta &= \int_0^{\pi/4} \int_0^2 \left[\frac{rz^2}{2} \right]_0^{2-r} dr \, d\theta \\
 &= \frac{1}{2} \int_0^{\pi/4} \int_0^2 (4r - 4r^2 + r^3) dr \, d\theta = \frac{1}{2} \int_0^{\pi/4} \left[2r^2 - \frac{4r^3}{3} + \frac{r^4}{4} \right]_0^2 d\theta = \frac{2}{3} \int_0^{\pi/4} d\theta = \frac{\pi}{6} \\
 4. \quad \int_0^{\pi/2} \int_0^\pi \int_0^2 e^{-\rho^3} \rho^2 \, d\rho \, d\theta \, d\phi &= \int_0^{\pi/2} \int_0^\pi \left[-\frac{1}{3} e^{-\rho^3} \right]_0^2 d\theta \, d\phi = \int_0^{\pi/2} \int_0^\pi \frac{1}{3} (1 - e^{-8}) d\theta \, d\phi = \frac{\pi^2}{6} (1 - e^{-8})
 \end{aligned}$$

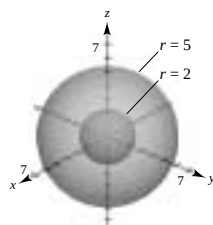
$$\begin{aligned}
 6. \int_0^{\pi/4} \int_0^{\pi/4} \int_0^{\cos \theta} \rho^2 \sin \phi \cos \phi \, d\rho \, d\theta \, d\phi &= \frac{1}{3} \int_0^{\pi/4} \int_0^{\pi/4} \cos^3 \theta \sin \phi \cos \phi \, d\theta \, d\phi \\
 &= \frac{1}{3} \int_0^{\pi/4} \int_0^{\pi/4} \sin \phi \cos \phi [\cos \theta (1 - \sin^2 \theta)] \, d\theta \, d\phi \\
 &= \frac{1}{3} \int_0^{\pi/4} \sin \phi \cos \phi \left[\sin \theta - \frac{\sin^3 \theta}{3} \right]_0^{\pi/4} \, d\phi \\
 &= \frac{5\sqrt{2}}{36} \int_0^{\pi/4} \sin \phi \cos \phi \, d\phi = \left[\frac{5\sqrt{2}}{36} \frac{\sin^2 \phi}{2} \right]_0^{\pi/4} = \frac{5\sqrt{2}}{144}
 \end{aligned}$$

$$8. \int_0^{\pi/2} \int_0^{\pi} \int_0^{\sin \theta} (2 \cos \phi) \rho^2 \, d\rho \, d\theta \, d\phi = \frac{8}{9}$$

$$\begin{aligned}
 10. \int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^{3-r^2} r \, dz \, dr \, d\theta &= \int_0^{2\pi} \int_0^{\sqrt{3}} r(3 - r^2) \, dr \, d\theta \\
 &= \int_0^{2\pi} \left(\frac{3r^2}{2} - \frac{r^4}{4} \right) \Big|_0^{\sqrt{3}} \, d\theta \\
 &= \int_0^{2\pi} \frac{9}{4} \, d\theta = \frac{9\pi}{2}
 \end{aligned}$$



$$\begin{aligned}
 12. \int_0^{2\pi} \int_0^{\pi} \int_2^5 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta &= \frac{117}{3} \int_0^{2\pi} \int_0^{\pi} \sin \phi \, d\phi \, d\theta \\
 &= \frac{117}{3} \int_0^{2\pi} \left[-\cos \phi \right]_0^{\pi} \, d\theta \\
 &= \frac{468\pi}{3}
 \end{aligned}$$



$$14. (a) \int_0^{\pi/2} \int_0^2 \int_0^{\sqrt{16-r^2}} r^2 \, dz \, dr \, d\theta = \frac{8\pi^2}{3} - 2\pi\sqrt{3}$$

$$(b) \int_0^{\pi/2} \int_0^{\pi/6} \int_0^4 \rho^3 \sin^2 \phi \, d\rho \, d\phi \, d\theta + \int_0^{\pi/2} \int_{\pi/6}^{\pi/2} \int_4^{2 \csc \phi} \rho^3 \sin^2 \phi \, d\rho \, d\phi \, d\theta = \frac{8\pi^2}{3} - 2\pi\sqrt{3}$$

$$16. (a) \int_0^{\pi/2} \int_0^1 \int_0^{\sqrt{1-r^2}} r \sqrt{r^2 + z^2} \, dz \, dr \, d\theta = \frac{\pi}{8}$$

$$(b) \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^3 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{\pi}{8}$$

$$18. V = \frac{2}{3} \pi (4)^3 + 4 \left[\int_0^{\pi/2} \int_0^{2\sqrt{2}} \int_0^r r \, dz \, dr \, d\theta + \int_0^{\pi/2} \int_{2\sqrt{2}}^4 \int_0^{\sqrt{16-r^2}} r \, dz \, dr \, d\theta \right]$$

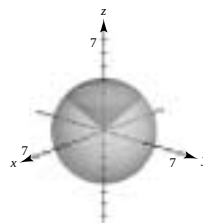
(Volume of lower hemisphere) + 4(Volume in the first octant)

$$V = \frac{128\pi}{3} + 4 \left[\int_0^{\pi/2} \int_0^{2\sqrt{2}} r^2 \, dr \, d\theta + \int_0^{\pi/2} \int_{2\sqrt{2}}^4 r \sqrt{16-r^2} \, dr \, d\theta \right]$$

$$= \frac{128\pi}{3} + 4 \left[\frac{8\sqrt{2}\pi}{3} + \int_0^{\pi/2} \left[-\frac{1}{3} (16-r^2)^{3/2} \right]_{2\sqrt{2}}^4 \, d\theta \right]$$

$$= \frac{128\pi}{3} + 4 \left[\frac{8\sqrt{2}\pi}{3} + \frac{8\sqrt{2}\pi}{3} \right]$$

$$= \frac{128\pi}{3} + \frac{64\sqrt{2}\pi}{3} = \frac{64\pi}{3} (2 + \sqrt{2})$$



$$\begin{aligned}
20. \quad V &= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} (r\sqrt{4-r^2} - r^2) \, dr \, d\theta \\
&= \int_0^{2\pi} \left[-\frac{1}{3}(4-r^2)^{3/2} - \frac{r^3}{3} \right]_0^{\sqrt{2}} d\theta \\
&= \frac{8\pi}{3}(2 - \sqrt{2})
\end{aligned}$$

$$\begin{aligned}
22. \quad \int_0^{\pi/2} \int_0^2 \int_0^{12e^{-r^2}} k r \, dz \, dr \, d\theta &= \int_0^{\pi/2} \int_0^2 12ke^{-r^2} r \, dr \, d\theta \\
&= \int_0^{\pi/2} \left[-6ke^{-r^2} \right]_0^2 d\theta \\
&= \int_0^{\pi/2} (-6ke^{-4} + 6k) \, d\theta \\
&= 3k\pi(1 - e^{-4})
\end{aligned}$$

24. $\bar{x} = \bar{y} = 0$ by symmetry

$$m = \frac{1}{3}\pi r_0^2 h k \text{ from Exercise 23}$$

$$\begin{aligned}
M_{xy} &= 4k \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} zr \, dz \, dr \, d\theta \\
&= \frac{2kh^2}{r_0^2} \int_0^{\pi/2} \int_0^{r_0} (r_0^2 r - 2r_0 r^2 + r^3) \, dr \, d\theta \\
&= \frac{2kh^2}{r_0^2} \left(\frac{r_0^4}{12} \right) \left(\frac{\pi}{2} \right) = \frac{kr_0^2 h^2 \pi}{12} \\
\bar{z} &= \frac{M_{xy}}{m} = \frac{kr_0^2 h^2 \pi}{12} \left(\frac{3}{\pi r_0^2 h k} \right) = \frac{h}{4}
\end{aligned}$$

26. $\rho = kz$

$\bar{x} = \bar{y} = 0$ by symmetry

$$\begin{aligned}
m &= 4k \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} zr \, dz \, dr \, d\theta \\
&= \frac{1}{12} k \pi r_0^2 h^2 \\
M_{xy} &= 4k \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} z^2 r \, dz \, dr \, d\theta \\
&= \frac{1}{30} k \pi r_0^2 h^3 \\
\bar{z} &= \frac{M_{xy}}{m} = \frac{k \pi r_0^2 h^3 / 30}{k \pi r_0^2 h^2 / 12} = \frac{2h}{5}
\end{aligned}$$

$$\begin{aligned}
28. \quad I_z &= \iiint_Q (x^2 + y^2) \rho(x, y, z) \, dV \\
&= 4k \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} r^4 \, dz \, dr \, d\theta \\
&= 4kh \int_0^{\pi/2} \int_0^{r_0} \frac{r_0 - r}{r_0} r^4 \, dr \, d\theta \\
&= 4kh \int_0^{\pi/2} \left[\frac{r^5}{5} - \frac{r^6}{6r_0} \right]_0^{r_0} d\theta \\
&= 4kh \int_0^{\pi/2} \left[\frac{r_0^5}{5} - \frac{r_0^5}{6} \right] d\theta \\
&= 4kh \int_0^{\pi/2} \frac{1}{30} r_0^5 \, d\theta \\
&= 4kh \frac{1}{30} r_0^5 \frac{\pi}{2} \\
&= \frac{1}{15} r_0^5 \pi k h
\end{aligned}$$

30. $m = k\pi a^2 h$

$$\begin{aligned}
I_z &= 2k \int_0^{\pi/2} \int_0^{2a \sin \theta} \int_0^h r^3 \, dz \, dr \, d\theta \\
&= \frac{3}{2} k \pi a^4 h \\
&= \frac{3}{2} m a^2
\end{aligned}$$

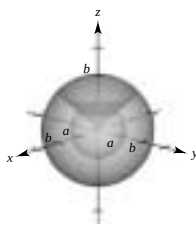
$$32. V = 8 \int_0^{\pi/4} \int_0^{\pi/2} \int_a^b \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \quad (\text{includes upper and lower cones})$$

$$= \frac{8}{3}(b^3 - a^3) \int_0^{\pi/4} \int_0^{\pi/2} \sin \phi \, d\theta \, d\phi$$

$$= \frac{4\pi}{3}(b^3 - a^3) \int_0^{\pi/4} \sin \phi \, d\phi$$

$$= \left[\frac{4\pi}{3}(b^3 - a^3)(-\cos \phi) \right]_0^{\pi/4}$$

$$= \left(1 - \frac{\sqrt{2}}{2} \right) \frac{4\pi}{3}(b^3 - a^3) = \frac{2\pi}{3}(2 - \sqrt{2})(b^3 - a^3)$$



$$34. m = 8k \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \rho^3 \sin^2 \phi \, d\rho \, d\theta \, d\phi$$

$$= 2ka^4 \int_0^{\pi/2} \int_0^{\pi/2} \sin^2 \phi \, d\theta \, d\phi$$

$$= k\pi a^4 \int_0^{\pi/2} \sin^2 \phi \, d\phi$$

$$= \left[k\pi a^4 \left(\frac{1}{2}\phi - \frac{1}{4}\sin 2\phi \right) \right]_0^{\pi/2}$$

$$= k\pi a^4 \frac{\pi}{4} = \frac{1}{4}k\pi^2 a^4$$

$$36. \quad \bar{x} = \bar{y} = 0 \text{ by symmetry}$$

$$m = k \left(\frac{2}{3}\pi R^3 - \frac{2}{3}\pi r^3 \right) = \frac{2}{3}k\pi(R^3 - r^3)$$

$$M_{xy} = 4k \int_0^{\pi/2} \int_0^{\pi/2} \int_r^R \rho^3 \cos \phi \sin \phi \, d\rho \, d\theta \, d\phi$$

$$= \frac{1}{2}k(R^4 - r^4) \int_0^{\pi/2} \int_0^{\pi/2} \sin 2\phi \, d\theta \, d\phi$$

$$= \frac{1}{4}k\pi(R^4 - r^4) \int_0^{\pi/2} \sin 2\phi \, d\phi$$

$$= \left[-\frac{1}{8}k\pi(R^4 - r^4) \cos 2\phi \right]_0^{\pi/2} = \frac{1}{4}k\pi(R^4 - r^4)$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi(R^4 - r^4)/4}{2k\pi(R^3 - r^3)/3} = \frac{3(R^4 - r^4)}{8(R^3 - r^3)}$$

$$38. I_z = 4k \int_0^{\pi/2} \int_0^{\pi/2} \int_r^R \rho^4 \sin^3 \phi \, d\rho \, d\theta \, d\phi$$

$$= \frac{4k}{5}(R^5 - r^5) \int_0^{\pi/2} \int_0^{\pi/2} \sin^3 \phi \, d\theta \, d\phi$$

$$= \frac{2k\pi}{5}(R^5 - r^5) \int_0^{\pi/2} \sin \phi (1 - \cos^2 \phi) \, d\phi$$

$$= \left[\frac{2k\pi}{5}(R^5 - r^5) \left(-\cos \phi + \frac{\cos^3 \phi}{3} \right) \right]_0^{\pi/2}$$

$$= \frac{4k\pi}{15}(R^5 - r^5)$$

$$40. \quad x = \rho \sin \phi \cos \theta \quad \rho^2 = x^2 + y^2 + z^2$$

$$y = \rho \sin \phi \sin \theta \quad \tan \theta = \frac{y}{x}$$

$$z = \rho \cos \phi \quad \cos \phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$42. \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

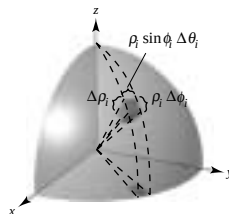
44. (a) You are integrating over a cylindrical wedge.

(b) You are integrating over a spherical block.

46. The volume of this spherical block can be determined as follows. One side is length $\Delta\rho$. Another side is $\rho\Delta\phi$. Finally, the third side is given by the length of an arc of angle $\Delta\theta$ in a circle of radius $\rho \sin \phi$. Thus:

$$\Delta V \approx (\Delta\rho)(\rho\Delta\phi)(\Delta\theta \rho \sin \phi)$$

$$= \rho^2 \sin \phi \Delta\rho \Delta\phi \Delta\theta$$



Section 13.8 Change of Variables: Jacobians

2. $x = au + bv$

$$y = cu + dv$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = ad - cb$$

6. $x = u + a$

$$y = v + a$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = (1)(1) - (0)(0) = 1$$

10. $x = \frac{1}{3}(4u - v)$

$$y = \frac{1}{3}(u - v)$$

$$u = x - y$$

$$v = x - 4y$$

(x, y)	(u, v)
$(0, 0)$	$(0, 0)$
$(4, 1)$	$(3, 0)$
$(2, 2)$	$(0, -6)$
$(6, 3)$	$(3, -6)$

4. $x = uv - 2u$

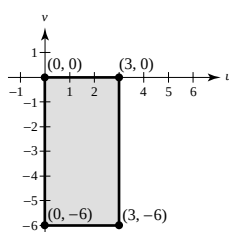
$$y = uv$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = (v - 2)u - vu = -2u$$

8. $x = \frac{u}{v}$

$$y = u + v$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \left(\frac{1}{v}\right)(1) - (1)\left(-\frac{u}{v^2}\right) = \frac{1}{v} + \frac{u}{v^2} = \frac{u + v}{v^2}$$



12. $x = \frac{1}{2}(u + v), \quad u = x - y$

$$y = -\frac{1}{2}(u - v), \quad v = x + y$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \frac{1}{2}\left(\frac{1}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2}$$

$$\iint_R 60xy \, dA$$

$$= \int_{-1}^1 \int_1^3 60\left(\frac{1}{2}(u + v)\right)\left(-\frac{1}{2}(u - v)\right)\left(\frac{1}{2}\right) dv du$$

$$= \int_{-1}^1 \int_1^3 -\frac{15}{2}(v^2 - u^2) dv du$$

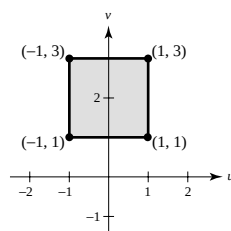
$$= \int_{-1}^1 \left[-\frac{15}{2}\left(\frac{v^3}{3} - u^2v\right)\right]_1^3 du$$

$$= \int_{-1}^1 \frac{15}{2}\left(2u^2 - \frac{26}{3}\right) du$$

$$= \left[\frac{15}{2}\left(\frac{2}{3}u^3 - \frac{26}{3}u\right)\right]_{-1}^1$$

$$= 15\left(\frac{2}{3} - \frac{26}{3}\right) = -120$$

(x, y)	(u, v)
$(0, 1)$	$(-1, 1)$
$(2, 1)$	$(1, 3)$
$(1, 2)$	$(-1, 3)$
$(1, 0)$	$(1, 1)$

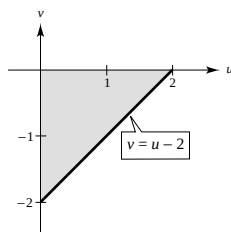


14. $x = \frac{1}{2}(u + v)$

$y = \frac{1}{2}(u - v)$

$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = -\frac{1}{2}$

$$\begin{aligned} \iint_R 4(x + y)e^{x-y} dA &= \int_0^2 \int_{u-2}^0 4ue^v \left(\frac{1}{2}\right) dv du \\ &= \int_0^2 2u(1 - e^{u-2}) du = 2 \left[\frac{u^2}{2} - ue^{u-2} + e^{u-2} \right]_0^2 = 2(1 - e^{-2}) \end{aligned}$$



16. $x = \frac{u}{v}$

$y = v$

$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \frac{1}{v}$

$$\iint_R y \sin xy dA = \int_1^4 \int_1^4 v(\sin u) \frac{1}{v} dv du = \int_1^4 3 \sin u du = \left[-3 \cos u \right]_1^4 = 3(\cos 1 - \cos 4) \approx 3.5818$$

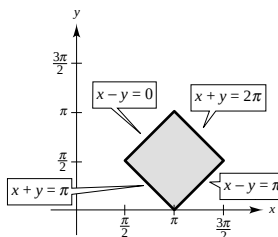
18. $u = x + y = \pi, \quad v = x - y = 0$

$u = x + y = 2\pi, \quad v = x - y = \pi$

$x = \frac{1}{2}(u + v), \quad y = \frac{1}{2}(u - v)$

$\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{2}$

$$\begin{aligned} \iint_R (x + y)^2 \sin^2(x - y) dA &= \int_0^\pi \int_\pi^{2\pi} u^2 \sin^2 v \left(\frac{1}{2}\right) dv du \\ &= \int_0^\pi \left[\frac{1}{2} \left(\frac{u^3}{3} \right) \frac{1 - \cos 2v}{2} \right]_\pi^{2\pi} dv = \left[\frac{7\pi^3}{12} \left(v - \frac{1}{2} \sin 2v \right) \right]_0^\pi = \frac{7\pi^4}{12} \end{aligned}$$



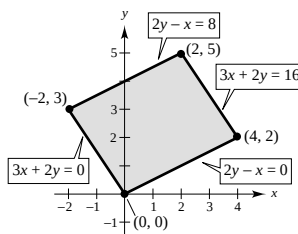
20. $u = 3x + 2y = 0, \quad v = 2y - x = 0$

$u = 3x + 2y = 16, \quad v = 2y - x = 8$

$x = \frac{1}{4}(u - v), \quad y = \frac{1}{8}(u + 3v)$

$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \frac{1}{4} \left(\frac{3}{8} \right) - \frac{1}{8} \left(-\frac{1}{4} \right) = \frac{1}{8}$

$$\begin{aligned} \iint_R (3x + 2y)(2y - x)^{3/2} dA &= \int_0^8 \int_0^{16} uv^{3/2} \left(\frac{1}{8}\right) dv du \\ &= \int_0^8 16v^{3/2} dv = \left(\frac{2}{5} \right) 16v^{5/2} \Big|_0^8 = \frac{4096}{5} \sqrt{2} \end{aligned}$$



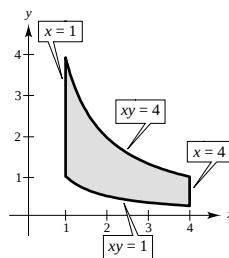
$$22. \quad u = x = 1, \quad v = xy = 1$$

$$u = x = 4, \quad v = xy = 4$$

$$x = u, \quad y = \frac{v}{u}$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \frac{1}{u}$$

$$\begin{aligned} \iint_R \frac{xy}{1+x^2y^2} dA &= \int_1^4 \int_1^4 \frac{v}{1+v^2} \left(\frac{1}{u}\right) dv du \\ &= \int_1^4 \left[\frac{1}{2} \ln(1+v^2) \right]_1^4 \frac{1}{u} du = \left[\frac{1}{2} [\ln 17 - \ln 2] \ln u \right]_1^4 = \frac{1}{2} \left(\ln \frac{17}{2} \right) (\ln 4) \end{aligned}$$



$$24. \quad (a) \quad f(x, y) = 16 - x^2 - y^2$$

$$R: \frac{x^2}{16} + \frac{y^2}{9} \leq 1$$

$$V = \iint_R f(x, y) dA$$

$$\text{Let } x = 4u \text{ and } y = 3v.$$

$$\begin{aligned} \iint_R (16 - x^2 - y^2) dA &= \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} (16 - 16u^2 - 9v^2) 12 dv du \quad (\text{Let } u = r \cos \theta, v = r \sin \theta.) \\ &= \int_0^{2\pi} \int_0^1 (16 - 16r^2 \cos^2 \theta - 9r^2 \sin^2 \theta) 12r dr d\theta \\ &= 12 \int_0^{2\pi} \left[8r^2 - 4r^4 \cos^2 \theta - \frac{9}{4} r^4 \sin^2 \theta \right]_0^1 d\theta = 12 \int_0^{2\pi} \left[8 - 4 \cos^2 \theta - \frac{9}{4} \sin^2 \theta \right] d\theta \\ &= 12 \int_0^{2\pi} \left[8 - 4 \left(\frac{1 + \cos 2\theta}{2} \right) - \frac{9}{4} \left(\frac{1 - \cos 2\theta}{2} \right) \right] d\theta = 12 \int_0^{2\pi} \left[\frac{39}{8} - \frac{7}{8} \cos 2\theta \right] d\theta \\ &= 12 \left[\frac{39}{8} \theta - \frac{7}{16} \sin 2\theta \right]_0^{2\pi} = 12 \left[\frac{39\pi}{4} \right] = 117\pi \end{aligned}$$

$$(b) \quad f(x, y) = A \cos \left[\frac{\pi}{2} \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}} \right]$$

$$R: \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$

$$\text{Let } x = au \text{ and } y = bv.$$

$$\iint_R f(x, y) dA = \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} A \cos \left[\frac{\pi}{2} \sqrt{u^2 + v^2} \right] ab dv du$$

$$\text{Let } u = r \cos \theta, v = r \sin \theta.$$

$$\begin{aligned} Aab \int_0^{2\pi} \int_0^1 \cos \left[\frac{\pi}{2} r \right] r dr d\theta &= Aab \left[\frac{2r}{\pi} \sin \left(\frac{\pi r}{2} \right) + \frac{4}{\pi^2} \cos \left(\frac{\pi r}{2} \right) \right]_0^1 (2\pi) \\ &= 2\pi Aab \left[\left(\frac{2}{\pi} + 0 \right) - \left(0 + \frac{4}{\pi^2} \right) \right] = \frac{4(\pi - 2)Aab}{\pi} \end{aligned}$$

26. See Theorem 13.5.

28. $x = 4u - v$, $y = 4v - w$, $z = u + w$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 4 & -1 & 0 \\ 0 & 4 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 17$$

30. $x = r \cos \theta$, $y = r \sin \theta$, $z = z$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1[r \cos^2 \theta + r \sin^2 \theta] = r$$

Review Exercises for Chapter 13

$$2. \int_y^{2y} (x^2 + y^2) dx = \left[\frac{x^3}{3} + xy^2 \right]_y^{2y} = \frac{10y^3}{3}$$

$$4. \int_0^2 \int_{x^2}^{2x} (x^2 + 2y) dy dx = \int_0^2 \left[x^2 y + y^2 \right]_{x^2}^{2x} dx = \int_0^2 (4x^2 + 2x^3 - 2x^4) dx = \left[\frac{4}{3}x^3 + \frac{1}{2}x^4 - \frac{2}{5}x^5 \right]_0^2 = \frac{88}{15}$$

$$6. \int_0^{\sqrt{3}} \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dx dy = 2 \int_0^{\sqrt{3}} \sqrt{4-y^2} dy = \left[y\sqrt{4-y^2} + 4 \arcsin \frac{y}{2} \right]_0^{\sqrt{3}} = \sqrt{3} + \frac{4\pi}{3}$$

$$8. \int_0^2 \int_0^x dy dx + \int_2^3 \int_0^{6-2x} dy dx = \int_0^2 \int_y^{(6-y)/2} dx dy$$

$$A = \int_0^2 \int_y^{(6-y)/2} dx dy$$

$$= \frac{1}{2} \int_0^2 (6-3y) dy = \left[\frac{1}{2} \left(6y - \frac{3}{2}y^2 \right) \right]_0^2 = 3$$

$$10. \int_0^4 \int_{x^2-2x}^{6x-x^2} dy dx = \int_{-1}^0 \int_{1-\sqrt{1+y}}^{1+\sqrt{1+y}} dy dx + \int_0^8 \int_{3-\sqrt{9-y}}^{1+\sqrt{1+y}} dx dy + \int_8^9 \int_{3-\sqrt{9-y}}^{3+\sqrt{9-y}} dx dy$$

$$A = \int_0^4 \int_{x^2-2x}^{6x-x^2} dy dx = \int_0^4 (8x-2x^2) dx = \left[4x^2 - \frac{2}{3}x^3 \right]_0^4 = \frac{64}{3}$$

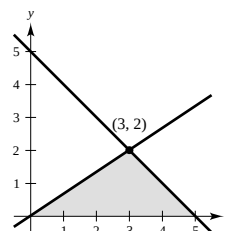
$$12. A = \int_0^2 \int_0^{y^2+1} dx dy = \int_0^1 \int_0^2 dy dx + \int_1^5 \int_{\sqrt{x-1}}^2 dy dx = \frac{14}{3}$$

$$14. A = \int_0^3 \int_{-y}^{2y-y^2} dx dy = \int_{-3}^0 \int_{-x}^{1+\sqrt{1-x}} dy dx + \int_0^1 \int_{1-\sqrt{1-x}}^{1+\sqrt{1-x}} dy dx = \frac{9}{2}$$

16. Both integrations are over the common region R shown in the figure. Analytically,

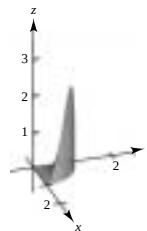
$$\int_0^2 \int_{3y/2}^{5-y} e^{x+y} dx dy = \frac{2}{5} + \frac{8}{5}e^5$$

$$\int_0^3 \int_0^{2x/3} e^{x+y} dy dx + \int_3^5 \int_0^{5-x} e^{x+y} dy dx = \left(\frac{3}{5}e^5 - e^3 + \frac{2}{5} \right) + (e^5 + e^3) = \frac{8}{5}e^5 + \frac{2}{5}$$



$$\begin{aligned}
 18. \quad V &= \int_0^3 \int_0^x (x+y) \, dy \, dx \\
 &= \int_0^3 \left[xy + \frac{1}{2}y^2 \right]_0^x \, dx \\
 &= \frac{3}{2} \int_0^3 x^2 \, dx \\
 &= \left[\frac{1}{2}x^3 \right]_0^3 = \frac{27}{2}
 \end{aligned}$$

20. Matches (c)



$$\begin{aligned}
 22. \quad \int_0^1 \int_0^x kxy \, dy \, dx &= \int_0^1 \left[\frac{kxy^2}{2} \right]_0^x \, dx \\
 &= \int_0^1 \frac{kx^3}{2} \, dx \\
 &= \left[\frac{kx^4}{8} \right]_0^1 = \frac{k}{8}
 \end{aligned}$$

Since $k/8 = 1$, we have $k = 8$.

$$P = \int_0^{0.5} \int_0^{0.25} 8xy \, dy \, dx = 0.03125$$

$$24. \text{ False, } \int_0^1 \int_0^1 x \, dy \, dx \neq \int_1^2 \int_1^2 x \, dy \, dx$$

$$26. \text{ True, } \int_0^1 \int_0^1 \frac{1}{1+x^2+y^2} \, dx \, dy < \int_0^1 \int_0^1 \frac{1}{1+x^2} \, dx \, dy = \frac{\pi}{4}$$

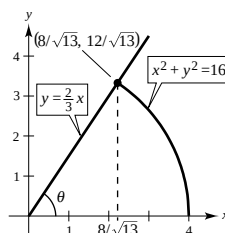
$$28. \quad \int_0^4 \int_0^{\sqrt{16-y^2}} (x^2 + y^2) \, dx \, dy = \int_0^{\pi/2} \int_0^4 r^3 \, dr \, d\theta = \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_0^4 \, d\theta = \int_0^{\pi/2} 64 \, d\theta = 32\pi$$

$$\begin{aligned}
 30. \quad V &= 8 \int_0^{\pi/2} \int_b^R \sqrt{R^2 - r^2} \, r \, dr \, d\theta \\
 &= -\frac{8}{3} \int_0^{\pi/2} \left[(R^2 - r^2)^{3/2} \right]_b^R \, d\theta \\
 &= \frac{8}{3} (R^2 - b^2)^{3/2} \int_0^{\pi/2} d\theta \\
 &= \frac{4}{3} \pi (R^2 - b^2)^{3/2}
 \end{aligned}$$

$$32. \quad \tan \theta = \frac{12\sqrt{13}}{8\sqrt{13}} = \frac{3}{2} \Rightarrow \theta \approx 0.9828$$

The polar region is given by $0 \leq r \leq 4$ and $0 \leq \theta \leq 0.9828$. Hence,

$$\int_0^{\arctan(3/2)} \int_0^4 (r \cos \theta)(r \sin \theta) r \, dr \, d\theta = \frac{288}{13}$$



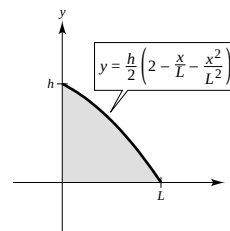
$$34. \quad m = k \int_0^L \int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} dy \, dx = \frac{kh}{2} \int_0^L \left(2 - \frac{x}{L} - \frac{x^2}{L^2} \right) dx = \frac{7khL}{12}$$

$$\begin{aligned} M_x &= k \int_0^L \int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} y \, dy \, dx \\ &= \frac{kh^2}{8} \int_0^L \left(2 - \frac{x}{L} - \frac{x^2}{L^2} \right)^2 dx \\ &= \frac{kh^2}{8} \int_0^L \left[4 - \frac{4x}{L} - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} + \frac{x^4}{L^4} \right] dx \\ &= \frac{kh^2}{8} \left[4x - \frac{2x^2}{L} - \frac{x^3}{L^2} + \frac{x^4}{2L^3} + \frac{x^5}{5L^4} \right]_0^L = \frac{kh^2}{8} \cdot \frac{17L}{10} = \frac{17kh^2L}{80} \end{aligned}$$

$$\begin{aligned} M_y &= k \int_0^L \int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} x \, dy \, dx \\ &= \frac{kh}{2} \int_0^L \left(2x - \frac{x^2}{L} - \frac{x^3}{L^2} \right) dx = \frac{kh}{2} \left[x^2 - \frac{x^3}{3L} - \frac{x^4}{4L^2} \right]_0^L = \frac{kh}{2} \cdot \frac{5L^2}{12} = \frac{5khL^2}{24} \end{aligned}$$

$$\bar{x} = \frac{M_y}{m} = \frac{5khL^2}{24} \cdot \frac{12}{7khL} = \frac{5L}{14}$$

$$\bar{y} = \frac{M_x}{m} = \frac{17kh^2L}{80} \cdot \frac{12}{7khL} = \frac{51h}{140}$$



$$36. \quad I_x = \int_R \int y^2 \rho(x, y) \, dA = \int_0^2 \int_0^{4-x^2} ky^3 \, dy \, dx = \frac{16,384}{315}k$$

$$I_y = \int_R \int x^2 \rho(x, y) \, dA = \int_0^2 \int_0^{4-x^2} kx^2y \, dy \, dx = \frac{512}{105}k$$

$$I_0 = I_x + I_y = \frac{16,384k}{315} + \frac{512k}{105} = \frac{17,920}{315}k = \frac{512}{9}k$$

$$m = \int_R \int \rho(x, y) \, dA = \int_0^2 \int_0^{4-x^2} ky \, dy \, dx = \frac{128}{15}k$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{512k/105}{128k/15}} = \sqrt{\frac{4}{7}}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{16,384k/315}{128k/15}} = \sqrt{\frac{128}{21}}$$

$$38. \quad f(x, y) = 16 - x - y^2$$

$$R = \{(x, y): 0 \leq x \leq 2, 0 \leq y \leq x\}$$

$$f_x = -1, f_y = -2y$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{2 + 4y^2}$$

$$S = \int_0^2 \int_y^2 \sqrt{2 + 4y^2} \, dx \, dy = \int_0^2 [2\sqrt{2 + 4y^2} - y\sqrt{2 + 4y^2}] \, dy$$

$$= \left[\frac{1}{2}(2y\sqrt{2 + 4y^2} + 2 \ln|2y + \sqrt{2 + 4y^2}|) - \frac{1}{12}(2 + 4y^2)^{3/2} \right]_0^2$$

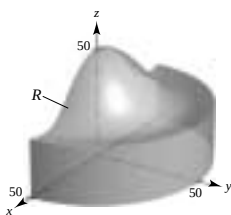
$$= \left[\frac{1}{2}(4\sqrt{18} + 2 \ln|4 + \sqrt{18}|) - \frac{1}{12}(18\sqrt{18}) \right] - \left[\ln\sqrt{2} - \frac{2\sqrt{2}}{12} \right]$$

$$= 6\sqrt{2} + \ln|4 + 3\sqrt{2}| - \frac{9\sqrt{2}}{2} - \ln\sqrt{2} + \frac{\sqrt{2}}{6} = \frac{5\sqrt{2}}{3} + \ln|2\sqrt{2} + 3|$$

40. (a) Graph of

$$f(x, y) = z$$

$$= 25 \left[1 + e^{-(x^2+y^2)/1000} \cos^2 \left(\frac{x^2 + y^2}{1000} \right) \right]$$

over region R 

$$(b) \text{ Surface area} = \iint_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2} dA$$

Using a symbolic computer program, you obtain surface area $\approx 4,540$ sq. ft.

$$42. \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{(x^2+y^2)/2} (x^2 + y^2) dz dy dx = \int_0^{2\pi} \int_0^2 \int_0^{r^2/2} r^3 dz dr d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^2 r^5 dr d\theta = \frac{16}{3} \int_0^{2\pi} d\theta = \frac{32\pi}{3}$$

$$44. \int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^{\sqrt{25-x^2-y^2}} \frac{1}{1+x^2+y^2+z^2} dz dy dx = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^5 \frac{\rho^2}{1+\rho^2} \sin \phi d\rho d\phi d\theta$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \left[\rho - \arctan \rho \right]_0^5 \sin \phi d\phi d\theta$$

$$= \int_0^{\pi/2} \left[(5 - \arctan 5)(-\cos \phi) \right]_0^{\pi/2} d\theta = \frac{\pi}{2} (5 - \arctan 5)$$

$$46. \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} xyz dz dy dx = \frac{4}{3}$$

$$48. V = 2 \int_0^{\pi/2} \int_0^{2 \sin \theta} \int_0^{16-r^2} r dz dr d\theta = 2 \int_0^{\pi/2} \int_0^{2 \sin \theta} r(16-r^2) dr d\theta$$

$$= 2 \int_0^{\pi/2} (32 \sin^2 \theta - 4 \sin^4 \theta) d\theta = 8 \int_0^{\pi/2} (8 \sin^2 \theta - \sin^4 \theta) d\theta$$

$$= 8 \left[4\theta - 2 \sin 2\theta + \frac{1}{4} \sin^3 \theta \cos \theta - \frac{3}{4} \left(\frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right) \right]_0^{\pi/2} = \frac{29\pi}{2}$$

$$50. m = 2k \int_0^{\pi/2} \int_0^a \int_0^{cr \sin \theta} r dz dr d\theta = 2kc \int_0^{\pi/2} \int_0^a r^2 \sin \theta dr d\theta = \frac{2}{3} kca^3 \int_0^{\pi/2} \sin \theta d\theta = \frac{2}{3} kca^3$$

$$M_{xz} = 2k \int_0^{\pi/2} \int_0^a \int_0^{cr \sin \theta} r^2 \sin \theta dz dr d\theta = 2kc \int_0^{\pi/2} \int_0^a r^3 \sin^2 \theta dr d\theta = \frac{1}{2} kca^4 \int_0^{\pi/2} \sin^2 \theta d\theta = \frac{1}{8} \pi kca^4$$

$$M_{xy} = 2k \int_0^{\pi/2} \int_0^a \int_0^{cr \sin \theta} rz dz dr d\theta = kc^2 \int_0^{\pi/2} \int_0^a r^3 \sin^2 \theta dr d\theta = \frac{1}{4} kc^2 a^4 \int_0^{\pi/2} \sin^2 \theta d\theta = \frac{1}{16} \pi kc^2 a^4$$

$$\bar{x} = 0$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{\pi kca^4/8}{2kca^3/3} = \frac{3\pi a}{16}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{\pi kc^2 a^4/16}{2kca^3/3} = \frac{3\pi ca}{32}$$

$$\begin{aligned}
 52. \quad m &= \frac{500\pi}{3} - \int_0^3 \int_0^{2\pi} \int_4^{\sqrt{25-r^2}} r \, dz \, d\theta \, dr = \frac{500\pi}{3} - \int_0^3 \int_0^{2\pi} (r\sqrt{25-r^2} - 4r) d\theta \, dr \\
 &= \frac{500\pi}{3} - 2\pi \left[-\frac{1}{3}(25-r^2)^{3/2} - 2r^2 \right]_0^3 = \frac{500\pi}{3} - 2\pi \left[-\frac{64}{3} - 18 + \frac{125}{3} \right] = \frac{500\pi}{3} - \frac{14\pi}{3} = 162\pi
 \end{aligned}$$

$\bar{x} = \bar{y} = 0$ by symmetry

$$\begin{aligned}
 M_{xy} &= \int_0^{2\pi} \int_0^3 \int_{-\sqrt{25-r^2}}^4 zr \, dz \, dr \, d\theta + \int_0^{2\pi} \int_3^5 \int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} zr \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^3 \left[8 - \frac{1}{2}(25-r^2) \right] r \, dr \, d\theta + 0 \\
 &= \int_0^{2\pi} \int_0^3 \left[\frac{1}{2}r^3 - \frac{9}{2}r \right] dr \, d\theta = \int_0^{2\pi} \left[\frac{1}{8}r^4 - \frac{9}{4}r^2 \right]_0^3 d\theta = \left[-\frac{81}{8}\theta \right]_0^{2\pi} = -\frac{81}{4}\pi \\
 \bar{z} &= \frac{M_{xy}}{m} = -\frac{81\pi}{4} \frac{1}{162\pi} = -\frac{1}{8}
 \end{aligned}$$

$$\begin{aligned}
 54. \quad I_z &= k \int_0^\pi \int_0^{2\pi} \int_0^a \rho^2 \sin^2 \phi(\rho) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\
 &= \frac{4k\pi a^6}{9}
 \end{aligned}$$

$$56. \quad x^2 + y^2 + \frac{z^2}{a^2} = 1$$

$$\begin{aligned}
 I_z &= \iiint_Q (x^2 + y^2) dV \\
 &= \int_{-a}^a \int_{-\sqrt{1-z^2-a^2}}^{\sqrt{1-z^2-a^2}} \int_{-\sqrt{1-y^2-z^2-a^2}}^{\sqrt{1-y^2-z^2-a^2}} (x^2 + y^2) dx \, dy \, dz \\
 &= \frac{8}{15}\pi a
 \end{aligned}$$

$$\begin{aligned}
 60. \quad \frac{\partial(x, y)}{\partial(u, v)} &= \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \\
 &= (2u)(-2v) - (2u)(2v) = -8uv
 \end{aligned}$$

$$62. \quad \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = 1 \left(\frac{1}{u} \right) - 0 = \frac{1}{u}$$

$$x = u, y = \frac{v}{u} \Rightarrow u = x, v = xy$$

Boundary in xy-plane

$$x = 1$$

$$x = 5$$

$$xy = 1$$

$$xy = 5$$

Boundary in uv-plane

$$u = 1$$

$$u = 5$$

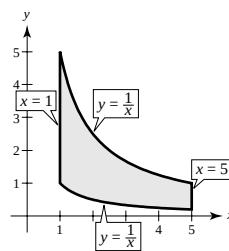
$$v = 1$$

$$v = 5$$

$$\begin{aligned}
 \iint_R \frac{x}{1+x^2y^2} dA &= \int_1^5 \int_1^5 \frac{u}{1+u^2(v/u)^2} \left(\frac{1}{u} \right) du \, dv = \int_1^5 \int_1^5 \frac{1}{1+v^2} du \, dv = \int_1^5 \frac{4}{1+v^2} dv \\
 &= 4 \arctan v \Big|_1^5 = 4 \arctan 5 - \pi
 \end{aligned}$$

$$58. \quad \int_0^\pi \int_0^2 \int_0^{1+r^2} r \, dz \, dr \, d\theta$$

Since $z = 1 + r^2$ represents a paraboloid with vertex $(0, 0, 1)$, this integral represents the volume of the solid below the paraboloid and above the semi-circle $y = \sqrt{4 - x^2}$ in the xy -plane.



Review Exercises for Chapter 13

$$1. \int_1^{x^2} x \ln y \, dy = \left[xy(-1 + \ln y) \right]_1^{x^2} = x^3(-1 + \ln x^2) + x = x - x^3 + x^3 \ln x^2$$

$$3. \int_0^1 \int_0^{1+x} (3x + 2y) \, dy \, dx = \int_0^1 \left[3xy + y^2 \right]_0^{1+x} dx = \int_0^1 (4x^2 + 5x + 1) \, dx = \left[\frac{4}{3}x^3 + \frac{5}{2}x^2 + x \right]_0^1 = \frac{29}{6}$$

$$5. \int_0^3 \int_0^{\sqrt{9-x^2}} 4x \, dy \, dx = \int_0^3 4x \sqrt{9-x^2} \, dx = \left[-\frac{4}{3}(9-x^2)^{3/2} \right]_0^3 = 36$$

$$7. \int_0^3 \int_0^{(3-x)/3} dy \, dx = \int_0^1 \int_0^{3-3y} dx \, dy$$

$$A = \int_0^1 \int_0^{3-3y} dx \, dy = \int_0^1 (3-3y) \, dy = \left[3y - \frac{3}{2}y^2 \right]_0^1 = \frac{3}{2}$$

$$9. \int_{-5}^3 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} dy \, dx = \int_{-5}^{-4} \int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx \, dy + \int_{-4}^4 \int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx \, dy + \int_4^5 \int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx \, dy$$

$$A = 2 \int_{-5}^3 \int_0^{\sqrt{25-x^2}} dy \, dx = 2 \int_{-5}^3 \sqrt{25-x^2} \, dx = \left[x\sqrt{25-x^2} + 25 \arcsin \frac{x}{5} \right]_{-5}^3 = \frac{25\pi}{2} + 12 + 25 \arcsin \frac{3}{5} \approx 67.36$$

$$11. A = 4 \int_0^1 \int_0^{x\sqrt{1-x^2}} dy \, dx = 4 \int_0^1 x\sqrt{1-x^2} \, dx = \left[-\frac{4}{3}(1-x^2)^{3/2} \right]_0^1 = \frac{4}{3}$$

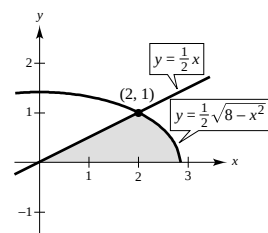
$$A = 4 \int_0^{1/2} \int_{\sqrt{(1-\sqrt{1-4y^2})/2}}^{\sqrt{(1+\sqrt{1-4y^2})/2}} dx \, dy$$

$$13. A = \int_2^5 \int_{x-3}^{\sqrt{x-1}} dy \, dx + 2 \int_1^2 \int_0^{\sqrt{x-1}} dy \, dx = \int_{-1}^2 \int_{y^2+1}^{y+3} dx \, dy = \frac{9}{2}$$

15. Both integrations are over the common region R shown in the figure. Analytically,

$$\int_0^1 \int_{2y}^{2\sqrt{2-y^2}} (x+y) \, dx \, dy = \frac{4}{3} + \frac{4}{3}\sqrt{2}$$

$$\int_0^2 \int_0^{x/2} (x+y) \, dy \, dx + \int_2^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}/2} (x+y) \, dy \, dx = \frac{5}{3} + \left(\frac{4}{3}\sqrt{2} - \frac{1}{3} \right) = \frac{4}{3} + \frac{4}{3}\sqrt{2}$$



$$17. V = \int_0^4 \int_0^{x^2+4} (x^2 - y + 4) \, dy \, dx$$

$$= \int_0^4 \left[x^2 y - \frac{1}{2} y^2 + 4y \right]_0^{x^2+4} dx$$

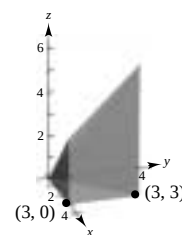
$$= \int_0^4 \left(\frac{1}{2} x^4 + 4x^2 + 8 \right) dx$$

$$= \left[\frac{1}{10} x^5 + \frac{4}{3} x^3 + 8x \right]_0^4 = \frac{3296}{15}$$

19. Volume $\approx$ (base)(height)

$$\approx \frac{9}{2}(3) = \frac{27}{2}$$

Matches (c)



$$21. \int_0^\infty \int_0^\infty kxye^{-(x+y)} dy dx = \int_0^\infty \left[-kxe^{-(x+y)}(y+1) \right]_0^\infty dx = \int_0^\infty kxe^{-x} dx = \left[-k(x+1)e^{-x} \right]_0^\infty = k$$

Therefore, $k = 1$.

$$P = \int_0^1 \int_0^1 xye^{-(x+y)} dy dx \approx 0.070$$

23. True

25. True

$$27. \int_0^h \int_0^x \sqrt{x^2 + y^2} dy dx = \int_0^{\pi/4} \int_0^{h \sec \theta} r^2 dr d\theta \\ = \frac{h^3}{3} \int_0^{\pi/4} \sec^3 \theta d\theta = \frac{h^3}{6} \left[\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right]_0^{\pi/4} = \frac{h^3}{6} [\sqrt{2} + \ln(\sqrt{2} + 1)]$$

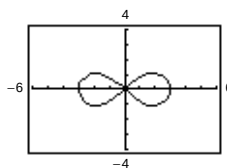
$$29. V = 4 \int_0^h \int_0^{\pi/2} \int_1^{\sqrt{1+z^2}} r dr d\theta dz \\ = 2 \int_0^h \int_0^{\pi/2} (1+z^2-1) d\theta dz \\ = \pi \int_0^h z^2 dz \\ = \left[\pi \left(\frac{1}{3} z^3 \right) \right]_0^h = \frac{\pi h^3}{3}$$

$$31. (a) (x^2 + y^2)^2 = 9(x^2 - y^2)$$

$$(r^2)^2 = 9(r^2 \cos^2 \theta - r^2 \sin^2 \theta)$$

$$r^2 = 9(\cos^2 \theta - \sin^2 \theta) = 9 \cos 2\theta$$

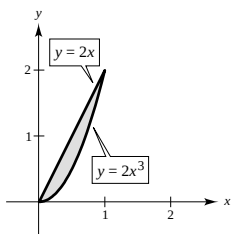
$$r = 3\sqrt{\cos 2\theta}$$



$$(b) A = 4 \int_0^{\pi/4} \int_0^{3\sqrt{\cos 2\theta}} r dr d\theta = 9$$

$$(c) V = 4 \int_0^{\pi/4} \int_0^{3\sqrt{\cos 2\theta}} \sqrt{9 - r^2} r dr d\theta \approx 20.392$$

$$33. (a) m = k \int_0^1 \int_{2x^3}^{2x} xy dy dx = \frac{k}{4} \\ M_x = k \int_0^1 \int_{2x^3}^{2x} xy^2 dy dx = \frac{16k}{55} \\ M_y = k \int_0^1 \int_{2x^3}^{2x} x^2 y dy dx = \frac{8k}{45} \\ \bar{x} = \frac{M_y}{m} = \frac{32}{45} \\ \bar{y} = \frac{M_x}{m} = \frac{64}{55}$$



$$(b) m = k \int_0^1 \int_{2x^3}^{2x} (x^2 + y^2) dy dx = \frac{17k}{30} \\ M_x = k \int_0^1 \int_{2x^3}^{2x} y(x^2 + y^2) dy dx = \frac{392k}{585} \\ M_y = k \int_0^1 \int_{2x^3}^{2x} x(x^2 + y^2) dy dx = \frac{156k}{385} \\ \bar{x} = \frac{M_y}{m} = \frac{936}{1309} \\ \bar{y} = \frac{M_x}{m} = \frac{784}{663}$$

$$35. I_x = \int_R \int y^2 \rho(x, y) dA = \int_0^a \int_0^b kxy^2 dy dx = \frac{1}{6} kb^3 a^2$$

$$I_y = \int_R \int x^2 \rho(x, y) dA = \int_0^a \int_0^b kx^3 dy dx = \frac{1}{4} kba^4$$

$$I_0 = I_x + I_y = \frac{1}{6} kb^3 a^2 + \frac{1}{4} kba^4 = \frac{ka^2 b}{12} (2b^2 + 3a^2)$$

$$m = \int_R \int \rho(x, y) dA = \int_0^a \int_0^b kx dy dx = \frac{1}{2} kba^2$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{(1/4)kba^4}{(1/2)kba^2}} = \sqrt{\frac{a^2}{2}} = \frac{a\sqrt{2}}{2}$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{(1/6)kb^3 a^2}{(1/2)kba^2}} = \sqrt{\frac{b^2}{3}} = \frac{b\sqrt{3}}{3}$$

$$37. S = \int_R \int \sqrt{1 + (f_x)^2 + (f_y)^2} dA$$

$$= 4 \int_0^4 \int_0^{\sqrt{16-x^2}} \sqrt{1 + 4x^2 + 4y^2} dy dx$$

$$= 4 \int_0^{\pi/2} \int_0^4 \sqrt{1 + 4r^2} r dr d\theta$$

$$= \left[\frac{1}{3} (65^{3/2} - 1) \theta \right]_0^{\pi/2} = \frac{\pi}{6} (65\sqrt{65} - 1)$$

$$39. f(x, y) = 9 - y^2$$

$$f_x = 0, f_y = -2y$$

$$S = \int_R \int \sqrt{1 + f_x^2 + f_y^2} dA$$

$$= \int_0^3 \int_{-y}^y \sqrt{1 + 4y^2} dx dy$$

$$= \int_0^3 \left[\sqrt{1 + 4y^2} x \right]_{-y}^y dy$$

$$= \int_0^3 2\sqrt{1 + 4y^2} dy = \frac{1}{4} \frac{2}{3} (1 + 4y^2)^{3/2} \Big|_0^3 = \frac{1}{6} [(37)^{3/2} - 1]$$

$$41. \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{x^2+y^2}^9 \sqrt{x^2 + y^2} dz dy dx = \int_0^{2\pi} \int_0^3 \int_{r^2}^9 r^2 dz dr d\theta$$

$$= \int_0^{2\pi} \int_0^3 (9r^2 - r^4) dr d\theta = \int_0^{2\pi} \left[3r^3 - \frac{r^5}{5} \right]_0^3 d\theta = \frac{162}{5} \int_0^{2\pi} d\theta = \frac{324\pi}{5}$$

$$43. \int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2) dx dy dz = \int_0^a \int_0^b \left(\frac{1}{3} c^3 + cy^2 + cz^2 \right) dy dz$$

$$= \int_0^a \left(\frac{1}{3} bc^3 + \frac{1}{3} b^3 c + bcz^2 \right) dz = \frac{1}{3} abc^3 + \frac{1}{3} ab^3 c + \frac{1}{3} a^3 bc = \frac{1}{3} abc(a^2 + b^2 + c^2)$$

$$45. \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} (x^2 + y^2) dz dy dx = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^3 dz dr d\theta = \frac{8\pi}{15}$$

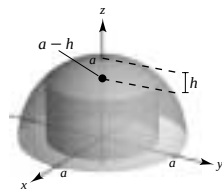
$$\begin{aligned}
 47. \quad V &= 4 \int_0^{\pi/2} \int_0^{2 \cos \theta} \int_0^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta \\
 &= 4 \int_0^{\pi/2} \int_0^{2 \cos \theta} r \sqrt{4-r^2} \, dr \, d\theta \\
 &= - \int_0^{\pi/2} \left[\frac{4}{3} (4-r^2)^{3/2} \right]_0^{2 \cos \theta} d\theta \\
 &= \frac{32}{3} \int_0^{\pi/2} (1 - \sin^3 \theta) \, d\theta \\
 &= \frac{32}{3} \left[\theta + \cos \theta - \frac{1}{3} \cos^3 \theta \right]_0^{\pi/2} = \frac{32}{3} \left(\frac{\pi}{2} - \frac{2}{3} \right)
 \end{aligned}$$

$$\begin{aligned}
 49. \quad m &= 4k \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \int_0^{\cos \phi} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\
 &= \frac{4}{3} k \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \cos^3 \phi \sin \phi \, d\theta \, d\phi = \frac{2}{3} k\pi \int_{\pi/4}^{\pi/2} \cos^3 \phi \sin \phi \, d\phi = \left[-\frac{2}{3} k\pi \left(\frac{1}{4} \cos^4 \phi \right) \right]_{\pi/4}^{\pi/2} = \frac{k\pi}{24} \\
 M_{xy} &= 4k \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \int_0^{\cos \phi} \rho^3 \cos \phi \sin \phi \, d\rho \, d\theta \, d\phi \\
 &= k \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \cos^5 \phi \sin \phi \, d\theta \, d\phi = \frac{1}{2} k\pi \int_{\pi/4}^{\pi/2} \cos^5 \phi \sin \phi \, d\phi = \left[-\frac{1}{12} k\pi \cos^6 \phi \right]_{\pi/4}^{\pi/2} = \frac{k\pi}{96} \\
 \bar{z} &= \frac{M_{xy}}{m} = \frac{k\pi/96}{k\pi/24} = \frac{1}{4} \\
 \bar{x} = \bar{y} &= 0 \quad \text{by symmetry}
 \end{aligned}$$

$$\begin{aligned}
 51. \quad m &= k \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \frac{k\pi a^3}{6} \\
 M_{xy} &= k \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \frac{k\pi a^4}{16} \\
 \bar{x} = \bar{y} = \bar{z} &= \frac{M_{xy}}{m} = \frac{k\pi a^4}{16} \left(\frac{6}{k\pi a^3} \right) = \frac{3a}{8}
 \end{aligned}$$

$$\begin{aligned}
 53. \quad I_z &= 4k \int_0^{\pi/2} \int_3^4 \int_0^{16-r^2} r^3 \, dz \, dr \, d\theta \\
 &= 4k \int_0^{\pi/2} \int_3^4 (16r^3 - r^5) \, dr \, d\theta = \frac{833\pi k}{3}
 \end{aligned}$$

$$\begin{aligned}
 55. \quad z &= f(x, y) = \sqrt{a^2 - x^2 - y^2} \\
 &= \sqrt{a^2 - r^2} \\
 0 &\leq r \leq \sqrt{2ah - h^2}
 \end{aligned}$$

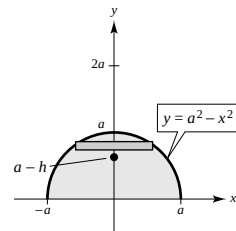


(a) Disc Method

$$\begin{aligned}
 V &= \pi \int_{a-h}^a (a^2 - y^2) dy \\
 &= \pi \left[a^2 y - \frac{y^3}{3} \right]_{a-h}^a = \pi \left[\left(a^3 - \frac{a^3}{3} \right) - \left(a^2(a-h) - \frac{(a-h)^3}{3} \right) \right] \\
 &= \pi \left[a^3 - \frac{a^3}{3} - a^3 + a^2 h + \frac{a^3}{3} - a^2 h + ah^2 - \frac{h^3}{3} \right] = \pi \left[ah^2 - \frac{h^3}{3} \right] = \frac{1}{3} \pi h^2 [3a - h]
 \end{aligned}$$

Equivalently, use spherical coordinates

$$V = \int_0^{2\pi} \int_{\cos^{-1}(a-h/a)}^{\cos^{-1}(a-h/a)} \int_{(a-h)\sec \phi}^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$



—CONTINUED—

55. —CONTINUED—

$$(b) M_{xy} = \int_0^{2\pi} \int_0^{\cos^{-1}(a-h/a)} \int_{(a-h)\sec\phi}^a (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{4} h^2 \pi (2a - h)^2$$

$$\bar{z} = \frac{M_{xy}}{V} = \frac{\frac{1}{4} h^2 \pi (2a - h)^2}{\frac{1}{3} h^2 \pi (3a - h)} = \frac{3}{4} \frac{(2a - h)^2}{3a - h}$$

$$\text{centroid: } \left(0, 0, \frac{3(2a - h)^2}{4(3a - h)}\right)$$

$$(c) \text{ If } h = a, \bar{z} = \frac{3(a)^2}{4(2a)} = \frac{3}{8}a$$

$$\text{centroid of hemisphere: } \left(0, 0, \frac{3}{8}a\right)$$

$$(d) \lim_{h \rightarrow 0} \bar{z} = \lim_{h \rightarrow 0} \frac{3(2a - h)^2}{4(3a - h)} = \frac{3(4a^2)}{12a} = a$$

$$(e) x^2 + y^2 = \rho^2 \sin^2 \phi$$

$$I_z = \int_0^{2\pi} \int_0^{\cos^{-1}(a-h/a)} \int_{(a-h)\sec\phi}^a (\rho^2 \sin^2 \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{h^3}{30} (20a^2 - 15ah + 3h^2) \pi$$

$$(f) \text{ If } h = a, I_z = \frac{a^3 \pi}{30} (20a^2 - 15a^2 + 3a^2) = \frac{4}{15} a^5 \pi$$

$$57. \int_0^{2\pi} \int_0^{\pi} \int_0^{6 \sin \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

Since $\rho = 6 \sin \phi$ represents (in the yz -plane) a circle of radius 3 centered at $(0, 3, 0)$, the integral represents the volume of the torus formed by revolving $(0 < \theta < 2\pi)$ this circle about the z -axis.

$$59. \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$$

$$= 1(-3) - 2(3) = -9$$

$$61. \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = \frac{1}{2} \left(-\frac{1}{2}\right) - \frac{1}{2} \left(\frac{1}{2}\right) = -\frac{1}{2}$$

$$x = \frac{1}{2}(u + v), y = \frac{1}{2}(u - v) \Rightarrow u = x + y, v = x - y$$

Boundaries in xy -plane

$$x + y = 3$$

$$x + y = 5$$

$$x - y = -1$$

$$x - y = 1$$

Boundaries in uv -plane

$$u = 3$$

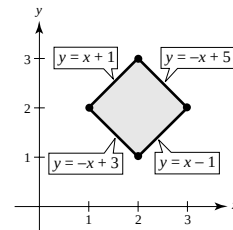
$$u = 5$$

$$v = -1$$

$$v = 1$$

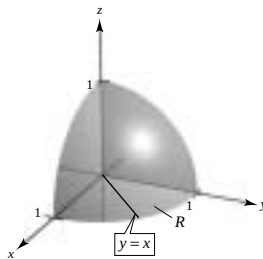
$$\iint_R \ln(x + y) dA = \int_3^5 \int_{-1}^1 \ln\left(\frac{1}{2}(u + v) + \frac{1}{2}(u - v)\right) dv du = \int_3^5 \int_{-1}^1 \frac{1}{2} \ln u \, dv du = \int_3^5 \ln u \, du = \left[u \ln u - u\right]_3^5$$

$$= (5 \ln 5 - 5) - (3 \ln 3 - 3) = 5 \ln 5 - 3 \ln 3 - 2 \approx 2.751$$



Problem Solving for Chapter 13

$$\begin{aligned}
 1. \quad (a) \quad V &= 16 \iint_R \sqrt{1-x^2} \, dA \\
 &= 16 \int_0^{\pi/4} \int_0^1 \sqrt{1-r^2 \cos^2 \theta} \, r \, dr \, d\theta \\
 &= -\frac{16}{3} \int_0^{\pi/4} \frac{1}{\cos^2 \theta} [(1 - \cos^2 \theta)^{3/2} - 1] \, d\theta \\
 &= -\frac{16}{3} \left[\sec \theta + \cos \theta - \tan \theta \right]_0^{\pi/4} \\
 &= 8(2 - \sqrt{2}) \approx 4.6863
 \end{aligned}$$



(b) Programs will vary.

$$3. \quad (a) \quad \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + c. \text{ Let } a^2 = 2 - u^2, u = v.$$

$$\text{Then } \int \frac{1}{(2 - u^2) + v^2} dv = \frac{1}{\sqrt{2 - u^2}} \arctan \frac{v}{\sqrt{2 - u^2}} + C.$$

$$\begin{aligned}
 (b) \quad I_1 &= \int_0^{\sqrt{2}/2} \left[\frac{2}{\sqrt{2 - u^2}} \arctan \frac{v}{\sqrt{2 - u^2}} \right]_{-u}^u du \\
 &= \int_0^{\sqrt{2}/2} \frac{2}{\sqrt{2 - u^2}} \left(\arctan \frac{u}{\sqrt{2 - u^2}} - \arctan \frac{-u}{\sqrt{2 - u^2}} \right) du \\
 &= \int_0^{\sqrt{2}/2} \frac{4}{\sqrt{2 - u^2}} \arctan \frac{u}{\sqrt{2 - u^2}} du
 \end{aligned}$$

$$\text{Let } u = \sqrt{2} \sin \theta, du = \sqrt{2} \cos \theta d\theta, 2 - u^2 = 2 - 2 \sin^2 \theta = 2 \cos^2 \theta.$$

$$\begin{aligned}
 I_1 &= 4 \int_0^{\pi/6} \frac{1}{\sqrt{2} \cos \theta} \arctan \left(\frac{\sqrt{2} \sin \theta}{\sqrt{2} \cos \theta} \right) \cdot \sqrt{2} \cos \theta d\theta \\
 &= 4 \int_0^{\pi/6} \arctan(\tan \theta) d\theta = \frac{4\theta^2}{2} \Big|_0^{\pi/6} = 2 \left(\frac{\pi}{6} \right)^2 = \frac{\pi^2}{18}
 \end{aligned}$$

$$\begin{aligned}
 (c) \quad I_2 &= \int_{\sqrt{2}/2}^{\sqrt{2}} \left[\frac{2}{\sqrt{2 - u^2}} \arctan \frac{v}{\sqrt{2 - u^2}} \right]_{u - \sqrt{2}}^{-u + \sqrt{2}} du \\
 &= \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{2}{\sqrt{2 - u^2}} \left[\arctan \left(\frac{-u + \sqrt{2}}{\sqrt{2 - u^2}} \right) - \arctan \left(\frac{u - \sqrt{2}}{\sqrt{2 - u^2}} \right) \right] du \\
 &= \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{4}{\sqrt{2 - u^2}} \arctan \left(\frac{\sqrt{2} - u}{\sqrt{2 - u^2}} \right) du
 \end{aligned}$$

$$\text{Let } u = \sqrt{2} \sin \theta.$$

$$\begin{aligned}
 I_2 &= 4 \int_{\pi/6}^{\pi/2} \frac{1}{\sqrt{2} \cos \theta} \arctan \left(\frac{\sqrt{2} - \sqrt{2} \sin \theta}{\sqrt{2} \cos \theta} \right) \cdot \sqrt{2} \cos \theta d\theta \\
 &= 4 \int_{\pi/6}^{\pi/2} \arctan \left(\frac{1 - \sin \theta}{\cos \theta} \right) d\theta
 \end{aligned}$$

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3. —CONTINUED—

$$\begin{aligned}
 \text{(d)} \quad \tan\left(\frac{1}{2}\left(\frac{\pi}{2} - \theta\right)\right) &= \sqrt{\frac{1 - \cos((\pi/2) - \theta)}{1 + \cos((\pi/2) - \theta)}} = \sqrt{\frac{1 - \sin \theta}{1 + \sin \theta}} \\
 &= \sqrt{\frac{(1 - \sin \theta)^2}{(1 + \sin \theta)(1 - \sin \theta)}} = \sqrt{\frac{(1 - \sin \theta)^2}{\cos^2 \theta}} = \frac{1 - \sin \theta}{\cos \theta}
 \end{aligned}$$

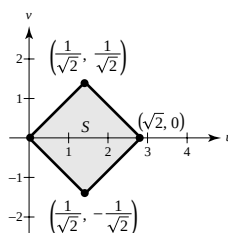
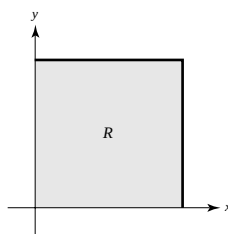
$$\begin{aligned}
 \text{(e)} \quad I_2 &= 4 \int_{\pi/6}^{\pi/2} \arctan\left(\frac{1 - \sin \theta}{\cos \theta}\right) d\theta = 4 \int_{\pi/6}^{\pi/2} \arctan\left(\tan\left(\frac{1}{2}\left(\frac{\pi}{2} - \theta\right)\right)\right) d\theta \\
 &= 4 \int_{\pi/6}^{\pi/2} \frac{1}{2}\left(\frac{\pi}{2} - \theta\right) d\theta = 2 \int_{\pi/6}^{\pi/2} \left(\frac{\pi}{2} - \theta\right) d\theta \\
 &= 2 \left[\frac{\pi}{2} \theta - \frac{\theta^2}{2} \right]_{\pi/6}^{\pi/2} = 2 \left[\left(\frac{\pi^2}{4} - \frac{\pi^2}{8} \right) - \left(\frac{\pi^2}{12} - \frac{\pi^2}{72} \right) \right] \\
 &= 2 \left[\frac{18 - 9 - 6 + 1}{72} \pi^2 \right] = \frac{4}{36} \pi^2 = \frac{\pi^2}{9}
 \end{aligned}$$

$$\text{(f)} \quad \frac{1}{1 - xy} = 1 + (xy) + (xy)^2 + \cdots \quad |xy| < 1$$

$$\begin{aligned}
 \int_0^1 \int_0^1 \frac{1}{1 - xy} dx dy &= \int_0^1 \int_0^1 [1 + (xy) + (xy)^2 + \cdots] dx dy \\
 &= \int_0^1 \int_0^1 \sum_{K=0}^{\infty} (xy)^K dx dy = \sum_{K=0}^{\infty} \int_0^1 \frac{x^{K+1} y^K}{K+1} \Big|_0^1 dy \\
 &= \sum_{K=0}^{\infty} \int_0^1 \frac{y^K}{K+1} dy = \sum_{K=0}^{\infty} \frac{y^{K+1}}{(K+1)^2} \Big|_0^1 \\
 &= \sum_{K=0}^{\infty} \frac{1}{(K+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{(g)} \quad u &= \frac{x+y}{\sqrt{2}}, v = \frac{y-x}{\sqrt{2}} \\
 u - v &= \frac{2x}{\sqrt{2}} \Rightarrow x = \frac{u-v}{\sqrt{2}} \\
 u + v &= \frac{2y}{\sqrt{2}} \Rightarrow y = \frac{u+v}{\sqrt{2}} \\
 \frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{vmatrix} = 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{R} \quad & \mathbf{S} \\
 (0, 0) & \leftrightarrow (0, 0) \\
 (1, 0) & \leftrightarrow \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \\
 (0, 1) & \leftrightarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\
 (1, 1) & \leftrightarrow (\sqrt{2}, 0)
 \end{aligned}$$



$$\begin{aligned}
 \int_0^1 \int_0^1 \frac{1}{1 - xy} dx dy &= \int_0^{\sqrt{2}/2} \int_{-u}^u \frac{1}{1 - \frac{u^2}{2} + \frac{v^2}{2}} dv du + \int_{\sqrt{2}/2}^{\sqrt{2}} \int_{u-\sqrt{2}}^{-u+\sqrt{2}} \frac{1}{1 - \frac{u^2}{2} + \frac{v^2}{2}} dv du \\
 &= I_1 + I_2 = \frac{\pi^2}{18} + \frac{\pi^2}{9} = \frac{\pi^2}{6}
 \end{aligned}$$

5. Boundary in xy -plane

$$y = \sqrt{x}$$

$$y = \sqrt{2x}$$

$$y = \frac{1}{3}x^2$$

$$y = \frac{1}{4}x^2$$

Boundary in uv -plane

$$u = 1$$

$$u = 2$$

$$v = 3$$

$$v = 4$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{3}\left(\frac{v}{u}\right)^{2/3} & \frac{2}{3}\left(\frac{u}{v}\right)^{1/3} \\ \frac{2}{3}\left(\frac{v}{u}\right)^{1/3} & \frac{1}{3}\left(\frac{u}{v}\right)^{2/3} \end{vmatrix} = -\frac{1}{3}$$

$$A = \iint_R 1 \, dA = \iint_S 1 \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA = \frac{1}{3}$$

9. From Exercise 55, Section 13.3,

$$\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$$

$$\text{Thus, } \int_0^{\infty} e^{-x^2/2} dx = \frac{\sqrt{2\pi}}{2} \text{ and } \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\int_0^{\infty} x^2 e^{-x^2} dx = \left[-\frac{1}{2} x e^{-x^2} \right]_0^{\infty} + \frac{1}{2} \int_0^{\infty} e^{-x^2} dx = \frac{1}{2} \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4}$$

$$11. f(x, y) = \begin{cases} k e^{-(x+y)/a} & x \geq 0, y \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dA &= \int_0^{\infty} \int_0^{\infty} k e^{-(x+y)/a} dx dy \\ &= k \int_0^{\infty} e^{-x/a} dx \cdot \int_0^{\infty} e^{-y/a} dy \end{aligned}$$

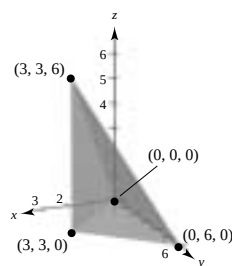
These two integrals are equal to

$$\int_0^{\infty} e^{-x/a} dx = \lim_{b \rightarrow \infty} \left[(-a) e^{-x/a} \right]_0^b = a.$$

Hence, assuming $a, k > 0$, you obtain

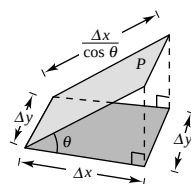
$$1 = ka^2 \quad \text{or} \quad a = \frac{1}{\sqrt{k}}.$$

7.



$$V = \int_0^3 \int_0^{2x} \int_x^{6-x} dy \, dz \, dx = 18$$

$$13. A = l \cdot w = \left(\frac{\Delta x}{\cos \theta} \right) \Delta y = \sec \theta \Delta x \Delta y$$



Area in xy -plane: $\Delta x \Delta y$

26. See Theorem 13.5.

28. $x = 4u - v$, $y = 4v - w$, $z = u + w$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 4 & -1 & 0 \\ 0 & 4 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 17$$

30. $x = r \cos \theta$, $y = r \sin \theta$, $z = z$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1[r \cos^2 \theta + r \sin^2 \theta] = r$$

Review Exercises for Chapter 13

$$2. \int_y^{2y} (x^2 + y^2) dx = \left[\frac{x^3}{3} + xy^2 \right]_y^{2y} = \frac{10y^3}{3}$$

$$4. \int_0^2 \int_{x^2}^{2x} (x^2 + 2y) dy dx = \int_0^2 \left[x^2 y + y^2 \right]_{x^2}^{2x} dx = \int_0^2 (4x^2 + 2x^3 - 2x^4) dx = \left[\frac{4}{3}x^3 + \frac{1}{2}x^4 - \frac{2}{5}x^5 \right]_0^2 = \frac{88}{15}$$

$$6. \int_0^{\sqrt{3}} \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dx dy = 2 \int_0^{\sqrt{3}} \sqrt{4-y^2} dy = \left[y\sqrt{4-y^2} + 4 \arcsin \frac{y}{2} \right]_0^{\sqrt{3}} = \sqrt{3} + \frac{4\pi}{3}$$

$$8. \int_0^2 \int_0^x dy dx + \int_2^3 \int_0^{6-2x} dy dx = \int_0^2 \int_y^{(6-y)/2} dx dy$$

$$A = \int_0^2 \int_y^{(6-y)/2} dx dy$$

$$= \frac{1}{2} \int_0^2 (6-3y) dy = \left[\frac{1}{2} \left(6y - \frac{3}{2}y^2 \right) \right]_0^2 = 3$$

$$10. \int_0^4 \int_{x^2-2x}^{6x-x^2} dy dx = \int_{-1}^0 \int_{1-\sqrt{1+y}}^{1+\sqrt{1+y}} dy dx + \int_0^8 \int_{3-\sqrt{9-y}}^{1+\sqrt{1+y}} dx dy + \int_8^9 \int_{3-\sqrt{9-y}}^{3+\sqrt{9-y}} dx dy$$

$$A = \int_0^4 \int_{x^2-2x}^{6x-x^2} dy dx = \int_0^4 (8x-2x^2) dx = \left[4x^2 - \frac{2}{3}x^3 \right]_0^4 = \frac{64}{3}$$

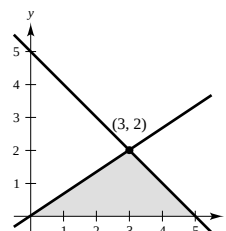
$$12. A = \int_0^2 \int_0^{y^2+1} dx dy = \int_0^1 \int_0^2 dy dx + \int_1^5 \int_{\sqrt{x-1}}^2 dy dx = \frac{14}{3}$$

$$14. A = \int_0^3 \int_{-y}^{2y-y^2} dx dy = \int_{-3}^0 \int_{-x}^{1+\sqrt{1-x}} dy dx + \int_0^1 \int_{1-\sqrt{1-x}}^{1+\sqrt{1-x}} dy dx = \frac{9}{2}$$

16. Both integrations are over the common region R shown in the figure. Analytically,

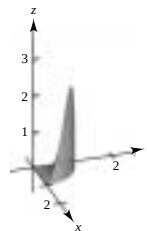
$$\int_0^2 \int_{3y/2}^{5-y} e^{x+y} dx dy = \frac{2}{5} + \frac{8}{5}e^5$$

$$\int_0^3 \int_0^{2x/3} e^{x+y} dy dx + \int_3^5 \int_0^{5-x} e^{x+y} dy dx = \left(\frac{3}{5}e^5 - e^3 + \frac{2}{5} \right) + (e^5 + e^3) = \frac{8}{5}e^5 + \frac{2}{5}$$



$$\begin{aligned}
 18. \quad V &= \int_0^3 \int_0^x (x+y) \, dy \, dx \\
 &= \int_0^3 \left[xy + \frac{1}{2}y^2 \right]_0^x \, dx \\
 &= \frac{3}{2} \int_0^3 x^2 \, dx \\
 &= \left[\frac{1}{2}x^3 \right]_0^3 = \frac{27}{2}
 \end{aligned}$$

20. Matches (c)



$$\begin{aligned}
 22. \quad \int_0^1 \int_0^x kxy \, dy \, dx &= \int_0^1 \left[\frac{kxy^2}{2} \right]_0^x \, dx \\
 &= \int_0^1 \frac{kx^3}{2} \, dx \\
 &= \left[\frac{kx^4}{8} \right]_0^1 = \frac{k}{8}
 \end{aligned}$$

Since $k/8 = 1$, we have $k = 8$.

$$P = \int_0^{0.5} \int_0^{0.25} 8xy \, dy \, dx = 0.03125$$

$$24. \text{ False, } \int_0^1 \int_0^1 x \, dy \, dx \neq \int_1^2 \int_1^2 x \, dy \, dx$$

$$26. \text{ True, } \int_0^1 \int_0^1 \frac{1}{1+x^2+y^2} \, dx \, dy < \int_0^1 \int_0^1 \frac{1}{1+x^2} \, dx \, dy = \frac{\pi}{4}$$

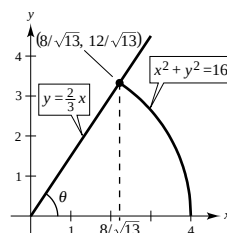
$$28. \quad \int_0^4 \int_0^{\sqrt{16-y^2}} (x^2 + y^2) \, dx \, dy = \int_0^{\pi/2} \int_0^4 r^3 \, dr \, d\theta = \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_0^4 \, d\theta = \int_0^{\pi/2} 64 \, d\theta = 32\pi$$

$$\begin{aligned}
 30. \quad V &= 8 \int_0^{\pi/2} \int_b^R \sqrt{R^2 - r^2} \, r \, dr \, d\theta \\
 &= -\frac{8}{3} \int_0^{\pi/2} \left[(R^2 - r^2)^{3/2} \right]_b^R \, d\theta \\
 &= \frac{8}{3} (R^2 - b^2)^{3/2} \int_0^{\pi/2} d\theta \\
 &= \frac{4}{3} \pi (R^2 - b^2)^{3/2}
 \end{aligned}$$

$$32. \quad \tan \theta = \frac{12\sqrt{13}}{8\sqrt{13}} = \frac{3}{2} \Rightarrow \theta \approx 0.9828$$

The polar region is given by $0 \leq r \leq 4$ and $0 \leq \theta \leq 0.9828$. Hence,

$$\int_0^{\arctan(3/2)} \int_0^4 (r \cos \theta)(r \sin \theta)r \, dr \, d\theta = \frac{288}{13}$$



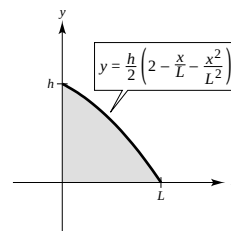
$$34. \quad m = k \int_0^L \int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} dy \, dx = \frac{kh}{2} \int_0^L \left(2 - \frac{x}{L} - \frac{x^2}{L^2}\right) dx = \frac{7khL}{12}$$

$$\begin{aligned} M_x &= k \int_0^L \int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} y \, dy \, dx \\ &= \frac{kh^2}{8} \int_0^L \left(2 - \frac{x}{L} - \frac{x^2}{L^2}\right)^2 dx \\ &= \frac{kh^2}{8} \int_0^L \left[4 - \frac{4x}{L} - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} + \frac{x^4}{L^4}\right] dx \\ &= \frac{kh^2}{8} \left[4x - \frac{2x^2}{L} - \frac{x^3}{L^2} + \frac{x^4}{2L^3} + \frac{x^5}{5L^4}\right]_0^L = \frac{kh^2}{8} \cdot \frac{17L}{10} = \frac{17kh^2L}{80} \end{aligned}$$

$$\begin{aligned} M_y &= k \int_0^L \int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} x \, dy \, dx \\ &= \frac{kh}{2} \int_0^L \left(2x - \frac{x^2}{L} - \frac{x^3}{L^2}\right) dx = \frac{kh}{2} \left[x^2 - \frac{x^3}{3L} - \frac{x^4}{4L^2}\right]_0^L = \frac{kh}{2} \cdot \frac{5L^2}{12} = \frac{5khL^2}{24} \end{aligned}$$

$$\bar{x} = \frac{M_y}{m} = \frac{5khL^2}{24} \cdot \frac{12}{7khL} = \frac{5L}{14}$$

$$\bar{y} = \frac{M_x}{m} = \frac{17kh^2L}{80} \cdot \frac{12}{7khL} = \frac{51h}{140}$$



$$36. \quad I_x = \int_R \int y^2 \rho(x, y) \, dA = \int_0^2 \int_0^{4-x^2} ky^3 \, dy \, dx = \frac{16,384}{315}k$$

$$I_y = \int_R \int x^2 \rho(x, y) \, dA = \int_0^2 \int_0^{4-x^2} kx^2y \, dy \, dx = \frac{512}{105}k$$

$$I_0 = I_x + I_y = \frac{16,384k}{315} + \frac{512k}{105} = \frac{17,920}{315}k = \frac{512}{9}k$$

$$m = \int_R \int \rho(x, y) \, dA = \int_0^2 \int_0^{4-x^2} ky \, dy \, dx = \frac{128}{15}k$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{512k/105}{128k/15}} = \sqrt{\frac{4}{7}}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{16,384k/315}{128k/15}} = \sqrt{\frac{128}{21}}$$

$$38. \quad f(x, y) = 16 - x - y^2$$

$$R = \{(x, y): 0 \leq x \leq 2, 0 \leq y \leq x\}$$

$$f_x = -1, f_y = -2y$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{2 + 4y^2}$$

$$S = \int_0^2 \int_y^2 \sqrt{2 + 4y^2} \, dx \, dy = \int_0^2 [2\sqrt{2 + 4y^2} - y\sqrt{2 + 4y^2}] \, dy$$

$$= \left[\frac{1}{2}(2y\sqrt{2 + 4y^2} + 2 \ln|2y + \sqrt{2 + 4y^2}|) - \frac{1}{12}(2 + 4y^2)^{3/2} \right]_0^2$$

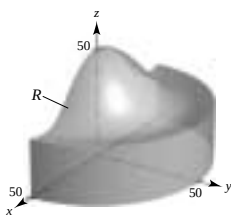
$$= \left[\frac{1}{2}(4\sqrt{18} + 2 \ln|4 + \sqrt{18}|) - \frac{1}{12}(18\sqrt{18}) \right] - \left[\ln\sqrt{2} - \frac{2\sqrt{2}}{12} \right]$$

$$= 6\sqrt{2} + \ln|4 + 3\sqrt{2}| - \frac{9\sqrt{2}}{2} - \ln\sqrt{2} + \frac{\sqrt{2}}{6} = \frac{5\sqrt{2}}{3} + \ln|2\sqrt{2} + 3|$$

40. (a) Graph of

$$f(x, y) = z$$

$$= 25 \left[1 + e^{-(x^2+y^2)/1000} \cos^2 \left(\frac{x^2 + y^2}{1000} \right) \right]$$

over region R 

$$(b) \text{ Surface area} = \iint_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2} dA$$

Using a symbolic computer program, you obtain surface area $\approx 4,540$ sq. ft.

$$42. \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{(x^2+y^2)/2} (x^2 + y^2) dz dy dx = \int_0^{2\pi} \int_0^2 \int_0^{r^2/2} r^3 dz dr d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^2 r^5 dr d\theta = \frac{16}{3} \int_0^{2\pi} d\theta = \frac{32\pi}{3}$$

$$44. \int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^{\sqrt{25-x^2-y^2}} \frac{1}{1+x^2+y^2+z^2} dz dy dx = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^5 \frac{\rho^2}{1+\rho^2} \sin \phi d\rho d\phi d\theta$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \left[\rho - \arctan \rho \right]_0^5 \sin \phi d\phi d\theta$$

$$= \int_0^{\pi/2} \left[(5 - \arctan 5)(-\cos \phi) \right]_0^{\pi/2} d\theta = \frac{\pi}{2} (5 - \arctan 5)$$

$$46. \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} xyz dz dy dx = \frac{4}{3}$$

$$48. V = 2 \int_0^{\pi/2} \int_0^{2 \sin \theta} \int_0^{16-r^2} r dz dr d\theta = 2 \int_0^{\pi/2} \int_0^{2 \sin \theta} r(16-r^2) dr d\theta$$

$$= 2 \int_0^{\pi/2} (32 \sin^2 \theta - 4 \sin^4 \theta) d\theta = 8 \int_0^{\pi/2} (8 \sin^2 \theta - \sin^4 \theta) d\theta$$

$$= 8 \left[4\theta - 2 \sin 2\theta + \frac{1}{4} \sin^3 \theta \cos \theta - \frac{3}{4} \left(\frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right) \right]_0^{\pi/2} = \frac{29\pi}{2}$$

$$50. m = 2k \int_0^{\pi/2} \int_0^a \int_0^{cr \sin \theta} r dz dr d\theta = 2kc \int_0^{\pi/2} \int_0^a r^2 \sin \theta dr d\theta = \frac{2}{3} kca^3 \int_0^{\pi/2} \sin \theta d\theta = \frac{2}{3} kca^3$$

$$M_{xz} = 2k \int_0^{\pi/2} \int_0^a \int_0^{cr \sin \theta} r^2 \sin \theta dz dr d\theta = 2kc \int_0^{\pi/2} \int_0^a r^3 \sin^2 \theta dr d\theta = \frac{1}{2} kca^4 \int_0^{\pi/2} \sin^2 \theta d\theta = \frac{1}{8} \pi kca^4$$

$$M_{xy} = 2k \int_0^{\pi/2} \int_0^a \int_0^{cr \sin \theta} rz dz dr d\theta = kc^2 \int_0^{\pi/2} \int_0^a r^3 \sin^2 \theta dr d\theta = \frac{1}{4} kc^2 a^4 \int_0^{\pi/2} \sin^2 \theta d\theta = \frac{1}{16} \pi kc^2 a^4$$

$$\bar{x} = 0$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{\pi kca^4/8}{2kca^3/3} = \frac{3\pi a}{16}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{\pi kc^2 a^4/16}{2kca^3/3} = \frac{3\pi ca}{32}$$

$$\begin{aligned}
 52. \quad m &= \frac{500\pi}{3} - \int_0^3 \int_0^{2\pi} \int_4^{\sqrt{25-r^2}} r \, dz \, d\theta \, dr = \frac{500\pi}{3} - \int_0^3 \int_0^{2\pi} (r\sqrt{25-r^2} - 4r) d\theta \, dr \\
 &= \frac{500\pi}{3} - 2\pi \left[-\frac{1}{3}(25-r^2)^{3/2} - 2r^2 \right]_0^3 = \frac{500\pi}{3} - 2\pi \left[-\frac{64}{3} - 18 + \frac{125}{3} \right] = \frac{500\pi}{3} - \frac{14\pi}{3} = 162\pi
 \end{aligned}$$

$\bar{x} = \bar{y} = 0$ by symmetry

$$\begin{aligned}
 M_{xy} &= \int_0^{2\pi} \int_0^3 \int_{-\sqrt{25-r^2}}^4 zr \, dz \, dr \, d\theta + \int_0^{2\pi} \int_3^5 \int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} zr \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^3 \left[8 - \frac{1}{2}(25-r^2) \right] r \, dr \, d\theta + 0 \\
 &= \int_0^{2\pi} \int_0^3 \left[\frac{1}{2}r^3 - \frac{9}{2}r \right] dr \, d\theta = \int_0^{2\pi} \left[\frac{1}{8}r^4 - \frac{9}{4}r^2 \right]_0^3 d\theta = \left[-\frac{81}{8}\theta \right]_0^{2\pi} = -\frac{81}{4}\pi \\
 \bar{z} &= \frac{M_{xy}}{m} = -\frac{81\pi}{4} \frac{1}{162\pi} = -\frac{1}{8}
 \end{aligned}$$

$$\begin{aligned}
 54. \quad I_z &= k \int_0^\pi \int_0^{2\pi} \int_0^a \rho^2 \sin^2 \phi(\rho) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\
 &= \frac{4k\pi a^6}{9}
 \end{aligned}$$

$$56. \quad x^2 + y^2 + \frac{z^2}{a^2} = 1$$

$$\begin{aligned}
 I_z &= \iiint_Q (x^2 + y^2) dV \\
 &= \int_{-a}^a \int_{-\sqrt{1-z^2-a^2}}^{\sqrt{1-z^2-a^2}} \int_{-\sqrt{1-y^2-z^2-a^2}}^{\sqrt{1-y^2-z^2-a^2}} (x^2 + y^2) dx \, dy \, dz \\
 &= \frac{8}{15}\pi a
 \end{aligned}$$

$$\begin{aligned}
 60. \quad \frac{\partial(x, y)}{\partial(u, v)} &= \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \\
 &= (2u)(-2v) - (2u)(2v) = -8uv
 \end{aligned}$$

$$62. \quad \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = 1 \left(\frac{1}{u} \right) - 0 = \frac{1}{u}$$

$$x = u, y = \frac{v}{u} \Rightarrow u = x, v = xy$$

Boundary in xy-plane

$$x = 1$$

$$x = 5$$

$$xy = 1$$

$$xy = 5$$

Boundary in uv-plane

$$u = 1$$

$$u = 5$$

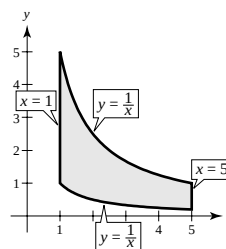
$$v = 1$$

$$v = 5$$

$$\begin{aligned}
 \iint_R \frac{x}{1+x^2y^2} dA &= \int_1^5 \int_1^5 \frac{u}{1+u^2(v/u)^2} \left(\frac{1}{u} \right) du \, dv = \int_1^5 \int_1^5 \frac{1}{1+v^2} du \, dv = \int_1^5 \frac{4}{1+v^2} dv \\
 &= 4 \arctan v \Big|_1^5 = 4 \arctan 5 - \pi
 \end{aligned}$$

$$58. \quad \int_0^\pi \int_0^2 \int_0^{1+r^2} r \, dz \, dr \, d\theta$$

Since $z = 1 + r^2$ represents a paraboloid with vertex $(0, 0, 1)$, this integral represents the volume of the solid below the paraboloid and above the semi-circle $y = \sqrt{4 - x^2}$ in the xy -plane.



Problem Solving for Chapter 13

$$2. z = \frac{1}{c}(d - ax - by) \quad \text{Plane}$$

$$f_x = -\frac{a}{c}, f_y = -\frac{b}{c}$$

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}}$$

$$S = \iint_R \sqrt{1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}} dA$$

$$= \frac{\sqrt{a^2 + b^2 + c^2}}{c} \iint_R dA$$

$$= \frac{\sqrt{a^2 + b^2 + c^2}}{c} A(R)$$

$$6. (a) V = \int_0^{2\pi} \int_0^2 \int_2^{\sqrt{8-r^2}} r dz dr d\theta = \frac{8\pi}{3}(4\sqrt{2} - 5)$$

$$(b) V = \int_0^{2\pi} \int_0^{\pi/4} \int_{2 \sec \phi}^{2\sqrt{2}} \rho^2 \sin \phi d\rho d\phi d\theta = \frac{8\pi}{3}(4\sqrt{2} - 5)$$

$$8. \text{Volume} \approx [5 + 6 + 5 + 5]4 = 84 \text{ m}^3$$

$$10. \text{Let } v = \ln\left(\frac{1}{x}\right), dv = -\frac{dx}{x}.$$

$$e^v = \frac{1}{x}, x = e^{-v}, dx = -e^{-v} dv$$

$$\int_0^1 \sqrt{\ln(1/x)} dx = \int_{-\infty}^0 \sqrt{v}(-e^{-v}) dv = \int_0^{\infty} \sqrt{v}e^{-v} dv$$

$$\text{Let } u = \sqrt{v}, u^2 = v, 2u du = dv.$$

$$\int_0^1 \sqrt{\ln(1/x)} dx = \int_0^{\infty} u e^{-u^2}(2u du) = 2 \int_0^{\infty} u^2 e^{-u^2} du = 2 \left(\frac{\sqrt{\pi}}{4} \right) = \frac{\sqrt{\pi}}{2} \quad (\text{PS \#9})$$

12. Essay

$$4. A: \int_0^{2\pi} \int_4^5 \left(\frac{r}{16} - \frac{r^2}{160} \right) r dr d\theta = \frac{1333\pi}{960} \approx 4.36 \text{ ft}^3$$

$$B = \int_0^{2\pi} \int_9^{10} \left(\frac{r}{16} - \frac{r^2}{160} \right) r dr d\theta = \frac{523\pi}{960} \approx 1.71 \text{ ft}^3$$

The distribution is not uniform. Less water in region of greater area.

In one hour, the entire lawn receives

$$\int_0^{2\pi} \int_0^{10} \left(\frac{r}{16} - \frac{r^2}{160} \right) r dr d\theta = \frac{125\pi}{12} \approx 32.72 \text{ ft}^3.$$

14. The greater the angle between the given plane and the xy -plane, the greater the surface area. Hence:

$$z_2 < z_1 < z_4 < z_3$$

CHAPTER 14

Vector Analysis

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CHAPTER 14

Vector Analysis

Section 14.1 Vector Fields

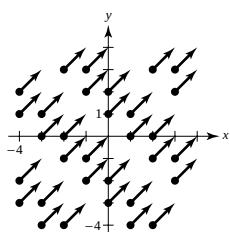
Solutions to Odd-Numbered Exercises

1. All vectors are parallel to y -axis.
Matches (c)

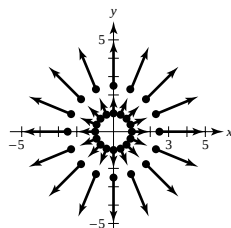
3. All vectors point outward.
Matches (b)

5. Vectors are parallel to x -axis for
 $y = n\pi$.
Matches (a)

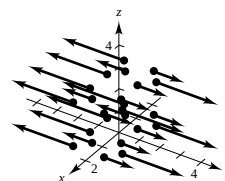
7. $\mathbf{F}(x, y) = \mathbf{i} + \mathbf{j}$
 $\|\mathbf{F}\| = \sqrt{2}$



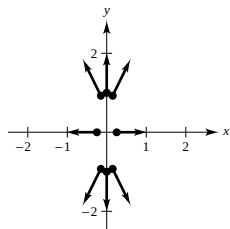
9. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$
 $\|\mathbf{F}\| = \sqrt{x^2 + y^2} = c$
 $x^2 + y^2 = c^2$



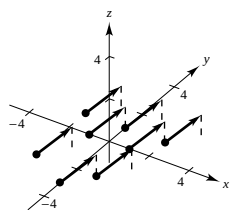
11. $\mathbf{F}(x, y, z) = 3y\mathbf{j}$
 $\|\mathbf{F}\| = 3|y| = c$



13. $\mathbf{F}(x, y) = 4x\mathbf{i} + y\mathbf{j}$
 $\|\mathbf{F}\| = \sqrt{16x^2 + y^2} = c$
 $\frac{x^2}{c^2/16} + \frac{y^2}{c^2} = 1$



15. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$
 $\|\mathbf{F}\| = \sqrt{3}$



- 17.

- 19.

21. $f(x, y) = 5x^2 + 3xy + 10y^2$
 $f_x(x, y) = 10x + 3y$
 $f_y(x, y) = 3x + 20y$
 $\mathbf{F}(x, y) = (10x + 3y)\mathbf{i} + (3x + 20y)\mathbf{j}$

23. $f(x, y, z) = z - ye^{x^2}$
 $f_x(x, y, z) = -2xye^{x^2}$
 $f_y(x, y, z) = -e^{x^2}$
 $f_z = 1$
 $\mathbf{F}(x, y, z) = -2xye^{x^2}\mathbf{i} - e^{x^2}\mathbf{j} + \mathbf{k}$

25. $g(x, y, z) = xy \ln(x + y)$

$$g_x(x, y, z) = y \ln(x + y) + \frac{xy}{x + y}$$

$$g_y(x, y, z) = x \ln(x + y) + \frac{xy}{x + y}$$

$$g_z(x, y, z) = 0$$

$$\mathbf{G}(x, y, z) = \left[\frac{xy}{x + y} + y \ln(x + y) \right] \mathbf{i} + \left[\frac{xy}{x + y} + x \ln(x + y) \right] \mathbf{j}$$

27. $\mathbf{F}(x, y) = 12xy\mathbf{i} + 6(x^2 + y)\mathbf{j}$

$M = 12xy$ and $N = 6(x^2 + y)$ have continuous first partial derivatives.

$$\frac{\partial N}{\partial x} = 12x = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} \text{ is conservative.}$$

29. $\mathbf{F}(x, y) = \sin y\mathbf{i} + x \cos y\mathbf{j}$

$M = \sin y$ and $N = x \cos y$ have continuous first partial derivatives.

$$\frac{\partial N}{\partial x} = \cos y = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} \text{ is conservative.}$$

31. $M = 15y^3, N = -5xy^2$

$$\frac{\partial N}{\partial x} = -5y^2 \neq \frac{\partial M}{\partial y} = 15y^2 \Rightarrow \text{Not conservative}$$

33. $M = \frac{2}{y}e^{2x/y}, N = \frac{-2x}{y^2}e^{2x/y}$

$$\frac{\partial N}{\partial x} = \frac{-2(y + 2x)}{y^3}e^{2x/y} = \frac{\partial M}{\partial y} \Rightarrow \text{Conservative}$$

35. $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

$$\frac{\partial}{\partial y}[2xy] = 2x$$

$$\frac{\partial}{\partial x}[x^2] = 2x$$

Conservative

$$f_x(x, y) = 2xy$$

$$f_y(x, y) = x^2$$

$$f(x, y) = x^2y + K$$

37. $\mathbf{F}(x, y) = xe^{x^2y}(2y\mathbf{i} + x\mathbf{j})$

$$\frac{\partial}{\partial y}[2xye^{x^2y}] = 2xe^{x^2y} + 2x^3ye^{x^2y}$$

$$\frac{\partial}{\partial x}[x^2e^{x^2y}] = 2xe^{x^2y} + 2x^3ye^{x^2y}$$

Conservative

$$f_x(x, y) = 2xye^{x^2y}$$

$$f_y(x, y) = x^2e^{x^2y}$$

$$f(x, y) = e^{x^2y} + K$$

39. $\mathbf{F}(x, y) = \frac{x}{x^2 + y^2}\mathbf{i} + \frac{y}{x^2 + y^2}\mathbf{j}$

$$\frac{\partial}{\partial y}\left[\frac{x}{x^2 + y^2}\right] = -\frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial}{\partial x}\left[\frac{y}{x^2 + y^2}\right] = -\frac{2xy}{(x^2 + y^2)^2}$$

Conservative

$$f_x(x, y) = \frac{x}{x^2 + y^2}$$

$$f_y(x, y) = \frac{y}{x^2 + y^2}$$

$$f(x, y) = \frac{1}{2} \ln(x^2 + y^2) + K$$

41. $\mathbf{F}(x, y) = e^x(\cos y\mathbf{i} + \sin y\mathbf{j})$

$$\frac{\partial}{\partial y}[e^x \cos y] = -e^x \sin y$$

$$\frac{\partial}{\partial x}[e^x \sin y] = e^x \sin y$$

Not conservative

43. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}, (1, 2, 1)$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

$$\text{curl } \mathbf{F}(1, 2, 1) = 2\mathbf{j} - \mathbf{k}$$

45. $\mathbf{F}(x, y, z) = e^x \sin y\mathbf{i} - e^x \cos y\mathbf{j}, (0, 0, 3)$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y & -e^x \cos y & 0 \end{vmatrix} = -2e^x \cos y\mathbf{k}$$

$$\text{curl } \mathbf{F}(0, 0, 3) = -2\mathbf{k}$$

47. $\mathbf{F}(x, y, z) = \arctan\left(\frac{x}{y}\right)\mathbf{i} + \ln\sqrt{x^2 + y^2}\mathbf{j} + \mathbf{k}$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \arctan\left(\frac{x}{y}\right) & \frac{1}{2} \ln(x^2 + y^2) & 1 \end{vmatrix} = \left[\frac{x}{x^2 + y^2} - \frac{(-x/y^2)}{1 + (x/y)^2} \right] \mathbf{k} = \frac{2x}{x^2 + y^2} \mathbf{k}$$

49. $\mathbf{F}(x, y, z) = \sin(x - y)\mathbf{i} + \sin(y - z)\mathbf{j} + \sin(z - x)\mathbf{k}$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(x - y) & \sin(y - z) & \sin(z - x) \end{vmatrix} = \cos(y - z)\mathbf{i} + \cos(z - x)\mathbf{j} + \cos(x - y)\mathbf{k}$$

51. $\mathbf{F}(x, y, z) = \sin y\mathbf{i} - x \cos y\mathbf{j} + \mathbf{k}$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y & -x \cos y & 1 \end{vmatrix} = -2 \cos y \mathbf{k} \neq \mathbf{0}$$

Not conservative

53. $\mathbf{F}(x, y, z) = e^z(y\mathbf{i} + x\mathbf{j} + xy\mathbf{k})$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & xe^z & xye^z \end{vmatrix} = \mathbf{0}$$

Conservative

$$f_x(x, y, z) = ye^z$$

$$f_y(x, y, z) = xe^z$$

$$f_z(x, y, z) = xye^z$$

$$f(x, y, z) = xye^z + K$$

55. $\mathbf{F}(x, y, z) = \frac{1}{y}\mathbf{i} - \frac{x}{y^2}\mathbf{j} + (2z - 1)\mathbf{k}$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{y} & -\frac{x}{y^2} & 2z - 1 \end{vmatrix} = \mathbf{0}$$

Conservative

$$f_x(x, y, z) = \frac{1}{y}$$

$$f_y(x, y, z) = -\frac{x}{y^2}$$

$$f_z(x, y, z) = 2z - 1$$

$$f(x, y, z) = \int \frac{1}{y} dx = \frac{x}{y} + g(y, z) + K_1$$

$$f(x, y, z) = \int -\frac{x}{y^2} dy = \frac{x}{y} + h(x, z) + K_2$$

$$\begin{aligned} f(x, y, z) &= \int (2z - 1) dz \\ &= z^2 - z + p(x, y) + K_3 \end{aligned}$$

$$f(x, y, z) = \frac{x}{y} + z^2 - z + K$$

57. $\mathbf{F}(x, y) = 6x^2\mathbf{i} - xy^2\mathbf{j}$

$$\begin{aligned} \operatorname{div} \mathbf{F}(x, y) &= \frac{\partial}{\partial x}[6x^2] + \frac{\partial}{\partial y}[-xy^2] \\ &= 12x - 2xy \end{aligned}$$

59. $\mathbf{F}(x, y, z) = \sin x \mathbf{i} + \cos y \mathbf{j} + z^2 \mathbf{k}$

$$\operatorname{div} \mathbf{F}(x, y, z) = \frac{\partial}{\partial x}[\sin x] + \frac{\partial}{\partial y}[\cos y] + \frac{\partial}{\partial z}[z^2] = \cos x - \sin y + 2z$$

61. $\mathbf{F}(x, y, z) = xyz \mathbf{i} + y \mathbf{j} + z \mathbf{k}$

$$\operatorname{div} \mathbf{F}(x, y, z) = yz + 1 + 1 = yz + 2$$

$$\operatorname{div} \mathbf{F}(1, 2, 1) = 4$$

63. $\mathbf{F}(x, y, z) = e^x \sin y \mathbf{i} - e^x \cos y \mathbf{j}$

$$\operatorname{div} \mathbf{F}(x, y, z) = e^x \sin y + e^x \sin y$$

$$\operatorname{div} \mathbf{F}(0, 0, 3) = 0$$

65. See the definition, page 1008. Examples include velocity fields, gravitational fields and magnetic fields.

67. See the definition on page 1014.

69. $\mathbf{F}(x, y, z) = \mathbf{i} + 2x \mathbf{j} + 3y \mathbf{k}$

$$\mathbf{G}(x, y, z) = x \mathbf{i} - y \mathbf{j} + z \mathbf{k}$$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 3y \\ x & -y & z \end{vmatrix} = (2xz + 3y^2) \mathbf{i} - (z - 3xy) \mathbf{j} + (-y - 2x^2) \mathbf{k}$$

$$\operatorname{curl}(\mathbf{F} \times \mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz + 3y^2 & 3xy - z & -y - 2x^2 \end{vmatrix} = (-1 + 1) \mathbf{i} - (-4x - 2x) \mathbf{j} + (3y - 6y) \mathbf{k} = 6x \mathbf{j} - 3y \mathbf{k}$$

71. $\mathbf{F}(x, y, z) = xyz \mathbf{i} + y \mathbf{j} + z \mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy \mathbf{j} - xz \mathbf{k}$$

$$\operatorname{curl}(\operatorname{curl} \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & xy & -xz \end{vmatrix} = z \mathbf{j} + y \mathbf{k}$$

73. $\mathbf{F}(x, y, z) = \mathbf{i} + 2x \mathbf{j} + 3y \mathbf{k}$

$$\mathbf{G}(x, y, z) = x \mathbf{i} - y \mathbf{j} + z \mathbf{k}$$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 3y \\ x & -y & z \end{vmatrix} = (2xz + 3y^2) \mathbf{i} - (z - 3xy) \mathbf{j} + (-y - 2x^2) \mathbf{k}$$

$$\operatorname{div}(\mathbf{F} \times \mathbf{G}) = 2z + 3x$$

75. $\mathbf{F}(x, y, z) = xyz \mathbf{i} + y \mathbf{j} + z \mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy \mathbf{j} - xz \mathbf{k}$$

$$\operatorname{div}(\operatorname{curl} \mathbf{F}) = x - x = 0$$

77. Let $\mathbf{F} = M \mathbf{i} + N \mathbf{j} + P \mathbf{k}$ and $\mathbf{G} = Q \mathbf{i} + R \mathbf{j} + S \mathbf{k}$ where M, N, P, Q, R , and S have continuous partial derivatives.

$$\mathbf{F} + \mathbf{G} = (M + Q) \mathbf{i} + (N + R) \mathbf{j} + (P + S) \mathbf{k}$$

$$\begin{aligned} \operatorname{curl}(\mathbf{F} + \mathbf{G}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M + Q & N + R & P + S \end{vmatrix} \\ &= \left[\frac{\partial}{\partial y}(P + S) - \frac{\partial}{\partial z}(N + R) \right] \mathbf{i} - \left[\frac{\partial}{\partial x}(P + S) - \frac{\partial}{\partial z}(M + Q) \right] \mathbf{j} + \left[\frac{\partial}{\partial x}(N + R) - \frac{\partial}{\partial y}(M + Q) \right] \mathbf{k} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} + \left(\frac{\partial S}{\partial y} - \frac{\partial R}{\partial z} \right) \mathbf{i} - \left(\frac{\partial S}{\partial x} - \frac{\partial Q}{\partial z} \right) \mathbf{j} + \left(\frac{\partial R}{\partial x} - \frac{\partial Q}{\partial y} \right) \mathbf{k} \\ &= \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G} \end{aligned}$$

79. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and $\mathbf{G} = R\mathbf{i} + S\mathbf{j} + T\mathbf{k}$.

$$\begin{aligned}\operatorname{div}(\mathbf{F} + \mathbf{G}) &= \frac{\partial}{\partial x}(M + R) + \frac{\partial}{\partial y}(N + S) + \frac{\partial}{\partial z}(P + T) = \frac{\partial M}{\partial x} + \frac{\partial R}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial S}{\partial y} + \frac{\partial P}{\partial z} + \frac{\partial T}{\partial z} \\ &= \left[\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right] + \left[\frac{\partial R}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial T}{\partial z} \right] \\ &= \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}\end{aligned}$$

81. $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$

$$\begin{aligned}\nabla \times [\nabla f + (\nabla \times \mathbf{F})] &= \operatorname{curl}(\nabla f + (\nabla \times \mathbf{F})) \\ &= \operatorname{curl}(\nabla f) + \operatorname{curl}(\nabla \times \mathbf{F}) \quad (\text{Exercise 77}) \\ &= \operatorname{curl}(\nabla \times \mathbf{F}) \quad (\text{Exercise 78}) \\ &= \nabla \times (\nabla \times \mathbf{F})\end{aligned}$$

83. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, then $f\mathbf{F} = fM\mathbf{i} + fN\mathbf{j} + fP\mathbf{k}$.

$$\begin{aligned}\operatorname{div}(f\mathbf{F}) &= \frac{\partial}{\partial x}(fM) + \frac{\partial}{\partial y}(fN) + \frac{\partial}{\partial z}(fP) = f\frac{\partial M}{\partial x} + M\frac{\partial f}{\partial x} + f\frac{\partial N}{\partial y} + N\frac{\partial f}{\partial y} + f\frac{\partial P}{\partial z} + P\frac{\partial f}{\partial z} \\ &= f\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) + \left(\frac{\partial f}{\partial x}M + \frac{\partial f}{\partial y}N + \frac{\partial f}{\partial z}P\right) \\ &= f\operatorname{div} \mathbf{F} + \nabla f \cdot \mathbf{F}\end{aligned}$$

In Exercises 85 and 87, $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $f(x, y, z) = \|\mathbf{F}(x, y, z)\| = \sqrt{x^2 + y^2 + z^2}$.

85. $\ln f = \frac{1}{2} \ln(x^2 + y^2 + z^2)$

$$\nabla(\ln f) = \frac{x}{x^2 + y^2 + z^2}\mathbf{i} + \frac{y}{x^2 + y^2 + z^2}\mathbf{j} + \frac{z}{x^2 + y^2 + z^2}\mathbf{k} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{x^2 + y^2 + z^2} = \frac{\mathbf{F}}{f^2}$$

87. $f^n = (\sqrt{x^2 + y^2 + z^2})^n$

$$\begin{aligned}\nabla f^n &= n(\sqrt{x^2 + y^2 + z^2})^{n-1} \frac{x}{\sqrt{x^2 + y^2 + z^2}}\mathbf{i} + n(\sqrt{x^2 + y^2 + z^2})^{n-1} \frac{y}{\sqrt{x^2 + y^2 + z^2}}\mathbf{j} \\ &\quad + n(\sqrt{x^2 + y^2 + z^2})^{n-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}}\mathbf{k} \\ &= n(\sqrt{x^2 + y^2 + z^2})^{n-2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = nf^{n-2}\mathbf{F}\end{aligned}$$

89. The winds are stronger over Phoenix. Although the winds over both cities are northeasterly, they are more towards the east over Atlanta.

Section 14.2 Line Integrals

1. $x^2 + y^2 = 9$

$$\frac{x^2}{9} + \frac{y^2}{9} = 1$$

$$\cos^2 t + \sin^2 t = 1$$

$$\cos^2 t = \frac{x^2}{9}$$

$$\sin^2 t = \frac{y^2}{9}$$

$$x = 3 \cos t$$

$$y = 3 \sin t$$

$$\mathbf{r}(t) = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j}$$

$$0 \leq t \leq 2\pi$$

3. $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \leq t \leq 3 \\ 3\mathbf{i} + (t-3)\mathbf{j}, & 3 \leq t \leq 6 \\ (9-t)\mathbf{i} + 3\mathbf{j}, & 6 \leq t \leq 9 \\ (12-t)\mathbf{j}, & 9 \leq t \leq 12 \end{cases}$

5. $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + \sqrt{t}\mathbf{j}, & 0 \leq t \leq 1 \\ (2-t)\mathbf{i} + (2-t)\mathbf{j}, & 1 \leq t \leq 2 \end{cases}$

7. $\mathbf{r}(t) = 4t\mathbf{i} + 3t\mathbf{j}$, $0 \leq t \leq 2$; $\mathbf{r}'(t) = 4\mathbf{i} + 3\mathbf{j}$

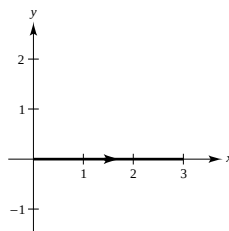
$$\int_C (x-y) ds = \int_0^2 (4t-3t)\sqrt{(4)^2 + (3)^2} dt = \int_0^2 5t dt = \left[\frac{5t^2}{2} \right]_0^2 = 10$$

9. $\mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + 8t \mathbf{k}$, $0 \leq t \leq \frac{\pi}{2}$; $\mathbf{r}'(t) = \cos t \mathbf{i} - \sin t \mathbf{j} + 8 \mathbf{k}$

$$\begin{aligned} \int_C (x^2 + y^2 + z^2) ds &= \int_0^{\pi/2} (\sin^2 t + \cos^2 t + 64t^2) \sqrt{(\cos t)^2 + (-\sin t)^2 + 64} dt \\ &= \int_0^{\pi/2} \sqrt{65}(1 + 64t^2) dt = \left[\sqrt{65} \left(t + \frac{64t^3}{3} \right) \right]_0^{\pi/2} = \sqrt{65} \left(\frac{\pi}{2} + \frac{8\pi^3}{3} \right) = \frac{\sqrt{65}\pi}{6} (3 + 16\pi^2) \end{aligned}$$

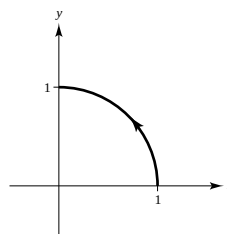
11. $\mathbf{r}(t) = t\mathbf{i}$, $0 \leq t \leq 3$

$$\begin{aligned} \int_C (x^2 + y^2) ds &= \int_0^3 [t^2 + 0^2] \sqrt{1 + 0} dt \\ &= \int_0^3 t^2 dt \\ &= \left[\frac{1}{3} t^3 \right]_0^3 = 9 \end{aligned}$$



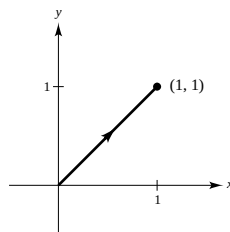
13. $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}$, $0 \leq t \leq \frac{\pi}{2}$

$$\begin{aligned} \int_C (x^2 + y^2) ds &= \int_0^{\pi/2} [\cos^2 t + \sin^2 t] \sqrt{(-\sin t)^2 + (\cos t)^2} dt \\ &= \int_0^{\pi/2} dt = \frac{\pi}{2} \end{aligned}$$



15. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}$, $0 \leq t \leq 1$

$$\begin{aligned} \int_C (x + 4\sqrt{y}) \, ds &= \int_0^1 (t + 4\sqrt{t}) \sqrt{1+1} \, dt \\ &= \left[\sqrt{2} \left(\frac{t^2}{2} + \frac{8}{3} t^{3/2} \right) \right]_0^1 = \frac{19\sqrt{2}}{6} \end{aligned}$$



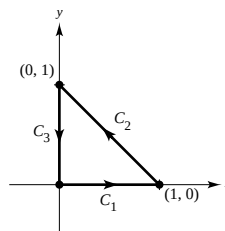
17. $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \leq t \leq 1 \\ (2-t)\mathbf{i} + (t-1)\mathbf{j}, & 1 \leq t \leq 2 \\ (3-t)\mathbf{j}, & 2 \leq t \leq 3 \end{cases}$

$$\int_{C_1} (x + 4\sqrt{y}) \, ds = \int_0^1 t \, dt = \frac{1}{2}$$

$$\begin{aligned} \int_{C_2} (x + 4\sqrt{y}) \, ds &= \int_1^2 [(2-t) + 4\sqrt{t-1}] \sqrt{1+1} \, dt \\ &= \sqrt{2} \left[2t - \frac{t^2}{2} + \frac{8}{3}(t-1)^{3/2} \right]_1^2 = \frac{19\sqrt{2}}{6} \end{aligned}$$

$$\int_{C_3} (x + 4\sqrt{y}) \, ds = \int_2^3 4\sqrt{3-t} \, dt = \left[-\frac{8}{3}(3-t)^{3/2} \right]_2^3 = \frac{8}{3}$$

$$\int_C (x + 4\sqrt{y}) \, ds = \frac{1}{2} + \frac{19\sqrt{2}}{6} + \frac{8}{3} = \frac{19 + 19\sqrt{2}}{6} = \frac{19(1 + \sqrt{2})}{6}$$



19. $\rho(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2)$

$$\mathbf{r}(t) = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j} + 2t \mathbf{k}, \quad 0 \leq t \leq 4\pi$$

$$\mathbf{r}'(t) = -3 \sin t \mathbf{i} + 3 \cos t \mathbf{j} + 2 \mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-3 \sin t)^2 + (3 \cos t)^2 + (2)^2} = \sqrt{13}$$

$$\begin{aligned} \text{Mass} &= \int_C \rho(x, y, z) \, ds = \int_0^{4\pi} \frac{1}{2} [(3 \cos t)^2 + (3 \sin t)^2 + (2t)^2] \sqrt{13} \, dt \\ &= \frac{\sqrt{13}}{2} \int_0^{4\pi} (9 + 4t^2) \, dt = \left[\frac{\sqrt{13}}{2} \left(9t + \frac{4t^3}{3} \right) \right]_0^{4\pi} \\ &= \frac{2\sqrt{13}\pi}{3} (27 + 64\pi^2) \approx 4973.8 \end{aligned}$$

21. $\mathbf{F}(x, y) = xy\mathbf{i} + y\mathbf{j}$

$$C: \mathbf{r}(t) = 4t\mathbf{i} + t\mathbf{j}, \quad 0 \leq t \leq 1$$

$$\mathbf{F}(t) = 4t^2\mathbf{i} + t\mathbf{j}$$

$$\mathbf{r}'(t) = 4\mathbf{i} + \mathbf{j}$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 (16t^2 + t) \, dt \\ &= \left[\frac{16}{3}t^3 + \frac{1}{2}t^2 \right]_0^1 = \frac{35}{6} \end{aligned}$$

23. $\mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{j}$

$$C: \mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2}$$

$$\mathbf{F}(t) = 6 \cos t \mathbf{i} + 8 \sin t \mathbf{j}$$

$$\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j}$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} (-12 \sin t \cos t + 16 \sin t \cos t) \, dt \\ &= \left[2 \sin^2 t \right]_0^{\pi/2} = 2 \end{aligned}$$

25. $\mathbf{F}(x, y, z) = x^2y\mathbf{i} + (x - z)\mathbf{j} + xyz\mathbf{k}$

$C: \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2\mathbf{k}, 0 \leq t \leq 1$

$\mathbf{F}(t) = t^4\mathbf{i} + (t - 2)\mathbf{j} + 2t^3\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 [t^4 + 2t(t - 2)] dt \\ &= \left[\frac{t^5}{5} + \frac{2t^3}{3} - 2t^2 \right]_0^1 = -\frac{17}{15} \end{aligned}$$

29. $\mathbf{F}(x, y) = -x\mathbf{i} - 2y\mathbf{j}$

$C: y = x^3$ from $(0, 0)$ to $(2, 8)$

$\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j}, 0 \leq t \leq 2$

$\mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{F}(t) = -t\mathbf{i} - 2t^3\mathbf{j}$

$\mathbf{F} \cdot \mathbf{r}' = -t - 6t^5$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 (-t - 6t^5) dt = \left[-\frac{1}{2}t^2 - t^6 \right]_0^2 = -66$$

31. $\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}$

C : counterclockwise around the triangle whose vertices are $(0, 0)$, $(1, 0)$, $(1, 1)$

$$\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \leq t \leq 1 \\ \mathbf{i} + (t - 1)\mathbf{j}, & 1 \leq t \leq 2 \\ (3 - t)\mathbf{i} + (3 - t)\mathbf{j}, & 2 \leq t \leq 3 \end{cases}$$

On C_1 : $\mathbf{F}(t) = 2t\mathbf{i}, \mathbf{r}'(t) = \mathbf{i}$

$$\text{Work} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 2t dt = 1$$

On C_2 : $\mathbf{F}(t) = 2\mathbf{i} + (t - 1)\mathbf{j}, \mathbf{r}'(t) = \mathbf{j}$

$$\text{Work} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_1^2 (t - 1) dt = \frac{1}{2}$$

On C_3 : $\mathbf{F}(t) = 2(3 - t)\mathbf{i} + (3 - t)\mathbf{j}, \mathbf{r}'(t) = -\mathbf{i} - \mathbf{j}$

$$\text{Work} = \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_2^3 [-2(3 - t) - (3 - t)] dt = -\frac{3}{2}$$

$$\text{Total work} = \int_C \mathbf{F} \cdot d\mathbf{r} = 1 + \frac{1}{2} - \frac{3}{2} = 0$$

33. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 5z\mathbf{k}$

$C: \mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 2\pi$

$\mathbf{r}'(t) = -2 \sin t\mathbf{i} + 2 \cos t\mathbf{j} + \mathbf{k}$

$\mathbf{F}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} - 5t\mathbf{k}$

$\mathbf{F} \cdot \mathbf{r}' = -5t$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} -5t dt = -10\pi^2$$

27. $\mathbf{F}(x, y, z) = x^2z\mathbf{i} + 6y\mathbf{j} + yz^2\mathbf{k}$

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k}, 1 \leq t \leq 3$

$\mathbf{F}(t) = t^2 \ln t\mathbf{i} + 6t^2\mathbf{j} + t^2 \ln^2 t\mathbf{k}$

$d\mathbf{r} = \left(\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k} \right) dt$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_1^3 [t^2 \ln t + 12t^3 + t(\ln t)^2] dt \\ &\approx 249.49 \end{aligned}$$

35. $\mathbf{r}(t) = 3 \sin t\mathbf{i} + 3 \cos t\mathbf{j} + \frac{10}{2\pi}t\mathbf{k}, 0 \leq t \leq 2\pi$

$\mathbf{F} = 150\mathbf{k}$

$d\mathbf{r} = \left(3 \cos t\mathbf{i} - 3 \sin t\mathbf{j} + \frac{10}{2\pi}\mathbf{k} \right) dt$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \frac{1500}{2\pi} dt = \left[\frac{1500}{2\pi}t \right]_0^{2\pi} = 1500 \text{ ft} \cdot \text{lb}$$

37. $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

(a) $\mathbf{r}_1(t) = 2t\mathbf{i} + (t - 1)\mathbf{j}, 1 \leq t \leq 3$

$\mathbf{r}_1'(t) = 2\mathbf{i} + \mathbf{j}$

$\mathbf{F}(t) = 4t^2\mathbf{i} + 2t(t - 1)\mathbf{j}$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_1^3 (8t^2 + 2t(t - 1)) dt = \frac{236}{3}$$

Both paths join (2, 0) and (6, 2). The integrals are negatives of each other because the orientations are different.

(b) $\mathbf{r}_2(t) = 2(3 - t)\mathbf{i} + (2 - t)\mathbf{j}, 0 \leq t \leq 2$

$\mathbf{r}_2'(t) = -2\mathbf{i} - \mathbf{j}$

$\mathbf{F}(t) = 4(3 - t)^2\mathbf{i} + 2(3 - t)(2 - t)\mathbf{j}$

$$\begin{aligned} \int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^2 [-8(3 - t)^2 - 2(3 - t)(2 - t)] dt \\ &= -\frac{236}{3} \end{aligned}$$

39. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} - 2t\mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} - 2\mathbf{j}$

$\mathbf{F}(t) = -2t\mathbf{i} - t\mathbf{j}$

$\mathbf{F} \cdot \mathbf{r}' = -2t + 2t = 0$

Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

41. $\mathbf{F}(x, y) = (x^3 - 2x^2)\mathbf{i} + \left(x - \frac{y}{2}\right)\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = (t^3 - 2t^2)\mathbf{i} + \left(t - \frac{t^2}{2}\right)\mathbf{j}$

$\mathbf{F} \cdot \mathbf{r}' = (t^3 - 2t^2) + 2t\left(t - \frac{t^2}{2}\right) = 0$

Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

43. $x = 2t, y = 10t, 0 \leq t \leq 1 \Rightarrow y = 5x \text{ or } x = \frac{y}{5}, 0 \leq y \leq 10$

$$\int_C (x + 3y^2) dy = \int_0^{10} \left(\frac{y}{5} + 3y^2\right) dy = \left[\frac{y^2}{10} + y^3\right]_0^{10} = 1010$$

45. $x = 2t, y = 10t, 0 \leq t \leq 1 \Rightarrow x = \frac{y}{5}, 0 \leq y \leq 10, dx = \frac{1}{5} dy$

$$\int_C xy dx + y dy = \int_0^{10} \left(\frac{y^2}{25} + y\right) dy = \left[\frac{y^3}{75} + \frac{y^2}{2}\right]_0^{10} = \frac{190}{3} \text{ OR}$$

$y = 5x, dy = 5 dx, 0 \leq x \leq 2$

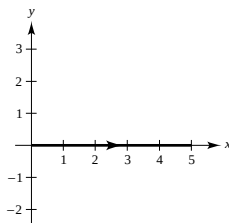
$$\int_C xy dx + y dy = \int_0^2 (5x^2 + 25x) dx = \left[\frac{5x^3}{3} + \frac{25x^2}{2}\right]_0^2 = \frac{190}{3}$$

47. $\mathbf{r}(t) = t\mathbf{i}, 0 \leq t \leq 5$

$x(t) = t, y(t) = 0$

$dx = dt, dy = 0$

$$\int_C (2x - y) dx + (x + 3y) dy = \int_0^5 2t dt = 25$$



$$49. \mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \leq t \leq 3 \\ 3\mathbf{i} + (t-3)\mathbf{j}, & 3 \leq t \leq 6 \end{cases}$$

$$C_1: x(t) = t, y(t) = 0,$$

$$dx = dt, dy = 0$$

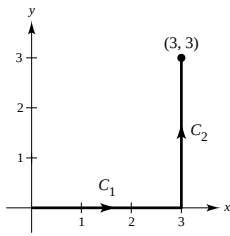
$$\int_{C_1} (2x - y) dx + (x + 3y) dy = \int_0^3 2t dt = 9$$

$$C_2: x(t) = 3, y(t) = t - 3$$

$$dx = 0, dy = dt$$

$$\int_{C_2} (2x - y) dx + (x + 3y) dy = \int_3^6 [3 + 3(t-3)] dt = \left[\frac{3t^2}{2} - 6t \right]_3^6 = \frac{45}{2}$$

$$\int_C (2x - y) dx + (x + 3y) dy = 9 + \frac{45}{2} = \frac{63}{2}$$



$$51. x(t) = t, y(t) = 1 - t^2, 0 \leq t \leq 1, dx = dt, dy = -2t dt$$

$$\begin{aligned} \int_C (2x - y) dx + (x + 3y) dy &= \int_0^1 [(2t - 1 + t^2) + (t + 3 - 3t^2)(-2t)] dt \\ &= \int_0^1 (6t^3 - t^2 - 4t - 1) dt = \left[\frac{3t^4}{2} - \frac{t^3}{3} - 2t^2 - t \right]_0^1 = -\frac{11}{6} \end{aligned}$$

$$53. x(t) = t, y(t) = 2t^2, 0 \leq t \leq 2$$

$$dx = dt, dy = 4t dt$$

$$\begin{aligned} \int_C (2x - y) dx + (x + 3y) dy &= \int_0^2 (2t - 2t^2) dt + \int_0^2 (t + 6t^2) 4t dt \\ &= \int_0^2 (24t^3 + 2t^2 + 2t) dt = \left[6t^4 + \frac{2}{3}t^3 + t^2 \right]_0^2 = \frac{316}{3} \end{aligned}$$

$$55. f(x, y) = h$$

C: line from (0, 0) to (3, 4)

$$\mathbf{r} = 3t\mathbf{i} + 4t\mathbf{j}, 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = 3\mathbf{i} + 4\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 5$$

Lateral surface area:

$$\int_C f(x, y) ds = \int_0^1 5h dt = 5h$$

$$57. f(x, y) = xy$$

C: $x^2 + y^2 = 1$ from (1, 0) to (0, 1)

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}, 0 \leq t \leq \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 1$$

Lateral surface area:

$$\begin{aligned} \int_C f(x, y) ds &= \int_0^{\pi/2} \cos t \sin t dt \\ &= \left[\frac{\sin^2 t}{2} \right]_0^{\pi/2} = \frac{1}{2} \end{aligned}$$

59. $f(x, y) = h$ C: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

$$\mathbf{r}(t) = (1 - t)\mathbf{i} + [1 - (1 - t)^2]\mathbf{j}, \quad 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = -\mathbf{i} + 2(1 - t)\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{1 + 4(1 - t)^2}$$

Lateral surface area:

$$\begin{aligned} \int_C f(x, y) \, ds &= \int_0^1 h \sqrt{1 + 4(1 - t)^2} \, dt \\ &= -\frac{h}{4} \left[2(1 - t) \sqrt{1 + 4(1 - t)^2} + \ln |2(1 - t) + \sqrt{1 + 4(1 - t)^2}| \right]_0^1 \\ &= \frac{h}{4} [2\sqrt{5} + \ln(2 + \sqrt{5})] \approx 1.4789h \end{aligned}$$

61. $f(x, y) = xy$ C: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$ You could parameterize the curve C as in Exercises 59 and 60. Alternatively, let $x = \cos t$, then:

$$y = 1 - \cos^2 t = \sin^2 t$$

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin^2 t \mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -\sin t \mathbf{i} + 2 \sin t \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + 4 \sin^2 t \cos^2 t} = \sin t \sqrt{1 + 4 \cos^2 t}$$

Lateral surface area:

$$\int_C f(x, y) \, ds = \int_0^{\pi/2} \cos t \sin^2 t (\sin t \sqrt{1 + 4 \cos^2 t}) \, dt = \int_0^{\pi/2} \sin^2 t [(1 + 4 \cos^2 t)^{1/2} \sin t \cos t] \, dt$$

Let $u = \sin^2 t$ and $dv = (1 + 4 \cos^2 t)^{1/2} \sin t \cos t$, then $du = 2 \sin t \cos t \, dt$ and $v = -\frac{1}{12}(1 + 4 \cos^2 t)^{3/2}$.

$$\begin{aligned} \int_C f(x, y) \, ds &= \left[-\frac{1}{12} \sin^2 t (1 + 4 \cos^2 t)^{3/2} \right]_0^{\pi/2} + \frac{1}{6} \int_0^{\pi/2} (1 + 4 \cos^2 t)^{3/2} \sin t \cos t \, dt \\ &= \left[-\frac{1}{12} \sin^2 t (1 + 4 \cos^2 t)^{3/2} - \frac{1}{120} (1 + 4 \cos^2 t)^{5/2} \right]_0^{\pi/2} \\ &= \left(-\frac{1}{12} - \frac{1}{120} \right) + \frac{1}{120} (5)^{5/2} = \frac{1}{120} (25\sqrt{5} - 11) \approx 0.3742 \end{aligned}$$

63. (a) $f(x, y) = 1 + y^2$

$$\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j}, \quad 0 \leq t \leq 2\pi$$

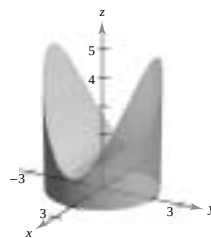
$$\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 2$$

$$\begin{aligned} S &= \int_C f(x, y) \, ds = \int_0^{2\pi} (1 + 4 \sin^2 t)(2) \, dt \\ &= \left[2t + 4(t - \sin t \cos t) \right]_0^{2\pi} = 12\pi \approx 37.70 \text{ cm}^2 \end{aligned}$$

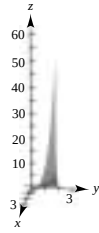
$$(b) \quad 0.2(12\pi) = \frac{12\pi}{5} \approx 7.54 \text{ cm}^3$$

(c)

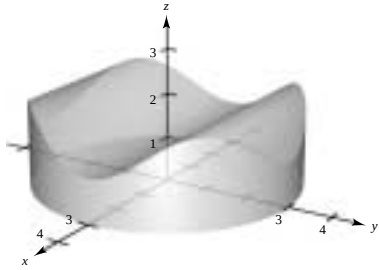


65. $S \approx 25$

Matches b



67. (a) Graph of: $\mathbf{r}(t) = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j} + (1 + \sin^2 2t) \mathbf{k}$ $0 \leq t \leq 2\pi$



(b) Consider the portion of the surface in the first quadrant. The curve $z = 1 + \sin^2 2t$ is over the curve $\mathbf{r}_1(t) = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j}$, $0 \leq t \leq \pi/2$. Hence, the total lateral surface area is

$$4 \int_C f(x, y) ds = 4 \int_0^{\pi/2} (1 + \sin^2 2t) 3 dt = 12 \left(\frac{3\pi}{4} \right) = 9\pi \text{ sq. cm}$$

(c) The cross sections parallel to the xz -plane are rectangles of height $1 + 4(y/3)^2(1 - y^2/9)$ and base $2\sqrt{9 - y^2}$. Hence,

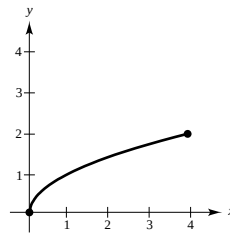
$$\text{Volume} = 2 \int_0^3 2\sqrt{9 - y^2} \left(1 + 4\frac{y^2}{9} \left(1 - \frac{y^2}{9} \right) \right) dy \approx 42.412 \text{ cm}^3$$

69. See the definition of Line Integral, page 1020.

See Theorem 14.4.

71. The greater the height of the surface over the curve, the greater the lateral surface area. Hence,

$$z_3 < z_1 < z_2 < z_4.$$



73. False

$$\int_C xy ds = \sqrt{2} \int_0^1 t^2 dt$$

75. False, the orientations are different.

Section 14.3 Conservative Vector Fields and Independence of Path

1. $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 1$

$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t^2 + 2t^4) dt = \frac{11}{15}$$

(b) $\mathbf{r}_2(\theta) = \sin \theta \mathbf{i} + \sin^2 \theta \mathbf{j}, 0 \leq \theta \leq \frac{\pi}{2}$

$\mathbf{r}_2'(\theta) = \cos \theta \mathbf{i} + 2 \sin \theta \cos \theta \mathbf{j}$

$\mathbf{F}(t) = \sin^2 \theta \mathbf{i} + \sin^3 \theta \mathbf{j}$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} (\sin^2 \theta \cos \theta + 2 \sin^4 \theta \cos \theta) d\theta \\ &= \left[\frac{\sin^3 \theta}{3} + \frac{2 \sin^5 \theta}{5} \right]_0^{\pi/2} = \frac{11}{15} \end{aligned}$$

3. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(\theta) = \sec \theta \mathbf{i} + \tan \theta \mathbf{j}, 0 \leq \theta \leq \frac{\pi}{3}$

$\mathbf{r}_1'(\theta) = \sec \theta \tan \theta \mathbf{i} + \sec^2 \theta \mathbf{j}$

$\mathbf{F}(\theta) = \tan \theta \mathbf{i} - \sec \theta \mathbf{j}$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/3} (\sec \theta \tan^2 \theta - \sec^3 \theta) d\theta = \int_0^{\pi/3} [\sec \theta (\sec^2 \theta - 1) - \sec^3 \theta] d\theta \\ &= - \int_0^{\pi/3} \sec \theta d\theta = [-\ln|\sec \theta + \tan \theta|]_0^{\pi/3} = -\ln(2 + \sqrt{3}) \approx -1.317 \end{aligned}$$

(b) $\mathbf{r}_2(t) = \sqrt{t+1}\mathbf{i} + \sqrt{t}\mathbf{j}, 0 \leq t \leq 3$

$\mathbf{r}_2'(t) = \frac{1}{2\sqrt{t+1}}\mathbf{i} + \frac{1}{2\sqrt{t}}\mathbf{j}$

$\mathbf{F}(t) = \sqrt{t}\mathbf{i} - \sqrt{t+1}\mathbf{j}$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^3 \left[\frac{\sqrt{t}}{2\sqrt{t+1}} - \frac{\sqrt{t+1}}{2\sqrt{t}} \right] dt = -\frac{1}{2} \int_0^3 \frac{1}{\sqrt{t}\sqrt{t+1}} dt = -\frac{1}{2} \int_0^3 \frac{1}{\sqrt{t^2 + t + (1/4) - (1/4)}} dt \\ &= -\frac{1}{2} \int_0^3 \frac{1}{\sqrt{[t + (1/2)]^2 - (1/4)}} dt = \left[-\frac{1}{2} \ln \left| \left(t + \frac{1}{2} \right) + \sqrt{t^2 + t} \right| \right]_0^3 \\ &= -\frac{1}{2} \left[\ln \left(\frac{7}{2} + 2\sqrt{3} \right) - \ln \left(\frac{1}{2} \right) \right] = -\frac{1}{2} \ln(7 + 4\sqrt{3}) \approx -1.317 \end{aligned}$$

5. $\mathbf{F}(x, y) = e^x \sin y \mathbf{i} + e^x \cos y \mathbf{j}$

$$\frac{\partial N}{\partial x} = e^x \cos y \quad \frac{\partial M}{\partial y} = e^x \cos y$$

Since $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$, $\mathbf{F}$ is conservative.

7. $\mathbf{F}(x, y) = \frac{1}{y}\mathbf{i} + \frac{x}{y^2}\mathbf{j}$

$$\frac{\partial N}{\partial x} = \frac{1}{y^2} \quad \frac{\partial M}{\partial y} = -\frac{1}{y^2}$$

Since $\frac{\partial N}{\partial x} \neq \frac{\partial M}{\partial y}$, $\mathbf{F}$ is not conservative.

9. $\mathbf{F}(x, y, z) = y^2z\mathbf{i} + 2xyz\mathbf{j} + xy^2\mathbf{k}$

$\text{curl } \mathbf{F} = \mathbf{0} \Rightarrow \mathbf{F}$ is conservative.

11. $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 1$

$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = 2t^3\mathbf{i} + t^2\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 4t^3 dt = 1$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t^3\mathbf{j}, 0 \leq t \leq 1$

$\mathbf{r}_2'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{F}(t) = 2t^4\mathbf{i} + t^2\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 5t^4 dt = 1$

13. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t\mathbf{j}, 0 \leq t \leq 1$

$\mathbf{r}_1'(t) = \mathbf{i} + \mathbf{j}$

$\mathbf{F}(t) = t\mathbf{i} - t\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 1$

$\mathbf{r}_2'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = t^2\mathbf{i} - t\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -t^2 dt = -\frac{1}{3}$

(c) $\mathbf{r}_3(t) = t\mathbf{i} + t^3\mathbf{j}, 0 \leq t \leq 1$

$\mathbf{r}_3'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{F}(t) = t^3\mathbf{i} - t\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -2t^3 dt = -\frac{1}{2}$

15. $\int_C y^2 dx + 2xy dy$

Since $\partial M/\partial y = \partial N/\partial x = 2y$, $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$ is conservative. The potential function is $f(x, y) = xy^2 + k$. Therefore, we can use the Fundamental Theorem of Line Integrals.

(a) $\int_C y^2 dx + 2xy dy = \left[x^2 y \right]_{(0,0)}^{(4,4)} = 64$

(b) $\int_C y^2 dx + 2xy dy = \left[x^2 y \right]_{(-1,0)}^{(1,0)} = 0$

(c) and (d) Since C is a closed curve, $\int_C y^2 dx + 2xy dy = 0$.

17. $\int_C 2xy dx + (x^2 + y^2) dy$

Since $\partial M/\partial y = \partial N/\partial x = 2x$,

$\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 + y^2)\mathbf{j}$ is conservative.

The potential function is $f(x, y) = x^2 y + \frac{y^3}{3} + k$.

(a) $\int_C 2xy dx + (x^2 + y^2) dy = \left[x^2 y + \frac{y^3}{3} \right]_{(5,0)}^{(0,4)} = \frac{64}{3}$

(b) $\int_C 2xy dx + (x^2 + y^2) dy = \left[x^2 y + \frac{y^3}{3} \right]_{(2,0)}^{(0,4)} = \frac{64}{3}$

19. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

Since $\text{curl } \mathbf{F} = \mathbf{0}$, $\mathbf{F}(x, y, z)$ is conservative. The potential function is $f(x, y, z) = xyz + k$.

(a) $\mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 4$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \left[xyz \right]_{(0,2,0)}^{(4,2,4)} = 32$

(b) $\mathbf{r}_2(t) = t^2\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}, 0 \leq t \leq 2$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \left[xyz \right]_{(0,0,0)}^{(4,2,4)} = 32$

21. $\mathbf{F}(x, y, z) = (2y + x)\mathbf{i} + (x^2 - z)\mathbf{j} + (2y - 4z)\mathbf{k}$

$\mathbf{F}(x, y, z)$ is not conservative.

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}, 0 \leq t \leq 1$

$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = (2t^2 + t)\mathbf{i} + (t^2 - 1)\mathbf{j} + (2t^2 - 4)\mathbf{k}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2t^3 + 2t^2 - t) dt = \frac{2}{3}$

—CONTINUED—

21. —CONTINUED—

$$(b) \mathbf{r}_2(t) = t\mathbf{i} + t\mathbf{j} + (2t - 1)^2\mathbf{k}, \quad 0 \leq t \leq 1$$

$$\mathbf{r}_2'(t) = \mathbf{i} + \mathbf{j} + 4(2t - 1)\mathbf{k}$$

$$\mathbf{F}(t) = 3t\mathbf{i} + [t^2 - (2t - 1)^2]\mathbf{j} + [2t - 4(2t - 1)^2]\mathbf{k}$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 [3t + t^2 - (2t - 1)^2 + 8t(2t - 1) - 16(2t - 1)^3] dt \\ &= \int_0^1 [17t^2 - 5t - (2t - 1)^2 - 16(2t - 1)^3] dt = \left[\frac{17t^3}{3} - \frac{5t^2}{2} - \frac{(2t - 1)^3}{6} - 2(2t - 1)^4 \right]_0^1 = \frac{17}{6} \end{aligned}$$

$$23. \mathbf{F}(x, y, z) = e^z(y\mathbf{i} + x\mathbf{j} + xy\mathbf{k})$$

$\mathbf{F}(x, y, z)$ is conservative. The potential function is $f(x, y, z) = xye^z + k$.

$$(a) \mathbf{r}_1(t) = 4 \cos t\mathbf{i} + 4 \sin t\mathbf{j} + 3\mathbf{k}, \quad 0 \leq t \leq \pi$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left[xye^z \right]_{(4, 0, 3)}^{(-4, 0, 3)} = 0$$

$$(b) \mathbf{r}_2(t) = (4 - 8t)\mathbf{i} + 3\mathbf{k}, \quad 0 \leq t \leq 1$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left[xye^z \right]_{(4, 0, 3)}^{(-4, 0, 3)} = 0$$

$$25. \int_C (y\mathbf{i} + x\mathbf{j}) \cdot d\mathbf{r} = \left[xy \right]_{(0, 0)}^{(3, 8)} = 24$$

$$27. \int_C \cos x \sin y \, dx + \sin x \cos y \, dy = \left[\sin x \sin y \right]_{(0, -\pi)}^{(3\pi/2, \pi/2)} = -1$$

$$29. \int_C e^x \sin y \, dx + e^x \cos y \, dy = \left[e^x \sin y \right]_{(0, 0)}^{(2\pi, 0)} = 0$$

$$31. \int_C (y + 2z) \, dx + (x - 3z) \, dy + (2x - 3y) \, dz$$

$\mathbf{F}(x, y, z)$ is conservative and the potential function is $f(x, y, z) = xy - 3yz + 2xz$.

$$(a) \left[xy - 3yz + 2xz \right]_{(0, 0, 0)}^{(1, 1, 1)} = 0 - 0 = 0$$

$$(b) \left[xy - 3yz + 2xz \right]_{(0, 0, 0)}^{(0, 0, 1)} + \left[xy - 3yz + 2xz \right]_{(0, 0, 1)}^{(1, 1, 1)} = 0 + 0 = 0$$

$$(c) \left[xy - 3yz + 2xz \right]_{(0, 0, 0)}^{(1, 0, 0)} + \left[xy - 3yz + 2xz \right]_{(1, 0, 0)}^{(1, 1, 0)} + \left[xy - 3yz + 2xz \right]_{(1, 1, 0)}^{(1, 1, 1)} = 0 + 1 + (-1) = 0$$

$$33. \int_C -\sin x \, dx + z \, dy + y \, dz = \left[\cos x + yz \right]_{(0, 0, 0)}^{(\pi/2, 3, 4)} = 12 - 1 = 11$$

$$35. \mathbf{F}(x, y) = 9x^2y^2\mathbf{i} + (6x^3y - 1)\mathbf{j} \text{ is conservative.}$$

$$\text{Work} = \left[3x^3y^2 - y \right]_{(0, 0)}^{(5, 9)} = 30,366$$

37. $\mathbf{r}(t) = 2 \cos 2\pi t \mathbf{i} + 2 \sin 2\pi t \mathbf{j}$

$$\mathbf{r}'(t) = -4\pi \sin 2\pi t \mathbf{i} + 4\pi \cos 2\pi t \mathbf{j}$$

$$\mathbf{a}(t) = -8\pi^2 \cos 2\pi t \mathbf{i} - 8\pi^2 \sin 2\pi t \mathbf{j}$$

$$\mathbf{F}(t) = m \cdot \mathbf{a}(t) = \frac{1}{32} \mathbf{a}(t) = -\frac{\pi^2}{4} (\cos 2\pi t \mathbf{i} + \sin 2\pi t \mathbf{j})$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C -\frac{\pi^2}{4} (\cos 2\pi t \mathbf{i} + \sin 2\pi t \mathbf{j}) \cdot 4\pi (-\sin 2\pi t \mathbf{i} + \cos 2\pi t \mathbf{j}) dt = -\pi^3 \int_C 0 dt = 0$$

39. Since the sum of the potential and kinetic energies remains constant from point to point, if the kinetic energy is decreasing at a rate of 10 units per minute, then the potential energy is increasing at a rate of 10 units per minute.

41. No. The force field is conservative.

43. See Theorem 14.5, page 1033.

45. (a) The direct path along the line segment joining $(-4, 0)$ to $(3, 4)$ requires less work than the path going from $(-4, 0)$ to $(-4, 4)$ and then to $(3, 4)$.

(b) The closed curve given by the line segments joining $(-4, 0)$, $(-4, 4)$, $(3, 4)$, and $(-4, 0)$ satisfies $\int_C \mathbf{F} \cdot d\mathbf{r} \neq 0$.

47. False, it would be true if $\mathbf{F}$ were conservative.

49. True

51. Let

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j} = \frac{\partial f}{\partial y} \mathbf{i} - \frac{\partial f}{\partial x} \mathbf{j}.$$

$$\text{Then } \frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left(-\frac{\partial f}{\partial x} \right) = -\frac{\partial^2 f}{\partial x^2}. \text{ Since}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \text{ we have } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Thus, $\mathbf{F}$ is conservative. Therefore, by Theorem 14.7, we have

$$\int_C \left(\frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial x} dy \right) = \int_C (M dx + N dy) = \int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

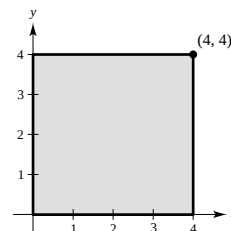
for every closed curve in the plane.

Section 14.4 Green's Theorem

$$1. \mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \leq t \leq 4 \\ 4\mathbf{i} + (t-4)\mathbf{j}, & 4 \leq t \leq 8 \\ (12-t)\mathbf{i} + 4\mathbf{j}, & 8 \leq t \leq 12 \\ (16-t)\mathbf{j}, & 12 \leq t \leq 16 \end{cases}$$

$$\begin{aligned} \int_C y^2 dx + x^2 dy &= \int_0^4 [0 dt + t^2(0)] + \int_4^8 [(t-4)^2(0) + 16 dt] \\ &\quad + \int_8^{12} [16(-dt) + (12-t)^2(0)] + \int_{12}^{16} [(16-t)^2(0) + 0(-dt)] \\ &= 0 + 64 - 64 + 0 = 0 \end{aligned}$$

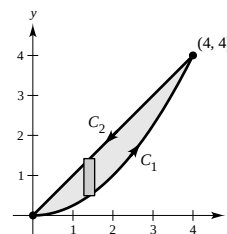
$$\text{By Green's Theorem, } \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \int_0^4 \int_0^4 (2x - 2y) dy dx = \int_0^4 (8x - 16) dx = 0.$$



$$\begin{aligned}
 3. \mathbf{r}(t) &= \begin{cases} t\mathbf{i} + t^2/4\mathbf{j}, & 0 \leq t \leq 4 \\ (8-t)\mathbf{i} + (8-t)\mathbf{j}, & 4 \leq t \leq 8 \end{cases} \\
 \int_C y^2 dx + x^2 dy &= \int_0^4 \left[\frac{t^4}{16}(dt) + t^2 \left(\frac{t}{2} dt \right) \right] + \int_4^8 [(8-t)^2(-dt) + (8-t)^2(-dt)] \\
 &= \int_0^4 \left[\frac{t^4}{16} + \frac{t^3}{2} \right] dt + \int_4^8 -2(8-t)^2 dt = \frac{224}{5} - \frac{128}{3} = \frac{32}{15}
 \end{aligned}$$

By Green's Theorem,

$$\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \int_0^4 \int_{x^2/4}^x (2x - 2y) dy dx = \int_0^4 \left(x^2 - \frac{x^3}{2} + \frac{x^4}{16} \right) dx = \frac{32}{15}.$$



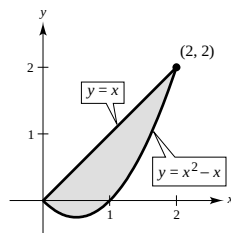
$$5. C: x^2 + y^2 = 4$$

Let $x = 2 \cos t$ and $y = 2 \sin t$, $0 \leq t \leq 2\pi$.

$$\begin{aligned}
 \int_C x e^y dx + e^x dy &= \int_0^{2\pi} [2 \cos t e^{2 \sin t} (-2 \sin t) + e^{2 \cos t} (2 \cos t)] dt \approx 19.99 \\
 \int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (e^x - x e^y) dy dx = \int_{-2}^2 \left[2\sqrt{4-x^2} e^x - x e^{\sqrt{4-x^2}} + x e^{-\sqrt{4-x^2}} \right] dx \approx 19.99
 \end{aligned}$$

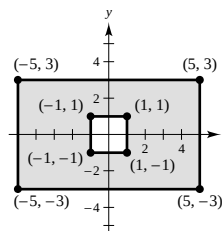
In Exercises 7 and 9, $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$.

$$\begin{aligned}
 7. \int_C (y-x) dx + (2x-y) dy &= \int_0^2 \int_{x^2-x}^x dy dx \\
 &= \int_0^2 (2x - x^2) dx \\
 &= \frac{4}{3}
 \end{aligned}$$



9. From the accompanying figure, we see that R is the shaded region. Thus, Green's Theorem yields

$$\begin{aligned}
 \int_C (y-x) dx + (2x-y) dy &= \iint_R 1 dA \\
 &= \text{Area of } R \\
 &= 6(10) - 2(2) \\
 &= 56.
 \end{aligned}$$



11. Since the curves $y = 0$ and $y = 4 - x^2$ intersect at $(-2, 0)$ and $(2, 0)$, Green's Theorem yields

$$\begin{aligned}
 \int_C 2xy dx + (x+y) dy &= \iint_R (1-2x) dA = \int_{-2}^2 \int_0^{4-x^2} (1-2x) dy dx \\
 &= \int_{-2}^2 \left[y - 2xy \right]_0^{4-x^2} dx \\
 &= \int_{-2}^2 (4 - 8x - x^2 + 2x^3) dx \\
 &= \left[4x - 4x^2 - \frac{x^3}{3} + \frac{x^4}{2} \right]_{-2}^2 \\
 &= -\frac{8}{3} - \frac{8}{3} + 16 = \frac{32}{3}.
 \end{aligned}$$

13. Since R is the interior of the circle $x^2 + y^2 = a^2$, Green's Theorem yields

$$\begin{aligned}\int_C (x^2 - y^2) dx + 2xy dy &= \iint_R (2y + 2y) dA \\ &= \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 4y dy dx = 4 \int_{-a}^a 0 dx = 0.\end{aligned}$$

15. Since $\frac{\partial M}{\partial y} = \frac{2x}{x^2 + y^2} = \frac{\partial N}{\partial x}$,

we have path independence and

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = 0.$$

17. By Green's Theorem,

$$\begin{aligned}\int_C \sin x \cos y dx + (xy + \cos x \sin y) dy &= \iint_R [(y - \sin x \sin y) - (-\sin x \sin y)] dA \\ &= \int_0^1 \int_x^{\sqrt{x}} y dy dx = \frac{1}{2} \int_0^1 (x - x^2) dx = \frac{1}{2} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{12}.\end{aligned}$$

19. By Green's Theorem,

$$\begin{aligned}\int_C xy dx + (x + y) dy &= \iint_R (1 - x) dA \\ &= \int_0^{2\pi} \int_1^3 (1 - r \cos \theta) r dr d\theta = \int_0^{2\pi} \left(4 - \frac{26}{3} \cos \theta \right) d\theta = 8\pi.\end{aligned}$$

21. $\mathbf{F}(x, y) = xy\mathbf{i} + (x + y)\mathbf{j}$

$C: x^2 + y^2 = 4$

$$\text{Work} = \int_C xy dx + (x + y) dy = \iint_R (1 - x) dA = \int_0^{2\pi} \int_0^2 (1 - r \cos \theta) r dr d\theta = \int_0^{2\pi} \left(2 - \frac{8}{3} \cos \theta \right) d\theta = 4\pi$$

23. $\mathbf{F}(x, y) = (x^{3/2} - 3y)\mathbf{i} + (6x + 5\sqrt{y})\mathbf{j}$

C : boundary of the triangle with vertices $(0, 0)$, $(5, 0)$, $(0, 5)$

$$\text{Work} = \int_C (x^{3/2} - 3y) dx + (6x + 5\sqrt{y}) dy = \iint_R 9 dA = 9\left(\frac{1}{2}\right)(5)(5) = \frac{225}{2}$$

25. C : let $x = a \cos t$, $y = a \sin t$, $0 \leq t \leq 2\pi$. By Theorem 14.9, we have

$$A = \frac{1}{2} \int_C x dy - y dx = \frac{1}{2} \int_0^{2\pi} [a \cos t(a \cos t) - a \sin t(-a \sin t)] dt = \frac{1}{2} \int_0^{2\pi} a^2 dt = \left[\frac{a^2}{2} t \right]_0^{2\pi} = \pi a^2.$$

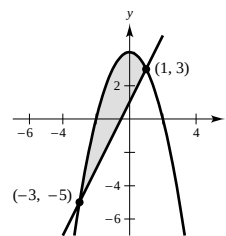
27. From the accompanying figure we see that

$$C_1: y = 2x + 1, \quad dy = 2 \, dx$$

$$C_2: y = 4 - x^2, \quad dy = -2x \, dx.$$

Thus, by Theorem 14.9, we have

$$\begin{aligned} A &= \frac{1}{2} \int_{-3}^1 [x(2) - (2x + 1)] \, dx + \frac{1}{2} \int_1^{-3} [x(-2x) - (4 - x^2)] \, dx \\ &= \frac{1}{2} \int_{-3}^1 (-1) \, dx + \frac{1}{2} \int_1^{-3} (-x^2 - 4) \, dx \\ &= \frac{1}{2} \int_{-3}^1 (-1) \, dx + \frac{1}{2} \int_{-3}^1 (x^2 + 4) \, dx = \frac{1}{2} \int_{-3}^1 (3 + x^2) \, dx = \frac{1}{2} \left[3x + \frac{x^3}{3} \right]_{-3}^1 = \frac{32}{3}. \end{aligned}$$



29. See Theorem 14.8, page 1042.

31. Answers will vary.

$$\mathbf{F}_1(x, y) = y\mathbf{i} + x\mathbf{j}$$

$$\mathbf{F}_2(x, y) = x^2\mathbf{i} + y^2\mathbf{j}$$

$$\mathbf{F}_3(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$$

$$33. A = \int_{-2}^2 (4 - x^2) \, dx = \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{32}{3}$$

$$\bar{x} = \frac{1}{2A} \int_{C_1} x^2 \, dy + \frac{1}{2A} \int_{C_2} x^2 \, dy$$

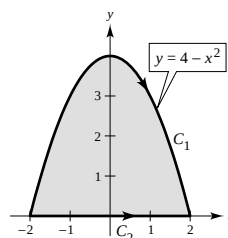
For C_1 , $dy = -2x \, dx$ and for C_2 , $dy = 0$. Thus,

$$\bar{x} = \frac{1}{2(32/3)} \int_2^{-2} x^2(-2x \, dx) = \left[\frac{3}{64} \left(-\frac{x^4}{2} \right) \right]_2^{-2} = 0.$$

To calculate $\bar{y}$, note that $y = 0$ along C_2 . Thus,

$$\bar{y} = \frac{-1}{2(32/3)} \int_2^{-2} (4 - x^2)^2 \, dx = \frac{3}{64} \int_{-2}^2 (16 - 8x^2 + x^4) \, dx = \frac{3}{64} \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_{-2}^2 = \frac{8}{5}.$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{5} \right)$$

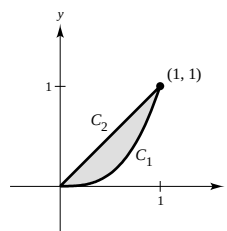


35. Since $A = \int_0^1 (x - x^3) \, dx = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{4}$, we have $\frac{1}{2A} = 2$. On C_1 we have $y = x^3$, $dy = 3x^2 \, dx$ and on C_2 we have $y = x$, $dy = dx$. Thus,

$$\begin{aligned} \bar{x} &= 2 \int_C x^2 \, dy = 2 \int_{C_1} x^2(3x^2 \, dx) + 2 \int_{C_2} x^2 \, dx \\ &= 6 \int_0^1 x^4 \, dx + 2 \int_1^0 x^2 \, dx = \frac{6}{5} - \frac{2}{3} = \frac{8}{15} \end{aligned}$$

$$\begin{aligned} \bar{y} &= -2 \int_C y^2 \, dx \\ &= -2 \int_0^1 x^6 \, dx - 2 \int_1^0 x^2 \, dx = -\frac{2}{7} + \frac{2}{3} = \frac{8}{21}. \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{15}, \frac{8}{21} \right)$$



$$\begin{aligned}
 37. A &= \frac{1}{2} \int_0^{2\pi} a^2(1 - \cos \theta)^2 d\theta \\
 &= \frac{a^2}{2} \int_0^{2\pi} \left(1 - 2\cos \theta + \frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta = \frac{a^2}{2} \left[\frac{3\theta}{2} - 2\sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = \frac{a^2}{2} (3\pi) = \frac{3\pi a^2}{2}
 \end{aligned}$$

39. In this case the inner loop has domain $\frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}$. Thus,

$$\begin{aligned}
 A &= \frac{1}{2} \int_{2\pi/3}^{4\pi/3} (1 + 4\cos \theta + 4\cos^2 \theta) d\theta \\
 &= \frac{1}{2} \int_{2\pi/3}^{4\pi/3} (3 + 4\cos \theta + 2\cos 2\theta) d\theta = \frac{1}{2} \left[3\theta + 4\sin \theta + \sin 2\theta \right]_{2\pi/3}^{4\pi/3} = \pi - \frac{3\sqrt{3}}{2}.
 \end{aligned}$$

$$41. I = \int_C \frac{y dx - x dy}{x^2 + y^2}$$

(a) Let $\mathbf{F} = \frac{y}{x^2 + y^2} \mathbf{i} - \frac{x}{x^2 + y^2} \mathbf{j}$.

$\mathbf{F}$ is conservative since $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$.

$\mathbf{F}$ is defined and has continuous first partials everywhere except at the origin. If C is a circle (a closed path) that does not contain the origin, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = 0.$$

(b) Let $\mathbf{r} = a \cos t \mathbf{i} - a \sin t \mathbf{j}$, $0 \leq t \leq 2\pi$ be a circle C_1 oriented clockwise inside C (see figure). Introduce line segments C_2 and C_3 as illustrated in Example 6 of this section in the text. For the region inside C and outside C_1 , Green's Theorem applies. Note that since C_2 and C_3 have opposite orientations, the line integrals over them cancel. Thus, $C_4 = C_1 + C_2 + C + C_3$ and

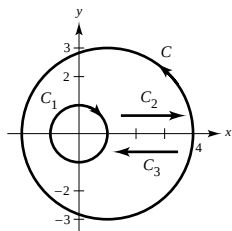
$$\int_{C_4} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

But,

$$\begin{aligned}
 \int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \left[\frac{(-a \sin t)(-a \sin t)}{a^2 \cos^2 t + a^2 \sin^2 t} + \frac{(-a \cos t)(-a \cos t)}{a^2 \cos^2 t + a^2 \sin^2 t} \right] dt \\
 &= \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = \left[t \right]_0^{2\pi} = 2\pi.
 \end{aligned}$$

Finally, $\int_C \mathbf{F} \cdot d\mathbf{r} = - \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = -2\pi$.

Note: If C were orientated clockwise, then the answer would have been 2π .



43. Pentagon: $(0, 0), (2, 0), (3, 2), (1, 4), (-1, 1)$

$$A = \frac{1}{2}[(0 - 0) + (4 - 0) + (12 - 2) + (1 + 4) + (0 - 0)] = \frac{19}{2}$$

$$45. \int_C y^n dx + x^n dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

For the line integral, use the two paths

$$C_1: \mathbf{r}_1(x) = x\mathbf{i}, \quad -a \leq x \leq a$$

$$C_2: \mathbf{r}_2(x) = x\mathbf{i} + \sqrt{a^2 - x^2}\mathbf{j}, \quad x = a \text{ to } x = -a$$

$$\int_{C_1} y^n dx + x^n dy = 0$$

$$\int_{C_2} y^n dx + x^n dy = \int_a^{-a} \left[(a^2 - x^2)^{n/2} + x^n \frac{-x}{\sqrt{a^2 - x^2}} \right] dx$$

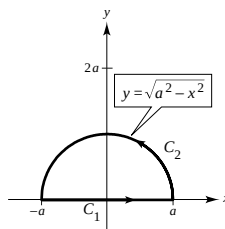
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \int_{-a}^a \int_0^{\sqrt{a^2 - x^2}} [nx^{n-1} - ny^{n-1}] dy dx$$

(a) For $n = 1, 3, 5, 7$, both integrals give 0.

(b) For n even, you obtain

$$n = 2: -\frac{4}{3}a^3 \quad n = 4: -\frac{16}{15}a^5 \quad n = 6: -\frac{32}{35}a^7 \quad n = 8: -\frac{256}{315}a^9$$

(c) If n is odd and $0 < a < 1$, then the integral equals 0.



$$47. \int_C (f D_{\mathbf{N}} g - g D_{\mathbf{N}} f) ds = \int_C f D_{\mathbf{N}} g ds - \int_C g D_{\mathbf{N}} f ds$$

$$= \iint_R (f \nabla^2 g + \nabla f \cdot \nabla g) dA - \iint_R (g \nabla^2 f + \nabla g \cdot \nabla f) dA = \iint_R (f \nabla^2 g - g \nabla^2 f) dA$$

$$49. \mathbf{F} = M\mathbf{i} + N\mathbf{j}$$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = 0 \Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_R (0) dA = 0$$

Section 14.5 Parametric Surfaces

$$1. \mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + uv\mathbf{k}$$

$$z = xy$$

Matches c.

$$3. \mathbf{r}(u, v) = 2 \cos v \cos u \mathbf{i} + 2 \cos v \sin u \mathbf{j} + 2 \sin v \mathbf{k}$$

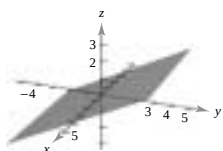
$$x^2 + y^2 + z^2 = 4$$

Matches b.

$$5. \mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \frac{v}{2}\mathbf{k}$$

$$y - 2z = 0$$

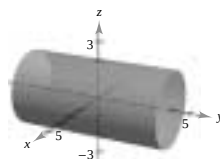
Plane



$$7. \mathbf{r}(u, v) = 2 \cos u \mathbf{i} + v\mathbf{j} + 2 \sin u \mathbf{k}$$

$$x^2 + z^2 = 4$$

Cylinder

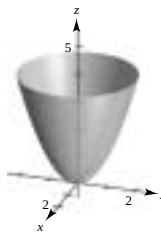


For Exercises 9 and 11,

$$\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + u^2 \mathbf{k}, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi.$$

Eliminating the parameter yields

$$z = x^2 + y^2, \quad 0 \leq z \leq 4.$$



9. $\mathbf{s}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} - u^2 \mathbf{k}, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi$
 $z = -(x^2 + y^2)$

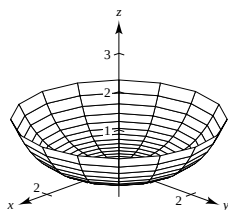
The paraboloid is reflected (inverted) through the xy -plane.

11. $\mathbf{s}(u, v) = u \cos v \mathbf{i} - u \sin v \mathbf{j} + u^2 \mathbf{k}, \quad 0 \leq u \leq 3, \quad 0 \leq v \leq 2\pi$
 The height of the paraboloid is increased from 4 to 9.

13. $\mathbf{r}(u, v) = 2u \cos v \mathbf{i} + 2u \sin v \mathbf{j} + u^4 \mathbf{k},$

$$0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi$$

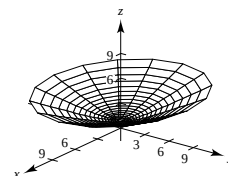
$$z = \frac{(x^2 + y^2)^2}{16}$$



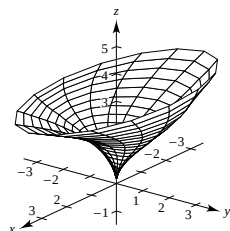
15. $\mathbf{r}(u, v) = 2 \sinh u \cos v \mathbf{i} + \sinh u \sin v \mathbf{j} + \cosh u \mathbf{k},$

$$0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi$$

$$\frac{z^2}{1} - \frac{x^2}{4} - \frac{y^2}{1} = 1$$



17. $\mathbf{r}(u, v) = (u - \sin u) \cos v \mathbf{i} + (1 - \cos u) \sin v \mathbf{j} + u \mathbf{k},$
 $0 \leq u \leq \pi, \quad 0 \leq v \leq 2\pi$



19. $z = y$

$$\mathbf{r}(u, v) = u \mathbf{i} + v \mathbf{j} + v \mathbf{k}$$

21. $x^2 + y^2 = 16$

$$\mathbf{r}(u, v) = 4 \cos u \mathbf{i} + 4 \sin u \mathbf{j} + v \mathbf{k}$$

23. $z = x^2$

$$\mathbf{r}(u, v) = u \mathbf{i} + v \mathbf{j} + u^2 \mathbf{k}$$

25. $z = 4$ inside $x^2 + y^2 = 9$.

$$\mathbf{r}(u, v) = v \cos u \mathbf{i} + v \sin u \mathbf{j} + 4 \mathbf{k}, \quad 0 \leq v \leq 3$$

27. Function: $y = \frac{x}{2}, \quad 0 \leq x \leq 6$

Axis of revolution: x -axis

$$x = u, \quad y = \frac{u}{2} \cos v, \quad z = \frac{u}{2} \sin v$$

$$0 \leq u \leq 6, \quad 0 \leq v \leq 2\pi$$

29. Function: $x = \sin z, \quad 0 \leq z \leq \pi$

Axis of revolution: z -axis

$$x = \sin u \cos v, \quad y = \sin u \sin v, \quad z = u$$

$$0 \leq u \leq \pi, \quad 0 \leq v \leq 2\pi$$

$$31. \mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + v\mathbf{k}, (1, -1, 1)$$

$$\mathbf{r}_u(u, v) = \mathbf{i} + \mathbf{j}, \mathbf{r}_v(u, v) = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

At $(1, -1, 1)$, $u = 0$ and $v = 1$.

$$\mathbf{r}_u(0, 1) = \mathbf{i} + \mathbf{j}, \mathbf{r}_v(0, 1) = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\mathbf{N} = \mathbf{r}_u(0, 1) \times \mathbf{r}_v(0, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

Tangent plane: $(x - 1) - (y + 1) - 2(z - 1) = 0$

$$x - y - 2z = 0$$

(The original plane!)

$$33. \mathbf{r}(u, v) = 2u \cos v \mathbf{i} + 3u \sin v \mathbf{j} + u^2 \mathbf{k}, (0, 6, 4)$$

$$\mathbf{r}_u(u, v) = 2 \cos v \mathbf{i} + 3 \sin v \mathbf{j} + 2u \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -2u \sin v \mathbf{i} + 3u \cos v \mathbf{j}$$

At $(0, 6, 4)$, $u = 2$ and $v = \pi/2$.

$$\mathbf{r}_u\left(2, \frac{\pi}{2}\right) = 3\mathbf{j} + 4\mathbf{k}, \mathbf{r}_v\left(2, \frac{\pi}{2}\right) = -4\mathbf{i}$$

$$\mathbf{N} = \mathbf{r}_u\left(2, \frac{\pi}{2}\right) \times \mathbf{r}_v\left(2, \frac{\pi}{2}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 4 \\ -4 & 0 & 0 \end{vmatrix} = -16\mathbf{j} + 12\mathbf{k}$$

Direction numbers: $0, 4, -3$

Tangent plane: $4(y - 6) - 3(z - 4) = 0$

$$4y - 3z = 12$$

$$35. \mathbf{r}(u, v) = 2u\mathbf{i} - \frac{v}{2}\mathbf{j} + \frac{v}{2}\mathbf{k}, 0 \leq u \leq 2, 0 \leq v \leq 1$$

$$\mathbf{r}_u(u, v) = 2\mathbf{i}, \mathbf{r}_v(u, v) = -\frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\mathbf{j} - \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{2}$$

$$A = \int_0^1 \int_0^2 \sqrt{2} \, du \, dv = 2\sqrt{2}$$

$$37. \mathbf{r}(u, v) = a \cos u \mathbf{i} + a \sin u \mathbf{j} + v\mathbf{k}, 0 \leq u \leq 2\pi, 0 \leq v \leq b$$

$$\mathbf{r}_u(u, v) = -a \sin u \mathbf{i} + a \cos u \mathbf{j}$$

$$\mathbf{r}_v(u, v) = \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin u & a \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = a \cos u \mathbf{i} + a \sin u \mathbf{j}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = a$$

$$A = \int_0^b \int_0^{2\pi} a \, du \, dv = 2\pi ab$$

$$39. \mathbf{r}(u, v) = au \cos v \mathbf{i} + au \sin v \mathbf{j} + u\mathbf{k}, 0 \leq u \leq b, 0 \leq v \leq 2\pi$$

$$\mathbf{r}_u(u, v) = a \cos v \mathbf{i} + a \sin v \mathbf{j} + \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -au \sin v \mathbf{i} + au \cos v \mathbf{j}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos v & a \sin v & 1 \\ -au \sin v & au \cos v & 0 \end{vmatrix} = -au \cos v \mathbf{i} - au \sin v \mathbf{j} + a^2 u \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = au\sqrt{1 + a^2}$$

$$A = \int_0^{2\pi} \int_0^b a\sqrt{1 + a^2} u \, du \, dv = \pi ab^2 \sqrt{1 + a^2}$$

41. $\mathbf{r}(u, v) = \sqrt{u} \cos v \mathbf{i} + \sqrt{u} \sin v \mathbf{j} + u \mathbf{k}, 0 \leq u \leq 4, 0 \leq v \leq 2\pi$

$$\mathbf{r}_u(u, v) = \frac{\cos v}{2\sqrt{u}} \mathbf{i} + \frac{\sin v}{2\sqrt{u}} \mathbf{j} + \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -\sqrt{u} \sin v \mathbf{i} + \sqrt{u} \cos v \mathbf{j}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\cos v}{2\sqrt{u}} & \frac{\sin v}{2\sqrt{u}} & 1 \\ -\sqrt{u} \sin v & \sqrt{u} \cos v & 0 \end{vmatrix} = -\sqrt{u} \cos v \mathbf{i} - \sqrt{u} \sin v \mathbf{j} + \frac{1}{2} \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{u + \frac{1}{4}}$$

$$A = \int_0^{2\pi} \int_0^4 \sqrt{u + \frac{1}{4}} du dv = \frac{\pi}{6} (17\sqrt{17} - 1) \approx 36.177$$

43. See the definition, page 1051.

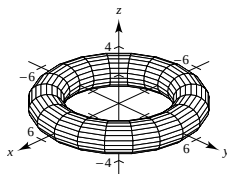
45. (a) From $(-10, 10, 0)$

(b) From $(10, 10, 10)$

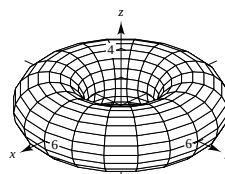
(c) From $(0, 10, 0)$

(d) From $(10, 0, 0)$

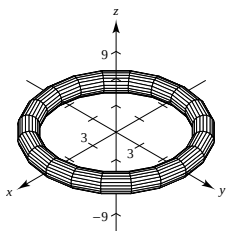
47. (a) $\mathbf{r}(u, v) = (4 + \cos v) \cos u \mathbf{i} + (4 + \cos v) \sin u \mathbf{j} + \sin v \mathbf{k},$
 $0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$



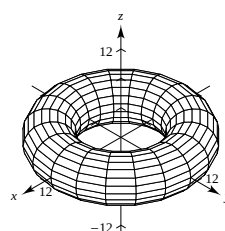
(b) $\mathbf{r}(u, v) = (4 + 2 \cos v) \cos u \mathbf{i} + (4 + 2 \cos v) \sin u \mathbf{j} + 2 \sin v \mathbf{k},$
 $0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$



(c) $\mathbf{r}(u, v) = (8 + \cos v) \cos u \mathbf{i} + (8 + \cos v) \sin u \mathbf{j} + \sin v \mathbf{k},$
 $0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$



(d) $\mathbf{r}(u, v) = (8 + 3 \cos v) \cos u \mathbf{i} + (8 + 3 \cos v) \sin u \mathbf{j} + 3 \sin v \mathbf{k},$
 $0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$



The radius of the generating circle that is revolved about the z -axis is b , and its center is a units from the axis of revolution.

49. $\mathbf{r}(u, v) = 20 \sin u \cos v \mathbf{i} + 20 \sin u \sin v \mathbf{j} + 20 \cos u \mathbf{k}$ $0 \leq u \leq \pi/3$, $0 \leq v \leq 2\pi$

$$\mathbf{r}_u = 20 \cos u \cos v \mathbf{i} + 20 \cos u \sin v \mathbf{j} - 20 \sin u \mathbf{k}$$

$$\mathbf{r}_v = -20 \sin u \sin v \mathbf{i} + 20 \sin u \cos v \mathbf{j}$$

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 20 \cos u \cos v & 20 \cos u \sin v & -20 \sin u \\ -20 \sin u \sin v & 20 \sin u \cos v & 0 \end{vmatrix} \\ &= 400 \sin^2 u \cos v \mathbf{i} + 400 \sin^2 u \sin v \mathbf{j} + 400(\cos u \sin u \cos^2 v + \cos u \sin u \sin^2 v) \mathbf{k} \\ &= 400[\sin^2 u \cos v \mathbf{i} + \sin^2 u \sin v \mathbf{j} + \cos u \sin u \mathbf{k}] \\ \|\mathbf{r}_u \times \mathbf{r}_v\| &= 400 \sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \cos^2 u \sin^2 u} \\ &= 400 \sqrt{\sin^4 u + \cos^2 u \sin^2 u} \\ &= 400 \sqrt{\sin^2 u} = 400 \sin u \end{aligned}$$

$$\begin{aligned} S &= \iint_S dS = \int_0^{2\pi} \int_0^{\pi/3} 400 \sin u \, du \, dv = \int_0^{2\pi} \left[-400 \cos u \right]_0^{\pi/3} dv \\ &= \int_0^{2\pi} 200 \, dv = 400\pi \text{ m}^2 \end{aligned}$$

51. $\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + 2v \mathbf{k}$, $0 \leq u \leq 3$, $0 \leq v \leq 2\pi$

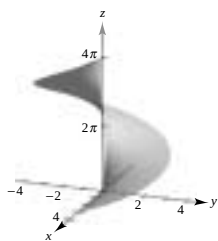
$$\mathbf{r}_u(u, v) = \cos v \mathbf{i} + \sin v \mathbf{j}$$

$$\mathbf{r}_v(u, v) = -u \sin v \mathbf{i} + u \cos v \mathbf{j} + 2 \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 2 \end{vmatrix} = 2 \sin v \mathbf{i} - 2 \cos v \mathbf{j} + u \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{4 + u^2}$$

$$A = \int_0^{2\pi} \int_0^3 \sqrt{4 + u^2} \, du \, dv = \pi \left[3\sqrt{13} + 4 \ln \left(\frac{3 + \sqrt{13}}{2} \right) \right]$$



53. Essay

Section 14.6 Surface Integrals

1. $S: z = 4 - x$, $0 \leq x \leq 4$, $0 \leq y \leq 4$, $\frac{\partial z}{\partial x} = -1$, $\frac{\partial z}{\partial y} = 0$

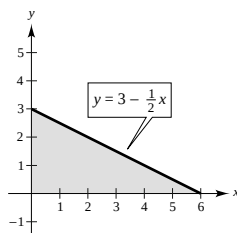
$$\begin{aligned} \iint_S (x - 2y + z) \, dS &= \int_0^4 \int_0^4 (x - 2y + 4 - x) \sqrt{1 + (-1)^2 + (0)^2} \, dy \, dx \\ &= \sqrt{2} \int_0^4 \int_0^4 (4 - 2y) \, dy \, dx = 0 \end{aligned}$$

3. $S: z = 10, x^2 + y^2 \leq 1, \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$

$$\begin{aligned} \iint_S (x - 2y + z) dS &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x - 2y + 10) \sqrt{1 + (0)^2 + (0)^2} dy dx \\ &= \int_0^{2\pi} \int_0^1 (r \cos \theta - 2r \sin \theta + 10) r dr d\theta \\ &= \int_0^{2\pi} \left(\frac{1}{3} \cos \theta - \frac{2}{3} \sin \theta + 5 \right) d\theta \\ &= \left[\frac{1}{3} \sin \theta + \frac{2}{3} \cos \theta + 5\theta \right]_0^{2\pi} = 10\pi \end{aligned}$$

5. $S: z = 6 - x - 2y$, (first octant) $\frac{\partial z}{\partial x} = -1, \frac{\partial z}{\partial y} = -2$

$$\begin{aligned} \iint_S xy dS &= \int_0^6 \int_0^{3-(x/2)} xy \sqrt{1 + (-1)^2 + (-2)^2} dy dx \\ &= \sqrt{6} \int_0^6 \left[\frac{xy^2}{2} \right]_0^{3-(x/2)} dx \\ &= \frac{\sqrt{6}}{2} \int_0^6 x \left(9 - 3x + \frac{1}{4}x^2 \right) dx \\ &= \frac{\sqrt{6}}{2} \left[\frac{9x^2}{2} - x^3 + \frac{x^4}{16} \right]_0^6 = \frac{27\sqrt{6}}{2} \end{aligned}$$



7. $S: z = 9 - x^2, 0 \leq x \leq 2, 0 \leq y \leq x,$

$$\frac{\partial z}{\partial x} = -2x, \frac{\partial z}{\partial y} = 0$$

$$\iint_S xy dS = \int_0^2 \int_y^2 xy \sqrt{1 + 4x^2} dx dy = \frac{391\sqrt{17} + 1}{240}$$

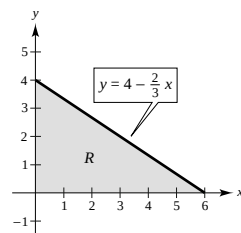
9. $S: z = 10 - x^2 - y^2, 0 \leq x \leq 2, 0 \leq y \leq 2$

$$\iint_S (x^2 - 2xy) dS = \int_0^2 \int_0^2 (x^2 - 2xy) \sqrt{1 + 4x^2 + 4y^2} dy dx \approx -11.47$$

11. $S: 2x + 3y + 6z = 12$ (first octant) $\Rightarrow z = 2 - \frac{1}{3}x - \frac{1}{2}y$

$$\rho(x, y, z) = x^2 + y^2$$

$$\begin{aligned} m &= \iint_R (x^2 + y^2) \sqrt{1 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{2}\right)^2} dA \\ &= \frac{7}{6} \int_0^6 \int_0^{4-(2x/3)} (x^2 + y^2) dy dx \\ &= \frac{7}{6} \int_0^6 \left[x^2 \left(4 - \frac{2}{3}x \right) + \frac{1}{3} \left(4 - \frac{2}{3}x \right)^3 \right] dx = \frac{7}{6} \left[\frac{4}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{8} \left(4 - \frac{2}{3}x \right)^4 \right]_0^6 = \frac{364}{3} \end{aligned}$$



13. $S: \mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \frac{v}{2}\mathbf{k}, 0 \leq u \leq 1, 0 \leq v \leq 2$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \left\| -\frac{1}{2}\mathbf{j} + \mathbf{k} \right\| = \frac{\sqrt{5}}{2}$$

$$\iint_S (y + 5) dS = \int_0^2 \int_0^1 (v + 5) \frac{\sqrt{5}}{2} du dv = 6\sqrt{5}$$

15. $S: \mathbf{r}(u, v) = 2 \cos u \mathbf{i} + 2 \sin u \mathbf{j} + v \mathbf{k}, 0 \leq u \leq \frac{\pi}{2}, 0 \leq v \leq 2$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \|2 \cos u \mathbf{i} + 2 \sin u \mathbf{j}\| = 2$$

$$\iint_S xy dS = \int_0^2 \int_0^{\pi/2} 8 \cos u \sin u du dv = 8$$

17. $f(x, y, z) = x^2 + y^2 + z^2$

$$S: z = x + 2, x^2 + y^2 \leq 1$$

$$\begin{aligned} \iint_S f(x, y, z) dS &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [x^2 + y^2 + (x + 2)^2] \sqrt{1 + (1)^2 + (0)^2} dy dx \\ &= \sqrt{2} \int_0^{2\pi} \int_0^1 [r^2 + (r \cos \theta + 2)^2] r dr d\theta \\ &= \sqrt{2} \int_0^{2\pi} \int_0^1 [r^2 + r^2 \cos^2 \theta + 4r \cos \theta + 4] r dr d\theta \\ &= \sqrt{2} \int_0^{2\pi} \left[\frac{r^4}{4} + \frac{r^4}{4} \cos^2 \theta + \frac{4r^3}{3} \cos \theta + 2r^2 \right]_0^1 d\theta \\ &= \sqrt{2} \int_0^{2\pi} \left[\frac{9}{4} + \left(\frac{1}{4} \right) \frac{1 + \cos 2\theta}{2} + \frac{4}{3} \cos \theta \right] d\theta \\ &= \sqrt{2} \left[\frac{9}{4} \theta + \frac{1}{8} \left(\theta + \frac{1}{2} \sin 2\theta \right) + \frac{4}{3} \sin \theta \right]_0^{2\pi} = \sqrt{2} \left[\frac{18\pi}{4} + \frac{\pi}{4} \right] = \frac{19\sqrt{2}\pi}{4} \end{aligned}$$

19. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$$S: z = \sqrt{x^2 + y^2}, x^2 + y^2 \leq 4$$

$$\begin{aligned} \iint_S f(x, y, z) dS &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2 + (\sqrt{x^2 + y^2})^2} \sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}} \right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}} \right)^2} dy dx \\ &= \sqrt{2} \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2} \sqrt{\frac{x^2 + y^2 + x^2 + y^2}{x^2 + y^2}} dy dx \\ &= 2 \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2} dy dx \\ &= 2 \int_0^{2\pi} \int_0^2 r^2 dr d\theta = 2 \int_0^{2\pi} \left[\frac{r^3}{3} \right]_0^2 d\theta = \left[\frac{16}{3} \theta \right]_0^{2\pi} = \frac{32\pi}{3} \end{aligned}$$

21. $f(x, y, z) = x^2 + y^2 + z^2$

$S: x^2 + y^2 = 9, 0 \leq x \leq 3, 0 \leq y \leq 3, 0 \leq z \leq 9$

Project the solid onto the yz -plane; $x = \sqrt{9 - y^2}, 0 \leq y \leq 3, 0 \leq z \leq 9$.

$$\begin{aligned} \iint_S f(x, y, z) dS &= \int_0^3 \int_0^9 [(9 - y^2) + y^2 + z^2] \sqrt{1 + \left(\frac{y}{\sqrt{9 - y^2}}\right)^2 + (0)^2} dz dy \\ &= \int_0^3 \int_0^9 (9 + z^2) \frac{3}{\sqrt{9 - y^2}} dz dy = \int_0^3 \left[\frac{3}{\sqrt{9 - y^2}} \left(9z + \frac{z^3}{3} \right) \right]_0^9 dy \\ &= 324 \int_0^3 \frac{3}{\sqrt{9 - y^2}} dy = \left[972 \arcsin\left(\frac{y}{3}\right) \right]_0^3 = 972 \left(\frac{\pi}{2} - 0 \right) = 486\pi \end{aligned}$$

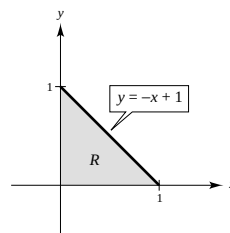
23. $\mathbf{F}(x, y, z) = 3z\mathbf{i} - 4\mathbf{j} + y\mathbf{k}$

$S: x + y + z = 1$ (first octant)

$G(x, y, z) = x + y + z - 1$

$\nabla G(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{N} dS &= \iint_R \mathbf{F} \cdot \nabla G dA = \int_0^1 \int_0^{1-x} (3z - 4 + y) dy dx \\ &= \int_0^1 \int_0^{1-x} [3(1 - x - y) - 4 + y] dy dx \\ &= \int_0^1 \int_0^{1-x} (-1 - 3x - 2y) dy dx \\ &= \int_0^1 \left[-y - 3xy - y^2 \right]_0^{1-x} dx \\ &= - \int_0^1 [(1 - x) + 3x(1 - x) + (1 - x)^2] dx \\ &= - \int_0^1 (2 - 2x^2) dx = -\frac{4}{3} \end{aligned}$$



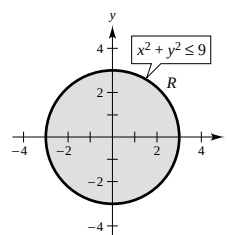
25. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$S: z = 9 - x^2 - y^2, 0 \leq z$

$G(x, y, z) = x^2 + y^2 + z - 9$

$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{N} dS &= \iint_R \mathbf{F} \cdot \nabla G dA = \iint_R (2x^2 + 2y^2 + z) dA \\ &= \iint_R [2x^2 + 2y^2 + (9 - x^2 - y^2)] dA \\ &= \iint_R (x^2 + y^2 + 9) dA \\ &= \int_0^{2\pi} \int_0^3 (r^2 + 9)r dr d\theta \\ &= \int_0^{2\pi} \left[\frac{r^4}{4} + \frac{9r^2}{2} \right]_0^3 d\theta = \frac{243\pi}{2} \end{aligned}$$

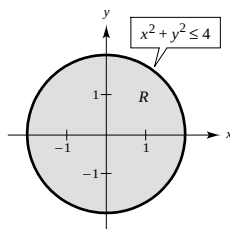


27. $\mathbf{F}(x, y, z) = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

$S: z = x^2 + y^2, x^2 + y^2 \leq 4$

$G(x, y, z) = -x^2 - y^2 + z$

$\nabla G(x, y, z) = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$



$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{N} \, dS &= \iint_R \mathbf{F} \cdot \nabla G \, dA = \iint_R (-8x + 6y + 5) \, dA \\ &= \int_0^{2\pi} \int_0^2 [-8r \cos \theta + 6r \sin \theta + 5] r \, dr \, d\theta \\ &= \int_0^{2\pi} \left[-\frac{8}{3} r^3 \cos \theta + 2r^3 \sin \theta + \frac{5}{2} r^2 \right]_0^2 d\theta \\ &= \int_0^{2\pi} \left[-\frac{64}{3} \cos \theta + 16 \sin \theta + 10 \right] d\theta \\ &= \left[-\frac{64}{3} \sin \theta - 16 \cos \theta + 10\theta \right]_0^{2\pi} = 20\pi \end{aligned}$$

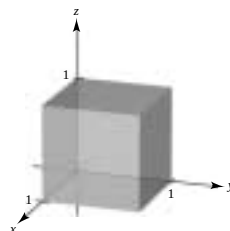
29. $\mathbf{F}(x, y, z) = 4xy\mathbf{i} + z^2\mathbf{j} + yz\mathbf{k}$

S : unit cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

S_1 : The top of the cube

$\mathbf{N} = \mathbf{k}, z = 1$

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{N} \, dS = \int_0^1 \int_0^1 y(1) \, dy \, dx = \frac{1}{2}$$



S_2 : The bottom of the cube

$\mathbf{N} = -\mathbf{k}, z = 0$

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{N} \, dS = \int_0^1 \int_0^1 -y(0) \, dy \, dx = 0$$

S_4 : The back of the cube

$\mathbf{N} = -\mathbf{i}, x = 0$

$$\iint_{S_4} \mathbf{F} \cdot \mathbf{N} \, dS = \int_0^1 \int_0^1 -4(0)y \, dy \, dx = 0$$

S_6 : The left side of the cube

$\mathbf{N} = -\mathbf{j}, y = 0$

$$\iint_{S_6} \mathbf{F} \cdot \mathbf{N} \, dS = \int_0^1 \int_0^1 -z^2 \, dz \, dx = -\frac{1}{3}$$

S_3 : The front of the cube

$\mathbf{N} = \mathbf{i}, x = 1$

$$\iint_{S_3} \mathbf{F} \cdot \mathbf{N} \, dS = \int_0^1 \int_0^1 4(1)y \, dy \, dz = 2$$

S_5 : The right side of the cube

$\mathbf{N} = \mathbf{j}, y = 1$

$$\iint_{S_5} \mathbf{F} \cdot \mathbf{N} \, dS = \int_0^1 \int_0^1 z^2 \, dz \, dx = \frac{1}{3}$$

Therefore,

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \frac{1}{2} + 0 + 2 + 0 + \frac{1}{3} - \frac{1}{3} = \frac{5}{2}.$$

31. The surface integral of f over a surface S , where S is given by $z = g(x, y)$, is defined as

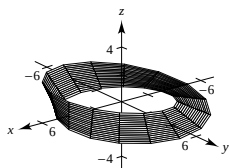
$$\iint_S f(x, y, z) \, dS = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(x_i, y_i, z_i) \Delta S_i. \quad (\text{page 1061})$$

See Theorem 14.10, page 1061.

33. See the definition, page 1067.

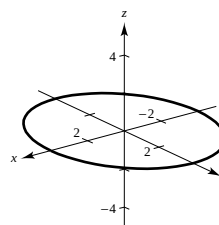
See Theorem 14.11, page 1067.

35. (a)


 (b) If a normal vector at a point P on the surface is moved around the Möbius strip once, it will point in the opposite direction.

 (c) $\mathbf{r}(u, 0) = 4 \cos(2u)\mathbf{i} + 4 \sin(2u)\mathbf{j}$

This is a circle.



(d) (construction)

(e) You obtain a strip with a double twist and twice as long as the original Möbius strip.

 37. $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq a$

$$m = \iint_S k \, dS = k \iint_R \sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2} \, dA = k \iint_R \sqrt{2} \, dA = \sqrt{2} k \pi a^2$$

$$I_z = \iint_S k(x^2 + y^2) \, dS = \iint_R k(x^2 + y^2) \sqrt{2} \, dA$$

$$= \sqrt{2} k \int_0^{2\pi} \int_0^a r^3 \, dr \, d\theta = \frac{\sqrt{2} k a^4}{4} (2\pi)$$

$$= \frac{\sqrt{2} k \pi a^4}{2} = \frac{a^2}{2} (\sqrt{2} k \pi a^2) = \frac{a^2 m}{2}$$

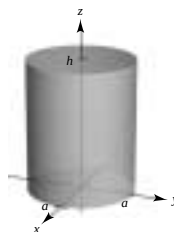
 39. $x^2 + y^2 = a^2$, $0 \leq z \leq h$

$$\rho(x, y, z) = 1$$

$$y = \pm \sqrt{a^2 - x^2}$$

 Project the solid onto the xz -plane.

$$\begin{aligned} I_z &= 4 \iint_S (x^2 + y^2)(1) \, dS \\ &= 4 \int_0^h \int_0^a [x^2 + (a^2 - x^2)] \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2 + (0)^2} \, dx \, dz \\ &= 4a^3 \int_0^h \int_0^a \frac{1}{\sqrt{a^2 - x^2}} \, dx \, dz \\ &= 4a^3 \int_0^h \left[\arcsin \frac{x}{a} \right]_0^a \, dz = 4a^3 \left(\frac{\pi}{2} \right) (h) = 2\pi a^3 h \end{aligned}$$


 41. $S: z = 16 - x^2 - y^2$, $z \geq 0$

$$\mathbf{F}(x, y, z) = 0.5z\mathbf{k}$$

$$\begin{aligned} \iint_S \rho \mathbf{F} \cdot \mathbf{N} \, dS &= \iint_R \rho \mathbf{F} \cdot (-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}) \, dA = \iint_R 0.5\rho z \mathbf{k} \cdot (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}) \, dA \\ &= \iint_R 0.5\rho z \, dA = \iint_R 0.5\rho(16 - x^2 - y^2) \, dA \\ &= 0.5\rho \int_0^{2\pi} \int_0^4 (16 - r^2)r \, dr \, d\theta = 0.5\rho \int_0^{2\pi} 64 \, d\theta = 64\pi\rho \end{aligned}$$

Section 14.7 Divergence Theorem

1. Surface Integral: There are six surfaces to the cube, each with $dS = \sqrt{1} dA$.

$$\begin{aligned}
 z = 0, \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = -z^2, \quad \int_{S_1} \int 0 dA &= 0 \\
 z = a, \quad \mathbf{N} = \mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = z^2, \quad \int_{S_2} \int a^2 dA &= \int_0^a \int_0^a a^2 dx dy = a^4 \\
 x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = -2x, \quad \int_{S_3} \int 0 dA &= 0 \\
 x = a, \quad \mathbf{N} = \mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = 2x, \quad \int_{S_4} \int 2a dy dz &= \int_0^a \int_0^a 2a dy dz = 2a^3 \\
 y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = 2y, \quad \int_{S_5} \int 0 dA &= 0 \\
 y = a, \quad \mathbf{N} = \mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = -2y, \quad \int_{S_6} \int -2a dA &= \int_0^a \int_0^a -2a dz dx = -2a^3
 \end{aligned}$$

Therefore, $\int_S \mathbf{F} \cdot \mathbf{N} dS = a^4 + 2a^3 - 2a^3 = a^4$.

Divergence Theorem: Since $\text{div } \mathbf{F} = 2z$, the Divergence Theorem yields

$$\iiint_Q \text{div } \mathbf{F} dV = \int_0^a \int_0^a \int_0^a 2z dz dy dx = \int_0^a \int_0^a a^2 dy dx = a^4.$$

3. Surface Integral: There are four surfaces to this solid.

$$z = 0, \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = -z$$

$$\int_{S_1} \int 0 dS = 0$$

$$y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = 2y - z, \quad dS = dA = dx dz$$

$$\int_{S_2} \int -z dS = \int_0^6 \int_0^{6-z} -z dx dz = \int_0^6 (z^2 - 6z) dz = -36$$

$$x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = y - 2x, \quad dS = dA = dz dy$$

$$\int_{S_3} \int y dS = \int_0^3 \int_0^{6-2y} y dz dy = \int_0^3 (6y - 2y^2) dy = 9$$

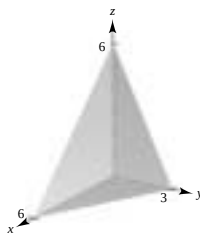
$$x + 2y + z = 6, \quad \mathbf{N} = \frac{\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{6}}, \quad \mathbf{F} \cdot \mathbf{N} = \frac{2x - 5y + 3z}{\sqrt{6}}, \quad dS = \sqrt{6} dA$$

$$\int_{S_4} \int (2x - 5y + 3z) dz dy = \int_0^3 \int_0^{6-2y} (18 - x - 11y) dx dy = \int_0^3 (90 - 90y + 20y^2) dy = 45$$

Therefore, $\int_S \mathbf{F} \cdot \mathbf{N} dS = 0 - 36 + 9 + 45 = 18$.

Divergence Theorem: Since $\text{div } \mathbf{F} = 1$, we have

$$\iiint_Q dV = (\text{Volume of solid}) = \frac{1}{3}(\text{Area of base}) \times (\text{Height}) = \frac{1}{3}(9)(6) = 18.$$



5. Since $\operatorname{div} \mathbf{F} = 2x + 2y + 2z$, we have

$$\begin{aligned}\iint_Q \operatorname{div} \mathbf{F} \, dV &= \int_0^a \int_0^a \int_0^a (2x + 2y + 2z) \, dz \, dy \, dx \\ &= \int_0^a \int_0^a (2ax + 2ay + a^2) \, dy \, dx = \int_0^a (2a^2x + 2a^3) \, dx = \left[a^2x^2 + 2a^3x \right]_0^a = 3a^4.\end{aligned}$$

7. Since $\operatorname{div} \mathbf{F} = 2x - 2x + 2xyz = 2xyz$

$$\begin{aligned}\iint_Q \operatorname{div} \mathbf{F} \, dV &= \iiint_Q 2xyz \, dV = \int_0^a \int_0^{2\pi} \int_0^{\pi/2} 2(\rho \sin \phi \cos \theta)(\rho \sin \phi \sin \theta)(\rho \cos \phi) \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho \\ &= \int_0^a \int_0^{2\pi} \int_0^{\pi/2} 2\rho^5 (\sin \theta \cos \theta)(\sin^3 \phi \cos \phi) \, d\phi \, d\theta \, d\rho \\ &= \int_0^a \int_0^{2\pi} \frac{1}{2} \rho^5 \sin \theta \cos \theta \, d\theta \, d\rho = \int_0^a \left[\left(\frac{\rho^5}{2} \right) \frac{\sin^2 \theta}{2} \right]_0^{2\pi} d\rho = 0.\end{aligned}$$

9. Since $\operatorname{div} \mathbf{F} = 3$, we have

$$\iiint_Q 3 \, dV = 3(\text{Volume of sphere}) = 3 \left[\frac{4}{3} \pi (2)^3 \right] = 32\pi.$$

11. Since $\operatorname{div} \mathbf{F} = 1 + 2y - 1 = 2y$, we have

$$\iiint_Q 2y \, dV = \int_0^4 \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} 2y \, dx \, dy \, dz = \int_0^4 \int_{-3}^3 4y \sqrt{9-y^2} \, dy \, dz = \int_0^4 \left[-\frac{4}{3} (9-y^2)^{3/2} \right]_{-3}^3 dz = 0.$$

13. Since $\operatorname{div} \mathbf{F} = 3x^2 + x^2 + 0 = 4x^2$, we have

$$\iiint_Q 4x^2 \, dV = \int_0^6 \int_0^4 \int_0^{4-y} 4x^2 \, dz \, dy \, dx = \int_0^6 \int_0^4 4x^2(4-y) \, dy \, dx = \int_0^6 32x^2 \, dx = 2304.$$

15. $\mathbf{F}(x, y, z) = xy\mathbf{i} + 4y\mathbf{j} + xz\mathbf{k}$

$$\operatorname{div} \mathbf{F} = y + 4 + x$$

$$\begin{aligned}\iint_S \mathbf{F} \cdot \mathbf{N} \, dS &= \iiint_Q \operatorname{div} \mathbf{F} \, dV = \iiint_Q (y + x + 4) \, dV \\ &= \int_0^3 \int_0^\pi \int_0^{2\pi} (\rho \sin \phi \sin \theta + \rho \sin \phi \cos \theta + 4) \rho^2 \sin \phi \, d\theta \, d\phi \, d\rho \\ &= \int_0^3 \int_0^\pi \int_0^{2\pi} [\rho^3 \sin^2 \phi \sin \theta + \rho^3 \sin^2 \phi \cos \theta + 4\rho^2 \sin \phi] \, d\theta \, d\phi \, d\rho \\ &= \int_0^3 \int_0^\pi \left[-\rho^3 \sin^2 \phi \cos \theta + \rho^3 \sin^2 \phi \sin \theta + 4\rho^2 \sin \phi \cdot \theta \right]_0^{2\pi} d\phi \, d\rho \\ &= \int_0^3 \int_0^\pi 8\pi \rho^2 \sin \phi \, d\phi \, d\rho \\ &= \int_0^3 \left[-8\pi \rho^2 \cos \phi \right]_0^\pi d\rho \\ &= \int_0^3 16\pi \rho^2 \, d\rho = \left[\frac{16\pi \rho^3}{3} \right]_0^3 = 144\pi.\end{aligned}$$

17. Using the Divergence Theorem, we have

$$\begin{aligned}\iint_S \mathbf{curl} \mathbf{F} \cdot \mathbf{N} \, dS &= \iiint_Q \operatorname{div}(\mathbf{curl} \mathbf{F}) \, dV \\ \mathbf{curl} \mathbf{F}(x, y, z) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy + z^2 & 2x^2 + 6yz & 2xz \end{vmatrix} = -6y\mathbf{i} - (2z - 2z)\mathbf{j} + (4x - 4x)\mathbf{k} = -6y\mathbf{i} \\ \operatorname{div}(\mathbf{curl} \mathbf{F}) &= 0.\end{aligned}$$

Therefore, $\iiint_Q \operatorname{div}(\mathbf{curl} \mathbf{F}) \, dV = 0.$

19. See Theorem 14.12, page 1073.

21. Using the triple integral to find volume, we need $\mathbf{F}$ so that

$$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = 1.$$

Hence, we could have $\mathbf{F} = x\mathbf{i}$, $\mathbf{F} = y\mathbf{j}$, or $\mathbf{F} = z\mathbf{k}$.

For $dA = dy \, dz$ consider $\mathbf{F} = x\mathbf{i}$, $x = f(y, z)$, then $\mathbf{N} = \frac{\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}}{\sqrt{1 + f_y^2 + f_z^2}}$ and $dS = \sqrt{1 + f_y^2 + f_z^2} \, dy \, dz$.

For $dA = dz \, dx$ consider $\mathbf{F} = y\mathbf{j}$, $y = f(x, z)$, then $\mathbf{N} = \frac{f_x\mathbf{i} + \mathbf{j} + f_z\mathbf{k}}{\sqrt{1 + f_x^2 + f_z^2}}$ and $dS = \sqrt{1 + f_x^2 + f_z^2} \, dz \, dx$.

For $dA = dx \, dy$ consider $\mathbf{F} = z\mathbf{k}$, $z = f(x, y)$, then $\mathbf{N} = \frac{f_x\mathbf{i} + f_y\mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}}$ and $dS = \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy$.

Correspondingly, we then have $V = \iiint_S \mathbf{F} \cdot \mathbf{N} \, dS = \iiint_S x \, dy \, dz = \iiint_S y \, dz \, dx = \iiint_S z \, dx \, dy.$

23. Using the Divergence Theorem, we have $\iint_S \mathbf{curl} \mathbf{F} \cdot \mathbf{N} \, dS = \iiint_Q \operatorname{div}(\mathbf{curl} \mathbf{F}) \, dV$. Let

$$\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$$

$$\mathbf{curl} \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}$$

$$\operatorname{div}(\mathbf{curl} \mathbf{F}) = \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 M}{\partial y \partial z} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 0.$$

Therefore, $\iint_S \mathbf{curl} \mathbf{F} \cdot \mathbf{N} \, dS = \iiint_Q 0 \, dV = 0.$

25. If $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\operatorname{div} \mathbf{F} = 3$.

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \iiint_Q \operatorname{div} \mathbf{F} \, dV = \iiint_Q 3 \, dV = 3V.$$

27. $\iint_S f D_{\mathbf{N}} g \, dS = \iint_S f \nabla g \cdot \mathbf{N} \, dS$

$$= \iiint_Q \operatorname{div}(f \nabla g) \, dV = \iiint_Q (f \operatorname{div} \nabla g + \nabla f \cdot \nabla g) \, dV = \iiint_Q (f \nabla^2 g + \nabla f \cdot \nabla g) \, dV$$

Section 14.8 Stokes's Theorem

1. $\mathbf{F}(x, y, z) = (2y - z)\mathbf{i} + xyz\mathbf{j} + e^z\mathbf{k}$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y - z & xyz & e^z \end{vmatrix} = -xy\mathbf{i} - \mathbf{j} + (yz - 2)\mathbf{k}$$

3. $\mathbf{F}(x, y, z) = 2z\mathbf{i} - 4x^2\mathbf{j} + \arctan x\mathbf{k}$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & -4x^2 & \arctan x \end{vmatrix} = \left(2 - \frac{1}{1+x^2}\right)\mathbf{j} - 8x\mathbf{k}$$

5. $\mathbf{F}(x, y, z) = e^{x^2+y^2}\mathbf{i} + e^{y^2+z^2}\mathbf{j} + xyz\mathbf{k}$

$$\begin{aligned} \mathbf{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x^2+y^2} & e^{y^2+z^2} & xyz \end{vmatrix} \\ &= (xz - 2ze^{y^2+z^2})\mathbf{i} - yz\mathbf{j} - 2ye^{x^2+y^2}\mathbf{k} \\ &= z(x - 2e^{y^2+z^2})\mathbf{i} - yz\mathbf{j} - 2ye^{x^2+y^2}\mathbf{k} \end{aligned}$$

7. In this case, $M = -y + z$, $N = x - z$, $P = x - y$ and C is the circle $x^2 + y^2 = 1$, $z = 0$, $dz = 0$.

Line Integral: $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C -y dx + x dy$

Letting $x = \cos t$, $y = \sin t$, we have $dx = -\sin t dt$, $dy = \cos t dt$ and

$$\int_C -y dx + x dy = \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = 2\pi.$$

Double Integral: Consider $F(x, y, z) = x^2 + y^2 + z^2 - 1$.

Then

$$\mathbf{N} = \frac{\nabla F}{\|\nabla F\|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Since

$$z^2 = 1 - x^2 - y^2, \quad z_x = \frac{-2x}{2z} = \frac{-x}{z}, \quad \text{and} \quad z_y = \frac{-y}{z}, \quad dS = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} dA = \frac{1}{z} dA.$$

Now, since $\mathbf{curl} \mathbf{F} = 2\mathbf{k}$, we have

$$\int_S (\mathbf{curl} \mathbf{F}) \cdot \mathbf{N} dS = \int_R \int 2z \left(\frac{1}{z}\right) dA = \int_R \int 2 dA = 2(\text{Area of circle of radius 1}) = 2\pi.$$

9. Line Integral: From the accompanying figure we see that for

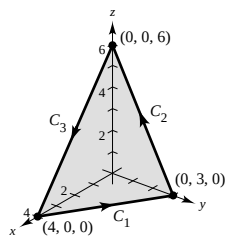
$$C_1: z = 0, \quad dz = 0$$

$$C_2: x = 0, \quad dx = 0$$

$$C_3: y = 0, \quad dy = 0.$$

$$\text{Hence, } \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C xyz \, dx + y \, dy + z \, dz$$

$$\begin{aligned} &= \int_{C_1} y \, dy + \int_{C_2} y \, dy + z \, dz + \int_{C_3} z \, dz \\ &= \int_0^3 y \, dy + \int_3^0 y \, dy + \int_0^6 z \, dz + \int_6^0 z \, dz = 0. \end{aligned}$$



Double Integral: $\text{curl } \mathbf{F} = xy\mathbf{j} - xz\mathbf{k}$

Considering $F(x, y, z) = 3x + 4y + 2z - 12$, then

$$\mathbf{N} = \frac{\nabla F}{\|\nabla F\|} = \frac{3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}}{\sqrt{29}} \text{ and } dS = \sqrt{29} \, dA.$$

Thus,

$$\begin{aligned} \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} \, dS &= \iint_R (4xy - 2xz) \, dy \, dx \\ &= \int_0^4 \int_0^{(-3x+12)/4} \left[4xy - 2x \left(6 - 2y - \frac{3}{2}x \right) \right] dy \, dx \\ &= \int_0^4 \int_0^{(12-3x)/4} (8xy + 3x^2 - 12x) \, dy \, dx \\ &= \int_0^4 0 \, dx = 0. \end{aligned}$$

11. Let $A = (0, 0, 0)$, $B = (1, 1, 1)$ and $C = (0, 2, 0)$. Then $\mathbf{U} = \overrightarrow{AB} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{V} = \overrightarrow{AC} = 2\mathbf{j}$. Thus,

$$\mathbf{N} = \frac{\mathbf{U} \times \mathbf{V}}{\|\mathbf{U} \times \mathbf{V}\|} = \frac{-2\mathbf{i} + 2\mathbf{k}}{2\sqrt{2}} = \frac{-\mathbf{i} + \mathbf{k}}{\sqrt{2}}.$$

Surface S has direction numbers $-1, 0, 1$, with equation $z - x = 0$ and $dS = \sqrt{2} \, dA$. Since $\text{curl } \mathbf{F} = -3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, we have

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} \, dS = \iint_R \frac{1}{\sqrt{2}} (\sqrt{2}) \, dA = \iint_R dA = (\text{Area of triangle with } a = 1, b = 2) = 1.$$

13. $\mathbf{F}(x, y, z) = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$, $S: z = 4 - x^2 - y^2$, $0 \leq z$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x^2 & y^2 \end{vmatrix} = 2y\mathbf{i} + 2z\mathbf{j} + 2x\mathbf{k}$$

$$G(x, y, z) = x^2 + y^2 + z - 4$$

$$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$$

$$\begin{aligned} \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} \, dS &= \iint_R (4xy + 4yz + 2x) \, dA = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [4xy + 4y(4 - x^2 - y^2) + 2x] \, dy \, dx \\ &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [4xy + 16y - 4x^2y - 4y^3 + 2x] \, dy \, dx \\ &= \int_{-2}^2 4x\sqrt{4-x^2} \, dx = 0 \end{aligned}$$

15. $\mathbf{F}(x, y, z) = z^2\mathbf{i} + y\mathbf{j} + xz\mathbf{k}$, $S: z = \sqrt{4 - x^2 - y^2}$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y & xz \end{vmatrix} = z\mathbf{j}$$

$$G(x, y, z) = z - \sqrt{4 - x^2 - y^2}$$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\int_S \int (\mathbf{curl} \mathbf{F}) \cdot \mathbf{F} \, dS = \int_R \int \frac{yz}{\sqrt{4 - x^2 - y^2}} \, dA = \int_R \int \frac{y\sqrt{4 - x^2 - y^2}}{\sqrt{4 - x^2 - y^2}} \, dA = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y \, dy \, dx = 0$$

17. $\mathbf{F}(x, y, z) = -\ln\sqrt{x^2 + y^2}\mathbf{i} + \arctan\frac{x}{y}\mathbf{j} + \mathbf{k}$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -1/2 \ln(x^2 + y^2) & \arctan x/y & 1 \end{vmatrix} = \left[\frac{(1/y)}{1 + (x^2/y^2)} + \frac{y}{x^2 + y^2} \right] \mathbf{k} = \left[\frac{2y}{x^2 + y^2} \right] \mathbf{k}$$

$S: z = 9 - 2x - 3y$ over one petal of $r = 2 \sin 2\theta$ in the first octant.

$$G(x, y, z) = 2x + 3y + z - 9$$

$$\nabla G(x, y, z) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\begin{aligned} \int_S \int (\mathbf{curl} \mathbf{F}) \cdot \mathbf{N} \, dS &= \int_R \int \frac{2y}{x^2 + y^2} \, dA \\ &= \int_0^{\pi/2} \int_0^{2 \sin 2\theta} \frac{2r \sin \theta}{r^2} r \, dr \, d\theta \\ &= \int_0^{\pi/2} \int_0^{4 \sin \theta \cos \theta} 2 \sin \theta \, dr \, d\theta \\ &= \int_0^{\pi/2} 8 \sin^2 \theta \cos \theta \, d\theta = \left[\frac{8 \sin^3 \theta}{3} \right]_0^{\pi/2} = \frac{8}{3} \end{aligned}$$

19. From Exercise 10, we have $\mathbf{N} = \frac{2x\mathbf{i} - \mathbf{k}}{\sqrt{1 + 4x^2}}$ and $dS = \sqrt{1 + 4x^2} \, dA$. Since $\mathbf{curl} \mathbf{F} = xy\mathbf{j} - xz\mathbf{k}$, we have

$$\int_S \int (\mathbf{curl} \mathbf{F}) \cdot \mathbf{N} \, dS = \int_R \int xz \, dA = \int_0^a \int_0^a x^3 \, dy \, dx = \int_0^a ax^3 \, dx = \left[\frac{ax^4}{4} \right]_0^a = \frac{a^5}{4}.$$

21. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 1 & -2 \end{vmatrix} = \mathbf{0}$$

Letting $\mathbf{N} = \mathbf{k}$, we have $\int_S \int (\mathbf{curl} \mathbf{F}) \cdot \mathbf{N} \, dS = 0$.

23. See Theorem 14.13, page 1081.

25. (a) $\int_C f \nabla g \cdot d\mathbf{r} = \int_S \int \mathbf{curl}[f \nabla g] \cdot \mathbf{N} dS$ (Stoke's Theorem)

$$\begin{aligned}
 f \nabla g &= f \frac{\partial g}{\partial x} \mathbf{i} + f \frac{\partial g}{\partial y} \mathbf{j} + f \frac{\partial g}{\partial z} \mathbf{k} \\
 \mathbf{curl}(f \nabla g) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(\partial g/\partial x) & f(\partial g/\partial y) & f(\partial g/\partial z) \end{vmatrix} \\
 &= \left[\left[f \left(\frac{\partial^2 g}{\partial y \partial z} \right) + \left(\frac{\partial f}{\partial y} \right) \left(\frac{\partial g}{\partial z} \right) \right] - \left[f \left(\frac{\partial^2 g}{\partial z \partial y} \right) + \left(\frac{\partial f}{\partial z} \right) \left(\frac{\partial g}{\partial y} \right) \right] \right] \mathbf{i} \\
 &\quad - \left[\left[f \left(\frac{\partial^2 g}{\partial x \partial z} \right) + \left(\frac{\partial f}{\partial x} \right) \left(\frac{\partial g}{\partial z} \right) \right] - \left[f \left(\frac{\partial^2 g}{\partial z \partial x} \right) + \left(\frac{\partial f}{\partial z} \right) \left(\frac{\partial g}{\partial x} \right) \right] \right] \mathbf{j} \\
 &\quad + \left[\left[f \left(\frac{\partial^2 g}{\partial x \partial y} \right) + \left(\frac{\partial f}{\partial x} \right) \left(\frac{\partial g}{\partial y} \right) \right] - \left[f \left(\frac{\partial^2 g}{\partial y \partial x} \right) + \left(\frac{\partial f}{\partial y} \right) \left(\frac{\partial g}{\partial x} \right) \right] \right] \mathbf{k} \\
 &= \left[\left(\frac{\partial f}{\partial y} \right) \left(\frac{\partial g}{\partial z} \right) - \left(\frac{\partial f}{\partial z} \right) \left(\frac{\partial g}{\partial y} \right) \right] \mathbf{i} - \left[\left(\frac{\partial f}{\partial x} \right) \left(\frac{\partial g}{\partial z} \right) - \left(\frac{\partial f}{\partial z} \right) \left(\frac{\partial g}{\partial x} \right) \right] \mathbf{j} + \left[\left(\frac{\partial f}{\partial x} \right) \left(\frac{\partial g}{\partial y} \right) - \left(\frac{\partial f}{\partial y} \right) \left(\frac{\partial g}{\partial x} \right) \right] \mathbf{k} \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix} = \nabla f \times \nabla g
 \end{aligned}$$

Therefore, $\int_C f \nabla g \cdot d\mathbf{r} = \int_S \int \mathbf{curl}[f \nabla g] \cdot \mathbf{N} dS = \int_S \int [\nabla f \times \nabla g] \cdot \mathbf{N} dS$.

(b) $\int_C (f \nabla f) \cdot d\mathbf{r} = \int_S \int (\nabla f \times \nabla f) \cdot \mathbf{N} dS$ (using part a.)
 $= 0$ since $\nabla f \times \nabla f = 0$.

(c) $\int_C (f \nabla g + g \nabla f) \cdot d\mathbf{r} = \int_C (f \nabla g) \cdot d\mathbf{r} + \int_C (g \nabla f) \cdot d\mathbf{r}$
 $= \int_S \int (\nabla f \times \nabla g) \cdot \mathbf{N} dS + \int_S \int (\nabla g \times \nabla f) \cdot \mathbf{N} dS$ (using part a.)
 $= \int_S \int (\nabla f \times \nabla g) \cdot \mathbf{N} dS + \int_S \int -(\nabla f \times \nabla g) \cdot \mathbf{N} dS = 0$

27. Let $\mathbf{C} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then

$$\frac{1}{2} \int_C (\mathbf{C} \times \mathbf{r}) \cdot d\mathbf{r} = \frac{1}{2} \int_S \int \mathbf{curl}(\mathbf{C} \times \mathbf{r}) \cdot \mathbf{N} dS = \frac{1}{2} \int_S \int 2\mathbf{C} \cdot \mathbf{N} dS = \int_S \int \mathbf{C} \cdot \mathbf{N} dS$$

since

$$\mathbf{C} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy)\mathbf{i} - (az - cx)\mathbf{j} + (ay - bx)\mathbf{k}$$

and

$$\mathbf{curl}(\mathbf{C} \times \mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz - cy & cx - az & ay - bx \end{vmatrix} = 2(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = 2\mathbf{C}.$$

CHAPTER 14

Vector Analysis

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CHAPTER 14

Vector Analysis

Section 14.1 Vector Fields

Solutions to Even-Numbered Exercises

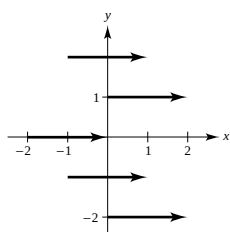
2. All vectors are parallel to x -axis.
Matches (d)

4. Vectors are in rotational pattern.
Matches (e)

6. Vectors along x -axis have no x -component.
Matches (f)

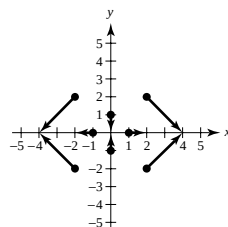
8. $\mathbf{F}(x, y) = 2\mathbf{i}$

$$\|\mathbf{F}\| = 2$$



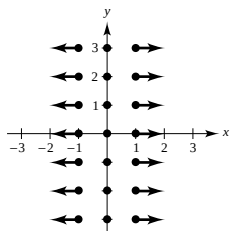
10. $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$

$$\|\mathbf{F}\| = \sqrt{x^2 + y^2}$$



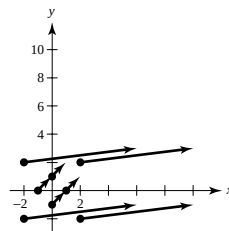
12. $\mathbf{F}(x, y) = x\mathbf{i}$

$$\|\mathbf{F}\| = |x| = c$$



14. $\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} + \mathbf{j}$

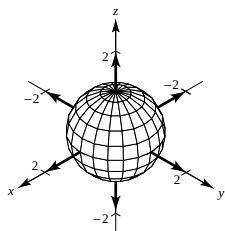
$$\|\mathbf{F}\| = \sqrt{1 + (x^2 + y^2)^2}$$



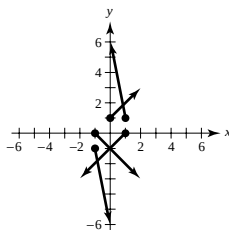
16. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\|\mathbf{F}\| = \sqrt{x^2 + y^2 + z^2} = c$$

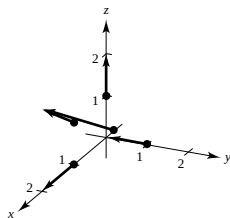
$$x^2 + y^2 + z^2 = c^2$$



18. $\mathbf{F}(x, y) = (2y - 3x)\mathbf{i} + (2y + 3x)\mathbf{j}$



20. $\mathbf{F}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$



24. $f(x, y, z) = \frac{y}{z} + \frac{z}{x} - \frac{xz}{y}$

$$f_x(x, y, z) = -\frac{z}{x^2} - \frac{z}{y}$$

$$f_y(x, y, z) = \frac{1}{z} + \frac{xz}{y^2}$$

$$f_z(x, y, z) = -\frac{y}{z^2} + \frac{1}{x} - \frac{x}{y}$$

$$\mathbf{F}(x, y, z) = \left(-\frac{z}{x^2} - \frac{z}{y}\right)\mathbf{i} + \left(\frac{1}{z} + \frac{xz}{y^2}\right)\mathbf{j} + \left(-\frac{y}{z^2} + \frac{1}{x} - \frac{x}{y}\right)\mathbf{k}$$

28. $\mathbf{F}(x, y) = \frac{1}{x^2}(y\mathbf{i} - x\mathbf{j}) = \frac{y}{x^2}\mathbf{i} - \frac{1}{x}\mathbf{j}$

$M = y/x^2$ and $N = -(1/x)$ have continuous first partial derivatives for all $x \neq 0$.

$$\frac{\partial N}{\partial x} = \frac{1}{x^2} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} \text{ is conservative.}$$

32. $M = \frac{x}{\sqrt{x^2 + y^2}}, N = \frac{y}{\sqrt{x^2 + y^2}}$

$$\frac{\partial N}{\partial x} = \frac{-xy}{(x^2 + y^2)^{3/2}} = \frac{\partial M}{\partial y} \Rightarrow \text{Conservative}$$

36. $\mathbf{F}(x, y) = \frac{1}{y^2}(y\mathbf{i} - 2x\mathbf{j})$

$$= \frac{1}{y}\mathbf{i} - \frac{2x}{y^2}\mathbf{j}$$

$$\frac{\partial}{\partial y}\left[\frac{1}{y}\right] = -\frac{1}{y^2}$$

$$\frac{\partial}{\partial x}\left[-\frac{2x}{y^2}\right] = -\frac{2}{y^2}$$

Not conservative

22. $f(x, y) = \sin 3x \cos 4y$

$$f_x(x, y) = 3 \cos 3x \cos 4y$$

$$f_y(x, y) = -4 \sin 3x \sin 4y$$

$$\mathbf{F}(x, y) = 3 \cos 3x \cos 4y\mathbf{i} - 4 \sin 3x \sin 4y\mathbf{j}$$

26. $g(x, y, z) = x \arcsin yz$

$$g_x(x, y, z) = \arcsin yz$$

$$g_y(x, y, z) = \frac{xz}{\sqrt{1 - y^2z^2}}$$

$$g_z(x, y, z) = \frac{xy}{\sqrt{1 - y^2z^2}}$$

$$\mathbf{G}(x, y, z) = (\arcsin yz)\mathbf{i} + \frac{xz}{\sqrt{1 - y^2z^2}}\mathbf{j} + \frac{xy}{\sqrt{1 - y^2z^2}}\mathbf{k}$$

30. $\mathbf{F}(x, y) = \frac{1}{xy}(y\mathbf{i} - x\mathbf{j}) = \frac{1}{x}\mathbf{i} - \frac{1}{y}\mathbf{j}$

$M = 1/x$ and $N = -1/y$ have continuous first partial derivatives for all $x, y \neq 0$.

$$\frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} \text{ is conservative.}$$

34. $M = \frac{y}{\sqrt{1 - x^2y^2}}, N = \frac{-x}{\sqrt{1 - x^2y^2}}$

$$\frac{\partial N}{\partial x} = \frac{-1}{(1 - x^2y^2)^{3/2}} \neq \frac{\partial M}{\partial y} = \frac{1}{(1 - x^2y^2)^{3/2}}$$

$\Rightarrow$ Not conservative

38. $\mathbf{F}(x, y) = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$

$$\frac{\partial}{\partial y}[3x^2y^2] = 6x^2y$$

$$\frac{\partial}{\partial x}[2x^3y] = 6x^2y$$

Conservative

$$f_x(x, y) = 3x^2y^2$$

$$f_y(x, y) = 2x^3y$$

$$f(x, y) = x^3y^2 + K$$

40. $\mathbf{F}(x, y) = \frac{2y}{x}\mathbf{i} - \frac{x^2}{y^2}\mathbf{j}$

$$\frac{\partial}{\partial y}\left[\frac{2y}{x}\right] = \frac{2}{x}$$

$$\frac{\partial}{\partial x}\left[-\frac{x^2}{y^2}\right] = -\frac{2x}{y^2}$$

Not conservative

$$42. \mathbf{F}(x, y) = \frac{2x}{(x^2 + y^2)^2} \mathbf{i} + \frac{2y}{(x^2 + y^2)^2} \mathbf{j}$$

$$\frac{\partial}{\partial y} \left[\frac{2x}{(x^2 + y^2)^2} \right] = -\frac{8xy}{(x^2 + y^2)^3}$$

$$\frac{\partial}{\partial x} \left[\frac{2y}{(x^2 + y^2)^2} \right] = -\frac{8xy}{(x^2 + y^2)^3}$$

Conservative

$$f_x(x, y) = \frac{2x}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{2y}{(x^2 + y^2)^2}$$

$$f(x, y) = -\frac{1}{x^2 + y^2} + K$$

$$44. \mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}, (2, -1, 3)$$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix}$$

$$= (z + 2x)\mathbf{i} - (0 - x^2)\mathbf{j} + (-2z - 0)\mathbf{k}$$

$$= (z + 2x)\mathbf{i} + x^2\mathbf{j} - 2z\mathbf{k}$$

$$\mathbf{curl} \mathbf{F}(2, -1, 3) = 7\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$$

$$46. \mathbf{F}(x, y, z) = e^{-xyz}(\mathbf{i} + \mathbf{j} + \mathbf{k}), (3, 2, 0)$$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{-xyz} & e^{-xyz} & e^{-xyz} \end{vmatrix} = (-xz + xy)e^{-xyz}\mathbf{i} - (-yz + xy)e^{-xyz}\mathbf{j} + (-yz + xz)e^{-xyz}\mathbf{k}$$

$$\mathbf{curl} \mathbf{F}(3, 2, 0) = 6\mathbf{i} - 6\mathbf{j}$$

$$48. \mathbf{F}(x, y, z) = \frac{yz}{y-z}\mathbf{i} + \frac{xz}{x-z}\mathbf{j} + \frac{xy}{x-y}\mathbf{k}$$

$$\begin{aligned} \mathbf{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{yz}{y-z} & \frac{xz}{x-z} & \frac{xy}{x-y} \end{vmatrix} \\ &= \left[\frac{x^2}{(x-y)^2} - \frac{x^2}{(x-z)^2} \right] \mathbf{i} - \left[\frac{-y^2}{(x-y)^2} - \frac{y^2}{(y-z)^2} \right] \mathbf{j} + \left[\frac{-z^2}{(x-z)^2} - \frac{-z^2}{(y-z)^2} \right] \mathbf{k} \\ &= x^2 \left[\frac{1}{(x-y)^2} - \frac{1}{(x-z)^2} \right] \mathbf{i} + y^2 \left[\frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} \right] \mathbf{j} + z^2 \left[\frac{1}{(y-z)^2} - \frac{1}{(x-z)^2} \right] \mathbf{k} \end{aligned}$$

$$50. \mathbf{F}(x, y, z) = \sqrt{x^2 + y^2 + z^2}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sqrt{x^2 + y^2 + z^2} & \sqrt{x^2 + y^2 + z^2} & \sqrt{x^2 + y^2 + z^2} \end{vmatrix} = \frac{(y-z)\mathbf{i} + (z-x)\mathbf{j} + (x-y)\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$$

$$52. \mathbf{F}(x, y, z) = e^z(y\mathbf{i} + x\mathbf{j} + \mathbf{k})$$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & xe^z & e^z \end{vmatrix} = -xe^z\mathbf{i} + ye^z\mathbf{j} \neq \mathbf{0}$$

Not conservative

$$54. \mathbf{F}(x, y, z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \mathbf{0}$$

Conservative

$$f(x, y, z) = xy^2z^3 + K$$

$$56. \mathbf{F}(x, y, z) = \frac{x}{x^2 + y^2} \mathbf{i} + \frac{y}{x^2 + y^2} \mathbf{j} + \mathbf{k}$$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2 + y^2} & \frac{y}{x^2 + y^2} & 1 \end{vmatrix} = \mathbf{0}$$

Conservative

$$f_x(x, y, z) = \frac{x}{x^2 + y^2}$$

$$f_y(x, y, z) = \frac{y}{x^2 + y^2}$$

$$f_z(x, y, z) = 1$$

$$\begin{aligned} f(x, y, z) &= \int \frac{x}{x^2 + y^2} dx \\ &= \frac{1}{2} \ln(x^2 + y^2) + g(y, z) + K_1 \end{aligned}$$

$$\begin{aligned} f(x, y, z) &= \int \frac{y}{x^2 + y^2} dy \\ &= \frac{1}{2} \ln(x^2 + y^2) + h(x, z) + K_2 \end{aligned}$$

$$f(x, y, z) = \int dz = z + p(x, y) + K_3$$

$$f(x, y, z) = \frac{1}{2} \ln(x^2 + y^2) + z + K$$

$$60. \mathbf{F}(x, y, z) = \ln(x^2 + y^2) \mathbf{i} + xy \mathbf{j} + \ln(y^2 + z^2) \mathbf{k}$$

$$\text{div } \mathbf{F}(x, y, z) = \frac{\partial}{\partial x} [\ln(x^2 + y^2)] + \frac{\partial}{\partial y} [xy] + \frac{\partial}{\partial z} [\ln(y^2 + z^2)] = \frac{2x}{x^2 + y^2} + x + \frac{2z}{y^2 + z^2}$$

$$62. \mathbf{F}(x, y, z) = x^2 z \mathbf{i} - 2xz \mathbf{j} + yz \mathbf{k}$$

$$\text{div } \mathbf{F}(x, y, z) = 2xz + y$$

$$\text{div } \mathbf{F}(2, -1, 3) = 11$$

$$58. \mathbf{F}(x, y) = xe^x \mathbf{i} + ye^y \mathbf{j}$$

$$\begin{aligned} \text{div } \mathbf{F}(x, y) &= \frac{\partial}{\partial x} [xe^x] + \frac{\partial}{\partial y} [ye^y] \\ &= xe^x + e^x + ye^y + e^y \\ &= e^x(x + 1) + e^y(y + 1) \end{aligned}$$

$$64. \mathbf{F}(x, y, z) = \ln(xyz)(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$\text{div } \mathbf{F}(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

$$\text{div } \mathbf{F}(3, 2, 1) = \frac{1}{3} + \frac{1}{2} + 1 = \frac{11}{6}$$

66. See the definition of Conservative Vector Field on page 1011. To test for a conservative vector field, see Theorem 14.1 and 14.2.

68. See the definition on page 1016.

$$70. \mathbf{F}(x, y, z) = x \mathbf{i} - z \mathbf{k}$$

$$\mathbf{G}(x, y, z) = x^2 \mathbf{i} + y \mathbf{j} + z^2 \mathbf{k}$$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & 0 & -z \\ x^2 & y & z^2 \end{vmatrix} = yz \mathbf{i} - (xz^2 + x^2 z) \mathbf{j} + xy \mathbf{k}$$

$$\begin{aligned} \text{curl}(\mathbf{F} \times \mathbf{G}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz^2 - x^2 z & xy \end{vmatrix} \\ &= (x + 2xz + x^2) \mathbf{i} - (y - y) \mathbf{j} + (-z^2 - 2xz - z) \mathbf{k} \\ &= x(x + 2z + 1) \mathbf{i} - z(z + 2x + 1) \mathbf{k} \end{aligned}$$

72. $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix} = (z + 2x)\mathbf{i} + x^2\mathbf{j} - 2z\mathbf{k}$$

$$\mathbf{curl}(\mathbf{curl} \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z + 2x & x^2 & -2z \end{vmatrix} = \mathbf{j} + 2x\mathbf{k}$$

74. $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{k}$

$$\mathbf{G}(x, y, z) = x^2\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & 0 & -z \\ x^2 & y & z^2 \end{vmatrix} = yz\mathbf{i} - (xz^2 + x^2z)\mathbf{j} + xy\mathbf{k}$$

$$\text{div}(\mathbf{F} \times \mathbf{G}) = 0$$

76. $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix} = (z + 2x)\mathbf{i} + x^2\mathbf{j} - 2z\mathbf{k}$$

$$\text{div}(\mathbf{curl} \mathbf{F}) = 2 - 2 = 0$$

78. Let $f(x, y, z)$ be a scalar function whose second partial derivatives are continuous.

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$\mathbf{curl}(\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k} = \mathbf{0}$$

80. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and $\mathbf{G} = R\mathbf{i} + S\mathbf{j} + T\mathbf{k}$.

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ M & N & P \\ R & S & T \end{vmatrix} = (NT - PS)\mathbf{i} - (MT - PR)\mathbf{j} + (MS - NR)\mathbf{k}$$

$$\begin{aligned} \text{div}(\mathbf{F} \times \mathbf{G}) &= \frac{\partial}{\partial x}(NT - PS) + \frac{\partial}{\partial y}(PR - MT) + \frac{\partial}{\partial z}(MS - NR) \\ &= N\frac{\partial T}{\partial x} + T\frac{\partial N}{\partial x} - P\frac{\partial S}{\partial x} - S\frac{\partial P}{\partial x} + P\frac{\partial R}{\partial y} + R\frac{\partial P}{\partial y} - M\frac{\partial T}{\partial y} - T\frac{\partial M}{\partial y} + M\frac{\partial S}{\partial z} + S\frac{\partial M}{\partial z} - N\frac{\partial R}{\partial z} - R\frac{\partial N}{\partial z} \\ &= \left[\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)R + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)S + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)T\right] - \left[M\left(\frac{\partial T}{\partial y} - \frac{\partial S}{\partial z}\right) + N\left(\frac{\partial R}{\partial z} - \frac{\partial T}{\partial x}\right) + P\left(\frac{\partial S}{\partial x} - \frac{\partial R}{\partial y}\right)\right] \\ &= (\mathbf{curl} \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\mathbf{curl} \mathbf{G}) \end{aligned}$$

82. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$.

$$\begin{aligned} \nabla \times (f\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fM & fN & fP \end{vmatrix} \\ &= \left(\frac{\partial f}{\partial y}P + f\frac{\partial P}{\partial y} - \frac{\partial f}{\partial z}N - f\frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial f}{\partial x}P + f\frac{\partial P}{\partial x} - \frac{\partial f}{\partial z}M - f\frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial f}{\partial x}N + f\frac{\partial N}{\partial x} - \frac{\partial f}{\partial y}M - f\frac{\partial M}{\partial y}\right)\mathbf{k} \\ &= f\left[\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}\right] + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ M & N & P \end{vmatrix} = f[\nabla \times \mathbf{F}] + (\nabla f) \times \mathbf{F} \end{aligned}$$

84. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$.

$$\begin{aligned}\operatorname{curl} \mathbf{F} &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} \\ \operatorname{div}(\operatorname{curl} \mathbf{F}) &= \frac{\partial}{\partial x} \left[\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right] - \frac{\partial}{\partial y} \left[\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right] + \frac{\partial}{\partial z} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] \\ &= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 M}{\partial y \partial z} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 0 \quad (\text{since the mixed partials are equal})\end{aligned}$$

In Exercises 86 and 88, $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $f(x, y, z) = \|\mathbf{F}(x, y, z)\| = \sqrt{x^2 + y^2 + z^2}$.

86. $\frac{1}{f} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$

$$\nabla\left(\frac{1}{f}\right) = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{i} + \frac{-y}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{j} + \frac{-z}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{k} = \frac{-(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(\sqrt{x^2 + y^2 + z^2})^3} = -\frac{\mathbf{F}}{f^3}$$

88. $w = \frac{1}{f} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$

$$\frac{\partial w}{\partial x} = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial w}{\partial y} = -\frac{y}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial w}{\partial z} = -\frac{z}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\frac{\partial^2 w}{\partial z^2} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$

Therefore $w = \frac{1}{f}$ is harmonic.

Section 14.2 Line Integrals

2. $\frac{x^2}{16} + \frac{y^2}{9} = 1$

$$\cos^2 t + \sin^2 t = 1$$

$$\cos^2 t = \frac{x^2}{16}$$

$$\sin^2 t = \frac{y^2}{9}$$

$$x = 4 \cos t$$

$$y = 3 \sin t$$

$$\mathbf{r}(t) = 4 \cos t \mathbf{i} + 3 \sin t \mathbf{j}$$

$$0 \leq t \leq 2\pi$$

4. $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + \frac{4}{5}t\mathbf{j}, & 0 \leq t \leq 5 \\ 5\mathbf{i} + (9-t)\mathbf{j}, & 5 \leq t \leq 9 \\ (14-t)\mathbf{i}, & 9 \leq t \leq 14 \end{cases}$

6. $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t^2\mathbf{j}, & 0 \leq t \leq 2 \\ (4-t)\mathbf{i} + 4\mathbf{j}, & 2 \leq t \leq 4 \\ (8-t)\mathbf{j}, & 4 \leq t \leq 8 \end{cases}$

8. $\mathbf{r}(t) = t\mathbf{i} + (2-t)\mathbf{j}$, $0 \leq t \leq 2$; $\mathbf{r}'(t) = \mathbf{i} - \mathbf{j}$

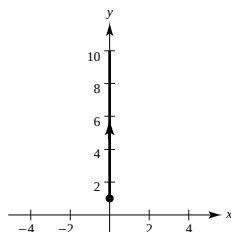
$$\int_C 4xy \, ds = \int_0^2 4t(2-t)\sqrt{1+1} \, dt = 4\sqrt{2} \int_0^2 (2t-t^2) \, dt = 4\sqrt{2} \left[t^2 - \frac{t^3}{3} \right]_0^2 = 4\sqrt{2} \left(4 - \frac{8}{3} \right) = \frac{16\sqrt{2}}{3}$$

10. $\mathbf{r}(t) = 12t\mathbf{i} + 5t\mathbf{j} + 3\mathbf{k}$, $0 \leq t \leq 2$; $\mathbf{r}'(t) = 12\mathbf{i} + 5\mathbf{j}$

$$\int_C 8xyz \, ds = \int_0^2 8(12t)(5t)(3)\sqrt{12^2 + 5^2 + 0^2} \, dt = \int_0^2 18,720t^2 \, dt = 18,720 \left[\frac{t^3}{3} \right]_0^2 = 49,920$$

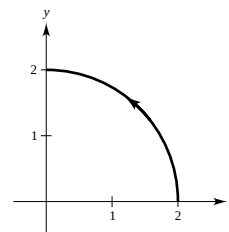
12. $\mathbf{r}(t) = t\mathbf{j}$, $1 \leq t \leq 10$

$$\begin{aligned} \int_C (x^2 + y^2) \, ds &= \int_1^{10} [0 + t^2]\sqrt{0 + 1} \, dt \\ &= \int_1^{10} t^2 \, dt \\ &= \left[\frac{1}{3}t^3 \right]_1^{10} = 333 \end{aligned}$$



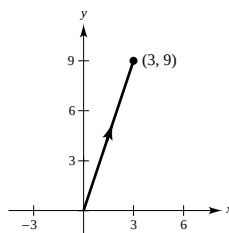
14. $\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j}$, $0 \leq t \leq \frac{\pi}{2}$

$$\begin{aligned} \int_C (x^2 + y^2) \, ds &= \int_0^{\pi/2} [4 \cos^2 t + 4 \sin^2 t] \sqrt{(-2 \sin t)^2 + (2 \cos t)^2} \, dt \\ &= \int_0^{\pi/2} 8 \, dt = 4\pi \end{aligned}$$



16. $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j}$, $0 \leq t \leq 3$

$$\begin{aligned} \int_C (x + 4\sqrt{y}) \, ds &= \int_0^3 (t + 4\sqrt{3t})\sqrt{1 + 9} \, dt \\ &= \left[\sqrt{10} \left(\frac{t^2}{2} + \frac{8\sqrt{3}}{3} t^{3/2} \right) \right]_0^3 \\ &= \frac{\sqrt{10}}{6} (27 + 144) = \frac{57\sqrt{10}}{2} \end{aligned}$$



18. $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \leq t \leq 2 \\ 2\mathbf{i} + (t-2)\mathbf{j}, & 2 \leq t \leq 4 \\ (6-t)\mathbf{i} + 2\mathbf{j}, & 4 \leq t \leq 6 \\ (8-t)\mathbf{j}, & 6 \leq t \leq 8 \end{cases}$

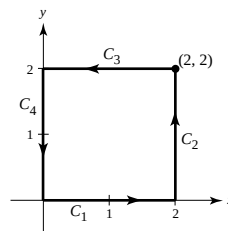
$$\int_{C_1} (x + 4\sqrt{y}) \, ds = \int_0^2 t \, dt = 2$$

$$\int_{C_2} (x + 4\sqrt{y}) \, ds = \int_2^4 (2 + 4\sqrt{t-2}) \, ds = 4 + \frac{16\sqrt{2}}{3}$$

$$\int_{C_3} (x + 4\sqrt{y}) \, ds = \int_4^6 ((6-t) + 4\sqrt{2}) \, ds = 2 + 8\sqrt{2}$$

$$\int_{C_4} (x + 4\sqrt{y}) \, ds = \int_6^8 4\sqrt{8-t} \, ds = \frac{16\sqrt{2}}{3}$$

$$\int_C (x + 4\sqrt{y}) \, ds = 2 + 4 + \frac{16\sqrt{2}}{3} + 2 + 8\sqrt{2} + \frac{16\sqrt{2}}{3} = 8 + \frac{56}{3}\sqrt{2}$$



20. $\rho(x, y, z) = z$

$$\mathbf{r}(t) = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j} + 2t \mathbf{k}, \quad 0 \leq t \leq 4\pi$$

$$\mathbf{r}'(t) = -3 \sin t \mathbf{i} + 3 \cos t \mathbf{j} + 2 \mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-3 \sin t)^2 + (3 \cos t)^2 + (2)^2} = \sqrt{13}$$

$$\text{Mass} = \int_C \rho(x, y, z) \, ds = \int_0^{4\pi} 2t \sqrt{13} \, dt = 16\pi^2 \sqrt{13}$$

24. $\mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{j}$

$$C: \mathbf{r}(t) = t\mathbf{i} + \sqrt{4-t^2}\mathbf{j}, \quad -2 \leq t \leq 2$$

$$\mathbf{F}(t) = 3t\mathbf{i} + 4\sqrt{4-t^2}\mathbf{j}$$

$$\mathbf{r}'(t) = \mathbf{i} - \frac{t}{\sqrt{4-t^2}}\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-2}^2 (3t - 4t) \, dt = \left[-\frac{t^2}{2} \right]_{-2}^2 = 0$$

28. $\mathbf{F}(x, y, z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + e^t\mathbf{k}, \quad 0 \leq t \leq 2$$

$$\mathbf{F}(t) = \frac{t\mathbf{i} + t\mathbf{j} + e^t\mathbf{k}}{\sqrt{2t^2 + e^{2t}}}$$

$$d\mathbf{r} = (\mathbf{i} + \mathbf{j} + e^t\mathbf{k}) \, dt$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \frac{1}{\sqrt{2t^2 + e^{2t}}} (2t + e^{2t}) \, dt \approx 6.91$$

30. $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$

$$C: x = \cos^3 t, y = \sin^3 t \text{ from } (1, 0) \text{ to } (0, 1)$$

$$\mathbf{r}(t) = \cos^3 t \mathbf{i} + \sin^3 t \mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -3 \cos^2 t \sin t \mathbf{i} + 3 \sin^2 t \cos t \mathbf{j}$$

$$\mathbf{F}(t) = \cos^6 t \mathbf{i} - \cos^3 t \sin^3 t \mathbf{j}$$

$$\mathbf{F} \cdot \mathbf{r}' = -3 \cos^8 t \sin t - 3 \cos^4 t \sin^5 t$$

$$= -3 \cos^4 t \sin t (\cos^4 t + \sin^4 t)$$

$$= -3 \cos^4 t \sin t [\cos^4 t + (1 - \cos^2 t)^2]$$

$$= -3 \cos^4 t \sin t (2 \cos^4 t - 2 \cos^2 t + 1)$$

$$= -6 \cos^8 t \sin t + 6 \cos^6 t \sin t - 3 \cos^4 t \sin t$$

$$\begin{aligned} \text{Work} &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} [-6 \cos^8 t \sin t + 6 \cos^6 t \sin t - 3 \cos^4 t \sin t] \, dt \\ &= \left[-\frac{2 \cos^9 t}{9} + \frac{6 \cos^7 t}{7} - \frac{3 \cos^5 t}{5} \right]_0^{\pi/2} = -\frac{43}{105} \end{aligned}$$

22. $\mathbf{F}(x, y) = xy\mathbf{i} + y\mathbf{j}$

$$C: \mathbf{r}(t) = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2}$$

$$\mathbf{F}(t) = 16 \sin t \cos t \mathbf{i} + 4 \sin t \mathbf{j}$$

$$\mathbf{r}'(t) = -4 \sin t \mathbf{i} + 4 \cos t \mathbf{j}$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} (-64 \sin^2 t \cos t + 16 \sin t \cos t) \, dt \\ &= \left[-\frac{64}{3} \sin^3 t + 8 \sin^2 t \right]_0^{\pi/2} = -\frac{40}{3} \end{aligned}$$

26. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

$$C: \mathbf{r}(t) = 2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + \frac{1}{2}t^2 \mathbf{k}, \quad 0 \leq t \leq \pi$$

$$\mathbf{F}(t) = 4 \sin^2 t \mathbf{i} + 4 \cos^2 t \mathbf{j} + \frac{1}{4}t^4 \mathbf{k}$$

$$\mathbf{r}'(t) = 2 \cos t \mathbf{i} - 2 \sin t \mathbf{j} + t \mathbf{k}$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi \left(8 \sin^2 t \cos t - 8 \cos^2 t \sin t + \frac{1}{4}t^5 \right) \, dt \\ &= \left[\frac{8}{3} \sin^3 t + \frac{8}{3} \cos^3 t + \frac{t^6}{24} \right]_0^\pi \\ &= -\frac{8}{3} + \frac{\pi^6}{24} - \frac{8}{3} = \frac{\pi^6}{24} - \frac{16}{3} \end{aligned}$$

32. $\mathbf{F}(x, y) = -y\mathbf{i} - x\mathbf{j}$

C: counterclockwise along the semicircle $y = \sqrt{4 - x^2}$ from $(2, 0)$ to $(-2, 0)$

$$\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j}, \quad 0 \leq t \leq \pi$$

$$\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j}$$

$$\mathbf{F}(t) = -2 \sin t \mathbf{i} - 2 \cos t \mathbf{j}$$

$$\mathbf{F} \cdot \mathbf{r}' = 4 \sin^2 t - 4 \cos^2 t = -4 \cos 2t$$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = -4 \int_0^\pi \cos 2t \, dt = \left[-2 \sin 2t \right]_0^\pi = 0$$

36. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \leq t \leq 1$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &\approx \frac{1-0}{3(4)} [5 + 4(4) + 2(4) + 4(6) + 11] \\ &= \frac{16}{3} \end{aligned}$$

38. $\mathbf{F}(x, y) = x^2y\mathbf{i} + xy^{3/2}\mathbf{j}$

(a) $\mathbf{r}_1(t) = (t+1)\mathbf{i} + t^2\mathbf{j}, \quad 0 \leq t \leq 2$

$$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{F}(t) = (t+1)^2t^2\mathbf{i} + (t+1)t^3\mathbf{j}$$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 [(t+1)^2t^2 + 2t^4(t+1)] \, dt = \frac{256}{3}$$

(b) $\mathbf{r}_2(t) = (1+2\cos t)\mathbf{i} + 4\cos^2 t\mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2}$

$$\mathbf{r}_2'(t) = -2\sin t\mathbf{i} - 8\cos t\sin t\mathbf{j}$$

$$\mathbf{F}(t) = (1+2\cos t)^2(4\cos^2 t)\mathbf{i} + (1+2\cos t)(8\cos^3 t)\mathbf{j}$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} [(1+2\cos t)^2(4\cos^2 t)(-2\sin t) - 8\cos t\sin t(1+2\cos t)(8\cos^3 t)] \, dt = -\frac{256}{5}$$

Both paths join $(1, 0)$ and $(3, 4)$. The integrals are negatives of each other because the orientations are different.

40. $\mathbf{F}(x, y) = -3y\mathbf{i} + x\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} - t^3\mathbf{j}$

$$\mathbf{r}'(t) = \mathbf{i} - 3t^2\mathbf{j}$$

$$\mathbf{F}(t) = 3t^3\mathbf{i} + t\mathbf{j}$$

$$\mathbf{F} \cdot \mathbf{r}' = 3t^3 - 3t^3 = 0$$

Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

44. $x = 2t, y = 10t, \quad 0 \leq t \leq 1 \Rightarrow y = 5x, \quad 0 \leq x \leq 2$

$$\int_C (x + 3y^2) \, dx = \int_0^2 (x + 75x^2) \, dx = \left[\frac{x^2}{2} + 25x^3 \right]_0^2 = 202$$

34. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

C: line from $(0, 0, 0)$ to $(5, 3, 2)$

$$\mathbf{r}(t) = 5t\mathbf{i} + 3t\mathbf{j} + 2t\mathbf{k}, \quad 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = 5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{F}(t) = 6t^2\mathbf{i} + 10t^2\mathbf{j} + 15t^2\mathbf{k}$$

$$\mathbf{F} \cdot \mathbf{r}' = 90t^2$$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 90t^2 \, dt = 30$$

(x, y)	$(0, 0)$	$(\frac{1}{4}, \frac{1}{16})$	$(\frac{1}{2}, \frac{1}{4})$	$(\frac{3}{4}, \frac{9}{16})$	$(1, 1)$
$\mathbf{F}(x, y)$	$5\mathbf{i}$	$3.5\mathbf{i} + \mathbf{j}$	$2\mathbf{i} + 2\mathbf{j}$	$1.5\mathbf{i} + 3\mathbf{j}$	$\mathbf{i} + 5\mathbf{j}$
$\mathbf{r}'(t)$	$\mathbf{i}$	$\mathbf{i} + 0.5\mathbf{j}$	$\mathbf{i} + \mathbf{j}$	$\mathbf{i} + 1.5\mathbf{j}$	$\mathbf{i} + 2\mathbf{j}$
$\mathbf{F} \cdot \mathbf{r}'$	5	4	4	6	11

42. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

C: $\mathbf{r}(t) = 3\sin t\mathbf{i} + 3\cos t\mathbf{j}$

$$\mathbf{r}'(t) = 3\cos t\mathbf{i} - 3\sin t\mathbf{j}$$

$$\mathbf{F}(t) = 3\sin t\mathbf{i} + 3\cos t\mathbf{j}$$

$$\mathbf{F} \cdot \mathbf{r}' = 9\sin t\cos t - 9\sin t\cos t = 0$$

Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

46. $x = 2t$, $y = 10t$, $0 \leq t \leq 1 \Rightarrow y = 5x$, $dy = 5 dx$, $0 \leq x \leq 2$

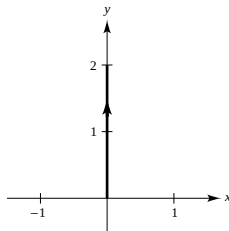
$$\begin{aligned}\int_C (3y - x) dx + y^2 dy &= \int_0^2 (3(5x) - x) dx + (5x)^2 5 dx = \int_0^2 (14x + 125x^2) dx \\ &= \left[7x^2 + \frac{125}{3}x^3 \right]_0^2 = 28 + \frac{125}{3}(8) = \frac{1084}{3}\end{aligned}$$

48. $\mathbf{r}(t) = t\mathbf{j}$, $0 \leq t \leq 2$

$x(t) = 0$, $y(t) = t$

$dx = 0$, $dy = dt$

$$\int_C (2x - y) dx + (x + 3y) dy = \int_0^2 3t dt = \left[\frac{3}{2}t^2 \right]_0^2 = 6$$



50. $\mathbf{r}(t) = \begin{cases} -t\mathbf{j}, & 0 \leq t \leq 3 \\ (t-3)\mathbf{i} - 3\mathbf{j}, & 3 \leq t \leq 5 \end{cases}$

C_1 : $x(t) = 0$, $y(t) = -t$

$dx = 0$, $dy = -dt$

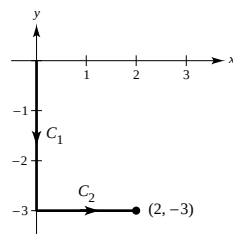
$$\int_{C_1} (2x - y) dx + (x + 3y) dy = \int_0^3 3t dt = \frac{27}{2}$$

C_2 : $x(t) = t - 3$, $y(t) = -3$

$dx = dt$, $dy = 0$

$$\int_{C_2} (2x - y) dx + (x + 3y) dy = \int_3^5 [2(t-3) + 3] dt = \left[(t-3)^2 + 3t \right]_3^5 = 10$$

$$\int_C (2x - y) dx + (x + 3y) dy = \frac{27}{2} + 10 = \frac{47}{2}$$



52. $x(t) = t$, $y(t) = t^{3/2}$, $0 \leq t \leq 4$, $dx = dt$, $dy = \frac{3}{2}t^{1/2} dt$

$$\begin{aligned}\int_C (2x - y) dx + (x + 3y) dy &= \int_0^4 \left[(2t - t^{3/2}) + (t + 3t^{3/2}) \left(\frac{3}{2}t^{1/2} \right) \right] dt \\ &= \int_0^4 \left(\frac{9}{2}t^2 + \frac{1}{2}t^{3/2} + 2t \right) dt = \left[\frac{3}{2}t^3 + \frac{1}{5}t^{5/2} + t^2 \right]_0^4 = 96 + \frac{1}{5}(32) + 16 = \frac{592}{5}\end{aligned}$$

54. $x(t) = 4 \sin t$, $y(t) = 3 \cos t$, $0 \leq t \leq \frac{\pi}{2}$

$dx = 4 \cos t dt$, $dy = -3 \sin t dt$

$$\begin{aligned}\int_C (2x - y) dx + (x + 3y) dy &= \int_0^{\pi/2} (8 \sin t - 3 \cos t)(4 \cos t) dt + (4 \sin t + 9 \cos t)(-3 \sin t) dt \\ &= \int_0^{\pi/2} (5 \sin t \cos t - 12 \cos^2 t - 12 \sin^2 t) dt \\ &= \left[\frac{5}{2} \sin^2 t - 12t \right]_0^{\pi/2} = \frac{5}{2} - 6\pi\end{aligned}$$

56. $f(x, y) = y$

C: line from $(0, 0)$ to $(4, 4)$

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \leq t \leq 4$$

$$\mathbf{r}'(t) = \mathbf{i} + \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{2}$$

Lateral surface area:

$$\int_C f(x, y) \, ds = \int_0^4 t(\sqrt{2}) \, dt = 8\sqrt{2}$$

58. $f(x, y) = x + y$

C: $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 1$$

Lateral surface area:

$$\begin{aligned} \int_C f(x, y) \, ds &= \int_0^{\pi/2} (\cos t + \sin t) \, dt \\ &= \left[\sin t - \cos t \right]_0^{\pi/2} = 2 \end{aligned}$$

60. $f(x, y) = y + 1$

C: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

$$\mathbf{r}(t) = (1 - t)\mathbf{i} + [1 - (1 - t)^2]\mathbf{j}, \quad 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = -\mathbf{i} + 2(1 - t)\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{1 + 4(1 - t)^2}$$

Lateral surface area:

$$\begin{aligned} \int_C f(x, y) \, ds &= \int_0^1 [2 - (1 - t)^2] \sqrt{1 + 4(1 - t)^2} \, dt \\ &= 2 \int_0^1 \sqrt{1 + 4(1 - t)^2} \, dt - \int_0^1 (1 - t)^2 \sqrt{1 + 4(1 - t)^2} \, dt \\ &= -\frac{1}{2} \left[2(1 - t) \sqrt{1 + 4(1 - t)^2} + \ln |2(1 - t) + \sqrt{1 + 4(1 - t)^2}| \right]_0^1 \\ &\quad + \frac{1}{64} \left[2(1 - t)[2(4)(1 - t)^2 + 1] \sqrt{1 + 4(1 - t)^2} - \ln |2(1 - t) + \sqrt{1 + 4(1 - t)^2}| \right]_0^1 \\ &= \frac{1}{2} [2\sqrt{5} + \ln(2 + \sqrt{5})] - \frac{1}{64} [18\sqrt{5} - \ln(2 + \sqrt{5})] \\ &= \frac{23}{32} \sqrt{5} + \frac{33}{64} \ln(2 + \sqrt{5}) = \frac{1}{64} [46\sqrt{5} + 33 \ln(2 + \sqrt{5})] \approx 2.3515 \end{aligned}$$

62. $f(x, y) = x^2 - y^2 + 4$

C: $x^2 + y^2 = 4$

$$\mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j}, \quad 0 \leq t \leq 2\pi$$

$$\mathbf{r}'(t) = -2 \sin t\mathbf{i} + 2 \cos t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 2$$

Lateral surface area:

$$\int_C f(x, y) \, ds = \int_0^{2\pi} (4 \cos^2 t - 4 \sin^2 t + 4)(2) \, dt = 8 \int_0^{2\pi} (1 + \cos 2t) \, dt = \left[8 \left(t + \frac{1}{2} \sin 2t \right) \right]_0^{2\pi} = 16\pi$$

64. $f(x, y) = 20 + \frac{1}{4}x$

$C: y = x^{3/2}, 0 \leq x \leq 40$

$\mathbf{r}(t) = t\mathbf{i} + t^{3/2}\mathbf{j}, 0 \leq t \leq 40$

$\mathbf{r}'(t) = \mathbf{i} + \frac{3}{2}t^{1/2}\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{1 + \left(\frac{9}{4}\right)t}$

Lateral surface area: $\int_C f(x, y) ds = \int_0^{40} \left(20 + \frac{1}{4}t\right) \sqrt{1 + \left(\frac{9}{4}\right)t} dt$

Let $u = \sqrt{1 + \left(\frac{9}{4}\right)t}$, then $t = \frac{4}{9}(u^2 - 1)$ and $dt = \frac{8}{9}u du$.

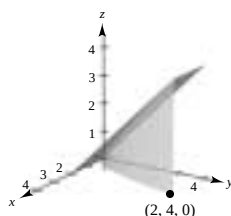
$$\begin{aligned} \int_0^{40} \left(20 + \frac{1}{4}t\right) \sqrt{1 + \left(\frac{9}{4}\right)t} dt &= \int_1^{\sqrt{91}} \left[20 + \frac{1}{9}(u^2 - 1)\right] \left(u\right) \left(\frac{8}{9}u\right) du = \frac{8}{81} \int_1^{\sqrt{91}} (u^4 + 179u^2) du \\ &= \frac{8}{81} \left[\frac{u^5}{5} + \frac{179u^3}{3} \right]_1^{\sqrt{91}} = \frac{850,304\sqrt{91} - 7184}{1215} \approx 6670.12 \end{aligned}$$

66. $f(x, y) = y$

$C: y = x^2$ from $(0, 0)$ to $(2, 4)$

$S \approx 8$

Matches c .



68. $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy$

$M = 15(4 - x^2y) = 60 - 15x^2(c - cx^2)$

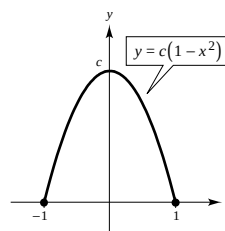
$N = -15xy = -15x(c - cx^2)$

$dx = dx, dy = -2cx dx$

$W = \int_{-1}^1 [60 - 15x^2(c - cx^2) + (-15x(c - cx^2))(-2cx)] dx$

$= 120 - 4c + 8c^2$ (parabola)

$w' = 16c - 4 = 0 \Rightarrow c = \frac{1}{4}$ yields the minimum work, 119.5. Along the straight line path, $y = 0$, the work is 120.



70. See the definition, page 1024.

72. (a) Work = 0

(b) Work is negative, since against force field.

(c) Work is positive, since with force field.

74. False, the orientation of C does not affect the form

$$\int_C f(x, y) ds.$$

76. False. For example, see Exercise 32.

Section 14.3 Conservative Vector Fields and Independence of Path

2. $\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + \sqrt{t}\mathbf{j}, 0 \leq t \leq 4$

$$\mathbf{r}_1'(t) = \mathbf{i} + \frac{1}{2\sqrt{t}}\mathbf{j}$$

$$\mathbf{F}(t) = (t^2 + t)\mathbf{i} - t\mathbf{j}$$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^4 \left(t^2 + t - \frac{1}{2}\sqrt{t} \right) dt \\ &= \left[\frac{t^3}{3} + \frac{t^2}{2} - \frac{t^{3/2}}{3} \right]_0^4 = \frac{80}{3}\end{aligned}$$

(b) $\mathbf{r}_2(w) = w^2\mathbf{i} + w\mathbf{j}, 0 \leq w \leq 2$

$$\mathbf{r}_2'(w) = 2w\mathbf{i} + \mathbf{j}$$

$$\mathbf{F}(w) = (w^4 + w^2)\mathbf{i} - w^2\mathbf{j}$$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^2 [2w(w^4 + w^2) - w^2] dw \\ &= \left[\frac{w^6}{3} + \frac{w^4}{2} - \frac{w^3}{3} \right]_0^2 = \frac{80}{3}\end{aligned}$$

4. $\mathbf{F}(x, y) = y\mathbf{i} + x^2\mathbf{j}$

(a) $\mathbf{r}_1(t) = (2 + t)\mathbf{i} + (3 - t)\mathbf{j}, 0 \leq t \leq 3$

$$\mathbf{r}_1'(t) = \mathbf{i} - \mathbf{j}$$

$$\mathbf{F}(t) = (3 - t)\mathbf{i} + (2 + t)^2\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 [(3 - t) - (2 + t)^2] dt = \left[-\frac{(3 - t)^2}{2} - \frac{(2 + t)^3}{3} \right]_0^3 = -\frac{69}{2}$$

(b) $\mathbf{r}_2(w) = (2 + \ln w)\mathbf{i} + (3 - \ln w)\mathbf{j}, 1 \leq w \leq e^3$

$$\mathbf{r}_2'(w) = \frac{1}{w}\mathbf{i} - \frac{1}{w}\mathbf{j}$$

$$\mathbf{F}(w) = (3 - \ln w)\mathbf{i} + (2 + \ln w)^2\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^{e^3} \left[(3 - \ln w)\left(\frac{1}{w}\right) - (2 + \ln w)^2\left(\frac{1}{w}\right) \right] dw = \left[-\frac{(3 - \ln w)^2}{2} - \frac{(2 + \ln w)^3}{3} \right]_1^{e^3} = -\frac{69}{2}$$

6. $\mathbf{F}(x, y) = 15x^2y^2\mathbf{i} + 10x^3y\mathbf{j}$

$$\frac{\partial N}{\partial x} = 30x^2y \quad \frac{\partial M}{\partial y} = 30x^2y$$

Since $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$, $\mathbf{F}$ is conservative.

8. $\mathbf{F}(x, y, z) = y \ln z \mathbf{i} - x \ln z \mathbf{j} + \frac{xy}{z} \mathbf{k}$

$\text{curl } \mathbf{F} \neq \mathbf{0}$ so $\mathbf{F}$ is not conservative.

$$\left(\frac{\partial P}{\partial y} = \frac{x}{z} \neq -\frac{x}{z} = \frac{\partial N}{\partial z} \right)$$

10. $\mathbf{F}(x, y, z) = \sin(yz)\mathbf{i} + xz \cos(yz)\mathbf{j} + xy \sin(yz)\mathbf{k}$

$\text{curl } \mathbf{F} \neq \mathbf{0}$, so $\mathbf{F}$ is not conservative.

12. $\mathbf{F}(x, y) = ye^{xy}\mathbf{i} + xe^{xy}\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} - (t - 3)\mathbf{j}, 0 \leq t \leq 3$

$$\mathbf{r}_1'(t) = \mathbf{i} - \mathbf{j}$$

$$\mathbf{F}(t) = -(t - 3)e^{3t-t^2}\mathbf{i} + te^{3t-t^2}\mathbf{j}$$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^3 [-(t - 3)e^{3t-t^2} - te^{3t-t^2}] dt \\ &= \int_0^3 e^{3t-t^2}(3 - 2t) dt \\ &= \left[e^{3t-t^2} \right]_0^3 = e^0 - e^0 = 0\end{aligned}$$

(b) $\mathbf{F}(x, y)$ is conservative since

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = xye^{xy} + e^{xy}.$$

The potential function is $f(x, y) = e^{xy} + k$.

14. $\mathbf{F}(x, y) = xy^2\mathbf{i} + 2x^2y\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + \frac{1}{t^2}\mathbf{j}, \quad 1 \leq t \leq 3$

$$\mathbf{r}_1'(t) = \mathbf{i} - \frac{1}{t^3}\mathbf{j}$$

$$\mathbf{F}(t) = \frac{1}{t}\mathbf{i} + 2t\mathbf{j}$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_1^3 -\frac{1}{t} dt \\ &= \left[-\ln|t| \right]_1^3 = -\ln 3 \end{aligned}$$

(b) $\mathbf{r}_2(t) = (t+1)\mathbf{i} - \frac{1}{3}(t-3)\mathbf{j}, \quad 0 \leq t \leq 2$

$$\mathbf{r}_2'(t) = \mathbf{i} - \frac{1}{3}\mathbf{j}$$

$$\mathbf{F}(t) = \frac{1}{9}(t+1)(t-3)^2\mathbf{i} - \frac{2}{3}(t+1)^2(t-3)\mathbf{j}$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^2 \left[\frac{1}{9}(t+1)(t-3)^2 + \frac{2}{9}(t+1)^2(t-3) \right] dt \\ &= \frac{1}{9} \int_0^2 (3t^3 - 7t^2 - 7t + 3) dt \\ &= \frac{1}{9} \left[\frac{3t^4}{4} - \frac{7t^3}{3} - \frac{7t^2}{2} + 3t \right]_0^2 = -\frac{44}{27} \end{aligned}$$

16. $\int_C (2x - 3y + 1) dx - (3x + y - 5) dy$

Since $\partial M/\partial y = \partial N/\partial x = -3$, $\mathbf{F}(x, y) = (2x - 3y + 1)\mathbf{i} - (3x + y - 5)\mathbf{j}$ is conservative. The potential function is $f(x, y) = x^2 - 3xy - (y^2/2) + x + 5y + k$.

(a) and (d) Since C is a closed curve, $\int_C (2x - 3y + 1) dx - (3x + y - 5) dy = 0$.

(b) $\int_C (2x - 3y + 1) dx - (3x + y - 5) dy = \left[x^2 - 3xy - \frac{y^2}{2} + x + 5y \right]_{(0, -1)}^{(0, 1)} = 10$

(c) $\int_C (2x - 3y + 1) dx - (3x + y - 5) dy = \left[x^2 - 3xy - \frac{y^2}{2} + x + 5y \right]_{(0, 1)}^{(2, e^2)} = \frac{1}{2}(3 - 2e^2 - e^4)$

18. $\int_C (x^2 + y^2) dx + 2xy dy$

Since $\partial M/\partial y = \partial N/\partial x = 2y$,

$$\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} + 2xy\mathbf{j}$$

is conservative. The potential function is

$$f(x, y) = (x^3/3) + xy^2 + k.$$

(a) $\int_C (x^2 + y^2) dx + 2xy dy = \left[\frac{x^3}{3} + xy^2 \right]_{(0, 0)}^{(8, 4)} = \frac{896}{3}$

(b) $\int_C (x^2 + y^2) dx + 2xy dy = \left[\frac{x^3}{3} + xy^2 \right]_{(2, 0)}^{(0, 2)} = -\frac{8}{3}$

20. $\mathbf{F}(x, y, z) = \mathbf{i} + z\mathbf{j} + y\mathbf{k}$

Since $\text{curl } \mathbf{F} = \mathbf{0}$, $\mathbf{F}(x, y, z)$ is conservative. The potential function is $f(x, y, z) = x + yz + k$.

(a) $\mathbf{r}_1(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t^2\mathbf{k}, \quad 0 \leq t \leq \pi$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left[x + yz \right]_{(1, 0, 0)}^{(-1, 0, \pi^2)} = -2$$

(b) $\mathbf{r}_2(t) = (1 - 2t)\mathbf{i} + \pi^2 t\mathbf{k}, \quad 0 \leq t \leq 1$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left[x + yz \right]_{(1, 0, 0)}^{(-1, 0, \pi^2)} = -2$$

22. $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + 3xz^2\mathbf{k}$

$\mathbf{F}(x, y, z)$ is not conservative.

(a) $\mathbf{r}_1(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq \pi$

$$\mathbf{r}_1'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$$

$$\mathbf{F}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 3t^2 \cos t\mathbf{k}$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi [\sin^2 t + \cos^2 t + 3t^2 \cos t] dt = \int_0^\pi [1 + 3t^2 \cos t] dt \\ &= \left[t \right]_0^\pi + 3 \left[t^2 \sin t \right]_0^\pi - 6 \int_0^\pi t \sin t dt = \left[t + 3t^2 \sin t - 6(\sin t - t \cos t) \right]_0^\pi = -5\pi \end{aligned}$$

—CONTINUED—

22. —CONTINUED—

$$(b) \mathbf{r}_2(t) = (1 - 2t)\mathbf{i} + \pi t\mathbf{k}, \quad 0 \leq t \leq 1$$

$$\mathbf{r}_2'(t) = -2\mathbf{i} + \pi\mathbf{k}$$

$$\mathbf{F}(t) = (1 - 2t)\mathbf{j} + 3\pi^2 t^2(1 - 2t)\mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 3\pi^2 t^2(1 - 2t) dt = 3\pi^2 \int_0^1 (t^2 - 2t^3) dt = 3\pi^2 \left[\frac{t^3}{3} - \frac{t^4}{2} \right]_0^1 = -\frac{\pi^3}{2}$$

$$24. \mathbf{F}(x, y, z) = y \sin z \mathbf{i} + x \sin z \mathbf{j} + xy \cos z \mathbf{k}$$

$$(a) \mathbf{r}_1(t) = t^2 \mathbf{i} + t^2 \mathbf{j}, \quad 0 \leq t \leq 2$$

$$\mathbf{r}_1'(t) = 2t \mathbf{i} + 2t \mathbf{j}$$

$$\mathbf{F}(t) = t^4 \cos t^2 \mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 0 dt = 0$$

$$(b) \mathbf{r}_2(t) = 4t \mathbf{i} + 4t \mathbf{j}, \quad 0 \leq t \leq 1$$

$$\mathbf{r}_2'(t) = 4 \mathbf{i} + 4 \mathbf{j}$$

$$\mathbf{F}(t) = 16t^2 \cos(4t) \mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 0 dt = 0$$

$$26. \int_C [2(x + y)\mathbf{i} + 2(x + y)\mathbf{j}] \cdot d\mathbf{r} = \left[(x + y)^2 \right]_{(-3, 2)}^{(4, 3)} = 49$$

$$28. \int_C \frac{y dx - x dy}{x^2 + y^2} = \left[\arctan\left(\frac{x}{y}\right) \right]_{(1, 1)}^{(2\sqrt{3}, 2)} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

$$30. \int_C \frac{2x}{(x^2 + y^2)^2} dx + \frac{2y}{(x^2 + y^2)^2} dy = \left[-\frac{1}{x^2 + y^2} \right]_{(7, 5)}^{(1, 5)} = -\frac{1}{26} + \frac{1}{74} = \frac{-12}{481}$$

$$32. \int_C zy dx + xz dy + xy dz$$

Note: Since $\mathbf{F}(x, y, z) = yz \mathbf{i} + xz \mathbf{j} + xy \mathbf{k}$ is conservative and the potential function is $f(x, y, z) = xyz + k$, the integral is independent of path as illustrated below.

$$(a) \left[xyz \right]_{(0, 0, 0)}^{(1, 1, 1)} = 1$$

$$(b) \left[xyz \right]_{(0, 0, 0)}^{(0, 0, 1)} + \left[xyz \right]_{(0, 0, 1)}^{(1, 1, 1)} = 0 + 1 = 1$$

$$(c) \left[xyz \right]_{(0, 0, 0)}^{(1, 0, 0)} + \left[xyz \right]_{(1, 0, 0)}^{(1, 1, 0)} + \left[xyz \right]_{(1, 1, 0)}^{(1, 1, 1)} = 0 + 0 + 1 = 1$$

$$34. \int_C 6x dx - 4z dy - (4y - 20z) dz = \left[3x^2 - 4yz + 10z^2 \right]_{(0, 0, 0)}^{(4, 3, 1)} = 46$$

$$36. \mathbf{F}(x, y) = \frac{2x}{y} \mathbf{i} - \frac{x^2}{y^2} \mathbf{j} \text{ is conservative.}$$

$$\text{Work} = \left[\frac{x^2}{y} \right]_{(-3, 2)}^{(1, 4)} = \frac{1}{4} - \frac{9}{2} = -\frac{17}{4}$$

38. $\mathbf{F}(x, y, z) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Since $\mathbf{F}(x, y, z)$ is conservative, the work done in moving a particle along any path from P to Q is

$$\begin{aligned} f(x, y, z) &= \left[a_1x + a_2y + a_3z \right]_{P=(p_1, p_2, p_3)}^{Q=(q_1, q_2, q_3)} \\ &= a_1(q_1 - p_1) + a_2(q_2 - p_2) + a_3(q_3 - p_3) = \mathbf{F} \cdot \overrightarrow{PQ}. \end{aligned}$$

40. $\mathbf{F} = -150\mathbf{j}$

(a) $\mathbf{r}(t) = t\mathbf{i} + (50 - t)\mathbf{j}$, $0 \leq t \leq 50$

$$d\mathbf{r} = (\mathbf{i} - \mathbf{j}) dt$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{50} 150 dt = 7500 \text{ ft} \cdot \text{lbs}$$

(b) $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{50}(50 - t)^2\mathbf{j}$

$$d\mathbf{r} = \left(\mathbf{i} - \frac{1}{25}(50 - t)\mathbf{j} \right) dt$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 6 \int_0^{50} (50 - t) dt = 7500 \text{ ft} \cdot \text{lbs}$$

42. $\mathbf{F}(x, y) = \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j}$

(a) $M = \frac{y}{x^2 + y^2}$

$$\frac{\partial M}{\partial y} = \frac{(x^2 + y^2)(1) - y(2y)}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$N = -\frac{x}{x^2 + y^2}$$

$$\frac{\partial N}{\partial x} = \frac{(x^2 + y^2)(-1) + x(2x)}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\text{Thus, } \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

(c) $\mathbf{r}(t) = \cos t\mathbf{i} - \sin t\mathbf{j}$, $0 \leq t \leq \pi$

$$\mathbf{F} = -\sin t\mathbf{i} - \cos t\mathbf{j}$$

$$d\mathbf{r} = (-\sin t\mathbf{i} - \cos t\mathbf{j}) dt$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi (\sin^2 t + \cos^2 t) dt \\ &= \left[t \right]_0^\pi = \pi \end{aligned}$$

(b) $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \leq t \leq \pi$

$$\mathbf{F} = \sin t\mathbf{i} - \cos t\mathbf{j}$$

$$d\mathbf{r} = (-\sin t\mathbf{i} + \cos t\mathbf{j}) dt$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi (-\sin^2 t - \cos^2 t) dt = \left[-t \right]_0^\pi = -\pi$$

(d) $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \leq t \leq 2\pi$

$$\mathbf{F} = \sin t\mathbf{i} - \cos t\mathbf{j}$$

$$d\mathbf{r} = (-\sin t\mathbf{i} + \cos t\mathbf{j}) dt$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (-\sin^2 t - \cos^2 t) dt \\ &= \left[-t \right]_0^{2\pi} = -2\pi \end{aligned}$$

This does not contradict Theorem 14.7 since $\mathbf{F}$ is not continuous at $(0, 0)$ in R enclosed by curve C .

(e) $\nabla\left(\arctan \frac{x}{y}\right) = \frac{1/y}{1 + (x/y)^2}\mathbf{i} + \frac{-x/y^2}{1 + (x/y)^2}\mathbf{j}$

$$= \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j} = \mathbf{F}$$

44. A line integral is independent of path if $\int_C \mathbf{F} \cdot d\mathbf{r}$ does not depend on the curve joining P and Q . See Theorem 14.6

46. No, the amount of fuel required depends on the flight path. Fuel consumption is dependent on wind speed and direction. The vector field is not conservative.

48. True

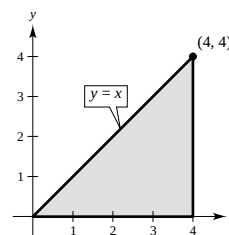
50. False, the requirement is $\partial M/\partial y = \partial N/\partial x$.

Section 14.4 Green's Theorem

$$2. \mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \leq t \leq 4 \\ 4\mathbf{i} + (t-4)\mathbf{j}, & 4 \leq t \leq 8 \\ (12-t)\mathbf{i} + (12-t)\mathbf{j}, & 8 \leq t \leq 12 \end{cases}$$

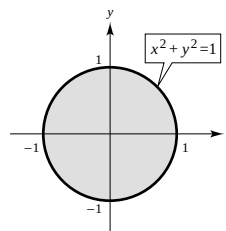
$$\begin{aligned} \int_C y^2 dx + x^2 dy &= \int_0^4 [0 dt + t^2(0)] + \int_4^8 [(t-4)^2(0) + 16 dt] \\ &\quad + \int_8^{12} [(12-t)^2(-dt) + (12-t)^2(-dt)] = 0 + 64 - \frac{128}{3} = \frac{64}{3} \end{aligned}$$

By Green's Theorem, $\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \int_0^4 \int_0^x (2x - 2y) dy dx = \int_0^4 x^2 dx = \frac{64}{3}.$



$$4. \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}, \quad 0 \leq t \leq 2\pi$$

$$\begin{aligned} \int_C y^2 dx + x^2 dy &= \int_0^{2\pi} [\sin^2 t (-\sin t dt) + \cos^2 t (\cos t dt)] \\ &= \int_0^{2\pi} (\cos^3 t - \sin^3 t) dt \\ &= \int_0^{2\pi} [\cos t (1 - \sin^2 t) - \sin t (1 - \cos^2 t)] dt \\ &= \left[\sin t - \frac{\sin^3 t}{3} + \cos t - \frac{\cos^3 t}{3} \right]_0^{2\pi} = 0 \end{aligned}$$



By Green's Theorem,

$$\begin{aligned} \int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2x - 2y) dy dx \\ &= \int_0^{2\pi} \int_0^1 (2r \cos \theta - 2r \sin \theta) r dr d\theta = \frac{2}{3} \int_0^{2\pi} (\cos \theta - \sin \theta) d\theta = \frac{2}{3}(0) = 0. \end{aligned}$$

6. C: boundary of the region lying between the graphs of $y = x$ and $y = x^3$

$$\begin{aligned} \int_C x e^y dx + e^x dy &= \int_0^1 (x e^{x^3} + 3x^2 e^x) dx + \int_1^0 (x e^x + e^x) dx \approx 2.936 - 2.718 \approx 0.22 \\ \int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA &= \int_0^1 \int_{x^3}^x (e^x - x e^y) dy dx = \int_0^1 (x e^{x^3} - x^3 e^x) dx \approx 0.22 \end{aligned}$$

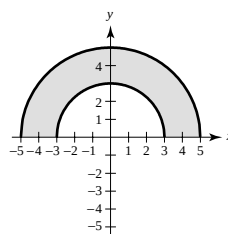
In Exercises 8 and 10, $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$.

8. Since C is an ellipse with $a = 2$ and $b = 1$, then R is an ellipse of area $\pi ab = 2\pi$. Thus, Green's Theorem yields

$$\int_C (y - x) dx + (2x - y) dy = \iint_R 1 dA = \text{Area of ellipse} = 2\pi.$$

10. R is the shaded region of the accompanying figure.

$$\begin{aligned} \int_C (y - x) dx + (2x - y) dy &= \iint_R 1 dA \\ &= \text{Area of shaded region} \\ &= \frac{1}{2} \pi [25 - 9] = 8\pi \end{aligned}$$



12. The given curves intersect at $(0, 0)$ and $(9, 3)$. Thus, Green's Theorem yields

$$\begin{aligned}\int_C y^2 dx + xy dy &= \iint_R (y - 2y) dA \\ &= \int_0^9 \int_0^{\sqrt{x}} -y dy dx = \int_0^9 \left[\frac{-y^2}{2} \right]_0^{\sqrt{x}} dx = \int_0^9 \frac{-x}{2} dx = \left[\frac{-x^2}{4} \right]_0^9 = -\frac{81}{4}\end{aligned}$$

14. In this case, let $y = r \sin \theta$, $x = r \cos \theta$. Then $dA = r dr d\theta$ and Green's Theorem yields

$$\begin{aligned}\int_C (x^2 - y^2) dx + 2xy dy &= \iint_R 4y dA = 4 \int_0^{2\pi} \int_0^{1+\cos\theta} r \sin \theta r dr d\theta \\ &= 4 \int_0^{2\pi} \int_0^{1+\cos\theta} r^2 \sin \theta dr d\theta \\ &= \frac{4}{3} \int_0^{2\pi} \sin \theta (1 + \cos \theta)^3 d\theta \\ &= \left[-\frac{(1 + \cos \theta)^4}{3} \right]_0^{2\pi} = 0.\end{aligned}$$

16. Since $\frac{\partial M}{\partial y} = -2e^x \sin 2y = \frac{\partial N}{\partial x}$ we have

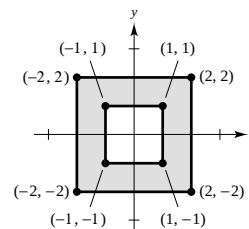
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = 0.$$

18. By Green's Theorem,

$$\int_C (e^{-x^2/2} - y) dx + (e^{-y^2/2} + x) dy = \iint_R 2 dA = 2(\text{Area of } R) = 2[\pi(6)^2 - \pi(2)(3)] = 60\pi.$$

20. By Green's Theorem,

$$\begin{aligned}\int_C 3x^2 e^y dx + e^y dy &= \iint_R -3x^2 e^y dA \\ &= \int_1^2 \int_{-2}^2 -3x^2 e^y dy dx + \int_{-1}^1 \int_1^2 -3x^2 e^y dy dx \\ &\quad + \int_{-2}^{-1} \int_{-2}^2 -3x^2 e^y dy dx + \int_{-1}^1 \int_{-2}^{-1} -3x^2 e^y dy dx \\ &= -7(e^2 - e^{-2}) - 2(e^2 - e) - 7(e^2 - e^{-2}) - 2(e^{-1} - e^{-2}) \\ &= -16e^2 + 16e^{-2} + 2e - 2e^{-1}.\end{aligned}$$



22. $\mathbf{F}(x, y) = (e^x - 3y)\mathbf{i} + (e^y + 6x)\mathbf{j}$

$C: r = 2 \cos \theta$

Work $= \int_C (e^x - 3y) dx + (e^y + 6x) dy = \iint_R 9 dA = 9\pi$ since $r = 2 \cos \theta$ is a circle with a radius of one.

24. $\mathbf{F}(x, y) = (3x^2 + y)\mathbf{i} + 4xy^2\mathbf{j}$

C : boundary of the region bounded by the graphs of $y = \sqrt{x}$, $y = 0$, $x = 4$

$$\text{Work} = \int_C (3x^2 + y) dx + 4xy^2 dy = \int_0^4 \int_0^{\sqrt{x}} (4y^2 - 1) dy dx = \int_0^4 \left(\frac{4}{3}x^{3/2} - x^{1/2} \right) dx = \frac{176}{15}$$

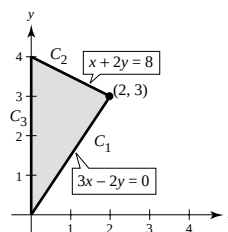
26. From the figure we see that

$$C_1: y = \frac{3}{2}x, \quad dy = \frac{3}{2}dx, \quad 0 \leq x \leq 2$$

$$C_2: y = -\frac{x}{2} + 4, \quad dy = -\frac{1}{2}dx$$

$$C_3: x = 0, \quad dx = 0.$$

$$\begin{aligned} A &= \frac{1}{2} \int_0^2 \left(\frac{3}{2}x - \frac{3}{2}x \right) dx + \frac{1}{2} \int_2^0 \left(-\frac{1}{2}x + \frac{x}{2} - 4 \right) dx + \frac{1}{2} \int_0^2 (-4) dx \\ &= \frac{1}{2} \int_2^0 (-4) dx = 2 \int_0^2 dx = 4 \end{aligned}$$



28. Since the loop of the folium is formed on the interval $0 \leq t \leq \infty$,

$$dx = \frac{3(1-2t^3)}{(t^3+1)^2} dt \quad \text{and} \quad dy = \frac{3(2t-t^4)}{(t^3+1)^2} dt,$$

we have

$$\begin{aligned} A &= \frac{1}{2} \int_0^\infty \left[\left(\frac{3t}{t^3+1} \right) \frac{3(2t-t^4)}{(t^3+1)^2} - \left(\frac{3t^2}{t^3+1} \right) \frac{3(1-2t^3)}{(t^3+1)^2} \right] dt \\ &= \frac{9}{2} \int_0^\infty \frac{t^5+t^2}{(t^3+1)^3} dt = \frac{9}{2} \int_0^\infty \frac{t^2(t^3+1)}{(t^3+1)^3} dt = \frac{3}{2} \int_0^\infty 3t^2(t^3+1)^{-2} dt = \left[\frac{-3}{2(t^3+1)} \right]_0^\infty = \frac{3}{2}. \end{aligned}$$

30. See Theorem 14.9: $A = \frac{1}{2} \int_C x dy - y dx$.

32. (a) For the moment about the x -axis, $M_x = \iint_R y dA$. Let $N = 0$ and $M = -y^2/2$. By Green's Theorem,

$$M_x = \int_C -\frac{y^2}{2} dx = -\frac{1}{2} \int_C y^2 dx \quad \text{and} \quad \bar{y} = \frac{M_x}{2A} = -\frac{1}{2A} \int_C y^2 dx.$$

For the moment about the y -axis, $M_y = \iint_R x dA$. Let $N = x^2/2$ and $M = 0$. By Green's Theorem,

$$M_y = \int_C \frac{x^2}{2} dy = \frac{1}{2} \int_C x^2 dy \quad \text{and} \quad \bar{x} = \frac{M_y}{2A} = \frac{1}{2A} \int_C x^2 dy.$$

(b) By Theorem 14.9 and the fact that $x = r \cos \theta$, $y = r \sin \theta$, we have

$$A = \frac{1}{2} \int_C x dy - y dx = \frac{1}{2} \int_C (r \cos \theta)(r \cos \theta) d\theta - (r \sin \theta)(-r \sin \theta) d\theta = \frac{1}{2} \int_C r^2 d\theta.$$

34. Since $A = \text{area of semicircle} = \frac{\pi a^2}{2}$, we have $\frac{1}{2A} = \frac{1}{\pi a^2}$. Note that $y = 0$ and $dy = 0$ along the boundary $y = 0$.

Let $x = a \cos t$, $y = a \sin t$, $0 \leq t \leq \pi$, then

$$\bar{x} = \frac{1}{\pi a^2} \int_0^\pi a^2 \cos^2 t (a \cos t) dt = \frac{a}{\pi} \int_0^\pi \cos^3 t dt = \frac{a}{\pi} \int_0^\pi (1 - \sin^2 t) \cos t dt = \frac{a}{\pi} \left[\sin t - \frac{\sin^3 t}{3} \right]_0^\pi = 0$$

$$\bar{y} = \frac{-1}{\pi a^2} \int_0^\pi a^2 \sin^2 t (-a \sin t dt) = \frac{a}{\pi} \int_0^\pi \sin^3 t dt = \frac{a}{\pi} \left[-\cos t + \frac{\cos^3 t}{3} \right]_0^\pi = \frac{4a}{3\pi}.$$

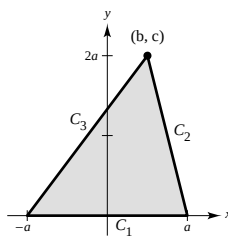
$$(\bar{x}, \bar{y}) = \left(0, \frac{4a}{3\pi} \right)$$

36. Since $A = \frac{1}{2}(2a)(c) = ac$, we have $\frac{1}{2A} = \frac{1}{2ac}$,

$$C_1: y = 0, \quad dy = 0$$

$$C_2: y = \frac{c}{b-a}(x-a), \quad dy = \frac{c}{b-a} dx$$

$$C_3: y = \frac{c}{b+a}(x+a), \quad dy = \frac{c}{b+a} dx.$$



Thus,

$$\bar{x} = \frac{1}{2ac} \int_C x^2 dy = \frac{1}{2ac} \left[\int_{-a}^a 0 + \int_a^b x^2 \frac{c}{b-a} dx + \int_b^{-a} x^2 \frac{c}{b+a} dx \right] = \frac{1}{2ac} \left[0 + \frac{2abc}{3} \right] = \frac{b}{3}$$

$$\begin{aligned} \bar{y} &= \frac{-1}{2ac} \int_C y^2 dx = \frac{-1}{2ac} \left[0 + \int_a^b \left(\frac{c}{b-a} \right)^2 (x-a)^2 dx + \int_b^{-a} \left(\frac{c}{b+a} \right)^2 (x+a)^2 dx \right] \\ &= \frac{-1}{2ac} \left[\frac{c^2(b-a)}{3} - \frac{c^2(b+a)}{3} \right] = \frac{c}{3}. \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{c}{3} \right)$$

38. $A = \frac{1}{2} \int_0^\pi a^2 \cos^2 3\theta d\theta = \frac{a^2}{2} \int_0^\pi \frac{1 + \cos 6\theta}{2} d\theta = \frac{a^2}{4} \left[\theta + \frac{\sin 6\theta}{6} \right]_0^\pi = \frac{\pi a^2}{4}$

Note: In this case R is enclosed by $r = a \cos 3\theta$ where $0 \leq \theta \leq \pi$.

40. In this case, $0 \leq \theta \leq 2\pi$ and we let

$$u = \frac{\sin \theta}{1 + \cos \theta}, \quad \cos \theta = \frac{1 - u^2}{1 + u^2}, \quad d\theta = \frac{2 du}{1 + u^2}.$$

Now $u \Rightarrow \infty$ as $\theta \Rightarrow \pi$ and we have

$$\begin{aligned} A &= 2 \left(\frac{1}{2} \right) \int_0^\pi \frac{9}{(2 - \cos \theta)^2} d\theta = 9 \int_0^\pi \frac{\frac{2du}{1+u^2}}{4 - 4 \left(\frac{1-u^2}{1+u^2} \right) + \frac{(1-u^2)^2}{(1+u^2)^2}} = 18 \int_0^\infty \frac{1+u^2}{(1+3u^2)^2} du \\ &= 18 \int_0^\infty \frac{1/3}{1+3u^2} du + 18 \int_0^\infty \frac{2/3}{(1+3u^2)^2} du = \left[\frac{6}{\sqrt{3}} \arctan \sqrt{3} u \right]_0^\infty + \frac{12}{\sqrt{3}} \left(\frac{1}{2} \right) \left[\frac{u}{1+3u^2} + \int \frac{\sqrt{3}}{1+3u^2} du \right]_0^\infty \\ &= \frac{6}{\sqrt{3}} \left(\frac{\pi}{2} \right) + \frac{6}{\sqrt{3}} \left[\frac{u}{1+3u^2} \right]_0^\infty + \left[\frac{6}{\sqrt{3}} \arctan \sqrt{3} u \right]_0^\infty = \frac{3\pi}{\sqrt{3}} + 0 + \frac{3\pi}{\sqrt{3}} = 2\sqrt{3}\pi. \end{aligned}$$

42. (a) Let C be the line segment joining (x_1, y_1) and (x_2, y_2) .

$$y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1$$

$$dy = \frac{y_2 - y_1}{x_2 - x_1} dx$$

$$\begin{aligned} \int_C -y dx + x dy &= \int_{x_1}^{x_2} \left[-\frac{y_2 - y_1}{x_2 - x_1}(x - x_1) - y_1 + x \left(\frac{y_2 - y_1}{x_2 - x_1} \right) \right] dx = \int_{x_1}^{x_2} \left[x_1 \left(\frac{y_2 - y_1}{x_2 - x_1} \right) - y_1 \right] dx \\ &= \left[\left[x_1 \left(\frac{y_2 - y_1}{x_2 - x_1} \right) - y_1 \right] x \right]_{x_1}^{x_2} = \left[x_1 \left(\frac{y_2 - y_1}{x_2 - x_1} \right) - y_1 \right] (x_2 - x_1) \\ &= x_1(y_2 - y_1) - y_1(x_2 - x_1) = x_1 y_2 - x_2 y_1 \end{aligned}$$

—CONTINUED—

42. —CONTINUED—

(b) Let C be the boundary of the region $A = \frac{1}{2} \int_C -y \, dx + x \, dy = \frac{1}{2} \iint_R (1 - (-1)) \, dA = \iint_R dA$.

Therefore,

$$\iint_R dA = \frac{1}{2} \left[\int_{C_1} -y \, dx + x \, dy + \int_{C_2} -y \, dx + x \, dy + \cdots + \int_{C_n} -y \, dx + x \, dy \right]$$

where C_1 is the line segment joining (x_1, y_1) and (x_2, y_2) , C_2 is the line segment joining (x_2, y_2) and (x_3, y_3) , . . . , and C_n is the line segment joining (x_n, y_n) and (x_1, y_1) . Thus,

$$\iint_R dA = \frac{1}{2} [(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \cdots + (x_{n-1} y_n - x_n y_{n-1}) + (x_n y_1 - x_1 y_n)].$$

44. Hexagon: $(0, 0), (2, 0), (3, 2), (2, 4), (0, 3), (-1, 1)$

$$A = \frac{1}{2} [(0 - 0) + (4 - 0) + (12 - 4) + (6 - 0) + (0 + 3) + (0 - 0)] = \frac{21}{2}$$

46. Since $\int_C \mathbf{F} \cdot \mathbf{N} \, ds = \int_R \int \operatorname{div} \mathbf{F} \, dA$, then

$$\begin{aligned} \int_C f D_{\mathbf{N}} g \, ds &= \int_C f \nabla g \cdot \mathbf{N} \, ds \\ &= \int_R \int \operatorname{div}(f \nabla g) \, dA = \int_R \int (f \operatorname{div}(\nabla g) + \nabla f \cdot \nabla g) \, dA = \int_R \int (f \nabla^2 g + \nabla f \cdot \nabla g) \, dA. \end{aligned}$$

48. $\int_C f(x) \, dx + g(y) \, dy = \int_R \int \left[\frac{\partial}{\partial x} g(y) - \frac{\partial}{\partial y} f(x) \right] \, dA = \int_R \int (0 - 0) \, dA = 0$

Section 14.5 Parametric Surfaces

2. $\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + u \mathbf{k}$

$$x^2 + y^2 = z^2$$

Matches d.

4. $\mathbf{r}(u, v) = 4 \cos u \mathbf{i} + 4 \sin u \mathbf{j} + v \mathbf{k}$

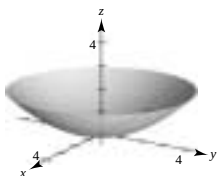
$$x^2 + y^2 = 16$$

Matches a.

6. $\mathbf{r}(u, v) = 2u \cos v \mathbf{i} + 2u \sin v \mathbf{j} + \frac{1}{2}u^2 \mathbf{k}$

$$z = \frac{1}{2}u^2, x^2 + y^2 = 4u^2 \Rightarrow z = \frac{1}{8}(x^2 + y^2)$$

Paraboloid



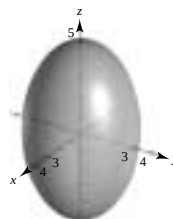
8. $\mathbf{r}(u, v) = 3 \cos v \cos u \mathbf{i} + 3 \cos v \sin u \mathbf{j} + 5 \sin v \mathbf{k}$

$$x^2 + y^2 = 9 \cos^2 v \cos^2 u + 9 \cos^2 v \sin^2 u = 9 \cos^2 v$$

$$\frac{x^2 + y^2}{9} + \frac{z^2}{25} = \cos^2 v + \sin^2 v = 1$$

$$\frac{x^2}{9} + \frac{y^2}{9} + \frac{z^2}{25} = 1$$

Ellipsoid

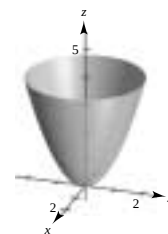


For Exercises 10 and 12,

$$\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + u^2 \mathbf{k}, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi.$$

Eliminating the parameter yields

$$z = x^2 + y^2, \quad 0 \leq z \leq 4.$$



10. $\mathbf{s}(u, v) = u \cos v \mathbf{i} + u^2 \mathbf{j} + u \sin v \mathbf{k}, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi$

$$y = x^2 + z^2$$

The paraboloid opens along the y-axis instead of the z-axis.

12. $\mathbf{s}(u, v) = 4u \cos v \mathbf{i} + 4u \sin v \mathbf{j} + u^2 \mathbf{k}, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi$

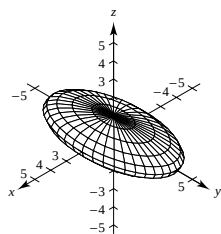
$$z = \frac{x^2 + y^2}{16}$$

The paraboloid is “wider.” The top is now the circle $x^2 + y^2 = 64$. It was $x^2 + y^2 = 4$.

14. $\mathbf{r}(u, v) = 2 \cos v \cos u \mathbf{i} + 4 \cos v \sin u \mathbf{j} + \sin v \mathbf{k},$

$$0 \leq u \leq 2\pi, \quad 0 \leq v \leq 2\pi$$

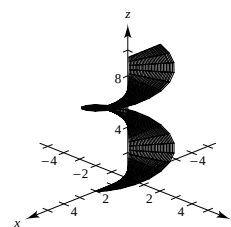
$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{1} = 1$$



16. $\mathbf{r}(u, v) = 2u \cos v \mathbf{i} + 2u \sin v \mathbf{j} + v \mathbf{k},$

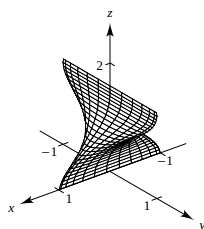
$$0 \leq u \leq 1, \quad 0 \leq v \leq 3\pi$$

$$z = \arctan\left(\frac{y}{x}\right)$$



18. $\mathbf{r}(u, v) = \cos^3 u \cos v \mathbf{i} + \sin^3 u \sin v \mathbf{j} + u \mathbf{k},$

$$0 \leq u \leq \frac{\pi}{2}, \quad 0 \leq v \leq 2\pi$$



20. $z = 6 - x - y$

$$\mathbf{r}(u, v) = u \mathbf{i} + v \mathbf{j} + (6 - u - v) \mathbf{k}$$

22. $4x^2 + y^2 = 16$

$$\mathbf{r}(u, v) = 2 \cos u \mathbf{i} + 4 \sin u \mathbf{j} + v \mathbf{k}$$

24. $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{1} = 1$

$$\mathbf{r}(u, v) = 3 \cos v \cos u \mathbf{i} + 2 \cos v \sin u \mathbf{j} + \sin v \mathbf{k}$$

26. $z = x^2 + y^2$ inside $x^2 + y^2 = 9$.

$$\mathbf{r}(u, v) = v \cos u \mathbf{i} + v \sin u \mathbf{j} + v^2 \mathbf{k}, \quad 0 \leq v \leq 3$$

28. Function: $y = x^{3/2}, \quad 0 \leq x \leq 4$

Axis of revolution: x-axis

$$x = u, \quad y = u^{3/2} \cos v, \quad z = u^{3/2} \sin v$$

$$0 \leq u \leq 4, \quad 0 \leq v \leq 2\pi$$

30. Function: $z = 4 - y^2, \quad 0 \leq y \leq 2$

Axis of revolution: y-axis

$$x = (4 - u^2) \cos v, \quad y = u, \quad z = (4 - u^2) \sin v$$

$$0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi$$

32. $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \sqrt{uv}\mathbf{k}, (1, 1, 1)$

$$\mathbf{r}_u(u, v) = \mathbf{i} + \frac{v}{2\sqrt{uv}}\mathbf{k}, \quad \mathbf{r}_v(u, v) = \mathbf{j} + \frac{u}{2\sqrt{uv}}\mathbf{k}$$

At $(1, 1, 1)$, $u = 1$ and $v = 1$.

$$\mathbf{r}_u(1, 1) = \mathbf{i} + \frac{1}{2}\mathbf{k}, \quad \mathbf{r}_v(1, 1) = \mathbf{j} + \frac{1}{2}\mathbf{k}$$

$$\mathbf{N} = \mathbf{r}_u(1, 1) \times \mathbf{r}_v(1, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}$$

Direction numbers: 1, 1, -2

Tangent plane: $(x - 1) + (y - 1) - 2(z - 1) = 0$

$$x + y - 2z = 0$$

34. $\mathbf{r}(u, v) = 2u \cosh v\mathbf{i} + 2u \sinh v\mathbf{j} + \frac{1}{2}u^2\mathbf{k}$

$$\mathbf{r}_u(u, v) = 2 \cosh v\mathbf{i} + 2 \sinh v\mathbf{j} + u\mathbf{k}$$

$$\mathbf{r}_v(u, v) = 2u \sinh v\mathbf{i} + 2u \cosh v\mathbf{j}$$

At $(-4, 0, 2)$, $u = -2$ and $v = 0$.

$$\mathbf{r}_u(-2, 0) = 2\mathbf{i} - 2\mathbf{k}, \quad \mathbf{r}_v(-2, 0) = -4\mathbf{j}$$

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = -8\mathbf{i} - 8\mathbf{k}$$

Direction numbers: 1, 0, 1

Tangent plane: $(x + 4) + (z - 2) = 0$

$$x + z = -2$$

36. $\mathbf{r}(u, v) = 4u \cos v\mathbf{i} + 4u \sin v\mathbf{j} + u^2\mathbf{k}, 0 \leq u \leq 2, 0 \leq v \leq 2\pi$

$$\mathbf{r}_u(u, v) = 4 \cos v\mathbf{i} + 4 \sin v\mathbf{j} + 2u\mathbf{k}$$

$$\mathbf{r}_v(u, v) = -4u \sin v\mathbf{i} + 4u \cos v\mathbf{j}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 \cos v & 4 \sin v & 2u \\ -4u \sin v & 4u \cos v & 0 \end{vmatrix} = -8u^2 \cos v\mathbf{i} - 8u^2 \sin v\mathbf{j} + 16u\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{64u^4 + 256u^2} = 8u\sqrt{u^2 + 4}$$

$$A = \int_0^{2\pi} \int_0^2 8u\sqrt{u^2 + 4} \, du \, dv = \int_0^{2\pi} \left(\frac{128\sqrt{2}}{3} - \frac{64}{3} \right) dv = \frac{128\pi}{3}(2\sqrt{2} - 1)$$

38. $\mathbf{r}(u, v) = a \sin u \cos v\mathbf{i} + a \sin u \sin v\mathbf{j} + a \cos u\mathbf{k}, 0 \leq u \leq \pi, 0 \leq v \leq 2\pi$

$$\mathbf{r}_u(u, v) = a \cos u \cos v\mathbf{i} + a \cos u \sin v\mathbf{j} - a \sin u\mathbf{k}$$

$$\mathbf{r}_v(u, v) = -a \sin u \sin v\mathbf{i} + a \sin u \cos v\mathbf{j}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos u \cos v & a \cos u \sin v & -a \sin u \\ -a \sin u \sin v & a \sin u \cos v & 0 \end{vmatrix} = a^2 \sin^2 u \cos v\mathbf{i} + a^2 \sin^2 u \sin v\mathbf{j} + a^2 \sin u \cos u\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = a^2 \sin u$$

$$A = \int_0^{2\pi} \int_0^\pi a^2 \sin u \, du \, dv = 4\pi a^2$$

40. $\mathbf{r}(u, v) = (a + b \cos v) \cos u\mathbf{i} + (a + b \cos v) \sin u\mathbf{j} + b \sin v\mathbf{k}, a > b, 0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$

$$\mathbf{r}_u(u, v) = -(a + b \cos v) \sin u\mathbf{i} + (a + b \cos v) \cos u\mathbf{j}$$

$$\mathbf{r}_v(u, v) = -b \sin v \cos u\mathbf{i} - b \sin v \sin u\mathbf{j} + b \cos v\mathbf{k}$$

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -(a + b \cos v) \sin u & (a + b \cos v) \cos u & 0 \\ -b \sin v \cos u & -b \sin v \sin u & b \cos v \end{vmatrix} \\ &= b \cos u \cos v(a + b \cos v)\mathbf{i} + b \sin u \cos v(a + b \cos v)\mathbf{j} + b \sin v(a + b \cos v)\mathbf{k} \end{aligned}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = b(a + b \cos v)$$

$$A = \int_0^{2\pi} \int_0^{2\pi} b(a + b \cos v) \, du \, dv = 4\pi^2 ab$$

42. $\mathbf{r}(u, v) = \sin u \cos v \mathbf{i} + u \mathbf{j} + \sin u \sin v \mathbf{k}$, $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$

$$\mathbf{r}_u(u, v) = \cos u \cos v \mathbf{i} + \mathbf{j} + \cos u \sin v \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -\sin u \sin v \mathbf{i} + \sin u \cos v \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \sin u \cos v \mathbf{i} - \cos u \sin v \mathbf{j} + \sin u \sin v \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sin u \sqrt{1 + \cos^2 u}$$

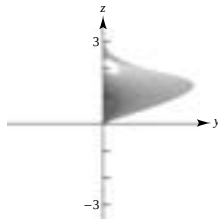
$$A = \int_0^{2\pi} \int_0^\pi \sin u \sqrt{1 + \cos^2 u} \, du \, dv = \pi \left[2\sqrt{2} + \ln \left| \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right| \right]$$

44. See the definition, page 1055.

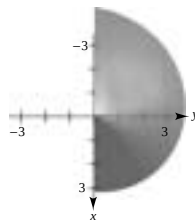
46. Graph of $\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + v \mathbf{k}$

$$0 \leq u \leq \pi, 0 \leq v \leq \pi \text{ from}$$

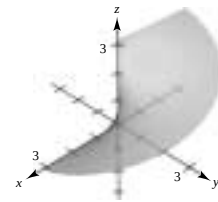
(a) $(10, 0, 0)$



(b) $(0, 0, 10)$



(c) $(10, 10, 10)$



48. $\mathbf{r}(u, v) = 2u \cos v \mathbf{i} + 2u \sin v \mathbf{j} + v \mathbf{k}$, $0 \leq u \leq 1$, $0 \leq v \leq 3\pi$

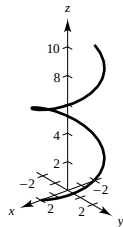
(a) If $u = 1$:

$$\mathbf{r}(1, v) = 2 \cos v \mathbf{i} + 2 \sin v \mathbf{j} + v \mathbf{k}$$

$$x^2 + y^2 = 4$$

$$0 \leq z \leq 3\pi$$

Helix



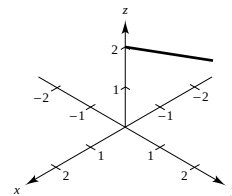
(b) If $v = \frac{2\pi}{3}$:

$$\mathbf{r}\left(u, \frac{2\pi}{3}\right) = -u \mathbf{i} + \sqrt{3}u \mathbf{j} + \frac{2\pi}{3} \mathbf{k}$$

$$y = -\sqrt{3}x$$

$$z = \frac{2\pi}{3}$$

Line



(c) If one parameter is held constant, the result is a **curve** in 3-space.

50. $x^2 + y^2 - z^2 = 1$

Let $x = u \cos v$, $y = u \sin v$, and $z = \sqrt{u^2 - 1}$. Then,

$$\mathbf{r}_u(u, v) = \cos v \mathbf{i} + \sin v \mathbf{j} + \frac{u}{\sqrt{u^2 - 1}} \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -u \sin v \mathbf{i} + u \cos v \mathbf{j}.$$

At $(1, 0, 0)$, $u = 1$ and $v = 0$. $\mathbf{r}_u(1, 0)$ is undefined and $\mathbf{r}_v(1, 0) = \mathbf{j}$. The tangent plane at $(1, 0, 0)$ is $x = 1$.

$$52. \mathbf{r}(u, v) = u\mathbf{i} + f(u) \cos v \mathbf{j} + f(u) \sin v \mathbf{k}, \quad a \leq u \leq b, \quad 0 \leq v \leq 2\pi$$

$$\mathbf{r}_u(u, v) = \mathbf{i} + f'(u) \cos v \mathbf{j} + f'(u) \sin v \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -f(u) \sin v \mathbf{j} + f(u) \cos v \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(u) \cos v & f'(u) \sin v \\ 0 & -f(u) \sin v & f(u) \cos v \end{vmatrix} = f(u)f'(u)\mathbf{i} - f(u) \cos v \mathbf{j} - f(u) \sin v \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = f(u)\sqrt{1 + [f'(u)]^2}$$

$$\begin{aligned} A &= \int_0^{2\pi} \int_a^b f(u)\sqrt{1 + [f'(u)]^2} \, du \, dv \\ &= 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2} \, dx \quad (\text{since } u = x) \end{aligned}$$

Section 14.6 Surface Integrals

$$2. S: z = 15 - 2x + 3y, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 4, \quad \frac{\partial z}{\partial x} = -2, \quad \frac{\partial z}{\partial y} = 3, \quad dS = \sqrt{1 + 4 + 9} \, dy \, dx = \sqrt{14} \, dy \, dx$$

$$\begin{aligned} \iint_S (x - 2y + z) \, dS &= \int_0^2 \int_0^4 (x - 2y + 15 - 2x + 3y) \sqrt{14} \, dy \, dx \\ &= \sqrt{14} \int_0^2 \int_0^4 (15 - x + y) \, dy \, dx = 128\sqrt{14} \end{aligned}$$

$$4. S: z = \frac{2}{3}x^{3/2}, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq x, \quad \frac{\partial z}{\partial x} = x^{1/2}, \quad \frac{\partial z}{\partial y} = 0$$

$$\begin{aligned} \iint_S (x - 2y + z) \, dS &= \int_0^1 \int_0^x \left(x - 2y + \frac{2}{3}x^{3/2} \right) \sqrt{1 + (x^{1/2})^2 + (0)^2} \, dy \, dx \\ &= \int_0^1 \int_0^x \left(x - 2y + \frac{2}{3}x^{3/2} \right) \sqrt{1 + x} \, dy \, dx \\ &= \frac{2}{3} \int_0^1 x^{5/2} \sqrt{x+1} \, dx \\ &= \frac{2}{3} \left[\frac{1}{4} x^{5/2} (1+x)^{3/2} \right]_0^1 - \frac{5}{12} \int_0^1 x^{3/2} \sqrt{1+x} \, dx \\ &= \left[\frac{1}{6} x^{5/2} (1+x)^{3/2} \right]_0^1 - \frac{5}{12} \left(\frac{1}{3} \right) \left[x^{3/2} (1+x)^{3/2} \right]_0^1 + \frac{5}{24} \int_0^1 x^{1/2} \sqrt{1+x} \, dx \\ &= \frac{\sqrt{2}}{3} - \frac{5\sqrt{2}}{18} + \frac{5}{24} \int_0^1 \sqrt{x+x^2} \, dx \\ &= \frac{\sqrt{2}}{18} + \frac{5}{24} \int_0^1 \sqrt{\left(x + \frac{1}{2}\right)^2 - \frac{1}{4}} \, dx \\ &= \frac{\sqrt{2}}{18} + \frac{5}{24} \left(\frac{1}{2} \right) \left[\left(x + \frac{1}{2}\right) \sqrt{x^2 + x} - \frac{1}{4} \ln \left| \left(x + \frac{1}{2}\right) + \sqrt{x^2 + x} \right| \right]_0^1 \\ &= \frac{\sqrt{2}}{18} + \frac{5}{48} \left[\frac{3}{2} \sqrt{2} - \frac{1}{4} \ln \left| \frac{3}{2} + \sqrt{2} \right| + \frac{1}{4} \ln \left| \frac{1}{2} \right| \right] \\ &= \frac{\sqrt{2}}{18} + \frac{15\sqrt{2}}{96} + \frac{5}{192} \ln \left| \frac{1}{3 + 2\sqrt{2}} \right| = \frac{61\sqrt{2}}{288} - \frac{5}{192} \ln|3 + 2\sqrt{2}| \approx 0.2536 \end{aligned}$$

6. $S: z = h, 0 \leq x \leq 2, 0 \leq y \leq \sqrt{4 - x^2}, \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$

$$\iint_S dx \, dS = \int_0^2 \int_0^{\sqrt{4-x^2}} xy \, dy \, dx = \frac{1}{2} \int_0^2 x(4 - x^2) \, dx = \frac{1}{2} \left[2x^2 - \frac{x^4}{4} \right]_0^2 = 2$$

8. $S: z = \frac{1}{2}xy, 0 \leq x \leq 4, 0 \leq y \leq 4, \frac{\partial z}{\partial x} = \frac{1}{2}y, \frac{\partial z}{\partial y} = \frac{1}{2}x$

$$\iint_S xy \, dS = \int_0^4 \int_0^4 xy \sqrt{1 + \frac{y^2}{4} + \frac{x^2}{4}} \, dy \, dx = \frac{3904}{15} - \frac{160\sqrt{5}}{3}$$

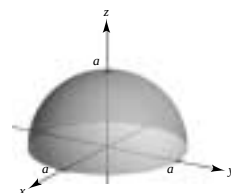
10. $S: z = \cos x, 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \frac{x}{2}$

$$\iint_S (x^2 - 2xy) \, dS = \int_0^{\pi/2} \int_0^{x/2} (x^2 - 2xy) \sqrt{1 + \sin^2 x} \, dy \, dx = \int_0^{\pi/2} \frac{x^3}{4} \sqrt{1 + \sin^2 x} \, dx \approx 0.52$$

12. $S: z = \sqrt{a^2 - x^2 - y^2}$

$$\rho(x, y, z) = kz$$

$$\begin{aligned} m &= \iint_S kz \, dS = \iint_R k \sqrt{a^2 - x^2 - y^2} \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2 - y^2}} \right)^2 + \left(\frac{-y}{\sqrt{a^2 - x^2 - y^2}} \right)^2} \, dA \\ &= \iint_R k \sqrt{a^2 - x^2 - y^2} \left(\frac{a}{\sqrt{a^2 - x^2 - y^2}} \right) \, dA \\ &= \iint_R ka \, dA = ka \iint_R dA = ka(2\pi a^2) = 2ka^3\pi \end{aligned}$$



14. $S: \mathbf{r}(u, v) = 2 \cos u \mathbf{i} + 2 \sin u \mathbf{j} + v \mathbf{k}, 0 \leq u \leq \frac{\pi}{2}, 0 \leq v \leq 2$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \|2 \cos u \mathbf{i} + 2 \sin u \mathbf{j}\| = 2$$

$$\iint_S (x + y) \, dS = \int_0^2 \int_0^{\pi/2} (2 \cos u + 2 \sin u) 2 \, du \, dv = 16$$

16. $S: \mathbf{r}(u, v) = 4u \cos v \mathbf{i} + 4u \sin v \mathbf{j} + 3u \mathbf{k}, 0 \leq u \leq 4, 0 \leq v \leq \pi$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \|-12u \cos v \mathbf{i} - 12u \sin v \mathbf{j} + 16u \mathbf{k}\| = 20u$$

$$\iint_S (x + y) \, dS = \int_0^\pi \int_0^4 (4u \cos v + 4u \sin v) 20u \, du \, dv = \frac{10,240}{3}$$

18. $f(x, y, z) = \frac{xy}{z}$

$$S: z = x^2 + y^2, 4 \leq x^2 + y^2 \leq 16$$

$$\begin{aligned} \iint_S f(x, y, z) \, dS &= \iint_S \frac{xy}{x^2 + y^2} \sqrt{1 + 4x^2 + 4y^2} \, dy \, dx = \int_0^{2\pi} \int_2^4 \frac{r^2 \sin \theta \cos \theta}{r^2} \sqrt{1 + 4r^2} r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_2^4 r \sqrt{1 + 4r^2} \sin \theta \cos \theta \, dr \, d\theta = \int_0^{2\pi} \left[\frac{1}{12} (1 + 4r^2)^{3/2} \right]_2^4 \sin \theta \cos \theta \, d\theta \\ &= \left[\frac{65\sqrt{65}}{12} - \frac{17\sqrt{17}}{12} \left(\frac{\sin^2 \theta}{2} \right) \right]_0^{2\pi} = 0 \end{aligned}$$

20. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$S: z = \sqrt{x^2 + y^2}, (x-1)^2 + y^2 \leq 1$

$$\begin{aligned} \iint_S f(x, y, z) dS &= \iint_S \sqrt{x^2 + y^2 + (\sqrt{x^2 + y^2})^2} \sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2} dy dx \\ &= \iint_S \sqrt{2(x^2 + y^2)} \sqrt{\frac{2(x^2 + y^2)}{x^2 + y^2}} dy dx \\ &= 2 \iint_S \sqrt{x^2 + y^2} dy dx = 2 \int_0^\pi \int_0^{2 \cos \theta} r^2 dr d\theta \\ &= \frac{16}{3} \int_0^\pi \cos^3 \theta d\theta = \frac{16}{3} \int_0^\pi (1 - \sin^2 \theta) \cos \theta d\theta \\ &= \left[\frac{16}{3} \left(\sin \theta - \frac{\sin^3 \theta}{3} \right) \right]_0^\pi = 0 \end{aligned}$$

22. $f(x, y, z) = x^2 + y^2 + z^2$

$S: x^2 + y^2 = 9, 0 \leq x \leq 3, 0 \leq z \leq x$

Project the solid onto the xz -plane; $y = \sqrt{9 - x^2}$.

$$\begin{aligned} \iint_S f(x, y, z) dS &= \int_0^3 \int_0^x [x^2 + (9 - x^2) + z^2] \sqrt{1 + \left(\frac{-x}{\sqrt{9 - x^2}}\right)^2 + (0)^2} dz dx \\ &= \int_0^3 \int_0^x (9 + z^2) \frac{3}{\sqrt{9 - x^2}} dz dx = \int_0^3 \left[\frac{3}{\sqrt{9 - x^2}} \left(9z + \frac{z^3}{3} \right) \right]_0^x dx \\ &= \int_0^3 \frac{3}{\sqrt{9 - x^2}} \left(9x + \frac{x^3}{3} \right) dx = \int_0^3 27x(9 - x^2)^{-1/2} dx + \int_0^3 x^3(9 - x^2)^{-1/2} dx \end{aligned}$$

Let $u = x^2$, $dv = x(9 - x^2)^{-1/2} dx$, then $du = 2x dx$, $v = -\sqrt{9 - x^2}$.

$$\begin{aligned} &= \left[-27\sqrt{9 - x^2} \right]_0^3 + \left[\left[-x^2\sqrt{9 - x^2} \right]_0^3 + \int_0^3 2x\sqrt{9 - x^2} dx \right] \\ &= \left[81 - \frac{2}{3}(9 - x^2)^{3/2} \right]_0^3 = 81 + 18 = 99 \end{aligned}$$

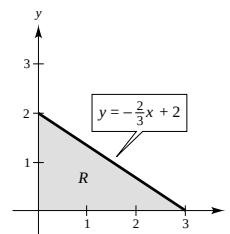
24. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j}$

$S: 2x + 3y + z = 6$ (first octant)

$G(x, y, z) = 2x + 3y + z - 6$

$\nabla G(x, y, z) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{N} dS &= \iint_R \mathbf{F} \cdot \nabla G dA = \int_0^3 \int_0^{-(2x/3)+2} (2x + 3y) dy dx \\ &= \int_0^3 \left[-\frac{4}{3}x^2 + 4x + \frac{3}{2} \left(-\frac{2}{3}x + 2 \right)^2 \right] dx \\ &= \left[-\frac{4}{9}x^3 + 2x^2 - \frac{3}{4} \left(-\frac{2}{3}x + 2 \right)^3 \right]_0^3 = 12 \end{aligned}$$



26. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$S: x^2 + y^2 + z^2 = 36 \quad (\text{first octant})$$

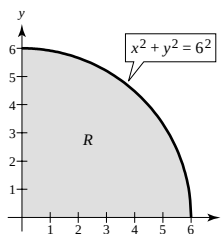
$$z = \sqrt{36 - x^2 - y^2}$$

$$G(x, y, z) = z - \sqrt{36 - x^2 - y^2}$$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{36 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{36 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\mathbf{F} \cdot \nabla G = \frac{x^2}{\sqrt{36 - x^2 - y^2}} + \frac{y^2}{\sqrt{36 - x^2 - y^2}} + z = \frac{36}{\sqrt{36 - x^2 - y^2}}$$

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{N} \, dS &= \iint_R \mathbf{F} \cdot \nabla G \, dA = \iint_R \frac{36}{\sqrt{36 - x^2 - y^2}} \, dA \\ &= \int_0^{\pi/2} \int_0^6 \frac{36}{\sqrt{36 - r^2}} r \, dr \, d\theta \quad (\text{improper}) \\ &= 108\pi \end{aligned}$$

28. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$

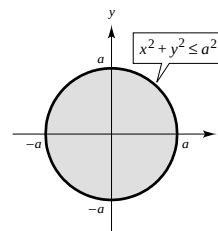
$$S: z = \sqrt{a^2 - x^2 - y^2}$$

$$G(x, y, z) = z - \sqrt{a^2 - x^2 - y^2}$$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{a^2 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{a^2 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\mathbf{F} \cdot \nabla G = \frac{x^2}{\sqrt{a^2 - x^2 - y^2}} + \frac{y^2}{\sqrt{a^2 - x^2 - y^2}} - 2\sqrt{a^2 - x^2 - y^2} = \frac{3x^2 + 3y^2 - 2a^2}{\sqrt{a^2 - x^2 - y^2}}$$

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{N} \, dS &= \iint_R \mathbf{F} \cdot \nabla G \, dA = \iint_R \frac{3x^2 + 3y^2 - 2a^2}{\sqrt{a^2 - x^2 - y^2}} \, dA \\ &= \int_0^{2\pi} \int_0^a \frac{3r^2 - 2a^2}{\sqrt{a^2 - r^2}} r \, dr \, d\theta \\ &= 3 \int_0^{2\pi} \int_0^a \frac{r^3}{\sqrt{a^2 - r^2}} \, dr \, d\theta - 2a^2 \int_0^{2\pi} \int_0^a \frac{r}{\sqrt{a^2 - r^2}} \, dr \, d\theta \\ &= 3 \left[\int_0^{2\pi} \left[-r^2 \sqrt{a^2 - r^2} - \frac{2}{3}(a^2 - r^2)^{3/2} \right]_0^a d\theta \right] - 2a^2 \int_0^{2\pi} \left[-\sqrt{a^2 - r^2} \right]_0^a d\theta \\ &= 3 \int_0^{2\pi} \frac{2}{3} a^3 \, d\theta - 2a^2 \int_0^{2\pi} a \, d\theta = 0 \end{aligned}$$

30. $\mathbf{F}(x, y, z) = (x + y)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$S: z = 1 - x^2 - y^2, \quad z = 0$$

$$G(x, y, z) = z + x^2 + y^2 - 1$$

$$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$$

$$\mathbf{F} \cdot \nabla G = 2x(x + y) + 2y(y) + (1 - x^2 - y^2) = x^2 + 2xy + y^2 + 1$$

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{N} \, dS &= \iint_R \mathbf{F} \cdot \nabla G \, dA = \iint_R (x^2 + 2xy + y^2 + 1) \, dA \\ &= \int_0^{2\pi} \int_0^1 (r^2 + 2r^2 \cos \theta \sin \theta + 1) r \, dr \, d\theta \\ &= \int_0^{2\pi} \left(\frac{3}{4} + \frac{1}{2} \sin \theta \cos \theta \right) d\theta = \left[\frac{3}{4}\theta + \frac{\sin^2 \theta}{4} \right]_0^{2\pi} = \frac{3\pi}{2} \end{aligned}$$

The flux across the bottom $z = 0$ is zero.

32. A surface is orientable if a unit normal vector N can be defined at every nonboundary point of S in such a way that the normal vectors vary continuously over the surface S .

34. Orientable

36. $\mathbf{E} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$$S: z = \sqrt{1 - x^2 - y^2}$$

$$\begin{aligned} \iint_S \mathbf{E} \cdot \mathbf{N} \, dS &= \iint_R \mathbf{E} \cdot (-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}) \, dA \\ &= \iint_R (yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) \cdot \left(\frac{x}{\sqrt{1-x^2-y^2}}\mathbf{i} + \frac{y}{\sqrt{1-x^2-y^2}}\mathbf{j} + \mathbf{k} \right) \, dA \\ &= \iint_R \left(\frac{2xyz}{\sqrt{1-x^2-y^2}} + xy \right) \, dA = \iint_R 3xy \, dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3xy \, dy \, dx = 0 \end{aligned}$$

38. $x^2 + y^2 + z^2 = a^2$

$$z = \pm \sqrt{a^2 - x^2 - y^2}$$

$$\begin{aligned} m &= 2 \iint_S k \, dS = 2k \iint_R \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2 - y^2}} \right)^2 + \left(\frac{-y}{\sqrt{a^2 - x^2 - y^2}} \right)^2} \, dA \\ &= 2k \iint_R \frac{a}{\sqrt{a^2 - x^2 - y^2}} \, dA = 2ka \int_0^{2\pi} \int_0^a \frac{r}{\sqrt{a^2 - r^2}} \, dr \, d\theta \\ &= 2ka \left[-\sqrt{a^2 - r^2} \right]_0^a (2\pi) = 4\pi ka^2 \end{aligned}$$

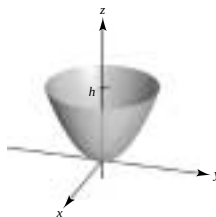
$$\begin{aligned} I_z &= 2 \iint_S k(x^2 + y^2) \, dS \\ &= 2k \iint_R (x^2 + y^2) \frac{a}{\sqrt{a^2 - x^2 - y^2}} \, dA = 2ka \int_0^{2\pi} \int_0^a \frac{r^3}{\sqrt{a^2 - r^2}} \, dr \, d\theta \quad (\text{use integration by parts}) \\ &= 2ka \left[-r^2 \sqrt{a^2 - r^2} - \frac{2}{3}(a^2 - r^2)^{3/2} \right]_0^a (2\pi) \\ &= 2ka \left(\frac{2}{3}a^3 \right) (2\pi) = \frac{2}{3}a^2(4\pi ka^2) = \frac{2}{3}a^2 m \end{aligned}$$

$$\text{Let } u = r^2, \, dv = r(a^2 - r^2)^{-1/2} \, dr, \, du = 2r \, dr, \, v = -\sqrt{a^2 - r^2}.$$

40. $z = x^2 + y^2, \, 0 \leq z \leq h$

Project the solid onto the xy -plane.

$$\begin{aligned} I_z &= \iint_S (x^2 + y^2)(1) \, dS \\ &= \int_{-\sqrt{h}}^{\sqrt{h}} \int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}} (x^2 + y^2) \sqrt{1 + 4x^2 + 4y^2} \, dy \, dx \\ &= \int_0^{2\pi} \int_0^{\sqrt{h}} r^2 \sqrt{1 + 4r^2} \, r \, dr \, d\theta \\ &= 2\pi \left[\frac{h}{12}(1 + 4h)^{3/2} - \frac{1}{120}(1 + 4h)^{5/2} \right] + \frac{2\pi}{120} \\ &= \frac{(1 + 4h)^{3/2}\pi}{60} [10h - (1 + 4h)] + \frac{\pi}{60} = \frac{\pi}{60} [(1 + 4h)^{3/2}(6h - 1) + 1] \end{aligned}$$



42. $S: z = \sqrt{16 - x^2 - y^2}$

$$\mathbf{F}(x, y, z) = 0.5z\mathbf{k}$$

$$\begin{aligned} \iint_S \rho \mathbf{F} \cdot \mathbf{N} \, dS &= \iint_R \rho \mathbf{F} \cdot (-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}) \, dA \\ &= \iint_R 0.5\rho z \mathbf{k} \cdot \left[\frac{x}{\sqrt{16 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{16 - x^2 - y^2}}\mathbf{j} + \mathbf{k} \right] dA \\ &= \iint_R 0.5\rho z \, dA = \iint_R 0.5\rho \sqrt{16 - x^2 - y^2} \, dA \\ &= 0.5\rho \int_0^{2\pi} \int_0^4 \sqrt{16 - r^2} r \, dr \, d\theta = 0.5\rho \int_0^{2\pi} \frac{64}{3} d\theta = \frac{64\pi\rho}{3} \end{aligned}$$

Section 14.7 Divergence Theorem

2. **Surface Integral:** There are three surfaces to the cylinder.

Bottom: $z = 0$, $\mathbf{N} = -\mathbf{k}$, $\mathbf{F} \cdot \mathbf{N} = -z^2$

$$\iint_{S_1} 0 \, dS = 0$$

Top: $z = h$, $\mathbf{N} = \mathbf{k}$, $\mathbf{F} \cdot \mathbf{N} = z^2$

$$\iint_{S_2} h^2 \, dS = h^2 (\text{Area of circle}) = 4\pi h^2$$

Side: $\mathbf{r}(u, v) = 2 \cos u \mathbf{i} + 2 \sin u \mathbf{j} + v \mathbf{k}$, $0 \leq u \leq 2\pi$, $0 \leq v \leq h$

$$\mathbf{r}_u = -2 \sin u \mathbf{i} + 2 \cos u \mathbf{j}, \mathbf{r}_v = \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = 2 \cos u \mathbf{i} + 2 \sin u \mathbf{j}$$

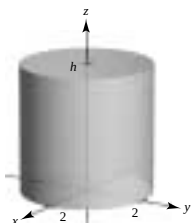
$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 8 \cos^2 u - 8 \sin^2 u$$

$$\iint_{S_3} \mathbf{F} \cdot \mathbf{N} \, dS = \int_0^h \int_0^{2\pi} (8 \cos^2 u - 8 \sin^2 u) \, du \, dv = 0$$

Therefore, $\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = 0 + 4\pi h^2 + 0 = 4\pi h^2$.

Divergence Theorem: $\text{div } \mathbf{F} = 2 - 2 + 2z = 2z$

$$\iiint_Q 2z \, dV = \int_0^{2\pi} \int_0^2 \int_0^h 2z \, dz \, dr \, d\theta = 4\pi h^2.$$



4. $\mathbf{F}(x, y, z) = xy\mathbf{i} + z\mathbf{j} + (x + y)\mathbf{k}$

S: surface bounded by the planes $y = 4$, $z = 4 - x$ and the coordinate planes

Surface Integral: There are five surfaces to this solid.

$$z = 0, \mathbf{N} = -\mathbf{k}, \mathbf{F} \cdot \mathbf{N} = -(x + y)$$

$$\int_{S_1} \int -(x + y) dS = \int_0^4 \int_0^4 -(x + y) dy dx = - \int_0^4 (4x + 8) dx = -64$$

$$y = 0, \mathbf{N} = -\mathbf{j}, \mathbf{F} \cdot \mathbf{N} = -z$$

$$\int_{S_2} \int -z dS = \int_0^4 \int_0^{4-x} -z dz dx = - \int_0^4 \frac{(4-x)^2}{2} dx = -\frac{32}{3}$$

$$y = 4, \mathbf{N} = \mathbf{j}, \mathbf{F} \cdot \mathbf{N} = z$$

$$\int_{S_3} \int z dS = \int_0^4 \int_0^{4-x} z dz dx = \int_0^4 \frac{(4-x)^2}{2} dx = \frac{32}{3}$$

$$x = 0, \mathbf{N} = -\mathbf{i}, \mathbf{F} \cdot \mathbf{N} = -xy$$

$$\int_{S_4} \int -xy dS = \int_0^4 \int_0^4 0 dS = 0$$

$$x + z = 4, \mathbf{N} = \frac{\mathbf{i} + \mathbf{k}}{\sqrt{2}}, \mathbf{F} \cdot \mathbf{N} = \frac{1}{\sqrt{2}}[xy + x + y], dS = \sqrt{2} dA$$

$$\int_{S_5} \int \frac{1}{\sqrt{2}}[xy + x + y]\sqrt{2} dA = \int_0^4 \int_0^4 (xy + x + y) dy dx = 128$$

$$\text{Therefore, } \int_S \mathbf{F} \cdot \mathbf{N} dS = -64 - \frac{32}{3} + \frac{32}{3} + 0 + 128 = 64.$$

Divergence Theorem: Since $\text{div } \mathbf{F} = y$, we have

$$\iiint_Q \text{div } \mathbf{F} dV = \int_0^4 \int_0^4 \int_0^{4-x} y dz dy dx = 64.$$

6. Since $\text{div } \mathbf{F} = 2xz^2 - 2 + 3xy$ we have

$$\begin{aligned} \iiint_Q \text{div } \mathbf{F} dV &= \int_0^a \int_0^a \int_0^a (2xz^2 - 2 + 3xy) dz dy dx = \int_0^a \int_0^a \left(\frac{2}{3}xa^3 - 2a + 3xya \right) dy dx \\ &= \int_0^a \left(\frac{2}{3}xa^4 - 2a^2 + \frac{3}{2}xa^3 \right) dx \\ &= \frac{1}{3}a^6 - 2a^3 + \frac{3}{4}a^5. \end{aligned}$$

8. Since $\text{div } \mathbf{F} = y + z - y = z$, we have

$$\begin{aligned} \iiint_Q \text{div } \mathbf{F} dV &= \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} z dz dy dx = \int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2-r^2}} zr dz dr d\theta \\ &= \int_0^{2\pi} \int_0^a \left[\frac{a^2r}{2} - \frac{r^3}{2} \right] dr d\theta = \int_0^{2\pi} \left[\frac{a^2r^2}{4} - \frac{r^4}{8} \right]_0^a d\theta = \int_0^{2\pi} \frac{a^4}{8} d\theta = \frac{\pi a^4}{4}. \end{aligned}$$

10. Since $\text{div } \mathbf{F} = xz$, we have

$$\iiint_Q xz dV = \int_0^4 \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xz dx dy dz = \int_0^4 \int_{-3}^3 \frac{z}{2}(0) dy dz = 0.$$

12. Since $\operatorname{div} \mathbf{F} = y^2 + x^2 + e^z$, we have

$$\begin{aligned} \iiint_Q (x^2 + y^2 + e^z) dV &= \int_0^{16} \int_{-\sqrt{256-x^2}}^{\sqrt{256-x^2}} \int_{(1/2)\sqrt{x^2+y^2}}^8 (x^2 + y^2 + e^z) dz dy dx \\ &= \int_0^{2\pi} \int_0^{16} \int_{r/2}^8 (r^2 + e^z) r dz dr d\theta = \int_0^{2\pi} \int_0^{16} \left(8r^3 + re^8 - \frac{1}{2}r^4 - re^{r/2} \right) dr d\theta \\ &= \int_0^{2\pi} \left(\frac{131,052}{5} + 100e^8 \right) d\theta = \frac{262,104}{5}\pi + 200e^8\pi \end{aligned}$$

14. Since $\operatorname{div} \mathbf{F} = e^z + e^z + e^z = 3e^z$, we have

$$\iiint_Q 3e^z dV = \int_0^6 \int_0^4 \int_0^{4-y} 3e^z dz dy dx = \int_0^6 \int_0^4 3[e^{4-y} - 1] dy dx = \int_0^6 3(e^4 - 5) dx = 18(e^4 - 5).$$

16. $\operatorname{div} \mathbf{F} = 2$

$$\iint_S \mathbf{F} \cdot \mathbf{N} dS = \iiint_Q \operatorname{div} \mathbf{F} dV = \iiint_Q 2 dV.$$

The surface S is the upper half of a hemisphere of radius 2. Since the volume is $\frac{1}{2}(\frac{4}{3}\pi(2^3)) = 16\pi/3$, you have

$$\iint_S \mathbf{F} \cdot \mathbf{N} dS = 2(\text{Volume}) = \frac{32\pi}{3}.$$

18. Using the Divergence Theorem, we have

$$\begin{aligned} \iint_S \mathbf{curl} \mathbf{F} \cdot \mathbf{N} dS &= \iiint_Q \operatorname{div}(\mathbf{curl} \mathbf{F}) dV \\ \mathbf{curl} \mathbf{F}(x, y, z) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy \cos z & yz \sin x & xyz \end{vmatrix} = (xz - y \sin x)\mathbf{i} - (yz + xy \sin z)\mathbf{j} + (yz \cos x - x \cos z)\mathbf{k}. \end{aligned}$$

Now, $\operatorname{div} \mathbf{curl} \mathbf{F}(x, y, z) = (z - y \cos x) - (z + x \sin z) + (y \cos x + x \sin z) = 0$. Therefore,

$$\iint_S \mathbf{curl} \mathbf{F} \cdot \mathbf{N} dS = \iiint_Q \operatorname{div}(\mathbf{curl} \mathbf{F}) dV = 0.$$

20. If $\operatorname{div} \mathbf{F}(x, y, z) > 0$, then source.

If $\operatorname{div} \mathbf{F}(x, y, z) < 0$, then sink.

If $\operatorname{div} \mathbf{F}(x, y, z) = 0$, then incompressible.

$$22. v = \int_0^a \int_0^a x dy dz = \int_0^a \int_0^a a dy dz = \int_0^a a^2 dz = a^3$$

$$\text{Similarly, } \int_0^a \int_0^a y dz dx = \int_0^a \int_0^a z dx dy = a^3.$$

24. If $\mathbf{F}(x, y, z) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, then $\operatorname{div} \mathbf{F} = 0$.

Therefore,

$$\iint_S \mathbf{F} \cdot \mathbf{N} dS = \iiint_Q \operatorname{div} \mathbf{F} dV = \iiint_Q 0 dV = 0.$$

26. If $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\operatorname{div} \mathbf{F} = 3$.

$$\frac{1}{\|\mathbf{F}\|} \iint_S \mathbf{F} \cdot \mathbf{N} dS = \frac{1}{\|\mathbf{F}\|} \iiint_Q \operatorname{div} \mathbf{F} dV = \frac{1}{\|\mathbf{F}\|} \iiint_Q 3 dV = \frac{3}{\|\mathbf{F}\|} \iiint_Q dV$$

$$28. \iint_S (f D_N g - g D_N f) dS = \iint_S f D_N g dS - \iint_S g D_N f dS$$

$$= \iiint_Q (f \nabla^2 g + \nabla f \cdot \nabla g) dV - \iiint_Q (g \nabla^2 f + \nabla g \cdot \nabla f) dV = \iiint_Q (f \nabla^2 g - g \nabla^2 f) dV$$

Section 14.8 Stokes's Theorem

2. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + x^2\mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & x^2 \end{vmatrix} = -2x\mathbf{j}$$

4. $\mathbf{F}(x, y, z) = x \sin y \mathbf{i} - y \cos x \mathbf{j} + yz^2 \mathbf{k}$

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x \sin y & -y \cos x & yz^2 \end{vmatrix} \\ &= z^2\mathbf{i} + (y \sin x - x \cos y)\mathbf{k} \end{aligned}$$

6. $\mathbf{F}(x, y, z) = \arcsin y \mathbf{i} + \sqrt{1-x^2} \mathbf{j} + y^2 \mathbf{k}$

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \arcsin y & \sqrt{1-x^2} & y^2 \end{vmatrix} \\ &= 2y\mathbf{i} + \left[\frac{-x}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}} \right] \mathbf{k} \\ &= 2y\mathbf{i} - \left[\frac{x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}} \right] \mathbf{k} \end{aligned}$$

8. In this case C is the circle $x^2 + y^2 = 4$, $z = 0$, $dz = 0$.

$$\text{Line Integral: } \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C -y dx + x dy$$

Let $x = 2 \cos t$, $y = 2 \sin t$, then $dx = -2 \sin t dt$, $dy = 2 \cos t dt$, and $\int_C -y dx + x dy = \int_0^{2\pi} 4 dt = 8\pi$.

$$\text{Double Integral: } F(x, y, z) = z + x^2 + y^2 - 4, \mathbf{N} = \frac{\nabla F}{\|\nabla F\|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{1+4x^2+4y^2}}, dS = \sqrt{1+4x^2+4y^2} dA$$

$\operatorname{curl} \mathbf{F} = 2\mathbf{k}$, therefore

$$\begin{aligned} \iint (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} dS &= \iint_R 2 dA = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2 dy dx = 2 \int_{-2}^2 2\sqrt{4-x^2} dx \\ &= 4 \int_{-2}^2 \sqrt{4-x^2} dx = 2 \left[x\sqrt{4-x^2} + 4 \arcsin \frac{x}{2} \right]_{-2}^2 = 8\pi. \end{aligned}$$

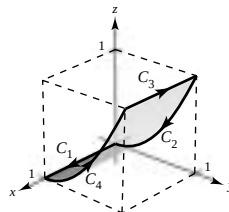
10. **Line Integral:** From the accompanying figure we see that for

$$C_1: y = 0, z = 0, dy = dz = 0$$

$$C_2: z = y^2, x = 0, dx = 0, dz = 2y dy$$

$$C_3: y = a, z = a^2, dy = dz = 0$$

$$C_4: z = y^2, x = a, dx = 0, dz = 2y dy.$$



Hence,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C z^2 dx + x^2 dy + y^2 dz \\ &= \int_{C_1} 0 + \int_{C_2} 2y^3 dy + \int_{C_3} a^4 dx + \int_{C_4} a^2 dy + y^2(2y) dy \\ &= \int_0^a 2y^3 dy + \int_0^a a^4 dx + \int_a^0 a^2 dy + \int_a^0 2y^3 dy = \left[a^4 x \right]_0^a + \left[a^2 y \right]_a^0 = a^5 - a^3 = a^3(a^2 - 1). \end{aligned}$$

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10. —CONTINUED—

Double Integral: Since $\mathbf{F}(x, y, z) = y^2 - z$, we have

$$\mathbf{N} = \frac{2y\mathbf{j} - \mathbf{k}}{\sqrt{1 + 4y^2}} \text{ and } dS = \sqrt{1 + 4y^2} dA.$$

Furthermore, $\text{curl } \mathbf{F} = 2y\mathbf{i} + 2z\mathbf{j} + 2x\mathbf{k}$. Therefore,

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} dS = \iint_R (4yz - 2x) dA = \int_0^a \int_0^a (4y^2 - 2x) dy dx = \int_0^a (a^4 - 2ax) dx = \left[a^4x - ax^2 \right]_0^a = a^3(a^2 - 1).$$

12. Let $A = (0, 0, 0)$, $B = (1, 1, 1)$, and $C = (0, 0, 2)$. Then $\mathbf{U} = \overrightarrow{AB} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\mathbf{V} = \overrightarrow{AC} = 2\mathbf{k}$, and

$$\mathbf{N} = \frac{\mathbf{U} \times \mathbf{V}}{\|\mathbf{U} \times \mathbf{V}\|} = \frac{2\mathbf{i} - 2\mathbf{j}}{2\sqrt{2}} = \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}.$$

Hence, $F(x, y, z) = x - y$ and $dS = \sqrt{2} dA$. Since $\text{curl } \mathbf{F} = \frac{2x}{x^2 + y^2} \mathbf{k}$, we have $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} dS = \iint_R 0 dS = 0$.

14. $\mathbf{F}(x, y, z) = 4xz\mathbf{i} + y\mathbf{j} + 4xy\mathbf{k}$, $S: 9 - x^2 - y^2$, $z \leq 0$

$$\text{curl } \mathbf{F} = 4x\mathbf{i} + (4x - 4y)\mathbf{j}$$

$$G(x, y, z) = x^2 + y^2 + z - 9$$

$$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$$

$$\begin{aligned} \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} dS &= \iint_R [8x^2 + 2y(4x - 4y)] dA \\ &= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} [8x^2 + 8xy - 8y^2] dy dx \\ &= \int_{-3}^3 \left(16x^2\sqrt{9-x^2} - \frac{16}{3}(9-x^2)^{3/2} \right) dx = 0 \end{aligned}$$

16. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + z^2\mathbf{j} - xyz\mathbf{k}$, $S: z = \sqrt{4 - x^2 - y^2}$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & z^2 & -xyz \end{vmatrix} = (-xz - 2z)\mathbf{i} + yz\mathbf{j}$$

$$G(x, y, z) = z - \sqrt{4 - x^2 - y^2}$$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\begin{aligned} \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} dS &= \iint_R \left[\frac{-z(x+2)x}{\sqrt{4-x^2-y^2}} + \frac{y^2z}{\sqrt{4-x^2-y^2}} \right] dA \\ &= \iint_R [-x(x+2) + y^2] dA = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (-x^2 - 2x + y^2) dy dx \\ &= \int_{-2}^2 \left[-x^2y - 2xy + \frac{y^3}{3} \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \\ &= \int_{-2}^2 \left[-2x^2\sqrt{4-x^2} - 4x\sqrt{4-x^2} + \frac{2}{3}(4-x^2)\sqrt{4-x^2} \right] dx \\ &= \int_{-2}^2 \left[-\frac{8}{3}x^2\sqrt{4-x^2} - 4x\sqrt{4-x^2} + \frac{8}{3}\sqrt{4-x^2} \right] dx \\ &= \left[-\frac{8}{3}\left(\frac{1}{8}\right) \left[x(2x^2-4)\sqrt{4-x^2} + 16 \arcsin \frac{x}{2} \right] + \frac{4}{3}(4-x^2)^{3/2} + \frac{8}{3}\left(\frac{1}{2}\right) \left[x\sqrt{4-x^2} + 4 \arcsin \frac{x}{2} \right] \right]_{-2}^2 \\ &= \left[\left(-\frac{1}{3}\right)(8\pi) + \frac{4}{3}(2\pi) + \frac{1}{3}(-8\pi) - \frac{4}{3}(-2\pi) \right] = 0 \end{aligned}$$

18. $\mathbf{F}(x, y, z) = yz\mathbf{i} + (2 - 3y)\mathbf{j} + (x^2 + y^2)\mathbf{k}$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 2 - 3y & x^2 + y^2 \end{vmatrix} = 2y\mathbf{i} + (y - 2x)\mathbf{j} - z\mathbf{k}$$

S: the first octant portion of $x^2 + z^2 = 16$ over $x^2 + y^2 = 16$

$$G(x, y, z) = z - \sqrt{16 - x^2}$$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{16 - x^2}}\mathbf{i} + \mathbf{k}$$

$$\begin{aligned} \iint_S (\mathbf{curl} \mathbf{F}) \cdot \mathbf{N} \, dS &= \iint_R \left[\frac{2xy}{\sqrt{16 - x^2}} - z \right] dA \\ &= \iint_R \left[\frac{2xy}{\sqrt{16 - x^2}} - \sqrt{16 - x^2} \right] dA \\ &= \int_0^4 \int_0^{\sqrt{16 - x^2}} \left[\frac{2xy}{\sqrt{16 - x^2}} - \sqrt{16 - x^2} \right] dy \, dx \\ &= \int_0^4 \left[\frac{x}{\sqrt{16 - x^2}} y^2 - \sqrt{16 - x^2} y \right]_0^{\sqrt{16 - x^2}} dx \\ &= \int_0^4 [x\sqrt{16 - x^2} - (16 - x^2)] dx \\ &= \left[-\frac{1}{3}(16 - x^2)^{3/2} - 16x + \frac{x^3}{3} \right]_0^4 \\ &= \left(-64 + \frac{64}{3} \right) - \left(-\frac{64}{3} \right) = -\frac{64}{3} \end{aligned}$$

20. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

S: the first octant portion of $z = x^2$ over $x^2 + y^2 = a^2$. We have

$$\begin{aligned} \mathbf{N} &= \frac{2x\mathbf{i} - \mathbf{k}}{\sqrt{1 + 4x^2}} \text{ and } dS = \sqrt{1 + 4x^2} \, dA. \\ \iint_S (\mathbf{curl} \mathbf{F}) \cdot \mathbf{N} \, dS &= \iint_R xz \, dA = \iint_R x^3 \, dA \\ &= \int_0^a \int_0^{\sqrt{a^2 - x^2}} x^3 \, dy \, dx \\ &= \int_0^a x^3 \sqrt{a^2 - x^2} \, dx \\ &= \left[-\frac{1}{3}x^2(a^2 - x^2)^{3/2} - \frac{2}{15}(a^2 - x^2)^{5/2} \right]_0^a \\ &= \frac{2}{15}a^5 \end{aligned}$$

22. $\mathbf{F}(x, y, z) = -z\mathbf{i} + y\mathbf{k}$

$S: x^2 + y^2 = 1$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & 0 & y \end{vmatrix} = \mathbf{i} - \mathbf{j}$$

Letting $\mathbf{N} = \mathbf{k}$, $\mathbf{curl} \mathbf{F} \cdot \mathbf{N} = 0$ and $\iint_S (\mathbf{curl} \mathbf{F}) \cdot \mathbf{N} \, dS = 0$.

24. $\mathbf{curl} \mathbf{F}$ measures the rotational tendency.

See page 1084.

26. $f(x, y, z) = xyz$, $g(x, y, z) = z$, $S: z = \sqrt{4 - x^2 - y^2}$

(a) $\nabla g(x, y, z) = \mathbf{k}$

$f(x, y, z)\nabla g(x, y, z) = xyz\mathbf{k}$

$\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + 0\mathbf{k}$, $0 \leq t \leq 2\pi$

$$\int_C [f(x, y, z)\nabla g(x, y, z)] \cdot d\mathbf{r} = 0$$

(b) $\nabla f(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$\nabla g(x, y, z) = \mathbf{k}$

$$\nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ yz & xz & xy \\ 0 & 0 & 1 \end{vmatrix} = xz\mathbf{i} - yz\mathbf{j}$$

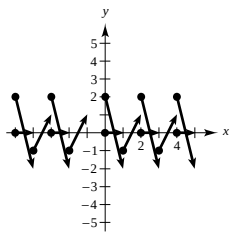
$$\mathbf{N} = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$dS = \sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{4 - x^2 - y^2}}\right)^2} dA = \frac{2}{\sqrt{4 - x^2 - y^2}} dA$$

$$\begin{aligned} \iint_S [\nabla f(x, y, z) \times \nabla g(x, y, z)] \cdot \mathbf{N} \, dS &= \iint_S \left[\frac{x^2 z}{\sqrt{4 - x^2 - y^2}} - \frac{y^2 z}{\sqrt{4 - x^2 - y^2}} \right] \frac{2}{\sqrt{4 - x^2 - y^2}} dA \\ &= \iint_S \frac{2(x^2 - y^2)}{\sqrt{4 - x^2 - y^2}} dA \\ &= \int_0^2 \int_0^{2\pi} \frac{2r^2(\cos^2 \theta - \sin^2 \theta)}{\sqrt{4 - r^2}} r \, d\theta \, dr \\ &= \int_0^2 \left[\frac{2r^3}{\sqrt{4 - r^2}} \left(\frac{1}{2} \sin 2\theta \right) \right]_0^{2\pi} dr = 0 \end{aligned}$$

Review Exercises for Chapter 14

2. $\mathbf{F}(x, y) = \mathbf{i} - 2y\mathbf{j}$



4. $f(x, y, z) = x^2 e^{yz}$

$$\begin{aligned} \mathbf{F}(x, y, z) &= 2xe^{yz}\mathbf{i} + x^2ze^{yz}\mathbf{j} + x^2ye^{yz}\mathbf{k} \\ &= xe^{yz}(2\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) \end{aligned}$$

6. Since $\partial M/\partial y = -1/x^2 = \partial N/\partial x$, $\mathbf{F}$ is conservative. From $M = \partial U/\partial x = -y/x^2$ and $N = \partial U/\partial y = 1/x$, partial integration yields $U = (y/x) + h(y)$ and $U = (y/x) + g(x)$ which suggests that $U(x, y) = (y/x) + C$.

8. Since $\partial M/\partial y = -6y^2 \sin 2x = \partial N/\partial x$, $\mathbf{F}$ is conservative. From $M = \partial U/\partial x = -2y^3 \sin 2x$ and $N = \partial U/\partial y = 3y^2(1 + \cos 2x)$, we obtain $U = y^3 \cos 2x + h(y)$ and $U = y^3(1 + \cos 2x) + g(x)$ which suggests that $h(y) = y^3$, $g(x) = C$, and $U(x, y) = y^3(1 + \cos 2x) + C$.

10. Since

$$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x},$$

$$\frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x},$$

$$\frac{\partial N}{\partial z} = 6y \neq \frac{\partial P}{\partial y}$$

$\mathbf{F}$ is not conservative.

14. Since $\mathbf{F} = xy^2\mathbf{j} - zx^2\mathbf{k}$;

(a) $\operatorname{div} \mathbf{F} = 2xy - x^2$

(b) $\operatorname{curl} \mathbf{F} = 2xz\mathbf{j} + y^2\mathbf{k}$

18. Since $\mathbf{F} = (x^2 - y)\mathbf{i} - (x + \sin^2 y)\mathbf{j}$;

(a) $\operatorname{div} \mathbf{F} = 2x - 2 \sin y \cos y$

(b) $\operatorname{curl} \mathbf{F} = \mathbf{0}$

12. Since

$$\frac{\partial M}{\partial y} = \sin z = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = y \cos z \neq \frac{\partial P}{\partial x},$$

$\mathbf{F}$ is not conservative.

16. Since $\mathbf{F} = (3x - y)\mathbf{i} + (y - 2z)\mathbf{j} + (z - 3x)\mathbf{k}$;

(a) $\operatorname{div} \mathbf{F} = 3 + 1 + 1 = 5$

(b) $\operatorname{curl} \mathbf{F} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

20. Since $\mathbf{F} = \frac{z}{x}\mathbf{i} + \frac{z}{y}\mathbf{j} + z^2\mathbf{k}$;

(a) $\operatorname{div} \mathbf{F} = -\frac{z}{x^2} - \frac{z}{y^2} + 2z = z\left(2 - \frac{1}{x^2} - \frac{1}{y^2}\right)$

(b) $\operatorname{curl} \mathbf{F} = -\frac{1}{y}\mathbf{i} + \frac{1}{x}\mathbf{j}$

22. (a) Let $x = 5t$, $y = 4t$, $0 \leq t \leq 1$, then $ds = \sqrt{41} dt$.

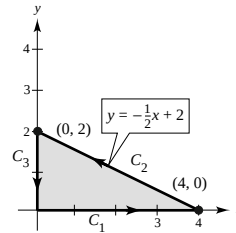
$$\int_C xy \, ds = \int_0^1 20t^2 \sqrt{41} \, dt = \frac{20\sqrt{41}}{3}$$

(b) $C_1: x = t, y = 0, 0 \leq t \leq 4, ds = dt$

$$C_2: x = 4 - 4t, y = 2t, 0 \leq t \leq 1, ds = 2\sqrt{5} \, dt$$

$$C_3: x = 0, y = 2 - t, 0 \leq t \leq 2, ds = dt$$

$$\begin{aligned} \text{Therefore, } \int_C xy \, ds &= \int_0^4 0 \, dt = \int_0^1 (8t - 8t^2) 2\sqrt{5} \, dt + \int_0^2 0 \, dt \\ &= 16\sqrt{5} \left[\frac{t^2}{2} - \frac{t^3}{3} \right]_0^1 = \frac{8\sqrt{5}}{3}. \end{aligned}$$



24. $x = t - \sin t$, $y = 1 - \cos t$, $0 \leq t \leq 2\pi$, $\frac{dx}{dt} = 1 - \cos t$, $\frac{dy}{dt} = \sin t$

$$\begin{aligned} \int_C x \, ds &= \int_0^{2\pi} (t - \sin t) \sqrt{(1 - \cos t)^2 + (\sin t)^2} \, dt = \int_0^{2\pi} (t - \sin t) \sqrt{2 - 2\cos t} \, dt \\ &= \sqrt{2} \int_0^{2\pi} [t\sqrt{1 - \cos t} - \sin t\sqrt{1 - \cos t}] \, dt = \sqrt{2} \left[-\frac{2}{3}(1 - \cos t)^{3/2} \right]_0^{2\pi} + \sqrt{2} \int_0^{2\pi} t\sqrt{1 - \cos t} \, dt \\ &= \sqrt{2} \int_0^{2\pi} t\sqrt{1 - \cos t} \, dt \\ &= 8\pi \end{aligned}$$

26. $x = \cos t + t \sin t, y = \sin t - t \sin t, 0 \leq t \leq \frac{\pi}{2}, dx = t \cos t dt, dy = (\cos t - t \cos t - \sin t) dt$

$$\int_C (2x - y) dx + (x + 3y) dy = \int_0^{\pi/2} [\sin t \cos t (5t^2 - 6t + 2) + \cos^2 t(t + 1) + \sin^2 t(2t - 3)] dt \approx 1.01$$

28. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^{3/2}\mathbf{k}, 0 \leq t \leq 4$

$$x'(t) = 1, y'(t) = 2t, z'(t) = \frac{3}{2}t^{1/2}$$

$$\int_C (x^2 + y^2 + z^2) ds = \int_0^4 (t^2 + t^4 + t^3) \sqrt{1 + 4t^2 + \frac{9}{4}t} dt \approx 2080.59$$

30. $f(x, y) = 12 - x - y$

$C: y = x^2$ from $(0, 0)$ to $(2, 4)$

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 2$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{1 + 4t^2}$$

Lateral surface area:

$$\int_C f(x, y) ds = \int_0^2 (12 - t - t^2) \sqrt{1 + 4t^2} dt \approx 41.532$$

32. $d\mathbf{r} = [(-4 \sin t)\mathbf{i} + 3 \cos t\mathbf{j}] dt$

$$\mathbf{F} = (4 \cos t - 3 \sin t)\mathbf{i} + (4 \cos t + 3 \sin t)\mathbf{j}, 0 \leq t \leq 2\pi$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (12 - 7 \sin t \cos t) dt = \left[12t - \frac{7 \sin^2 t}{2} \right]_0^{2\pi} = 24\pi$$

34. $x = 2 - t, y = 2 - t, z = \sqrt{4t - t^2}, 0 \leq t \leq 2$

$$d\mathbf{r} = \left[-\mathbf{i} - \mathbf{j} + \frac{2 - t}{\sqrt{4t - t^2}} \mathbf{k} \right] dt$$

$$\mathbf{F} = (4 - 2t - \sqrt{4t - t^2})\mathbf{i} + (\sqrt{4t - t^2} - 2 + t)\mathbf{j} + 0\mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 (t - 2) dt = \left[\frac{t^2}{2} - 2t \right]_0^2 = -2$$

36. Let $x = 2 \sin t, y = -2 \cos t, z = 4 \sin^2 t, 0 \leq t \leq \pi$.

$$d\mathbf{r} = [(2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + (8 \sin t \cos t)\mathbf{k}] dt$$

$$\mathbf{F} = 0\mathbf{i} + 4\mathbf{j} + (2 \sin t)\mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi (8 \sin t + 16 \sin^2 t \cos t) dt = \left[-8 \cos t + \frac{16}{3} \sin^3 t \right]_0^\pi = 16$$

38. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (2x - y) dx + (2y - x) dy$

$$\mathbf{r}(t) = (2 \cos t + 2t \sin t)\mathbf{i} + (2 \sin t - 2t \cos t)\mathbf{j}, 0 \leq t \leq \pi$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 4\pi^2 + 4\pi$$

$$\begin{aligned}
 40. \mathbf{r}(t) &= 10 \sin t \mathbf{i} + 10 \cos t \mathbf{j} + \frac{2000/5280}{\pi/2} t \mathbf{k}, \quad 0 \leq t \leq \frac{\pi}{2} \\
 &= 10 \sin t \mathbf{i} + 10 \cos t \mathbf{j} + \frac{25}{33\pi} t \mathbf{k}
 \end{aligned}$$

$$\mathbf{F} = 20\mathbf{k}$$

$$d\mathbf{r} = \left(10 \cos t \mathbf{i} - 10 \sin t \mathbf{j} + \frac{25}{33\pi} \mathbf{k} \right)$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \frac{500}{33\pi} dt = \frac{250}{33} \text{ mi} \cdot \text{ton}$$

$$42. \int_C y \, dx + x \, dy + \frac{1}{z} \, dz = \left[xy + \ln|z| \right]_{(0,0,1)}^{(4,4,4)} = 16 + \ln 4$$

$$44. x = a(\theta - \sin \theta), y = a(1 - \cos \theta), 0 \leq \theta \leq 2\pi$$

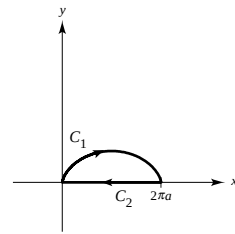
$$(a) A = \frac{1}{2} \int_C x \, dy - y \, dx.$$

Since these equations orient the curve backwards, we will use

$$\begin{aligned}
 A &= \frac{1}{2} \int (y \, dx - x \, dy) \\
 &= \frac{1}{2} \int_0^{2\pi} [a^2(1 - \cos \theta)(1 - \cos \theta) - a^2(\theta - \sin \theta)(\sin \theta)] \, d\theta + \frac{1}{2} \int_0^{2\pi} (0 - 0) \, d\theta \\
 &= \frac{a^2}{2} \int_0^{2\pi} [1 - 2 \cos \theta + \cos^2 \theta - \theta \sin \theta + \sin^2 \theta] \, d\theta \\
 &= \frac{a^2}{2} \int_0^{2\pi} (2 - 2 \cos \theta - \theta \sin \theta) \, d\theta = \frac{a^2}{2} (6\pi) = 3\pi a^2.
 \end{aligned}$$

(b) By symmetry, $\bar{x} = \pi a$. From Section 14.4,

$$\bar{y} = -\frac{1}{2A} \int_C y^2 \, dx = \frac{1}{2A} \int_0^{2\pi} a^3(1 - \cos \theta)^2(1 - \cos \theta) \, d\theta = \frac{1}{2(3\pi a^2)} a^3(5\pi) = \frac{5}{6}a$$



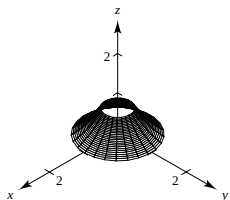
$$\begin{aligned}
 46. \int_C xy \, dx + (x^2 + y^2) \, dy &= \int_0^2 \int_0^2 (2x - x) \, dy \, dx \\
 &= \int_0^2 2x \, dx = 4
 \end{aligned}$$

$$\begin{aligned}
 48. \int_C (x^2 - y^2) \, dx + 2xy \, dy &= \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 4y \, dy \, dx \\
 &= \int_{-a}^a 0 \, dx = 0
 \end{aligned}$$

$$\begin{aligned}
 50. \int_C y^2 \, dx + x^{4/3} \, dy &= \int_{-1}^1 \int_{-(1-x^{2/3})^{3/2}}^{(1-x^{2/3})^{3/2}} \left(\frac{4}{3} x^{1/3} - 2y \right) \, dy \, dx \\
 &= \int_{-1}^1 \left[\frac{4}{3} x^{1/3} y - y^2 \right]_{-(1-x^{2/3})^{3/2}}^{(1-x^{2/3})^{3/2}} \, dx \\
 &= \int_{-1}^1 \frac{8}{3} x^{1/3} (1 - x^{2/3})^{3/2} \, dx \\
 &= \left[-\frac{8}{7} x^{2/3} (1 - x^{2/3})^{5/2} - \frac{16}{35} (1 - x^{2/3})^{5/2} \right]_{-1}^1 \\
 &= 0
 \end{aligned}$$

52. $\mathbf{r}(u, v) = e^{-u/4} \cos v \mathbf{i} + e^{-u/4} \sin v \mathbf{j} + \frac{u}{6} \mathbf{k}$

$$0 \leq u \leq 4, \quad 0 \leq v \leq 2\pi$$



54. $S: \mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + \sin v \mathbf{k}, \quad 0 \leq u \leq 2, 0 \leq v \leq \pi$

$$\mathbf{r}_u(u, v) = \mathbf{i} + \mathbf{j}$$

$$\mathbf{r}_v(u, v) = \mathbf{i} - \mathbf{j} + \cos v \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & \cos v \end{vmatrix} = \cos v \mathbf{i} - \cos v \mathbf{j} - 2\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{2 \cos^2 v + 4}$$

$$\iint_S z \, dS = \int_0^\pi \int_0^2 \sin v \sqrt{2 \cos^2 v + 4} \, du \, dv = 2 \left[\sqrt{6} + \sqrt{2} \ln \left(\frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}} \right) \right]$$

56. (a) $z = a(a - \sqrt{x^2 + y^2}), 0 \leq z \leq a^2$

$$z = 0 \Rightarrow x^2 + y^2 = a^2$$

(b) $S: g(x, y) = z = a^2 - a\sqrt{x^2 + y^2}$

$$\rho(x, y) = k\sqrt{x^2 + y^2}$$

$$m = \iint_S e(x, y, z) \, dS$$

$$= \iint_R k\sqrt{x^2 + y^2} \sqrt{1 + g_x^2 + g_y^2} \, dA$$

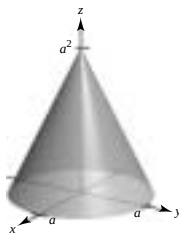
$$= k \iint_R \sqrt{x^2 + y^2} \sqrt{1 + \frac{a^2 x^2}{x^2 + y^2} + \frac{a^2 y^2}{x^2 + y^2}} \, dA$$

$$= k \iint_R \sqrt{a^2 + 1} (\sqrt{x^2 + y^2}) \, dA$$

$$= k\sqrt{a^2 + 1} \int_0^{2\pi} \int_0^a r^2 \, dr \, d\theta$$

$$= k\sqrt{a^2 + 1} \int_0^{2\pi} \frac{a^3}{3} \, d\theta$$

$$= \frac{2}{3} k\sqrt{a^2 + 1} a^3 \pi$$



58. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Q : solid region bounded by the coordinate planes and the plane $2x + 3y + 4z = 12$

Surface Integral: There are four surfaces for this solid.

$$z = 0 \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = -z, \quad \iint_{S_1} 0 \, dS = 0$$

$$y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = -y, \quad \iint_{S_2} 0 \, dS = 0$$

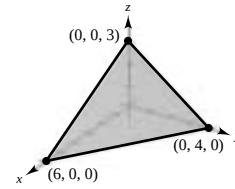
$$x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = -x, \quad \iint_{S_3} 0 \, dS = 0$$

$$2x + 3y + 4z = 12, \quad \mathbf{N} = \frac{2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}}{\sqrt{29}}, \quad dS = \sqrt{1 + \left(\frac{1}{4}\right) + \left(\frac{9}{16}\right)} dA = \frac{\sqrt{29}}{4} dA$$

$$\begin{aligned} \iint_{S_4} \mathbf{N} \cdot \mathbf{F} \, dS &= \frac{1}{4} \iint_R (2x + 3y + 4z) \, dy \, dx \\ &= \frac{1}{4} \int_0^6 \int_0^{(12-2x)/3} 12 \, dy \, dx = 3 \int_0^6 \left(4 - \frac{2x}{3}\right) dx = 3 \left[4x - \frac{x^2}{3}\right]_0^6 = 36 \end{aligned}$$

Triple Integral: Since $\text{div } \mathbf{F} = 3$, the Divergence Theorem yields.

$$\iiint_Q \text{div } \mathbf{F} \, dV = \iiint_Q 3 \, dV = 3(\text{Volume of solid}) = 3 \left[\frac{1}{3} (\text{Area of base})(\text{Height}) \right] = \frac{1}{2} (6)(4)(3) = 36.$$

60. $\mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - z)\mathbf{j} + x^2\mathbf{k}$

S : first octant portion of the plane $3x + y + 2z = 12$

Line Integral:

$$C_1: y = 0, \quad dy = 0, \quad z = \frac{12 - 3x}{2}, \quad dz = -\frac{3}{2} dx$$

$$C_2: x = 0, \quad dx = 0, \quad z = \frac{12 - y}{2}, \quad dz = -\frac{1}{2} dy$$

$$C_3: z = 0, \quad dz = 0, \quad y = 12 - 3x, \quad dy = -3 dx$$

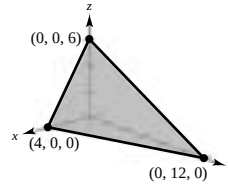
$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (x - z) dx + (y - z) dy + x^2 dz \\ &= \int_{C_1} \left[x - \frac{12 - 3x}{2} + x^2 \left(-\frac{3}{2}\right) \right] dx + \int_{C_2} \left[y - \frac{12 - y}{2} \right] dy + \int_{C_3} [x + (12 - 3x)(-3)] dx \\ &= \int_4^0 \left(-\frac{3}{2}x^2 + \frac{5}{2}x - 6 \right) dx + \int_0^{12} \left(\frac{3}{2}y - 6 \right) dy + \int_0^4 (10x - 36) dx = 8 \end{aligned}$$

Double Integral: $G(x, y, z) = \frac{12 - 3x - y}{2} - z$

$$\nabla G(x, y, z) = -\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} - \mathbf{k}$$

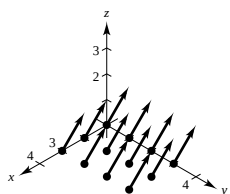
$$\text{curl } \mathbf{F} = \mathbf{i} - (2x + 1)\mathbf{j}$$

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} \, dS = \int_0^4 \int_0^{12-3x} (x - 1) \, dy \, dx = \int_0^4 (-3x^2 + 15x - 12) \, dx = 8$$



Review Exercises for Chapter 14

1. $\mathbf{F}(x, y, z) = x\mathbf{i} + \mathbf{j} + 2\mathbf{k}$



3. $f(x, y, z) = 8x^2 + xy + z^2$

$$\mathbf{F}(x, y, z) = (16x + y)\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$$

5. Since $\partial M/\partial y = -1/y^2 \neq \partial N/\partial x$, $\mathbf{F}$ is not conservative.

7. Since $\partial M/\partial y = 12xy = \partial N/\partial x$, $\mathbf{F}$ is conservative. From $M = \partial U/\partial x = 6xy^2 - 3x^2$ and $N = \partial U/\partial y = 6x^2y + 3y^2 - 7$, partial integration yields $U = 3x^2y^2 - x^3 + h(y)$ and $U = 3x^2y^2 + y^3 - 7y + g(x)$ which suggests $h(y) = y^3 - 7y$, $g(x) = -x^3$, and $U(x, y) = 3x^2y^2 - x^3 + y^3 - 7y + C$.

9. Since

$$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x},$$

$$\frac{\partial M}{\partial z} = 1 \neq \frac{\partial P}{\partial x}.$$

$\mathbf{F}$ is not conservative.

11. Since

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2z} = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = \frac{-1}{yz^2} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{x}{y^2z^2} = \frac{\partial P}{\partial y},$$

$\mathbf{F}$ is conservative. From

$$M = \frac{\partial U}{\partial x} = \frac{1}{yz}, \quad N = \frac{\partial U}{\partial y} = \frac{-x}{y^2z}, \quad P = \frac{\partial U}{\partial z} = \frac{-x}{yz^2}$$

we obtain

$$U = \frac{x}{yz} + f(y, z), \quad U = \frac{x}{yz} + g(x, z), \quad U = \frac{x}{yz} + h(x, y) \Rightarrow f(x, y, z) = \frac{x}{yz} + K$$

13. Since $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$:

(a) $\operatorname{div} \mathbf{F} = 2x + 2y + 2z$

(b) $\operatorname{curl} \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} = 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

15. Since $\mathbf{F} = (\cos y + y \cos x)\mathbf{i} + (\sin x - x \sin y)\mathbf{j} + xyz\mathbf{k}$:

(a) $\operatorname{div} \mathbf{F} = -y \sin x - x \cos y + xy$

(b) $\operatorname{curl} \mathbf{F} = xz\mathbf{i} - yz\mathbf{j} + (\cos x - \sin y + \sin y - \cos x)\mathbf{k} = xz\mathbf{i} - yz\mathbf{j}$

17. Since $\mathbf{F} = \arcsin x \mathbf{i} + xy^2 \mathbf{j} + yz^2 \mathbf{k}$:

(a) $\operatorname{div} \mathbf{F} = \frac{1}{\sqrt{1-x^2}} + 2xy + 2yz$

(b) $\operatorname{curl} \mathbf{F} = z^2 \mathbf{i} + y^2 \mathbf{k}$

19. Since $\mathbf{F} = \ln(x^2 + y^2) \mathbf{i} + \ln(x^2 + y^2) \mathbf{j} + z \mathbf{k}$:

(a) $\operatorname{div} \mathbf{F} = \frac{2x}{x^2 + y^2} + \frac{2y}{x^2 + y^2} + 1$

$$= \frac{2x + 2y}{x^2 + y^2} + 1$$

(b) $\operatorname{curl} \mathbf{F} = \frac{2x - 2y}{x^2 + y^2} \mathbf{k}$

21. (a) Let $x = t, y = t, -1 \leq t \leq 2$, then $ds = \sqrt{2} dt$.

$$\int_C (x^2 + y^2) ds = \int_{-1}^2 2t^2 \sqrt{2} dt = \left[2\sqrt{2} \left(\frac{t^3}{3} \right) \right]_{-1}^2 = 6\sqrt{2}$$

(b) Let $x = 4 \cos t, y = 4 \sin t, 0 \leq t \leq 2\pi$, then $ds = 4 dt$.

$$\int_C (x^2 + y^2) ds = \int_0^{2\pi} 16(4 dt) = 128\pi$$

23. $x = \cos t + t \sin t, y = \sin t - t \cos t, 0 \leq t \leq 2\pi, \frac{dx}{dt} = t \cos t, \frac{dy}{dt} = t \sin t$

$$\begin{aligned} \int_C (x^2 + y^2) ds &= \int_0^{2\pi} [(\cos t + t \sin t)^2 + (\sin t - t \cos t)^2] \sqrt{t^2 \cos^2 t + t^2 \sin^2 t} dt = \int_0^{2\pi} [t^3 + t] dt \\ &= 2\pi^2(1 + 2\pi^2) \end{aligned}$$

25. (a) Let $x = 2t, y = -3t, 0 \leq t \leq 1$

$$\int_C (2x - y) dx + (x + 3y) dy = \int_0^1 [7t(2) + (-7t)(-3)] dt = \int_0^1 35t dt = \frac{35}{2}$$

(b) $x = 3 \cos t, y = 3 \sin t, dx = -3 \sin t dt, dy = 3 \cos t dt, 0 \leq t \leq 2\pi$

$$\int_C (2x - y) dx + (x + 3y) dy = \int_0^{2\pi} (9 + 9 \sin t \cos t) dt = 18\pi$$

27. $\int_C (2x + y) ds, \mathbf{r}(t) = a \cos^3 t \mathbf{i} + a \sin^3 t \mathbf{j}, 0 \leq t \leq \frac{\pi}{2}$

$$x'(t) = -3a \cdot \cos^2 t \sin t$$

$$y'(t) = 3a \cdot \sin^2 t \cos t$$

$$\int_C (2x + y) ds = \int_0^{\pi/2} (2(a \cdot \cos^3 t) + a \cdot \sin^3 t) \sqrt{x'(t)^2 + y'(t)^2} dt = \frac{9a^2}{5}$$

29. $f(x, y) = 5 + \sin(x + y)$ $C: y = 3x$ from $(0, 0)$ to $(2, 6)$

$$\mathbf{r}(t) = t \mathbf{i} + 3t \mathbf{j}, 0 \leq t \leq 2$$

$$\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{10}$$

Lateral surface area:

$$\int_{C_2} f(x, y) ds = \int_0^2 [5 + \sin(t + 3t)] \sqrt{10} dt = \sqrt{10} \int_0^2 (5 + \sin 4t) dt = \frac{\sqrt{10}}{4} (41 - \cos 8) \approx 32.528$$

$$31. d\mathbf{r} = (2t\mathbf{i} + 3t^2\mathbf{j}) dt$$

$$\mathbf{F} = t^5\mathbf{i} + t^4\mathbf{j}, 0 \leq t \leq 1$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 5t^6 dt = \frac{5}{7}$$

$$33. d\mathbf{r} = [(-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + \mathbf{k}] dt$$

$$\mathbf{F} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 2\pi$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} t dt = 2\pi^2$$

$$35. \text{ Let } x = t, y = -t, z = 2t^2, -2 \leq t \leq 2, d\mathbf{r} = [\mathbf{i} - \mathbf{j} + 4t\mathbf{k}] dt.$$

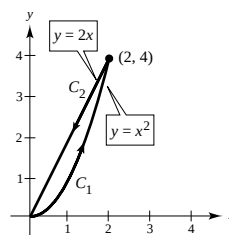
$$\mathbf{F} = (-t - 2t^2)\mathbf{i} + (2t^2 - t)\mathbf{j} + (2t)\mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-2}^2 4t^2 dt = \left[\frac{4t^3}{3} \right]_{-2}^2 = \frac{64}{3}$$

$$37. \text{ For } y = x^2, \mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 2$$

$$\text{For } y = 2x, \mathbf{r}_2(t) = (2 - t)\mathbf{i} + (4 - 2t)\mathbf{j}, 0 \leq t \leq 2$$

$$\begin{aligned} \int_C xy dx + (x^2 + y^2) dy &= \int_{C_1} xy dx + (x^2 + y^2) dy + \int_{C_2} xy dx + (x^2 + y^2) dy \\ &= \frac{100}{3} + (-32) = \frac{4}{3} \end{aligned}$$



$$39. \mathbf{F} = x\mathbf{i} - \sqrt{y}\mathbf{j} \text{ is conservative.}$$

$$\text{Work} = \left[\frac{1}{2}x^2 - \frac{2}{3}y^{3/2} \right]_{(0,0)}^{(4,8)} = \frac{1}{2}(16) - \left(\frac{2}{3} \right) 8^{3/2} = \frac{8}{3}(3 - 4\sqrt{2})$$

$$41. \int_C 2xyz dx + x^2z dy + x^2y dz = \left[x^2yz \right]_{(0,0,0)}^{(1,4,3)} = 12$$

$$\begin{aligned} 43. (a) \int_C y^2 dx + 2xy dy &= \int_0^1 [(1+t)^2(3) + 2(1+3t)(1+t)] dt \\ &= \int_0^1 3(t^2 + 2t + 1) + 2(3t^2 + 4t + 1) dt \\ &= \int_0^1 (9t^2 + 14t + 5) dt \\ &= \left[3t^3 + 7t^2 + 5t \right]_0^1 = 15 \end{aligned}$$

$$\begin{aligned} (b) \int_C y^2 dx + 2xy dy &= \int_1^4 \left[t(1) + 2(t)(\sqrt{t}) \frac{1}{2\sqrt{t}} \right] dt \\ &= \int_1^4 (t + t) dt \\ &= \left[t^2 \right]_1^4 = 15 \end{aligned}$$

$$(c) \mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j} = \nabla f \text{ where } f(x, y) = xy^2.$$

Hence,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 4(2)^2 - 1(1)^2 = 15$$

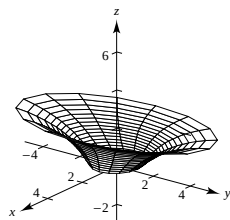
$$45. \int_C y dx + 2x dy = \int_0^2 \int_0^2 (2 - 1) dy dx = \int_0^2 2 dx = 4$$

$$47. \int_C xy^2 dx + x^2y dy = \iint_R (2xy - 2xy) dA = 0$$

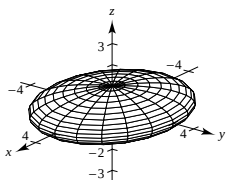
$$49. \int_C xy \, dx + x^2 \, dy = \int_0^1 \int_{x^2}^x x \, dy \, dx = \int_0^1 (x^2 - x^3) \, dx = \frac{1}{12}$$

$$51. \mathbf{r}(u, v) = \sec u \cos v \mathbf{i} + (1 + 2 \tan u) \sin v \mathbf{j} + 2u \mathbf{k}$$

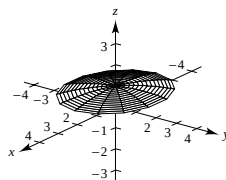
$$0 \leq u \leq \frac{\pi}{3}, \quad 0 \leq v \leq 2\pi$$



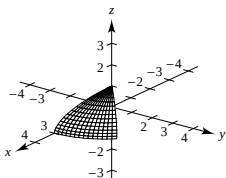
53. (a)



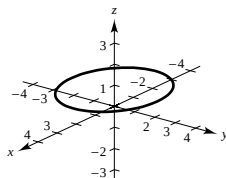
(b)



(c)



(d)



The space curve is a circle:

$$\mathbf{r}\left(u, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} \cos u \mathbf{i} + \frac{3\sqrt{2}}{2} \sin u \mathbf{j} + \frac{\sqrt{2}}{2} \mathbf{k}$$

$$(e) \mathbf{r}_u = -3 \cos v \sin u \mathbf{i} + 3 \cos v \cos u \mathbf{j}$$

$$\mathbf{r}_v = -3 \sin v \cos u \mathbf{i} - 3 \sin v \sin u \mathbf{j} + \cos v \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 \cos v \sin u & 3 \cos v \cos u & 0 \\ -3 \sin v \cos u & -3 \sin v \sin u & \cos v \end{vmatrix}$$

$$= (3 \cos^2 v \cos u) \mathbf{i} + (3 \cos^2 v \sin u) \mathbf{j} + (9 \cos v \sin v \sin^2 u + 9 \cos v \sin v \cos^2 u) \mathbf{k}$$

$$= (3 \cos^2 v \cos u) \mathbf{i} + (3 \cos^2 v \sin u) \mathbf{j} + (9 \cos v \sin v) \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{9 \cos^4 v \cos^2 u + 9 \cos^4 v \sin^2 u + 81 \cos^2 v \sin^2 v}$$

$$= \sqrt{9 \cos^4 v + 81 \cos^2 v \sin^2 v}$$

Using a Symbolic integration utility,

$$\int_{\pi/4}^{\pi/2} \int_0^{2\pi} \|\mathbf{r}_u \times \mathbf{r}_v\| \, du \, dv \approx 14.44$$

(f) Similarly,

$$\int_0^{\pi/4} \int_0^{\pi/2} \|\mathbf{r}_u \times \mathbf{r}_v\| \, dv \, du \approx 4.27$$

55. $S: \mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + (u - 1)(2 - u)\mathbf{k}, \quad 0 \leq u \leq 2, 0 \leq v \leq 2\pi$

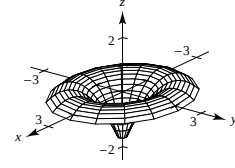
$$\mathbf{r}_u(u, v) = \cos v \mathbf{i} + \sin v \mathbf{j} + (3 - 2u)\mathbf{k}$$

$$\mathbf{r}_v(u, v) = -u \sin v \mathbf{i} + u \cos v \mathbf{j}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 3 - 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix} = (2u - 3)u \cos v \mathbf{i} + (2u - 3)u \sin v \mathbf{j} + u \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = u\sqrt{(2u - 3)^2 + 1}$$

$$\begin{aligned} \iint_S (x + y) dS &= \int_0^{2\pi} \int_0^2 (u \cos v + u \sin v) u \sqrt{(2u - 3)^2 + 1} du dv \\ &= \int_0^{2\pi} \int_0^2 (\cos v + \sin v) u^2 \sqrt{(2u - 3)^2 + 1} dv du = 0 \end{aligned}$$



57. $\mathbf{F}(x, y, z) = x^2 \mathbf{i} + xy \mathbf{j} + z \mathbf{k}$

Q : solid region bounded by the coordinates planes and the plane $2x + 3y + 4z = 12$

Surface Integral: There are four surfaces for this solid.

$$z = 0, \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = -z, \quad \iint_{S_1} 0 dS = 0$$

$$y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = -xy, \quad \iint_{S_2} 0 dS = 0$$

$$x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = -x^2, \quad \iint_{S_3} 0 dS = 0$$

$$2x + 3y + 4z = 12, \quad \mathbf{N} = \frac{2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}}{\sqrt{29}}, \quad dS = \sqrt{1 + \left(\frac{1}{4}\right) + \left(\frac{9}{16}\right)} dA = \frac{\sqrt{29}}{4} dA$$

$$\begin{aligned} \iint_{S_4} \mathbf{F} \cdot \mathbf{N} dS &= \frac{1}{4} \iint_R (2x^2 + 3xy + 4z) dA \\ &= \frac{1}{4} \int_0^6 \int_0^{4-(2x/3)} (2x^2 + 3xy + 12 - 2x - 3y) dy dx \\ &= \frac{1}{4} \int_0^6 \left[2x^2 \left(\frac{12 - 2x}{3} \right) + \frac{3x}{2} \left(\frac{12 - 2x}{3} \right)^2 + 12 \left(\frac{12 - 2x}{3} \right) - 2x \left(\frac{12 - 2x}{3} \right) - \frac{3}{2} \left(\frac{12 - 2x}{3} \right)^2 \right] dx \\ &= \frac{1}{6} \int_0^6 (-x^3 + x^2 + 24x + 36) dx = \frac{1}{6} \left[-\frac{x^4}{4} + \frac{x^3}{3} + 12x^2 + 36x \right]_0^6 = 66 \end{aligned}$$

Divergence Theorem: Since $\text{div } \mathbf{F} = 2x + x + 1 = 3x + 1$, Divergence Theorem yields

$$\begin{aligned} \iiint_Q \text{div } \mathbf{F} dV &= \int_0^6 \int_0^{(12-2x)/3} \int_0^{(12-2x-3y)/4} (3x + 1) dz dy dx \\ &= \int_0^6 \int_0^{(12-2x)/3} (3x + 1) \left(\frac{12 - 2x - 3y}{4} \right) dy dx \\ &= \frac{1}{4} \int_0^6 (3x + 1) \left[12y - 2xy - \frac{3}{2}y^2 \right]_0^{(12-2x)/3} dx \\ &= \frac{1}{4} \int_0^6 (3x + 1) \left[4(12 - 2x) - 2x \left(\frac{12 - 2x}{3} \right) - \frac{3}{2} \left(\frac{12 - 2x}{3} \right)^2 \right] dx \\ &= \frac{1}{4} \int_0^6 \frac{2}{3} (3x^3 - 35x^2 + 96x + 36) dx = \frac{1}{6} \left[\frac{3x^4}{4} - \frac{35x^3}{3} + 48x^2 + 36x \right]_0^6 = 66. \end{aligned}$$

59. $\mathbf{F}(x, y, z) = (\cos y + y \cos x)\mathbf{i} + (\sin x - x \sin y)\mathbf{j} + xyz\mathbf{k}$

S: portion of $z = y^2$ over the square in the xy -plane with vertices $(0, 0)$, $(a, 0)$, (a, a) , $(0, a)$

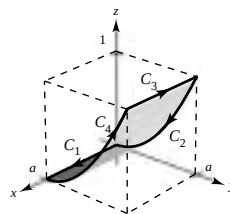
Line Integral: Using the line integral we have:

$$C_1: y = 0, \quad dy = 0$$

$$C_2: x = 0, \quad dx = 0, \quad z = y^2, \quad dz = 2y \, dy$$

$$C_3: y = a, \quad dy = 0, \quad z = a^2, \quad dz = 0$$

$$C_4: x = a, \quad dx = 0, \quad z = y^2, \quad dz = 2y \, dy$$



$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (\cos y + y \cos x) \, dx + (\sin x - x \sin y) \, dy + xyz \, dz \\ &= \int_{C_1} dx + \int_{C_2} 0 + \int_{C_3} (\cos a + a \cos x) \, dx + \int_{C_4} (\sin a - a \sin y) \, dy + ay^3(2y \, dy) \\ &= \int_0^a dx + \int_a^0 (\cos a + a \cos x) \, dx + \int_0^a (\sin a - a \sin y) \, dy + \int_0^a 2ay^4 \, dy \\ &= a + \left[x \cos a + a \sin x \right]_a^0 + \left[y \sin a + a \cos y \right]_0^a + \left[2a \frac{y^5}{5} \right]_0^a \\ &= a - a \cos a - a \sin a + a \sin a + a \cos a - a + \frac{2a^6}{5} = \frac{2a^6}{5} \end{aligned}$$

Double Integral: Considering $f(x, y, z) = z - y^2$, we have:

$$\mathbf{N} = \frac{\nabla f}{\|\nabla f\|} = \frac{-2y\mathbf{j} + \mathbf{k}}{\sqrt{1 + 4y^2}}, \quad dS = \sqrt{1 + 4y^2} \, dA, \quad \text{and } \mathbf{curl} \, \mathbf{F} = xz\mathbf{i} - yz\mathbf{j}.$$

Hence,

$$\iint_S (\mathbf{curl} \, \mathbf{F}) \cdot \mathbf{N} \, dS = \int_0^a \int_0^a 2y^2 z \, dy \, dx = \int_0^a \int_0^a 2y^4 \, dy \, dx = \int_0^a \frac{2a^5}{5} \, dx = \frac{2a^6}{5}.$$

Problem Solving for Chapter 14

1. (a) $\nabla T = \frac{-25}{(x^2 + y^2 + z^2)^{3/2}} [x\mathbf{i} + y\mathbf{j} + z\mathbf{k}]$

$$\mathbf{N} = x\mathbf{i} + \sqrt{1 - x^2} \mathbf{k}$$

$$dS = \frac{1}{\sqrt{1 - x^2}} \, dy \, dx$$

$$\text{Flux} = \iint_S -k \nabla T \cdot \mathbf{N} \, dS$$

$$= 25k \iint_R \left[\frac{x^2}{(x^2 + y^2 + z^2)^{3/2}(1 - x^2)^{1/2}} + \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right] dA$$

$$= 25k \int_{-1/2}^{1/2} \int_0^1 \left[\frac{x^2}{(x^2 + y^2 + z^2)^{3/2}(1 - x^2)^{1/2}} + \frac{1 - x^2}{(x^2 + y^2 + z^2)^{3/2}(1 - x^2)^{1/2}} \right] dy \, dx$$

$$= 25k \int_{-1/2}^{1/2} \int_0^1 \frac{1}{(1 + y^2)^{3/2}(1 - x^2)^{1/2}} \, dy \, dx$$

$$= 25k \int_0^1 \frac{1}{(1 + y^2)^{3/2}} \, dy \int_{-1/2}^{1/2} \frac{1}{(1 - x^2)^{1/2}} \, dx$$

$$= 25k \left(\frac{\sqrt{2}}{2} \right) \left(\frac{\pi}{3} \right) = 25k \frac{\sqrt{2}\pi}{6}$$

—CONTINUED—

1. —CONTINUED—

$$(b) \mathbf{r}(u, v) = \langle \cos u, v, \sin u \rangle$$

$$\mathbf{r}_u = \langle -\sin u, 0, \cos u \rangle, \mathbf{r}_v = \langle 0, 1, 0 \rangle$$

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -\cos u, 0, -\sin u \rangle$$

$$\nabla T = \frac{-25}{(x^2 + y^2 + z^2)^{3/2}} [x\mathbf{i} + y\mathbf{j} + z\mathbf{k}]$$

$$= \frac{-25}{(v^2 + 1)^{3/2}} [\cos u\mathbf{i} + v\mathbf{j} + \sin u\mathbf{k}]$$

$$\nabla T \cdot (\mathbf{r}_u \times \mathbf{r}_v) = \frac{-25}{(v^2 + 1)^{3/2}} (-\cos^2 u - \sin^2 u) = \frac{25}{(v^2 + 1)^{3/2}}$$

$$\text{Flux} = \int_0^1 \int_{\pi/3}^{2\pi/3} \frac{25k}{(v^2 + 1)^{3/2}} du dv = 25k \frac{\sqrt{2}\pi}{6}$$

$$3. \mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, 2t \rangle$$

$$\mathbf{r}'(t) = \langle -3 \sin t, 3 \cos t, 2 \rangle, \|\mathbf{r}'(t)\| = \sqrt{13}$$

$$I_x = \int_C (y^2 + z^2) \rho ds = \int_0^{2\pi} (9 \sin^2 t + 4t^2) \sqrt{13} dt = \frac{1}{3} \sqrt{13} \pi (32\pi^2 + 27)$$

$$I_y = \int_C (x^2 + z^2) \rho ds = \int_0^{2\pi} (9 \cos^2 t + 4t^2) \sqrt{13} dt = \frac{1}{3} \sqrt{13} \pi (32\pi^2 + 27)$$

$$I_z = \int_C (x^2 + y^2) \rho ds = \int_0^{2\pi} (9 \cos^2 t + 9 \sin^2 t) \sqrt{13} dt = 18\pi \sqrt{13}$$

$$\begin{aligned} 5. \frac{1}{2} \int_C x dy - y dx &= \frac{1}{2} \int_0^{2\pi} [a(\theta - \sin \theta)(a \sin \theta) d\theta - a(1 - \cos \theta)(a(1 - \cos \theta)) d\theta] \\ &= \frac{1}{2} a^2 \int_0^{2\pi} [\theta \sin \theta - \sin^2 \theta - 1 + 2 \cos \theta - \cos^2 \theta] d\theta \\ &= \frac{1}{2} a^2 \int_0^{2\pi} (\theta \sin \theta + 2 \cos \theta - 2) d\theta \\ &= -3\pi a^2 \end{aligned}$$

Hence, the area is $3\pi a^2$.

$$7. (a) \mathbf{r}(t) = t\mathbf{j}, 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = \mathbf{j}$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t\mathbf{i} + \mathbf{j}) \cdot \mathbf{j} dt = \int_0^1 dt = 1$$

$$(b) \mathbf{r}(t) = (t - t^2)\mathbf{i} + t\mathbf{j}, 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = (1 - 2t)\mathbf{i} + \mathbf{j}$$

$$\begin{aligned} W &= \mathbf{F} \cdot d\mathbf{r} = \int_0^1 ((2t - t^2)\mathbf{i} + [(t - t^2)^2 + 1]\mathbf{j}) \cdot ((1 - 2t)\mathbf{i} + \mathbf{j}) dt \\ &= \int_0^1 [(1 - 2t)(2t - t^2) + (t^4 - 2t^3 + t^2 + 1)] dt \\ &= \int_0^1 (t^4 - 4t^2 + 2t + 1) dt = \frac{13}{15} \end{aligned}$$

—CONTINUED—

7. —CONTINUED—

$$(c) \quad \mathbf{r}(t) = c(t - t^2)\mathbf{i} + t\mathbf{j}, 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = c(1 - 2t)\mathbf{i} + \mathbf{j}$$

$$\begin{aligned} \mathbf{F} \cdot d\mathbf{r} &= (c(t - t^2) + t)(c(1 - 2t)) + (c^2(t - t^2)^2 + 1)(1) \\ &= c^2t^4 - 2c^2t^2 + c^2t - 2ct^2 + ct + 1 \end{aligned}$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{30}c^2 - \frac{1}{6}c + 1$$

$$\frac{dW}{dc} = \frac{1}{15}c - \frac{1}{6} = 0 \Rightarrow c = \frac{5}{2}$$

$$\frac{d^2W}{dc^2} = \frac{1}{15} > 0 \quad c = \frac{5}{2} \text{ minimum.}$$

$$9. \quad \mathbf{v} \times \mathbf{r} = \langle a_1, a_2, a_3 \rangle \times \langle x, y, z \rangle$$

$$= \langle a_2z - a_3y, -a_1z + a_3x, a_1y - a_2x \rangle$$

$$\mathbf{curl}(\mathbf{v} \times \mathbf{r}) = \langle 2a_1, 2a_2, 2a_3 \rangle = 2\mathbf{v}$$

By Stoke's Theorem,

$$\begin{aligned} \int_C (\mathbf{v} \times \mathbf{r}) \cdot d\mathbf{r} &= \iint_S \mathbf{curl}(\mathbf{v} \times \mathbf{r}) \cdot \mathbf{N} \, dS \\ &= \iint_S 2\mathbf{v} \cdot \mathbf{N} \, dS. \end{aligned}$$

$$11. \quad \mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} = \frac{m}{(x^2 + y^2)^{5/2}}[3xy\mathbf{i} + (2y^2 - x^2)\mathbf{j}]$$

$$M = \frac{3mxy}{(x^2 + y^2)^{5/2}} = 3mxy(x^2 + y^2)^{-5/2}$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= 3mxy \left[-\frac{5}{2}(x^2 + y^2)^{-7/2}(2y) \right] + (x^2 + y^2)^{-5/2}(3mx) \\ &= 3mx(x^2 + y^2)^{-7/2}[-5y^2 + (x^2 + y^2)] = \frac{3mx(x^2 - 4y^2)}{(x^2 + y^2)^{7/2}} \end{aligned}$$

$$N = \frac{m(2y^2 - x^2)}{(x^2 + y^2)^{5/2}} = m(2y^2 - x^2)(x^2 + y^2)^{-5/2}$$

$$\begin{aligned} \frac{\partial N}{\partial x} &= m(2y^2 - x^2) \left[-\frac{5}{2}(x^2 + y^2)^{-7/2}(2x) \right] + (x^2 + y^2)^{-5/2}(-2mx) \\ &= mx(x^2 + y^2)^{-7/2}[(2y^2 - x^2)(-5) + (x^2 + y^2)(-2)] \\ &= mx(x^2 + y^2)^{-7/2}(3x^2 - 12y^2) = \frac{3mx(x^2 - 4y^2)}{(x^2 + y^2)^{7/2}} \end{aligned}$$

Therefore, $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ and $\mathbf{F}$ is conservative.

22. $\mathbf{F}(x, y, z) = -z\mathbf{i} + y\mathbf{k}$

$S: x^2 + y^2 = 1$

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & 0 & y \end{vmatrix} = \mathbf{i} - \mathbf{j}$$

Letting $\mathbf{N} = \mathbf{k}$, $\mathbf{curl} \mathbf{F} \cdot \mathbf{N} = 0$ and $\iint_S (\mathbf{curl} \mathbf{F}) \cdot \mathbf{N} \, dS = 0$.

24. $\mathbf{curl} \mathbf{F}$ measures the rotational tendency.

See page 1084.

26. $f(x, y, z) = xyz$, $g(x, y, z) = z$, $S: z = \sqrt{4 - x^2 - y^2}$

(a) $\nabla g(x, y, z) = \mathbf{k}$

$f(x, y, z)\nabla g(x, y, z) = xyz\mathbf{k}$

$\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + 0\mathbf{k}$, $0 \leq t \leq 2\pi$

$$\int_C [f(x, y, z)\nabla g(x, y, z)] \cdot d\mathbf{r} = 0$$

(b) $\nabla f(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$\nabla g(x, y, z) = \mathbf{k}$

$$\nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ yz & xz & xy \\ 0 & 0 & 1 \end{vmatrix} = xz\mathbf{i} - yz\mathbf{j}$$

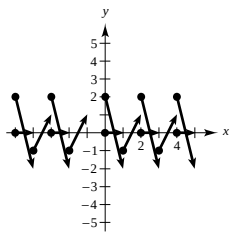
$$\mathbf{N} = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$dS = \sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{4 - x^2 - y^2}}\right)^2} dA = \frac{2}{\sqrt{4 - x^2 - y^2}} dA$$

$$\begin{aligned} \iint_S [\nabla f(x, y, z) \times \nabla g(x, y, z)] \cdot \mathbf{N} \, dS &= \iint_S \left[\frac{x^2 z}{\sqrt{4 - x^2 - y^2}} - \frac{y^2 z}{\sqrt{4 - x^2 - y^2}} \right] \frac{2}{\sqrt{4 - x^2 - y^2}} dA \\ &= \iint_S \frac{2(x^2 - y^2)}{\sqrt{4 - x^2 - y^2}} dA \\ &= \int_0^2 \int_0^{2\pi} \frac{2r^2(\cos^2 \theta - \sin^2 \theta)}{\sqrt{4 - r^2}} r \, d\theta \, dr \\ &= \int_0^2 \left[\frac{2r^3}{\sqrt{4 - r^2}} \left(\frac{1}{2} \sin 2\theta \right) \right]_0^{2\pi} dr = 0 \end{aligned}$$

Review Exercises for Chapter 14

2. $\mathbf{F}(x, y) = \mathbf{i} - 2y\mathbf{j}$



4. $f(x, y, z) = x^2 e^{yz}$

$$\begin{aligned} \mathbf{F}(x, y, z) &= 2xe^{yz}\mathbf{i} + x^2ze^{yz}\mathbf{j} + x^2ye^{yz}\mathbf{k} \\ &= xe^{yz}(2\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) \end{aligned}$$

6. Since $\partial M/\partial y = -1/x^2 = \partial N/\partial x$, $\mathbf{F}$ is conservative. From $M = \partial U/\partial x = -y/x^2$ and $N = \partial U/\partial y = 1/x$, partial integration yields $U = (y/x) + h(y)$ and $U = (y/x) + g(x)$ which suggests that $U(x, y) = (y/x) + C$.

8. Since $\partial M/\partial y = -6y^2 \sin 2x = \partial N/\partial x$, $\mathbf{F}$ is conservative. From $M = \partial U/\partial x = -2y^3 \sin 2x$ and $N = \partial U/\partial y = 3y^2(1 + \cos 2x)$, we obtain $U = y^3 \cos 2x + h(y)$ and $U = y^3(1 + \cos 2x) + g(x)$ which suggests that $h(y) = y^3$, $g(x) = C$, and $U(x, y) = y^3(1 + \cos 2x) + C$.

10. Since

$$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x},$$

$$\frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x},$$

$$\frac{\partial N}{\partial z} = 6y \neq \frac{\partial P}{\partial y}$$

$\mathbf{F}$ is not conservative.

14. Since $\mathbf{F} = xy^2\mathbf{j} - zx^2\mathbf{k}$;

(a) $\operatorname{div} \mathbf{F} = 2xy - x^2$

(b) $\operatorname{curl} \mathbf{F} = 2xz\mathbf{j} + y^2\mathbf{k}$

18. Since $\mathbf{F} = (x^2 - y)\mathbf{i} - (x + \sin^2 y)\mathbf{j}$;

(a) $\operatorname{div} \mathbf{F} = 2x - 2 \sin y \cos y$

(b) $\operatorname{curl} \mathbf{F} = \mathbf{0}$

12. Since

$$\frac{\partial M}{\partial y} = \sin z = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = y \cos z \neq \frac{\partial P}{\partial x},$$

$\mathbf{F}$ is not conservative.

16. Since $\mathbf{F} = (3x - y)\mathbf{i} + (y - 2z)\mathbf{j} + (z - 3x)\mathbf{k}$;

(a) $\operatorname{div} \mathbf{F} = 3 + 1 + 1 = 5$

(b) $\operatorname{curl} \mathbf{F} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

20. Since $\mathbf{F} = \frac{z}{x}\mathbf{i} + \frac{z}{y}\mathbf{j} + z^2\mathbf{k}$;

(a) $\operatorname{div} \mathbf{F} = -\frac{z}{x^2} - \frac{z}{y^2} + 2z = z\left(2 - \frac{1}{x^2} - \frac{1}{y^2}\right)$

(b) $\operatorname{curl} \mathbf{F} = -\frac{1}{y}\mathbf{i} + \frac{1}{x}\mathbf{j}$

22. (a) Let $x = 5t$, $y = 4t$, $0 \leq t \leq 1$, then $ds = \sqrt{41} dt$.

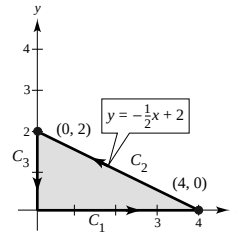
$$\int_C xy \, ds = \int_0^1 20t^2 \sqrt{41} \, dt = \frac{20\sqrt{41}}{3}$$

(b) $C_1: x = t, y = 0, 0 \leq t \leq 4, ds = dt$

$$C_2: x = 4 - 4t, y = 2t, 0 \leq t \leq 1, ds = 2\sqrt{5} \, dt$$

$$C_3: x = 0, y = 2 - t, 0 \leq t \leq 2, ds = dt$$

$$\begin{aligned} \text{Therefore, } \int_C xy \, ds &= \int_0^4 0 \, dt = \int_0^1 (8t - 8t^2) 2\sqrt{5} \, dt + \int_0^2 0 \, dt \\ &= 16\sqrt{5} \left[\frac{t^2}{2} - \frac{t^3}{3} \right]_0^1 = \frac{8\sqrt{5}}{3}. \end{aligned}$$



24. $x = t - \sin t$, $y = 1 - \cos t$, $0 \leq t \leq 2\pi$, $\frac{dx}{dt} = 1 - \cos t$, $\frac{dy}{dt} = \sin t$

$$\begin{aligned} \int_C x \, ds &= \int_0^{2\pi} (t - \sin t) \sqrt{(1 - \cos t)^2 + (\sin t)^2} \, dt = \int_0^{2\pi} (t - \sin t) \sqrt{2 - 2 \cos t} \, dt \\ &= \sqrt{2} \int_0^{2\pi} [t \sqrt{1 - \cos t} - \sin t \sqrt{1 - \cos t}] \, dt = \sqrt{2} \left[-\frac{2}{3} (1 - \cos t)^{3/2} \right]_0^{2\pi} + \sqrt{2} \int_0^{2\pi} t \sqrt{1 - \cos t} \, dt \\ &= \sqrt{2} \int_0^{2\pi} t \sqrt{1 - \cos t} \, dt \\ &= 8\pi \end{aligned}$$

26. $x = \cos t + t \sin t, y = \sin t - t \sin t, 0 \leq t \leq \frac{\pi}{2}, dx = t \cos t dt, dy = (\cos t - t \cos t - \sin t) dt$

$$\int_C (2x - y) dx + (x + 3y) dy = \int_0^{\pi/2} [\sin t \cos t (5t^2 - 6t + 2) + \cos^2 t(t + 1) + \sin^2 t(2t - 3)] dt \approx 1.01$$

28. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^{3/2}\mathbf{k}, 0 \leq t \leq 4$

$$x'(t) = 1, y'(t) = 2t, z'(t) = \frac{3}{2}t^{1/2}$$

$$\int_C (x^2 + y^2 + z^2) ds = \int_0^4 (t^2 + t^4 + t^3) \sqrt{1 + 4t^2 + \frac{9}{4}t} dt \approx 2080.59$$

30. $f(x, y) = 12 - x - y$

$C: y = x^2$ from $(0, 0)$ to $(2, 4)$

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 2$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{1 + 4t^2}$$

Lateral surface area:

$$\int_C f(x, y) ds = \int_0^2 (12 - t - t^2) \sqrt{1 + 4t^2} dt \approx 41.532$$

32. $d\mathbf{r} = [(-4 \sin t)\mathbf{i} + 3 \cos t\mathbf{j}] dt$

$$\mathbf{F} = (4 \cos t - 3 \sin t)\mathbf{i} + (4 \cos t + 3 \sin t)\mathbf{j}, 0 \leq t \leq 2\pi$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (12 - 7 \sin t \cos t) dt = \left[12t - \frac{7 \sin^2 t}{2} \right]_0^{2\pi} = 24\pi$$

34. $x = 2 - t, y = 2 - t, z = \sqrt{4t - t^2}, 0 \leq t \leq 2$

$$d\mathbf{r} = \left[-\mathbf{i} - \mathbf{j} + \frac{2 - t}{\sqrt{4t - t^2}} \mathbf{k} \right] dt$$

$$\mathbf{F} = (4 - 2t - \sqrt{4t - t^2})\mathbf{i} + (\sqrt{4t - t^2} - 2 + t)\mathbf{j} + 0\mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 (t - 2) dt = \left[\frac{t^2}{2} - 2t \right]_0^2 = -2$$

36. Let $x = 2 \sin t, y = -2 \cos t, z = 4 \sin^2 t, 0 \leq t \leq \pi$.

$$d\mathbf{r} = [(2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + (8 \sin t \cos t)\mathbf{k}] dt$$

$$\mathbf{F} = 0\mathbf{i} + 4\mathbf{j} + (2 \sin t)\mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi (8 \sin t + 16 \sin^2 t \cos t) dt = \left[-8 \cos t + \frac{16}{3} \sin^3 t \right]_0^\pi = 16$$

38. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (2x - y) dx + (2y - x) dy$

$$\mathbf{r}(t) = (2 \cos t + 2t \sin t)\mathbf{i} + (2 \sin t - 2t \cos t)\mathbf{j}, 0 \leq t \leq \pi$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 4\pi^2 + 4\pi$$

$$\begin{aligned}
 40. \mathbf{r}(t) &= 10 \sin t \mathbf{i} + 10 \cos t \mathbf{j} + \frac{2000/5280}{\pi/2} t \mathbf{k}, \quad 0 \leq t \leq \frac{\pi}{2} \\
 &= 10 \sin t \mathbf{i} + 10 \cos t \mathbf{j} + \frac{25}{33\pi} t \mathbf{k}
 \end{aligned}$$

$$\mathbf{F} = 20\mathbf{k}$$

$$d\mathbf{r} = \left(10 \cos t \mathbf{i} - 10 \sin t \mathbf{j} + \frac{25}{33\pi} \mathbf{k} \right)$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \frac{500}{33\pi} dt = \frac{250}{33} \text{ mi} \cdot \text{ton}$$

$$42. \int_C y \, dx + x \, dy + \frac{1}{z} \, dz = \left[xy + \ln|z| \right]_{(0,0,1)}^{(4,4,4)} = 16 + \ln 4$$

$$44. x = a(\theta - \sin \theta), y = a(1 - \cos \theta), 0 \leq \theta \leq 2\pi$$

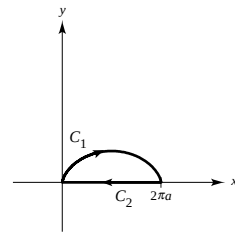
$$(a) A = \frac{1}{2} \int_C x \, dy - y \, dx.$$

Since these equations orient the curve backwards, we will use

$$\begin{aligned}
 A &= \frac{1}{2} \int (y \, dx - x \, dy) \\
 &= \frac{1}{2} \int_0^{2\pi} [a^2(1 - \cos \theta)(1 - \cos \theta) - a^2(\theta - \sin \theta)(\sin \theta)] \, d\theta + \frac{1}{2} \int_0^{2\pi} (0 - 0) \, d\theta \\
 &= \frac{a^2}{2} \int_0^{2\pi} [1 - 2 \cos \theta + \cos^2 \theta - \theta \sin \theta + \sin^2 \theta] \, d\theta \\
 &= \frac{a^2}{2} \int_0^{2\pi} (2 - 2 \cos \theta - \theta \sin \theta) \, d\theta = \frac{a^2}{2} (6\pi) = 3\pi a^2.
 \end{aligned}$$

(b) By symmetry, $\bar{x} = \pi a$. From Section 14.4,

$$\bar{y} = -\frac{1}{2A} \int_C y^2 \, dx = \frac{1}{2A} \int_0^{2\pi} a^3(1 - \cos \theta)^2(1 - \cos \theta) \, d\theta = \frac{1}{2(3\pi a^2)} a^3(5\pi) = \frac{5}{6}a$$



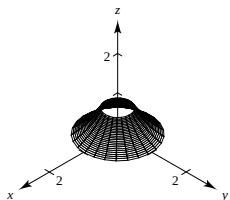
$$\begin{aligned}
 46. \int_C xy \, dx + (x^2 + y^2) \, dy &= \int_0^2 \int_0^2 (2x - x) \, dy \, dx \\
 &= \int_0^2 2x \, dx = 4
 \end{aligned}$$

$$\begin{aligned}
 48. \int_C (x^2 - y^2) \, dx + 2xy \, dy &= \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 4y \, dy \, dx \\
 &= \int_{-a}^a 0 \, dx = 0
 \end{aligned}$$

$$\begin{aligned}
 50. \int_C y^2 \, dx + x^{4/3} \, dy &= \int_{-1}^1 \int_{-(1-x^{2/3})^{3/2}}^{(1-x^{2/3})^{3/2}} \left(\frac{4}{3} x^{1/3} - 2y \right) \, dy \, dx \\
 &= \int_{-1}^1 \left[\frac{4}{3} x^{1/3} y - y^2 \right]_{-(1-x^{2/3})^{3/2}}^{(1-x^{2/3})^{3/2}} \, dx \\
 &= \int_{-1}^1 \frac{8}{3} x^{1/3} (1 - x^{2/3})^{3/2} \, dx \\
 &= \left[-\frac{8}{7} x^{2/3} (1 - x^{2/3})^{5/2} - \frac{16}{35} (1 - x^{2/3})^{5/2} \right]_{-1}^1 \\
 &= 0
 \end{aligned}$$

52. $\mathbf{r}(u, v) = e^{-u/4} \cos v \mathbf{i} + e^{-u/4} \sin v \mathbf{j} + \frac{u}{6} \mathbf{k}$

$$0 \leq u \leq 4, \quad 0 \leq v \leq 2\pi$$



54. $S: \mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + \sin v \mathbf{k}, \quad 0 \leq u \leq 2, 0 \leq v \leq \pi$

$$\mathbf{r}_u(u, v) = \mathbf{i} + \mathbf{j}$$

$$\mathbf{r}_v(u, v) = \mathbf{i} - \mathbf{j} + \cos v \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & \cos v \end{vmatrix} = \cos v \mathbf{i} - \cos v \mathbf{j} - 2\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{2 \cos^2 v + 4}$$

$$\iint_S z \, dS = \int_0^\pi \int_0^2 \sin v \sqrt{2 \cos^2 v + 4} \, du \, dv = 2 \left[\sqrt{6} + \sqrt{2} \ln \left(\frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}} \right) \right]$$

56. (a) $z = a(a - \sqrt{x^2 + y^2}), 0 \leq z \leq a^2$

$$z = 0 \Rightarrow x^2 + y^2 = a^2$$

(b) $S: g(x, y) = z = a^2 - a\sqrt{x^2 + y^2}$

$$\rho(x, y) = k\sqrt{x^2 + y^2}$$

$$m = \iint_S e(x, y, z) \, dS$$

$$= \iint_R k\sqrt{x^2 + y^2} \sqrt{1 + g_x^2 + g_y^2} \, dA$$

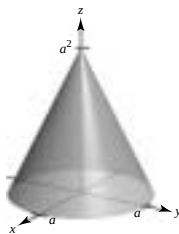
$$= k \iint_R \sqrt{x^2 + y^2} \sqrt{1 + \frac{a^2 x^2}{x^2 + y^2} + \frac{a^2 y^2}{x^2 + y^2}} \, dA$$

$$= k \iint_R \sqrt{a^2 + 1} (\sqrt{x^2 + y^2}) \, dA$$

$$= k\sqrt{a^2 + 1} \int_0^{2\pi} \int_0^a r^2 \, dr \, d\theta$$

$$= k\sqrt{a^2 + 1} \int_0^{2\pi} \frac{a^3}{3} \, d\theta$$

$$= \frac{2}{3} k\sqrt{a^2 + 1} a^3 \pi$$



58. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Q : solid region bounded by the coordinate planes and the plane $2x + 3y + 4z = 12$

Surface Integral: There are four surfaces for this solid.

$$z = 0 \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = -z, \quad \iint_{S_1} 0 \, dS = 0$$

$$y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = -y, \quad \iint_{S_2} 0 \, dS = 0$$

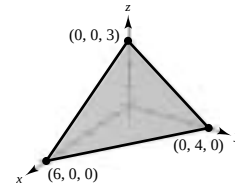
$$x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = -x, \quad \iint_{S_3} 0 \, dS = 0$$

$$2x + 3y + 4z = 12, \quad \mathbf{N} = \frac{2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}}{\sqrt{29}}, \quad dS = \sqrt{1 + \left(\frac{1}{4}\right) + \left(\frac{9}{16}\right)} dA = \frac{\sqrt{29}}{4} dA$$

$$\begin{aligned} \iint_{S_4} \mathbf{N} \cdot \mathbf{F} \, dS &= \frac{1}{4} \iint_R (2x + 3y + 4z) \, dy \, dx \\ &= \frac{1}{4} \int_0^6 \int_0^{(12-2x)/3} 12 \, dy \, dx = 3 \int_0^6 \left(4 - \frac{2x}{3}\right) dx = 3 \left[4x - \frac{x^2}{3}\right]_0^6 = 36 \end{aligned}$$

Triple Integral: Since $\text{div } \mathbf{F} = 3$, the Divergence Theorem yields.

$$\iiint_Q \text{div } \mathbf{F} \, dV = \iiint_Q 3 \, dV = 3(\text{Volume of solid}) = 3 \left[\frac{1}{3} (\text{Area of base})(\text{Height}) \right] = \frac{1}{2} (6)(4)(3) = 36.$$

60. $\mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - z)\mathbf{j} + x^2\mathbf{k}$

S : first octant portion of the plane $3x + y + 2z = 12$

Line Integral:

$$C_1: y = 0, \quad dy = 0, \quad z = \frac{12 - 3x}{2}, \quad dz = -\frac{3}{2} dx$$

$$C_2: x = 0, \quad dx = 0, \quad z = \frac{12 - y}{2}, \quad dz = -\frac{1}{2} dy$$

$$C_3: z = 0, \quad dz = 0, \quad y = 12 - 3x, \quad dy = -3 dx$$

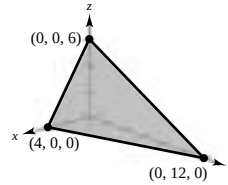
$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (x - z) dx + (y - z) dy + x^2 dz \\ &= \int_{C_1} \left[x - \frac{12 - 3x}{2} + x^2 \left(-\frac{3}{2}\right) \right] dx + \int_{C_2} \left[y - \frac{12 - y}{2} \right] dy + \int_{C_3} [x + (12 - 3x)(-3)] dx \\ &= \int_4^0 \left(-\frac{3}{2}x^2 + \frac{5}{2}x - 6 \right) dx + \int_0^{12} \left(\frac{3}{2}y - 6 \right) dy + \int_0^4 (10x - 36) dx = 8 \end{aligned}$$

Double Integral: $G(x, y, z) = \frac{12 - 3x - y}{2} - z$

$$\nabla G(x, y, z) = -\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} - \mathbf{k}$$

$$\text{curl } \mathbf{F} = \mathbf{i} - (2x + 1)\mathbf{j}$$

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} \, dS = \int_0^4 \int_0^{12-3x} (x - 1) \, dy \, dx = \int_0^4 (-3x^2 + 15x - 12) \, dx = 8$$



Problem Solving for Chapter 14

$$2. (a) \quad z = \sqrt{1 - x^2 - y^2}, \quad \frac{\partial z}{\partial x} = \frac{-x}{\sqrt{1 - x^2 - y^2}}, \quad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{1 - x^2 - y^2}}$$

$$dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA = \frac{1}{\sqrt{1 - x^2 - y^2}}$$

$$\nabla T = \frac{-25}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = -25(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\begin{aligned} \mathbf{N} &= \frac{-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}} \\ &= \left(\frac{x}{\sqrt{1 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{1 - x^2 - y^2}}\mathbf{j} + \mathbf{k}\right)\sqrt{1 - x^2 - y^2} \\ &= x\mathbf{i} + y\mathbf{j} + \sqrt{1 - x^2 - y^2}\mathbf{k} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \end{aligned}$$

$$\begin{aligned} \text{Flux} &= \iint_S -k\nabla T \cdot \mathbf{N} dS \\ &= k \iint_R 25(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \frac{1}{\sqrt{1 - x^2 - y^2}} dA \\ &= k \iint_R \frac{25}{\sqrt{1 - x^2 - y^2}} dA \\ &= 25k \int_0^{2\pi} \int_0^1 \frac{1}{\sqrt{1 - r^2}} r dr d\theta = 50\pi k \end{aligned}$$

$$(b) \quad \mathbf{r}(u, v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle$$

$$\mathbf{r}_u = \langle \cos u \cos v, \cos u \sin v, -\sin u \rangle$$

$$\mathbf{r}_v = \langle -\sin u \sin v, \sin u \cos v, 0 \rangle$$

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u \sin^2 v + \sin u \cos u \cos^2 v \rangle$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sin u$$

$$\text{Flux} = 25k \int_0^{2\pi} \int_0^{\pi/2} \sin u du dv = 50\pi k$$

$$4. \quad \mathbf{r}(t) = \left\langle \frac{t^2}{2}, t, \frac{2\sqrt{2}t^{3/2}}{3} \right\rangle$$

$$\mathbf{r}'(t) = \langle t, 1, \sqrt{2}t^{1/2} \rangle, \quad \|\mathbf{r}'(t)\| = t + 1$$

$$\rho ds = \frac{1}{1+t}(t+1) dt = 1$$

$$I_y = \int_C (x^2 + z^2) \rho ds = \int_0^1 \left(\frac{t^4}{4} + \frac{8}{9}t^3 \right) dt = \frac{49}{180}$$

$$I_x = \int_C (y^2 + z^2) \rho ds = \int_0^1 \left(t^2 + \frac{8}{9}t^3 \right) dt = \frac{5}{9}$$

$$I_z = \int_C (x^2 + y^2) \rho ds = \int_0^1 \left(\frac{t^4}{4} + t^2 \right) dt = \frac{23}{60}$$

$$6. \frac{1}{2} \int_C x \, dy - y \, dx = 2 \int_0^{\pi/2} \left[\frac{1}{2} \sin 2t \cos t - \sin t \cos 2t \right] dt = 2 \left(\frac{2}{3} \right)$$

Hence, the area is $4/3$.

$$8. F(x, y) = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j} \text{ is conservative.}$$

$$f(x, y) = x^3y^2 \text{ potential function.}$$

$$\text{Work} = f(2, 4) - f(1, 1) = 8(16) - 1 = 127$$

$$10. \text{Area} = \pi ab$$

$$\mathbf{r}(t) = a \cos t \mathbf{i} + b \sin t \mathbf{j}, 0 \leq t \leq 2\pi$$

$$\mathbf{r}'(t) = -a \sin t \mathbf{i} + b \cos t \mathbf{j}$$

$$\mathbf{F} = -\frac{1}{2}b \sin t \mathbf{i} + \frac{1}{2}a \cos t \mathbf{j}$$

$$\mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{2}ab \sin^2 t + \frac{1}{2}ab \cos^2 t \right] dt = \frac{1}{2}ab$$

$$W = \int_0^{2\pi} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}ab(2\pi) = \pi ab$$

Same as area.