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Chapter 0

Problems 0.1

1. True; –13 is a negative integer.

2. True, because –2 and 7 are integers and $7 \neq 0$.

3. False, because the natural numbers are 1, 2, 3, and so on.

4. False, because $0 = \frac{0}{1}$.

5. True, because $5 = \frac{5}{1}$.

6. False, since a rational number cannot have denominator of zero. In fact, $\frac{7}{0}$ is not a number at all because we cannot divide by 0.

7. False, because $\sqrt{25} = 5$, which is a positive integer.

8. True; $\sqrt{2}$ is an irrational real number.

9. False; we cannot divide by 0.

10. False, because the natural numbers are 1, 2, 3, and so on, and $\sqrt{3}$ lies between 1 and 2.

11. True

12. False, since the integer 0 is neither positive nor negative.

Problems 0.2

1. False, because 0 does not have a reciprocal.

2. True, because $\frac{7}{3} \cdot \frac{3}{7} = \frac{21}{21} = 1$.

3. False; the negative of 7 is –7 because $7 + (-7) = 0$.

4. False; $2(3 \cdot 4) = 2(12) = 24$, but $(2 \cdot 3)(2 \cdot 4) = 6 \cdot 8 = 48$.

5. False; $-x + y = y + (-x) = y - x$.

6. True; $(x + 2)(4) = (x)(4) + (2)(4) = 4x + 8$.

7. True; $\frac{x+2}{2} = \frac{x}{2} + \frac{2}{2} = \frac{x}{2} + 1$.

8. True, because $a\left(\frac{b}{c}\right) = \frac{ab}{c}$.

9. False; the left side is $5xy$, but the right side is $5x^2y$.

10. True; by the associative and commutative properties, $x(4y) = (x \cdot 4)y = (4 \cdot x)y = 4xy$.

11. distributive

12. commutative

13. associative

14. definition of division

15. commutative and distributive

16. associative

17. definition of subtraction

18. commutative

19. distributive

20. distributive

21. $2x(y - 7) = (2x)y - (2x)7 = 2xy - (7)(2x)$
$= 2xy - (7 \cdot 2)x = 2xy - 14x$

22. $(a - b) + c = [a + (-b)] + c = a + (-b + c)$
$= a + [c + (-b)] = a + (c - b)$

23. $(x + y)(2) = 2(x + y) = 2x + 2y$

24. $2[27 + (x + y)] = 2[27 + (y + x)] = 2[(27 + y) + x]$
$= 2[(y + 27) + x]$

25. $x[(2y + 1) + 3] = x[2y + (1 + 3)] = x[2y + 4]$
$= x(2y) + x(4) = (x \cdot 2)y + 4x = (2x)y + 4x$
$= 2xy + 4x$

26. $(1 + a)(b + c) = 1(b + c) + a(b + c)$
$= 1(b) + 1(c) + a(b) + a(c) = b + c + ab + ac$

27. $x(y - z + w) = x[(y - z) + w] = x(y - z) + x(w)$
$= x[y + (-z)] + xw = x(y) + x(-z) + xw$
$= xy - xz + xw$

28. $-2 + (-4) = -6$

29. $-6 + 2 = -4$

30. $6 + (-4) = 2$

31. $7 - 2 = 5$

32. $7 - (-4) = 7 + 4 = 11$

33. $-5 - (-13) = -5 + 13 = 8$

34. $-a - (-b) = -a + b$

35. $(-2)(9) = -(2 \cdot 9) = -18$

36. $7(-9) = -(7 \cdot 9) = -63$

37. $(-2)(-12) = 2(12) = 24$

38. $19(-1) = (-1)19 = -(1 \cdot 19) = -19$

39. $\frac{-1}{-\frac{1}{9}} = -1\left(-\frac{9}{1}\right) = 9$

40. $-(-6 + x) = -(-6) - x = 6 - x$

41. $-7(x) = -(7x) = -7x$

42. $-12(x - y) = (-12)x - (-12)(y) = -12x + 12y$
(or $12y - 12x$)

43. $-[-6 + (-y)] = -(-6) - (-y) = 6 + y$

44. $-3 \div 15 = \frac{-3}{15} = -\frac{3}{15} = -\frac{1 \cdot 3}{5 \cdot 3} = -\frac{1}{5}$

45. $-9 \div (-27) = \frac{-9}{-27} = \frac{9}{27} = \frac{9 \cdot 1}{9 \cdot 3} = \frac{1}{3}$

46. $(-a) \div (-b) = \frac{-a}{-b} = \frac{a}{b}$

47. $2(-6 + 2) = 2(-4) = -8$

48. $3[-2(3) + 6(2)] = 3[-6 + 12] = 3[6] = 18$

49. $(-2)(-4)(-1) = 8(-1) = -8$

50. $(-12)(-12) = (12)(12) = 144$

51. $X(1) = X$

52. $3(x - 4) = 3(x) - 3(4) = 3x - 12$

53. $4(5 + x) = 4(5) + 4(x) = 20 + 4x$

54. $-(x - 2) = -x + 2$

55. $0(-x) = 0$

56. $8\left(\frac{1}{11}\right) = \frac{8 \cdot 1}{11} = \frac{8}{11}$

57. $\frac{5}{1} = 5$

58. $\frac{14x}{21y} = \frac{2 \cdot 7 \cdot x}{3 \cdot 7 \cdot y} = \frac{2x}{3y}$

59. $\frac{3}{-2x} = \frac{3}{-(2x)} = -\frac{3}{2x}$

60. $\frac{2}{3} \cdot \frac{1}{x} = \frac{2 \cdot 1}{3 \cdot x} = \frac{2}{3x}$

61. $\frac{a}{c}(3b) = \frac{a(3b)}{c} = \frac{3ab}{c}$

62. $(5a)\left(\frac{7}{5a}\right) = 7$

63. $\frac{-aby}{-ax} = \frac{-a \cdot by}{-a \cdot x} = \frac{by}{x}$

64. $\frac{7}{y} \cdot \frac{1}{x} = \frac{7 \cdot 1}{y \cdot x} = \frac{7}{xy}$

65. $\frac{2}{x} \cdot \frac{5}{y} = \frac{2 \cdot 5}{x \cdot y} = \frac{10}{xy}$

66. $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{3+2}{6} = \frac{5}{6}$

67. $\frac{5}{12} + \frac{3}{4} = \frac{5}{12} + \frac{9}{12} = \frac{5+9}{12} = \frac{14}{12} = \frac{2 \cdot 7}{2 \cdot 6} = \frac{7}{6}$

68. $\frac{3}{10} - \frac{7}{15} = \frac{9}{30} - \frac{14}{30} = \frac{9-14}{30} = \frac{-5}{30} = -\frac{5 \cdot 1}{5 \cdot 6} = -\frac{1}{6}$

69. $\frac{4}{5}+\frac{6}{5}=\frac{4+6}{5}=\frac{10}{5}=2$

70. $\frac{X}{\sqrt{5}}-\frac{Y}{\sqrt{5}}=\frac{X-Y}{\sqrt{5}}$

71. $\frac{3}{2}-\frac{1}{4}+\frac{1}{6}=\frac{18}{12}-\frac{3}{12}+\frac{2}{12}=\frac{18-3+2}{12}=\frac{17}{12}$

72. $\frac{2}{5}-\frac{3}{8}=\frac{16}{40}-\frac{15}{40}=\frac{16-15}{40}=\frac{1}{40}$

73. $\frac{6}{\frac{x}{y}}=6\div\frac{x}{y}=6\cdot\frac{y}{x}=\frac{6y}{x}$

74. $\frac{\frac{l}{3}}{m}=\frac{l}{3}\div\frac{m}{1}=\frac{l}{3}\cdot\frac{1}{m}=\frac{l}{3m}$

75. $\frac{\frac{-x}{y^2}}{\frac{z}{xy}}=-\frac{x}{y^2}\div\frac{z}{xy}=-\frac{x}{y^2}\cdot\frac{xy}{z}=-\frac{x^2}{yz}$

76. $\frac{7}{0}$ is not defined (we cannot divide by 0).

77. $\frac{0}{7}=0$

78. $\frac{0}{0}$ is not defined (we cannot divide by 0).

79. $0\cdot 0=0$

Problems 0.3

1. $(2^3)(2^2)=2^{3+2}=2^5(=32)$

2. $x^6x^9=x^{6+9}=x^{15}$

3. $w^4w^8=w^{4+8}=w^{12}$

4. $z^3zz^2=z^{3+1+2}=z^6$

5. $\frac{x^3x^5}{y^9y^5}=\frac{x^{3+5}}{y^{9+5}}=\frac{x^8}{y^{14}}$

6. $(x^{12})^4=x^{12\cdot 4}=x^{48}$

7. $\frac{(a^3)^7}{(b^4)^5}=\frac{a^{3\cdot 7}}{b^{4\cdot 5}}=\frac{a^{21}}{b^{20}}$

8. $\left(\frac{x^2}{y^3}\right)^5=\frac{(x^2)^5}{(y^3)^5}=\frac{x^{2\cdot 5}}{y^{3\cdot 5}}=\frac{x^{10}}{y^{15}}$

9. $(2x^2y^3)^3=2^3(x^2)^3(y^3)^3=8x^{2\cdot 3}y^{3\cdot 3}=8x^6y^9$

10. $\left(\frac{w^2s^3}{y^2}\right)^2=\frac{(w^2s^3)^2}{(y^2)^2}=\frac{(w^2)^2(s^3)^2}{y^{2\cdot 2}}=\frac{w^{2\cdot 2}s^{3\cdot 2}}{y^4}$
$=\frac{w^4s^6}{y^4}$

11. $\frac{x^9}{x^5}=x^{9-5}=x^4$

12. $\left(\frac{2a^4}{7b^5}\right)^6=\frac{(2a^4)^6}{(7b^5)^6}$
$=\frac{2^6(a^4)^6}{7^6(b^5)^6}$
$=\frac{64a^{4\cdot 6}}{117,649b^{5\cdot 6}}$
$=\frac{64a^{24}}{117,649b^{30}}$

13. $\frac{(x^3)^6}{x(x^3)}=\frac{x^{3\cdot 6}}{x^{1+3}}=\frac{x^{18}}{x^4}=x^{18-4}=x^{14}$

14. $\frac{(x^2)^3(x^3)^2}{(x^3)^4}=\frac{x^{2\cdot 3}x^{3\cdot 2}}{x^{3\cdot 4}}=\frac{x^6x^6}{x^{12}}=\frac{x^{12}}{x^{12}}$
$x^{12-12}=x^0=1$

15. $\sqrt{25}=5$

16. $\sqrt[4]{81}=3$

17. $\sqrt[7]{-128}=-2$

18. $\sqrt{0.04}=0.2$

19. $\sqrt[4]{\frac{1}{16}}=\frac{\sqrt[4]{1}}{\sqrt[4]{16}}=\frac{1}{2}$

20. $\sqrt[3]{\frac{-8}{27}} = \frac{\sqrt[3]{-8}}{\sqrt[3]{27}} = \frac{-2}{3} = -\frac{2}{3}$

21. $(49)^{1/2} = \sqrt{49} = 7$

22. $(64)^{1/3} = \sqrt[3]{64} = 4$

23. $9^{3/2} = \left(\sqrt{9}\right)^3 = (3)^3 = 27$

24. $(9)^{-5/2} = \frac{1}{(9)^{5/2}} = \frac{1}{\left(\sqrt{9}\right)^5} = \frac{1}{3^5} = \frac{1}{243}$

25. $(32)^{-2/5} = \frac{1}{(32)^{2/5}} = \frac{1}{\left(\sqrt[5]{32}\right)^2} = \frac{1}{(2)^2} = \frac{1}{4}$

26. $(0.09)^{-1/2} = \frac{1}{(0.09)^{1/2}} = \frac{1}{\sqrt{0.09}} = \frac{1}{0.3}$
$= \frac{1}{\frac{3}{10}} = \frac{10}{3}$

27. $\left(\frac{1}{32}\right)^{4/5} = \left(\sqrt[5]{\frac{1}{32}}\right)^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

28. $\left(-\frac{64}{27}\right)^{2/3} = \left(\sqrt[3]{-\frac{64}{27}}\right)^2 = \left(-\frac{4}{3}\right)^2 = \frac{16}{9}$

29. $\sqrt{50} = \sqrt{25\cdot 2} = \sqrt{25}\cdot\sqrt{2} = 5\sqrt{2}$

30. $\sqrt[3]{54} = \sqrt[3]{27\cdot 2} = \sqrt[3]{27}\,\sqrt[3]{2} = 3\sqrt[3]{2}$

31. $\sqrt[3]{2x^3} = \sqrt[3]{2}\,\sqrt[3]{x^3} = x\,\sqrt[3]{2}$

32. $\sqrt{4x} = \sqrt{4}\sqrt{x} = 2\sqrt{x}$

33. $\sqrt{16x^4} = \sqrt{16}\sqrt{x^4} = 4x^2$

34. $\sqrt[4]{\frac{x}{16}} = \frac{\sqrt[4]{x}}{\sqrt[4]{16}} = \frac{\sqrt[4]{x}}{2}$

35. $2\sqrt{8} - 5\sqrt{27} + \sqrt[3]{128} = 2\sqrt{4\cdot 2} - 5\sqrt{9\cdot 3} + \sqrt[3]{64\cdot 2}$
$= 2\cdot 2\sqrt{2} - 5\cdot 3\sqrt{3} + 4\sqrt[3]{2}$
$= 4\sqrt{2} - 15\sqrt{3} + 4\sqrt[3]{2}$

36. $\sqrt{\frac{3}{13}} = \sqrt{\frac{3}{13}\cdot\frac{13}{13}} = \sqrt{\frac{39}{13^2}} = \frac{\sqrt{39}}{\sqrt{13^2}} = \frac{\sqrt{39}}{13}$

37. $(9z^4)^{1/2} = \sqrt{9z^4} = \sqrt{3^2(z^2)^2} = \sqrt{3^2}\sqrt{(z^2)^2}$
$= 3z^2$

38. $(16y^8)^{3/4} = \left[\sqrt[4]{16y^8}\right]^3 = \left[\sqrt[4]{(2y^2)^4}\right]^3 = (2y^2)^3$
$= 8y^6$

39. $\left(\frac{27t^3}{8}\right)^{2/3} = \left(\left[\frac{3t}{2}\right]^3\right)^{2/3} = \left[\frac{3t}{2}\right]^2 = \frac{9t^2}{4}$

40. $\left(\frac{256}{x^{12}}\right)^{-3/4} = \left(\left[\frac{4}{x^3}\right]^4\right)^{-3/4} = \left[\frac{4}{x^3}\right]^{-3} = \frac{4^{-3}}{(x^3)^{-3}}$
$= \frac{4^{-3}}{x^{-9}} = \frac{x^9}{4^3} = \frac{x^9}{64}$

41. $\frac{a^5b^{-3}}{c^2} = a^5\cdot b^{-3}\cdot\frac{1}{c^2} = a^5\cdot\frac{1}{b^3}\cdot\frac{1}{c^2} = \frac{a^5}{b^3c^2}$

42. $\sqrt[5]{x^2y^3z^{-10}} = x^{2/5}y^{3/5}z^{-10/5} = \frac{x^{2/5}y^{3/5}}{z^2}$

43. $5m^{-2}m^{-7} = 5m^{-2+(-7)} = 5m^{-9} = \frac{5}{m^9}$

44. $x + y^{-1} = x + \frac{1}{y}$

45. $(3t)^{-2} = \frac{1}{(3t)^2} = \frac{1}{9t^2}$

46. $(3-z)^{-4} = \frac{1}{(3-z)^4}$

47. $\sqrt[5]{5x^2} = (5x^2)^{1/5} = 5^{1/5}(x^2)^{1/5} = 5^{1/5}x^{2/5}$

48. $(X^3Y^{-3})^{-3} = (X^3)^{-3}(Y^{-3})^{-3}$
$= X^{-9}Y^9$
$= \frac{Y^9}{X^9}$

49. $\sqrt{x}-\sqrt{y}=x^{1/2}-y^{1/2}$

50. $\frac{u^{-2}v^{-6}w^{3}}{vw^{-5}}=\frac{w^{3-(-5)}}{u^{2}v^{1-(-6)}}=\frac{w^{8}}{u^{2}v^{7}}$

51. $x^{2}\sqrt[4]{xy^{-2}z^{3}}=x^{2}(xy^{-2}z^{3})^{1/4}=x^{2}x^{1/4}y^{-2/4}z^{3/4}$
$=\frac{x^{9/4}z^{3/4}}{y^{1/2}}$

52. $\sqrt[4]{a^{-3}b^{-2}}\,a^{5}b^{-4}=(a^{-3}b^{-2})^{1/4}a^{5}b^{-4}$
$=a^{-3/4}b^{-1/2}a^{5}b^{-4}$
$=a^{17/4}b^{-9/2}$
$=\frac{a^{17/4}}{b^{9/2}}$

53. $(2a-b+c)^{2/3}=\sqrt[3]{(2a-b+c)^{2}}$

54. $(ab^{2}c^{3})^{3/4}=\sqrt[4]{(ab^{2}c^{3})^{3}}=\sqrt[4]{a^{3}b^{6}c^{9}}$

55. $x^{-4/5}=\frac{1}{x^{4/5}}=\frac{1}{\sqrt[5]{x^{4}}}$

56. $2x^{1/2}-(2y)^{1/2}=2\sqrt{x}-\sqrt{2y}$

57. $3w^{-3/5}-(3w)^{-3/5}=\frac{3}{w^{3/5}}-\frac{1}{(3w)^{3/5}}$
$=\frac{3}{\sqrt[5]{w^{3}}}-\frac{1}{\sqrt[5]{(3w)^{3}}}=\frac{3}{\sqrt[5]{w^{3}}}-\frac{1}{\sqrt[5]{27w^{3}}}$

58. $[(x^{-4})^{1/5}]^{1/6}=[x^{-4/5}]^{1/6}=x^{-4/30}=x^{-2/15}$
$=\frac{1}{x^{2/15}}=\frac{1}{\sqrt[15]{x^{2}}}$

59. $\frac{6}{\sqrt{5}}=\frac{6}{5^{1/2}}=\frac{6\cdot 5^{1/2}}{5^{1/2}\cdot 5^{1/2}}=\frac{6\sqrt{5}}{5}$

60. $\frac{3}{\sqrt[4]{8}}=\frac{3}{8^{1/4}}=\frac{3\cdot 2^{1/4}}{8^{1/4}\cdot 2^{1/4}}=\frac{3\sqrt[4]{2}}{\sqrt[4]{16}}=\frac{3\sqrt[4]{2}}{2}$

61. $\frac{4}{\sqrt{2x}}=\frac{4}{(2x)^{1/2}}=\frac{4(2x)^{1/2}}{(2x)^{1/2}(2x)^{1/2}}=\frac{4\sqrt{2x}}{2x}$
$=\frac{2\sqrt{2x}}{x}$

62. $\frac{y}{\sqrt{2y}}=\frac{y}{(2y)^{1/2}}=\frac{y(2y)^{1/2}}{(2y)^{1/2}(2y)^{1/2}}=\frac{y\sqrt{2y}}{2y}$
$=\frac{\sqrt{2y}}{2}$

63. $\frac{1}{\sqrt[3]{3x}}=\frac{1}{(3x)^{1/3}}=\frac{1(3x)^{2/3}}{(3x)^{1/3}(3x)^{2/3}}=\frac{\sqrt[3]{(3x)^{2}}}{3x}$
$=\frac{\sqrt[3]{9x^{2}}}{3x}$

64. $\frac{2}{3\sqrt[3]{y^{2}}}=\frac{2}{3y^{2/3}}=\frac{2\cdot y^{1/3}}{3y^{2/3}\cdot y^{1/3}}=\frac{2y^{1/3}}{3y}=\frac{2\sqrt[3]{y}}{3y}$

65. $\frac{\sqrt{12}}{\sqrt{3}}=\sqrt{\frac{12}{3}}=\sqrt{4}=2$

66. $\sqrt{\frac{18}{2}}=\sqrt{9}=3$

67. $\frac{\sqrt[5]{2}}{\sqrt[4]{a^{2}b}}=\frac{\sqrt[5]{2}}{a^{2/4}b^{1/4}}=\frac{\sqrt[5]{2}\cdot a^{1/2}b^{3/4}}{a^{1/2}b^{1/4}\cdot a^{1/2}b^{3/4}}$
$=\frac{2^{1/5}a^{1/2}b^{3/4}}{ab}=\frac{2^{4/20}a^{10/20}b^{15/20}}{ab}$
$=\frac{(2^{4}a^{10}b^{15})^{1/20}}{ab}=\frac{\sqrt[20]{16a^{10}b^{15}}}{ab}$

68. $\frac{\sqrt{2}}{\sqrt[3]{3}}=\frac{\sqrt{2}}{3^{1/3}}=\frac{2^{1/2}\cdot 3^{2/3}}{3^{1/3}\cdot 3^{2/3}}=\frac{2^{3/6}3^{4/6}}{3}$
$=\frac{(2^{3}3^{4})^{1/6}}{3}=\frac{\sqrt[6]{648}}{3}$

69. $2x^{2}y^{-3}x^{4}=2x^{6}y^{-3}=\frac{2x^{6}}{y^{3}}$

70. $\frac{3}{u^{5/2}v^{1/2}}=\frac{3\cdot u^{1/2}v^{1/2}}{u^{5/2}v^{1/2}\cdot u^{1/2}v^{1/2}}=\frac{3u^{1/2}v^{1/2}}{u^{3}v}$

71. $\frac{\sqrt{243}}{\sqrt{3}}=\sqrt{\frac{243}{3}}=\sqrt{81}=9$

72. $\{[(3a^3)^2]^{-5}\}^{-2}=\{[3^2a^6]^{-5}\}^{-2}$
$=\{3^{-10}a^{-30}\}^{-2}$
$=3^{20}a^{60}$

73. $\frac{2^0}{(2^{-2}x^{1/2}y^{-2})^3}=\frac{1}{2^{-6}x^{3/2}y^{-6}}=\frac{2^6y^6}{x^{3/2}}$
$=\frac{64y^6\cdot x^{1/2}}{x^{3/2}\cdot x^{1/2}}=\frac{64y^6x^{1/2}}{x^2}$

74. $\frac{\sqrt{s^5}}{\sqrt[3]{s^2}}=\frac{s^{5/2}}{s^{2/3}}=\frac{s^{15/6}}{s^{4/6}}=s^{11/6}$

75. $\sqrt[3]{x^2yz^3}\sqrt[3]{xy^2}=\sqrt[3]{(x^2yz^3)(xy^2)}=\sqrt[3]{x^3y^3z^3}$
$=xyz$

76. $\left(\sqrt[4]{3}\right)^8=(3^{1/4})^8=3^{8/4}=3^2=9$

77. $3^2(32)^{-2/5}=3^2(2^5)^{-2/5}$
$=3^2(2^{-2})$
$=3^2\cdot\frac{1}{2^2}$
$=\frac{9}{4}$

78. $\left(\sqrt[5]{x^2y}\right)^{2/5}=[(x^2y)^{1/5}]^{2/5}=(x^2y)^{2/25}$
$=x^{4/25}y^{2/25}$

79. $(2x^{-1}y^2)^2=2^2x^{-2}y^4=\frac{4y^4}{x^2}$

80. $\frac{3}{\sqrt[3]{y}\sqrt[4]{x}}=\frac{3}{y^{1/3}x^{1/4}}=\frac{3\cdot y^{2/3}x^{3/4}}{y^{1/3}x^{1/4}\cdot y^{2/3}x^{3/4}}$
$=\frac{3x^{3/4}y^{2/3}}{xy}$

81. $\sqrt{x}\sqrt{x^2y^3}\sqrt{xy^2}=x^{1/2}(x^2y^3)^{1/2}(xy^2)^{1/2}$
$=x^{1/2}(xy^{3/2})(x^{1/2}y)=x^2y^{5/2}$

82. $\sqrt{75k^4}=(75k^4)^{1/2}=[(25k^4)(3)]^{1/2}$
$=[(5k^2)^2 3]^{1/2}=5k^2 3^{1/2}$

83. $\frac{(ab^{-3}c)^8}{(a^{-1}c^2)^{-3}}=\frac{a^8b^{-24}c^8}{a^3c^{-6}}=\frac{a^5c^{14}}{b^{24}}$

84. $\sqrt[3]{7(49)}=\sqrt[3]{7\cdot 7^2}=\sqrt[3]{7^3}=7$

85. $\frac{(x^2)^3}{x^4}\div\left[\frac{x^3}{(x^3)^2}\right]^2=\frac{x^6}{x^4}\div\frac{(x^3)^2}{(x^6)^2}$
$=x^2\div\frac{x^6}{x^{12}}=x^2\div x^{6-12}=x^2\div x^{-6}$
$=x^2\div\frac{1}{x^6}=x^2\cdot x^6=x^8$

86. $\sqrt{(-6)(-6)}=\sqrt{36}=6$
Note that $\sqrt{(-6)^2}\neq -6$ since $-6<0$.

87. $-\frac{8s^{-2}}{2s^3}=-\frac{4}{s^3s^2}=-\frac{4}{s^5}$

88. $\left(a^5b^{-3}\sqrt{c}\right)^3=(a^5)^3(b^{-3})^3(c^{1/2})^3$
$=a^{15}b^{-9}c^{3/2}$
$=\frac{a^{15}c^{3/2}}{b^9}$

89. $(3x^3y^2\div 2y^2z^{-3})^4=\left(\frac{3x^3y^2}{2y^2z^{-3}}\right)^4$
$=\left(\frac{3x^3z^3}{2}\right)^4$
$=\frac{(3x^3z^3)^4}{(2)^4}$
$=\frac{3^4x^{12}z^{12}}{2^4}$
$=\frac{81x^{12}z^{12}}{16}$

90. $\frac{1}{\left(\frac{\sqrt{2}x^{-2}}{\sqrt{16}x^3}\right)^2}=\frac{1}{\frac{\left(2^{1/2}\right)^2(x^{-2})^2}{(16^{1/2})^2(x^3)^2}}=\frac{1}{\frac{2x^{-4}}{16x^6}}=\frac{1}{\frac{1}{8x^{10}}}=8x^{10}$

Problems 0.4

1. $8x - 4y + 2 + 3x + 2y - 5 = 11x - 2y - 3$

2. $6x^2 - 10xy + 2 + 2z - xy + 4$
$= 6x^2 - 11xy + 2z + 6$

3. $8t^2 - 6s^2 + 4s^2 - 2t^2 + 6 = 6t^2 - 2s^2 + 6$

4. $\sqrt{x} + 2\sqrt{x} + \sqrt{x} + 3\sqrt{x} = 7\sqrt{x}$

5. $\sqrt{a} + 2\sqrt{3b} - \sqrt{c} + 3\sqrt{3b}$
$= \sqrt{a} + 5\sqrt{3b} - \sqrt{c}$

6. $3a + 7b - 9 - 5a - 9b - 21 = -2a - 2b - 30$

7. $6x^2 - 10xy + \sqrt{2} - 2z + xy - 4$
$= 6x^2 - 9xy - 2z + \sqrt{2} - 4$

8. $\sqrt{x} + 2\sqrt{x} - \sqrt{x} - 3\sqrt{x} = -\sqrt{x}$

9. $\sqrt{x} + \sqrt{2y} - \sqrt{x} - \sqrt{3z} = \sqrt{2y} - \sqrt{3z}$

10. $8z - 4w - 3w + 6z = 14z - 7w$

11. $9x + 9y - 21 - 24x + 6y - 6 = -15x + 15y - 27$

12. $u - 3v - 5u - 4v + u - 3 = -3u - 7v - 3$

13. $5x^2 - 5y^2 + xy - 3x^2 - 8xy - 28y^2$
$= 2x^2 - 33y^2 - 7xy$

14. $2 - [3 + 4s - 12] = 2 - [4s - 9] = 2 - 4s + 9$
$= 11 - 4s$

15. $2\{3[3x^2 + 6 - 2x^2 + 10]\} = 2\{3[x^2 + 16]\}$
$= 2\{3x^2 + 48\} = 6x^2 + 96$

16. $4\{3t + 15 - t[1 - t - 1]\} = 4\{3t + 15 - t[-t]\}$
$= 4\{3t + 15 + t^2\} = 4t^2 + 12t + 60$

17. $-5(8x^3 + 8x^2 - 2(x^2 - 5 + 2x))$
$= -5(8x^3 + 8x^2 - 2x^2 + 10 - 4x)$
$= -5(8x^3 + 6x^2 - 4x + 10)$
$= -40x^3 - 30x^2 + 20x - 50$

18. $-\{-6a - 6b + 6 + 10a + 15b - a[2b + 10]\}$
$= -\{4a + 9b + 6 - 2ab - 10a\}$
$= -\{-6a + 9b + 6 - 2ab\}$
$= 6a - 9b - 6 + 2ab$

19. $x^2 + (4 + 5)x + 4(5) = x^2 + 9x + 20$

20. $u^2 + (5 + 2)u + 2(5) = u^2 + 7u + 10$

21. $(w + 2)(w - 5) = w^2 + (-5 + 2)x + 2(-5)$
$= w^2 - 3w - 10$

22. $z^2 + (-7 - 3)z + (-7)(-3) = z^2 - 10z + 21$

23. $(2x)(5x) + [(2)(2) + (3)(5)]x + 3(2)$
$= 10x^2 + 19x + 6$

24. $(t)(2t) + [(1)(7) + (-5)(2)]t + (-5)(7)$
$= 2t^2 - 3t - 35$

25. $X^2 + 2(X)(2Y) + (2Y)^2 = X^2 + 4XY + 4Y^2$

26. $(2x)^2 - 2(2x)(1) + 1^2 = 4x^2 - 4x + 1$

27. $x^2 - 2(5)x + 5^2 = x^2 - 10x + 25$

28. $(1 \cdot 2)\left(\sqrt{x}\right)^2 + [(1)(5) + (-1)(2)]\sqrt{x} + (-1)(5)$
$= 2x + 3\sqrt{x} - 5$

29. $\left(\sqrt{3x}\right)^2 + 2\left(\sqrt{3x}\right)(5) + (5)^2$
$= 3x + 10\sqrt{3x} + 25$

30. $\left(\sqrt{y}\right)^2 - 3^2 = y - 9$

31. $(2s)^2 - 1^2 = 4s^2 - 1$

32. $(z^2)^2 - (3w)^2 = z^4 - 9w^2$

33. $x^2(x + 4) - 3(x + 4)$
$= x^3 + 4x^2 - 3x - 12$

34. $x(x^2 + x + 3) + 1(x^2 + x + 3)$
$= x^3 + x^2 + 3x + x^2 + x + 3$
$= x^3 + 2x^2 + 4x + 3$

35. $x^2(3x^2+2x-1)-4(3x^2+2x-1)$
$=3x^4+2x^3-x^2-12x^2-8x+4$
$=3x^4+2x^3-13x^2-8x+4$

36. $3y(4y^3+2y^2-3y)-2(4y^3+2y^2-3y)$
$=12y^4+6y^3-9y^2-8y^3-4y^2+6y$
$=12y^4-2y^3-13y^2+6y$

37. $x\{2(x^2-2x-35)+4[2x^2-12x]\}$
$=x\{2x^2-4x-70+8x^2-48x\}$
$=x\{10x^2-52x-70\}$
$=10x^3-52x^2-70x$

38. $[(2z)^2-1^2](4z^2+1)=[4z^2-1](4z^2+1)$
$=(4z^2)^2-1^2=16z^4-1$

39. $x(3x+2y-4)+y(3x+2y-4)+2(3x+2y-4)$
$=3x^2+2xy-4x+3xy+2y^2-4y+6x+4y-8$
$=3x^2+2y^2+5xy+2x-8$

40. $[x^2+(x+1)]^2$
$=(x^2)^2+2x^2(x+1)+(x+1)^2$
$=x^4+2x^3+2x^2+x^2+2x+1$
$=x^4+2x^3+3x^2+2x+1$

41. $(2a)^3+3(2a)^2(3)+3(2a)(3)^2+(3)^3$
$=8a^3+36a^2+54a+27$

42. $(3y)^3-3(3y)^2(2)+3(3y)(2)^2-(2)^3$
$=27y^3-54y^2+36y-8$

43. $(2x)^3-3(2x)^2(3)+3(2x)(3)^2-3^3$
$=8x^3-36x^2+54x-27$

44. $x^3+3x^2(2y)+3x(2y)^2+(2y)^3$
$=x^3+6x^2y+12xy^2+8y^3$

45. $\frac{z^2}{z}-\frac{18z}{z}=z-18$

46. $\frac{2x^3}{x}-\frac{7x}{x}+\frac{4}{x}=2x^2-7+\frac{4}{x}$

47. $\frac{6x^5}{2x^2}+\frac{4x^3}{2x^2}-\frac{1}{2x^2}=3x^3+2x-\frac{1}{2x^2}$

48. $\frac{3y-4-9y-5}{3y}$
$=\frac{-6y-9}{3y}$
$=\frac{-6y}{3y}-\frac{9}{3y}$
$=-2-\frac{3}{y}$

49.
$$\begin{array}{r|l} & x \\ \hline x+5 & x^2+5x-3 \\ & \underline{x^2+5x} \\ & \qquad\quad -3 \end{array}$$
Answer: $x+\frac{-3}{x+5}$

50.
$$\begin{array}{r|l} & x-1 \\ \hline x-4 & x^2-5x+4 \\ & \underline{x^2-4x} \\ & \quad -x+4 \\ & \quad \underline{-x+4} \\ & \qquad\quad 0 \end{array}$$
Answer: $x-1$

51.
$$\begin{array}{r|l} & 3x^2-8x+17 \\ \hline x+2 & 3x^3-2x^2+\ \ x-3 \\ & \underline{3x^3+6x^2} \\ & \quad -8x^2+\ \ x \\ & \quad \underline{-8x^2-16x} \\ & \qquad\qquad 17x-3 \\ & \qquad\qquad \underline{17x+34} \\ & \qquad\qquad\quad -37 \end{array}$$
Answer: $3x^2-8x+17+\frac{-37}{x+2}$

52.
$$\begin{array}{r} x^3+x^2+3x+3 \\ x-1\overline{\big)\,x^4+0x^3+2x^2+0x+1} \\ \underline{x^4-x^3} \\ x^3+2x^2 \\ \underline{x^3-x^2} \\ 3x^2+0x \\ \underline{3x^2-3x} \\ 3x+1 \\ \underline{3x-3} \\ 4 \end{array}$$

Answer: $x^3+x^2+3x+3+\frac{4}{x-1}$

53.
$$\begin{array}{r} x^2-2x+4 \\ x+2\overline{\big)\,x^3+0x^2+0x+0} \\ \underline{x^3+2x^2} \\ -2x^2+0 \\ \underline{-2x^2-4x} \\ 4x+0 \\ \underline{4x+8} \\ -8 \end{array}$$

Answer: $x^2-2x+4-\frac{8}{x+2}$

54.
$$\begin{array}{r} 3x-\frac{1}{2} \\ 2x+3\overline{\big)\,6x^2+8x+1} \\ \underline{6x^2+9x} \\ -x+1 \\ \underline{-x-\frac{3}{2}} \\ \frac{5}{2} \end{array}$$

Answer: $3x-\frac{1}{2}+\frac{\frac{5}{2}}{2x+3}$

55.
$$\begin{array}{r} x-2 \\ 3x+2\overline{\big)\,3x^2-4x+3} \\ \underline{3x^2+2x} \\ -6x+3 \\ \underline{-6x-4} \\ 7 \end{array}$$

Answer: $x-2+\frac{7}{3x+2}$

56.
$$\begin{array}{r} z+2 \\ z^2-z+1\overline{\big)\,z^3+z^2+z} \\ \underline{z^3-z^2+z} \\ 2z^2 \\ \underline{2z^2-2z+2} \\ 2z-2 \end{array}$$

Answer: $z+2+\frac{2z-2}{z^2-z+1}$

Problems 0.5

1. $2(ax+b)$

2. $2y(3y-2)$

3. $5x(2y+z)$

4. $3x^2y(1-3xy^2)$

5. $4bc(2a^3-3ab^2d+b^3cd^2)$

6. $6u^2v(uv^2+3w^4-12v^2)$

7. $z^2-7^2=(z+7)(z-7)$

8. $(x+2)(x-3)$

9. $(p+3)(p+1)$

10. $(s-4)(s-2)$

11. $(4x)^2-3^2=(4x+3)(4x-3)$

12. $(x+6)(x-4)$

13. $(a+7)(a+5)$

14. $(2t)^2-(3s)^2=(2t+3s)(2t-3s)$

15. $x^2+2(3)(x)+3^2=(x+3)^2$

16. $(y-10)(y-5)$

17. $5(x^2+5x+6)$
$=5(x+3)(x+2)$

18. $3(t^2+4t-5)$
$=3(t-1)(t+5)$

19. $3(x^2 - 1^2) = 3(x+1)(x-1)$

20. $(3y - 4)(3y - 2)$

21. $6y^2 + 13y + 2 = (6y+1)(y+2)$

22. $(4x + 3)(x - 1)$

23. $2s(6s^2 + 5s - 4) = 2s(3s+4)(2s-1)$

24. $(3z)^2 + 2(3z)(5) + 5^2 = (3z+5)^2$

25. $u^{3/5}v(u^2 - 4v^2) = u^{3/5}v(u+2v)(u-2v)$

26. $(3x^{2/7})^2 - 1^2 = (3x^{2/7}+1)(3x^{2/7}-1)$

27. $2x(x^2 + x - 6) = 2x(x+3)(x-2)$

28. $(xy)^2 - 2(xy)(2) + 2^2 = (xy-2)^2$

29. $[2(2x+1)]^2 = 2^2(2x+1)^2$
$= 4(2x+1)^2$

30. $2x^2[2x(1-2x)]^2$
$= 2x^2(2x)^2(1-2x)^2$
$= 2x^2(4x^2)(1-2x)^2$
$= 8x^4(1-2x)^2$

31. $x(x^2y^2 - 14xy + 49) = x[(xy)^2 - 2(xy)(7) + 7^2]$
$= x(xy-7)^2$

32. $x(5x + 2) + 2(5x + 2) = (5x + 2)(x + 2)$

33. $x(x^2 - 4) + 2(4 - x^2)$
$= x(x^2 - 4) - 2(x^2 - 4)$
$= (x^2 - 4)(x - 2)$
$= (x + 2)(x - 2)(x - 2)$
$= (x+2)(x-2)^2$

34. $(x + 1)(x - 1) + (x - 2)(x + 1)$
$= (x + 1)[(x - 1) + (x - 2)]$
$= (x + 1)(2x - 3)$

35. $y^2(y^2 + 8y + 16) - (y^2 + 8y + 16)$
$= (y^2 + 8y + 16)(y^2 - 1)$
$= (y+4)^2(y+1)(y-1)$

36. $xy(x^2 - 4) + z^2(x^2 - 4) = (x^2 - 4)(xy + z^2)$
$= (x+2)(x-2)(xy+z^2)$

37. $b^3 + 4^3 = (b+4)(b^2 - 4(b) + 4^2)$
$= (b+4)(b^2 - 4b + 16)$

38. $x^3 - 1^3 = (x-1)[x^2 + 1(x) + 1^2]$
$= (x-1)(x^2 + x + 1)$

39. $(x^3)^2 - 1^2 = (x^3+1)(x^3-1)$
$= (x+1)(x^2 - x + 1)(x-1)(x^2 + x + 1)$

40. $3^3 + (2x)^3 = (3+2x)[3^2 - 3(2x) + (2x)^2]$
$= (3+2x)(9 - 6x + 4x^2)$

41. $(x+3)^2(x-1)[(x+3) + (x-1)]$
$= (x+3)^2(x-1)[2x+2]$
$= (x+3)^2(x-1)[2(x+1)]$
$= 2(x+3)^2(x-1)(x+1)$

42. $(a+5)^2(a+1)^2[(a+5) + (a+1)]$
$= (a+5)^2(a+1)^2(2a+6)$
$= 2(a+5)^2(a+1)^2(a+3)$

43. $[P(1 + r)] + [P(1 + r)]r = [P(1 + r)](1 + r)$
$= P(1+r)^2$

44. $(3X + 5I)[(X - 3I) - (X + 2I)] = (3X + 5I)(-5I)$
$= -5I(3X + 5I)$

45. $(x^2)^2 - 4^2 = (x^2 + 4)(x^2 - 4)$
$= (x^2 + 4)(x+2)(x-2)$

46. $(9x^2)^2 - (y^2)^2 = (9x^2 + y^2)(9x^2 - y^2)$
$= (9x^2 + y^2)(3x + y)(3x - y)$

47. $(y^4)^2 - 1^2 = (y^4+1)(y^4-1)$
$= (y^4+1)(y^2+1)(y^2-1)$
$= (y^4+1)(y^2+1)(y+1)(y-1)$

48. $(t^2)^2 - 2^2 = (t^2+2)(t^2-2)$
$= (t^2+2)\left[t^2 - \left(\sqrt{2}\right)^2\right]$
$= (t^2+2)\left(t+\sqrt{2}\right)\left(t-\sqrt{2}\right)$

49. $(X^2+5)(X^2-1) = (X^2+5)(X+1)(X-1)$

50. $(x^2-9)(x^2-1) = (x+3)(x-3)(x+1)(x-1)$

51. $y(x^4-2x^2+1) = y(x^2-1)^2 = y[(x+1)(x-1)]^2$
$= y(x+1)^2(x-1)^2$

52. $2x(2x^2-3x-2) = 2x(2x+1)(x-2)$

Problems 0.6

1. $\dfrac{a^2-9}{a^2-3a} = \dfrac{(a-3)(a+3)}{a(a-3)} = \dfrac{a+3}{a}$

2. $\dfrac{x^2-3x-10}{x^2-4} = \dfrac{(x+2)(x-5)}{(x+2)(x-2)} = \dfrac{x-5}{x-2}$

3. $\dfrac{x^2-9x+20}{x^2+x-20} = \dfrac{(x-5)(x-4)}{(x+5)(x-4)} = \dfrac{x-5}{x+5}$

4. $\dfrac{3x^2-27x+24}{2x^3-16x^2+14x} = \dfrac{3(x-8)(x-1)}{2x(x-7)(x-1)}$
$= \dfrac{3(x-8)}{2x(x-7)}$

5. $\dfrac{6x^2+x-2}{2x^2+3x-2} = \dfrac{(3x+2)(2x-1)}{(x+2)(2x-1)} = \dfrac{3x+2}{x+2}$

6. $\dfrac{12x^2-19x+4}{6x^2-17x+12} = \dfrac{(4x-1)(3x-4)}{(2x-3)(3x-4)} = \dfrac{4x-1}{2x-3}$

7. $\dfrac{y^2(-1)}{(y-3)(y+2)} = -\dfrac{y^2}{(y-3)(y+2)}$

8. $\dfrac{(t+3)(t-3)t^2}{t(t+3)(t-3)^2} = \dfrac{t}{t-3}$

9. $\dfrac{(ax-b)(c-x)}{(x-c)(ax+b)} = \dfrac{(ax-b)(-1)(x-c)}{(x-c)(ax+b)}$
$= \dfrac{(ax-b)(-1)}{ax+b}$
$= \dfrac{b-ax}{ax+b}$

10. $\dfrac{(x+y)(x-y)(x+y)^2}{(x+y)(y-x)} = \dfrac{(x-y)(x+y)^2}{(-1)(x-y)}$
$= -(x+y)^2$

11. $\dfrac{2(x-1)}{(x-4)(x+2)} \cdot \dfrac{(x+4)(x+1)}{(x+1)(x-1)}$
$= \dfrac{2(x-1)(x+4)(x+1)}{(x-4)(x+2)(x+1)(x-1)}$
$= \dfrac{2(x+4)}{(x-4)(x+2)}$

12. $\dfrac{x(x+2)}{3(x-4)(x-2)} \cdot \dfrac{(x-2)^2}{(x-3)(x+2)}$
$= \dfrac{x(x+2)(x-2)^2}{3(x-4)(x-2)(x-3)(x+2)}$
$= \dfrac{x(x-2)}{3(x-4)(x-3)}$

13. $\dfrac{X^2}{8} \cdot \dfrac{4}{X} = \dfrac{4X^2}{8X} = \dfrac{X}{2}$

14. $\dfrac{3x^2}{7x} \cdot \dfrac{14}{x} = \dfrac{3x}{7} \cdot \dfrac{14}{x} = \dfrac{3(14)x}{7x} = 6$

15. $\dfrac{2m}{n^2} \cdot \dfrac{n^3}{6m} = \dfrac{2mn^3}{6mn^2} = \dfrac{n}{3}$

16. $\dfrac{c+d}{c} \cdot \dfrac{2c}{c-d} = \dfrac{2c(c+d)}{c(c-d)} = \dfrac{2(c+d)}{c-d}$

17. $\dfrac{4x}{3} \div 2x = \dfrac{4x}{3} \cdot \dfrac{1}{2x} = \dfrac{4x}{6x} = \dfrac{2}{3}$

18. $\dfrac{4x}{1} \cdot \dfrac{2x}{3} = \dfrac{4x(2x)}{3} = \dfrac{8x^2}{3}$

19. $\frac{-9x^3}{1} \cdot \frac{3}{x} = \frac{-27x^3}{x} = -27x^2$

20. $\frac{-12Y^4}{Y} \div 4 = \frac{-12Y^3}{1} \cdot \frac{1}{4} = \frac{-12Y^3}{4} = -3Y^3$

21. $\frac{x-3}{1} \cdot \frac{x-4}{(x-3)(x-4)} = \frac{x-3}{1} \cdot \frac{1}{x-3} = \frac{x-3}{x-3} = 1$

22. $\frac{(x+3)^2}{x} \div (x+3) = \frac{(x+3)^2}{x} \cdot \frac{1}{x+3}$

$= \frac{(x+3)^2}{x(x+3)} = \frac{x+3}{x}$

23. $\frac{10x^3}{(x+1)(x-1)} \cdot \frac{x+1}{5x} = \frac{10x^3(x+1)}{5x(x+1)(x-1)} = \frac{2x^2}{x-1}$

24. $\frac{(x-3)(x+2)}{(x+3)(x-3)} \cdot \frac{(x+3)(x-1)}{(x+2)(x-2)}$

$= \frac{x+2}{x+3} \cdot \frac{(x+3)(x-1)}{(x+2)(x-2)}$

$= \frac{(x+2)(x+3)(x-1)}{(x+3)(x+2)(x-2)}$

$= \frac{x-1}{x-2}$

25. $\frac{(x+2)(x+5)}{(x+5)(x+1)} \cdot \frac{(x-4)(x+1)}{(x-4)(x+2)}$

$= \frac{x+2}{x+1} \cdot \frac{x+1}{x+2}$

$= \frac{(x+2)(x+1)}{(x+1)(x+2)}$

$= 1$

26. $\frac{(x+3)^2}{4x-3} \cdot \frac{(3+4x)(3-4x)}{7(x+3)}$

$= \frac{(x+3)^2(3+4x)(3-4x)}{7(4x-3)(x+3)}$

$= \frac{(x+3)(3+4x)(-1)(4x-3)}{7(4x-3)}$

$= -\frac{(x+3)(3+4x)}{7}$

27. $\frac{(2x+3)(2x-3)}{(x+4)(x-1)} \cdot \frac{(1+x)(1-x)}{2x-3}$

$= \frac{(2x+3)(2x-3)(1+x)(1-x)}{(x+4)(x-1)(2x-3)}$

$= \frac{(2x+3)(1+x)(-1)(x-1)}{(x+4)(x-1)}$

$= -\frac{(2x+3)(1+x)}{x+4}$

28. $\frac{y(6x^2+7x-3)}{x(y-1)+5(y-1)} \cdot \frac{x(y-1)+4(y-1)}{x^2y(x+4)}$

$= \frac{y(3x-1)(2x+3)(y-1)(x+4)}{(y-1)(x+5)x^2y(x+4)}$

$= \frac{(3x-1)(2x+3)}{x^2(x+5)}$

29. $\frac{x^2+5x+6}{x+3} = \frac{(x+3)(x+2)}{x+3} = x+2$

30. $\frac{2+x}{x+2} = \frac{x+2}{x+2} = 1$

31. LCD = $3t$

$\frac{2}{t} + \frac{1}{3t} = \frac{6}{3t} + \frac{1}{3t} = \frac{6+1}{3t} = \frac{7}{3t}$

32. LCD = X^3

$\frac{9}{X^3} - \frac{1}{X^2} = \frac{9}{X^3} - \frac{X}{X^3} = \frac{9-X}{X^3}$

33. LCD = $x^3 - 1$

$1 - \frac{x^3}{x^3-1} = \frac{x^3-1}{x^3-1} - \frac{x^3}{x^3-1}$

$= \frac{x^3-1-x^3}{x^3-1}$

$= \frac{-1}{x^3-1}$

$= \frac{1}{1-x^3}$

34. LCD = $s + 4$

$\frac{4}{s+4} + s = \frac{4}{s+4} + \frac{s(s+4)}{s+4} = \frac{4+s(s+4)}{s+4}$

$= \frac{s^2+4s+4}{s+4} = \frac{(s+2)^2}{s+4}$

35. LCD = $(2x-1)(x+3)$

$$\frac{4}{2x-1}+\frac{x}{x+3}=\frac{4(x+3)}{(2x-1)(x+3)}+\frac{x(2x-1)}{(x+3)(2x-1)}$$
$$=\frac{4(x+3)+x(2x-1)}{(2x-1)(x+3)}=\frac{2x^2+3x+12}{(2x-1)(x+3)}$$

36. LCD = $(x-1)(x+1)$

$$\frac{x+1}{x-1}-\frac{x-1}{x+1}=\frac{(x+1)(x+1)}{(x-1)(x+1)}-\frac{(x-1)(x-1)}{(x-1)(x+1)}$$
$$=\frac{(x+1)^2-(x-1)^2}{(x+1)(x-1)}$$
$$=\frac{x^2+2x+1-(x^2-2x+1)}{(x+1)(x-1)}=\frac{4x}{(x+1)(x-1)}$$

37. LCD = $(x-3)(x+1)(x+3)$

$$\frac{1}{(x-3)(x+1)}+\frac{1}{(x+3)(x-3)}$$
$$=\frac{x+3}{(x-3)(x+1)(x+3)}+\frac{x+1}{(x-3)(x+1)(x+3)}$$
$$=\frac{(x+3)+(x+1)}{(x-3)(x+1)(x+3)}$$
$$=\frac{2x+4}{(x-3)(x+1)(x+3)}$$
$$=\frac{2(x+2)}{(x-3)(x+1)(x+3)}$$

38. LCD = $(x-4)(2x+1)(2x-1)$

$$\frac{4}{(x-4)(2x+1)}-\frac{x}{(x-4)(2x-1)}$$
$$=\frac{4(2x-1)}{(x-4)(2x+1)(2x-1)}-\frac{x(2x+1)}{(x-4)(2x+1)(2x-1)}$$
$$=\frac{4(2x-1)-x(2x+1)}{(x-4)(2x+1)(2x-1)}$$
$$=\frac{-2x^2+7x-4}{(x-4)(2x+1)(2x-1)}$$

39. LCD = $(x-1)(x+5)$

$$\frac{4}{x-1}-3+\frac{-3x^2}{-(x-1)(x+5)}$$
$$=\frac{4(x+5)}{(x-1)(x+5)}-\frac{3(x-1)(x+5)}{(x-1)(x+5)}+\frac{3x^2}{(x-1)(x+5)}$$
$$=\frac{4x+20-3(x^2+4x-5)+3x^2}{(x-1)(x+5)}$$
$$=\frac{35-8x}{(x-1)(x+5)}$$

40. LCD = $(2x-1)(x+6)(3x-2)$

$$\frac{2x-3}{(2x-1)(x+6)}-\frac{3x+1}{(3x-2)(x+6)}+\frac{1}{3x-2}$$
$$=\frac{(2x-3)(3x-2)-(3x+1)(2x-1)+(2x-1)(x+6)}{(2x-1)(x+6)(3x-2)}$$
$$=\frac{6x^2-13x+6-(6x^2-x-1)+2x^2+11x-6}{(2x-1)(x+6)(3x-2)}$$
$$=\frac{2x^2-x+1}{(2x-1)(x+6)(3x-2)}$$

41. $\left(1+\frac{1}{x}\right)^2=\left(\frac{x}{x}+\frac{1}{x}\right)^2=\left(\frac{x+1}{x}\right)^2=\frac{x^2+2x+1}{x^2}$

42. $\left(\frac{1}{x}+\frac{1}{y}\right)^2=\left(\frac{y}{xy}+\frac{x}{xy}\right)^2=\left(\frac{y+x}{xy}\right)^2$

$$=\frac{y^2+2xy+x^2}{x^2y^2}$$

43. $\left(\frac{1}{x}-y\right)^{-1}=\left(\frac{1}{x}-\frac{xy}{x}\right)^{-1}=\left(\frac{1-xy}{x}\right)^{-1}=\frac{x}{1-xy}$

44. $\left(a+\frac{1}{b}\right)^2=\left(\frac{ab}{b}+\frac{1}{b}\right)^2=\left(\frac{ab+1}{b}\right)^2$

$$=\frac{a^2b^2+2ab+1}{b^2}$$

45. Multiplying the numerator and denominator of the given fraction by x gives $\frac{7x+1}{5x}$.

46. Multiplying numerator and denominator by x gives $\frac{x+3}{x^2-9}=\frac{x+3}{(x+3)(x-3)}=\frac{1}{x-3}$.

47. Multiplying numerator and denominator by $2x(x + 2)$ gives

$$\frac{3(2x)(x+2)-1(x+2)}{x(2x)(x+2)+x(2x)} = \frac{(x+2)[3(2x)-1]}{2x^2[(x+2)+1]}$$

$$= \frac{(x+2)(6x-1)}{2x^2(x+3)}.$$

48. Multiplying numerator and denominator by $3(x + 3)(x + 2)$ gives

$$\frac{3(x-1)-1(3)(x+3)}{3(3)(x+3)(x+2)+(x-7)(x+3)(x+2)}$$

$$= \frac{-12}{(x+3)(x+2)[9+(x-7)]} = -\frac{12}{(x+3)(x+2)^2}.$$

49. $\text{LCD} = \sqrt[3]{x+h}\cdot\sqrt[3]{x}$

$$\frac{3}{\sqrt[3]{x+h}} - \frac{3}{\sqrt[3]{x}} = \frac{3\sqrt[3]{x}}{\sqrt[3]{x+h}\sqrt[3]{x}} - \frac{3\sqrt[3]{x+h}}{\sqrt[3]{x+h}\sqrt[3]{x}}$$

$$= \frac{3\left(\sqrt[3]{x}-\sqrt[3]{x+h}\right)}{\sqrt[3]{x+h}\sqrt[3]{x}}$$

50. $\text{LCD} = \sqrt{5+a}\sqrt{a}$

$$\frac{a\sqrt{a}}{\sqrt{5+a}} + \frac{1}{\sqrt{a}} = \frac{a\sqrt{a}\left(\sqrt{a}\right)}{\sqrt{5+a}\sqrt{a}} + \frac{1\left(\sqrt{5+a}\right)}{\sqrt{5+a}\sqrt{a}}$$

$$= \frac{a^2+\sqrt{5+a}}{\sqrt{a}\sqrt{5+a}}$$

51. $\dfrac{1}{2+\sqrt{3}}\cdot\dfrac{2-\sqrt{3}}{2-\sqrt{3}} = \dfrac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}$

52. $\dfrac{1}{1-\sqrt{2}}\cdot\dfrac{1+\sqrt{2}}{1+\sqrt{2}} = \dfrac{1+\sqrt{2}}{1-2} = \dfrac{1+\sqrt{2}}{-1} = -1-\sqrt{2}$

53. $\dfrac{\sqrt{2}}{\sqrt{3}-\sqrt{6}}\cdot\dfrac{\sqrt{3}+\sqrt{6}}{\sqrt{3}+\sqrt{6}}$

$$= \frac{\sqrt{2}\left(\sqrt{3}+\sqrt{6}\right)}{3-6} = \frac{\sqrt{6}+\sqrt{12}}{-3} = -\frac{\sqrt{6}+2\sqrt{3}}{3}$$

54. $\dfrac{5}{\sqrt{6}+\sqrt{7}}\cdot\dfrac{\sqrt{6}-\sqrt{7}}{\sqrt{6}-\sqrt{7}} = \dfrac{5\left(\sqrt{6}-\sqrt{7}\right)}{6-7}$

$$= \frac{5\left(\sqrt{6}-\sqrt{7}\right)}{-1} = 5\left(\sqrt{7}-\sqrt{6}\right)$$

55. $\dfrac{2\sqrt{2}}{\sqrt{2}-\sqrt{3}}\cdot\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt{3}} = \dfrac{2\sqrt{2}\left(\sqrt{2}+\sqrt{3}\right)}{2-3}$

$$= \frac{4+2\sqrt{6}}{-1} = -4-2\sqrt{6}$$

56. $\dfrac{2\sqrt{5}}{\sqrt{3}-\sqrt{7}}\cdot\dfrac{\sqrt{3}+\sqrt{7}}{\sqrt{3}+\sqrt{7}}$

$$= \frac{2\sqrt{5}\left(\sqrt{3}+\sqrt{7}\right)}{3-7}$$

$$= \frac{2\left(\sqrt{15}+\sqrt{35}\right)}{-4}$$

$$= -\frac{\sqrt{15}+\sqrt{35}}{2}$$

57. $\dfrac{3}{t+\sqrt{7}}\cdot\dfrac{t-\sqrt{7}}{t-\sqrt{7}} = \dfrac{3t-3\sqrt{7}}{t^2-7}$

58. $\dfrac{(x-3)+4}{\sqrt{x}-1} = \dfrac{x+1}{\sqrt{x}-1}\cdot\dfrac{\sqrt{x}+1}{\sqrt{x}+1} = \dfrac{(x+1)\left(\sqrt{x}+1\right)}{x-1}$

59. $\dfrac{5\left(2-\sqrt{3}\right)}{\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)} - \dfrac{4\left(1+\sqrt{2}\right)}{\left(1-\sqrt{2}\right)\left(1+\sqrt{2}\right)}$

$$= \frac{5\left(2-\sqrt{3}\right)}{4-3} - \frac{4\left(1+\sqrt{2}\right)}{1-2}$$

$$= \frac{5\left(2-\sqrt{3}\right)}{1} - \frac{4\left(1+\sqrt{2}\right)}{-1}$$

$$= 5\left(2-\sqrt{3}\right)+4\left(1+\sqrt{2}\right) = 4\sqrt{2}-5\sqrt{3}+14$$

60. $\dfrac{4x^2}{3\left(\sqrt{x}+2\right)} = \dfrac{4x^2\left(\sqrt{x}-2\right)}{3\left(\sqrt{x}+2\right)\left(\sqrt{x}-2\right)} = \dfrac{4x^2\left(\sqrt{x}-2\right)}{3(x-4)}$

Problems 0.7

1. $9x - x^2 = 0$
Set $x = 1$:
$9(1) - (1)^2 \stackrel{?}{=} 0$
$9 - 1 \stackrel{?}{=} 0$
$8 \neq 0$
Set $x = 0$:
$9(0) - (0)^2 \stackrel{?}{=} 0$
$0 - 0 \stackrel{?}{=} 0$
$0 = 0$
Thus, 0 satisfies the equation, but 1 does not.

2. $12 - 7x = -x^2$; 4, 3
Set $x = 4$:
$12 - 7(4) \stackrel{?}{=} -(4)^2$
$12 - 28 \stackrel{?}{=} -16$
$-16 = -16$
Set $x = 3$:
$12 - 7(3) \stackrel{?}{=} -(3)^2$
$12 - 21 \stackrel{?}{=} -9$
$-9 = -9$
Thus, 4 and 3 satisfy the equation.

3. $z + 3(z - 4) = 5;\ \frac{17}{4},\ 4$
Set $z = \frac{17}{4}$:
$\frac{17}{4} + 3\left(\frac{17}{4} - 4\right) \stackrel{?}{=} 5$
$\frac{17}{4} + \frac{51}{4} - 12 \stackrel{?}{=} 5$
$5 = 5$
Set $z = 4$:
$4 + 3(4 - 4) \stackrel{?}{=} 5$
$4 + 0 \stackrel{?}{=} 5$
$4 \neq 5$
Thus, $\frac{17}{4}$ satisfies the equation, but 4 does not.

4. $2x + x^2 - 8 = 0$
Set $x = 2$:
$2 \cdot 2 + 2^2 - 8 \stackrel{?}{=} 0$
$4 + 4 - 8 \stackrel{?}{=} 0$
$0 = 0$
Set $x = -4$:
$2(-4) + (-4)^2 - 8 \stackrel{?}{=} 0$
$-8 + 16 - 8 \stackrel{?}{=} 0$
$0 = 0$
Thus, 2 and –4 satisfy the equation.

5. $x(6 + x) - 2(x + 1) - 5x = 4$
Set $x = -2$:
$(-2)(6 - 2) - 2(-2 + 1) - 5(-2) \stackrel{?}{=} 4$
$-2(4) - 2(-1) + 10 \stackrel{?}{=} 4$
$-8 + 2 + 10 \stackrel{?}{=} 4$
$4 = 4$
Set $x = 0$:
$0(6) - 2(1) - 5(0) \stackrel{?}{=} 4$
$-2 \neq 4$
Thus, –2 satisfies the equation, but 0 does not.

6. $x(x + 1)^2(x + 2) = 0$
Set $x = 0$:
$0(1)^2(2) \stackrel{?}{=} 0$
$0 = 0$
Set $x = -1$:
$(-1)(0)^2(1) \stackrel{?}{=} 0$
$0 = 0$
Set $x = 2$:
$2(3)^2(4) \stackrel{?}{=} 0$
$72 \neq 0$
Thus, 0 and –1 satisfy the equation, but 2 does not.

7. Adding 5 to both sides; equivalence guaranteed

8. Dividing both sides by 8; equivalence guaranteed

9. Raising both sides to the third power; equivalence not guaranteed.

10. Dividing both sides by 2; equivalence guaranteed

11. Dividing both sides by x; equivalence not guaranteed

12. Multiplying both sides by $x - 2$; equivalence not guaranteed

13. Multiplying both sides by $x - 1$; equivalence not guaranteed

14. Dividing both sides by $(x + 3)$; equivalence not guaranteed.

15. Multiplying both sides by $\frac{2x-3}{2x}$; equivalence not guaranteed

16. Adding 9 – x to both sides and then dividing both sides by 2; equivalence guaranteed

17. $4x = 10$
$x = \frac{10}{4} = \frac{5}{2}$

18. $0.2x = 7$
$x = \frac{7}{0.2} = 35$

19. $3y = 0$
$y = \frac{0}{3} = 0$

20. $2x - 4x = -5$
$-2x = -5$
$x = \frac{-5}{-2} = \frac{5}{2}$

21. $-8x = 12 - 20$
$-8x = -8$
$x = \frac{-8}{-8} = 1$

22. $4 - 7x = 3$
$-7x = -1$
$x = \frac{-1}{-7} = \frac{1}{7}$

23. $5x - 3 = 9$
$5x = 12$
$x = \frac{12}{5}$

24. $\sqrt{2}x + 3 = 8$
$\sqrt{2}x = 5$
$x = \frac{5}{\sqrt{2}}\left(\text{or } \frac{5\sqrt{2}}{2}\right)$

25. $7x + 7 = 2(x + 1)$
$7x + 7 = 2x + 2$
$5x + 7 = 2$
$5x = -5$
$x = \frac{-5}{5} = -1$

26. $4s + 3s - 1 = 41$
$7s - 1 = 41$
$7s = 42$
$s = \frac{42}{7} = 6$

27. $5(p-7) - 2(3p-4) = 3p$
$5p - 35 - 6p + 8 = 3p$
$-p - 27 = 3p$
$-27 = 4p$
$p = -\frac{27}{4}$

28. $t = 2 - 2[2t - 3(1 - t)]$
$t = 2 - 2[2t - 3 + 3t]$
$t = 2 - 2[5t - 3]$
$t = 2 - 10t + 6$
$11t = 8$
$t = \frac{8}{11}$

29. $\frac{x}{5} = 2x - 6$
$x = 5(2x - 6)$
$x = 10x - 30$
$30 = 9x$
$x = \frac{30}{9} = \frac{10}{3}$

30. $\frac{5y}{7} - \frac{6}{7} = 2 - 4y$
$5y - 6 = 14 - 28y$
$33y = 20$
$y = \frac{20}{33}$

31. $7 + \frac{4x}{9} = \frac{x}{2}$
Multiplying both sides by $9 \cdot 2$ gives
$9 \cdot 2 \cdot 7 + 2(4x) = 9(x)$
$126 + 8x = 9x$
$x = 126$

32. $\frac{x}{3} - 4 = \frac{x}{5}$
$5x - 60 = 3x$
$2x = 60$
$x = 30$

33. $r = \frac{4}{3}r - 5$

Multiplying both sides by 3 gives

$3r = 4r - 15$

$-r = -15$

$r = 15$

34. $\frac{3x}{5} + \frac{5x}{3} = 9$

$9x + 25x = 135$

$34x = 135$

$x = \frac{135}{34}$

35. $3x + \frac{x}{5} - 5 = \frac{1}{5} + 5x$

Multiplying both sides by 5 gives

$15x + x - 25 = 1 + 25x$

$16x - 25 = 1 + 25x$

$-9x = 26$

$x = -\frac{26}{9}$

36. $y - \frac{y}{2} + \frac{y}{3} - \frac{y}{4} = \frac{y}{5}$

$60y - 30y + 20y - 15y = 12y$

$35y = 12y$

$23y = 0$

$y = 0$

37. $\frac{2y-3}{4} = \frac{6y+7}{3}$

Multiplying both sides by 12 gives

$3(2y - 3) = 4(6y + 7)$

$6y - 9 = 24y + 28$

$-18y = 37$

$y = -\frac{37}{18}$

38. $\frac{t}{4} + \frac{5}{3}t = \frac{7}{2}(t-1)$

Multiplying both sides by 12 gives

$3t + 20t = 42(t-1)$

$23t = 42t - 42$

$42 = 19t$

$t = \frac{42}{19}$

39. $w - \frac{w}{2} + \frac{w}{6} - \frac{w}{24} = 120$

Multiplying both sides by 24 gives

$24w - 12w + 4w - w = 2880$

$15w = 2880$

$w = \frac{2880}{15} = 192$

40. $\frac{7+2(x+1)}{3} = \frac{6x}{5}$

$35 + 10(x + 1) = 18x$

$35 + 10x + 10 = 18x$

$45 = 8x$

$x = \frac{45}{8}$

41. $\frac{x+2}{3} - \frac{2-x}{6} = x - 2$

Multiplying both sides by 6 gives

$2(x + 2) - (2 - x) = 6(x - 2)$

$2x + 4 - 2 + x = 6x - 12$

$3x + 2 = 6x - 12$

$2 = 3x - 12$

$14 = 3x$

$x = \frac{14}{3}$

42. $\frac{x}{5} + \frac{2(x-4)}{10} = 7$

$2x + 2(x - 4) = 70$

$2x + 2x - 8 = 70$

$4x = 78$

$x = \frac{78}{4} = \frac{39}{2}$

43. $\frac{9}{5}(3-x) = \frac{3}{4}(x-3)$

Multiplying both sides by 20 gives

$36(3 - x) = 15(x - 3)$

$108 - 36x = 15x - 45$

$153 = 51x$

$x = 3$

44. $\frac{2y-7}{3} + \frac{8y-9}{14} = \frac{3y-5}{21}$

$14(2y - 7) + 3(8y - 9) = 2(3y - 5)$

$28y - 98 + 24y - 27 = 6y - 10$

$46y = 115$

$y = \frac{115}{46} = \frac{5}{2}$

45. $\frac{4}{3}(5x-2) = 7[x-(5x-2)]$

$4(5x-2) = 21(x-5x+2)$

$20x-8 = -84x+42$

$104x = 50$

$x = \frac{50}{104} = \frac{25}{52}$

46. $(2x-5)^2 + (3x-3)^2 = 13x^2 - 5x + 7$

$4x^2 - 20x + 25 + 9x^2 - 18x + 9 = 13x^2 - 5x + 7$

$13x^2 - 38x + 34 = 13x^2 - 5x + 7$

$-33x = -27$

$x = \frac{-27}{-33} = \frac{9}{11}$

47. $\frac{5}{x} = 25$

Multiplying both sides by x gives

$5 = 25x$

$x = \frac{5}{25}$

$x = \frac{1}{5}$

48. $\frac{4}{x-1} = 2$

$4 = 2(x-1)$

$4 = 2x-2$

$6 = 2x$

$x = 3$

49. Multiplying both sides by $3-x$ gives $7 = 0$, which is false. Thus there is no solution, so the solution set is $\varnothing$.

50. $\frac{3x-5}{x-3} = 0$

$3x-5 = 0$

$3x = 5$

$x = \frac{5}{3}$

51. $\frac{3}{5-2x} = \frac{7}{2}$

$3(2) = 7(5-2x)$

$6 = 35-14x$

$14x = 29$

$x = \frac{29}{14}$

52. $\frac{x+3}{x} = \frac{2}{5}$

$5(x+3) = 2x$

$5x + 15 = 2x$

$3x = -15$

$x = -5$

53. $\frac{q}{5q-4} = \frac{1}{3}$

$3q = 5q-4$

$-2q = -4$

$q = 2$

54. $\frac{4p}{7-p} = 1$

$4p = 7-p$

$5p = 7$

$p = \frac{7}{5}$

55. $\frac{1}{p-1} = \frac{2}{p-2}$

$p-2 = 2(p-1)$

$p-2 = 2p-2$

$p = 0$

56. $\frac{2x-3}{4x-5} = 6$

$2x-3 = 24x-30$

$27 = 22x$

$x = \frac{27}{22}$

57. $\frac{1}{x} + \frac{1}{7} = \frac{3}{7}$

$\frac{1}{x} = \frac{3}{7} - \frac{1}{7}$

$\frac{1}{x} = \frac{2}{7}$

$x = \frac{7}{2}$

58. $\frac{2}{x-1} = \frac{3}{x-2}$

$2(x-2) = 3(x-1)$

$2x-4 = 3x-3$

$-x = 1$

$x = -1$

59. $\frac{3x-2}{2x+3}=\frac{3x-1}{2x+1}$
$(3x-2)(2x+1)=(3x-1)(2x+3)$
$6x^2-x-2=6x^2+7x-3$
$1=8x$
$x=\frac{1}{8}$

60. $\frac{x+2}{x-1}+\frac{x+1}{3-x}=0$
$(x+2)(3-x)+(x+1)(x-1)=0$
$3x-x^2+6-2x+x^2-1=0$
$x+5=0$
$x=-5$

61. $\frac{y-6}{y}-\frac{6}{y}=\frac{y+6}{y-6}$
Multiplying both sides by $y(y-6)$ gives
$(y-6)^2-6(y-6)=y(y+6)$
$y^2-12y+36-6y+36=y^2+6y$
$y^2-18y+72=y^2+6y$
$72=24y$
$y=3$

62. $\frac{y-2}{y+2}=\frac{y-2}{y+3}$
$(y-2)(y+3)=(y-2)(y+2)$
$y^2+y-6=y^2-4$
$y=2$

63. $\frac{-5}{2x-3}=\frac{7}{3-2x}+\frac{11}{3x+5}$
Multiplying both sides by $(2x-3)(3x+5)$ gives
$-5(3x+5)=-7(3x+5)+11(2x-3)$
$-15x-25=-21x-35+22x-33$
$-15x-25=x-68$
$-16x=-43$
$x=\frac{43}{16}$

64. $\frac{1}{x-3}-\frac{3}{x-2}=\frac{4}{1-2x}$
$(x-2)(1-2x)-3(x-3)(1-2x)$
$=4(x-3)(x-2)$
$-2x^2+5x-2-3(-2x^2+7x-3)=4(x^2-5x+6)$
$4x^2-16x+7=4x^2-20x+24$
$4x=17$
$x=\frac{17}{4}$

65. $\frac{9}{x-3}=\frac{3x}{x-3}$
$9=3x$
$x=3$
But the given equation is not defined for $x=3$, so there is no solution. The solution set is $\varnothing$.

66. $\frac{x}{x+3}-\frac{x}{x-3}=\frac{3x-4}{x^2-9}$
$x(x-3)-x(x+3)=3x-4$
$x^2-3x-x^2-3x=3x-4$
$-6x=3x-4$
$-9x=-4$
$x=\frac{4}{9}$

67. $\sqrt{x+5}=4$
$\left(\sqrt{x+5}\right)^2=4^2$
$x+5=16$
$x=11$

68. $\sqrt{z-2}=3$
$\left(\sqrt{z-2}\right)^2=3^2$
$z-2=9$
$z=11$

69. $\sqrt{3x-4}-8=0$
$\sqrt{3x-4}=8$
$\left(\sqrt{3x-4}\right)^2=(8)^2$
$3x-4=64$
$3x=68$
$x=\frac{68}{3}$

70. $4-\sqrt{3x+1}=0$

$4=\sqrt{3x+1}$

$4^2=\left(\sqrt{3x+1}\right)^2$

$16=3x+1$

$15=3x$

$x=5$

71. $\sqrt{\frac{x}{2}+1}=\frac{2}{3}$

$\left(\sqrt{\frac{x}{2}+1}\right)^2=\left(\frac{2}{3}\right)^2$

$\frac{x}{2}+1=\frac{4}{9}$

$\frac{x}{2}=-\frac{5}{9}$

$x=2\left(-\frac{5}{9}\right)=-\frac{10}{9}$

72. $(x+6)^{1/2}=7$

$[(x+6)^{1/2}]^2=7^2$

$x+6=49$

$x=43$

73. $\sqrt{4x-6}=\sqrt{x}$

$\left(\sqrt{4x-6}\right)^2=\left(\sqrt{x}\right)^2$

$4x-6=x$

$3x=6$

$x=2$

74. $\sqrt{4+3x}=\sqrt{2x+5}$

$\left(\sqrt{4+3x}\right)^2=\left(\sqrt{2x+5}\right)^2$

$4+3x=2x+5$

$x=1$

75. $(x-5)^{3/4}=27$

$[(x-5)^{3/4}]^{4/3}=27^{4/3}$

$x-5=81$

$x=86$

76. $\sqrt{y^2-9}=9-y$

$\left(\sqrt{y^2-9}\right)^2=(9-y)^2$

$y^2-9=81-18y+y^2$

$18y=90$

$y=\frac{90}{18}=5$

77. $\sqrt{y}+\sqrt{y+2}=3$

$\sqrt{y+2}=3-\sqrt{y}$

$\left(\sqrt{y+2}\right)^2=\left(3-\sqrt{y}\right)^2$

$y+2=9-6\sqrt{y}+y$

$6\sqrt{y}=7$

$\left(6\sqrt{y}\right)^2=7^2$

$36y=49$

$y=\frac{49}{36}$

78. $\sqrt{x}-\sqrt{x+1}=1$

$\sqrt{x}=\sqrt{x+1}+1$

$\left(\sqrt{x}\right)^2=\left(\sqrt{x+1}+1\right)^2$

$x=x+1+2\sqrt{x+1}+1$

$-2=2\sqrt{x+1}$

$-1=\sqrt{x+1}$, which is impossible because $\sqrt{a}\geq 0$ for all a. Thus there is no solution. The solution set is $\varnothing$.

79. $\sqrt{z^2+2z}=3+z$

$\left(\sqrt{z^2+2z}\right)^2=(3+z)^2$

$z^2+2z=9+6z+z^2$

$-9=4z$

$z=-\frac{9}{4}$

80. $\sqrt{\frac{1}{w}}-\sqrt{\frac{2}{5w-2}}=0$

$\sqrt{\frac{1}{w}}=\sqrt{\frac{2}{5w-2}}$

$\left(\sqrt{\frac{1}{w}}\right)^2=\left(\sqrt{\frac{2}{5w-2}}\right)^2$

$\frac{1}{w}=\frac{2}{5w-2}$

$5w-2=2w$

$3w=2$

$w=\frac{2}{3}$

81. $I=Prt$

$r=\frac{I}{Pt}$

82. $P\left(1+\frac{p}{100}\right)-R=0$

$P\left(1+\frac{p}{100}\right)=R$

$P=\frac{R}{1+\frac{p}{100}}$

83. $p=8q-1$

$p+1=8q$

$q=\frac{p+1}{8}$

84. $p=-3q+6$

$p-6=-3q$

$q=\frac{p-6}{-3}=\frac{6-p}{3}$

85. $S=P(1+rt)$

$S=P+Prt$

$S-P=r(Pt)$

$r=\frac{S-P}{Pt}$

86. $r=\frac{2mI}{B(n+1)}$

$\frac{r[B(n+1)]}{2m}=I$

$I=\frac{rB(n+1)}{2m}$

87. $A=\frac{R[1-(1+i)^{-n}]}{i}$

$R=\frac{Ai}{1-(1+i)^{-n}}$

88. $S=\frac{R[(1+i)^n-1]}{i}$

$Si=R[(1+i)^n-1]$

$R=\frac{Si}{(1+i)^n-1}$

89. $r=\frac{d}{1-dt}$

$r(1-dt)=d$

$r-rdt=d$

$-rdt=-r+d$

$t=-\frac{d-r}{rd}=\frac{r-d}{rd}$

90. $\frac{x-a}{b-x}=\frac{x-b}{a-x}$

Multiplying both sides by $(b-x)(a-x)$ gives

$(x-a)(a-x)=(x-b)(b-x)$

$(x-a)(a-x)(-1)=(x-b)(b-x)(-1)$

$(x-a)(x-a)=(x-b)(x-b)$

$x^2-2ax+a^2=x^2-2bx+b^2$

$a^2-b^2=2ax-2bx$

$(a+b)(a-b)=2x(a-b)$

$a+b=2x$ (for $a\neq b$)

$\frac{a+b}{2}=x$

91. $r=\frac{2mI}{B(n+1)}$

$r(n+1)=\frac{2mI}{B}$

$n+1=\frac{2mI}{rB}$

$n=\frac{2mI}{rB}-1$

92. $\frac{1}{p}+\frac{1}{q}=\frac{1}{f}$

$\frac{1}{q}=\frac{1}{f}-\frac{1}{p}$

$\frac{1}{q}=\frac{p-f}{pf}$

$q=\frac{pf}{p-f}$

93. $P=2l+2w$
$660=2l+2(160)$
$660=2l+320$
$340=2l$
$l=\frac{340}{2}=170$
The length of the rectangle is 170 m.

94. $V=\pi r^2h$
$355=\pi(2)^2h$
$355=4\pi h$
$h=\frac{355}{4\pi}$
The height of the can is
$\frac{355}{4\pi}\approx 28.25$ centimeters.

95. $c = x + 0.0825x = 1.0825x$

96. Revenue equals cost when $450x = 380x + 3500$.
$450x = 380x + 3500$
$70x = 3500$
$x = 50$
50 toddlers need to be enrolled.

97. $V=C\left(1-\frac{n}{N}\right)$

$2000=3200\left(1-\frac{n}{8}\right)$

$2000 = 3200 - 400n$
$400n = 1200$
$n = 3$
The furniture will have a value of $2000 after 3 years.

98. $F=\frac{vf}{334.8}$

$495=\frac{v(2500)}{334.8}$

$165,726=2500v$

$v=\frac{165,726}{2500}=66.2904$

Since the car is traveling at 66.2904 mi/h on a 65 mi/h highway, the officer can claim that you were speeding.

99. Bronwyn's weekly salary for working h hours is $27h + 18$. Steve's weekly salary for working h hours is $35h$.

$\frac{1}{5}(27h+18+35h)=550$
$62h+18=2750$
$62h=2732$
$h=\frac{2732}{62}\approx 44.1$

They must each work 44 hours each week.

100. $y = a(1 - by)x$
$y = ax(1 - by)$
$y = ax - abxy$
$y + abxy = ax$
$y(1 + abx) = ax$
$y=\frac{ax}{1+abx}$

101. $y=\frac{1.4x}{1+0.09x}$

With $y = 10$ the equation is

$10=\frac{1.4x}{1+0.09x}$

$10(1 + 0.09x) = 1.4x$
$10 + 0.9x = 1.4x$
$10 = 0.5x$
$x = 20$
The prey density should be 20.

102. Let x = the maximum number of customers.

$\frac{8}{x-92}=\frac{10}{x-46}$

$8(x - 46) = 10(x - 92)$
$8x - 368 = 10x - 920$
$552 = 2x$
$x = 276$
The maximum number of customers is 276.

103. $t = \frac{d}{r-c}$
$t(r-c) = d$
$tr - tc = d$
$tr - d = tc$
$c = \frac{tr-d}{t} = r - \frac{d}{t}$

104. Let x = the horizontal distance from the base of the tower to the house. By the Pythagorean theorem, $x^2 + 100^2 = (x+1)^2$.
$x^2 + 10{,}000 = x^2 + 2x + 1$
$10{,}000 = 2x + 1$
$9999 = 2x$
$x = \frac{9999}{2} = 4999.5$
The distance from the top of the tower to the house is $x + 1 = 4999.5 + 1 = 5000.5$ meters.

105. $s = \sqrt{30\,fd}$
Set $s = 45$ and (for dry concrete) $f = 0.8$.
$45 = \sqrt{30(0.8)d}$
$45 = \sqrt{24d}$
$(45)^2 = \left(\sqrt{24d}\right)^2$
$2025 = 24d$
$d = \frac{2025}{24} = \frac{675}{8} = 84\frac{3}{8} \approx 84$ ft

106. Let P be the amount in the account one year ago. Then the interest earned is $0.073P$ and $P + 0.073P = 1257$.
$1.073P = 1257$
$P = \frac{1257}{1.073} \approx 1171.48$
The amount in the account one year ago was \$1171.48, and the interest earned is \$1171.48(0.073) = \$85.52.

107. Let e be Tom's expenses in Nova Scotia before the HST tax. Then the HST tax is $0.15e$ and the total receipts are $e + 0.15e = 1.15e$. The percentage of the total that is HST is
$\frac{0.15e}{1.15e} = \frac{0.15}{1.15} = \frac{15}{115} = \frac{3}{23}$ or approximately 13%.

108. $\frac{1}{8}$ and $-\frac{1}{14}$ are roots.

109. $-\frac{1}{2}$ is a root.

110. $\frac{14}{61}$ is a root.

111. 0 is a root.

Problems 0.8

1. $x^2 - 4x + 4 = 0$
$(x-2)^2 = 0$
$x - 2 = 0$
$x = 2$

2. $(t+1)(t+2) = 0$
$t + 1 = 0$ or $t + 2 = 0$
$t = -1$ or $t = -2$

3. $t^2 - 8t + 15 = 0$
$(t-3)(t-5) = 0$
$t - 3 = 0$ or $t - 5 = 0$
$t = 3$ or $t = 5$

4. $(x-2)(x+5) = 0$
$x - 2 = 0$ or $x + 5 = 0$
$x = 2$ or $x = -5$

5. $x^2 - 2x - 3 = 0$
$(x-3)(x+1) = 0$
$x - 3 = 0$ or $x + 1 = 0$
$x = 3$ or $x = -1$

6. $(x-4)(x+4) = 0$
$x - 4 = 0$ or $x + 4 = 0$
$x = 4$ or $x = -4$

7. $u^2 - 13u = -36$
$u^2 - 13u + 36 = 0$
$(u-4)(u-9) = 0$
$u - 4 = 0$ or $u - 9 = 0$
$u = 4$ or $u = 9$

8. $3(w^2 - 4w + 4) = 0$
$3(w-2)^2 = 0$
$w - 2 = 0$
$w = 2$

9. $x^2 - 4 = 0$
$(x-2)(x+2) = 0$
$x - 2 = 0$ or $x + 2 = 0$
$x = 2$ or $x = -2$

10. $3u(u-2) = 0$
$u = 0$ or $u - 2 = 0$
$u = 0$ or $u = 2$

11. $t^2 - 5t = 0$
$t(t-5) = 0$
$t = 0$ or $t - 5 = 0$
$t = 0$ or $t = 5$

12. $x^2 + 9x + 14 = 0$
$(x+7)(x+2) = 0$
$x + 7 = 0$ or $x + 2 = 0$
$x = -7$ or $x = -2$

13. $4x^2 + 1 = 4x$
$4x^2 - 4x + 1 = 0$
$(2x-1)^2 = 0$
$2x - 1 = 0$
$x = \frac{1}{2}$

14. $2z^2 + 9z - 5 = 0$
$(2z-1)(z+5) = 0$
$2z - 1 = 0$ or $z + 5 = 0$
$z = \frac{1}{2}$ or $z = -5$

15. $v(3v-5) = -2$
$3v^2 - 5v = -2$
$3v^2 - 5v + 2 = 0$
$(3v-2)(v-1) = 0$
$3v - 2 = 0$ or $v - 1 = 0$
$v = \frac{2}{3}$ or $v = 1$

16. $-6x^2 + x + 2 = 0$
$6x^2 - x - 2 = 0$
$(2x+1)(3x-2) = 0$
$2x + 1 = 0$ or $3x - 2 = 0$
$x = -\frac{1}{2}$ or $x = \frac{2}{3}$

17. $-x^2 + 3x + 10 = 0$
$x^2 - 3x - 10 = 0$
$(x-5)(x+2) = 0$
$x - 5 = 0$ or $x + 2 = 0$
$x = 5$ or $x = -2$

18. $\frac{1}{7}y^2 - \frac{3}{7}y = 0$
$\frac{1}{7}y(y-3) = 0$
$y = 0$ or $y - 3 = 0$
$y = 0$ or $y = 3$

19. $2p^2 = 3p$
$2p^2 - 3p = 0$
$p(2p-3) = 0$
$p = 0$ or $2p - 3 = 0$
$p = 0$ or $p = \frac{3}{2}$

20. $r^2 + r - 12 = 0$
$(r-3)(r+4) = 0$
$r - 3 = 0$ or $r + 4 = 0$
$r = 3$ or $r = -4$

21. $x(x+4)(x-1) = 0$
$x = 0$ or $x + 4 = 0$ or $x - 1 = 0$
$x = 0$ or $x = -4$ or $x = 1$

22. $(w-3)^2(w+1)^2 = 0$
$w - 3 = 0$ or $w + 1 = 0$
$w = 3$ or $w = -1$

23. $t^3 - 49t = 0$
$t(t^2 - 49) = 0$
$t(t+7)(t-7) = 0$
$t = 0$ or $t + 7 = 0$ or $t - 7 = 0$
$t = 0$ or $t = -7$ or $t = 7$

24. $x(x^2 - 4x - 5) = 0$
$x(x-5)(x+1) = 0$
$x = 0$ or $x - 5 = 0$ or $x + 1 = 0$
$x = 0$ or $x = 5$ or $x = -1$

25. $6x^3 + 5x^2 - 4x = 0$
$x(6x^2 + 5x - 4) = 0$
$x(2x-1)(3x+4) = 0$

$x = 0$ or $2x - 1 = 0$ or $3x + 4 = 0$

$x = 0$ or $x = \frac{1}{2}$ or $x = -\frac{4}{3}$

26. $x^2 + 2x + 1 - 5x + 1 = 0$

$x^2 - 3x + 2 = 0$

$(x - 1)(x - 2) = 0$

$x - 1 = 0$ or $x - 2 = 0$

$x = 1$ or $x = 2$

27. $(x-3)(x^2-4) = 0$

$(x-3)(x-2)(x+2) = 0$

$x - 3 = 0$ or $x - 2 = 0$ or $x + 2 = 0$

$x = 3$ or $x = 2$ or $x = -2$

28. $5(x + 4)(x - 3)(x - 8) = 0$

$x + 4 = 0$ or $x - 3 = 0$ or $x - 8 = 0$

$x = -4$ or $x = 3$ or $x = 8$

29. $p(p-3)^2 - 4(p-3)^3 = 0$

$(p-3)^2[p - 4(p-3)] = 0$

$(p-3)^2(12-3p) = 0$

$3(p-3)^2(4-p) = 0$

$p - 3 = 0$ or $4 - p = 0$

$p = 3$ or $p = 4$

30. $(x^2-1)(x^2-2) = 0$

$(x+1)(x-1)\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right) = 0$

$x + 1 = 0$ or $x - 1 = 0$

or $x+\sqrt{2} = 0$ or $x-\sqrt{2} = 0$

$x = -1$ or $x = 1$

or $x = -\sqrt{2}$ or $x = \sqrt{2}$

31. $x^2 + 2x - 24 = 0$

$a = 1, b = 2, c = -24$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - 4(1)(-24)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{100}}{2}$$

$$= \frac{-2 \pm 10}{2}$$

$x = \frac{-2+10}{2} = 4$ or $x = \frac{-2-10}{2} = -6$

32. $x^2 - 2x - 15 = 0$

$a = 1, b = -2, c = -15$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{4 - 4(1)(-15)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{64}}{2}$$

$$= \frac{2 \pm 8}{2}$$

$x = \frac{2+8}{2} = 5$ or $x = \frac{2-8}{2} = -3$

33. $4x^2 - 12x + 9 = 0$

$a = 4, b = -12, c = 9$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-12) \pm \sqrt{144 - 4(4)(9)}}{2(4)}$$

$$= \frac{12 \pm \sqrt{0}}{8}$$

$$= \frac{12 \pm 0}{8}$$

$$= \frac{3}{2}$$

34. $q^2 - 5q = 0$

$a = 1, b = -5, c = 0$

$$q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{5 \pm \sqrt{25 - 4(1)(0)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{25}}{2}$$

$$= \frac{5 \pm 5}{2}$$

$q = \frac{5+5}{2} = 5$ or $q = \frac{5-5}{2} = 0$

35. $p^2 - 2p - 7 = 0$

$a = 1$, $b = -2$, $c = -7$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-7)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{32}}{2}$$

$$= 1 \pm 2\sqrt{2}$$

$p = 1 + 2\sqrt{2}$ or $p = 1 - 2\sqrt{2}$

36. $2 - 2x + x^2 = 0$

$x^2 - 2x + 2 = 0$

$a = 1$, $b = -2$, $c = 2$

$$x = \frac{-(-2) \pm \sqrt{4 - 4(1)(2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

no real roots

37. $4 - 2n + n^2 = 0$

$n^2 - 2n + 4 = 0$

$a = 1$, $b = -2$, $c = 4$

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{4 - 4(1)(4)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-12}}{2}$$

no real roots

38. $2x^2 + x = 5$

$2x^2 + x - 5 = 0$

$a = 2$, $b = 1$, $c = -5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1 - 4(2)(-5)}}{2(2)}$$

$$= \frac{-1 \pm \sqrt{41}}{4}$$

$x = \frac{-1 + \sqrt{41}}{4}$ or $x = \frac{-1 - \sqrt{41}}{4}$

39. $4x^2 + 5x - 2 = 0$

$a = 4$, $b = 5$, $c = -2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{25 - 4(4)(-2)}}{2(4)}$$

$$= \frac{-5 \pm \sqrt{57}}{8}$$

$x = \frac{-5 + \sqrt{57}}{8}$ or $x = \frac{-5 - \sqrt{57}}{8}$

40. $w^2 - 2w + 1 = 0$

$a = 1$, $b = -2$, $c = 1$

$$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{0}}{2}$$

$$= 1$$

41. $0.02w^2 - 0.3w = 20$

$0.02w^2 - 0.3w - 20 = 0$

$a = 0.02$, $b = -0.3$, $c = -20$

$$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-0.3) \pm \sqrt{0.09 - 4(0.02)(-20)}}{2(0.02)}$$

$$= \frac{0.3 \pm \sqrt{1.69}}{0.04}$$

$$= \frac{0.3 \pm 1.3}{0.04}$$

$w = \frac{0.3 + 1.3}{0.04} = \frac{1.6}{0.04} = 40$ or

$w = \frac{0.3 - 1.3}{0.04} = \frac{-1.0}{0.04} = -25$

42. $0.01x^2 + 0.2x - 0.6 = 0$
$a = 0.01,\ b = 0.2,\ c = -0.6$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-0.2 \pm \sqrt{0.04 - 4(0.01)(-0.6)}}{2(0.01)}$$
$$= \frac{-0.2 \pm \sqrt{0.064}}{0.02}$$
$$= \frac{-0.2 \pm \sqrt{(0.0064)(10)}}{0.02}$$
$$= \frac{-0.2 \pm 0.08\sqrt{10}}{0.02}$$
$$= -10 \pm 4\sqrt{10}$$
$x = -10 + 4\sqrt{10}$ or $x = -10 - 4\sqrt{10}$

43. $2x^2 + 4x = 5$
$2x^2 + 4x - 5 = 0$
$a = 2,\ b = 4,\ c = -5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-4 \pm \sqrt{16 - 4(2)(-5)}}{2(2)}$$
$$= \frac{-4 \pm \sqrt{56}}{4}$$
$$= \frac{-4 \pm 2\sqrt{14}}{4}$$
$$= \frac{-2 \pm \sqrt{14}}{2}$$
$x = \frac{-2 + \sqrt{14}}{2}$ or $x = \frac{-2 - \sqrt{14}}{2}$

44. $-2x^2 - 6x + 5 = 0$
$a = -2,\ b = -6,\ c = 5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-6) \pm \sqrt{36 - 4(-2)(5)}}{2(-2)}$$
$$= \frac{6 \pm \sqrt{76}}{-4}$$
$$= \frac{6 \pm 2\sqrt{19}}{-4}$$
$$= \frac{-3 \pm \sqrt{19}}{2}$$
$x = \frac{-3 + \sqrt{19}}{2}$ or $x = \frac{-3 - \sqrt{19}}{2}$

45. $(x^2)^2 - 5(x^2) + 6 = 0$
Let $w = x^2$. Then
$w^2 - 5w + 6 = 0$
$(w - 3)(w - 2) = 0$
$w = 3, 2$
Thus $x^2 = 3$ or $x^2 = 2$, so $x = \pm\sqrt{3}, \pm\sqrt{2}$.

46. $(X^2)^2 - 3(X)^2 - 10 = 0$
Let $w = X^2$. Then
$w^2 - 3w - 10 = 0$
$(w - 5)(w + 2) = 0$
$w = 5, -2$
Thus $X^2 = 5$ or $X^2 = -2$, so the real solutions are $X = \pm\sqrt{5}$.

47. $3\left(\frac{1}{x}\right)^2 - 7\left(\frac{1}{x}\right) + 2 = 0$
Let $w = \frac{1}{x}$. Then
$3w^2 - 7w + 2 = 0$
$(3w - 1)(w - 2) = 0$
$w = \frac{1}{3}, 2$
Thus, $x = 3, \frac{1}{2}$.

48. $(x^{-1})^2 + x^{-1} - 12 = 0$

Let $w = x^{-1}$. Then

$w^2 + w - 12 = 0$

$(w + 4)(w - 3) = 0$

$w = -4, 3$

Thus, $x = -\frac{1}{4}, \frac{1}{3}$.

49. $(x^{-2})^2 - 9(x^{-2}) + 20 = 0$

Let $w = x^{-2}$. Then

$w^2 - 9w + 20 = 0$

$(w - 5)(w - 4) = 0$

$w = 5, 4$

Thus, $\frac{1}{x^2} = 5$ or $\frac{1}{x^2} = 4$, so $x^2 = \frac{1}{5}$ or $x^2 = \frac{1}{4}$.

$x = \pm\frac{\sqrt{5}}{5}, \pm\frac{1}{2}$.

50. $\left(\frac{1}{x^2}\right)^2 - 9\left(\frac{1}{x^2}\right) + 8 = 0$

Let $w = \frac{1}{x^2}$. Then

$w^2 - 9w + 8 = 0$

$(w - 8)(w - 1) = 0$

$w = 8, 1$

Thus, $\frac{1}{x^2} = 8$ or $\frac{1}{x^2} = 1$, so $x^2 = \frac{1}{8}$ or $x^2 = 1$.

$x = \pm\frac{\sqrt{2}}{4}, \pm 1$.

51. $(X - 5)^2 + 7(X - 5) + 10 = 0$

Let $w = X - 5$. Then

$w^2 + 7w + 10 = 0$

$(w + 2)(w + 5) = 0$

$w = -2, -5$

Thus, $X - 5 = -2$ or $X - 5 = -5$, so $X = 3, 0$.

52. $(3x + 2)^2 - 5(3x + 2) = 0$

Let $w = 3x + 2$. Then

$w^2 - 5w = 0$

$w(w - 5) = 0$

$w = 0, 5$

Thus $3x + 2 = 0$ or $3x + 2 = 5$, so $x = -\frac{2}{3}, 1$.

53. $\left(\frac{1}{x-2}\right)^2 - 12\left(\frac{1}{x-2}\right) + 35 = 0$

Let $w = \frac{1}{x-2}$, then

$w^2 - 12w + 35 = 0$

$(w - 7)(w - 5) = 0$

$w = 7, 5$

Thus, $\frac{1}{x-2} = 7$ or $\frac{1}{x-2} = 5$.

$x = \frac{15}{7}, \frac{11}{5}$.

54. $2\left(\frac{1}{x+4}\right)^2 + 7\left(\frac{1}{x+4}\right) + 3 = 0$

Let $w = \frac{1}{x+4}$. Then

$2w^2 + 7w + 3 = 0$

$(2w + 1)(w + 3) = 0$

$w = -\frac{1}{2}, -3$

Thus, $\frac{1}{x+4} = -\frac{1}{2}$ or $\frac{1}{x+4} = -3$.

$x = -6, -\frac{13}{3}$

55. $x^2 = \frac{x+3}{2}$

$2x^2 = x + 3$

$2x^2 - x - 3 = 0$

$(2x - 3)(x + 1) = 0$

Thus, $x = \frac{3}{2}, -1$.

56. $\frac{x}{2} = \frac{7}{x} - \frac{5}{2}$

Multiplying both sides by the LCD, $2x$, gives

$x^2 = 14 - 5x$

$x^2 + 5x - 14 = 0$

$(x - 2)(x + 7) = 0$

Thus, $x = 2, -7$.

57. $\frac{3}{x-4}+\frac{x-3}{x}=2$

Multiplying both sides by the LCD, $x(x-4)$, gives

$3x+(x-3)(x-4)=2x(x-4)$

$3x+x^2-7x+12=2x^2-8x$

$x^2-4x+12=2x^2-8x$

$0=x^2-4x-12$

$0=(x-6)(x+2)$

Thus, $x=6, -2$.

58. $\frac{2}{2x+1}-\frac{6}{x-1}=5$

Multiplying both sides by the LCD, $(2x+1)(x-1)$, gives

$2(x-1)-6(2x+1)=5(2x+1)(x-1)$

$-10x-8=10x^2-5x-5$

$0=10x^2+5x+3$

$a=10$, $b=5$, $c=3$

$b^2-4ac=25-4(10)(3)=-95<0$, thus there are no real roots.

59. $\frac{3x+2}{x+1}-\frac{2x+1}{2x}=1$

Multiplying both sides by the LCD, $2x(x+1)$, gives

$2x(3x+2)-(2x+1)(x+1)=2x(x+1)$

$6x^2+4x-(2x^2+3x+1)=2x^2+2x$

$4x^2+x-1=2x^2+2x$

$2x^2-x-1=0$

$(2x+1)(x-1)=0$

Thus, $x=-\frac{1}{2}, 1$.

60. $\frac{6(w+1)}{2-w}+\frac{w}{w-1}=3$

Multiplying both sides by the LCD, $(2-w)(w-1)$, gives

$6(w+1)(w-1)+w(2-w)=3(2-w)(w-1)$

$6(w^2-1)+2w-w^2=3(-w^2+3w-2)$

$5w^2+2w-6=-3w^2+9w-6$

$8w^2-7w=0$

$w(8w-7)=0$

Thus, $w=0, \frac{7}{8}$.

61. $\frac{2}{r-2}-\frac{r+1}{r+4}=0$

Multiplying both sides by the LCD, $(r-2)(r+4)$, gives

$2(r+4)-(r-2)(r+1)=0$

$2r+8-(r^2-r-2)=0$

$-r^2+3r+10=0$

$r^2-3r-10=0$

$(r-5)(r+2)=0$

Thus, $r=5, -2$.

62. $\frac{2x-3}{2x+5}+\frac{2x}{3x+1}=1$

Multiplying both sides by the LCD, $(2x+5)(3x+1)$, gives

$(2x-3)(3x+1)+2x(2x+5)=(2x+5)(3x+1)$

$6x^2-7x-3+4x^2+10x=6x^2+17x+5$

$10x^2+3x-3=6x^2+17x+5$

$4x^2-14x-8=0$

$2x^2-7x-4=0$

$(2x+1)(x-4)=0$

Thus, $x=-\frac{1}{2}, 4$.

63. $\frac{t+1}{t+2}+\frac{t+3}{t+4}=\frac{t+5}{t^2+6t+8}$

Multiplying both sides by the LCD, $(t+2)(t+4)$, gives

$(t+1)(t+4)+(t+3)(t+2)=t+5$

$t^2+5t+4+t^2+5t+6=t+5$

$2t^2+10t+10=t+5$

$2t^2+9t+5=0$

$a=2$, $b=9$, $c=5$

$$t=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-9\pm\sqrt{81-4(2)(5)}}{2(2)}=\frac{-9\pm\sqrt{41}}{4}$$

Thus $t=\frac{-9+\sqrt{41}}{4}, \frac{-9-\sqrt{41}}{4}$.

64. $\frac{2}{x+1}+\frac{3}{x}=\frac{4}{x+2}$

Multiplying both sides by the LCD, $x(x+1)(x+2)$, gives

$$2x(x+2)+3(x+1)(x+2)=4x(x+1)$$
$$2x^2+4x+3x^2+9x+6=4x^2+4x$$
$$5x^2+13x+6=4x^2+4x$$
$$x^2+9x+6=0$$

$a = 1$, $b = 9$, $c = 6$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-9\pm\sqrt{9^2-4(1)(6)}}{2(1)}$$
$$=\frac{-9\pm\sqrt{57}}{2}$$

Thus, $x=\frac{-9+\sqrt{57}}{2}, \frac{-9-\sqrt{57}}{2}$.

65. $\frac{2}{x^2-1}-\frac{1}{x(x-1)}=\frac{2}{x^2}$

Multiplying both sides by the LCD, $x^2(x+1)(x-1)$, gives

$$2x^2-x(x+1)=2(x+1)(x-1)$$
$$2x^2-x^2-x=2x^2-2$$
$$x^2-x=2x^2-2$$
$$0=x^2+x-2$$
$$0=(x+2)(x-1)$$

$x = -2$ or $x = 1$

But $x = 1$ does not check. The solution is -2.

66. If $x \neq -3$, the equation is $5-\frac{3}{x}=\frac{1-x}{x}$.

Multiplying both sides by x gives

$$5x-3=1-x$$
$$6x=4$$
$$x=\frac{2}{3}$$

67. $\left(\sqrt{2x-3}\right)^2=(x-3)^2$

$$2x-3=x^2-6x+9$$
$$0=x^2-8x+12$$
$$0=(x-6)(x-2)$$

$x = 6$ or $x = 2$

Only $x = 6$ checks.

68. $\left(3\sqrt{x+4}\right)^2=(x-6)^2$

$$9x+36=x^2-12x+36$$
$$0=x^2-21x$$
$$0=x(x-21)$$

$x = 0$ or $x = 21$

Only $x = 21$ checks.

69. $(q+2)^2=\left(2\sqrt{4q-7}\right)^2$

$$q^2+4q+4=16q-28$$
$$q^2-12q+32=0$$
$$(q-4)(q-8)=0$$

Thus, $q = 4, 8$.

70. $\left(\sqrt{x}\right)^2+2\left(\sqrt{x}\right)-5=0$

Let $w=\sqrt{x}$, then $w^2+2w-5=0$

$a = 1$, $b = 2$, $c = -5$

$$w=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$w=\frac{-2\pm\sqrt{4-4(1)(-5)}}{2(1)}$$
$$=\frac{-2\pm\sqrt{24}}{2}$$
$$=\frac{-2\pm2\sqrt{6}}{2}$$
$$=-1\pm\sqrt{6}$$

Since $w=\sqrt{x}$ and $-1-\sqrt{6}<0$, $w=-1-\sqrt{6}$ does not check. Thus $w=-1+\sqrt{6}$, so

$$x=\left(-1+\sqrt{6}\right)^2=7-2\sqrt{6}.$$

71. $\sqrt{z+3}=\sqrt{3z}+1$

$$\left(\sqrt{z+3}\right)^2=\left(\sqrt{3z}+1\right)^2$$
$$z+3=3z+2\sqrt{3z}+1$$
$$-2z+2=2\sqrt{3z}$$
$$-z+1=\sqrt{3z}$$
$$(-z+1)^2=\left(\sqrt{3z}\right)^2$$
$$z^2-2z+1=3z$$
$$z^2-5z+1=0$$

$a = 1$, $b = -5$, $c = 1$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(1)}}{2(1)}$$
$$= \frac{5 \pm \sqrt{21}}{2}$$

Only $z = \frac{5 - \sqrt{21}}{2}$ checks.

72. $\sqrt{x} - 2 = \sqrt{2x - 8}$

$\left(\sqrt{x} - 2\right)^2 = \left(\sqrt{2x - 8}\right)^2$

$x - 4\sqrt{x} + 4 = 2x - 8$

$-4\sqrt{x} = x - 12$

$\left(-4\sqrt{x}\right)^2 = (x - 12)^2$

$16x = x^2 - 24x + 144$

$0 = x^2 - 40x + 144$

$0 = (x - 4)(x - 36)$

$x = 4$ or $x = 36$

Only $x = 4$ checks.

73. $\sqrt{x} + 1 = \sqrt{2x + 1}$

$\left(\sqrt{x} + 1\right)^2 = \left(\sqrt{2x + 1}\right)^2$

$x + 2\sqrt{x} + 1 = 2x + 1$

$2\sqrt{x} = x$

$\left(2\sqrt{x}\right)^2 = x^2$

$4x = x^2$

$0 = x^2 - 4x$

$0 = x(x - 4)$

Thus, $x = 0, 4$.

74. $\left(\sqrt{y - 2} + 2\right)^2 = \left(\sqrt{2y + 3}\right)^2$

$y - 2 + 4\sqrt{y - 2} + 4 = 2y + 3$

$4\sqrt{y - 2} = y + 1$

$\left(4\sqrt{y - 2}\right)^2 = (y + 1)^2$

$16y - 32 = y^2 + 2y + 1$

$0 = y^2 - 14y + 33$

$0 = (y - 11)(y - 3)$

Thus, $y = 11, 3$.

75. $\left(\sqrt{x + 3} + 1\right)^2 = \left(3\sqrt{x}\right)^2$

$x + 3 + 2\sqrt{x + 3} + 1 = 9x$

$2\sqrt{x + 3} = 8x - 4$

$\sqrt{x + 3} = 4x - 2$

$\left(\sqrt{x + 3}\right)^2 = (4x - 2)^2$

$x + 3 = 16x^2 - 16x + 4$

$0 = 16x^2 - 17x + 1$

$0 = (16x - 1)(x - 1)$

$x = \frac{1}{16}$ or $x = 1$

Only $x = 1$ checks.

76. $\left(\sqrt{\sqrt{t} + 2}\right)^2 = \left(\sqrt{3t + 1}\right)^2$

$\sqrt{t} + 2 = 3t + 1$

$\sqrt{t} = 3t - 1$

$\left(\sqrt{t}\right)^2 = (3t - 1)^2$

$t = 9t^2 - 6t + 1$

$0 = 9t^2 - 7t + 1$

$a = 9$, $b = -7$, $c = 1$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(9)(1)}}{2(9)}$$
$$= \frac{7 \pm \sqrt{13}}{18}$$

Only $\frac{7 + \sqrt{13}}{18}$ checks.

77. $x = \frac{-(-2.7) \pm \sqrt{(-2.7)^2 - 4(0.04)(8.6)}}{2(0.04)}$

≈ 64.15 or 3.35

78. $x = \frac{-0.2 \pm \sqrt{(0.2)^2 - 4(0.01)(-0.6)}}{2(0.01)}$

≈ 2.65 or -22.65

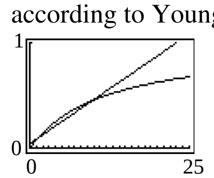
79. Let l be the length of the picture, then its width is $l - 2$.
$l(l - 2) = 48$
$l^2 - 2l - 48 = 0$
$(l - 8)(l + 6) = 0$
$l - 8 = 0$ or $l + 6 = 0$
$l = 8$ or $l = -6$
Since length cannot be negative, $l = 8$. The width of the picture is $l - 2 = 8 - 2 = 6$ inches.
The dimensions of the picture are 6 inches by 8 inches.

80. The amount that the temperature has risen over the X days is
(X degrees per day)(X days) = X^2 degrees.
$X^2 + 15 = 51$
$X^2 = 36$
$X = \pm\sqrt{36}$
$X = 6$ or $X = -6$
The temperature has been rising 6 degrees per day for 6 days.

81. $\overline{M} = \frac{Q(Q+10)}{44}$
$44\overline{M} = Q^2 + 10Q$
$0 = Q^2 + 10Q - 44\overline{M}$
From the quadratic formula with $a = 1$, $b = 10$, $c = -44\overline{M}$,
$Q = \frac{-10 \pm \sqrt{100 - 4(1)(-44\overline{M})}}{2(1)}$
$= \frac{-10 + 2\sqrt{25 + 44\overline{M}}}{2}$
$= -5 \pm \sqrt{25 + 44\overline{M}}$
Thus, $-5 + \sqrt{25 + 44\overline{M}}$ is a root.

82. $g = -200P^2 + 200P + 20$
Set $g = 60$.
$60 = -200P^2 + 200P + 20$
$200P^2 - 200P + 40 = 0$
$5P^2 - 5P + 1 = 0$
From the quadratic formula with $a = 5$, $b = -5$, $c = 1$,
$P = \frac{5 \pm \sqrt{25 - 4(5)(1)}}{2(5)} = \frac{5 \pm \sqrt{5}}{10}$
$P \approx 0.28$ or $P \approx 0.72$
28% and 72% of yeast gave an average weight gain of 60 grams.

83. $\frac{A}{A+12}d = \frac{A+1}{24}d.$
Dividing both sides by d and then multiplying both sides by $24(A + 12)$ gives
$24A = (A + 12)(A + 1)$
$24A = A^2 + 13A + 12$
$0 = A^2 - 11A + 12$
From the quadratic formula,
$A = \frac{11 \pm \sqrt{121 - 48}}{2} = \frac{11 \pm \sqrt{73}}{2}.$
$A = \frac{11 + \sqrt{73}}{2} \approx 10$ or $A = \frac{11 - \sqrt{73}}{2} \approx 1.$
The doses are the same at 1 year and 10 years.
$c = d$ in Cowling's rule when $\frac{A+1}{24} = 1,$ which occurs when $A = 23$. Thus, adulthood is achieved at age 23 according to Cowling's rule.
$c = d$ in Young's rule when $\frac{A}{A+12} = 1,$ which is never true. Thus, adulthood is never reached according to Young's rule.



Young's rule prescribes less than Cowling's for ages less than one year and greater than 10 years. Cowling's rule prescribes less for ages between 1 and 10.

84. a. $(2n-1)v^2 - 2nv + 1 = 0$
From the quadratic formula with $a = 2n - 1$, $b = -2n$, $c = 1$,
$v = \frac{-(-2n) \pm \sqrt{4n^2 - 4(2n-1)(1)}}{2(2n-1)}$
$v = \frac{2n \pm \sqrt{4n^2 - 8n + 4}}{2(2n-1)}$
$v = \frac{2n \pm 2\sqrt{n^2 - 2n + 1}}{2(2n-1)} = \frac{n \pm \sqrt{(n-1)^2}}{2n-1}$
Because of the condition that $n \geq 1$, it follows that $n - 1$ is nonnegative. Thus, $\sqrt{(n-1)^2} = n - 1$ and we have
$v = \frac{n \pm (n-1)}{2n-1}.$
$v = 1$ or $v = \frac{1}{2n-1}.$

b. $nv^2 - (2n+1)v + 1 = 0$

From the quadratic formula with $a = n$, $b = -(2n + 1)$, and $c = 1$,

$$v = \frac{-[-(2n+1)] \pm \sqrt{[-(2n+1)]^2 - 4(n)(1)}}{2n}$$

$$v = \frac{2n+1 \pm \sqrt{4n^2+1}}{2n}$$

Because $\sqrt{4n^2+1}$ is greater than $2n$, choosing the plus sign gives a numerator greater than $2n + 1 + 2n$, or $4n + 1$, so v is greater than $\frac{4n+1}{2n} = 2 + \frac{1}{2n}$. Thus v is greater than 2. This contradicts the restriction on v. On the other hand, because $\sqrt{4n^2+1}$ is greater than 1, choosing the minus sign gives a numerator less than $2n$, so v is less than $\frac{2n}{2n} = 1$. This meets the condition on v. Thus we choose

$$v = \frac{2n+1-\sqrt{4n^2+1}}{2n}.$$

85. a. When the object strikes the ground, h must be 0, so

$$0 = 39.2t - 4.9t^2 = 4.9t(8-t)$$

$t = 0$ or $t = 8$

The object will strike the ground 8 s after being thrown.

b. Setting $h = 68.2$ gives

$$68.2 = 39.2t - 4.9t^2$$

$$4.9t^2 - 39.2t + 68.2 = 0$$

$$t = \frac{39.2 \pm \sqrt{(-39.2)^2 - 4(4.9)(68.2)}}{2(4.9)}$$

$$\approx \frac{39.2 \pm 14.1}{9.8}$$

$t \approx 5.4$ s or $t \approx 2.6$ s.

86. By a program, roots are 4.5 and –3.

Algebraically:

$$2x^2 - 3x - 27 = 0$$

$$(2x - 9)(x + 3) = 0$$

Thus, $2x - 9 = 0$ or $x + 3 = 0$

So $x = \frac{9}{2} = 4.5$ or $x = -3$.

87. By a program, roots are 1.5 and 0.75.

Algebraically:

$$8x^2 - 18x + 9 = 0$$

$$(2x - 3)(4x - 3) = 0$$

Thus, $2x - 3 = 0$ or $4x - 3 = 0$.

So $x = \frac{3}{2} = 1.5$ or $x = \frac{3}{4} = 0.75$.

88. By a program, roots are –0.762 and 0.262.

89. By a program, there are no real roots.

90. $\frac{9}{2}z^2 - 6.3 = \frac{z}{3}(1.1 - 7z)$

$$\frac{9}{2}z^2 - 6.3 = \frac{1.1}{3}z - \frac{7}{3}z^2$$

$$\left(\frac{9}{2} + \frac{7}{3}\right)z^2 - \frac{1.1}{3}z - 6.3 = 0$$

Roots: 0.987, –0.934

91. $(\pi t - 4)^2 = 4.1t - 3$

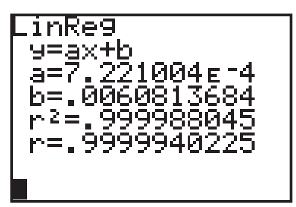
$$\pi^2 t^2 - 8\pi t + 16 = 4.1t - 3$$

$$\pi^2 t^2 + (-8\pi - 4.1)t + 19 = 0$$

Roots: 1.999, 0.963

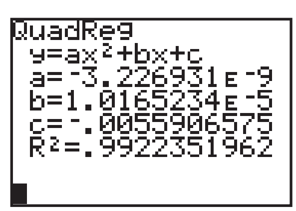
Mathematical Snapshot Chapter 0

1.



2. The procedure works because multiplying a list by a number is the same as multiplying each element in the list by the number, adding a number to a list has the effect of adding the number to each element of the list, and subtracting one list from another is the same as subtracting corresponding elements. The plots match.

3.



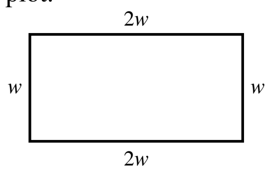
The results agree.

4. The smaller quadratic residuals indicate a better fit. The fairly random pattern suggests that the model cannot be improved any further. The slight deviations from the quadratic model are presumably due to random measurement errors.

Chapter 1

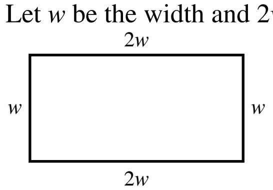
Problems 1.1

1. Let w be the width and $2w$ be the length of the plot.



Then area = 800.
$(2w)w = 800$
$2w^2 = 800$
$w^2 = 400$
$w = 20$ ft
Thus the length is 40 ft, so the amount of fencing needed is $2(40) + 2(20) = 120$ ft.

2. Let w be the width and $2w$ be the length.



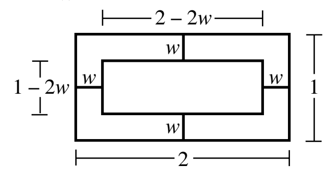
The perimeter $P = 2w + 2l = 2w + 2(2w) = 6w$.
Thus $6w = 300$.
$w = \frac{300}{6} = 50$ ft
Thus the length is $2(50) = 100$ ft.
The dimensions are 50 ft by 100 ft.

3. Let n = number of ounces in each part. Then we have
$4n + 5n = 145$
$9n = 145$
$n = 16\frac{1}{9}$
Thus there should be $4\left(16\frac{1}{9}\right) = 64\frac{4}{9}$ ounces of A and $5\left(16\frac{1}{9}\right) = 80\frac{5}{9}$ ounces of B.

4. Let n = number of cubic feet in each part. Then we have
$1n + 3n + 5n = 765$
$9n = 765$
$n = 85$
Thus he needs $1n = 1(85) = 85$ ft^3 of portland cement, $3n = 3(85) = 255$ ft^3 of sand, and $5n = 5(85) = 425$ ft^3 of crushed stone.

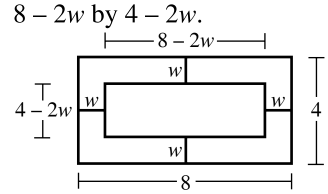
5. Let n = number of ounces in each part. Then we have
$2n + 1n = 16$
$3n = 16$
$n = \frac{16}{3}$
Thus the turpentine needed is
$(1)n = \frac{16}{3} = 5\frac{1}{3}$ ounces.

6. Let w = width (in miles) of strip to be cut. Then the remaining forest has dimensions $2 - 2w$ by $1 - 2w$.



Considering the area of the remaining forest, we have
$(2-2w)(1-2w) = \frac{3}{4}$
$2 - 6w + 4w^2 = \frac{3}{4}$
$8 - 24w + 16w^2 = 3$
$16w^2 - 24w + 5 = 0$
$(4w - 1)(4w - 5) = 0$
Hence $w = \frac{1}{4}, \frac{5}{4}$. But $w = \frac{5}{4}$ is impossible since one dimension of original forest is 1 mi. Thus the width of the strip should be $\frac{1}{4}$ mi.

7. Let w = width (in meters) of pavement. The remaining plot for flowers has dimensions $8 - 2w$ by $4 - 2w$.



Thus
$(8 - 2w)(4 - 2w) = 12$
$32 - 24w + 4w^2 = 12$
$4w^2 - 24w + 20 = 0$
$w^2 - 6w + 5 = 0$
$(w - 1)(w - 5) = 0$

Hence $w = 1, 5$. But $w = 5$ is impossible since one dimension of the original plot is 4 m. Thus the width of the pavement should be 1 m.

8. Since diameter of circular end is 140 mm, the radius is 70 mm. Area of circular end is $\pi(\text{radius})^2 = \pi(70)^2$. Area of square end is x^2.
Equating areas, we have $x^2 = \pi(70)^2$.
Thus $x = \pm\sqrt{\pi(70)^2} = \pm 70\sqrt{\pi}$. Since x must be positive, $x = 70\sqrt{\pi} \approx 124$ mm.

9. Let q = number of tons for \$560,000 profit.
Profit = Total Revenue – Total Cost
$560,000 = 134q - (82q + 120,000)$
$560,000 = 52q - 120,000$
$680,000 = 52q$
$\frac{680,000}{52} = q$
$q \approx 13,076.9 \approx 13,077$ tons.

10. Let q = required number of units.
Profit = Total Revenue – Total Cost
$150,000 = 50q - (25q + 500,000)$
$150,000 = 25q - 500,000$
$650,000 = 25q$, from which
$q = 26,000$

11. Let x = amount at 6% and
$20,000 - x$ = amount at $7\frac{1}{2}\%$.
$x(0.06) + (20,000 - x)(0.075) = 1440$
$-0.015x + 1500 = 1440$
$-0.015x = -60$
$x = 4000$, so $20,000 - x = 16,000$. Thus the investment should be \$4000 at 6% and \$16,000 at $7\frac{1}{2}\%$.

12. Let x = amount at 6% and
$20,000 - x$ = amount at 7%.
$x(0.06) + (20,000 - x)(0.07) = 20,000(0.0675)$
$-0.01x + 1400 = 1350$
$-0.01x = -50$, so $x = 5000$
The investment consisted of \$5000 at 6% and \$15,000 at 7%.

13. Let p = selling price. Then profit = $0.2p$.
selling price = cost + profit
$p = 3.40 + 0.2p$
$0.8p = 3.40$
$p = \frac{3.40}{0.8} = \$4.25$

14. Following the procedure in Example 6 we obtain the total value at the end of the second year to be $1,000,000(1+r)^2$.
So at the end of the third year, the accumulated amount will be $1,000,000(1+r)^2$ plus the interest on this, which is $1,000,000(1+r)^2 r$.
Thus the total value at the end of the third year will be $1,000,000(1+r)^2 + 1,000,000(1+r)^2 r$
$= 1,000,000(1+r)^3$.
This must equal \$1,125,800.
$1,000,000(1+r)^3 = 1,125,800$
$(1+r)^3 = \frac{1,125,800}{1,000,000} = 1.1258$
$1 + r \approx 1.04029$
$r \approx 0.04029$
Thus $r \approx 0.04029 \approx 4\%$.

15. Following the procedure in Example 6 we obtain
$3,000,000(1+r)^2 = 3,245,000$
$(1+r)^2 = \frac{649}{600}$
$1 + r = \pm\sqrt{\frac{649}{600}}$
$r = -1 \pm \sqrt{\frac{649}{600}}$
$r \approx -2.04$ or 0.04
We choose $r \approx 0.04 = 4\%$.

16. Total revenue = variable cost + fixed cost
$100\sqrt{q} = 2q + 1200$
$50\sqrt{q} = q + 600$
$2500q = q^2 + 1200q + 360,000$
$0 = q^2 - 1300q + 360,000$
$0 = (q - 400)(q - 900)$
$q = 400$ or $q = 900$

17. Let n = number of room applications sent out.
$0.95n = 76$
$$n = \frac{76}{0.95} = 80$$

18. Let n = number of people polled.
$0.20p = 700$
$$p = \frac{700}{0.20} = 3500$$

19. Let s = monthly salary of deputy sheriff.
$0.30s = 200$
$$s = \frac{200}{0.30}$$
$$\text{Yearly salary} = 12s = 12\left(\frac{200}{0.30}\right) = \$8000$$

20. Yearly salary before strike = (7.50)(8)(260) = \$15,600
Lost wages = (7.50)(8)(46) = \$2760
Let P be the required percentage increase (as a decimal).
$P(15{,}600) = 2760$
$$P = \frac{2760}{15{,}600} \approx 0.177 = 17.7\%$$

21. Let q = number of cartridges sold to break even.
total revenue = total cost
$$21.95q = 14.92q + 8500$$
$$7.03q = 8500$$
$$q \approx 1209.10$$
1209 cartridges must be sold to approximately break even.

22. Let n = number of shares of stock to be bought.
total investment = $4000 + 15n$
total yield (goal) = 6% of total investment $= 0.06(4000 + 15n)$
total yield = bond yield + stock yield $= 0.07(4000) + 0.60n$
Thus,
$$0.06(4000 + 15n) = 0.07(4000) + 0.60n$$
$$240 + 0.9n = 280 + 0.6n$$
$$0.3n = 40$$
$$n = 133\frac{1}{3}$$

23. Let v = total annual vision-care expenses (in dollars) covered by program. Then
$35 + 0.80(v - 35) = 100$
$0.80v + 7 = 100$
$0.80v = 93$
$v = \$116.25$

24. **a.** $0.031c$

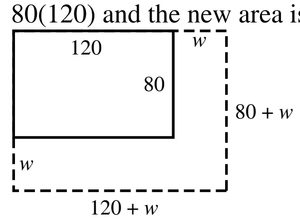
b. $c - 0.031c = 600{,}000{,}000$
$0.969c = 600{,}000{,}000$
$c \approx 619{,}195{,}046$
Approximately 619,195,046 bars will have to be made.

25. Revenue = (number of units sold)(price per unit)
Thus
$$400 = q\left[\frac{80 - q}{4}\right]$$
$$1600 = 80q - q^2$$
$$q^2 - 80q + 1600 = 0$$
$$(q - 40)^2 = 0$$
$q = 40$ units

26. If I = interest, P = principal, r = rate, and t = time, then $I = Prt$. To triple an investment of P at the end of t years, the interest earned during that time must equal $2P$. Thus
$2P = P(0.045)t$
$2 = 0.045t$
$$t = \frac{2}{0.045} \approx 44.4 \text{ years}$$

27. Let q = required number of units. We equate incomes under both proposals.
$2000 + 0.50q = 25{,}000$
$0.50q = 23{,}000$
$q = 46{,}000$ units

28. Let w = width of strip. The original area is 80(120) and the new area is $(120 + w)(80 + w)$.



Thus
$(120 + w)(80 + w) = 2(80)(120)$
$9600 + 200w + w^2 = 19,200$
$w^2 + 200w - 9600 = 0$
$(w + 240)(w - 40) = 0$
$w = -240$ or $w = 40$
We choose $w = 40$ ft.

29. Let n = number of \$20 increases. Then at the rental charge of $400 + 20n$ dollars per unit, the number of units that can be rented is $50 - 2n$. The total of all monthly rents is $(400 + 20n)(50 - 2n)$, which must equal 20,240.
$20,240 = (400 + 20n)(50 - 2n)$
$20,240 = 20,000 + 200n - 40n^2$
$40n^2 - 200n + 240 = 0$
$n^2 - 5n + 6 = 0$
$(n - 2)(n - 3) = 0$
$n = 2, 3$
Thus the rent should be either
\$400 + 2(\$20) = \$440 or \$400 + 3(\$20) = \$460.

30. Let x = original value of the blue-chip investment, then $3,100,000 - x$ is the original value of the glamour stocks. Then the current value of the blue-chip stock is $x + \frac{1}{10}x$, or $\frac{11}{10}x$.
For the glamour stocks the current value is
$(3,100,000 - x) - \frac{1}{10}(3,100,000 - x)$, which simplifies to $\frac{9}{10}(3,100,000 - x)$.
Thus for the current value of the portfolio,
$\frac{11}{10}x + \frac{9}{10}(3,100,000 - x) = 3,240,000$
$11x + 27,900,000 - 9x = 32,400,000$
$2x = 4,500,000$
$x = 2,250,000$
Thus the current value of the blue chip investment is $\frac{11}{10}(2,250,000)$ or \$2,475,000.

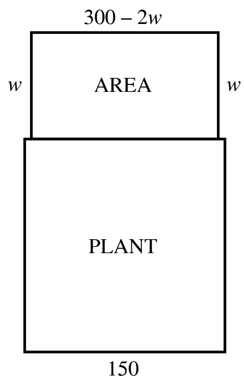
31. $10,000 = 800p - 7p^2$
$7p^2 - 800p + 10,000 = 0$
$p = \frac{800 \pm \sqrt{640,000 - 280,000}}{14}$
$= \frac{800 \pm \sqrt{360,000}}{14} = \frac{800 \pm 600}{14}$
For $p > 50$ we choose $p = \frac{800 + 600}{14} = \$100$.

32. Let p be the percentage increase in market value. Then
$1.1\left(\frac{P}{E}\right) = \frac{(1+p)P}{(1.2)E}$
$1.1 = \frac{1+p}{1.2}$
$1.32 = 1 + p$
$p = 0.32 = 32\%$

33. To have supply = demand,
$2p - 10 = 200 - 3p$
$5p = 210$
$p = 42$

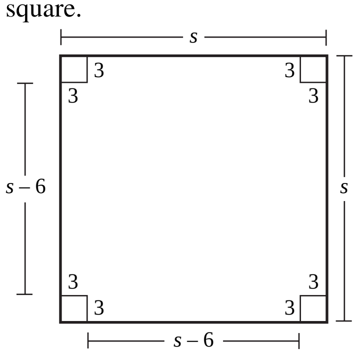
34. $2p^2 - 3p = 20 - p^2$
$3p^2 - 3p - 20 = 0$
$a = 3$, $b = -3$, $c = -20$
$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(3)(-20)}}{2(3)}$
$= \frac{3 \pm \sqrt{249}}{6}$
$p \approx 3.130$ or $p \approx -2.130$
The equilibrium price is $p \approx 3.13$.

35. Let w = width (in ft) of enclosed area. Then length of enclosed area is
$300 - w - w = 300 - 2w$.



Thus
$w(300 - 2w) = 11{,}200$
$2w(150 - w) = 11{,}200$
$w(150 - w) = 5600$
$0 = w^2 - 150w + 5600$
$0 = (w - 80)(w - 70)$
Hence $w = 80, 70$. If $w = 70$, then length is $300 - 2w = 300 - 2(70) = 160$. Since the building has length of only 150 ft, we reject $w =$ 70. If
$w = 80$, then length is $300 - 2w = 300 - 2(80) = 140$. Thus the dimensions are 80 ft by 140 ft.

36. Let s = length in inches of side of original square.



Considering the volume of the box, we have (length)(width)(height) = volume
$(s - 4)(s - 4)(2) = 50$
$(s-4)^2 = 25$
$s - 4 = \pm\sqrt{25} = \pm 5$
$s = 4 \pm 5$
Hence $s = -1, 9$. We reject $s = -1$ and choose $s = 9$. The dimensions are 9 in. by 9 in.

37. Original volume = (10)(5)(2) = 100 cm^3
Volume cut from bar = 0.28(100) = 28 cm^3
Volume of new bar = 100 – 28 = 72 cm^3
Let x = number of centimeters that the length and width are each reduced. Then
$(10 - x)(5 - x)2 = 72$
$(10 - x)(5 - x) = 36$
$x^2 - 15x + 50 = 36$
$x^2 - 15x + 14 = 0$
$(x - 1)(x - 14) = 0$
$x = 1$ or 14
Because of the length and width of the original bar, we reject $x = 14$ and choose $x = 1$. The new bar has length $10 - x = 10 - 1 = 9$ cm and width is $5 - x = 5 - 1 = 4$ cm.

38. Volume of old style candy
$= \pi(7.1)^2(2.1) - \pi(2)^2(2.1)$
$= 97.461\pi \text{ mm}^3$
Let r = inner radius (in millimeters) of new style candy. Considering the volume of the new style candy, we have
$$\pi(7.1)^2(2.1) - \pi r^2(2.1) = 0.78(97.461\pi)$$
$$29.84142\pi = 2.1\pi r^2$$
$$14.2102 = r^2$$
$$r \approx \pm 3.7696$$
Since r is a radius, we choose $r = 3.77$ mm.

39. Let x = amount of loan. Then the amount actually received is $x - 0.16x$. Hence,
$$x - 0.16x = 195{,}000$$
$$0.84x = 195{,}000$$
$$x \approx 232{,}142.86$$
To the nearest thousand, the loan amount is \$232,000. In the general case, the amount received from a loan of L with a compensating balance of p% is $L - \frac{p}{100}L$.
$$L - \frac{p}{100}L = E$$
$$\frac{100 - p}{100}L = E$$
$$L = \frac{100E}{100 - p}$$

40. Let n = number of machines sold over 600. Then the commission on each of $600 + n$ machines is $40 + 0.04n$. Equating total commissions to 30,800 we obtain
$(600 + n)(40 + 0.04n) = 30{,}800$
$24{,}000 + 24n + 40n + 0.04n^2 = 30{,}800$
$0.02n^2 + 32n - 3400 = 0$
$$n = \frac{-32 \pm \sqrt{1024 + 272}}{0.04} = \frac{-32 \pm 36}{0.04}$$

We choose $n = \frac{-32+36}{0.04} = 100.$ Thus the number of machines that must be sold is $600 + 100 = 700$.

41. Let n = number of acres sold. Then $n + 20$ acres were originally purchased at a cost of $\frac{7200}{n+20}$ each. The price of each acre sold was $30 + \left[\frac{7200}{n+20}\right]$. Since the revenue from selling n acres is \$7200 (the original cost of the parcel), we have

$$n\left[30 + \frac{7200}{n+20}\right] = 7200$$

$$n\left[\frac{30n + 600 + 7200}{n+20}\right] = 7200$$

$$n(30n + 600 + 7200) = 7200(n + 20)$$

$$30n^2 + 7800n = 7200n + 144,000$$

$$30n^2 + 600n - 144,000 = 0$$

$$n^2 + 20n - 4800 = 0$$

$$(n + 80)(n - 60) = 0$$

$n = 60$ acres (since $n > 0$), so 60 acres were sold.

42. Let q = number of units of product sold last year and $q + 2000$ = the number sold this year. Then the revenue last year was $3q$ and this year it is $3.5(q + 2000)$. By the definition of margin of profit, it follows that

$$\frac{7140}{3.5(q+2000)} = \frac{4500}{3q} + 0.02$$

$$\frac{2040}{q+2000} = \frac{1500}{q} + 0.02$$

$$2040q = 1500(q + 2000) + 0.02q(q + 2000)$$

$$2040q = 1500q + 3,000,000 + 0.02q^2 + 40q$$

$$0 = 0.02q^2 - 500q + 3,000,000$$

$$q = \frac{500 \pm \sqrt{250,000 - 240,000}}{0.04}$$

$$= \frac{500 \pm \sqrt{10,000}}{0.04}$$

$$= \frac{500 \pm 100}{0.04}$$

$= 10,000$ or $15,000$

So that the margin of profit this year is not greater than 0.15, we choose $q = 15,000$. Thus 15,000 units were sold last year and 17,000 this year.

43. Let q = number of units of B and $q + 25$ = number of units of A produced.

Each unit of B costs $\frac{1000}{q}$, and each unit of A costs $\frac{1500}{q+25}$. Therefore,

$$\frac{1500}{q+25} = \frac{1000}{q} + 2$$

$$1500q = 1000(q + 25) + 2(q)(q + 25)$$

$$0 = 2q^2 - 450q + 25,000$$

$$0 = q^2 - 225q + 12,500$$

$$0 = (q - 100)(q - 125)$$

$q = 100$ or $q = 125$

If $q = 100$, then $q + 25 = 125$; if $q = 125$, $q + 25 = 150$. Thus the company produces either 125 units of A and 100 units of B, or 150 units of A and 125 units of B.

Principles in Practice 1.2

1. $200 + 0.8S \geq 4500$

$0.8S \geq 4300$

$S \geq 5375$

He must sell at least 5375 products per month.

2. Since $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,$ and $x_4 \geq 0,$ we have the inequalities

$$150 - x_4 \geq 0$$

$$3x_4 - 210 \geq 0$$

$$x_4 + 60 \geq 0$$

$$x_4 \geq 0$$

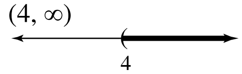
Problems 1.2

1. $3x > 12$

$x > \frac{12}{3}$

$x > 4$

$(4, \infty)$

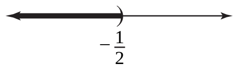


2. $4x < -2$

$x < \frac{-2}{4}$

$x < -\frac{1}{2}$

$\left(-\infty, -\frac{1}{2}\right)$

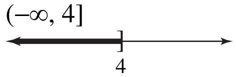


3. $5x - 11 \le 9$

$5x \le 20$

$x \le 4$

$(-\infty, 4]$

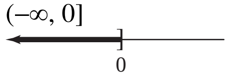


4. $5x \le 0$

$x \le \frac{0}{5}$

$x \le 0$

$(-\infty, 0]$

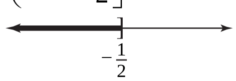


5. $-4x \ge 2$

$x \le \frac{2}{-4}$

$x \le -\frac{1}{2}$

$\left(-\infty, -\frac{1}{2}\right]$

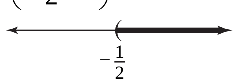


6. $2y + 1 > 0$

$2y > -1$

$y > -\frac{1}{2}$

$\left(-\frac{1}{2}, \infty\right)$

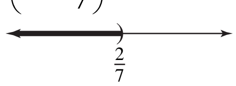


7. $5 - 7s > 3$

$-7s > -2$

$s < \frac{2}{7}$

$\left(-\infty, \frac{2}{7}\right)$

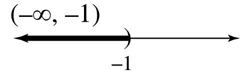


8. $4s - 1 < -5$

$4s < -4$

$s < -1$

$(-\infty, -1)$



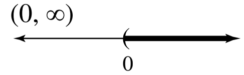
9. $3 < 2y + 3$

$0 < 2y$

$0 < y$

$y > 0$

$(0, \infty)$



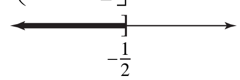
10. $4 \le 3 - 2y$

$1 \le -2y$

$-\frac{1}{2} \ge y$

$y \le -\frac{1}{2}$

$\left(-\infty, -\frac{1}{2}\right]$

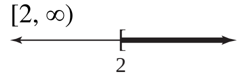


11. $x + 5 \le 3 + 2x$

$-x \le -2$

$x \ge 2$

$[2, \infty)$



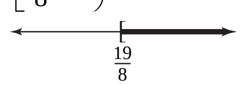
12. $-3 \ge 8(2 - x)$

$-3 \ge 16 - 8x$

$8x \ge 19$

$x \ge \frac{19}{8}$

$\left[\frac{19}{8}, \infty\right)$



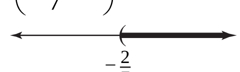
13. $3(2 - 3x) > 4(1 - 4x)$

$6 - 9x > 4 - 16x$

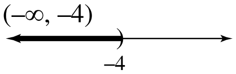
$7x > -2$

$x > -\frac{2}{7}$

$\left(-\frac{2}{7}, \infty\right)$



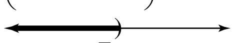
14. $8(x+1)+1<3(2x)+1$
$8x+9<6x+1$
$2x<-8$
$x<-4$
$(-\infty, -4)$



15. $2(4x-2)>4(2x+1)$
$8x-4>8x+4$
$-4>4,$ which is false for all x.
Thus the solution set is $\varnothing$.

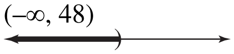
16. $4-(x+3)\le 3(3-x)$
$1-x\le 9-3x$
$2x\le 8$
$x\le 4$
$(-\infty, 4]$

17. $x+2<\sqrt{3}-x$
$2x<\sqrt{3}-2$
$x<\frac{\sqrt{3}-2}{2}$
$\left(-\infty, \frac{\sqrt{3}-2}{2}\right)$

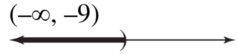


18. $\sqrt{2}(x+2)>\sqrt{8}(3-x)$
$\sqrt{2}(x+2)>2\sqrt{2}(3-x)$
$x+2>2(3-x)$
$x+2>6-2x$
$3x>4$
$x>\frac{4}{3}$
$\left(\frac{4}{3}, \infty\right)$

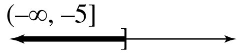
19. $\frac{5}{6}x<40$
$5x<240$
$x<48$
$(-\infty, 48)$



20. $-\frac{2}{3}x>6$
$-x>9$
$x<-9$
$(-\infty, -9)$



21. $\frac{9y+1}{4}\le 2y-1$
$9y+1\le 8y-4$
$y\le -5$
$(-\infty, -5]$



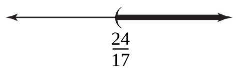
22. $\frac{3y-2}{3}\ge\frac{1}{4}$
$12y-8\ge 3$
$12y\ge 11$
$y\ge\frac{11}{12}$
$\left[\frac{11}{12}, \infty\right)$

23. $-3x+1\le -3(x-2)+1$
$-3x+1\le -3x+7$
$1\le 7$, which is true for all x. The solution is $-\infty<x<\infty$.
$(-\infty, \infty)$

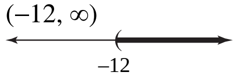
24. $0x\le 0$
$0\le 0$, which is true for all x. The solution is $-\infty<x<\infty$.
$(-\infty, \infty)$

25. $\frac{1-t}{2}<\frac{3t-7}{3}$
$3(1-t)<2(3t-7)$
$3-3t<6t-14$
$-9t<-17$
$t>\frac{17}{9}$
$\left(\frac{17}{9}, \infty\right)$

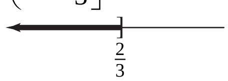
26. $\frac{3(2t-2)}{2} > \frac{6t-3}{5} + \frac{t}{10}$
$15(2t - 2) > 2(6t - 3) + t$
$30t - 30 > 13t - 6$
$17t > 24$
$t > \frac{24}{7}$
$\left(\frac{24}{7}, \infty\right)$



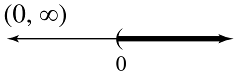
27. $2x + 13 \geq \frac{1}{3}x - 7$
$6x + 39 \geq x - 21$
$5x \geq -60$
$x \geq -12$
$(-12, \infty)$



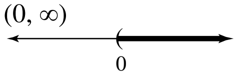
28. $3x - \frac{1}{3} \leq \frac{5}{2}x$
$18x - 2 \leq 15x$
$3x \leq 2$
$x \leq \frac{2}{3}$
$\left(-\infty, \frac{2}{3}\right]$



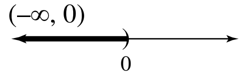
29. $\frac{2}{3}r < \frac{5}{6}r$
$4r < 5r$
$0 < r$
$r > 0$
$(0, \infty)$



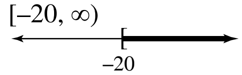
30. $\frac{7}{4}t > -\frac{8}{3}t$
$21t > -32t$
$53t > 0$
$t > 0$
$(0, \infty)$



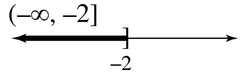
31. $\frac{y}{2} + \frac{y}{3} > y + \frac{y}{5}$
$15y + 10y > 30y + 6y$
$25y > 36y$
$0 > 11y$
$0 > y$
$y < 0$
$(-\infty, 0)$



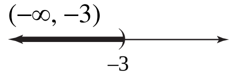
32. $9 - 0.1x \leq \frac{2 - 0.01x}{0.2}$
$1.8 - 0.02x \leq 2 - 0.01x$
$-0.01x \leq 0.2$
$x \geq -20$
$[-20, \infty)$



33. $0.1(0.03x + 4) \geq 0.02x + 0.434$
$0.003x + 0.4 \geq 0.02x + 0.434$
$-0.017x \geq 0.034$
$x \leq -2$
$(-\infty, -2]$



34. $\frac{3y-1}{-3} < \frac{5(y+1)}{-3}$
$3y - 1 > 5y + 5$
$-6 > 2y$
$-3 > y$
$y < -3$
$(-\infty, -3)$



35. $12(50) < S < 12(150)$
$600 < S < 1800$

36. $2\frac{1}{2} \leq x \leq 4$

37. The measures of the acute angles of a right triangle sum to 90°. If x is the measure of one acute angle, the other angle has measure $90 - x$.
$x < 3(90 - x) + 10$
$x < 270 - 3x + 10$
$4x < 280$
$x < 70$
The measure of the angle is less than 70°.

38. Let d be the number of disks. The stereo plus d disks will cost $219 + 18.95d$.
$219 + 18.95d \le 360$
$18.95d \le 141$
$d \le \frac{141}{18.95} \approx 7.44$
The student can buy at most 7 disks.

Problems 1.3

1. Let q = number of units sold.
Profit > 0
Total revenue – Total cost > 0
$20q - (15q + 600{,}000) > 0$
$5q - 600{,}000 > 0$
$5q > 600{,}000$
$q > 120{,}000$
Thus at least 120,001 units must be sold.

2. Let q = number of units sold.
Total revenue – Total cost = Profit
We want Profit > 0.
$7.40q - [(2.50 + 4)q + 5000] > 0$
$0.9q - 5000 > 0$
$0.9q > 5000$
$q > \frac{5000}{0.9} = 5555\frac{5}{9}$
Thus at least 5556 units must be sold.

3. Let x = number of miles driven per year.
If the auto is leased, the annual cost is $12(420) + 0.06x$.
If the auto is purchased, the annual cost is $4700 + 0.08x$. We want Rental cost $\le$ Purchase cost.
$12(420) + 0.06x \le 4700 + 0.08x$
$5040 + 0.06x \le 4700 + 0.08x$
$340 \le 0.02x$
$17{,}000 \le x$
The number of miles driven per year must be at least 17,000.

4. Let N = required number of shirts. Then
Total revenue = $3.5N$ and
Total cost = $1.3N + 0.4N + 6500$.

$$\begin{aligned} \text{Profit} &> 0 \\ 3.5N - (1.3N + 0.4N + 6500) &> 0 \\ 1.8N - 6500 &> 0 \\ 1.8N &> 6500 \\ N &> 3611.1 \end{aligned}$$

At least 3612 shirts must be sold.

5. Let q be the number of magazines printed. Then the cost of publication is $0.55q$. The number of magazines sold is $0.90q$. The revenue from dealers is $(0.60)(0.90q)$. If fewer than 30,000 magazines are sold, the only revenue is from the sales to dealers, while if more than 30,000 are sold, there are advertising revenues of $0.10(0.60)(0.90q - 30{,}000)$. Thus,

$$\text{Revenue} = \begin{cases} 0.6(0.9)q & \text{if} \quad 0.9q \le 30,000 \\ 0.6(0.9)q + 0.1(0.6)(0.9q - 30,000) & \text{if} \quad 0.9q > 30,000 \end{cases}$$
$$= \begin{cases} 0.54q & q \le 33,333 \\ 0.594q - 1800 & q > 33,333 \end{cases}$$

$$\text{Profit} = \text{Revenue} - \text{Cost}$$
$$= \begin{cases} 0.54q - 0.55q & q \le 33,333 \\ 0.594q - 1800 - 0.55q & q > 33,333 \end{cases}$$
$$= \begin{cases} -0.01q & q \le 33,333 \\ 0.044q - 1800 & q > 33,333 \end{cases}$$

Clearly, the profit is negative if fewer than 33,334 magazines are sold.

$$0.044q - 1800 \ge 0$$
$$0.044q \ge 1800$$
$$q \ge 40,910$$

Thus, at least 40,910 magazines must be printed in order to avoid a loss.

6. Let q = number of clocks produced during regular work week, so $11,000 - q$ = number produced in overtime. Then

$2q + 3(11,000 - q) \le 25,000$

$-q + 33,000 \le 25,000$

$8000 \le q$

At least 8000 clocks must be produced during the regular workweek.

7. Let x = amount at $6\frac{3}{4}\%$ and $30,000 - x$ = amount at 5%. Then

interest at $6\frac{3}{4}\%$ + interest at 5% $\ge$ interest at $6\frac{1}{2}\%$

$x(0.0675) + (30,000 - x)(0.05) \ge (0.065)(30,000)$

$0.0175x + 1500 \ge 1950$

$0.0175x \ge 450$

$x \ge 25,714.29$

Thus at least \$25,714.29 must be invested at $6\frac{3}{4}\%$.

8. Let L be current liabilities. Then

$$\text{Current ratio} = \frac{\text{current assets}}{\text{current liabilities}}$$

$$3.8 = \frac{570,000}{L}$$

$3.8L = 570,000$

$L = \$150,000$

Let x = amount of money they can borrow, where $x \ge 0$.

$$\frac{570,000 + x}{150,000 + x} \ge 2.6$$

$570,000 + x \ge 390,000 + 2.6x$

$180,000 \ge 1.6x$

$112,500 \ge x$

Thus current liabilities are \$150,000 and the maximum amount they can borrow is \$112,500.

9. Let q be the number of units sold this month at \$4.00 each. Then $2500 - q$ will be sold at \$4.50 each. Then
Total revenue $\geq$ 10,750
$4q + 4.5(2500 - q) \geq 10{,}750$
$-0.5q + 11{,}250 \geq 10{,}750$
$500 \geq 0.5q$
$1000 \geq q$
The maximum number of units that can be sold this month is 1000.

10. Revenue = (no. of units)(price per unit)
$q\left(\frac{100}{q} + 1\right) > 5000$
$100 + q > 5000$
$q > 4900$
At least 4901 units must be sold.

11. For $t < 40$, we want
income on hourly basis
> income on per-job basis
$9t > 320 + 3(40 - t)$
$9t > 440 - 3t$
$12t > 440$
$t > 36.7$ hr

12. Let s = yearly sales. With the first method, the salary is $35{,}000 + 0.03s$, and with the second method it is $0.05s$.
$35{,}000 + 0.03s > 0.05s$
$35{,}000 > 0.02s$
$1{,}750{,}000 > s$
The first method is better for yearly sales less than \$1,750,000.

13. Let x = accounts receivable. Then
Acid test ratio = $\frac{450{,}000 + x}{398{,}000}$
$1.3 \leq \frac{450{,}000 + x}{398{,}000}$
$517{,}400 \leq 450{,}000 + x$
$x \geq 67{,}400$
The company must have at least \$67,400 in accounts receivable.

Principles in Practice 1.4

1. $|w - 22| \leq 0.3$

Problems 1.4

1. $|-13| = 13$

2. $|2^{-1}| = \left|\frac{1}{2}\right| = \frac{1}{2}$

3. $|8 - 2| = |6| = 6$

4. $\left|\frac{-4-6}{2}\right| = \left|\frac{-10}{2}\right| = |-5| = 5$

5. $\left|2\left(-\frac{7}{2}\right)\right| = |-7| = 7$

6. $|3 - 5| - |5 - 3| = |-2| - |2| = 2 - 2 = 0$

7. $|x| < 4$, $-4 < x < 4$

8. $|x| < 10$, $-10 < x < 10$

9. Because $2 - \sqrt{5} < 0$,
$\left|2 - \sqrt{5}\right| = -\left(2 - \sqrt{5}\right) = \sqrt{5} - 2$.

10. Because $\sqrt{5} - 2 > 0$, $\left|\sqrt{5} - 2\right| = \sqrt{5} - 2$.

11. **a.** $|x - 7| < 3$

b. $|x - 2| < 3$

c. $|x - 7| \leq 5$

d. $|x - 7| = 4$

e. $|x + 4| < 2$

f. $|x| < 3$

g. $|x| > 6$

h. $|x - 105| < 3$

i. $|x - 850| < 100$

12. $|f(x) - L| < \varepsilon$

13. $|p_1 - p_2| \leq 9$

14. $|x-\mu| \le 2\sigma$
$-2\sigma \le x-\mu \le 2\sigma$
$\mu-2\sigma \le x \le \mu+2\sigma$

15. $|x| = 7$
$x = \pm 7$

16. $|-x| = 2$
$-x = 2$ or -2
$x = \pm 2$

17. $\left|\frac{x}{5}\right| = 7$
$\frac{x}{5} = \pm 7$
$x = \pm 35$

18. $\left|\frac{5}{x}\right| = 12$
$\frac{5}{x} = \pm 12$
$x = \pm\frac{5}{12}$

19. $|x-5| = 8$
$x - 5 = \pm 8$
$x = 5 \pm 8$
$x = 13$ or $x = -3$

20. $|4+3x| = 6$
$4 + 3x = \pm 6$
$3x = -4 \pm 6$
$3x = -10$ or 2
$x = -\frac{10}{3}$ or $x = \frac{2}{3}$

21. $|5x-2| = 0$
$5x - 2 = 0$
$x = \frac{2}{5}$

22. $|7x+3| = x$
Here we must have $x \ge 0$.
$7x + 3 = x$ or $-(7x + 3) = x$
$6x = -3$ $\quad -7x - 3 = x$
$x = -\frac{1}{2} < 0$ $\quad x = -\frac{3}{8} < 0$
There is no solution.

23. $|7-4x| = 5$
$7 - 4x = \pm 5$
$-4x = -7 \pm 5$
$-4x = -2$ or -12
$x = \frac{1}{2}$ or $x = 3$

24. $|5-3x| = 2$
$5-3x = \pm 2$
$-3x = -5 \pm 2$
$-3x = -3$ or -7
$x = 1$ or $x = \frac{7}{3}$

25. $|x| < M$
$-M < x < M$
$(-M, M)$
Note that $M > 0$ is required.

26. $|-x| < 3$
$|x| < 3$
$-3 < x < 3$
$(-3, 3)$

27. $\left|\frac{x}{4}\right| > 2$
$\frac{x}{4} < -2$ or $\frac{x}{4} > 2$
$x < -8$ or $x > 8$, so the solution is $(-\infty, -8) \cup (8, \infty)$.

28. $\left|\frac{x}{3}\right| > \frac{1}{2}$
$\frac{x}{3} < -\frac{1}{2}$ or $\frac{x}{3} > \frac{1}{2}$
$x < -\frac{3}{2}$ or $x > \frac{3}{2}$, so the solution is
$\left(-\infty, -\frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$.

29. $|x+9| < 5$
$-5 < x+9 < 5$
$-14 < x < -4$
$(-14, -4)$

30. $|2x - 17| < -4$
Because $-4 < 0$, the solution set is $\varnothing$.

31. $\left|x-\frac{1}{2}\right|>\frac{1}{2}$

$x-\frac{1}{2}<-\frac{1}{2}$ or $x-\frac{1}{2}>\frac{1}{2}$

$x<0$ or $x>1$

$(-\infty, 0) \cup (1, \infty)$

32. $|1-3x|>2$

$1-3x>2$ or $1-3x<-2$

$-3x>1$ or $-3x<-3$

$x<-\frac{1}{3}$ or $x>1$

The solution is $\left(-\infty, -\frac{1}{3}\right)\cup(1, \infty)$.

33. $|5-8x|\le 1$

$-1 \le 5-8x \le 1$

$-6 \le -8x \le -4$

$\frac{3}{4} \ge x \ge \frac{1}{2}$, which may be rewritten as

$\frac{1}{2} \le x \le \frac{3}{4}$.

The solution is $\left[\frac{1}{2}, \frac{3}{4}\right]$.

34. $|4x-1|\ge 0$ is true for all x because $|a|\ge 0$ for all a. Thus $-\infty < x < \infty$, or $(-\infty,\infty)$.

35. $\left|\frac{3x-8}{2}\right|\ge 4$

$\frac{3x-8}{2}\le -4$ or $\frac{3x-8}{2}\ge 4$

$3x-8 \le -8$ or $3x-8 \ge 8$

$3x \le 0$ or $3x \ge 16$

$x \le 0$ or $x\ge\frac{16}{3}$

The solution is $(-\infty, 0]\cup\left[\frac{16}{3}, \infty\right)$.

36. $\left|\frac{x-7}{3}\right|\le 5$

$-5\le\frac{x-7}{3}\le 5$

$-15\le x-7\le 15$

$-8\le x\le 22$

$[-8, 22]$

37. $|d-35.2\text{ m}|\le 20\text{ cm}$ or $|d-35.2|\le 0.20$

38. Let T_1 and T_2 be the temperatures of the two chemicals.

$5\le|T_1-T_2|\le 10$

39. $|x-\mu|>h\sigma$

Either $x-\mu<-h\sigma$, or $x-\mu>h\sigma$. Thus either $x<\mu-h\sigma$ or $x>\mu+h\sigma$, so the solution is $(-\infty, \mu-h\sigma)\cup(\mu+h\sigma, \infty)$.

40. $|x-0.01|\le 0.005$

Problems 1.5

1. The bounds of summation are 12 and 17; the index of summation is *t*.

2. The bounds of summation are 3 and 450; the index of summation is *m*.

3. $\sum_{i=1}^{7}6i$

$=6(1)+6(2)+6(3)+6(4)+6(5)+6(6)+6(7)$

$=6+12+18+24+30+36+42$

$=168$

4. $\sum_{p=0}^{4}10p=10(0)+10(1)+10(2)+10(3)+10(4)$

$=0+10+20+30+40$

$=100$

5. $\sum_{k=3}^{9}(10k+16) = [10(3)+16]+[10(4)+16]+[10(5)+16]+[10(6)+16]+[10(7)+16]+[10(8)+16]+[10(9)+16]$

$= 46+56+66+76+86+96+106$

$= 532$

6. $\sum_{n=7}^{11}(2n-3) = [2(7)-3]+[2(8)-3]+[2(9)-3]+[2(10)-3]+[2(11)-3]$

$= 11+13+15+17+19$

$= 75$

7. $36+37+38+39+\cdots+60 = \sum_{i=36}^{60} i$

8. $1+4+9+16+25 = \sum_{k=1}^{5} k^2$

9. $5^3+5^4+5^5+5^6+5^7+5^8 = \sum_{i=3}^{8} 5^i$

10. $11+15+19+23+\cdots+71 = \sum_{i=1}^{16}(7+4i)$

11. $2+4+8+16+32+64+128+256 = \sum_{i=1}^{8} 2^i$

12. $10+100+1000+\cdots+100{,}000{,}000 = \sum_{j=1}^{8} 10^j$

13. $\sum_{k=1}^{43} 10 = 10\sum_{k=1}^{43} 1 = 10(43) = 430$

14. $\sum_{k=35}^{135} 2 = 2\sum_{k=35}^{135} 1 = 2\sum_{i=1}^{101} 1 = 2(101) = 202$

15. $\sum_{k=1}^{n}\left(5\cdot\frac{1}{n}\right) = \left(5\cdot\frac{1}{n}\right)\sum_{k=1}^{n} 1 = \left(5\cdot\frac{1}{n}\right)(n) = 5$

16. $\sum_{k=1}^{200}(k-100) = \sum_{k=1}^{200} k - 100\sum_{k=1}^{200} 1 = \frac{200(201)}{2} - 100(200) = 20{,}100 - 20{,}000 = 100$

17. $\sum_{k=51}^{100} 10k = 10\sum_{i=1}^{50}(i+50)$
$= 10\sum_{i=1}^{50} i + (10)(50)\sum_{i=1}^{50} 1$
$= 10\cdot\frac{50(51)}{2} + 500(50) = 12,750 + 25,000$
$= 37,750$

18. $\sum_{k=1}^{n} \frac{n}{n+1}k^2 = \frac{n}{n+1}\sum_{k=1}^{n} k^2$
$= \frac{n}{n+1}\cdot\frac{n(n+1)(2n+1)}{6}$
$= \frac{n^2(2n+1)}{6}$

19. $\sum_{k=1}^{20}(5k^2+3k) = 5\sum_{k=1}^{20}k^2 + 3\sum_{k=1}^{20}k$
$= 5\cdot\frac{20(21)(41)}{6} + 3\frac{20(21)}{2}$
$= 5(2870) + 3(210) = 14,980$

20. $\sum_{k=1}^{100}\frac{3k^2-200k}{101} = \frac{3}{101}\sum_{k=1}^{100}k^2 - \frac{200}{101}\sum_{k=1}^{100}k$
$= \frac{3}{101}\cdot\frac{100(101)(201)}{6} - \frac{200}{101}\cdot\frac{100\cdot 101}{2}$
$= 10,050 - 10,000 = 50$

21. $\sum_{k=51}^{100}k^2 = \sum_{i=1}^{50}(i+50)^2 = \sum_{i=1}^{50}(i^2+100i+2500)$
$= \sum_{i=1}^{50}i^2 + 100\sum_{i=1}^{50}i + 2500\sum_{i=1}^{50}1$
$= \frac{50(51)(101)}{6} + 100\frac{50(51)}{2} + 2500(50)$
$= 42,925 + 127,500 + 125,000 = 295,425$

22. $\sum_{k=1}^{50}(k+50)^2 = \sum_{k=1}^{50}(k^2+100k+2500)$
$= \sum_{k=1}^{50}k^2 + 100\sum_{k=1}^{50}k + 2500\sum_{k=1}^{50}1$
$= \frac{50(51)(101)}{6} + 100\frac{50(51)}{2} + 2500(50)$
$= 42,925 + 127,500 + 125,000 = 295,425$

23. $\sum_{k=1}^{10}\left\{\left[4-\left(\frac{2k}{10}\right)^2\right]\left(\frac{2}{10}\right)\right\} = \frac{1}{5}\sum_{k=1}^{10}\left(4-\frac{1}{25}k^2\right)$
$= \frac{1}{5}(4)\sum_{k=1}^{10}1 - \frac{1}{5}\left(\frac{1}{25}\right)\sum_{k=1}^{10}k^2$
$= \frac{4}{5}(10) - \frac{1}{125}\cdot\frac{10(11)(21)}{6} = 8 - \frac{1}{125}\cdot 385$
$= 8 - \frac{77}{25} = \frac{123}{25} = 4\frac{23}{25}$

24. $\sum_{k=1}^{100}\left\{\left[4-\left(\frac{2}{100}k\right)^2\right]\left(\frac{2}{100}\right)\right\}$
$= \frac{1}{50}\sum_{k=1}^{100}\left(4-\frac{1}{2500}k^2\right)$
$= \frac{1}{50}(4)\sum_{k=1}^{100}1 - \frac{1}{50}\left(\frac{1}{2500}\right)\sum_{k=1}^{100}k^2$
$= \frac{2}{25}(100) - \frac{1}{125,000}\cdot\frac{100(101)(201)}{6}$
$= 8 - \frac{1}{125,000}\cdot 338,350 = 8 - \frac{6767}{2500}$
$= \frac{13,233}{2500} = 5\frac{733}{2500}$

25. $\sum_{k=1}^{n}\left\{\left[5-\left(\frac{3}{n}\cdot k\right)^2\right]\frac{3}{n}\right\}$
$= \frac{3}{n}\sum_{k=1}^{n}\left(5-\frac{9}{n^2}k^2\right)$
$= \frac{3}{n}(5)\sum_{k=1}^{n}1 - \frac{3}{n}\left(\frac{9}{n^2}\right)\sum_{k=1}^{n}k^2$
$= \frac{15}{n}(n) - \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$
$= 15 - \frac{9(n+1)(2n+1)}{2n^2}$

26. $\sum_{k=1}^{n}\frac{k^2}{(n+1)(2n+1)} = \frac{1}{(n+1)(2n+1)}\sum_{k=1}^{n}k^2$
$= \frac{1}{(n+1)(2n+1)}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{n}{6}$

Chapter 1 Review Problems

1. $5x-2 \geq 2(x-7)$
$5x-2 \geq 2x-14$
$3x \geq -12$
$x \geq -4$
$[-4, \infty)$

2. $2x-(7+x) \leq x$
$2x-7-x \leq x$
$-7 \leq 0$, which is true for all x, so $-\infty < x < \infty$, or $(-\infty, \infty)$.

3. $-(5x+2) < -(2x+4)$
$-5x-2 < -2x-4$
$-3x < -2$
$x > \frac{2}{3}$
$\left(\frac{2}{3}, \infty\right)$

4. $-2(x+6) > x+4$
$-2x-12 > x+4$
$-3x > 16$
$x < -\frac{16}{3}$
$\left(-\infty, -\frac{16}{3}\right)$

5. $3p(1-p) > 3(2+p)-3p^2$
$3p-3p^2 > 6+3p-3p^2$
$0 > 6$, which is false for all x. The solution set is $\varnothing$.

6. $3\left(5-\frac{7}{3}q\right) < 9$
$15-7q < 9$
$-7q < -6$
$q > \frac{6}{7}$
$\left(\frac{6}{7}, \infty\right)$

7. $\frac{x+5}{3}-\frac{1}{2} \leq 2$
$2(x+5)-3(1) \leq 6(2)$
$2x+10-3 \leq 12$
$2x \leq 5$
$x \leq \frac{5}{2}$
$\left(-\infty, \frac{5}{2}\right]$

8. $\frac{x}{3}-\frac{x}{4} > \frac{x}{5}$
$20x-15x > 12x$
$5x > 12x$
$0 > 7x$
$0 > x$
$(-\infty, 0)$

9. $\frac{1}{4}s-3 \leq \frac{1}{8}(3+2s)$
$2s-24 \leq 3+2s$
$0 \leq 27$, which is true for all s. Thus $-\infty < s < \infty$, or $(-\infty,\infty)$.

10. $\frac{1}{3}(t+2) \geq \frac{1}{4}$
$4(t+2) \geq 3t+48$
$4t+8 \geq 3t+48$
$t \geq 40$
$[40, \infty)$

11. $|3-2x| = 7$
$3-2x = 7$ or $3-2x = -7$
$-2x = 4$ or $-2x = -10$
$x = -2$ or $x = 5$

12. $\left|\frac{5x-6}{13}\right| = 0$
$\frac{5x-6}{13} = 0$
$5x-6 = 0$
$x = \frac{6}{5}$

13. $|2z-3| < 5$
$-5 < 2z-3 < 5$
$-2 < 2z < 8$
$-1 < z < 4$
$(-1, 4)$

14. $4 < \left|\frac{2}{3}x + 5\right|$

$\frac{2}{3}x + 5 < -4$ or $\frac{2}{3}x + 5 > 4$

$\frac{2}{3}x < -9$ or $\frac{2}{3}x > -1$

$x < -\frac{27}{2}$ or $x > -\frac{3}{2}$

The solution is $\left(-\infty, -\frac{27}{2}\right) \cup \left(-\frac{3}{2}, \infty\right)$.

15. $|3 - 2x| \geq 4$

$3 - 2x \geq 4$ or $3 - 2x \leq -4$

$-2x \geq 1$ or $-2x \leq -7$

$x \leq -\frac{1}{2}$ or $x \geq \frac{7}{2}$

The solution is $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{7}{2}, \infty\right)$.

16. $\sum_{i=1}^{5}(i+2)^3 = \sum_{i=1}^{5}(i^3 + 6i^2 + 12i + 8)$

$= \sum_{i=1}^{5} i^3 + 6\sum_{i=1}^{5} i^2 + 12\sum_{i=1}^{5} i + 8\sum_{i=1}^{5} 1$

$= \frac{5^2(6)^2}{4} + 6\frac{5(6)(11)}{6} + 12\frac{5(6)}{2} + 8(5)$

$= 225 + 330 + 180 + 40$

$= 775$

17. $\sum_{i=3}^{7} i^3 = \sum_{i=1}^{7} i^3 - \sum_{i=1}^{2} i^3$

$= \frac{7^2(8)^2}{4} - \frac{2^2(3)^2}{4}$

$= 784 - 9$

$= 775$

This uses Equation (1.9). By Equation (1.8),

$\sum_{i=3}^{7} i^3 = \sum_{i=1}^{5}(i+2)^3$.

18. Let p = selling price, c = cost. Then

$p - 0.40p = c$

$0.6p = c$

$p = \frac{c}{0.6} = \frac{5c}{3} = c + \left(\frac{2}{3}\right)c$

Thus the profit is $\frac{2}{3}$, or $66\frac{2}{3}\%$, of the cost.

19. Let x be the number of issues with a decline, and $x + 48$ be the number of issues with an increase. Then

$x + (x + 48) = 1132$

$2x = 1084$

$x = 542$

20. Let x = purchase amount excluding tax.

$x + 0.065x = 3039.29$

$1.065x = 3039.29$

$x = 2853.79$

Thus tax is $3039.29 - 2853.79 = \$185.50$.

21. Let q units be produced at A and $10{,}000 - q$ at B.

Cost at A + Cost at B $\leq$ 117,000

$[5q + 30{,}000] + [5.50(10{,}000 - q) + 35{,}000] \leq 117{,}000$

$-0.5q + 120{,}000 \leq 117{,}000$

$-0.5q \leq -3000$

$q \geq 6000$

Thus at least 6000 units must be produced at plant A.

22. Total volume of old tanks

$= \pi(10)^2(25) + \pi(20)^2(25)$

$= 2500\pi + 10{,}000\pi$

$= 12{,}500\pi \text{ ft}^3$

Let r be the radius (in feet) of the new tank. Then

$\frac{4}{3}\pi r^3 = 12{,}500\pi$

$r^3 = 9375$

$r = \sqrt[3]{9375} \approx 21.0858$

The radius is approximately 21.0858 feet.

23. Let c = operating costs

$\frac{c}{236{,}460} < 0.90$

$c < \$212{,}814$

Mathematical Snapshot Chapter 1

1. Here $m = 120$ and $M = 2\frac{1}{2}(60) = 150$. For LP, $r = 2$, so the first t minutes take up $\frac{t}{2}$ of the 120 available minutes. For SP, $r = 1$, so the remaining $150 - t$ minutes take up $\frac{150-t}{1}$ of the 120 available.

$$\frac{t}{2}+\frac{150-t}{1}=120$$
$$t+300-2t=240$$
$$-t=-60$$
$$t=60$$

Switch after 1 hour.

2. Here $m = 120$ and $M = 2\frac{1}{2}(60)=150.$ For EP, $r = 3$, so the first t minutes will take up $\frac{t}{3}$ of the 120 available minutes. For SP, $r = 1$, so the remaining $150 - t$ minutes take up $\frac{150-t}{1}$ of the 120 available.

$$\frac{t}{3}+\frac{150-t}{1}=120$$
$$t+450-3t=360$$
$$-2t=-90$$
$$t=45$$

Switch after 45 minutes.

3. Use the reasoning in Exercise 1, with M unknown and $m = 120$.

$$\frac{t}{2}+\frac{M-t}{1}=120$$
$$t+2M-2t=240$$
$$-t=240-2M$$
$$t=2M-240$$

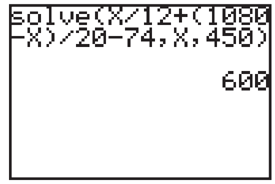
The switch should be made after $2M - 240$ minutes.

4. Use the reasoning in Exercise 2, with M unknown and $m = 120$.

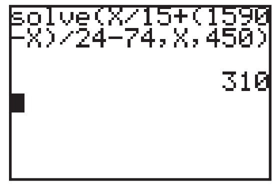
$$\frac{t}{3}+\frac{M-t}{1}=120$$
$$t+3M-3t=360$$
$$-2t=360-3M$$
$$t=\frac{1}{2}(3M-360)$$

The switch should be made after $\frac{1}{2}(3M-360)$ minutes.

5.



$x = 600$



$x = 310$

6. Both equations represent audio being written onto 74-minute CDs. In the first equation, 18 hours (1080 minutes) are being written to a CD using a combination of 12-to-1 and 20-to-1 compression ratios. Here, x gives the maximum amount of audio (600 minutes or 10 hours) that can be written using the 12-to-1 compression ratio. In the second equation, 26.5 hours (1590 minutes) is being written using 15-to-1 and 24-to-1 compression ratios. A maximum of 310 minutes can be written at 15-to-1.

7. The first t minutes use $\frac{t}{R}$ of the m available minutes, the remaining $M - t$ minutes use $\frac{M-t}{r}$ of the m available.

$$\frac{t}{R}+\frac{M-t}{r}=m$$
$$\frac{t}{R}+\frac{M}{r}-\frac{t}{r}=m$$
$$t\left(\frac{1}{R}-\frac{1}{r}\right)=m-\frac{M}{r}$$
$$t\left(\frac{r-R}{rR}\right)=\frac{mr-M}{r}$$
$$t=\frac{R(mr-M)}{r-R}$$

Chapter 2

Principles in Practice 2.1

1. **a.** The formula for the area of a circle is πr^2, where r is the radius.
   $a(r) = \pi r^2$

   b. The domain of $a(r)$ is all real numbers.

   c. Since a radius cannot be negative or zero, the domain for the function, in context, is $r > 0$.

2. **a.** The formula relating distance, time, and speed is $d = rt$ where d is the distance, r is the speed, and t is the time. This can also be written as $t = \frac{d}{r}$. When $d = 300$, we have
   $t(r) = \frac{300}{r}$.

   b. The domain of $t(r)$ is all real numbers except 0.

   c. Since speed is not negative, the domain for the function, in context, is $r > 0$.

   d. Replacing r by x: $t(x) = \frac{300}{x}$.
   Replacing r by $\frac{x}{2}$: $t\left(\frac{x}{2}\right) = \frac{300}{\frac{x}{2}} = \frac{600}{x}$.
   Replacing r by $\frac{x}{4}$: $t\left(\frac{x}{4}\right) = \frac{300}{\frac{x}{4}} = \frac{1200}{x}$.

   e. When the speed is reduced (divided) by a constant, the time is scaled (multiplied) by the same constant; $t\left(\frac{r}{c}\right) = \frac{300c}{r}$.

3. **a.** If the price is \$18.50 per large pizza, $p = 18.5$.
   $18.5 = 26 - \frac{q}{40}$
   $-7.5 = -\frac{q}{40}$
   $300 = q$
   At a price of \$18.50 per large pizza, 300 pizzas are sold each week.

   b. If 200 large pizzas are being sold each week, $q = 200$.
   $p = 26 - \frac{200}{40}$
   $p = 26 - 5$
   $p = 21$
   The price is \$21 per pizza if 200 large pizzas are being sold each week.

   c. To double the number of large pizzas sold, use $q = 400$.
   $p = 26 - \frac{400}{40}$
   $p = 26 - 10$
   $p = 16$
   To sell 400 large pizzas each week, the price should be \$16 per pizza.

Problems 2.1

1. The functions are not equal because $f(x) \geq 0$ for all values of x, while $g(x)$ can be less than 0. For example, $f(-2) = \sqrt{(-2)^2} = \sqrt{4} = 2$ and $g(-2) = -2$, thus $f(-2) \neq g(-2)$.

2. The functions are different because they have different domains. The domain of $G(x)$ is $[-1, \infty)$ (all real numbers ≥ -1) because you can only take the square root of a non-negative number, while the domain of $H(x)$ is all real numbers.

3. The functions are not equal because they have different domains. $h(x)$ is defined for all non-zero real numbers, while $k(x)$ is defined for all real numbers.

4. The functions are equal. For $x = 3$ we have $f(3) = 2$ and $g(3) = 3 - 1 = 2$, hence $f(3) = g(3)$. For $x \neq 3$, we have
   $$f(x) = \frac{x^2 - 4x + 3}{x - 3} = \frac{(x-3)(x-1)}{x-3} = x - 1.$$
   Note that we can cancel the $x - 3$ because we are assuming $x \neq 3$ and so $x - 3 \neq 0$. Thus for $x \neq 3$ $f(x) = x - 1 = g(x)$.
   $f(x) = g(x)$ for all real numbers and they have the same domains, thus the functions are equal.

5. The denominator is zero when $x = 0$. Any other real number can be used for x.
   Answer: all real numbers except 0

6. Any real number can be used for x.
Answer: all real numbers

7. For $\sqrt{x-3}$ to be real, $x - 3 \geq 0$, so $x \geq 3$.
Answer: all real numbers ≥ 3

8. For $\sqrt{z-1}$ to be real, $z - 1 \geq 0$, so $z \geq 1$. We exclude values of z for which $\sqrt{z-1} = 0$, so $z - 1 = 0$, thus $z = 1$.
Answer: all real numbers > 1

9. Any real number can be used for z.
Answer: all real numbers

10. We exclude values of x for which
$x + 8 = 0$
$x = -8$
Answer: all real numbers except –8

11. We exclude values of x where
$2x + 7 = 0$
$2x = -7$
$x = -\frac{7}{2}$
Answer: all real numbers except $-\frac{7}{2}$

12. For $\sqrt{4x+3}$ to be real,
$4x + 3 \geq 0$
$4x \geq -3$
$x \geq -\frac{3}{4}$
Answer: all real numbers $\geq -\frac{3}{4}$

13. We exclude values of y for which $y^2 - 4y + 4 = 0$. $y^2 - 4y + 4 = (y-2)^2$, so we exclude values of y for which $y - 2 = 0$, thus $y = 2$.
Answer: all real numbers except 2.

14. We exclude values of x for which
$x^2 + x - 6 = 0$
$(x+3)(x-2) = 0$
$x = -3, 2$
Answer: all real numbers except –3 and 2

15. We exclude all values of s for which
$2s^2 - 7s - 4 = 0$
$(s - 4)(2s + 1) = 0$
$s = 4, -\frac{1}{2}$
Answer: all real numbers except 4 and $-\frac{1}{2}$

16. $r^2 + 1$ is never 0.
Answer: all real numbers

17. $f(x) = 2x + 1$
$f(0) = 2(0) + 1 = 1$
$f(3) = 2(3) + 1 = 7$
$f(-4) = 2(-4) + 1 = -7$

18. $H(s) = 5s^2 - 3$
$H(4) = 5(4)^2 - 3 = 80 - 3 = 77$
$H\left(\sqrt{2}\right) = 5\left(\sqrt{2}\right)^2 - 3 = 10 - 3 = 7$
$H\left(\frac{2}{3}\right) = 5\left(\frac{2}{3}\right)^2 - 3 = \frac{20}{9} - 3 = -\frac{7}{9}$

19. $G(x) = 2 - x^2$
$G(-8) = 2 - (-8)^2 = 2 - 64 = -62$
$G(u) = 2 - u^2$
$G(u^2) = 2 - (u^2)^2 = 2 - u^4$

20. $F(x) = -5x$
$F(s) = -5s$
$F(t + 1) = -5(t + 1) = -5t - 5$
$F(x + 3) = -5(x + 3) = -5x - 15$

21. $\gamma(u) = 2u^2 - u$
$\gamma(-2) = 2(-2)^2 - (-2) = 8 + 2 = 10$
$\gamma(2v) = 2(2v)^2 - (2v) = 8v^2 - 2v$
$\gamma(x+a) = 2(x+a)^2 - (x+a)$
$= 2x^2 + 4ax + 2a^2 - x - a$

22. $h(v) = \frac{1}{\sqrt{v}}$
$h(16) = \frac{1}{\sqrt{16}} = \frac{1}{4}$
$h\left(\frac{1}{4}\right) = \frac{1}{\sqrt{\frac{1}{4}}} = \frac{1}{\frac{1}{2}} = 2$
$h(1-x) = \frac{1}{\sqrt{1-x}}$

23. $f(x) = x^2 + 2x + 1$

$f(1) = 1^2 + 2(1) + 1 = 1 + 2 + 1 = 4$

$f(-1) = (-1)^2 + 2(-1) + 1 = 1 - 2 + 1 = 0$

$f(x+h) = (x+h)^2 + 2(x+h) + 1$
$= x^2 + 2xh + h^2 + 2x + 2h + 1$

24. $H(x) = (x+4)^2$

$H(0) = (0+4)^2 = 16$

$H(2) = (2+4)^2 = 6^2 = 36$

$H(t-4) = [(t-4)+4]^2 = t^2$

25. $k(x) = \frac{x-7}{x^2+2}$

$k(5) = \frac{5-7}{5^2+2} = -\frac{2}{27}$

$k(3x) = \frac{3x-7}{(3x)^2+2} = \frac{3x-7}{9x^2+2}$

$k(x+h) = \frac{(x+h)-7}{(x+h)^2+2} = \frac{x+h-7}{x^2+2xh+h^2+2}$

26. $k(x) = \sqrt{x-3}$

$k(4) = \sqrt{4-3} = \sqrt{1} = 1$

$k(3) = \sqrt{3-3} = \sqrt{0} = 0$

$k(x+1) - k(x) = \sqrt{(x+1)-3} - \sqrt{x-3}$
$= \sqrt{x-2} - \sqrt{x-3}$

27. $f(x) = x^{4/3}$

$f(0) = 0^{4/3} = 0$

$f(64) = 64^{4/3} = \left(\sqrt[3]{64}\right)^4 = (4)^4 = 256$

$f\left(\frac{1}{8}\right) = \left(\frac{1}{8}\right)^{4/3} = \left(\sqrt[3]{\frac{1}{8}}\right)^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

28. $g(x) = x^{2/5}$

$g(32) = 32^{2/5} = \left(\sqrt[5]{32}\right)^2 = (2)^2 = 4$

$g(-64) = (-64)^{2/5} = \left(\sqrt[5]{-64}\right)^2$

$= \left(\sqrt[5]{-32}\sqrt[5]{2}\right)^2 = \left(-2\sqrt[5]{2}\right)^2 = 4\sqrt[5]{4}$

$g(t^{10}) = (t^{10})^{2/5} = t^4$

29. $f(x) = 4x - 5$

a. $f(x + h) = 4(x + h) - 5 = 4x + 4h - 5$

b. $\frac{f(x+h) - f(x)}{h}$
$= \frac{(4x+4h-5)-(4x-5)}{h} = \frac{4h}{h} = 4$

30. $f(x) = \frac{x}{2}$

a. $f(x+h) = \frac{x+h}{2}$

b. $\frac{f(x+h) - f(x)}{h} = \frac{\frac{x+h}{2} - \frac{x}{2}}{h} = \frac{\frac{h}{2}}{h} = \frac{1}{2}$

31. $f(x) = x^2 + 2x$

a. $f(x+h) = (x+h)^2 + 2(x+h)$
$= x^2 + 2xh + h^2 + 2x + 2h$

b. $\frac{f(x+h) - f(x)}{h}$
$= \frac{(x^2+2xh+h^2+2x+2h)-(x^2+2x)}{h}$
$= \frac{2xh+h^2+2h}{h} = 2x + h + 2$

32. $f(x) = 3x^2 - 2x - 1$

a. $f(x+h) = 3(x+h)^2 - 2(x+h) - 1$
$= 3(x^2+2xh+h^2) - 2x - 2h - 1$
$= 3x^2 + 6xh + 3h^2 - 2x - 2h - 1$

b. $\dfrac{f(x+h)-f(x)}{h}=\dfrac{(3x^2+6xh+3h^2-2x-2h-1)-(3x^2-2x-1)}{h}$

$=\dfrac{6xh+3h^2-2h}{h}$

$=6x+3h-2$

33. $f(x)=3-2x+4x^2$

a. $f(x+h)=3-2(x+h)+4(x+h)^2$

$=3-2x-2h+4(x^2+2xh+h^2)$

b. $\dfrac{f(x+h)-f(x)}{h}=\dfrac{3-2x-2h+4x^2+8xh+4h^2-(3-2x+4x^2)}{h}$

$=\dfrac{-2h+8xh+4h^2}{h}$

$=-2+8x+4h$

34. $f(x)=x^3$

a. $f(x+h)=(x+h)^3=x^3+3x^2h+3xh^2+h^3$

b. $\dfrac{f(x+h)-f(x)}{h}=\dfrac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h}=\dfrac{3x^2h+3xh^2+h^3}{h}=3x^2+3xh+h^2$

35. $f(x)=\dfrac{1}{x}$

a. $f(x+h)=\dfrac{1}{x+h}$

b. $\dfrac{f(x+h)-f(x)}{h}=\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}=\dfrac{\frac{x-(x+h)}{x(x+h)}}{h}=\dfrac{-h}{x(x+h)h}=-\dfrac{1}{x(x+h)}$

36. $f(x)=\dfrac{x+8}{x}$

a. $f(x+h)=\dfrac{(x+h)+8}{x+h}=\dfrac{x+h+8}{x+h}$

b. $\dfrac{f(x+h)-f(x)}{h}=\dfrac{\frac{x+h+8}{x+h}-\frac{x+8}{x}}{h}=\dfrac{x(x+h)\left(\frac{x+h+8}{x+h}-\frac{x+8}{x}\right)}{x(x+h)h}=\dfrac{x(x+h+8)-(x+h)(x+8)}{x(x+h)h}$

$=\dfrac{x^2+xh+8x-x^2-hx-8x-8h}{x(x+h)h}=\dfrac{-8h}{x(x+h)h}=-\dfrac{8}{x(x+h)}$

37. $\frac{f(3+h)-f(3)}{h} = \frac{[5(3+h)+3]-[5(3)+3]}{h}$
$= \frac{[15+5h+3]-[15+3]}{h}$
$= \frac{18+5h-18}{h}$
$= \frac{5h}{h}$
$= 5$

38. $\frac{f(x)-f(2)}{x-2} = \frac{2x^2-x+1-(8-2+1)}{x-2}$
$= \frac{2x^2-x+1-7}{x-2}$
$= \frac{2x^2-x-6}{x-2}$
$= 2x+3$

39. $9y - 3x - 4 = 0$

The equivalent form $y = \frac{3x+4}{9}$ shows that for each input x there is exactly one output, $\frac{3x+4}{9}$. Thus y is a function of x. Solving for x gives $x = \frac{9y-4}{3}$. This shows that for each input y there is exactly one output, $\frac{9y-4}{3}$. Thus x is a function of y.

40. $x^2 + y = 0$

The form $y = -x^2$ shows that for each input x there is exactly one output, $-x^2$. Thus y is a function of x. Solving for x gives $x = \pm\sqrt{-y}$. If, for example, $y = -1$, then $x = \pm 1$, so x is not a function of y.

41. $y = 7x^2$

For each input x, there is exactly one output $7x^2$. Thus y is a function of x. Solving for x gives $x = \pm\sqrt{\frac{y}{7}}$. If, for example, $y = 7$, then $x = \pm 1$, so x is not a function of y.

42. $x^2 + y^2 = 1$

Solving for y we have $y = \pm\sqrt{1-x^2}$. If $x = 0$, then $y = \pm 1$, so y is not a function of x. Solving for x gives $x = \pm\sqrt{1-y^2}$. If $y = 0$, then $x = \pm 1$, so x is not a function of y.

43. Yes, because corresponding to each input r there is exactly one output, πr^2.

44. a. $f(a) = a^2a^3 + a^3a^2 = a^5 + a^5 = 2a^5$

b. $f(ab) = a^2(ab)^3 + a^3(ab)^2$
$= a^2a^3b^3 + a^3a^2b^2$
$= a^5b^3 + a^5b^2$
$= a^5b^2(b+1)$

45. Weekly excess of income over expenses is $6500 - 4800 = 1700$.
After t weeks the excess accumulates to $1700t$. Thus the value of V of the business at the end of t weeks is given by $V = f(t) = 25{,}000 + 1700t$.

46. Depreciation at the end of t years is $0.02t(30{,}000)$, so value V of machine is $V = f(t) = 30{,}000 - 0.02t(30{,}000)$, or $V = f(t) = 30{,}000(1 - 0.02t)$.

47. Yes; for each input q there corresponds exactly one output, $1.25q$, so P is a function of q. The dependent variable is P and the independent variable is q.

48. Charging \$600,000 per film corresponds to $p = 600{,}000$.

$600{,}000 = \frac{1{,}200{,}000}{q}$

$q = 2$

The actor will star in 2 films per year. To star in 4 films per year the actor should charge $p = \frac{1{,}200{,}000}{4} = \$300{,}000$ per film.

49. The function can be written as $q = 48p$.
At \$8.39 per pound, the coffee house will supply $q = 48(8.39) = 402.72$ pounds per week.
At \$19.49 per pound, the coffee house will supply $q = 48(19.49) = 935.52$ pounds per week.
The amount the coffee house supplies increases as the price increases.

50. a. $f(0) = 1 - 1 = 0$

b. $f(100) = 1 - \left(\frac{300}{400}\right)^3 = 1 - \left(\frac{3}{4}\right)^3 = 1 - \frac{27}{64}$

$= \frac{37}{64}$

c. $f(900) = 1 - \left(\frac{300}{1200}\right)^3$

$= 1 - \left(\frac{1}{4}\right)^3$

$= 1 - \frac{1}{64}$

$= \frac{63}{64}$

d. We solve

$$0.500 = 1 - \left(\frac{300}{300+t}\right)^3$$

$$\left(\frac{300}{300+t}\right)^3 = 0.5$$

$$\frac{300}{300+t} = \sqrt[3]{0.5}$$

$$300 = 300\sqrt[3]{0.5} + t\sqrt[3]{0.5}$$

$$t = \frac{300 - 300\sqrt[3]{0.5}}{\sqrt[3]{0.5}} \approx 77.98$$

78 days

51. a. $f(1000) = \frac{\left(\sqrt[3]{1000}\right)^4}{2500} = \frac{10^4}{2500} = \frac{10,000}{2500} = 4$

b. $f(2000) = \frac{\left[\sqrt[3]{1000(2)}\right]^4}{2500} = \frac{\left(10\sqrt[3]{2}\right)^4}{2500}$

$= \frac{10,000\sqrt[3]{2^4}}{2500} = 4\sqrt[3]{2^3 \cdot 2} = 8\sqrt[3]{2}$

c. $f(2I_0) = \frac{(2I_0)^{4/3}}{2500} = \frac{2^{4/3} I_0^{4/3}}{2500}$

$= 2\sqrt[3]{2}\left[\frac{I_0^{4/3}}{2500}\right] = 2\sqrt[3]{2}\, f(I_0)$

Thus $f(2I_0) = 2\sqrt[3]{2}\, f(I_0)$, which means that doubling the intensity increases the response by a factor of $2\sqrt[3]{2}$.

52. $P(1) = 1 - \frac{1}{2}(1-0.344)^0 = 1 - \frac{1}{2}(1) = \frac{1}{2}$

$P(2) = 1 - \frac{1}{2}(1-0.344)^1 = 1 - \frac{1}{2}(0.656) = 0.672$

53. a. Domain: 3000, 2900, 2300, 2000
$f(2900) = 12$, $f(3000) = 10$

b. Domain: 10, 12, 17, 20
$g(10) = 3000$, $g(17) = 2300$

54. a. –18.97

b. –581.77

c. –18.51

55. a. –5.13

b. 2.64

c. –17.43

56. a. 1,997,723.57

b. 1,287,532.35

c. 2,964,247.40

57. a. 7.89

b. 63.85

c. 1.21

Principles in Practice 2.2

1. a. Let n = the number of visits and $p(n)$ be the premium amount.
$p(n) = 125$

b. The premiums do not change regardless of the number of doctor visits.

c. This is a constant function.

2. a. $d(t) = 3t^2$ is a quadratic function.

b. The degree of $d(t) = 3t^2$ is 2.

c. The leading coefficient of $d(t) = 3t^2$ is 3.

3. The price for n pairs of socks is given by

$$c(n) = \begin{cases} 3.5n & 0 \le n \le 5 \\ 3n & 5 < n \le 10 \\ 2.75n & 10 < n \end{cases}.$$

4. Think of the bookshelf having 7 slots, from left to right. You have a choice of 7 books for the first slot. Once a book has been put in the first slot, you have 6 choices for which book to put in the second slot, etc. The number of arrangements is $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7! = 5040$.

Problems 2.2

1. yes

2. $f(x) = \frac{x^3+7x-3}{3} = \frac{1}{3}x^3 + \frac{7}{3}x - 1$, which is a polynomial function.

3. no

4. yes

5. yes

6. yes

7. no

8. $g(x) = 4x^{-4} = \frac{4}{x^4}$, which is a rational function.

9. all real numbers

10. all real numbers

11. all real numbers

12. all x such that $1 \le x \le 3$

13. **a.** 3

b. 7

14. **a.** 1

b. 7

15. **a.** 7

b. 1

16. **a.** 0

b. 9

17. $f(x) = 8$
$f(2) = 8$
$f(t+8) = 8$
$f\left(-\sqrt{17}\right) = 8$

18. $g(x) = |x-3|$
$g(10) = |10-3| = |7| = 7$
$g(3) = |3-3| = |0| = 0$
$g(-3) = |-3-3| = |-6| = 6$

19. $F(10) = 1$
$F\left(-\sqrt{3}\right) = -1$
$F(0) = 0$
$F\left(-\frac{18}{5}\right) = -1$

20. $f(3) = 4$
$f(-4) = 3$
$f(0) = 4$

21. $G(8) = 8 - 1 = 7$
$G(3) = 3 - 1 = 2$
$G(-1) = 3 - (-1)^2 = 2$
$G(1) = 3 - (1)^2 = 2$

22. $F(3) = 3^2 - 3(3) + 1 = 1$
$F(-3) = 2(-3) - 5 = -11$
$F(2)$ is not defined.

23. $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

24. $0! = 1$

25. $(4 - 2)! = 2! = 2 \cdot 1 = 2$

26.
$$\begin{aligned} 6! \cdot 2! &= (6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)(2\cdot 1) \\ &= (720)(2) \\ &= 1440 \end{aligned}$$

27. $\frac{n!}{(n-1)!} = \frac{n\cdot(n-1)!}{(n-1)!} = n$

28.
$$\begin{aligned} \frac{8!}{5!(8-5)!} &= \frac{8!}{5!\cdot 3!} \\ &= \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(5\cdot 4\cdot 3\cdot 2\cdot 1)(3\cdot 2\cdot 1)} \\ &= \frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1} \\ &= 8\cdot 7 \\ &= 56 \end{aligned}$$

29. Let i = the passenger's income and
$c(i)$ = the cost for the ticket.
$c(i) = 4.5$
This is a constant function.

30. Let w = the width of the prism, then
$w + 3$ = the length of the prism, and
$2w - 1$ = the height of the prism. The formula for the volume of a rectangular prism is
V = length · width · height.
$V(w) = (w+3)(w)(2w-1) = 2w^3 + 5w^2 - 3w$
This is a cubic function.

31. **a.** $C = 850 + 3q$

b. $1600 = 850 + 3q$
$750 = 3q$
$q = 250$

32. The interest is Prt, so principal and interest amount to $f(t) = P + Prt$, or $f(t) = P(1 + rt)$. Since $f(t) = at + b$ where a (= Pr) and b (= P) are constants, f is a linear function of t.

33. The cost for buying n tickets is

$$c(n) = \begin{cases} 9.5n & 0 \le n < 12 \\ 8.75n & 12 \le n \end{cases}$$

34. For a committee of four, there are 4 choices for who will be member A. For each choice of member A, there are 3 choices for member G. Once members A and G have been chosen, there are two choices for member M, then one choice for member S once members A, G, and M have been chosen. Thus, there are
$4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$ ways to label the members. Similarly, a committee of five can label itself with five labels in
$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$ ways.

35. $$P(2) = \frac{3!\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^1}{2!(1!)} = \frac{6\left(\frac{1}{16}\right)\left(\frac{3}{4}\right)}{2(1)} = \frac{9}{64}$$

36. $$P(5) = \frac{5!\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^0}{5!(0!)} = \frac{5!\left(\frac{1}{1024}\right)(1)}{5!(1)} = \frac{1}{1024}$$

37. **a.** all T such that $30 \le T \le 39$

b. $$f(30) = \frac{1}{24}(30) + \frac{11}{4} = \frac{5}{4} + \frac{11}{4} = \frac{16}{4} = 4$$
$$f(36) = \frac{1}{24}(36) + \frac{11}{4} = \frac{6}{4} + \frac{11}{4} = \frac{17}{4}$$
$$f(39) = \frac{4}{3}(39) - \frac{175}{4} = 52 - \frac{175}{4} = \frac{33}{4}$$

38. **a.** 742.50

b. –20.28

c. 1218.60

39. **a.** 1182.74

b. 4985.27

c. 252.15

40. **a.** 19.12

b. –62.94

c. 57.69

41. **a.** 2.21

b. 9.98

c. –14.52

Principles in Practice 2.3

1. The customer's price is
$(c \circ s)(x) = c(s(x)) = c(x+3) = 2(x+3)$
$= 2x + 6$

2. $g(x) = (x+3)^2$ can be written as
$g(x) = a(l(x)) = (a \circ l)(x)$ where $a(x) = x^2$ and $l(x) = x + 3$. Then $l(x)$ represents the length of the sides of the square, while $a(x)$ is the area of a square with side of length x.

Problems 2.3

1. $f(x) = x + 3$, $g(x) = x + 5$

a. $(f+g)(x) = f(x) + g(x)$
$= (x+3) + (x+5)$
$= 2x + 8$

b. $(f+g)(0) = 2(0) + 8 = 8$

c. $(f-g)(x) = f(x) - g(x) = (x+3)-(x+5) = -2$

d. $(fg)(x) = f(x)g(x) = (x+3)(x+5) = x^2+8x+15$

e. $(fg)(-2) = (-2)^2 + 8(-2) + 15 = 3$

f. $\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x+3}{x+5}$

g. $(f \circ g)(x) = f(g(x)) = f(x+5) = (x+5)+3 = x+8$

h. $(f \circ g)(3) = 3+8 = 11$

i. $(g \circ f)(x) = g(f(x)) = g(x+3) = (x+3)+5 = x+8$

j. $(g \circ f)(3) = 3+8 = 11$

2. $f(x) = 2x$, $g(x) = 6 + x$

a. $(f+g)(x) = f(x)+g(x) = (2x)+(6+x) = 3x+6$

b. $(f-g)(x) = f(x)-g(x) = (2x)-(6+x) = x-6$

c. $(f-g)(4) = (4) - 6 = -2$

d. $(fg)(x) = f(x)g(x) = 2x(6+x) = 12x + 2x^2$

e. $\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{2x}{6+x}$

f. $\frac{f}{g}(2) = \frac{2(2)}{6+2} = \frac{4}{8} = \frac{1}{2}$

g. $(f \circ g)(x) = f(g(x)) = f(6+x) = 2(6+x) = 12+2x$

h. $(g \circ f)(x) = g(f(x)) = g(2x) = 6+2x$

i. $(g \circ f)(2) = 6+2(2) = 6+4 = 10$

3. $f(x) = x^2+1, \quad g(x) = x^2 - x$

a. $(f+g)(x) = f(x)+g(x) = (x^2+1)+(x^2-x) = 2x^2 - x + 1$

b. $(f-g)(x) = f(x)-g(x) = (x^2+1)-(x^2-x) = x+1$

c. $(f-g)\left(-\frac{1}{2}\right) = -\frac{1}{2}+1 = \frac{1}{2}$

d. $(fg)(x) = f(x)g(x) = (x^2+1)(x^2-x) = x^4 - x^3 + x^2 - x$

e. $\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x^2+1}{x^2-x}$

f. $\frac{f}{g}\left(-\frac{1}{2}\right) = \frac{\left(-\frac{1}{2}\right)^2+1}{\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)} = \frac{5}{3}$

g. $(f \circ g)(x) = f(g(x)) = f(x^2-x) = (x^2-x)^2+1 = x^4 - 2x^3 + x^2 + 1$

h. $(g \circ f)(x) = g(f(x)) = g(x^2+1) = (x^2+1)^2 - (x^2+1) = x^4 + x^2$

i. $(g \circ f)(-3) = (-3)^4 + (-3)^2 = 90$

4. $f(x) = x^2 + 1, g(x) = 5$

a. $(f+g)(x) = f(x) + g(x) = (x^2+1)+5 = x^2+6$

b. $(f+g)\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^2 + 6 = \frac{58}{9}$

c. $(f-g)(x) = f(x) - g(x) = (x^2+1)-5 = x^2-4$

d. $(fg)(x) = f(x)g(x) = (x^2+1)(5) = 5x^2+5$

e. $(fg)(7) = 5(7^2)+5 = 245+5 = 250$

f. $\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x^2+1}{5}$

g. $(f \circ g)(x) = f(g(x)) = f(5) = 5^2+1 = 26$

h. $(f \circ g)(12{,}003) = 26$

i. $(g \circ f)(x) = g(f(x)) = g(x^2+1) = 5$

5. $f(g(2)) = f(4-4) = f(0) = 0 + 6 = 6$
$g(f(2)) = g(12+6) = g(18) = 4 - 36 = -32$

6. $(f \circ g)(p) = f(g(p)) = f\left(\frac{p-2}{3}\right) = \frac{4}{\frac{p-2}{3}} = \frac{12}{p-2}$

$(g \circ f)(p) = g(f(p)) = g\left(\frac{4}{p}\right) = \frac{\frac{4}{p}-2}{3} = \frac{4-2p}{3p}$

7. $(F \circ G)(t) = F(G(t)) = F\left(\frac{2}{t-1}\right) = \left(\frac{2}{t-1}\right)^2 + 7\left(\frac{2}{t-1}\right) + 1 = \frac{4}{(t-1)^2} + \frac{14}{t-1} + 1$

$(G \circ F)(t) = G(F(t)) = G(t^2+7t+1) = \frac{2}{(t^2+7t+1)-1} = \frac{2}{t^2+7t}$

8. $(F \circ G)(t) = F(G(t)) = F(3t^2+4t+2) = \sqrt{3t^2+4t+2}$

$(G \circ F)(t) = G(F(t)) = G\left(\sqrt{t}\right) = 3\left(\sqrt{t}\right)^2 + 4\left(\sqrt{t}\right) + 2 = 3t + 4\sqrt{t} + 2$

9. $(f \circ g)(v) = f(g(v)) = f\left(\sqrt{v+2}\right) = \frac{1}{\left(\sqrt{v+2}\right)^2 + 1} = \frac{1}{v+2+1} = \frac{1}{v+3}$

$(g \circ f)(v) = g(f(v)) = g\left(\frac{1}{v^2+1}\right) = \sqrt{\frac{1}{v^2+1} + 2} = \sqrt{\frac{1+2(v^2+1)}{v^2+1}} = \sqrt{\frac{2v^2+3}{v^2+1}}$

10. $(f \circ f)(x) = f(f(x))$
$= f(x^2 + 2x - 1)$
$= (x^2 + 2x - 1)^2 + 2(x^2 + 2x - 1) - 1$
$= x^4 + 4x^3 + 4x^2 - 2$

11. Let $g(x) = 11x$ and $f(x) = x - 7$. Then
$h(x) = g(x) - 7 = f(g(x))$

12. Let $g(x) = x^2 - 2$ and $f(x) = \sqrt{x}$. Then
$h(x) = \sqrt{x^2 - 2} = \sqrt{g(x)} = f(g(x))$

13. Let $g(x) = x^2 - 2$ and $f(x) = \frac{1}{x}$. Then
$h(x) = \frac{1}{x^2 - 2} = \frac{1}{g(x)} = f(g(x))$

14. Let $g(x) = 9x^3 - 5x$ and $f(x) = x^3 - x^2 + 11$.
Then $h(x) = [g(x)]^3 - [g(x)]^2 + 11 = f(g(x))$

15. Let $g(x) = \frac{x^2 - 1}{x + 3}$ and $f(x) = \sqrt[4]{x}$.
Then $h(x) = \sqrt[4]{g(x)} = f(g(x))$.

16. Let $g(x) = 3x - 5$ and $f(x) = \frac{2 - x}{x^2 + 2}$. Then
$h(x) = \frac{2 - (3x - 5)}{(3x - 5)^2 + 2} = f(g(x))$.

17. **a.** The revenue is \$9.75 per pound of coffee sold, so $r(x) = 9.75x$.

b. The expenses are $e(x) = 4500 + 4.25x$.

c. Profit = revenue – expenses.
$(r - e)(x) = 9.75x - (4500 + 4.25x)$
$= 5.5x - 4500.$

18. $v(x) = (4x - 2)^3$ can be written as
$v(x) = f(l(x)) = (f \circ l)(x)$ where $f(x) = x^3$ and $l(x) = 4x - 2$. Then $l(x)$ represents the length of the sides of the cube, while $f(x)$ is the volume of a cube with sides of length x.

19. $(g \circ f)(m) = g(f(m))$
$= g\left(\frac{40m - m^2}{4}\right)$
$= 40\left(\frac{40m - m^2}{4}\right)$
$= 10(40m - m^2)$
$= 400m - 10m^2$
This represents the total revenue received when the total output of m employees is sold.

20. $(f \circ g)(E) = f(g(E))$
$= f(7202 + 0.29E^{3.68})$
$= 0.45(7202 + 0.29E^{3.68} - 1000)^{0.53}$
$= 0.45(6202 + 0.29E^{3.68})^{0.53}$
This represents status based on years of education.

21. **a.** 14.05

b. 1169.64

22. **a.** –0.13

b. 18.85

23. **a.** 194.47

b. 0.29

24. **a.** 0.45

b. 1.61

Problems 2.4

1. $f^{-1}(x) = \frac{x}{3} - \frac{7}{3}$

2. $g^{-1}(x) = \frac{x}{2} - \frac{1}{2}$

3. $F^{-1}(x) = 2x + 14$

4. $f^{-1}(x) = \frac{\sqrt{x}}{4} + \frac{5}{4}$

5. $r(A) = \sqrt{\frac{A}{\pi}}$

6. $r(V)=\sqrt[3]{\frac{3V}{4\pi}}$

7. $f(x) = 5x + 12$ is one-to-one, for if $f(x_1)=f(x_2)$ then $5x_1+12=5x_2+12,$ so $5x_1=5x_2$ and thus $x_1=x_2$.

8. $g(x)=(5x+12)^2$ is not one-to-one, because $g(x_1)=g(x_2)$ does not imply $x_1=x_2$. For example, $g\left(-\frac{11}{5}\right)=g\left(-\frac{13}{5}\right)=1.$

9. $h(x)=(5x+12)^2,$ for $x\ge-\frac{5}{12},$ is one-to-one. If $h(x_1)=h(x_2)$ then $(5x_1+12)^2=(5x_2+12)^2.$ Since $x\ge-\frac{5}{12}$ we have $5x + 12 \ge 0$, and thus $(5x_1+12)^2=(5x_2+12)^2$ only if $5x_1+12=5x_2+12,$ and hence $x_1=x_2$.

10. $F(x)=|x-9|$ is not one-to-one, because $F(x_1)=F(x_2)$ does not imply $x_1=x_2$. For example, $F(8) = F(10) = 1$.

11. The inverse of $f(x)=(4x-5)^2$ for $x\ge\frac{5}{4}$ is $f^{-1}(x)=\frac{\sqrt{x}}{4}+\frac{5}{4},$ so to find the solution, we find $f^{-1}(23)$.

$f^{-1}(23)=\frac{\sqrt{23}}{4}+\frac{5}{4}$

The solution is $x=\frac{\sqrt{23}}{4}+\frac{5}{4}.$

12. The inverse of $V(r)=\frac{4}{3}\pi r^3$ is $r(V)=\sqrt[3]{\frac{3V}{4\pi}},$ so the solution is $r(100)=\sqrt[3]{\frac{3(100)}{4\pi}}.$

13. From $p=\frac{1,200,000}{q},$ we get $q=\frac{1,200,000}{p}.$ Since $q > 0$, p is also greater than 0, so q as a function of p is $q=q(p)=\frac{1,200,000}{p},\ p>0.$

$$\begin{aligned}p(q(p)) &= p\left(\frac{1,200,000}{p}\right)\\ &= \frac{1,200,000}{\frac{1,200,000}{p}}\\ &= 1,200,000\cdot\frac{p}{1,200,000}\\ &= p\end{aligned}$$

Similarly, $q(p(q)) = q$, so the functions are inverses.

14. From $p=\frac{q}{48},$ we get $q = 48p$. Since $q > 0$, p is also greater than 0, so q as a function of p is $q = q(p) = 48p$, $p > 0$.

$$q(p(q))=q\left(\frac{q}{48}\right)=48\cdot\frac{q}{48}=q$$

$$p(q(p))=p(48p)=\frac{48p}{48}=p$$

Thus, $p(q)$ and $q(p)$ are inverses.

Principles in Practice 2.5

1. Let y = the amount of money in the account. Then, after one month, $y = 7250 - (1 \cdot 600) = \$6650$, and after two months $y = 7250 - (2 \cdot 600) = \$6050$. Thus, in general, if we let x = the number of months during which Rachel spends from this account, $y = 7250 - 600x$. To identify the x-intercept, we set $y = 0$ and solve for x.

$y = 7250 - 600x$

$0 = 7250 - 600x$

$600x = 7250$

$x=12\frac{1}{12}$

The x-intercept is $\left(12\frac{1}{12},0\right).$

Therefore, after 12 months and approximately 2.5 days Rachel will deplete her savings. To identify the y-intercept, we set $x = 0$ and solve for y.

$y = 7250 - 600x$

$y = 7250 - 600(0)$

$y = 7250$

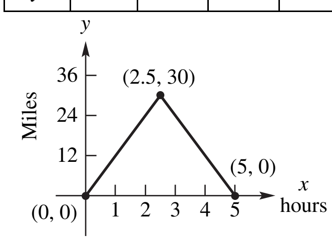
The y-intercept is (0, 7250).

Therefore, before any months have gone by, Rachel has $7250 in her account.

2. Let y = the cost to the customer and let x = the number of rides he or she takes. Since the cost does not change, regardless of the number of rides taken, the equation $y = 24.95$ represents this situation. The graph of $y = 24.95$ is a horizontal line whose y-intercept is (0, 24.95). Since the line is parallel to the x-axis, there is no x-intercept.

3. The formula relating distance, time, and speed is $d = rt$, where d is the distance, r is the speed, and t is the time. Let x = the time spent biking (in hours). Then, $12x$ = the distance traveled. Brett bikes $12 \cdot 2.5 = 30$ miles and then turns around and bikes the same distance back to the rental shop. Therefore, we can represent the distance from the turn-around point at any time x as $|30-12x|$. Similarly, the distance from the rental shop at any time x can be represented by the function $y = 30-|30-12x|$.

x	0	1	2	2.5	3	4	5
y	0	12	24	30	24	12	0



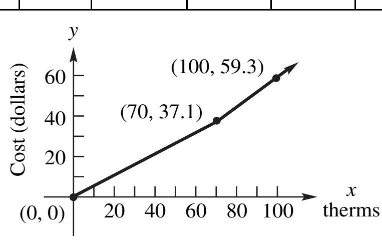
4. The monthly cost of x therms of gas is

$$y = \begin{cases} 0.53x, & \text{if } 0 \le x \le 70 \\ 0.53(70) + 0.74(x-70), & \text{if } x>70 \end{cases}$$

or

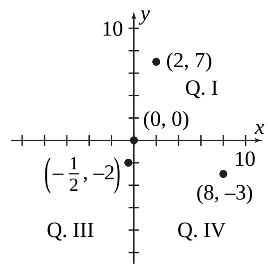
$$y = \begin{cases} 0.53x, & \text{if } 0 \le x \le 70 \\ 0.74x - 14.7, & \text{if } x > 70 \end{cases}$$

x	0	10	30	50	70	80	90	100
x	0	5.3	15.9	26.5	37.1	44.5	51.9	59.3

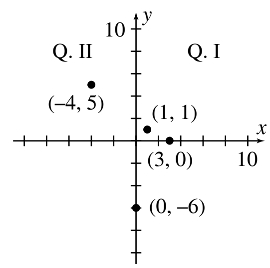


Problems 2.5

1.



2.



3. a. $f(0) = 1, f(2) = 2, f(4) = 3, f(-2) = 0$

b. Domain: all real numbers

c. Range: all real numbers

d. $f(x) = 0$ for $x = -2$. So a real zero is -2.

4. a. $f(0) = 2, f(2) = 0$

b. Domain: all $x \ge 0$

c. Range: all $y \ge 2$

d. $f(x) = 0$ for $x = 2$. So a real zero is 2.

5. a. $f(0) = 0, f(1) = 1, f(-1) = 1$

b. Domain: all real numbers

c. Range: all nonnegative real numbers

d. $f(x) = 0$ for $x = 0$. So a real zero is 0.

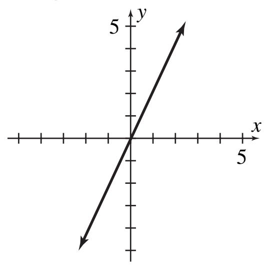
6. a. $f(0) = 0, f(2) = 1, f(3) = 3, f(4) = 2$

b. Domain: all x such that $0 \le x \le 4$

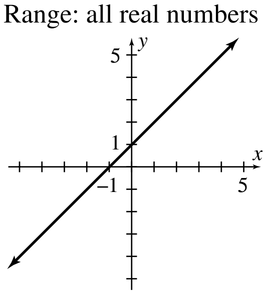
c. Range: all y such that $0 \le y \le 3$

d. $f(x) = 0$ for $x = 0$. So a real zero is 0.

7. $y = 2x$
If $y = 0$, then $x = 0$. If $x = 0$, then $y = 0$.
Intercept: (0, 0)
y is a function of x. One-to-one.
Domain: all real numbers
Range: all real numbers



8. $y = x + 1$
If $y = 0$, then $x = -1$.
If $x = 0$, then $y = 1$.
Intercepts: (–1, 0), (0, 1)
y is a function of x. One-to-one.
Domain: all real numbers
Range: all real numbers

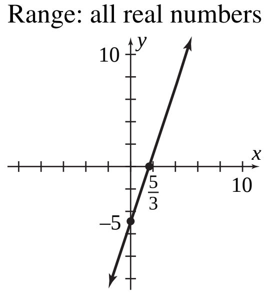


9. $y = 3x - 5$

If $y = 0$, then $0 = 3x - 5$, $x = \frac{5}{3}$.

If $x = 0$, then $y = -5$. Intercepts: $\left(\frac{5}{3}, 0\right)$, $(0, -5)$

y is a function of x. One-to-one.
Domain: all real numbers
Range: all real numbers

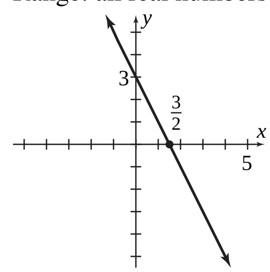


10. $y = 3 - 2x$

If $y = 0$, then $0 = 3 - 2x$, $x = \frac{3}{2}$.

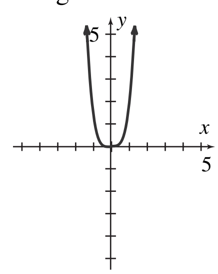
If $x = 0$, then, $y = 3$. Intercepts: $\left(\frac{3}{2}, 0\right)$, $(0, 3)$

y is a function of x. One-to-one.
Domain: all real numbers
Range: all real numbers



11. $y = x^4$

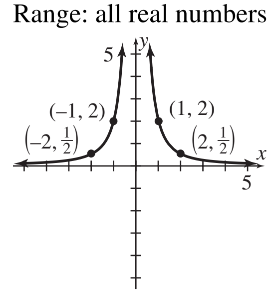
If $y = 0$, then $0 = x^4$, $x = 0$. If $x = 0$, then $y = 0$.
Intercept: (0, 0)
y is a function of x. Not one-to-one.
Domain: all real numbers
Range: all real numbers ≥ 0



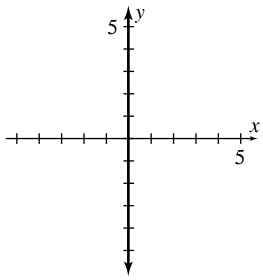
12. $y = \frac{2}{x^2}$

If $y = 0$, then $0 = \frac{2}{x^2}$, which has no solution.

Thus there is no x-intercept. Because $x \neq 0$,
Not one-to-one.
Domain: all real numbers except 0
Range: all real numbers > 0



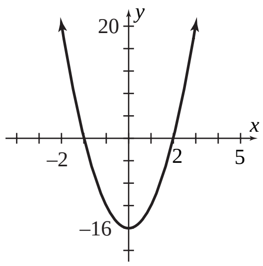
13. $x = 0$
If $y = 0$, then $x = 0$. If $x = 0$, then y can be any real number. Intercepts: every point on y-axis
y is not a function of x.



14. $y = 4x^2 - 16$

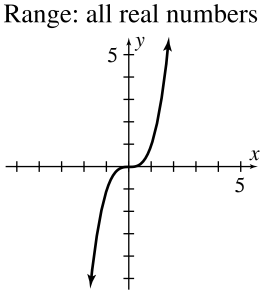
If $y = 0$, then $0 = 4x^2 - 16 = 4(x^2 - 4)$,

$0 = 4(x+2)(x-2)$, $x = \pm 2$.
If $x = 0$, then $y = -16$.
Intercepts: $(\pm 2, 0)$, $(0, -16)$
y is a function of x. Not one-to-one.
Domain: all real numbers
Range: all real numbers ≥ -16

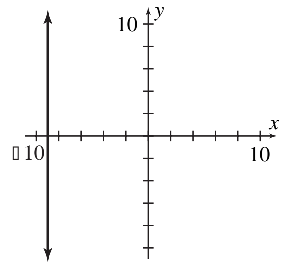


15. $y = x^3$

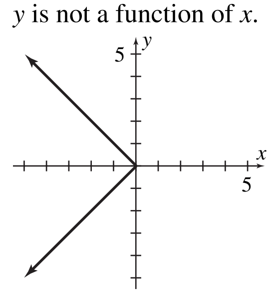
If $y = 0$, then $0 = x^3$, $x = 0$. If $x = 0$, then $y = 0$.
Intercept: $(0, 0)$. y is a function of x. One-to-one.
Domain: all real numbers
Range: all real numbers



16. $x = -9$
If $y = 0$ then $x = -9$. Because x cannot be 0, there is no y-intercept. Intercept: $(-9, 0)$.
y is not a function of x.



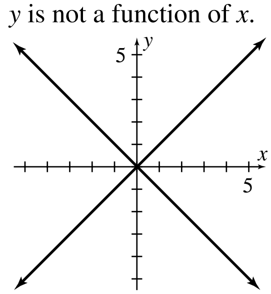
17. $x = -|y|$
If $y = 0$, then $x = 0$. If $x = 0$, then $0 = -|y|$, $y = 0$.
Intercept: $(0, 0)$
y is not a function of x.



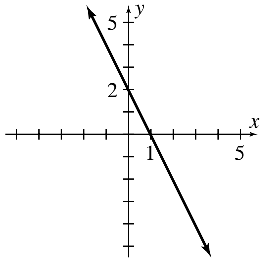
18. $x^2 = y^2$

If $y = 0$, then $x^2 = 0$, $x = 0$. If $x = 0$, then

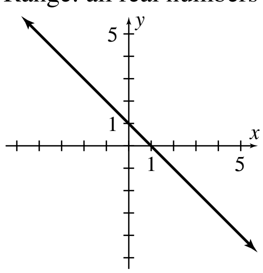
$0 = y^2$, $y = 0$. Intercept: $(0, 0)$
y is not a function of x.



19. $2x + y - 2 = 0$
If $y = 0$, then $2x - 2 = 0$, $x = 1$. If $x = 0$, then $y - 2 = 0$, $y = 2$. Intercepts: $(1, 0)$, $(0, 2)$
Note that $y = 2 - 2x$. y is a function of x.
One-to-one.
Domain: all real numbers
Range: all real numbers

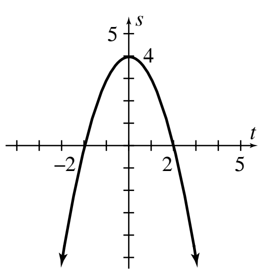


20. $x + y = 1$
If $y = 0$, then $x = 1$. If $x = 0$, then $y = 1$.
Intercepts: (1, 0), (0, 1)
Note that $y = 1 - x$.
y is a function of x. One-to-one.
Domain: all real numbers
Range: all real numbers



21. $s = f(t) = 4 - t^2$

If $s = 0$, then $0 = 4 - t^2$
$0 = (2 + t)(2 - t)$
$t = \pm 2$. If $t = 0$, then $s = 4$.
Intercepts: (2, 0), (–2, 0), (0, 4)
Domain: all real numbers
Range: all real numbers ≤ 4



22. $f(x) = 5 - 2x^2$. If $f(x) = 0$, then $0 = 5 - 2x^2$

$2x^2 = 5$

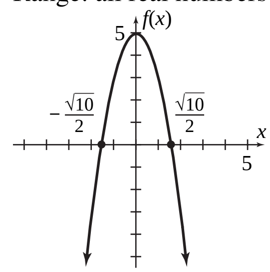
$x^2 = \frac{5}{2}$

$x = \pm\sqrt{\frac{5}{2}} = \pm\frac{\sqrt{10}}{2}$.

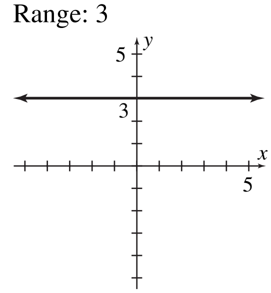
If $x = 0$, then $f(x) = 5$.

Intercepts: $\left(\pm\frac{\sqrt{10}}{2}, 0\right), (0, 5)$

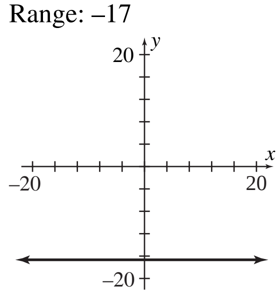
Domain: all real numbers
Range: all real numbers ≤ 5



23. $y = h(x) = 3$
Because y cannot be 0, there is no x-intercept. If $x = 0$, then $y = 3$. Intercept: (0, 3)
Domain: all real numbers
Range: 3

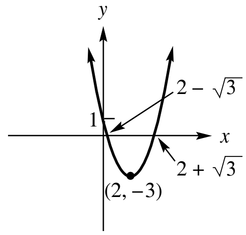


24. $g(s) = -17$
Because $g(s)$ cannot be 0, there is no s-intercept.
If $s = 0$, then $g(s) = -17$.
Intercept: (0, –17)
Domain: all real numbers
Range: –17



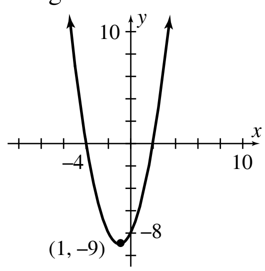
25. $y = h(x) = x^2 - 4x + 1$

If $y = 0$, then $0 = x^2 - 4x + 1$, and by the quadratic formula, $x = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$. If $x = 0$, then $y = 1$. Intercepts: $(2 \pm \sqrt{3}, 0), (0, 1)$
Domain: all real numbers
Range: all real numbers ≥ -3



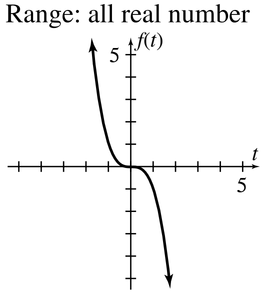
26. $y = f(x) = x^2 + 2x - 8$

If $y = 0$, then $0 = x^2 + 2x - 8$
$0 = (x + 4)(x - 2)$, so $x = -4, 2$.
If $x = 0$, then $y = -8$.
Intercepts: (−4, 0), (2, 0), (0, −8).
Domain: all real numbers
Range: all real numbers ≥ -9



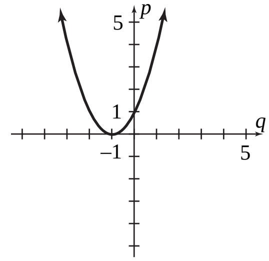
27. $f(t) = -t^3$

If $f(t) = 0$, then $0 = -t^3$, $t = 0$.
If $t = 0$, then $f(t) = 0$. Intercept: (0, 0)
Domain: all real numbers
Range: all real number



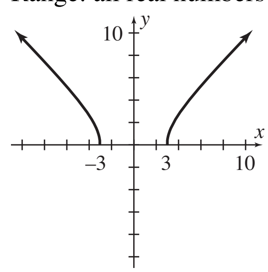
28. $p = h(q) = 1 + 2q + q^2$

If $p = 0$, then $1 + 2q + q^2 = 0$, $(1 + q)^2 = 0$, so $q = -1$. If $q = 0$ then $p = 1$.
Intercepts: (−1, 0), (0, 1)
Domain: all real numbers
Range: all real numbers ≥ 0



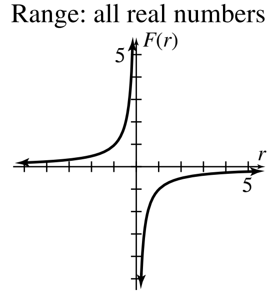
29. $s = f(t) = \sqrt{t^2 - 9}$

Note that for $\sqrt{t^2 - 9}$ to be a real number, $t^2 - 9 \geq 0$, so $t^2 \geq 9$, and $|t| \geq 3$. If $s = 0$, then $0 = \sqrt{t^2 - 9}$, $0 = t^2 - 9$, or $t = \pm 3$. Because $|t| \geq 3$, we know $t \neq 0$, so no *s*-intercept exists.
Intercepts: (−3, 0), (3, 0)
Domain: all real numbers $t \leq -3$ and ≥ 3
Range: all real numbers ≥ 0



30. $F(r) = -\dfrac{1}{r}$

If $F(r) = 0$, then $0 = -\dfrac{1}{r}$, which has no solution.
Because $r \neq 0$, there is no vertical-axis intercept. Intercept: none.
Domain: all real numbers $\neq 0$
Range: all real numbers $\neq 0$



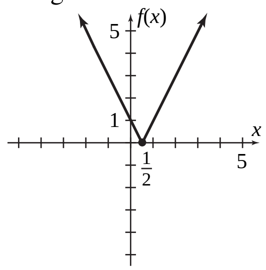
31. $f(x) = |2x - 1|$

If $f(x) = 0$, then $0 = |2x - 1|$, $2x - 1 = 0$, so $x = \dfrac{1}{2}$.
If $x = 0$, then $f(x) = |-1| = 1$.

Intercepts: $\left(\frac{1}{2}, 0\right)$, (0,1)

Domain: all real numbers

Range: all real numbers ≥ 0



32. $v = H(u) = |u - 3|$

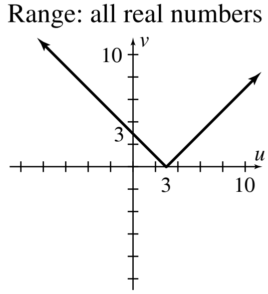
If $v = 0$, then $0 = |u - 3|$, $u - 3 = 0$, so $u = 3$.

If $u = 0$, then $v = |-3| = 3$.

Intercepts: (3, 0), (0, 3).

Domain: all real numbers

Range: all real numbers ≥ 0



33. $F(t) = \dfrac{16}{t^2}$

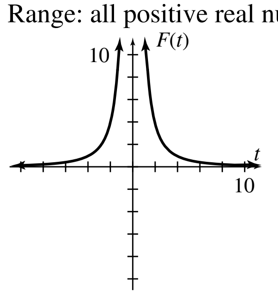
If $F(t) = 0$, then $0 = \dfrac{16}{t^2}$, which has no solution.

Because $t \neq 0$, there is no vertical-axis intercept.

No intercepts

Domain: all nonzero real numbers

Range: all positive real numbers



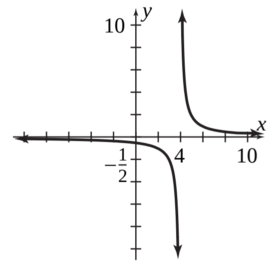
34. $y = f(x) = \dfrac{2}{x - 4}$

Note that the denominator is 0 when $x = 4$. Thus $x \neq 4$. If $y = 0$, then $0 = \dfrac{2}{x - 4}$, which has no solution. If $x = 0$, then $y = -\dfrac{1}{2}$.

Intercept: $\left(0, -\frac{1}{2}\right)$

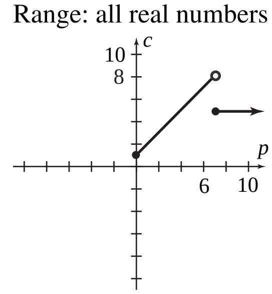
Domain: all real numbers except 4

Range: all real numbers except 0



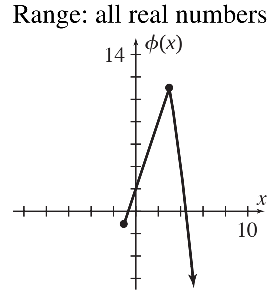
35. Domain: all real numbers ≥ 0

Range: all real numbers $1 \leq c < 8$



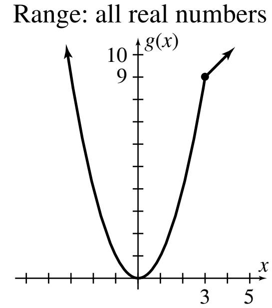
36. Domain: all real numbers ≥ -1

Range: all real numbers ≤ 11

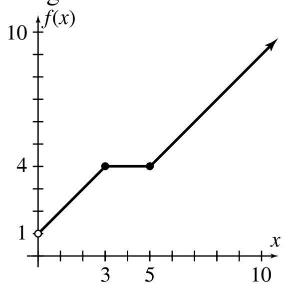


37. Domain: all real numbers

Range: all real numbers ≥ 0



38. Domain: all positive real numbers
Range: all real numbers > 1



39. From the vertical-line test, the graphs that represent functions of x are (a), (b), and (d).

40. From the horizontal line test, the graphs which represent one-to-one functions of x are (c) and (d).

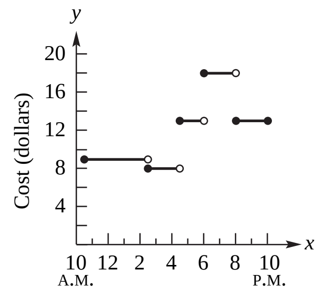
41. Let y = the amount that is owed and let x = the number of monthly payments made. Then, the amount Tara owes is represented by the equation $y = 2400 - 275x$.
To determine the x-intercept, we set $y = 0$ and solve for x.

$$y = 2400 - 275x$$
$$0 = 2400 - 275x$$
$$275x = 2400$$
$$x = 8\frac{8}{11}$$

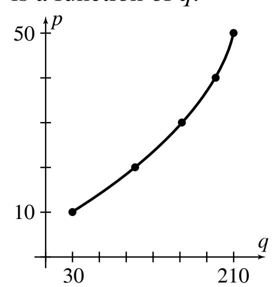
The x-intercept is $\left(8\frac{8}{11}, 0\right)$. Therefore, Tara will have paid off her debt after 9 months.
To determine the y-intercept, we set $x = 0$ and solve for y.
$y = 2400 - 275x$
$y = 2400 - 275(0)$
$y = 2400$
The y-intercept is (0, 2400). Therefore, before any payments are made, Tara owes \$2400.

42. The cost of an item as a function of the time of day, x is

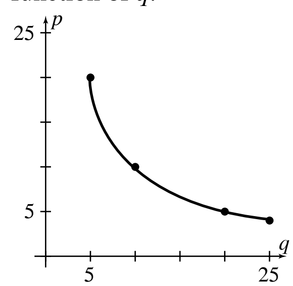
$$y = \begin{cases} 9, \text{ if } 10{:}30 \text{ A.M.} \le x < 2{:}30 \text{ P.M.} \\ 8, \text{ if } 2{:}30 \text{ P.M.} \le x < 4{:}30 \text{ P.M.} \\ 13, \text{ if } 4{:}30 \text{ P.M.} \le x < 6{:}00 \text{ P.M.} \\ 18, \text{ if } 6{:}00 \text{ P.M.} \le x < 8{:}00 \text{ P.M.} \\ 13, \text{ if } 8{:}00 \text{ P.M.} \le x \le 10{:}00 \text{ P.M.} \end{cases}$$



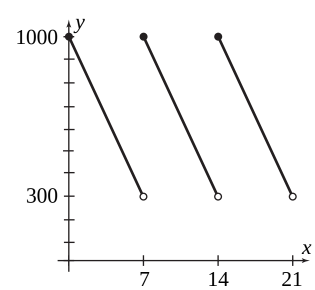
43. As price increases, quantity supplied increases; p is a function of q.



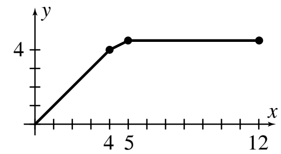
44. As price decreases, quantity increases; p is a function of q.



45.



46.



47. 0.39

48. –0.50, 0.57

49. –0.61, –0.04

50. 0.62, 1.73, 4.65

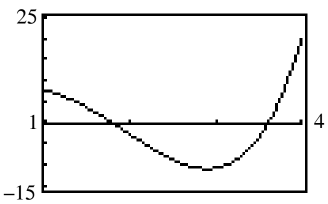
51. –1.12

52. No real zeros

53. –1.70, 0

54. –0.49, 0.52, 1.25

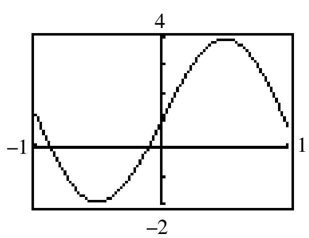
55.



a. maximum value of $f(x)$: 19.60

b. minimum value of $f(x)$: –10.86

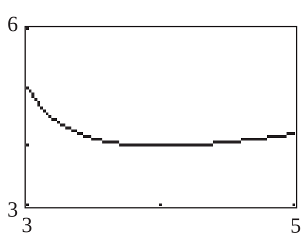
56.



a. maximum value of $f(x)$: 3.94

b. minimum value of $f(x)$: –1.94

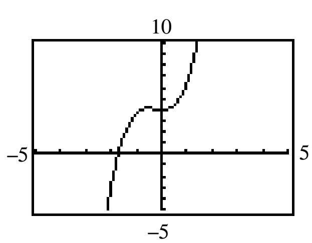
57.



a. maximum value of $f(x)$: 5

b. minimum value of $f(x)$: 4

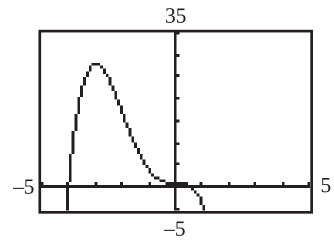
58.



a. range: $(-\infty, \infty)$

b. intercepts: (–1.73, 0), (0, 4)

59.

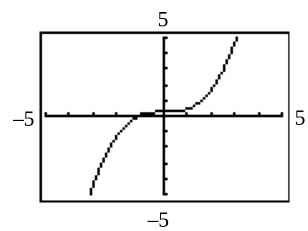


a. maximum value of $f(x)$: 28

b. range: $(-\infty, 28]$

c. real zeros: –4.02, 0.60

60.

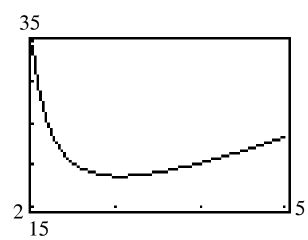


a. range: $(-\infty, \infty)$

b. intercepts: (0, 0.29), (–1.03, 0)

c. real zero: –1.03

61.



a. maximum value of $f(x)$: 34.21

b. minimum value of $f(x)$: 18.68

c. range: [18.68, 34.21]

d. no intercept

Problems 2.6

1. $y = 5x$
Intercepts: If $y = 0$, then $5x = 0$, or $x = 0$; if $x = 0$, then $y = 5 \cdot 0 = 0$.
Testing for symmetry gives:
x-axis: $-y = 5x$
$y = -5x$
y-axis: $y = 5(-x) = -5x$
origin: $-y = 5(-x)$
$y = 5x$
line $y = x$: (a, b) on graph, then $b = 5a$, and $a = \frac{1}{5}b \neq 5b$ for all b, so (b, a) is not on the graph.

Answer: (0, 0); symmetry about origin

2. $y = f(x) = x^2 - 4$
Intercepts: If $y = 0$, then $0 = x^2 - 4 = (x+2)(x-2)$, or $x = \pm 2$; if $x = 0$, then $y = 0^2 - 4 = -4$.
Testing for symmetry gives:
x-axis: $-y = x^2 - 4$
$y = -x^2 + 4$
y-axis: $y = (-x)^2 - 4 = x^2 - 4$
origin: $-y = (-x)^2 - 4$
$y = -x^2 + 4$
line $y = x$: (a, b) on graph, then $b = a^2 - 4$, and $a = \pm\sqrt{b+4} \neq b^2 - 4$ for all b, so (b, a) is not on the graph.
Answer: $(\pm 2, 0)$, $(0, -4)$; symmetry about y-axis

3. $2x^2 + y^2x^4 = 8 - y$
Intercepts: If $y = 0$, then $2x^2 = 8$, $x^2 = 4$, or $x = \pm 2$;
if $x = 0$, then $0 = 8 - y$, so $y = 8$.
Testing for symmetry gives:
x-axis: $2x^2 + (-y)^2x^4 = 8 - (-y)$
$2x^2 + y^2x^4 = 8 + y$
y-axis: $2(-x)^2 + y^2(-x)^4 = 8 - y$
$2x^2 + y^2x^4 = 8 - y$
origin: $2(-x)^2 + (-y)^2(-x)^4 = 8 - (-y)$
$2x^2 + y^2x^4 = 8 + y$
line $y = x$: (a, b) on graph, then $2a^2 + b^2a^4 = 8 - b$, but $2b^2 + a^2b^4 = 8 - a$ will not necessarily be true, so (b, a) is not on the graph.
Answer: $(\pm 2, 0)$, $(0, 8)$; symmetry about y-axis

4. $x = y^3$
Intercepts: If $y = 0$, then $x = 0$; if $x = 0$, then $0 = y^3$, so $y = 0$.
Testing for symmetry gives:
x-axis: $x = (-y)^3 = -y^3$
y-axis: $-x = y^3$
$x = -y^3$
origin: $-x = (-y)^3$
$x = y^3$
line $y = x$: (a, b) on graph, then $a = b^3$, and $b = \sqrt[3]{a} \neq a^3$ for all a, so (b, a) is not on the graph.
Answer: (0, 0); symmetry about origin

5. $16x^2 - 9y^2 = 25$
Intercepts: If $y = 0$, then $16x^2 = 25$, $x^2 = \frac{25}{16}$, so $x = \pm\frac{5}{4}$;
if $x = 0$, then $-9y^2 = 25$, $y^2 = -\frac{25}{9}$, which has no real root.
Testing for symmetry gives:
x-axis: $16x^2 - 9(-y)^2 = 25$
$16x^2 - 9y^2 = 25$
y-axis: $16(-x)^2 - 9y^2 = 25$
$16x^2 - 9y^2 = 25$
origin: Since the graph has symmetry about x- and y-axes, there is also symmetry about the origin.
line $y = x$: (a, b) on graph, then $16a^2 - 9b^2 = 25$, and $a^2 = \frac{1}{16}(9b^2 + 25)$. (b, a) on graph, then $16b^2 - 9a^2 = 25$ and $a^2 = \frac{1}{9}(16b^2 - 25) \neq \frac{1}{16}(9b^2 + 25)$

for all b, so (b, a) and (a, b) are not always both on the graph.

Answer: $\left(\pm\frac{5}{4}, 0\right)$; symmetry about x-axis, y-axis, and origin.

6. $y = 57$

Intercepts: Because $y \neq 0$, there is no x-intercept; if $x = 0$, then $y = 57$.

Testing for symmetry gives:

x-axis: $(-y) = 57$
$y = -57$

y-axis: $y = 57$

origin: $(-y) = 57$
$y = -57$

line $y = x$: (a, b) on graph, then $b = 57$, but a can be any value, so $(b, a) = (57, a)$ is not necessarily on the graph.

Answer: (0, 57); symmetry about y–axis

7. $x = -2$

Intercepts: If $y = 0$, then $x = -2$; because $x \neq 0$, there is no y-intercept.

Testing for symmetry gives:

x-axis: $x = -2$

y-axis: $-x = -2$
$x = 2$

origin: $-x = -2$
$x = 2$

line $y = x$: (a, b) on graph, then $a = -2$, but b can be any value, so $(b, a) = (b, -2)$ is not necessarily on the graph.

Answer: (–2, 0); symmetry about x-axis

8. $y = |2x| - 2$

Intercepts: If $y = 0$, then $|2x| = 2,\ 2|x| = 2,$ $|x| = 1,$ so $x = \pm 1$; if $x = 0$, then $y = -2$.

Testing for symmetry gives:

x-axis: $-y = |2x| - 2$
$y = -|2x| + 2$

y-axis: $y = |2(-x)| - 2$
$y = |2x| - 2$

origin: $-y = |2(-x)| - 2$
$y = -|2x| + 2$

line $y = x$: (a, b) on graph, then $b = |2a| - 2$ and
$a = \pm\frac{b+2}{2} \neq |2b| - 2$ for all b, so
(b, a) is not on the graph.

Answer: (±1, 0), (0, –2); symmetry about y-axis

9. $x = -y^{-4}$

Intercepts: Because $y \neq 0$, there is no x-intercept; if $x = 0$, then $0 = -\frac{1}{y^4}$, which has no solution.

Testing for symmetry gives:

x-axis: $x = -(-y)^{-4}$
$x = -y^{-4}$

y-axis: $-x = -y^{-4}$
$x = y^{-4}$

origin: $-x = -(-y)^{-4}$
$x = y^{-4}$

line $y = x$: (a, b) on graph, then $a = -b^{-4}$ and
$b = (-a)^{-1/4} \neq -a^{-4}$ for all a, so
(b, a) is not on the graph.

Answer: no intercepts; symmetry about x-axis

10. $y = \sqrt{x^2 - 25}$

Intercepts: If $y = 0$, then $\sqrt{x^2 - 25} = 0,$
$x^2 - 25 = 0,\ x^2 = 25,$ so $x = \pm 5$;
if $x = 0$, then $y = \sqrt{-25}$, which has no real root.

Testing for symmetry gives:

x-axis: $-y = \sqrt{x^2 - 25}$
$y = -\sqrt{x^2 - 25}$

y-axis: $y = \sqrt{(-x)^2 - 25}$
$y = \sqrt{x^2 - 25}$

origin: $-y = \sqrt{(-x)^2 - 25}$
$y = -\sqrt{x^2 - 25}$.

line $y = x$: (a, b) on graph, then $b = \sqrt{a^2 - 25}$ or
$b^2 = a^2 - 25$ and
$a^2 = b^2 + 25 \neq b^2 - 25$ for all b, so
(b, a) is not on the graph.

Answer: (±5, 0); symmetry about y-axis

11. $x - 4y - y^2 + 21 = 0$

Intercepts: If $y = 0$, then $x + 21 = 0$, so $x = -21$;
if $x = 0$, then $-4y - y^2 + 21 = 0,$
$y^2 + 4y - 21 = 0,\ (y + 7)(y - 3) = 0$, so $y = -7$ or $y = 3$.

Testing for symmetry gives:

x-axis: $x-4(-y)-(-y)^2+21=0$

$x+4y-y^2+21=0$

y-axis: $(-x)-4y-y^2+21=0$

$-x-4y-y^2+21=0$

origin: $(-x)-4(-y)-(-y)^2+21=0$

$-x+4y-y^2+21=0$

line $y = x$: (a, b) on graph, then $a-4b-b^2+21=0$ and $a=b^2+4b-21$, but $b=a^2+4a-21$ will not necessarily be true, so (b, a) is not on the graph.

Answer: (–21, 0), (0, –7), (0, 3); no symmetry

12. $x^2+xy+y^3=0$

Intercepts: If $y = 0$, then $x^2=0$, so $x = 0$;
if $x = 0$, then $y^3=0$, so $y = 0$.

Testing for symmetry gives:

x-axis: $x^2+x(-y)+(-y)^3=0$

$x^2-xy-y^3=0$

y-axis: $(-x)^2+(-x)y+y^3=0$

$x^2-xy+y^3=0$

origin: $(-x)^2+(-x)(-y)+(-y)^3=0$

$x^2+xy-y^3=0$

line $y = x$: (a, b) on graph, then $a^2+ab+b^3=0$, but $b^2+ab+a^3=0$ will not necessarily be true, so (b, a) is not on the graph.

Answer: (0, 0); no symmetry.

13. $y=f(x)=\dfrac{x^3-2x^2+x}{x^2+1}$

Intercepts: If $y = 0$, then
$\dfrac{x^3-2x^2+x}{x^2+1}=\dfrac{x(x-1)^2}{x^2+1}=0$, so $x=0, 1$;
if $x = 0$, then $y = 0$.

Testing for symmetry gives:

x-axis: Because *f* is not the zero function, there is no *x*-axis symmetry

y-axis: $y=\dfrac{(-x)^3-2(-x)^2+(-x)}{(-x)^2+1}$

$y=\dfrac{-x^3-2x^2-x}{x^2+1}$

origin: $-y=\dfrac{(-x)^3-2(-x)^2+(-x)}{(-x)^2+1}$

$y=\dfrac{x^3+2x^2+x}{x^2+1}$

line $y = x$: (a, b) on graph, then $b=\dfrac{a^3-2a^2+a}{a^2+1}$, but $a=\dfrac{b^3-2b^2+b}{b^2+1}$ is not necessarily true, so (b, a) is not on the graph.

Answer: (1, 0), (0, 0); no symmetry of the given types

14. $x^2+xy+y^2=0$

Intercepts: If $y = 0$, then $x^2=0$, so $x = 0$;
if $x = 0$, then $y^2=0$, so $y = 0$.

Testing for symmetry gives:

x-axis: $x^2+x(-y)+(-y)^2=0$

$x^2-xy+y^2=0$

y-axis: $(-x)^2+(-x)y+y^2=0$

$x^2-xy+y^2=0$

origin: $(-x)^2+(-x)(-y)+(-y)^2=0$

$x^2+xy+y^2=0$

line $y = x$: (a, b) on graph, then $a^2+ab+b^2=0$ and $b^2+ba+a^2=0$, so (b, a) is on the graph.

Answer: (0, 0); symmetry about origin, symmetry about $y = x$

15. $y=\dfrac{3}{x^3+8}$

Intercepts: If $y = 0$, then $\dfrac{3}{x^3+8}=0$, which has no solution; if $x = 0$, then $y=\dfrac{3}{8}$.

Testing for symmetry gives:

x-axis: $-y = \frac{3}{x^3+8}$

$y = -\frac{3}{x^3+8}$

y-axis: $y = \frac{3}{(-x)^3+8}$

$y = \frac{3}{-x^3+8}$

origin: $-y = \frac{3}{(-x)^3+8}$

$-y = \frac{3}{-x^3+8}$

$y = \frac{3}{x^3-8}$

line $y = x$: (a, b) on graph, then $b = \frac{3}{a^3+8}$ and

$a = \sqrt[3]{\frac{3}{b}-8} \neq \frac{3}{b^3+8}$ for all b, so (b, a) is not on the graph.

Answer: $\left(0, \frac{3}{8}\right)$; no symmetry of the given types

16. $y = \frac{x^4}{x+y}$

Intercepts: If $y = 0$, then $\frac{x^4}{x} = 0$, which has no solution; if $x = 0$, then $y = \frac{0}{y}$, which has no solution.

Testing for symmetry gives:

x-axis: $-y = \frac{x^4}{x+(-y)}$

$y = \frac{x^4}{-x+y}$

y-axis: $y = \frac{(-x)^4}{(-x)+y}$

$y = \frac{x^4}{-x+y}$

origin: $-y = \frac{(-x)^4}{(-x)+(-y)}$

$y = \frac{x^4}{x+y}$

line $y = x$: (a, b) on graph, then $b = \frac{a^4}{a+b}$, and

$a+b = \frac{a^4}{b}$, but $a+b = \frac{b^4}{a}$ will not necessarily be true, so (b, a) is not on the graph.

Answer: no intercepts; symmetry about origin

17. $3x + y^2 = 9$

Intercepts: If $y = 0$, then $3x = 9$, so $x = 3$; if $x = 0$, then $y^2 = 9$, so $y = \pm 3$.

Testing for symmetry gives:

x-axis: $3x + (-y)^2 = 9$

$3x + y^2 = 9$

y-axis: $3(-x) + y^2 = 9$

$-3x + y^2 = 9$

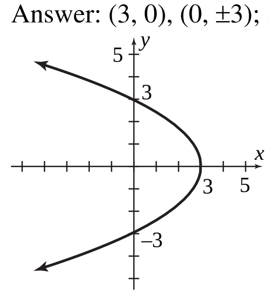
origin: $3(-x) + (-y)^2 = 9$

$-3x + y^2 = 9$

line $y = x$: (a, b) on graph, then $3a + b^2 = 9$ and

$a = \frac{1}{3}(9 - b^2)$, but $b = \frac{1}{3}(9 - a^2)$ will not necessarily be true, so (b, a) is not on the graph.

Answer: $(3, 0)$, $(0, \pm 3)$; symmetry about x-axis



18. $x - 1 = y^4 + y^2$ or $x = y^4 + y^2 + 1$

Intercepts: If $y = 0$, then $x = 1$; if $x = 0$, then $y^4 + y^2 = -1$, so no y-intercept

Testing for symmetry gives:

x-axis: $x - 1 = (-y)^4 + (-y)^2$

$x - 1 = y^4 + y^2$

y-axis: $-x = y^4 + y^2 + 1$

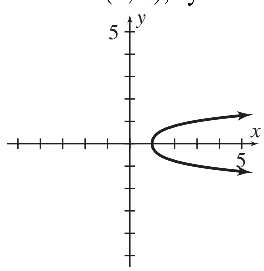
$x = -y^4 - y^2 - 1$

origin: $-x = (-y)^4 + (-y)^2 + 1$

$x = -y^4 - y^2 - 1$

line $y = x$: (a, b) on graph, then $a = b^4 + b^2 + 1$ and $b \neq a^4 + a^2 + 1$ for all a so (b, a) is not on the graph.

Answer: (1, 0); symmetry about x-axis.



19. $y = f(x) = x^3 - 4x$

Intercepts: If $y = 0$, then $x^3 - 4x = 0$, $x(x + 2)(x - 2) = 0$, so $x = 0$ or $x = \pm 2$; if $x = 0$, then $y = 0$.

Testing for symmetry gives:

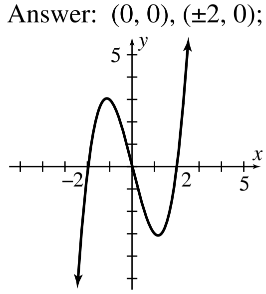
x-axis: Because f is not the zero function, there is no x-axis symmetry.

y-axis: $y = (-x)^3 - 4(-x)$
$y = -x^3 + 4x$

origin: $-y = (-x)^3 - 4(-x)$
$y = x^3 - 4x$

line $y = x$: (a, b) on graph, then $b = a^3 - 4a$, but $a = b^3 - 4b$ will not necessarily be true, so (b, a) is not on the graph.

Answer: (0, 0), (±2, 0); symmetry about origin.



20. $3y = 5x - x^3$

Intercepts: If $y = 0$, then $5x - x^3 = 0$, $x\left(\sqrt{5} + x\right)\left(\sqrt{5} - x\right) = 0$, so $x = 0$ or $x = \pm\sqrt{5}$; if $x = 0$, then $y = 0$.

Testing for symmetry gives:

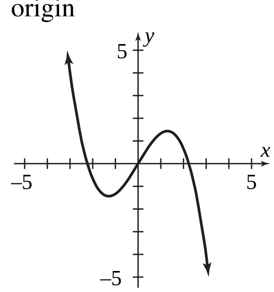
x-axis: $3(-y) = 5x - x^3$
$3y = -5x + x^3$

y-axis: $3y = 5(-x) - (-x)^3$
$3y = -5x + x^3$

origin: $3(-y) = 5(-x) - (-x)^3$
$3y = 5x - x^3.$

line $y = x$: (a, b) on graph, then $3b = 5a - a^3$, but $3a = 5b - b^3$ will not necessarily be true so (b, a) is not on the graph.

Answer: (0, 0), $\left(\pm\sqrt{5}, 0\right)$; symmetry about origin



21. $|x| - |y| = 0$

Intercepts: If $y = 0$, then $|x| = 0$, so $x = 0$; if $x = 0$, then $-|y| = 0$, so $y = 0$.

Testing for symmetry gives:

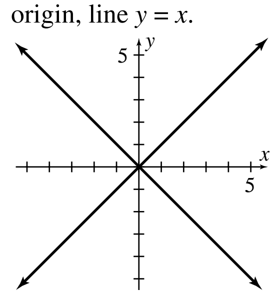
x-axis: $|x| - |-y| = 0$
$|x| - |y| = 0$

y-axis: $|-x| - |y| = 0$
$|x| - |y| = 0$

origin: Since there is symmetry about the x- and y-axes, symmetry about origin exists.

line $y = x$: (a, b) on graph, then $|a| - |b| = 0$, thus $|a| = |b|$, and $|b| - |a| = 0$, so (b, a) is on the graph.

Answer: (0, 0); symmetry about x-axis, y-axis, origin, line $y = x$.



22. $x^2 + y^2 = 16$

Intercepts: If $y = 0$, then $x^2 = 16$, so $x = \pm 4$; if $x = 0$, then $y^4 = 16$, so $y = \pm 4$.

Testing for symmetry gives:

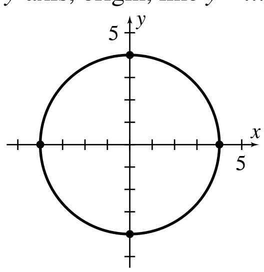
x-axis: $x^2 + (-y)^2 = 16$
$x^2 + y^2 = 16$

y-axis: $(-x)^2 + y^2 = 16$
$x^2 + y^2 = 16$

origin: Since there is symmetry about *x*- and *y*-axes, symmetry about origin exists.

line $y = x$: (a, b) on graph, then $a^2 + b^2 = 16$ and $b^2 + a^2 = 16$, so (b, a) is on the graph.

Answer: $(\pm 4, 0)$, $(0, \pm 4)$; symmetry about *x*-axis, *y*-axis, origin, line $y = x$.



23. $9x^2 + 4y^2 = 25$

Intercepts: If $y = 0$, then $9x^2 = 25$, $x^2 = \frac{25}{9}$, so $x = \pm\frac{5}{3}$; if $x = 0$, then $4y^2 = 25$, so $y = \pm\frac{5}{2}$.

Testing for symmetry gives:

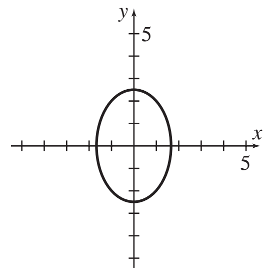
x-axis: $9x^2 + 4(-y)^2 = 25$
$9x^2 + 4y^2 = 25$

y-axis: $9(-x)^2 + 4y^2 = 25$
$9x^2 + 4y^2 = 25$

origin: Since there is symmetry about *x*- and *y*-axes, symmetry about origin exists.

line $y = x$: (a, b) on graph, then $9a^2 + 4b^2 = 25$ and $b^2 = \frac{1}{4}(25 - 9a^2)$. (b, a) on graph, then $9b^2 + 4a^2 = 25$ and $b^2 = \frac{1}{9}(25 - 4a^2)$, so (a, b) and (b, a) are not always both on the graph.

Answer: $\left(\pm\frac{5}{3}, 0\right)$, $\left(0, \pm\frac{5}{2}\right)$; symmetry about *x*-axis, *y*-axis, origin



24. $x^2 - y^2 = 4$

Intercepts: If $y = 0$, then $x^2 = 4$, so $x = \pm 2$; if $x = 0$, then $-y^2 = 4$, $y^2 = -4$, which has no real roots.

Testing for symmetry gives:

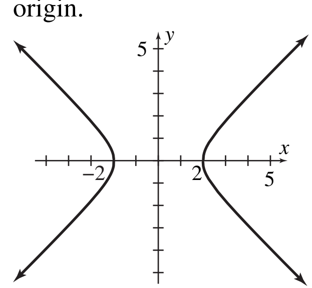
x-axis: $x^2 - (-y)^2 = 4$
$x^2 - y^2 = 4$

y-axis: $(-x)^2 - y^2 = 4$
$x^2 - y^2 = 4$

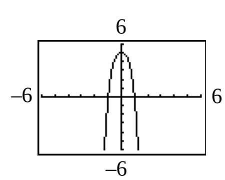
origin: Since there is symmetry about *x*-and *y*-axes, symmetry about origin exists.

line $y = x$: (a, b) on graph, then $a^2 - b^2 = 4$ and $a^2 = 4 + b^2 \neq b^2 - 4$ for all *b*, so (b, a) is not on the graph.

Answer: $(\pm 2, 0)$; symmetry about *x*-axis, *y*-axis, origin.



25.



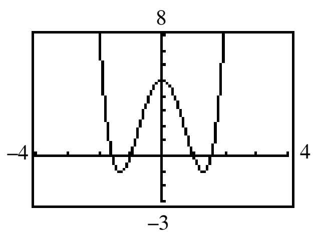
$y = f(x) = 5 - 1.96x^2 - \pi x^4$. Replacing *x* by $-x$ gives $y = 5 - 1.96(-x)^2 - \pi(-x)^4$ or $y = 5 - 1.96x^2 - \pi x^4$, which is equivalent to original equation. Thus the graph is symmetric about the y-axis.

a. Intercepts: $(\pm 0.99, 0)$, $(0, 5)$

b. Maximum value of $f(x)$: 5

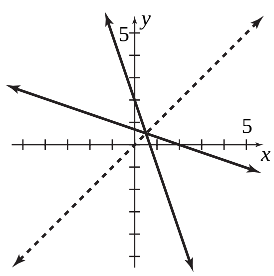
c. Range: $(-\infty, 5]$

26.



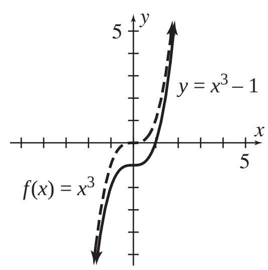
$y = f(x) = 2x^4 - 7x^2 + 5$. Replacing x by $-x$ gives $y = 2(-x)^4 - 7(-x)^2 + 5$ or $y = 2x^4 - 7x^2 + 5$, which is equivalent to original equation. Thus the graph is symmetric about y-axis.
Real zeros of f: ± 1, ± 1.58

27.



Problems 2.7

1.



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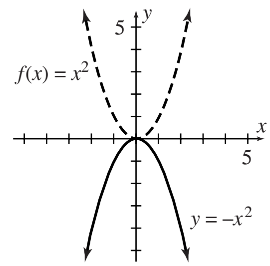

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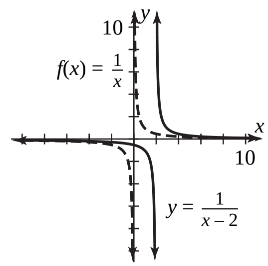

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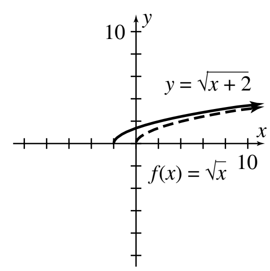

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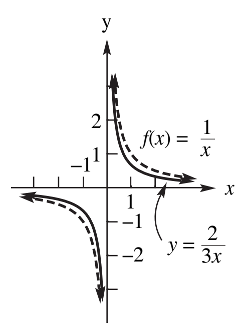

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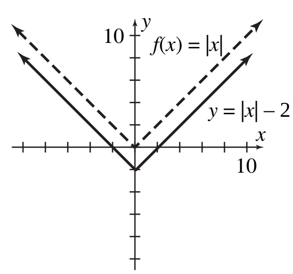

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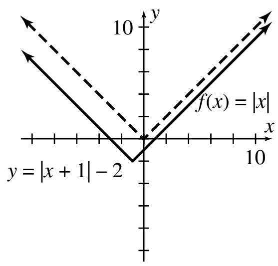

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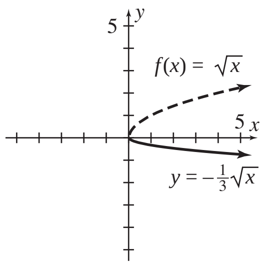

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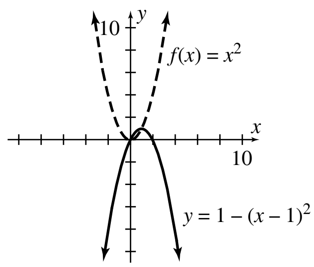

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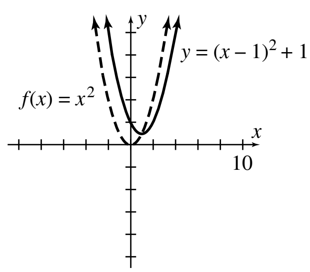

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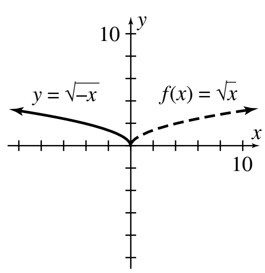

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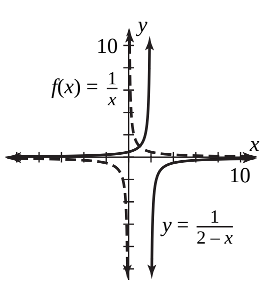


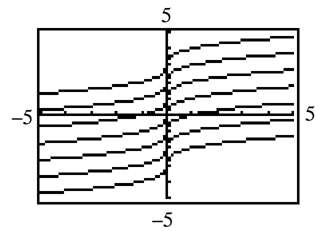
13. Translate 3 units to the left, stretch vertically away from the x-axis by a factor of 2, reflect about the x-axis, and move 2 units upward.

14. Translate 3 units to the left and 4 units downward.

15. Reflect about the y-axis and translate 5 units downward.

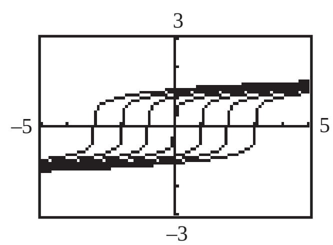
16. Shrink horizontally toward the y-axis by a factor of 3.

17.



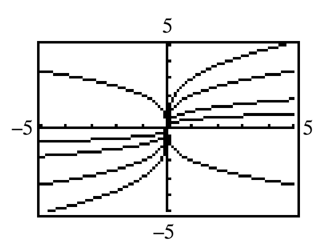
Compared to the graph for $k = 0$, the graphs for $k = 1$, 2, and 3 are vertical shifts upward of 1, 2, and 3 units, respectively. The graphs for $k = -1$, -2, and -3 are vertical shifts downward of 1, 2, and 3 units, respectively.

18.



Compared to the graph for $k = 0$, the graphs for $k = 1$, 2, and 3 are horizontal shifts to the left of 1, 2, and 3 units, respectively. The graphs for $k = -1$, -2, and -3 are horizontal shifts to the right of 1, 2, and 3 units, respectively.

19.



Compared to the graph for $k = 1$, the graphs for $k = 2$ and 3 are vertical stretches away from the x-axis by factors of 2 and 3, respectively. The graph for $k = \frac{1}{2}$ is a vertical shrinking toward the x-axis.

Chapter 2 Review Problems

1. Denominator is 0 when
$x^2 - 6x + 5 = 0$
$(x-1)(x-5) = 0$
$x = 1, 5$
Domain: all real numbers except 1 and 5.

2. all real numbers

3. all real numbers

4. all real numbers

5. For $\sqrt{x}$ to be real, x must be nonnegative. For the denominator $x - 1$ to be different from 0, x cannot be 1. Both conditions are satisfied by all nonnegative numbers except 1.
Domain: all nonnegative real numbers except 1.

6. $s - 5 \geq 0$
$s \geq 5$
Domain: all real numbers s such that $s \geq 5$.

7. $f(x) = 3x^2 - 4x + 7$
$f(0) = 3(0)^2 - 4(0) + 7 = 7$
$f(-3) = 3(-3)^2 - 4(-3) + 7 = 27 + 12 + 7 = 46$
$f(5) = 3(5)^2 - 4(5) + 7 = 75 - 20 + 7 = 62$
$f(t) = 3t^2 - 4t + 7$

8. $h(x) = 7$; all function values are 7.
Answer: 7, 7, 7, 7

9. $G(x) = \sqrt[4]{x-3}$
$G(3) = \sqrt[4]{3-3} = \sqrt[4]{0} = 0$
$G(19) = \sqrt[4]{19-3} = \sqrt[4]{16} = 2$
$G(t+1) = \sqrt[4]{(t+1)-3} = \sqrt[4]{t-2}$
$G(x^3) = \sqrt[4]{x^3 - 3}$

10. $F(x) = \frac{x-3}{x+4}$
$F(-1) = \frac{-1-3}{-1+4} = -\frac{4}{3}$
$F(0) = \frac{0-3}{0+4} = -\frac{3}{4}$
$F(5) = \frac{5-3}{5+4} = \frac{2}{9}$
$F(x+3) = \frac{(x+3)-3}{(x+3)+4} = \frac{x}{x+7}$

11. $h(u) = \frac{\sqrt{u+4}}{u}$
$h(5) = \frac{\sqrt{5+4}}{5} = \frac{\sqrt{9}}{5} = \frac{3}{5}$
$h(-4) = \frac{\sqrt{-4+4}}{-4} = \frac{0}{-4} = 0$
$h(x) = \frac{\sqrt{x+4}}{x}$
$h(u-4) = \frac{\sqrt{(u-4)+4}}{u-4} = \frac{\sqrt{u}}{u-4}$

12. $H(s) = \frac{(s-4)^2}{3}$
$H(-2) = \frac{(-2-4)^2}{3} = \frac{36}{3} = 12$
$H(7) = \frac{(7-4)^2}{3} = \frac{9}{3} = 3$
$H\left(\frac{1}{2}\right) = \frac{\left[\left(\frac{1}{2}\right)-4\right]^2}{3} = \frac{\left(-\frac{7}{2}\right)^2}{3} = \frac{\frac{49}{4}}{3} = \frac{49}{12}$
$H\left(x^2\right) = \frac{(x^2-4)^2}{3} = \frac{x^4 - 8x^2 + 16}{3}$

13. $f(4) = 4 + 16 = 20$
$f(-2) = -3$
$f(0) = -3$
$f(1)$ is not defined.

14. $f\left(-\frac{1}{2}\right) = -\left(-\frac{1}{2}\right) + 1 = \frac{1}{2} + 1 = \frac{3}{2}$
$f(0) = 0^2 + 1 = 1$
$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 + 1 = \frac{1}{4} + 1 = \frac{5}{4}$
$f(5) = 5^3 - 99 = 125 - 99 = 26$
$f(6) = 6^3 - 99 = 216 - 99 = 117$

15. a. $f(x+h) = 3 - 7(x+h) = 3 - 7x - 7h$

b. $\frac{f(x+h)-f(x)}{h} = \frac{(3-7x-7h)-(3-7x)}{h} = \frac{-7h}{h} = -7$

16. a. $f(x+h) = 11(x+h)^2+4$
$= 11x^2+22xh+11h^2+4$

b. $\frac{f(x+h)-f(x)}{h} = \frac{(11x^2+22xh+11h^2+4)-(11x^2+4)}{h} = \frac{22xh+11h^2}{h} = 22x+11h$

17. a. $f(x+h) = 4(x+h)^2+2(x+h)-5 = 4x^2+8xh+4h^2+2x+2h-5$

b. $\frac{f(x+h)-f(x)}{h} = \frac{(4x^2+8xh+4h^2+2x+2h-5)-(4x^2+2x-5)}{h}$
$= \frac{8xh+4h^2+2h}{h}$
$= 8x+4h+2$

18. a. $f(x+h) = \frac{7}{(x+h)+1} = \frac{7}{x+h+1}$

b. $\frac{f(x+h)-f(x)}{h} = \frac{\frac{7}{x+h+1}-\frac{7}{x+1}}{h} = \frac{\frac{7(x+1)-7(x+h+1)}{(x+h+1)(x+1)}}{h} = \frac{-7h}{(x+h+1)(x+1)h} = \frac{-7}{(x+h+1)(x+1)}$

19. $f(x) = 3x-1$, $g(x) = 2x+3$

a. $(f+g)(x) = f(x)+g(x) = (3x-1)+(2x+3) = 5x+2$

b. $(f+g)(4) = 5(4)+2 = 22$

c. $(f-g)(x) = f(x)-g(x) = (3x-1)-(2x+3) = x-4$

d. $(fg)(x) = f(x)g(x) = (3x-1)(2x+3) = 6x^2+7x-3$

e. $(fg)(1) = 6(1)^2+7(1)-3 = 10$

f. $\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{3x-1}{2x+3}$

g. $(f\circ g)(x) = f(g(x)) = f(2x+3) = 3(2x+3)-1 = 6x+8$

h. $(f\circ g)(5) = 6(5)+8 = 38$

i. $(g\circ f)(x) = g(f(x)) = g(3x-1) = 2(3x-1)+3 = 6x+1$

20. $f(x) = -x^2$, $g(x) = 3x-2$

a. $(f+g)(x) = f(x)+g(x) = -x^2+3x-2$

b. $(f-g)(x) = f(x)-g(x) = -x^2-(3x-2) = -x^2-3x+2$

c. $(f-g)(-3) = -(-3)^2 - 3(-3) + 2 = 2$

d. $(fg)(x) = f(x)g(x)$
$= (-x^2)(3x-2)$
$= -3x^3 + 2x^2$

e. $\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)} = \dfrac{-x^2}{3x-2}$

f. $\dfrac{f}{g}(2) = \dfrac{-(2)^2}{3(2)-2} = -1$

g. $(f \circ g)(x) = f(g(x))$
$= f(3x-2)$
$= -(3x-2)^2$
$= -9x^2 + 12x - 4$

h. $(g \circ f)(x) = g(f(x))$
$= g(-x^2)$
$= 3(-x^2) - 2$
$= -3x^2 - 2$

i. $(g \circ f)(-4) = -3(-4)^2 - 2 = -48 - 2 = -50$

21. $f(x) = \dfrac{1}{x^2}$, $g(x) = x + 1$

$(f \circ g)(x) = f(g(x)) = f(x+1) = \dfrac{1}{(x+1)^2}$

$(g \circ f)(x) = g(f(x)) = g\left(\dfrac{1}{x^2}\right) = \dfrac{1}{x^2} + 1 = \dfrac{1+x^2}{x^2}$

22. $f(x) = \dfrac{x+1}{4}$, $g(x) = \sqrt{x}$

$(f \circ g)(x) = f(g(x)) = f\left(\sqrt{x}\right) = \dfrac{\sqrt{x}+1}{4}$

$(g \circ f)(x) = g(f(x)) = g\left(\dfrac{x+1}{4}\right) = \sqrt{\dfrac{x+1}{4}}$

$= \dfrac{\sqrt{x+1}}{2}$

23. $f(x) = \sqrt{x+2}$, $g(x) = x^3$

$(f \circ g)(x) = f(g(x)) = f(x^3) = \sqrt{x^3+2}$

$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x+2}\right) = \left(\sqrt{x+2}\right)^3$

$= (x+2)^{3/2}$

24. $f(x) = 2$, $g(x) = 3$
$(f \circ g)(x) = f(g(x)) = f(3) = 2$
$(g \circ f)(x) = g(f(x)) = g(2) = 3$

25. $y = 3x - x^3$

Intercepts: If $y = 0$, then $0 = 3x - x^3$,
$x(3 - x^2) = 0$, $x = 0, \pm\sqrt{3}$.
If $x = 0$, then $y = 0$.
Testing for symmetry gives:

x-axis: $-y = 3x - x^3$
$y = -3x + x^3$, which is not the original equation.

y-axis: $y = 3(-x) - (-x)^3$
$y = -3x + x^3$

origin: $-y = 3(-x) - (-x)^3$
$y = 3x - x^3$, which is the original equation.

line $y = x$: (a, b) on graph, then $b = 3a - a^3$, but $a = 3b - b^3$ is not necessarily true, so (b, a) is not on the graph.

Answer: $(0, 0)$, $\left(\pm\sqrt{3}, 0\right)$; symmetry about origin

26. $\dfrac{x^2y^2}{x^2 + y^2 + 1} = 4$

Intercepts: If $y = 0$, then $0 = 4$, which is impossible; if $x = 0$, then $0 = 4$, which is impossible.
Testing for symmetry gives:

x-axis: $\dfrac{x^2(-y)^2}{x^2 + (-y)^2 + 1} = 4$

$\dfrac{x^2y^2}{x^2 + y^2 + 1} = 4$, which is the original equation.

y-axis: $\frac{(-x)^2 y^2}{(-x)^2 + y^2 + 1} = 4$

$\frac{x^2 y^2}{x^2 + y^2 + 1} = 4$, which is the original equation.

origin: $\frac{(-x)^2 (-y)^2}{(-x)^2 + (-y)^2 + 1} = 4$

$\frac{x^2 y^2}{x^2 + y^2 + 1} = 4$, which is the original equation.

line $y = x$: (a, b) on graph, then $\frac{a^2 b^2}{a^2 + b^2 + 1} = 4$

and $b^2 = \frac{4(a^2 + 1)}{a^2 - 4}$. (b, a) on graph,

then $\frac{b^2 a^2}{b^2 + a^2 + 1} = 4$ and

$b^2 = \frac{4(a^2 + 1)}{a^2 - 4}$, so (a, b) and (b, a) are both on the graph.

Answer: no intercepts; symmetry about x-axis, y-axis, origin, and $y = x$.

27. $y = 9 - x^2$

Intercepts: If $y = 0$, then

$0 = 9 - x^2 = (3 + x)(3 - x)$, or $x = \pm 3$

If $x = 0$, then $y = 9$.

Testing for symmetry gives:

x-axis: $-y = 9 - x^2$

$y = -9 + x^2$, which is not the original equation.

y-axis: $y = 9 - (-x)^2$

$y = 9 - x^2$, which is the original equation.

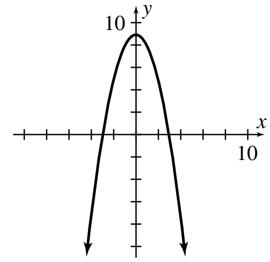
origin: $-y = 9 - (-x)^2$

$y = -9 + x^2$, which is not the original equation.

line $y = x$: (a, b) on graph, then $b = 9 - a^2$ and

$a = \pm\sqrt{9 - b} \neq 9 - b^2$ for all b, so (b, a) is not on the graph.

Answer: $(0, 9)$, $(\pm 3, 0)$; symmetry about y-axis.



28. $y = 3x - 7$

Intercepts: If $y = 0$, then $0 = 3x - 7$, or $x = \frac{7}{3}$.

If $x = 0$, then $y = -7$.

Testing for symmetry gives:

x-axis: $-y = 3x - 7$

$y = -3x + 7$, which is not the original equation.

y-axis: $y = 3(-x) - 7$

$y = -3x - 7$, which is not the original equation.

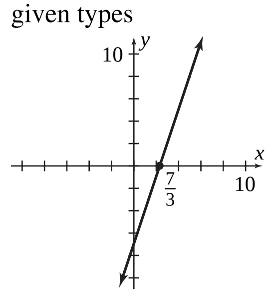
origin: $-y = 3(-x) - 7$

$y = 3x + 7$, which is not the original equation.

line $y = x$: (a, b) on graph, then $b = 3a - 7$ and

$a = \frac{1}{3}(b + 7) \neq 3b - 7$ for all b, so (b, a) is not on the graph.

Answer: $(0, -7)$, $\left(\frac{7}{3}, 0\right)$; no symmetry of the given types



29. $G(u) = \sqrt{u + 4}$

If $G(u) = 0$, then $0 = \sqrt{u + 4}$.

$0 = u + 4$,

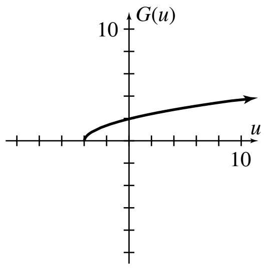
$u = -4$

If $u = 0$, then $G(u) = \sqrt{4} = 2$.

Intercepts: $(0, 2)$, $(-4, 0)$

Domain: all real numbers u such that $u \geq -4$

Range: all real numbers ≥ 0

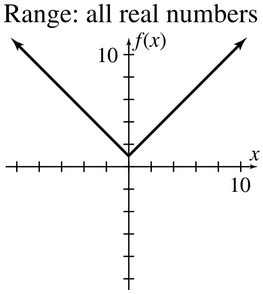


30. $f(x) = |x| + 1$

If $f(x) = 0$, then $0 = |x| + 1$.

$|x| = -1$, which has no solution.

If $x = 0$, then $f(x) = 1$.
Intercept: (0, 1)
Domain: all real numbers
Range: all real numbers ≥ 1



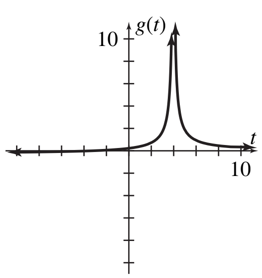
31. $y = g(t) = \dfrac{2}{|t-4|}$

If $y = 0$, then $0 = \dfrac{2}{|t-4|}$, which has no solution.

If $t = 0$, then $y = \dfrac{2}{4} = \dfrac{1}{2}$.

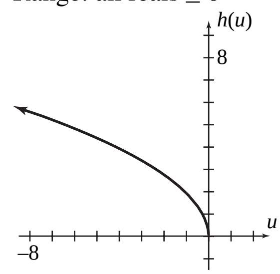
Intercept: $\left(0, \dfrac{1}{2}\right)$

Domain: all real numbers t such that $t \neq 4$
Range: all real numbers > 0

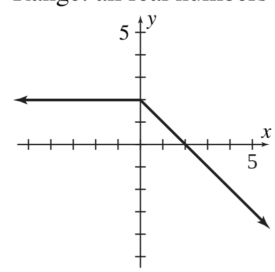


32. $h(u) = \sqrt{-5u}$

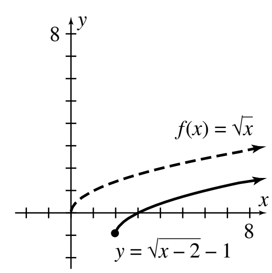
If $h(u) = 0$, then $0 = \sqrt{-5u}$,
$u = 0$.
If $u = 0$, $h(u) = 0$.
Intercept: (0, 0)
Domain: all reals ≤ 0
Range: all reals ≥ 0



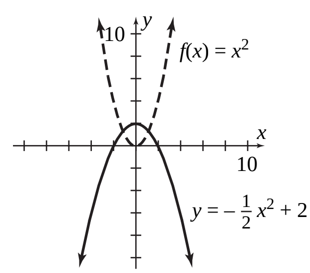
33. Domain: all real numbers.
Range: all real numbers ≤ 2



34.



35.



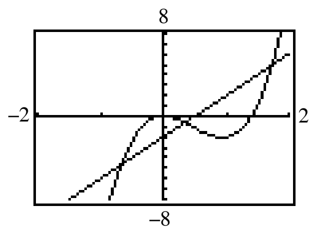
36. For 2006, $t = 5$. Hence
$S = 150{,}000 + 3000(5) = \$165{,}000$.
S is a function of t.

37. From the vertical-line test, the graphs that represent functions of x are (a) and (c).

38. **a.** 729

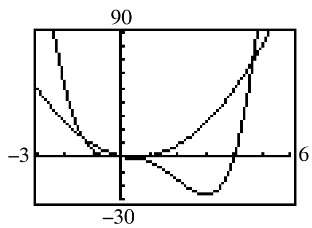
b. 359.43

39.



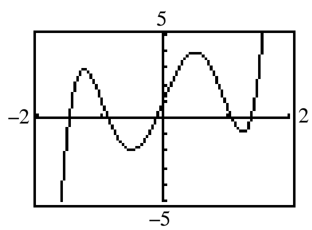
–0.67; 0.34, 1.73

40.



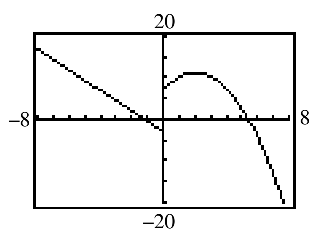
–1.38, 4.68

41.



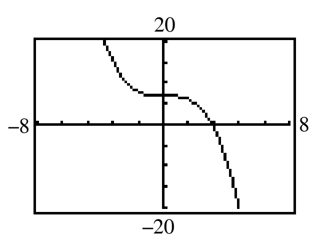
–1.50, –0.88, –0.11, 1.09, 1.40

42.



$(-\infty,\infty)$

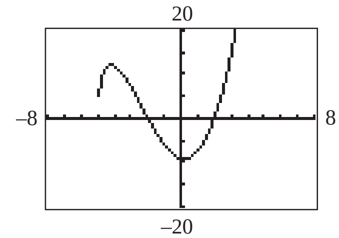
43.



a. $(-\infty,\infty)$

b. (1.92, 0), (0, 7)

44.

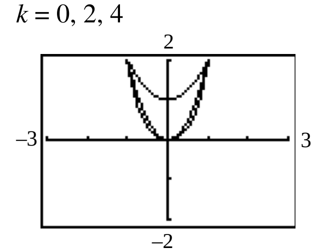


a. –9.03

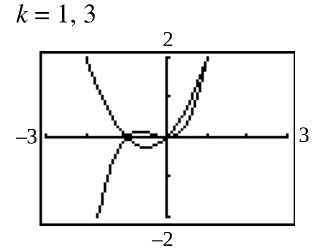
b. all real numbers ≥ -9.03

c. –5, ±2.

45. $k = 0, 2, 4$



$k = 1, 3$



a. 0, 2, 4

b. none

Mathematical Snapshot Chapter 2

1. $$\begin{aligned} f(23{,}000) &= 1510 + 0.15(23{,}000 - 15{,}100) \\ &= 2695 \end{aligned}$$
The tax on $23,000 is $2695.

2. $$\begin{aligned} f(85{,}000) &= 8440 + 0.25(85{,}000 - 61{,}300) \\ &= 14{,}365 \end{aligned}$$
The tax on $85,000 is $14,365.

3. $$\begin{aligned} f(290{,}000) &= 42{,}170 + 0.33(290{,}000 - 188{,}450) \\ &= 75{,}681.5 \end{aligned}$$
The tax on $290,000 is $75,681.50.

4. $f(462,000) = 91,043 + 0.35(462,000 - 336,550)$
$= 134,950.5$

The tax on \$462,000 is \$134,950.50.

5. Answers may vary.

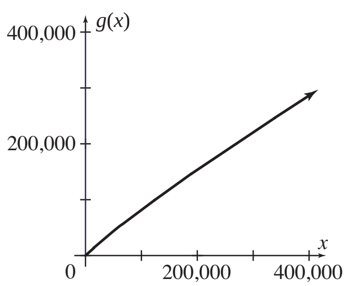
6. There should be no jump in tax as one moves from one tax bracket to the next, since it would be unfair for two couples whose incomes differ by only \$0.01 to pay substantially different taxes.

7. $g(x) = x - f(x)$

$$= \begin{cases} x - 0.10x & \text{if } 0 \le x \le 15,100 \\ x - [1510 + 0.15(x - 15,100)] & \text{if } 15,100 < x \le 61,300 \\ x - [8440 + 0.25(x - 61,300)] & \text{if } 61,300 < x \le 123,700 \\ x - [24,040 + 0.28(x - 123,700)] & \text{if } 123,700 < x \le 188,450 \\ x - [42,170 + 0.33(x - 188,450)] & \text{if } 188,450 < x \le 336,550 \\ x - [91,043 + 0.35(x - 336,550)] & \text{if } x > 336,550 \end{cases}$$

$$= \begin{cases} 0.90x & \text{if } 0 \le x \le 15,100 \\ 0.85x + 755 & \text{if } 15,100 < x \le 61,300 \\ 0.75x + 6885 & \text{if } 61,300 < x \le 123,700 \\ 0.72x + 10,596 & \text{if } 123,700 < x \le 188,450 \\ 0.67x + 20,018.50 & \text{if } 188,450 < x \le 336,550 \\ 0.65x + 26,749.50 & \text{if } x > 336,550 \end{cases}$$

8.



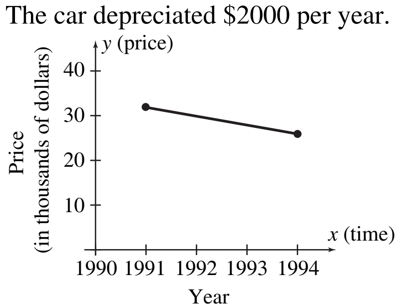
Chapter 3

Principles in Practice 3.1

1. Let x = the time (in years) and let y = the selling price. Then,
In 1991: $x_1 = 1991$ and $y_1 = 32,000$
In 1994: $x_2 = 1994$ and $y_2 = 26,000$
The slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{26,000 - 32,000}{1994 - 1991} = \frac{-6000}{3} = -2000$$

The car depreciated \$2000 per year.



2. An equation relating the growth in enrollment to the number of years can be found by using the point-slope form of an equation of a line. If S = the number of students enrolled, and T = the number of years, then the point-slope form can be written as

$S - S_1 = m(T - T_1)$

Let $m = 14$, $S_1 = 50$, and $T_1 = 3$.

$S - 50 = 14(T - 3)$
$S - 50 = 14T - 42$
$S = 14T + 8$

3. A linear function relating Fahrenheit temperature to Celsius temperature can be found by using the point-slope form of an equation of a line.

$$m = \frac{F_2 - F_1}{C_2 - C_1} = \frac{77 - 41}{25 - 5} = \frac{36}{20} = \frac{9}{5}$$

$$F - F_1 = m(C - C_1)$$
$$F - 41 = \frac{9}{5}(C - 5)$$
$$F - 41 = \frac{9}{5}C - 9$$
$$F = \frac{9}{5}C + 32$$

4. To find the slope and y-intercept, let $a = 1000$, then write the equation in slope-intercept form.

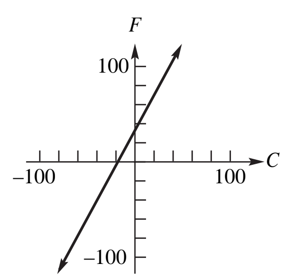
$$y = \frac{1}{24}(t+1)a$$
$$y = \frac{1}{24}(t+1)1000$$
$$y = \frac{1000}{24}t + \frac{1000}{24}$$
$$y = \frac{125}{3}t + \frac{125}{3}$$

Thus the slope, m, is $\frac{125}{3}$ and the y-intercept, b, is $\frac{125}{3}$.

5.
$$F = \frac{9}{5}C + 32$$
$$5(F) = 5\left(\frac{9}{5}C + 32\right)$$
$$5F = 9C + 160$$
$$0 = 9C - 5F + 160$$

Thus, $9C - 5F + 160 = 0$ is a general linear form of $F = \frac{9}{5}C + 32$.

6.



To convert Celsius to Fahrenheit, locate the Celsius temperature on the horizontal axis, move vertically to the line, then move horizontally to read the Fahrenheit temperature of the vertical axis.

7. Right angles are formed by perpendicular lines. The slopes of the sides of the triangle are:

$$\overline{AB}\left\{m = \frac{0-0}{6-0} = \frac{0}{6} = 0\right.$$

$$\overline{BC}\left\{m = \frac{7-0}{7-6} = \frac{7}{1} = 7\right.$$

$$\overline{AC}\left\{m = \frac{7-0}{7-0} = \frac{7}{7} = 1\right.$$

Since none of the slopes are negative reciprocals of each other, there are no perpendicular lines. Therefore, the points do not define a right triangle.

Problems 3.1

1. $m = \frac{10-1}{7-4} = \frac{9}{3} = 3$

2. $m = \frac{10-3}{-2-5} = \frac{7}{-7} = -1$

3. $m = \frac{-3-(-2)}{8-6} = \frac{-1}{2} = -\frac{1}{2}$

4. $m = \frac{-4-(-4)}{3-2} = \frac{0}{1} = 0$

5. The difference in the x-coordinates is 5 – 5 = 0, so the slope is undefined.

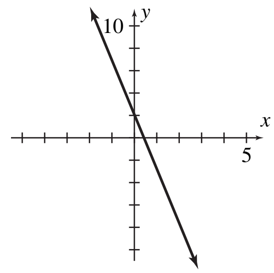
6. $m = \frac{0-(-6)}{3-0} = \frac{6}{3} = 2$

7. $m = \frac{-2-(-2)}{4-5} = \frac{0}{-1} = 0$

8. $m = \frac{0-(-7)}{9-1} = \frac{7}{8}$

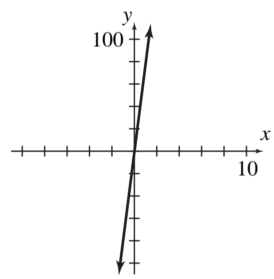
9.

$$\begin{aligned} y-7 &= -5[x-(-1)] \\ y-7 &= -5(x+1) \\ y-7 &= -5x-5 \\ 5x+y-2 &= 0 \end{aligned}$$



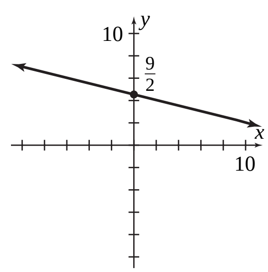
10.

$$\begin{aligned} y-0 &= 75(x-0) \\ y &= 75x \\ 75x-y &= 0 \end{aligned}$$

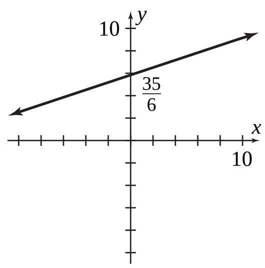


11.

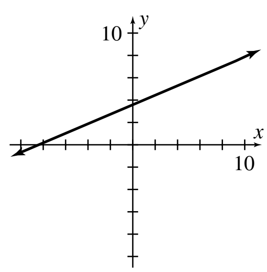
$$\begin{aligned} y-5 &= -\frac{1}{4}[x-(-2)] \\ 4(y-5) &= -(x+2) \\ 4y-20 &= -x-2 \\ x+4y-18 &= 0 \end{aligned}$$



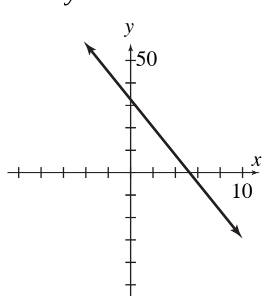
12. $$y-5=\frac{1}{3}\left[x-\left(-\frac{5}{2}\right)\right]$$
$$6(y-5)=2\left[x+\frac{5}{2}\right]$$
$$6y-30=2x+5$$
$$2x-6y+35=0$$



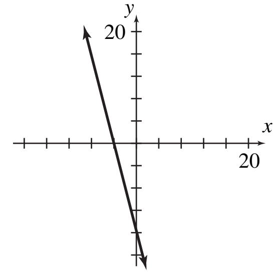
13. $m=\frac{4-1}{1-(-6)}=\frac{3}{7}$
$$y-4=\frac{3}{7}(x-1)$$
$$7(y-4)=3(x-1)$$
$$7y-28=3x-3$$
$$3x-7y+25=0$$



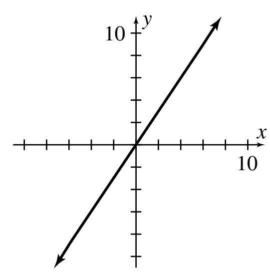
14. $m=\frac{2-(-4)}{5-6}=\frac{6}{-1}=-6$
$$y-2=-6(x-5)$$
$$y-2=-6x+30$$
$$6x+y-32=0$$



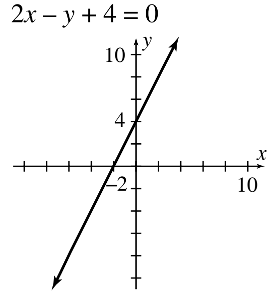
15. $m=\frac{-8-(-4)}{-2-(-3)}=\frac{-4}{1}=-4$
$$y-(-4)=-4[x-(-3)]$$
$$y+4=-4x-12$$
$$4x+y+16=0$$



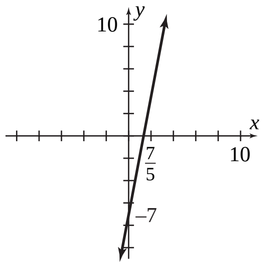
16. $m=\frac{3-0}{2-0}=\frac{3}{2}$
$$y-0=\frac{3}{2}(x-0)$$
$$y=\frac{3}{2}x$$
$$2y=3x$$
$$3x-2y=0$$



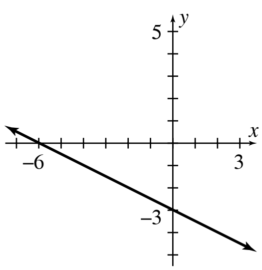
17. $$y=2x+4$$
$$2x-y+4=0$$



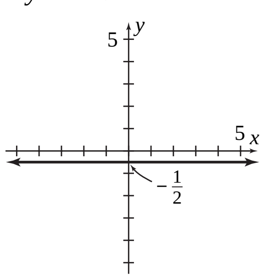
18. $y = 5x - 7$
$5x - y - 7 = 0$



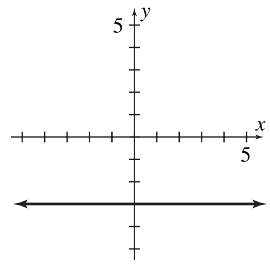
19.
$$y = -\frac{1}{2}x - 3$$
$$2y = 2\left(-\frac{1}{2}x - 3\right)$$
$$2y = -x - 6$$
$$x + 2y + 6 = 0$$



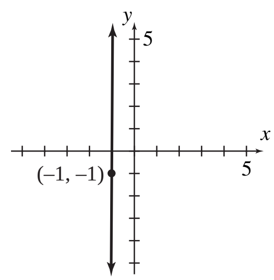
20.
$$y = 0x - \frac{1}{2}$$
$$y = -\frac{1}{2}$$
$$2y = -1$$
$$2y + 1 = 0$$



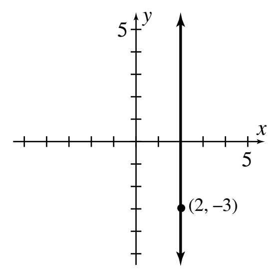
21. A horizontal line has the form $y = b$. Thus $y = -3$, or $y + 3 = 0$.



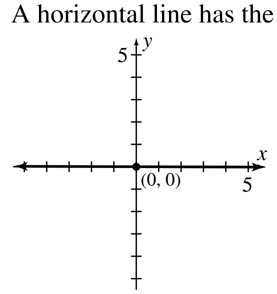
22. A vertical line has the form $x = a$. Thus $x = -1$, or $x + 1 = 0$.



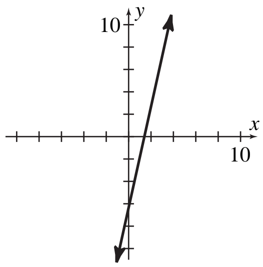
23. A vertical line has the form $x = a$. Thus $x = 2$, or $x - 2 = 0$.



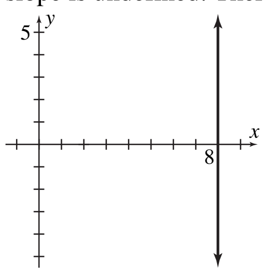
24. A horizontal line has the form $y = b$. Thus $y = 0$.



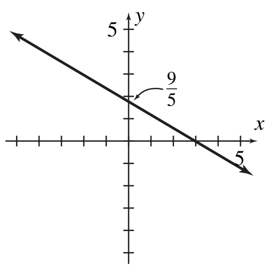
25. $y = 4x - 6$ has the form $y = mx + b$, where $m = 4$ and $b = -6$.



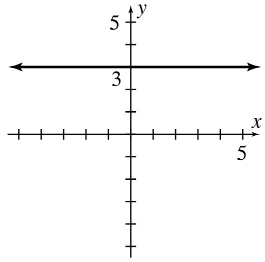
26. $x - 2 = 6$ or $x = 8$, is a vertical line. Thus the slope is undefined. There is no y-intercept.



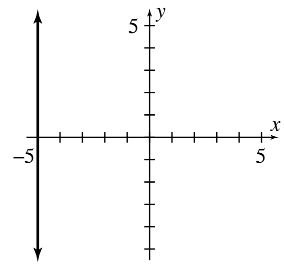
27.
$$3x + 5y - 9 = 0$$
$$5y = -3x + 9$$
$$y = -\frac{3}{5}x + \frac{9}{5}$$
$$m = -\frac{3}{5},\ b = \frac{9}{5}$$



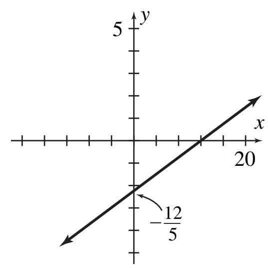
28.
$$y + 4 = 7$$
$$y = 3$$
$$y = 0x + 3$$
$$m = 0,\ b = 3$$



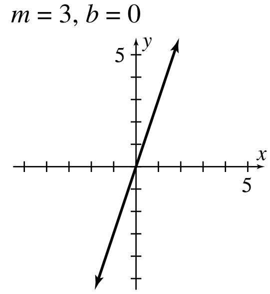
29. $x = -5$ is a vertical line. Thus the slope is undefined. There is no y-intercept.



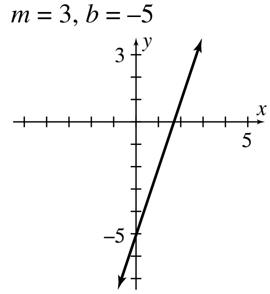
30.
$$x - 9 = 5y + 3$$
$$5y = x - 12$$
$$y = \frac{1}{5}x - \frac{12}{5}$$
$$m = \frac{1}{5},\ b = -\frac{12}{5}$$



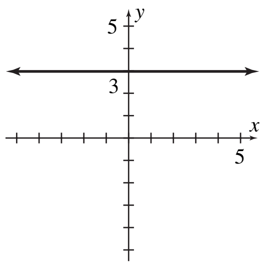
31. $y = 3x$
$y = 3x + 0$
$m = 3,\ b = 0$



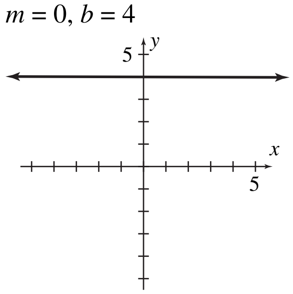
32.
$$y - 7 = 3(x - 4)$$
$$y - 7 = 3x - 12$$
$$y = 3x - 5$$
$$m = 3,\ b = -5$$



33. $y = 3$
$y = 0x + 3$
$m = 0,\ b = 3$



34. $6y - 24 = 0$
$y = 4$
$y = 0x + 4$
$m = 0,\ b = 4$



35. $2x = 5 - 3y$, or $2x + 3y - 5 = 0$ (general form)
$3y = -2x + 5$, or $y = -\frac{2}{3}x + \frac{5}{3}$ (slope-intercept form)

36. $3x + 2y = 6$, or $3x + 2y - 6 = 0$ (general form)
$2y = -3x + 6$, or $y = -\frac{3}{2}x + 3$ (slope-intercept form)

37. $4x + 9y - 5 = 0$ is a general form.
$9y = -4x + 5$, or $y = -\frac{4}{9}x + \frac{5}{9}$ (slope-intercept form)

38. $3(x-4) - 7(y+1) = 2$
$3x - 12 - 7y - 7 = 2$
$3x - 7y - 21 = 0$ (general form)
$-7y = -3x + 21$, or $y = \frac{3}{7}x - 3$ (slope-intercept form)

39. $-\frac{x}{2} + \frac{2y}{3} = -4\frac{3}{4}$
$12\left(-\frac{x}{2} + \frac{2y}{3}\right) = 12\left(-\frac{19}{4}\right)$
$-6x + 8y = -57$
$6x - 8y - 57 = 0$ (general form)
$-8y = -6x + 57$
$y = \frac{3}{4}x - \frac{57}{8}$ (slope-intercept form)

40. $y = \frac{1}{300}x + 8$ is in slope-intercept form.
$300y = 300\left(\frac{1}{300}x + 8\right)$
$300y = x + 2400$
$x - 300y + 2400 = 0$ (general form)

41. The lines $y = 7x + 2$ and $y = 7x - 3$ have the same slope, 7. Thus they are parallel.

42. The lines $y = 4x + 3$ and $y = 5 + 4x$ (or $y = 4x + 5$) have the same slope, 4. Thus they are parallel.

43. The lines $y = 5x + 2$ and $-5x + y - 3 = 0$ (or $y = 5x + 3$) have the same slope, 5. Thus they are parallel.

44. The line $y = x$ has slope $m_1 = 1$ and the line $y = -x$ has slope $m_2 = -1$. $m_1 = -\frac{1}{m_2}$ so the lines are perpendicular.

45. The line $x + 3y + 5 = 0$ $\left(\text{or } y = -\frac{1}{3}x - \frac{5}{3}\right)$ has slope $m_1 = -\frac{1}{3}$ and the line $y = -3x$ has slope $m_2 = -3$. Since $m_1 \neq m_2$ and $m_1 \neq -\frac{1}{m_2}$, the lines are neither parallel nor perpendicular.

46. The line $x + 3y = 0$ $\left(\text{or } y = -\frac{1}{3}x\right)$ has slope $m_1 = -\frac{1}{3}$ and the line $x + 6y - 4 = 0$ (or $y = -\frac{1}{6}x + \frac{2}{3}$) has slope $m_2 = -\frac{1}{6}$. Since

$m_1 \neq m_2$ and $m_1 \neq -\frac{1}{m_2}$, the lines are neither parallel nor perpendicular.

47. The line $y = 3$ is horizontal and the line $x = -\frac{1}{3}$ is vertical, so the lines are perpendicular.

48. Both lines are vertical and thus parallel.

49. The line $3x + y = 4$ (or $y = -3x + 4$) has slope $m_1 = -3$, and the line $x - 3y + 1 = 0$ $\left(\text{or } y = \frac{1}{3}x + \frac{1}{3}\right)$ has slope $m_2 = \frac{1}{3}$. Since $m_2 = -\frac{1}{m_1}$, the lines are perpendicular.

50. The line $x - 2 = 3$ (or $x = 5$) is vertical and the line $y = 2$ is horizontal, so the lines are perpendicular.

51. The slope of $y = -\frac{x}{4} - 2$ is $-\frac{1}{4}$, so the slope of a line parallel to it must also be $-\frac{1}{4}$. An equation of the desired line is $y - 1 = -\frac{1}{4}(x - 1)$ or $y = -\frac{1}{4}x + \frac{5}{4}$.

52. $x = -4$ is a vertical line. A line parallel to $x = -4$ has the form $x = a$. Since the line must pass through $(2, -8)$, its equation is $x = 2$.

53. $y = 2$ is a horizontal line. A line parallel to it has the form $y = b$. Since the line must pass through $(2, 1)$ its equation is $y = 1$.

54. The slope of $y = 3 + 2x$ is 2, so the slope of a line parallel to it must also be 2. An equation of the desired line is $y - (-4) = 2(x - 3)$, or $y = 2x - 10$.

55. The slope of $y = 3x - 5$ is 3, so the slope of a line perpendicular to it must have slope $-\frac{1}{3}$. An equation of the desired line is $y - 4 = -\frac{1}{3}(x - 3)$, or $y = -\frac{1}{3}x + 5$.

56. $y = -4$ is a horizontal line. The perpendicular line must be vertical and has an equation of the form $x = a$. Since that line passes through $(1, 1)$, its equation is $x = 1$.

57. $y = -3$ is a horizontal line, so the perpendicular line must be vertical with equation of the form $x = a$. Since that line passes through $(5, 2)$, its equation is $x = 5$.

58. The line $3y = -\frac{2x}{5} + 3$ $\left(\text{or } y = -\frac{2x}{15} + 1\right)$ has slope $-\frac{2}{15}$, so the slope of a line perpendicular to it must have slope $\frac{15}{2}$. An equation of the desired line is $y - (-5) = \frac{15}{2}(x - 4)$ or $y = \frac{15}{2}x - 35$.

59. The line $2x + 3y + 6 = 0$ has slope $-\frac{2}{3}$, so the slope of a line parallel to it must also be $-\frac{2}{3}$. An equation of the desired line is $y - (-5) = -\frac{2}{3}[x - (-7)]$, or $y = -\frac{2}{3}x - \frac{29}{3}$.

60. The y-axis is vertical. A parallel line is also vertical and has an equation of the form $x = a$. Since it passes through $(-4, 10)$, its equation is $x = -4$.

61. $(1, 2), (-3, 8)$

$$m = \frac{8-2}{-3-1} = \frac{6}{-4} = -\frac{3}{2}$$

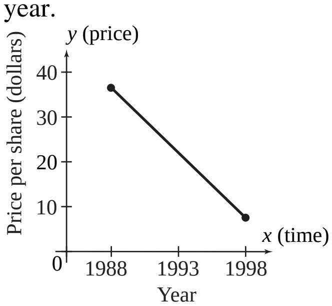
Point-slope form: $y - 2 = -\frac{3}{2}(x - 1)$. When the x-coordinate is 5,

$$y - 2 = -\frac{3}{2}(5 - 1)$$
$$y - 2 = -\frac{3}{2}(4)$$
$$y - 2 = -6$$
$$y = -4$$

Thus the point is $(5, -4)$.

62. $m = 3$, $b = 1$
Slope-intercept form: $y = 3x + 1$. The point $(-1, -2)$ lies on the line if its coordinates satisfy the equation. If $x = -1$ and $y = -2$, then $-2 = 3(-1) + 1$ or $-2 = -2$, which is true. Thus $(-1, -2)$ lies on the line.

63. Let x = the time (in years) and y = the price per share. Then,
In 1988: $x_1 = 1988$ and $y_1 = 37$
In 1998: $x_2 = 1998$ and $y_2 = 8$
The slope is
$$m = \frac{8-37}{1998-1988} = \frac{-29}{10} = -2.9$$
The stock price dropped an average of \$2.90 per year.



64. The number of home runs hit increased as a function of time (in months). The given points are $(x_1, y_1) = (3, 14)$ and $(x_2, y_2) = (5, 20)$.
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{20-14}{5-3} = \frac{6}{2} = 3$$
Using the point-slope form with $m = 3$ and $(x_1, y_1) = (3, 14)$ gives
$$y - y_1 = m(x - x_1)$$
$$y - 14 = 3(x - 3)$$
$$y - 14 = 3x - 9$$
$$y = 3x + 5$$

65. The owner's profits increased as a function of time. Let x = the time (in years) and let y = the profit (in dollars). The given points are $(x_1, y_1) = (0, -100{,}000)$ and $(x_2, y_2) = (5, 40{,}000)$.
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{40{,}000 - (-100{,}000)}{5-0} = \frac{140{,}000}{5}$$
$= 28{,}000$
Using the point-slope form with $m = 28{,}000$ and $(x_1, y_1) = (0, -100{,}000)$ gives
$$y - y_1 = m(x - x_1)$$
$$y - (-100{,}000) = 28{,}000(x - 0)$$
$$y + 100{,}000 = 28{,}000x$$
$$y = 28{,}000x - 100{,}000$$

66. Solve the equation for t.
$L = 1.53t - 6.7$
$L + 6.7 = 1.53t$
$$\frac{(L+6.7)}{1.53} = t$$
$0.65L + 4.38 = t$
The slope is approximately 0.65 and the y-intercept is approximately 4.38.

67. A general linear form of $d = 184 + t$ is $-t + d - 184 = 0$.

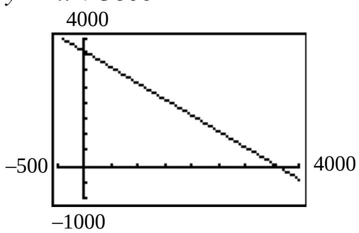
68. **a.** Using the points $(3.5, -1.5)$ and $(0.5, 0.5)$ gives a slope of $m = \frac{-1.5-0.5}{3.5-0.5} = -\frac{2}{3}$.
An equation is $y - 0.5 = -\frac{2}{3}(x - 0.5)$ or
$y = -\frac{2}{3}x + \frac{5}{6}$.

b. Using the points $(0.5, 0.5)$ and $(-1, -2.5)$ gives a slope of $m = \frac{-2.5-0.5}{-1-0.5} = \frac{-3}{-1.5} = 2$.
An equation is $y - 0.5 = 2(x - 0.5)$ or
$y = 2x - \frac{1}{2}$.

These two paths are not perpendicular to each other because the slopes are not negative reciprocals of each other.

69. The slopes of the sides of the figure are:
$$\overline{AB}\left\{ m = \frac{4-0}{0-0} = \frac{4}{0} = \text{ undefined (vertical)}\right.$$
$$\overline{CD}\left\{ m = \frac{7-3}{2-2} = \frac{4}{0} = \text{ undefined (vertical)}\right.$$
$$\overline{AC}\left\{ m = \frac{3-0}{2-0} = \frac{3}{2}\right.$$
$$\overline{BD}\left\{ m = \frac{7-4}{2-0} = \frac{3}{2}\right.$$
Since $\overline{AB}$ is parallel to $\overline{CD}$ and $\overline{AC}$ is parallel to $\overline{BC}$, $ABCD$ is a parallelogram.

70. Let x = the distance traveled and let y = the altitude. The path of descent is a straight line with a slope of –1 and y-intercept of 3600. Therefore, using the slope-intercept form with $m = -1$ and $b = 3600$ gives
$y = mx + b$
$y = (-1)x + 3600$
$y = -x + 3600$

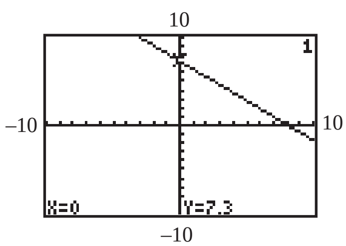


If the airport is located 3800 feet from where the plane begins its landing approach, the plane will crash 200 feet short of the airport.

71. The line has slope 59.82 and passes through (6, 1128.50). Thus $C - 1128.50 = 59.82(T - 6)$ or
$C = 59.82T + 769.58$.

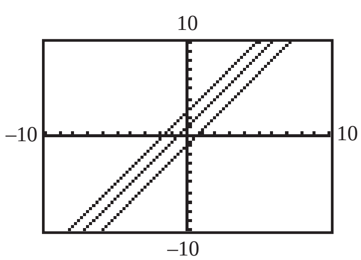
72. The line has slope 50,000 and passes through (5, 330,000). Thus $R - 330{,}000 = 50{,}000(T - 5)$ or $R = 50{,}000T + 80{,}000$.

73.



The graph of the equation $y = -0.9x - 7.3$ shows that when $x = 0$, $y = 7.3$. Thus, the y-intercept is 7.3.

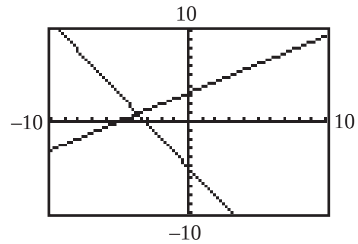
74.

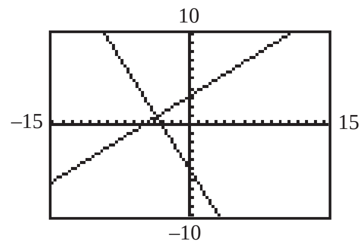


The lines are parallel, which is expected because they have the same slope, 1.5.

75. The slope is 7.1.

76.





The slope of the first line is

$m_1 = \frac{0.1875}{0.3} = 0.625$, and the slope of the second line is $m_2 = -\frac{0.32}{0.2} = -1.6$. Since $m_1 = -\frac{1}{m_2}$, the lines are perpendicular.

Principles in Practice 3.2

1. Let x = the number of skis that are produced and let y = the number of boots that are produced. Then, the equation $8x + 14y = 1000$ describes all possible production levels of the two products.

2. The quantity and price are linearly related such that $p = 575$ when $q = 1200$, and $p = 725$ when $q = 800$. Thus $(q_1,\ p_1) = (1200,\ 575)$ and $(q_2,\ p_2) = (800,\ 725)$. The slope is

$m = \frac{725 - 575}{800 - 1200} = -\frac{3}{8}$.

An equation of the line is

$p - p_1 = m(q - q_1)$

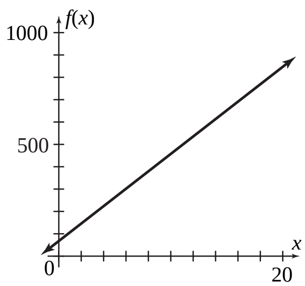
$p - 575 = -\frac{3}{8}(q - 1200)$

$p - 575 = -\frac{3}{8}q + 450$

$p = -\frac{3}{8}q + 1025$

3. Answers may vary, but two possible points are (0, 60) and (2, 140).
$f(x) = 40x + 60$

x	$f(x)$
0	60
2	140

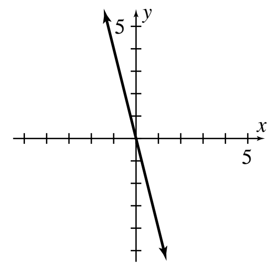


4. If t = the age of the child, then $f(t)$ = the height of the child at any age t. The height and age are linearly related such that $f(8) = 50.6$. Since $f(t)$ is a linear function it has the form $f(t) = at + b$. Since the height changes by 2.3 inches per year, $a = 2.3$. Then,
$f(t) = at + b$
$f(8) = 2.3(8) + b$
$50.6 = 18.4 + b$
$32.2 = b$
Thus, $f(t) = 2.3t + 32.2$ is a function that describes the height of the child at age t.

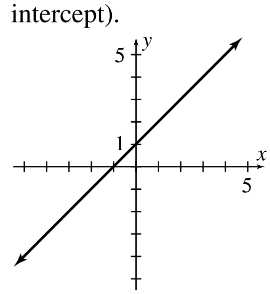
5. Let $y = f(x)$ = a linear function that describes the value of the necklace after x years. The problem states that $f(3) = 360$ and $f(7) = 640$. Thus, $(x_1, y_1) = (3, 360)$ and $(x_2, y_2) = (7, 640)$. The slope is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{640 - 360}{7 - 3} = \frac{280}{4} = 70$
Using the point-slope form with $m = 70$ and $(x_1, y_1) = (3, 360)$ gives
$y - y_1 = m(x - x_1)$
$y - 360 = 70(x - 3)$
$y = f(x) = 70x + 150$

Problems 3.2

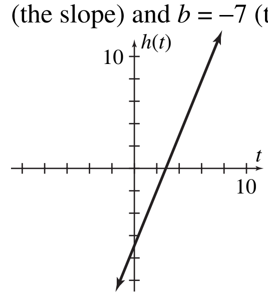
1. $y = f(x) = -4x = -4x + 0$ has the form $f(x) = ax + b$ where $a = -4$ (the slope) and $b = 0$ (the vertical-axis intercept).



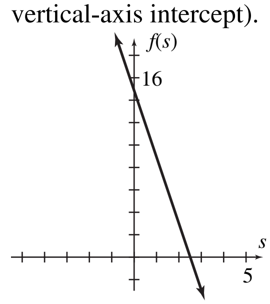
2. $y = f(x) = x + 1$ has the form $f(x) = ax + b$ where $a = 1$ (the slope) and $b = 1$ (the vertical-axis intercept).



3. $h(t) = 5t - 7$ has the form $h(t) = at + b$ with $a = 5$ (the slope) and $b = -7$ (the vertical-axis intercept).



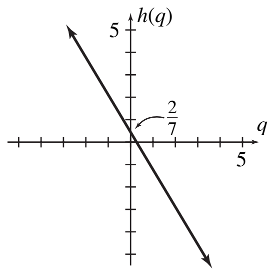
4. $f(s) = 3(5 - 2s) = 15 - 6s$ has the form $f(s) = as + b$ where $a = -6$ (slope) and $b = 15$ (the vertical-axis intercept).



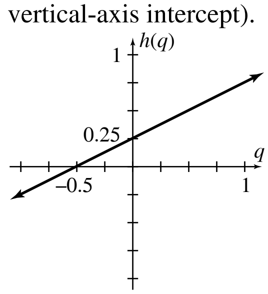
5. $h(q) = \frac{2-q}{7} = \frac{2}{7} - \frac{1}{7}q$ has the form

$h(q) = aq + b$ where $a = -\frac{1}{7}$ (the slope) and

$b = \frac{2}{7}$ (the vertical-axis intercept).



6. $h(q) = 0.5q + 0.25$ has the form $h(q) = aq + b$ with $a = 0.5$ (the slope) and $b = 0.25$ (the vertical-axis intercept).



7. $f(x) = ax + b = 4x + b$. Since $f(2) = 8$, $8 = 4(2) + b$, $8 = 8 + b$, $b = 0 \Rightarrow f(x) = 4x$.

8. Let $y = f(x)$. The points (0, 3) and (4, –5) lie on the graph of f. $m = \frac{-5-3}{4-0} = -2$. Thus $y - 3 = -2(x - 0)$, so $y = -2x + 3 \Rightarrow f(x) = -2x + 3$.

9. Let $y = f(x)$. The points (1, 2) and (–2, 8) lie on the graph of f. $m = \frac{8-2}{-2-1} = -2$. Thus $y - 2 = -2(x - 1)$, so $y = -2x + 4 \Rightarrow f(x) = -2x + 4$.

10. $f(x) = ax + b = -2x + b$.

Since $f\left(\frac{2}{5}\right) = -7$, we have

$$-7 = -2\left(\frac{2}{5}\right) + b$$

$$b = -7 + \frac{4}{5} = -\frac{31}{5}$$

so $f(x) = -2x - \frac{31}{5}$.

11. $f(x) = ax + b = -\frac{2}{3}x + b$. Since $f\left(-\frac{2}{3}\right) = -\frac{2}{3}$, we have

$$-\frac{2}{3} = -\frac{2}{3}\left(-\frac{2}{3}\right) + b$$

$$b = -\frac{2}{3} - \frac{4}{9} = -\frac{10}{9},$$

so $f(x) = -\frac{2}{3}x - \frac{10}{9}$.

12. Let $y = f(x)$. The points (1, 1) and (2, 2) lie on the graph of f. $m = \frac{2-1}{2-1} = 1$.
Thus $y - 1 = 1(x - 1) \Rightarrow y = x$, so $f(x) = x$.

13. Let $y = f(x)$. The points (–2, –1) and (–4, –3) lie on the graph of f. $m = \frac{-3+1}{-4+2} = 1$. Thus $y + 1 = 1(x + 2)$, so $y = x + 1 \Rightarrow f(x) = x + 1$.

14. $f(x) = ax + b = 0.01x + b$. Since $f(0.1) = 0.01$, we have $0.01 = (0.01)(0.1) + b \Rightarrow b = 0.009$ $\Rightarrow f(x) = 0.01x + 0.009$.

15. The points (40, 12.75) and (25, 18.75) lie on the graph of the equation, which is a line.
$m = \frac{18.75-12.75}{25-40} = -\frac{2}{5}$. Hence an equation of the line is $p - 12.75 = -\frac{2}{5}(q - 40)$, which can be written $p = -\frac{2}{5}q + 28.75$. When $q = 37$, then

$$p = -\frac{2}{5}(37) + 28.75 = \$13.95.$$

16. The line passes through (26,000, 12) and (10,000, 18), so
$m = \frac{18-12}{10,000-26,000} = -0.000375$. Then
$p - 18 = -0.000375(q - 10,000)$ or
$p = -0.000375q + 21.75$.

17. The line passes through (3000, 940) and (2200, 740), so $m = \frac{740-940}{2200-3000} = 0.25$. Then $p - 740 = 0.25(q - 2200)$ or $p = 0.25q + 190$.

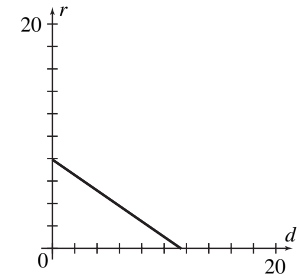
18. The points (50, 35) and (35, 30) lie on the graph of the equation, which is a line.
$m = \frac{30-35}{35-50} = \frac{-5}{-15} = \frac{1}{3}$. Hence an equation of the line is
$$p - 35 = \frac{1}{3}(q - 50)$$
$$p = \frac{1}{3}q + \frac{55}{3}$$

19. The line passing through (10, 40) and (20, 70) has slope $\frac{70-40}{20-10} = 3$, so an equation for the line is
$c - 40 = 3(q - 10)$
$c = 3q + 10$
If $q = 35$, then $c = 3(35) + 10 = 105 + 10 = \$115$.

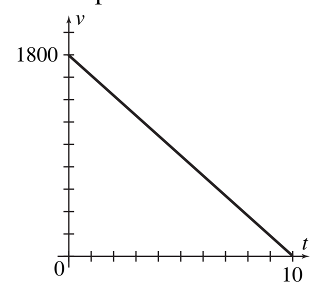
20. The line passing through (100, 79) and (400, 88) has slope $\frac{88-79}{400-100} = 0.03$, so an equation for the line is
$c - 79 = 0.03(x - 100)$
$c = 0.03x + 76$

21. If x = the number of kilowatt hours used in a month, then $f(x)$ = the total monthly charges for x kilowatt hours of electricity. If $f(x)$ is a linear function it has the form $f(x) = ax + b$. The problem states that $f(380) = 51.65$. Since 12.5 cents are charged per kilowatt hour used, $a = 0.125$.
$f(x) = ax + b$
$51.65 = 0.125(380) + b$
$51.65 = 47.5 + b$
$4.15 = b$
Hence, $f(x) = 0.125x + 4.15$ is a linear function that describes the total monthly charges for any number of kilowatt hours x.

22. The number of curative units from d cubic centimeters of the drug is $210d$, and the number of curative units from r minutes of radiation is $305r$. Thus
$210d + 305r = 2410$
$42d + 61r = 482$



23. Each year the value decreases by 0.10(1800). After t years the total decrease is $0.10(1800)t$. Thus
$v = 1800 - 0.10(1800)t$
$v = -180t + 1800$
The slope is –180.



24. The line has slope –120 and passes through (4, 340). Thus $y - 340 = -120(x - 4)$ or $y = f(x) = -120x + 820$.

25. The line has slope 45,000 and passes through (5, 960,000). Thus
$y - 960{,}000 = 45{,}000(x - 5)$ or
$y = f(x) = 45{,}000x + 735{,}000$.

26. The line has slope $\frac{245{,}000}{15} = \frac{49{,}000}{3}$ and y-intercept 245,000. So
$$y = f(x) = \frac{49{,}000}{3}x + 245{,}000.$$

27. If x = the number of hours of service, then $f(x)$ = the price of x hours of service. Let $y = f(x)$. $f(1) = 159$ and $f(3) = 287$, so (1, 159) and (3, 287) lie on the graph of f which has slope $a = \frac{287-159}{3-1} = 64$. Using (1, 159), we get $y - 159 = 64(x - 1)$ or $y = 64x + 95$, so $f(x) = 64x + 95$.

28. a. Suppose r = respiratory rate, l = wool length, and (l, r) lies on the graph, which is a line. The points (2, 160) and (4, 125) are on the line, so its slope is $\frac{125-160}{4-2} = -\frac{35}{2}$. Thus

$$r - 160 = -\frac{35}{2}(l-2)$$

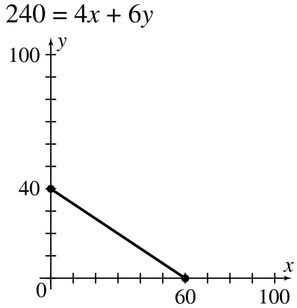
$$r = -\frac{35}{2}l + 195$$

b. If $l = 1$, then $r = -\frac{35}{2}(1) + 195 = 177.5$

29. At \$200/ton, x tons cost $200x$, and at \$2000/acre, y acres cost $2000y$. Hence the required equation is $200x + 2000y = 20,000$, which can be written as $x + 10y = 100$.

30. $P = 4x + 6y$ where $x, y \geq 0$.

a. $240 = 4x + 6y$



b. Since the equation can be written $y = -\frac{2}{3}x + 40$, slope $= -\frac{2}{3}$.

c. $600 = 4x + 6y$. Since the equation can be written $y = -\frac{2}{3}x + 100$,

slope $= -\frac{2}{3}$.

d. Solving $P = 4x + 6y$ for y gives $y = -\frac{2}{3}x + \frac{P}{6}$. Thus any isoprofit line has slope $-\frac{2}{3}$, and lines with the same slope are parallel. Hence isoprofit lines are parallel.

31. a. $m = \frac{100-65}{100-56} = \frac{35}{44}$

$$y - 100 = \frac{35}{44}(x - 100)$$

$$y = \frac{35}{44}x - \frac{3500}{44} + 100$$

$$y = \frac{35}{44}x + \frac{225}{11}$$

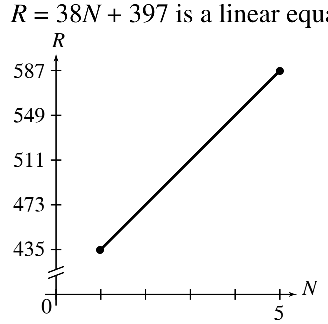
b. $62 = \frac{35}{44}x + \frac{225}{11}$

$$\frac{35}{44}x = 62 - \frac{225}{11}$$

$$x = \frac{1828}{35} \approx 52.2$$

52.2 is the lowest passing score on original scale.

32. $R = 38N + 397$ is a linear equation. Slope = 38.



33. $p = f(t) = at + b$, $f(5) = 0.32$, a = slope = 0.059.

a. $p = f(t) = 0.059t + b$. Since $f(5) = 0.32$, $0.32 = 0.059(5) + b$, $0.32 = 0.295 + b$, so $b = 0.025$. Thus $p = 0.059t + 0.025$.

b. When $t = 9$, then $p = 0.059(9) + 0.025 = 0.556$.

34. $w = f(d) = ad + b$, $f(0) = 21$, a = slope $= \frac{6.3}{10} = 0.63$. Thus $w = f(d) = 0.63d + b$. Since $f(0) = 21$, we have $20 = 0.63(0) + b$, so $b = 21$. Hence $w = 0.63d + 21$.
When $d = 55$, then $w = 0.63(55) + 21 = 34.65 + 21 = 55.65$ kg.

35. a. $m = \frac{t_2 - t_1}{c_2 - c_1} = \frac{80-68}{172-124} = \frac{12}{48} = \frac{1}{4}$.

$t - 68 = \frac{1}{4}(c - 124)$, $t - 68 = \frac{1}{4}c - 31$, or

$t = \frac{1}{4}c + 37$.

b. Since c is the number of chirps per minute, then $\frac{1}{4}c$ is the number of chirps in $\frac{1}{4}$ minute or 15 seconds. Thus from part (a), to estimate temperature add 37 to the number of chirps in 15 seconds.

Principles in Practice 3.3

1. In the quadratic function $y = P(x) = -x^2 + 2x + 399$, $a = -1$, $b = 2$, $c = 399$. Since $a < 0$, the parabola opens downward. The x-coordinate of the vertex is

$-\frac{b}{2a} = -\frac{2}{2(-1)} = 1$.

The y-coordinate of the vertex is

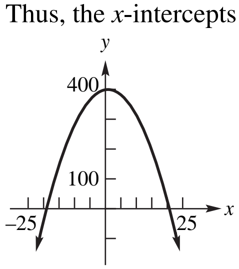
$P(1) = -\left(1^2\right) + 2(1) + 399 = 400$. Thus, the vertex is (1, 400). Since $c = 399$, the y-intercept is (0, 399). To find the x-intercepts we set $y = p(x) = 0$.

$0 = -x^2 + 2x + 399$

$0 = -\left(x^2 - 2x - 399\right)$

$0 = -(x + 19)(x - 21)$

Thus, the x-intercepts are (–19, 0) and (21, 0).



If the model is correct, this is not a good business, since it will lose money if more than 21 minivans are sold.

2. In the quadratic function $h(t) = -16t^2 + 32t + 8$, $a = -16$, $b = 32$, and $c = 8$. Since $a < 0$, the parabola opens downward. The x-coordinate of the vertex is $-\frac{b}{2a} = -\frac{32}{2(-16)} = 1$. The y-coordinate of the vertex is

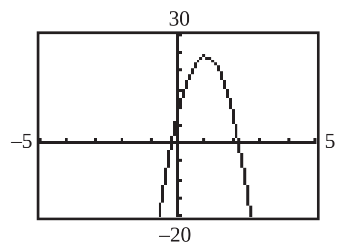
$h(1) = -16\left(1^2\right) + 32(1) + 8 = 24$. Thus, the vertex is (1, 24). Since $c = 8$, the y-intercept is (0, 8). To find the x-intercepts we set $y = h(t) = 0$.

$0 = -16t^2 + 32t + 8$

$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-32 \pm \sqrt{32^2 - 4(-16)(8)}}{2(-16)}$

$= \frac{-32 \pm \sqrt{1536}}{-32} = \frac{-32 \pm 16\sqrt{6}}{-32} = 1 \pm \frac{\sqrt{6}}{2}$

Thus, the x-intercepts are $\left(1 + \frac{\sqrt{6}}{2}, 0\right)$ and $\left(1 - \frac{\sqrt{6}}{2}, 0\right)$.



3. If we express the revenue r as a function of the quantity produced q, we obtain

$r = pq$

$r = (6 - 0.003q)q$

$r = 6q - 0.003q^2$

We note that this is a quadratic function with $a = -0.003$, $b = 6$, and $c = 0$. Since $a < 0$, the graph of the function is a parabola that opens downward, and r is maximum at the vertex (q, r).

$q = -\frac{b}{2a} = -\frac{6}{2(-0.003)} = 1000$

$r = 6(1000) - 0.003(1000)^2 = 3000$

Thus, the maximum revenue that the manufacturer can receive is \$3000, which occurs at a production level of 1000 units.

Problems 3.3

1. $f(x) = 5x^2$ has the form $f(x) = ax^2 + bx + c$ where $a = 5$, $b = 0$, and $c = 0 \Rightarrow$ quadratic.

2. $g(x) = \frac{1}{2x^2 - 4}$ cannot be put in the form $g(x) = ax^2 + bx + c$ where $a \neq 0 \Rightarrow$ not quadratic.

3. $g(x) = 7 - 6x$ cannot be put in the form $g(x) = ax^2 + bx + c$ where $a \neq 0 \Rightarrow$ not quadratic.

4. $k(v) = 3v^2(v^2 + 2) = 3v^4 + 6v^2$ cannot be put in the form $k(v) = av^2 + bv + c$ where $a \neq 0 \Rightarrow$ not quadratic.

5. $h(q) = (3 - q)^2 = 9 - 6q + q^2$ has form $h(q) = aq^2 + bq + c$ where $a = 1$, $b = -6$, and $c = 9 \Rightarrow$ quadratic.

6. $f(t) = 2t(3 - t) + 4t = -2t^2 + 10t$ has the form $f(t) = at^2 + bt + c$ where $a = -2$, $b = 10$, and $c = 0 \Rightarrow$ quadratic.

7. $f(s) = \frac{s^2 - 9}{2} = \frac{1}{2}s^2 - \frac{9}{2}$ has the form $f(s) = as^2 + bs + c$ where $a = \frac{1}{2}$, $b = 0$, and $c = -\frac{9}{2} \Rightarrow$ quadratic.

8. $g(t) = \left(t^2 - 1\right)^2 = t^4 - 2t^2 + 1$ cannot be put in the form $g(t) = at^2 + bt + c$ where $a \neq 0 \Rightarrow$ not quadratic.

9. $y = f(x) = -4x^2 + 8x + 7$
$a = -4$, $b = 8$, $c = 7$

a. Vertex occurs when $x = -\frac{b}{2a} = -\frac{8}{2(-4)} = 1$.
When $x = 1$, then
$y = f(1) = -4(1)^2 + 8(1) + 7 = 11$.
Vertex: (1, 11)

b. $a = -4 < 0$, so the vertex corresponds to the highest point.

10. $y = f(x) = 8x^2 + 4x - 1$
$a = 8$, $b = 4$, $c = -1$

a. $-\frac{b}{2a} = -\frac{4}{2 \cdot 8} = -\frac{1}{4}$

$$f\left(-\frac{1}{4}\right) = 8\left(-\frac{1}{4}\right)^2 + 4\left(-\frac{1}{4}\right) - 1 = -\frac{3}{2}$$

Vertex: $\left(-\frac{1}{4}, -\frac{3}{2}\right)$

b. $a = 8 > 0$, so the vertex corresponds to the lowest point.

11. $y = x^2 + x - 6$
$a = 1$, $b = 1$, $c = -6$

a. $c = -6$. Thus the y-intercept is -6.

b. $x^2 + x - 6 = (x - 2)(x + 3) = 0$, so $x = 2, -3$.
x-intercepts: 2, -3

c. $-\frac{b}{2a} = -\frac{1}{2}$

$$f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^2 - \frac{1}{2} - 6 = -\frac{25}{4}$$

Vertex: $\left(-\frac{1}{2}, -\frac{25}{4}\right)$

12. $y = f(x) = 5 - x - 3x^2$
$a = -3$, $b = -1$, $c = 5$

a. $c = 5$. Thus the y-intercept is 5.

b.
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-3)(5)}}{2(-3)} = \frac{1 \pm \sqrt{61}}{-6} = \frac{-1 \pm \sqrt{61}}{6}$$

x-intercepts: $\frac{-1 + \sqrt{61}}{6}, \frac{-1 - \sqrt{61}}{6}$

c. $-\frac{b}{2a} = -\frac{-1}{2(-3)} = -\frac{1}{6}$

$f\left(-\frac{1}{6}\right) = 5 - \left(-\frac{1}{6}\right) - 3\left(-\frac{1}{6}\right)^2 = \frac{61}{12}$

Vertex: $\left(-\frac{1}{6}, \frac{61}{12}\right)$

13. $y = f(x) = x^2 - 6x + 5$

$a = 1$, $b = -6$, $c = 5$

Vertex: $-\frac{b}{2a} = -\frac{-6}{2 \cdot 1} = 3$

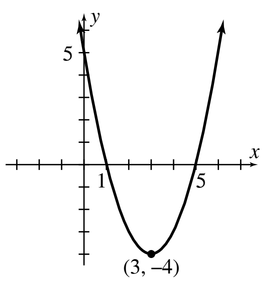
$f(3) = 3^2 - 6(3) + 5 = -4$

Vertex = (3, –4)

y-intercept: $c = 5$

x-intercepts: $x^2 - 6x + 5 = (x - 1)(x - 5) = 0$, so $x = 1, 5$.

Range: all $y \geq -4$



14. $y = f(x) = -4x^2$

$a = -4$, $b = 0$, $c = 0$

Vertex: $-\frac{b}{2a} = -\frac{0}{2(-4)} = 0$

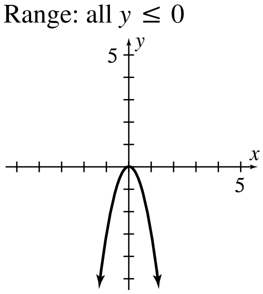
$f(0) = -4(0)^2 = 0$

Vertex = (0, 0)

y-intercept: $c = 0$

x-intercepts: $-4x^2 = 0$, so $x = 0$.

Range: all $y \leq 0$



15. $y = g(x) = -2x^2 - 6x$

$a = -2$, $b = -6$, $c = 0$

Vertex: $-\frac{b}{2a} = -\frac{-6}{2(-2)} = -\frac{6}{4} = -\frac{3}{2}$

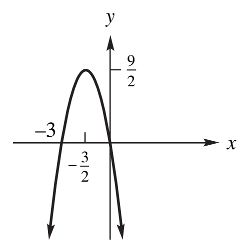
$f\left(\frac{-3}{2}\right) = -2\left(\frac{-3}{2}\right)^2 - 6\left(\frac{-3}{2}\right) = \frac{-9}{2} + 9 = \frac{9}{2}$

Vertex: $\left(-\frac{3}{2}, \frac{9}{2}\right)$

y-intercept: $c = 0$

x-intercepts: $-2x^2 - 6x = -2x(x + 3) = 0$, so $x = 0, -3$.

Range: all $y \leq \frac{9}{2}$



16. $y = f(x) = x^2 - 4$

$a = 1$, $b = 0$, $c = -4$

Vertex: $-\frac{b}{2a} = -\frac{0}{2 \cdot 1} = 0$

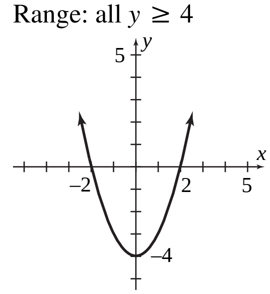
$f(0) = 0^2 - 4 = -4$

Vertex = (0, –4)

y-intercept: $c = -4$

x-intercepts: $x^2 - 4 = (x + 2)(x - 2) = 0$, so $x = -2, 2$.

Range: all $y \geq 4$



17. $s = h(t) = t^2 + 6t + 9$

$a = 1$, $b = 6$, $c = 9$

Vertex: $-\frac{b}{2a} = -\frac{6}{2 \cdot 1} = -3$

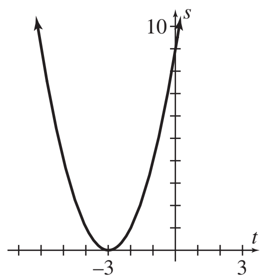
$h(-3) = (-3)^2 + 6(-3) + 9 = 0$

Vertex = (–3, 0)

s-intercept: $c = 9$

t-intercepts: $t^2 + 6t + 9 = (t+3)^2 = 0$, so $t = -3$.

Range: all $s \geq 0$



18. $s = h(t) = 2t^2 + 3t - 2$

$a = 2$, $b = 3$, $c = -2$

Vertex: $-\frac{b}{2a} = -\frac{3}{2 \cdot 2} = -\frac{3}{4}$

$h\left(-\frac{3}{4}\right) = 2\left(-\frac{3}{4}\right)^2 + 3\left(-\frac{3}{4}\right) - 2$

$= \frac{9}{8} - \frac{9}{4} - 2 = -\frac{25}{8}$

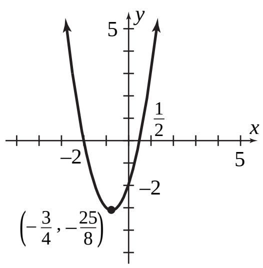
Vertex $= \left(-\frac{3}{4}, -\frac{25}{8}\right)$

s-intercept: $c = -2$

t-intercepts: $2t^2 + 3t - 2 = (2t-1)(t+2) = 0$, so

$t = \frac{1}{2}, -2.$

Range: all $s \geq -\frac{25}{8}$



19. $y = f(x) = -9 + 8x - 2x^2$

$a = -2$, $b = 8$, $c = -9$

Vertex: $-\frac{b}{2a} = -\frac{8}{2(-2)} = 2$

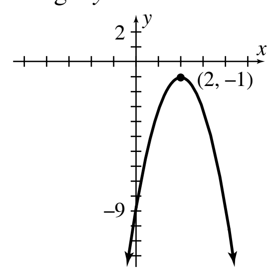
$f(2) = -9 + 8(2) - 2(2)^2 = -1$

Vertex = (2, –1)

y-intercept: $c = -9$

x-intercepts: Because the parabola opens downward ($a < 0$) and the vertex is below the *x*-axis, there is no *x*-intercept.

Range: $y \leq -1$



20. $y = H(x) = 1 - x - x^2$

$a = -1$, $b = -1$, $c = 1$

Vertex: $-\frac{b}{2a} = -\frac{-1}{2(-1)} = -\frac{1}{2}$

$f\left(-\frac{1}{2}\right) = 1 - \left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right)^2 = \frac{5}{4}$

Vertex $= \left(-\frac{1}{2}, \frac{5}{4}\right)$

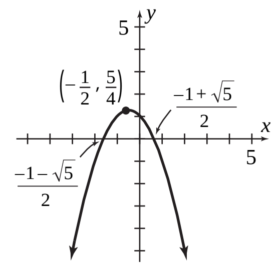
y-intercept: $c = 1$

x-intercepts: Solving $1 - x - x^2 = 0$ by the quadratic formula gives

$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-1)(1)}}{2(-1)} = \frac{1 \pm \sqrt{5}}{-2}$

$= \frac{-1 \pm \sqrt{5}}{2}$

Range: all $y \leq \frac{5}{4}$



21. $t = f(s) = s^2 - 8s + 14$

$a = 1$, $b = -8$, $c = 14$

Vertex: $-\frac{b}{2a} = -\frac{-8}{2 \cdot 1} = 4$

$f(4) = 4^2 - 8(4) + 14 = -2$

Vertex = (4, –2)

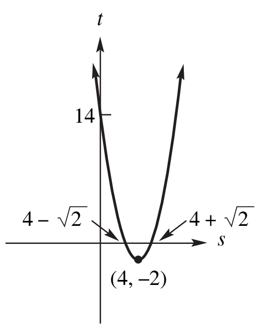
t-intercept: $c = 14$

s-intercepts: Solving $s^2 - 8s + 14 = 0$ by the quadratic formula:

$$s = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(14)}}{2(1)}$$

$$= \frac{8 \pm \sqrt{8}}{2} = \frac{8 \pm 2\sqrt{2}}{2} = 4 \pm \sqrt{2}$$

Range: all $t \geq -2$



22. $t = f(s) = s^2 + 6s + 11$

$a = 1$, $b = 6$, $c = 11$

Vertex: $-\frac{b}{2a} = -\frac{6}{2 \cdot 1} = -3$

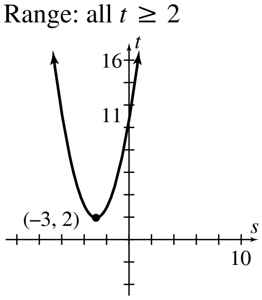
$f(-3) = (-3)^2 + 6(-3) + 11 = 2$

Vertex: (–3, 2)

t-intercept: $c = 11$

s-intercepts: Because the parabola opens upward ($a > 0$) and the vertex is above the s-axis, there is no s-intercept.

Range: all $t \geq 2$



23. $f(x) = 49x^2 - 10x + 17$

Since $a = 49 > 0$, the parabola opens upward and $f(x)$ has a minimum value that occurs when $x = -\frac{b}{2a} = -\frac{-10}{2 \cdot 49} = \frac{5}{49}$. The minimum value is

$$f\left(\frac{5}{49}\right) = 49\left(\frac{5}{49}\right)^2 - 10\left(\frac{5}{49}\right) + 17 = \frac{808}{49}.$$

24. $f(x) = -3x^2 - 18x + 7$

Since $a = -3 < 0$, the parabola opens downward and $f(x)$ has a maximum value that occurs when

$$x = -\frac{b}{2a} = -\frac{-18}{2(-3)} = -3$$

The maximum value is

$f(-3) = -3(-3)^2 - 18(-3) + 7 = 34.$

25. $f(x) = 4x - 50 - 0.1x^2$

Since $a = -0.1 < 0$, the parabola opens downward and $f(x)$ has a maximum value that occurs when $x = -\frac{b}{2a} = -\frac{4}{2(-0.1)} = 20$. The maximum value is

$f(20) = 4(20) - 50 - 0.1(20)^2 = -10$.

26. $f(x) = x(x+3) - 12 = x^2 + 3x - 12$

Because $a = 1 > 0$, the parabola opens upward and $f(x)$ has a minimum value that occurs when $x = -\frac{b}{2a} = -\frac{3}{2 \cdot 1} = -\frac{3}{2}$. The minimum value is

$$f\left(-\frac{3}{2}\right) = \left(-\frac{3}{2}\right)^2 + 3\left(-\frac{3}{2}\right) - 12 = -\frac{57}{4}$$

27. $f(x) = x^2 - 2x + 4$

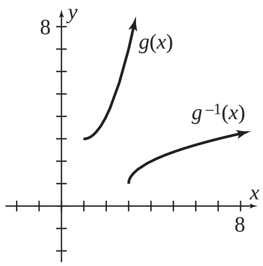
$a = 1$, $b = -2$, $c = 4$

$$v = -\frac{b}{2a} = -\frac{-2}{2(1)} = 1$$

The restricted function is $g(x) = x^2 - 2x + 4$, $x \geq 1$. From the quadratic formula applied to $x^2 - 2x + 4 - y = 0$, we get

$$x = \frac{2 \pm \sqrt{4 - 4(1)(4 - y)}}{2(1)} = 1 \pm \sqrt{1 - (4 - y)}$$

So the inverse of $g(x)$ is $g^{-1}(x) = 1 + \sqrt{x - 3}$, $x \geq 3$.



28. $f(x) = -x^2 + 4x - 3$

$a = -1$, $b = 4$, $c = -3$

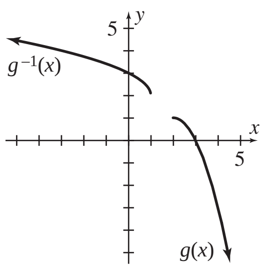
$$v = -\frac{b}{2a} = -\frac{4}{2(-1)} = 2$$

The restricted function is $g(x) = -x^2 + 4x - 3$, $x \geq 2$. From the quadratic formula applied to $-x^2 + 4x - 3 - y = 0$, we get

$$x = \frac{-4 \pm \sqrt{16 - 4(-1)(-3 - y)}}{2(-1)}$$

$$= 2 \pm (-1)\sqrt{4 + (-3 - y)}$$

So the inverse of $g(x)$ is $g^{-1}(x) = 2 + \sqrt{1 - x}$, $x \leq 1$.



29. If we express the revenue r as a function of the quantity produced q, we obtain

$r = pq$

$r = (200 - 5q)q$

$r = 200q - 5q^2$

This is a quadratic function with $a = -5$, $b = 200$, and $c = 0$. Since $a < 0$, the graph of the function is a parabola that opens downward, and r is maximum at the vertex (q, r).

$$q = -\frac{b}{2a} = -\frac{200}{2(-5)} = 20$$

$$r = 200(20) - 5(20)^2 = 2000$$

Thus, the maximum revenue that the manufacturer can receive is \$2000, which occurs at a production level of 20 units.

30. If we express the revenue r as a function of the quantity produced q, we obtain

$r = pq$

$r = (0.85 - 0.00045q)q$

$r = 0.85q - 0.00045q^2$

This is a quadratic function with $a = -0.00045$, $b = 0.85$, and $c = 0$. Since $a < 0$, the graph of the function is a parabola that opens downward, and r is a maximum at the vertex (q, r).

$$q = -\frac{b}{2a} = -\frac{0.85}{2(-0.00045)} = \frac{8500}{9} \approx 944$$

$$r = 0.85(944) - 0.00045(944)^2 = 401.39$$

Thus, the maximum revenue that the manufacturer can receive is \$401.39, which occurs at a production level of 944 units.

31. If we express the revenue r as a function of the quantity produced q, we obtain

$r = pq$

$r = (2400 - 6q)q$

$r = 2400q - 6q^2$

This is a quadratic function with $a = -6$, $b = 2400$, and $c = 0$. Since $a < 0$, the graph of the function is a parabola that opens downward, and r is maximum at the vertex (q, r).

$$q = -\frac{b}{2a} = -\frac{2400}{2(-6)} = 200$$

$$r = 2400(200) - 6(200)^2 = 240{,}000$$

Thus, the maximum revenue that the manufacturer can receive is \$240,000, which occurs at a production level of 200 units.

32. $f(n) = \frac{10}{9}n(12 - n) = \frac{40}{3}n - \frac{10}{9}n^2$, where $0 \leq n \leq 12$. Since $a = -\frac{10}{9} < 0$, $f(n)$ has a maximum value that occurs at the vertex.

$$-\frac{b}{2a} = -\frac{\frac{40}{3}}{2\left(-\frac{10}{9}\right)} = 6$$

The maximum value of $f(n)$ is

$f(6) = \frac{40}{3}(6) - \frac{10}{9}(6)^2 = 80 - 40 = 40$, which corresponds to 40,000 households.

33. In the quadratic function

$P(x) = -x^2 + 18x + 144$,

$a = -1$, $b = 18$, and $c = 144$. Since $a < 0$, the graph of the function is a parabola that opens downward. The x-coordinate of the vertex is $-\frac{b}{2a} = -\frac{18}{2(-1)} = 9$. The y-coordinate of the vertex is $P(9) = -\left(9^2\right) + 18(9) + 144 = 225$.

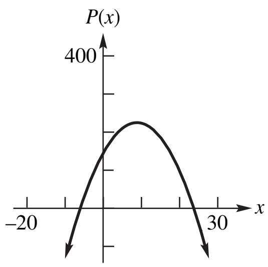
Thus, the vertex is (9, 225). Since $c = 144$, the y-intercept is (0, 144). To find the x-intercepts, let $y = P(x) = 0$.

$0 = -x^2 + 18x + 144$

$0 = -\left(x^2 - 18x - 144\right)$

$0 = -(x - 24)(x + 6)$

Thus, the x-intercepts are (24, 0) and (–6, 0).



34. If $k = 2$, then

$y = kx^2$

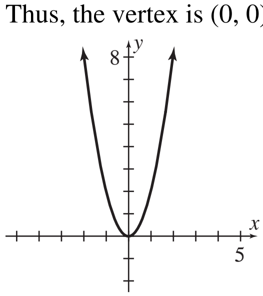
$y = 2x^2$

This is a quadratic equation with $a = 2$, $b = 0$ and $c = 0$. Since $a > 0$, the graph of the function is a parabola that opens upward. The x-coordinate of the vertex is $-\frac{b}{2a} = -\frac{0}{2(2)} = 0$.

The y-coordinate is

$y = 2(0)^2 = 0$

Thus, the vertex is (0, 0).



35. $f(P) = -\frac{1}{50}P^2 + 2P + 20$, where $0 \le P \le 100$.

Because $a = -\frac{1}{50} < 0$, $f(P)$ has a maximum value that occurs at the vertex.

$-\frac{b}{2a} = -\frac{2}{2\left(-\frac{1}{50}\right)} = 50$. The maximum value of $f(P)$ is

$f(50) = \frac{-1}{50}(50)^2 + 2(50) + 20 = 70$ grams.

36. $s = -4.9t^2 + 62.3t + 1.8$

Since $a = -4.9 < 0$, s has a maximum value that occurs at the vertex where

$t = -\frac{b}{2a} = -\frac{62.3}{2(-4.9)} = \frac{62.3}{9.8} = \frac{89}{14} \approx 6.36$ sec.

When $t = \frac{89}{14}$, then

$s = -4.9\left(\frac{89}{14}\right)^2 + 62.3\left(\frac{89}{14}\right) + 1.8$

$= 199.825$ meters.

37. $h(t) = -16t^2 + 85t + 22$

Since $a = -16 < 0$, $h(t)$ has a maximum value that occurs at the vertex where

$t = -\frac{b}{2a} = -\frac{85}{2(-16)} \approx 2.7$ sec. When $t = 2.7$, then

$h(t) = -16(2.7)^2 + 85(2.7) + 22 = 134.86$ feet.

38. $h(t) = -16t^2 + 16t + 4$

Since $a = -16 < 0$, $h(t)$ has a maximum value that occurs at the vertex where

$t = -\frac{b}{2a} = -\frac{16}{2(-16)} = \frac{1}{2}$ sec. When $t = \frac{1}{2}$,

then, $h(t) = -16\left(\frac{1}{2}\right)^2 + 16\left(\frac{1}{2}\right) + 4 = 8$ feet.

39. In the quadratic function $h(t) = -16t^2 + 80t + 16$, $a = -16$, $b = 80$, and $c = 16$. Since $a < 0$, the graph of the function is a parabola that opens downward. The x-coordinate of the vertex is

$-\frac{b}{2a} = -\frac{80}{2(-16)} = \frac{5}{2}$.

The y-coordinate of the vertex is

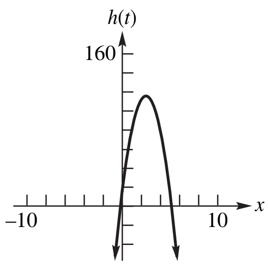
$h\left(\frac{5}{2}\right) = -16\left(\frac{5}{2}\right)^2 + 80\left(\frac{5}{2}\right) + 16 = 116$

Thus, the vertex is $\left(\frac{5}{2}, 116\right)$. Since $c = 16$, the y-intercept is (0, 16). To find the x-intercepts, we let $y = h(t) = 0$.

$0 = -16t^2 + 80t + 16$

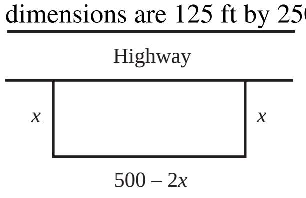
$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-80 \pm \sqrt{80^2 - 4(-16)(16)}}{2(-16)}$$
$$= \frac{-80 \pm \sqrt{7424}}{-32} = \frac{5 \pm \sqrt{29}}{2}$$

Thus, the x-intercepts are $\left(\frac{5+\sqrt{29}}{2}, 0\right)$ and $\left(\frac{5-\sqrt{29}}{2}, 0\right)$.



40. $A = x(11 - x) = 11x - x^2$, so A is a quadratic function of x where $a = -1 < 0$. A has maximum value at the vertex where
$x = -\frac{b}{2a} = -\frac{11}{2(-1)} = \frac{11}{2}$.

41. Since the total length of fencing is 500, the side opposite the highway has length 500 – 2x. The area A is given by
$A = x(500 - 2x) = 500x - 2x^2$,
which is quadratic with $a = -2 < 0$. Thus A is maximum when $x = -\frac{500}{2(-2)} = 125.$ Then the side opposite the highway is
500 – 2x = 500 – 2(125) = 250. Thus the dimensions are 125 ft by 250 ft.



42. Let x, y be two numbers whose sum is 78. Thus $x + y = 78$ and $y = 78 - x$. Their product is then $p(x) = x(78 - x) = 78x - x^2.$ Since $a = -1 < 0$, $p(x)$ has a maximum value that occurs at the vertex where $x = -\frac{b}{2a} = -\frac{78}{2(-1)} = 39$ and $y = 78 - x = 78 - 39 = 39$. Thus, two numbers whose sum is 78 and whose product is a maximum are 39 and 39.

43. (1.11, 2.88)

44. –1.61, 3.73

45. **a.** none

b. one

c. two

46. 14.18

47. 4.89

Principles in Practice 3.4

1. Let x = the number invested at 9% and let y = the amount invested at 8%. Then, the problem states
$$\begin{cases} x + y = 200,000, \\ 0.09x + 0.08y = 17,200. \end{cases}$$
We eliminate x by multiplying the first equation by –0.09 and then adding
$$\begin{cases} -0.09x - 0.09y = -18,000, \\ 0.09x + 0.08y = 17,200. \end{cases}$$
$-0.01y = -800,$

$y = 80,000.$

Therefore,
$$\begin{cases} x = 120,000, \\ y = 80,000. \end{cases}$$
Thus, \$120,000 is invested at 9% and \$80,000 is invested at 8%.

2. Let A = the number of deer of species A, and let B = the number of deer of species B. Then, the number of pounds of food pellets that will be consumed is $4A + 2B = 4000$. The number of pounds of hay that will be consumed is $5A + 7B = 9500$. Then, we have
$$\begin{cases} 4A + 2B = 4000, \\ 5A + 7B = 9500. \end{cases}$$
If we solve the first equation for B, we obtain
$$\begin{cases} B = 2000 - 2A \\ 5A + 7B = 9500. \end{cases}$$
Substituting 2000 – 2A for B in the second equation gives
$5A + 7(2000 - 2A) = 9500$
$A = 500$

Thus
$$\begin{cases} B = 2000 - 2A, \\ A = 500. \end{cases}$$
and
$$\begin{cases} B = 1000, \\ A = 500. \end{cases}$$
The food will support 500 of species *A* and 1000 of species *B*.

3. Let *A* = the number of fish of species A, and let *B* = the number of fish of species B. Then, the number of milligrams of the first supplement that will be consumed is 15*A* + 20*B* = 100,000. The number of milligrams of the second supplement that will be consumed is 30*A* + 40*B* = 200,000.
$$\begin{cases} 15A + 20B = 100,000, \\ 30A + 40B = 200,000. \end{cases}$$
We multiply the second equation by $-\frac{1}{2}$ and then add.
$$\begin{cases} 15A + 20B = 100,000, \\ -15A - 20B = -100,000, \end{cases}$$
$$0 = 0$$
Thus, there are infinitely many solutions of the form $A = \frac{20,000}{3} - \frac{4}{3}r$, *B* = *r*, where $0 \le r \le 5000$.

4. Let *A* = the amount of type A used, let *B* = the amount of type B used, and let *C* = the amount of type C used. If the final blend will sell for \$8.50 per pound, then 12*A* + 9*B* + 7*C* = 8.50, and *A* + *B* + *C* = 1. Furthermore, since the amount of type *B* is to be twice the amount of type *A*, *B* = 2*A*. Thus, the system of equations is
$$\begin{cases} 12A + 9B + 7C = 8.50, \\ A + B + C = 1, \\ B = 2A. \end{cases}$$
Simplifying gives
$$\begin{cases} 30A + 7C = 8.50, \\ 3A + C = 1, \\ B = 2A. \end{cases}$$
$$\begin{cases} A = \frac{1}{6}, \\ C = \frac{1}{2}, \\ B = \frac{1}{3}. \end{cases}$$
Thus, the final mixture will consist of $\frac{1}{6}$ lb of A, $\frac{1}{3}$ lb of B, and $\frac{1}{2}$ lb of C.

Problems 3.4

1. $\begin{cases} x + 4y = 3, & (1) \\ 3x - 2y = -5. & (2) \end{cases}$

From Eq. (1), *x* = 3 – 4*y*. Substituting in Eq. (2) gives
3(3 – 4*y*) – 2*y* = –5
9 – 12*y* – 2*y* = –5
–14*y* = –14,
or *y* = 1 ⇒ *x* = 3 – 4*y* = 3 – 4(1) = –1.
Thus *x* = –1, *y* = 1.

2. $\begin{cases} 4x + 2y = 9, & (1) \\ 5y - 4x = 5. & (2) \end{cases}$

Rewriting the system gives
$$\begin{cases} 4x + 2y = 9, \\ -4x + 5y = 5. \end{cases}$$
Adding the equations gives
7*y* = 14
y = 2
From Eq. (1) we have
4*x* + 2(2) = 9
4*x* = 5
$$x = \frac{5}{4}$$
Thus $x = \frac{5}{4}$, *y* = 2.

3. $\begin{cases} 3x - 4y = 13, & (1) \\ 2x + 3y = 3. & (2) \end{cases}$

Multiplying Eq. (1) by 3 and Eq. (2) by 4 gives
$$\begin{cases} 9x - 12y = 39, \\ 8x + 12y = 12. \end{cases}$$

Adding gives
$17x = 51$
$x = 3$
From Eq. (2) we have
$2(3) + 3y = 3$
$3y = -3$
$y = -1$
Thus $x = 3$, $y = -1$.

4. $\begin{cases} 2x - y = 1, & (1) \\ -x + 2y = 7. & (2) \end{cases}$

From Eq. (1), $y = 2x - 1$. Substituting in Eq. (2) gives
$-x + 2(2x - 1) = 7$
$3x = 9$
$x = 3 \Rightarrow y = 2x - 1 = 2(3) - 1 = 5$.
Thus $x = 3$, $y = 5$.

5. $\begin{cases} u + v = 5 \\ u - v = 7 \end{cases}$

From the first equation, $v = 5 - u$. Substituting in the second equation gives
$u - (5 - u) = 7$
$2u - 5 = 7$
$2u = 12$
or $u = 6$ so $v = 5 - u = 5 - 6 = -1$.
Thus, $u = 6$, $v = -1$.

6. $\begin{cases} 2p + q = 16, & (1) \\ 3p + 3q = 33. & (2) \end{cases}$

From Eq. (1), $q = 16 - 2p$. Substituting in Eq. (2) gives
$3p + 3(16 - 2p) = 33$
$-3p = -15$
$p = 5 \Rightarrow q = 16 - 2p = 16 - 10 = 6$.
Thus, $p = 5$, $q = 6$.

7. $\begin{cases} x - 2y = -7, & (1) \\ 5x + 3y = -9. & (2) \end{cases}$

From Eq. (1), $x = 2y - 7$. Substituting in Eq. (2) gives
$5(2y - 7) + 3y = -9$
$13y = 26$
$y = 2 \Rightarrow x = 2y - 7 = 2(2) - 7 = -3$.
Thus $x = -3$, $y = 2$.

8. $\begin{cases} 3x + 5y = 7, & (1) \\ 5x + 9y = 7. & (2) \end{cases}$

Multiplying Eq. (1) by 5 and Eq. (2) by –3 gives
$\begin{cases} 15x + 25y = 35, \\ -15x - 27y = -21. \end{cases}$
Adding gives $-2y = 14$, or $y = -7$. From Eq. (2) we have
$5x + 9(-7) = 7$
$5x = 70$
$x = 14$
Thus $x = 14$, $y = -7$.

9. $\begin{cases} 4x - 3y - 2 = 3x - 7y, \\ x + 5y - 2 = y + 4. \end{cases}$

Simplifying, we have
$\begin{cases} x + 4y = 2, \\ x + 4y = 6. \end{cases}$
Subtracting the second equation from the first gives $0 = -4$, which is never true. Thus there is no solution.

10. $\begin{cases} 5x + 7y + 2 = 9y - 4x + 6, \\ \frac{21}{2}x - \frac{4}{3}y - \frac{11}{4} = \frac{3}{2}x + \frac{2}{3}y + \frac{5}{4}. \end{cases}$

By simplifying, we have
$\begin{cases} 9x - 2y = 4, \\ 9x - 2y = 4. \end{cases}$
Both equations represent the same line, so we have infinitely many solutions. Let $y = r$. Then $9x - 2r = 4 \Rightarrow x = \frac{2}{9}r + \frac{4}{9}$. Thus a parametric solution is $x = \frac{2}{9}r + \frac{4}{9}$, $y = r$, where r is any real number.

11. $\begin{cases} \frac{2}{3}x + \frac{1}{2}y = 2, \\ \frac{3}{8}x + \frac{5}{6}y = -\frac{11}{2}. \end{cases}$

Clearing fractions gives the system
$\begin{cases} 4x + 3y = 12, \\ 9x + 20y = -132. \end{cases}$
Multiplying the first equation by 9 and the second equation by –4 gives

$$\begin{cases} 36x + 27y = 108, \\ -36x - 80y = 528. \end{cases}$$

Adding gives
$-53y = 636$
$y = -12$
From $4x + 3y = 12$, we have
$4x + 3(-12) = 12$
$4x = 48 \Rightarrow x = 12$. Thus $x = 12$, $y = -12$.

12. $$\begin{cases} \frac{1}{2}z - \frac{1}{4}w = \frac{1}{6} \\ \frac{1}{2}z + \frac{1}{4}w = \frac{1}{6} \end{cases}$$

Multiplying both equations by 12 gives

$$\begin{cases} 6z - 3w = 2 \\ 6z + 3w = 2 \end{cases}$$

Adding gives $12z = 4$ and so $z = \frac{1}{3}$.

From the first equation we have $6\left(\frac{1}{3}\right) - 3w = 2$,

from which $w = 0$. Thus $z = \frac{1}{3}$, $w = 0$.

13. $$\begin{cases} 5p + 11q = 7, & (1) \\ 10p + 22q = 33. & (2) \end{cases}$$

Multiplying Eq. (1) by –2 gives

$$\begin{cases} -10p - 22q = -14, \\ \ \ 10p + 22q = 33. \end{cases}$$

Adding gives 0 = 19, which is never true, so the system has no solution.

14. $$\begin{cases} 5x - 3y = 2, & (1) \\ -10x + 6y = 4. & (2) \end{cases}$$

Multiplying Eq. (1) by 2 gives

$$\begin{cases} 10x - 6y = 4, \\ -10x + 6y = 4. \end{cases}$$

Adding gives 0 = 8, which is never true, so the system has no solution.

15. $$\begin{cases} 2x + y + 6z = 3, & (1) \\ x - y + 4z = 1, & (2) \\ 3x + 2y - 2z = 2. & (3) \end{cases}$$

Adding Eq. (1) and (2), and adding 2 times Eq. (2) to Eq. (3) gives

$$\begin{cases} 3x + 10z = 4, \\ 5x + 6z = 4. \end{cases}$$

Multiplying the first equation by 5 and the second equation by –3 gives

$$\begin{cases} 15x + 50z = 20, \\ -15x - 18z = -12. \end{cases}$$

Adding gives $32z = 8$, or $z = \frac{1}{4}$. From $3x + 10z = 4$, we have

$3x + 10\left(\frac{1}{4}\right) = 4$

$3x = \frac{3}{2}$

$x = \frac{1}{2}$

From $2x + y + 6z = 3$, we have

$2\left(\frac{1}{2}\right) + y + 6\left(\frac{1}{4}\right) = 3$

$y = \frac{1}{2}$

Therefore $x = \frac{1}{2}$, $y = \frac{1}{2}$, $z = \frac{1}{4}$.

16. $$\begin{cases} x + y + z = -1, & (1) \\ 3x + y + z = 1, & (2) \\ 4x - 2y + 2z = 0. & (3) \end{cases}$$

Subtracting Eq. (2) from Eq. (1) gives $-2x = -2$, or $x = 1$. Substituting $x = 1$ in Eqs. (2) and (3) and simplifying gives

$$\begin{cases} y + z = -2, \\ -2y + 2z = -4. \end{cases}$$

Multiplying the first equation by 2 gives

$$\begin{cases} 2y + 2z = -4, \\ -2y + 2z = -4. \end{cases}$$

By adding, we have
$4z = -8$
$z = -2$
From $y + z = -2$, we have
$y + (-2) = -2$
$y = 0$
Thus $x = 1$, $y = 0$, $z = -2$.

17. $$\begin{cases} x + 4y + 3z = 10 \\ 4x + 2y - 2z = -2 \\ 3x - y + z = 11 \end{cases}$$

From the third equation, $y = 3x + z - 11$. Substituting in the first two equations gives

$$\begin{cases} x + 4(3x + z - 11) + 3z = 10 \\ 4x + 2(3x + z - 11) - 2z = -2 \end{cases}$$

or

$$\begin{cases} 13x + 7z = 54 \\ 10x = 20 \end{cases}$$

From the last equation we have $x = 2$.
Thus $13(2) + 7z = 54$, and $7z = 28$, hence $z = 4$.
Substitute these two values to solve for y:
$y = 3(2) + 4 - 11 = -1$
Therefore, $x = 2$, $y = -1$, $z = 4$.

18. $$\begin{cases} x + y + z = 18 & (1) \\ x - y - z = 12 & (2) \\ 3x + y + 4z = 4 & (3) \end{cases}$$

Adding Eq. (2) to both Eq. (1) and Eq. (3) gives

$$\begin{cases} 2x = 30 \\ 4x + 3z = 16 \end{cases}$$

From the first equation, $x = 15$. Substituting in the second equation gives

$$4(15) + 3z = 16$$
$$3z = -44$$
$$z = -\frac{44}{3}$$

From $x + y + z = 18$

$$15 + y - \frac{44}{3} = 18$$
$$y = \frac{53}{3}$$

Thus, $x = 15$, $y = \frac{53}{3}$, $z = -\frac{44}{3}$.

19. $$\begin{cases} x - 2z = 1, & (1) \\ y + z = 3. & (2) \end{cases}$$

From Eq. (1), $x = 1 + 2z$; from Eq. (2), $y = 3 - z$. Setting $z = r$ gives the parametric solution $x = 1 + 2r$, $y = 3 - r$, $z = r$, where r is any real number.

20. $$\begin{cases} 2y + 3z = 1, & (1) \\ 3x - 4z = 0. & (2) \end{cases}$$

From Eq. (1), $y = \frac{1}{2} - \frac{3}{2}z$; from Eq. (2), $x = \frac{4}{3}z$. Setting $z = r$ gives the parametric solution $x = \frac{4}{3}r$, $y = \frac{1}{2} - \frac{3}{2}r$, $z = r$, where r is any real number.

21. $$\begin{cases} x - y + 2z = 0, & (1) \\ 2x + y - z = 0 & (2) \\ x + 2y - 3z = 0 & (3) \end{cases}$$

Adding Eq. (1) to Eq. (3) gives

$$\begin{cases} x - y + 2z = 0, \\ 2x + y - z = 0 \\ 2x + y - z = 0 \end{cases}$$

We can ignore the third equation because the second equation can be used to reduce it to $0 = 0$. We have

$$\begin{cases} x - y + 2z = 0, \\ 2x + y - z = 0. \end{cases}$$

Adding the first equation to the second gives
$3x + z = 0$

$$x = -\frac{1}{3}z$$

Substituting in the first equation we have

$$-\frac{1}{3}z - y + 2z = 0$$
$$y = \frac{5}{3}z$$

Letting $z = r$ gives the parametric solution $x = -\frac{1}{3}r$, $y = \frac{5}{3}r$, $z = r$, where r is any real number.

22. $$\begin{cases} x - 2y - z = 0, & (1) \\ 2x - 4y - 2z = 0 & (2) \\ -x + 2y + z = 0 & (3) \end{cases}$$

Adding Eq. (1) to Eq. (3) gives

$$\begin{cases} x - 2y - z = 0, \\ 2x - 4y - 2z = 0 \\ 0 = 0 \end{cases}$$

We can ignore the third equation, so we have

$$\begin{cases} x - 2y - z = 0, \\ 2x - 4y - 2z = 0. \end{cases}$$

Multiplying the first equation by –2 gives

$$\begin{cases} -2x + 4y + 2z = 0, \\ 2x - 4y - 2z = 0. \end{cases}$$

Adding the first equation to the second, we have

$$\begin{cases} -2x + 4y + 2z = 0, \\ 0 = 0. \end{cases}$$

From the first equation, $x = 2y + z$. Setting $y = r$ and $z = s$ gives the parametric solution $x = 2r + s$, $y = r$, $z = s$, where r and s are any real numbers.

23. $\begin{cases} 2x+2y-z=3, & (1) \\ 4x+4y-2z=6. & (2) \end{cases}$

Multiplying Eq. (2) by $-\frac{1}{2}$ gives

$\begin{cases} 2x+2y-z=3, \\ -2x-2y+z=-3. \end{cases}$

Adding the first equation to the second equation gives

$\begin{cases} 2x+2y-z=3, \\ 0=0. \end{cases}$

Solving the first equation for *x*, we have $x=\frac{3}{2}-y+\frac{1}{2}z$. Letting $y = r$ and $z = s$ gives the parametric solution $x=\frac{3}{2}-r+\frac{1}{2}s$, $y = r$, $z = s$, where *r* and *s* are any real numbers.

24. $\begin{cases} 5x+y+z=17 \\ 4x+y+z=14 \end{cases}$

Subtracting the second equation from the first gives $x = 3$.
From the first equation we have
$y + z = 17 - 5x = 17 - 5(3) = 2$
Letting $z = r$ we have the parametric solution $x = 3$, $y = 2 - r$, $z = r$, where *r* is any real number.

25. Let x = number of gallons of 20% solution and y = number of gallons of 35% solution. Then

$\begin{cases} x+y=800, & (1) \\ 0.20x+0.35y=0.25(800). & (2) \end{cases}$

From Eq. (1), $y = 800 - x$. Substituting in Eq. (2) gives

$0.20x + 0.35(800 - x) = 0.25(800)$
$-0.15x + 280 = 200$
$-0.15x = -80$
$x=\frac{1600}{3}\approx 533.3$

$y=800-x=800-\frac{1600}{3}=\frac{800}{3}\approx 266.7.$ Thus 533.3 gal of 20% solution and 266.7 gal of 35% solution must be mixed.

26. Let x = the number of pounds of 3% nitrogen fertilizer, and let y = the number of pounds of 11% nitrogen fertilizer. Then

$\begin{cases} 0.03x+0.11y=0.09(20), \\ x+y=20. \end{cases}$

$\begin{cases} 0.03x+0.11y=1.8, \\ y=20-x. \end{cases}$

By substituting $20 - x$ for *y* in the first equation, and then simplifying, we obtain

$\begin{cases} x=5, \\ y=15. \end{cases}$

Thus, the final mixture should contain 5 lb of 3% nitrogen fertilizer, and 15 lb of 11% nitrogen fertilizer.

27. Let C = the number of pounds of cotton, let P = the number of pounds of polyester, and let N = the number of pounds of nylon. If the final blend will cost \$3.25 per pound to make, then $4C + 3P + 2N = 3.25$. Furthermore, if we use the same amount of nylon as polyester to prepare, say, 1 pound of fabric, then $N = P$ and $C + P + N = 1$. Thus, the system of equations is

$\begin{cases} 4C+3P+2N=3.25, \\ C+P+N=1, \\ N=P. \end{cases}$

Simplifying gives

$\begin{cases} 4C+5N=3.25, \\ C+2N=1, \\ N=P. \end{cases}$

$\begin{cases} N=0.25, \\ C=0.5, \\ P=0.25. \end{cases}$

Thus, each pound of the final fabric will contain 0.25 lb each of nylon and polyester, and 0.5 lb of cotton.

28. Let F = federal tax and S = state tax. Now solve the system

$\begin{cases} F=0.25(312{,}000-S), \\ S=0.10(312{,}000-F), \end{cases}$

which is equivalent to

$\begin{cases} 4F+S=312{,}000 \\ F+10S=312{,}000, \end{cases}$

and has solution

$\begin{cases} F=72{,}000, \\ S=24{,}000. \end{cases}$

Federal tax is \$72,000 and state tax is \$24,000.

29. Let p = speed of airplane in still air and w = wind speed. Now convert the time into minutes and solve the system

$$\begin{cases} p+w=\dfrac{900}{175} \\ p-w=\dfrac{900}{206}, \end{cases}$$

Thus $2p=\dfrac{900}{175}+\dfrac{900}{206}=\dfrac{36}{7}+\dfrac{450}{103}$

$p=\dfrac{3429}{721}$ miles per minute

$w=\dfrac{279}{721}$ miles per minute

Multiplying by 60 to get miles per hour we have
$p \approx 285$ and $w \approx 23.2$
Plane speed in still air is about 285 mph and wind speed is about 23.2 mph.

30. Let r = speed of raft in still water and c = speed of current. Then rate of raft downstream is $r + c$, and rate upstream is $r - c$. Since (rate)(time) = distance, we have

$$\begin{cases} (r+c)\left(\dfrac{1}{2}\right)=10, \\ (r-c)\left(\dfrac{3}{4}\right)=10, \end{cases}$$

or, more simply,

$$\begin{cases} r+c=20, \\ r-c=\dfrac{40}{3}. \end{cases}$$

Adding the equations gives

$2r=\dfrac{100}{3}$

$r=\dfrac{50}{3}$

Since $r + c = 20$, we have $c=\dfrac{10}{3}$. Thus the speed of the raft in still water is $16\dfrac{2}{3}$ mi/h; speed of the current is $3\dfrac{1}{3}$ mi/h.

31. Let x = number of early American units and y = number of Contemporary units. The fact that 20% more of early American styles are sold than Contemporary styles means that
$x = y + 0.20y$
$x = 1.20y$
An analysis of profit gives
$250x + 350y = 130{,}000$. Thus we have the system

$$\begin{cases} x=1.20y, & (1) \\ 250x+350y=130{,}000. & (2) \end{cases}$$

Substituting $1.20y$ for x in Eq. (2) gives
$250(1.20y) + 350y = 130{,}000$
$300y + 350y = 130{,}000$
$650y = 130{,}000$
$y = 200$
Thus $x = 1.20y = 1.20(200) = 240$. Therefore 240 units of early American and 200 units of Contemporary must be sold.

32. Let x = number of favorable comments, y = number of unfavorable comments, and z = number of no comments. Then

$$\begin{cases} x+y+z=250, & (1) \\ x=1.625y, & (2) \\ z=0.16(250). & (3) \end{cases}$$

From Eq. (3), $z = 40$. Substituting for x and z in Eq. (1), we obtain
$(1.625y) + y + (40) = 250$
$2.625y = 210$
$y = 80$
Thus $x = 1.625y = 1.625(80) = 130$. Therefore 130 liked, 80 disliked, and 40 had no comment.

33. Let x = number of calculators produced at Exton, and y = number of calculators produced at Whyton. The total cost of Exton is $7.50x + 7000$, and the total cost at Whyton is $6.00y + 8800$. Thus $7.50x + 7000 = 6.00y + 8800$. Also, $x + y = 1500$. This gives the system

$$\begin{cases} x+y=1500, & (1) \\ 7.50x+7000=6.00y+8800. & (2) \end{cases}$$

From Eq. (1), $y = 1500 - x$. Substituting in Eq. (2) gives
$7.50x + 7000 = 6.00(1500 - x) + 8800$
$7.50x + 7000 = 9000 - 6x + 8800$
$13.5x = 10{,}800$
$x = 800$
Thus $y = 1500 - x = 1500 - 800 = 700$.
Therefore 800 calculators must be made at the Exton plant and 700 calculators at the Whyton plant.

34. Let x, y, and z be the amounts of 2.20, 2.30, and 2.60 dollars/lb coffee, respectively. Then

$$\begin{cases} x+y+z=100, & (1) \\ 2.20x+2.30y+2.60z=2.40(100), & (2) \\ y=z. & (3) \end{cases}$$

From Eq. (3), $y = z$. Substituting for y in Eqs. (1) and (2) gives

$$\begin{cases} x+z+z=100, \\ 2.20x+2.30z+2.60z=240. \end{cases}$$

or, by simplifying,

$$\begin{cases} x+2z=100, \\ 2.20x+4.90z=240. \end{cases}$$

From the first equation, $x = 100 - 2z$.
Substituting in the second equation gives
$2.20(100 - 2z) + 4.90z = 240$
$0.50z = 20$
$z = 40$
From $x = 100 - 2z$, $x = 100 - 2(40) = 20$. From $y = z$, $y = 40$. Thus, 20, 40, and 40 lb of \$2.20, \$2.30, and \$2.60 per lb coffee must be used, respectively.

35. Let x = rate on first \$100,000 and y = rate on sales over \$100,000. Then

$$\begin{cases} 100,000x+75,000y=8500, & (1) \\ 100,000x+180,000y=14,800. & (2) \end{cases}$$

Subtracting Eq. (1) from Eq. (2) gives
$105,000y = 6300$
$y = 0.06$
Substituting in Eq. (1) gives
$100,000x + 75,000(0.06) = 8500$
$100,000x + 4500 = 8500$, $100,000x = 4000$, or $x = 0.04$. Thus the rate is 4% on the first \$100,000 and 6% on the remainder.

36. A system that describes the situation is

$$\begin{cases} T=L+25,000,000 \\ T=L+0.30L \end{cases}$$

We can rewrite this as

$$\begin{cases} T=L+25,000,000 \\ T=1.30L \end{cases}$$

Thus $T = 1.3L$ and we can substitute this in the first equation:
$1.3L = L + 25,000,000$. Solving for L
$0.3L = 25,000,000$
$L = 83,333,333$
$T = 1.3L = 1.3(83,333,333) = 108,333,333$ thus $T = \$108,333,333$ and $L = \$83,333,333$.

37. Let x = number of loose-filled boxes and y = number of boxes of clam-shells that will be filled. Then $8y$ clam-shells will be used. This will take $20x + 2.2(8y)$ pounds of peaches.

$$\begin{cases} x=y & (1) \\ 20x+17.6y=3600 & (2) \end{cases}$$

Substitute $x = y$ in Eq. (2).
$20x+17.6x=3600$
$37.6x=3600$
$x \approx 95.74$
$y = x \approx 95.74$
Thus, 95 boxes will be loose-filled and 8(95) = 760 clam-shells will be used, for a total of 190 boxes.

38. Let p_1 and p_2 be the amounts of the two investments, respectively. Then the total amount invested was $p_1 + p_2$, and from the statement of the problem we can write $\frac{3}{10}(p_1+p_2)+600=p_1$. The return on the second investment was $1120 - 384 = 736$. Since the percentage return on each was the same, and since $\text{rate} = \frac{\text{interest}}{\text{amt. invested}}$, we can write $\frac{384}{p_1}=\frac{736}{p_2}$. This can also be written as $\frac{p_1}{384}=\frac{p_2}{736}$. Hence we have the system

$$\begin{cases} \frac{3}{10}(p_1+p_2)+600=p_1, \\ \frac{p_1}{384}=\frac{p_2}{736}. \end{cases}$$

Simplifying, we have

$$\begin{cases} -\frac{7}{10}p_1+\frac{3}{10}p_2=-600, \\ p_1=\frac{12}{23}p_2. \end{cases}$$

Substituting $p_1=\frac{12}{23}p_2$ in first equation gives

$$-\frac{7}{10}\left(\frac{12}{23}p_2\right)+\frac{3}{10}p_2=-600$$

$$-\frac{3}{46}p_2=-600$$

$$p_2=9200$$

Thus $p_1=\frac{12}{23}p_2=\frac{12}{23}(9200)=4800$. The total amount invested was $p_1+p_2=4800+9200=\$14,000$.

39. Let c = number of chairs company makes, r = number of rockers, and l = number of chaise lounges.
Wood used: $(1)c + (1)r + (1)l = 400$
Plastic used: $(1)c + (1)r + (2)l = 600$

Aluminum used: $(2)c + (3)r + (5)l = 1500$
Thus we have the system

$$\begin{cases} c+r+l=400, & (1) \\ c+r+2l=600, & (2) \\ 2c+3r+5l=1500. & (3) \end{cases}$$

Subtracting Eq. (1) from Eq. (2) gives $l = 200$.
Adding –2 times Eq. (1) to Eq. (3) gives
$r + 3l = 700$, from which
$r + 3(200) = 700$,
$r = 100$
From Eq. (1) we have $c + 100 + 200 = 400$, or
$c = 100$. Thus 100 chairs, 100 rockers and 200 chaise lounges should be made.

40. Let x, y, and z, be the amounts originally invested at 7%, 8%, and 9%, respectively. Then

$$\begin{cases} x+y+z=35,000, & (1) \\ 0.07x+0.08y+0.09z=2830, & (2) \\ 0.07x+0.08y+0.10z=2960. & (3) \end{cases}$$

Subtracting Eq. (2) from Eq. (3) gives
$0.01z = 130$
$z = 13,000$
Subtracting 0.07 times Eq. (1) from Eq. (2) gives
$0.01y + 0.02z = 380$. Letting $z = 13,000$, we have $0.01y + 0.02(13,000) = 380$
$0.01y = 120$
$y = 12,000$
From Eq. (1),
$x + 12,000 + 13,000 = 35,000$
$x = 10,000$
The investments are \$10,000 at 7%, \$12,000 at 8%, \$13,000 at 9% (later 10%).

41. Let x = number of skilled workers employed, y = number of semiskilled workers employed, z = number of shipping clerks employed.
Then we have the system

$$\begin{cases} \text{number of workers:} & x+y+z=70, & (1) \\ \text{wages:} & 16x+9.5y+10z=725 & (2) \\ \text{semiskilled:} & y=2x & (3) \end{cases}$$

From the last equation, $y = 2x$ so substitute into the first two equations:

$$\begin{cases} x+2x+z=70 \\ 16x+9.5(2x)+10z=725 \end{cases}$$

or

$$\begin{cases} 3x+z=70 \\ 35x+10z=725 \end{cases}$$

Adding –10 times the first equation to the second gives:
$5x = 25$
$x = 5$
So $y = 2x = 10$
$z = 70 - 3x = 70 - 15 = 55$
The company should hire 5 skilled workers, 10 semiskilled workers, and 55 shipping clerks.

42. *Method 1.* Let a = number of minutes that pump for tank A operates, and b = number of minutes that pump for tank B operates. Then $b = a + 5$. $25a$ gallons are pumped from tank A and $35b$ from tank B.

$$\begin{cases} b=a+5, & (1) \\ 25a+35b=10,000. & (2) \end{cases}$$

Since $b = a + 5$, substituting in Eq. (2) gives
$25a + 35(a + 5) = 10,000$
$60a = 9825$
$a = 163.75$
$b = a + 5$, $b = 163.75 + 5 = 168.75$. Thus
$25(163.75) = 4093.75$ gallons are pumped from A, and $35(168.75) = 5906.25$ gallons are pumped from B.
Method 2. Let a = number of gallons from A, and let b = number of gallons from B. Then $a + b = 10,000$. The number of minutes the pump on A operates is $\frac{a}{25}$. For the pump on B, it is $\frac{b}{35}$. Thus

$$\begin{cases} \frac{a}{25}+5=\frac{b}{35} & (1) \\ a+b=10,000. & (2) \end{cases}$$

From Eq. (2), $a = 10,000 - b$. Substituting in Eq. (1) gives

$$\frac{10,000-b}{25}+5=\frac{b}{35}$$
$$400-\frac{b}{25}+5=\frac{b}{35}$$
$$405=\frac{12b}{175}$$
$$5906.25=b$$

Thus
$a = 10,000 - b = 10,000 - 5906.25 = 4093.75$.

45. $x = 3$, $y = 2$

46. $x = 1.33$, $y = 0.67$

47. $x = 8.3$, $y = 14.0$

Problems 3.5

In the following solutions, any reference to Eq. (1) or Eq. (2) refers to the first or second equation, respectively, in the given system.

1. From Eq. (2), $y = 3 - 2x$. Substituting in Eq. (1) gives

$$3-2x = x^2 - 9$$
$$0 = x^2 + 2x - 12$$
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4(1)(-12)}}{2(1)} = \frac{-2 \pm \sqrt{52}}{2} = -1 \pm \sqrt{13}$$

From $y = 3 - 2x$, if $x = -1+\sqrt{13}$, then $y = 5-2\sqrt{13}$; if $x = -1-\sqrt{13}$, then $y = 5+2\sqrt{13}$.
There are two solutions:
$x = -1+\sqrt{13}$, $y = 5-2\sqrt{13}$;
$x = -1-\sqrt{13}$, $y = 5+2\sqrt{13}$.

2. From Eq. (2), $y = x$. Substituting in Eq. (1) gives

$$x = x^3$$
$$x - x^3 = 0$$
$$x\left(1-x^2\right) = 0$$

$x(1 + x)(1 - x) = 0$. Thus $x = 0, \pm 1$. From $y = x$, if $x = 0$, then $y = 0$; if $x = 1$, then $y = 1$; if $x = -1$, then $y = -1$. There are three solutions: $x = 0$, $y = 0$; $x = 1$, $y = 1$; $x = -1$, $y = -1$.

3. From Eq. (2), $q = p - 1$. Substituting in Eq. (1) gives

$$p^2 = 5-(p-1)$$
$$p^2 + p - 6 = 0$$
$$(p + 3)(p - 2) = 0$$

Thus $p = -3, 2$. From $q = p - 1$, if $p = -3$, we have $q = -3 - 1 = -4$; if $p = 2$, then $q = 2 - 1 = 1$. There are two solutions: $p = -3$, $q = -4$; $p = 2$, $q = 1$.

4. From Eq. (2), $y = x - 14$. Substituting in Eq. (1) gives

$$(x-14)^2 - x^2 = 28$$
$$-28x + 196 = 28$$
$$-28x = -168$$
$$x = 6$$

If $x = 6$, then $y = x - 14 = 6 - 14 = -8$. The only solution is $x = 6$, $y = -8$.

5. Substituting $y = x^2$ into $x = y^2$ gives $x = x^4$,

$$x^4 - x = 0$$
$$x\left(x^3-1\right) = 0$$

Thus $x = 0, 1$. From $y = x^2$, if $x = 0$, then $y = 0^2 = 0$; $x = 1$, then $y = 1^2 = 1$. There are two solutions: $x = 0$, $y = 0$; $x = 1$, $y = 1$.

6. $\begin{cases} p^2 - q + 1 = 0 \\ 5q - 3p - 2 = 0 \end{cases}$

From the first equation $q = p^2 + 1$. Substituting into the second equation gives

$$5(p^2+1) - 3p - 2 = 0$$
$$5p^2 - 3p + 3 = 0$$
$$p = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{3 \pm \sqrt{(-3)^2 - 4(5)(3)}}{2(5)} = \frac{3 \pm \sqrt{-51}}{10}$$

Since $\sqrt{-51}$ is not a real number, there are no real solutions.

7. Substituting $y = x^2 - 2x$ in Eq. (1) gives

$$x^2 - 2x = 4x - x^2 + 8$$
$$2x^2 - 6x - 8 = 0$$
$$x^2 - 3x - 4 = 0$$
$$(x - 4)(x + 1) = 0$$

Thus $x = 4, -1$. From $y = x^2 - 2x$, if $x = 4$, then we have $y = 4^2 - 2(4) = 8$; if $x = -1$, then $y = (-1)^2 - 2(-1) = 3$. There are two solutions: $x = 4$, $y = 8$; $x = -1$, $y = 3$.

8. From Eq. (1), $y = x^2 + 4x + 4$. Substituting in Eq. (2) gives

$$x^2 + 4x + 4 - x^2 - 4x + 3 = 0$$
$$7 = 0$$

Since this is never true, the system has no solution.

9. Substituting $p = \sqrt{q}$ in Eq. (2) gives $\sqrt{q} = q^2$. Squaring both sides gives

$$q = q^4$$
$$q^4 - q = 0$$
$$q\left(q^3 - 1\right) = 0$$

Thus $q = 0, 1$. From $p = \sqrt{q}$, if $q = 0$, then $p = \sqrt{0} = 0$; if $q = 1$, then $p = \sqrt{1} = 1$. There are two solutions: $p = 0, q = 0$; $p = 1, q = 1$.

10. Substituting $z = \frac{4}{w}$ in Eq. (2) gives

$$3\left(\frac{4}{w}\right) = 2w + 2$$
$$12 = 2w^2 + 2w$$
$$w^2 + w - 6 = 0$$
$$(w + 3)(w - 2) = 0$$

Thus $w = -3, 2$. From $z = \frac{4}{w}$, if $w = -3$, then $z = -\frac{4}{3}$; if $w = 2$, then $z = \frac{4}{2} = 2$. There are two solutions: $w = -3$, $z = -\frac{4}{3}$; $w = 2, z = 2$.

11. Replacing x^2 by $y^2 + 13$ in Eq. (2) gives

$$y = \left(y^2 + 13\right) - 15$$
$$y^2 - y - 2 = 0$$
$$(y - 2)(y + 1) = 0$$

Thus $y = 2, -1$. If $y = 2$, then $x^2 = y^2 + 13 = 2^2 + 13 = 17$, so $x = \pm\sqrt{17}$.

If $y = -1$, then $x^2 = y^2 + 13 = (-1)^2 + 13 = 14$, so $x = \pm\sqrt{14}$. The system has four solutions: $x = \sqrt{17}$, $y = 2$; $x = -\sqrt{17}$, $y = 2$; $x = \sqrt{14}$, $y = -1$; $x = -\sqrt{14}$, $y = -1$.

12. From Eq. (2), $y = 3x - 5$. Substituting in Eq. (1) gives

$$x^2 + (3x - 5)^2 - 2x(3x - 5) = 1$$
$$4x^2 - 20x + 24 = 0$$
$$x^2 - 5x + 6 = 0$$
$$(x - 3)(x - 2) = 0$$

Thus $x = 3, 2$. From $y = 3x - 5$, if $x = 3$, then $y = 3(3) - 5 = 4$; if $x = 2$, then $y = 3(2) - 5 = 1$. Thus there are two solutions: $x = 3, y = 4$; $x = 2, y = 1$.

13. From Eq. (1), $y = x - 1$. Substituting in Eq. (2) gives

$$x - 1 = 2\sqrt{x + 2}$$
$$(x - 1)^2 = 4(x + 2)$$
$$x^2 - 2x + 1 = 4x + 8$$
$$x^2 - 6x - 7 = 0$$
$$(x + 1)(x - 7) = 0$$

Thus $x = -1$ or 7.
From $y = x - 1$, if $x = -1$, then $y = -2$; if $x = 7$, then $y = 6$. However, from Eq. (2), $y \geq 0$. The only solution is $x = 7$, $y = 6$.

14. Substituting $y = \frac{1}{x - 1}$ in Eq. (1) gives

$$\frac{1}{x - 1} = \frac{x^2}{x - 1} + 1$$
$$1 = x^2 + (x - 1)$$
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$

Thus $x = -2, 1$. But x cannot equal 1 in either of the original equations (division by zero). From $y = \frac{1}{x - 1}$, if $x = -2$, then $y = \frac{1}{-2 - 1} = -\frac{1}{3}$. The solution is $x = -2$, $y = -\frac{1}{3}$.

15. We can write the following system of equations.

$$\begin{cases} y = 0.01x^2 + 0.01x + 7, \\ y = 0.01x + 8.0. \end{cases}$$

By substituting $0.01x + 8.0$ for y in the first equation and simplifying, we obtain

$$0.01x + 8.0 = 0.01x^2 + 0.01x + 7$$
$$0 = 0.01x^2 - 1$$
$$0 = (0.1x + 1)(0.1x - 1)$$
$$x = -10 \quad \text{or} \quad x = 10$$

If $x = -10$ then $y = 7.9$, and if $x = 10$ then $y = 8.1$.

The rope touches the streamer twice, 10 feet away from center on each side at (–10, 7.9) and (10, 8.1).

16. We can write the following system of equations.

$$\begin{cases} y = 0.06x^2 + 0.012x + 8, \\ y = 0.912x + 5. \end{cases}$$

By substituting 0.912*x* + 5 for *y* in the first equation and then simplifying, we obtain

$$0.912x + 5 = 0.06x^2 + 0.012x + 8$$
$$0 = 0.06x^2 - 0.9x + 3$$
$$0 = 0.06\left(x^2 - 15x + 50\right)$$
$$0 = 0.06(x - 10)(x - 5)$$
$$x = 10 \quad \text{or} \quad x = 5$$

If $x = 10$ then $y = 14.12$, and if $x = 5$ then $y = 9.56$. The two holes are located at (10, 14.12) and (5, 9.56).

17. The system has 3 solutions.

18. $x = 2$, $y = 4$

19. $x = -1.3$, $y = 5.1$

20. $x = -1.9$, $y = -3.6$; $x = -0.3$, $y = 1.2$; $x = 2.1$, $y = 8.3$

21. $x = 1.76$

22. $x = 2.81$

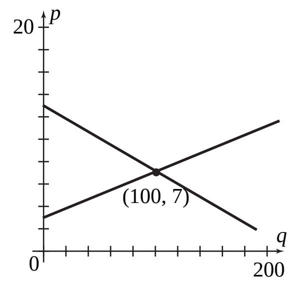
23. $x = -1.46$

Problems 3.6

1. Equating *p*-values gives

$$\frac{4}{100}q + 3 = -\frac{6}{100} + 13$$
$$\frac{10}{100}q = 10$$
$$q = 100$$
$$p = \frac{4}{100}(100) + 3 = 7$$

Thus, the equilibrium point is (100, 7).



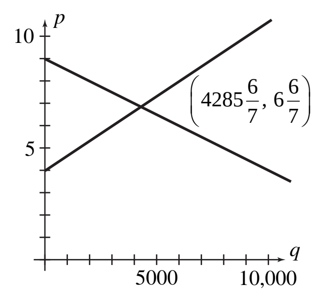
2. Equating *p*-values gives

$$\frac{1}{1500}q + 4 = -\frac{1}{2000}q + 9$$
$$\frac{7}{6000}q = 5$$
$$q = \frac{30{,}000}{7} = 4285\frac{5}{7} \approx 4285.71$$

When $q = 4285\frac{5}{7}$, then

$$p = \frac{1}{1500}q + 4 = \frac{1}{1500}\left(4285\frac{5}{7}\right) + 4 = 6\frac{6}{7} \approx 6.86$$

The equilibrium point is $\left(4285\frac{5}{7},\ 6\frac{6}{7}\right)$.



3.
$$\begin{cases} 35q - 2p + 250 = 0, & (1) \\ 65q + p - 537.5 = 0. & (2) \end{cases}$$

Multiplying Eq. (2) by 2 and adding equations gives

$165q - 825 = 0$

$q = 5$

From Eq. (2),

$65(5) + p - 537.5 = 0$

$p = 212.50$

Thus the equilibrium point is (5, 212.50).

4. $\begin{cases} 246p - 3.25q - 2460 = 0, & (1) \\ 410p + 3q - 14,452.5 = 0. & (2) \end{cases}$

Multiplying Eq. (1) by 3 and Eq. (2) by 3.25 gives

$\begin{cases} 738p - 9.75q - 7380 = 0, \\ 1332.5p + 9.75q - 46,970.625 = 0. \end{cases}$

Adding gives

$2070.5p - 54,350.625 = 0$

$p = \frac{54,350.625}{2070.5} = 26.25$

From Eq. (2) in original system,

$q = \frac{14,452.5 - 410p}{3} = \frac{14,452.5 - 410(26.25)}{3}$

$= \frac{14,452.5 - 10,762.5}{3} = \frac{3690}{3} = 1230$

The equilibrium point is (1230, 26.25).

5. Equating p-values:

$2q + 20 = 200 - 2q^2$

$2q^2 + 2q - 180 = 0$

$q^2 + q - 90 = 0$

$(q + 10)(q - 9) = 0$

Thus $q = -10, 9$. Since $q \geq 0$, choose $q = 9$. Then $p = 2q + 20 = 2(9) + 20 = 38$. The equilibrium point is (9, 38).

6. Equating p-values gives

$(q + 10)^2 = 388 - 16q - q^2$

$2q^2 + 36q - 288 = 0$

$q^2 + 18q - 144 = 0$

$(q + 24)(q - 6) = 0$

Thus $q = -24, 6$. Since $q \geq 0$, choose $q = 6$. Then $p = (q + 10)^2 = (6 + 10)^2 = 16^2 = 256$. The equilibrium point is (6, 256).

7. Equating p-values gives $20 - q = \sqrt{q + 10}$.

Squaring both sides gives

$400 - 40q + q^2 = q + 10$

$q^2 - 41q + 390 = 0$

$(q - 26)(q - 15) = 0$

Thus $q = 26, 15$. If $q = 26$, then $p = 20 - q = 20 - 26 = -6$. But p cannot be negative. If $q = 15$, then $p = 20 - q = 20 - 15 = 5$. The equilibrium point is (15, 5).

8. Equating p-values gives

$$\frac{q}{4} + 6 = \frac{2240}{q + 2}$$

$(q + 24)(q + 2) = 2240(4)$

$q^2 + 26q + 48 = 8960$

$q^2 + 26q - 8912 = 0$

$q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-26 \pm \sqrt{(26)^2 - 4(1)(-8912)}}{2(1)}$

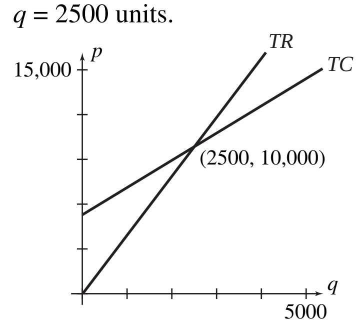
$q \approx 82.29$ or -108.29

$q \geq 0$ so choose $q \approx 82.29$.

Then $p \approx \frac{82.29}{4} + 6 \approx 26.57$.

The equilibrium point is (82.29, 26.57).

9. Letting $y_{TR} = y_{TC}$ gives $4q = 2q + 5000$, or $q = 2500$ units.

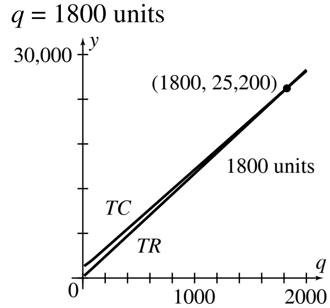


10. Letting $y_{TR} = y_{TC}$ gives

$14q = \frac{40}{3}q + 1200$

$\frac{2}{3}q = 1200$

$q = 1800$ units



11. Letting $y_{TR} = y_{TC}$ gives

$0.05q = 0.85q + 600$

$-0.80q = 600$

$q = -750$, which is negative. Thus one cannot break even at any level of production.

12. Letting $y_{TR} = y_{TC}$ gives
$0.25q = 0.16q + 360$
$0.09q = 360$
$q = 4000$ units

13. Letting $y_{TR} = y_{TC}$ gives $90 - \frac{900}{q+3} = 1.1q + 37.3$

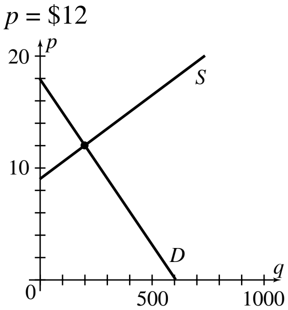
$$90(q+3) - 900 = (1.1q + 37.3)(q+3)$$
$$90q + 270 - 900 = 1.1q^2 + 40.6q + 111.9$$
$$1.1q^2 - 49.4q + 741.9 = 0$$
$$q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{49.4 \pm \sqrt{(-49.4)^2 - 4(1.1)(741.9)}}{2(1.1)} = \frac{49.4 \pm \sqrt{-824}}{2.2}$$

There are no real solutions, therefore one cannot break even at any level of production.

14. Letting $y_{TR} = y_{TC}$ gives
$$0.1q^2 + 9q = 3q + 400$$
$$0.1q^2 + 6q - 400 = 0$$
$$q^2 + 60q - 4000 = 0$$
$$(q+100)(q-40) = 0$$
Thus $q = -100, 40$. Since $q \geq 0$, choose $q = 40$ units.

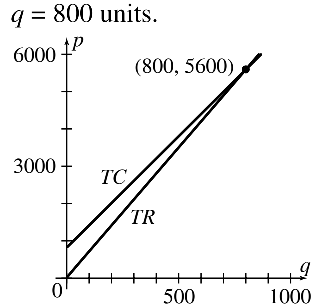
15. $\begin{cases} 3q - 200p + 1800 = 0, & (1) \\ 3q + 100p - 1800 = 0. & (2) \end{cases}$

a. Subtracting Eq. (2) from Eq. (1) gives
$-300p + 3600 = 0$
$p = \$12$



b. Before the tax, the supply equation is
$3q - 200p + 1800 = 0$
$-200p = -3q - 1800$
$$p = \frac{3}{200}q + 9$$
After the tax, the supply equation is
$$p = \frac{3}{200}q + 9 + 0.27$$
$$p = \frac{3}{200}q + 9.27$$
This equation can be written $-3q + 200p - 1854 = 0$, and the new system to solve is
$\begin{cases} -3q + 200p - 1854 = 0, \\ 3q + 100p - 1800 = 0. \end{cases}$
Adding gives
$$300p - 3654 = 0 \Rightarrow p = \frac{3654}{300} = \$12.18.$$

16. a. Letting $y_{TR} = y_{TC}$ gives $7q = 6q + 800$, or $q = 800$ units.



b. The new total cost equation is
$$y_{TC} = 1.05(6q + 800)$$
$$y_{TC} = 6.3q + 840$$
Letting $y_{TR} = y_{TC}$ gives
$7q = 6.3q + 840$
$0.7q = 840$
$q = 1200$ units

17. Since profit = total revenue – total cost, then $4600 = 8.35q - (2116 + 7.20q)$. Solving gives
$4600 = 1.15q - 2116$
$1.15q = 6716$
$$q = \frac{6716}{1.15} = 5840 \text{ units}$$
For a loss (negative profit) of \$1150, we solve $-1150 = 8.35q - (2116 + 7.20q)$. Thus
$-1150 = 1.15q - 2116$
$1.15q = 966$
$q = 840$ units
To break even, we have $y_{TR} = y_{TC}$, or
$8.35q = 2116 + 7.20q$
$1.15q = 2116$
$q = 1840$ units

18. For the supply equation we fit the points (0, 1) and (13,500, 4.50) to a straight line. We have

$$m = \frac{4.50-1}{13,500-0} = \frac{3.5}{13,500} = \frac{\frac{7}{2}}{13,500} = \frac{7}{27,000},$$

so the line is

$$p-1 = \frac{7}{27,000}(q-0)$$

27,000(p – 1) = 7q

7q – 27,000p + 27,000 = 0

For the demand equation, we fit the points (0, 20) and (13,500, 4.50) to a straight line. We have

$$m = \frac{4.50-20}{13,500-0} = -\frac{15.5}{13,500} = -\frac{\frac{31}{2}}{13,500}$$

$= -\frac{31}{27,000}$, so the line is

$$p-20 = -\frac{31}{27,000}(q-0)$$

27,000(p – 20) = –31q

31q + 27,000p – 540,000 = 0

19. Let q = break-even quantity. Since total revenue is 5q, we have 5q = 200,000, which yields q = 40,000. Let c be the variable cost per unit. Then at the break even point,

Tot. Rev. = Tot. Cost

= Variable Cost + Fixed Cost.

Thus

200,000 = 40,000c + 40,000

160,000 = 40,000c

c = \$4.

20. Let q = number of pairs sold.

Total Revenue = 2.63q

Total Cost = 0.85q + 0.96q + 0.32q + 70,500

At the break-even point,

Total Revenue = Total cost, or

2.63q = 0.85q + 0.96q + 0.32q + 70,500

Solving for q gives

2.63q = 2.13q + 70,500 or 0.5q = 70,500

q = 141,000

21. $y_{TC} = 3q + 1250$: $y_{TR} = 60\sqrt{q}$. Letting $y_{TR} = y_{TC}$ gives

$$60\sqrt{q} = 3q + 1250$$

$$20\sqrt{q} = q + \frac{1250}{3}$$

Squaring gives

$$400q = q^2 + \frac{2500}{3}q + \left(\frac{1250}{3}\right)^2$$

$$q^2 + \frac{1300}{3}q + \frac{1,562,500}{9} = 0$$

Using the quadratic formula,

$$q = \frac{-\frac{1300}{3} \pm \sqrt{\left(\frac{1300}{3}\right)^2 - 4(1)\left(\frac{1,562,500}{9}\right)}}{2},$$

which is not real. Thus total cost always exceeds total revenue; there is no break-even point.

22. $p = \frac{1000}{q}$

a. $4 = \frac{1000}{q}$ gives $q = \frac{1000}{4} = 250$ units

b. $2 = \frac{1000}{q}$ gives $q = \frac{1000}{2} = 500$ units

c. $0.50 = \frac{1000}{q}$ gives $q = \frac{1000}{0.50} = 2000$ units

The revenue is $qp = q\left(\frac{1000}{q}\right) = 1000$, so revenue of \$1000 is received regardless of price.

23. After the subsidy the supply equation is

$$p = \left[\frac{8}{100}q + 50\right] - 1.50$$

$$p = \frac{8}{100}q + 48.50$$

The system to consider is

$$\begin{cases} p = \frac{8}{100}q + 48.50, \\ p = -\frac{7}{100}q + 65. \end{cases}$$

Equating p-values gives

$$\frac{8}{100}q + 48.50 = -\frac{7}{100}q + 65$$

$$\frac{15}{100}q = 16.5$$

q = 110

When q = 110, then

$$p = \frac{8}{100}q + 48.50 = \frac{8}{100}(110) + 48.50$$

$= 8.8 + 48.50 = 57.30$.

Thus the original equilibrium price decreases by \$0.70.

24. a. Profit = Total Revenue – Total Cost
= 280,000(2.00) – [110,000 + 280,000(1.75)]
= 560,000 – 600,000 = –40,000.
There is a net loss of $40,000.

b. Let q = unit sales volume. Then
$40{,}000 = 2.00q - [110{,}000 + 1.75q]$
$150{,}000 = 0.25q$
$q = 600{,}000$ units

25. Equating q_A-values gives
$$7 - p_A + p_B = -3 + 4p_A - 2p_B$$
$$10 = 5p_A - 3p_B$$
Equating q_B-values gives
$$21 + p_A - p_B = -5 - 2p_A + 4p_B$$
$$26 = -3p_A + 5p_B$$
Now we solve
$$\begin{cases} 10 = 5p_A - 3p_B \\ 26 = -3p_A + 5p_B \end{cases}$$
Adding 3 times the first equation to 5 times the second equation gives
$$160 = 16p_B$$
$$p_B = 10$$
From $5p_A - 3p_B = 10$, $5p_A - 3(10) = 10$ or $p_A = 8$.
Thus $p_A = 8$ and $p_B = 10$.

26. $17.80; 2.6 thousand units

27. 2.4 and 11.3

Chapter 3 Review Problems

1. Solving $\frac{k-5}{3-2} = 4$ gives $k - 5 = 4$, $k = 9$.

2. The equation $\frac{4-4}{5-k} = 0$ is true for any real number $k \neq 5$.

3. (–2, 3) and (0, –1) lie on the line, so
$m = \frac{-1-3}{0-(-2)} = -2$. Slope-intercept form:
$y = mx + b \Rightarrow y = -2x - 1$. A general form:
$2x + y + 1 = 0$.

4. Slope of $y = 3x - 4$ is $m = 3$, so slope of parallel line is also $m = 3$. Thus
$y - (-1) = 3[x - (-1)]$
$y + 1 = 3x + 3$,
Slope-intercept form: $y = 3x + 2$. General form:
$3x - y + 2 = 0$.

5. $y - 4 = \frac{1}{2}(x - 10)$
$y - 4 = \frac{1}{2}x - 5$
$y = \frac{1}{2}x - 1$, which is slope-intercept form.
Clearing fractions, we have
$2y = 2\left(\frac{1}{2}x - 1\right)$
$2y = x - 2$
$x - 2y - 2 = 0$, which is a general form.

6. Slope of a vertical line is undefined, so slope-intercept form does not exist. An equation of the vertical line is $x = 3$. General form: $x - 3 = 0$.

7. Slope of a horizontal line is 0. Thus
$y - 4 = 0[x - (-2)]$
$y - 4 = 0$,
so slope-intercept form is $y = 4$. A general form is $y - 4 = 0$.

8. $-3y + 5x = 7$ $\left(\text{or } y = \frac{5}{3}x - \frac{7}{3}\right)$ has slope $\frac{5}{3}$.
Thus the line perpendicular to it has slope $-\frac{3}{5}$
and its equation is $y - 2 = -\frac{3}{5}(x - 1)$, or
$y = -\frac{3}{5}x + \frac{13}{5}$. A general form is $3x + 5y - 13 = 0$.

9. The line $2y + 5x = 2$ $\left(\text{or } y = -\frac{5}{2}x + 1\right)$ has slope
$-\frac{5}{2}$, so the line perpendicular to it has slope $\frac{2}{5}$.
Since the y-intercept is –3, the equation is
$y = \frac{2}{5}x - 3$. A general form is $2x - 5y - 15 = 0$.

10. The line has slope $\frac{8-2}{1-(-1)} = \frac{6}{2} = 3$, so an equation of the line is $y - 8 = 3(x - 1)$. If $x = 3$, then
$y - 8 = 3(3 - 1)$
$y - 8 = 6$
$y = 14$
Thus (3, 13) does not lie on the line.

In Problems 11–16, $m_1 =$ slope of first line, and $m_2 =$ slope of second line.

11. $x + 4y + 2 = 0$ $\left(\text{or } y = -\frac{1}{4}x - \frac{1}{2}\right)$ has slope $m_1 = -\frac{1}{4}$ and $8x - 2y - 2 = 0$ (or $y = 4x - 1$) has slope $m_2 = 4$. Since $m_1 = -\frac{1}{m_2}$, the lines are perpendicular to each other.

12. $y - 2 = 2(x - 1)$ (or $y = 2x$) has slope $m_1 = 2$, and $2x + 4y - 3 = 0$ $\left(\text{or } y = -\frac{1}{2}x + \frac{3}{4}\right)$ has slope $m_2 = -\frac{1}{2}$. Since $m_1 = -\frac{1}{m_2}$, the lines are perpendicular.

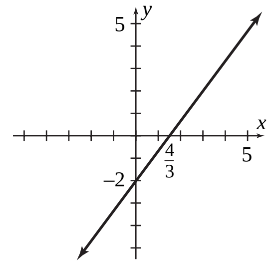
13. $x - 3 = 2(y + 4)$ $\left(\text{or } y = \frac{1}{2}x - \frac{11}{2}\right)$ has slope $m_1 = \frac{1}{2}$, and $y = 4x + 2$ has slope $m_2 = 4$. Since $m_1 \neq m_2$ and $m_1 \neq -\frac{1}{m_2}$, the lines are neither parallel nor perpendicular to each other.

14. $2x + 7y - 4 = 0$ $\left(\text{or } y = -\frac{2}{7}x + \frac{4}{7}\right)$ has slope $m_1 = -\frac{2}{7}$, and $6x + 21y = 90$ $\left(\text{or } y = -\frac{2}{7}x + \frac{30}{7}\right)$ has slope $m_2 = -\frac{2}{7}$. Since $m_1 = m_2$, the lines are parallel.

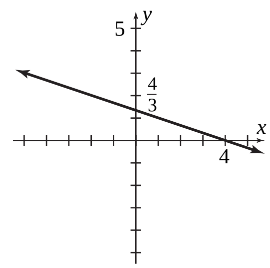
15. $y = 3x + 5$ has slope 3, and $6x - 2y = 7$ $\left(\text{or } y = 3x - \frac{7}{2}\right)$ has slope 3. Since $m_1 = m_2$, the lines are parallel.

16. $y = 7x$ has slope $m_1 = 7$, and $y = 7$ has slope $m_2 = 0$. Since $m_1 \neq m_2$ and $m_1 \neq -\frac{1}{m_2}$, the lines are neither parallel nor perpendicular.

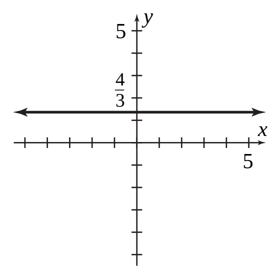
17. $3x - 2y = 4$
$-2y = -3x + 4$
$y = \frac{3}{2}x - 2$
$m = \frac{3}{2}$



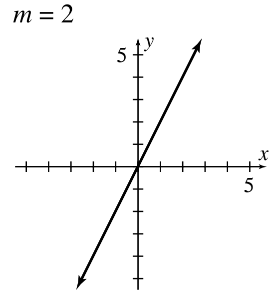
18. $x = -3y + 4$
$3y = -x + 4$
$y = -\frac{1}{3}x + \frac{4}{3}$
$m = -\frac{1}{3}$



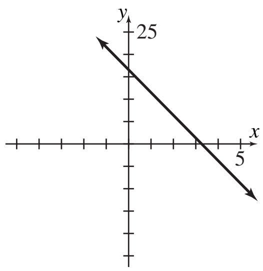
19. $4 - 3y = 0$
$-3y = -4$
$y = \frac{4}{3}$
$m = 0$



20. $y = 2x$
$m = 2$



21. $y = f(x) = 17 - 5x$ has the linear form $f(x) = ax + b$, where $a = -5$ and $b = 17$.
Slope = –5; y-intercept (0, 17).



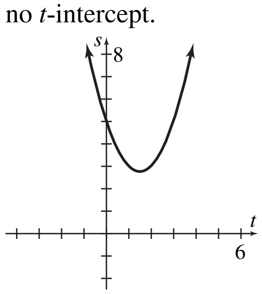
22. $s = g(t) = 5 - 3t + t^2$ has the quadratic form $g(t) = at^2 + bt + c$, where $a = 1$, $b = -3$, $c = 5$.

Vertex: $-\frac{b}{2a} = -\frac{-3}{2(1)} = \frac{3}{2}$

$$g\left(\frac{3}{2}\right) = 5 - 3\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 = \frac{11}{4}$$

$\Rightarrow$ Vertex $= \left(\frac{3}{2}, \frac{11}{4}\right)$

s-intercept: $c = 5$
t-intercepts: Because the parabola opens upward ($a > 0$) and the vertex is above the t-axis, there is no t-intercept.

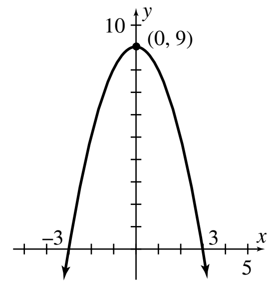


23. $y = f(x) = 9 - x^2$ has the quadratic form $f(x) = ax^2 + bx + c$, where $a = -1$, $b = 0$ and $c = 9$.

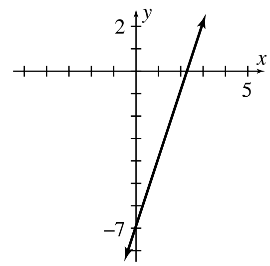
Vertex: $-\frac{b}{2a} = -\frac{0}{2(-1)} = 0$

$f(0) = 9 - 0^2 = 9$

$\Rightarrow$ Vertex = (0, 9)
y-intercept: $c = 9$
x-intercepts: $9 - x^2 = (3 - x)(3 + x) = 0$, so $x = 3, -3$.



24. $y = f(x) = 3x - 7$ has the linear form $f(x) = ax + b$, where $a = 3$, $b = -7$.
Slope = 3; y-intercept (0, –7)

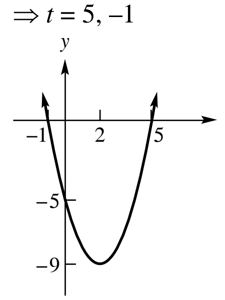


25. $y = h(t) = t^2 - 4t - 5$ has the quadratic form $h(t) = at^2 + bt + c$, where $a = 1$, $b = -4$, and $c = -5$.

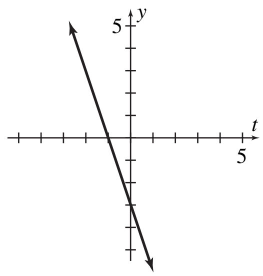
Vertex: $-\frac{b}{2a} = -\frac{-4}{2 \cdot 1} = 2$

$h(2) = 2^2 - 4(2) - 5 = -9$

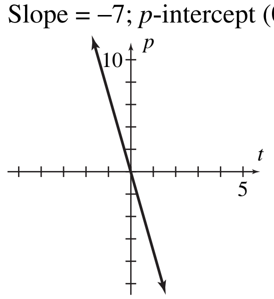
$\Rightarrow$ Vertex = (2, –9)
y-intercept: $c = -5$
t-intercepts: $t^2 - 4t - 5 = (t - 5)(t + 1) = 0$
$\Rightarrow t = 5, -1$



26. $y = k(t) = -3 - 3t$ has the linear form $k(t) = at + b$, where $a = -3$, $b = -3$.
Slope = -3, y-intercept $(0, -3)$



27. $p = g(t) = -7t$ has the linear form $g(t) = at + b$, where $a = -7$ and $b = 0$.
Slope = -7; p-intercept $(0, 0)$



28. $y = F(x) = (2x-1)^2 = 4x^2 - 4x + 1$ has the quadratic form $F(x) = ax^2 + bx + c$, where $a = 4$, $b = -4$, $c = 1$.

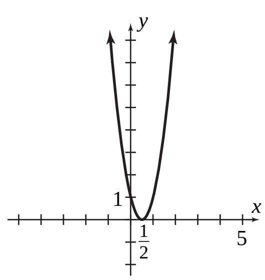
Vertex: $-\dfrac{b}{2a} = -\dfrac{-4}{2\cdot 4} = \dfrac{1}{2}$

$$F\left(\frac{1}{2}\right) = \left[2\left(\frac{1}{2}\right) - 1\right]^2 = 0$$

$\Rightarrow$ Vertex $= \left(\dfrac{1}{2}, 0\right)$

y-intercept: $c = 1$

x-intercepts: $(2x-1)^2 = 0$, so $x = \dfrac{1}{2}$

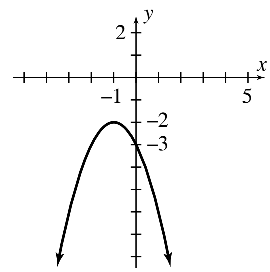


29. $y = F(x) = -\left(x^2 + 2x + 3\right) = -x^2 - 2x - 3$ has the quadratic form $F(x) = ax^2 + bx + c$, where $a = -1$, $b = -2$, and $c = -3$

Vertex: $-\dfrac{b}{2a} = -\dfrac{-2}{2(-1)} = -1$

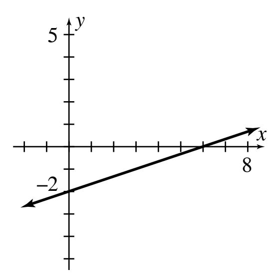
$$F(-1) = -\left[(-1)^2 + 2(-1) + 3\right] = -2$$

$\Rightarrow$ Vertex = $(-1, -2)$
y-intercept: $c = -3$
x-intercepts: Because the parabola opens downward ($a < 0$) and the vertex is below the x-axis, there is no x-intercept.



30. $y = f(x) = \dfrac{x}{3} - 2 = \dfrac{1}{3}x - 2$ has the linear form $f(x) = ax + b$, where $a = \dfrac{1}{3}$, $b = -2$.

Slope $= \dfrac{1}{3}$; y-intercept $(0, -2)$



31. $\begin{cases} 2x - y = 6, & (1) \\ 3x + 2y = 5. & (2) \end{cases}$

From Eq. (1), $y = 2x - 6$. Substituting in Eq. (2) gives
$3x + 2(2x - 6) = 5$
$7x - 12 = 5$, $7x = 17$

$x = \dfrac{17}{7} \Rightarrow y = 2x - 6 = 2\cdot\dfrac{17}{7} - 6 = -\dfrac{8}{7}$.

Thus $x = \dfrac{17}{7}$, $y = -\dfrac{8}{7}$.

32. $\begin{cases} 8x - 4y = 7, & (1) \\ y = 2x - 4. & (2) \end{cases}$

Replacing y by $2x - 4$ in Eq. (1) gives
$8x - 4(2x - 4) = 7$
$16 = 7$, which is never true.
There is no solution.

33. $\begin{cases} 7x + 5y = 5 \\ 6x + 5y = 3 \end{cases}$

Subtracting the second equation from the first equation gives $x = 2$. Then $7(2) + 5y = 5$, or $5y = -9$, so $y = -\frac{9}{5}$. Thus $x = 2$, $y = -\frac{9}{5}$.

34. $\begin{cases} 2x + 4y = 8 & (1) \\ 3x + 6y = 12 & (2) \end{cases}$

Multiplying Eq. (1) by 3 and Eq. (2) by −2 gives
$\begin{cases} 6x + 12y = 24 \\ -6x - 12y = -24. \end{cases}$

Adding gives 0 = 0. Thus, the equations are equivalent. From EQ. (1), $x = -2y + 4$. Letting $y = r$ gives the parametric solution $x = -2r + 4$, $y = r$, where r is any real number.

35. $\begin{cases} \frac{1}{4}x - \frac{3}{2}y = -4, & (1) \\ \frac{3}{4}x + \frac{1}{2}y = 8. & (2) \end{cases}$

Multiplying Eq. (2) by 3 gives
$\begin{cases} \frac{1}{4}x - \frac{3}{2}y = -4, \\ \frac{9}{4}x + \frac{3}{2}y = 24. \end{cases}$

Adding the first equation to the second gives
$\frac{5}{2}x = 20$
$x = 8$
From Eq. (1),
$\frac{1}{4}(8) - \frac{3}{2}y = -4$
$-\frac{3}{2}y = -6$
$y = 4$
Thus
$x = 8$, $y = 4$.

36. $\begin{cases} \frac{1}{3}x - \frac{1}{4}y = \frac{1}{12}, & (1) \\ \frac{4}{3}x + 3y = \frac{5}{3}. & (2) \end{cases}$

Multiplying Eq. (1) by −4 gives
$\begin{cases} -\frac{4}{3}x + y = -\frac{1}{3}, \\ \frac{4}{3}x + 3y = \frac{5}{3}. \end{cases}$

Adding gives $4y = \frac{4}{3} \Rightarrow y = \frac{1}{3}$. From Eq. (2),
$\frac{4}{3}x + 3\left(\frac{1}{3}\right) = \frac{5}{3}$
$\frac{4}{3}x = \frac{2}{3}$
$x = \frac{1}{2}$
Thus $x = \frac{1}{2}$, $y = \frac{1}{3}$.

37. $\begin{cases} 3x - 2y + z = -2, & (1) \\ 2x + y + z = 1, & (2) \\ x + 3y - z = 3. & (3) \end{cases}$

Subtracting Eq. (2) from Eq. (1) and adding Eq. (2) to Eq. (3) gives
$\begin{cases} x - 3y = -3, \\ 3x + 4y = 4. \end{cases}$

Multiplying the first equation by −3 gives
$\begin{cases} -3x + 9y = 9, \\ 3x + 4y = 4. \end{cases}$

Adding the first equation to the second gives
$13y = 13$
$y = 1$
From the equation $x - 3y = -3$, we get
$x - 3(1) = -3$
$x = 0$
From $3x - 2y + z = -2$, we get
$3(0) - 2(1) + z = -2$
$z = 0$
Thus $x = 0$, $y = 1$, $z = 0$.

38. $\begin{cases} 2x+\frac{3y+x}{3}=9 \\ y+\frac{5x+2y}{4}=7 \end{cases}$

simplifies to

$\begin{cases} 7x+3y=27 & (1) \\ 5x+6y=28 & (2) \end{cases}$

Multiplying Eq. (1) by –2 gives

$\begin{cases} -14x-6y=-54 \\ 5x+6y=28 \end{cases}$

Adding the equations gives

$-9x=-26$

$x=\frac{26}{9}$

Multiplying Eq. (1) by –5 and Eq. (2) by 7 gives

$\begin{cases} -35x-15y=-135 \\ 35x+42y=196 \end{cases}$

Adding the equations gives

$27y=61$

$y=\frac{61}{27}$

Thus, $x=\frac{26}{9}$, $y=\frac{61}{27}$.

39. $\begin{cases} x^2-y+5x=2, & (1) \\ x^2+y=3. & (2) \end{cases}$

From Eq. (2), $y=3-x^2$. Substituting in Eq. (1) gives

$x^2-(3-x^2)+5x=2$

$2x^2+5x-5=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-5\pm\sqrt{5^2-4(2)(-5)}}{2(2)}=\frac{-5\pm\sqrt{65}}{4}$$

Since $y=3-x^2$, if $x=\frac{-5+\sqrt{65}}{4}$, then $y=\frac{-21+5\sqrt{65}}{8}$; if $x=\frac{-5-\sqrt{65}}{4}$, then $y=\frac{-21-5\sqrt{65}}{8}$.

Thus, the two solutions are

$x=\frac{-5+\sqrt{65}}{4}$, $y=\frac{-21+5\sqrt{65}}{8}$, and

$x=\frac{-5-\sqrt{65}}{4}$, $y=\frac{-21-5\sqrt{65}}{8}$.

40. $\begin{cases} y=\frac{18}{x+4}, & (1) \\ x-y+7=0. & (2) \end{cases}$

From Eq. (2), $y = x + 7$. Substituting in Eq. (1) we have

$x+7=\frac{18}{x+4}$

$(x + 7)(x + 4) = 18$

$x^2+11x+28=18$

$x^2+11x+10=0$

$(x + 1)(x + 10) = 0$

Thus $x = -1, -10$. From $y = x + 7$, if $x = -1$, then $y = -1 + 7 = 6$; if $x = -10$, then $y = -10 + 7 = -3$. Thus the two solutions are $x = -1$, $y = 6$, and $x = -10$, $y = -3$.

41. $\begin{cases} x+2z=-2, & (1) \\ x+y+z=5. & (2) \end{cases}$

From Eq. (1) we have $x = -2 - 2z$. Substituting in Eq. (2) gives $-2 - 2z + y + z = 5$, so $y = 7 + z$. Letting $z = r$ gives the parametric solution $x = -2 - 2r$, $y = 7 + r$, $z = r$, where r is any real number.

42. $\begin{cases} x+y+z=0, & (1) \\ x-y+z=0, & (2) \\ x+z=0. & (3) \end{cases}$

Subtracting Eq. (3) from both Eqs. (1) and (2) gives

$\begin{cases} y=0, \\ -y=0, \\ x+z=0. \end{cases}$

The first two equations state that $y = 0$, and the third implies that $x = -z$. Letting $z = r$ gives the parametric solution $x = -r$, $y = 0$, $z = r$, where r is any real number.

43. $\begin{cases} x - y - z = 0, & (1) \\ 2x - 2y + 3z = 0. & (2) \end{cases}$

Multiplying Eq. (1) by –2 gives
$\begin{cases} -2x + 2y + 2z = 0, \\ 2x - 2y + 3z = 0. \end{cases}$
Adding the first equation to the second gives
$\begin{cases} -2x + 2y + 2z = 0, \\ 5z = 0. \end{cases}$
From the second equation, $z = 0$. Substituting in Eq. (1) gives $x - y - 0 = 0$, so $x = y$. Letting $y = r$ gives the parametric solution $x = r$, $y = r$, $z = 0$, where r is any real number.

44. $\begin{cases} 2x - 5y + 6z = 1, & (1) \\ 4x - 10y + 12z = 2. & (2) \end{cases}$

Multiplying Eq. (1) by –2 gives
$\begin{cases} -4x + 10y - 12z = -2, \\ 4x - 10y + 12z = 2. \end{cases}$
Adding the first equation to the second gives
$\begin{cases} -4x + 10y - 12z = -2, \\ 0 = 0. \end{cases}$
Solving the first equation for x, we have $x = \frac{1}{2} + \frac{5}{2}y - 3z$. Letting $y = r$ and $z = s$ gives the parametric solution $x = \frac{1}{2} + \frac{5}{2}r - 3s$, $y = r$, $z = s$, where r and s are any real numbers.

45. $a = 1$ when $b = 2$; $a = 5$ when $b = 3$, so
$m = \frac{a_2 - a_1}{b_2 - b_1} = \frac{5-1}{3-2} = \frac{4}{1} = 4.$
Thus an equation relating a and b is
$a - 1 = 4(b - 2)$
$a - 1 = 4b - 8$
$a - 4b = -7$
When $b = 5$, then $a = 4b - 7 = 4(5) - 7 = 13$.

46. **a.** $r = 206$ when $T = 36$; $r = 122$ when $T = 30$.
Thus $m = \frac{r_2 - r_1}{T_2 - T_1} = \frac{122 - 206}{30 - 36} = \frac{-84}{-6} = 14$
$r - 206 = 14(T - 36)$
$r = 14T - 298$

b. If $T = 27$, then
$r = 14T - 298 = 14(27) - 298 = 80$.

47. Slope is $\frac{-4}{3} \Rightarrow f(x) = ax + b = -\frac{4}{3}x + b$. Since $f(1) = 5$,
$5 = -\frac{4}{3}(1) + b$
$b = \frac{19}{3}$
Thus $f(x) = -\frac{4}{3}x + \frac{19}{3}$.

48. The slope of f is $\frac{5-8}{2-(-1)} = \frac{-3}{3} = -1$. Thus
$f(x) = ax + b = -x + b$. Since $f(2) = 5$,
$5 = -2 + b$
$b = 7$
Thus $f(x) = -x + 7$.

49. $r = pq = (200 - 2q)q = 200q - 2q^2$, which is a quadratic function with $a = -2$, $b = 200$, $c = 0$. Since $a < 0$, r has a maximum value when
$q = -\frac{b}{2a} = -\frac{200}{-4} = 50$ units. If $q = 50$, then
$r = [200 - 2(50)](50) = \$5000$.

50. Let p_1 and p_2 be the prices (in dollars) of the two items, respectively, before the tax. At the time the difference in prices is $p_1 - p_2 = 3.5$. After the tax, the prices are $1.05p_1$ and $1.05p_2$, so their difference is $1.05p_1 - 1.05p_2$, or 4.1. This gives the system
$\begin{cases} p_1 - p_2 = 3.5 \\ 1.05p_1 - 1.05p_2 = 4.1 \end{cases}$
Adding –1.05 times the first equation to the second equation gives $0 = 0.425$, which indicates that the system does not have a solution. Thus this scenario is not possible.

51. $\begin{cases} 120p - q - 240 = 0, \\ 100p + q - 1200 = 0. \end{cases}$

Adding gives $220p - 1440 = 0$, or
$p = \frac{1440}{220} \approx 6.55$.

52. a. $R = aL + b$. If $L = 0$, then $R = 1310$. Thus we have $1310 = 0 \cdot L + b$, or $b = 1310$. So $R = aL + 1310$. Since $R = 1460$ when $L = 2$,
$1460 = a(2) + 1310$
$150 = 2a$
$a = 75$
Thus $R = 75L + 1310$.

b. If $L = 1$, then
$R = 75(1) + 1310 = 1385$ milliseconds.

c. Since $R = 75L + 1310$, the slope is 75. The slope gives the change in R for each 1-unit increase in L. Thus the time necessary to travel from one level to the next level is 75 milliseconds.

53. $y_{TR} = 16q$; $y_{TC} = 8q + 10{,}000$. Letting $y_{TR} = y_{TC}$ gives
$16q = 8q + 10{,}000$
$8q = 10{,}000$
$q = 1250$
If $q = 1250$, then $y_{TR} = 16(1250) = 20{,}000$.
Thus the break-even point is (1250, 20,000) or 1250 units, $20,000.

54. $C = aF + b$. The points (32, 0) and (212, 100) lie on the graph of the function. Thus its slope is
$\frac{100-0}{212-32} = \frac{100}{180} = \frac{5}{9}$, so $C = \frac{5}{9}F + b$. Since
$C = 0$ when $F = 32$, $0 = \frac{5}{9}(32) + b$, so
$b = -\frac{160}{9}$. Thus $C = \frac{5}{9}F - \frac{160}{9}$ or
$C = \frac{5}{9}(F - 32)$. When
$F = 50$, then $C = \frac{5}{9}(50-32) = \frac{5}{9}(18) = 10$.

55. Equating L-values gives

$$0.0183 - \frac{0.0042}{p} = 0.0005 + \frac{0.0378}{p}$$
$$0.0178 = \frac{0.042}{p}$$
$$0.0178p = 0.042$$
$$p \approx 2.36$$

The equilibrium pollution level is about 2.36 tons per square kilometer.

56. $x = 12$, $y = -4$

57. $x = 7.29$, $y = -0.78$

58. $x = 3.02$, $y = 0.14$

59. $x = 0.75$, $y = 1.43$

60. $x = 2.68$

Mathematical Snapshot Chapter 3

1. $P_1(6000) = 39.99 + 0.45(6000 - 450)$
$= 2537.49$
$P_6(6000) = 199.99$
He loses $2537.49 – $199.99 = $2337.50 by using P_1.

2. The graph shows that P_2 and P_3 intersect when the second branch of P_2 crosses the first branch of P_3. Thus

$$59.99 + 0.40(t - 900) = 79.99$$
$$t = 950$$

P_2 is best for usage between 494.44 and 950 minutes.

3. The graph shows that P_3 and P_4 intersect when the second branch of P_3 crosses the first branch of P_4 Thus

$$79.99 + 0.35(t - 1350) = 99.99$$
$$t \approx 1407.14$$

P_3 is best for usage between 950 and 1407.14 minutes.

4. The graph shows that P_4 and P_5 intersect when the second branch of P_4 crosses the first branch of P_5 Thus

$$99.99 + 0.25(t - 2000) = 149.99$$
$$t = 2200$$

P_4 is best for usage between 1407.14 and 2200 minutes.

5. The graph shows that P_5 and P_6 intersect when the second branch of P_5 crosses the first branch of P_6 Thus

$$149.99 + 0.25(t - 4000) = 199.99$$
$$t = 4200$$

P_5 is best for usage between 2200 and 4200 minutes.

6. P_6 is best for usage of greater than 4200 minutes.

7. No; answers may vary.

Chapter 4

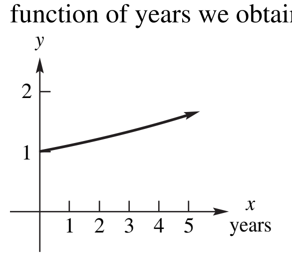
Principles in Practice 4.1

1. The shapes of the graphs are the same. The value of A scales the value of any point by A.

2. If P = the amount of money invested and r = the annual rate at which P increases, then after 1 year, the investment has grown from P to $P + Pr = P(1 + r)$. Since $r = 0.10$, the factor by which P increases for the first year is $1 + r = 1 + 0.1 = 1.1$. Similarly, during the second year the investment grows from $P(1 + r)$ to $P(1+r)+r[P(1+r)] = P(1+r)^2$. Again, since $r = 0.10$, the multiplicative increase for the second year is $(1+0.10)^2 = (1.1)^2 = 1.21$. This pattern will continue as shown in the table.

Year	Multiplicative Increase	Expression
0	1	1.1^0
1	1.1	1.1^1
2	1.21	1.1^2
3	1.33	1.1^3
4	1.46	1.1^4

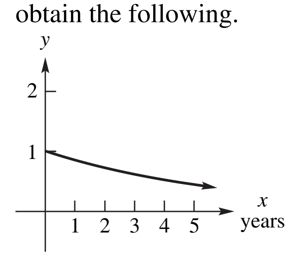
Thus, the growth of the initial investment is exponential with a base of $1 + r = 1 + 0.1 = 1.1$. If we graph the multiplicative increase as a function of years we obtain the following.



3. If V = the value of the car and r = the annual rate at which V depreciates, then after 1 year the value of the car is $V - rV = V(1 - r)$. Since $r = 0.15$, the factor by which V decreases for the first year is $1 - r = 1 - 0.15 = 0.85$. Similarly, after the second year the value of the car is $V(1-r)-r[V(1-r)] = V(1-r)^2$. Again, since $r = 0.15$, the multiplicative decrease for the second year is $(1-r)^2 = (1-0.15)^2 = 0.72$. This pattern will continue as shown in the table.

Year	Multiplicative Decrease	Expression
0	1	0.85^0
1	0.85	0.85^1
2	0.72	0.85^2
3	0.61	0.85^3

Thus, the depreciation is exponential with a base of $1 - r = 1 - 0.15 = 0.85$. If we graph the multiplicative decrease as a function of years, we obtain the following.



4. Let t = the time at which George's sister began saving, then since George is 3 years behind, $t - 3$ = the time when George began saving. Therefore, if $y = 1.08^t$ represents the multiplicative increase in George's sister's account $y = 1.08^{t-3}$ represents the multiplicative increase in George's account. A graph showing the projected increase in George's money will have the same shape as the graph of the projected increase in his sister's account, but will be shifted 3 units to the right.

5. $S = P(1+r)^n$

   $S = 2000(1+0.13)^5 = 2000(1.13)^5 \approx 3684.87$

   The value of the investment after 5 years will be \$3684.87. The interest earned over the first 5 years is 3684.87 – 2000 = \$1684.87.

6. Let $N(t)$ = the number of employees at time t, where t is in years. Then,

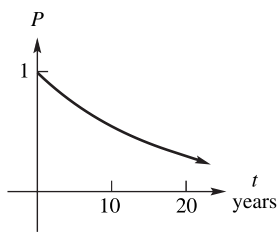
   $N(4) = 5(1+1.2)^4 = 5(2.2)^4 = 117.128$

   Thus, there will be 117 employees at the end of 4 years.

7. $P = e^{-0.06t} = \left(\frac{1}{e}\right)^{0.06t}$

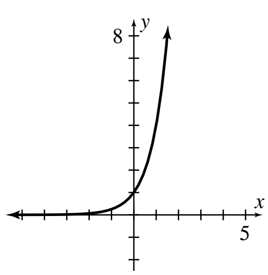
Since $0 < \frac{1}{e} < 1$, the graph is that of an exponential function falling from left to right.

x	y
0	1
2	0.89
4	0.79
6	0.70
8	0.62
10	0.55



Problems 4.1

1.



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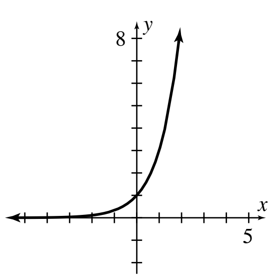

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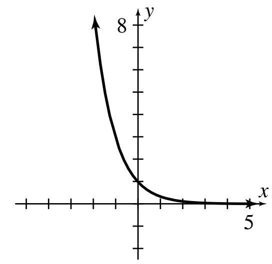

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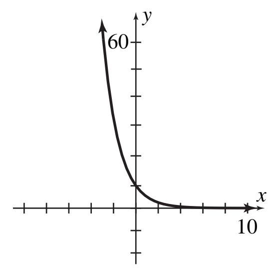

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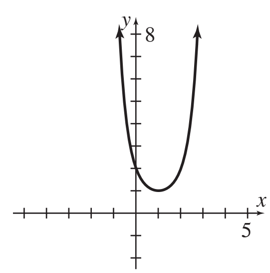

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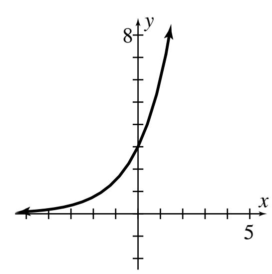

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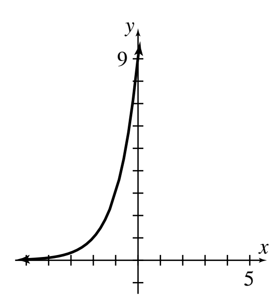

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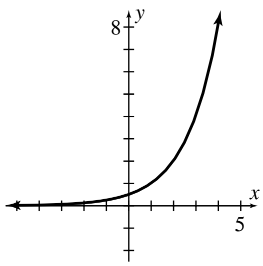

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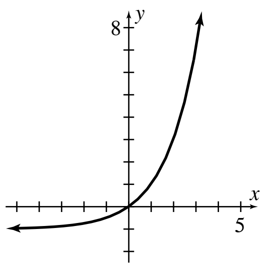

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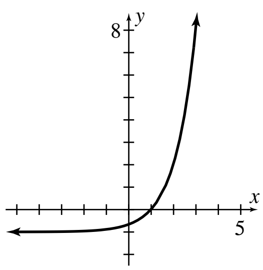

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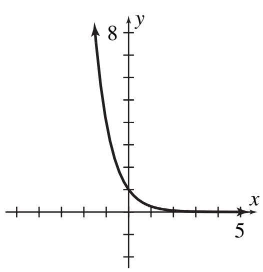

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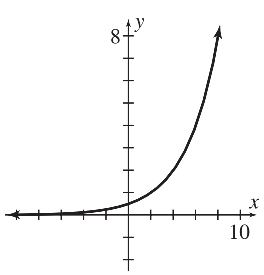


13. For the curves, the bases involved are 0.4, 2, and 5. For base 5, the curve rises from left to right, and in the first quadrant it rises faster than the curve for base 2. Thus the graph of $y = 5^x$ is *B*.

14. $y = 0.4^x$ has base $b = 0.4$ and $0 < b < 1$, so its graph falls from left to right. Thus the graph is *A*.

15. For 2015 we have $t = 20$, so
$P = 125,000(1.11)^{\frac{20}{20}} = 125,000(1.11)^1$
$= 138,750$.

16. a. For 1999, $t = 1$ and
$P = 1,527,000(1.015)^1 = 1,549,905$

b. For 2000, $t = 2$ and
$P = 1,527,000(1.015)^2 \approx 1,573,154$

17. With $c = \frac{1}{2}$, $P = 1 - \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = 1 - \left(\frac{1}{2}\right)^n$.

$n = 1$: $P = 1 - \left(\frac{1}{2}\right)^1 = 1 - \frac{1}{2} = \frac{1}{2}$

$n = 2$: $P = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$

$n = 3$: $P = 1 - \left(\frac{1}{2}\right)^3 = 1 - \frac{1}{8} = \frac{7}{8}$

18. $y = 2^{3x} = \left(2^3\right)^x = 8^x$. Thus $y = 8^x$.

19. a. $4000(1.06)^7 \approx \$6014.52$

b. $6014.52 - 4000 = \$2014.52$

20. a. $5000(1.05)^{20} \approx \$13,266.49$

b. $13,266.49 - 5000 = \$8266.49$

21. a. $700(1.035)^{30} \approx \$1964.76$

b. $1964.76 - 700 = \$1264.76$

22. a. $4000(1.0375)^{24} \approx \$9677.75$

b. $9677.75 - 4000 = \$5677.75$

23. a. $3000\left(1 + \frac{0.0875}{4}\right)^{64} \approx 11,983.37$

b. $11,983.37 - 3000 = \$8983.37$

24. **a.** $2000\left(1+\frac{0.07}{4}\right)^{48} \approx \$4599.20$

b. $4599.20 - 2000 = \$2599.20$

25. **a.** $5000(1.0075)^{30} \approx \$6256.36$

b. $6256.36 - 5000 = \$1256.36$

26. **a.** $500\left(1+\frac{0.11}{2}\right)^{10} \approx \$854.07$

b. $854.07 - 500 = \$354.07$

27. **a.** $8000\left(1+\frac{0.0625}{365}\right)^{3(365)} \approx \$9649.69$

b. $9649.69 - 8000 = \$1649.69$

28. **a.** $900(1.0225)^{10} \approx \$1124.28$

b. $900(1.045)^{5} \approx \$1121.56$

29. $6500\left(1+\frac{0.04}{4}\right)^{24} \approx \$8253.28$

30. **a.** $P = 5000(1.03)^t$

b. When $t = 3$, then $P = 5000(1.03)^3 \approx 5464$.

31. **a.** $N = 400(1.05)^t$

b. When $t = 1$, then $N = 400(1.05)^1 = 420$.

c. When $t = 4$, then $N = 400(1.05)^4 \approx 486$.

32. If $N = N(t)$ = the number of bacteria present at any time t, where t is in hours, and if r = the rate at which the bacteria are reduced, then, after the first hour, the number of bacteria remaining is
$N - rN = N(1 - r) = 100{,}000(1 - 0.1)$
$= 100{,}000(0.9) = 90{,}000$.
Similarly, after the second hour, the number of bacteria remaining is
$N(1 - r) - r[N(1 - r)] = N(1-r)^2$
$= 100{,}000(1-0.1)^2 = 100{,}000(0.9)^2 = 81{,}000$
This pattern will continue as shown in the table.

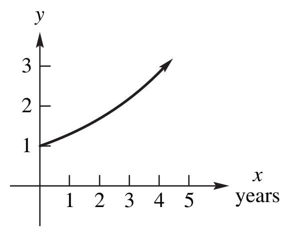
Hours	Bacteria	Expression
0	100,000	$100{,}000\left(\frac{9}{10}\right)^0$
1	90,000	$100{,}000\left(\frac{9}{10}\right)^1$
2	81,000	$100{,}000\left(\frac{9}{10}\right)^2$
3	72,900	$100{,}000\left(\frac{9}{10}\right)^3$
4	65,610	$100{,}000\left(\frac{9}{10}\right)^4$
t		$100{,}000\left(\frac{9}{10}\right)^t$

Thus, in general, the number of bacteria present after t hours is given by $N(t) = 100{,}000\left(\frac{9}{10}\right)^t$.

33. Let P = the amount of plastic recycled and let r = the rate at which P increases each year. Then after the first year, the amount of plastic recycled, increases from P to $P + rP = P(1 + r)$, since $r = 0.3$, the factor by which P increases for the first year, is $1 + r = 1 + 0.3 = 1.3$. Similarly, during the second year, the amount of plastic recycled increases from $P(1 + r)$ to $P(1 + r) + r[P(1+r)] = P(1+r)^2$. Again, since $r = 0.3$, the multiplicative increase for the second year is $(1+r)^2 = (1+0.3)^2 = (1.3)^2 = 1.69$. This pattern will continue as shown in the table.

Year	Multiplicative Increase	Expression
0	1	1.3^0
1	1.3	1.3^1
2	1.69	1.3^2
3	2.20	1.3^3

Thus, the increase in recycling is exponential with a base = 1 + r = 1 + 0.3 = 1.3. If we graph the multiplicative increase as function of years, we obtaining the following.



From the graph it appears that recycling will triple after about 4 years.

34. Population of city A after 5 years: $70,000(1.04)^5$.
Population of city B after 5 years: $60,000(1.05)^5$.
Difference in populations: $\left|70,000(1.05)^5 - 60,000(1.05)^5\right| \approx 8589$.

35. $P = 350,000(1-0.015)^t = 350,000(0.985)^t$, where P is the population after t years. When $t = 3$, $P = 350,000(0.985)^3 \approx 334,485$.

36. $E = 14,000(1-0.03)^t = 14,000(0.97)^t$, where E is the enrollment after t years. When $t = 12$, $E = 14,000(0.97)^{12} \approx 9714$.

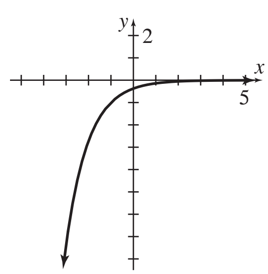
37. 4.4817

38. 29.9641

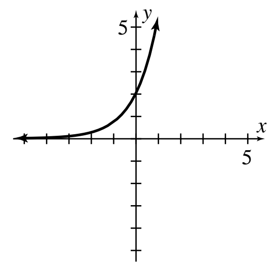
39. 0.4966

40. 0.5134

41.



42.



43. For $x = 3$, $P = \dfrac{e^{-3}3^3}{3!} \approx 0.2240$

44. $f(0) \approx 0.399$; $f(-1) = f(1) \approx 0.242$

45. $e^{kt} = \left(e^k\right)^t = b^t$, where $b = e^k$

46. $\dfrac{1}{e^x} = \left(\dfrac{1}{e}\right)^x = b^x$, where $b = \dfrac{1}{e}$

47. **a.** When $t = 0$, $N = 12e^{-0.031(0)} = 12 \cdot 1 = 12$.

b. When $t = 10$, $N = 12e^{-0.031(10)} = 12e^{-0.31} = 8.8$.

c. When $t = 44$, $N = 12e^{-0.031(44)} = 12e^{-1.364} \approx 3.1$.

d. After 44 hours, approximately $\dfrac{1}{4}$ of the initial amount remains. Because $\dfrac{1}{4} = \left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)$, 44 hours corresponds to 2 half-lives. Thus the half-life is approximately 22 hours.

48. $N = 75e^{-0.045(10)} \approx 48$

49. After one half-life, $\dfrac{1}{2}$ gram remains. After two half-lives, $\dfrac{1}{2} \cdot \dfrac{1}{2} = \left(\dfrac{1}{2}\right)^2 = \dfrac{1}{4}$ gram remains. Continuing in this manner, after n half-lives, $\left(\dfrac{1}{2}\right)^n$ gram remains. Because $\dfrac{1}{16} = \left(\dfrac{1}{2}\right)^4$, after 4 half-lives, $\dfrac{1}{16}$ gram remains. This corresponds to $4 \cdot 8 = 32$ years.

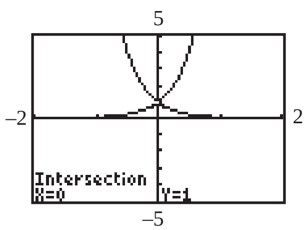
50. $f(x)=\frac{e^{-0.5}(0.5)^x}{x!}$

$f(2)=\frac{e^{-0.5}(0.5)^2}{2!}\approx 0.0758$

51. $f(x)=\frac{e^{-4}4^x}{x!}$

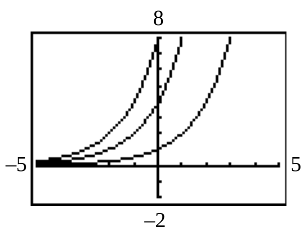
$f(2)=\frac{e^{-4}4^2}{2!}\approx 0.1465$

52.



The intersection point is (0, 1).

53.



If $f(x)=2^x$, then

$y=2^a\cdot 2^x=2^{x+a}=f(x+a)$. Thus, the graph of $y=2^a\cdot 2^x$ is the graph of $y=2^x$ shifted a units to the left.

54. 0.71

55. 3.17

56. The first integer t for which the graph of $P=1000(1.07)^t$ lies on or above the horizontal line $P = 3000$ is 17.

57. $300\left(\frac{4}{3}\right)^{4.1}\approx 976$

$300\left(\frac{4}{3}\right)^{4.2}\approx 1004$

4.2 minutes

58. a. When $p = 10$, then
$q=10{,}000(0.95123)^{10}\approx 6065$.

b. Using a graphics calculator, $0.95123=e^{-x}$ when $x\approx 0.05$. Thus, $0.95123\approx e^{-0.05}$.

$q=10{,}000(0.95123)^p\approx 10{,}000\left(e^{-0.05}\right)^p$.

$=10{,}000e^{-0.05p}$

c. $q=10{,}000e^{-0.05(10)}\approx 6065$.

59. The first integer t for which the graph of $P = 2500(1.043)^t$ lies on or above the horizontal line $P = 5000$ is 17.

Principles in Practice 4.2

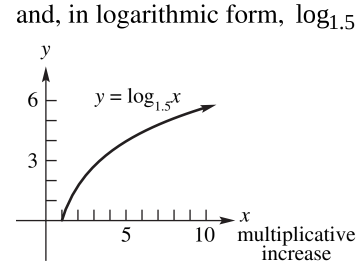
1. If $16=2^t$ is the exponential form then $t=\log_2 16$ is the logarithmic form, where t represents the number of times the bacteria have doubled.

2. If $8.3=\log_{10}\left(\frac{I}{I_0}\right)$ is the logarithmic form, then

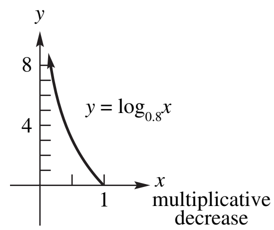
$\frac{I}{I_0}=10^{8.3}$ is the exponential form.

3. Let R = the amount of material recycled every year. If the amount being recycled increases by 50% every year, then the amount recycled at the end of y years is
$R(1+r)^y=R(1+0.5)^y=R(1.5)^y$ Thus, the multiplicative increase in recycling at the end of y years is $(1.5)^y$. If we let

x = the multiplicative increase, then $x=(1.5)^y$ and, in logarithmic form, $\log_{1.5}x=y$.



4. Let V = the value of the boat. If the value depreciates by 20% every year, then at the end of y years the value of the boat is $V(1-r)^y = V(1.02)^y = V(0.8)^y$. Thus, the multiplicative decrease in value at the end of y years is $(0.8)^y$. If we let x = the multiplicative decrease, then $x = (0.8)^y$ and, in logarithmic form, $\log_{0.8} x = y$



5. The equation $t(r) = \frac{\ln 4}{r}$ can be rewritten as $r = \frac{\ln 4}{t(r)}$. When this equation is graphed we find that the annual rate r needed to quadruple the investment in 10 years is approximately 13.9%. Alternatively, we can solve for r by setting $t(r) = 10$.

$$r = \frac{\ln(4)}{t(r)}$$

$$r = \frac{\ln(4)}{10} \approx 0.139 \text{ or } \approx 13.9\%$$

6. Since $m = e^{rt}$, then $\ln m = rt$.

$\ln m = rt$

$\frac{\ln m}{t} = r$

Let $m = 3$ and $t = 12$.

$\frac{\ln 3}{12} = r$

$0.092 = r$

Thus, to triple your investment in 12 years, invest at an annual percentage rate of 9.2%.

Problems 4.2

1. $\log 10{,}000 = 4$

2. $(12)^2 = 144$

3. $2^6 = 64$

4. $\log_8 4 = \frac{2}{3}$

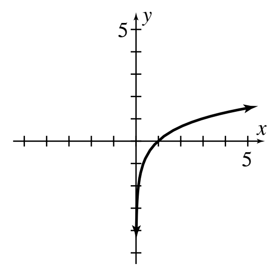
5. $\ln 20.0855 = 3$

6. $\ln 1.4 = 0.33647$

7. $e^{1.09861} = 3$

8. $10^{0.6990} = 5$

9.



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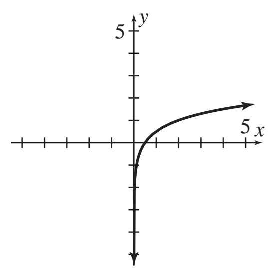

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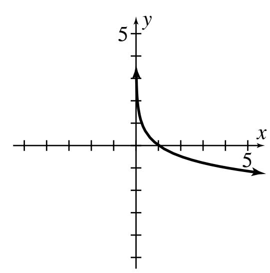

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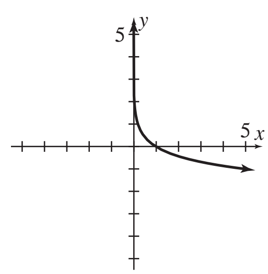

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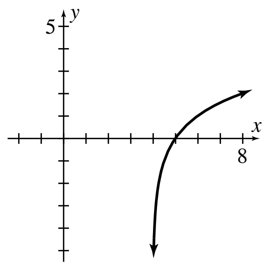

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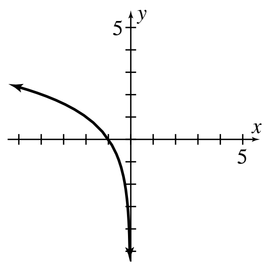

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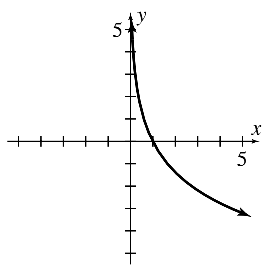

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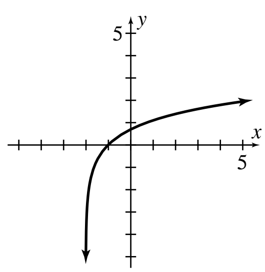


17. Because $6^2 = 36$, $\log_6 36 = 2$

18. Because $2^6 = 64$, $\log_2 64 = 6$.

19. Because $3^3 = 27$, $\log_3 27 = 3$

20. Because $16^{1/2} = 4$, $\log_{16} 4 = \frac{1}{2}$

21. Because $7^1 = 7$, $\log_7 7 = 1$

22. Because $10^4 = 10,000$, $\log 10,000 = 4$

23. Because $10^{-2} = 0.01$, $\log 0.01 = -2$

24. Because $2^{1/3} = \sqrt[3]{2}$, $\log_2 \sqrt[3]{2} = \frac{1}{3}$.

25. Because $5^0 = 1$, $\log_5 1 = 0$

26. Because $5^{-2} = \frac{1}{25}$, $\log_5 \frac{1}{25} = -2$

27. Because $2^{-3} = \frac{1}{8}$, $\log_2 \frac{1}{8} = -3$

28. Because $4^{1/5} = \sqrt[5]{4}$, $\log_4 \sqrt[5]{4} = \frac{1}{5}$.

29. $3^4 = x$
$x = 81$

30. $2^8 = x$
$x = 256$

31. $5^3 = x$
$x = 125$

32. $4^0 = x$
$x = 1$

33. $10^{-1} = x$
$x = \frac{1}{10}$

34. $e^1 = x$
$x = e$

35. $e^{-3} = x$

36. $x^2 = 25$
Since $x > 0$, we choose $x = 5$.

37. $x^3 = 8$
$x = 2$

38. $x^{1/2} = 3$
$x = 9$

39. $x^{-1} = \frac{1}{6}$
$x = 6$

40. $y = x^1$
$x = y$

41. $3^{-3} = x$
$x = \frac{1}{27}$

42. $x^1 = 2x - 3$
$x = 3$

43. $x^2 = 6 - x$
$x^2 + x - 6 = 0$
$(x + 3)(x - 2) = 0$
The roots of this equation are –3 and 2. But since $x > 0$, we choose $x = 2$.

44. $\log_8 64 = x - 1$
$8^{x-1} = 64$
$x - 1 = 2$
$x = 3$

45. $2 + \log_2 4 = 3x - 1$
$2 + 2 = 3x - 1$
$5 = 3x$
$x = \frac{5}{3}$

46. $3^{-2} = x + 2$
$\frac{1}{9} = x + 2$
$x = -\frac{17}{9}$

47. $x^2 = 2x + 8$
$x^2 - 2x - 8 = 0$
$(x - 4)(x + 2) = 0$
The roots of this equation are 4 and –2. But since $x > 0$, we choose $x = 4$.

48. $x^2 = 6 + 4x - x^2$
$2x^2 - 4x - 6 = 0$
$x^2 - 2x - 3 = 0$
$(x - 3)(x + 1) = 0$
The roots of the equation are 3 and –1. But since $x > 0$, we choose $x = 3$.

49. $e^{3x} = 2$
$3x = \ln 2$
$x = \frac{\ln 2}{3}$

50. $0.1e^{0.1x} = 0.5$
$e^{0.1x} = 5$
$0.1x = \ln 5$
$x = 10 \ln 5$

51. $e^{2x-5} + 1 = 4$
$e^{2x-5} = 3$
$2x - 5 = \ln 3$
$x = \frac{5 + \ln 3}{2}$

52. $6e^{2x} - 1 = \frac{1}{2}$
$6e^{2x} = \frac{3}{2}$
$e^{2x} = \frac{1}{4}$
$2x = \ln \frac{1}{4}$
$x = \frac{1}{2} \ln \frac{1}{4}$

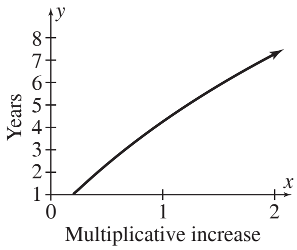
53. 1.60944

54. 1.45161

55. 2.00013

56. 2.30058

57. If V = the value of the antique. If the value appreciates by 10% every year, then at the end of y years the value of the antique is $V(1 + r)^y = V(1 + 0.10)^y = V(1.10)^y$. Thus, the multiplicative increase in value at the end of y years is $(1.10)^y$. If we let x = the multiplicative increase, then $x = (1.10)^y$, and, in logarithm form, $\log_{1.10} x = y$.



58. $c = 3(6) \ln 6 + 12 \approx 44.25$

59. $p = \log\left[10 + \frac{1980}{2}\right] = \log[10 + 990] = \log 1000$
$= 3$

60. $1.5M = \log\left(\frac{E}{2.5 \times 10^{11}}\right)$

$10^{1.5M} = \frac{E}{2.5 \times 10^{11}}$

$E = \left(2.5 \times 10^{11}\right)\left(10^{1.5M}\right)$

$E = 2.5 \times 10^{11+1.5M}$

61. **a.** If $t = k$, then $N = N_0\left(2^1\right) = 2N_0$

b. From part (a), $N = 2N_0$ when $t = k$. Thus k is the time it takes for the population to double.

c. $N_1 = N_0 2^{\frac{t}{k}}$

$\frac{N_1}{N_0} = 2^{\frac{t}{k}}$

$\frac{t}{k} = \log_2 \frac{N_1}{N_0}$

$t = k \log_2 \frac{N_1}{N_0}$

62. $u_0 = A\ln\left(x_1\right) + \frac{x_2^2}{2}$

$u_0 - \frac{x_2^2}{2} = A\ln\left(x_1\right)$

$\ln\left(x_1\right) = \frac{u_0 - \frac{x_2^2}{2}}{A}$

$x_1 = e^{\frac{u_0 - \left(x_2^2/2\right)}{A}}$

63. $T = \frac{\ln 2}{0.01920} \approx 36.1$ minutes

64. $T = \frac{\ln 2}{0.03194} \approx 21.7$ years

65. From $\log_y x = 3$, $y^3 = x$; from $\log_z x = 2$, $z^2 = x$. Thus $z^2 = y^3$ or $z = y^{\frac{3}{2}}$.

66. $x + 3e^{2y} - 8 = 0$

$3e^{2y} = 8 - x$

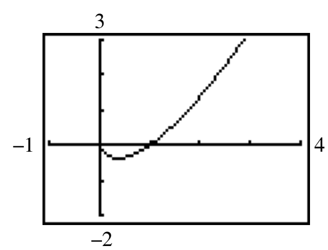
$e^{2y} = \frac{8-x}{3}$

$\ln[e^{2y}] = \ln\left[\frac{8-x}{3}\right]$

$2y = \ln\left[\frac{8-x}{3}\right]$

$y = \frac{1}{2}\ln\left[\frac{8-x}{3}\right]$

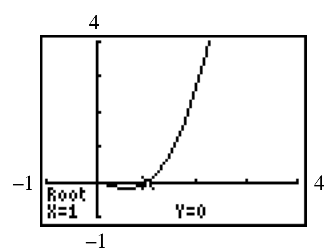
67.



a. (0, 1)

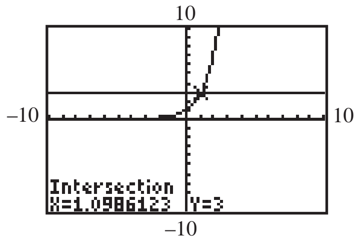
b. [–0.37, ∞)

68.



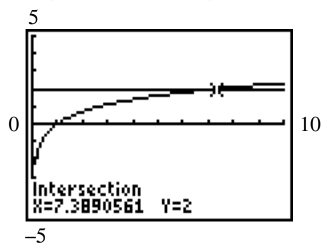
(1, 0)

69. For $y = e^x$, if $y = 3$, then $3 = e^x$ or $x = \ln 3$.



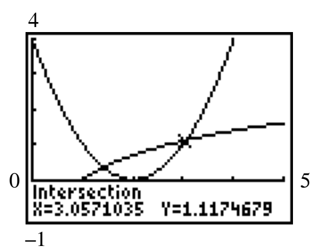
From the graph of $y = e^x$, when $y = 3$, then $x = \ln 3 \approx 1.10$.

70. For $y = \ln x$, when $y = 2$, then $2 = \ln x$ or $x = e^2$.



From the graph of $y = \ln x$, when $y = 2$, then $x = e^2 \approx 7.39$.

71.



1.41, 3.06

Principles in Practice 4.3

1. The magnitude (Richter Scale) of an earthquake is given by $R = \log\left(\frac{I}{I_0}\right)$ where I is the intensity of the earthquake and I_0 is the intensity of a zero-level reference earthquake. $\frac{I}{I_0} =$ how many times greater the earthquake is than a zero-level earthquake. Thus, when $\frac{I}{I_0} = 900{,}000$,

$R_1 = \log(900{,}000)$

When $\frac{I}{I_0} = 9000$

$R_2 = \log(9000)$

$R_1 - R_2 = \log(900{,}000) - \log 9000$

$= \log\frac{900{,}000}{9000} = \log 100 = \log 10^2 = 2 \log 10$

$= 2$

Thus, the two earthquakes differ by 2 on the Richter scale.

2. The magnitude (Richter Scale) of an earthquake is given by $R = \log\left(\frac{I}{I_0}\right)$ where I is the intensity of the earthquake and I_0 is the intensity of a zero-level reference earthquake. $\frac{I}{I_0} =$ how many times greater the earthquake is than a zero-level earthquake. Thus, if $\frac{I}{I_0} = 10{,}000$, then

$R = \log 10{,}000 = \log 10^4 = 4 \log 10 = 4$

The earthquake measures 4 on the Richter scale.

Problems 4.3

1. $\log 30 = \log(2\cdot 3\cdot 5)$
$= \log 2 + \log 3 + \log 5$
$= a + b + c$

2. $\log 16 = \log 2^4 = 4\log 2 = 4a$

3. $\log\frac{2}{3} = \log 2 - \log 3 = a - b$

4. $\log\frac{5}{2} = \log 5 - \log 2 = c - a$

5. $\log\frac{8}{3} = \log 8 - \log 3 = \log 2^3 - \log 3$
$= 3 \log 2 - \log 3 = 3a - b$

6. $\log\frac{6}{25} = \log\frac{2\cdot 3}{5^2}$
$= \log 2 + \log 3 - 2\log 5$
$= a + b - 2c$

7. $\log 36 = \log(2\cdot 3)^2 = 2\log(2\cdot 3)$
$= 2(\log 2 + \log 3) = 2(a + b)$

8. $\log 0.00003 = \log(3\cdot 10^{-5})$
$= \log 3 + \log 10^{-5}$
$= \log 3 - 5\log 10$
$= \log 3 - 5\log(2\cdot 5)$
$= \log 3 - 5(\log 2 + \log 5)$
$= b - 5(a+c)$
$= -5a + b - 5c$

9. $\log_2 3 = \frac{\log_{10} 3}{\log_{10} 2} = \frac{\log 3}{\log 2} = \frac{b}{a}$

10. $\log_3 5 = \frac{\log_{10} 5}{\log_{10} 3} = \frac{\log 5}{\log 3} = \frac{c}{b}$

11. $\log_7 7^{48} = 48$

12. $\log_5 \left(5\sqrt{5}\right)^5 = \log_5 \left(5^{\frac{3}{2}}\right)^5 = \log_5 5^{\frac{15}{2}} = \frac{15}{2}$

13. $\log 0.0000001 = \log 10^{-7} = -7$

14. $10^{\log 3.4} = 10^{\log_{10} 3.4} = 3.4$

15. $\ln e^{5.01} = \log_e e^{5.01} = 5.01$

16. $\ln e = \log_e e = 1$

17. $\ln \frac{1}{e^2} = -\ln e^2 = -\log_e e^2 = -2$

18. $\log_3 81 = \log_3 3^4 = 4$

19. $\log \frac{1}{10} + \ln e^3 = \log_{10} \frac{1}{10} + \log_e e^3 = -1 + 3 = 2$

20. $e^{\ln \pi} = e^{\log_e \pi} = \pi$

21. $\ln\left[x(x+1)^2\right] = \ln x + \ln(x+1)^2$
$= \ln x + 2\ln(x+1)$

22. $\ln \frac{\sqrt{x}}{x+1} = \ln x^{\frac{1}{2}} - \ln(x+1) = \frac{1}{2}\ln x - \ln(x+1)$

23. $\ln \frac{x^2}{(x+1)^3} = \ln x^2 - \ln(x+1)^3$
$= 2\ln x - 3\ln(x+1)$

24. $\ln[x(x+1)]^3 = 3\ln[x(x+1)] = 3[\ln x + \ln(x+1)]$

25. $\ln\left(\frac{x+1}{x+2}\right)^4 = 4\ln\frac{x+1}{x+2} = 4[\ln(x+1) - \ln(x+2)]$

26. $\ln\sqrt{x(x+1)(x+2)} = \ln[x(x+1)(x+2)]^{1/2}$
$= \frac{1}{2}[\ln x(x+1)(x+2)]$
$= \frac{1}{2}[\ln x + \ln(x+1)\ln(x+2)]$

27. $\ln \frac{x}{(x+1)(x+2)} = \ln x - \ln[(x+1)(x+2)]$
$= \ln x - [\ln(x+1) + \ln(x+2)]$
$= \ln x - \ln(x+1) - \ln(x+2)$

28. $\ln \frac{x^2(x+1)}{x+2} = \ln\left[x^2(x+1)\right] - \ln(x+2)$
$= \ln x^2 + \ln(x+1) - \ln(x+2)$
$= 2\ln x + \ln(x+1) - \ln(x+2)$

29. $\ln \frac{\sqrt{x}}{(x+1)^2(x+2)^3} = \ln x^{\frac{1}{2}} - \ln\left[(x+1)^2(x+2)^3\right]$
$= \frac{1}{2}\ln x - \left[\ln(x+1)^2 + \ln(x+2)^3\right]$
$= \frac{1}{2}\ln x - [2\ln(x+1) + 3\ln(x+2)]$
$= \frac{1}{2}\ln x - 2\ln(x+1) - 3\ln(x+2)$

30. $\ln \frac{x}{(x+1)(x+2)} = \ln x - [\ln(x+1) + \ln(x+2)]$
$= \ln x - \ln(x+1) - \ln(x+2)$

31. $\ln\left[\frac{1}{x+2}\sqrt[5]{\frac{x^2}{x+1}}\right] = \ln\left[\frac{1}{x+2}\left(\frac{x^2}{x+1}\right)^{\frac{1}{5}}\right]$
$= \ln \frac{x^{\frac{2}{5}}}{(x+2)(x+1)^{\frac{1}{5}}}$
$= \ln x^{\frac{2}{5}} - \ln\left[(x+2)(x+1)^{\frac{1}{5}}\right]$
$= \frac{2}{5}\ln x - \left[\ln(x+2) + \ln(x+1)^{\frac{1}{5}}\right]$
$= \frac{2}{5}\ln x - \ln(x+2) - \frac{1}{5}\ln(x+1)$

32. $\ln\sqrt[3]{\frac{x^3(x+2)^2}{(x+1)^3}} = \frac{1}{3}\ln\frac{x^3(x+2)^2}{(x+1)^3}$
$= \frac{1}{3}\left\{\ln[x^3(x+2)^2] - \ln(x+1)^3\right\}$
$= \frac{1}{3}[\ln x^3 + \ln(x+2)^2 - \ln(x+1)^3]$
$= \frac{1}{3}[3\ln x + 2\ln(x+2) - 3\ln(x+1)]$
$= \ln x + \frac{2}{3}\ln(x+2) - \ln(x+1)$

33. $\log(6\cdot 4) = \log 24$

34. $\log_3\left(\frac{10}{5}\right) = \log_3 2$

35. $\log_2\frac{2x}{x+1}$

36. $\log x^2 - \log\sqrt{x-2} = \log\frac{x^2}{\sqrt{x-2}}$

37. $5\log_2 10 + 2\log_2 13 = \log_2 10^5 + \log_2 13^2$
$= \log_2(10^5 \cdot 13^2)$

38. $5(\log x^2 + \log y^3 - \log z^2)$
$= 5\log\left(\frac{x^2y^3}{z^2}\right)$
$= \log\left[\left(\frac{x^2y^3}{z^2}\right)^5\right]$

39. $\log 100 + \log(1.05)^{10} = \log\left[100(1.05)^{10}\right]$

40. $\frac{1}{2}\left(\log 215 + \log 6^8 - \log 169^3\right) = \frac{1}{2}\log\frac{215\left(6^8\right)}{169^3}$
$= \log\sqrt{\frac{215(6)^8}{169^3}}$

41. $e^{4\ln 3 - 3\ln 4} = e^{\ln 3^4 - \ln 4^3} = e^{\ln\left(\frac{3^4}{4^3}\right)} = \frac{3^4}{4^3} = \frac{81}{64}$

42. $\log_2\left[\ln\left(\sqrt{5+e^2}+\sqrt{5}\right) + \ln\left(\sqrt{5+e^2}-\sqrt{5}\right)\right]$
$= \log_2\left[\ln\left(\sqrt{5+e^2}+\sqrt{5}\right)\left(\sqrt{5+e^2}-\sqrt{5}\right)\right]$
$= \log_2[\ln(5+e^2-5)]$
$= \log_2[\ln e^2]$
$= \log_2(2)$
$= 1$

43. $\log_6 54 - \log_6 9 = \log_6\frac{54}{9} = \log_6 6 = 1$

44. $\log_3\sqrt{3} + \log_2\sqrt[3]{2} - \log_5\sqrt[4]{5}$
$= \log_3 3^{1/2} + \log_2 2^{1/3} - \log_5 5^{1/4}$
$= \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$
$= \frac{7}{12}$

45. $e^{\ln(2x)} = 5$
$2x = 5$
$x = \frac{5}{2}$

46. $4^{\log_4(x)+\log_4(2)} = 3$
$4^{\log_4(2x)} = 3$
$2x = 3$
$x = \frac{3}{2}$

47. $10^{\log x^2} = 4$
$x^2 = 4$
$x = \pm 2$

48. $e^{3\ln x} = 8$
$e^{\ln x^3} = 8$
$x^3 = 8$
$x = 2$

49. From the change of base formula with $b = 2$, $m = 2x + 1$, and $a = e$, we have
$\log_2(2x+1) = \frac{\log_e(2x+1)}{\log_e 2} = \frac{\ln(2x+1)}{\ln 2}$

50. From the change of base formula with $b = 3$, $m = x^2 + 2x + 2$ and $a = e$,

$$\log_3(x^2 + 2x + 2) = \frac{\log_e(x^2 + 2x + 2)}{\log_e 3} = \frac{\ln(x^2 + 2x + 2)}{\ln 3}$$

51. From the change of base formula with $b = 3$, $m = x^2 + 1$, and $a = e$, we have

$$\log_3\left(x^2 + 1\right) = \frac{\log_e\left(x^2 + 1\right)}{\log_e 3} = \frac{\ln\left(x^2 + 1\right)}{\ln 3}.$$

52. From the change of base formula with $b = 5$, $m = 9 - x^2$, and $a = e$, we have

$$\log_5\left(9 - x^2\right) = \frac{\log_e\left(9 - x^2\right)}{\log_3 5} = \frac{\ln\left(9 - x^2\right)}{\ln 5}$$

53. $e^{\ln z} = 7e^y$

$z = 7e^y$

$\frac{z}{7} = e^y$

$y = \ln\frac{z}{7}$

54. $y = ab^x$ so

$\log y = \log(ab^x)$

$= \log a + \log b^x$

$= \log a + x \log b.$

This is a linear expression because it is in the form $Ax + B$, where $A = \log b$ and $B = \log a$.

55. $C = B + E$

$C = B\left(1 + \frac{E}{B}\right)$

$\ln C = \ln\left[B\left(1 + \frac{E}{B}\right)\right]$

$\ln C = \ln B + \ln\left(1 + \frac{E}{B}\right)$

56. $M = \log(A) + 3$

a. $M = \log(10) + 3 = 1 + 3 = 4$

b. Given $M_1 = \log\left(A_1\right) + 3$, let

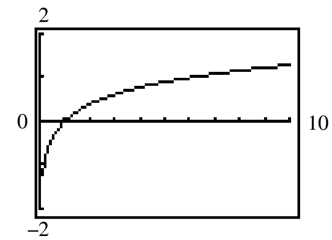
$M = \log\left(10A_1\right) + 3$

$M = \log 10 + \log\left(A_1\right) + 3$

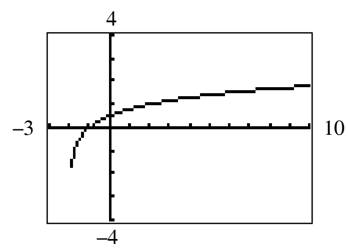
$M = 1 + \left[\log\left(A_1\right) + 3\right]$

$M = 1 + M_1$

57. $y = \log_6 x = \frac{\ln x}{\ln 6}$



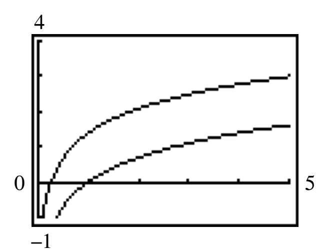
58. $y = \log_4(x + 2) = \frac{\ln(x + 2)}{\ln 4}$



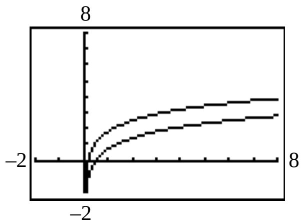
59. By the change of base formula, $\log x = \frac{\ln x}{\ln 10}$.

Thus the graphs of $y = \log x$ and $y = \frac{\ln x}{\ln 10}$ are identical.

60.



$y = \ln(4x) = \ln 4 + \ln x$. If $f(x) = \ln x$, then $y = \ln(4x) = f(x) + \ln 4$. Thus the graph of $y = \ln(4x)$ is the graph of $y = \ln x$ shifted $\ln 4$ units upward.

61. 

$\ln(6x) = \ln(3 \cdot 2x) = \ln 3 + \ln(2x)$.
If $f(x) = \ln(2x)$, then $y = \ln(6x) = f(x) + \ln 3$.
Thus, the graph of $y = \ln(6x)$ is the graph of $y = \ln(2x)$ shifted ln 3 units upward.

Principles in Practice 4.4

1. Let x = the number and let y = the unknown exponent. Then

$$x \cdot 32^y = x \cdot 4^{(3y-9)}$$
$$32^y = 4^{(3y-9)}$$
$$\log 32^y = \log 4^{(3y-9)}$$
$$y \log 32 = (3y - 9) \log 4$$
$$y \log 32 = 3y \log 4 - 9 \log 4$$
$$y(\log 32 - 3 \log 4) = -9 \log 4$$
$$y = \frac{-9\log 4}{\log \frac{32}{4^3}} = \frac{-18\log 2}{\log \frac{1}{2}} = \frac{-18\log 2}{-\log 2}$$
$$y = 18$$

Thus, Greg used 32 to the power of 18.

2. Let $S = 450$.

$$S = 800\left(\frac{4}{3}\right)^{-0.1d}$$
$$450 = 800\left(\frac{4}{3}\right)^{-0.1d}$$
$$\frac{450}{800} = \left(\frac{4}{3}\right)^{-0.1d}$$
$$\log\frac{450}{800} = -0.1d \log\left(\frac{4}{3}\right)$$
$$\frac{\log \frac{450}{800}}{-0.1\log\left(\frac{4}{3}\right)} = d$$
$$20 = d$$

Thus, he should start the new campaign 20 days after the last one ends.

3. The magnitude (Richter Scale) of an earthquake is given by $R = \log\left(\frac{I}{I_0}\right)$ where I is the intensity of the earthquake and I_0 is the intensity of a zero-level reference earthquake. $\frac{I}{I_0} =$ how many times greater the earthquake is than a zero-level earthquake.

$$R_1 = \log(675{,}000)$$
$$R_2 = \log\left(\frac{I}{I_0}\right)$$

Since $R_1 - 4 = R_2$

$$\log(675{,}000) - 4 = \log\left(\frac{I}{I_0}\right)$$
$$\log\left(6.75\times 10^5\right) - 4 = \log\left(\frac{I}{I_0}\right)$$
$$\log 6.75 + 5\log 10 - 4 = \log\left(\frac{I}{I_0}\right)$$
$$1.829 = \log\left(\frac{I}{I_0}\right)$$
$$10^{1.829} = \frac{I}{I_0}$$
$$67.5 = \frac{I}{I_0}$$

Thus, the other earthquake is 67.5 times as intense as a zero-level earthquake.

Problems 4.4

1.
$$\log(3x + 2) = \log(2x + 5)$$
$$3x + 2 = 2x + 5$$
$$x = 3$$

2.
$$\log x - \log 5 = \log 7$$
$$\log x = \log 5 + \log 7$$
$$\log x = \log 35$$
$$x = 35$$

3. $\log 7 - \log(x-1) = \log 4$

$\log \frac{7}{x-1} = \log 4$

$\frac{7}{x-1} = 4$

$7 = 4x - 4$

$4x = 11$

$x = \frac{11}{4} = 2.75$

4. $\log_2 x + \log_2 2^3 = \log_2 \frac{2}{x}$

$\log_2(8x) = \log_2 \frac{2}{x}$

$8x = \frac{2}{x}$

$8x^2 = 2$

$x^2 = \frac{1}{4}$

$x = \frac{1}{2} = 0.5$ since $x > 0$

5. $\ln(-x) = \ln\left(x^2 - 6\right)$

$-x = x^2 - 6$

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$x = -3$ or $x = 2$

However, $x = -3$ is the only value that satisfies the original equation.

$x = -3$

6. $\ln(4-x) + \ln 2 = 2\ln x$

$\ln[(4-x)2] = \ln x^2$

$(4-x)2 = x^2$

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$

$x = -4$ or $x = 2$

However, $x = 2$ is the only value that satisfies the original equation.

$x = 2$

7. $e^{2x}e^{5x} = e^{14}$

$e^{7x} = e^{14}$

$7x = 14$

$x = 2$

8. $(e^{3x-2})^3 = e^3$

$e^{3(3x-2)} = e^3$

$3(3x-2) = 3$

$3x - 2 = 1$

$3x = 3$

$x = 1$

9. $(81)^{4x} = 9$

$(3^4)^{4x} = 3^2$

$3^{16x} = 3^2$

$16x = 2$

$x = \frac{2}{16} = \frac{1}{8} = 0.125$

10. $(27)^{2x+1} = 3^{-1}$

$\left(3^3\right)^{2x+1} = 3^{-1}$

$3^{6x+3} = 3^{-1}$

$6x + 3 = -1$

$6x = -4$

$x = -\frac{2}{3} \approx -0.667$

11. $e^{2x} = 9$

$(e^x)^2 = 3^2$

$e^x = 3$

$x = \ln 3 \approx 1.099$

12. $e^{4x} = \frac{3}{4}$

$4x = \ln\frac{3}{4}$

$x = \frac{\ln\left(\frac{3}{4}\right)}{4} \approx -0.072$

13. $2e^{5x+2} = 17$

$e^{5x+2} = \frac{17}{2}$

$5x + 2 = \ln\left(\frac{17}{2}\right)$

$5x = \ln\left(\frac{17}{2}\right) - 2$

$x = \frac{1}{5}\left[\ln\left(\frac{17}{2}\right) - 2\right] \approx 0.028$

14. $5e^{2x-1}-2=23$

$5e^{2x-1}=25$

$e^{2x-1}=5$

$2x-1=\ln 5$

$x=\dfrac{1+\ln 5}{2}\approx 1.305$

15. $10^{\frac{4}{x}}=6$

$\dfrac{4}{x}=\log 6$

$x=\dfrac{4}{\log 6}\approx 5.140$

16. $\dfrac{4(10)^{0.2x}}{5}=3$

$(10)^{0.2x}=\dfrac{15}{4}$

$0.2x=\log\dfrac{15}{4}$

$x=\dfrac{\log\left(\frac{15}{4}\right)}{0.2}\approx 2.870$

17. $\dfrac{5}{10^{2x}}=7$

$10^{2x}=\dfrac{5}{7}$

$2x=\log\dfrac{5}{7}$

$x=\dfrac{\log\left(\frac{5}{7}\right)}{2}\approx -0.073$

18. $2(10)^{x}+(10)^{x+1}=4$

$2(10)^{x}+10(10)^{x}=4$

$12(10)^{x}=4$

$(10)^{x}=\dfrac{1}{3}$

$x=\log\dfrac{1}{3}\approx -0.477$

19. $2^{x}=5$

$\ln 2^{x}=\ln 5$

$x\ln 2=\ln 5$

$x=\dfrac{\ln 5}{\ln 2}\approx 2.322$

20. $7^{2x+3}=9$

$\ln(7^{2x+3})=\ln 9$

$(2x+3)\ln 7=\ln 9$

$2x+3=\dfrac{\ln 9}{\ln 7}$

$2x=\dfrac{\ln 9}{\ln 7}-3$

$x=\dfrac{1}{2}\left(\dfrac{\ln 9}{\ln 7}-3\right)\approx -0.935$

21. $7^{3x-2}=5$

$\ln 7^{3x-2}=\ln 5$

$(3x-2)\ln 7=\ln 5$

$3x-2=\dfrac{\ln 5}{\ln 7}$

$3x=\dfrac{\ln 5}{\ln 7}+2$

$x=\dfrac{\frac{\ln 5}{\ln 7}+2}{3}\approx 0.942$

22. $4^{\frac{x}{2}}=20$

$\ln 4^{\frac{x}{2}}=\ln 20$

$\dfrac{x}{2}\ln 4=\ln 20$

$\dfrac{x}{2}=\dfrac{\ln 20}{\ln 4}$

$x=\dfrac{2\ln 20}{\ln 4}\approx 4.322$

23. $2^{-\frac{2x}{3}}=\dfrac{4}{5}$

$\ln 2^{-\frac{2x}{3}}=\ln\dfrac{4}{5}$

$-\dfrac{2x}{3}\ln 2=\ln\dfrac{4}{5}$

$-\dfrac{2x}{3}=\dfrac{\ln\left(\frac{4}{5}\right)}{\ln 2}$

$x=-\dfrac{3\ln\left(\frac{4}{5}\right)}{2\ln 2}\approx 0.483$

24. $5\left(3^x - 6\right) = 10$

$3^x - 6 = 2$

$3^x = 8$

$\ln 3^x = \ln 8$

$x \ln 3 = \ln 8$

$x = \dfrac{\ln 8}{\ln 3} \approx 1.893$

25. $(4)5^{3-x} - 7 = 2$

$5^{3-x} = \dfrac{9}{4}$

$\ln 5^{3-x} = \ln \dfrac{9}{4}$

$(3-x)\ln 5 = \ln \dfrac{9}{4}$

$3 - x = \dfrac{\ln\left(\frac{9}{4}\right)}{\ln 5}$

$x = 3 - \dfrac{\ln\left(\frac{9}{4}\right)}{\ln 5} \approx 2.496$

26. $\dfrac{7}{3^x} = 13$

$\dfrac{7}{13} = 3^x$

$\ln\left(\dfrac{7}{13}\right) = \ln(3^x)$

$\ln\left(\dfrac{7}{13}\right) = x \ln 3$

$x = \dfrac{\ln\left(\frac{7}{13}\right)}{\ln 3} \approx -0.563$

27. $\log(x - 3) = 3$

$10^3 = x - 3$

$x = 10^3 + 3 = 1003$

28. $\log_2(x+1) = 4$

$2^4 = x + 1$

$x = 2^4 - 1 = 15$

29. $\log_4(9x - 4) = 2$

$4^2 = 9x - 4$

$9x = 4^2 + 4$

$x = \dfrac{4^2 + 4}{9} = \dfrac{20}{9} \approx 2.222$

30. $\log_4(2x + 4) - 3 = \log_4 3$

$\log_4(2x + 4) - \log_4 3 = 3$

$\log_4 \dfrac{2x + 4}{3} = 3$

$4^3 = \dfrac{2x + 4}{3}$

$2x + 4 = 3 \cdot 4^3$

$x = \dfrac{3 \cdot 4^3 - 4}{2} = \dfrac{188}{2} = 94$

31. $\log(3x - 1) - \log(x - 3) = 2$

$\log \dfrac{3x - 1}{x - 3} = 2$

$10^2 = \dfrac{3x - 1}{x - 3}$

$100(x - 3) = 3x - 1$

$97x = 299$

$x = \dfrac{299}{97} \approx 3.082$

32. $\log(x - 3) + \log(x - 5) = 1$

$\log[(x - 3)(x - 5)] = 1$

$x^2 - 8x + 15 = 10$

$x^2 - 8x + 5 = 0$

$x = \dfrac{8 \pm \sqrt{(-8)^2 - 4(1)(5)}}{2(1)} = 4 \pm \sqrt{11}$

However, $x = 4 + \sqrt{11} \approx 7.317$ is the only value that satisfies the original equation.

$x \approx 7.317$

33. $\log_2(5x + 1) = 4 - \log_2(3x - 2)$

$\log_2(5x + 1) + \log_2(3x - 2) = 4$

$\log[(5x + 1)(3x - 2)] = 4$

$(5x + 1)(3x - 2) = 2^4$

$15x^2 - 7x - 2 = 16$

$15x^2 - 7x - 18 = 0$

$x \approx 1.353$ or $x \approx -0.887$

However, $x \approx 1.353$ is the only value that satisfies the original equation.

$x \approx 1.353$

34. $\log(x+2)^2 = 2$

$2\log(x+2) = 2$

$\log(x+2) = 1$

$10^1 = x+2$

$x = 8$

35. $\log_2\left(\frac{2}{x}\right) = 3 + \log_2 x$

$\log_2\left(\frac{2}{x}\right) - \log_2 x = 3$

$\log_2 \frac{\frac{2}{x}}{x} = 3$

$\log_2 \frac{2}{x^2} = 3$

$2^3 = \frac{2}{x^2}$

$x^2 = \frac{1}{4}$

$x = \pm\frac{1}{2}$

However, $x = \frac{1}{2}$ is the only value that satisfies the original equation.

$x = \frac{1}{2} = 0.5$

36.

$$\ln(x-2) = \ln(2x-1) + 3$$
$$\ln(x-2) - \ln(2x-1) = 3$$
$$\ln\left(\frac{x-2}{2x-1}\right) = 3$$
$$\frac{x-2}{2x-1} = e^3$$
$$e^3(2x-1) = x-2$$
$$2e^3x - e^3 = x-2$$
$$x(2e^3-1) = -2+e^3$$
$$x = \frac{-2+e^3}{2e^3-1} \approx 0.462$$

However, this value does not satisfy the original equation. The equation has no solution.

37. $\log S = \log 12.4 + 0.26\log A$

$\log S = \log 12.4 + \log A^{0.26}$

$\log S = \log\left[12.4A^{0.26}\right]$

$S = 12.4A^{0.26}$

38. $\log T = 1.7 + 0.2068 \log P - 0.1334(\log P)^2$
$\log T = \log 50 + 0.2068 \log P - 0.1334(\log P)(\log P)$
$\log T = \log 50 + 0.2068 \log P + [-0.1334 \log P] \log P$
$\log T = \log 50 + \log P^{0.2068} + \log P^{[-0.1334 \log P]}$
$\log T = \log\left[(50)\left(P^{0.2068}\right)\left(P^{-0.1334 \log P}\right)\right]$
$T = 50P^{0.2068-(0.1334 \log P)}$
$(\log_b x)^2 = (\log_b x)(\log_b x) = \log_b (x^{\log_b x})$

39. a. When $t = 0$, $Q = 100e^{-0.035(0)} = 100e^0 = 100 \cdot 1 = 100$.

b. If $Q = 20$, then $20 = 100e^{-0.035t}$. Solving for t gives
$\frac{20}{100} = e^{-0.035t}$
$\frac{1}{5} = e^{-0.035t}$
$\ln \frac{1}{5} = -0.035t$
$-\ln 5 = -0.035t$
$t = \frac{\ln 5}{0.035} \approx 46$

40. $100 = 225e^{-\frac{N}{225}}$
$e^{\frac{N}{225}} = \frac{225}{100} = \frac{9}{4}$
$\frac{N}{225} = \ln \frac{9}{4}$
$N = 225 \ln \frac{9}{4} \approx 182$

41. If $P = 1{,}500{,}000$, then $1{,}500{,}000 = 1{,}000{,}000(1.02)^t$. Solving for t gives
$\frac{1{,}500{,}000}{1{,}000{,}000} = (1.02)^t$
$1.5 = (1.02)^t$
$\ln 1.5 = \ln(1.02)^t$
$\ln 1.5 = t \ln 1.02$
$t = \frac{\ln 1.5}{\ln 1.02} \approx 20.5$

42. If $F(0) = 0$, then $0 = \dfrac{q - pe^{-C(p+q)}}{q\left[1+e^{C(p+q)}\right]}$. Thus

$$q - pe^{-C(p+q)} = 0$$
$$-pe^{-C(p+q)} = -q$$
$$e^{-C(p+q)} = \frac{q}{p}$$
$$-C(p+q) = \ln\frac{q}{p}$$
$$C = -\frac{1}{p+q}\ln\frac{q}{p}.$$

43. $q = 80 - 2^p$

$$2^p = 80 - q$$
$$\log 2^p = \log(80-q)$$
$$p\log 2 = \log(80-q)$$
$$p = \frac{\log(80-q)}{\log 2}$$

When $q = 60$, then $p = \dfrac{\log 20}{\log 2} \approx 4.32$.

44. The investment doubles when $A = 2P$. Thus $2P = P(1.105)^t$, or $2 = (1.105)^t$. Solving for t gives

$$\ln 2 = \ln(1.105)^t$$
$$\ln 2 = t\ln 1.105$$
$$t = \frac{\ln 2}{\ln 1.105} \approx 7$$

45. $q = 1000\left(\dfrac{1}{2}\right)^{0.8^t}$

$$\log q = \log 1000 + \log\left(\frac{1}{2}\right)^{0.8^t}$$
$$\log q = 3 + 0.8^t \log\frac{1}{2}$$
$$\log q = 3 + 0.8^t(-\log 2)$$
$$\log(q) - 3 = 0.8^t(-\log 2)$$

Thus

$$0.8^t = \frac{\log(q)-3}{-\log 2} = \frac{3-\log q}{\log 2}$$
$$t\log(0.8) = \log\left(\frac{3-\log q}{\log 2}\right).$$
$$t = \frac{\log\left(\frac{3-\log q}{\log 2}\right)}{\log(0.8)}$$

$$y = Ab^{a^x}$$
$$\log y = \log A + \log b^{a^x}$$
$$\log y = \log A + a^x \log b$$
$$\log y - \log A = a^x \log b$$
$$a^x = \frac{\log y - \log A}{\log b}$$
$$\log a^x = \log\left(\frac{\log y - \log A}{\log b}\right)$$
$$x\log a = \log\left(\frac{\log y - \log A}{\log b}\right)$$
$$x = \frac{\log\left(\frac{\log y - \log A}{\log b}\right)}{\log a}$$

The previous solution was the special case $y = q$, $A = 1000$, $b = \dfrac{1}{2}$, $a = 0.8$, and $x = t$.

46. $q = 500\left(1-e^{-0.2t}\right)$

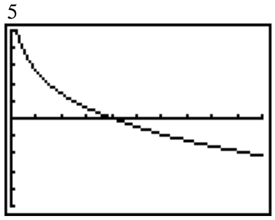
a. If $t = 1$, then $q = 500\left(1-e^{-0.2}\right) \approx 91$.

b. If $t = 10$, then $q = 500\left(1-e^{-2}\right) \approx 432$.

c. We solve the equation

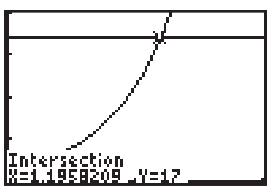
$$400 = 500\left(1-e^{-0.2t}\right)$$
$$\frac{4}{5} = 1 - e^{-0.2t}$$
$$e^{-0.2t} = \frac{1}{5}$$
$$-0.2t = \ln\frac{1}{5} = -\ln 5$$
$$t = \frac{\ln 5}{0.2} \approx 8$$

47. $\log_2 x = 5 - \log_2(x+4)$ is equivalent to $0 = 5 - \log_2(x+4) - \log_2 x$, or $0 = 5 - \frac{\ln(x+4)}{\ln 2} - \frac{\ln x}{\ln 2}$. Thus the solutions of the original equation are the zeros of the function $y = 5 - \frac{\ln(x+4)}{\ln 2} - \frac{\ln x}{\ln 2}$.



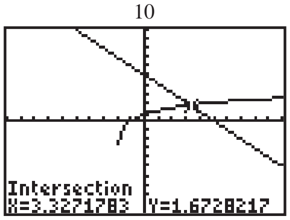
From the graph of this function, the only zero is $x = 4$. Thus 4 is the only solution of the original equation.

48.



1.20

49.

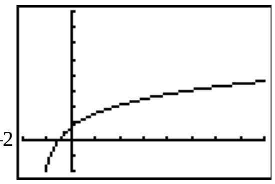


3.33

50. $(3)2^y - 4x = 5$

$$(3)2^y = 4x + 5$$
$$2^y = \frac{4x+5}{3}$$
$$\ln 2^y = \ln\left(\frac{4x+5}{3}\right)$$
$$y \ln 2 = \ln\left(\frac{4x+5}{3}\right)$$
$$y = \frac{\ln\left(\frac{4x+5}{3}\right)}{\ln 2}$$

The graph of the original equation is the graph of $y = \frac{\ln\left(\frac{4x+5}{3}\right)}{\ln 2}$.



Chapter 4 Review Problems

1. $\log_3 243 = 5$

2. $5^4 = 625$

3. $81^{\frac{1}{4}} = 3$

4. $\log 100{,}000 = 5$

5. $\ln 54.598 = 4$

6. $9^1 = 9$

7. Because $5^3 = 125$, $\log_5 125 = 3$

8. Because $4^2 = 16$, $\log_4 16 = 2$

9. Because $3^{-4} = \frac{1}{81}$, $\log_3 \frac{1}{81} = -4$

10. Because $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$, $\log_{\frac{1}{4}} \frac{1}{64} = 3$.

11. Because $\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$, $\log_{\frac{1}{3}} 9 = -2$

12. Because $4^{\frac{1}{2}} = 2$, $\log_4 2 = \frac{1}{2}$

13. $5^x = 625$
$x = 4$

14. $\log_x \frac{1}{81} = -4$

$$x^{-4} = \frac{1}{81}$$
$$\frac{1}{x^4} = \frac{1}{81}$$
$$x^4 = 81$$
$$x = 3$$

15. $2^{-5} = x$

$$x = \frac{1}{2^5} = \frac{1}{32}$$

16. $e^x = \frac{1}{e} = e^{-1}$

$x = -1$

17. $\ln(2x+3) = 0$

$$\begin{aligned} e^0 &= 2x+3 \\ 1 &= 2x+3 \\ 2x &= -2 \\ x &= -1 \end{aligned}$$

18. Because $e^{\ln(x+4)} = x+4$,

$x + 4 = 7$

$x = 3$

19. $\log 8000 = \log(2\cdot 10)^3 = 3\log(2\cdot 10)$

$= 3(\log 2 + \log 10) = 3(a+1)$

20. $\log\frac{9}{\sqrt{2}} = \log\frac{3^2}{2^{\frac{1}{2}}} = \log 3^2 - \log 2^{\frac{1}{2}}$

$= 2\log 3 - \frac{1}{2}\log 2 = 2b - \frac{1}{2}a = 2b - \frac{a}{2}$

21. $3\log 7 - 2\log 5 = \log 7^3 - \log 5^2 = \log\frac{7^3}{5^2}$

22. $5\ln x + 2\ln y + \ln z = \ln x^5 + \ln y^2 + \ln z$

$= \ln(x^5 y^2 z)$

23. $2\ln x + \ln y - 3\ln z = \ln x^2 + \ln y - \ln z^3$

$= \ln x^2 y - \ln z^3 = \ln\frac{x^2 y}{z^3}$

24. $\log_6 2 - \log_6 4 - 9\log_6 3$

$= \log_6 2 - \left[\log_6 4 + \log_6 3^9\right]$

$= \log_6 2 - \log_6\left(4\cdot 3^9\right) = \log_6\frac{2}{4\cdot 3^9} = \log_6\frac{1}{39,366}$

25. $\frac{1}{2}\log_2 x + 2\log_2 x^2 - 3\log_2(x+1) - 4\log_2(x+2)$

$= \log_2 x^{\frac{1}{2}} + \log_2\left(x^2\right)^2 - \left[\log_2(x+1)^3 + \log_2(x+2)^4\right]$

$= \log_2\left(x^{\frac{1}{2}}x^4\right) - \log_2\left[(x+1)^3(x+2)^4\right]$

$= \log_2\frac{x^{\frac{9}{2}}}{(x+1)^3(x+2)^4}$

26. $4\log x + 2\log y - 3(\log z + \log w)$

$= \log x^4 + \log y^2 - 3\log zw$

$= \log x^4 + \log y^2 - \log(zw)^3$

$= \log x^4 y^2 - \log z^3 w^3$

$= \log\frac{x^4 y^2}{z^3 w^3}$

27. $\ln\frac{x^3 y^2}{z^{-5}} = \ln x^3 y^2 - \ln z^{-5}$

$= \ln x^3 + \ln y^2 - \ln z^{-5}$

$= 3\ln x + 2\ln y + 5\ln z$

28. $\ln\frac{\sqrt{x}}{(yz)^2} = \ln\sqrt{x} - \ln(yz)^2 = \ln x^{\frac{1}{2}} - 2\ln(yz)$

$= \frac{1}{2}\ln x - 2(\ln y + \ln z)$

29. $\ln\sqrt[3]{xyz} = \ln(xyz)^{\frac{1}{3}} = \frac{1}{3}\ln(xyz)$

$= \frac{1}{3}(\ln x + \ln y + \ln z)$

30. $\ln\left[\frac{xy^3}{z^2}\right]^4 = 4\ln\frac{xy^3}{z^2} = 4\left(\ln xy^3 - \ln z^2\right)$

$= 4\left(\ln x + \ln y^3 - \ln z^2\right)$

$= 4(\ln x + 3\ln y - 2\ln z)$

31. $\ln\left[\frac{1}{x}\sqrt{\frac{y}{z}}\right] = \ln\frac{\left(\frac{y}{z}\right)^{1/2}}{x} = \ln\left(\frac{y}{z}\right)^{\frac{1}{2}} - \ln x$

$= \frac{1}{2}\ln\frac{y}{z} - \ln x = \frac{1}{2}(\ln y - \ln z) - \ln x$

32. $\ln\left[\left(\frac{x}{y}\right)^2\left(\frac{x}{z}\right)^3\right] = \ln\frac{x^5}{y^2z^3} = \ln x^5 - \ln y^2z^3$

$= \ln x^5 - \left(\ln y^2 + \ln z^3\right) = 5\ln x - 2\ln y - 3\ln z$

33. $\log_3(x+5) = \frac{\log_e(x+5)}{\log_e 3} = \frac{\ln(x+5)}{\ln 3}$

34. $\log_2(7x^3+5) = \frac{\log_{10}(7x^3+5)}{\log_{10} 2}$

$= \frac{\log(7x^3+5)}{\log 2}$

35. $\log_5 19 = \frac{\log_2 19}{\log_2 5} = \frac{4.2479}{2.3219} \approx 1.8295$

36. $\log_4 5 = \frac{\ln 5}{\ln 4} \approx 1.1610$

37. $\ln\left(16\sqrt{3}\right) = \ln 4^2 + \ln\sqrt{3} = 2\ln 4 + \frac{1}{2}\ln 3$

$= 2y + \frac{1}{2}x$

38. $\log\frac{x^3\sqrt[3]{x+1}}{\sqrt[5]{x^2+2}}$

$= \log x^3\sqrt[3]{x+1} - \log\sqrt[5]{x^2+2}$

$= \log x^3 + \log\sqrt[3]{x+1} - \log\sqrt[5]{x^2+2}$

$= 3\log x + \frac{1}{3}\log(x+1) - \frac{1}{5}\log(x^2+2)$

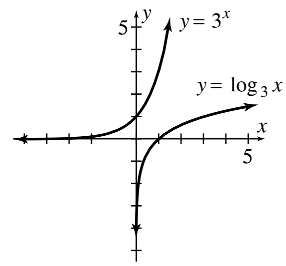
39. $10^{\log x} + \log 10^x + \log 10 = x + x + 1 = 2x + 1$

40. $\log 10^2 + \log(1000) - 5 = \log 10^2 + \log\left(10^3\right) - 5$

$= 2 + 3 - 5 = 0$

41. In exponential form, $y = e^{x^2+2}$.

42.



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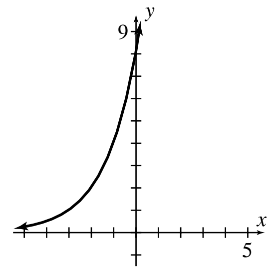

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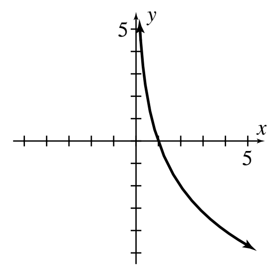


45. $\log(5x+1) = \log(4x+6)$

$5x + 1 = 4x + 6$

$x = 5$

46. $\log 3x + \log 3 = 2$

$\log 9x = 2$

$9x = 10^2$

$9x = 100$

$x = \frac{100}{9}$

47. $3^{4x} = 9^{x+1}$

$3^{4x} = \left(3^2\right)^{x+1}$

$3^{4x} = 3^{2(x+1)}$

$4x = 2(x + 1)$

$4x = 2x + 2$

$2x = 2$

$x = 1$

48. $4^{3-x} = \frac{1}{16}$
$4^{3-x} = 4^{-2}$
$3 - x = -2$
$x = 5$

49. $\log x + \log(10x) = 3$
$\log x + \log 10 + \log x = 3$
$2\log(x) + 1 = 3$
$2\log(x) = 2$
$\log x = 1$
$x = 10^1 = 10$

50. $\log_2(x+4) = \log_2(x-2) + 3$
$\log_2\left(\frac{x+4}{x-2}\right) = 3$
$\frac{x+4}{x-2} = 2^3 = 8$
$x + 4 = 8(x-2) = 8x - 16$
$20 = 7x$
$x = \frac{20}{7}$

51. $\ln(\log_x 3) = 2$
$\log_x 3 = e^2$
$x^{e^2} = 3$
$(x^{e^2})^{-e^2} = 3^{-e^2}$
$x^{e^2 \cdot -e^2} = 3^{-e^2}$
$x^1 = 3^{-e^2}$
$x = \frac{1}{3^{e^2}}$

52. $\log_2 x + \log_4 x = 3$
$\log_2 x + \frac{\log_2 x}{\log_2 4} = 3$
$\log_2 x + \frac{\log_2 x}{2} = 3$
$\frac{3}{2}\log_2 x = 3$
$\log_2 x = 2$
$x = 2^2$
$x = 4$

53. $e^{3x} = 14$
$3x = \ln 14$
$x = \frac{\ln 14}{3} \approx 0.880$

54. $10^{\frac{3x}{2}} = 5$
$\frac{3x}{2} = \log 5$
$x = \frac{2}{3}\log 5 \approx 0.466$

55. $3\left(10^{x+4} - 3\right) = 9$
$10^{x+4} - 3 = 3$
$10^{x+4} = 6$
$x + 4 = \log 6$
$x = \log(6) - 4 \approx -3.222$

56. $7e^{3x-1} - 2 = 1$
$7e^{3x-1} = 3$
$e^{3x-1} = \frac{3}{7}$
$3x - 1 = \ln\frac{3}{7}$
$3x = \ln\frac{3}{7} + 1$
$x = \frac{\ln\frac{3}{7} + 1}{3} \approx 0.051$

57. $4^{x+3} = 7$
$\ln 4^{x+3} = \ln 7$
$(x+3)\ln 4 = \ln 7$
$x + 3 = \frac{\ln 7}{\ln 4}$
$x = \frac{\ln 7}{\ln 4} - 3 \approx -1.596$

58. $3^{5/x} = 2$
$\frac{5}{x}\ln 3 = \ln 2$
$x = \frac{5\ln 3}{\ln 2} \approx 7.925$

59. Quarterly rate $= \frac{0.06}{4} = 0.015$
$6\frac{1}{2}$ yr = 26 quarters

a. $2600(1.015)^{26} \approx \$3829.04$

b. $3829.04 - 2600 = \$1229.04$

60. Monthly rate $= \frac{0.11}{12}$

5 yr = 60 mo.

$$4000\left(1+\frac{0.11}{12}\right)^{60} \approx \$6915.66$$

61. $12\left(1\frac{1}{6}\%\right) = 14\%$

62. a. $N = 600(1.05)^t$

b. When $t = 1$, $N = 600(1.05)^1 = 630$.

c. When $t = 5$, $N = 600(1.05)^5 \approx 766$.

63. a. $P = 6000[1+(-0.005)]^t$ or

$P = 6000(0.995)^t$

b. When $t = 10$, then

$P = 6000(0.995)^{10} \approx 5707$.

64. If $t = 2$, $R = 200,000e^{-0.4} \approx 134,064$

If $t = 3$, $R = 200,000e^{-0.6} \approx 109,762$

65. $N = 10e^{-0.41t}$

a. When $t = 0$, then $N = 10e^0 = 10 \cdot 1 = 10$ mg

b. When $t = 2$, then $N = 10e^{-0.82} \approx 4.4$ mg

c. When $t = 10$, then $N = 10e^{-4.1} \approx 0.2$ mg

d. $\frac{\ln 2}{0.41} \approx 1.7$

e. If $N = 1$, then $1 = 10e^{-0.41t}$. Solving for t gives

$$\frac{1}{10} = e^{-0.41t}$$

$$-0.41t = \ln\frac{1}{10} = -\ln 10$$

$$t = \frac{\ln 10}{0.41} \approx 5.6$$

66. Because $\frac{1}{8} = \left(\frac{1}{2}\right)^3$, it will take 3 · 10 = 30 days for $\frac{1}{8}$ of the initial amount to be present.

67. $R = 10e^{-\frac{t}{40}}$

a. If $t = 20$, $R = 10e^{-\frac{20}{40}} = 10e^{-\frac{1}{2}} \approx 6$.

b. $5 = 10e^{-\frac{t}{40}}$, $\frac{1}{2} = e^{-\frac{t}{40}}$. Thus

$$-\frac{t}{40} = \ln\frac{1}{2} = -\ln 2$$

$t = 40 \ln 2 \approx 28$.

68. Let d = depth in centimeters.

$$(0.9)^{\frac{d}{20}} = 0.0017$$

$$\ln(0.9)^{\frac{d}{20}} = \ln 0.0017$$

$$\frac{d}{20}\ln 0.9 = \ln 0.0017$$

$$d = \frac{20\ln 0.0017}{\ln 0.9} \approx 1210 \text{ cm}$$

69. $T_t - T_e = \left(T_t - T_e\right)_o e^{-at}$

$$e^{-at} = \frac{T_t - T_e}{\left(T_t - T_e\right)_o}$$

$$-at = \ln\frac{T_t - T_e}{\left(T_t - T_e\right)_o}$$

$$a = -\frac{1}{t}\ln\frac{T_t - T_e}{\left(T_t - T_e\right)_o}$$

$$a = \frac{1}{t}\ln\frac{\left(T_t - T_e\right)_o}{T_t - T_e}$$

70. For double-declining balance depreciation, the equation is $V = C\left(1-\frac{2}{N}\right)^n$.

$$700 = 1800\left(1-\frac{2}{48}\right)^n$$

$$\frac{700}{1800} = \left(\frac{46}{48}\right)^n$$

$$\frac{7}{18} = \left(\frac{23}{24}\right)^n$$

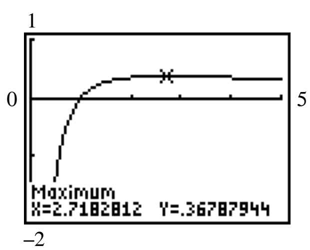
$$\ln\left(\frac{7}{18}\right) = \ln\left(\frac{23}{24}\right)^n$$

$$\ln\left(\frac{7}{8}\right) = n\ln\left(\frac{23}{24}\right)$$

$$n = \frac{\ln\left(\frac{7}{18}\right)}{\ln\left(\frac{23}{24}\right)} \approx 22$$

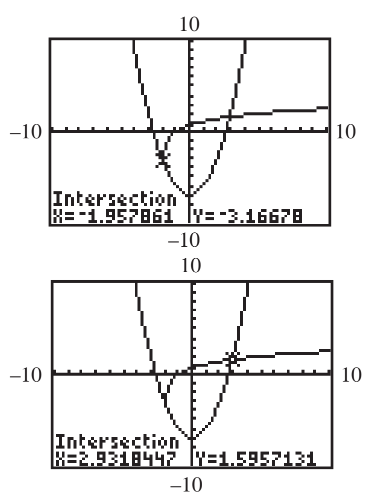
The value drops below \$700 at about 22 months.

71.



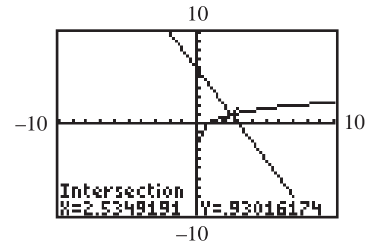
$(-\infty, 0.37]$

72.



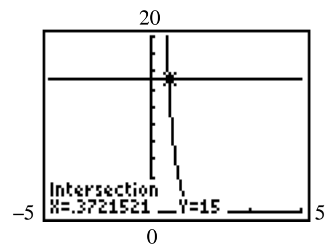
(–1.96, –3.17), (2.93, 1.60)

73.



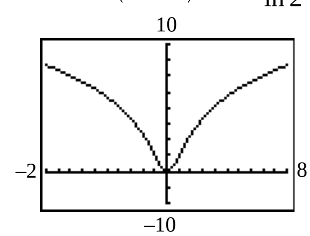
2.53

74.



0.37

75. $y = \log_2\left(x^2+1\right) = \frac{\ln\left(x^2+1\right)}{\ln 2}$



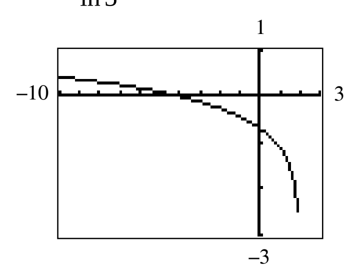
76. $(6)5^y + x = 2$

$$5^y = \frac{2-x}{6}$$

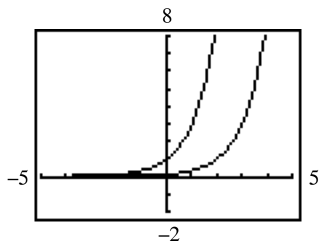
$$\ln 5^y = \ln\frac{2-x}{6}$$

$$y\ln 5 = \ln\frac{2-x}{6}$$

$$y = \frac{\ln\frac{2-x}{6}}{\ln 5}$$



77.



$y = \frac{3^x}{9} = \frac{3^x}{3^2} = 3^{x-2}$.

If $f(x) = 3^x$, then we have $y = 3^{x-2} = f(x-2)$.

Thus the graph of $y = \frac{3^x}{9}$ is the graph of $y = 3^x$ shifted 2 units to the right.

Mathematical Snapshot Chapter 4

1. $T = \frac{P\left(1-e^{-dkI}\right)}{e^{kI}-1}$

a. $T\left(e^{kI}-1\right) = P\left(1-e^{-dkI}\right)$

$\frac{T\left(e^{kI}-1\right)}{1-e^{-dkI}} = P$ or $P = \frac{T\left(e^{kI}-1\right)}{1-e^{-dkI}}$

b. $T\left(e^{kI}-1\right) = P - Pe^{-dkI}$

$Pe^{-dkI} = P - T\left(e^{kI}-1\right)$

$e^{-dkI} = \frac{P-T\left(e^{kI}-1\right)}{P}$

$-dkI = \ln\left[\frac{P-T\left(e^{kI}-1\right)}{P}\right]$

$d = -\frac{1}{kI}\ln\left[\frac{P-T\left(e^{kI}-1\right)}{P}\right]$

$d = \frac{1}{kI}\ln\left[\frac{P}{P-T\left(e^{kI}-1\right)}\right]$

2. From the text, the half-life H is given by $H = \frac{\ln 2}{k}$ or, equivalently, $k = \frac{\ln 2}{H}$. If $H = I$, then $k = \frac{\ln 2}{I}$. Thus

$$T = \frac{P\left(1-e^{-dkI}\right)}{e^{kI}-1} = \frac{P\left(1-e^{-d\cdot\frac{\ln 2}{I}\cdot I}\right)}{e^{\frac{\ln 2}{I}\cdot I}-1}$$

$$= \frac{P\left(1-\left[e^{\ln 2}\right]^{-d}\right)}{e^{\ln 2}-1} = \frac{P\left(1-2^{-d}\right)}{2-1}$$

$$= P\left(1-2^{-d}\right) = \left(1-\frac{1}{2^d}\right)P.$$

3. $P = 100$, $I = 4$, $d = 3$, $H = 8$, $k = \frac{\ln 2}{H} = \frac{\ln 2}{8}$

a. $$T = \frac{P\left(1-e^{-dkI}\right)}{e^{kI}-1} = \frac{100\left(1-e^{-3\cdot\frac{\ln 2}{8}\cdot 4}\right)}{e^{\frac{\ln 2}{8}\cdot 4}-1}$$

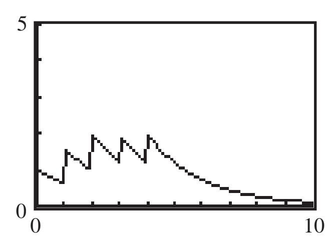
$$= \frac{100\left(1-\left[e^{\ln 2}\right]^{-\frac{3}{2}}\right)}{\left[e^{\ln 2}\right]^{\frac{1}{2}}-1} = \frac{100\left(1-2^{-\frac{3}{2}}\right)}{2^{\frac{1}{2}}-1} \approx 156$$

b. $R = P\left(1-e^{-dkI}\right)$. From part (a),

$P\left(1-e^{-dkI}\right) = 100\left(1-2^{-\frac{3}{2}}\right)$. Thus

$R = 100\left(1-2^{-\frac{3}{2}}\right) \approx 65.$

4.



As d changes, some of the coefficients need to change from P to Y1 or vice versa.

Chapter 5

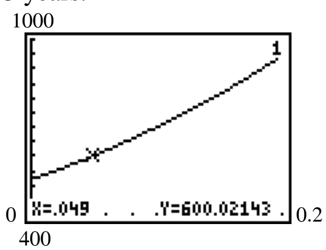
Principles in Practice 5.1

1. Let $P = 518$ and let $n = 3(365) = 1095$.

$S = P(1+r)^n$

$S = 518\left(1+\frac{r}{365}\right)^{1095}$

By graphing S as a function of the nominal rate r, we find that when $r = 0.049$, $S = 600$. Thus, at the nominal rate of 4.9% compounded daily, the initial amount of \$518 will grow to \$600 after 3 years.



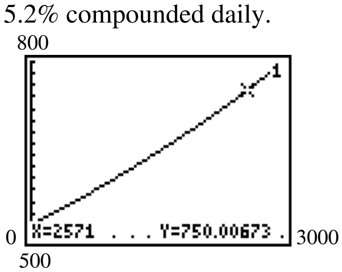
2. Let $P = 520$ and let $r = 0.052$.

$S = P(1+r)^n$

$S = 520\left(1+\frac{0.052}{365}\right)^n$

$S = 520\left(\frac{365.052}{365}\right)^n$

By graphing S as a function of n, we find that when $n = 2571$, $S = 750$. Thus, it will take $\frac{2571}{365} \approx 7.044$ years, or 7 years and 16 days for \$520 to grow to \$750 at the nominal rate of 5.2% compounded daily.

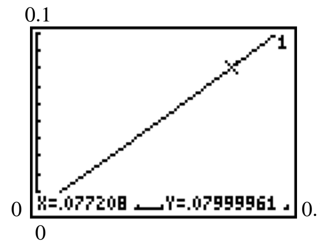


3. Let $n = 12$.

$r_e = \left(1+\frac{r}{n}\right)^n - 1$

$r_e = \left(1+\frac{r}{12}\right)^{12} - 1$

By graphing r_e as a function of r, we find that, when the nominal rate $r = 0.077208$ or 7.7208%, the effective rate $r_e = 0.08$ or 8%.



4. The respective effective rates of interest are found using the formula $r_e = \left(1+\frac{r}{n}\right)^n - 1$.

Let $n = 12$ when $r = 0.11$:

$r_e = \left(1+\frac{0.11}{12}\right)^{12} - 1 \approx 0.1157$. Hence, when the nominal rate $r = 11\%$ is compounded monthly, the effective rate is $r_e = 11.57\%$. When $r = 0.1125$: $r_e = \left(1+\frac{0.1125}{4}\right)^4 - 1 \approx 0.1173$.

Hence in the second case when the nominal rate $r = 11.25\%$ is compounded quarterly, the effective rate is $r_e = 11.73\%$. This is the better effective rate of interest. To find the better investment, compare the compound amounts, S at the end of n years. With $P = 10{,}000$ and $r_e = 0.1157$,

$S_1 = P(1+r)^n = 10,000(1+0.1157)^n$, and, in the second case, when $P = 9700$ and $r_e = 0.1173$

$S_2 = P(1+r)^n = 9700(1+0.1173)^n$.

$S_1(20) = 10,000(1.1157)^{20} \approx 89,319.99$

$S_2(20) = 9700(1.1173)^{20} \approx 89,159.52$

The \$10,000 investment is slightly better over 20 years.

Problems 5.1

1. a. $6000(1.08)^8 \approx \$11,105.58$

b. $11,105.58 - 6000 = \$5105.58$

2. **a.** 750(1.07) = \$802.50

b. 802.5 – 750 = \$52.50

3. $(1.015)^2 - 1 \approx 0.030225$ or 3.023%

4. $\left(1+\frac{0.05}{4}\right)^4 - 1 = (1.0125)^4 - 1 \approx 0.05095$ or 5.095%

5. $\left(1+\frac{0.04}{365}\right)^{365} - 1 \approx 0.04081$ or 4.081%

6. $\left(1+\frac{0.06}{365}\right)^{365} - 1 \approx 0.06183$ or 6.183%

7. **a.** A nominal rate compounded yearly is the same as the effective rate, so the effective rate is 10%.

b. $\left(1+\frac{0.10}{2}\right)^2 - 1 = 0.1025$ or 10.25%

c. $\left(1+\frac{0.10}{4}\right)^4 - 1 \approx 0.10381$ or 10.381%

d. $\left(1+\frac{0.10}{12}\right)^{12} - 1 \approx 0.10471$ or 10.471%

e. $\left(1+\frac{0.10}{365}\right)^{365} - 1 \approx 0.10516$ or 10.516%

8. **a.** (i) $1000\left(1+\frac{0.07}{4}\right)^{4(5)} - 1000 \approx \$414.78$

(ii) $\left(1+\frac{0.07}{4}\right)^4 - 1 \approx 0.07186$ or 7.186%

b. (i) $1000\left(1+\frac{0.07}{12}\right)^{12(5)} - 1000 \approx \$417.63$

(ii) $\left(1+\frac{0.07}{12}\right)^{12} - 1 \approx 0.07229$ or 7.229%

c. (i) $1000\left(1+\frac{0.07}{52}\right)^{52(5)} - 1000 \approx \$418.73$

(ii) $\left(1+\frac{0.07}{52}\right)^{52} - 1 \approx 0.07246$ or 7.246%

d. (i) $1000\left(1+\frac{0.07}{365}\right)^{365(5)} - 1000 \approx \$419.02$

(ii) $\left(1+\frac{0.07}{365}\right)^{365} - 1 \approx 0.07250$ or 7.250%

9. Let r_e be the effective rate. Then

$$2000(1+r_e)^5 = 2950$$
$$(1+r_e)^5 = \frac{2950}{2000}$$
$$1+r_e = \sqrt[5]{\frac{2950}{2000}}$$
$$r_e = \sqrt[5]{\frac{2950}{2000}} - 1$$
$$r_e \approx 0.0808 \text{ or } 8.08\%.$$

10. Let r be the monthly rate. Then

$$(1+r)^{84} = 1835$$
$$(1+r)^{84} = \frac{1835}{1000}$$
$$1+r = \sqrt[84]{\frac{1835}{1000}}$$
$$r = \sqrt[84]{\frac{1835}{1000}} - 1$$
$$r = 0.0072529$$

This gives a nominal rate of approximately 12(0.0072529) = 0.0870 ≈ 8.70% compounded monthly.

11. From Example 6, the number of years, *n*, is given by $n = \frac{\ln 2}{\ln(1.09)} \approx 8.0$ years.

12. From Example 6, the number of years, *n*, is given by $n = \frac{\ln 2}{\ln(1.05)} \approx 14.2$ years.

13. $6000(1.08)^7 \approx \$10,282.95$

14. $3P = P(1+r)^n$

$3 = (1+r)^n$

$\ln 3 = n \ln(1 + r)$

$n = \frac{\ln 3}{\ln(1+r)}$

15. $21,500(1.06)^{10} \approx \$38,503.23$

16. $21,500\left(1+\frac{0.02}{4}\right)^{40} \approx \$26,247.08$

17. **a.** $(0.015)(12) = 0.18$ or 18%

b. $(1.015)^{12} - 1 \approx 0.1956$ or 19.56%

18. $2P = P(1.01)^n$

$2 = (1.01)^n$

$\ln 2 = n \ln(1.01)$

$n = \frac{\ln 2}{\ln(1.01)} \approx 70$ months

19. The compound amount after the first four years is $2000(1.06)^4$. After the next four years the compound amount is $\left[2000(1.06)^4\right](1.03)^8 \approx \$3198.54$.

20. $700 = 500(1.02)^n$

$1.4 = (1.02)^n$

$\ln(1.4) = n \ln(1.02)$

$n = \frac{\ln(1.4)}{\ln(1.02)} \approx 17$ quarters or 4 years, 3 months

21. 7.8% compounded semiannually is equivalent to an effective rate of $(1.039)^2 - 1 = 0.079521$ or 7.9521%. Thus 8% compounded annually, which is the effective rate, is the better rate.

22. Let r be the required nominal rate.

$\left(1+\frac{r}{12}\right)^{12} - 1 = 0.045$

$\left(1+\frac{r}{12}\right)^{12} = 1.045$

$1+\frac{r}{12} = \sqrt[12]{1.045}$

$\frac{r}{12} = \sqrt[12]{1.045} - 1$

$r = 12\left[\sqrt[12]{1.045} - 1\right] \approx 0.0441$

or 4.41%.

23. **a.** $\left(1+\frac{0.0475}{360}\right)^{365} - 1 \approx 0.0493$ or 4.93%

b. $\left(1+\frac{0.0475}{365}\right)^{365} - 1 \approx 0.0486$ or 4.86%

24. Let r be the nominal rate.

$801.06 = 700\left(1+\frac{r}{4}\right)^8$

$1+\frac{r}{4} = \sqrt[8]{\frac{801.06}{700}}$

$r = 4\left(\sqrt[8]{\frac{801.06}{700}} - 1\right) \approx 0.0680$ or 6.80%

25. Let r_e = effective rate.

$300,000 = 100,000(1+r_e)^{10}$

$(1+r_e)^{10} = 3$

$1+r_e = \sqrt[10]{3}$

$r_e = \sqrt[10]{3} - 1 \approx 0.1161$ or 11.61%.

26. Let P = average price of such a good,
n = number of days.

$$2P = P\left(1+\frac{0.0725}{365}\right)^n$$

$$2 = \left(1+\frac{0.0725}{365}\right)^n$$

$$\ln 2 = n\ln\left(1+\frac{0.0725}{365}\right)$$

$$n = \frac{\ln 2}{\ln\left(1+\frac{0.0725}{365}\right)} \approx 3489.98 \text{ days}$$

or ≈ 9.56 years

27. Let r = the required nominal rate.

$$420\left(1+\frac{r}{2}\right)^{28} = 1000$$

$$\left(1+\frac{r}{2}\right)^{28} = \frac{1000}{420} = \frac{50}{21}$$

$$1+\frac{r}{2} = \sqrt[28]{\frac{50}{21}}$$

$$r = 2\left[\sqrt[28]{\frac{50}{21}}-1\right] \approx 0.0629 \text{ or } 6.29\%$$

28. $1000(1-0.01)^{20} = 1000(0.99)^{20} \approx \$817.91$

Problems 5.2

1. $6000(1.05)^{-20} \approx \$2261.34$

2. $3500(1.06)^{-8} \approx \$2195.94$

3. $4000(1.035)^{-24} \approx \$1751.83$

4. $1740(1.015)^{-24} = \$1217.21$

5. $9000\left(1+\frac{0.08}{4}\right)^{-22} \approx \$5821.55$

6. $6000\left(1+\frac{0.10}{2}\right)^{-13} \approx \$3181.93$

7. $8000\left(1+\frac{0.10}{12}\right)^{-60} \approx \$4862.31$

8. $500\left(1+\frac{0.0875}{4}\right)^{-12} \approx \$385.65$

9. $10,000\left(1+\frac{0.095}{365}\right)^{-4(365)} \approx \$6838.95$

10. $1250\left(1+\frac{0.135}{52}\right)^{-78} \approx \$1021.13$

11. $12,000\left(1+\frac{0.053}{12}\right)^{-12} \approx \$11,381.89$

12. $12,000\left(1+\frac{0.071}{2}\right)^{-2} \approx \$11,191.31$

13. $27,000(1.03)^{-22} \approx \$14,091.10$

14. $550(1.025)^{-16} + 550(1.025)^{-20} \approx \$706.14$

15. Let x be the payment 2 years from now. The equation of value at year 2 is

$$x = 600(1.04)^{-2} + 800(1.04)^{-4}$$

$$x \approx \$1238.58$$

16. Let x be the payment at the end of 5 years. The equation of value at year 5 is

$$3000\left(1+\frac{0.08}{12}\right)^{60}+x=7000$$

$$x=7000-3000\left(1+\frac{0.08}{12}\right)^{60}$$

$$x\approx \$2530.46$$

17. Let x be the payment at the end of 6 years. The equation of value at year 6 is

$$2000(1.025)^4+4000(1.025)^2+x=5000(1.025)+5000(1.025)^{-4}$$

$$x=5000(1.025)+5000(1.025)^{-4}-2000(1.025)^4-4000(1.025)^2$$

$x\approx$ \$3244.63.

18. Let x be the amount of each of the equal payments. The equation of value at year 3 is

$$1500(1.07)^3+x(1.07)^2+x(1.07)+x=3500(1.07)^{-1}+5000(1.07)^{-3}$$

$$x[(1.07)^2+1.07+1]=3500(1.07)^{-1}+5000(1.07)^{-3}-1500(1.07)^3$$

$$x=\frac{3500(1.07)^{-1}+5000(1.07)^{-3}-1500(1.07)^3}{(1.07)^2+2.07}$$

$$x\approx \$1715.44$$

19. **a.** $NPV=8000(1.025)^{-6}+10,000(1.025)^{-8}+14,000(1.025)^{-12}-25,000\approx \$515.62$

b. Since $NPV>0$, the investment is profitable.

20. **a.** $NPV=8000(1.03)^{-6}+10,000(1.03)^{-8}+14,000(1.03)^{-12}-25,000\approx -\$586.72$

b. Since $NPV<0$, the investment is not profitable.

21. We consider the value of each investment at the end of eight years. The savings account has a value of $10,000(1.03)^{16}\approx \$16,047.06$.
The business investment has a value of \$16,000. Thus the better choice is the savings account.

22. The payments due B are $1000(1.07)^5$ at year 5 and $2000(1.04)^{14}$ at year 7. Let x be the payment at the end of 6 years. The equation of value at year 6 is $x=1000(1.07)^5(1.015)^4+2000(1.04)^{14}(1.015)^{-4}$ $x\approx$ \$4751.73

23. $1000\left(1+\frac{0.075}{4}\right)^{-80}\approx \$226.25$

24. $6500\left(1+\frac{0.058}{360}\right)^{-1460}\approx \$5137.67$

25. Let r be the nominal discount rate, compounded quarterly. Then

$$4700 = 10,000\left(1+\frac{r}{4}\right)^{-32}$$

$$4700 = \frac{10,000}{\left(1+\frac{r}{4}\right)^{32}}$$

$$\left(1+\frac{r}{4}\right)^{32} = \frac{10,000}{4700} = \frac{100}{47}$$

$$1+\frac{r}{4} = \sqrt[32]{\frac{100}{47}}$$

$$r = 4\left[\sqrt[32]{\frac{100}{47}} - 1\right] \approx 0.0955 \text{ or } 9.55\%$$

Problems 5.3

1. $S = 4000e^{0.0625(6)} \approx \$5819.97$
$5819.97 - 4000 = \$1819.97$

2. $S = 4000e^{0.09(6)} \approx \$6864.03$
$6864.03 - 4000 = \$2864.03$

3. $P = 2500e^{-0.0675(8)} \approx \$1456.87$

4. $P = 2500e^{-0.08(8)} \approx \$1318.23$

5. $e^{0.04} - 1 \approx 0.0408$
Answer: 4.08%

6. $e^{0.08} - 1 \approx 0.0833$
Answer: 8.33%

7. $e^{0.03} - 1 \approx 0.0305$
Answer: 3.05%

8. $e^{0.11} - 1 = 0.1163$
Answer: 11.63%

9. $S = 100e^{0.045(2)} \approx \$109.42$

10. $S = 1000e^{0.03(8)} \approx \$1271.25$

11. $P = 1,000,000e^{-0.05(5)} \approx \$778,800.78$

12. $P = 50,000e^{-0.06(30)} \approx \$8264.94$

13. **a.** $25,000(1+0.035)^{25} = \$59,081$

b. $P = 59,081e^{-(0.045)(25)} \approx \$19,181$

14. With option (a), after 18 months they have $50,000(1+0.0125)^6 \approx \$53,869.16$ with option (b), they have $50,000e^{(0.045)(1.5)} \approx \$53,491.51$.

15. Effective rate $= e^r - 1$. Thus $0.05 = e^r - 1$, $e^r = 1.05$, $r = \ln 1.05 \approx 0.0488$.
Answer: 4.88%

16. If r is the annual rate compounded continuously, then at the end of 1 year the compound amount of a principal of P dollars is $Pe^{r(1)} = Pe^r$. This amount must equal the compound amount of P dollars at a nominal rate of 6% compounded semiannually, which is $P(1.03)^2$. Thus

$$Pe^r = P(1.03)^2$$

$$e^r = (1.03)^2$$

$$r = \ln(1.03)^2$$

$$r = 2\ln 1.03 \approx 0.0591$$

Answer: 5.91%

17. $3P = Pe^{0.07t}$

$$3 = e^{0.07t}$$

$$0.07t = \ln 3$$

$$t = \frac{\ln 3}{0.07} \approx 16$$

Answer: 16 years

18. $4P = Pe^{r(30)}$

$$4 = e^{30r}$$

$$30r = \ln 4$$

$$r = \frac{\ln 4}{30} \approx 0.046$$

Answer: 5%

19. The accumulated amounts under each option are:

a. $1000e^{(0.035)(2)} \approx \$1072.51$

b. $1020(1.0175)^4 \approx \$1093.30$

c. $500e^{(0.035)(2)} + 500(1.0175)^4$
$\approx 536.25 + 535.93 = \$1072.18$

20. **a.** On Nov. 1, 2006 the accumulated amount is $10,000e^{(0.04)(10)} \approx \$14,918.25$.
On Nov. 1, 2011 the accumulated amount is $14,918.25(1.05)^5 \approx \$19,039.89$.

b. $10,000(1.045)^{15} \approx \$19,352.82$, which is \$312.93 more than the amount in part (a).

21. **a.** $9000(1.0125)^4 \approx \$9458.51$

b. After one year the accumulated amount of the investment is
$10,000e^{0.055} \approx \$10,565.41$. The payoff for the loan (including interest) is
1000 + 1000(0.08) = \$1080. The net return is 10,565.41 – 1080 = \$9485.41.
Thus, this strategy is better by
9485.41 – 9458.51 = \$26.90.

Principles in Practice 5.4

1. Let $a = 64$ and let $r = \frac{3}{4}$. Then, the next five heights of the ball are $64\left(\frac{3}{4}\right)$, $64\left(\frac{3}{4}\right)^2$, $64\left(\frac{3}{4}\right)^3$, $64\left(\frac{3}{4}\right)^4$, $64\left(\frac{3}{4}\right)^5$, or 48 ft, 36 ft, 27 ft, $20\frac{1}{4}$ ft, and $15\frac{3}{16}$ ft.

2. Let $a = 500$ and let $r = 1.5$. Then, the number of bacteria at the end of each minute for the first six minutes is 500(1.5), $500(1.5)^2$, $500(1.5)^3$, $500(1.5)^4$, $500(1.5)^5$, $500(1.5)^6$, or 750, 1125, 1688, 2531, 3797, 5695.

3. The total vertical distance traveled in the air after n bounces is equal to 2 times the sum of heights. If $a = 6$ and $r = \frac{2}{3}$, then when the ball hits the ground for the twelfth time, $n = 12$ and the distance traveled in the air is

$$2s = 2\left[\frac{a\left(1-r^n\right)}{1-r}\right] = 2\left[\frac{6\left(1-\left(\frac{2}{3}\right)^{12}\right)}{1-\frac{2}{3}}\right]$$

$\approx$ 35.72 meters

4. The amount of profit earned in the first two years is the sum of the monthly profits.
Let $a = 2000$,
$r = 1.1$, and $n = 24$.

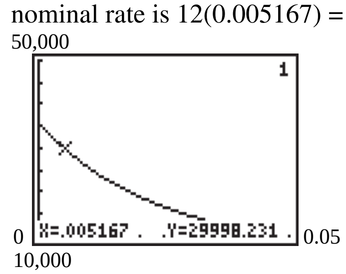
$$s = \frac{2000\left(1-(1.1)^{24}\right)}{1-1.1} \approx 176,994.65$$

Thus, the company earned \$176,994.65 in the first two years.

5. Let $R = 500$ and let $n = 72$. Then, the present value A of the annuity is given by

$$A = R\left(\frac{1-(1+r)^{-n}}{r}\right) = 500\left(\frac{1-(1+r)^{-72}}{r}\right)$$

By graphing A as a function of r, we find that when $r \approx 0.005167$, $A = 30,000$. Thus, if the present value of the annuity is \$30,000, the monthly interest rate is 0.5167%, and the nominal rate is 12(0.005167) = 0.062 or 6.2%.



6. Since the man pays \$2000 for 6 years and \$3500 for 8 years, we can consider the payments to be an annuity of \$3500 for 14 years minus an annuity of \$1500 for 6 years so that the first 24 payments are \$2000 each. Thus, the present value is

$$3500a_{\overline{56}|0.015} - 1500a_{\overline{24}|0.015}$$
$$\approx 3500(37.705879) - 1500(20.030405)$$
$$= 101,924.97$$

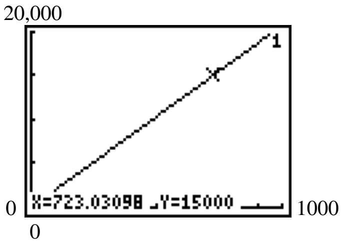
Thus, the present value of the payments is \$101,925. Since the man made an initial down payment of \$20,000, list price was 101,925 + 20,000 = \$121,925.

7. Let $r = \frac{0.048}{4} = 0.012$, and $n = 24$.

$$A = R\left(\frac{1-(1+r)^{-n}}{r}\right)$$

$$A = R\left(\frac{1-(1+0.012)^{-24}}{0.012}\right) = R\left(\frac{1-(1.012)^{-24}}{0.012}\right)$$

By Graphing A as a function of R, we find that when $R = 723.03$, $A = 15,000$. Thus the monthly payment is \$723.03 if the present value of the annuity is $15,000.



8. Find the annuity due. The man makes an initial payment of $1200 followed by an ordinary annuity of $1200 for 11 months. Thus, let $R = 1200$, $n = 11$, and $r = \frac{0.068}{12}$. The present value of the annuity due is

$$1200\left(1 + a_{\overline{11}|\frac{0.068}{12}}\right) \approx 1200(1 + 10.635005)$$

$$\approx 13{,}962.01$$

Thus, he should pay $13,962.01.

9. Let $R = 2000$ and let $r = 0.057$. Then, the value of the IRA at the end of 15 years, when $n = 15$, is given by

$$S = R\left(\frac{(1+r)^n - 1}{r}\right)$$

$$S = 2000\left(\frac{(1+0.057)^{15} - 1}{0.057}\right) \approx 45{,}502.06$$

Thus, at the end of 15 years the IRA will be worth $45,502.06.

10. Let $R = 2000$ and let $r = 0.057$. Since the deposits are made at the beginning of each year, the value of the IRA at the end of 15 years is given by

$$S = R\left(\frac{(1+r)^{n+1} - 1}{r}\right) - R.$$

Let $n = 15$.

$$S = 2000\left(\frac{(1+0.057)^{16} - 1}{0.057}\right) - 2000 \approx 48{,}095.67$$

Thus, the IRA is worth $48,095.67 at the end of 15 years.

Problems 5.4

1. 64

$$64\left(\frac{1}{2}\right) = 32$$

$$64\left(\frac{1}{2}\right)^2 = 16$$

$$64\left(\frac{1}{2}\right)^3 = 8$$

$$64\left(\frac{1}{2}\right)^4 = 4$$

2. 2

$$2(-3) = -6$$

$$2(-3)^2 = 18$$

$$2(-3)^3 = -54$$

3. 100

$$100(1.02) = 102$$

$$100(1.02)^2 = 104.04$$

4. 81

$$81\left(\frac{1}{3}\right) = 27$$

$$81\left(\frac{1}{3}\right)^2 = 9$$

$$81\left(\frac{1}{3}\right)^3 = 3$$

5. $$s = \frac{\frac{4}{7}\left[1 - \left(\frac{4}{7}\right)^5\right]}{1 - \frac{4}{7}} = \frac{\frac{4}{7}\left[\frac{15{,}783}{16{,}807}\right]}{\frac{3}{7}} = \frac{21{,}044}{16{,}807}$$

6. $$s = \frac{1\left[1 - \left(\frac{1}{5}\right)^7\right]}{1 - \frac{1}{5}} = \frac{\frac{78{,}124}{78{,}125}}{\frac{4}{5}} = \frac{19{,}531}{15{,}625}$$

7. $s = \dfrac{1\left[1-(0.1)^6\right]}{1-0.1} = 1.11111$

8. Observe that $(1.1)^{-1} = \dfrac{1}{1.1} = \dfrac{10}{11}$. Thus $a = \dfrac{10}{11}$ and $r = \dfrac{10}{11}$.

$$s = \frac{\frac{10}{11}\left[1-\left(\frac{10}{11}\right)^6\right]}{1-\frac{10}{11}} = 10\left[1-\left(\frac{10}{11}\right)^6\right] \approx 4.355$$

9. $a_{\overline{35}|0.04} \approx 18.664613$

10. $a_{\overline{15}|0.07} \approx 9.107914$

11. $s_{\overline{8}|0.0075} \approx 8.213180$

12. $s_{\overline{11}|0.0125} \approx 11.713937$

13. $600a_{\overline{6}|0.06} \approx 600(4.917324) \approx \$2950.39$

14. $1000a_{\overline{8}|0.05} \approx 1000(6.463213) \approx 6463.21$

15. $2000a_{\overline{18}|0.02} \approx 2000(14.992031) \approx \$29,984.06$

16. $1500a_{\overline{15}|0.0075} \approx 1500(14.136995) \approx \$21,205.49$

17. $800 + 800a_{\overline{11}|0.035} \approx 800 + 800(9.001551)$
$\approx \$8001.24$

18. $150 + 150a_{\overline{59}|\frac{0.07}{12}} \approx 150 + 150(49.796588)$
$\approx \$7619.49$

19. $2000s_{\overline{36}|0.0125} \approx 2000(45.115505)$
$\approx \$90,231.01$

20. $600s_{\overline{16}|0.02} \approx 600(18.639285) \approx \$11,183.57$

21. $5000s_{\overline{20}|0.07} \approx 5000(40.995492) \approx \$204,977.46$

22. $2000s_{\overline{20}|0.03} \approx 2000(26.870374) \approx 53,740.75$

23. $1200\left(s_{\overline{13}|0.08} - 1\right) \approx 1200(21.495297 - 1)$
$\approx \$24,594.36$

24. $600\left(s_{\overline{31}|0.025} - 1\right) \approx 600(46.000271 - 1)$
$\approx \$27,000.16$

25. $175a_{\overline{32}|\frac{0.04}{12}} - 25a_{\overline{8}|\frac{0.04}{12}}$
$\approx 175(30.304595) - 25(7.881321)$
$\approx \$5106.27$

26. $1500 + 1500a_{\overline{5}|0.0075} \approx 1500 + 1500(4.889440)$
$\approx \$8834.16$

27. $R = \dfrac{5000}{a_{\overline{12}|0.015}} \approx \dfrac{5000}{10.907505} \approx \$458.40$

28. $3000 + 250a_{\overline{12}|0.04} \approx 3000 + 250(9.385074)$
$\approx \$5346.27$

29. a. $\left(50s_{\overline{48}|0.005}\right)(1.005)^{24}$
$\approx 50(54.097832)(1.005)^{24}$
$\approx \$3048.85$

b. $3048.85 - 48(50) = \$648.85$

30. Let R be the yearly payment.
$275,000 = R + Ra_{\overline{9}|0.035}$

$275,000 = R\left(1 + a_{\overline{9}|0.035}\right)$

$275,000 \approx R(8.607687)$,
$R \approx \$31,948.19$

31. $R = \dfrac{48,000}{s_{\overline{10}|0.07}} \approx \dfrac{48,000}{13.816448} \approx \$3474.12$

32. Let x be the purchase price. In the same manner as in Example 12,

$$[50,000-0.08x]s_{\overline{10}|0.06}=x$$

$$50,000-0.08x=\frac{x}{s_{\overline{10}|0.06}}$$

$$50,000=0.08x+\frac{x}{s_{\overline{10}|0.06}}$$

$$50,000=x\left(0.08+\frac{1}{s_{\overline{10}|0.06}}\right)$$

$$x=\frac{50,000}{0.08+\frac{1}{s_{\overline{10}|0.06}}}\approx\frac{50,000}{0.08+\frac{1}{13.180795}}\approx\$320,800.$$

33. The original annual payment is $\frac{25,000}{s_{\overline{10}|0.06}}$**.** After six years the value of the fund is

$$\frac{25,000}{s_{\overline{10}|0.06}}s_{\overline{6}|0.06}.$$

This accumulates to

$$\left[\frac{25,000}{s_{\overline{10}|0.06}}s_{\overline{6}|0.06}\right](1.07)^4.$$

Let x be the amount of the new payment.

$$xs_{\overline{4}|0.07}=25,000-\left[\frac{25,000}{s_{\overline{10}|0.06}}s_{\overline{6}|0.06}(1.07)^4\right]$$

$$x=\frac{25,000-\left[\frac{25,000}{s_{\overline{10}|0.06}}s_{\overline{6}|0.06}(1.07)^4\right]}{s_{\overline{4}|0.07}}$$

$$x\approx\frac{25,000-\left[\frac{25,000}{13.180795}(6.975319)(1.07)^4\right]}{4.439943}$$

$x\approx\$1725$

34. Let x be the final payment.

$$5000=1000a_{\overline{5}|0.08}+x(1.08)^{-6}$$

$$5000-1000a_{\overline{5}|0.08}=x(1.08)^{-6}$$

Thus

$$x=(1.08)^6\left(5000-1000a_{\overline{5}|0.08}\right)$$

$$\approx(1.08)^6[5000-1000(3.992710)]\approx\$1598.44$$

35. $s_{\overline{60}|0.017}=\frac{(1.017)^{60}-1}{0.017}\approx102.91305$

36. $a_{\overline{9}|0.052}=\frac{1-(1.052)^{-9}}{0.052}\approx7.04494$

37. $750a_{\overline{480}|0.0135}=750\left[\frac{1-(1.0135)^{-480}}{0.0135}\right]$

$$\approx55,466.57$$

38. $1000s_{\overline{120}|0.01}=1000\left[\frac{(1.01)^{120}-1}{0.01}\right]$

$$\approx230,038.69$$

39. $R=\frac{3000}{s_{\overline{20}|0.01375}}=\frac{3000(0.01375)}{(1.01375)^{20}-1}\approx\$131.34$

40. $R=\frac{25,000}{a_{\overline{60}|\frac{0.1}{12}}}=\frac{25,000\left(\frac{0.1}{12}\right)}{1-\left(1+\frac{0.1}{12}\right)^{-60}}\approx\$531.18$

41. $200,000+200,000a_{\overline{19}|0.10}$

$$=200,000+200,000\left[\frac{1-(1.10)^{-19}}{0.10}\right]$$

$$\approx\$1,872,984.02$$

42. **a.** $\$650(12)(15)=\$117,000$

b. $650a_{\overline{180}|\frac{0.055}{12}}=650\left[\frac{1-\left(1+\frac{0.055}{12}\right)^{-180}}{\frac{0.055}{12}}\right]$

$$\approx\$79,551.24$$

43. For the first situation, the compound amount is

$$\left[2000\left(s_{\overline{11}|0.07}-1\right)\right](1.07)^{30}$$

$$=2000\left[\frac{(1.07)^{11}-1}{0.07}-1\right](1.07)^{30}$$

≈ \$225,073,
so the net earnings are
225,073 – 20,000 = \$205,073.

For the second situation, the compound amount is

$$2000\left(s_{\overline{31}|0.07}-1\right)=2000\left[\frac{(1.07)^{31}-1}{0.07}-1\right]$$

$\approx \$202,146$,

so the net earnings are 202,146 – 60,000 = \$142,146.

44. $100\dfrac{1-e^{-0.05(20)}}{0.05}\approx \$1264$

45. $40,000\dfrac{1-e^{-0.04(5)}}{0.04}\approx \$181,269.25$

Problems 5.5

1. $R=\dfrac{8000}{a_{\overline{36}|\frac{0.14}{12}}}\approx\dfrac{8000}{29.258904}\approx \$273.42$

2. $A=50a_{\overline{36}|0.01}\approx 50(30.107505)\approx \$1505.38$

3. $R=\dfrac{8000}{a_{\overline{36}|\frac{0.04}{12}}}\approx\dfrac{8000}{33.870766}\approx \$236.19$

Finance charge = 36(236.19) – 8000 = \$502.84

4. a. $R=\dfrac{500}{a_{\overline{12}|0.0125}}\approx\dfrac{500}{11.079312}\approx \$45.13$

b. 12(45.13) – 500 = \$41.56

5. a. $R=\dfrac{7500}{a_{\overline{36}|\frac{0.04}{12}}}\approx\dfrac{7500}{33.870766}\approx \$221.43$

b. $7500\dfrac{0.04}{12}=\$25$

c. 221.43 – 25 = \$196.43

6. a. $R=\dfrac{35,000}{a_{\overline{48}|\frac{0.078}{12}}}\approx\dfrac{35,000}{41.119856}\approx \$851.17$

b. $35,000\dfrac{0.078}{12}=\$227.50$

c. 851.17 – 227.50 = \$623.67

7. $R = \frac{5000}{a_{\overline{4}|0.07}} \approx \frac{5000}{3.387211} \approx \$1476.14$

The interest for the first period is (0.07)(5000) = \$350, so the principal repaid at the end of that period is 1476.14 – 350 = \$1126.14. The principal outstanding at the beginning of period 2 is 5000 – 1126.14 = \$3873.86. The interest for period 2 is (0.07)(3873.86) = \$271.17, so the principal repaid at the end of that period is 1476.14 – 271.17 = \$1204.97. The principal outstanding at beginning of period 3 is 3873.86 – 1204.97 = \$2668.89. Continuing in this manner, we construct the following amortization schedule.

Period	Prin. Outs. at Beginning	Int. for Period	Pmt. at End	Prin. Repaid at End
1	5000.00	350.00	1476.14	1126.14
2	3873.86	271.17	1476.14	1204.97
3	2668.89	186.82	1476.14	1289.32
4	1379.57	96.57	1476.14	1379.57
Total		904.56	5904.56	5000.00

8. $R = \frac{9000}{a_{\overline{8}|0.0475}} \approx \frac{9000}{6.529036} \approx \$1378.46$

The interest for the first period is (0.0475)(9000) = \$427.50, so the principal repaid at the end of that period is 1378.46 – 427.50 = \$950.96. The principal outstanding at the beginning of period 2 is 9000 – 950.96 = \$8049.04. The interest for period 2 is (0.0475)(8049.04) = \$382.33, so the principal repaid at the end of that period is 1378.46 – 382.33 = \$996.13. The principal outstanding at beginning of period 3 is 8049.04 – 996.13 = \$7052.91. Continuing in this manner, we construct the following amortization schedule. Note the adjustment in the final payment.

Period	Prin. Outs. at Beginning	Int. for Period	Pmt. at End	Prin. Repaid at End
1	9000.00	427.50	1378.46	950.96
2	8049.04	382.33	1378.46	996.13
3	7052.91	335.01	1378.46	1043.45
4	6009.46	285.45	1378.46	1093.01
5	4916.45	233.53	1378.46	1144.93
6	3771.52	179.15	1378.46	1199.31
7	2572.21	122.18	1378.46	1256.28
8	1315.93	62.51	1378.44	1315.93
Total		2027.66	11,027.66	9000.00

9. $R = \frac{900}{a_{\overline{5}|0.025}} \approx \frac{900}{4.645828} \approx \$193.72$

The interest for period 1 is (0.025)(900) = \$22.50, so the principal repaid at the end of that period is 193.72 – 22.50 = \$171.22. The principal outstanding at the beginning of period 2 is 900 – 171.22 = \$728.78. The interest for that period is (0.025)(728.78) = \$18.22, so the principal repaid at the end of that period is 193.72 – 18.22 = \$175.50. The principal outstanding at the beginning of period 3 is 728.78 – 175.50 = \$553.28. Continuing in this manner, we obtain the following amortization schedule. Note the adjustment in the final payment.

Period	Prin. Outs. at Beginning	Int. for Period	Pmt. at End	Prin. Repaid at End
1	900.00	22.50	193.72	171.22
2	728.78	18.22	193.72	175.50
3	553.28	13.83	193.72	179.89
4	313.39	9.33	193.72	184.39
5	189.00	4.73	193.73	189.00
Total		68.61	968.61	900.00

10. $R = \frac{10{,}000}{a_{\overline{5}|0.0075}} \approx \frac{10{,}000}{4.889440} \approx \$2045.22$

The interest for period 1 is (0.0075)(10,000) = \$75, so the principal repaid at the end of that period is 2045.22 – 75 = \$1970.22. The principal outstanding at the beginning of period 2 is 10,000 – 1970.22 = \$8029.78. The interest for period 2 is (0.0075)(8029.78) = \$60.22, so the principal repaid at the end of that period is 2045.22 – 60.22 = \$1985. The principal outstanding at the beginning of period 3 is 8029.78 – 1985 = \$6044.78. Continuing in this manner, we construct the following amortization schedule. Note the adjustment in the final payment.

Period	Prin. Outs. at Beginning	Int. for Period	Pmt. at End	Prin. Repaid at End
1	10,000.00	75.00	2045.22	1970.22
2	8029.78	60.22	2045.22	1985.00
3	6044.78	45.34	2045.22	1999.88
4	4044.90	30.34	2045.22	2014.88
5	2030.02	15.23	2045.25	2030.02
Total		226.13	10,226.13	10,000.00

11. From Eq. (1),

$$n = \frac{\ln\left[\frac{100}{100-1000(0.02)}\right]}{\ln(1.02)} \approx 11.268\,.$$

Thus the number of full payments is 11.

12. a. $\frac{2000}{a_{\overline{48}|0.01}} \approx \frac{2000}{37.973959} \approx \$52.67$

b. $52.67a_{\overline{13}|0.01} \approx 52.67(12.133740)$

$\approx \$639.08$

c. $(639.08)(0.01) \approx \$6.39$

d. $52.67 - 6.39 = \$46.28$

e. $48(52.67) - 2000 = \$528.16$

13. Each of the original payments is $\dfrac{18,000}{a_{\overline{15}|0.035}}$.

After two years the value of the remaining payments is $\left[\dfrac{18,000}{a_{\overline{15}|0.035}}\right]a_{\overline{11}|0.035}$. Thus the new semi-annual payment is

$$\frac{18,000a_{\overline{11}|0.035}}{a_{\overline{15}|0.035}} \cdot \frac{1}{a_{\overline{11}|0.04}}$$

$$= \frac{18,000(9.001551)}{11.517411} \cdot \frac{1}{8.760477}$$

$\approx \$1606.$

14. $R = \dfrac{2000}{a_{\overline{60}|0.014}} = \dfrac{2000(0.014)}{1-(1.014)^{-60}} \approx \$49.49$

15. a. Monthly interest rate is $\dfrac{0.092}{12}$.

Monthly payment is

$$\frac{245,000}{a_{\overline{300}|\frac{0.092}{12}}} = 245,000\left[\frac{\frac{0.092}{12}}{1-\left(1+\frac{0.092}{12}\right)^{-300}}\right]$$

$\approx \$2089.69$

b. $245,000\left(\dfrac{0.092}{12}\right) = \$1878.33$

c. $2089.69 - 1878.33 = \$211.36$

d. $300(2089.69) - 245,000 = \$381,907$

16. a. Monthly interest rate is $\dfrac{0.132}{12} = 0.011$.

Monthly payment is

$$\frac{8500}{a_{\overline{48}|0.011}} = 8500\left[\frac{0.011}{1-(1.011)^{-48}}\right]$$

$\approx \$228.88$

b. $48(228.88) - 8500 = \$2486.24$

17. $n = \dfrac{\ln\left[\frac{100}{100-2000(0.015)}\right]}{\ln 1.015} \approx 23.956$. Thus the number of full payments is 23.

18. $R = \dfrac{9500}{a_{\overline{60}|0.0077}} = 9500\left[\dfrac{0.0077}{1-(1.0077)^{-60}}\right]$

$\approx \$198.31$

19. Present value of mortgage payments is

$$600a_{\overline{360}|\frac{0.076}{12}} = 600\left[\frac{1-\left(1+\frac{0.076}{12}\right)^{-360}}{\frac{0.076}{12}}\right]$$

$\approx \$84,976.84$

This amount is 75% of the purchase price x.

$0.75x = 84,976.84$

$x = \$113,302.45 \approx \$113,302$

20. For the 15-year mortgage, the monthly payment is

$$\frac{240,000}{a_{\overline{180}|0.005}} = 240,000\left[\frac{0.005}{1-(1+0.005)^{-180}}\right]$$

$\approx \$2025.26$

The finance charge is

$180(2025.26) - 240,000 = \$124,546.80$

For the 25-year mortgage, the monthly payment is

$$\frac{240,000}{a_{\overline{300}|0.005}} = 240,000\left[\frac{0.005}{1-(1+0.005)^{-300}}\right]$$

$\approx \$1546.32$

The finance charge is

$300(1546.32) - 240,000 = 223,896.00$

Thus the savings is

$223,896.00 - 124,546.80 = \$99,349.20$

21. $\dfrac{25,000}{a_{\overline{60}|0.0125}} - \dfrac{25,000}{a_{\overline{60}|0.01}}$

$$= 25,000\left[\frac{1}{a_{\overline{60}|0.0125}} - \frac{1}{a_{\overline{60}|0.01}}\right]$$

$$= 25,000\left[\frac{0.0125}{1-(1.0125)^{-60}} - \frac{0.01}{1-(1.01)^{-60}}\right]$$

$\approx \$38.64$

22. The government's payment is

$(y-x)a_{\overline{60}|\frac{0.0925}{12}}$

$$= \left[\frac{5000}{a_{\overline{60}|\frac{0.0925}{12}}} - \frac{5000}{a_{\overline{60}|\frac{0.04}{12}}}\right] a_{\overline{60}|\frac{0.0925}{12}}$$

$$= 5000\left[1 - \frac{a_{\overline{60}|\frac{0.0925}{12}}}{a_{\overline{60}|\frac{0.04}{12}}}\right]$$

$$= 5000\left[1 - \frac{\frac{1-\left(1+\frac{0.0925}{12}\right)^{-60}}{\frac{0.0925}{12}}}{\frac{1-\left(1+\frac{0.04}{12}\right)^{-60}}{\frac{0.04}{12}}}\right]$$

$$= 5000\left[1 - \frac{1-\left(1+\frac{0.0925}{12}\right)^{-60}}{1-\left(1+\frac{0.04}{12}\right)^{-60}} \cdot \frac{0.04}{0.0925}\right]$$

$\approx$ \$589.89

Chapter 5 Review Problems

1. $s = 3 + 2 + 2 \cdot \frac{2}{3} + \cdots + 3\left(\frac{2}{3}\right)^5$

$$= \frac{3\left[1-\left(\frac{2}{3}\right)^6\right]}{1-\frac{2}{3}} = \frac{3\left[\frac{665}{729}\right]}{\frac{1}{3}} = \frac{665}{81}$$

2. $\left(1+\frac{0.05}{12}\right)^{12} - 1 \approx 0.0512$ or 5.12%

3. 8.2% compounded semiannually corresponds to an effective rate of $(1.041)^2 - 1 = 0.083681$ or 8.37%. Thus the better choice is 8.5% compounded annually.

4. NPV $= 3400(1.035)^{-4} + 3500(1.035)^{-8} - 7000$

$\approx -\$1379.16$

5. Let x be the payment at the end of 2 years. The equation of value at the end of year 2 is

$1000(1.04)^4 + x = 1200(1.04)^{-4} + 1000(1.04)^{-8}$

$x = 1200(1.04)^{-4} + 1000(1.04)^{-8} - 1000(1.04)^4$

$\approx \$586.60$

6. $250a_{\overline{48}|0.005} \approx 250(42.580318) \approx \$10,645.08$

7. a. $A = 200a_{\overline{13}|0.04} \approx 200(9.985648)$

$\approx \$1997.13$

b. $S = 200s_{\overline{13}|0.04} \approx 200(16.626838)$

$\approx \$3325.37$

8. $150s_{\overline{14}|0.04} - 150 = 150(18.291911) - 150$

≈ 2593.79

9. $200s_{\overline{13}|\frac{0.08}{12}} - 200 \approx 200(13.532926) - 200$

$\approx \$2506.59$

10. $250a_{\overline{20}|0.025} \approx 250(15.589162) \approx \$3897.29$

11. $\frac{5000}{s_{\overline{5}|0.06}} \approx \frac{5000}{5.637093} \approx \$886.98$

12. a. $\frac{7000}{a_{\overline{36}|\frac{0.04}{12}}} \approx \frac{7000}{33.870766} \approx \$206.67$

b. 36(206.67) – 7000 = \$440.12

13. Let x be the first payment. The equation of value now is

$x + 2x(1.07)^{-3} = 500(1.05)^{-3} + 500(1.03)^{-8}$

$x\left[1 + 2(1.07)^{-3}\right] = 500(1.05)^{-3} + 500(1.03)^{-8}$

$$x = \frac{500(1.05)^{-3} + 500(1.03)^{-8}}{1 + 2(1.07)^{-3}}$$

$x \approx \$314.00$

14. $R = \frac{3500}{a_{\overline{3}|0.01375}} = 3500\left[\frac{0.01375}{1-(1.01375)^{-3}}\right]$

$\approx \$1198.90$

The interest for the first period is (0.01375)(3500) = \$48.13, so the principal repaid at the end of that period is 1198.90 – 48.13 = \$1150.77. The principal outstanding at the beginning of period 2 is 3500 – 1150.77 = \$2349.23. The interest for that period is (0.01375)(2349.23) = \$32.30. The principal repaid at the end of that period is 1198.90 – 32.30 = \$1166.60. The principal outstanding at the beginning of period 3 is 2349.23 – 1166.60 = \$1182.63. Continuing, we obtain the following amortization schedule. Note the adjustment in the final payment.

Period	Prin. Outs. at Beginning	Int. for Period	Pmt. at End	Prin. Repaid at End
1	3500.00	48.13	1198.90	1150.77
2	2349.23	32.30	1198.90	1166.60
3	1182.63	16.26	1198.89	1182.63
Total		96.69	3596.69	3500.00

15. $R = \frac{15,000}{a_{\overline{5}|0.0075}} \approx \frac{15,000}{4.889440} \approx \$3067.84$

The interest for period 1 is (0.0075)(15,000) = \$112.50, so the principal repaid at the end of that period is 3067.84 – 112.50 = \$2955.34. The principal outstanding at beginning of period 2 is 15,000 – 2955.34 = \$12,044.66. The interest for period 2 is 0.0075(12,044.66) = \$90.33, so the principal repaid at the end of that period is 3067.84 – 90.33 = \$2977.51. Principal outstanding at the beginning of period 3 is 12,044.66 – 2977.51 = \$9067.15. Continuing, we obtain the following amortization schedule. Note the adjustment in the final payment.

Period	Prin. Outs. at Beginning	Int. for Period	Pmt. at End	Prin. Repaid at End
1	15,000	112.50	3067.84	2955.34
2	12,044.66	90.33	3067.84	2977.51
3	9067.15	68.00	3067.84	2999.84
4	6067.31	45.50	3067.84	3022.34
5	3044.97	22.84	3067.81	3044.97
Total		339.17	15,339.17	15,000.00

16. $540a_{\overline{84}|\frac{0.10}{12}} = 540\left[\frac{1-\left(1+\frac{0.10}{12}\right)^{-84}}{\frac{0.10}{12}}\right] \approx \$32,527.80$

17. The monthly payment is

$$\frac{11,000}{a_{\overline{48}|\frac{0.055}{12}}} = 11,000\left[\frac{\frac{0.055}{12}}{1-\left(1+\frac{0.055}{12}\right)^{-48}}\right] \approx \$255.82$$

The finance charge is 48(255.82) – 11,000 = \$1279.36

Mathematical Snapshot Chapter 5

1. $\frac{0.085}{2} = 0.0425$, thus $R = 0.0425(25{,}000) = 1062.50$.

$$P = 25{,}000(1.0825)^{-25} + 1062.50 \cdot \frac{1-(1.0825)^{-25}}{\sqrt{1.0825}-1}$$
$$\approx \$26{,}102.13$$

2. $\frac{0.065}{2} = 0.0325$, thus $R = 0.0325(10{,}000) = 325$.

On a graphics calculator, let $Y_1 = 10{,}389$ and $Y_2 = 10{,}000(1 + x)$^ − 7 + 325(1 − (1 + x)^ − 7)/$\left(\sqrt{\ }(1+x)-1\right)$.

The curves intersect at 0.0590. The yield is 5.9%.

3. The normal yield curve assumes a stable economic climate. By contrast, if investors are expecting a drop in interest rates, and with it a drop in yields from future investments, they will gladly give up liquidity for long-term investment at current, more favorable, interest rates. T-bills, which force the investor to find a new investment in a short time, are correspondingly less attractive, and so prices drop and yields rise.

Chapter 6

Principles in Practice 6.1

1. There are 3 rows, one for each source. There are two columns, one for each raw material. Thus, the size of the matrix is 3 × 2. Alternatively, she could use a 2 × 3 matrix.

2. The first column consists of 1's each representing the 1 hour needed for each phase of project 1. The second column consists of 2's for each phase of project 2 and so on. In general the *n*th column will consist of 2^n's, each representing the 2^n hours needed for each phase of project *n*. The time-analysis matrix is as follows.
$$\begin{bmatrix} 1 & 2 & 4 & 8 & 16 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 2 & 4 & 8 & 16 \end{bmatrix}$$

Problems 6.1

1. **a.** The size is the number of rows by the columns. Thus **A** is 2 × 3, **B** is 3 × 3, **C** is 3 × 2, **D** is 2 × 2, **E** is 4 × 4, **F** is 1 × 2, **G** is 3 × 1, **H** is 3 × 3, and **J** is 1 × 1.

   b. A square matrix has the same number of rows as columns. Thus the square matrices are **B**, **D**, **E**, **H**, and **J**.

   c. An upper triangular matrix is a square matrix where all entries below the main diagonal are zeros. Thus **H** and **J** are upper triangular. A lower triangular matrix is a square matrix where all entries above the main diagonal are zeros. Thus **D** and **J** are lower triangular.

   d. A row vector (or row matrix) has only one row. Thus **F** and **J** are row vectors.

   e. A column vector (or column matrix) has only one column. Thus **G** and **J** are column vectors.

2. **A** has 4 rows and 4 columns. Thus the order of **A** is 4.

3. a_{21} is the entry in the 2nd row and 1st column, namely 6.

4. a_{14} is the entry in the 1st row and 4th column, namely 6.

5. a_{32} is the entry in the 3rd row and 2nd column, namely 4.

6. a_{34} is the entry in the 3rd row and 4th column, namely 0.

7. a_{44} is the entry in the 4th row and 4th column, namely 0.

8. a_{55} is the entry in the 5th row and 5th column. But **A** has only 4 rows and 4 columns. Thus a_{55} does not exist.

9. The main diagonal entries are the entries on the diagonal extending from the upper left corner to the lower right corner. Thus the main diagonal entries are 7, 2, 1, 0.

10. $$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

11. $\begin{bmatrix} -2\cdot 1+3\cdot 1 & -2\cdot 1+3\cdot 2 & -2\cdot 1+3\cdot 3 & -2\cdot 1+3\cdot 4 & -2\cdot 1+3\cdot 5 \\ -2\cdot 2+3\cdot 1 & -2\cdot 2+3\cdot 2 & -2\cdot 2+3\cdot 3 & -2\cdot 2+3\cdot 4 & -2\cdot 2+3\cdot 5 \\ -2\cdot 3+3\cdot 1 & -2\cdot 3+3\cdot 2 & -2\cdot 3+3\cdot 3 & -2\cdot 3+3\cdot 4 & -2\cdot 3+3\cdot 5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 & 10 & 13 \\ -1 & 2 & 5 & 8 & 11 \\ -3 & 0 & 3 & 6 & 9 \end{bmatrix}$

12. $\begin{bmatrix} (-1)^{1+1}\left(1^2+1^2\right) & (-1)^{1+2}\left(1^2+2^2\right) \\ (-1)^{2+1}\left(2^2+1^2\right) & (-1)^{2+2}\left(2^2+2^2\right) \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -5 & 8 \end{bmatrix}$

13. 12 · 10 = 120, so **A** has 120 entries. For a_{33}, $i = 3 = j$, so $a_{33} = 1$. Since 5 ≠ 2, $a_{52} = 0$. For $a_{10,\,10}$, $i = 10 = j$, so $a_{10,\,10} = 1$. Since 12 ≠ 10, $a_{12,\,10} = 0$.

14. The main diagonal is the diagonal extending from the upper left corner to the lower right corner.

a. 1, 0, –5, 2

b. x, y, z

15. A zero matrix is a matrix in which all entries are zeros.

a. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

16. If **A** is 7 × 9, then $\mathbf{A}^T$ is 9 × 7.

17. $\mathbf{A}^T = \begin{bmatrix} 6 & -3 \\ 2 & 4 \end{bmatrix}^T = \begin{bmatrix} 6 & 2 \\ -3 & 4 \end{bmatrix}$

18. $\mathbf{A}^T = \begin{bmatrix} 2 & 4 & 6 & 8 \end{bmatrix}^T = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$

19. $\mathbf{A}^T = \begin{bmatrix} 1 & 3 & 7 & 3 \\ 3 & 2 & -2 & 0 \\ -4 & 5 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 5 \\ 7 & -2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

20. $\mathbf{A}^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 5 & 1 \\ 0 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 5 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

21. a. **A** and **C** are diagonal matrices.

b. All are them are triangular matrices.

22. $\mathbf{A}^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 5 & 1 \\ 0 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 5 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

Since $\mathbf{A}^T = \mathbf{A}$, the matrix of Problem 20 is *symmetric*.

23. $\mathbf{A}^T = \begin{bmatrix} 1 & 0 & -1 \\ 7 & 0 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 7 \\ 0 & 0 \\ -1 & 9 \end{bmatrix}$

$(\mathbf{A}^T)^T = \begin{bmatrix} 1 & 7 \\ 0 & 0 \\ -1 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & -1 \\ 7 & 0 & 9 \end{bmatrix} = \mathbf{A}$

24. Equating corresponding entries gives $2x = 4$, $y = 6$, $z = 0$, and $3w = 7$. Thus $x = 2$, $y = 6$, $z = 0$, $w = \frac{7}{3}$.

25. Equating corresponding entries gives $6 = 6$, $2 = 2$, $x = 6$, $7 = 7$, $3y = 2$, and $2z = 7$. Thus $x = 6$, $y = \frac{2}{3}$, $z = \frac{7}{2}$.

26. Equating entries in the 3rd row and 3rd column gives $7 = 8$, which is never true, so there is no solution.

27. Equating corresponding entries gives $2x = y$, $7 = 7$, $7 = 7$, and $2y = y$. Now $2y = y$ yields $y = 0$. Thus from $2x = y$ we get $2x = 0$, so $x = 0$. The solution is $x = 0$, $y = 0$.

28. $\begin{bmatrix} 125 & 275 & 400 \end{bmatrix}$

$\begin{bmatrix} 0.95 \\ 1.03 \\ 1.25 \end{bmatrix}$

29. a. From **J**, the entry in row 3 (super-duper) and column 2 (white) is 7. Thus in January, 7 white super-duper models were sold.

b. From **F**, the entry in row 2 (deluxe) and column 3 (blue) is 3. Thus in February, 3 blue deluxe models were sold.

c. The entries in row 1 (regular) and column 4 (purple) give the number of purple regular models sold. For **J** the entry is 2 and for **F** the entry is 4. Thus more purple regular models were sold in February.

d. In both January and February, the deluxe blue models (row 2, column 3) sold the same number of units (3).

e. In January a total of $0 + 1 + 3 + 5 = 9$ deluxe models were sold. In February a total of
$2 + 3 + 3 + 2 = 10$ deluxe models were sold. Thus more deluxe models were sold in February.

f. In January a total of $2 + 0 + 2 = 4$ red widgets were sold, while in February a total of $0 + 2 + 4 = 6$ red widgets were sold. Thus more red widgets were sold in February.

g. Adding all entries in matrix **J** yields that a total of 38 widgets were sold in January.

30. The sums of the entries in the columns are 680, 710, 1510, and 6690. The sum of the entries in the rows are 680, 710, 1510, and 6690. The amount an industry consumes is equal to the amount of its output. Industry B has to increase output by (0.20)(90) = 18 units and industry C has to increase output by (0.20)(120) = 24 units. All other producers have to increase it by (0.20)(420) = 84 units.

31. By equating entries we find that x must satisfy $x^2 + 2000x = 2001$ and $\sqrt{x^2} = -x$.
The second equation implies that $x < 0$. From the first equation, $x^2 + 2000x - 2001 = 0$, $(x + 2001)(x - 1) = 0$, so $x = -2001$.

32. $\begin{bmatrix} 3 & -2 \\ -4 & 1 \\ 5 & 6 \end{bmatrix}$

33. $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 7 & 4 \\ 4 & 3 & 1 \\ 2 & 6 & 2 \end{bmatrix}$

Principles in Practice 6.2

1. $T = J + F = \begin{bmatrix} 120 & 80 \\ 105 & 130 \end{bmatrix} + \begin{bmatrix} 110 & 140 \\ 85 & 125 \end{bmatrix}$

$= \begin{bmatrix} 120+110 & 80+140 \\ 105+85 & 130+125 \end{bmatrix} = \begin{bmatrix} 230 & 220 \\ 190 & 255 \end{bmatrix}$

2. $0.8\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 40 \\ 30 \\ 60 \end{bmatrix} = 2\begin{bmatrix} 248 \\ 319 \\ 532 \end{bmatrix}$

$\begin{bmatrix} 0.8x_1 \\ 0.8x_2 \\ 0.8x_3 \end{bmatrix} - \begin{bmatrix} 40 \\ 30 \\ 60 \end{bmatrix} = \begin{bmatrix} 496 \\ 638 \\ 1064 \end{bmatrix}$

$\begin{bmatrix} 0.8x_1 - 40 \\ 0.8x_2 - 30 \\ 0.8x_3 - 60 \end{bmatrix} = \begin{bmatrix} 496 \\ 638 \\ 1064 \end{bmatrix}$

Solve $0.8x_1 - 40 = 496$ to get $x_1 = 670$.
Solve $0.8x_2 - 30 = 638$ to get $x_2 = 835$.
Solve $0.8x_3 - 60 = 1064$ to get $x_3 = 1405$.

Problems 6.2

1. $\begin{bmatrix} 2 & 0 & -3 \\ -1 & 4 & 0 \\ 1 & -6 & 5 \end{bmatrix} + \begin{bmatrix} 2 & -3 & 4 \\ -1 & 6 & 5 \\ 9 & 11 & -2 \end{bmatrix} = \begin{bmatrix} 2+2 & 0+(-3) & -3+4 \\ -1+(-1) & 4+6 & 0+5 \\ 1+9 & -6+11 & 5+(-2) \end{bmatrix} = \begin{bmatrix} 4 & -3 & 1 \\ -2 & 10 & 5 \\ 10 & 5 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 2 & -7 \\ -6 & 4 \end{bmatrix} + \begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 2+7+2 & -7+(-4)+7 \\ -6+(-2)+7 & 4+1+2 \end{bmatrix} = \begin{bmatrix} 11 & -4 \\ -1 & 7 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 4 \\ -2 & 7 \\ 6 & 9 \end{bmatrix} - \begin{bmatrix} 6 & -1 \\ 7 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1-6 & 4-(-1) \\ -2-7 & 7-2 \\ 6-1 & 9-0 \end{bmatrix} = \begin{bmatrix} -5 & 5 \\ -9 & 5 \\ 5 & 9 \end{bmatrix}$

4. $\frac{1}{2}\begin{bmatrix} 4 & -2 & 6 \\ 2 & 10 & -12 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\cdot 4 & \frac{1}{2}(-2) & \frac{1}{2}\cdot 6 \\ \frac{1}{2}\cdot 2 & \frac{1}{2}\cdot 10 & \frac{1}{2}(-12) \\ \frac{1}{2}\cdot 0 & \frac{1}{2}\cdot 0 & \frac{1}{2}\cdot 8 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 5 & -6 \\ 0 & 0 & 4 \end{bmatrix}$

5. $2[2 \quad -1 \quad 3] + 4[-2 \quad 0 \quad 1] - 0[2 \quad 3 \quad 1]$
$= [4 \quad -2 \quad 6] + [-8 \quad 0 \quad 4] - [0 \quad 0 \quad 0]$
$= [4 - 8 - 0 \quad -2 + 0 - 0 \quad 6 + 4 - 0]$
$= [-4 \quad -2 \quad 10]$

6. $\begin{bmatrix} 7 & 7 \end{bmatrix}$ is a matrix and 66 is a number, so the sum is not defined.

7. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ has size 2 × 2, and $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ has size 2 × 1. Thus the sum is not defined.

8. $\begin{bmatrix} 2 & -1 \\ 7 & 4 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 7 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 7 & 4 \end{bmatrix}$

9. $-6\begin{bmatrix} 2 & -6 & 7 & 1 \\ 7 & 1 & 6 & -2 \end{bmatrix} = \begin{bmatrix} -6\cdot 2 & -6(-6) & -6\cdot 7 & -6\cdot 1 \\ -6\cdot 7 & -6\cdot 1 & -6\cdot 6 & -6(-2) \end{bmatrix} = \begin{bmatrix} -12 & 36 & -42 & -6 \\ -42 & -6 & -36 & 12 \end{bmatrix}$

10. $\begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -6 \\ 4 & 9 \end{bmatrix} - 3\begin{bmatrix} -6 & 9 \\ 2 & 6 \\ 1 & -2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -6 \\ 4 & 9 \end{bmatrix} - \begin{bmatrix} -18 & 27 \\ 6 & 18 \\ 3 & -6 \\ 12 & 15 \end{bmatrix} = \begin{bmatrix} 19 & -28 \\ -4 & -18 \\ 0 & 0 \\ -8 & -6 \end{bmatrix}$

11. $\begin{bmatrix} 1 & -5 & 0 \\ -2 & 7 & 0 \\ 4 & 6 & 10 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 10 & 0 & 30 \\ 0 & 5 & 0 \\ 5 & 20 & 25 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 0 \\ -2 & 7 & 0 \\ 4 & 6 & 10 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 6 \\ 0 & 1 & 0 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -5 & 6 \\ -2 & 8 & 0 \\ 5 & 10 & 15 \end{bmatrix}$

12.
$$3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 3\left(\begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -2 & 2 \\ -3 & 21 & -9 \\ 0 & 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - 3\begin{bmatrix} -3 & 4 & -2 \\ 3 & -23 & 10 \\ 0 & -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} -9 & 12 & -6 \\ 9 & -69 & 30 \\ 0 & -3 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 12 & -12 & 6 \\ -9 & 72 & -30 \\ 0 & 3 & 0 \end{bmatrix}$$

13. $-\mathbf{B} = -\begin{bmatrix} -6 & -5 \\ 2 & -3 \end{bmatrix} = (-1)\begin{bmatrix} -6 & -5 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} -1(-6) & -1(-5) \\ -1(2) & -1(-3) \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ -2 & 3 \end{bmatrix}$

14. $-(\mathbf{A}-\mathbf{B}) = -\begin{bmatrix} 2-(-6) & 1-(-5) \\ 3-2 & -3-(-3) \end{bmatrix} = -\begin{bmatrix} 8 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -8 & -6 \\ -1 & 0 \end{bmatrix}$

15. $2\mathbf{O} = 2\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2\cdot 0 & 2\cdot 0 \\ 2\cdot 0 & 2\cdot 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}$

16. $\mathbf{A} - \mathbf{B} + \mathbf{C} = \begin{bmatrix} 2-(-6)+(-2) & 1-(-5)+(-1) \\ 3-2+(-3) & -3-(-3)+3 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ -2 & 3 \end{bmatrix}$

17. $3(2\mathbf{A} - 3\mathbf{B}) = 3\left\{2\begin{bmatrix} 2 & 1 \\ 3 & -3 \end{bmatrix} - 3\begin{bmatrix} -6 & -5 \\ 2 & -3 \end{bmatrix}\right\} = 3\left\{\begin{bmatrix} 4 & 2 \\ 6 & -6 \end{bmatrix} - \begin{bmatrix} -18 & -15 \\ 6 & -9 \end{bmatrix}\right\} = 3\begin{bmatrix} 22 & 17 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 66 & 51 \\ 0 & 9 \end{bmatrix}$

18. $0(\mathbf{A} + \mathbf{B}) = 0\begin{bmatrix} -4 & -4 \\ 5 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}$

19. 3(**A** – **C**) is a 2 × 2 matrix and 6 is a number. Therefore 3(**A** – **C**) + 6 is not defined.

20. $\mathbf{A}+(\mathbf{C}+\mathbf{B})=\begin{bmatrix}2 & 1\\ 3 & -3\end{bmatrix}+\begin{bmatrix}-2+(-6) & -1+(-5)\\ -3+2 & 3+(-3)\end{bmatrix}=\begin{bmatrix}2+(-8) & 1+(-6)\\ 3+(-1) & -3+0\end{bmatrix}=\begin{bmatrix}-6 & -5\\ 2 & -3\end{bmatrix}$

21. $2\mathbf{B}-3\mathbf{A}+2\mathbf{C}=2\begin{bmatrix}-6 & -5\\ 2 & -3\end{bmatrix}-3\begin{bmatrix}2 & 1\\ 3 & -3\end{bmatrix}+2\begin{bmatrix}-2 & -1\\ -3 & 3\end{bmatrix}$

$=\begin{bmatrix}-12 & -10\\ 4 & -6\end{bmatrix}-\begin{bmatrix}6 & 3\\ 9 & -9\end{bmatrix}+\begin{bmatrix}-4 & -2\\ -6 & 6\end{bmatrix}$

$=\begin{bmatrix}-18 & -13\\ -5 & 3\end{bmatrix}+\begin{bmatrix}-4 & -2\\ -6 & 6\end{bmatrix}=\begin{bmatrix}-22 & -15\\ -11 & 9\end{bmatrix}$

22. $3\mathbf{C}-2\mathbf{B}=\begin{bmatrix}-6 & -3\\ -9 & 9\end{bmatrix}-\begin{bmatrix}-12 & -10\\ 4 & -6\end{bmatrix}=\begin{bmatrix}6 & 7\\ -13 & 15\end{bmatrix}$

23. $\frac{1}{2}\mathbf{A}-2(\mathbf{B}+2\mathbf{C})=\frac{1}{2}\begin{bmatrix}2 & 1\\ 3 & -3\end{bmatrix}-2\left\{\begin{bmatrix}-6 & -5\\ 2 & -3\end{bmatrix}+2\begin{bmatrix}-2 & -1\\ -3 & 3\end{bmatrix}\right\}$

$=\begin{bmatrix}1 & \frac{1}{2}\\ \frac{3}{2} & -\frac{3}{2}\end{bmatrix}-2\left\{\begin{bmatrix}-6 & -5\\ 2 & -3\end{bmatrix}+\begin{bmatrix}-4 & -2\\ -6 & 6\end{bmatrix}\right\}$

$=\begin{bmatrix}1 & \frac{1}{2}\\ \frac{3}{2} & -\frac{3}{2}\end{bmatrix}-2\begin{bmatrix}-10 & -7\\ -4 & 3\end{bmatrix}=\begin{bmatrix}1 & \frac{1}{2}\\ \frac{3}{2} & -\frac{3}{2}\end{bmatrix}-\begin{bmatrix}-20 & -14\\ -8 & 6\end{bmatrix}=\begin{bmatrix}21 & \frac{29}{2}\\ \frac{19}{2} & -\frac{15}{2}\end{bmatrix}$

24. $\frac{1}{2}\mathbf{A}-5(\mathbf{B}+\mathbf{C})=\begin{bmatrix}1 & \frac{1}{2}\\ \frac{3}{2} & -\frac{3}{2}\end{bmatrix}-5\begin{bmatrix}-8 & -6\\ -1 & 0\end{bmatrix}=\begin{bmatrix}1 & \frac{1}{2}\\ \frac{3}{2} & -\frac{3}{2}\end{bmatrix}+\begin{bmatrix}40 & 30\\ 5 & 0\end{bmatrix}=\begin{bmatrix}41 & \frac{61}{2}\\ \frac{13}{2} & -\frac{3}{2}\end{bmatrix}$

25. $3(\mathbf{A}+\mathbf{B})=3\begin{bmatrix}-4 & -4\\ 5 & -6\end{bmatrix}=\begin{bmatrix}-12 & -12\\ 15 & -18\end{bmatrix}$

$3\mathbf{A}+3\mathbf{B}=\begin{bmatrix}6 & 3\\ 9 & -9\end{bmatrix}+\begin{bmatrix}-18 & -15\\ 6 & -9\end{bmatrix}=\begin{bmatrix}-12 & -12\\ 15 & -18\end{bmatrix}$

Thus 3(**A** + **B**) = 3**A** + 3**B**.

26. $(2+3)\mathbf{A}=5\mathbf{A}=\begin{bmatrix}10 & 5\\ 15 & -15\end{bmatrix}$

$2\mathbf{A}+3\mathbf{A}=\begin{bmatrix}4 & 2\\ 6 & -6\end{bmatrix}+\begin{bmatrix}6 & 3\\ 9 & -9\end{bmatrix}=\begin{bmatrix}10 & 5\\ 15 & -15\end{bmatrix}$

Thus (2 + 3)**A** = 2**A** + 3**A**.

27. $k_1(k_2\mathbf{A})=k_1\begin{bmatrix}2k_2 & k_2\\ 3k_2 & -3k_2\end{bmatrix}=\begin{bmatrix}2k_1k_2 & k_1k_2\\ 3k_1k_2 & -3k_1k_2\end{bmatrix}$

$(k_1k_2)\mathbf{A}=\begin{bmatrix}2k_1k_2 & k_1k_2\\ 3k_1k_2 & -3k_1k_2\end{bmatrix}$

Thus $k_1(k_2\mathbf{A})=(k_1k_2)\mathbf{A}$.

28. $k(\mathbf{A}-2\mathbf{B}+\mathbf{C})=k\left(\begin{bmatrix}2 & 1\\ 3 & -3\end{bmatrix}-\begin{bmatrix}-12 & -10\\ 4 & -6\end{bmatrix}+\begin{bmatrix}-2 & -1\\ -3 & 3\end{bmatrix}\right)=k\begin{bmatrix}12 & 10\\ -4 & 6\end{bmatrix}=\begin{bmatrix}12k & 10k\\ -4k & 6k\end{bmatrix}$

$k\mathbf{A}-2k\mathbf{B}+k\mathbf{C}=\begin{bmatrix}2k & k\\ 3k & -3k\end{bmatrix}-\begin{bmatrix}-12k & -10k\\ 4k & -6k\end{bmatrix}+\begin{bmatrix}-2k & -k\\ -3k & 3k\end{bmatrix}=\begin{bmatrix}12k & 10k\\ -4k & 6k\end{bmatrix}$

Thus $k(\mathbf{A}-2\mathbf{B}+\mathbf{C})=k\mathbf{A}-2k\mathbf{B}+k\mathbf{C}$.

29. $3\mathbf{A}+\mathbf{D}^{\mathrm{T}}=3\begin{bmatrix}1 & 2\\ 0 & -1\\ 7 & 0\end{bmatrix}+\begin{bmatrix}1 & 1\\ 2 & 0\\ -1 & 2\end{bmatrix}=\begin{bmatrix}3 & 6\\ 0 & -3\\ 21 & 0\end{bmatrix}+\begin{bmatrix}1 & 1\\ 2 & 0\\ -1 & 2\end{bmatrix}=\begin{bmatrix}4 & 7\\ 2 & -3\\ 20 & 2\end{bmatrix}$

30. $(\mathbf{B}-\mathbf{C})^{\mathrm{T}}=\left\{\begin{bmatrix}1 & 3\\ 4 & -1\end{bmatrix}-\begin{bmatrix}1 & 0\\ 1 & 2\end{bmatrix}\right\}^{\mathrm{T}}=\begin{bmatrix}0 & 3\\ 3 & -3\end{bmatrix}^{\mathrm{T}}=\begin{bmatrix}0 & 3\\ 3 & -3\end{bmatrix}$

31. $2\mathbf{B}^{\mathrm{T}}-3\mathbf{C}^{\mathrm{T}}=2\begin{bmatrix}1 & 4\\ 3 & -1\end{bmatrix}-3\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}=\begin{bmatrix}2 & 8\\ 6 & -2\end{bmatrix}-\begin{bmatrix}3 & 3\\ 0 & 6\end{bmatrix}=\begin{bmatrix}-1 & 5\\ 6 & -8\end{bmatrix}$

32. $2\mathbf{B}+\mathbf{B}^{\mathrm{T}}=2\begin{bmatrix}1 & 3\\ 4 & -1\end{bmatrix}+\begin{bmatrix}1 & 4\\ 3 & -1\end{bmatrix}=\begin{bmatrix}2 & 6\\ 8 & -2\end{bmatrix}+\begin{bmatrix}1 & 4\\ 3 & -1\end{bmatrix}=\begin{bmatrix}3 & 10\\ 11 & -3\end{bmatrix}$

33. $\mathbf{C}^{\mathrm{T}}-\mathbf{D}=\begin{bmatrix}1 & 0\\ 1 & 2\end{bmatrix}^{\mathrm{T}}-\begin{bmatrix}1 & 2 & -1\\ 1 & 0 & 2\end{bmatrix}$ is impossible because $\mathbf{C}^{\mathrm{T}}$ and **D** are not of the same size.

34. $\left(\mathbf{D}-2\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}}=\left\{\begin{bmatrix}1 & 2 & -1\\ 1 & 0 & 2\end{bmatrix}-2\begin{bmatrix}1 & 0 & 7\\ 2 & -1 & 0\end{bmatrix}\right\}^{\mathrm{T}}$

$=\left\{\begin{bmatrix}1 & 2 & -1\\ 1 & 0 & 2\end{bmatrix}-\begin{bmatrix}2 & 0 & 14\\ 4 & -2 & 0\end{bmatrix}\right\}^{\mathrm{T}}=\begin{bmatrix}-1 & 2 & -15\\ -3 & 2 & 2\end{bmatrix}^{\mathrm{T}}$

$=\begin{bmatrix}-1 & -3\\ 2 & 2\\ -15 & 2\end{bmatrix}$

35. $x\begin{bmatrix}3\\ 2\end{bmatrix}-y\begin{bmatrix}-4\\ 7\end{bmatrix}=3\begin{bmatrix}2\\ 4\end{bmatrix}$

$\begin{bmatrix}3x\\ 2x\end{bmatrix}-\begin{bmatrix}-4y\\ 7y\end{bmatrix}=\begin{bmatrix}6\\ 12\end{bmatrix}=\begin{bmatrix}3x+4y\\ 2x-7y\end{bmatrix}=\begin{bmatrix}6\\ 12\end{bmatrix}$

Equating corresponding entries gives

$\begin{cases}3x+4y=6\\ 2x-7y=12\end{cases}$

Multiply the first equation by 2 and the second equation by −3 to get

$\begin{cases}6x+\ 8y=12\\ -6x+21y=-36\end{cases}$

Now add the two equations to get

$29y = -24$

$y = -\frac{24}{29}$

Therefore

$3x = 6 - 4\left(-\frac{24}{29}\right) = \frac{270}{29}$

$x = \frac{90}{29}$

The solution is $x = \frac{90}{29}$, $y = -\frac{24}{29}$.

36. $\begin{bmatrix} 2x-4y \\ 5x+7y \end{bmatrix} = \begin{bmatrix} 16 \\ -3 \end{bmatrix}$

$\begin{bmatrix} 2x \\ 5x \end{bmatrix} + \begin{bmatrix} -4y \\ 7y \end{bmatrix} = \begin{bmatrix} 16 \\ -3 \end{bmatrix}$

$x\begin{bmatrix} 2 \\ 5 \end{bmatrix} + y\begin{bmatrix} -4 \\ 7 \end{bmatrix} = \begin{bmatrix} 16 \\ -3 \end{bmatrix}$

37. $3\begin{bmatrix} x \\ y \end{bmatrix} - 3\begin{bmatrix} -2 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 6 \\ -2 \end{bmatrix}$

$\begin{bmatrix} 3x+6 \\ 3y-12 \end{bmatrix} = \begin{bmatrix} 24 \\ -8 \end{bmatrix}$

$3x + 6 = 24$, $3x = 18$, or $x = 6$.

$3y - 12 = -8$, $3y = 4$, or $y = \frac{4}{3}$.

Thus $x = 6$, $y = \frac{4}{3}$.

38. $3\begin{bmatrix} x \\ 2 \end{bmatrix} - 4\begin{bmatrix} 7 \\ -y \end{bmatrix} = \begin{bmatrix} -x \\ 2y \end{bmatrix}$

$\begin{bmatrix} 3x-28 \\ 6+4y \end{bmatrix} = \begin{bmatrix} -x \\ 2y \end{bmatrix}$

$3x - 28 = -x$, $4x = 28$, or $x = 7$.

$6 + 4y = 2y$, $2y = -6$, or $y = -3$.

Thus $x = 7$, $y = -3$.

39. $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + 2\begin{bmatrix} x \\ y \\ 4z \end{bmatrix} = \begin{bmatrix} -10 \\ -24 \\ 14 \end{bmatrix}$

$\begin{bmatrix} 2+2x \\ 4+2y \\ 6+8z \end{bmatrix} = \begin{bmatrix} -10 \\ -24 \\ 14 \end{bmatrix}$

$2 + 2x = -10$, $2x = -12$, or $x = -6$.

$4 + 2y = -24$, $2y = -28$, or $y = -14$.

$6 + 8z = 14$, $8z = 8$, or $z = 1$.

Thus $x = -6$, $y = -14$, $z = 1$.

40. $x\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix} + y\begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 2x+12-5y \end{bmatrix}$

$\begin{bmatrix} 2x-2 \\ 2y \\ 2x+12-5y \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 2x+12-5y \end{bmatrix}$

$2x - 2 = 10$, $2x = 12$, or $x = 6$.

$2y = 6$ or $y = 3$.

$2x + 12 - 5y = 2x + 12 - 5y$, which is true for all values of x and y. Thus $x = 6$, $y = 3$.

41. $\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 30 & 50 \\ 800 & 720 \\ 25 & 30 \end{bmatrix} + \begin{bmatrix} 15 & 25 \\ 960 & 800 \\ 10 & 5 \end{bmatrix}$

$= \begin{bmatrix} 30+15 & 50+25 \\ 800+960 & 720+800 \\ 25+10 & 30+5 \end{bmatrix} = \begin{bmatrix} 45 & 75 \\ 1760 & 1520 \\ 35 & 35 \end{bmatrix}$

42. $2\mathbf{B} - \mathbf{A} = 2\begin{bmatrix} 380 & 330 & 220 \\ 460 & 320 & 750 \end{bmatrix} - \begin{bmatrix} 400 & 350 & 150 \\ 450 & 280 & 850 \end{bmatrix}$

$= \begin{bmatrix} 2\cdot380 & 2\cdot330 & 2\cdot220 \\ 2\cdot460 & 2\cdot320 & 2\cdot750 \end{bmatrix} - \begin{bmatrix} 400 & 350 & 150 \\ 450 & 280 & 850 \end{bmatrix}$

$= \begin{bmatrix} 760 & 660 & 440 \\ 920 & 640 & 1500 \end{bmatrix} - \begin{bmatrix} 400 & 350 & 150 \\ 450 & 280 & 850 \end{bmatrix}$

$= \begin{bmatrix} 360 & 310 & 290 \\ 470 & 360 & 650 \end{bmatrix}$

43. $\mathbf{P} + 0.1\mathbf{P} = [p_1 \ \ p_2 \ \ p_3] + [0.1p_1 \ \ 0.1p_2 \ \ 0.1p_3]$

$= [1.1p_1 \ \ 1.1p_2 \ \ 1.1p_3] = 1.1\mathbf{P}$

Thus P must be multiplied by 1.1.

44. $(\mathbf{A} - \mathbf{B})^T = [\mathbf{A} + (-1)\mathbf{B}]^T$ [definition of subtraction]

$= \mathbf{A}^T + [(-1)\mathbf{B}]^T$ [transpose of a sum]

$= \mathbf{A}^T + (-1)\mathbf{B}^T$ [transpose of a scalar multiple]

$= \mathbf{A}^T - \mathbf{B}^T$ [definition of subtraction]

45. $\begin{bmatrix} 15 & -4 & 26 \\ 4 & 7 & 30 \end{bmatrix}$

46. $\begin{bmatrix} -16 & -11 & -24 \\ -16 & -3 & -36 \end{bmatrix}$

47. $\begin{bmatrix} -10 & 22 & 12 \\ 24 & 36 & -44 \end{bmatrix}$

Principles in Practice 6.3

1. Represent the value of each book by $\begin{bmatrix} 28 & 22 & 16 \end{bmatrix}$ and the number of each book by $\begin{bmatrix} 100 \\ 70 \\ 90 \end{bmatrix}$.
   The total value is given by the following matrix product.
   $$\begin{bmatrix} 28 & 22 & 16 \end{bmatrix}\begin{bmatrix} 100 \\ 70 \\ 90 \end{bmatrix} = [2800 + 1540 + 1440]$$
   $$= [5780]$$
   The total value is \$5780.

2. The total cost is given by the matrix product **PQ**.
   $$\mathbf{PQ} = \begin{bmatrix} 26.25 & 34.75 & 28.50 \end{bmatrix}\begin{bmatrix} 250 \\ 325 \\ 175 \end{bmatrix}$$
   $$= \begin{bmatrix} 6562.5 + 11,293.75 + 4987.5 \end{bmatrix} = [22,843.75]$$
   The total cost is \$22,843.75.

3. First, write the equations with the variable terms on the left-hand side.
   $$\begin{cases} y + \frac{8}{5}x = \frac{8}{5} \\ y + \frac{1}{3}x = \frac{5}{3} \end{cases}$$
   Let $\mathbf{A} = \begin{bmatrix} 1 & \frac{8}{5} \\ 1 & \frac{1}{3} \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} y \\ x \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} \frac{8}{5} \\ \frac{5}{3} \end{bmatrix}$.
   Then the pair of lines is equivalent to the matrix equation **AX** = **B** or $\begin{bmatrix} 1 & \frac{8}{5} \\ 1 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} \frac{8}{5} \\ \frac{5}{3} \end{bmatrix}$.

Problems 6.3

1. $c_{11} = 1(0) + 3(-2) + (-2)(3) = -12$

2. $c_{23} = -2(3) + 1(-2) + (-1)(-1) = -7$

3. $c_{32} = 0(-2) + 4(4) + 3(1) = 19$

4. $c_{33} = 0(3) + 4(-2) + 3(-1) = -11$

5. $c_{31} = 0(0) + 4(-2) + 3(3) = 1$

6. $c_{12} = 1(-2) + 3(4) + (-2)(1) = 8$

7. **A** is 2 × 3 and **E** is 3 × 2, so **AE** is 2 × 2;
   2 · 2 = 4 entries.

8. **D** is 4 × 3 and **E** is 3 × 2, so **DE** is 4 × 2;
   4 · 2 = 8 entries.

9. **E** is 3 × 2 and **C** is 2 × 5, so **EC** is 3 × 5;
   3 · 5 = 15 entries.

10. **D** is 4 × 3 and B is 3 × 1, so **DB** is 4 × 1;
    4 · 1 = 4 entries.

11. **F** is 2 × 3 and **B** is 3 × 1, so **FB** is 2 × 1;
    2 · 1 = 2 entries.

12. **B** is 3 × 1 and **C** is 2 × 5. Because the number of columns of **B** does not equal the number of rows of **C**, **BC** is not defined.

13. **E** is 3 × 2, $\mathbf{E}^T$ is 2 × 3, and **B** is 3 × 1, so $\mathbf{EE}^T\mathbf{B}$ is 3 × 1; 3 · 1 = 3 entries.

14. **A** is 2 × 3 and **E** is 3 × 2, so **AE** is 2 × 2. Thus **E**(**AE**) is 3 × 2; 3 · 2 = 6 entries.

15. **E** is 3 × 2. **F** is 2 × 3 and **B** is 3 × 1, so **FB** is 2 × 1. Thus **E**(**FB**) is 3 × 1; 3 · 1 = 3 entries.

16. Both **F** and **A** are 2 × 3, so **F** + **A** is 2 × 3. Because **B** is 3 × 1, (**F** + **A**)**B** is 2 × 1; 2 · 1 = 2 entries.

17. An identity matrix is a square matrix (in this case 4 × 4) with 1's on the main diagonal and all other entries 0's.
    $$\mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

18. $$\mathbf{I}_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

19. $\begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(4)+(-4)(-1) & 2(0)+(-4)(3) \\ 3(4)+2(-1) & 3(0)+2(3) \end{bmatrix} = \begin{bmatrix} 12 & -12 \\ 10 & 6 \end{bmatrix}$

20. $\begin{bmatrix} -1 & 1 \\ 0 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1(1)+1(3) & -1(-2)+1(4) \\ 0(1)+4(3) & 0(-2)+4(4) \\ 2(1)+1(3) & 2(-2)+1(4) \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 12 & 16 \\ 5 & 0 \end{bmatrix}$

21. $\begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 2(1)+0(4)+3(7) \\ -1(1)+4(4)+5(7) \end{bmatrix} = \begin{bmatrix} 23 \\ 50 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 0 & 6 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = [1(0)+0(1)+6(2)+2(3)] = [18]$

23. $\begin{bmatrix} 1 & 4 & -1 \\ 0 & 0 & 2 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1(2)+4(0)+(-1)1 & 1(1)+4(-1)+(-1)(1) & 1(0)+4(1)+(-1)(2) \\ 0(2)+0(0)+2(1) & 0(1)+0(-1)+2(1) & 0(0)+0(1)+2(2) \\ -2(2)+1(0)+1(1) & -2(1)+1(-1)+1(1) & -2(0)+1(1)+1(2) \end{bmatrix} = \begin{bmatrix} 1 & -4 & 2 \\ 2 & 2 & 4 \\ -3 & -2 & 3 \end{bmatrix}$

24. $\begin{bmatrix} 4 & 2 & -2 \\ 3 & 10 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4(3)+2(0)+(-2)(0) & 4(1)+2(0)+(-2)(1) & 4(1)+2(0)+(-2)(0) & 4(0)+2(0)+(-2)(1) \\ 3(3)+10(0)+0(0) & 3(1)+10(0)+0(1) & 3(1)+10(0)+0(0) & 3(0)+10(0)+0(1) \\ 1(3)+0(0)+2(0) & 1(1)+0(0)+2(1) & 1(1)+0(0)+2(0) & 1(0)+0(0)+2(1) \end{bmatrix}$

$= \begin{bmatrix} 12 & 2 & 4 & -2 \\ 9 & 3 & 3 & 0 \\ 3 & 3 & 1 & 2 \end{bmatrix}$

25. $[1 \quad -2 \quad 5]\begin{bmatrix} 1 & 5 & -2 & -1 \\ 0 & 0 & 2 & 1 \\ -1 & 0 & 1 & -3 \end{bmatrix}$

$= [1+0-5 \quad 5+0+0 \quad -2-4+5 \quad -1-2-15]$

$= [-4 \quad 5 \quad -1 \quad -18]$

26. The first matrix is 1×2 and the second is 3×2, so the product is not defined.

27. $\begin{bmatrix} 2 \\ 3 \\ -4 \\ 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 2(2) & 2(3) & 2(-2) & 2(3) \\ 3(2) & 3(3) & 3(-2) & 3(3) \\ -4(2) & -4(3) & -4(-2) & -4(3) \\ 1(2) & 1(3) & 1(-2) & 1(3) \end{bmatrix} = \begin{bmatrix} 4 & 6 & -4 & 6 \\ 6 & 9 & -6 & 9 \\ -8 & -12 & 8 & -12 \\ 2 & 3 & -2 & 3 \end{bmatrix}$

28. $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\left\{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0(1)+1(1) & 0(1)+1(1) & 0(1)+1(1) \\ 2(1)+3(1) & 2(1)+3(1) & 2(1)+3(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 5 & 5 \end{bmatrix}$

29. $3\left\{\begin{bmatrix} -2 & 0 & 2 \\ 3 & -1 & 1 \end{bmatrix} + 2\begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & -2 \end{bmatrix}\right\}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$= 3\left\{\begin{bmatrix} -2 & 0 & 2 \\ 3 & -1 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 & 4 \\ 2 & 2 & -4 \end{bmatrix}\right\}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$= 3\left\{\begin{bmatrix} -4 & 0 & 6 \\ 5 & 1 & -3 \end{bmatrix}\right\}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} -12 & 0 & 18 \\ 15 & 3 & -9 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$= \begin{bmatrix} -12(1)+0(3)+18(5) & -12(2)+0(4)+18(6) \\ 15(1)+3(3)+(-9)(5) & 15(2)+3(4)+(-9)(6) \end{bmatrix} = \begin{bmatrix} 78 & 84 \\ -21 & -12 \end{bmatrix}$

30. $\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 & -1 & 0 & 0 \\ 2 & 1 & 2 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1(-1)+(-1)(2) & 1(0)+(-1)(1) & 1(-1)+(-1)(2) & 1(0)+(-1)(1) & 1(0)+(-1)(1) \\ 0(-1)+3(2) & 0(0)+3(1) & 0(-1)+3(2) & 0(0)+3(1) & 0(0)+3(1) \end{bmatrix}$

$= \begin{bmatrix} -3 & 1 & -3 & -1 & -1 \\ 6 & 3 & 6 & 3 & 3 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\left\{\begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}\right\} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\left\{\begin{bmatrix} 2+0+3 & -4+0+0 \\ 1+0-6 & -2+0+0 \end{bmatrix}\right\}$

$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & -4 \\ -5 & -2 \end{bmatrix} = \begin{bmatrix} 5-10 & -4-4 \\ 15-20 & -12-8 \end{bmatrix} = \begin{bmatrix} -5 & -8 \\ -5 & -20 \end{bmatrix}$

32. $3\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} - 4\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 6 & 1 \end{bmatrix}\right\} = \begin{bmatrix} 3 & 6 \\ -3 & 12 \end{bmatrix} - 4\left\{I\begin{bmatrix} -2 & 4 \\ 6 & 1 \end{bmatrix}\right\}$

$= \begin{bmatrix} 3 & 6 \\ -3 & 12 \end{bmatrix} - 4\left\{\begin{bmatrix} -2 & 4 \\ 6 & 1 \end{bmatrix}\right\} = \begin{bmatrix} 3 & 6 \\ -3 & 12 \end{bmatrix} - \begin{bmatrix} -8 & 16 \\ 24 & 4 \end{bmatrix} = \begin{bmatrix} 11 & -10 \\ -27 & 8 \end{bmatrix}$

33. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0\cdot x+0\cdot y+1\cdot z \\ 0\cdot x+1\cdot y+0\cdot z \\ 1\cdot x+0\cdot y+0\cdot z \end{bmatrix} = \begin{bmatrix} z \\ y \\ x \end{bmatrix}$

34. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$

35. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 9 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 + 3x_3 \\ 4x_1 + 9x_2 + 7x_3 \end{bmatrix}$

36. $\begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - 3x_2 \\ x_2 \\ 2x_1 + x_2 \end{bmatrix}$

37. $\mathbf{D} - \frac{1}{3}\mathbf{EI} = \mathbf{D} - \frac{1}{3}\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

38. $\mathbf{DD} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+1 & 0+1+2 & 0+1+1 \\ 1+0+1 & 0+2+2 & 0+2+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$

39. $3\mathbf{A} - 2\mathbf{BC} = 3\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} - 2\begin{bmatrix} -2 & 3 & 0 \\ 1 & -4 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 3 \\ 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 3 & -6 \\ 0 & 9 \end{bmatrix} - 2\begin{bmatrix} 2+0+0 & -2+9+0 \\ -1+0+2 & 1-12+4 \end{bmatrix}$

$= \begin{bmatrix} 3 & -6 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 14 \\ 2 & -14 \end{bmatrix} = \begin{bmatrix} -1 & -20 \\ -2 & 23 \end{bmatrix}$

40. $\mathbf{B}(\mathbf{D} + \mathbf{E}) = \begin{bmatrix} -2 & 3 & 0 \\ 1 & -4 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} -8+0+0 & 0+21+0 & 0+3+0 \\ 4+0+1 & 0-28+2 & 0-4+4 \end{bmatrix}$

$= \begin{bmatrix} -8 & 21 & 3 \\ 5 & -26 & 0 \end{bmatrix}$

41. $3\mathbf{I}-\frac{2}{3}\mathbf{FE}=3\mathbf{I}-\frac{2}{3}\begin{bmatrix}\frac{1}{3}&0&0\\0&\frac{1}{6}&0\\0&0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}3&0&0\\0&6&0\\0&0&3\end{bmatrix}$

$$=3\mathbf{I}-\frac{2}{3}\begin{bmatrix}\frac{1}{3}\cdot 3+0+0&0+0+0&0+0+0\\0+0+0&\frac{1}{6}\cdot 6+0&0+0+0\\0+0+0&0+0+0&0+0+\frac{1}{3}\cdot 3\end{bmatrix}$$

$$=3\mathbf{I}-\frac{2}{3}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}-\begin{bmatrix}\frac{2}{3}&0&0\\0&\frac{2}{3}&0\\0&0&\frac{2}{3}\end{bmatrix}=\begin{bmatrix}\frac{7}{3}&0&0\\0&\frac{7}{3}&0\\0&0&\frac{7}{3}\end{bmatrix}$$

42. $\mathbf{FE}(\mathbf{D}-\mathbf{I})=\begin{bmatrix}\frac{1}{3}&0&0\\0&\frac{1}{6}&0\\0&0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}3&0&0\\0&6&0\\0&0&3\end{bmatrix}\left(\begin{bmatrix}1&0&0\\0&1&1\\1&2&1\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)$

$$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&1\\1&2&0\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&1\\1&2&0\end{bmatrix}$$

43. $(\mathbf{DC})\mathbf{A}=\left\{\begin{bmatrix}1&0&0\\0&1&1\\1&2&1\end{bmatrix}\begin{bmatrix}-1&1\\0&3\\2&4\end{bmatrix}\right\}\mathbf{A}=\begin{bmatrix}-1+0+0&1+0+0\\0+0+2&0+3+4\\-1+0+2&1+6+4\end{bmatrix}\mathbf{A}$

$$=\begin{bmatrix}-1&1\\2&7\\1&11\end{bmatrix}\begin{bmatrix}1&-2\\0&3\end{bmatrix}=\begin{bmatrix}-1+0&2+3\\2+0&-4+21\\1+0&-2+33\end{bmatrix}=\begin{bmatrix}-1&5\\2&17\\1&31\end{bmatrix}$$

44. $\mathbf{A}(\mathbf{BC})=\mathbf{A}\left\{\begin{bmatrix}-2&3&0\\1&-4&1\end{bmatrix}\begin{bmatrix}-1&1\\0&3\\2&4\end{bmatrix}\right\}=\mathbf{A}\begin{bmatrix}2+0+0&-2+9+0\\-1+0+2&1-12+4\end{bmatrix}=\begin{bmatrix}1&-2\\0&3\end{bmatrix}\begin{bmatrix}2&7\\1&-7\end{bmatrix}$

$$=\begin{bmatrix}2-2&7+14\\0+3&0-21\end{bmatrix}=\begin{bmatrix}0&21\\3&-21\end{bmatrix}$$

45. Impossible: **A** is not a square matrix, so $\mathbf{A}^2$ is not defined.

46. $\mathbf{A}^{\mathrm{T}}\mathbf{A}=\begin{bmatrix}1&0\\-1&1\\0&1\end{bmatrix}\begin{bmatrix}1&-1&0\\0&1&1\end{bmatrix}=\begin{bmatrix}1&-1&0\\-1&2&1\\0&1&1\end{bmatrix}$

47. $\mathbf{B}^3 = \left(\mathbf{B}^2\right)\mathbf{B} = \begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^2 \mathbf{B} = \begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\mathbf{B}$

$= \begin{bmatrix} 0 & 0 & -2 \\ -2 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -4 \\ 2 & -1 & -2 \\ 0 & 0 & 8 \end{bmatrix}$

48. $\mathbf{A}(\mathbf{B}^{\mathrm{T}})^2\mathbf{C} = \mathbf{A}\begin{bmatrix} 0 & 2 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 2 \end{bmatrix}\mathbf{C}$

$= \mathbf{A}\begin{bmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 4 \end{bmatrix}\mathbf{C}$

$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 4 \end{bmatrix}\mathbf{C}$

$= \begin{bmatrix} 0 & -3 & 0 \\ -2 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -6 & 3 \\ -4 & 5 \end{bmatrix}$

49. $(\mathbf{AIC})^{\mathrm{T}} = \left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}\right)^{\mathrm{T}}$

$= \left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}\right)^{\mathrm{T}}$

$= \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}^{\mathrm{T}}$

$= \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$

50. $\mathbf{A}^{\mathrm{T}}\left(2\mathbf{C}^{\mathrm{T}}\right) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & 0 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ -2 & -6 & 2 \\ 0 & -2 & 2 \end{bmatrix}$

51. $\left(\mathbf{BA}^{\mathrm{T}}\right)^{\mathrm{T}} = \left\{\begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}\right\}^{\mathrm{T}} = \begin{bmatrix} 0 & -1 \\ 3 & -1 \\ 0 & 2 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

52. $(2\mathbf{B})^{\mathrm{T}} = \left\{2\begin{bmatrix}0 & 0 & -1\\ 2 & -1 & 0\\ 0 & 0 & 2\end{bmatrix}\right\}^{\mathrm{T}} = \begin{bmatrix}0 & 0 & -2\\ 4 & -2 & 0\\ 0 & 0 & 4\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}0 & 4 & 0\\ 0 & -2 & 0\\ -2 & 0 & 4\end{bmatrix}$

53. $(2\mathbf{I})^2 - 2\mathbf{I}^2 = (2\mathbf{I})^2 - 2\mathbf{I} = \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}^2 - \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$

$$= \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix} - \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix} = \begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix} - \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix} = \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$$

54. $\mathbf{A}^{\mathrm{T}}$ is 3 × 2, $\mathbf{C}^{\mathrm{T}}$ is 2 × 3, and **B** is 3 × 3, so $\mathbf{A}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{B}$ is 3 × 3 and $(\mathbf{A}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{B})^0 = \mathbf{I} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$.

55. **A**(**I** – **O**) = **A**(**I**) = **AI**. Since **I** is 3 × 3 and **A** has three columns, **AI** = **A**. Thus $\mathbf{A}(\mathbf{I} - \mathbf{O}) = \mathbf{A} = \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 1\end{bmatrix}$.

56. $\mathbf{I}^{\mathrm{T}}\mathbf{O} = \mathbf{IO} = \mathbf{O} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

57. $(\mathbf{AB})(\mathbf{AB})^{\mathrm{T}} = \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & -1\\ 2 & -1 & 0\\ 0 & 0 & 2\end{bmatrix}(\mathbf{AB})^{\mathrm{T}} = \begin{bmatrix}-2 & 1 & -1\\ 2 & -1 & 2\end{bmatrix}(\mathbf{AB})^{\mathrm{T}}$

$$= \begin{bmatrix}-2 & 1 & -1\\ 2 & -1 & 2\end{bmatrix}\begin{bmatrix}-2 & 2\\ 1 & -1\\ -1 & 2\end{bmatrix} = \begin{bmatrix}6 & -7\\ -7 & 9\end{bmatrix}$$

58. $\mathbf{B}^2 - 3\mathbf{B} + 2\mathbf{I}$

$$= \begin{bmatrix}0 & 0 & -1\\ 2 & -1 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}0 & 0 & -1\\ 2 & -1 & 0\\ 0 & 0 & 2\end{bmatrix} - 3\begin{bmatrix}0 & 0 & -1\\ 2 & -1 & 0\\ 0 & 0 & 2\end{bmatrix} + 2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0 & -2\\ -2 & 1 & -2\\ 0 & 0 & 4\end{bmatrix} - \begin{bmatrix}0 & 0 & -3\\ 6 & -3 & 0\\ 0 & 0 & 6\end{bmatrix} + \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0 & 1\\ -8 & 4 & -2\\ 0 & 0 & -2\end{bmatrix} + \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix} = \begin{bmatrix}2 & 0 & 1\\ -8 & 6 & -2\\ 0 & 0 & 0\end{bmatrix}$$

59. **AX** = **B**

$\mathbf{A} = \begin{bmatrix}3 & 1\\ 2 & -9\end{bmatrix}$

$\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\mathbf{B} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$

The system is represented by $\begin{bmatrix} 3 & 1 \\ 2 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$.

60. **AX = B**

$\mathbf{A} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & -1 & 1 \\ 5 & -1 & 2 \end{bmatrix}$

$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\mathbf{B} = \begin{bmatrix} 2 \\ 4 \\ 12 \end{bmatrix}$

The system is represented by $\begin{bmatrix} 3 & 1 & 1 \\ 1 & -1 & 1 \\ 5 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 12 \end{bmatrix}$.

61. **AX = B**

$\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 5 & -1 & 2 \\ 3 & -2 & 2 \end{bmatrix}$

$\mathbf{X} = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$

$\mathbf{B} = \begin{bmatrix} 9 \\ 5 \\ 11 \end{bmatrix}$

The system is represented by $\begin{bmatrix} 2 & -1 & 3 \\ 5 & -1 & 2 \\ 3 & -2 & 2 \end{bmatrix}\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 11 \end{bmatrix}$.

62. "the/falcon/has/landed" converted to corresponding numbers and slashes is "20, 8, 5/ 6, 1, 12, 3, 15, 14/ 8, 1, 19/ 12, 1, 14, 4, 5, 4." Taking the numbers two at a time as 2 × 1 matrices and multiplying them by **E** gives:

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 1\cdot 20 + 3\cdot 8 \\ 2\cdot 20 + 4\cdot 8 \end{bmatrix} = \begin{bmatrix} 20+24 \\ 40+32 \end{bmatrix} = \begin{bmatrix} 44 \\ 72 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1\cdot 5 + 3\cdot 6 \\ 2\cdot 5 + 4\cdot 6 \end{bmatrix} = \begin{bmatrix} 5+18 \\ 10+24 \end{bmatrix} = \begin{bmatrix} 23 \\ 34 \end{bmatrix}$$

$$\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}\begin{bmatrix}1\\12\end{bmatrix}=\begin{bmatrix}1\cdot1+3\cdot12\\2\cdot1+4\cdot12\end{bmatrix}=\begin{bmatrix}1+36\\2+48\end{bmatrix}=\begin{bmatrix}37\\50\end{bmatrix}$$
$$\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}\begin{bmatrix}3\\15\end{bmatrix}=\begin{bmatrix}1\cdot3+3\cdot15\\2\cdot3+4\cdot15\end{bmatrix}=\begin{bmatrix}3+45\\6+60\end{bmatrix}=\begin{bmatrix}48\\66\end{bmatrix}$$
$$\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}\begin{bmatrix}14\\8\end{bmatrix}=\begin{bmatrix}1\cdot14+3\cdot8\\2\cdot14+4\cdot8\end{bmatrix}=\begin{bmatrix}14+24\\28+32\end{bmatrix}=\begin{bmatrix}38\\60\end{bmatrix}$$
$$\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}\begin{bmatrix}1\\19\end{bmatrix}=\begin{bmatrix}1\cdot1+3\cdot19\\2\cdot1+4\cdot19\end{bmatrix}=\begin{bmatrix}1+57\\2+76\end{bmatrix}=\begin{bmatrix}58\\78\end{bmatrix}$$
$$\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}\begin{bmatrix}12\\1\end{bmatrix}=\begin{bmatrix}1\cdot12+3\cdot1\\2\cdot12+4\cdot1\end{bmatrix}=\begin{bmatrix}12+3\\24+4\end{bmatrix}=\begin{bmatrix}15\\28\end{bmatrix}$$
$$\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}\begin{bmatrix}14\\4\end{bmatrix}=\begin{bmatrix}1\cdot14+3\cdot4\\2\cdot14+4\cdot4\end{bmatrix}=\begin{bmatrix}14+12\\28+16\end{bmatrix}=\begin{bmatrix}26\\44\end{bmatrix}$$
$$\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}\begin{bmatrix}5\\4\end{bmatrix}=\begin{bmatrix}1\cdot5+3\cdot4\\2\cdot5+4\cdot4\end{bmatrix}=\begin{bmatrix}5+12\\10+16\end{bmatrix}=\begin{bmatrix}17\\26\end{bmatrix}$$

The encoded message is
44, 72, 23/ 34, 37, 50, 48, 66, 38/ 60, 58, 78/ 15, 28, 26, 44, 17, 26.

63. $\begin{bmatrix}6 & 10 & 7\end{bmatrix}\begin{bmatrix}55\\150\\35\end{bmatrix}=[6\cdot55+10\cdot150+7\cdot35]$

= [330 + 1500 + 245]
= [2075]
The value of the inventory is $2075.

64. $\begin{bmatrix}200 & 300 & 500 & 250\end{bmatrix}\begin{bmatrix}100\\150\\200\\300\end{bmatrix}=[240,000]$

The total cost of the stocks is $240,000.

65. $\mathbf{Q}=[5 \quad 2 \quad 4]$

$$\mathbf{R}=\begin{bmatrix}5 & 20 & 16 & 7 & 17\\7 & 18 & 12 & 9 & 21\\6 & 25 & 8 & 5 & 13\end{bmatrix}$$

$$\mathbf{C}=\begin{bmatrix}2500\\1200\\800\\150\\1500\end{bmatrix}$$

$$\mathbf{QRC} = \mathbf{Q}(\mathbf{RC}) = \mathbf{Q}\begin{bmatrix} 5\cdot 2500 + 20\cdot 1200 + 16\cdot 800 + 7\cdot 150 + 17\cdot 1500 \\ 7\cdot 2500 + 18\cdot 1200 + 12\cdot 800 + 9\cdot 150 + 21\cdot 1500 \\ 6\cdot 2500 + 25\cdot 1200 + 8\cdot 800 + 5\cdot 150 + 13\cdot 1500 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 & 4 \end{bmatrix}\begin{bmatrix} 75,850 \\ 81,550 \\ 71,650 \end{bmatrix}$$

$$= [5(75,850) + 2(81,550) + 4(71,650)]$$

$$= [828,950]$$

The total cost of raw materials is $828,950.

66. a. $$\mathbf{RC} = \begin{bmatrix} 5 & 20 & 16 & 7 & 17 \\ 7 & 18 & 12 & 9 & 21 \\ 6 & 25 & 8 & 5 & 13 \end{bmatrix}\begin{bmatrix} 3500 & 50 \\ 1500 & 50 \\ 1000 & 100 \\ 250 & 10 \\ 3500 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 17,500 + 30,000 + 16,000 + 1750 + 59,500 & 250 + 1000 + 1600 + 70 + 0 \\ 24,500 + 27,000 + 12,000 + 2250 + 73,500 & 350 + 900 + 1200 + 90 + 0 \\ 21,000 + 37,500 + 8000 + 1250 + 45,500 & 300 + 1250 + 800 + 50 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 124,750 & 2920 \\ 139,250 & 2540 \\ 113,250 & 2400 \end{bmatrix}$$

b. $$\mathbf{QRC} = \mathbf{Q}(\mathbf{RC}) = \begin{bmatrix} 5 & 7 & 12 \end{bmatrix}\begin{bmatrix} 124,750 & 2920 \\ 139,250 & 2540 \\ 113,250 & 2400 \end{bmatrix}$$

$$= \begin{bmatrix} 623,750 + 974,750 + 1,359,000 & 14,600 + 17,780 + 28,800 \end{bmatrix}$$

$$= \begin{bmatrix} 2,957,500 & 61,180 \end{bmatrix}$$

c. $$\mathbf{QRCZ} = (\mathbf{QRC})\mathbf{Z} = \begin{bmatrix} 2,957,500 & 61,180 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

= [2,957,500 + 61,180] = [3,018,680]

67. a. Amount spent on goods:

coal industry: $\mathbf{D}_C\mathbf{P} = \begin{bmatrix} 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 10,000 \\ 20,000 \\ 40,000 \end{bmatrix} = [180,000]$

elec. industry: $\mathbf{D}_E\mathbf{P} = \begin{bmatrix} 20 & 0 & 8 \end{bmatrix}\begin{bmatrix} 10,000 \\ 20,000 \\ 40,000 \end{bmatrix} = [520,000]$

steel industry: $\mathbf{D}_S\mathbf{P} = \begin{bmatrix} 30 & 5 & 0 \end{bmatrix}\begin{bmatrix} 10,000 \\ 20,000 \\ 40,000 \end{bmatrix} = [400,000]$

The coal industry spends $180,000, the electric industry spends $520,000, and the steel industry spends

$400,000.

consumer 1: $\mathbf{D}_1\mathbf{P} = \begin{bmatrix} 3 & 2 & 5 \end{bmatrix}\begin{bmatrix} 10,000 \\ 20,000 \\ 40,000 \end{bmatrix} = [270,000]$

consumer 2: $\mathbf{D}_2\mathbf{P} = \begin{bmatrix} 0 & 17 & 1 \end{bmatrix}\begin{bmatrix} 10,000 \\ 20,000 \\ 40,000 \end{bmatrix} = [380,000]$

consumer 3: $\mathbf{D}_3\mathbf{P} = \begin{bmatrix} 4 & 6 & 12 \end{bmatrix}\begin{bmatrix} 10,000 \\ 20,000 \\ 40,000 \end{bmatrix} = [640,000]$

Consumer 1 pays $270,000, consumer 2 pays $380,000, and consumer 3 pays $640,000.

b. From Example 3 of Sec. 6.2, the number of units sold of coal, electricity, and steel are 57, 31, and 30, respectively. Thus the profit for coal is 10,000(57) – 180,000 = $390,000, the profit for elec. is 20,000(31) – 520,000 = $100,000, and the profit for steel is 40,000(30) – 400,000 = $800,000.

c. From (a), the total amount of money that is paid out by all the industries and consumers is 180,000 + 520,000 + 400,000 + 270,000 + 380,000 + 640,000 = $2,390,000.

d. The proportion of the total amount in (c) paid out by the industries is

$$\frac{180,000+520,000+400,000}{2,390,000} = \frac{110}{239}.$$

The proportion of the total amount in (c) paid by consumers is

$$\frac{270,000+380,000+640,000}{2,390,000} = \frac{129}{239}.$$

68. $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}(\mathbf{A}-\mathbf{B}) + \mathbf{B}(\mathbf{A}-\mathbf{B})$ [dist. prop.]

$= \mathbf{A}^2 - \mathbf{AB} + \mathbf{BA} - \mathbf{B}^2$ [dist prop.]

$= \mathbf{A}^2 - \mathbf{BA} + \mathbf{BA} - \mathbf{B}^2$ [$\mathbf{AB} = \mathbf{BA}$, given]

$= \mathbf{A}^2 - \mathbf{B}^2$

69. $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1(2)+(2)(-1) & 1(-3)+2\left(\frac{3}{2}\right) \\ 1(2)+2(-1) & 1(-3)+2\left(\frac{3}{2}\right) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

70. Let $\mathbf{D}_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ and $\mathbf{D}_2 = \begin{bmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{bmatrix}$.

a. $\mathbf{D}_1\mathbf{D}_2 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} ad & 0 & 0 \\ 0 & be & 0 \\ 0 & 0 & cf \end{bmatrix}$

$\mathbf{D}_2\mathbf{D}_1 = \begin{bmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} ad & 0 & 0 \\ 0 & be & 0 \\ 0 & 0 & cf \end{bmatrix}$

Both $\mathbf{D}_1\mathbf{D}_2$ and $\mathbf{D}_2\mathbf{D}_1$ are diagonal matrices.

b. From part (a), $\mathbf{D}_1\mathbf{D}_2 = \mathbf{D}_2\mathbf{D}_1$. Thus $\mathbf{D}_1$ and $\mathbf{D}_2$ commute. [In fact, all $n \times n$ diagonal matrices commute.]

71. $\begin{bmatrix} 72.82 & -9.8 \\ 51.32 & -36.32 \end{bmatrix}$

72. $\begin{bmatrix} 23.994 & -20.832 & -12.648 \\ 26.164 & 7.44 & -168.64 \end{bmatrix}$

73. $\begin{bmatrix} 15.606 & 64.08 \\ -739.428 & 373.056 \end{bmatrix}$

74. $\begin{bmatrix} 11.952 & 54.06 \\ 86.496 & 278.648 \end{bmatrix}$

Principles in Practice 6.4

1. The corresponding system is

$$\begin{cases} 6A+B+3C=35 \\ 3A+2B+3C=22 \\ A+5B+3C=18 \end{cases}$$

Reduce the augmented coefficient matrix of the system.

$$\left[\begin{array}{ccc|c} 6 & 1 & 3 & 35 \\ 3 & 2 & 3 & 22 \\ 1 & 5 & 3 & 18 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 5 & 3 & 18 \\ 3 & 2 & 3 & 22 \\ 6 & 1 & 3 & 35 \end{array}\right]$$

$$\xrightarrow[-6R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & 5 & 3 & 18 \\ 0 & -13 & -6 & -32 \\ 0 & -29 & -15 & -73 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{13}R_2} \left[\begin{array}{ccc|c} 1 & 5 & 3 & 18 \\ 0 & 1 & \frac{6}{13} & \frac{32}{13} \\ 0 & -29 & -15 & -73 \end{array}\right]$$

$$\xrightarrow[29R_2+R_3]{-5R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & \frac{9}{13} & \frac{74}{13} \\ 0 & 1 & \frac{6}{13} & \frac{32}{13} \\ 0 & 0 & -\frac{21}{13} & -\frac{21}{13} \end{array}\right]$$

$$\xrightarrow{-\frac{13}{21}R_3} \left[\begin{array}{ccc|c} 1 & 0 & \frac{9}{13} & \frac{74}{13} \\ 0 & 1 & \frac{6}{13} & \frac{32}{13} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[-\frac{6}{13}R_3+R_2]{-\frac{9}{13}R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus there should be 5 blocks of A, 2 blocks of B, and 1 block of C suggested.

2. Let x be the number of tablets of X, y be the number of tablets of Y, and z be the number of tablets of Z. The system is

$40x + 10y + 10z = 180$
$20x + 10y + 50z = 200$
$10x + 30y + 20z = 190$

Reduce the augmented coefficient matrix of the system.

$$\left[\begin{array}{ccc|c} 40 & 10 & 10 & 180 \\ 20 & 10 & 50 & 200 \\ 10 & 30 & 20 & 190 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 10 & 30 & 20 & 190 \\ 20 & 10 & 50 & 200 \\ 40 & 10 & 10 & 180 \end{array}\right]$$

$$\xrightarrow[\substack{\frac{1}{10}R_2 \\ \frac{1}{10}R_3}]{\frac{1}{10}R_1} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 19 \\ 2 & 1 & 5 & 20 \\ 4 & 1 & 1 & 18 \end{array}\right]$$

$$\xrightarrow[-4R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 19 \\ 0 & -5 & 1 & -18 \\ 0 & -11 & -7 & -58 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 19 \\ 0 & 1 & -\frac{1}{5} & \frac{18}{5} \\ 0 & -11 & -7 & -58 \end{array}\right]$$

$$\xrightarrow[11R_2+R_3]{-3R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & \frac{13}{5} & \frac{41}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{18}{5} \\ 0 & 0 & -\frac{46}{5} & -\frac{92}{5} \end{array}\right]$$

$$\xrightarrow{-\frac{5}{46}R_3} \left[\begin{array}{ccc|c} 1 & 0 & \frac{13}{5} & \frac{41}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{18}{5} \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[\frac{1}{5}R_3+R_2]{-\frac{13}{5}R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

She should take 3 tablets of X, 4 tablets of Y, and 2 tablets of Z.

3. Let a, b, c, and d be the number of bags of foods A, B, C, and D, respectively. The corresponding system is

$$\begin{cases} 5a+5b+10c+5d=10{,}000 \\ 10a+5b+30c+10d=20{,}000 \\ 5a+15b+10c+25d=20{,}000 \end{cases}$$

Reduce the augmented coefficient matrix of the system.

$$\left[\begin{array}{cccc|c} 5 & 5 & 10 & 5 & 10{,}000 \\ 10 & 5 & 30 & 10 & 20{,}000 \\ 5 & 15 & 10 & 25 & 20{,}000 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2000 \\ 10 & 5 & 30 & 10 & 20{,}000 \\ 5 & 15 & 10 & 25 & 20{,}000 \end{array}\right]$$

$$\xrightarrow[-5R_1+R_3]{-10R_1+R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2000 \\ 0 & -5 & 10 & 0 & 0 \\ 0 & 10 & 0 & 20 & 10{,}000 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2000 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 10 & 0 & 20 & 10{,}000 \end{array}\right]$$

$$\xrightarrow[-10R_2+R_3]{-R_2+R_1} \left[\begin{array}{cccc|c} 1 & 0 & 4 & 1 & 2000 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 20 & 20 & 10{,}000 \end{array}\right]$$

$$\xrightarrow{\frac{1}{20}R_3} \left[\begin{array}{cccc|c} 0 & 0 & 4 & 0 & 2000 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 500 \end{array}\right]$$

$$\xrightarrow[2R_3+R_2]{-4R_3+R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 2 & 1000 \\ 0 & 0 & 1 & 1 & 500 \end{array}\right]$$

This reduced matrix corresponds to the system

$$\begin{cases} a-3d=0 \\ b+2d=1000 \\ c+d=500 \end{cases}$$

Letting $d = r$, we get the general solution of the system:

$a = 3r$

$b = -2r + 1000$

$c = -r + 500$

$d = r$

Note that a, b, c, and d cannot be negative, given the context, hence $0 \le r \le 500$. One specific solution is when $r = 250$, then $a = 750$, $b = 500$, $c = 250$, and $d = 250$.

Problems 6.4

1. The first nonzero entry in row 2 is not to the right of the first nonzero entry in row 1, hence not reduced.

2. Reduced.

3. Reduced.

4. In row 2, the first nonzero entry is in column 2, but not all other entries in column 2 are zeros, hence not reduced.

5. The first row consists entirely of zeros and is not below each row containing a nonzero entry, hence not reduced.

6. The first nonzero entry of row 2 is to the left of the first nonzero entry of row 1, hence not reduced.

7. $$\begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix} \xrightarrow{-4R_1+R_2} \begin{bmatrix} 1 & 3 \\ 0 & -12 \end{bmatrix} \xrightarrow{-\frac{1}{12}R_2} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow{-3R_2+R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

8. $$\begin{bmatrix} 0 & -3 & 0 & 2 \\ 1 & 5 & 0 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 5 & 0 & 2 \\ 0 & -3 & 0 & 2 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 5 & 0 & 2 \\ 0 & 1 & 0 & -\frac{2}{3} \end{bmatrix} \xrightarrow{-5R_2+R_1} \begin{bmatrix} 1 & 0 & 0 & \frac{16}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \end{bmatrix}$$

9. $$\begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

10. $\begin{bmatrix} 2 & 3 \\ 1 & -6 \\ 4 & 8 \\ 1 & 7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -6 \\ 2 & 3 \\ 4 & 8 \\ 1 & 7 \end{bmatrix} \xrightarrow[\substack{-4R_1+R_3 \\ -R_1+R_4}]{-2R_1+R_2} \begin{bmatrix} 1 & -6 \\ 0 & 15 \\ 0 & 32 \\ 0 & 13 \end{bmatrix} \xrightarrow{\frac{1}{15}R_2} \begin{bmatrix} 1 & -6 \\ 0 & 1 \\ 0 & 32 \\ 0 & 13 \end{bmatrix} \xrightarrow[\substack{-32R_2+R_3 \\ -13R_2+R_4}]{6R_2+R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 2 & 0 & 3 & 1 \\ 1 & 4 & 2 & 2 \\ -1 & 3 & 1 & 4 \\ 0 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 4 & 2 & 2 \\ 2 & 0 & 3 & 1 \\ -1 & 3 & 1 & 4 \\ 0 & 2 & 1 & 0 \end{bmatrix}$

$\xrightarrow[R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 4 & 2 & 2 \\ 0 & -8 & -1 & -3 \\ 0 & 7 & 3 & 6 \\ 0 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{8}R_2} \begin{bmatrix} 1 & 4 & 2 & 2 \\ 0 & 1 & \frac{1}{8} & \frac{3}{8} \\ 0 & 7 & 3 & 6 \\ 0 & 2 & 1 & 0 \end{bmatrix}$

$\xrightarrow[\substack{-7R_2+R_3 \\ -2R_2+R_4}]{-4R_2+R_1} \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{8} & \frac{3}{8} \\ 0 & 0 & \frac{17}{8} & \frac{27}{8} \\ 0 & 0 & \frac{3}{4} & -\frac{3}{4} \end{bmatrix} \xrightarrow{\frac{8}{17}R_3} \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{8} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{27}{17} \\ 0 & 0 & \frac{3}{4} & -\frac{3}{4} \end{bmatrix}$

$\xrightarrow[\substack{-\frac{1}{8}R_3+R_2 \\ -\frac{3}{4}R_3+R_4}]{-\frac{3}{2}R_3+R_1} \begin{bmatrix} 1 & 0 & 0 & -\frac{32}{17} \\ 0 & 1 & 0 & \frac{3}{17} \\ 0 & 0 & 1 & \frac{27}{17} \\ 0 & 0 & 0 & -\frac{33}{17} \end{bmatrix}$

$\xrightarrow{-\frac{17}{33}R_4} \begin{bmatrix} 1 & 0 & 0 & -\frac{32}{17} \\ 0 & 1 & 0 & \frac{3}{17} \\ 0 & 0 & 1 & \frac{27}{17} \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{-\frac{3}{17}R_4+R_2 \\ -\frac{27}{17}R_4+R_3}]{\frac{32}{17}R_4+R_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 1 \end{bmatrix}$

$\xrightarrow{-R_2} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{-4R_2+R_4} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[-R_3+R_4]{-\frac{3}{2}R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

13. $\left[\begin{array}{cc|c} 2 & -7 & 50 \\ 1 & 3 & 10 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 10 \\ 2 & -7 & 50 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 1 & 3 & 10 \\ 0 & -13 & 30 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 10 \\ 0 & 1 & -\frac{30}{13} \end{array}\right]$

$\to \left[\begin{array}{cc|c} 1 & 0 & \frac{220}{13} \\ 0 & 1 & -\frac{30}{13} \end{array}\right]$

Thus $x = \dfrac{220}{13}$ and $y = -\dfrac{30}{13}$.

14. $\left[\begin{array}{cc|c} 1 & -3 & -11 \\ 4 & 3 & 9 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & -11 \\ 0 & 15 & 53 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & -11 \\ 0 & 1 & \frac{53}{15} \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -\frac{2}{5} \\ 0 & 1 & \frac{53}{15} \end{array}\right]$

Thus $x = -\dfrac{2}{5}$, $y = \dfrac{53}{15}$.

15. $\left[\begin{array}{cc|c} 3 & 1 & 4 \\ 12 & 4 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 1 & 4 \\ 0 & 0 & -14 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & 0 & -14 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{array}\right]$

The last row indicates 0 = 1, which is never true, so there is no solution.

16. $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ -2 & -4 & 6 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

The last row indicates that 0 = 1, which is never true. There is no solution.

17. $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 0 & 2 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -6 & -1 & -7 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & \frac{1}{6} & \frac{7}{6} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & \frac{5}{3} \\ 0 & 1 & \frac{1}{6} & \frac{7}{6} \end{array}\right]$,

which gives $\begin{cases} x + \frac{2}{3}z = \frac{5}{3} \\ y + \frac{1}{6}z = \frac{7}{6} \end{cases}$.

Thus, $x = -\dfrac{2}{3}r + \dfrac{5}{3}$, $y = -\dfrac{1}{6}r + \dfrac{7}{6}$, $z = r$, where r is any real number.

18. $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 1 & 1 & 5 & 10 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -2 & 3 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & -\frac{3}{2} & -\frac{9}{2} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{13}{2} & \frac{29}{2} \\ 0 & 1 & -\frac{3}{2} & -\frac{9}{2} \end{array}\right]$

Thus $x = -\dfrac{13}{2}r + \dfrac{29}{2}$, $y = \dfrac{3}{2}r - \dfrac{9}{2}$, $z = r$, where r is any real number.

19. $\left[\begin{array}{cc|c} 1 & -3 & 0 \\ 2 & 2 & 3 \\ 5 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 8 & 3 \\ 0 & 14 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 1 & \frac{3}{8} \\ 0 & 14 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \frac{9}{8} \\ 0 & 1 & \frac{3}{8} \\ 0 & 0 & -\frac{17}{4} \end{array}\right]$

From the third row, $0 = -\dfrac{17}{4}$, which is never true, so there is no solution.

20. $\left[\begin{array}{cc|c} 1 & 4 & 9 \\ 3 & -1 & 6 \\ 1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 4 & 9 \\ 0 & -13 & -21 \\ 0 & -5 & -7 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 1 & 4 & 9 \\ 0 & 1 & \frac{21}{13} \\ 0 & -5 & -7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \frac{33}{13} \\ 0 & 1 & \frac{21}{13} \\ 0 & 0 & \frac{14}{13} \end{array}\right]$

$\to \left[\begin{array}{cc|c} 1 & 0 & \frac{33}{13} \\ 0 & 1 & \frac{21}{13} \\ 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$

The last row indicates that 0 = 1, which is never true. There is no solution.

21. $\left[\begin{array}{ccc|c} 1 & -1 & -3 & -5 \\ 2 & -1 & -4 & -8 \\ 1 & 1 & -1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & -3 & -5 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 2 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & 0 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$

Thus, $x = -3$, $y = 2$, $z = 0$.

22. $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 2 & -3 & -2 & 4 \\ 1 & -1 & -5 & 23 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & -5 & 0 & -10 \\ 0 & -2 & -4 & 16 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & -2 & -4 & 16 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & 20 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -5 \end{array}\right]$

Thus $x = 0$, $y = 2$, $z = -5$.

23. $\left[\begin{array}{ccc|c} 2 & 0 & -4 & 8 \\ 1 & -2 & -2 & 14 \\ 1 & 1 & -2 & -1 \\ 3 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 4 \\ 1 & -2 & -2 & 14 \\ 1 & 1 & -2 & -1 \\ 3 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 4 \\ 0 & -2 & 0 & 10 \\ 0 & 1 & 0 & -5 \\ 0 & 1 & 7 & -12 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & -2 & 0 & 10 \\ 0 & 1 & 7 & -12 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & -7 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Thus $x = 2$, $y = -5$, $z = -1$.

24. $\left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 3 & 2 & 11 & 1 \\ 1 & 1 & 4 & 1 \\ 2 & -3 & 3 & -8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & -3 & -3 & -6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -3 & -3 & -6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Thus $x = -3r - 1$, $y = -r + 2$, $z = r$, where r is any real number.

25. $\begin{bmatrix}1&-1&-1&-1&-1&|&0\\1&1&-1&-1&-1&|&0\\1&1&1&-1&-1&|&0\\1&1&1&1&-1&|&0\end{bmatrix}\to\begin{bmatrix}1&-1&-1&-1&-1&|&0\\0&2&0&0&0&|&0\\0&2&2&0&0&|&0\\0&2&2&2&0&|&0\end{bmatrix}$

$\to\begin{bmatrix}1&-1&-1&-1&-1&|&0\\0&1&0&0&0&|&0\\0&2&2&0&0&|&0\\0&2&2&2&0&|&0\end{bmatrix}\to\begin{bmatrix}1&0&-1&-1&-1&|&0\\0&1&0&0&0&|&0\\0&0&2&0&0&|&0\\0&0&2&2&0&|&0\end{bmatrix}$

$\to\begin{bmatrix}1&0&-1&-1&-1&|&0\\0&1&0&0&0&|&0\\0&0&1&0&0&|&0\\0&0&2&2&0&|&0\end{bmatrix}\to\begin{bmatrix}1&0&0&-1&-1&|&0\\0&1&0&0&0&|&0\\0&0&1&0&0&|&0\\0&0&0&2&0&|&0\end{bmatrix}$

$\to\begin{bmatrix}1&0&0&-1&-1&|&0\\0&1&0&0&0&|&0\\0&0&1&0&0&|&0\\0&0&0&1&0&|&0\end{bmatrix}\to\begin{bmatrix}1&0&0&0&-1&|&0\\0&1&0&0&0&|&0\\0&0&1&0&0&|&0\\0&0&0&1&0&|&0\end{bmatrix}$

Thus, $x_1 = r$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, and $x_5 = r$, where r is any number.

26. $\begin{bmatrix}1&1&1&-1&|&0\\1&-1&-1&1&|&0\\1&1&-1&-1&|&0\\1&1&-1&1&|&0\end{bmatrix}\to\begin{bmatrix}1&1&1&-1&|&0\\0&-2&-2&2&|&0\\0&0&-2&0&|&0\\0&0&-2&2&|&0\end{bmatrix}\to\begin{bmatrix}1&1&1&-1&|&0\\0&1&1&-1&|&0\\0&0&1&0&|&0\\0&0&1&-1&|&0\end{bmatrix}\to\begin{bmatrix}1&1&1&-1&|&0\\0&1&1&-1&|&0\\0&0&1&0&|&0\\0&0&0&1&|&0\end{bmatrix}$

$\to\begin{bmatrix}1&1&1&-1&|&0\\0&1&0&0&|&0\\0&0&1&0&|&0\\0&0&0&1&|&0\end{bmatrix}\to\begin{bmatrix}1&0&0&0&|&0\\0&1&0&0&|&0\\0&0&1&0&|&0\\0&0&0&1&|&0\end{bmatrix}$

Thus $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$.

27. Let x = federal tax and y = state tax. Then $x = 0.25(312{,}000 - y)$ and $y = 0.10(312{,}000 - x)$. Equivalently,

$$\begin{cases}x + 0.25y = 78{,}000\\0.10x + y = 31{,}200.\end{cases}$$

$\begin{bmatrix}1&0.25&|&78{,}000\\0.10&1&|&31{,}200\end{bmatrix}\to\begin{bmatrix}1&0.25&|&78{,}000\\0&0.975&|&23{,}400\end{bmatrix}\to\begin{bmatrix}1&0.25&|&78{,}000\\0&1&|&24{,}000\end{bmatrix}\to\begin{bmatrix}1&0&|&72{,}000\\0&1&|&24{,}000\end{bmatrix}.$

Thus x = 72,000 and y = 24,000, so the federal tax is $72,000 and the state tax is $24,000.

28. x = no. of units of A to be sold and y = no. of units of B to be sold. Then $x = 1.25y$ and $8x + 11y = 42{,}000$. Equivalently,

$$\begin{cases}x - 1.25y = 0,\\8x + 11y = 42{,}000.\end{cases}$$

$\begin{bmatrix}1&-1.25&|&0\\8&11&|&42{,}000\end{bmatrix}\to\begin{bmatrix}1&-1.25&|&0\\0&21&|&42{,}000\end{bmatrix}\to\begin{bmatrix}1&-1.25&|&0\\0&1&|&2000\end{bmatrix}\to\begin{bmatrix}1&0&|&2500\\0&1&|&2000\end{bmatrix}.$

Thus x = 2500 and y = 2000, so 2500 units of A and 2000 units of B must be sold.

29. Let x = number of units of A produced, y = number of units of B produced, and z = number of units of C produced. Then
no. of units: $x + y + z = 11{,}000$
total cost: $4x + 5y + 7z + 17{,}000 = 80{,}000$

total profit: $x + 2y + 3z = 25{,}000$
Equivalently,

$$\begin{cases} x+y+z=11{,}000 \\ 4x+5y+7z=63{,}000 \\ x+2y+3z=25{,}000 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 11{,}000 \\ 4 & 5 & 7 & 63{,}000 \\ 1 & 2 & 3 & 25{,}000 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 11{,}000 \\ 0 & 1 & 3 & 19{,}000 \\ 0 & 1 & 2 & 14{,}000 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -8{,}000 \\ 0 & 1 & 3 & 19{,}000 \\ 0 & 0 & -1 & -5{,}000 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -8{,}000 \\ 0 & 1 & 3 & 19{,}000 \\ 0 & 0 & 1 & 5{,}000 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2000 \\ 0 & 1 & 0 & 4000 \\ 0 & 0 & 1 & 5000 \end{array}\right]$$

Thus $x = 2000$, $y = 4000$, and $z = 5000$, so 2000 units of A, 4000 units of B and 5000 units of C should be produced.

30. Let x = number of desks to be produced at the East Coast plant and y = number of desks to be produced at the West Coast plant. Then $x + y = 800$ and $90x + 20{,}000 = 95y + 18{,}000$.
Equivalently,

$$\begin{cases} x+y=800 \\ 90x-95y=-2000. \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 800 \\ 90 & -95 & -2000 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 800 \\ 0 & -185 & -74{,}000 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 800 \\ 0 & 1 & 400 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 400 \\ 0 & 1 & 400 \end{array}\right]$$

$x = 400$ and $y = 400$
Thus the production order is 400 units at the East Coast plant and 400 units at the West Coast plant.

31. Let x = number of brand X pills, y = number of brand Y pills, and z = number of brand Z pills. Considering the unit requirements gives the system

$$\begin{cases} 2x+1y+1z=10 & \text{(vitamin A)} \\ 3x+3y+0z=9 & \text{(vitamin D)} \\ 5x+4y+1z=19 & \text{(vitamin E)} \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 3 & 0 & 9 \\ 5 & 4 & 1 & 19 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 5 \\ 3 & 3 & 0 & 9 \\ 5 & 4 & 1 & 19 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 5 \\ 0 & \frac{3}{2} & -\frac{3}{2} & -6 \\ 0 & \frac{3}{2} & -\frac{3}{2} & -6 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 5 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 7 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\begin{cases} x=7-r \\ y=r-4 \\ z=r \end{cases}$ where $r = 4, 5, 6, 7$.

The only solutions for the problem are $z = 4$, $x = 3$, and $y = 0$; $z = 5$, $x = 2$, and $y = 1$; $z = 6$, $x = 1$, and $y = 2$; $z = 7$, $x = 0$, and $y = 3$. Their respective costs (in cents) are 15, 23, 31, and 39.

a. The possible combinations are 3 of X, 4 of Z; 2 of X, 1 of Y, 5 of Z; 1 of X, 2 of Y, 6 of Z; 3 of Y, 7 of Z.

b. The combination 3 of X, 4 of Z costs 15 cents a day.

c. The least expensive combination is 3 of X, 4 of Z; the most expensive is 3 of Y, 7 of Z.

32. Let x, y, and z be the numbers of units of A, B, and C, respectively.

$$\begin{cases} 3x+1y+2z=490 & \text{(machine I)} \\ 1x+2y+1z=310 & \text{(machine (II)} \\ 2x+4y+1z=560 & \text{(machine III)} \end{cases}$$

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 490 \\ 1 & 2 & 1 & 310 \\ 2 & 4 & 1 & 560 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 310 \\ 3 & 1 & 2 & 490 \\ 2 & 4 & 1 & 560 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 310 \\ 0 & -5 & -1 & -440 \\ 0 & 0 & -1 & -60 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 310 \\ 0 & 1 & \frac{1}{5} & 88 \\ 0 & 0 & -1 & -60 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{5} & 134 \\ 0 & 1 & \frac{1}{5} & 88 \\ 0 & 0 & -1 & -60 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{5} & 134 \\ 0 & 1 & \frac{1}{5} & 88 \\ 0 & 0 & 1 & 60 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 98 \\ 0 & 1 & 0 & 76 \\ 0 & 0 & 1 & 60 \end{array}\right]$$

$x = 98$, $y = 76$, $z = 60$

Thus, 98 units of A, 76 units of B, and 60 units of C should be produced.

33. a. Let s, d, and g represent the number of units of S, D, and G, respectively. Then

$$\begin{cases} 12s+20d+32g=220 & \text{(stock A)} \\ 16s+12d+28g=176 & \text{(stock B)} \\ 8s+28d+36g=264 & \text{(stock C)} \end{cases}$$

$$\left[\begin{array}{ccc|c} 12 & 20 & 32 & 220 \\ 16 & 12 & 28 & 176 \\ 8 & 28 & 36 & 264 \end{array}\right] \xrightarrow[\left(\frac{1}{4}\right)R_2,\ \left(\frac{1}{8}\right)R_3]{\left(\frac{1}{4}\right)R_1} \left[\begin{array}{ccc|c} 3 & 5 & 8 & 55 \\ 4 & 3 & 7 & 44 \\ 1 & \frac{7}{2} & \frac{9}{2} & 33 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & \frac{7}{2} & \frac{9}{2} & 33 \\ 4 & 3 & 7 & 44 \\ 3 & 5 & 8 & 55 \end{array}\right]$$

$$\xrightarrow[-3R_1+R_3]{-4R_1+R_2} \left[\begin{array}{ccc|c} 1 & \frac{7}{2} & \frac{9}{2} & 33 \\ 0 & -11 & -11 & -88 \\ 0 & -\frac{11}{2} & -\frac{11}{2} & -44 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{11}R_2} \left[\begin{array}{ccc|c} 1 & \frac{7}{2} & \frac{9}{2} & 33 \\ 0 & 1 & 1 & 8 \\ 0 & -\frac{11}{2} & -\frac{11}{2} & -44 \end{array}\right]$$

$$\xrightarrow[\frac{11}{2}R_2+R_3]{-\frac{7}{2}R_2+R_1}\left[\begin{array}{ccc|c}1&0&1&5\\0&1&1&8\\0&0&0&0\end{array}\right]$$

Thus $s = 5 - r$, $d = 8 - r$, and $g = r$, where $r = 0, 1, 2, 3, 4, 5$.
The six possible combinations are given by

COMBINATION						
r	0	1	2	3	4	5
S	5	4	3	2	1	0
D	8	7	6	5	4	3
G	0	1	2	3	4	5

b. Computing the cost of each combination, we find that they are 4700, 4600, 4500, 4400, 4300, and 4200 dollars, respectively. Buying 3 units of Deluxe and 5 units of Gold Star ($s = 0$, $d = 3$, $g = 5$) minimizes the cost.

Principles in Practice 6.5

1. Write the coefficients matrix and reduce.

$$\begin{bmatrix}5&3&4\\6&8&7\\3&1&2\end{bmatrix}\xrightarrow{\frac{1}{5}R_1}\begin{bmatrix}1&\frac{3}{5}&\frac{4}{5}\\6&8&7\\3&1&2\end{bmatrix}\xrightarrow[-3R_1+R_3]{-6R_1+R_2}\begin{bmatrix}1&\frac{3}{5}&\frac{4}{5}\\0&\frac{22}{5}&\frac{11}{5}\\0&-\frac{4}{5}&-\frac{2}{5}\end{bmatrix}$$

$$\xrightarrow{\frac{5}{22}R_2}\begin{bmatrix}1&\frac{3}{5}&\frac{4}{5}\\0&1&\frac{1}{2}\\0&-\frac{4}{5}&-\frac{2}{5}\end{bmatrix}\xrightarrow[\frac{4}{5}R_2+R_3]{-\frac{3}{5}R_2+R_1}\begin{bmatrix}1&0&\frac{1}{2}\\0&1&\frac{1}{2}\\0&0&0\end{bmatrix}$$

The system has infinitely many solutions since there are two nonzero rows in the reduced coefficient matrix.

$$x+\frac{1}{2}z=0$$

$$y+\frac{1}{2}z=0$$

Let $z = r$, so $x=-\frac{1}{2}r$ and $y=-\frac{1}{2}r$, where r is any real number.

Problems 6.5

1. $$\left[\begin{array}{cccc|c}1&1&-1&-9&-3\\2&3&2&15&12\\2&1&2&5&8\end{array}\right]\to\left[\begin{array}{cccc|c}1&1&-1&-9&3\\0&1&4&33&18\\0&-1&4&23&14\end{array}\right]\to\left[\begin{array}{cccc|c}1&1&-1&-9&-3\\0&1&4&33&18\\0&0&8&56&32\end{array}\right]\to\left[\begin{array}{cccc|c}1&1&-1&-9&-3\\0&1&4&33&18\\0&0&1&7&4\end{array}\right]$$

$$\to\left[\begin{array}{cccc|c}1&1&0&-2&1\\0&1&0&5&2\\0&0&1&7&4\end{array}\right]\to\left[\begin{array}{cccc|c}1&0&0&-7&-1\\0&1&0&5&2\\0&0&1&7&4\end{array}\right]$$

Thus $w = -1 + 7r$, $x = 2 - 5r$, $y = 4 - 7r$, $z = r$ (where r is any real number).

2. $\left[\begin{array}{cccc|c} 2 & 1 & 10 & 15 & -5 \\ 1 & -5 & 2 & 15 & -10 \\ 1 & 1 & 6 & 12 & 9 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -5 & 2 & 15 & -10 \\ 2 & 1 & 10 & 15 & -5 \\ 1 & 1 & 6 & 12 & 9 \end{array}\right]$

$\to \left[\begin{array}{cccc|c} 1 & -5 & 2 & 15 & -10 \\ 0 & 11 & 6 & -15 & 15 \\ 0 & 6 & 4 & -3 & 19 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -5 & 2 & 15 & -10 \\ 0 & 1 & \frac{6}{11} & -\frac{15}{11} & \frac{15}{11} \\ 0 & 6 & 4 & -3 & 19 \end{array}\right]$

$\to \left[\begin{array}{cccc|c} 1 & 0 & \frac{52}{11} & \frac{90}{11} & -\frac{35}{11} \\ 0 & 1 & \frac{6}{11} & -\frac{15}{11} & \frac{15}{11} \\ 0 & 0 & \frac{8}{11} & \frac{57}{11} & \frac{119}{11} \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & \frac{52}{11} & \frac{90}{11} & -\frac{35}{11} \\ 0 & 1 & \frac{6}{11} & -\frac{15}{11} & \frac{15}{11} \\ 0 & 0 & 1 & \frac{57}{8} & \frac{119}{8} \end{array}\right]$

$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{51}{2} & -\frac{147}{2} \\ 0 & 1 & 0 & -\frac{21}{4} & -\frac{27}{4} \\ 0 & 0 & 1 & \frac{57}{8} & \frac{119}{8} \end{array}\right]$

Thus, $w = \frac{51}{2}r - \frac{147}{2}$, $x = \frac{21}{4}r - \frac{27}{4}$, $y = -\frac{57}{8}r + \frac{119}{8}$, $z = r$ (where r is any real number).

3. $\left[\begin{array}{cccc|c} 3 & -1 & -3 & -1 & -2 \\ 2 & -2 & -6 & -6 & -4 \\ 2 & -1 & -3 & -2 & -2 \\ 3 & 1 & 3 & 7 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & -1 & -\frac{1}{3} & -\frac{2}{3} \\ 2 & -2 & -6 & -6 & -4 \\ 2 & -1 & -3 & -2 & -2 \\ 3 & 1 & 3 & 7 & 2 \end{array}\right]$

$\to \left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & -1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & -\frac{4}{3} & -4 & -\frac{16}{3} & -\frac{8}{3} \\ 0 & -\frac{1}{3} & -1 & -\frac{4}{3} & -\frac{2}{3} \\ 0 & 2 & 6 & 8 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & -1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 3 & 4 & 2 \\ 0 & -\frac{1}{3} & -1 & -\frac{4}{3} & -\frac{2}{3} \\ 0 & 2 & 6 & 8 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Thus, $w = -s$, $x = -3r - 4s + 2$, $y = r$, $z = s$ (where r and s are any real numbers).

4. $\left[\begin{array}{cccc|c} 1 & 1 & 0 & 5 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & -3 & 4 & -7 & 1 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 5 & 1 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & -4 & 4 & -12 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 5 & 1 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & -4 & 4 & -12 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Thus, $w = -r - 2s + 1$, $x = r - 3s$, $y = r$, $z = s$ (where r and s are any real numbers).

5. $\left[\begin{array}{cccc|c} 1 & 1 & 3 & -1 & 2 \\ 2 & 1 & 5 & -2 & 0 \\ 2 & -1 & 3 & -2 & -8 \\ 3 & 2 & 8 & -3 & 2 \\ 1 & 0 & 2 & -1 & -2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 3 & -1 & 2 \\ 0 & -1 & -1 & 0 & -4 \\ 0 & -3 & -3 & 0 & -12 \\ 0 & -1 & -1 & 0 & -4 \\ 0 & -1 & -1 & 0 & -4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 3 & -1 & 2 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & -3 & -3 & 0 & -12 \\ 0 & -1 & -1 & 0 & -4 \\ 0 & -1 & -1 & 0 & -4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & -2 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Thus, $w = -2r + s - 2$, $x = -r + 4$, $y = r$, $z = s$ (where r and s are any real numbers).

6. $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 4 \\ 2 & 1 & 2 & 2 & 7 \\ 1 & 2 & 1 & 4 & 5 \\ 3 & -2 & 3 & -4 & 7 \\ 4 & -3 & 4 & -6 & 9 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 4 \\ 0 & -1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & -5 & 0 & -10 & -5 \\ 0 & -7 & 0 & -14 & -7 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & -5 & 0 & -10 & -5 \\ 0 & -7 & 0 & -14 & -7 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Thus, $w = -r + 3$, $x = -2s + 1$, $y = r$, $z = s$ (where r and s are any real numbers).

7. $\left[\begin{array}{ccccc|c} 4 & -3 & 5 & -10 & 11 & -8 \\ 2 & 1 & 5 & 0 & 3 & 6 \end{array}\right] \to \left[\begin{array}{ccccc|c} 0 & -5 & -5 & -10 & 5 & -20 \\ 2 & 1 & 5 & 0 & 3 & 6 \end{array}\right]$

$\to \left[\begin{array}{ccccc|c} 2 & 1 & 5 & 0 & 3 & 6 \\ 0 & -5 & -5 & -10 & 5 & -20 \end{array}\right] \to \left[\begin{array}{ccccc|c} 2 & 1 & 5 & 0 & 3 & 6 \\ 0 & 1 & 1 & 2 & -1 & 4 \end{array}\right]$

$\to \left[\begin{array}{ccccc|c} 2 & 0 & 4 & -2 & 4 & 2 \\ 0 & 1 & 1 & 2 & -1 & 4 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 1 \\ 0 & 1 & 1 & 2 & -1 & 4 \end{array}\right]$

Thus, $x_1 = -2r + s - 2t + 1$, $x_2 = -r - 2s + t + 4$, $x_3 = r$, $x_4 = s$, $x_5 = t$ (where r, s, and t are any real numbers).

8. $\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 4 & 1 \\ 0 & 1 & 1 & -2 & 0 & 0 \\ 2 & -2 & 3 & 10 & 15 & 10 \\ 1 & 2 & 3 & -2 & 2 & -2 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 4 & 1 \\ 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & -2 & -3 & 8 & 7 & 8 \\ 0 & 2 & 0 & -3 & -2 & -3 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 4 & 1 \\ 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 4 & 7 & 8 \\ 0 & 0 & -2 & 1 & -2 & -3 \end{array}\right]$

$\to \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 4 & 1 \\ 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -4 & -7 & -8 \\ 0 & 0 & 0 & -7 & -16 & -19 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & \frac{12}{7} & -\frac{12}{7} \\ 0 & 1 & 1 & 0 & \frac{32}{7} & \frac{38}{7} \\ 0 & 0 & 1 & 0 & \frac{15}{7} & \frac{20}{7} \\ 0 & 0 & 0 & 1 & \frac{16}{7} & \frac{19}{7} \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -\frac{33}{7} & -\frac{72}{7} \\ 0 & 1 & 0 & 0 & \frac{17}{7} & \frac{18}{7} \\ 0 & 0 & 1 & 0 & \frac{15}{7} & \frac{20}{7} \\ 0 & 0 & 0 & 1 & \frac{16}{7} & \frac{19}{7} \end{array}\right]$

Thus $x_1 = -\frac{72}{7} + \frac{33}{7}r$, $x_2 = \frac{18}{7} - \frac{17}{7}r$, $x_3 = \frac{20}{7} - \frac{15}{7}r$, $x_4 = \frac{19}{7} - \frac{16}{7}r$, and $x_5 = r$, where r is any real number.

9. The system is homogeneous with fewer equations than unknowns (2 < 3), so there are infinitely many solutions.

10. The system is homogeneous with fewer equations than unknowns (2 < 4), so there are infinitely many solutions.

11. $\begin{bmatrix} 3 & -4 \\ 1 & 5 \\ 4 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 \\ 3 & -4 \\ 4 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 \\ 0 & -19 \\ 0 & -21 \end{bmatrix} \to \begin{bmatrix} 1 & 5 \\ 0 & 1 \\ 0 & -21 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \mathbf{A}$

A has $k = 2$ nonzero rows. Number of unknowns is $n = 2$. Thus $k = n$, so the system has the trivial solution only.

12. $\begin{bmatrix} 2 & 3 & 12 \\ 3 & -2 & 5 \\ 4 & 1 & 14 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & 6 \\ 3 & -2 & 5 \\ 4 & 1 & 14 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & 6 \\ 0 & -\frac{13}{2} & -13 \\ 0 & -5 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & 6 \\ 0 & 1 & 2 \\ 0 & -5 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{A}$

A has $k = 2$ nonzero rows. Number of unknowns is $n = 3$. Thus $k < n$, so the system has infinitely many solutions.

13. $\begin{bmatrix}1&1&1\\1&0&-1\\1&-2&-5\end{bmatrix}\to\begin{bmatrix}1&1&1\\0&-1&-2\\0&-3&-6\end{bmatrix}\to\begin{bmatrix}1&1&1\\0&1&2\\0&-3&-6\end{bmatrix}\to\begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix}=\mathbf{A}$

A has $k = 2$ nonzero rows. Number of unknowns is $n = 3$. Thus $k < n$, so the system has infinitely many solutions.

14. $\begin{bmatrix}3&2&-2\\2&2&-2\\0&-4&5\end{bmatrix}\to\begin{bmatrix}1&\frac{2}{3}&-\frac{2}{3}\\2&2&-2\\0&-4&5\end{bmatrix}\to\begin{bmatrix}1&\frac{2}{3}&-\frac{2}{3}\\0&\frac{2}{3}&-\frac{2}{3}\\0&-4&5\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&-1\\0&0&1\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\mathbf{A}$

A has $k = 3$ nonzero rows. Number of unknowns is $n = 3$. Thus $k = n$, so the system has the trivial solution only.

15. $\begin{bmatrix}1&1\\3&-4\end{bmatrix}\to\begin{bmatrix}1&1\\0&-7\end{bmatrix}\to\begin{bmatrix}1&1\\0&1\end{bmatrix}\to\begin{bmatrix}1&0\\0&1\end{bmatrix}$

The solution is $x = 0$, $y = 0$.

16. $\begin{bmatrix}2&-5\\8&-20\end{bmatrix}\to\begin{bmatrix}2&-5\\0&0\end{bmatrix}\to\begin{bmatrix}1&-\frac{5}{2}\\0&0\end{bmatrix}$

The solution is $x=\frac{5}{2}r$, $y = r$.

17. $\begin{bmatrix}1&6&-2\\2&-3&4\end{bmatrix}\to\begin{bmatrix}1&6&-2\\0&-15&8\end{bmatrix}\to\begin{bmatrix}1&6&-2\\0&1&-\frac{8}{15}\end{bmatrix}\to\begin{bmatrix}1&0&\frac{6}{5}\\0&1&-\frac{8}{15}\end{bmatrix}$

The solution is $x=-\frac{6}{5}r$, $y=\frac{8}{15}r$, $z = r$.

18. $\begin{bmatrix}4&7\\2&3\end{bmatrix}\to\begin{bmatrix}1&\frac{7}{4}\\2&3\end{bmatrix}\to\begin{bmatrix}1&\frac{7}{4}\\0&-\frac{1}{2}\end{bmatrix}\to\begin{bmatrix}1&\frac{7}{4}\\0&1\end{bmatrix}\to\begin{bmatrix}1&0\\0&1\end{bmatrix}$

The solution is $x = 0$, $y = 0$.

19. $\begin{bmatrix}1&1\\3&-4\\5&-8\end{bmatrix}\to\begin{bmatrix}1&1\\0&-7\\0&-13\end{bmatrix}\to\begin{bmatrix}1&1\\0&1\\0&-13\end{bmatrix}\to\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$

The solution is $x = 0$, $y = 0$.

20. $\begin{bmatrix}2&-3&1\\1&2&-1\\1&1&1\end{bmatrix}\to\begin{bmatrix}0&-5&-1\\0&1&-2\\1&1&1\end{bmatrix}\to\begin{bmatrix}1&1&1\\0&1&-2\\0&-5&-1\end{bmatrix}\to\begin{bmatrix}1&0&3\\0&1&-2\\0&0&-11\end{bmatrix}\to\begin{bmatrix}1&0&3\\0&1&-2\\0&0&1\end{bmatrix}\to\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$

The solution is $x = 0$, $y = 0$, $z = 0$.

21. $\begin{bmatrix}1&1&1\\0&-7&-14\\0&-2&-4\\0&-5&-10\end{bmatrix}\to\begin{bmatrix}1&1&1\\0&1&2\\0&-2&-4\\0&-5&-10\end{bmatrix}\to\begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\\0&0&0\end{bmatrix}$

The solution is $x = r$, $y = -2r$, $z = r$.

22. $\begin{bmatrix} 1 & 1 & 7 \\ 1 & -1 & -1 \\ 2 & -3 & -6 \\ 3 & 1 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 7 \\ 0 & -2 & -8 \\ 0 & -5 & -20 \\ 0 & -2 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 7 \\ 0 & 1 & 4 \\ 0 & -5 & -20 \\ 0 & -2 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The solution is $x = -3r$, $y = -4r$, $z = r$.

23. $\begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 1 & 0 & 5 \\ 2 & 1 & 3 & 4 \\ 1 & -3 & 2 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & -4 \\ 0 & -4 & 1 & -13 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -1 & 1 & -4 \\ 0 & 0 & -1 & 1 \\ 0 & -4 & 1 & -13 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -1 & 1 \\ 0 & -4 & 1 & -13 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The solution is $w = -2r$, $x = -3r$, $y = r$, $z = r$.

24. $\begin{bmatrix} 1 & 1 & 2 & 7 \\ 1 & -2 & -1 & 1 \\ 1 & 2 & 3 & 9 \\ 2 & -3 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 7 \\ 0 & -3 & -3 & -6 \\ 0 & 1 & 1 & 2 \\ 0 & -5 & -5 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -5 & -5 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The solution is $w = -r - 5s$, $x = -r - 2s$, $y = r$, $z = s$.

Principles in Practice 6.6

1. $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Yes, they are inverses.

2. $\begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} 28 \\ 46 \end{bmatrix} = \begin{bmatrix} 13 \\ 5 \end{bmatrix} = \begin{bmatrix} \text{M} \\ \text{E} \end{bmatrix}$

$\begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} 65 \\ 90 \end{bmatrix} = \begin{bmatrix} 5 \\ 20 \end{bmatrix} = \begin{bmatrix} \text{E} \\ \text{T} \end{bmatrix}$

$\begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} 61 \\ 82 \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix} = \begin{bmatrix} \text{A} \\ \text{T} \end{bmatrix}$

$\begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} 59 \\ 88 \end{bmatrix} = \begin{bmatrix} 14 \\ 15 \end{bmatrix} = \begin{bmatrix} \text{N} \\ \text{O} \end{bmatrix}$

$\begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} 57 \\ 86 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \end{bmatrix} = \begin{bmatrix} \text{O} \\ \text{N} \end{bmatrix}$

$\begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} 60 \\ 84 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix} = \begin{bmatrix} \text{F} \\ \text{R} \end{bmatrix}$

$\begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} 21 \\ 34 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} \text{I} \\ \text{D} \end{bmatrix}$

$\begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} 76 \\ 102 \end{bmatrix} = \begin{bmatrix} 1 \\ 25 \end{bmatrix} = \begin{bmatrix} \text{A} \\ \text{Y} \end{bmatrix}$

The message is “MEET AT NOON FRIDAY.”

3. $[\mathbf{E}\,|\,\mathbf{I}] = \left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 2 & 2 & 2 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & -2 & -1 & 1 & -\frac{3}{2} & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & -2 & -1 & 1 & -\frac{3}{2} & 0 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & -2 & -1 & 1 & -\frac{3}{2} & 0 \end{array}\right]$$

$$\xrightarrow[2R_2+R_3]{-R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & -\frac{1}{2} & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & -3 & 1 & \frac{1}{2} & -2 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & -\frac{1}{2} & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{array}\right]$$

$$\xrightarrow[R_3+R_2]{-2R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{5}{6} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{array}\right]$$

$$\mathbf{E}^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{bmatrix}$$

$$[\mathbf{F}\,|\,\mathbf{I}] = \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 4 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 4 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-4R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{3}{2} & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{2R_2} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-R_2+R_3]{-\frac{1}{2}R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

F does not reduce to **I** so **F** is not invertible.

4. Let x be the number of shares of A, y be the number of shares of B, and z be the number of shares of C. We get the following equations from the given conditions.

$50x + 20y + 80z = 500{,}000$

$x = 2z$

$0.13(50x) + 0.15(20y) + 0.10(80z)$
$= 0.12(50x + 20y + 80z)$

Simplify the first equation.

$5x + 2y + 8z = 50{,}000$

Simplify the second equation.

$x - 2z = 0$

Simplify the third equation.

$6.5x + 3y + 8z = 6x + 2.4y + 9.6z$

$0.5x + 0.6y - 1.6z = 0$

$5x + 6y - 16z = 0$

Thus, we solve the following system of equations.

$x - 2z = 0$

$5x + 6y - 16z = 0$

$5x + 2y + 8z = 50{,}000$

The coefficient matrix is $\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 \\ 5 & 6 & -16 \\ 5 & 2 & 8 \end{bmatrix}$.

$$[\mathbf{A}\,|\,\mathbf{I}] = \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 5 & 6 & -16 & 0 & 1 & 0 \\ 5 & 2 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-5R_1+R_3]{-5R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 6 & -6 & -5 & 1 & 0 \\ 0 & 2 & 18 & -5 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{6}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -\frac{5}{6} & \frac{1}{6} & 0 \\ 0 & 2 & 18 & -5 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -\frac{5}{6} & \frac{1}{6} & 0 \\ 0 & 0 & 20 & -\frac{10}{3} & -\frac{1}{3} & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{20}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -\frac{5}{6} & \frac{1}{6} & 0 \\ 0 & 0 & 1 & -\frac{1}{6} & -\frac{1}{60} & \frac{1}{20} \end{array}\right]$$

$$\xrightarrow[R_3+R_2]{2R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & -\frac{1}{30} & \frac{1}{10} \\ 0 & 1 & 0 & -1 & \frac{3}{20} & \frac{1}{20} \\ 0 & 0 & 1 & -\frac{1}{6} & -\frac{1}{60} & \frac{1}{20} \end{array}\right]$$

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{30} & \frac{1}{10} \\ -1 & \frac{3}{20} & \frac{1}{20} \\ -\frac{1}{6} & -\frac{1}{60} & \frac{1}{20} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{30} & \frac{1}{10} \\ -1 & \frac{3}{20} & \frac{1}{20} \\ -\frac{1}{6} & -\frac{1}{60} & \frac{1}{20} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 50{,}000 \end{bmatrix} = \begin{bmatrix} 5000 \\ 2500 \\ 2500 \end{bmatrix}$$

They should buy 5000 shares of Company A, 2500 shares of Company B, and 2500 shares of Company C.

Problems 6.6

1. $\left[\begin{array}{cc|cc} 6 & 1 & 1 & 0 \\ 7 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{1}{6} & \frac{1}{6} & 0 \\ 7 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & -\frac{1}{6} & -\frac{7}{6} & 1 \end{array}\right]$

$\to \left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & -\frac{1}{6} & -\frac{7}{6} & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 7 & -6 \end{array}\right]$

The inverse is $\begin{bmatrix} -1 & 1 \\ 7 & -6 \end{bmatrix}$.

2. $\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 3 & 6 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 1 & 2 & 0 & \frac{1}{3} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{3} \end{array}\right]$

The given matrix is not invertible.

3. $\left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & 1 \end{array}\right]$

The given matrix is not invertible.

4. $\left[\begin{array}{cc|cc} \frac{1}{4} & \frac{3}{8} & 1 & 0 \\ 0 & -\frac{1}{6} & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & 4 & 0 \\ 0 & 1 & 0 & -6 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 4 & 9 \\ 0 & 1 & 0 & -6 \end{array}\right]$

The inverse is $\begin{bmatrix} 4 & 9 \\ 0 & -6 \end{bmatrix}$.

5. $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{4} \end{array}\right]$

The inverse is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$.

6. $\left[\begin{array}{ccc|ccc} 2 & 0 & 8 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ -1 & 4 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 4 & 4 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -8 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 1 & -8 & -1 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 0 & -9 & -\frac{9}{8} & -\frac{1}{4} & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{1}{9} & \frac{4}{9} \\ 0 & 1 & 0 & 0 & \frac{2}{9} & \frac{1}{9} \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{array}\right]$

The inverse is $\begin{bmatrix} 0 & -\frac{1}{9} & \frac{4}{9} \\ 0 & \frac{2}{9} & \frac{1}{9} \\ \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{bmatrix}$.

7. $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & -\frac{5}{4} & 1 \end{array}\right]$

The given matrix is not invertible.

8. $\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$

The given matrix is not invertible.

9. The matrix is not square, so it is not invertible.

10. For any 3 × 3 matrix **B**, $\mathbf{B}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \mathbf{I}.$

Thus the matrix is not invertible.

11. $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

The inverse is $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$

12. $\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -3 & 3 & -1 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -9 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 15 & -1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -9 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{15} & \frac{1}{5} & \frac{1}{15} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{5} & -\frac{1}{5} & \frac{3}{5} \\ 0 & 1 & 0 & \frac{4}{15} & \frac{1}{5} & -\frac{4}{15} \\ 0 & 0 & 1 & -\frac{1}{15} & \frac{1}{5} & \frac{1}{15} \end{array}\right]$

The inverse is $\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} & \frac{3}{5} \\ \frac{4}{15} & \frac{1}{5} & -\frac{4}{15} \\ -\frac{1}{15} & \frac{1}{5} & \frac{1}{15} \end{bmatrix}.$

13. $$\left[\begin{array}{ccc|ccc} 7 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{2}{7} & \frac{1}{7} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{2}{7} & \frac{1}{7} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{7} & \frac{3}{7} & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{7} & \frac{3}{7} & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 7 \end{array}\right]$$

The inverse is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & 7 \end{bmatrix}$.

14. $$\left[\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ -1 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & -3 & 0 & 0 & -1 \\ 2 & 3 & -1 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & -4 & 0 & -1 & -1 \\ 0 & -1 & -3 & 1 & -2 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 & 1 & 1 \\ 0 & -1 & -3 & 1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & -4 & 5 & -3 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -8 & 5 \\ 0 & 1 & 0 & -4 & 5 & -3 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

The inverse is $\begin{bmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix}$.

15. $$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 4 & -1 & 5 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 4 & -1 & 5 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 3 & -3 & 0 & 1 & -4 \\ 0 & 3 & -4 & 1 & 0 & -2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & \frac{1}{3} & -\frac{4}{3} \\ 0 & 3 & -4 & 1 & 0 & -2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & -1 & 0 & \frac{1}{3} & -\frac{4}{3} \\ 0 & 0 & -1 & 1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & -1 & 0 & \frac{1}{3} & -\frac{4}{3} \\ 0 & 0 & 1 & -1 & 1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{2}{3} & \frac{5}{3} \\ 0 & 1 & 0 & -1 & \frac{4}{3} & -\frac{10}{3} \\ 0 & 0 & 1 & -1 & 1 & -2 \end{array}\right].$$

The inverse is $\begin{bmatrix} 1 & -\frac{2}{3} & \frac{5}{3} \\ -1 & \frac{4}{3} & -\frac{10}{3} \\ -1 & 1 & -2 \end{bmatrix}$.

16. $$\left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 1 & -\frac{6}{5} & \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & -\frac{6}{5} & \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{8}{5} & -\frac{6}{5} & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & -\frac{6}{5} & \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 4 & -3 \\ 0 & 1 & 0 & 10 & -7 & 6 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right]$$

The inverse is $\begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$.

17. $$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 1 & 0 \\ 1 & 5 & 12 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 9 & -1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 3 & -2 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 3 & 2 & -3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 3 & -2 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & -1 & \frac{1}{3} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{11}{3} & -3 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{7}{3} & 3 & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -1 & \frac{1}{3} \end{array}\right]$$

The inverse is $\begin{bmatrix} \frac{11}{3} & -3 & \frac{1}{3} \\ -\frac{7}{3} & 3 & -\frac{2}{3} \\ \frac{2}{3} & -1 & \frac{1}{3} \end{bmatrix}$.

18. $$\left[\begin{array}{ccc|ccc} 2 & -1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & -1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & -2 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

The inverse is $\begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

19. $\mathbf{X}=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\mathbf{A}^{-1}\mathbf{B}=\begin{bmatrix}1&2\\8&1\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}10\\20\end{bmatrix}\Rightarrow x_1=10,\ x_2=20$

20. $\mathbf{X}=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\mathbf{A}^{-1}\mathbf{B}=\begin{bmatrix}1&0&1\\0&3&0\\2&0&4\end{bmatrix}\begin{bmatrix}10\\2\\-1\end{bmatrix}=\begin{bmatrix}9\\6\\16\end{bmatrix}\Rightarrow x_1=9,\ x_2=6,\ x_3=16$

21. $\left[\begin{array}{cc|cc}6&5&1&0\\1&1&0&1\end{array}\right]\to\left[\begin{array}{cc|cc}1&1&0&1\\6&5&1&0\end{array}\right]\to\left[\begin{array}{cc|cc}1&1&0&1\\0&-1&1&-6\end{array}\right]\to\left[\begin{array}{cc|cc}1&1&0&1\\0&1&-1&6\end{array}\right]\to\left[\begin{array}{cc|cc}1&0&1&-5\\0&1&-1&6\end{array}\right]$

$\begin{bmatrix}x\\y\end{bmatrix}=\mathbf{A}^{-1}\mathbf{B}=\begin{bmatrix}1&-5\\-1&6\end{bmatrix}\begin{bmatrix}2\\-3\end{bmatrix}=\begin{bmatrix}17\\-20\end{bmatrix}\Rightarrow x=17,\ y=-20$

22. $\left[\begin{array}{cc|cc}2&4&1&0\\-1&3&0&1\end{array}\right]\to\left[\begin{array}{cc|cc}1&2&\frac{1}{2}&0\\-1&3&0&1\end{array}\right]\to\left[\begin{array}{cc|cc}1&2&\frac{1}{2}&0\\0&5&\frac{1}{2}&1\end{array}\right]\to\left[\begin{array}{cc|cc}1&0&\frac{3}{10}&-\frac{2}{5}\\0&1&\frac{1}{10}&\frac{1}{5}\end{array}\right]$

$\begin{bmatrix}x\\y\end{bmatrix}=\mathbf{A}^{-1}\mathbf{B}=\begin{bmatrix}\frac{3}{10}&-\frac{2}{5}\\\frac{1}{10}&\frac{1}{5}\end{bmatrix}\begin{bmatrix}5\\-2\end{bmatrix}=\begin{bmatrix}\frac{23}{10}\\\frac{1}{10}\end{bmatrix}\Rightarrow x=\frac{23}{10},\ y=\frac{1}{10}$

23. $\left[\begin{array}{cc|cc}3&1&1&0\\3&-1&0&1\end{array}\right]\to\left[\begin{array}{cc|cc}1&\frac{1}{3}&\frac{1}{3}&0\\3&-1&0&1\end{array}\right]\to\left[\begin{array}{cc|cc}1&\frac{1}{3}&\frac{1}{3}&0\\0&-2&-1&1\end{array}\right]\to\left[\begin{array}{cc|cc}1&\frac{1}{3}&\frac{1}{3}&0\\0&1&\frac{1}{2}&-\frac{1}{2}\end{array}\right]$

$\to\left[\begin{array}{cc|cc}1&0&\frac{1}{6}&\frac{1}{6}\\0&1&\frac{1}{2}&-\frac{1}{2}\end{array}\right]$

$\begin{bmatrix}x\\y\end{bmatrix}=\mathbf{A}^{-1}\mathbf{B}=\begin{bmatrix}\frac{1}{6}&\frac{1}{6}\\\frac{1}{2}&-\frac{1}{2}\end{bmatrix}\begin{bmatrix}5\\7\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}\Rightarrow x=2,\ y=-1$

24. $\left[\begin{array}{cc|cc}3&2&1&0\\4&3&0&1\end{array}\right]\to\left[\begin{array}{cc|cc}1&\frac{2}{3}&\frac{1}{3}&0\\4&3&0&1\end{array}\right]\to\left[\begin{array}{cc|cc}1&\frac{2}{3}&\frac{1}{3}&0\\0&\frac{1}{3}&-\frac{4}{3}&1\end{array}\right]\to\left[\begin{array}{cc|cc}1&0&3&-2\\0&\frac{1}{3}&-\frac{4}{3}&1\end{array}\right]\to\left[\begin{array}{cc|cc}1&0&3&-2\\0&1&-4&3\end{array}\right]$

$\begin{bmatrix}x\\y\end{bmatrix}=\mathbf{A}^{-1}\mathbf{B}=\begin{bmatrix}3&-2\\-4&3\end{bmatrix}\begin{bmatrix}26\\37\end{bmatrix}=\begin{bmatrix}4\\7\end{bmatrix}\Rightarrow x=4,\ y=7$

25. The coefficient matrix is not invertible. The method of reduction yields

$\left[\begin{array}{cc|c}2&6&2\\3&9&3\end{array}\right]\to\left[\begin{array}{cc|c}1&3&1\\3&9&3\end{array}\right]\to\left[\begin{array}{cc|c}1&3&1\\0&0&0\end{array}\right].$

Thus $x=-3r+1$, $y=r$.

26. The coefficient matrix is not invertible. The method of reduction yields

$\left[\begin{array}{cc|c}2&6&8\\3&9&7\end{array}\right]\to\left[\begin{array}{cc|c}1&3&4\\3&9&7\end{array}\right]\to\left[\begin{array}{cc|c}1&3&4\\0&0&-5\end{array}\right].$

Second row indicates $0=-5$, which is never true, so there is no solution.

27. $\left[\begin{array}{rrr|rrr} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -6 & -2 & -3 & 1 & 0 \\ 0 & -3 & 0 & -1 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{rrr|rrr} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} & 0 \\ 0 & -3 & 0 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right]$

$\to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{1}{6} & \frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right]$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{2} & -\frac{1}{3} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

Thus, $x = 0$, $y = 1$, $z = 2$.

28. $\left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 & -1 & 1 \end{array}\right]$

$\to \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 6 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ \frac{7}{2} \\ -\frac{5}{2} \end{bmatrix}$

Thus, $x = 5,\ y = \frac{7}{2},\ z = -\frac{5}{2}.$

29. $\left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 & -1 & 1 \end{array}\right]$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Thus, $x = 1$, $y = \dfrac{1}{2}$, $z = \dfrac{1}{2}$.

30. $$\left[\begin{array}{ccc|ccc} 2 & 0 & 8 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -4 & 0 & 0 & -1 & 0 \\ 2 & 0 & 8 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & -4 & 0 & 0 & -1 & 0 \\ 0 & 8 & 8 & 1 & 2 & 0 \\ 0 & 9 & 0 & 0 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -4 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 9 & 0 & 0 & 2 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 0 & -9 & -\frac{9}{8} & -\frac{1}{4} & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{1}{9} & \frac{4}{9} \\ 0 & 1 & 0 & 0 & \frac{2}{9} & \frac{1}{9} \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{array}\right]$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 0 & -\frac{1}{9} & \frac{4}{9} \\ 0 & \frac{2}{9} & \frac{1}{9} \\ \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{bmatrix} \begin{bmatrix} 8 \\ 36 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 1 \end{bmatrix}$$

Thus, $x = 0$, $y = 9$, $z = 1$.

31. The coefficient matrix is not invertible. The method of reduction yields

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 7 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 1 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 3 & 7 \\ 0 & -5 & -5 & -10 \\ 0 & -2 & -2 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 3 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & -2 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

The third row indicates that $0 = 1$, which is never true, so there is no solution.

32. The coefficient matrix is not invertible. The method of reduction yields

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 7 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 3 & 7 \\ 0 & -5 & -5 & -10 \\ 0 & -2 & -2 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 3 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & -2 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, $x = 1$, $y = -r + 2$, $z = r$.

33. $\left[\begin{array}{rrrr|rrrr} 1 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrrr|rrrr} 1 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -4 & -1 & -2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{rrrr|rrrr} 1 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -6 & 0 & -3 & 1 & 1 & 0 \\ 0 & 0 & -5 & 2 & -3 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrrr|rrrr} 1 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{6} & -\frac{1}{6} & 0 \\ 0 & 0 & -5 & 2 & -3 & 2 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -1 & 0 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{6} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 2 & -\frac{1}{2} & \frac{7}{6} & -\frac{5}{6} & 1 \end{array}\right] \to \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -1 & 0 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{6} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & \frac{7}{12} & -\frac{5}{12} & \frac{1}{2} \end{array}\right]$

$\to \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & -\frac{1}{4} & -\frac{1}{12} & -\frac{1}{12} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{6} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & \frac{7}{12} & -\frac{5}{12} & \frac{1}{2} \end{array}\right]$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{12} & -\frac{1}{12} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{6} & -\frac{1}{6} & 0 \\ -\frac{1}{4} & \frac{7}{12} & -\frac{5}{12} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 7 \end{bmatrix}$$

Thus, $w = 1$, $x = 3$, $y = -2$, $z = 7$.

34. $\left[\begin{array}{cccc|cccc} 1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array}\right]$

$\to \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 2 & -\frac{1}{2} & 1 & 0 & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 2 & -\frac{1}{2} & 1 & 0 & \frac{1}{2} \end{array}\right]$

$\to \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$

$\to \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & \frac{1}{2} & 1 & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

Thus, $w = -2$, $x = 1$, $y = -2$, $z = 1$.

35. $\mathbf{I} - \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -1 & -1 \end{bmatrix}$

$\left[\begin{array}{cc|cc} -4 & 2 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} -1 & -1 & 0 & 1 \\ -4 & 2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 0 & -1 \\ -4 & 2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 0 & -1 \\ 0 & 6 & 1 & -4 \end{array}\right]$

$\to \left[\begin{array}{cc|cc} 1 & 1 & 0 & -1 \\ 0 & 1 & \frac{1}{6} & -\frac{2}{3} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{6} & -\frac{1}{3} \\ 0 & 1 & \frac{1}{6} & -\frac{2}{3} \end{array}\right]$

Thus, $(\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} -\frac{1}{6} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{2}{3} \end{bmatrix}$.

36. $\mathbf{I}-\mathbf{A}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}-\begin{bmatrix}-3 & 2\\4 & 3\end{bmatrix}=\begin{bmatrix}4 & -2\\-4 & -2\end{bmatrix}$

$\left[\begin{array}{cc|cc}4 & -2 & 1 & 0\\-4 & -2 & 0 & 1\end{array}\right]\to\left[\begin{array}{cc|cc}1 & -\frac{1}{2} & \frac{1}{4} & 0\\-4 & -2 & 0 & 1\end{array}\right]$

$\to\left[\begin{array}{cc|cc}1 & -\frac{1}{2} & \frac{1}{4} & 0\\0 & -4 & 1 & 1\end{array}\right]\to\left[\begin{array}{cc|cc}1 & -\frac{1}{2} & \frac{1}{4} & 0\\0 & 1 & -\frac{1}{4} & -\frac{1}{4}\end{array}\right]\to\left[\begin{array}{cc|cc}1 & 0 & \frac{1}{8} & -\frac{1}{8}\\0 & 1 & -\frac{1}{4} & -\frac{1}{4}\end{array}\right]$

Thus $(\mathbf{I}-\mathbf{A})^{-1}=\begin{bmatrix}\frac{1}{8} & -\frac{1}{8}\\-\frac{1}{4} & -\frac{1}{4}\end{bmatrix}$

37. Let x = number of model A and y = number of model B.

a. The system is

$\begin{cases}x+y=100 & \text{(painting)}\\ \frac{1}{2}x+y=80 & \text{(polishing)}\end{cases}$

Let $\mathbf{A}=\begin{bmatrix}1 & 1\\\frac{1}{2} & 1\end{bmatrix}$.

$\left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\\frac{1}{2} & 1 & 0 & 1\end{array}\right]\to\left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\0 & \frac{1}{2} & -\frac{1}{2} & 1\end{array}\right]\to\left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\0 & 1 & -1 & 2\end{array}\right]\to\left[\begin{array}{cc|cc}1 & 0 & 2 & -2\\0 & 1 & -1 & 2\end{array}\right]$

$\begin{bmatrix}x\\y\end{bmatrix}=\mathbf{A}^{-1}\begin{bmatrix}100\\80\end{bmatrix}=\begin{bmatrix}2 & -2\\-1 & 2\end{bmatrix}\begin{bmatrix}100\\80\end{bmatrix}=\begin{bmatrix}40\\60\end{bmatrix}$

Thus 40 of model A and 60 of model B can be produced.

b. The system is

$\begin{cases}10x+7y=800 & \text{(widgets)}\\ 14x+10y=1130 & \text{(shims)}\end{cases}$

Let $\mathbf{A}=\begin{bmatrix}10 & 7\\14 & 10\end{bmatrix}$.

$\left[\begin{array}{cc|cc}10 & 7 & 1 & 0\\14 & 10 & 0 & 1\end{array}\right]\to\left[\begin{array}{cc|cc}1 & \frac{7}{10} & \frac{1}{10} & 0\\14 & 10 & 0 & 1\end{array}\right]$

$\to\left[\begin{array}{cc|cc}1 & \frac{7}{10} & \frac{1}{10} & 0\\0 & \frac{1}{5} & -\frac{7}{5} & 1\end{array}\right]\to\left[\begin{array}{cc|cc}1 & \frac{7}{10} & \frac{1}{10} & 0\\0 & 1 & -7 & 5\end{array}\right]\to\left[\begin{array}{cc|cc}1 & 0 & 5 & -\frac{7}{2}\\0 & 1 & -7 & 5\end{array}\right]$

$\begin{bmatrix}x\\y\end{bmatrix}=\mathbf{A}^{-1}\begin{bmatrix}800\\1130\end{bmatrix}=\begin{bmatrix}5 & -\frac{7}{2}\\-7 & 5\end{bmatrix}\begin{bmatrix}800\\1130\end{bmatrix}=\begin{bmatrix}45\\50\end{bmatrix}$

Thus 45 of model A and 50 of model B can be produced.

38. $\begin{bmatrix}\frac{1}{a} & 0 & 0\\0 & \frac{1}{b} & 0\\0 & 0 & \frac{1}{c}\end{bmatrix}\begin{bmatrix}a & 0 & 0\\0 & b & 0\\0 & 0 & c\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=\mathbf{I}$

39. a. $\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)(\mathbf{AB}) = \mathbf{B}^{-1}\left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{B} = \mathbf{B}^{-1}\mathbf{IB} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$

Since an invertible matrix has exactly one inverse, $\mathbf{B}^{-1}\mathbf{A}^{-1}$ is the inverse of **AB**.

b. From Part (a),

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 7 & 10 \end{bmatrix}$$

40. Left side: $\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. We find that $(\mathbf{A}^{\mathrm{T}})^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$.

Right side: $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$, so $(\mathbf{A}^{-1})^{\mathrm{T}} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$.

Thus $(\mathbf{A}^{\mathrm{T}})^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}}$.

41. $\mathbf{P}^{\mathrm{T}}\mathbf{P} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$, so $\mathbf{P}^{\mathrm{T}} = \mathbf{P}^{-1}$. Yes, **P** is orthogonal.

42. a. $\mathbf{A}^{-1} = \begin{bmatrix} 14 & -2 & 9 \\ -6 & 1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$

$$\mathbf{R}_1\mathbf{A}^{-1} = \begin{bmatrix} 33 & 87 & 70 \end{bmatrix}\begin{bmatrix} 14 & -2 & 9 \\ -6 & 1 & -4 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 21 & 19 \end{bmatrix}$$

$$\mathbf{R}_2\mathbf{A}^{-1} = \begin{bmatrix} 57 & 133 & 20 \end{bmatrix}\begin{bmatrix} 14 & -2 & 9 \\ -6 & 1 & -4 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 19 & 1 \end{bmatrix}$$

$$\mathbf{R}_3\mathbf{A}^{-1} = \begin{bmatrix} 38 & 90 & 33 \end{bmatrix}\begin{bmatrix} 14 & -2 & 9 \\ -6 & 1 & -4 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 25 & 14 & 15 \end{bmatrix}$$

b. Just say no.

43. Let x be the number of shares of D, y be the number of shares of E, and z be the number of shares of F. We get the following equations.

$60x + 80y + 30z = 500{,}000$

$0.16(60x) + 0.12(80y) + 0.09(30z) = 0.1368(60x + 80y + 30z)$

$z = 4y$

Simplify the first equation.

$6x + 8y + 3z = 50{,}000$

Simplify the second equation.

$9.6x + 9.6y + 2.7z = 8.208x + 10.944y + 4.104z$

$1.392x - 1.344y - 1.404z = 0$

$1392x - 1344y - 1404z = 0$

$116x - 112y - 117z = 0$

Simplify the third equation.
$4y - z = 0$
Thus we solve the following system of equations.
$6x + 8y + 3z = 50{,}000$
$116x - 112y - 117z = 0$
$4y - z = 0$

The coefficient matrix is $\mathbf{A} = \begin{bmatrix} 6 & 8 & 3 \\ 116 & -112 & -117 \\ 0 & 4 & -1 \end{bmatrix}$.

$$[\mathbf{A}|\mathbf{I}] = \left[\begin{array}{ccc|ccc} 6 & 8 & 3 & 1 & 0 & 0 \\ 116 & -112 & -117 & 0 & 1 & 0 \\ 0 & 4 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{6}R_1} \left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 116 & -112 & -117 & 0 & 1 & 0 \\ 0 & 4 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-116R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & -\frac{800}{3} & -175 & -\frac{58}{3} & 1 & 0 \\ 0 & 4 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{4}R_3]{-\frac{3}{800}R_2} \left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & 1 & \frac{21}{32} & \frac{29}{400} & -\frac{3}{800} & 0 \\ 0 & -1 & \frac{1}{4} & 0 & 0 & -\frac{1}{4} \end{array}\right]$$

$$\xrightarrow{R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & 1 & \frac{21}{32} & \frac{29}{400} & -\frac{3}{800} & 0 \\ 0 & 0 & \frac{29}{32} & \frac{29}{400} & -\frac{3}{800} & -\frac{1}{4} \end{array}\right]$$

$$\xrightarrow{\frac{32}{29}R_3} \left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & 1 & \frac{21}{32} & \frac{29}{400} & -\frac{3}{800} & 0 \\ 0 & 0 & 1 & \frac{2}{25} & -\frac{3}{725} & -\frac{8}{29} \end{array}\right]$$

$$\xrightarrow[-\frac{21}{32}R_3 + R_2]{-\frac{1}{2}R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & 0 & \frac{19}{150} & \frac{3}{1450} & \frac{4}{29} \\ 0 & 1 & 0 & \frac{1}{50} & -\frac{3}{2900} & \frac{21}{116} \\ 0 & 0 & 1 & \frac{2}{25} & -\frac{3}{725} & -\frac{8}{29} \end{array}\right]$$

$$\xrightarrow{-\frac{4}{3}R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{10} & \frac{1}{290} & -\frac{3}{29} \\ 0 & 1 & 0 & \frac{1}{50} & -\frac{3}{2900} & \frac{21}{116} \\ 0 & 0 & 1 & \frac{2}{25} & -\frac{3}{725} & -\frac{8}{29} \end{array}\right]$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & \frac{1}{290} & -\frac{3}{29} \\ \frac{1}{50} & -\frac{3}{2900} & \frac{21}{116} \\ \frac{2}{25} & -\frac{3}{725} & -\frac{8}{29} \end{bmatrix} \begin{bmatrix} 50{,}000 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5000 \\ 1000 \\ 4000 \end{bmatrix}$$

They should buy 5000 shares of company D, 1000 shares of company E, and 4000 shares of company F.

44. Let x be the number of shares of D, y be the number of shares of E, and z be the number of shares of F. We get the following conditions.
$60x + 80y + 30z = 500{,}000$
$0.16(60x) + 0.12(80y) + 0.09(30z) = 0.1452(60x + 80y + 30z)$
$z = 2y$
Simplify the first equation.
$6x + 8x + 3z = 50{,}000$
Simplify the second equation.
$9.6x + 9.6y + 2.7z = 8.712x + 11.616y + 4.356z$
$0.888x - 2.016y - 1.656z = 0$
$888x - 2016y - 1656z = 0$
$111x - 252y - 207z = 0$
Simplify the third equation.
$2y - z = 0$
Thus we solve the following system of equations.
$6x + 8y + 3z = 50{,}000$
$111x - 252y - 207z = 0$
$2y - z = 0$

The coefficient matrix is $\mathbf{A} = \begin{bmatrix} 6 & 8 & 3 \\ 111 & -252 & -207 \\ 0 & 2 & -1 \end{bmatrix}$.

$$[\mathbf{A}|\mathbf{I}] = \left[\begin{array}{ccc|ccc} 6 & 8 & 3 & 1 & 0 & 0 \\ 111 & -252 & -207 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{6}R_1} \left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 111 & -252 & -207 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-111R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & -400 & -\frac{525}{2} & -\frac{37}{2} & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{2}R_3]{-\frac{1}{400}R_2} \left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & 1 & \frac{21}{32} & \frac{37}{800} & -\frac{1}{400} & 0 \\ 0 & -1 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_2+R_3}\left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & 1 & \frac{21}{32} & \frac{37}{800} & -\frac{1}{400} & 0 \\ 0 & 0 & \frac{37}{32} & \frac{37}{800} & -\frac{1}{400} & -\frac{1}{2} \end{array}\right]$$

$$\xrightarrow{\frac{32}{37}R_3}\left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & 1 & \frac{21}{32} & \frac{37}{800} & -\frac{1}{400} & 0 \\ 0 & 0 & 1 & \frac{1}{25} & -\frac{2}{925} & -\frac{16}{37} \end{array}\right]$$

$$\xrightarrow[-\frac{21}{32}R_3+R_2]{-\frac{1}{2}R_3+R_1}\left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & 0 & \frac{11}{75} & \frac{1}{925} & \frac{8}{37} \\ 0 & 1 & 0 & \frac{1}{50} & -\frac{1}{925} & \frac{21}{74} \\ 0 & 0 & 1 & \frac{1}{25} & -\frac{2}{925} & -\frac{16}{37} \end{array}\right]$$

$$\xrightarrow{-\frac{4}{3}R_2+R_1}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{25} & \frac{7}{2775} & -\frac{6}{37} \\ 0 & 1 & 0 & \frac{1}{50} & -\frac{1}{925} & \frac{21}{74} \\ 0 & 0 & 1 & \frac{1}{25} & -\frac{2}{925} & -\frac{16}{37} \end{array}\right]$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{25} & \frac{7}{2775} & -\frac{6}{37} \\ \frac{1}{50} & -\frac{1}{925} & \frac{21}{74} \\ \frac{1}{25} & -\frac{2}{925} & -\frac{16}{37} \end{bmatrix}\begin{bmatrix} 50,000 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6000 \\ 1000 \\ 2000 \end{bmatrix}$$

They should buy 6000 shares of company D, 1000 shares of company E, and 2000 shares of company F.

45. a. $\begin{bmatrix} 2.05 & 1.28 \\ 0.73 & 1.71 \end{bmatrix}$

b. $\begin{bmatrix} \frac{84}{41} & \frac{105}{82} \\ \frac{30}{41} & \frac{70}{41} \end{bmatrix}$

46. a. $\begin{bmatrix} -0.03 & 0.06 & -0.12 \\ 0.13 & 0.02 & 0.05 \\ -0.10 & 0.07 & 0.01 \end{bmatrix}$

b. $\begin{bmatrix} -\frac{11}{323} & \frac{18}{323} & -\frac{39}{323} \\ \frac{83}{646} & \frac{11}{646} & \frac{15}{323} \\ -\frac{32}{323} & \frac{23}{323} & \frac{4}{323} \end{bmatrix}$

47. $\begin{bmatrix} 2.75 & -1.59 & -1.11 \\ -0.48 & 1.43 & 0.00 \\ -1.22 & 0.32 & 2.22 \end{bmatrix}$

48. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.9 & 3 & -4.7 \\ 2 & -0.4 & 2 \\ 1 & -0.8 & -0.5 \end{bmatrix}^{-1}\begin{bmatrix} 13 \\ 4.7 \\ 7.2 \end{bmatrix} = \begin{bmatrix} 4.78 \\ -1.33 \\ -2.70 \end{bmatrix}$

$x = 4.78$, $y = -1.33$, $z = -2.70$

49. $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & 4 & \frac{1}{2} & -\frac{3}{7} \\ \frac{5}{9} & -\frac{2}{3} & -4 & -1 \\ 0 & 1 & -\frac{4}{9} & \frac{5}{6} \\ \frac{1}{2} & 0 & 4 & -\frac{1}{3} \end{bmatrix}^{-1}\begin{bmatrix} \frac{14}{13} \\ \frac{7}{8} \\ 9 \\ \frac{4}{7} \end{bmatrix} = \begin{bmatrix} 14.44 \\ 0.03 \\ -0.80 \\ 10.33 \end{bmatrix}$

$w = 14.44$, $x = 0.03$, $y = -0.80$, $z = 10.33$

Problems 6.7

1. $\mathbf{A} = \begin{bmatrix} \frac{200}{1200} & \frac{500}{1500} \\ \frac{400}{1200} & \frac{200}{1500} \end{bmatrix}$

$\mathbf{D} = \begin{bmatrix} 600 \\ 805 \end{bmatrix}$

$\mathbf{X} = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{D} = \begin{bmatrix} 1290 \\ 1425 \end{bmatrix}$

The total value of other production costs is

$\mathbf{P}_A + \mathbf{P}_B = \frac{600}{1200}(1290) + \frac{800}{1500}(1425) = 1405$

2. $\mathbf{A} = \begin{bmatrix} \frac{40}{200} & \frac{120}{300} \\ \frac{120}{200} & \frac{90}{300} \end{bmatrix}$

a. $\mathbf{D} = \begin{bmatrix} 200 \\ 300 \end{bmatrix}$

$\mathbf{X} = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{D} = \begin{bmatrix} 812.5 \\ 1125 \end{bmatrix}$

b. $\mathbf{D} = \begin{bmatrix} 64 \\ 64 \end{bmatrix}$

$\mathbf{X} = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{D} = \begin{bmatrix} 220 \\ 280 \end{bmatrix}$

3. $\mathbf{A} = \begin{bmatrix} \frac{15}{100} & \frac{30}{120} & \frac{45}{180} \\ \frac{25}{100} & \frac{30}{120} & \frac{60}{180} \\ \frac{50}{100} & \frac{40}{120} & \frac{60}{180} \end{bmatrix}$

a. $\mathbf{D} = \begin{bmatrix} 15 \\ 10 \\ 35 \end{bmatrix}$

$\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{D} = \begin{bmatrix} 134.29 \\ 162.25 \\ 234.35 \end{bmatrix}$

b. $\mathbf{D} = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$

$\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{D} = \begin{bmatrix} 68.59 \\ 84.50 \\ 108.69 \end{bmatrix}$

4. $\mathbf{A} = \begin{bmatrix} \frac{100}{1000} & \frac{400}{800} & \frac{240}{1200} \\ \frac{100}{1000} & \frac{80}{800} & \frac{480}{1200} \\ \frac{300}{1000} & \frac{160}{800} & \frac{240}{1200} \end{bmatrix}$

$\mathbf{D} = \begin{bmatrix} 500 \\ 150 \\ 700 \end{bmatrix}$

$\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{D} = \begin{bmatrix} 1559.81 \\ 1112.44 \\ 1738.04 \end{bmatrix}$

5. $\mathbf{A} = \begin{bmatrix} \frac{400}{1000} & \frac{200}{1000} & \frac{200}{1000} \\ \frac{200}{1000} & \frac{400}{1000} & \frac{100}{1000} \\ \frac{200}{1000} & \frac{100}{1000} & \frac{300}{1000} \end{bmatrix}$

$\mathbf{D} = \begin{bmatrix} 300 \\ 350 \\ 450 \end{bmatrix}$

$\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{D} = \begin{bmatrix} 1301 \\ 1215 \\ 1188 \end{bmatrix}$

6. $\mathbf{A} = \begin{bmatrix} \frac{400}{1000} & \frac{200}{1000} & \frac{200}{1000} \\ \frac{200}{1000} & \frac{400}{1000} & \frac{100}{1000} \\ \frac{200}{1000} & \frac{100}{1000} & \frac{300}{1000} \end{bmatrix}$

$\mathbf{D} = \begin{bmatrix} 250 \\ 300 \\ 350 \end{bmatrix}$

$\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{D} = \begin{bmatrix} 1073 \\ 1016 \\ 952 \end{bmatrix}$

7. $\mathbf{A} = \begin{bmatrix} \frac{400}{1000} & \frac{200}{1000} & \frac{200}{1000} \\ \frac{200}{1000} & \frac{400}{1000} & \frac{100}{1000} \\ \frac{200}{1000} & \frac{100}{1000} & \frac{300}{1000} \end{bmatrix}$

$\mathbf{D} = \begin{bmatrix} 300 \\ 400 \\ 500 \end{bmatrix}$

$\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{D} = \begin{bmatrix} 1382 \\ 1344 \\ 1301 \end{bmatrix}$

8. $\mathbf{A} = \begin{bmatrix} \frac{1}{3} & \frac{3}{4} \\ \frac{1}{4} & 0 \end{bmatrix}$

$\mathbf{D} = \begin{bmatrix} 300 \\ 500 \end{bmatrix}$

$(\mathbf{I} - \mathbf{A})\mathbf{X} = \mathbf{D}$

Reducing $\left[\begin{array}{cc|c} \frac{2}{3} & -\frac{3}{4} & 300 \\ -\frac{1}{4} & 1 & 500 \end{array}\right]$ with a calculator results in $\left[\begin{array}{cc|c} 1 & 0 & 1408.70 \\ 0 & 1 & 852.17 \end{array}\right]$.

Thus 1408.70 units of agriculture and 852.17 units of milling need to be produced.

9. $\mathbf{A} = \begin{bmatrix} \frac{1}{10} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{3} \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \end{bmatrix}$

$\mathbf{D} = \begin{bmatrix} 300 \\ 200 \\ 500 \end{bmatrix}$

$(\mathbf{I} - \mathbf{A})\mathbf{X} = \mathbf{D}$

Reducing $\left[\begin{array}{ccc|c} \frac{9}{10} & -\frac{1}{3} & -\frac{1}{4} & 300 \\ -\frac{1}{10} & \frac{9}{10} & -\frac{1}{3} & 200 \\ -\frac{1}{10} & -\frac{1}{10} & \frac{9}{10} & 500 \end{array}\right]$ with a calculator results in $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 736.39 \\ 0 & 1 & 0 & 563.29 \\ 0 & 0 & 1 & 699.96 \end{array}\right]$.

Thus 736.39 units of coal, 563.29 units of steel, and 699.96 units of railroad services need to be produced.

Chapter 6 Review Problems

1. $2\begin{bmatrix} 3 & 4 \\ -5 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ -10 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 6 & 12 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ -16 & -10 \end{bmatrix}$

2. $8\begin{bmatrix} 1 & 2 \\ 7 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 16 \\ 56 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 16 \\ 56 & -2 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 7 \\ 2 & -3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+42 & -2+7 \\ 2+0 & 0-18 & -4-3 \\ 1+0 & 0+0 & -2+0 \end{bmatrix} = \begin{bmatrix} 1 & 42 & 5 \\ 2 & -18 & -7 \\ 1 & 0 & -2 \end{bmatrix}$

4. $[2 \quad 3 \quad 7]\begin{bmatrix} 2 & 3 \\ 0 & -1 \\ 5 & 2 \end{bmatrix} = [2(2)+3(0)+7(5) \quad 2(3)+3(-1)+7(2)] = [39 \quad 17]$

5. $\begin{bmatrix} 2 & 3 \\ -1 & 3 \end{bmatrix}\left(\begin{bmatrix} 2 & 3 \\ 7 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 8 \\ 4 & 4 \end{bmatrix}\right) = \begin{bmatrix} 2 & 3 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 11 & -4 \\ 8 & 11 \end{bmatrix}$

6. $-\left\{\begin{bmatrix} 2 & 0 \\ 7 & 8 \end{bmatrix} + 2\begin{bmatrix} 0 & -5 \\ 6 & -4 \end{bmatrix}\right\} = -\left\{\begin{bmatrix} 2 & 0 \\ 7 & 8 \end{bmatrix} + \begin{bmatrix} 0 & -10 \\ 12 & -8 \end{bmatrix}\right\}$

$= -\begin{bmatrix} 2 & -10 \\ 19 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 10 \\ -19 & 0 \end{bmatrix}$

7. $2\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}^2 [1 \quad -2]^{\text{T}} = 2\begin{bmatrix} -5 & -4 \\ 6 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 3 \\ 16 \end{bmatrix} = \begin{bmatrix} 6 \\ 32 \end{bmatrix}$

8. $\frac{1}{3}\begin{bmatrix} 3 & 0 \\ 3 & 6 \end{bmatrix}\left\{\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}^{\text{T}}\right\}^2 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 22 \end{bmatrix}$

9. $(2\mathbf{A})^{\text{T}} - 3\mathbf{I}^2 = 2\mathbf{A}^{\text{T}} - 3\mathbf{I} = 2\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2 & -2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}$

10. $\mathbf{A}(2\mathbf{I}) - \mathbf{A}\mathbf{O}^{\text{T}} = 2(\mathbf{A}\mathbf{I}) - \mathbf{A}\mathbf{O} = 2\mathbf{A} - \mathbf{O} = 2\mathbf{A} = \begin{bmatrix} 2 & 2 \\ -2 & 4 \end{bmatrix}$

11. $\mathbf{B}^3+\mathbf{I}^5=\begin{bmatrix}1&0\\0&2\end{bmatrix}^3+\begin{bmatrix}1&0\\0&1\end{bmatrix}^5=\begin{bmatrix}1&0\\0&8\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}2&0\\0&9\end{bmatrix}$

12. $(\mathbf{ABA})^{\mathrm{T}}-\mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}=\mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}-\mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}=\mathbf{O}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

13. $\begin{bmatrix}5x\\7x\end{bmatrix}=\begin{bmatrix}15\\y\end{bmatrix}$

$5x = 15$, or $x = 3$

$7x = y$, $7 \cdot 3 = y$, or $y = 21$

14. $\begin{bmatrix}2+x^2&1+3x\\4+xy&2+3y\end{bmatrix}=\begin{bmatrix}3&4\\3&y\end{bmatrix}$

$2 + 3y = y$, $2y = -2$, or $y = -1$

$1 + 3x = 4$, $3x = 3$, or $x = 1$

For these values of x and y, $2+x^2=3$ is true, and $4 + xy = 3$ is true. Thus $x = 1$, $y = -1$.

15. $\begin{bmatrix}1&4\\5&8\end{bmatrix}\to\begin{bmatrix}1&4\\0&-12\end{bmatrix}\to\begin{bmatrix}1&4\\0&1\end{bmatrix}\to\begin{bmatrix}1&0\\0&1\end{bmatrix}$

16. $\begin{bmatrix}0&0&7\\0&5&9\end{bmatrix}\to\begin{bmatrix}0&5&9\\0&0&7\end{bmatrix}\to\begin{bmatrix}0&1&\frac{9}{5}\\0&0&1\end{bmatrix}\to\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}$

17. $\begin{bmatrix}2&4&7\\1&2&4\\5&8&2\end{bmatrix}\to\begin{bmatrix}1&2&4\\2&4&7\\5&8&2\end{bmatrix}\to\begin{bmatrix}1&2&4\\0&0&-1\\0&-2&-18\end{bmatrix}\to\begin{bmatrix}1&2&4\\0&-2&-18\\0&0&-1\end{bmatrix}$

$\to\begin{bmatrix}1&2&4\\0&1&9\\0&0&-1\end{bmatrix}\to\begin{bmatrix}1&0&-14\\0&1&9\\0&0&-1\end{bmatrix}\to\begin{bmatrix}1&0&-14\\0&1&9\\0&0&1\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

18. $\begin{bmatrix}0&0&0&1\\0&0&0&0\\1&0&0&0\end{bmatrix}\to\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$

19. $\left[\begin{array}{cc|c}2&-5&0\\4&3&0\end{array}\right]\to\left[\begin{array}{cc|c}2&-5&0\\0&13&0\end{array}\right]\to\left[\begin{array}{cc|c}1&-\frac{5}{2}&0\\0&1&0\end{array}\right]\to\left[\begin{array}{cc|c}1&0&0\\0&1&0\end{array}\right]$

Thus $x = 0$, $y = 0$.

20. $\left[\begin{array}{ccc|c}1&-1&2&3\\3&1&1&5\end{array}\right]\to\left[\begin{array}{ccc|c}1&-1&2&3\\0&4&-5&-4\end{array}\right]\to\left[\begin{array}{ccc|c}1&-1&2&3\\0&1&-\frac{5}{4}&-1\end{array}\right]\to\left[\begin{array}{ccc|c}1&0&\frac{3}{4}&2\\0&1&-\frac{5}{4}&-1\end{array}\right]$

Thus $x=-\frac{3}{4}r+2$, $y=\frac{5}{4}r-1$, $z = r$.

21. $$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 3 & -2 & -4 & -7 \\ 2 & -1 & -2 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -5 & -10 & -10 \\ 0 & -3 & -6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & -3 & -6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6 \end{array}\right]$$

Row three indicates that 0 = 6, which is never true, so there is no solution.

22. $$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 1 & 1 & 2 & -3 \\ 2 & 0 & 2 & -7 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 2 & 3 & -4 \\ 0 & 2 & 4 & -9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 1 & \frac{3}{2} & -2 \\ 0 & 2 & 4 & -9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{2} & -2 \\ 0 & 0 & 1 & -5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{11}{2} \\ 0 & 0 & 1 & -5 \end{array}\right]$$

Thus $x = \frac{3}{2},\ y = \frac{11}{2},\ z = -5.$

23. $$\left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & -6 & -3 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{6} \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & -\frac{3}{2} & \frac{5}{6} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{6} \end{array}\right] \Rightarrow \mathbf{A}^{-1} = \begin{bmatrix} -\frac{3}{2} & \frac{5}{6} \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix}$$

24. $$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \Rightarrow \mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

25. $$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 1 & 0 \\ 3 & -2 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -11 & 8 & -4 & 1 & 0 \\ 0 & -11 & 8 & -3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -11 & 8 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{8}{11} & \frac{4}{11} & -\frac{1}{11} & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{11} & -\frac{1}{11} & \frac{3}{11} & 0 \\ 0 & 1 & -\frac{8}{11} & \frac{4}{11} & -\frac{1}{11} & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right] \Rightarrow \text{ no inverse exists}$$

26. $$\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 0 & 0 \\ -5 & 2 & 1 & 0 & 1 & 0 \\ -5 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{5}{2} & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{5} & \frac{3}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{2}{5} \end{array}\right] \Rightarrow \mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ \frac{2}{5} & \frac{3}{5} & -\frac{1}{5} \\ \frac{1}{5} & -\frac{1}{5} & \frac{2}{5} \end{bmatrix}.$$

27. $\left[\begin{array}{ccc|ccc} 3 & 1 & 4 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 1 & 4 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -3 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -3 & 0 \\ 0 & 0 & -1 & -2 & 6 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -3 & 0 \\ 0 & 0 & 1 & 2 & -6 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 7 & 1 \\ 0 & 1 & 0 & -1 & 3 & 1 \\ 0 & 0 & 1 & 2 & -6 & -1 \end{array}\right]$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} -2 & 7 & 1 \\ -1 & 3 & 1 \\ 2 & -6 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Thus $x = 0$, $y = 1$, $z = 0$.

28. We found $\mathbf{A}^{-1}$ in Exercise 26, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ \frac{2}{5} & \frac{3}{5} & -\frac{1}{5} \\ \frac{1}{5} & -\frac{1}{5} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \\ \frac{7}{5} \end{bmatrix}$$

29. $\mathbf{A}^2 = \mathbf{AA} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}$

Since $\mathbf{A}^3 = \mathbf{O}$, every higher power of **A** is also **O**, so $\mathbf{A}^{1000} = \mathbf{O}$.

Looking at $\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$, it is clear that there is no way of transforming the left side into $\mathbf{I}_3$, since there is no way to get a nonzero entry in the first column. Thus **A** does not have an inverse.

30. $\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

$\left(\mathbf{A}^{\mathrm{T}}\right)^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$

$\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$

$\left(\mathbf{A}^{-1}\right)^{\mathrm{T}} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$

Thus $(\mathbf{A}^{\mathrm{T}})^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}}$.

31. **a.** Let x, y, and z represent the weekly doses of capsules of brand I, II, and III, respectively. Then

$$\begin{cases} x+y+4z=13 & \text{(vitamin A)} \\ x+2y+7z=22 & \text{(vitamin B)} \\ x+3y+10z=31 & \text{(vitamin C)} \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & 13 \\ 1 & 2 & 7 & 22 \\ 1 & 3 & 10 & 31 \end{array}\right] \xrightarrow[-R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 13 \\ 0 & 1 & 3 & 9 \\ 0 & 2 & 6 & 18 \end{array}\right]$$

$$\xrightarrow[-2R_2+R_3]{-R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $x = 4 - r$, $y = 9 - 3r$, and $z = r$, where $r = 0, 1, 2, 3$.
The four possible combinations are

Combination	x	y	z
1	4	9	0
2	3	6	1
3	2	3	2
4	1	0	3

b. Computing the cost of each combination, we find that they are 83, 77, 71, and 65 cents, respectively. Thus combination 4, namely $x = 1$, $y = 0$, $z = 3$, minimizes weekly cost.

32. **a.** $\left(\mathbf{A}^{-1}\right)^3 \mathbf{A}^3 = \left(\mathbf{A}^{-1}\right)^2 \left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{A}^2 = \left(\mathbf{A}^{-1}\right)^2 \mathbf{I}\mathbf{A}^2 = \left(\mathbf{A}^{-1}\right)^2 \mathbf{A}^2$

$= \mathbf{A}^{-1}\left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{A} = \mathbf{A}^{-1}\mathbf{I}\mathbf{A} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

Thus $\mathbf{A}^3$ is invertible.

b. **AB** = **AC**. Thus $\mathbf{A}^{-1}(\mathbf{AB}) = \mathbf{A}^{-1}(\mathbf{AC})$, $\left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{B} = \left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{C}$, **IB** = **IC**, **B** = **C**.

c. $\mathbf{AA} = \mathbf{A} \Rightarrow \mathbf{A}^{-1}\mathbf{AA} = \mathbf{A}^{-1}\mathbf{A}$, **IA** = **I**, **A** = **I**. Thus $\mathbf{A} = \mathbf{I}_n$.

33. $\begin{bmatrix} 215 & 87 \\ 89 & 141 \end{bmatrix}$

34. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7.9 & -4.3 & 2.7 \\ 3.4 & 5.8 & -7.6 \\ 4.5 & -6.2 & -7.4 \end{bmatrix}^{-1} \begin{bmatrix} 11.1 \\ 10.8 \\ 15.9 \end{bmatrix} = \begin{bmatrix} 1.57 \\ -0.30 \\ -0.95 \end{bmatrix}$

Thus $x = 1.57$, $y = -0.30$, $z = -0.95$.

35. $\mathbf{A} = \begin{bmatrix} \frac{10}{34} & \frac{20}{39} \\ \frac{15}{34} & \frac{14}{39} \end{bmatrix}$; $\mathbf{D} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$; $\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{D} = \begin{bmatrix} 39.7 \\ 35.1 \end{bmatrix}$

Mathematical Snapshot Chapter 6

1. $\mathbf{A} = \begin{bmatrix} 20 & 40 & 30 & 10 \\ 30 & 0 & 10 & 10 \\ 10 & 0 & 30 & 50 \end{bmatrix}$

$\mathbf{T} = \begin{bmatrix} 7 \\ 10 \\ 7 \\ 5 \end{bmatrix}$

$\mathbf{C} = \begin{bmatrix} 9 \\ 8 \\ 10 \end{bmatrix}$

$$\mathbf{C}^{\mathrm{T}}(\mathbf{AT}) = \mathbf{C}^{\mathrm{T}}\left\{\begin{bmatrix} 20 & 40 & 30 & 10 \\ 30 & 0 & 10 & 10 \\ 10 & 0 & 30 & 50 \end{bmatrix}\begin{bmatrix} 7 \\ 10 \\ 7 \\ 5 \end{bmatrix}\right\} = \mathbf{C}^{\mathrm{T}}\begin{bmatrix} 800 \\ 330 \\ 530 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 10 \end{bmatrix}\begin{bmatrix} 800 \\ 330 \\ 530 \end{bmatrix} = [15{,}140]$$

The cost is \$151.40.

2. To the linear system, add $x_1 + x_2 + x_3 + x_4 = 52$.

$\mathbf{A} = \begin{bmatrix} 20 & 40 & 30 & 10 \\ 30 & 0 & 10 & 10 \\ 10 & 0 & 30 & 50 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$\mathbf{B} = \begin{bmatrix} 1180 \\ 580 \\ 1500 \\ 52 \end{bmatrix}$

$\mathbf{T} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 8 \\ 10 \\ 14 \\ 20 \end{bmatrix}$

Guest 1: 8 days; guest 2: 10 days;
guest 3: 14 days; guest 4: 20 days

3. It is not possible. Different combinations of lengths of stays can cost the same. For example, guest 1 staying for 20 days and guest 3 staying for 17 days costs the same as guest 1 staying for 15 days and guest 3 staying for 21 days (each costs \$214.50).

Chapter 7

Principles in Practice 7.1

1. Let x = the number of type A magnets and y = the number of type B magnets.
The cost for producing x type A magnets and y type B magnets is $50 + 0.90x + 0.70y$. The revenue for selling x type A magnets and y type B magnets is $2.00x + 1.50y$.
Revenue is greater than cost when
$2x + 1.5y > 50 + 0.9x + 0.7y$.
$0.8y > -1.1x + 50$
$y > -1.375x + 62.5$
Sketch the dashed line $y = -1.375x + 62.5$ and shade the half plane above the line. In order to make a profit, the number of magnets of types A and B must correspond to an ordered pair in the shaded region. Also, to take reality into account, both x and y must be positive (negative numbers of magnets are not feasible).

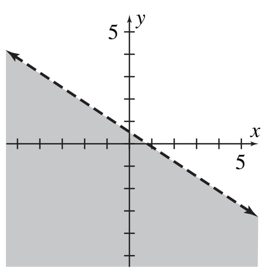
2. Since negative numbers of cameras cannot be sold, $x \geq 0$ and $y \geq 0$. Selling at least 50 cameras per week corresponds to $x + y \geq 50$. Selling twice as many of type I as of type II corresponds to
$x \geq 2y$. The system of inequalities is

$$\begin{cases} x + y \geq 50, \\ x \geq 2y, \\ x \geq 0, \\ y \geq 0. \end{cases}$$

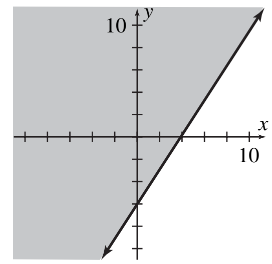
The region consists of points on or above the x-axis and on or to the right of the y-axis. In addition, the points must be on or above the line $x + y = 50$ and on or below the line $x = 2y$.

Problems 7.1

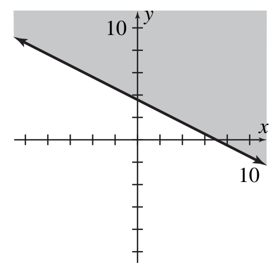
1.



2.



3.



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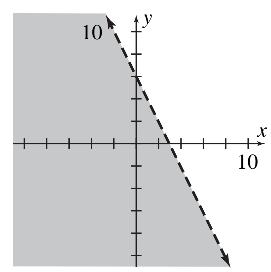

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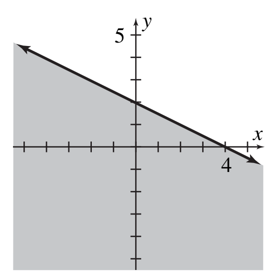

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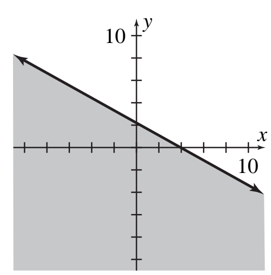

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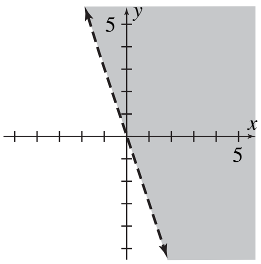
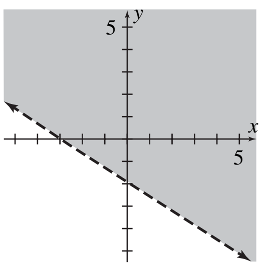
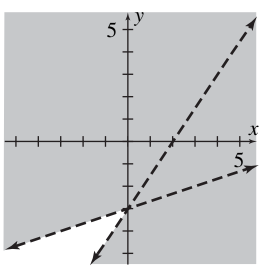


8.



9.



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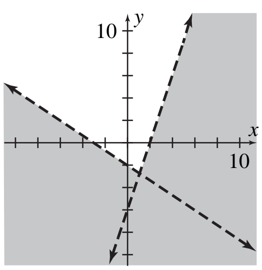

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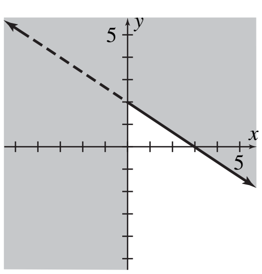

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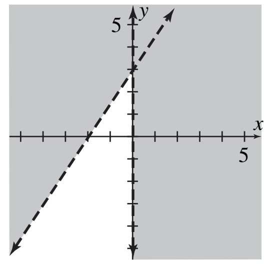

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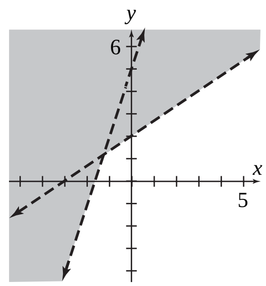

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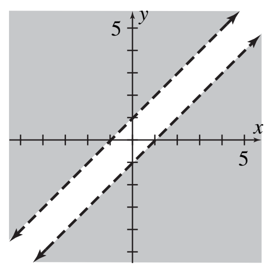

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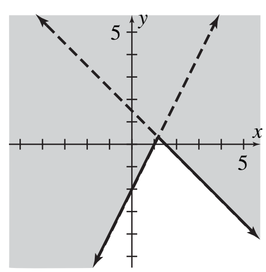

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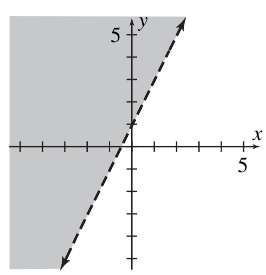

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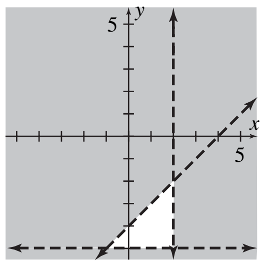

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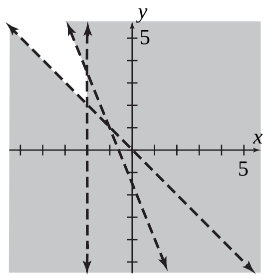

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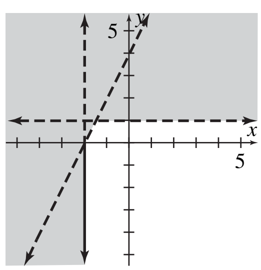

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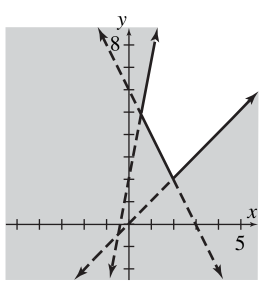

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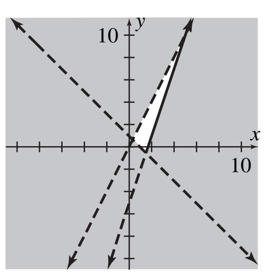

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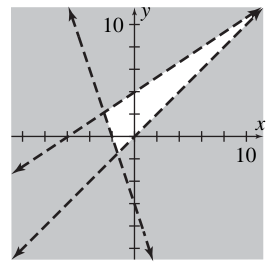

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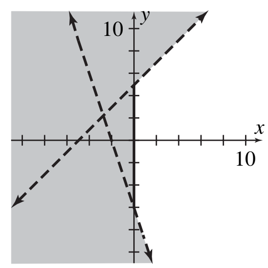

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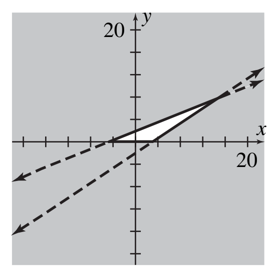
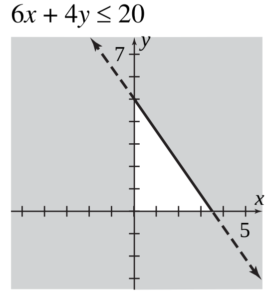
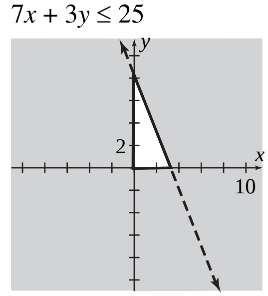


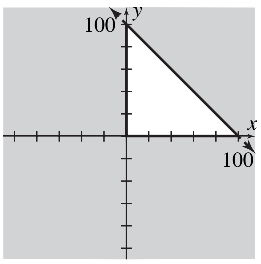
25. $6x + 4y \leq 20$



26. $7x + 3y \leq 25$



27. Let x be the amount purchased from supplier A, and y the amount purchased from B. The system of inequalities is
$x + y \leq 100$,
$x \geq 0$,
$y \geq 0$.



28. Since negative numbers of computers cannot be produced, $x \geq 0$ and $y \geq 0$. Producing at most 650 computers per week corresponds to $x + y \leq 650$. The system of inequalities is

$$\begin{cases} x + y \leq 650, \\ x \geq 0, \\ y \geq 0. \end{cases}$$

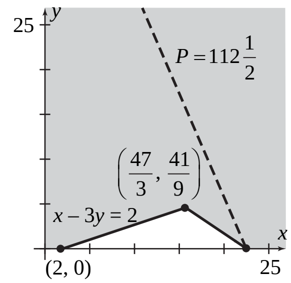
29. Since negative numbers of chairs cannot be produced, $x \geq 0$ and $y \geq 0$. The inequality for assembly time is $3x + 2y \leq 240$. The inequality for painting time is $\frac{1}{2}x + y \leq 80$. The system of inequalities is

$$\begin{cases} 3x + 2y \leq 240, \\ \frac{1}{2}x + y \leq 80, \\ x \geq 0, \\ y \geq 0. \end{cases}$$

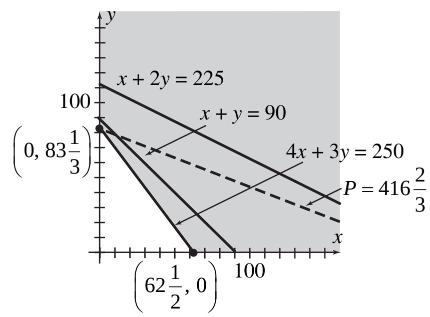
The region consists of points on or above the x-axis and on or to the right of the y-axis. In addition, the points must be on or below the line $3x + 2y = 240$ and on or below the line $\frac{1}{2}x + y = 80$ (or, equivalently $x + 2y = 160$).

Problems 7.2

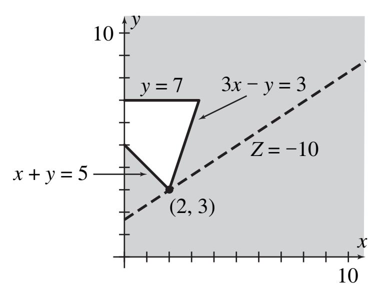
1. The feasible region appears below. The corner points are (2, 0), $\left(\frac{47}{3}, \frac{41}{9}\right)$, $\left(\frac{45}{2}, 0\right)$. Evaluating P at each corner point, we find that P has a maximum value of $112\frac{1}{2}$ when $x = \frac{45}{2}$ and $y = 0$.



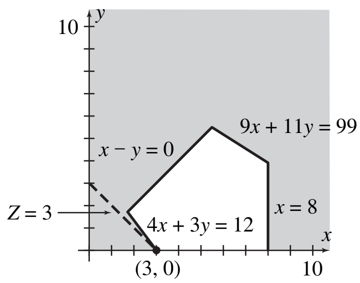
2. The feasible region appears below. The corner points are (0, 0), $\left(0, 83\frac{1}{3}\right)$, $\left(62\frac{1}{2}, 0\right)$. Evaluating P at each corner point, we find that P has a maximum value of $416\frac{2}{3}$ when $x = 0$ and $y = 83\frac{1}{3}$.



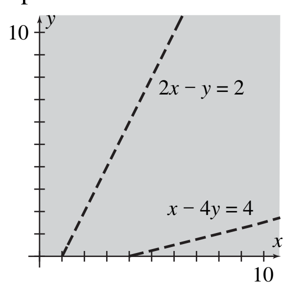
3. The feasible region appears below. The corner points are (2, 3), (0, 5), (0, 7) and $\left(\frac{10}{3}, 7\right)$. Evaluating Z at each point, we find that Z has a maximum value of −10 when $x = 2$ and $y = 3$.



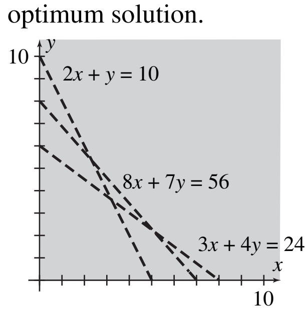
4. The feasible region appears below. The corner points are (8, 0), (3, 0), $\left(\frac{12}{7}, \frac{12}{7}\right)$, $\left(\frac{99}{20}, \frac{99}{20}\right)$ and $\left(8, \frac{27}{11}\right)$. Evaluating Z at each point, we find that Z has a minimum value of 3 when $x = 3$ and $y = 0$.



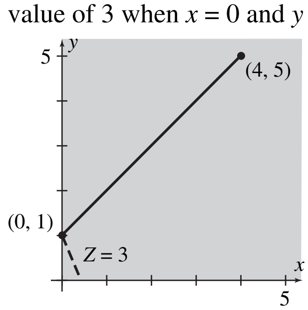
5. The feasible region is empty, so there is no optimum solution.



6. The feasible region is empty, so there is no optimum solution.

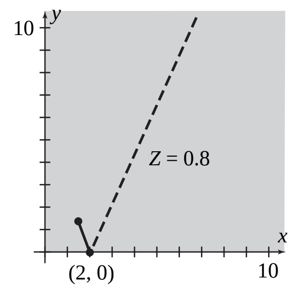


7. The feasible region is a line segment. The corner points are (0, 1) and (4, 5). Z has a minimum value of 3 when $x = 0$ and $y = 1$.

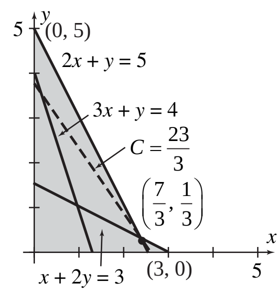


8. The feasible region is a line segment. The corner points are (2, 0) and $\left(\frac{27}{17}, \frac{21}{17}\right)$.

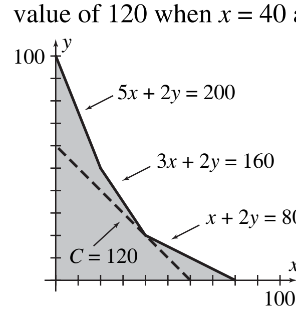
Z has a maximum value of 0.8 for $x = 2$ and $y = 0$.



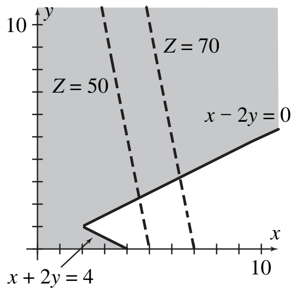
9. The feasible region is unbounded with 3 corner points. The member (see dashed line) of the family of lines $C = 3x + 2y$ which gives a minimum value of C, subject to the constraints, intersects the feasible region at corner point $\left(\frac{7}{3}, \frac{1}{3}\right)$ where $C = \frac{23}{3}$ Thus C has a minimum value of $\frac{23}{3}$ when $x = \frac{7}{3}$ and $y = \frac{1}{3}$. [Note: Here we chose the member of the family $y = \frac{1}{2}(-3x + C)$ whose y-intercept was *closest* to the origin and which had at least one point in common with the feasible region.]



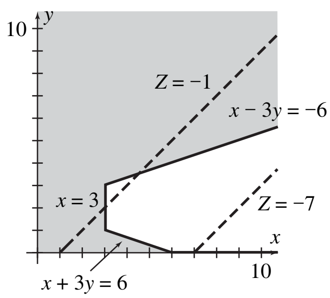
10. The feasible region is unbounded with 4 corner points. The member (see dashed line) of the family of lines $y = -x + \frac{C}{2}$ which gives a minimum value of C, subject to the constraints, intersects the feasible region at corner point (40, 20) where $C = 120$. Thus C has a minimum value of 120 when $x = 40$ and $y = 20$.



11. The feasible region is unbounded with 2 corner points. The family of lines given by $Z = 10x + 2y$ has members (see dashed lines for two sample members) that have arbitrarily large values of Z and that also intersect the feasible region. Thus no optimum solution exists.



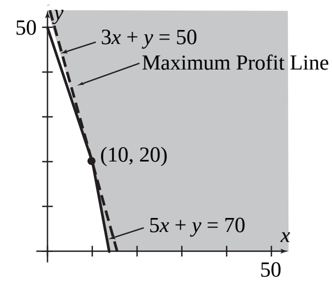
12. The feasible region is unbounded with 3 corner points. The family of lines given by $Z = y - x$ has members (see dashed lines for sample members) that have arbitrarily small values for Z and also intersect the feasible region. Thus no optimum solution exists.



13. Let x and y be the number of trucks and spinning tops made per week, respectively. Then we are to maximize $P = 7x + 2y$ where

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x + y \leq 80 \text{ (for machine A)} \\ 3x + y \leq 50 \text{ (for machine B)} \\ 5x + y \leq 70 \text{ (for finishing)} \end{cases}$$

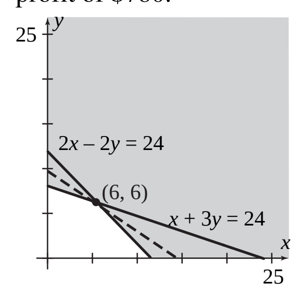
The feasible region is bounded. The corner points are (0, 50), (14, 0) and (10, 20). Evaluating P at each corner point, we find that P is maximized at corner point (10, 20), where its value is 110. Thus10 trucks and 20 spinning tops should be made each week to give a maximum profit of $110.



14. Let x and y be the numbers of Vista and Xtreme models made each day. Then we are to maximize
$P = 50x + 80y$, where

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ x + 3y \leq 24 \text{ (for machine A)} \\ 2x + 2y \leq 24 \text{ (for machine B)} \end{cases}$$

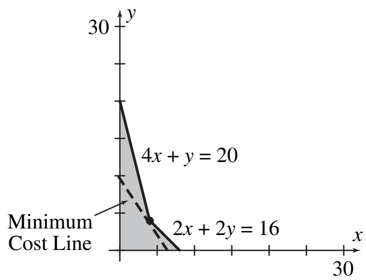
The feasible region is bounded. The corner points are (0, 0), (0, 8), (6, 6), and (12, 0). Evaluating P at each corner point, we find that P is maximized at corner point (6, 6) where its value is 780. Thus 6 of each model should be made each day in order to give a maximum profit of $780.



15. Let x and y be the numbers of units of Food A and Food B, respectively, that are purchased. Then we are to minimize $C = 1.20x + 0.80y$, where

$$\begin{cases} x \geq 0, \\ y \geq 0, \\ 2x + 2y \geq 16 \text{ (for carbohydrates),} \\ 4x + y \geq 20 \text{ (for protein).} \end{cases}$$

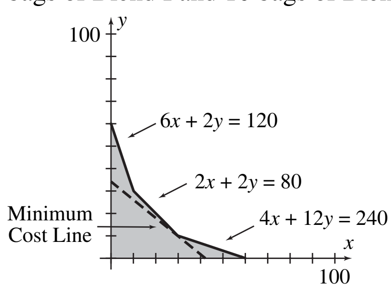
The feasible region is unbounded. The corner points are (8, 0), (4, 4) and (0, 20). C is minimized at corner point (4, 4) where $C = 8$ (see the minimum cost line). Thus 4 units of Food A and 4 units of Food B gives a minimum cost of $8.



16. Let x and y be the numbers of units of Blend I and Blend II, respectively, that are bought each week. Then we are to minimize $C = 8x + 10y$ where

$$\begin{cases} x \geq 0, \\ y \geq 0, \\ 2x + 2y \geq 80 & \text{(for Nutrient A)}, \\ 6x + 2y \geq 120 & \text{(for Nutrient B)}, \\ 4x + 12y \geq 240 & \text{(for Nutrient C)}. \end{cases}$$

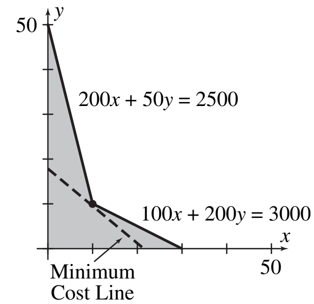
The feasible region is unbounded with 4 corner points. C is minimized at the corner point (30, 10) where $C = 340$ (see the minimum cost line). thus each week the grower should buy 30 bags of Blend I and 10 bags of Blend II.



17. Let x and y be the numbers of tons of ores I and II, respectively, that are processed. Then we are to minimize $C = 50x + 60y$, where

$$\begin{cases} x \geq 0, \\ y \geq 0, \\ 100x + 200y \geq 3000 & \text{(for mineral A)}, \\ 200x + 50y \geq 2500 & \text{(for mineral B)}. \end{cases}$$

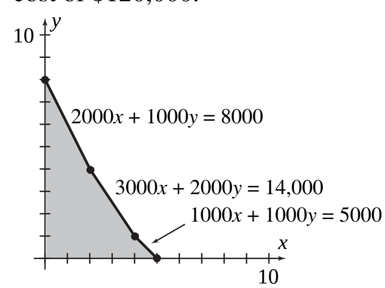
The feasible region is unbounded with 3 corner points. C is minimized at the corner point (10, 10) where $C = 1100$ (see the minimum cost line). Thus 10 tons of ore I and 10 tons of ore II give a minimum cost of $1100.



18. Let x and y be the number of days Refinery I and Refinery II are operated, respectively. Then we are to minimize $C = 25{,}000x + 20{,}000y$ where

$$\begin{cases} x \geq 0, \\ y \geq 0, \\ 2000x + 1000y \geq 8000 & \text{(for low grade)}, \\ 3000x + 2000y \geq 14{,}000 & \text{(for medium grade)}, \\ 1000x + 1000y \geq 5000 & \text{(for high grade)}. \end{cases}$$

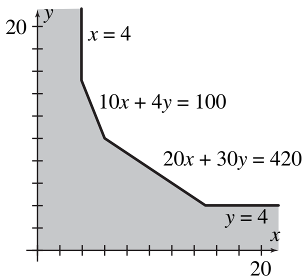
The feasible region is unbounded with 4 corner points. Evaluating C at each corner point, we find that C is minimized at corner point (4, 1) where $C = 120{,}000$. Thus, operate Refinery I for 4 days and Refinery II for 1 day for a minimum cost of $120,000.



19. Let x and y be the number of chambers of type A and B, respectively. Then we are to minimize $C = 600{,}000x + 300{,}000y$, where

$$\begin{cases} x \geq 4, \\ y \geq 4, \\ 10x + 4y \geq 100 & \text{(for polymer } P_1\text{)}, \\ 20x + 30y \geq 420 & \text{(for polymer } P_2\text{)}. \end{cases}$$

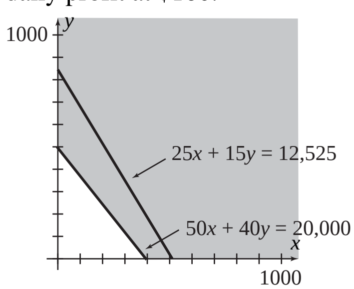
The feasible region is unbounded with 3 corner points. Evaluating C at each corner point, we find C is minimized at corner point (6, 10) where $C = 6{,}600{,}000$. Thus the solution is 6 chambers of type A and 10 chambers of type B.



20. Let x and y be the number of liters produced by the old and new processes, respectively. We want to maximize $P = 0.4x + 0.15y$, where

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ 25x + 15y \leq 12{,}525 \text{ (for carbon dioxide)} \\ 50x + 40y \leq 20{,}000 \text{ (for particulate matter)} \end{cases}$$

The feasible region is bounded with three corner points. Evaluating P at each corner point, we find that P is maximized at the corner point (400, 0), where $P = 160$. Thus daily production of 400 liters by only the old process maximizes daily profit at \$160.



21. **a.** A builds x km of highway and y km of expressway, so B builds $(300 - x)$ km of highway and $(200 - y)$ km of expressway. Thus
$D = 2x + 6y + 3(300 - x) + 5(200 - y)$
$= 1900 - x + y.$

b. The first constraint is company A's construction limit.
The second constraint is company B's construction limit, which arises as follows:

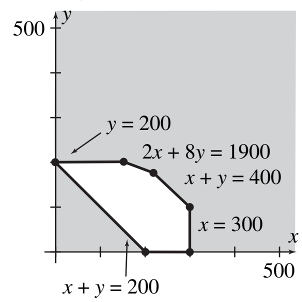
$$(300 - x) + (200 - y) \leq 300,$$
$$500 - x - y \leq 300,$$
$$-x - y \leq -200,$$
$$x + y \geq 200.$$

The third constraint is the minimum contract for A.
The fourth constraint is the minimum contract for B, which arises as follows:

$$2(300 - x) + 8(200 - y) \geq 300,$$
$$2200 - 2x - 8y \geq 300,$$
$$-2x - 8y \geq -1900,$$
$$2x + 8y \leq 1900.$$

The fifth constraint reflects the fact that company A will not build more than 300 km of highway, since 300 km is the total being built; the sixth constraint is the corresponding constraint for the amount of expressway.

c. The feasible region (see below) is bounded. The corner points are (0, 200), (150, 200), $\left(\frac{650}{3}, \frac{550}{3}\right)$, (300, 100), (300, 0), and (200, 0).
Evaluating D at each corner point, we find that D is maximized at point (0, 200), where $D = 2100$. That is, D is maximized when $x = 0$, $y = 200$.



22. $Z = 2.71$ when $x = 1.14$, $y = 1.43$

23. $Z = 15.54$ when $x = 2.56$, $y = 6.74$

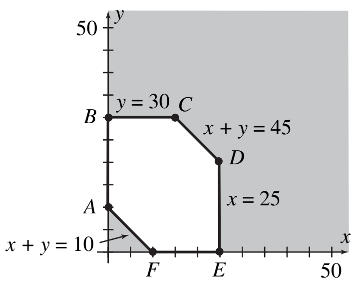
24. The feasible region is empty, so there is no optimum solution.

25. $Z = -75.98$ when $x = 9.48$, $y = 16.67$

Principles in Practice 7.3

1. Using the hint, the cost of shipping the TV sets is $Z = 18x + 24(25 - x) + 9y + 15(30 - y)$ $= 1050 - 6x - 6y$.
Since negative numbers of TV sets cannot be shipped, $x \geq 0$, $y \geq 0$, $25 - x \geq 0$, and $30 - y \geq 0$. Since warehouse C has only 45 TV sets,
$x + y \leq 45$. Similarly, since warehouse D has only 40 TV sets, $25 - x + 30 - y \leq 45$ or $x + y \geq 10$.
We need to minimize $Z = 1050 - 6x - 6y$ subject

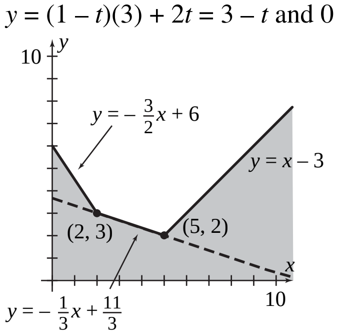
to the constraints
$x + y \le 45,$
$x + y \ge 10,$
$x \le 25,$
$y \le 30,$
$x \ge 0, y \ge 0.$



The feasible region shown has corners
$A = (0, 10)$, $B = (0, 30)$, $C = (15, 30)$,
$D = (25, 20)$, $E = (25, 0)$, and $F = (10, 0)$.
Evaluating the cost function at the corners gives
$Z(A) = 1050 - 6(0) - 6(10) = 990$
$Z(B) = 1050 - 6(0) - 6(30) = 870$
$Z(C) = 1050 - 6(15) - 6(30) = 780$
$Z(D) = 1050 - 6(25) - 6(20) = 780$
$Z(E) = 1050 - 6(25) - 6(0) = 900$
$Z(F) = 1050 - 6(10) - 6(0) = 990$
The minimum value of Z is 780 which occurs at all points on the line segment joining C and D. This is $x = (1 - t)(15) + t(25) = 15 + 10t$ and $y = (1 - t)(30) + t(20) = 30 - 10t$ for $0 \le t \le 1$. Thus, ship $10t + 15$ TV sets from C to A, $-10t + 30$ TV sets from C to B, $25 - (10t + 15) = -10t + 10$ TV sets from D to A, and $30 - (-10t + 30) = 10t$ TV sets from D to B, for $0 \le t \le 1$. The minimum cost is $780.

Problems 7.3

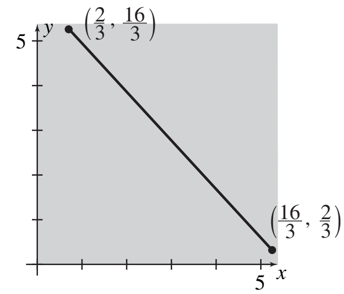
1. The feasible region is unbounded. Z is minimized at corner points (2, 3) and (5, 2), where its value is 33. Z is also minimized at all points on the line segment joining (2, 3) and (5, 2), so the solution is $Z = 33$ when
$x = (1 - t)(2) + 5t = 2 + 3t$
$y = (1 - t)(3) + 2t = 3 - t$ and $0 \le t \le 1$.



2. The feasible region is a line segment. The corner points are $\left(\frac{2}{3}, \frac{16}{3}\right)$ and $\left(\frac{16}{3}, \frac{2}{3}\right)$. At each of these points $Z = 12$. Thus Z is maximized at both corner points, as well as at all points on the line segment. Thus the solution is $Z = 12$ when

$$x = (1-t)\left(\frac{2}{3}\right) + \frac{16}{3}t = \frac{2}{3} + \frac{14}{3}t,$$

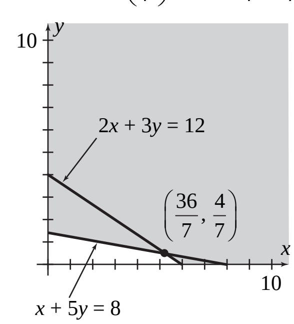
$$x = (1-t)\left(\frac{16}{3}\right) + \frac{2}{3}t = \frac{16}{3} - \frac{14}{3}t, \text{ and } 0 \le t \le 1.$$



3. The feasible region appears below. The corner points are (0, 0), $\left(0, \frac{8}{5}\right)$, $\left(\frac{36}{7}, \frac{4}{7}\right)$ and (6, 0). Z is maximized at $\left(\frac{36}{7}, \frac{4}{7}\right)$ and (6, 0), where its value is 84. Thus Z is also maximized at all points on the line segment joining $\left(\frac{36}{7}, \frac{4}{7}\right)$ and (6, 0). The solution is $Z = 84$ when

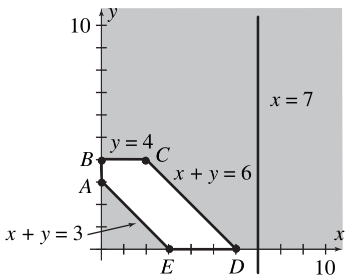
$$x = (1-t)\left(\frac{36}{7}\right) + 6t = \frac{6}{7}t + \frac{36}{7},$$

$$y = (1-t)\left(\frac{4}{7}\right) + 0t = \frac{4}{7} - \frac{4}{7}t \text{ and } 0 \le t \le 1.$$



4. Using the hint, the cost of delivering the cars is
$Z = 60x + 45y + 50(7 - x) + 35(4 - y)$
$= 490 + 10x + 10y.$
Since negative numbers of cars is not possible, $x \ge 0$, $y \ge 0$, $7 - x \ge 0$, and $4 - y \ge 0$. Since

the warehouse in Concord has only 6 cars,
$x + y \le 6$.
Similarly, since the Dublin warehouse has only 8 cars, $7 - x + 4 - y \le 8$ or $3 \le x + y$.
We need to minimize $Z = 490 + 10x + 10y$ subject to the constraints
$x + y \le 6$,
$x + y \ge 3$,
$x \le 7$,
$y \le 4$,
$x \ge 0, y \ge 0$.



The feasible region shown has corners $A = (0, 3)$, $B = (0, 4)$, $C = (2, 4)$, $D = (6, 0)$, and $E = (3, 0)$. Evaluating the cost function at the corners gives
$Z(A) = 490 + 10(0) + 10(3) = 520$
$Z(B) = 490 + 10(0) + 10(4) = 530$
$Z(C) = 490 + 10(2) + 10(4) = 550$
$Z(D) = 490 + 10(6) + 10(0) = 550$
$Z(E) = 490 + 10(3) + 10(0) = 520$
The minimum value of Z is 520 which occurs at all points on the line segment joining A and E. This is $x = (1 - t)(0) + t(3) = 3t$ and $y = (1 - t)(3) + t(0) = -3t + 3$ for $0 \le t \le 1$. Thus have $3t$ cars delivered from Concord to Atherton, $-3t + 3$ delivered from Concord to Berkeley, $7 - 3t$ delivered from Dublin to Atherton, and $4 - (-3t + 3) = 3t + 1$ delivered from Dublin to Berkeley, for $0 \le t \le 1$. The minimum cost is \$520.

Principles in Practice 7.4

In these problems, the pivot entry is underlined.

1. Let x_1, x_2, and x_3 be the numbers of Type 1, Type 2, and Type 3 players, respectively, that the company produces. The situation is to maximize the profit $P = 150x_1 + 250x_2 + 200x_3$, subject to the constraints

$$300x_1 + 300x_2 + 400x_3 \le 30{,}000$$
$$15x_1 + 15x_2 + 10x_3 \le 1200$$
$$2x_1 + 2x_2 + 3x_3 \le 180$$
$$x_1, x_2, x_3 \ge 0$$

The constraint inequalities can be simplified by dividing by the greatest common factor of the numbers involved. Thus, we will use

$$3x_1 + 3x_2 + 4x_3 \le 300$$
$$3x_1 + 3x_2 + 2x_3 \le 240$$
$$2x_1 + 2x_2 + 3x_3 \le 180$$
$$x_1, x_2, x_3 \ge 0$$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ P \end{array}\begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P & b \end{array} \\ \left[\begin{array}{ccccccc|c} 3 & 3 & 4 & 1 & 0 & 0 & 0 & 300 \\ 3 & \underline{3} & 2 & 0 & 1 & 0 & 0 & 240 \\ 2 & 2 & 3 & 0 & 0 & 1 & 0 & 180 \\ \hline -150 & -250 & -200 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \\ s_1 \\ x_2 \\ s_3 \\ P \end{array}\begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P & b \end{array} \\ \left[\begin{array}{ccccccc|c} 0 & 0 & 2 & 1 & -1 & 0 & 0 & 60 \\ 1 & 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 80 \\ 0 & 0 & \underline{\frac{5}{3}} & 0 & -\frac{2}{3} & 1 & 0 & 20 \\ \hline 100 & 0 & -\frac{100}{3} & 0 & \frac{250}{3} & 0 & 1 & 20{,}000 \end{array}\right] \end{array}$$

$$\begin{array}{c} \\ s_1 \\ x_2 \\ x_3 \\ P \end{array}\begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P & b \end{array} \\ \left[\begin{array}{ccccccc|c} 0 & 0 & 0 & 1 & -\frac{1}{5} & -\frac{6}{5} & 0 & 36 \\ 1 & 1 & 0 & 0 & \frac{3}{5} & -\frac{2}{5} & 0 & 72 \\ 0 & 0 & 1 & 0 & -\frac{2}{5} & \frac{3}{5} & 0 & 12 \\ \hline 100 & 0 & 0 & 0 & 70 & 20 & 1 & 20{,}400 \end{array}\right] \end{array}$$

The maximum value of P is 20,400 when $x_1 = 0, x_2 = 72$, and $x_3 = 12$. The maximum profit is \$20,400 when 72 Type 2 players and 12 Type 3 players are produced and sold.

Problems 7.4

In these problems, the pivot entry is underlined.

1.

$$\begin{array}{c} \\ s_1 \\ s_2 \\ Z \end{array}\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & Z & \end{array} \\ \left[\begin{array}{ccccc|c} 2 & 1 & 1 & 0 & 0 & 8 \\ 2 & \underline{3} & 0 & 1 & 0 & 12 \\ \hline -1 & -2 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}\begin{array}{c} \\ 8 \\ 4 \\ \\ \end{array}$$

$$\begin{array}{c} \\ s_1 \\ x_2 \\ Z \end{array}\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & Z & \end{array} \\ \left[\begin{array}{ccccc|c} \frac{4}{3} & 0 & 1 & -\frac{1}{3} & 0 & 4 \\ \frac{2}{3} & 1 & 0 & \frac{1}{3} & 0 & 4 \\ \hline \frac{1}{3} & 0 & 0 & \frac{2}{3} & 1 & 8 \end{array}\right] \end{array}$$

The solution is $Z = 8$ when $x_1 = 0, x_2 = 4$.

2.

$$\begin{array}{c|cccccc|c|l} & x_1 & x_2 & s_1 & s_2 & Z & & \\ \hline s_1 & -1 & 1 & 1 & 0 & 0 & 4 & \\ s_2 & \underline{1} & 1 & 0 & 1 & 0 & 6 & 6 \\ \hdashline Z & -2 & -1 & 0 & 0 & 1 & 0 & \end{array}$$

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & s_1 & s_2 & Z & \\ \hline s_1 & 0 & 2 & 1 & 1 & 0 & 10 \\ x_1 & 1 & 1 & 0 & 1 & 0 & 6 \\ \hdashline Z & 0 & 1 & 0 & 2 & 1 & 12 \end{array}$$

The solution is $Z = 12$ when $x_1 = 6, x_2 = 0$.

3.

$$\begin{array}{c|cccccc|c|l} & x_1 & x_2 & s_1 & s_2 & Z & & \\ \hline s_1 & 3 & 2 & 1 & 0 & 0 & 5 & \frac{5}{2} \\ s_2 & -1 & \underline{3} & 0 & 1 & 0 & 3 & 1 \\ \hdashline Z & 1 & -2 & 0 & 0 & 1 & 0 & \end{array}$$

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & s_1 & s_2 & Z & \\ \hline s_1 & \frac{11}{3} & 0 & 1 & -\frac{2}{3} & 0 & 3 \\ x_2 & -\frac{1}{3} & 1 & 0 & \frac{1}{3} & 0 & 1 \\ \hdashline Z & \frac{1}{3} & 0 & 0 & \frac{2}{3} & 1 & 2 \end{array}$$

The solution is $Z = 2$ when $x_1 = 0, x_2 = 1$.

4.

$$\begin{array}{c|cccccc|c|l} & x_1 & x_2 & s_1 & s_2 & Z & & \\ \hline s_1 & 2 & 3 & 1 & 0 & 0 & 9 & 3 \\ s_2 & 1 & \underline{5} & 0 & 1 & 0 & 10 & 2 \\ \hdashline Z & -4 & -7 & 0 & 0 & 1 & 0 & \end{array}$$

$$\begin{array}{c|cccccc|c|l} & x_1 & x_2 & s_1 & s_2 & Z & & \\ \hline s_1 & \frac{7}{5} & 0 & 1 & -\frac{3}{5} & 0 & 3 & \frac{15}{7} \\ x_2 & \frac{1}{5} & 1 & 0 & \frac{1}{5} & 0 & 2 & 10 \\ \hdashline Z & -\frac{13}{5} & 0 & 0 & \frac{7}{5} & 1 & 14 & \end{array}$$

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & s_1 & s_2 & Z & \\ \hline x_1 & 1 & 0 & \frac{5}{7} & -\frac{3}{7} & 0 & \frac{15}{7} \\ x_2 & 0 & 1 & -\frac{1}{7} & \frac{2}{7} & 0 & \frac{11}{7} \\ \hdashline Z & 0 & 0 & \frac{13}{7} & \frac{2}{7} & 1 & \frac{137}{7} \end{array}$$

The solution is $Z = \frac{137}{7}$ when $x_1 = \frac{15}{7}, x_2 = \frac{11}{7}$.

5.

$$\begin{array}{c|ccccccc|c|l} & x_1 & x_2 & s_1 & s_2 & s_3 & Z & & \\ \hline s_1 & \underline{1} & -1 & 1 & 0 & 0 & 0 & 1 & 1 \\ s_2 & 1 & 2 & 0 & 1 & 0 & 0 & 8 & 8 \\ s_3 & 1 & 1 & 0 & 0 & 1 & 0 & 5 & 5 \\ \hdashline Z & -8 & -2 & 0 & 0 & 0 & 1 & 0 & \end{array}$$

$$\begin{array}{c|ccccccc|c|l} & x_1 & x_2 & s_1 & s_2 & s_3 & Z & & \\ \hline x_1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 & \\ s_2 & 0 & 3 & -1 & 1 & 0 & 0 & 7 & \frac{7}{3} \\ s_3 & 0 & \underline{2} & -1 & 0 & 1 & 0 & 4 & 2 \\ \hdashline Z & 0 & -10 & 8 & 0 & 0 & 1 & 8 & \end{array}$$

$$\begin{array}{c|ccccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & Z & \\ \hline x_1 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 3 \\ s_2 & 0 & 0 & \frac{1}{2} & 1 & -\frac{3}{2} & 0 & 1 \\ x_2 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 2 \\ \hdashline Z & 0 & 0 & 3 & 0 & 5 & 1 & 28 \end{array}$$

The solution is $Z = 28$ when $x_1 = 3, x_2 = 2$.

6.

$$\begin{array}{c|ccccccc|c|l} & x_1 & x_2 & s_1 & s_2 & s_3 & Z & & \\ \hline s_1 & \underline{1} & -1 & 1 & 0 & 0 & 0 & 4 & 4 \\ s_2 & -1 & 1 & 0 & 1 & 0 & 0 & 4 & \\ s_3 & 1 & 1 & 0 & 0 & 1 & 0 & 6 & 6 \\ \hdashline Z & -2 & 6 & 0 & 0 & 0 & 1 & 0 & \end{array}$$

$$\begin{array}{c|ccccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & Z & \\ \hline x_1 & 1 & -1 & 1 & 0 & 0 & 0 & 4 \\ s_2 & 0 & 0 & 1 & 1 & 0 & 0 & 8 \\ s_3 & 0 & 2 & -1 & 0 & 1 & 0 & 2 \\ \hdashline Z & 0 & 4 & 2 & 0 & 0 & 1 & 8 \end{array}$$

The solution is $Z = 8$ when $x_1 = 4, x_2 = 0$

7.

$$\begin{array}{c|cccccc|c|c} & x_1 & x_2 & x_3 & s_1 & s_2 & Z & & \\ \hline s_1 & 1 & 2 & 0 & 1 & 0 & 0 & 10 & 5 \\ s_2 & 2 & \underline{2} & 1 & 0 & 1 & 0 & 10 & 5 \\ \hline Z & -3 & -4 & -\frac{3}{2} & 0 & 0 & 1 & 0 & \end{array}$$

choosing s_2 as departing variable

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & Z & \\ \hline s_1 & -1 & 0 & -1 & 1 & -1 & 0 & 0 \\ x_2 & 1 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 5 \\ \hline Z & 1 & 0 & \frac{1}{2} & 0 & 2 & 1 & 20 \end{array}$$

The solution is $Z = 20$ when $x_1 = 0, x_2 = 5, x_3 = 0$

8. If s_1 is the departing variable, then

$$\begin{array}{c|cccccc|c|c} & x_1 & x_2 & x_3 & s_1 & s_2 & Z & & \\ \hline s_1 & \underline{2} & 1 & -1 & 1 & 0 & 0 & 4 & 2 \\ s_2 & 1 & 1 & 1 & 0 & 1 & 0 & 2 & 2 \\ \hline Z & -2 & 1 & -1 & 0 & 0 & 1 & 0 & \end{array}$$

$$\begin{array}{c|cccccc|c|c} & x_1 & x_2 & x_3 & s_1 & s_2 & Z & & \\ \hline x_1 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 2 & \\ s_2 & 0 & \frac{1}{2} & \underline{\frac{3}{2}} & -\frac{1}{2} & 1 & 0 & 0 & 0 \\ \hline Z & 0 & 2 & -2 & 1 & 0 & 1 & 4 & \end{array}$$

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & Z & \\ \hline x_1 & 1 & \frac{2}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 2 \\ x_3 & 0 & \frac{1}{3} & 1 & -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \hline Z & 0 & \frac{8}{3} & 0 & \frac{1}{3} & \frac{4}{3} & 1 & 4 \end{array}$$

The solution is $Z = 4$ when $x_1 = 2, x_2 = 0, x_3 = 0$.

Choosing s_2 as the departing variable

$$\begin{array}{c|cccccc|c|c} & x_1 & x_2 & x_3 & s_1 & s_2 & Z & & \\ \hline s_1 & 2 & 1 & -1 & 1 & 0 & 0 & 4 & 2 \\ s_2 & \underline{1} & 1 & 1 & 0 & 1 & 0 & 2 & 2 \\ \hline Z & -2 & 1 & -1 & 0 & 0 & 1 & 0 & \end{array}$$

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & Z & \\ \hline s_1 & 0 & -1 & -3 & 1 & -2 & 0 & 0 \\ x_1 & 1 & 1 & 1 & 0 & 1 & 0 & 2 \\ \hline Z & 0 & 3 & 1 & 0 & 2 & 1 & 4 \end{array}$$

Thus the maximum value of Z is 4, when $x_1 = 2, x_2 = 0, x_3 = 0$.

9. To obtain a standard linear programming problem, we write the second constraint as $-x_1 + 2x_2 + x_3 \le 2$.

$$\begin{array}{c|cccccc|c|c} & x_1 & x_2 & x_3 & s_1 & s_2 & Z & & \\ \hline s_1 & \underline{1} & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ s_2 & -1 & 2 & 1 & 0 & 1 & 0 & 2 & \\ \hline Z & -2 & -1 & 1 & 0 & 0 & 1 & 0 & \end{array}$$

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & Z & \\ \hline x_1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ s_2 & 0 & 3 & 1 & 1 & 1 & 0 & 3 \\ \hline Z & 0 & 1 & 1 & 2 & 0 & 1 & 2 \end{array}$$

The solution is $Z = 2$ when $x_1 = 1, x_2 = 0, x_3 = 0$.

10. To obtain a standard linear programming problem, we write the third constraint as $-x_1 + x_2 \le 3$.

$$\begin{array}{c|ccccccc|c|c} & x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & Z & & \\ \hline s_1 & 1 & \underline{1} & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ s_2 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 2 & \\ s_3 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 3 & 3 \\ s_4 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 5 & \\ \hline Z & 2 & -3 & 0 & 0 & 0 & 0 & 1 & 0 & \end{array}$$

$$\begin{array}{c|ccccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & Z & \\ \hline x_2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ s_2 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 3 \\ s_3 & -2 & 0 & -1 & 0 & 1 & 0 & 0 & 2 \\ s_4 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 5 \\ \hline Z & 5 & 0 & 3 & 0 & 0 & 0 & 1 & 3 \end{array}$$

The solution is $Z = 3$ when $x_1 = 0, x_2 = 1$.

11.

$$\begin{array}{c|ccccccc|c|c} & x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & Z & & \\ \hline s_1 & \underline{2} & -1 & 1 & 0 & 0 & 0 & 0 & 4 & 2 \\ s_2 & -1 & 2 & 0 & 1 & 0 & 0 & 0 & 6 & \\ s_3 & 5 & 3 & 0 & 0 & 1 & 0 & 0 & 20 & 4 \\ s_4 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 10 & 5 \\ \hline Z & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & \end{array}$$

choosing x_1 as entering variable

$$\begin{array}{c|ccccccc|c|c} & x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & Z & & \\ \hline x_1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 2 & \\ s_2 & 0 & \frac{3}{2} & \frac{1}{2} & 1 & 0 & 0 & 0 & 8 & \frac{16}{3} \\ s_3 & 0 & \frac{11}{2} & -\frac{5}{2} & 0 & 1 & 0 & 0 & 10 & \frac{20}{11} \\ s_4 & 0 & 2 & -1 & 0 & 0 & 1 & 0 & 6 & 3 \\ \hline Z & 0 & -\frac{3}{2} & \frac{1}{2} & 0 & 0 & 0 & 1 & 2 & \end{array}$$

$$\begin{array}{c|ccccccc|c|c} & x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & Z & & \\ \hline x_1 & 1 & 0 & \frac{3}{11} & 0 & \frac{1}{11} & 0 & 0 & \frac{32}{11} & \frac{32}{3} \\ s_2 & 0 & 0 & \frac{13}{11} & 1 & -\frac{3}{11} & 0 & 0 & \frac{58}{11} & \frac{58}{13} \\ x_2 & 0 & 1 & -\frac{5}{11} & 0 & \frac{2}{11} & 0 & 0 & \frac{20}{11} & \\ s_4 & 0 & 0 & -\frac{1}{11} & 0 & -\frac{4}{11} & 1 & 0 & \frac{26}{11} & \\ \hline Z & 0 & 0 & -\frac{2}{11} & 0 & \frac{3}{11} & 0 & 1 & \frac{52}{11} & \end{array}$$

$$\begin{array}{c|ccccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & Z & \\ \hline x_1 & 1 & 0 & 0 & -\frac{3}{13} & \frac{2}{13} & 0 & 0 & \frac{22}{13} \\ s_1 & 0 & 0 & 1 & \frac{11}{13} & -\frac{3}{13} & 0 & 0 & \frac{58}{13} \\ x_2 & 0 & 1 & 0 & \frac{5}{13} & \frac{1}{13} & 0 & 0 & \frac{50}{13} \\ s_4 & 0 & 0 & 0 & \frac{1}{13} & -\frac{5}{13} & 1 & 0 & \frac{36}{13} \\ \hline Z & 0 & 0 & 0 & \frac{2}{13} & \frac{3}{13} & 0 & 1 & \frac{72}{13} \end{array}$$

Thus the maximum value of Z is $\frac{72}{13}$, when $x_1 = \frac{22}{13}$, $x_2 = \frac{50}{13}$. If we choose x_2 as the entering variable, then we have:

$$\begin{array}{c|ccccccc|c|c} & x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & Z & & \\ \hline s_1 & 2 & -1 & 1 & 0 & 0 & 0 & 0 & 4 & \\ s_2 & -1 & 2 & 0 & 1 & 0 & 0 & 0 & 6 & 3 \\ s_3 & 5 & 3 & 0 & 0 & 1 & 0 & 0 & 20 & \frac{20}{3} \\ s_4 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 10 & 10 \\ \hline Z & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & \end{array}$$

$$\begin{array}{c|ccccccc|c|c} & x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & Z & & \\ \hline s_1 & \frac{3}{2} & 0 & 1 & \frac{1}{2} & 0 & 0 & 0 & 7 & \frac{14}{3} \\ x_2 & -\frac{1}{2} & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 3 & \\ s_3 & \frac{13}{2} & 0 & 0 & -\frac{3}{2} & 1 & 0 & 0 & 11 & \frac{22}{13} \\ s_4 & \frac{5}{2} & 0 & 0 & -\frac{1}{2} & 0 & 1 & 0 & 7 & \frac{14}{5} \\ \hline Z & -\frac{3}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & 3 & \end{array}$$

$$\begin{array}{c|ccccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & Z & \\ \hline s_1 & 0 & 0 & 1 & \frac{11}{13} & -\frac{3}{13} & 0 & 0 & \frac{58}{13} \\ x_2 & 0 & 1 & 0 & \frac{5}{13} & \frac{1}{13} & 0 & 0 & \frac{50}{13} \\ x_1 & 1 & 0 & 0 & -\frac{3}{13} & \frac{2}{13} & 0 & 0 & \frac{22}{13} \\ s_4 & 0 & 0 & 0 & \frac{1}{13} & -\frac{5}{13} & 1 & 0 & \frac{36}{13} \\ \hline Z & 0 & 0 & 0 & \frac{2}{13} & \frac{3}{13} & 0 & 1 & \frac{72}{13} \end{array}$$

The solution is $Z = \frac{72}{13}$ when $x_1 = \frac{22}{13}$, $x_2 = \frac{50}{13}$.

12. To obtain a standard linear programming problem, we write the first constraint as $2x_1 - x_2 - x_3 \le 2$.

$$\begin{array}{c|ccccccc|c|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & W & & \\ \hline s_1 & 2 & -1 & -1 & 1 & 0 & 0 & 0 & 2 & 1 \\ s_2 & 1 & -1 & 1 & 0 & 1 & 0 & 0 & 4 & 4 \\ s_3 & 1 & 1 & 2 & 0 & 0 & 1 & 0 & 6 & 6 \\ \hline W & -2 & -1 & 2 & 0 & 0 & 0 & 1 & 0 & \end{array}$$

$$\begin{array}{c|ccccccc|c|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & W & & \\ \hline x_1 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 1 & \\ s_2 & 0 & -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 1 & 0 & 0 & 3 & \\ s_3 & 0 & \frac{3}{2} & \frac{5}{2} & -\frac{1}{2} & 0 & 1 & 0 & 5 & \frac{10}{3} \\ \hline W & 0 & -2 & 1 & 1 & 0 & 0 & 1 & 2 & \end{array}$$

$$\begin{array}{c|ccccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & W & \\ \hline x_1 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{8}{3} \\ s_2 & 0 & 0 & \frac{7}{3} & -\frac{2}{3} & 1 & \frac{1}{3} & 0 & \frac{14}{3} \\ x_2 & 0 & 1 & \frac{5}{3} & -\frac{1}{3} & 0 & \frac{2}{3} & 0 & \frac{10}{3} \\ \hline W & 0 & 0 & \frac{13}{3} & \frac{1}{3} & 0 & \frac{4}{3} & 1 & \frac{26}{3} \end{array}$$

The solution is $W = \frac{26}{3}$ when $x_1 = \frac{8}{3}, x_2 = \frac{10}{3}, x_3 = 0$.

13. To obtain a standard linear programming problem, we write the second constraint as $-x_1 - x_2 + x_3 \le 2$ and the third constraint as $x_1 - x_2 - x_3 \le 1$.

$$\begin{array}{c|ccccccc|c|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & W & & \\
\hline
s_1 & 4 & 3 & -1 & 1 & 0 & 0 & 0 & 1 & \\
s_2 & -1 & -1 & \underline{1} & 0 & 1 & 0 & 0 & 2 & 2 \\
s_3 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 1 & \\
\hline
W & -1 & 12 & -4 & 0 & 0 & 0 & 1 & 0 &
\end{array}$$

$$\begin{array}{c|ccccccc|c|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & W & & \\
\hline
s_1 & \underline{3} & 2 & 0 & 1 & 1 & 0 & 0 & 3 & 1 \\
x_3 & -1 & -1 & 1 & 0 & 1 & 0 & 0 & 2 & \\
s_3 & 0 & -2 & 0 & 0 & 1 & 1 & 0 & 3 & \\
\hline
W & -5 & 8 & 0 & 0 & 4 & 0 & 1 & 8 &
\end{array}$$

$$\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & W & \\
\hline
x_1 & 1 & \frac{2}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 1 \\
x_3 & 0 & -\frac{1}{3} & 1 & \frac{1}{3} & \frac{4}{3} & 0 & 0 & 3 \\
s_3 & 0 & -2 & 0 & 0 & 1 & 1 & 0 & 3 \\
\hline
W & 0 & \frac{34}{3} & 0 & \frac{5}{3} & \frac{17}{3} & 0 & 1 & 13
\end{array}$$

The solution is $W = 13$ when $x_1 = 1, x_2 = 0, x_3 = 3$.

14.

$$\begin{array}{c|ccccccc|c|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & W & & \\
\hline
s_1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 6 & 6 \\
s_2 & 1 & -1 & 1 & 0 & 1 & 0 & 0 & 10 & 10 \\
s_3 & \underline{1} & -1 & -1 & 0 & 0 & 1 & 0 & 4 & 4 \\
\hline
W & -4 & 0 & 1 & 0 & 0 & 0 & 1 & 0 &
\end{array}$$

$$\begin{array}{c|ccccccc|c|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & W & & \\
\hline
s_1 & 0 & \underline{2} & 2 & 1 & 0 & -1 & 0 & 2 & 1 \\
s_2 & 0 & 0 & 2 & 0 & 1 & -1 & 0 & 6 & \\
x_1 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 4 & \\
\hline
W & 0 & -4 & -3 & 0 & 0 & 4 & 1 & 16 &
\end{array}$$

$$\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & W & \\
\hline
x_2 & 0 & 1 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 1 \\
s_2 & 0 & 0 & 2 & 0 & 1 & -1 & 0 & 6 \\
x_1 & 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 5 \\
\hline
W & 0 & 0 & 1 & 2 & 0 & 2 & 1 & 20
\end{array}$$

The solution is $W = 20$ when $x_1 = 5, x_2 = 1, x_3 = 0$.

15.

$$\begin{array}{c|ccccccccc|c|c}
 & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & s_4 & Z & & \\
\hline
s_1 & 1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & \\
s_2 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 5 & \\
s_3 & 0 & 0 & \underline{1} & 1 & 0 & 0 & 1 & 0 & 0 & 4 & 4 \\
s_4 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 & 7 & 7 \\
\hline
Z & -60 & 0 & -90 & 0 & 0 & 0 & 0 & 0 & 1 & 0 &
\end{array}$$

$$\begin{array}{c|ccccccccc|c|c}
 & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & s_4 & Z & & \\
\hline
s_1 & \underline{1} & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 2 \\
s_2 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 5 & 5 \\
x_3 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 4 & \\
s_4 & 0 & 0 & 0 & -3 & 0 & 0 & -1 & 1 & 0 & 3 & \\
\hline
Z & -60 & 0 & 0 & 90 & 0 & 0 & 90 & 0 & 1 & 360 &
\end{array}$$

$$\begin{array}{c|ccccccccc|c|c}
 & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & s_4 & Z & & \\
\hline
x_1 & 1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & \\
s_2 & 0 & \underline{3} & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 3 & 1 \\
x_3 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 4 & \\
s_4 & 0 & 0 & 0 & -3 & 0 & 0 & -1 & 1 & 0 & 3 & \\
\hline
Z & 0 & -120 & 0 & 90 & 60 & 0 & 90 & 0 & 1 & 480 &
\end{array}$$

$$\begin{array}{c|ccccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & s_4 & Z & \\
\hline
x_1 & 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 4 \\
x_2 & 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 1 \\
x_3 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 4 \\
s_4 & 0 & 0 & 0 & -3 & 0 & 0 & -1 & 1 & 0 & 3 \\
\hline
Z & 0 & 0 & 0 & 90 & 20 & 40 & 90 & 0 & 1 & 600
\end{array}$$

The solution is $Z = 600$ for $x_1 = 4, x_2 = 1, x_3 = 4, x_4 = 0$.

16.

$$\begin{array}{c|cccccccc|c|c}
 & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & Z & & \\
\hline
s_1 & \underline{1} & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 3 & 3 \\
s_2 & 1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 6 & 6 \\
s_3 & 1 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 5 & 5 \\
\hline
Z & -3 & -2 & 2 & 1 & 0 & 0 & 0 & 1 & 0 &
\end{array}$$

$$\begin{array}{c|cccccccc|c|c}
 & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & Z & & \\
\hline
x_1 & 1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 3 & \\
s_2 & 0 & -1 & -1 & 2 & -1 & 1 & 0 & 0 & 3 & \\
s_3 & 0 & \underline{1} & -2 & 2 & -1 & 0 & 1 & 0 & 2 & 2 \\
\hline
Z & 0 & -2 & 5 & -2 & 3 & 0 & 0 & 1 & 9 &
\end{array}$$

choosing x_2 as the entering variable.

$$\begin{array}{c|cccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & Z & \\
\hline
x_1 & 1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 3 \\
s_2 & 0 & 0 & -3 & 4 & -2 & 1 & 1 & 0 & 5 \\
x_2 & 0 & 1 & -2 & 2 & -1 & 0 & 1 & 0 & 2 \\
\hline
Z & 0 & 0 & 1 & 2 & 1 & 0 & 2 & 1 & 13
\end{array}$$

Choosing x_4 as the entering variable in the second table, we have:

$$\begin{array}{c|cccccccc|c|c}
 & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & Z & & \\
\hline
x_1 & 1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 3 & \\
s_2 & 0 & -1 & -1 & 2 & -1 & 1 & 0 & 0 & 3 & \frac{3}{2} \\
s_3 & 0 & 1 & -2 & \underline{2} & -1 & 0 & 1 & 0 & 2 & 1 \\
\hline
Z & 0 & -2 & 5 & -2 & 3 & 0 & 0 & 1 & 9 &
\end{array}$$

$$\begin{array}{c|cccccccc|c|c}
 & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & Z & & \\
\hline
x_1 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 4 & 8 \\
s_2 & 0 & -2 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & \\
x_4 & 0 & \underline{\frac{1}{2}} & -1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 1 & 2 \\
\hline
Z & 0 & -1 & 3 & 0 & 2 & 0 & 1 & 1 & 11 &
\end{array}$$

$$\begin{array}{c|cccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & Z & \\
\hline
x_1 & 1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 3 \\
s_2 & 0 & 0 & -3 & 4 & -2 & 1 & 1 & 0 & 5 \\
x_2 & 0 & 1 & -2 & 2 & -1 & 0 & 1 & 0 & 2 \\
\hline
Z & 0 & 0 & 1 & 2 & 1 & 0 & 2 & 1 & 13
\end{array}$$

The solution is $Z = 13$ when
$x_1 = 3,\ x_2 = 2,\ x_3 = 0,\ x_4 = 0.$

17. Let x_1 and x_2 denote the numbers of boxes transported from A and B, respectively. The revenue received is $R = 0.75x_1 + 0.50x_2$. We want to maximize R subject to

$2x_1 + x_2 \le 2400$ (volume),
$3x_1 + 5x_2 \le 36{,}800$ (weight),
$x_1,\ x_2 \ge 0.$

$$\begin{array}{c|ccccc|c|c}
 & x_1 & x_2 & s_1 & s_2 & R & & \\
\hline
s_1 & \underline{2} & 1 & 1 & 0 & 0 & 2400 & 1200 \\
s_2 & 3 & 5 & 0 & 1 & 0 & 36{,}800 & 12{,}266\frac{2}{3} \\
\hline
R & -\frac{3}{4} & -\frac{1}{2} & 0 & 0 & 1 & 0 &
\end{array}$$

$$\begin{array}{c|ccccc|c|c}
 & x_1 & x_2 & s_1 & s_2 & R & & \\
\hline
x_1 & 1 & \underline{\frac{1}{2}} & \frac{1}{2} & 0 & 0 & 1200 & 2400 \\
s_2 & 0 & \frac{7}{2} & -\frac{3}{2} & 1 & 0 & 33{,}200 & 9485\frac{5}{7} \\
\hline
R & 0 & -\frac{1}{8} & \frac{3}{8} & 0 & 1 & 900 &
\end{array}$$

$$\begin{array}{c|ccccc|c}
 & x_1 & x_2 & s_1 & s_2 & R & \\
\hline
x_2 & 2 & 1 & 1 & 0 & 0 & 2400 \\
s_2 & -7 & 0 & -5 & 1 & 0 & 24{,}800 \\
\hline
R & \frac{1}{4} & 0 & \frac{1}{2} & 0 & 1 & 1200
\end{array}$$

Thus 0 boxes from A and 2400 from B give a maximum revenue of \$1200.

18. Let x, y, and z denote the numbers of units of X, Y, and Z produced, respectively. We want to maximize $P = 6x + 8y + 12z$ subject to

$x + 2y + 3z \le 900,$
$4x + 4y + 8z \le 5000,$
$x,\ y,\ z \ge 0.$

$$\begin{array}{c|cccccc|c|c}
 & x & y & z & s_1 & s_2 & P & & \\
\hline
s_1 & 1 & 2 & \underline{3} & 1 & 0 & 0 & 900 & 300 \\
s_2 & 4 & 4 & 8 & 0 & 1 & 0 & 5000 & 625 \\
\hline
P & -6 & -8 & -12 & 0 & 0 & 1 & 0 &
\end{array}$$

$$\begin{array}{c|cccccc|c|c}
 & x & y & z & s_1 & s_2 & P & & \\
\hline
z & \frac{1}{3} & \frac{2}{3} & 1 & \frac{1}{3} & 0 & 0 & 300 & 900 \\
s_2 & \frac{4}{3} & -\frac{4}{3} & 0 & -\frac{8}{3} & 1 & 0 & 2600 & 1950 \\
\hline
P & -2 & 0 & 0 & 4 & 0 & 1 & 3600 &
\end{array}$$

$$\begin{array}{c|cccccc|c}
 & x & y & z & s_1 & s_2 & P & \\
\hline
x & 1 & 2 & 3 & 1 & 0 & 0 & 900 \\
s_2 & 0 & -4 & -4 & -4 & 1 & 0 & 1400 \\
\hline
P & 0 & 4 & 6 & 6 & 0 & 1 & 5400
\end{array}$$

P is maximum when $x = 900$, $y = 0$, $z = 0$. This maximum profit is \$5400.

19. Let $x_1,\ x_2,$ and x_3 denote the numbers of chairs, rockers, and chaise lounges produced, respectively. We want to maximize $R = 21x_1 + 24x_2 + 36x_3$ subject to

$x_1 + x_2 + x_3 \le 400,$
$x_1 + x_2 + 2x_3 \le 500,$
$2x_1 + 3x_2 + 5x_3 \le 1450,$
$x_1, x_2, x_3 \ge 0.$

$$\begin{array}{c|ccccccc|c|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & R & & \\
\hline
s_1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 400 & 400 \\
s_2 & 1 & 1 & \underline{2} & 0 & 1 & 0 & 0 & 500 & 250 \\
s_3 & 2 & 3 & 5 & 0 & 0 & 1 & 0 & 1450 & 290 \\
\hline
R & -21 & -24 & -36 & 0 & 0 & 0 & 1 & 0 &
\end{array}$$

	x_1	x_2	x_3	s_1	s_2	s_3	R		
s_1	$\frac{1}{2}$	$\underline{\frac{1}{2}}$	0	1	$-\frac{1}{2}$	0	0	150	300
x_3	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	250	500
s_3	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{5}{2}$	1	0	200	400
R	–3	–6	0	0	18	0	1	9000	

	x_1	x_2	x_3	s_1	s_2	s_3	R	
x_2	1	1	0	2	–1	0	0	300
x_3	0	0	1	–1	1	0	0	100
s_3	–1	0	0	–1	–2	1	0	50
R	3	0	0	12	12	0	1	10,800

The production of 0 chairs, 300 rockers, and 100 chaise lounges gives the maximum revenue of $10,800.

Principles in Practice 7.5

1. Let x_1, x_2, x_3 be the numbers of device 1, device 2, and device 3, respectively, that the company produces. The situation is to maximize the profit $P = 50x_1 + 50x_2 + 50x_3$ subject to the constraints

$5.5x_1 + 5.5x_2 + 6.5x_3 \le 190,$
$3.5x_1 + 6.5x_2 + 7.5x_3 \le 180,$
$4.5x_1 + 6.0x_2 + 6.5x_3 \le 165,$

and $x_1, x_2, x_3 \ge 0$.

The matrices are shown rounded to 2 decimal places, although the exact values are used in the row operations.
Since the indicators are equal, we choose the first column as the pivot column.

	x_1	x_2	x_3	s_1	s_2	s_3	P	b
s_1	$\underline{5.5}$	5.5	6.5	1	0	0	0	190
s_2	3.5	6.5	7.5	0	1	0	0	180
s_3	4.5	6.0	6.5	0	0	1	0	165
P	–50	–50	–50	0	0	0	1	0

	x_1	x_2	x_2	s_1	s_2	s_3	P	b
x_1	1	1	1.18	0.18	0	0	0	34.55
s_2	0	3	3.36	–0.64	1	0	0	59.09
s_3	0	$\underline{1.50}$	1.18	–0.82	0	1	0	9.55
P	0	0	9.09	9.09	0	0	1	1727.27

An optimum solution is
$x_1 = 35, x_2 = 0, x_3 = 0,$ and $P = 1727.$ However, x_2 is a nonbasic variable and its indicator is 0, so we check for multiple solutions. Treating x_2 as an entering variable, the following table is obtained:

	x_1	x_2	x_3	s_1	s_2	s_3	P	b
x_1	1	0	0.39	0.73	0	–0.67	0	28.18
s_2	0	0	1.00	1.00	1	–2.00	0	40.00
x_2	0	1	0.79	–0.55	0	0.67	0	6.36
P	0	0	9.09	9.09	0	0	1	1727.27

Another optimum solution is
$x_1 = 28, x_2 = 6, x_3 = 0,$ and $P = 1727.$
Thus, the optimum solution is for the company to produce $(1 - t)35 + 28t = 35 - 7t$ of device 1, $(1 - t)0 + 6t = 6t$ of device 2, and none of device 3, for $0 \le t \le 1$.

Problems 7.5

1. Yes; for the table, x_2 is the entering variable and the quotients $\frac{6}{2}$ and $\frac{3}{1}$ tie for being the smallest.

2. Yes; the B.F.S. corresponding to the given table has the basic variable x_2 equal to 0.

3.

	x_1	x_2	s_1	s_2	s_3	Z	
s_1	4	–3	1	0	0	0	4
s_2	3	–1	0	1	0	0	6
s_3	5	0	0	0	1	0	8
Z	–2	–7	0	0	0	1	0

The entering variable is x_2. Since no quotients exist, the problem has an unbounded solution. Thus, no optimum solution (unbounded).

4.

	x_1	x_2	s_1	s_2	s_3	s_4	Z		
s_1	1	–1	1	0	0	0	0	7	7
s_2	–1	1	0	1	0	0	0	5	
s_3	8	5	0	0	1	0	0	40	5
s_4	$\underline{2}$	1	0	0	0	1	0	6	3
Z	–2	–1	0	0	0	0	1	0	

$$\begin{array}{c|ccccccc|c|c}
 & x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & Z & & \\
\hline
s_1 & 0 & -\frac{3}{2} & 1 & 0 & 0 & -\frac{1}{2} & 0 & 4 & \\
s_2 & 0 & \frac{3}{2} & 0 & 1 & 0 & \frac{1}{2} & 0 & 8 & \frac{16}{3} \\
s_3 & 0 & 1 & 0 & 0 & 1 & -4 & 0 & 16 & 16 \\
x_1 & 1 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 3 & 6 \\
\hline
Z & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 6 &
\end{array}$$

The maximum value of Z is 6 when $x_1 = 3$ and $x_2 = 0$. Since x_2 is nonbasic for the last table and its indicator is 0, there may be multiple optimum solutions. Treating x_2 as an entering variable and continuing, we have

$$\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & Z & \\
\hline
s_1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 12 \\
x_2 & 0 & 1 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & \frac{16}{3} \\
s_3 & 0 & 0 & 0 & -\frac{2}{3} & 1 & -\frac{13}{3} & 0 & \frac{32}{3} \\
x_1 & 1 & 0 & 0 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\
\hline
Z & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 6
\end{array}$$

Here $Z = 6$ when $x_1 = \frac{1}{3}$ and $x_2 = \frac{16}{3}$. Thus multiple optimum solutions exist. Hence Z is a maximum when $x_1 = (1-t)(3) + \frac{1}{3}t = 3 - \frac{8}{3}t$,

$x_2 = (1-t)(0) + \frac{16}{3}t = \frac{16}{3}t$, and $0 \le t \le 1$. For the last table, s_2 is nonbasic and its indicator is 0. If we continue the process for determining other optimum solutions, we return to the second table.

5.

$$\begin{array}{c|cccccc|c|c}
 & x_1 & x_2 & s_1 & s_2 & s_3 & Z & & \\
\hline
s_1 & 2 & -2 & 1 & 0 & 0 & 0 & 4 & \\
s_2 & -1 & 2 & 0 & 1 & 0 & 0 & 4 & 2 \\
s_3 & 3 & 1 & 0 & 0 & 1 & 0 & 6 & 6 \\
\hline
Z & 4 & -8 & 0 & 0 & 0 & 1 & 0 &
\end{array}$$

$$\begin{array}{c|cccccc|c|c}
 & x_1 & x_2 & s_1 & s_2 & s_3 & Z & & \\
\hline
s_1 & 1 & 0 & 1 & 1 & 0 & 0 & 8 & 8 \\
x_2 & -\frac{1}{2} & 1 & 0 & \frac{1}{2} & 0 & 0 & 2 & \\
s_3 & \frac{7}{2} & 0 & 0 & -\frac{1}{2} & 1 & 0 & 4 & \frac{8}{7} \\
\hline
Z & 0 & 0 & 0 & 4 & 0 & 1 & 16 &
\end{array}$$

Z has a maximum of 16 when $x_1 = 0$, $x_2 = 2$.

Since x_1 is nonbasic for the last table and its indicator is 0, there may be multiple optimum solutions. Treating x_1 as an entering variable, we have

$$\begin{array}{c|cccccc|c}
 & x_1 & x_2 & s_1 & s_2 & s_3 & Z & \\
\hline
s_1 & 0 & 0 & 1 & \frac{8}{7} & -\frac{2}{7} & 0 & \frac{48}{7} \\
x_2 & 0 & 1 & 0 & \frac{3}{7} & \frac{1}{7} & 0 & \frac{18}{7} \\
x_1 & 1 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & 0 & \frac{8}{7} \\
\hline
Z & 0 & 0 & 0 & 4 & 0 & 1 & 16
\end{array}$$

Here $Z = 16$ when $x_1 = \frac{8}{7}$, $x_2 = \frac{18}{7}$. Thus multiple optimum solutions exist. Hence Z is maximum when $x_1 = (1-t)(0) + \frac{8}{7}t = \frac{8}{7}t$,

$x_2 = (1-t)(2) + \frac{18}{7}t = 2 + \frac{4}{7}t$, and $0 \le t \le 1$. For the last table s_3 is nonbasic and its indicator is 0. If we continue the process for determining other optimum solutions, we return to the second table.

6. To obtain a standard linear programming problem, we write the second constraint as $-x_1 + x_2 + x_3 \le 4$.

$$\begin{array}{c|ccccccc|c|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & Z & & \\
\hline
s_1 & 1 & -1 & 4 & 1 & 0 & 0 & 0 & 6 & 6 \\
s_2 & -1 & 1 & 1 & 0 & 1 & 0 & 0 & 4 & \\
s_3 & 1 & -6 & 1 & 0 & 0 & 1 & 0 & 8 & 8 \\
\hline
Z & -8 & -2 & -4 & 0 & 0 & 0 & 1 & 0 &
\end{array}$$

$$\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & Z & \\
\hline
x_1 & 1 & -1 & 4 & 1 & 0 & 0 & 0 & 6 \\
s_2 & 0 & 0 & 5 & 1 & 1 & 0 & 0 & 10 \\
s_3 & 0 & -5 & -3 & -1 & 0 & 1 & 0 & 2 \\
\hline
Z & 0 & -10 & 28 & 8 & 0 & 0 & 1 & 48
\end{array}$$

For the last table, x_2 is the entering variable. Since no quotients exist, the problem has an unbounded solution.
Thus, no optimum solution (unbounded).

7.

	x_1	x_2	x_3	s_1	s_2	s_3	Z		
s_1	9	3	–2	1	0	0	0	5	$\frac{5}{3}$
s_2	4	2	–1	0	1	0	0	2	1
s_3	1	–4	1	0	0	1	0	3	
Z	–5	–6	–1	0	0	0	1	0	

	x_1	x_2	x_3	s_1	s_2	s_3	Z	
s_1	3	0	$-\frac{1}{2}$	1	$-\frac{3}{2}$	0	0	2
x_2	2	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	1
s_3	9	0	–1	0	2	1	0	7
Z	7	0	–4	0	3	0	1	6

For the last table, x_3 is the entering variable. Since no quotients exist, the problem has an unbounded solution.
Thus, no optimum solution (unbounded).

8.

	x_1	x_2	x_3	s_1	s_2	s_3	Z		
s_1	6	3	–3	1	0	0	0	10	$\frac{5}{3}$
s_2	1	–1	1	0	1	0	0	1	1
s_3	2	–1	2	0	0	1	0	12	6
Z	–2	–1	4	0	0	0	1	0	

	x_1	x_2	x_3	s_1	s_2	s_3	Z		
s_1	0	9	–9	1	–6	0	0	4	$\frac{4}{9}$
x_1	1	–1	1	0	1	0	0	1	
s_3	0	1	0	0	–2	1	0	10	10
Z	0	–3	6	0	2	0	1	2	

	x_1	x_2	x_3	s_1	s_2	s_3	Z		
x_2	0	1	–1	$\frac{1}{9}$	$-\frac{2}{3}$	0	0	$\frac{4}{9}$	
x_1	1	0	0	$\frac{1}{9}$	$\frac{1}{3}$	0	0	$\frac{13}{9}$	$\frac{13}{3}$
s_3	0	0	1	$-\frac{1}{9}$	$-\frac{4}{3}$	1	0	$\frac{86}{9}$	
Z	0	0	3	$\frac{1}{3}$	0	0	1	$\frac{10}{3}$	

Z has a maximum value of $\frac{10}{3}$ when

$x_1 = \frac{13}{9}, x_2 = \frac{4}{9}, x_3 = 0.$ Since s_2 is nonbasic for the last table and its indicator is 0, there may be multiple optimum solutions. Treating s_2 as an entering variable, we have

	x_1	x_2	x_3	s_1	s_2	s_3	Z	
x_2	2	1	–1	$\frac{1}{3}$	0	0	0	$\frac{10}{3}$
s_2	3	0	0	$\frac{1}{3}$	1	0	0	$\frac{13}{3}$
s_3	4	0	1	$\frac{1}{3}$	0	1	0	$\frac{46}{3}$
Z	0	0	3	$\frac{1}{3}$	0	0	1	$\frac{10}{3}$

Here $Z = \frac{10}{3}$ when $x_1 = 0, x_2 = \frac{10}{3}, x_3 = 0.$

Thus multiple optimum solutions exist. Hence Z is maximum when

$$x_1 = (1-t)\left(\frac{13}{9}\right) + 0t = \frac{13}{9} - \frac{13}{9}t,$$

$$x_2 = (1-t)\left(\frac{4}{9}\right) + \left(\frac{10}{3}\right)t = \frac{4}{9} + \frac{26}{9}t,$$

$$x_3 = (1-t)(0) + 0t = 0,$$

and $0 \le t \le 1$. For the last table, x_1 is nonbasic and its indicator is 0. if we continue the process for determining other optimum solutions, we return to the third table.

9. To obtain a standard linear programming problem, we write the second constraint as $4x_1 + x_2 \le 6$.

	x_1	x_2	x_3	s_1	s_2	Z		
s_1	2	1	1	1	0	0	7	$\frac{7}{2}$
s_2	4	1	0	0	1	0	6	$\frac{3}{2}$
Z	–6	–2	–1	0	0	1	0	

	x_1	x_2	x_3	s_1	s_2	Z		
s_1	0	$\frac{1}{2}$	1	1	$-\frac{1}{2}$	0	4	4
x_1	1	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	$\frac{3}{2}$	
Z	0	$-\frac{1}{2}$	–1	0	$\frac{3}{2}$	1	9	

	x_1	x_2	x_3	s_1	s_2	Z		
x_3	0	$\frac{1}{2}$	1	1	$-\frac{1}{2}$	0	4	8
x_1	1	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	$\frac{3}{2}$	6
Z	0	0	0	1	1	1	13	

Z has a maximum value of 13 when

$x_1 = \frac{3}{2}, x_2 = 0, x_3 = 4.$ Since x_2 is nonbasic for the last table and its indicator is 0, there may be multiple optimum solutions. Treating x_2 as an entering variable, we have

	x_1	x_2	x_3	s_1	s_2	Z	
x_3	−2	0	1	1	−1	0	1
x_2	4	1	0	0	1	0	6
Z	0	0	0	1	1	1	13

Here $Z = 13$ when $x_1 = 0, x_2 = 6, x_3 = 1$. Thus multiple optimum solutions exist. Hence Z is maximum when

$x_1 = (1-t)\left(\frac{3}{2}\right) + 0t = \frac{3}{2} - \frac{3}{2}t,$

$x_2 = (1-t)(0) + 6t = 6t,$

$x_3 = (1-t)(4) + (1)t = 4 - 3t,$ and $0 \le t \le 1$. For the last table, x_1 is nonbasic and its indicator is 0. If we continue the process for determining other optimum solutions, we return to the third table.

10.

	x_1	x_2	x_3	x_4	s_1	s_2	s_3	P		
s_1	1	−1	0	0	1	0	0	0	2	
s_2	0	1	−1	0	0	1	0	0	3	3
s_3	0	1	−3	1	0	0	1	0	4	4
P	−1	−2	−1	−2	0	0	0	1	0	

	x_1	x_2	x_3	x_4	s_1	s_2	s_3	P	
s_1	1	0	−1	0	1	1	0	0	5
x_2	0	1	−1	0	0	1	0	0	3
s_3	0	0	−2	1	0	−1	1	0	1
P	−1	0	−3	−2	0	2	0	1	6

Now x_3 is the entering variable but no quotients exist. Thus, the feasible region is unbounded and, hence, there is no optimum solution.

11. Let x_1, x_2, and x_3 denote the numbers of chairs, rockers, and chaise lounges produced, respectively. We want to maximize $R = 24x_1 + 32x_2 + 48x_3$ subject to

$x_1 + x_2 + x_3 \le 400,$

$x_1 + x_2 + 2x_3 \le 600,$

$2x_1 + 3x_2 + 5x_3 \le 1500,$

$x_1, x_2, x_3 \ge 0.$

	x_1	x_2	x_3	s_1	s_2	s_3	R		
s_1	1	1	1	1	0	0	0	400	400
s_2	1	1	2	0	1	0	0	600	300
s_3	2	3	5	0	0	1	0	1500	300
R	−24	−32	−48	0	0	0	1	0	

choosing s_3 as departing variable

	x_1	x_2	x_3	s_1	s_2	s_3	R		
s_1	$\frac{3}{5}$	$\frac{2}{5}$	0	1	0	$-\frac{1}{5}$	0	100	$\frac{500}{3}$
s_2	$\frac{1}{5}$	$-\frac{1}{5}$	0	0	1	$-\frac{2}{5}$	0	0	0
x_3	$\frac{2}{5}$	$\frac{3}{5}$	1	0	0	$\frac{1}{5}$	0	300	750
R	$-\frac{24}{5}$	$-\frac{16}{5}$	0	0	0	$\frac{48}{5}$	1	14,400	

	x_1	x_2	x_3	s_1	s_2	s_3	R		
s_1	0	1	0	1	−3	1	0	100	100
x_1	1	−1	0	0	5	−2	0	0	0
x_3	0	1	1	0	−2	1	0	300	300
R	0	−8	0	0	24	0	1	14,400	

	x_1	x_2	x_3	s_1	s_2	s_3	R		
x_2	0	1	0	1	−3	1	0	100	
x_1	1	0	0	1	2	−1	0	100	50
x_3	0	0	1	−1	1	0	0	200	200
R	0	0	0	8	0	8	1	15,200	

The maximum value of R is 15,200 when $x_1 = 100, x_2 = 100, x_3 = 200$. Since s_2 is nonbasic for the last table and its indicator is 0, there may be multiple optimum solutions. Treating s_2 as an entering variable, we have

	x_1	x_2	x_3	s_1	s_2	s_3	R	
x_2	$\frac{3}{2}$	1	0	$\frac{5}{2}$	0	$-\frac{1}{2}$	0	250
s_2	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	50
x_3	$-\frac{1}{2}$	0	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	0	150
R	0	0	0	8	0	8	1	15,200

Here $R = 15{,}200$ when
$x_1 = 0, x_2 = 250, x_3 = 150.$
Thus multiple optimum solutions exist.
Hence R is maximum when

$x_1 = (1-t)(100) + 0t = 100 - 100t,$

$x_2 = (1-t)(100) + 250t = 100 + 150t$

$x_3 = (1-t)(200) + 150t = 200 - 50t,$ and $0 \le t \le 1$. For the last table, x_1 is nonbasic and its indicator is 0. If we continue the process for determining other optimum solutions, we return to the fourth table. If we were to initially choose s_2 as the departing variable, then

	x_1	x_2	x_3	s_1	s_2	s_3	R		
s_1	1	1	1	1	0	0	0	400	400
s_2	1	1	2	0	1	0	0	600	300
s_3	2	3	5	0	0	1	0	1500	300
R	–24	–32	–48	0	0	0	1	0	

	x_1	x_2	x_3	s_1	s_2	s_3	R		
s_1	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	0	0	100	200
x_3	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	300	600
s_3	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{5}{2}$	1	0	0	0
R	0	–8	0	0	24	0	1	14,400	

	x_1	x_2	x_3	s_1	s_2	s_3	R		
s_1	1	0	0	1	2	–1	0	100	50
x_3	1	0	1	0	3	–1	0	300	100
x_2	–1	1	0	0	–5	2	0	0	
R	–8	0	0	0	–16	16	1	14,400	

	x_1	x_2	x_3	s_1	s_2	s_3	R		
s_2	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	50	100
x_3	$-\frac{1}{2}$	0	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	0	150	
x_2	$\frac{3}{2}$	1	0	$\frac{5}{2}$	0	$-\frac{1}{2}$	0	250	$\frac{500}{3}$
R	0	0	0	8	0	8	1	15,200	

the maximum value of R is 15,200 when $x_1 = 0, x_2 = 250, x_3 = 150.$ For the last table, x_1 is nonbasic and its indicator is 0. Treating x_1 as an entering variable, we have

	x_1	x_2	x_3	s_1	s_2	s_3	R	
x_1	1	0	0	1	2	–1	0	100
x_3	0	0	1	–1	1	0	0	200
x_2	0	1	0	1	–3	1	0	100
R	0	0	0	8	0	8	1	15,200

Here R = 15,200 when $x_1 = 100, x_2 = 100, x_3 = 200.$ For the last table, s_2 is nonbasic and its indicator is 0. If we continue the process of determining other optimum solutions, we return to the table corresponding to the solution $x_1 = 0, x_2 = 250, x_3 = 150.$

Thus, the maximum revenue is \$15,200 when $x_1 = 100 - 100t,\ \ x_2 = 100 + 150t,$ $x_3 = 200 - 50t,$ and $0 \le t \le 1$

Principles in Practice 7.6

1. Using the hint, $1000 - x_1$ standard and $800 - x_2$ deluxe snowboards must be manufactured at plant II. The constraints for plant I are $x_1 + x_2 \le 1200$ and $x_2 - x_1 \le 200.$ The constraints for plant II are $(1000 - x_1) + (800 - x_2) \le 1000$ or $x_1 + x_2 \ge 800.$ The quantity to be maximized is the profit
$P = 40x_1 + 60x_2 + 45(1000 - x_1) + 50(800 - x_2)$
$= -5x_1 + 10x_2 + 85{,}000$ subject to the constraints

$x_1 + x_2 \le 1200,$
$-x_1 + x_2 \le 200,$
$x_1 + x_2 \ge 800,$
and $x_1, x_2 \ge 0$.

Note that maximizing $Z = -5x_1 + 10x_2$ also maximizes the profit. The corresponding equations are:

$$\begin{aligned} x_1 + x_2 + s_1 &= 1200, \\ -x_1 + x_2 + s_2 &= 200, \\ x_1 + x_2 - s_3 + t &= 800. \end{aligned}$$

The artificial objective equation is
$W = -5x_1 + 10x_2 - Mt.$

The augmented coefficient matrix is:

$$\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\left[\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & t & W & \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1200 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 200 \\ 1 & 1 & 0 & 0 & -1 & 1 & 0 & 800 \\ \hline 5 & -10 & 0 & 0 & 0 & M & 1 & 0 \end{array}\right]$$

The simplex tables follow.

$$\begin{array}{c|ccccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & t & W & \\ \hline s_1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1200 \\ s_2 & -1 & \underline{1} & 0 & 1 & 0 & 0 & 0 & 200 \\ t & 1 & 1 & 0 & 0 & -1 & 1 & 0 & 800 \\ \hline W & 5-M & -10-M & 0 & 0 & M & 0 & 1 & -800M \end{array}$$

$$\begin{array}{c|ccccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & t & W & \\ \hline s_1 & 2 & 0 & 1 & -1 & 0 & 0 & 0 & 1000 \\ x_2 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 200 \\ t & \underline{2} & 0 & 0 & -1 & -1 & 1 & 0 & 600 \\ \hline W & -5-2M & 0 & 0 & 10+M & M & 0 & 1 & 2000-600M \end{array}$$

$$\begin{array}{c|ccccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & t & W & \\ \hline s_1 & 0 & 0 & 1 & 0 & \underline{1} & -1 & 0 & 400 \\ x_2 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 500 \\ x_1 & 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 300 \\ \hline W & 0 & 0 & 0 & \frac{15}{2} & -\frac{5}{2} & \frac{5}{2}+M & 1 & 3500 \end{array}$$

Delete the *t*-column since *t* = 0 and return to *Z*.

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & Z & \\ \hline s_3 & 0 & 0 & 1 & 0 & 1 & 0 & 400 \\ x_2 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 700 \\ x_1 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 500 \\ \hline Z & 0 & 0 & \frac{5}{2} & \frac{15}{2} & 0 & 0 & 4500 \end{array}$$

Thus, $x_1 = 500, x_2 = 700,$ and $Z = 4500.$ Plant I should manufacture 500 standard and 700 deluxe snowboards. Plant II should manufacture 1000 – 500 = 500 standard and 800 – 700 = 100 deluxe snowboards. The maximum profit is *P* = –5(500) + 10(700) + 85,000 = $89,500.

Problems 7.6

1.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & t_2 & W \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 6 \\ -1 & 1 & 0 & -1 & 1 & 0 & 4 \\ \hline -2 & -1 & 0 & 0 & M & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c|cccccc|c|c} & x_1 & x_2 & s_1 & s_2 & t_2 & W & & \\ \hline s_1 & 1 & 1 & 1 & 0 & 0 & 0 & 6 & 6 \\ t_2 & -1 & \underline{1} & 0 & -1 & 1 & 0 & 4 & 4 \\ \hline W & -2+M & -1-M & 0 & M & 0 & 1 & -4M & \end{array}$$

$$\begin{array}{c|cccccc|c|c} & x_1 & x_2 & s_1 & s_2 & t_2 & W & & \\ \hline s_1 & \underline{2} & 0 & 1 & 1 & -1 & 0 & 2 & 1 \\ x_2 & -1 & 1 & 0 & -1 & 1 & 0 & 4 & \\ \hline W & -3 & 0 & 0 & -1 & M+1 & 1 & 4 & \end{array}$$

$$\begin{array}{c|ccccc|c} & x_1 & x_2 & s_1 & s_2 & Z & \\ \hline x_1 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 \\ x_2 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 5 \\ \hline Z & 0 & 0 & \frac{3}{2} & \frac{1}{2} & 1 & 7 \end{array}$$

The maximum is $Z = 7$ when $x_1 = 1, x_2 = 5$.

2.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & t_2 & W \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 2 & 1 & 0 & 0 & 0 & 8 \\ 1 & 6 & 0 & -1 & 1 & 0 & 12 \\ \hline -3 & -4 & 0 & 0 & M & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c|cccccc|c|c} & x_1 & x_2 & s_1 & s_2 & t_2 & W & & \\ \hline s_1 & 1 & 2 & 1 & 0 & 0 & 0 & 8 & 4 \\ t_2 & 1 & \underline{6} & 0 & -1 & 1 & 0 & 12 & 2 \\ \hline W & -3-M & -4-6M & 0 & M & 0 & 1 & -12M & \end{array}$$

$$\begin{array}{c|cccccc|c|c} & x_1 & x_2 & s_1 & s_2 & t_2 & W & & \\ \hline s_1 & \underline{\frac{2}{3}} & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 0 & 4 & 6 \\ x_2 & \frac{1}{6} & 1 & 0 & -\frac{1}{6} & \frac{1}{6} & 0 & 2 & 12 \\ \hline W & -\frac{7}{3} & 0 & 0 & -\frac{2}{3} & \frac{2}{3}+M & 1 & 8 & \end{array}$$

$$\begin{array}{c|ccccc|c} & x_1 & x_2 & s_1 & s_2 & Z & \\ \hline x_1 & 1 & 0 & \frac{3}{2} & \frac{1}{2} & 0 & 6 \\ x_2 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 1 \\ \hline Z & 0 & 0 & \frac{7}{2} & \frac{1}{2} & 1 & 22 \end{array}$$

The maximum is $Z = 22$ when $x_1 = 6, x_2 = 1$.

3.

$$\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & t_2 & W & \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 2 & 1 & 1 & 0 & 0 & 0 & 5 \\ -1 & 1 & 1 & 0 & -1 & 1 & 0 & 1 \\ \hline -2 & -1 & 1 & 0 & 0 & M & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \\ s_1 \\ t_2 \\ W \end{array}\begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & t_2 & W & \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 2 & 1 & 1 & 0 & 0 & 0 & 5 \\ -1 & \underline{1} & 1 & 0 & -1 & 1 & 0 & 1 \\ \hline -2+M & -1-M & 1-M & 0 & M & 0 & 1 & -M \end{array}\right] \end{array}\begin{array}{c} \\ \frac{5}{2} \\ 1 \\ \\ \end{array}$$

$$\begin{array}{c} \\ s_1 \\ x_2 \\ W \end{array}\begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & t_2 & W & \end{array} \\ \left[\begin{array}{ccccccc|c} \underline{3} & 0 & -1 & 1 & 2 & -2 & 0 & 3 \\ -1 & 1 & 1 & 0 & -1 & 1 & 0 & 1 \\ \hline -3 & 0 & 2 & 0 & -1 & 1+M & 1 & 1 \end{array}\right] \end{array}\begin{array}{c} \\ 1 \\ \\ \\ \end{array}$$

$$\begin{array}{c} \\ x_1 \\ x_2 \\ Z \end{array}\begin{array}{c} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & Z & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 0 & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 & 1 \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 2 \\ \hline 0 & 0 & 1 & 1 & 1 & 1 & 4 \end{array}\right] \end{array}$$

The maximum is $Z = 4$ when $x_1 = 1, x_2 = 2, x_3 = 0$.

4.

$$\begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & t_2 & W & \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 9 \\ 1 & -2 & 1 & 0 & -1 & 1 & 0 & 6 \\ \hline -1 & 1 & -4 & 0 & 0 & M & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \\ s_1 \\ t_2 \\ W \end{array}\begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & t_2 & W & \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 9 \\ 1 & -2 & \underline{1} & 0 & -1 & 1 & 0 & 6 \\ \hline -1-M & 1+2M & -4-M & 0 & M & 0 & 1 & -6M \end{array}\right] \end{array}\begin{array}{c} \\ 9 \\ 6 \\ \\ \end{array}$$

$$\begin{array}{c} \\ s_1 \\ x_3 \\ W \end{array}\begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & t_2 & W & \end{array} \\ \left[\begin{array}{ccccccc|c} 0 & \underline{3} & 0 & 1 & 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 0 & -1 & 1 & 0 & 6 \\ \hline 3 & -7 & 0 & 0 & -4 & 4+M & 1 & 24 \end{array}\right] \end{array}\begin{array}{c} \\ 1 \\ \\ \\ \end{array}$$

$$\begin{array}{c} \\ x_2 \\ x_3 \\ Z \end{array}\begin{array}{c} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & Z & \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 1 & 0 & \frac{1}{3} & \underline{\frac{1}{3}} & 0 & 1 \\ 1 & 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 0 & 8 \\ \hline 3 & 0 & 0 & \frac{7}{3} & -\frac{5}{3} & 1 & 31 \end{array}\right] \end{array}\begin{array}{c} \\ 3 \\ \\ \\ \end{array}$$

	x_1	x_2	x_3	s_1	s_2	Z	
s_2	0	3	0	1	1	0	3
x_3	1	1	1	1	0	0	9
Z	3	5	0	4	0	1	36

The maximum is $Z = 36$ when $x_1 = 0, x_2 = 0, x_3 = 9$.

5.

x_1	x_2	x_3	s_1	t_2	W	
1	1	1	1	0	0	10
1	–1	–1	0	1	0	6
–3	–2	–1	0	M	1	0

	x_1	x_2	x_3	s_1	t_2	W		
s_1	1	1	1	1	0	0	10	10
t_2	1	–1	–1	0	1	0	6	6
W	$-3-M$	$-2+M$	$-1+M$	0	0	1	$-6M$	

	x_1	x_2	x_3	s_1	t_2	W		
s_1	0	2	2	1	–1	0	4	2
x_1	1	–1	–1	0	1	0	6	
W	0	–5	–4	0	$3+M$	1	18	

	x_1	x_2	x_3	s_1	W	
x_2	0	1	1	$\frac{1}{2}$	0	2
x_1	1	0	0	$\frac{1}{2}$	0	8
Z	0	0	1	$\frac{5}{2}$	1	28

The maximum is $Z = 28$ when $x_1 = 8$, $x_2 = 2$, and $x_3 = 0$.

6.

x_1	x_2	x_3	s_1	t_1	t_2	W	
0	1	–2	–1	1	0	0	5
1	1	1	0	0	1	0	7
–2	–1	–3	0	M	M	1	0

	x_1	x_2	x_3	s_1	t_1	t_2	W		
t_1	0	1	–2	–1	1	0	0	5	5
t_2	1	1	1	0	0	1	0	7	7
W	$-2-M$	$-1-2M$	$-3+M$	M	0	0	1	$-12M$	

	x_1	x_2	x_3	s_1	t_1	t_2	W		
x_2	0	1	–2	–1	1	0	0	5	
t_2	1	0	3	1	–1	1	0	2	$\frac{2}{3}$
W	$-2-M$	0	$-5-3M$	$-1-M$	$1+2M$	0	1	$5-2M$	

$$
\begin{array}{c}
\\ x_2 \\ x_3 \\ W
\end{array}
\begin{array}{c}
\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & t_1 & t_2 & W & \end{array} \\
\left[\begin{array}{ccccccc|c}
\frac{2}{3} & 1 & 0 & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 & \frac{19}{3} \\
\underline{\frac{1}{3}} & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 2 \\
\hdashline
-\frac{1}{3} & 0 & 0 & \frac{2}{3} & -\frac{2}{3}+M & \frac{5}{3}+M & 1 & \frac{25}{3}
\end{array}\right]
\end{array}
\begin{array}{c} \\ \frac{19}{2} \\ 6 \\ \\ \end{array}
$$

$$
\begin{array}{c}
\\ x_2 \\ x_1 \\ W
\end{array}
\begin{array}{c}
\begin{array}{ccccc|c} x_1 & x_2 & x_3 & s_1 & Z & \end{array} \\
\left[\begin{array}{ccccc|c}
0 & 1 & -2 & -1 & 0 & 5 \\
1 & 0 & 3 & 1 & 0 & 2 \\
\hdashline
0 & 0 & 1 & 1 & 1 & 9
\end{array}\right]
\end{array}
$$

The maximum is $Z = 9$ when $x_1 = 2,\ x_2 = 5,\ x_3 = 0.$

7.

$$
\begin{array}{c}
\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & t_3 & W & \end{array} \\
\left[\begin{array}{ccccccc|c}
1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 2 & 0 & 1 & 0 & 0 & 0 & 8 \\
1 & 1 & 0 & 0 & -1 & 1 & 0 & 5 \\
\hdashline
-1 & 10 & 0 & 0 & 0 & M & 1 & 0
\end{array}\right]
\end{array}
$$

$$
\begin{array}{c}
\\ s_1 \\ s_2 \\ t_3 \\ W
\end{array}
\begin{array}{c}
\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & t_3 & W & \end{array} \\
\left[\begin{array}{ccccccc|c}
\underline{1} & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 2 & 0 & 1 & 0 & 0 & 0 & 8 \\
1 & 1 & 0 & 0 & -1 & 1 & 0 & 5 \\
\hdashline
-1-M & 10-M & 0 & 0 & M & 0 & 1 & -5M
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 8 \\ 5 \\ \\ \end{array}
$$

$$
\begin{array}{c}
\\ x_1 \\ s_2 \\ t_3 \\ W
\end{array}
\begin{array}{c}
\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & t_3 & W & \end{array} \\
\left[\begin{array}{ccccccc|c}
1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 3 & -1 & 1 & 0 & 0 & 0 & 7 \\
0 & \underline{2} & -1 & 0 & -1 & 1 & 0 & 4 \\
\hdashline
0 & 9-2M & 1+M & 0 & M & 0 & 1 & 1-4M
\end{array}\right]
\end{array}
\begin{array}{c} \\ \\ \frac{7}{3} \\ 2 \\ \\ \end{array}
$$

$$
\begin{array}{c}
\\ x_1 \\ s_2 \\ x_2 \\ W
\end{array}
\begin{array}{c}
\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & t_3 & W & \end{array} \\
\left[\begin{array}{ccccccc|c}
1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 3 \\
0 & 0 & \frac{1}{2} & 1 & \frac{3}{2} & -\frac{3}{2} & 0 & 1 \\
0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 2 \\
\hdashline
0 & 0 & \frac{11}{2} & 0 & \frac{9}{2} & -\frac{9}{2}+M & 1 & -17
\end{array}\right]
\end{array}
$$

For the above table, $t_3 = 0.$ Thus $W = Z.$

The maximum is $Z = -17$ when $x_1 = 3, x_2 = 2$.

8.

$$\begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & t_1 & t_3 & W & \\ \hline 1 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & 5 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 3 \\ 1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 7 \\ \hline -1 & -4 & 1 & 0 & 0 & M & M & 1 & 0 \end{array}$$

$$\begin{array}{c|cccccccc|c|c} & x_1 & x_2 & x_3 & s_1 & s_2 & t_1 & t_3 & W & & \\ \hline t_1 & 1 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & 5 & 5 \\ s_2 & \underline{1} & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 3 & 3 \\ t_3 & 1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 7 & 7 \\ \hline W & -1-2M & -4 & 1 & M & 0 & 0 & 0 & 1 & -12M & \end{array}$$

$$\begin{array}{c|cccccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & t_1 & t_3 & W & \\ \hline t_1 & 0 & 0 & -2 & -1 & -1 & 1 & 0 & 0 & 2 \\ x_1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 3 \\ t_3 & 0 & -2 & 0 & 0 & -1 & 0 & 1 & 0 & 4 \\ \hline W & 0 & -3+2M & 2+2M & M & 1+2M & 0 & 0 & 1 & 3-6M \end{array}$$

There is no solution (empty feasible region).

9. We write the third constraint as $-x_1 + x_2 + x_3 \geq 6.$

$$\begin{array}{ccccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & t_2 & t_3 & W & \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 2 \\ -1 & 1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 6 \\ \hline -3 & 2 & -1 & 0 & 0 & 0 & M & M & 1 & 0 \end{array}$$

$$\begin{array}{c|ccccccccc|c|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & t_2 & t_3 & W & & \\ \hline s_1 & 1 & 1 & \underline{1} & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ t_2 & 1 & -1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 2 & 2 \\ t_3 & -1 & 1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 6 & 6 \\ \hline W & -3 & 2 & -1-2M & 0 & M & M & 0 & 0 & 1 & -8M & \end{array}$$

$$\begin{array}{c|ccccccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & t_2 & t_3 & W & \\ \hline x_3 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ t_2 & 0 & -2 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 1 \\ t_3 & -2 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 5 \\ \hline W & -2+2M & 3+2M & 0 & 1+2M & M & M & 0 & 0 & 1 & 1-6M \end{array}$$

There is no solution (empty feasible region).

10.

$$\begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & t_2 & t_3 & W & \\ \hline 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 8 \\ 1 & 6 & 0 & -1 & 0 & 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 2 \\ \hdashline -1 & -4 & 0 & 0 & 0 & M & M & 1 & 0 \end{array}$$

$$\begin{array}{c|cccccccc|c|c} & x_1 & x_2 & s_1 & s_2 & s_3 & t_2 & t_3 & W & & \\ \hline s_1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 8 & 4 \\ t_2 & 1 & 6 & 0 & -1 & 0 & 1 & 0 & 0 & 12 & 2 \\ t_3 & 0 & \underline{1} & 0 & 0 & -1 & 0 & 1 & 0 & 2 & 2 \\ \hdashline W & -1-M & -4-7M & 0 & M & M & 0 & 0 & 1 & -14M & \end{array}$$

Here we choose t_3 as the departing variable.

$$\begin{array}{c|cccccccc|c|c} & x_1 & x_2 & s_1 & s_2 & s_3 & t_2 & t_3 & W & & \\ \hline s_1 & 1 & 0 & 1 & 0 & 2 & 0 & -2 & 0 & 4 & 2 \\ t_2 & 1 & 0 & 0 & -1 & \underline{6} & 1 & -6 & 0 & 0 & 0 \\ x_2 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 2 & \\ \hdashline W & -1-M & 0 & 0 & M & -4-6M & 0 & 4+7M & 1 & 8 & \end{array}$$

$$\begin{array}{c|cccccccc|c|c} & x_1 & x_2 & s_1 & s_2 & s_3 & t_2 & t_3 & W & & \\ \hline s_1 & \frac{2}{3} & 0 & 1 & \underline{\frac{1}{3}} & 0 & -\frac{1}{3} & 0 & 0 & 4 & 12 \\ s_3 & \frac{1}{6} & 0 & 0 & -\frac{1}{6} & 1 & \frac{1}{6} & -1 & 0 & 0 & \\ x_2 & \frac{1}{6} & 1 & 0 & -\frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 2 & \\ \hdashline W & -\frac{1}{3} & 0 & 0 & -\frac{2}{3} & 0 & \frac{2}{3}+M & M & 1 & 8 & \end{array}$$

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & Z & \\ \hline s_2 & 2 & 0 & 3 & 1 & 0 & 0 & 12 \\ s_3 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 1 & 0 & 2 \\ x_2 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 4 \\ \hdashline Z & 1 & 0 & 2 & 0 & 0 & 1 & 16 \end{array}$$

Thus the maximum value of Z is 16, when $x_1 = 0, x_2 = 4$.

If we choose t_2 as the original departing variable, then

$$\begin{array}{c|cccccccc|c|c} & x_1 & x_2 & s_1 & s_2 & s_3 & t_2 & t_3 & W & & \\ \hline s_1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 8 & 4 \\ t_2 & 1 & \underline{6} & 0 & -1 & 0 & 1 & 0 & 0 & 12 & 2 \\ t_3 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 2 & 2 \\ \hdashline W & -1-M & -4-7M & 0 & M & M & 0 & 0 & 1 & -14M & \end{array}$$

	x_1	x_2	s_1	s_2	s_3	t_2	t_3	W		
s_1	$\frac{2}{3}$	0	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	0	4	12
x_2	$\frac{1}{6}$	1	0	$-\frac{1}{6}$	0	$\frac{1}{6}$	0	0	2	
t_3	$-\frac{1}{6}$	0	0	$\underline{\frac{1}{6}}$	–1	$-\frac{1}{6}$	1	0	0	0
W	$-\frac{1}{3}+\frac{1}{6}M$	0	0	$-\frac{2}{3}-\frac{1}{6}M$	M	$\frac{2}{3}+\frac{7}{6}M$	0	1	8	

	x_1	x_2	s_1	s_2	s_3	t_2	t_3	W		
s_1	1	0	1	0	$\underline{2}$	0	–2	0	4	2
x_2	0	1	0	0	–1	0	1	0	2	
s_2	–1	0	0	1	–6	–1	6	0	0	
W	–1	0	0	0	–4	M	$4+M$	1	8	

	x_1	x_2	s_1	s_2	s_3	Z	
s_3	$\frac{1}{2}$	0	$\frac{1}{2}$	0	1	0	2
x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	0	4
s_2	2	0	3	1	0	0	12
Z	1	0	2	0	0	1	16

The maximum is $Z = 16$ when $x_1 = 0, x_2 = 4$.

11.

x_1	x_2	s_1	s_3	t_2	t_3	W	
1	–1	1	0	0	0	0	4
–1	1	0	0	1	0	0	4
1	0	0	–1	0	1	0	6
3	–2	0	0	M	M	1	0

	x_1	x_2	s_1	s_3	t_2	t_3	W		
s_1	1	–1	1	0	0	0	0	4	
t_2	–1	$\underline{1}$	0	0	1	0	0	4	4
t_3	1	0	0	–1	0	1	0	6	
W	3	$-2-M$	0	M	0	0	1	$-10M$	

	x_1	x_2	s_1	s_3	t_2	t_3	W		
s_1	0	0	1	0	1	0	0	8	
x_2	–1	1	0	0	1	0	0	4	
t_3	$\underline{1}$	0	0	–1	0	1	0	6	6
W	$1-M$	0	0	M	$2+M$	0	1	$8-6M$	

	x_1	x_2	s_1	s_3	t_2	t_3	W	
s_1	0	0	1	0	1	0	0	8
x_2	0	1	0	–1	1	1	0	10
x_1	1	0	0	–1	0	1	0	6
W	0	0	0	1	$2+M$	$-1+M$	1	2

For the above table, $t_2 = t_3 = 0$. Thus $W = Z$.

The maximum is $Z = 2$ when $x_1 = 6, x_2 = 10$.

12. We write the first constraint as $-x_1 + 2x_2 \le 12$.

$$\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & t_2 & t_3 & W & \\ \hline -1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 12 \\ -1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 10 \\ \hline -2 & 8 & 0 & 0 & 0 & M & M & 1 & 0 \end{array}$$

$$\begin{array}{c|cccccccc|c|c} & x_1 & x_2 & s_1 & s_2 & s_3 & t_2 & t_3 & W & & \\ \hline s_1 & -1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 12 & 6 \\ t_2 & -1 & \underline{1} & 0 & -1 & 0 & 1 & 0 & 0 & 2 & 2 \\ t_3 & 1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 10 & 10 \\ \hline W & -2 & 8-2M & 0 & M & M & 0 & 0 & 1 & -12M & \end{array}$$

$$\begin{array}{c|cccccccc|c|c} & x_1 & x_2 & s_1 & s_2 & s_3 & t_2 & t_3 & W & & \\ \hline s_1 & 1 & 0 & 1 & 2 & 0 & -2 & 0 & 0 & 8 & 8 \\ x_2 & -1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 2 & \\ t_3 & \underline{2} & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 8 & 4 \\ \hline W & 6-2M & 0 & 0 & 8-M & M & -8+2M & 0 & 1 & -16-8M & \end{array}$$

$$\begin{array}{c|cccccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & t_2 & t_3 & W & \\ \hline s_1 & 0 & 0 & 1 & \frac{3}{2} & \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 4 \\ x_2 & 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 6 \\ x_1 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 4 \\ \hline W & 0 & 0 & 0 & 5 & 3 & -5+M & -3+M & 1 & -40 \end{array}$$

For the above table, $t_2 = t_3 = 0$. Thus $W = Z$. The maximum is $Z = -40$ when $x_1 = 4$ and $x_2 = 6$.

13. Let x_1 and x_2 denote the numbers of Standard and Executive bookcases produced, respectively, each week. We want to maximize the profit function $P = 35x_1 + 40x_2$ subject to

$2x_1 + 3x_2 \le 400,$
$3x_1 + 4x_2 \le 500,$
$3x_1 + 4x_2 \ge 250,$
$x_1, x_2 \ge 0.$

The artificial objective function is $W = P - Mt_3$.

$$\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & t_3 & W & \\ \hline 2 & 3 & 1 & 0 & 0 & 0 & 0 & 400 \\ 3 & 4 & 0 & 1 & 0 & 0 & 0 & 500 \\ 3 & 4 & 0 & 0 & -1 & 1 & 0 & 250 \\ \hline -35 & -40 & 0 & 0 & 0 & M & 1 & 0 \end{array}$$

$$\begin{array}{c|ccccccc|c|c} & x_1 & x_2 & s_1 & s_2 & s_3 & t_3 & W & & \\ \hline s_1 & 2 & 3 & 1 & 0 & 0 & 0 & 0 & 400 & \frac{400}{3} \\ s_2 & 3 & 4 & 0 & 1 & 0 & 0 & 0 & 500 & 125 \\ t_3 & 3 & \underline{4} & 0 & 0 & -1 & 1 & 0 & 250 & \frac{125}{2} \\ \hline W & -35-3M & -40-4M & 0 & 0 & M & 0 & 1 & -250M & \end{array}$$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ x_2 \\ W \end{array} \begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & t_3 & W & \end{array} \\ \left[\begin{array}{ccccccc|c} -\frac{1}{4} & 0 & 1 & 0 & \frac{3}{4} & -\frac{3}{4} & 0 & \frac{425}{2} \\ 0 & 0 & 0 & 1 & \underline{1} & -1 & 0 & 250 \\ \frac{3}{4} & 1 & 0 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & \frac{125}{2} \\ \hline -5 & 0 & 0 & 0 & -10 & 10+M & 1 & 2500 \end{array}\right] \end{array} \begin{array}{c} \frac{850}{3} \\ 250 \\ \\ \\ \end{array}$$

$$\begin{array}{c} \\ s_1 \\ s_3 \\ x_2 \\ P \end{array} \begin{array}{c} \begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & P & \end{array} \\ \left[\begin{array}{cccccc|c} -\frac{1}{4} & 0 & 1 & -\frac{3}{4} & 0 & 0 & 25 \\ 0 & 0 & 0 & 1 & 1 & 0 & 250 \\ \underline{\frac{3}{4}} & 1 & 0 & \frac{1}{4} & 0 & 0 & 125 \\ \hline -5 & 0 & 0 & 10 & 0 & 1 & 5000 \end{array}\right] \end{array} \begin{array}{c} \\ \\ \frac{500}{3} \\ \\ \end{array}$$

$$\begin{array}{c} \\ s_1 \\ s_3 \\ x_1 \\ P \end{array} \begin{array}{c} \begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & P & \end{array} \\ \left[\begin{array}{cccccc|c} 0 & \frac{1}{3} & 1 & -\frac{2}{3} & 0 & 0 & \frac{200}{3} \\ 0 & 0 & 0 & 1 & 1 & 0 & 250 \\ 1 & \frac{4}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{500}{3} \\ \hline 0 & \frac{20}{3} & 0 & \frac{35}{3} & 0 & 1 & \frac{17,500}{3} \end{array}\right] \end{array}$$

This table indicates that, to maximize profit, the company should produce $\frac{500}{3} = 166\frac{2}{3}$ Standard and 0 Executive bookcases. Since an integer answer is preferable, note that $x_1 = 167,\ x_2 = 0$ does not satisfy the constraint $3x_1 + 4x_2 \le 500,$ while $x_1 = 166,\ x_2 = 0$ satisfies all of the constraints. Thus the company should produce 166 Standard and 0 Executive bookcases each week.

14. Let *x*, *y* and *z* denote the numbers of units of products X, Y, and Z produced each week, respectively. We want to maximize the profit function $P = 50x + 60y + 75z$ subject to

$$\begin{aligned} x + 2y + 2z &\le 40, \\ x + y + 2z &\le 30, \\ z &\ge 5, \\ x, y, z &\ge 0. \end{aligned}$$

The artificial objective function is $W = P - Mt_3.$

$$\begin{array}{c} \begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & t_3 & W & \end{array} \\ \left[\begin{array}{cccccccc|c} 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 40 \\ 1 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 30 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 5 \\ \hline -50 & -60 & -75 & 0 & 0 & 0 & M & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ t_3 \\ W \end{array} \begin{array}{c} \begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & t_3 & W & \end{array} \\ \left[\begin{array}{cccccccc|c} 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 40 \\ 1 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 30 \\ 0 & 0 & \underline{1} & 0 & 0 & -1 & 1 & 0 & 5 \\ \hline -50 & -60 & -75-M & 0 & 0 & M & 0 & 1 & -5M \end{array}\right] \end{array} \begin{array}{c} 20 \\ 15 \\ 5 \\ \\ \end{array}$$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ z \\ W \end{array}\begin{array}{c}\begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & t_3 & W & \end{array}\\ \left[\begin{array}{cccccccc|c} 1 & 2 & 0 & 1 & 0 & 2 & -2 & 0 & 30 \\ 1 & 1 & 0 & 0 & 1 & \underline{2} & -2 & 0 & 20 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 5 \\ \hline -50 & -60 & 0 & 0 & 0 & -75 & 75+M & 1 & 375 \end{array}\right]\end{array}\begin{array}{c} \\ 15 \\ 10 \\ \\ \\ \end{array}$$

$$\begin{array}{c} \\ s_1 \\ s_3 \\ z \\ P \end{array}\begin{array}{c}\begin{array}{ccccccc|c} x & y & z & s_1 & s_2 & s_3 & P & \end{array}\\ \left[\begin{array}{ccccccc|c} 0 & \underline{1} & 0 & 1 & -1 & 0 & 0 & 10 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 1 & 0 & 10 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & 0 & 0 & 15 \\ \hline -\frac{25}{2} & -\frac{45}{2} & 0 & 0 & \frac{75}{2} & 0 & 1 & 1125 \end{array}\right]\end{array}\begin{array}{c} \\ 10 \\ 20 \\ 30 \\ \\ \end{array}$$

$$\begin{array}{c} \\ y \\ s_3 \\ z \\ P \end{array}\begin{array}{c}\begin{array}{ccccccc|c} x & y & z & s_1 & s_2 & s_3 & P & \end{array}\\ \left[\begin{array}{ccccccc|c} 0 & 1 & 0 & 1 & -1 & 0 & 0 & 10 \\ \underline{\frac{1}{2}} & 0 & 0 & -\frac{1}{2} & 1 & 1 & 0 & 5 \\ \frac{1}{2} & 0 & 1 & -\frac{1}{2} & 1 & 0 & 0 & 10 \\ \hline -\frac{25}{2} & 0 & 0 & \frac{45}{2} & 15 & 0 & 1 & 1350 \end{array}\right]\end{array}\begin{array}{c} \\ \\ 10 \\ 20 \\ \\ \end{array}$$

$$\begin{array}{c} \\ y \\ x \\ z \\ P \end{array}\begin{array}{c}\begin{array}{ccccccc|c} x & y & z & s_1 & s_2 & s_3 & P & \end{array}\\ \left[\begin{array}{ccccccc|c} 0 & 1 & 0 & 1 & -1 & 0 & 0 & 10 \\ 1 & 0 & 0 & -1 & 2 & 2 & 0 & 10 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 5 \\ \hline 0 & 0 & 0 & 10 & 40 & 25 & 1 & 1475 \end{array}\right]\end{array}$$

The production order should be 10 units of X, 10 units of Y, and 5 units of Z for a maximum profit of \$1475

15. Suppose I is the total investment. Let x_1, x_2, and x_3 be the proportions invested in A, AA, and AAA bonds, respectively. If Z is the total annual yield expressed as a proportion of I, then $ZI = 0.08x_1I + 0.07x_2I + 0.06x_3I$, or equivalently, $Z = 0.08x_1 + 0.07x_2 + 0.06x_3$. We want to maximize Z subject to

$x_1 + x_2 + x_3 = 1,$

$x_2 + x_3 \ge 0.50,$

$x_1 + x_2 \le 0.30,$

$x_1, x_2, x_3 \ge 0.$

The artificial objective function is $W = Z - Mt_1 - Mt_2$.

$$\begin{array}{c}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & s_2 & s_3 & t_1 & t_2 & W & \end{array}\\ \left[\begin{array}{cccccccc|c} 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0.5 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0.3 \\ \hline -0.08 & -0.07 & -0.06 & 0 & 0 & M & M & 1 & 0 \end{array}\right]\end{array}$$

$$\begin{array}{c} \\ t_1 \\ t_2 \\ s_3 \\ W \end{array}\begin{array}{c}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & s_2 & s_3 & t_1 & t_2 & W & \end{array}\\ \left[\begin{array}{cccccccc|c} 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0.5 \\ 1 & \underline{1} & 0 & 0 & 1 & 0 & 0 & 0 & 0.3 \\ \hline -0.08-M & -0.07-2M & -0.06-2M & M & 0 & 0 & 0 & 1 & -1.5M \end{array}\right]\end{array}\begin{array}{c} \\ 1 \\ 0.5 \\ 0.3 \\ \\ \end{array}$$

$$\begin{array}{c|cccccccc|c|c}
 & x_1 & x_2 & x_3 & s_2 & s_3 & t_1 & t_2 & W & & \\
\hline
t_1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0.7 & 0.7 \\
t_2 & -1 & 0 & \underline{1} & -1 & -1 & 0 & 1 & 0 & 0.2 & 0.2 \\
x_2 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0.3 & \\
\hline
W & -0.01+M & 0 & -0.06-2M & M & 0.07+2M & 0 & 0 & 1 & 0.021-0.9M &
\end{array}$$

$$\begin{array}{c|cccccccc|c|c}
 & x_1 & x_2 & x_3 & s_2 & s_3 & t_1 & t_2 & W & & \\
\hline
t_1 & 1 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0.5 & 0.5 \\
x_3 & -1 & 0 & 1 & -1 & -1 & 0 & 1 & 0 & 0.2 & \\
x_2 & \underline{1} & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0.3 & 0.3 \\
\hline
W & -0.07-M & 0 & 0 & -0.06-M & 0.01 & 0 & 0.06+2M & 1 & 0.033-0.5M &
\end{array}$$

$$\begin{array}{c|cccccccc|c|c}
 & x_1 & x_2 & x_3 & s_2 & s_3 & t_1 & t_2 & W & & \\
\hline
t_1 & 0 & -1 & 0 & \underline{1} & -1 & 1 & -1 & 0 & 0.2 & 0.2 \\
x_3 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0.5 & \\
x_1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0.3 & \\
\hline
W & 0 & 0.07+M & 0 & -0.06-M & 0.08+M & 0 & 0.06+2M & 1 & 0.054-0.2M &
\end{array}$$

$$\begin{array}{c|cccccccc|c}
 & x_1 & x_2 & x_3 & s_2 & s_3 & t_1 & t_2 & W & \\
\hline
s_2 & 0 & -1 & 0 & 1 & -1 & 1 & -1 & 0 & 0.2 \\
x_3 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0.7 \\
x_1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0.3 \\
\hline
W & 0 & 0.01 & 0 & 0 & 0.02 & 0.06+M & M & 1 & 0.066
\end{array}$$

For the above table, $t_1 = t_2 = 0$. Thus $W = Z$.

The fund should put 30% in A bonds, 0% in AA, and 70% in AAA for a yield of 6.6%.

Problems 7.7

1.

$$\begin{array}{ccccccc|c}
x_1 & x_2 & s_1 & s_2 & t_1 & t_2 & W & \\
\hline
1 & -1 & -1 & 0 & 1 & 0 & 0 & 7 \\
2 & 1 & 0 & -1 & 0 & 1 & 0 & 9 \\
\hline
2 & 5 & 0 & 0 & M & M & 1 & 0
\end{array}$$

$$\begin{array}{c|ccccccc|c|c}
 & x_1 & x_2 & s_1 & s_2 & t_1 & t_2 & W & & \\
\hline
t_1 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & 7 & 7 \\
t_2 & \underline{2} & 1 & 0 & -1 & 0 & 1 & 0 & 9 & \frac{9}{2} \\
\hline
W & 2-3M & 5 & M & M & 0 & 0 & 1 & -16M &
\end{array}$$

$$\begin{array}{c|ccccccc|c|c}
 & x_1 & x_2 & s_1 & s_2 & t_1 & t_2 & W & & \\
\hline
t_1 & 0 & -\frac{3}{2} & -1 & \underline{\frac{1}{2}} & 1 & -\frac{1}{2} & 0 & \frac{5}{2} & 5 \\
x_1 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{9}{2} & \\
\hline
W & 0 & 4+\frac{3}{2}M & M & 1-\frac{1}{2}M & 0 & -1+\frac{3}{2}M & 1 & -9-\frac{5}{2}M &
\end{array}$$

$$\begin{array}{c} \\ s_2 \\ x_1 \\ W \end{array}\begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & t_1 & t_2 & W & \end{array} \\ \left[\begin{array}{ccccccc|c} 0 & -3 & -2 & 1 & 2 & -1 & 0 & 5 \\ 1 & -1 & -1 & 0 & 1 & 0 & 0 & 7 \\ \hline 0 & 7 & 2 & 0 & -2+M & M & 1 & -14 \end{array}\right] \end{array}$$

The minimum is $Z = 14$ when $x_1 = 7,\ x_2 = 0$.

2.

$$\begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & t_1 & t_2 & W & \end{array} \\ \left[\begin{array}{ccccccc|c} 2 & 2 & -1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & -1 & 0 & 1 & 0 & 2 \\ \hline 8 & 12 & 0 & 0 & M & M & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \\ t_1 \\ t_2 \\ W \end{array}\begin{array}{c} \begin{array}{cccccccc|c} x_1 & x_2 & x_3 & s_2 & s_3 & t_1 & t_2 & W & \end{array} \\ \left[\begin{array}{ccccccc|c} 2 & \underline{2} & -1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & -1 & 0 & 1 & 0 & 2 \\ \hline 8-3M & 12-5M & M & M & 0 & 0 & 1 & -3M \end{array}\right] \end{array}\begin{array}{c} \\ \frac{1}{2} \\ \frac{2}{3} \\ \\ \end{array}$$

$$\begin{array}{c} \\ x_2 \\ t_2 \\ W \end{array}\begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & t_1 & t_2 & W & \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ -2 & 0 & \underline{\frac{3}{2}} & -1 & -\frac{3}{2} & 1 & 0 & \frac{1}{2} \\ \hline -4+2M & 0 & 6-\frac{3}{2}M & M & -6+\frac{5}{2}M & 0 & 1 & -6-\frac{1}{2}M \end{array}\right] \end{array}\begin{array}{c} \\ \\ \frac{1}{3} \\ \\ \end{array}$$

$$\begin{array}{c} \\ x_2 \\ s_1 \\ W \end{array}\begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & t_1 & t_2 & W & \end{array} \\ \left[\begin{array}{ccccccc|c} \frac{1}{3} & 1 & 0 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ -\frac{4}{3} & 0 & 1 & -\frac{2}{3} & -1 & \frac{2}{3} & 0 & \frac{1}{3} \\ \hline 4 & 0 & 0 & 4 & M & -4+M & 1 & -8 \end{array}\right] \end{array}$$

The minimum is $Z = 8$ when $x_1 = 0, x_2 = \frac{2}{3}$.

3.

$$\begin{array}{c} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & s & t & W & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & -1 & -1 & -1 & 1 & 0 & 18 \\ \hline 12 & 6 & 3 & 0 & M & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \\ t \\ W \end{array}\begin{array}{c} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & s & t & W & \end{array} \\ \left[\begin{array}{cccccc|c} \underline{1} & -1 & -1 & -1 & 1 & 0 & 18 \\ \hline 12-M & 6+M & 3+M & M & 0 & 1 & -18M \end{array}\right] \end{array}\begin{array}{c} \\ 18 \\ \\ \end{array}$$

$$\begin{array}{c} \\ x_1 \\ W \end{array}\begin{array}{c} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & s & t & W & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & -1 & -1 & -1 & 1 & 0 & 18 \\ \hline 0 & 18 & 15 & 12 & -12+M & 1 & -216 \end{array}\right] \end{array}$$

The minimum is $Z = 216$ when $x_1 = 18, x_2 = 0, x_3 = 0$.

4.
$$\begin{array}{c} \\ \\ \end{array}\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s & t & W & \\ \hline 1 & 2 & -1 & -1 & 1 & 0 & 4 \\ \hline 1 & 1 & 2 & 0 & M & 1 & 0 \end{array}$$

$$\begin{array}{c|cccccc|c|c} & x_1 & x_2 & x_3 & s & t & W & & \\ \hline t & 1 & \underline{2} & -1 & -1 & 1 & 0 & 4 & 2 \\ \hline W & 1-M & 1-2M & 2+M & M & 0 & 1 & -4M & \end{array}$$

$$\begin{array}{c|cccccc|c} x_2 & \frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 2 \\ \hline W & \frac{1}{2} & 0 & \frac{5}{2} & \frac{1}{2} & -\frac{1}{2}+M & 1 & -2 \end{array}$$

The minimum is $Z = 2$ when $x_1 = 0, x_2 = 2, x_3 = 0$.

5. We write the second constraint as $-x_1 + x_3 \geq 4$.

$$\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & t_2 & W & \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 6 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 4 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 5 \\ \hline 2 & 3 & 1 & 0 & 0 & 0 & M & 1 & 0 \end{array}$$

$$\begin{array}{c|cccccccc|c|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & t_2 & W & & \\ \hline s_1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 6 & 6 \\ t_2 & -1 & 0 & \underline{1} & 0 & -1 & 0 & 1 & 0 & 4 & 4 \\ s_3 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 5 & 5 \\ \hline W & 2+M & 3 & 1-M & 0 & M & 0 & 0 & 1 & -4M & \end{array}$$

$$\begin{array}{c|cccccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & t_2 & W & \\ \hline s_1 & 2 & 1 & 0 & 1 & 1 & 0 & -1 & 0 & 2 \\ x_3 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 4 \\ s_3 & 1 & 1 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \\ \hline W & 3 & 3 & 0 & 0 & 1 & 0 & -1+M & 1 & -4 \end{array}$$

The minimum is $Z = 4$ when $x_1 = 0, x_2 = 0, x_3 = 4$.

6.
$$\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & t_3 & W & \\ \hline 3 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 4 \\ 0 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 5 \\ 1 & 1 & 1 & 0 & 0 & -1 & 1 & 0 & 2 \\ \hline 5 & 1 & 3 & 0 & 0 & 0 & M & 1 & 0 \end{array}$$

$$\begin{array}{c|cccccccc|c|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & t_3 & W & & \\ \hline s_1 & 3 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 4 & 4 \\ s_2 & 0 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 5 & \frac{5}{2} \\ t_3 & 1 & \underline{1} & 1 & 0 & 0 & -1 & 1 & 0 & 2 & 2 \\ \hline W & 5-M & 1-M & 3-M & 0 & 0 & M & 0 & 1 & -2M & \end{array}$$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ x_2 \\ W \end{array}\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & t_3 & W \end{array} \\ \left[\begin{array}{cccccccc|c} 2 & 0 & -2 & 1 & 0 & 1 & -1 & 0 & 2 \\ -2 & 0 & 0 & 0 & 1 & 2 & -2 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & -1 & 1 & 0 & 2 \\ \hdashline 4 & 0 & 2 & 0 & 0 & 1 & -1+M & 1 & -2 \end{array}\right] \end{array}$$

The minimum is $Z = 2$ when $x_1 = 0$, $x_2 = 2$, and $x_3 = 0$.

7.

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_3 & t_1 & t_2 & W \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 6 \\ \hdashline 1 & -1 & -3 & 0 & M & M & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \\ t_1 \\ t_2 \\ s_3 \\ W \end{array}\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_3 & t_1 & t_2 & W \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & \underline{1} & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 6 \\ \hdashline 1-M & -1-3M & -3-2M & 0 & 0 & 0 & 1 & -5M \end{array}\right] \end{array}\begin{array}{c} \\ 2 \\ 1 \\ 6 \\ \\ \end{array}$$

$$\begin{array}{c} \\ t_1 \\ x_2 \\ s_3 \\ W \end{array}\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_3 & t_1 & t_2 & W \end{array} \\ \left[\begin{array}{ccccccc|c} \underline{1} & 0 & -1 & 0 & 1 & -2 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 & -1 & 0 & 5 \\ \hdashline 1-M & 0 & -2+M & 0 & 0 & 1+3M & 1 & 1-2M \end{array}\right] \end{array}\begin{array}{c} \\ 2 \\ \\ 5 \\ \\ \end{array}$$

$$\begin{array}{c} \\ x_1 \\ x_2 \\ s_3 \\ W \end{array}\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_3 & t_1 & t_2 & W \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 0 & -1 & 0 & 1 & -2 & 0 & 2 \\ 0 & 1 & \underline{1} & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 3 \\ \hdashline 0 & 0 & -1 & 0 & -1+M & 3+M & 1 & -1 \end{array}\right] \end{array}\begin{array}{c} \\ \\ 1 \\ \\ \\ \end{array}$$

$$\begin{array}{c} \\ x_1 \\ x_3 \\ s_3 \\ -Z \end{array}\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & s_3 & -Z \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ \hdashline 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The minimum is $Z = 0$ when $x_1 = 3$, $x_2 = 0$, $x_3 = 1$.

8.

$$\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccccc|c} x_1 & x_2 & s_1 & t_1 & t_2 & W & \\ \hline -1 & 1 & -1 & 1 & 0 & 0 & 4 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline 1 & -1 & 0 & M & M & 1 & 0 \end{array}$$

$$\begin{array}{c|cccccc|c|c} & x_1 & x_2 & s_1 & t_1 & t_2 & W & & \\ \hline t_1 & -1 & 1 & -1 & 1 & 0 & 0 & 4 & 4 \\ t_2 & 1 & \underline{1} & 0 & 0 & 1 & 0 & 1 & 1 \\ \hline W & 1 & -1-2M & M & 0 & 0 & 1 & -5M & \end{array}$$

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & s_1 & t_1 & t_2 & W & \\ \hline t_1 & -2 & 0 & -1 & 1 & -1 & 0 & 3 \\ x_2 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline W & 2+2M & 0 & M & 0 & 1+2M & 1 & 1-3M \end{array}$$

Since all of the indicators in the last table are positive, but the artificial variable t_1 is 3, the feasible region is empty. (This can also be seen graphically.)

9.

$$\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & t_1 & t_2 & W & \\ \hline 1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 & 8 \\ -1 & 2 & 1 & 0 & -1 & 0 & 1 & 0 & 2 \\ \hline 1 & 8 & 5 & 0 & 0 & M & M & 1 & 0 \end{array}$$

$$\begin{array}{c|cccccccc|c|c} & x_1 & x_2 & x_3 & s_1 & s_2 & t_1 & t_2 & W & & \\ \hline t_1 & 1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 & 8 & 8 \\ t_2 & -1 & \underline{2} & 1 & 0 & -1 & 0 & 1 & 0 & 2 & 1 \\ \hline W & 1 & 8-3M & 5-2M & M & M & 0 & 0 & 1 & -10M & \end{array}$$

$$\begin{array}{c|cccccccc|c|c} & x_1 & x_2 & x_3 & s_1 & s_2 & t_1 & t_2 & W & & \\ \hline t_1 & \underline{\frac{3}{2}} & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 & 7 & \frac{14}{3} \\ x_2 & -\frac{1}{2} & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 1 & \\ \hline W & 5-\frac{3}{2}M & 0 & 1-\frac{1}{2}M & M & 4-\frac{1}{2}M & 0 & -4+\frac{3}{2}M & 1 & -8-7M & \end{array}$$

$$\begin{array}{c|cccccccc|c|c} & x_1 & x_2 & x_3 & s_1 & s_2 & t_1 & t_2 & W & & \\ \hline x_1 & 1 & 0 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & \frac{14}{3} & 14 \\ x_2 & 0 & 1 & \underline{\frac{2}{3}} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \frac{10}{3} & 5 \\ \hline W & 0 & 0 & -\frac{2}{3} & \frac{10}{3} & \frac{7}{3} & -\frac{10}{3}+M & -\frac{7}{3}+M & 1 & -\frac{94}{3} & \end{array}$$

$$\begin{array}{c|cccccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & t_1 & t_2 & W & \\ \hline x_1 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 3 \\ x_3 & 0 & \frac{3}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 5 \\ \hline W & 0 & 1 & 0 & 3 & 2 & -3+M & -2+M & 1 & -28 \end{array}$$

The minimum is $Z = 28$ when $x_1 = 3, x_2 = 0, x_3 = 5$.

10.
$$\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} x_1 & x_2 & x_3 & s_1 & s_2 & t_2 & W & \\ \end{array}$$
$$\left[\begin{array}{ccccccc|c} 1 & -1 & -1 & 1 & 0 & 0 & 0 & 3 \\ 1 & -1 & 1 & 0 & -1 & 1 & 0 & 3 \\ \hline 4 & 4 & 6 & 0 & 0 & M & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ s_1 \\ t_2 \\ W \end{array}\left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & t_2 & W & \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 3 \\ \underline{1} & -1 & 1 & 0 & -1 & 1 & 0 & 3 \\ \hline 4-M & 4+M & 6-M & 0 & M & 0 & 1 & -3M \end{array}\right]\begin{array}{c} \\ 3 \\ 3 \\ \\ \end{array}$$

Here we choose t_2 as the departing variable.

$$\begin{array}{c} \\ s_1 \\ x_1 \\ W \end{array}\left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & t_2 & W & \\ 0 & 0 & -2 & 1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 1 & 0 & -1 & 1 & 0 & 3 \\ \hline 0 & 8 & 2 & 0 & 4 & -4+M & 1 & -12 \end{array}\right]$$

Thus Z has a minimum value of 12 when $x_1 = 3, x_2 = 0, x_3 = 0.$

If we choose s_1 as the departing variable, then

$$\begin{array}{c} \\ s_1 \\ t_2 \\ W \end{array}\left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & t_2 & W & \\ \underline{1} & -1 & -1 & 1 & 0 & 0 & 0 & 3 \\ 1 & -1 & 1 & 0 & -1 & 1 & 0 & 3 \\ \hline 4-M & 4+M & 6-M & 0 & M & 0 & 1 & -3M \end{array}\right]\begin{array}{c} \\ 3 \\ 3 \\ \\ \end{array}$$

$$\begin{array}{c} \\ x_1 \\ t_2 \\ W \end{array}\left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & t_2 & W & \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & \underline{2} & -1 & -1 & 1 & 0 & 0 \\ \hline 0 & 8 & 10-2M & -4+M & M & 0 & 1 & -12 \end{array}\right]\begin{array}{c} \\ \\ 0 \\ \\ \end{array}$$

$$\begin{array}{c} \\ x_1 \\ x_3 \\ W \end{array}\left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & t_2 & W & \\ 1 & -1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 3 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \hline 0 & 8 & 0 & 1 & 5 & -5+M & 1 & -12 \end{array}\right]$$

The minimum is $Z = 12$ when $x_1 = 3, x_2 = 0, x_3 = 0$.

11. Let $x_1, x_2,$ and x_3 denote the annual numbers of barrels of cement produced in kilns that use device A, device B, and no device, respectively. We want to minimize the annual emission control cost C (C in dollars) where $C = \frac{1}{4}x_1 + \frac{2}{5}x_2 + 0x_3$ subject to

$$x_1 + x_2 + x_3 = 3{,}300{,}000,$$
$$\frac{1}{2}x_1 + \frac{1}{4}x_2 + 2x_3 \le 1{,}000{,}000,$$
$$x_1, x_2, x_3 \ge 0.$$

$$\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_2 & t_1 & W & \\ \hline 1 & 1 & 1 & 0 & 1 & 0 & 3{,}300{,}000 \\ \frac{1}{2} & \frac{1}{4} & 2 & 1 & 0 & 0 & 1{,}000{,}000 \\ \hline \frac{1}{4} & \frac{2}{5} & 0 & 0 & M & 1 & 0 \end{array}$$

$$\begin{array}{c|cccccc|c|c} & x_1 & x_2 & x_3 & s_2 & t_1 & W & & \\ \hline t_1 & 1 & 1 & 1 & 0 & 1 & 0 & 3{,}300{,}000 & 3{,}300{,}000 \\ s_2 & \frac{1}{2} & \frac{1}{4} & \underline{2} & 1 & 0 & 0 & 1{,}000{,}000 & 500{,}000 \\ \hline W & \frac{1}{4}-M & \frac{2}{5}-M & -M & 0 & 0 & 1 & -3{,}300{,}000M & \end{array}$$

$$\begin{array}{c|cccccc|c|c} & x_1 & x_2 & x_3 & s_2 & t_1 & W & & \\ \hline t_1 & \frac{3}{4} & \underline{\frac{7}{8}} & 0 & -\frac{1}{2} & 1 & 0 & 2{,}800{,}000 & 3{,}200{,}000 \\ x_3 & \frac{1}{4} & \frac{1}{8} & 1 & \frac{1}{2} & 0 & 0 & 500{,}000 & 4{,}000{,}000 \\ \hline W & \frac{1}{4}-\frac{3}{4}M & \frac{2}{5}-\frac{7}{8}M & 0 & \frac{1}{2}M & 0 & 1 & -2{,}800{,}000M & \end{array}$$

$$\begin{array}{c|cccccc|c|c} & x_1 & x_2 & x_3 & s_2 & t_1 & W & & \\ \hline x_2 & \frac{6}{7} & 1 & 0 & -\frac{4}{7} & \frac{8}{7} & 0 & 3{,}200{,}000 & \frac{11{,}200{,}000}{3} \\ x_3 & \underline{\frac{1}{7}} & 0 & 1 & \frac{4}{7} & -\frac{1}{7} & 0 & 100{,}000 & 700{,}000 \\ \hline W & -\frac{13}{140} & 0 & 0 & \frac{8}{35} & -\frac{16}{35}+M & 1 & -1{,}280{,}000 & \end{array}$$

$$\begin{array}{c|ccccc|c} & x_1 & x_2 & x_3 & s_2 & -C & \\ \hline x_2 & 0 & 1 & -6 & -4 & 0 & 2{,}600{,}000 \\ x_1 & 1 & 0 & 7 & 4 & 0 & 700{,}000 \\ \hline -C & 0 & 0 & \frac{13}{20} & \frac{3}{5} & 1 & -1{,}215{,}000 \end{array}$$

Thus the minimum value of C is 1,215,000 when $x_1 = 700{,}000$, $x_2 = 2{,}600{,}000$, $x_3 = 0$.
The plant should install device A on kilns producing 700,000 barrels annually, and device B on kilns producing 2,600,000 barrels annually.

12. Let x_1 = number of type A trucks rented,

x_2 = number of type B trucks rented.

We want to minimize $C = 0.40\,x_1 + 0.60\,x_2$ subject to

$$\begin{aligned} 2x_1 + 2x_2 &\ge 12, \\ x_1 + 3x_2 &\ge 12, \\ x_1, x_2 &\ge 0. \end{aligned}$$

$$\begin{array}{c} \\ \\ \\ \end{array}
\begin{array}{c} x_1 \quad x_2 \quad s_1 \quad s_2 \quad t_1 \quad t_2 \quad W \\ \left[\begin{array}{ccccccc|c} 2 & 2 & -1 & 0 & 1 & 0 & 0 & 12 \\ 1 & 3 & 0 & -1 & 0 & 1 & 0 & 12 \\ \hline 0.40 & 0.60 & 0 & 0 & M & M & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \\ t_1 \\ t_2 \\ W \end{array}
\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & s_1 & s_2 & t_1 & t_2 & W \end{array} \\ \left[\begin{array}{ccccccc|c} 2 & 2 & -1 & 0 & 1 & 0 & 0 & 12 \\ 1 & \underline{3} & 0 & -1 & 0 & 1 & 0 & 12 \\ \hline \frac{2}{5}-3M & \frac{3}{5}-5M & M & M & 0 & 0 & 1 & -24M \end{array}\right] \end{array}
\begin{array}{c} \\ 6 \\ 4 \\ \\ \end{array}$$

$$\begin{array}{c} \\ t_1 \\ x_2 \\ W \end{array}
\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & s_1 & s_2 & t_1 & t_2 & W \end{array} \\ \left[\begin{array}{ccccccc|c} \underline{\frac{4}{3}} & 0 & -1 & \frac{2}{3} & 1 & -\frac{2}{3} & 0 & 4 \\ \frac{1}{3} & 1 & 0 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 4 \\ \hline \frac{1}{5}-\frac{4}{3}M & 0 & M & \frac{1}{5}-\frac{2}{3}M & 0 & -\frac{1}{5}+\frac{5}{3}M & 1 & -\frac{12}{5}-4M \end{array}\right] \end{array}
\begin{array}{c} \\ 3 \\ 12 \\ \\ \end{array}$$

$$\begin{array}{c} \\ x_1 \\ x_2 \\ W \end{array}
\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & s_1 & s_2 & t_1 & t_2 & W \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 0 & -\frac{3}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} & 0 & 3 \\ 0 & 1 & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} & \frac{1}{2} & 0 & 3 \\ \hline 0 & 0 & \frac{3}{20} & \frac{1}{10} & -\frac{3}{20}+M & -\frac{1}{10}+M & 1 & -3 \end{array}\right] \end{array}$$

The minimum value of C is 3 when $x_1 = 3$ and $x_2 = 3$. They should rent 3 of type A and 3 of type B. The cost per mile is \$3.00.

13. Let x_1 = number of DVD players shipped from Akron to Columbus,

x_2 = number of DVD players shipped from Springfield to Columbus,

x_3 = number of DVD players shipped from Akron to Dayton,

x_4 = number of DVD players shipped from Springfield to Dayton.

We want to minimize $C = 5x_1 + 3x_2 + 7x_3 + 2x_4$ subject to

$$\begin{aligned} x_1 + x_2 &= 150, \\ x_3 + x_4 &= 150, \\ x_1 + x_3 &\le 200, \\ x_2 + x_4 &\le 150, \\ x_1, x_2, x_3, x_4 &\ge 0. \end{aligned}$$

x_1	x_2	x_3	x_4	s_3	s_4	t_1	t_2	W	
1	1	0	0	0	0	1	0	0	150
0	0	1	1	0	0	0	1	0	150
1	0	1	0	1	0	0	0	0	200
0	1	0	1	0	1	0	0	0	150
5	3	7	2	0	0	M	M	1	0

	x_1	x_2	x_3	x_4	s_3	s_4	t_1	t_2	W		
t_1	1	1	0	0	0	0	1	0	0	150	
t_2	0	0	1	$\underline{1}$	0	0	0	1	0	150	150
s_3	1	0	1	0	1	0	0	0	0	200	
s_4	0	1	0	1	0	1	0	0	0	150	150
W	$5-M$	$3-M$	$7-M$	$2-M$	0	0	0	0	1	$-300M$	

	x_1	x_2	x_3	x_4	s_3	s_4	t_1	t_2	W		
t_1	1	1	0	0	0	0	1	0	0	150	150
x_4	0	0	1	1	0	0	0	1	0	150	
s_3	1	0	1	0	1	0	0	0	0	200	
s_4	0	$\underline{1}$	−1	0	0	1	0	−1	0	0	0
W	$5-M$	$3-M$	5	0	0	0	0	$-2+M$	1	$-300-150M$	

	x_1	x_2	x_3	x_4	s_3	s_4	t_1	t_2	W		
t_1	$\underline{1}$	0	1	0	0	−1	1	1	0	150	150
x_4	0	0	1	1	0	0	0	1	0	150	
s_3	1	0	1	0	1	0	0	0	0	200	200
x_2	0	1	−1	0	0	1	0	−1	0	0	
W	$5-M$	0	$8-M$	0	0	$-3+M$	0	1	1	$-300-150M$	

	x_1	x_2	x_3	x_4	s_3	s_4	t_1	t_2	W	
x_1	1	0	1	0	0	−1	1	1	0	150
x_4	0	0	1	1	0	0	0	1	0	150
s_3	0	0	0	0	1	1	−1	−1	0	50
x_2	0	1	−1	0	0	1	0	−1	0	0
W	0	0	3	0	0	2	$-5+M$	$-4+M$	1	−1050

The retailer should ship as follows: to Columbus, 150 from Akron and 0 from Springfield; to Dayton, 0 from Akron and 150 from Springfield. The transportation cost is \$1050.

If s_4 is chosen as the departing variable in the second table, the result is the same, although the final table is different:

	x_1	x_2	x_3	x_4	s_3	s_4	t_1	t_2	W	
x_1	1	1	0	0	0	0	1	0	0	150
x_3	0	−1	1	0	0	−1	0	1	0	0
s_3	0	0	0	0	1	1	−1	−1	0	50
x_4	0	1	0	1	0	1	0	0	0	150
W	0	3	0	0	0	5	$-5+M$	$-7+M$	1	−1050

14. Let x_A = number of alternators from supplier X to plant A
x_B = number of alternators from supplier X to plant B
y_A = number of alternators from supplier Y to plant A
y_B = number of alternators from supplier Y to plant B

We want to minimize $C = 300x_A + 320x_B + 340y_A + 280y_B$ subject to

$$x_A + y_A = 7000$$
$$x_B + y_B = 5000$$
$$x_A + x_B \geq 3000$$
$$x_A + x_B \leq 5000$$
$$y_A + y_B \geq 7000$$
$$x_A,\ x_B,\ y_A,\ y_B \geq 0$$

x_A	x_B	y_A	y_B	s_3	s_4	s_5	t_1	t_2	t_3	t_5	W	
1	0	1	0	0	0	0	1	0	0	0	0	7000
0	1	0	1	0	0	0	0	1	0	0	0	5000
1	1	0	0	−1	0	0	0	0	1	0	0	3000
1	1	0	0	0	1	0	0	0	0	0	0	5000
0	0	1	1	0	0	−1	0	0	0	1	0	7000
300	320	340	280	0	0	0	M	M	M	M	1	0

	x_A	x_B	y_A	y_B	s_3	s_4	s_5	t_1	t_2	t_3	t_5	W		
t_1	1	0	1	0	0	0	0	1	0	0	0	0	7000	
t_2	0	1	0	**1**	0	0	0	0	1	0	0	0	5000	5000
t_3	1	1	0	0	−1	0	0	0	0	1	0	0	3000	
s_4	1	1	0	0	0	1	0	0	0	0	0	0	5000	
t_5	0	0	1	1	0	0	−1	0	0	0	1	0	7000	7000
W	$300-2M$	$320-2M$	$340-2M$	$280-2M$	M	0	M	0	0	0	0	1	$-22{,}000M$	

	x_A	x_B	y_A	y_B	s_3	s_4	s_5	t_1	t_2	t_3	t_5	W		
t_1	1	0	1	0	0	0	0	1	0	0	0	0	7000	7000
y_B	0	1	0	1	1	0	0	0	1	0	0	0	5000	
t_3	**1**	1	0	0	−1	0	0	0	0	1	0	0	3000	3000
s_4	1	1	0	0	0	1	0	0	0	0	0	0	5000	5000
t_5	0	−1	1	0	0	0	−1	0	−1	0	1	0	2000	
W	$300-2M$	40	$340-2M$	0	M	0	M	0	$-280+2M$	0	0	1	$-1{,}400{,}000-12{,}000M$	

	x_A	x_B	y_A	y_B	s_3	s_4	s_5	t_1	t_2	t_3	t_5	W		
t_1	0	−1	1	0	1	0	0	1	0	−1	0	0	4000	4000
y_B	0	1	0	1	0	0	0	0	1	0	0	0	5000	
x_A	1	1	0	0	−1	0	0	0	0	1	0	0	3000	
s_4	0	0	0	0	1	1	0	0	0	−1	0	0	2000	
t_5	0	−1	**1**	0	0	0	−1	0	−1	0	1	0	2000	2000
W	0	$-260+2M$	$340-2M$	0	$300-M$	0	M	0	$-280+2M$	$-300+2M$	0	1	$-2{,}300{,}000-6000M$	

	x_A	x_B	y_A	y_B	s_3	s_4	s_5	t_1	t_2	t_3	t_5	W		
t_1	0	0	0	0	**1**	0	1	1	1	−1	−1	0	2000	2000
y_B	0	1	0	1	0	0	0	0	1	0	0	0	5000	
x_A	1	1	0	0	−1	0	0	0	0	1	0	0	3000	
s_4	0	0	0	0	1	1	0	0	0	−1	0	0	2000	2000
y_A	0	−1	1	0	0	0	−1	0	−1	0	1	0	2000	
W	0	80	0	0	$300-M$	0	$340-M$	0	60	$-300+2M$	$-340+2M$	1	$2{,}980{,}000-2000M$	

We choose t_1 as the departing variable.

	x_A	x_B	y_A	y_B	s_3	s_4	s_5	t_1	t_2	t_3	t_5	W	
s_3	0	0	0	0	1	0	1	1	1	–1	–1	0	2000
y_B	0	1	0	1	0	0	0	0	1	0	0	0	5000
x_A	1	1	0	0	0	0	1	1	1	0	–1	0	5000
s_4	0	0	0	0	0	1	–1	–1	–1	0	1	0	0
y_A	0	–1	1	0	0	0	–1	0	–1	0	1	0	2000
W	0	80	0	0	0	0	40	$-300+M$	$-240+M$	M	$-40+M$	1	$-3,580,000$

The manufacturer should order 5000 alternators from *X* to be shipped to *A*, 2000 from *Y* to *A*, and 5000 from *Y* to *B*. The minimum cost is $3,580,000. (Note that the same result is obtained if s_4 is chosen as the departing variable in the fifth table.)

15. **a.** Roll width

Roll width	15"	3	2	1	0
	10"	0	1	3	4
Trim loss		3	8	3	8

b. We want to minimize $L = 3x_1 + 8x_2 + 3x_3 + 8x_4$ subject to

$$3x_1 + 2x_2 + x_3 \geq 50,$$
$$x_2 + 3x_3 + 4x_4 \geq 60,$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

x_1	x_2	x_3	x_4	s_1	s_2	t_1	t_2	W	
3	2	1	0	–1	0	1	0	0	50
0	1	3	4	0	–1	0	1	0	60
3	8	3	8	0	0	M	M	1	0

	x_1	x_2	x_3	x_4	s_1	s_2	t_1	t_2	W		
t_1	3	2	1	0	–1	0	1	0	0	50	50
t_2	0	1	3	4	0	–1	0	1	0	60	20
W	$3-3M$	$8-3M$	$3-4M$	$8-4M$	M	M	0	0	1	$-110M$	

	x_1	x_2	x_3	x_4	s_1	s_2	t_1	t_2	W		
t_1	3	$\frac{5}{3}$	0	$-\frac{4}{3}$	–1	$\frac{1}{3}$	1	$-\frac{1}{3}$	0	30	10
x_3	0	$\frac{1}{3}$	1	$\frac{4}{3}$	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	20	
W	$3-3M$	$7-\frac{5}{3}M$	0	$4+\frac{4}{3}M$	M	$1-\frac{1}{3}M$	0	$-1+\frac{4}{3}M$	1	$-60-30M$	

	x_1	x_2	x_3	x_4	s_1	s_2	t_1	t_2	W	
x_1	1	$\frac{5}{9}$	0	$-\frac{4}{9}$	$-\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{3}$	$-\frac{1}{9}$	0	10
x_3	0	$\frac{1}{3}$	1	$\frac{4}{3}$	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	20
W	0	$\frac{16}{3}$	0	$\frac{16}{3}$	1	$\frac{2}{3}$	$-1+M$	$-\frac{2}{3}+M$	1	–90

$x_1 = 10, x_2 = 0, x_3 = 20, x_4 = 0.$

c. 90 in.

Principles in Practice 7.8

1. Let x_1, x_2, and x_3 be the numbers respectively, of Type 1, Type 2, and Type 3 gadgets produced. The original problem is to maximize
$P = 300x_1 + 200x_2 + 200x_3$, subject to
$300x_1 + 220x_2 + 180x_3 \le 60,000,$
$20x_1 + 40x_2 + 20x_3 \le 2000,$
$3x_1 + x_2 + 2x_3 \le 120,$
and $x_1, x_2, x_3 \ge 0.$
The dual problem is to minimize
$W = 60,000y_1 + 2000y_2 + 120y_3,$
subject to
$300y_1 + 20y_2 + 3y_3 \ge 300,$
$220y_1 + 40y_2 + y_3 \ge 200,$
$180y_1 + 20y_2 + 2y_3 \ge 200,$
and $y_1, y_2, y_3 \ge 0.$

2. Let x_1 and x_2 be the amounts, respectively of supplement 1 and supplement 2. The original problem is to minimize $C = 6x_1 + 2x_2$, subject to
$20x_1 + 6x_2 \ge 98,$
$8x_1 + 16x_2 \ge 80,$
and $x_1, x_2 \ge 0.$
The dual problem is to maximize
$W = 98y_1 + 80y_2$, subject to
$20y_1 + 8y_2 \le 6,$
$6y_1 + 16y_2 \le 2,$
and $y_1, y_2 \ge 0.$

3. Let x_1, x_2, and x_3 be the numbers, respectively, of devices 1, 2, and 3 produced.
The original problem is to maximize $P = 30x_1 + 20x_2 + 20x_3$, subject to
$30x_1 + 15x_2 + 10x_3 \le 300,$ $20x_1 + 30x_2 + 20x_3 \le 400,$ $40x_1 + 30x_2 + 25x_3 \le 600,$ and $x_1, x_2, x_3 \ge 0.$
The dual problem is to minimize $W = 300y_1 + 400y_2 + 600y_3$, subject to
$30y_1 + 20y_2 + 40y_3 \ge 30,$ $15y_1 + 30y_2 + 30y_3 \ge 20,$
$10y_1 + 20y_2 + 25y_3 \ge 20,$
and $y_1, y_2, y_3 \ge 0.$
The tablex to maximize $Z = -W = -300y_1 - 400y_2 - 600y_3$ follow.

$$\begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cccccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & t_1 & t_2 & t_3 & Z & \\ \hline 30 & 20 & 40 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 30 \\ 15 & 30 & 30 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 20 \\ 10 & 20 & 25 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 20 \\ \hline 300 & 400 & 600 & 0 & 0 & 0 & M & M & M & 1 & 0 \end{array}$$

	y_1	y_2	y_3	s_1	s_2	s_3	t_1	t_2	t_3	Z	
t_1	30	20	40	–1	0	0	1	0	0	0	30
t_2	15	30	30	0	–1	0	0	1	0	0	20
t_3	10	20	25	0	0	–1	0	0	1	0	20
Z	$300-55M$	$400-70M$	$600-95M$	M	M	M	0	0	0	1	$-70M$

	y_1	y_2	y_3	s_1	s_2	s_3	t_1	t_2	t_3	Z	
t_1	10	–20	0	–1	$\frac{4}{3}$	0	1	$-\frac{4}{3}$	0	0	$\frac{10}{3}$
y_3	$\frac{1}{2}$	1	1	0	$-\frac{1}{30}$	0	0	$\frac{1}{30}$	0	0	$\frac{2}{3}$
t_3	$-\frac{5}{2}$	–5	0	0	$\frac{5}{6}$	–1	0	$-\frac{5}{6}$	1	0	$\frac{10}{3}$
Z	$-\frac{15}{2}M$	$-200+25M$	0	M	$20-\frac{13}{6}M$	M	0	$-20+\frac{19}{6}M$	0	1	$-400-\frac{20}{3}M$

	y_1	y_2	y_3	s_1	s_2	s_3	t_1	t_2	t_3	Z	
y_1	1	–2	0	$-\frac{1}{10}$	$\frac{2}{15}$	0	$\frac{1}{10}$	$-\frac{2}{15}$	0	0	$\frac{1}{3}$
y_3	0	2	1	$\frac{1}{20}$	$-\frac{1}{10}$	0	$-\frac{1}{20}$	$\frac{1}{10}$	0	0	$\frac{1}{2}$
t_3	0	–10	0	$-\frac{1}{4}$	$\frac{7}{6}$	–1	$\frac{1}{4}$	$-\frac{7}{6}$	1	0	$\frac{25}{6}$
Z	0	$-200+10M$	0	$\frac{1}{4}M$	$20-\frac{7}{6}M$	M	$\frac{3}{4}M$	$-20+\frac{13}{6}M$	0	1	$-400-\frac{25}{6}M$

	y_1	y_2	y_3	s_1	s_2	s_3	t_1	t_2	t_3	Z	
s_2	$\frac{15}{2}$	–15	0	$-\frac{3}{4}$	1	0	$\frac{3}{4}$	–1	0	0	$\frac{5}{2}$
y_3	$\frac{3}{4}$	$\frac{1}{2}$	1	$-\frac{1}{40}$	0	0	$\frac{1}{40}$	0	0	0	$\frac{3}{4}$
t_3	$-\frac{35}{4}$	$\frac{15}{2}$	0	$\frac{5}{8}$	0	–1	$-\frac{5}{8}$	0	1	0	$\frac{5}{4}$
Z	$-150+\frac{35}{4}M$	$100-\frac{15}{2}M$	0	$15-\frac{5}{8}M$	0	M	$-15+\frac{13}{8}M$	M	0	1	$-450-\frac{5}{4}M$

	y_1	y_2	y_3	s_1	s_2	s_3	t_1	t_2	t_3	Z	
s_2	–10	0	0	$\frac{1}{2}$	1	–2	$-\frac{1}{2}$	–1	2	0	5
y_3	$\frac{4}{3}$	0	1	$-\frac{1}{15}$	0	$\frac{1}{15}$	$\frac{1}{15}$	0	$-\frac{1}{15}$	0	$\frac{2}{3}$
y_2	$-\frac{7}{6}$	1	0	$\frac{1}{12}$	0	$-\frac{2}{15}$	$-\frac{1}{12}$	0	$\frac{2}{15}$	0	$\frac{1}{6}$
Z	$-\frac{100}{3}$	0	0	$\frac{20}{3}$	0	$\frac{40}{3}$	$-\frac{20}{3}+M$	M	$-\frac{40}{3}+M$	1	$-\frac{1400}{3}$

The t_1, t_2, and t_3 columns are no longer needed.

	y_1	y_2	y_3	s_1	s_2	s_3	Z	
s_2	0	0	$\frac{15}{2}$	0	1	$-\frac{3}{2}$	0	10
y_1	1	0	$\frac{3}{4}$	$-\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{2}$
y_2	0	1	$\frac{7}{8}$	$\frac{1}{40}$	0	$-\frac{3}{40}$	0	$\frac{3}{4}$
Z	0	0	25	5	0	15	1	–450

From this table, the maximum profit of $450 corresponds to $x_1 = 5$, $x_2 = 0$, and $x_3 = 15$. The company should produce 5 of device 1 and 15 of device 3.

Problems 7.8

1. Minimize $W = 5y_1 + 3y_2$
subject to $y_1 - y_2 \ge 1$
$y_1 + y_2 \ge 2$
$y_1,\ y_2 \ge 0$

2. Minimize $W = 3y_1 + 5y_2$ subject to
$2y_1 - y_2 \ge 2,$
$2y_1 + 4y_2 \ge 1,$
$2y_2 \ge -1,$
$y_1, y_2 \ge 0.$

3. Maximize $W = 8y_1 + 2y_2$ subject to
$y_1 - y_2 \le 1,$
$y_1 + 2y_2 \le 8,$
$y_1 + y_2 \le 5,$
$y_1, y_2 \ge 0.$

4. Maximize $W = y_1 + 2y_2$ subject to
$2y_1 + y_2 \le 8,$
$2y_1 + 3y_2 \le 12,$
$y_1, y_2 \ge 0.$

5. The second and third constraints can be written as $x_1 - x_2 \le -3$ and $-x_1 - x_2 \le -11.$ Minimize $W = 13y_1 - 3y_2 - 11y_3$ subject to
$-y_1 + y_2 - y_3 \ge 1,$
$2y_1 - y_2 - y_3 \ge -1,$
$y_1, y_2, y_3 \ge 0.$

6. The second constraint can be written as $-x_1 + 2x_2 - x_3 \le -6.$ Minimize $W = 9y_1 - 6y_2$ subject to
$y_1 - y_2 \ge 1,$
$y_1 + 2y_2 \ge -1,$
$y_1 - y_2 \ge 4,$
$y_1, y_2 \ge 0.$

7. The first constraint can be written as $-x_1 + x_2 + x_3 \ge -3.$ Maximize $W = -3y_1 + 3y_2$ subject to
$-y_1 + y_2 \le 4,$
$y_1 - y_2 \le 4,$
$y_1 + y_2 \le 6,$
$y_1, y_2 \ge 0.$

8. The second constraint can be written as $-8x_1 + 10x_2 \ge -80.$
Maximize $W = -10y_1 - 80y_2$
subject to $-4y_1 - 8y_2 \le 5$
$3y_1 + 10y_2 \le 4$
$y_1, y_2 \ge 0$

9. The dual is: Maximize $W = 2y_1 + 3y_2$ subject to
$y_1 - y_2 \le 2,$
$-y_1 + 2y_2 \le 2,$
$2y_1 + y_2 \le 5,$
$y_1, y_2 \ge 0.$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ W \end{array}\begin{array}{cccccc|c} y_1 & y_2 & s_1 & s_2 & s_3 & W & \\ \hline 1 & -1 & 1 & 0 & 0 & 0 & 2 \\ -1 & \underline{2} & 0 & 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 & 1 & 0 & 5 \\ \hline -2 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\begin{array}{c} \\ \\ 1 \\ 5 \\ \\ \end{array}$$

$$\begin{array}{c} \\ s_1 \\ y_2 \\ s_3 \\ W \end{array}\begin{array}{cccccc|c} y_1 & y_2 & s_1 & s_2 & s_3 & W & \\ \hline \frac{1}{2} & 0 & 1 & \frac{1}{2} & 0 & 0 & 3 \\ -\frac{1}{2} & 1 & 0 & \frac{1}{2} & 0 & 0 & 1 \\ \underline{\frac{5}{2}} & 0 & 0 & -\frac{1}{2} & 1 & 0 & 4 \\ \hline -\frac{7}{2} & 0 & 0 & \frac{3}{2} & 0 & 1 & 3 \end{array}\begin{array}{c} \\ 6 \\ \\ \frac{8}{5} \\ \\ \end{array}$$

$$\begin{array}{c} \\ s_1 \\ y_2 \\ y_1 \\ W \end{array}\begin{array}{cccccc|c} y_1 & y_2 & s_1 & s_2 & s_3 & W & \\ \hline 0 & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{11}{5} \\ 0 & 1 & 0 & \frac{2}{5} & \frac{1}{5} & 0 & \frac{9}{5} \\ 1 & 0 & 0 & -\frac{1}{5} & \frac{2}{5} & 0 & \frac{8}{5} \\ \hline 0 & 0 & 0 & \frac{4}{5} & \frac{7}{5} & 1 & \frac{43}{5} \end{array}$$

The minimum is $Z = \frac{43}{5}$ when $x_1 = 0,\ \ x_2 = \frac{4}{5},$ $x_3 = \frac{7}{5}.$

10. The dual is: Maximize $W = 28y_1 + 2y_2 + 16y_3$ subject to
$y_1 + 2y_2 - 3y_3 \le 2,$
$4y_1 - y_2 + 8y_3 \le 2,$
$y_1, y_2, y_3 \ge 0.$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ W \end{array}\begin{array}{cccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & W & \\ \hline 1 & 2 & -3 & 1 & 0 & 0 & 2 \\ \underline{4} & -1 & 8 & 0 & 1 & 0 & 2 \\ \hline -28 & -2 & -16 & 0 & 0 & 1 & 0 \end{array}\begin{array}{c} \\ 2 \\ \frac{1}{2} \\ \\ \end{array}$$

$$\begin{array}{c} \\ s_1 \\ \\ y_1 \\ W \end{array}\begin{array}{c} \begin{array}{cccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & W & \\ \end{array} \\ \left[\begin{array}{cccccc|c} 0 & \underline{\frac{9}{4}} & -5 & 1 & -\frac{1}{4} & 0 & \frac{3}{2} \\ 1 & -\frac{1}{4} & 2 & 0 & \frac{1}{4} & 0 & \frac{1}{2} \\ \hdashline 0 & -9 & 40 & 0 & 7 & 1 & 14 \end{array}\right] \begin{array}{c} \frac{2}{3} \\ \\ \\ \end{array} \end{array}$$

$$\begin{array}{c} \\ y_2 \\ \\ y_1 \\ W \end{array}\begin{array}{c} \begin{array}{cccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & W & \\ \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 1 & -\frac{20}{9} & \frac{4}{9} & -\frac{1}{9} & 0 & \frac{2}{3} \\ 1 & 0 & \frac{13}{9} & \frac{1}{9} & \frac{2}{9} & 0 & \frac{2}{3} \\ \hdashline 0 & 0 & 20 & 4 & 6 & 1 & 20 \end{array}\right] \end{array}$$

The minimum is $Z = 20$ when $x_1 = 4, x_2 = 6$.

11. The dual is: Minimize $W = 8y_1 + 12y_2$ subject to

$$y_1 + y_2 \geq 3,$$
$$2y_1 + 6y_2 \geq 8,$$
$$y_1, y_2 \geq 0.$$

$$\begin{array}{c} \begin{array}{ccccccc|c} y_1 & y_2 & s_1 & s_2 & t_1 & t_2 & U & \\ \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 1 & -1 & 0 & 1 & 0 & 0 & 3 \\ 2 & 6 & 0 & -1 & 0 & 1 & 0 & 8 \\ \hdashline 8 & 12 & 0 & 0 & M & M & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \\ t_1 \\ t_2 \\ U \end{array}\begin{array}{c} \begin{array}{ccccccc|c} y_1 & y_2 & s_1 & s_2 & t_1 & t_2 & U & \\ \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 1 & -1 & 0 & 1 & 0 & 0 & 3 \\ 2 & \underline{6} & 0 & -1 & 0 & 1 & 0 & 8 \\ \hdashline 8-3M & 12-7M & M & M & 0 & 0 & 1 & -11M \end{array}\right] \begin{array}{c} 3 \\ \frac{4}{3} \\ \\ \end{array} \end{array}$$

$$\begin{array}{c} \\ t_1 \\ y_2 \\ U \end{array}\begin{array}{c} \begin{array}{ccccccc|c} y_1 & y_2 & s_1 & s_2 & t_1 & t_2 & U & \\ \end{array} \\ \left[\begin{array}{ccccccc|c} \underline{\frac{2}{3}} & 0 & -1 & \frac{1}{6} & 1 & -\frac{1}{6} & 0 & \frac{5}{3} \\ \frac{1}{3} & 1 & 0 & -\frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{4}{3} \\ \hdashline 4-\frac{2}{3}M & 0 & M & 2-\frac{1}{6}M & 0 & -2+\frac{7}{6}M & 1 & -16-\frac{5}{3}M \end{array}\right] \begin{array}{c} \frac{5}{2} \\ 4 \\ \\ \end{array} \end{array}$$

$$\begin{array}{c} \\ y_1 \\ y_2 \\ U \end{array}\begin{array}{c} \begin{array}{ccccccc|c} y_1 & y_2 & s_1 & s_2 & t_1 & t_2 & U & \\ \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 0 & -\frac{3}{2} & \frac{1}{4} & \frac{3}{2} & -\frac{1}{4} & 0 & \frac{5}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{2} \\ \hdashline 0 & 0 & 6 & 1 & -6+M & -1+M & 1 & -26 \end{array}\right] \end{array}$$

The maximum is $Z = 26$ when $x_1 = 6, x_2 = 1$.

12. The dual is: Minimize $W = 12y_1 + 8y_2$ subject to

$$\begin{aligned} 3y_1 + y_2 &\geq 2, \\ y_1 + y_2 &\geq 6, \\ y_1, y_2 &\geq 0. \end{aligned}$$

$$\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccccccc|c} y_1 & y_2 & s_1 & s_2 & t_1 & t_2 & U & \\ 3 & 1 & -1 & 0 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & -1 & 0 & 1 & 0 & 6 \\ \hline 12 & 8 & 0 & 0 & M & M & 1 & 0 \end{array}$$

$$\begin{array}{c|ccccccc|c|c} & y_1 & y_2 & s_1 & s_2 & t_1 & t_2 & U & & \\ t_1 & \underline{3} & 1 & -1 & 0 & 1 & 0 & 0 & 2 & \frac{2}{3} \\ t_2 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 6 & 6 \\ \hline U & 12-4M & 8-2M & M & M & 0 & 0 & 1 & -8M & \end{array}$$

$$\begin{array}{c|ccccccc|c|c} & y_1 & y_2 & s_1 & s_2 & t_1 & t_2 & U & & \\ y_1 & 1 & \underline{\frac{1}{3}} & -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} & 2 \\ t_2 & 0 & \frac{2}{3} & \frac{1}{3} & -1 & -\frac{1}{3} & 1 & 0 & \frac{16}{3} & 8 \\ \hline U & 0 & 4-\frac{2}{3}M & 4-\frac{1}{3}M & M & -4+\frac{4}{3}M & 0 & 1 & -8-\frac{16}{3}M & \end{array}$$

$$\begin{array}{c|ccccccc|c|c} & y_1 & y_2 & s_1 & s_2 & t_1 & t_2 & U & & \\ y_2 & 3 & 1 & -1 & 0 & 1 & 0 & 0 & 2 & \\ t_2 & -2 & 0 & \underline{1} & -1 & -1 & 1 & 0 & 4 & 4 \\ \hline U & -12+2M & 0 & 8-M & M & -8+2M & 0 & 1 & -16-4M & \end{array}$$

$$\begin{array}{c|ccccccc|c} & y_1 & y_2 & s_1 & s_2 & t_1 & t_2 & U & \\ y_2 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 6 \\ s_1 & -2 & 0 & 1 & -1 & -1 & 1 & 0 & 4 \\ \hline U & 4 & 0 & 0 & 8 & M & -8+M & 1 & -48 \end{array}$$

The maximum is $Z = 48$ when $x_1 = 0, x_2 = 8$.

13. The first constraint can be written as $x_1 - x_2 \geq -1$. The dual is: Maximize $W = -y_1 + 3y_2$ subject to

$$\begin{aligned} y_1 + y_2 &\leq 6, \\ -y_1 + y_2 &\leq 4, \\ y_1, y_2 &\geq 0. \end{aligned}$$

$$\begin{array}{c|ccccc|c|c} & y_1 & y_2 & s_1 & s_2 & W & & \\ s_1 & 1 & 1 & 1 & 0 & 0 & 6 & 6 \\ s_2 & -1 & \underline{1} & 0 & 1 & 0 & 4 & 4 \\ \hline W & 1 & -3 & 0 & 0 & 1 & 0 & \end{array}$$

$$\begin{array}{c|ccccc|c|c} & y_1 & y_2 & s_1 & s_2 & W & & \\ s_1 & \underline{2} & 0 & 1 & -1 & 0 & 2 & 1 \\ y_2 & -1 & 1 & 0 & 1 & 0 & 4 & \\ \hline W & -2 & 0 & 0 & 3 & 1 & 12 & \end{array}$$

$$\begin{array}{c} \\ y_1 \\ y_2 \\ W \end{array}\begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & s_1 & s_2 & W \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 5 \\ \hdashline 0 & 0 & 1 & 2 & 1 & 14 \end{array}\right] \end{array}$$

The minimum is $Z = 14$ when $x_1 = 1, x_2 = 2$.

14. The first constraint can be written as $-2x_1 + x_2 + x_3 \geq -2.$ The dual is:
Maximize $W = -2y_1 + 4y_2$
subject to $-2y_1 - y_2 \leq 2$
$y_1 - y_2 \leq 1$
$y_1 + 2y_2 \leq 1$
$y_1,\ y_2 \geq 0$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ W \end{array}\begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & s_1 & s_2 & s_3 & W \end{array} \\ \left[\begin{array}{cccccc|c} -2 & -1 & 1 & 0 & 0 & 0 & 2 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 \\ 1 & \underline{2} & 0 & 0 & 1 & 0 & 1 \\ \hdashline 2 & -4 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}\begin{array}{c} \\ \\ \\ \frac{1}{2} \\ \\ \end{array}$$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ y_2 \\ W \end{array}\begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & s_1 & s_2 & s_3 & W \end{array} \\ \left[\begin{array}{cccccc|c} -\frac{3}{2} & 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{5}{2} \\ \frac{3}{2} & 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{3}{2} \\ \frac{1}{2} & 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \hdashline 4 & 0 & 0 & 0 & 2 & 1 & 2 \end{array}\right] \end{array}$$

The minimum is $Z = 2$ when $x_1 = 0,\ \ x_2 = 0,$ $x_3 = 2.$

15. Let x_1 = amount spent on newspaper advertising,
x_2 = amount spent on radio advertising.
We want to minimize $C = x_1 + x_2$ subject to
$40x_1 + 50x_2 \geq 80{,}000,$
$100x_1 + 25x_2 \geq 60{,}000,$
$x_1, x_2 \geq 0.$
The dual is: Maximize
$W = 80{,}000y_1 + 60{,}000y_2$ subject to
$40y_1 + 100y_2 \leq 1,$
$50y_1 + 25y_2 \leq 1,$
$y_1, y_2 \geq 0.$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ W \end{array}\begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & s_1 & s_2 & W \end{array} \\ \left[\begin{array}{ccccc|c} 40 & 100 & 1 & 0 & 0 & 1 \\ \underline{50} & 25 & 0 & 1 & 0 & 1 \\ \hdashline -80{,}000 & -60{,}000 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}\begin{array}{c} \\ \frac{1}{40} \\ \frac{1}{50} \\ \\ \end{array}$$

$$\begin{array}{c} \\ s_1 \\ y_1 \\ W \end{array}\begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & s_1 & s_2 & W \end{array} \\ \left[\begin{array}{ccccc|c} 0 & \underline{80} & 1 & -\frac{4}{5} & 0 & \frac{1}{5} \\ 1 & \frac{1}{2} & 0 & \frac{1}{50} & 0 & \frac{1}{50} \\ \hdashline 0 & -20{,}000 & 0 & 1600 & 1 & 1600 \end{array}\right] \end{array}\begin{array}{c} \\ \frac{1}{400} \\ \frac{1}{25} \\ \\ \end{array}$$

$$\begin{array}{c} \\ y_2 \\ y_1 \\ W \end{array}\begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & s_1 & s_2 & W \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & \frac{1}{80} & -\frac{1}{100} & 0 & \frac{1}{400} \\ 1 & 0 & -\frac{1}{160} & \frac{1}{40} & 0 & \frac{3}{160} \\ \hdashline 0 & 0 & 250 & 1400 & 1 & 1650 \end{array}\right] \end{array}$$

The firm should spend \$250 on newspaper advertising and \$1400 on radio advertising for a cost of \$1650.

16. Let x_1 = number of type A trucks rented,
x_2 = number of type B trucks rented.
We want to minimize $C = 0.40x_1 + 0.60x_2$
subject to
$2x_1 + 2x_2 \geq 12,$
$x_1 + 3x_2 \geq 12,$
$x_1, x_2 \geq 0.$
The dual is: Maximize $W = 12y_1 + 12y_2$ subject to
$2y_1 + y_2 \leq 0.40,$
$2y_1 + 3y_2 \leq 0.60,$
$y_1, y_2 \geq 0.$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ W \end{array}\begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & s_1 & s_2 & W \end{array} \\ \left[\begin{array}{ccccc|c} \underline{2} & 1 & 1 & 0 & 0 & \frac{2}{5} \\ 2 & 3 & 0 & 1 & 0 & \frac{3}{5} \\ \hdashline -12 & -12 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}\begin{array}{c} \\ \frac{1}{5} \\ \frac{3}{10} \\ \\ \end{array}$$

Here we choose y_1 as the entering variable.

$$\begin{array}{c} \\ y_1 \\ s_2 \\ W \end{array}\begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & s_1 & s_2 & W \end{array} \\ \left[\begin{array}{ccccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{5} \\ 0 & \underline{2} & -1 & 1 & 0 & \frac{1}{5} \\ \hdashline 0 & -6 & 6 & 0 & 1 & \frac{12}{5} \end{array}\right] \end{array}\begin{array}{c} \\ \frac{2}{5} \\ \frac{1}{10} \\ \\ \end{array}$$

$$\begin{array}{c} \\ y_1 \\ y_2 \\ W \end{array}\begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & s_1 & s_2 & W \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{3}{20} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{10} \\ \hdashline 0 & 0 & 3 & 3 & 1 & 3 \end{array}\right] \end{array}$$

Thus the maximum value of *W*, and hence the minimum value of *C,* is 3.

If we choose y_2 as the entering variable, then

$$\begin{array}{c} \\ s_1 \\ s_2 \\ W \end{array}\begin{array}{c} \begin{array}{ccccc|c} y_1 & y_2 & s_1 & s_2 & W & \end{array} \\ \left[\begin{array}{ccccc|c} 2 & 1 & 1 & 0 & 0 & \frac{2}{5} \\ 2 & \underline{3} & 0 & 1 & 0 & \frac{3}{5} \\ \hline -12 & -12 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}\begin{array}{c} \\ \frac{2}{5} \\ \frac{1}{5} \\ \\ \end{array}$$

$$\begin{array}{c} \\ s_1 \\ y_2 \\ W \end{array}\begin{array}{c} \begin{array}{ccccc|c} y_1 & y_2 & s_1 & s_2 & W & \end{array} \\ \left[\begin{array}{ccccc|c} \underline{\frac{4}{3}} & 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{5} \\ \frac{2}{3} & 1 & 0 & \frac{1}{3} & 0 & \frac{1}{5} \\ \hline -4 & 0 & 0 & 4 & 1 & \frac{12}{5} \end{array}\right] \end{array}\begin{array}{c} \\ \frac{3}{20} \\ \frac{3}{10} \\ \\ \end{array}$$

$$\begin{array}{c} \\ y_1 \\ y_2 \\ W \end{array}\begin{array}{c} \begin{array}{ccccc|c} y_1 & y_2 & s_1 & s_2 & W & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{3}{20} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{10} \\ \hline 0 & 0 & 3 & 3 & 1 & 3 \end{array}\right] \end{array}$$

The minimum total cost per mile is \$3.

17. Let y_1 = number of shipping clerk apprentices,
y_2 = number of shipping clerks,
y_3 = number of semiskilled workers,
y_4 = number of skilled workers.
We want to minimize $W = 6y_1 + 9y_2 + 8y_3 + 14y_4$ subject to

$$\begin{aligned} y_1 + y_2 &\geq 60, \\ -2y_1 + y_2 &\geq 0, \\ y_3 + y_4 &\geq 90, \\ y_3 - 2y_4 &\geq 0, \\ y_1, y_2, y_3, y_4 &\geq 0. \end{aligned}$$

The dual is: Maximize $Z = 60x_1 + 0x_2 + 90x_3 + 0x_4$ subject to

$$\begin{aligned} x_1 - 2x_2 &\leq 6, \\ x_1 + x_2 &\leq 9, \\ x_3 + x_4 &\leq 8, \\ x_3 - 2x_4 &\leq 14, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ Z \end{array}\begin{array}{c} \begin{array}{ccccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & s_4 & Z & \end{array} \\ \left[\begin{array}{ccccccccc|c} 1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 6 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 9 \\ 0 & 0 & \underline{1} & 1 & 0 & 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 & 14 \\ \hline -60 & 0 & -90 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}\begin{array}{c} \\ \\ \\ 8 \\ 14 \\ \\ \end{array}$$

	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	Z		
s_1	<u>1</u>	–2	0	0	1	0	0	0	0	6	6
s_2	1	1	0	0	0	1	0	0	0	9	9
x_3	0	0	1	1	0	0	1	0	0	8	
s_4	0	0	0	–3	0	0	–1	1	0	6	
Z	–60	0	0	90	0	0	90	0	1	720	

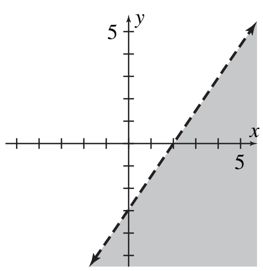
	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	Z		
x_1	1	–2	0	0	1	0	0	0	0	6	
s_2	0	<u>3</u>	0	0	–1	1	0	0	0	3	1
x_3	0	0	1	1	0	0	1	0	0	8	
s_4	0	0	0	–3	0	0	–1	1	0	6	
Z	0	–120	0	90	60	0	90	0	1	1080	

	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	Z	
x_1	1	0	0	0	$\frac{1}{3}$	$\frac{2}{3}$	0	0	0	8
x_2	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	0	0	1
x_3	0	0	1	1	0	0	1	0	0	8
s_4	0	0	0	–3	0	0	–1	1	0	6
Z	0	0	0	90	20	40	90	0	1	1200

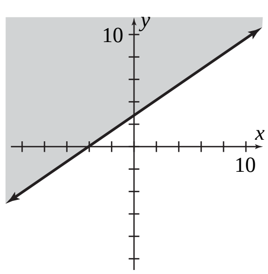
The company should employ 20 shipping clerk apprentices, 40 shipping clerks, 90 semiskilled workers, and 0 skilled workers for a total hourly wage of $1200.

Chapter 7 Review Problems

1.



2.



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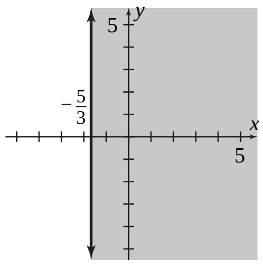

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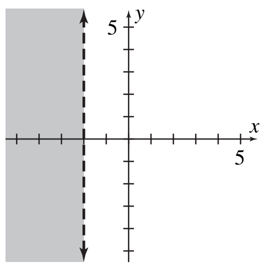

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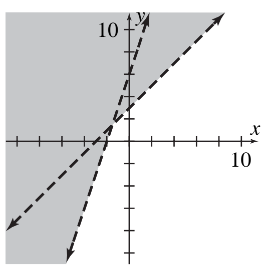

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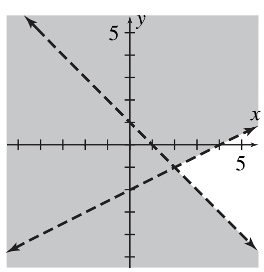

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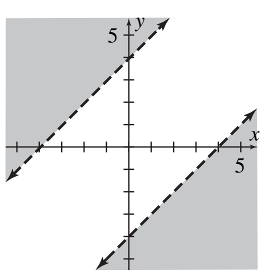

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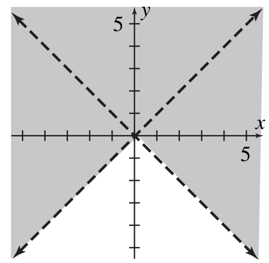
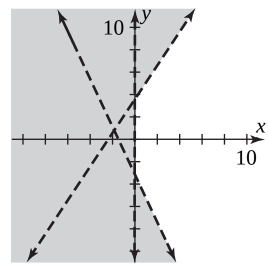
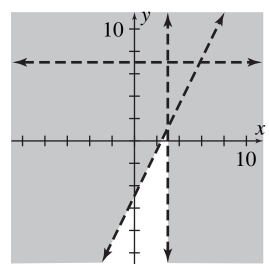


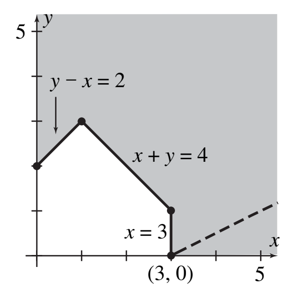
9.



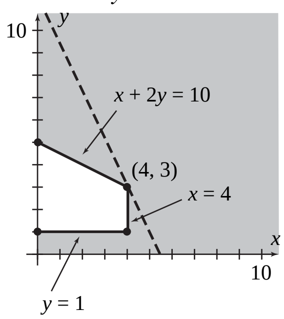
10.



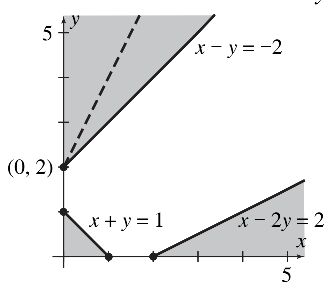
11. Feasible region follows. Corner points are (0, 0), (0, 2), (1, 3), (3, 1), (3, 0). Z is maximized at (3, 0) where its value is 3.
Thus $Z = 3$ when $x = 3$ and $y = 0$.



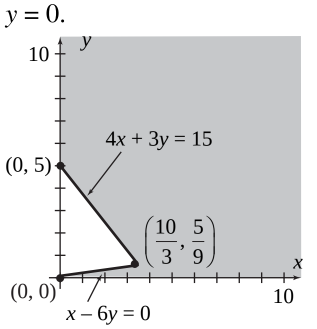
12. Feasible region follows. Corner points are (0, 1), (0, 5), (4, 3), and (4, 1). Z is maximized at (4, 3) where its value is 22. Thus $Z = 22$ when $x = 4$ and $y = 3$.



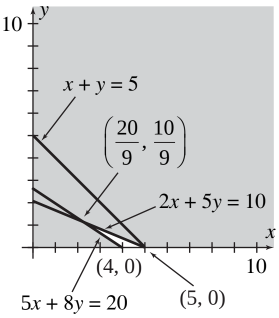
13. Feasible region is unbounded. Z is minimized at the corner point (0, 2) where its value is –2. Thus $Z = –2$ when $x = 0$ and $y = 2$.



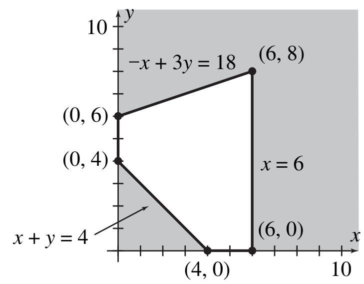
14. Feasible region follows. Corner points are (0, 0), $\left(\frac{10}{3}, \frac{5}{9}\right)$, and (0, 5). Z is minimized at (0, 0) where its value is 0. Thus $Z = 0$ when $x = 0$ and $y = 0$.



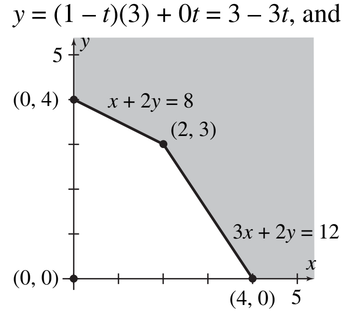
15. Feasible region follows. Corner points are $\left(\frac{20}{9}, \frac{10}{9}\right)$, (5, 0), and (4, 0). Z is minimized at $\left(\frac{20}{9}, \frac{10}{9}\right)$ where its value is $\frac{70}{9}$. Thus $Z = \frac{70}{9}$ when $x = \frac{20}{9}$ and $y = \frac{10}{9}$.



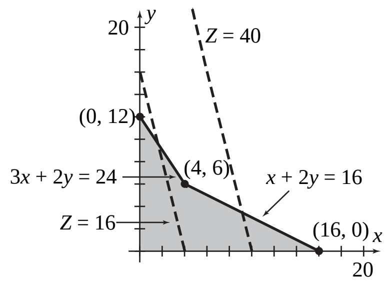
16. Feasible region follows. Corner points are (0, 4), (0, 6), (6, 8), (6, 0), and (4, 0). Z is minimized at (0, 4) and (4, 0) where its value is 8. Thus Z is minimized at all points on the line segment joining (0, 4) and (4, 0). The solution is $Z = 8$ when $x = (1 - t)(0) + 4t = 4t$, $y = (1 - t)(4) + 0t = 4 - 4t$, and $0 \le t \le 1$.



17. Feasible region follows. Corner points are (0, 0), (0, 4), (2, 3), and (4, 0). Z is maximized at (2, 3) and (4, 0) where its value is 36. Thus Z is maximized at all points on the line segment joining (2, 3) and (4, 0). The solution is $Z = 36$ when $x = (1 - t)(2) + 4t = 2 + 2t$, $y = (1 - t)(3) + 0t = 3 - 3t$, and $0 \le t \le 1$.



18. Feasible region is unbounded. The family of lines given by $Z = 4x + y$ has members having arbitrarily large values of Z and that also intersect the feasible region. Thus no optimum solution exists.



19.

$$\begin{array}{c} \\ s_1 \\ s_2 \\ Z \end{array}\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & Z & & \\ 1 & \underline{6} & 1 & 0 & 0 & & 12 \\ 1 & 2 & 0 & 1 & 0 & & 8 \\ \hline -4 & -5 & 0 & 0 & 1 & & 0 \end{array}\begin{array}{c} \\ 2 \\ 4 \\ \\ \end{array}$$

$$\begin{array}{c} \\ x_2 \\ s_2 \\ Z \end{array}\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & Z & \\ \frac{1}{6} & 1 & \frac{1}{6} & 0 & 0 & 2 \\ \underline{\frac{2}{3}} & 0 & -\frac{1}{3} & 1 & 0 & 4 \\ \hline -\frac{19}{6} & 0 & \frac{5}{6} & 0 & 1 & 10 \end{array}\begin{array}{c} \\ 12 \\ 6 \\ \\ \end{array}$$

$$\begin{array}{c} \\ x_2 \\ x_1 \\ Z \end{array}\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & Z & \\ 0 & 1 & \underline{\frac{1}{4}} & -\frac{1}{4} & 0 & 1 \\ 1 & 0 & -\frac{1}{2} & \frac{3}{2} & 0 & 6 \\ \hline 0 & 0 & -\frac{3}{4} & \frac{19}{4} & 1 & 29 \end{array}\begin{array}{c} \\ 4 \\ \\ \\ \end{array}$$

$$\begin{array}{c} \\ s_1 \\ x_1 \\ Z \end{array}\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & Z & \\ 0 & 4 & 1 & -1 & 0 & 4 \\ 1 & 2 & 0 & 1 & 0 & 8 \\ \hline 0 & 3 & 0 & 4 & 1 & 32 \end{array}$$

Thus $Z = 32$ when $x_1 = 8$ and $x_2 = 0$.

20.

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ Z \end{array}\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & Z & \\ 2 & 3 & 1 & 0 & 0 & 0 & 18 \\ 4 & 3 & 0 & 1 & 0 & 0 & 24 \\ 0 & \underline{1} & 0 & 0 & 1 & 0 & 5 \\ \hline -18 & -20 & 0 & 0 & 0 & 1 & 0 \end{array}\begin{array}{c} \\ 6 \\ 8 \\ 5 \\ \\ \end{array}$$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ x_2 \\ Z \end{array}\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & Z & \\ \underline{2} & 0 & 1 & 0 & -3 & 0 & 3 \\ 4 & 0 & 0 & 1 & -3 & 0 & 9 \\ 0 & 1 & 0 & 0 & 1 & 0 & 5 \\ \hline -18 & 0 & 0 & 0 & 20 & 1 & 100 \end{array}\begin{array}{c} \\ \frac{3}{2} \\ \frac{9}{4} \\ \\ \\ \end{array}$$

$$\begin{array}{c} \\ x_1 \\ s_2 \\ x_2 \\ Z \end{array}\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & Z & \\ 1 & 0 & \frac{1}{2} & 0 & -\frac{3}{2} & 0 & \frac{3}{2} \\ 0 & 0 & -2 & 1 & \underline{3} & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & 0 & 5 \\ \hline 0 & 0 & 9 & 0 & -7 & 1 & 127 \end{array}\begin{array}{c} \\ \\ 1 \\ 5 \\ \\ \end{array}$$

$$\begin{array}{c} \\ x_1 \\ s_3 \\ x_2 \\ Z \end{array}\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & Z & \\ 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 3 \\ 0 & 0 & -\frac{2}{3} & \frac{1}{3} & 1 & 0 & 1 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 4 \\ \hline 0 & 0 & \frac{13}{3} & \frac{7}{3} & 0 & 1 & 134 \end{array}$$

Thus $Z = 134$ when $x_1 = 3$ and $x_2 = 4$.

21.
$$\begin{array}{c} \\ \\ \end{array}\begin{array}{c} x_1 \quad x_2 \quad x_3 \quad s_1 \quad t_1 \quad W \\ \left[\begin{array}{cccccc|c} 1 & 2 & 3 & -1 & 1 & 0 & 5 \\ \hline 3 & 2 & 1 & 0 & M & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c|cccccc|c|c} & x_1 & x_2 & x_3 & s_1 & t_1 & W & & \\ \hline t_1 & 1 & 2 & \underline{3} & -1 & 1 & 0 & 5 & \frac{5}{3} \\ W & 3-M & 2-2M & 1-3M & M & 0 & 1 & -5M & \end{array}$$

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & x_3 & s_1 & t_1 & W & \\ \hline x_3 & \frac{1}{3} & \frac{2}{3} & 1 & -\frac{1}{3} & \frac{1}{3} & 0 & \frac{5}{3} \\ W & \frac{8}{3} & \frac{4}{3} & 0 & \frac{1}{3} & -\frac{1}{3}+M & 1 & -\frac{5}{3} \end{array}$$

Thus $Z = \frac{5}{3}$ when $x_1 = 0$, $x_2 = 0$, and $x_3 = \frac{5}{3}$.

22.
$$\begin{array}{c} x_1 \quad x_2 \quad s_1 \quad s_2 \quad t_1 \quad t_2 \quad W \\ \left[\begin{array}{ccccccc|c} 3 & 5 & -1 & 0 & 1 & 0 & 0 & 20 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 & 5 \\ \hline 1 & 2 & 0 & 0 & M & M & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c|ccccccc|c|c} & x_1 & x_2 & s_1 & s_2 & t_1 & t_2 & W & & \\ \hline t_1 & 3 & \underline{5} & -1 & 0 & 1 & 0 & 0 & 20 & 4 \\ t_2 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 5 & \\ W & 1-4M & 2-5M & M & M & 0 & 0 & 1 & -25M & \end{array}$$

$$\begin{array}{c|ccccccc|c|c} & x_1 & x_2 & s_1 & s_2 & t_1 & t_2 & W & & \\ \hline x_2 & \frac{3}{5} & 1 & -\frac{1}{5} & 0 & \frac{1}{5} & 0 & 0 & 4 & \frac{20}{3} \\ t_2 & \underline{1} & 0 & 0 & -1 & 0 & 1 & 0 & 5 & 5 \\ W & -\frac{1}{5}-M & 0 & \frac{2}{5} & M & -\frac{2}{5}+M & 0 & 1 & -8-5M & \end{array}$$

$$\begin{array}{c|ccccccc|c|c} & x_1 & x_2 & s_1 & s_2 & t_1 & t_2 & W & & \\ \hline x_2 & 0 & 1 & -\frac{1}{5} & \underline{\frac{3}{5}} & \frac{1}{5} & -\frac{3}{5} & 0 & 1 & \frac{5}{3} \\ x_1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 5 & \\ W & 0 & 0 & \frac{2}{5} & -\frac{1}{5} & -\frac{2}{5}+M & \frac{1}{5}+M & 1 & -7 & \end{array}$$

$$\begin{array}{c|ccccc|c} & x_1 & x_2 & s_1 & s_2 & Z & \\ \hline s_2 & 0 & \frac{5}{3} & -\frac{1}{3} & 1 & 0 & \frac{5}{3} \\ x_1 & 1 & \frac{5}{3} & -\frac{1}{3} & 0 & 0 & \frac{20}{3} \\ Z & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 1 & -\frac{20}{3} \end{array}$$

Thus $Z = \frac{20}{3}$ when $x_1 = \frac{20}{3}$, $x_2 = 0$.

23.
$$\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & t_2 & W & \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 & 12 \\ 1 & 1 & 0 & -1 & 0 & 1 & 0 & 5 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 10 \\ \hline -1 & -2 & 0 & 0 & 0 & M & 1 & 0 \end{array}$$

$$\begin{array}{c|ccccccc|c|c} & x_1 & x_2 & s_1 & s_2 & s_3 & t_2 & W & & \\ \hline s_1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 12 & 12 \\ t_2 & 1 & \underline{1} & 0 & -1 & 0 & 1 & 0 & 5 & 5 \\ s_3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 10 & \\ \hline W & -1-M & -2-M & 0 & M & 0 & 0 & 1 & -5M & \end{array}$$

$$\begin{array}{c|ccccccc|c|c} & x_1 & x_2 & s_1 & s_2 & s_3 & t_2 & W & & \\ \hline s_1 & 0 & 0 & 1 & \underline{1} & 0 & -1 & 0 & 7 & 7 \\ x_2 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 5 & \\ s_3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 10 & \\ \hline W & 1 & 0 & 0 & -2 & 0 & 2+M & 1 & 10 & \end{array}$$

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & Z & \\ \hline s_2 & 0 & 0 & 1 & 1 & 0 & 0 & 7 \\ x_2 & 1 & 1 & 1 & 0 & 0 & 0 & 12 \\ s_3 & 1 & 0 & 0 & 0 & 1 & 0 & 10 \\ \hline Z & 1 & 0 & 2 & 0 & 0 & 1 & 24 \end{array}$$

Thus $Z = 24$ when $x_1 = 0$ and $x_2 = 12$.

24.
$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & t_2 & W & \\ \hline 1 & 2 & 1 & 0 & 0 & 0 & 6 \\ 1 & 1 & 0 & -1 & 1 & 0 & 1 \\ \hline 2 & 1 & 0 & 0 & M & 1 & 0 \end{array}$$

$$\begin{array}{c|cccccc|c|c} & x_1 & x_2 & s_1 & s_2 & t_2 & W & & \\ \hline s_1 & 1 & 2 & 1 & 0 & 0 & 0 & 6 & 3 \\ t_2 & 1 & \underline{1} & 0 & -1 & 1 & 0 & 1 & 1 \\ \hline W & 2-M & 1-M & 0 & M & 0 & 1 & -M & \end{array}$$

$$\begin{array}{c|cccccc|c} s_1 & -1 & 0 & 1 & 2 & -2 & 0 & 4 \\ x_2 & 1 & 1 & 0 & -1 & 1 & 0 & 1 \\ \hline W & 1 & 0 & 0 & 1 & -1+M & 1 & -1 \end{array}$$

Thus $Z = 1$ when $x_1 = 0$ and $x_2 = 1$.

25. We write the first constraint as $-x_1 + x_2 + x_3 \geq 1$.

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & t_1 & t_2 & W & \\ \hline -1 & 1 & 1 & -1 & 1 & 0 & 0 & 1 \\ 6 & 3 & 2 & 0 & 0 & 1 & 0 & 12 \\ \hline 1 & 2 & 1 & 0 & M & M & 1 & 0 \end{array}$$

$$\begin{array}{c} \\ t_1 \\ t_2 \\ W \end{array}\begin{array}{c}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & t_1 & t_2 & W & \end{array}\\ \left[\begin{array}{ccccccc|c} -1 & 1 & 1 & -1 & 1 & 0 & 0 & 1 \\ \underline{6} & 3 & 2 & 0 & 0 & 1 & 0 & 12 \\ \hline 1-5M & 2-4M & 1-3M & M & 0 & 0 & 1 & -13M \end{array}\right]\end{array}\begin{array}{c} \\ \\ 2 \\ \\ \end{array}$$

$$\begin{array}{c} \\ t_1 \\ x_1 \\ W \end{array}\begin{array}{c}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & t_1 & t_2 & W & \end{array}\\ \left[\begin{array}{ccccccc|c} 0 & \underline{\frac{3}{2}} & \frac{4}{3} & -1 & 1 & \frac{1}{6} & 0 & 3 \\ 1 & \frac{1}{2} & \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 & 2 \\ \hline 0 & \frac{3}{2}-\frac{3}{2}M & \frac{2}{3}-\frac{4}{3}M & M & 0 & -\frac{1}{6}+\frac{5}{6}M & 1 & -2-3M \end{array}\right]\end{array}\begin{array}{c} \\ 2 \\ 4 \\ \\ \end{array}$$

$$\begin{array}{c} \\ x_2 \\ x_1 \\ W \end{array}\begin{array}{c}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & t_1 & t_2 & W & \end{array}\\ \left[\begin{array}{ccccccc|c} 0 & 1 & \underline{\frac{8}{9}} & -\frac{2}{3} & \frac{2}{3} & \frac{1}{9} & 0 & 2 \\ 1 & 0 & -\frac{1}{9} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{9} & 0 & 1 \\ \hline 0 & 0 & -\frac{2}{3} & 1 & -1+M & -\frac{1}{3}+M & 1 & -5 \end{array}\right]\end{array}\begin{array}{c} \\ \frac{9}{4} \\ \\ \\ \end{array}$$

$$\begin{array}{c} \\ x_3 \\ x_1 \\ -Z \end{array}\begin{array}{c}\begin{array}{ccccc|c} x_1 & x_2 & x_3 & s_1 & -Z & \end{array}\\ \left[\begin{array}{ccccc|c} 0 & \frac{9}{8} & 1 & -\frac{3}{4} & 0 & \frac{9}{4} \\ 1 & \frac{1}{8} & 0 & \frac{1}{4} & 0 & \frac{5}{4} \\ \hline 0 & \frac{3}{4} & 0 & \frac{1}{2} & 1 & -\frac{7}{2} \end{array}\right]\end{array}$$

Thus $Z = \frac{7}{2}$ when $x_1 = \frac{5}{4}$, $x_2 = 0$, and $x_3 = \frac{9}{4}$.

26.

$$\begin{array}{c}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & t_1 & W & \end{array}\\ \left[\begin{array}{ccccccc|c} 1 & 1 & 3 & -1 & 0 & 1 & 0 & 5 \\ 2 & 1 & 4 & 0 & 1 & 0 & 0 & 5 \\ \hline -2 & -3 & -5 & 0 & 0 & M & 1 & 0 \end{array}\right]\end{array}$$

$$\begin{array}{c} \\ t_1 \\ s_2 \\ W \end{array}\begin{array}{c}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & t_1 & W & \end{array}\\ \left[\begin{array}{ccccccc|c} 1 & 1 & 3 & -1 & 0 & 1 & 0 & 5 \\ 2 & 1 & \underline{4} & 0 & 1 & 0 & 0 & 5 \\ \hline -2-M & -3-M & -5-3M & M & 0 & 0 & 1 & -5M \end{array}\right]\end{array}\begin{array}{c} \\ \frac{5}{3} \\ \frac{5}{4} \\ \\ \end{array}$$

$$\begin{array}{c} \\ t_1 \\ x_3 \\ W \end{array}\begin{array}{c}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & t_1 & W & \end{array}\\ \left[\begin{array}{ccccccc|c} -\frac{1}{2} & \underline{\frac{1}{4}} & 0 & -1 & -\frac{3}{4} & 1 & 0 & \frac{5}{4} \\ \frac{1}{2} & \frac{1}{4} & 1 & 0 & \frac{1}{4} & 0 & 0 & \frac{5}{4} \\ \hline \frac{1}{2}+\frac{1}{2}M & -\frac{7}{4}-\frac{1}{4}M & 0 & M & \frac{5}{4}+\frac{3}{4}M & 0 & 1 & \frac{25}{4}-\frac{5}{4}M \end{array}\right]\end{array}\begin{array}{c} \\ 5 \\ 5 \\ \\ \end{array}$$

We choose t_1 as the departing variable.

$$\begin{array}{c|ccccccc|c|c} & x_1 & x_2 & x_3 & s_1 & s_2 & t_1 & W & & \\ \hline x_2 & -2 & 1 & 0 & -4 & -3 & 4 & 0 & 5 & \\ x_3 & 1 & 0 & 1 & \underline{1} & 1 & -1 & 0 & 0 & 0 \\ \hline W & -3 & 0 & 0 & -7 & -4 & 7+M & 1 & 15 & \end{array}$$

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & Z & \\ \hline x_2 & 2 & 1 & 4 & 0 & 1 & 0 & 5 \\ s_1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ \hline Z & 4 & 0 & 7 & 0 & 3 & 1 & 15 \end{array}$$

Thus $Z = 15$ when $x_1 = 0$, $x_2 = 5$, and $x_3 = 0$. Note that choosing x_3 as the departing variable results in the same solution.

27.

$$\begin{array}{c|cccccc|c|c} & x_1 & x_2 & x_3 & s_1 & s_2 & Z & & \\ \hline s_1 & 4 & -1 & 0 & 1 & 0 & 0 & 2 & \\ s_2 & -8 & \underline{2} & 5 & 0 & 1 & 0 & 2 & 1 \\ \hline Z & -1 & -4 & -2 & 0 & 0 & 1 & 0 & \end{array}$$

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & Z & \\ \hline s_1 & 0 & 0 & \frac{5}{2} & 1 & \frac{1}{2} & 0 & 3 \\ x_2 & -4 & 1 & \frac{5}{2} & 0 & \frac{1}{2} & 0 & 1 \\ \hline Z & -17 & 0 & 8 & 0 & 2 & 1 & 4 \end{array}$$

For the last table, x_1 is the entering variable. Since no quotients exist, the problem has an unbounded solution. That is, no optimum solution (unbounded).

28.

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & t_2 & W & \\ \hline 1 & 1 & 2 & 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 & M & 1 & 0 \end{array}$$

$$\begin{array}{c|ccccccc|c|c} & x_1 & x_2 & x_3 & s_1 & s_2 & t_2 & W & & \\ \hline s_1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 & 4 & 2 \\ t_2 & 0 & 0 & \underline{1} & 0 & -1 & 1 & 0 & 1 & 1 \\ \hline W & 1 & 1 & -M & 0 & M & 0 & 1 & -M & \end{array}$$

$$\begin{array}{c|ccccccc|c|c} & x_1 & x_2 & x_3 & s_1 & s_2 & t_2 & W & & \\ \hline s_1 & 1 & 1 & 0 & 1 & \underline{2} & -2 & 0 & 2 & 1 \\ x_3 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 1 & \\ \hline W & 1 & 1 & 0 & 0 & 0 & M & 1 & 0 & \end{array}$$

The minimum value of Z is 0 for $x_1 = 0$, $x_2 = 0$, and $x_3 = 1$.

Since s_2 is nonbasic for the last table and its indicator is 0, there may be multiple optimum solutions. Treating s_2 as an entering variable, deleting the t_2 -column, changing W to $-Z$, and continuing, we have

$$\begin{array}{c|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & -Z & \\ \hline s_2 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 1 & 0 & 1 \\ x_3 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 2 \\ \hline -Z & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Here $Z = 0$ when $x_1 = 0$, $x_2 = 0$, and $x_3 = 2$. Thus multiple optimum solutions exist. Hence Z is minimum when

$x_1 = (1-t)(0) + 0t = 0,$
$x_2 = (1-t)(0) + 0t = 0,$
$x_3 = (1-t)(1) + 2t = 1 + t,$

and $0 \le t \le 1$. For the last table, s_1 is nonbasic and its indicator is 0. If we continue the process for determining other optimum solutions, we return to a table corresponding to the third table.

29. The dual is: Maximize $W = 35y_1 + 25y_2$ subject to

$y_1 + y_2 \le 2,$
$2y_1 + y_2 \le 7,$
$3y_1 + y_2 \le 8,$
$y_1, y_2 \ge 0.$

$$\begin{array}{c|cccccc|c|c} & y_1 & y_2 & s_1 & s_2 & s_3 & W & & \\ \hline s_1 & \underline{1} & 1 & 1 & 0 & 0 & 0 & 2 & 2 \\ s_2 & 2 & 1 & 0 & 1 & 0 & 0 & 7 & \frac{7}{2} \\ s_3 & 3 & 1 & 0 & 0 & 1 & 0 & 8 & \frac{8}{3} \\ \hline W & -35 & -25 & 0 & 0 & 0 & 1 & 0 & \end{array}$$

$$\begin{array}{c|cccccc|c} & y_1 & y_2 & s_1 & s_2 & s_3 & W & \\ \hline y_1 & 1 & 1 & 1 & 0 & 0 & 0 & 2 \\ s_2 & 0 & -1 & -2 & 1 & 0 & 0 & 3 \\ s_3 & 0 & -2 & -3 & 0 & 1 & 0 & 2 \\ \hline W & 0 & 10 & 35 & 0 & 0 & 1 & 70 \end{array}$$

Thus $Z = 70$ when $x_1 = 35$, $x_2 = 0$, and $x_3 = 0$.

30. We write the third constraint as $-4x_1 - x_2 \le -2$. The dual is: Minimize $W = 3y_1 + 4y_2 - 2y_3$ subject to

$$\begin{aligned} y_1 + y_2 - 4y_3 &\ge 1, \\ -y_1 + 2y_2 - y_3 &\ge -2, \\ y_1, y_2, y_3 &\ge 0. \end{aligned}$$

We write the second constraint of the dual as $y_1 - 2y_2 + y_3 \le 2$.

$$\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccccccc} y_1 & y_2 & y_3 & s_1 & s_2 & t_1 & U & \\ \end{array}$$
$$\left[\begin{array}{ccccccc|c} 1 & 1 & -4 & -1 & 0 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 & 1 & 0 & 0 & 2 \\ \hline 3 & 4 & -2 & 0 & 0 & M & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ t_1 \\ s_2 \\ U \end{array}\begin{array}{c} \begin{array}{cccccccc} y_1 & y_2 & y_3 & s_1 & s_2 & t_1 & U & \end{array} \\ \left[\begin{array}{ccccccc|c} \underline{1} & 1 & -4 & -1 & 0 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 & 1 & 0 & 0 & 2 \\ \hline 3-M & 4-M & -2+4M & M & 0 & 0 & 1 & -M \end{array}\right] \end{array}\begin{array}{c} \\ 1 \\ 2 \\ \\ \end{array}$$

$$\begin{array}{c} \\ y_1 \\ s_2 \\ U \end{array}\begin{array}{c} \begin{array}{cccccccc} y_1 & y_2 & y_3 & s_1 & s_2 & t_1 & U & \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 1 & -4 & -1 & 0 & 1 & 0 & 1 \\ 0 & -3 & 5 & 1 & 1 & -1 & 0 & 1 \\ \hline 0 & 1 & 10 & 3 & 0 & -3+M & 1 & -3 \end{array}\right] \end{array}$$

Thus $Z = 3$ when $x_1 = 3$ and $x_2 = 0$.

31. Let x, y, and z denote the numbers of units of X, Y, and Z produced weekly, respectively. If P is the total profit obtained, we want to maximize

$P = 10x + 15y + 22z$ subject to

$$\begin{aligned} x + 2y + 2z &\le 40, \\ x + y + 2z &\le 34, \\ x, y, z &\ge 0. \end{aligned}$$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ P \end{array}\begin{array}{c} \begin{array}{ccccccc} x & y & z & s_1 & s_2 & P & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 2 & 2 & 1 & 0 & 0 & 40 \\ 1 & 1 & \underline{2} & 0 & 1 & 0 & 34 \\ \hline -10 & -15 & -22 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}\begin{array}{c} \\ 20 \\ 17 \\ \\ \end{array}$$

$$\begin{array}{c} \\ s_1 \\ z \\ P \end{array}\begin{array}{c} \begin{array}{ccccccc} x & y & z & s_1 & s_2 & P & \end{array} \\ \left[\begin{array}{cccccc|c} 0 & \underline{1} & 0 & 1 & -1 & 0 & 6 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & 0 & 17 \\ \hline 1 & -4 & 0 & 0 & 11 & 1 & 374 \end{array}\right] \end{array}\begin{array}{c} \\ 6 \\ 34 \\ \\ \end{array}$$

$$\begin{array}{c} y \\ z \\ P \end{array}\left[\begin{array}{cccccc|c} 0 & 1 & 0 & 1 & -1 & 0 & 6 \\ \frac{1}{2} & 0 & 1 & -\frac{1}{2} & 1 & 0 & 14 \\ \hline 1 & 0 & 0 & 4 & 7 & 1 & 398 \end{array}\right]$$

Thus 0 units of X, 6 units of Y, and 14 units of Z give a maximum profit of \$398.

32. We want to maximize $P = 10x + 15y + 22z$ subject to

$$\begin{aligned} x+2y+2z &\le 40, \\ x+y+2z &\le 34, \\ x+y+z &\ge 24, \\ x, y, z &\ge 0. \end{aligned}$$

$$\begin{array}{c} \begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & t_3 & W & \\ \end{array} \\ \left[\begin{array}{cccccccc|c} 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 40 \\ 1 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 34 \\ 1 & 1 & 1 & 0 & 0 & -1 & 1 & 0 & 24 \\ \hline -10 & -15 & -22 & 0 & 0 & 0 & M & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ t_3 \\ W \end{array} \begin{array}{c} \begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & t_3 & W & \end{array} \\ \left[\begin{array}{cccccccc|c} 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 40 \\ 1 & 1 & \underline{2} & 0 & 1 & 0 & 0 & 0 & 34 \\ 1 & 1 & 1 & 0 & 0 & -1 & 1 & 0 & 24 \\ \hline -10-M & -15-M & -22-M & 0 & 0 & M & 0 & 1 & -24M \end{array}\right] \end{array} \begin{array}{c} \\ 20 \\ 17 \\ 24 \\ \end{array}$$

$$\begin{array}{c} \\ s_1 \\ z \\ t_3 \\ W \end{array} \begin{array}{c} \begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & t_3 & W & \end{array} \\ \left[\begin{array}{cccccccc|c} 0 & \underline{1} & 0 & 1 & -1 & 0 & 0 & 0 & 6 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 17 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & -1 & 1 & 0 & 7 \\ \hline 1-\frac{M}{2} & -4-\frac{M}{2} & 0 & 0 & 11+\frac{M}{2} & M & 0 & 1 & 374-7M \end{array}\right] \end{array} \begin{array}{c} \\ 6 \\ 34 \\ 14 \\ \end{array}$$

$$\begin{array}{c} \\ y \\ z \\ t_3 \\ W \end{array} \begin{array}{c} \begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & t_3 & W & \end{array} \\ \left[\begin{array}{cccccccc|c} 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 6 \\ \frac{1}{2} & 0 & 1 & -\frac{1}{2} & 1 & 0 & 0 & 0 & 14 \\ \underline{\frac{1}{2}} & 0 & 0 & -\frac{1}{2} & 0 & -1 & 1 & 0 & 4 \\ \hline 1-\frac{M}{2} & 0 & 0 & 4+\frac{M}{2} & 7 & M & 0 & 1 & 398-4M \end{array}\right] \end{array} \begin{array}{c} \\ \\ 28 \\ 8 \\ \end{array}$$

$$\begin{array}{c} \\ y \\ z \\ x \\ W \end{array} \begin{array}{c} \begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & t_3 & W & \end{array} \\ \left[\begin{array}{cccccccc|c} 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 & 0 & 10 \\ 1 & 0 & 0 & -1 & 0 & -2 & 2 & 0 & 8 \\ \hline 0 & 0 & 0 & 5 & 7 & 2 & -2+M & 1 & 390 \end{array}\right] \end{array}$$

The company should produce 8 units of X, 6 units of Y, 10 units of Z, for a profit of \$390.

33. Let x_{AC}, x_{AD}, x_{BC}, and x_{BD} denote the amounts (in hundreds of thousands of gallons) transported from A to C, A to D, B to C, and B to D, respectively. If C is the total transportation cost in thousands of dollars, we want to minimize $C = x_{AC} + 2x_{AD} + 2x_{BC} + 4x_{BD}$ subject to

$$\begin{aligned} x_{AC} + x_{AD} &\le 6, \\ x_{BC} + x_{BD} &\le 6, \\ x_{AC} + x_{BC} &= 5, \\ x_{AD} + x_{BD} &= 5, \\ x_{AC}, x_{AD}, x_{BC}, x_{BD} &\ge 0. \end{aligned}$$

$$\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{c} x_{AC} & x_{AD} & x_{BC} & x_{BD} & s_1 & s_2 & t_3 & t_4 & W & \end{array}$$

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 6 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 5 \\ \hline 1 & 2 & 2 & 4 & 0 & 0 & M & M & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ t_3 \\ t_4 \\ W \end{array}\left[\begin{array}{ccccccccc|c} x_{AC} & x_{AD} & x_{BC} & x_{BD} & s_1 & s_2 & t_3 & t_4 & W & \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 6 \\ \underline{1} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 5 \\ \hline 1-M & 2-M & 2-M & 4-M & 0 & 0 & 0 & 0 & 1 & -10M \end{array}\right]\begin{array}{c} \\ 6 \\ \\ 5 \\ \\ \\ \end{array}$$

$$\begin{array}{c} \\ s_1 \\ s_2 \\ x_{AC} \\ t_4 \\ W \end{array}\left[\begin{array}{ccccccccc|c} x_{AC} & x_{AD} & x_{BC} & x_{BD} & s_1 & s_2 & t_3 & t_4 & W & \\ 0 & \underline{1} & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 6 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 5 \\ \hline 0 & 2-M & 1 & 4-M & 0 & 0 & -1+M & 0 & 1 & -5-5M \end{array}\right]\begin{array}{c} \\ 1 \\ \\ \\ 5 \\ \\ \end{array}$$

$$\begin{array}{c} \\ x_{AD} \\ s_2 \\ x_{AC} \\ t_4 \\ W \end{array}\left[\begin{array}{ccccccccc|c} x_{AC} & x_{AD} & x_{BC} & x_{BD} & s_1 & s_2 & t_3 & t_4 & W & \\ 0 & 1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 6 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & \underline{1} & 1 & -1 & 0 & 1 & 1 & 0 & 4 \\ \hline 0 & 0 & 3-M & 4-M & -2+M & 0 & 1 & 0 & 1 & -7-4M \end{array}\right]\begin{array}{c} \\ \\ 6 \\ 5 \\ 4 \\ \\ \end{array}$$

$$\begin{array}{c} \\ x_{AD} \\ s_2 \\ x_{AC} \\ x_{BC} \\ W \end{array}\left[\begin{array}{ccccccccc|c} x_{AC} & x_{AD} & x_{BC} & x_{BD} & s_1 & s_2 & t_3 & t_4 & W & \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 2 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & 1 & 0 & 4 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & -2+M & -3+M & 1 & -19 \end{array}\right]$$

The minimum value of C is 19, when $x_{AC} = 1, x_{AD} = 5, x_{BC} = 4,$ and $x_{BD} = 0.$ Thus 100,000 gal from A to C, 500,000 gal from A to D, and 400,000 gal from B to C give a minimum cost of $19,000.

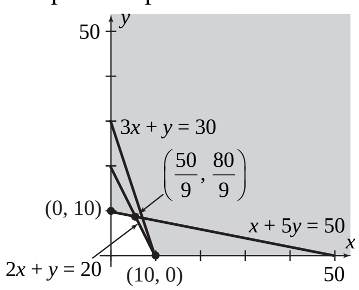
34. Let x and y be the weekly sales of Space Traders and Green Dwarf, respectively. We want to maximize $P = 5x + 9y$ subject to the constraints

$$\begin{aligned} 30x + 10y &\le 300 \\ 20x + 10y &\le 200 \\ 10x + 50y &\le 500 \\ x,\ y &\ge 0 \end{aligned}$$

The constraints can be written as

$$\begin{aligned} 3x + y &\le 30 \\ 2x + y &\le 20 \\ x + 5y &\le 50 \\ x,\ y &\ge 0 \end{aligned}$$

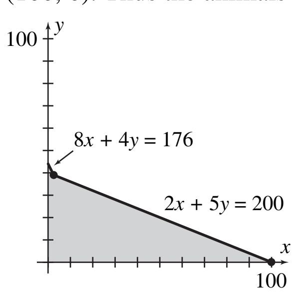
The feasible region has corner points (0, 0), (0, 10), $\left(\frac{50}{9}, \frac{80}{9}\right)$, and (10, 0). *P* has a maximum of $\frac{970}{9}$ at $\left(\frac{50}{9}, \frac{80}{9}\right) = \left(5\frac{5}{9}, 8\frac{8}{9}\right)$. The possible integer values are (5, 8), (5, 9), (6, 8), and (6, 9). However, the point (6, 9) does not satisfy the second or third constraints. Evaluating *P* at the other three points gives that Jason should sell 5 copies of Space Trader and 9 copies of Green Dwarf, for a weekly profit of $106.



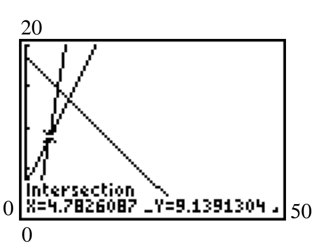
35. Let *x* and *y* represent daily consumption of foods *A* and *B* in 100-gram units. We want to minimize $C = 8x + 22y$ subject to the constraints

$$\begin{aligned} 8x + 4y &\ge 176, \\ 16x + 32y &\ge 1024, \\ 2x + 5y &\ge 200, \\ x &\ge 0, \\ y &\ge 0. \end{aligned}$$

The feasible region is unbounded with corner points (100, 0), $\left(\frac{5}{2}, 39\right)$ and (0, 44). *C* has a minimum value at (100, 0). Thus the animals should be fed $100 \times 100 = 10{,}000$ grams = 10 kilograms of food A each day.

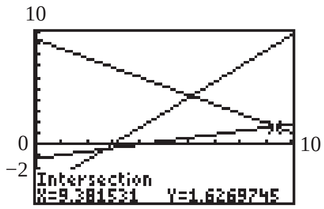


36.



$Z = 0.89$ when $x = 4.78$, $y = 9.14$

37.



$Z = 129.83$ when $x = 9.38$, $y = 1.63$

Mathematical Snapshot Chapter 7

1.

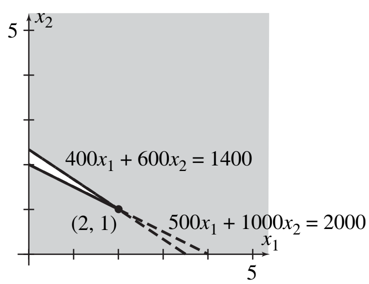
	CURATIVE UNITS	TOXIC UNITS	RELATIVE DISCOMFORT
Drug (per ounce)	500	400	1
Radiation (per min)	1000	600	1
Requirement	≥ 2000	≤1400	

Let x_1 = number of ounces of drug and let x_2 = number of minutes of radiation. We want to minimize the discomfort D, where $D = x_1 + x_2$, subject to

$500x_1 + 1000x_2 \geq 2000,$

$400x_1 + 600x_2 \leq 1400,$

where $x_1, x_2 \geq 0.$



The corner points are (0, 2), $\left(0, \frac{7}{3}\right)$, and (2, 1).

At (0, 2), $D = 0 + 2 = 2$;

at $\left(0, \frac{7}{3}\right)$, $D = 0 + \frac{7}{3} = \frac{7}{3}$;

at (2, 1), $D = 2 + 1 = 3$.

Thus D is minimum at (0, 2).

The patient should get 0 ounces of drug and 2 minutes of radiation.

2.

	CURATIVE UNITS	TOXIC UNITS	RELATIVE DISCOMFORT
Drug A (per ounce)	600	500	1
Drug B (per ounce)	500	100	2
Radiation (per min)	1000	1000	1
Requirement	≥ 3000	≤ 2000	

Let x_1 = number of ounces of drug A,
x_2 = number of ounces of drug B, and
x_3 = number of minutes of radiation.

We want to minimize the discomfort D, where $D = x_1 + 2x_2 + x_3$, subject to

$$
\begin{aligned}
600x_1 + 500x_2 + 1000x_3 &\geq 3000,\\
500x_1 + 100x_2 + 1000x_3 &\leq 2000,\\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

To minimize D, we maximize $-D$ by considering the artificial objective function $W = -D - Mt_1$.

$$
\begin{array}{c} \\ \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & t_1 & W & \end{array}\\
\left[\begin{array}{ccccccc|c}
600 & 500 & 1000 & -1 & 0 & 1 & 0 & 3000\\
500 & 100 & 1000 & 0 & 1 & 0 & 0 & 2000\\
\hline
1 & 2 & 1 & 0 & 0 & M & 1 & 0
\end{array}\right]
\end{array}
$$

$$
\begin{array}{c|ccccccc|c|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & t_1 & W & & \\
\hline
t_1 & 600 & 500 & 1000 & -1 & 0 & 1 & 0 & 3000 & 3\\
s_2 & 500 & 100 & \underline{1000} & 0 & 1 & 0 & 0 & 2000 & 2\\
\hline
W & 1-600M & 2-500M & 1-1000M & M & 0 & 0 & 1 & -3000M &
\end{array}
$$

$$
\begin{array}{c|ccccccc|c|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & t_1 & W & & \\
\hline
t_1 & 100 & \underline{400} & 0 & -1 & 1 & 1 & 0 & 1000 & 2.5\\
x_3 & 0.5 & 0.1 & 1 & 0 & 0.001 & 0 & 0 & 2 & 20\\
\hline
W & 0.5-100M & 1.9-400M & 0 & M & -0.001+M & 0 & 1 & -2-1000M &
\end{array}
$$

$$
\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & x_3 & s_1 & s_2 & t_1 & W & \\
\hline
x_2 & 0.25 & 1 & 0 & -0.0025 & -0.0025 & 0.0025 & 0 & 2.5\\
x_3 & 0.475 & 0 & 1 & 0.00025 & 0.00125 & -0.00025 & 0 & 1.75\\
\hline
W & 0.025 & 0 & 0 & 0.00475 & 0.00375 & -0.00475+M & 1 & -6.75
\end{array}
$$

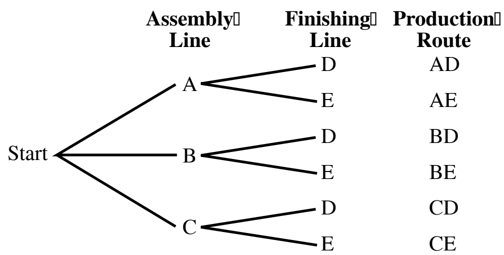
The minimum value of D is 6.75 when $x_1 = 0, x_2 = 2.5,$ and $x_3 = 1.75$.
The patient should get 0 ounces of drug A, 2.5 ounces of drug B, and 1.75 minutes of radiation.

3. Answers may vary.

Chapter 8

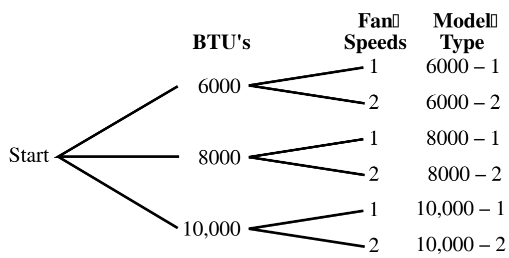
Problems 8.1

1.



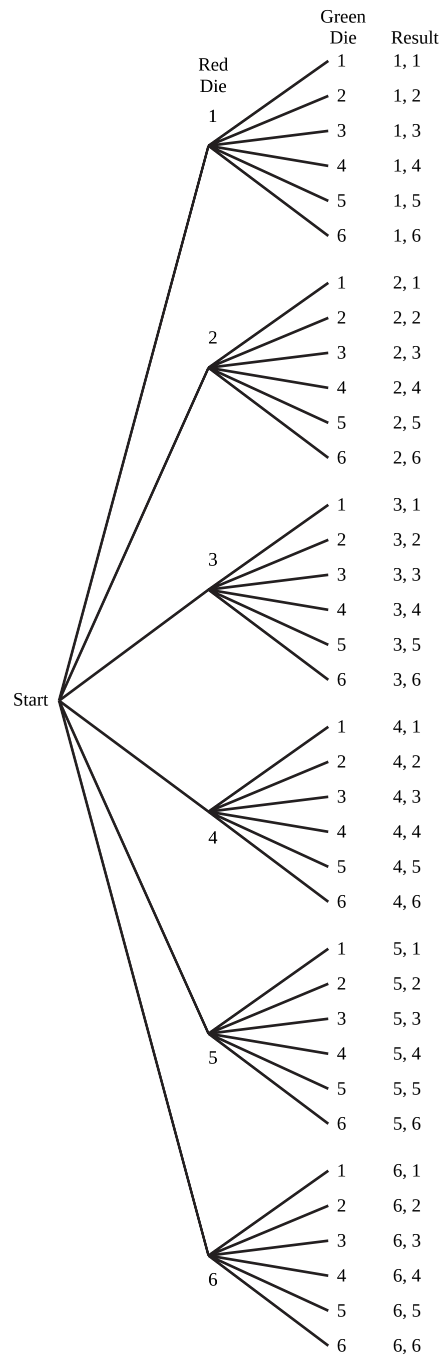
6 possible production routes

2.



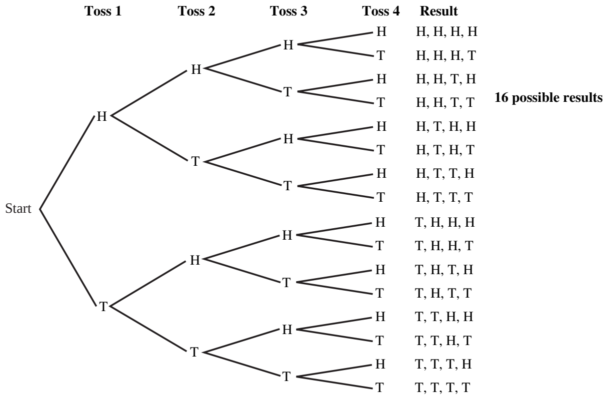
6 model types

3.



36 possible results

4.



5. There are 5 science courses and 4 humanities courses. By the basic counting principle, the number of selections is $5 \cdot 4 = 20$.

6. a. There are 5 roads from A to B, and 5 roads from B to A. By the basic counting principle, the number of possible routes for a round trip is $5 \cdot 5 = 25$.

b. There are 5 possible roads from A to B. Since a different road is to be used for the return trip, there are only 4 possible roads from B to A. By the basic counting principle, the number of possible round-trip routes is $5 \cdot 4 = 20$.

7. There are 2 appetizers, 4 entrees, 4 desserts, and 3 beverages. By the basic counting principle, the number of possible complete dinners is $2 \cdot 4 \cdot 4 \cdot 3 = 96$.

8. For each of the 6 questions, there are 4 choices. By the basic counting principle, the number of ways to answer the questions is $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^6 = 4096$.

9. For each of the 10 questions, there are 2 choices. By the basic counting principle, the number of ways to answer the examination is $2 \cdot 2 \cdot \cdots \cdot 2 = 2^{10} = 1024$.

10. Since there are 26 letters, there are 26 choices for the first, third and fifth symbols. There are 10 possible digits (0 through 9) for the second, fourth, and sixth symbols. By the basic counting principle, the number of codes is $26 \cdot 10 \cdot 26 \cdot 10 \cdot 26 \cdot 10 = 17{,}576{,}000$.

11. ${}_6P_3 = \dfrac{6!}{(6-3)!} = \dfrac{6!}{3!} = 6 \cdot 5 \cdot 4 = 120$

12. ${}_{95}P_1 = \dfrac{95!}{(95-1)!} = \dfrac{95!}{94!} = 95$

13. ${}_6P_6 = \dfrac{6!}{(6-6)!} = \dfrac{6!}{0!} = \dfrac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1} = 720$

14. ${}_9P_4 = \dfrac{9!}{(9-4)!} = \dfrac{9!}{5!} = 9 \cdot 8 \cdot 7 \cdot 6 = 3024$

15. ${}_4P_2 \cdot {}_5P_3 = (4 \cdot 3)(5 \cdot 4 \cdot 3) = (12)(60) = 720$

16. $\dfrac{{}_{99}P_5}{{}_{99}P_4} = \dfrac{99 \cdot 98 \cdot 97 \cdot 96 \cdot 95}{99 \cdot 98 \cdot 97 \cdot 96} = 95$

17. $\dfrac{1000!}{999!} = \dfrac{1000 \cdot 999!}{999!} = 1000$

For most calculators, attempting to evaluate $\dfrac{1000!}{999!}$ results in an error message (because of the magnitude of the numbers involved).

18. $\dfrac{{}_nP_r}{n!} = \dfrac{\frac{n!}{(n-r)!}}{n!} = \dfrac{1}{(n-r)!}$

19. A name for the firm is an ordered arrangement of the three last names. Thus the number of possible firm names is ${}_3P_3 = 3! = 3 \cdot 2 \cdot 1 = 6$.

20. The number of ways to arrange 6 teams in an order is ${}_6P_6 = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.

21. The number of ways of selecting 3 of 8 contestants in an order is ${}_8P_3 = 8 \cdot 7 \cdot 6 = 336$.

22. Six out of eight items in column 2 must be selected in an order. Thus the number of ways the matching can be done is ${}_8P_6 = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 20{,}160$.

23. On each roll of a die, there are 6 possible outcomes. By the basic counting principle, on 4 rolls the number of possible results is $6 \cdot 6 \cdot 6 \cdot 6 = 6^4 = 1296$.

24. On each toss there are 2 possible outcomes. By the basic counting principle, the number of possible results on 8 tosses is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^8 = 256$.

25. The number of ways of selecting 3 of the 12 students in an order is ${}_{12}P_3 = 12 \cdot 11 \cdot 10 = 1320$.

26. Three of the 26 letters must be selected (without repetition) in an order. Thus the number of possible lock combinations is ${}_{26}P_3 = 15,600$.

27. The number of ways a student can choose 4 of the 6 items in an order is ${}_6P_4 = 6 \cdot 5 \cdot 4 \cdot 3 = 360$.

28. On the second roll, there are 2 possible outcomes (a 1 or a 2). For each of the other two rolls, there are 6 possible outcomes. By the basic counting principle, the number of possible results for the three rolls is 6 · 2 · 6 = 72.

29. The number of ways to select six of the six different letters in the word MEADOW in an order is ${}_6P_6 = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.

30. The number of ways to select four of the four different letters in the word DISC in an order is ${}_4P_4 = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.

31. For an arrangement of books, order is important. The number of ways to arrange 5 of 7 books is ${}_7P_5 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$.
All 7 books can be arranged in ${}_7P_7 = 7! = 5040$ ways.

32. **a.** A student can enter by any of 5 doors. After a door is chosen, the student can exit by any of the 4 remaining doors. By the basic counting principle, the number of ways to enter by one door and exit by a different door is 5 · 4 = 20.

b. There are 5 doors by which to enter and 5 doors by which to exit. By the basic counting principle, the total number of ways to enter and exit is 5 · 5 = 25.

33. After a “four of a kind” hand is dealt, the cards can be arranged so that the first four have the same face value, and order is not important, There are 13 possibilities for the first four cards (all 2’s, all 3’s, ..., all aces). The fifth card can be any one of the 48 cards that remain. By the basic counting principle, the number of “four of a kind” hands is 13 · 48 = 624.

34. Five colors are available, and two are selected so that order is important. Thus the number of ways of placing an order is ${}_5P_2 = 5 \cdot 4 = 20$.

35. The number of ways the waitress can place five of the five different sandwiches (and order is important) is ${}_5P_5 = 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

36. Because order is important, the number of ways that the 5 people can line up is ${}_5P_5 = 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.
If a woman is to be at each end, then the number of ways to place one of the two women on the left side is ${}_2P_1$. Once a woman is chosen for the left side, the other woman must be on the right side. The number of ways to line the three men in the middle is ${}_3P_3$. By the basic counting principle, the number of line ups is ${}_2P_1 \cdot {}_3P_3 = (2)(3 \cdot 2 \cdot 1) = 12$.

37. **a.** To fill the four offices by different people, 4 of 12 members must be selected, and order is important. This can be done in ${}_{12}P_4 = 12 \cdot 11 \cdot 10 \cdot 9 = 11,880$ ways.

b. If the president and vice president must be different members, then there are 12 choices for president, 11 for vice president, 12 for secretary, and 12 for treasurer. By the basic counting principle, the offices can be filled in 12 · 11 · 12 · 12 = 19,008 ways.

38. **a.** There are 24 possibilities for each of the three letters in a name. By the basic counting principle, the number of names is $24 \cdot 24 \cdot 24 = 24^3 = 13,824$.

b. Since the order of letters is important and no letter is used more than one time, the number of names is ${}_{24}P_3 = 24 \cdot 23 \cdot 22 = 12,144$.

39. There are 2 choices for the center position. After that choice is made, to fill the remaining four positions (and order is important), there are ${}_4P_4$ ways. By the basic counting principle, to assign positions to the five-member team there are $2 \cdot {}_4P_4 = 2(4!) = 2(24) = 48$ ways.

40. For the first letter there are two possibilities. For the second and third letters there are 26 possibilities, and for the last letter there are 25 possibilities. By the basic counting principle, the number of possible identifications is 2 · 26 · 26 · 25 = 33,800.

41. There are ${}_3P_3$ ways to select the first three batters (order is important) and there are ${}_6P_6$ ways to select the remaining batters. By the basic counting principle, the number of possible batting orders is

${}_3P_3 \cdot {}_6P_6 = 3! \cdot 6! = 6 \cdot 720 = 4320.$

42. a. Four of four flags can be arranged (order is important) in ${}_4P_4 = 4! = 24$ ways. Thus 24 different signals are possible.

b. If only one of the four flags is used, there are ${}_4P_1$ possible signals. If exactly two flags are used, there are ${}_4P_2$ possible signals. Similarly, for exactly three and exactly four flags, there are ${}_3P_4$ and ${}_4P_4$ possible signals, respectively. Thus if at least one flag is used, the number of possible signals is

${}_4P_1 + {}_4P_2 + {}_4P_3 + {}_4P_4$
$= 4 + 4 \cdot 3 + 4 \cdot 3 \cdot 2 + 4 \cdot 3 \cdot 2 \cdot 1$
$= 4 + 12 + 24 + 24 = 64.$

Problems 8.2

1. $${}_6C_4 = \frac{6!}{4!(6-4)!} = \frac{6!}{4! \cdot 2!} = \frac{6 \cdot 5 \cdot 4!}{4!(2 \cdot 1)} = \frac{6 \cdot 5}{2 \cdot 1} = 15$$

2. $${}_6C_2 = \frac{6!}{2!(6-2)!} = \frac{6!}{2! \cdot 4!} = \frac{6 \cdot 5 \cdot 4!}{(2 \cdot 1)4!} = \frac{6 \cdot 5}{2 \cdot 1} = 15$$

3. $${}_{100}C_{100} = \frac{100!}{100!(100-100)!} = \frac{1}{0!} = \frac{1}{1} = 1$$

4. $${}_{1001}C_1 = \frac{1001!}{1!(1001-1)!} = \frac{1001!}{1! \cdot 1000!} = \frac{1001 \cdot 1000!}{1000!} = 1001$$

5. $${}_5P_3 \cdot {}_4C_2 = 5 \cdot 4 \cdot 3 \frac{4!}{2!(4-2)!} = 5 \cdot 4 \cdot 3 \frac{4 \cdot 3 \cdot 2!}{2!2!} = 60 \cdot 6 = 360$$

6. $${}_4P_2 \cdot {}_5C_3 = (4 \cdot 3)\frac{5!}{3!(5-3)!} = (4 \cdot 3)\frac{5 \cdot 4 \cdot 3!}{3! \cdot 2!} = (12) \cdot 10 = 120$$

7. $${}_nC_r = \frac{n!}{r!(n-r)!}$$

$${}_nC_{n-r} = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!}.$$

Thus ${}_nC_r = {}_nC_{n-r}$.

8. $${}_nC_n = \frac{n!}{n!(n-n)!} = \frac{1}{0!} = \frac{1}{1} = 1.$$

9. The number of ways of selecting 4 of 17 people so that order is not important is

$${}_{17}C_4 = \frac{17!}{4!(17-4)!} = \frac{17!}{4! \cdot 13!} = \frac{17 \cdot 16 \cdot 15 \cdot 14 \cdot 13!}{4 \cdot 3 \cdot 2 \cdot 1(13!)} = 2380$$

10. If horses A, B, and C finish in the money, then it does not matter if A finishes in first, second, or third place. Similarly for B and C. Thus order is not important. The number of ways in which 3 of 8 horses finish in the money is the number of ways of selecting 3 of the 8 without regard to order, namely

$${}_8C_3 = \frac{8!}{3!(8-3)!} = \frac{8!}{3! \cdot 5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3 \cdot 2 \cdot 1 \cdot 5!} = 56.$$

11. The number of ways of selecting 9 out of 13 questions (without regard to order) is

$${}_{13}C_9 = \frac{13!}{9!(13-9)!} = \frac{13!}{9! \cdot 4!} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9!}{9! \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 715.$$

12. In a deck of 52 cards, 26 of the cards are red. In a four-card hand, the order is not important. Thus, the number of four-card hands from the 26 red cards is

$${}_{26}C_4 = \frac{26!}{4!(26-4)!} = \frac{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22!}{4!22!} = 14{,}950$$

13. The order of selecting 10 of the 74 dresses is of no concern. Thus the number of possible samples $_{74}C_{10} = \frac{74!}{10!\cdot(74-10)!} = \frac{74!}{10!\cdot 64!}$.

14. This situation can be considered as a two-stage process. In the first stage, one of the 3 boxes is selected. In the second stage, 4 of the 7 types of jelly are selected (and order is not important), which can be done in $_7C_4$ ways. By the basic counting principle, the number of different gift boxes that are possible is

$$3\cdot{}_7C_4 = 3\cdot\frac{7!}{4!(7-4)!} = 3\cdot\frac{7!}{4!\cdot 3!}$$
$$= 3\cdot\frac{7\cdot 6\cdot 5\cdot 4!}{4!(3\cdot 2\cdot 1)} = 3\cdot 35 = 105.$$

15. To score 80, 90, or 100, exactly 8, 9, or 10 questions must be correct, respectively. The number of ways in which 8 of 10 questions can be correct is

$$_{10}C_8 = \frac{10!}{8!(10-8)!} = \frac{10!}{8!\cdot 2!} = \frac{10\cdot 9\cdot 8!}{8!\cdot 2\cdot 1} = 45.$$

For 9 of 10 questions, the number of ways is

$$_{10}C_9 = \frac{10!}{9!(10-9)!} = \frac{10!}{9!\cdot 1!} = \frac{10\cdot 9!}{9!\cdot 1} = 10,$$

and for 10 of 10 questions, it is

$$_{10}C_{10} = \frac{10!}{10!(10-10)!} = \frac{10!}{10!\cdot 0!} = 1.$$

Thus the number of ways to score 80 or better is 45 + 10 + 1 = 56.

16. Each of the 11 games can be assigned to one of three cells: a win cell, a loss cell, or a tie cell. The number of ways to have 4 wins, 5 losses, and 3 ties is

$$\frac{11!}{4!\cdot 5!\cdot 2!} = \frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5!}{4\cdot 3\cdot 2\cdot 1\cdot 5!\cdot 2\cdot 1} = 6930.$$

17. The word MISSISSAUGA has 11 letters with repetition: one M, two I's, four S's, two A's, one U, and one G. Thus the number of distinguishable arrangements is

$$\frac{11!}{1!\cdot 2!\cdot 4!\cdot 2!\cdot 1!\cdot 1!} = \frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4!}{(2)4!(2)}$$
$$= 415{,}800.$$

18. The word STREETSBORO has 11 letters with repetition: two S's, two T's, two R's, two E's, one B, and two O's. Thus the number of distinguishable arrangements is

$$\frac{11!}{2!\cdot 2!\cdot 2!\cdot 2!\cdot 1!\cdot 2!} = \frac{11!}{32} = 1{,}247{,}400.$$

19. The number of ways 4 heads and 3 tails can occur in 7 tosses of a coin is the same as the number of distinguishable permutations in the "word" HHHHTTT, which is

$$\frac{7!}{4!\cdot 3!} = \frac{7\cdot 6\cdot 5\cdot 4!}{4!(6)} = 35.$$

20. The number of ways for the given outcome to occur is the number of distinguishable permutations of six numbers such that two are 2's, three are 3's, and one is 4, which is

$$\frac{6!}{2!\cdot 3!\cdot 1!} = \frac{6\cdot 5\cdot 4\cdot 3!}{(2)3!} = 60.$$

21. Since the order in which the calls are made is important, the number of possible schedules for the 6 calls is $_6P_6 = 6! = 720$.

22. The number of ways to place the 12 members in three specific cars (cells), with 4 members in each car, is $\frac{12!}{4!\cdot 4!\cdot 4!} = 34{,}650$.

23. The number of ways to assign 9 scientists so 3 work on project A, 3 work on B, and 3 work on C is $\frac{9!}{3!3!3!} = 1680$.

24. There are 9 holly bushes, 5 of which are female, and 4 of which are male. Then the number of possible distinguishable arrangements is

$$\frac{9!}{5!\cdot 4!} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5!}{5!\cdot 4\cdot 3\cdot 2\cdot 1} = 126.$$

25. A response to the true-false questions can be considered an ordered arrangement of 10 letters, 5 of which are T's and 5 of which are F's. The number of different responses is

$$\frac{10!}{5!\cdot 5!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5!}{5!(5\cdot 4\cdot 3\cdot 2\cdot 1)} = 252.$$

26. The order in which the 7 food items are placed is important. However, there are 3 hamburgers (type 1), 2 cheeseburgers (type 2), and 2 steak sandwiches (type 3). Then the number of possible distinguishable ways of placing the items is $\frac{7!}{3!\cdot 2!\cdot 2!}=210$.

27. The number of ways to assign 15 clients to 3 caseworkers (cells) with 5 clients to each caseworker is $\frac{15!}{5!\cdot 5!\cdot 5!}=756,756.$

28. The number of ways of selecting 5 of the 10 remaining members so that order is important is

$${}_{10}P_5=\frac{10!}{(10-5)!}=\frac{10!}{5!}=\frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5!}{5!}$$
$$=30,240.$$

29. a. Seven flags must be arranged: two are red (type 1), three are green (type 2), and two are yellow (type 3). Thus the number of distinguishable arrangements (messages) is

$$\frac{7!}{2!\cdot 3!\cdot 2!}=210.$$

b. If exactly two yellow flags are used, then seven flags are involved and the number of different messages is $\frac{7!}{2!\cdot 3!\cdot 2!}=210.$ If all three yellow flags are used, then eight flags are involved and the number of different messages is $\frac{8!}{2!\cdot 3!\cdot 3!}=560.$ Thus if at least two yellow flags are used, the number of different messages is 210 + 560 = 770.

30. Of the 10 applicants, 4 will be hired for the assembly department (cell 1), 2 for the shipping department (cell 2), and 4 will not be hired (cell 3). Thus the number of ways to fill the positions is $\frac{10!}{4!\cdot 2!\cdot 4!}=3150$.

31. The order in which the securities go into the portfolio is not important. The number of ways to select 8 of 12 stocks is ${}_{12}C_8$. The number of ways to select 4 of 7 bonds is ${}_7C_4$. By the basic counting principle, the number of ways to create the portfolio is

$${}_{12}C_8\cdot {}_7C_4=\frac{12!}{8!(12-8)!}\cdot\frac{7!}{4!(7-4)!}$$
$$=\frac{12!}{8!\cdot 4!}\cdot\frac{7!}{4!\cdot 3!}=\frac{12\cdot 11\cdot 10\cdot 9\cdot 8!}{8!\cdot 4\cdot 3\cdot 2\cdot 1}\cdot\frac{7\cdot 6\cdot 5\cdot 4!}{4!\cdot 3\cdot 2\cdot 1}$$
$$=495\cdot 35=17,325.$$

32. Suppose the possible games are numbered 1, 2, 3, ..., 7. The order in which four games are won is not important. The number of ways that 4 of the possible 7 games can be won is

$${}_7C_4=\frac{7!}{4!(7-4)!}=\frac{7!}{4!\cdot 3!}=35.$$

33. a. Selecting 3 of the 3 males can be done in only 1 way.

b. Selecting 4 of the 4 females can be done in only 1 way.

c. Selecting 2 males and 2 females can be considered as a two-stage process. In the first stage, 2 of the 3 males are selected (and order is not important), which can be done in ${}_3C_2$ ways. In the second stage, 2 of the 4 females are selected, which can be done in ${}_4C_2$ ways. By the basic counting principle, the ways of selecting the subcommittee is

$${}_3C_2\cdot {}_4C_2=\frac{3!}{2!(3-2)!}\cdot\frac{4!}{2!(4-2)!}$$
$$=\frac{3!}{2!\cdot 1!}\cdot\frac{4!}{2!\cdot 2!}=3\cdot 6=18$$

34. Exactly 2, 3, or 4 females can serve on the subcommittee. Following the procedure in Problem 33(c), the number of ways exactly 2 females can serve is ${}_4C_2\cdot {}_4C_2$. The number of ways exactly 3 females can serve is ${}_4C_3\cdot {}_4C_1$. The number of ways exactly four females can serve is 1. Thus the number of ways that at least 2 females can serve on the subcommittee is

$${}_4C_2\cdot {}_4C_2+{}_4C_3\cdot {}_4C_1+1$$
$$=\frac{4!}{2!\cdot 2!}\cdot\frac{4!}{2!\cdot 2!}+\frac{4!}{3!\cdot 1!}\cdot\frac{4!}{1!\cdot 3!}+1$$

= 6 · 6 + 4 · 4 + 1 = 36 + 16 + 1 = 53.

35. There are 4 cards of a given denomination and the number of ways of selecting 3 cards of that denomination is ${}_4C_3$.
Since there are 13 denominations, the number of ways of selecting 3 cards of one denomination is

$13 \cdot {}_4C_3$. After that selection is made, the 2 other cards must be of the same denomination (of which 12 denominations remain). Thus for the remaining 2 cards there are $12 \cdot {}_4C_2$ selections. By the basic counting principle, the number of possible full-house hands is

$$13 \cdot {}_4C_3 \cdot 12 \cdot {}_4C_2 = 13 \cdot \frac{4!}{3! \cdot 1!} \cdot 12 \cdot \frac{4!}{2! \cdot 2!}$$

$= 13 \cdot 4 \cdot 12 \cdot 6 = 3744$.

36. There are 13 denominations and four cards of each denomination. The number of ways to get a pair of 8's is ${}_4C_2$. For the other pair, there are 12 denominations left to choose from, so ${}_{12}C_1$ possibilities, with ${}_4C_2$ ways to get such a pair. For the last card there are 11 denominations left, with 4 cards in each denomination. By the basic counting principle the number of two-pair hands where one pair is 8's is

$${}_4C_2 \cdot {}_{12}C_1 \cdot {}_4C_2 \cdot 11 \cdot 4$$
$$= \frac{4!}{2! \cdot 2!} \cdot \frac{12!}{1! \cdot 11!} \cdot \frac{4!}{2! \cdot 2!} \cdot 11 \cdot 4$$
$$= 19,008.$$

37. This situation can be considered as placing 18 tourists into 3 cells: 7 tourist go to the 7-passenger tram, 8 go to the 8-passenger tram, and 3 tourists remain at the bottom of the mountain. This can be done in

$$\frac{18!}{7! \cdot 8! \cdot 3!} = 5,250,960 \text{ ways.}$$

38. a. The 10 students are to be placed in 3 groups, with 4 in group A, 3 in group B, and 3 in group C. This can be done in

$$\frac{10!}{4! \cdot 3! \cdot 3!} = 4200 \text{ ways.}$$

b. For a given assignment of students to the three groups, the number of ways of selecting a group leader and a secretary for group A (order is important) is ${}_4P_2$; for group B, it is ${}_3P_2$; and for group C it is ${}_3P_2$. Thus the number of ways that the instructor can split the class into 3 groups and designate a group leader and secretary in each group is

$$4200 \cdot {}_4P_2 \cdot {}_3P_2 \cdot {}_3P_2$$
$$= 4200(4 \cdot 3)(3 \cdot 2)(3 \cdot 2) = 1,814,400.$$

Principles in Practice 8.3

1. This is a combination problem because the order in which the videos are selected is not important. The number of possible choices is the number of ways 3 videos can be selected from 400 without regard to order.

$${}_{400}C_3 = \frac{400!}{3!(400-3)!} = \frac{400!}{3!397!}$$
$$= \frac{400 \cdot 399 \cdot 398 \cdot 397!}{3!397!}$$
$$= \frac{400 \cdot 399 \cdot 398}{3 \cdot 2}$$
$$= 10,586,800$$

Problems 8.3

1. {9D, 9H, 9C, 9S]

2. {HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}

3. {1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T}

4. {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

5. [64, 69, 60, 61, 46, 49, 40, 41, 96, 94, 90, 91, 06, 04, 09, 01, 16, 14, 19, 10]

6. {BBBB, BBBG, BBGB, BBGG, BGBB, BGBG, BGGB, BGGG, GBBB, GBBG, GBGB, GBGG, GGBB, GGBG, GGGB, GGGG}

7. a. {RR, RW, RB, WR, WW, WB, BR, BW, BB};

b. {RW, RB, WR, WB, BR, BW}

8. {ADF, ADG, AEF, AEG, BDF, BDG, BEF, BEG, CDF, CDG, CEF, CEG}

9. Sample space consists of ordered sets of six elements and each element is H or T. Since there are two possibilities for each toss (H or T), and there are six tosses, by he basic counting principle, the number of sample points is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 64$.

10. Sample space consists of ordered sets of five elements where each element is an integer between 1 and 6 inclusive. Since there are six possibilities for each die, and there are 5 dice, by the basic counting principle, the number of sample points is $6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 = 6^5 = 7776$.

11. Sample space consists of ordered pairs where the first element indicates the card drawn (52 possibilities) and the second element indicates the number on the die (6 possibilities). By the basic counting principle, the number of sample points is $52 \cdot 6 = 312$.

12. Sample space consists of ordered sets of four elements where the elements and their position indicate the rabbit selected on the respective draw. Since the rabbits are not replaced, for the first draw there are 9 possibilities, for the second draw there are 8 possibilities, and for the third and fourth there are 7 and 6 possibilities, respectively. By the basic counting principle, the number of sample points is $9 \cdot 8 \cdot 7 \cdot 6 = 3024$.

13. Sample space consists of combinations of 52 cards taken 10 at a time. Thus the number of sample points is ${}_{52}C_{10}$.

14. Sample space consists of all four letter "words." For each of the four letters there are 26 possibilities. By the basic counting principle, the number of sample points is
$26 \cdot 26 \cdot 26 \cdot 26 = 26^4 = 456,976$.

15. The sample points that are either in E, or in F, or in both E and F are 1, 3, 5, 7, and 9. Thus $E \cup F = \{1, 3, 5, 7, 9\}$.

16. The sample points in S that are not in G are 1, 3, 5, 7, 9, and 10. Thus $G' = \{1, 3, 5, 7, 9, 10\}$.

17. The sample points in S that are not in E are 2, 4, 6, 7, 8, 9, and 10. Thus $E' = \{2, 4, 6, 7, 8, 9, 10\}$. The sample points common to both E' and F are 7 and 9. Thus $E' \cap F = \{7, 9\}$.

18. $F' = \{1, 2, 4, 6, 8, 10\}$ and $G' = \{1, 3, 5, 7, 9, 10\}$, so $F' \cap G' = \{1, 10\}$.

19. The sample points in S that are not in F are 1, 2, 4, 6, 8, and 10. Thus $F' = \{1, 2, 4, 6, 8, 10\}$.

20. $(E \cup F)' = \{1, 3, 5, 7, 9\}' = \{2, 4, 6, 8, 10\}$

21. $(F \cap G)' = \varnothing' = S$

22. $(E \cup G) \cap F'$
$= \{1, 2, 3, 4, 5, 6, 8\} \cap \{1, 2, 4, 6, 8, 10\}$
$= \{1, 2, 4, 6, 8\}$

23. $E_1 \cap E_2 \neq \varnothing$; $E_1 \cap E_3 \neq \varnothing$; $E_1 \cap E_4 = \varnothing$; $E_2 \cap E_3 = \varnothing$; $E_2 \cap E_4 \neq \varnothing$; $E_3 \cap E_4 = \varnothing$. Thus E_1 and E_4, E_2 and E_3, and E_3 and E_4 are mutually exclusive.

24. If both cards are jacks, then both cards can neither be clubs nor 3's. Thus $E_J \cap E_C = \varnothing$ and $E_J \cap E_3 = \varnothing$. If both cards are clubs, then both cards cannot be 3's. Thus $E_C \cap E_3 = \varnothing$. E_J and E_C, E_J and E_3, E_C and E_3 are mutually exclusive.

25. $E \cap F \neq \varnothing$, $E \cap G = \varnothing$, $E \cap H \neq \varnothing$, $E \cap I \neq \varnothing$, $F \cap G \neq \varnothing$, $F \cap H \neq \varnothing$ $F \cap I = \varnothing$, $G \cap H = \varnothing$, $G \cap I = \varnothing$, $H \cap I \neq \varnothing$. Thus E and G, F and I, G and H, and G and I are mutually exclusive.

26. $E \cap F = \varnothing$, $E \cap G = \varnothing$, $E \cap H \neq \varnothing$, $E \cap I \neq \varnothing$, $F \cap G \neq \varnothing$, $F \cap H \neq \varnothing$, $F \cap I = \varnothing$, $G \cap H = \varnothing$, $G \cap I = \varnothing$, $H \cap I = \varnothing$.
Thus E and F, E and G, F and I, G and H, G and I, H and I are mutually exclusive.

27. a. S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

b. E_1 = {HHH, HHT, HTH, HTT, THH, THT, TTH}

c. E_2 = {HHT, HTH, HTT, THH, THT, TTH, TTT}

d. $E_1 \cup E_2$ = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} = S

e. $E_1 \cap E_2$ = {HHT, HTH, HTT, THH, THT, TTH}

f. $(E_1 \cup E_2)' = S' = \varnothing$

g. $(E_1 \cap E_2)'$ = {HHT, HTH, HTT, THH, THT, TTH}′ = {HHH, TTT}

28. a. {BB, BG, GB, GG}

b. {BG, GB, GG}

c. {BB, BG, GB}

d. No; {BG, GB, GG}′ = {BB} $\neq$ event in (c)

29. a. {ABC, ACB, BAC, BCA, CAB, CBA}

b. {ABC, ACB}

c. {BAC, BCA, CAB, CBA}

30. a. {UUV, UUW, UUX, UUZ, UVV, UVW, UVX, UVZ, UXV, UXW, UXX, UXZ, UYV, UYW, UYX, UYZ, VUV, VUW, VUX, VUZ, VVV, VVW, VVX, VVZ, VXV, VXW, VXX, VXZ, VYV, VYW, VYX, VYZ, WUV, WUW, WUX, WUZ, WVV, WVW, WVX, WVZ, WXV, WXW, WXX, WXZ, WYV, WYW, WYX, WYZ}

b. {VVV}

c. {UUV, UUW, UUX, UUZ, UVV, UVW, UVX, UVZ, UXV, UXW, UXX, UXZ, UYV, UYW, UYX, UYZ, VUV, VUW, VUX, VUZ, VVW, VVX, VVZ, VXV, VXW, VXX, VXZ, VYV, VYW, VYX, VYZ, WUV, WUW, WUX, WUZ, WVV, WVW, WVX, WVZ, WXV, WXW, WXX, WXZ, WYV, WYW, WYX, WYZ}
More than one supplier is used.

31. Using the properties in Table 8.1, we have

$(E\cap F)\cap(E\cap F')$
$=(E\cap F\cap E)\cap F'$ [property 15]
$=(E\cap E\cap F)\cap F'$ [property 11]
$=(E\cap E)\cap(F\cap F')$ [porperty 15]
$=E\cap\varnothing$ [property 5]
$=\varnothing$ [property 9]

Thus
$(E\cap F)\cap(E\cap F')=\varnothing,$ so $E\cap F$ and $E\cap F'$ are mutually exclusive.

32. Using the properties in Table 8.1, we have

$(E\cap F)\cup(E\cap F')$
$=E\cap(F\cup F')$ [property 16]
$=E\cap S$ [property 4]
$=E$ [property 7]

Problems 8.4

1. $3000P(E) = 3000(0.25) = 750$

2. $3000P(E) = 3000[1 - P(E')] = 3000(1 - 0.45)$
$= 3000(0.55) = 1650$

3. a. $P(E') = 1 - P(E) = 1 - 0.2 = 0.8$

b. $P(E\cup F)=P(E)+P(F)-P(E\cap F)$
$= 0.2 + 0.3 - 0.1 = 0.4$

4. a. $P(E') = 1 - P(E) = 1 - \frac{1}{4} = \frac{3}{4}$

b. $P(E\cup F)=P(E)+P(F)-P(E\cap F)$
$=\frac{1}{4}+\frac{1}{2}-\frac{1}{8}=\frac{5}{8}$

5. If E and F are mutually exclusive, then $E\cap F=\varnothing$.
Thus $P(E\cap F)=P(\varnothing)=0$. Since it is given that $P(E\cap F)=0.831\neq 0$, E and F are not mutually exclusive.

6. $P(E\cup F)=P(E)+P(F)-P(E\cap F)$
Thus $P(F)=P(E\cup F)+P(E\cap F)-P(E)$
$=\frac{13}{20}+\frac{1}{10}-\frac{1}{2}=\frac{1}{4}.$

7. a. $E_8 = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$
$P(E_8)=\frac{n(E_8)}{n(S)}=\frac{5}{36}$

b. $E_{2\text{ or }3} = \{(1, 1), (1, 2), (2, 1)\}$
$P(E_{2\text{ or }3})=\frac{n(E_{2\text{ or }3})}{n(S)}=\frac{3}{36}=\frac{1}{12}$

c. $E_{3,4,\text{or }5}=\{(1,2),(2,1),(1,3),(2,2),(3,1),(1,4),(2,3),(3,2),(4,1)\}$
$P(E_{3,4,\text{or }5})=\frac{n(E_{3,4,\text{ or }5})}{n(S)}=\frac{9}{36}=\frac{1}{4}$

d. $E_{12\text{ or }13}=E_{12}$, since E_{13} is an impossible event.
$E_{12}=\{(6,6)\}$
$P(E_{12\text{ or }13})=\frac{n(E_{12\text{ or }13})}{n(S)}=\frac{1}{36}$

e. $E_2 = \{(1,1)\}$
$E_4 = \{(1,3),(2,2),(3,1)\}$
$E_6 = \{(1,5),(2,4),(3,3,),(4,2),(5,1)\}$
$E_8 = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$
$E_{10} = \{(4,6),(5,5),(6,4)\}$
$E_{12} = \{(6,6)\}$
$P(E_{\text{even}}) = P(E_2) + P(E_4)$
$+P(E_6) + P(E_8) + P(E_{10}) + P(E_{12})$
$= \frac{1}{36} + \frac{3}{36} + \frac{5}{36} + \frac{5}{36} + \frac{3}{36} + \frac{1}{36} = \frac{18}{36} = \frac{1}{2}$

f. $P(E_{\text{odd}}) = 1 - P(E_{\text{even}}) = 1 - \frac{1}{2} = \frac{1}{2}$

g. $E'_{\text{less than 10}} = E_{10} \cup E_{11} \cup E_{12}$
$= \{(4,6),(5,5),(6,4)\} \cup \{(5,6),(6,5)\} \cup \{(6,6)\}$
$= \{(4,6),(5,5),(6,4),(5,6),(6,5),(6,6)\}$.
$P(E_{\text{less than 10}}) = 1 - P(E'_{\text{less than 10}})$
$= 1 - \frac{6}{36} = \frac{30}{36} = \frac{5}{6}$.

8. $E_{\text{2 or 3 shows}} = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (1, 2), (4, 2), (5, 2), (6, 2), (1, 3), (4, 3), (5, 3), (6, 3)\}$

$P(E_{\text{2 or 3 shows}}) = \frac{n(E_{\text{2 or 3 shows}})}{n(S)} = \frac{20}{36} = \frac{5}{9}$

9. $n(S) = 52$.

a. $P(\text{king of hearts}) = \frac{n(E_{\text{king of hearts}})}{n(S)} = \frac{1}{52}$

b. $P(\text{diamond}) = \frac{n(E_{\text{diamond}})}{n(S)} = \frac{13}{52} = \frac{1}{4}$

c. $P(\text{jack}) = \frac{n(E_{\text{jack}})}{n(S)} = \frac{4}{52} = \frac{1}{13}$

d $P(\text{red}) = \frac{n(E_{\text{red}})}{n(S)} = \frac{26}{52} = \frac{1}{2}$

e. Because a heart is not a club,
$E_{\text{heart}} \cap E_{\text{club}} = \varnothing$.
Thus
$P(E_{\text{heart or club}}) = P(E_{\text{heart}} \cup E_{\text{club}})$
$= P(E_{\text{heart}}) + P(E_{\text{club}})$
$= \frac{n(E_{\text{heart}})}{n(S)} + \frac{n(E_{\text{club}})}{n(S)} = \frac{13}{52} + \frac{13}{52}$
$= \frac{26}{52} = \frac{1}{2}$

f. $E_{\text{club and 4}} = \{4C\}$
$P(E_{\text{club and 4}}) = \frac{n(E_{\text{club and 4}})}{n(S)} = \frac{1}{52}$

g. $P(\text{club or 4})$
$= P(\text{club}) + P(4) - P(\text{club and 4})$
$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

h. $E_{\text{red and king}} = \{\text{KH}, \text{KD}\}$
$P(\text{red and king}) = \frac{n(E_{\text{red and king}})}{n(S)}$
$= \frac{2}{52} = \frac{1}{26}$

i. $E_{\text{spade and heart}} = \varnothing$
Thus $P(\text{spade and heart}) = 0$

10. $n(S) = 2 \cdot 6 = 12$

a. $E_{\text{H,5}} = \{\text{H5}\}$
$P(\text{head and 5}) = \frac{n(E_{\text{H,5}})}{n(S)} = \frac{1}{12}$

b. $n(E_{\text{head}}) = 1 \cdot 6 = 6$.
$P(\text{head}) = \frac{n(E_{\text{head}})}{n(S)} = \frac{6}{12} = \frac{1}{2}$

c. $n(E_3) = 2 \cdot 1 = 2$
$P(3) = \frac{n(E_3)}{n(S)} = \frac{2}{12} = -\frac{1}{6}$

d. $n(E_{\text{head and even}}) = 1 \cdot 3 = 3$
$P(\text{head and even})$
$= \frac{n(E_{\text{head and even}})}{n(S)} = \frac{3}{12} = \frac{1}{4}$

11. $n(S) = 2 \cdot 6 \cdot 52 = 624$

a. P(tail, 3, queen of hearts)
$= \frac{n(E_{\text{T,3,QH}})}{n(S)} = \frac{1 \cdot 1 \cdot 1}{624} = \frac{1}{624}$

b. P(tail, 3, queen)
$= \frac{n(E_{\text{T,3,Q}})}{n(S)} = \frac{1 \cdot 1 \cdot 4}{624} = \frac{1}{156}$

c. P(head, 2 or 3, queen)
$= \frac{n(E_{\text{H,2 or 3,Q}})}{n(S)} = \frac{1 \cdot 2 \cdot 4}{624} = \frac{1}{78}$

d. P(head, even, diamond)
$= \frac{n(E_{\text{H,E,D}})}{n(S)} = \frac{1 \cdot 3 \cdot 13}{624} = \frac{1}{16}$

12. $n(S) = 8$

a. $E_{\text{3 heads}} = \{\text{HHH}\}$
$P(\text{3 heads}) = \frac{n(E_{\text{3 heads}})}{n(S)} = \frac{1}{8}$

b. $E_{\text{1 tail}} = \{\text{HHT,HTH,THH}\}$.
$P(\text{1 tail}) = \frac{n(E_{\text{1 tail}})}{n(S)} = \frac{3}{8}$

c. P(no more than 2 heads) = 1 – P(3 heads)
$= 1 - \frac{1}{8} = \frac{7}{8}$

d. $E_{\text{no more than 1 tail}} = E_{\text{0 tails}} \cup E_{\text{1 tail}}$
$= \{\text{HHH}\} \cup \{\text{HHT,HTH,THH}\}$
= {HHH, HHT, HTH, THH}.
P(no more than 1 tail)
$= \frac{n(E_{\text{no more than 1 tail}})}{n(S)} = \frac{4}{8} = \frac{1}{2}$

13. $n(S) = 52 \cdot 51 \cdot 50 = 132{,}600$

a. $P(\text{all kings}) = \frac{4 \cdot 3 \cdot 2}{132{,}600} = \frac{1}{5525}$

b. $P(\text{all hearts}) = \frac{13 \cdot 12 \cdot 11}{132{,}600} = \frac{11}{850}$

14. $n(S) = 52 \cdot 52 = 2704$

a. $P(\text{both kings}) = \frac{n(E_{\text{both kings}})}{n(S)} = \frac{4 \cdot 4}{2704}$
$= \frac{1}{169}$

b. Number of ways both cards are king of hearts: 1. Number of ways either first card is king of hearts and second card is a different heart, or vice versa: $2(1 \cdot 12) = 24$. Number of ways either first card is king of diamonds, clubs, or spades, and second card is a heart, or vice versa: $2(3 \cdot 13) = 78$. Thus, number ways one card is a king and the other is a heart is 1 + 24 + 78 = 103, so probability of given event is $\frac{103}{2704}$.

15. $n(S) = 2 \cdot 2 \cdot 2 = 8$

a. $E_{\text{3 girls}} = \{\text{GGG}\}$
$P(\text{3 girls}) = \frac{n(E_{\text{3 girls}})}{n(S)} = \frac{1}{8}$

b. $E_{\text{1 boy}} = \{\text{BGG,GBG,GGB}\}$
$P(\text{1 boy}) = \frac{n(E_{\text{1 boy}})}{n(S)} = \frac{3}{8}$

c. $E_{\text{no girl}} = \{\text{BBB}\}$
$P(\text{no girl}) = \frac{n(E_{\text{no girl}})}{n(S)} = \frac{1}{8}$

d. P(at least 1 girl) = 1 – P(no girl)
$= 1 - \frac{1}{8} = \frac{7}{8}$

16. The sample space consists of 18 jelly beans. Thus $n(S) = 18$.

a. $P(\text{blue}) = \frac{n(E_{\text{blue}})}{n(S)} = \frac{8}{18} = \frac{4}{9}$

b. P(not red) = 1 – P(red)
$= 1 - \frac{n(E_{\text{red}})}{n(S)} = 1 - \frac{7}{18} = \frac{11}{18}$

c. The events of drawing a red jelly bean and drawing a white jelly bean are mutually exclusive. Thus
$P(\text{red or white}) = P(\text{red}) + P(\text{white})$
$= \frac{7}{18} + \frac{3}{18} = \frac{10}{18} = \frac{5}{9}$

d. $P(\text{neither red nor blue}) = P(\text{white}) = \frac{3}{18} = \frac{1}{6}$

e. $E_{\text{yellow}} = \varnothing$. Thus $P(\text{yellow}) = 0$

f. $E_{\text{red}} \cap E_{\text{yellow}} = \varnothing$
Thus $P(\text{red or yellow}) = P(\text{red}) + P(\text{yellow})$
$= \frac{7}{18} + 0 = \frac{7}{18}$.

17. The sample space consists of 60 stocks. Thus $n(S) = 60$.

a. $P(6\% \text{ or more}) = \frac{n(E_{6\% \text{ or more}})}{n(S)}$
$= \frac{48}{60} = \frac{4}{5}$

b. $P(\text{less than } 6\%) = 1 - P(6\% \text{ or more})$
$= 1 - \frac{4}{5} = \frac{1}{5}$

18. Let N = number of ties. Then the number of pure silk ties is $0.4N$.

a. $P(100\% \text{ pure silk}) = \frac{0.4N}{N} = 0.4$

b. $P(\text{not } 100\% \text{ silk}) = 1 - P(100\% \text{ pure silk})$
$= 1 - 0.4 = 0.6$

19. $n(S) = 40$
Of the 40 students, 4 received an A, 10 a B, 14 a C, 10 a D, and 2 an F.

a. $P(\text{A}) = \frac{n(E_{\text{A}})}{n(S)} = \frac{4}{40} = \frac{1}{10} = 0.1$

b. $P(\text{A or B}) = \frac{n(E_{\text{A or B}})}{n(S)} = \frac{4+10}{40}$
$= \frac{14}{40} = 0.35$

c. $P(\text{neither D nor F}) = P(\text{A, B, or C})$
$= \frac{n(E_{\text{A,B, or C}})}{n(S)} = \frac{4+10+14}{40} = \frac{28}{40} = 0.7$

d. $P(\text{no F}) = 1 - P(\text{F}) = 1 - \frac{n(E_{\text{F}})}{n(S)}$
$= 1 - \frac{2}{40} = \frac{38}{40} = 0.95$

e. Let N = number of students. Then $n(S) = N$. Of the N students, $0.10N$ received an A, $0.25N$ a B, $0.35N$ a C, $0.25N$ a D, $0.05N$ an F.
$P(\text{A}) = \frac{0.10N}{N} = 0.1$
$P(\text{A or B}) = \frac{0.10N + 0.25N}{N}$
$= \frac{0.35N}{N} = 0.35$
$P(\text{neither D nor F}) = P(\text{A, B, or C})$
$= \frac{0.10N + 0.25N + 0.35N}{N}$
$= \frac{0.70N}{N} = 0.7$
$P(\text{no F}) = 1 - P(\text{F})$
$= 1 - \frac{0.05N}{N} = 1 - 0.05 = 0.95$

20. Bag 1 contains 5 jelly beans, and Bag 2 contains 9.
$n(S) = 5 \cdot 9 = 45$.

a. $P(\text{both red}) = \frac{n(E_{\text{R,R}})}{n(S)} = \frac{3 \cdot 4}{45} = \frac{4}{15}$

b. $P(\text{one red and other green})$
$= \frac{n(E_{R,G}) + n(E_{G,R})}{n(S)} = \frac{3 \cdot 5 + 2 \cdot 4}{45}$
$= \frac{15+8}{45} = \frac{23}{45}$

21. The sample space consists of combinations of 2 people selected from 5. Thus

$n(S) = {}_5C_2 = \frac{5!}{2!\cdot 3!} = \frac{5\cdot 4}{2} = 10$. Because there are only 2 women in the group, the number of possible 2-woman committees is 1. Thus

$P(2\text{ women}) = \frac{n(E_{2\text{ women}})}{n(S)} = \frac{1}{10}$.

22. Because there are 3 men and 2 women, the number of possible committees consisting of a man and a woman is 3 · 2 = 6.
Thus

$P(\text{man and woman}) = \frac{n(E_{\text{man and woman}})}{n(S)}$.

$= \frac{6}{10} = \frac{3}{5}$.

23. Number of ways to answer exam is
$2^{10} = 1024 = n(S)$.

a. There is only one way to achieve 100 points, namely to answer each question correctly. Thus

$P(100\text{ points}) = \frac{n(E_{100\text{ points}})}{n(S)} = \frac{1}{1024}$.

b. Number of ways to score
90 points = number of ways that exactly one question is answered incorrectly = 10.
Thus
P(90 or more points)
= P(90 points) + P(100 points)

$= \frac{10}{1024} + \frac{1}{1024} = \frac{11}{1024}$.

24. Number of ways to answer exam is
$4^8 = 65,536 = n(S)$.

a. $P(\text{all correct}) = \frac{n(E_{\text{all correct}})}{n(S)} = \frac{1}{65,536}$

b. The probability of answering one question correctly when answering in a random fashion is $\frac{1}{4}$ and the probability of answering incorrectly is $\frac{3}{4}$. Thus, the probability of answering the first four questions correctly and the last four incorrectly is $\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^4 = \frac{3^4}{4^8}$. Since there are ${}_8C_4$ distinguishable orders in which one can arrange 4 correct and 4 incorrect answers, and since each arrangement has the same overall probability of occurring, the probability of 4 correct and 4 incorrect answers is $\frac{3^4}{4^8}\cdot {}_8C_4 = \frac{3^4}{4^8}\cdot\frac{8!}{4!4!}$

$= \frac{3^4}{4^8}\cdot\frac{8\cdot 7\cdot 6\cdot 5\cdot 4!}{4\cdot 3\cdot 2\cdot 1\cdot 4!} = \frac{3^4}{4^8}\cdot\frac{2\cdot 7\cdot 5}{1}$

$= \frac{2835}{32,768}$.

25. A poker hand is a 5-card deal from 52 cards. Thus $n(S) = {}_{52}C_5$. In 52 cards, there are 4 cards of a particular denomination. Thus, for a four of a kind, the number of ways of selecting 4 of 4 cards of a particular denomination is ${}_4C_4$. Since there are 13 denominations, 4 cards of the same denomination can be dealt in $13\cdot {}_4C_4$ ways. For the remaining card, there are 12 denominations that are possible, and for each denomination there are ${}_4C_1$ ways of dealing a card. Thus

$P(\text{four of a kind}) = \frac{n(E_{\text{four of a kind}})}{n(S)}$

$= \frac{13\cdot {}_4C_4\cdot 12\cdot {}_4C_1}{{}_{52}C_5}$

$= \frac{13\cdot 12\cdot 4}{{}_{52}C_5}$

26. a. $P(E\cup F) = P(E) + P(F) - P(E\cap F)$
Thus $P(F) = P(E\cup F) + P(E\cap F) - P(E)$

$= \frac{5}{14} + \frac{1}{7} - \frac{1}{4} = \frac{1}{4}.$

b. $P(E'\cup F) = P(E') + P(F) - P(E'\cap F)$

$= \left(1 - \frac{1}{4}\right) + \frac{1}{4} - P(E'\cap F)$

$= 1 - P(E'\cap F)$

Since $F = (E\cap F)\cup(E'\cap F)$ and $E\cap F$ and $E'\cap F$ are mutually exclusive $P(F) = P(E\cap F) + P(E'\cap F)$,

$\frac{1}{4} = \frac{1}{7} + P(E'\cap F)$

Thus $P(E' \cap F) = \frac{1}{4} - \frac{1}{7} = \frac{3}{28}$. Hence,

$P(E' \cup F) = 1 - \frac{3}{28} = \frac{25}{28}$.

27. $n(S) = {}_{100}C_3 = \frac{100!}{3! \cdot 97!} = 161{,}700$

a. $n(E_{3\text{ females}}) = {}_{35}C_3 = \frac{35!}{3! \cdot 32!} = 6545$

$P(E_{3\text{ females}}) = \frac{n(E_{3\text{ females}})}{n(S)}$

$= \frac{6545}{161{,}700} \approx 0.040$

b. The number of ways of selecting one professor is 15; the number of ways of selecting two associate professors is ${}_{24}C_2$.

Thus $n(E_{1\text{ professor \& 2 associate professors}})$

$= 15 \cdot \frac{24!}{2! \cdot 22!} = 15 \cdot 276 = 4140$.

Therefore,

$P(E_{1\text{ professor \& 2 associate professers}})$

$= \frac{4140}{161{,}700} \approx 0.026$.

28. $P(\text{even number}) = P(2) + P(4) + P(6)$

$= \frac{2}{10} + \frac{1}{10} + \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$

29. Shiloh needs to win 3 more rounds to win the game and Caitlin needs to win 5 more rounds. Shiloh's probability of winning is

$$\sum_{k=0}^{4} \frac{{}_7C_k}{2^7} = \frac{1}{2^7}\sum_{k=0}^{4} {}_7C_k$$

$$= \frac{1}{2^7}({}_7C_0 + {}_7C_1 + {}_7C_2 + {}_7C_3 + {}_7C_4)$$

$$= \frac{1}{2^7}(1 + 7 + 21 + 35 + 35)$$

$$= \frac{99}{128}$$

Shiloh's share of the pot is then

$\frac{99}{128}(\$25) \approx \$19.34.$

30. Here Shiloh needs to win 5 more rounds to win the game and Caitlin needs to win 8 more rounds. Shiloh's probability of winning is

$\sum_{k=0}^{7} \frac{{}_{12}C_k}{2^{12}} = \frac{3302}{4096} = \frac{1651}{2048}$. Thus Shiloh's share of the pot is $\frac{1651}{2048}(\$50) \approx \$40.31$.

31. Let $p = P(1) = P(3) = P(5)$. Then $2p = P(2) = P(4) = P(6)$. Since $P(S) = 1$, then $3(p) + 3(2p) = 1$, $9p = 1$, $p = p(1) = \frac{1}{9}$.

32. Let $p_1 = P(a) = P(b) = P(c) = P(d) = P(e)$, and $p_2 = P(f) = P(g)$. Then

$P(S) = 5(p_1) + 2(p_2) = 1, \quad p_2 = \frac{1}{2} - \frac{5}{2}p_1.$

Since p_1 is not known, it is not possible to determine $P(f) = p_2$. If it is also known that $P(\{a, f\}) = \frac{1}{3}$, then we have

$P(\{a, f\}) = P(a) + P(f) = p_1 + p_2 = \frac{1}{3}.$

Thus $p_1 = \frac{1}{3} - p_2$ and $p_2 = \frac{1}{2} - \frac{5}{2}\left(\frac{1}{3} - p_2\right).$

$-\frac{3}{2}p_2 = -\frac{1}{3}$ or $p_2 = \frac{2}{9}$ and so $P(f) = \frac{2}{9}.$

33. a. Of the 100 voters, 51 favor the tax increase. Thus $P(\text{favors tax increase}) = \frac{51}{100} = 0.51$.

b. Of the 100 voters, 44 oppose the tax increase. Thus

$P(\text{opposes tax increase}) = \frac{44}{100} = 0.44$.

c. Of the 100 voters, 3 are Republican with no opinion. Thus

$P(\text{is a Republican with no opinion}) = \frac{3}{100}$

$= 0.03$.

34. a. For the chain, the total average number of sales is 170 units. For brand B, 65 units per month are sold. Thus

$P(\text{sale is for brand B}) = \frac{65}{170} = \frac{13}{34} \approx 0.38$.

b. Since 95 units per month are sold at the Exton store, and 30 are of brand C, P(sale is for brand C given that it is at Exton store) $=\frac{30}{95}=\frac{6}{19}\approx 0.32$.

35. $\frac{P(E)}{P(E')}=\frac{P(E)}{1-P(E)}=\frac{\frac{4}{5}}{1-\left(\frac{4}{5}\right)}=\frac{\frac{4}{5}}{\frac{1}{5}}=\frac{4}{1}$

The odds are 4:1.

36. $\frac{P(E)}{P(E')}=\frac{P(E)}{1-P(E)}=\frac{\frac{1}{6}}{1-\left(\frac{1}{6}\right)}=\frac{\frac{1}{6}}{\frac{5}{6}}=\frac{1}{5}$

The odds are 1:5.

37. $\frac{P(E)}{P(E')}=\frac{P(E)}{1-P(E)}=\frac{0.7}{1-0.7}=\frac{0.7}{0.3}=\frac{7}{3}$

The odds are 7:3.

38. $\frac{P(E)}{P(E')}=\frac{P(E)}{1-P(E)}=\frac{0.001}{1-0.001}=\frac{0.001}{0.999}=\frac{1}{999}$

The odds are 1:999.

39. $P(E)=\frac{7}{7+5}=\frac{7}{12}$

40. $P(E)=\frac{100}{100+1}=\frac{100}{101}$

41. $P(E)=\frac{4}{4+10}=\frac{4}{14}=\frac{2}{7}$

42. $P(E)=\frac{a}{a+a}=\frac{a}{2a}=\frac{1}{2}$

43. Odds that it will rain tomorrow $=\frac{P(\text{rain})}{P(\text{no rain})}=\frac{0.75}{1-0.75}=\frac{0.75}{0.25}=3$.

The odds are 3:1.

44. The odds of E *not* occurring are the odds of event E' which is $\frac{P(E')}{P(E'')}=\frac{P(E')}{P(E)}=\frac{3}{5}$. Then $\frac{P(E)}{P(E')}=\frac{1}{\frac{P(E')}{P(E)}}=\frac{1}{\frac{3}{5}}=\frac{5}{3}$, so the odds that E does occur are 5:3.

In general, if the odds of E *not* occurring are a:b, then the odds that E does occur are b:a.

Problems 8.5

1. a. $P\left(E\middle|F\right)=\frac{n(E\cap F)}{n(F)}=\frac{1}{5}$

b. Using the result of part (a), $P(E'\mid F)=1-P(E\mid F)=1-\frac{1}{5}=\frac{4}{5}$.

c. $F'=\{3,7,8,9\}$ so $P(E\mid F')=\frac{n(E\cap F')}{n(F')}=\frac{1}{4}$.

d. $P(F\mid E)=\frac{n(F\cap E)}{n(E)}=\frac{1}{2}$

e. $F\cap G=\{5,6\}$ so $P(E\mid F\cap G)=\frac{n(E\cap(F\cap G))}{n(F\cap G)}=\frac{0}{2}=0.$

2. a. $P(E)=\frac{n(E)}{n(S)}=\frac{2}{5}$

b. $P(E\mid F)=\frac{n(E\cap F)}{n(F)}=\frac{0}{2}=0$

c. $P(E\mid G)=\frac{n(E\cap G)}{n(G)}=\frac{2}{3}$

d. $P(G\mid E)=\frac{n(G\cap E)}{n(E)}=\frac{2}{2}=1$

e. $F'=\{1,2,5\}$

$P(G\mid F')=\frac{n(G\cap F')}{n(F')}=\frac{2}{3}$

f. $E'=\{3,4,5\}$

$P(E'\mid F')=\frac{n(E'\cap F')}{n(F')}=\frac{1}{3}$

3. $P(E\mid E)=\frac{P(E\cap E)}{P(E)}=\frac{P(E)}{P(E)}=1$

4. $P(\varnothing\mid E)=\frac{P(\varnothing\cap E)}{P(E)}=\frac{P(\varnothing)}{P(E)}=\frac{0}{P(E)}=0$

5. $P(E'\mid F)=1-P(E\mid F)=1-0.57=0.43$

6. $P(F \mid G) = \dfrac{P(F \cap G)}{P(G)} = \dfrac{P(\varnothing)}{P(G)} = \dfrac{0}{P(G)} = 0$

7. a. $P(E \mid F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{1/6}{1/3} = \dfrac{1}{2}$

b. $P(F \mid E) = \dfrac{P(F \cap E)}{P(E)} = \dfrac{1/6}{1/4} = \dfrac{2}{3}$

8. First we find $P(E \cap F)$:

$P(E \mid F) = \dfrac{P(E \cap F)}{P(F)}$,

$P(E \cap F) = P\left(E \middle| F\right) P(F) = \dfrac{3}{4} \cdot \dfrac{1}{3} = \dfrac{1}{4}.$

Then

$P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$= \dfrac{1}{4} + \dfrac{1}{3} - \dfrac{1}{4}$

$= \dfrac{1}{3}.$

9. a. $P(F \mid E) = \dfrac{P(F \cap E)}{P(E)} = \dfrac{1/6}{1/4} = \dfrac{2}{3}$

b. $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$\dfrac{7}{12} = \dfrac{1}{4} + P(F) - \dfrac{1}{6}$

Thus $P(F) = \dfrac{7}{12} - \dfrac{1}{4} + \dfrac{1}{6} = \dfrac{1}{2}..$

c. From part (b) $P(F) = \dfrac{1}{2}.$

Then $P(E \mid F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{1/6}{1/2} = \dfrac{1}{3}.$

d. $P(E) = P(E \cap F) + P(E \cap F')$

$\dfrac{1}{4} = \dfrac{1}{6} + P(E \cap F')$

so $P(E \cap F') = \dfrac{1}{4} - \dfrac{1}{6} = \dfrac{1}{12}.$

Then $P(E \mid F') = \dfrac{P(E \cap F')}{P(F')}$

$= \dfrac{1/12}{1 - 1/2} = \dfrac{1/12}{1/2} = \dfrac{1}{6}.$

10. $P(E \cup F) = P(E) + P(F) - P(E \cap F)$, so

$P(E \cap F) = P(E) + P(F) - P(E \cup F)$

$= \dfrac{3}{5} + \dfrac{3}{10} - \dfrac{7}{10} = \dfrac{1}{5}$

Then $P(E|F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{1/5}{3/10} = \dfrac{2}{3}.$

11. a. $P(F) = \dfrac{125}{200} = \dfrac{5}{8}$

b. $P(F \mid \text{II}) = \dfrac{n(F \cap \text{II})}{n(\text{II})} = \dfrac{35}{58}$

c. $P(O \mid \text{I}) = \dfrac{n(O \cap \text{I})}{n(\text{I})} = \dfrac{22}{78} = \dfrac{11}{39}$

d. $P(\text{III}) = \dfrac{64}{200} = \dfrac{8}{25}$

e. $P(\text{III} \mid O) = \dfrac{n(\text{III} \cap O)}{n(O)} = \dfrac{10}{47}$

f. $P(\text{II} \mid N') = \dfrac{n(\text{II} \cap N')}{n(N')}$

$= \dfrac{35 + 15}{125 + 47} = \dfrac{50}{172} = \dfrac{25}{86}$

12. a. $P(\text{Public}|\text{Middle}) = \dfrac{n(\text{Public} \cap \text{Middle})}{n(\text{Middle})}$

$= \dfrac{55}{80} = \dfrac{11}{16}$

b. $P(\text{High}|\text{Private}) = \dfrac{n(\text{High} \cap \text{Private})}{n(\text{Private})}$

$= \dfrac{14}{49} = \dfrac{2}{7}$

c. $P(\text{Private}|\text{High}) = \dfrac{n(\text{Private} \cap \text{High})}{n(\text{High})}$

$= \dfrac{14}{25}$

d. $P(\text{Public} \cup \text{Low})$

$= P(\text{Public}) + P(\text{Low}) - P(\text{Public} \cap \text{Low})$

$= \dfrac{126}{175} + \dfrac{70}{175} - \dfrac{60}{175} = \dfrac{136}{175}$

13. **a.** $P(\text{A}\mid\text{B})=\dfrac{P(\text{A}\cap\text{B})}{P(\text{B})}=\dfrac{0.20}{0.40}=\dfrac{1}{2}$

b. $P(\text{B}\mid\text{A})=\dfrac{P(\text{B}\cap\text{A})}{P(\text{A})}=\dfrac{0.20}{0.45}=\dfrac{4}{9}$

14. P(scratched screen|def. ear pieces)

$=\dfrac{P(\text{scratched screen}\cap\text{def. ear pieces})}{P(\text{def. ear pieces})}$

$=\dfrac{0.13}{0.19}=\dfrac{13}{19}$

15. S = {BB, BG, GG, GB}
Let E = {at least one girl} = {BG, GG, GB},
F = {at least one boy} = {BB, BG, GB}.
Thus $P(E\mid F)=\dfrac{n(E\cap F)}{n(F)}=\dfrac{2}{3}$.

16. S = {BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}
Let
E = {at least two girls}
= {BGG, GBG, GGB, GGG},
F = {at least one boy}
= {BBB, BBG, BGB, BGG, GBB, GBG, GGB},
G = {oldest is a girl}
= {GBB, GBG, GGB, GGG}.

a. $P(E\mid F)=\dfrac{n(E\cap F)}{n(F)}=\dfrac{3}{7}$

b. $P(E\mid G)=\dfrac{n(E\cap G)}{n(G)}=\dfrac{3}{4}$

17. S = {HHH, HHT, HTH, THH, THT, TTH, TTT}.
Let E = {exactly two tails}
= {HTT, THT, TTH},
F = {second toss is a tail}
= {HTH, HTT, TTH, TTT},
G = {second toss is a head}
= {HHH, HHT, THH, THT}.

a. $P(E\mid F)=\dfrac{n(E\cap F)}{n(F)}=\dfrac{2}{4}=\dfrac{1}{2}$

b. $P(E\mid G)=\dfrac{n(E\cap G)}{n(G)}=\dfrac{1}{4}$

18. S = {HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}.
Let E = {four tails } = {TTTT}, F = {first toss is a tail} = {THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}.
Since $P(E\mid F)=\dfrac{n(E\cap F)}{n(F)}=\dfrac{1}{8}$, the corresponding odds are
$\dfrac{P(E\mid F)}{P(E'\mid F)}=\dfrac{1/8}{1-(1/8)}=\dfrac{1}{7}$; that is, 1 to 7.

19. $P(<4\mid\text{odd})=\dfrac{n(<4\cap\text{odd})}{n(\text{odd})}=\dfrac{n(\{1,3\})}{n(\{1,3,5\})}=\dfrac{2}{3}$

20. Let F denote face card. There are 3 face cards for each suit. Let R denote red card. Half the cards are red, so there are 26.
$P\left(F\middle|R\right)=\dfrac{n(F\cap R)}{n(R)}=\dfrac{6}{26}=\dfrac{3}{13}$.

21. *Method 1.* The usual sample space has 36 outcomes, where the event "two 1's" is {(1, 1)}. Note that $\{\text{at least one 1}\}'=\{\text{no 1's}\}$, and the event "no 1's" occurs in 5 · 5 = 25 ways. Thus
P(two 1's | at least one 1)
$=\dfrac{n(\text{two 1's }\cap\text{at least one 1})}{n(\text{at least one 1})}=\dfrac{n(\{(1,1)\})}{36-25}=\dfrac{1}{11}$

Method 2. From the usual sample space, we find that the reduced sample space for "at least one 1" (which has 11 outcomes) is {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (3, 1), (4, 1), (5, 1)}.
Thus P(two 1's | at least one 1) = $\dfrac{1}{11}$.

22. *Method 1*. The reduced sample space, having 6 outcomes, is {(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)}, where, in each pair, the outcome 5 on the red die is given first. Two pairs have a sum greater than 9, namely (5, 5) and (5, 6). Thus $P(\text{sum}>9|\text{5 on red})=\dfrac{2}{6}=\dfrac{1}{3}$.

Method 2. The usual sample space has 36 outcomes. Let E = {5 on red}. Then $n(E)$ = 6. Let F = {sum > 9}. Then $n(E\cap F)=2$, namely (red 5, green 5) and (red 5, green 6). Thus
$P(F\mid E)=\dfrac{n(E\cap F)}{n(E)}=\dfrac{2}{6}=\dfrac{1}{3}$.

23. The usual sample space consists of ordered pairs (R, G), where R = no. on red die and G = no. on green die. Now, n(green is even) = 6 · 3 = 18, because the red die can show any of six numbers and the green any of three: 2, 4, or 6. Also,
$n(\text{total of } 7 \cap \text{green even})$
$= n(\{(5, 2), (3, 4), (1, 6)\}) = 3.$
Thus
$P(\text{total of } 7 | \text{green even})$

$$= \frac{n(\text{total of } 7 \cap \text{green even})}{n(\text{green even})}$$

$$= \frac{3}{18} = \frac{1}{6}.$$

24. The usual sample space S consists of 36 ordered pairs. Let E = {sum is 6} and F = {second toss is neither 2 nor 4}. Then $n(F) = 6 \cdot 4 = 24$ and $n(E \cap F) = n\{(5, 1), (3, 3), (1, 5)\} = 3.$

a. $P(E | F) = \dfrac{n(E \cap F)}{n(F)} = \dfrac{3}{24} = \dfrac{1}{8}$

b. $P(E \cap F) = \dfrac{n(E \cap F)}{n(S)} = \dfrac{3}{36} = \dfrac{1}{12}$

25. The usual sample space consists of 36 ordered pairs. Let E = {total > 7} and F = {first toss > 3}. Then $n(F) = 3 \cdot 6 = 18$ and
$n(E \cap F)$
$= n(\{(4, 4), (4, 5), (4, 6), (5, 3), (5, 4), (5, 5), (5, 6), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\})$
$= 12$

Thus $P\left(E \middle| F\right) = \dfrac{n(E \cap F)}{n(F)} = \dfrac{12}{18} = \dfrac{2}{3}.$

26. Let the sample space consist of ordered pairs (c, d), where c is T or H, and d is the number showing on the die. Let E = {tails shows} and F = {die shows odd number). Then $N(F) = 2 \cdot 3 = 6$ and $n(E \cap F) = 1 \cdot 3 = 3$. Thus

$$P(E | F) = \frac{n(E \cap F)}{n(F)} = \frac{3}{6} = \frac{1}{2}.$$

27. $P(K | H) = \dfrac{n(K \cap H)}{n(H)} = \dfrac{1}{13}$

28. $P(H | F) = \dfrac{n(H \cap F)}{n(F)} = \dfrac{3}{12} = \dfrac{1}{4}$

29. Let E = {second card is not a face card} and F = {first card is a face card}.

$$P(E | F) = \frac{n(E \cap F)}{n(F)} = \frac{12 \cdot \frac{51-11}{51}}{12} = \frac{40}{51}$$

30. a. $P(F_1 \cap F_2) = P(F_1)P(F_2 | F_1)$

$$= \frac{12}{52} \cdot \frac{11}{51} = \frac{11}{221}$$

b. $P(F_1 \cap F_2) = P(F_1)P(F_2 | F_1)$

$$= \frac{12}{52} \cdot \frac{12}{52} = \frac{3}{13} \cdot \frac{3}{13} = \frac{9}{169}$$

31. $P(K_1 \cap Q_2 \cap J_3)$
$= P(K_1)P(Q_2 | K_1)P(J_3 | (K_1 \cap Q_2))$

$$= \frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50} = \frac{8}{16{,}575}$$

32. $P(AS_1 \cap AH_2 \cap AD_2)$
$= P(AS_1)P\left(AH_2 \middle| AS_1\right)P\left(AD_2 \middle| (AS_1 \cap AH_2)\right)$

$$= \frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50} = \frac{1}{132{,}600}.$$

33. $P(J_1 \cap J_2 \cap J_3)$
$= P(J_1)P(J_2 | J_1)P(J_3 | (J_1 \cap J_2))$

$$= \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} = \frac{1}{5525}$$

34. Using a probability tree, we find that there are two possible paths such that the second card is a heart, namely, a heart followed by a heart, or a nonheart followed by a heart. Thus
$P(H_2) = P(H_1 \cap H_2) + P(H_1' \cap H_2)$
$= P(H_1)P(H_2 | H_1) + P(H_1')P(H_2 | H_1')$

$$= \frac{13}{52} \cdot \frac{12}{51} + \frac{39}{52} \cdot \frac{13}{51} = \frac{1}{4}.$$

35. Let D = {two diamonds} and R = {first card red}. We have $D \cap R = \{\text{two diamonds}\} = D$ and

$$P(D) = \frac{13}{52} \cdot \frac{12}{51}.$$

Thus $P(D | R) = \dfrac{P(D \cap R)}{P(R)} = \dfrac{\frac{13}{52} \cdot \frac{12}{51}}{\frac{26}{52}} = \dfrac{2}{17}.$

36. Using a probability tree, we find that there are two possible paths such that she will be on time, namely, she gets the call and she is on time, or she doesn't get the call and she is on time.

$P(T) = P(C \cap T) + P(C' \cap T)$

$= P(C)P(T \mid C) + P(C')P(T \mid C')$

$= (0.9)(0.9) + (0.1)(0.4) = 0.85$

37. a. $P(U) = P(F \cap U) + P(O \cap U) + P(N \cap U)$

$= P(F)P(U|F) + P(O)P(U|O) + P(N)P(U|N)$

$= (0.60)(0.45) + (0.30)(0.55) + (0.10)(0.35)$

$= 0.47 = \frac{47}{100}$

b. $P(F \mid U) = \frac{P(F \cap U)}{P(U)} = \frac{(0.60)(0.45)}{0.47} = \frac{27}{47}$

38. a. $P(\text{contact} \cap \text{purchase}) = P(\text{contact})P(\text{purchase}|\text{contact})$

$= (0.02)(0.014) = 0.00028$

b. $100,000(0.00028) = 28$

39. a. After the first draw, if the rabbit drawn is red, then 4 rabbits remain, 3 of which are yellow.

$P(\text{second is yellow} \mid \text{first is red}) = \frac{3}{4}$

b. After red rabbit is replaced, 5 rabbits remain, 3 of which are yellow.

$P(\text{second is yellow} \mid \text{first is red}) = \frac{3}{5}$

40. $P(G_2) = P(G_1 \cap G_2) + P(R_1 \cap G_2) = P(G_1)P(G_2 \mid G_1) + P(R_1)P(G_2 \mid R_1) = \frac{4}{7} \cdot \frac{4}{7} + \frac{3}{7} \cdot \frac{3}{7} = \frac{25}{49}$

41. $P(W) = P(\text{Box 1} \cap \text{W}) + P(\text{Box 2} \cap \text{W}) = P(\text{Box 1})P(W \mid \text{Box 1}) + P(\text{Box 2})P(W \mid \text{Box 2}) = \frac{1}{2} \cdot \frac{2}{5} + \frac{1}{2} \cdot \frac{2}{4} = \frac{9}{20}$

42. a. $P(W) = P(B1 \cap W) + P(B2 \cap W) + P(B3 \cap W)$

$= P(B1)P(W \mid B1) + P(B2)P(W \mid B2) + P(B3)P(W \mid B3)$

$= \frac{1}{3} \cdot \frac{3}{5} + \frac{1}{3} \cdot \frac{4}{7} + \frac{1}{3} \cdot \frac{2}{6} = \frac{158}{315}$

b. $P(R) = P(B1 \cap R) + P(B2 \cap R) + P(B3 \cap R)$

$= P(B1)P(R \mid B1) + P(B2)P(R \mid B2) + P(B3)P(R \mid B3)$

$= \frac{1}{3} \cdot \frac{2}{5} + \frac{1}{3} \cdot \frac{3}{7} + \frac{1}{3} \cdot \frac{2}{6} = \frac{122}{315}$

c. $P(G) = P(B3 \cap G) = P(B3)P(G \mid B3) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

43. $P(W_2) = P(B1 \cap G_1 \cap W_2) + P(B1 \cap R_1 \cap W_2) + P(B2 \cap W_1 \cap W_2)$

$= P(B1)P\left(G_1 \middle| B1\right)P\left(W_2 \middle| (G_1 \cap B1)\right) + P(B1)P\left(R_1 \middle| B1\right)P\left(W_2 \middle| (R_1 \cap B1)\right) + P(B2)P\left(W_1 \middle| B2\right)P\left(W_2 \middle| (W_1 \cap B2)\right)$

$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{4}$

44. $P(D_1 \cap D_2 \cap D_3 \cap D_4) = P(D_1)P\left(D_2 \middle| D_1\right)P\left(D_3 \middle| (D_1 \cap D_2)\right)P\left(D_4 \middle| (D_1 \cap D_2 \cap D_3)\right)$

$= \frac{5}{10} \cdot \frac{4}{9} \cdot \frac{3}{8} \cdot \frac{2}{7} = \frac{1}{42}$

45. $P(\text{Und.}) = P(\text{MS} \cap \text{Und.}) + P(\text{DS} \cap \text{Und.})$

$= P(\text{MS})P(\text{Und.}|\text{MS}) + P(\text{DS})P(\text{Und}|\text{DS})$

$= \frac{20,000}{60,000} \cdot \frac{1}{100} + \frac{40,000}{60,000} \cdot \frac{3}{100}$

$= \frac{7}{300}$

46. $P(5000) = P(B1 \cap 5000) + P(B2 \cap 5000) + P(B3 \cap 5000)$

$= P(B1)P(5000|B1) + P(B2)P(5000|B2) + P(B3)P(5000|B3)$

$= \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{2}{8} + \frac{1}{3} \cdot \frac{1}{6} = \frac{11}{36}$

47. $P(\text{Def}) = P(\text{A} \cap \text{Def}) + P(\text{B} \cap \text{Def}) + P(\text{C} \cap \text{Def})$

$= P(\text{A})P(\text{Def} \mid \text{A}) + P(\text{B})P(\text{Def} \mid \text{B}) + P(\text{C})P(\text{Def} \mid \text{C})$

$= (0.10)(0.06) + (0.20)(0.04) + (0.70)(0.05) = 0.049$

48. $P(\text{Def}) = P(\text{A} \cap \text{Def}) + P(\text{B} \cap \text{Def}) + P(\text{C} \cap \text{Def}) + P(\text{D} \cap \text{Def})$

$= P(\text{A})P(\text{Def} \mid \text{A}) + P(\text{B})P(\text{Def} \mid \text{B}) + P(\text{C})P(\text{Def} \mid \text{C}) + P(\text{D})P(\text{Def} \mid \text{D})$

$= (0.30)(0.06) + (0.20)(0.03) + (0.35)(0.02) + (0.15)(0.05)$

$= 0.0385$

49. a. $P(D \cap V) = P(D)P(V \mid D) = (0.40)(0.15) = 0.06$

b. $P(V) = P(D \cap V) + P(R \cap V) + P(I \cap V)$

$= P(D)P(V \mid D) + P(R)P(V \mid R) + P(I)P(V \mid I)$

$= (0.40)(0.15) + (0.35)(0.20) + (0.25)(0.10)$

$= 0.155$

50. Because Richard was not hired, the number of sample points in the reduced sample space is ${}_7C_4 = 35,$ of which Allison, Lesley, Tom, and Bronwyn form one sample point. Thus

$P(\text{Allison, Lesley, Tom, and Bronwyn were hired}) = \frac{1}{35}.$

51. $P(\text{3 Fem}|\text{at least one Fem})$

$$= \frac{P(\text{3 Fem} \cap \text{at least one Fem})}{P(\text{at least one Fem})}$$

$$= \frac{P(\text{3 Fem})}{1-P(\text{no Fem})} = \frac{\frac{{}_6C_3}{{}_{11}C_3}}{1-\frac{{}_5C_3}{{}_{11}C_3}} = \frac{\frac{4}{33}}{1-\frac{2}{33}} = \frac{4}{31}$$

Problems 8.6

1. a. $P(E \cap F) = P(E)P(F) = \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$

b. $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$= \frac{1}{3} + \frac{3}{4} - \frac{1}{4} = \frac{5}{6}$

c. $P(E \mid F) = P(E) = \frac{1}{3}$

d. $P(E' \mid F) = 1 - P(E \mid F) = 1 - \frac{1}{3} = \frac{2}{3}$

e. $P(E \cap F') = P(E)P(F') = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$

f. $P(E \cup F') = P(E) + P(F') - P(E \cap F')$

$= \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{1}{2}$

g. $P(E \mid F') = \frac{P(E \cap F')}{P(F')} = \frac{1/12}{1/4} = \frac{1}{3}$

2. a. $P(E \cap F) = P(E)P(F) = (0.1)(0.3) = 0.03$

b. $P(F \cap G) = P(F)P(G) = (0.3)(0.6) = 0.18$

c. $P(E \cap F \cap G) = P(E)P(F)P(G)$

$= (0.1)(0.3)(0.6) = 0.018$

d. $P\left(E \middle| (F \cap G)\right) = \frac{P(E \cap F \cap G)}{P(F \cap G)}$

$= \frac{0.018}{0.18} = 0.1$

e. $P(E' \cap F \cap G') = P(E')P(F)P(G')$

$= (0.9)(0.3)(0.4) = 0.108$

3. $P(E \cap F) = P(E)P(F)$,

$\frac{1}{9} = \frac{2}{7} \cdot P(F)$ so $P(F) = \frac{1}{9} \cdot \frac{7}{2} = \frac{7}{18}$

4. $P(E) = P(E \mid F) = \frac{1}{3}$,

so $P(E') = 1 - P(E) = 1 - \frac{1}{3} = \frac{2}{3}$.

5. $P(E)P(F) = \frac{3}{4} \cdot \frac{8}{9} = \frac{2}{3} = P(E \cap F)$

Since $P(E)P(F) = P(E \cap F)$, events E and F are independent.

6. $P(E)P(F) = (0.28)(0.15) = 0.042 \neq P(E \cap F)$, so E and F are dependent events.

7. Let F = {full service} and I = {increase in value}.

$P(F) = \frac{400}{600} = \frac{2}{3}$

and $P(F \mid I) = \frac{n(F \cap I)}{n(I)} = \frac{320}{480} = \frac{2}{3}$

Since $P(F \mid I) = P(F)$, events F and I are independent.

8. Let M = {male} and C = {cruncher}.

$P(M) = \frac{130}{175} = \frac{26}{35}$ and

$P(M \mid C) = \frac{n(M \cap C)}{n(C)} = \frac{55}{80} = \frac{11}{16}$

Since $P(M \mid C) \neq P(M)$, events M and C are dependent.

9. Let S be the usual sample space consisting of ordered pairs of the form (R, G), where the first component of each pair represents the number showing on the red die, and the second component represents the number on the green die. Then $n(S) = 6 \cdot 6 = 36$. For E, any number of four can occur on the red die, and any number on the green die. Thus $n(E) = 4 \cdot 6 = 24$. For F we have F = {(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)}, so $n(F) = 5$.
Also, $E \cap F = \{(4, 4), (5, 3), (6, 2)\}$, so $n(E \cap F) = 3$. Thus $P(E)P(F) = \frac{24}{36} \cdot \frac{5}{36} = \frac{5}{54}$ and $P(E \cap F) = \frac{3}{36} = \frac{1}{12}$. Since

$P(E)P(F) \neq P(E \cap F)$, events E and F are dependent.

10. $P(E) = \frac{26}{52} = \frac{1}{2}$

$P(F) = \frac{12}{52} = \frac{3}{13}$, and $P(E \cap F) = \frac{6}{52} = \frac{3}{26}$.

Because $P(E)P(F) = \frac{1}{2} \cdot \frac{3}{13} = \frac{3}{26} = P(E \cap F)$, events E and F are independent.

11. S = {HH, HT, TH, TT},
E = {HT, TH, TT},
F = {HT, TH}, and $E \cap F = \{\text{HT}, \text{TH}\}$.

Thus $P(E) = \frac{3}{4}$

$P(F) = \frac{2}{4} = \frac{1}{2}$, and

$P(E \cap F) = \frac{2}{4} = \frac{1}{2}$. We have

$P(E)P(F) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8} \neq P(E \cap F)$, so events E and F are dependent.

12. S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}
and $n(S) = 8$.
E = {HTT, THT, TTH, TTT} and $n(E) = 4$.
F = {HHT, HTH, THH, HTT, THT, TTH} and $n(F) = 6$.
$E \cap F = \{\text{HTT}, \text{THT}, \text{TTH}\}$ and $n(E \cap F) = 3$.

Thus $P(E)P(F) = \frac{4}{8} \cdot \frac{6}{8} = \frac{3}{8} = P(E \cap F)$, so E and F are independent.

13. Let S be the set of ordered pairs whose first (*second*) component represents the number on the first (*second*) chip. Then $n(S) = 7 \cdot 7 = 49$, $n(E) = 1 \cdot 7 = 7$, and $n(F) = 7 \cdot 1 = 7$. For G, if the first chip is 1, 3, 5 or 7, then the second chip must be 2, 4 or 6; if the first chip is 2, 4 or 6, the second must be 1, 3, 5 or 7. Thus $n(G) = 4 \cdot 3 + 3 \cdot 4 = 24$.

a. $E \cap F = \{(3,3)\}$, so $P(E \cap F) = \frac{1}{49}$. Since

$P(E)P(F) = \frac{7}{49} \cdot \frac{7}{49} = \frac{1}{49} = P(E \cap F)$,
events E and F are independent.

b. $E \cap G = \{(3,2), (3,4), (3,6)\}$,

so $P(E \cap G) = \frac{3}{49}$. Since

$P(E)P(G) = \frac{7}{49} \cdot \frac{24}{49} = \frac{24}{343} \neq P(E \cap G)$,
events E and G are dependent.

c. $F \cap G = \{(2,3), (4,3), (6,3)\}$

so $P(F \cap G) = \frac{3}{49}$.

Since

$P(F)P(G) = \frac{7}{49} \cdot \frac{24}{49} = \frac{24}{343} \neq P(F \cap G)$.
Events F and G are dependent.

d. $E \cap F \cap G = \varnothing$, so $P(E \cap F \cap G) = 0$.
However,
$P(E)P(F)P(G) \neq 0 = P(E \cap F \cap G)$,
so events E, F and G are not independent.

14. a. $E = \{3\}$
$F = \{5\}$
$E \cap F = \varnothing$, so E and F are mutually exclusive.

b. $P(E) = P(F) = \frac{1}{6}$

$P(E \cap F) = 0$

$P(E)P(F) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} \neq P(E \cap F)$
Thus E and F are not independent.

15. $P(E \cap F) = P(E)P(F \mid E)$, thus

$P(E) = \frac{P(E \cap F)}{P(F \mid E)} = \frac{0.3}{0.4} = 0.75$

Since $P(E) = 0.75 \neq 0.5 = P(E \mid F)$, E and F are dependent.

16. $P(E \cap F) = P(F)P(E \mid F)$,

thus $P(F) = \dfrac{P(E \cap F)}{P(E \mid F)} = \dfrac{\frac{5}{9}}{\frac{2}{3}} = \dfrac{5}{6}$

$P(E \cup F) = P(E) + P(F) - P(E \cap F)$, so

$P(E) = P(E \cup F) - P(F) + P(E \cap F) = \dfrac{17}{18} - \dfrac{5}{6} + \dfrac{5}{9} = \dfrac{2}{3}$

Since $P(E) = \dfrac{2}{3} = P(E \mid F)$, events E and F are independent.

17. Let E = {red 4} and F = {green > 4}. Assume E and F are independent.

$P(E \cap F) = P(E)P(F) = \dfrac{1}{6} \cdot \dfrac{1}{3} = \dfrac{1}{18}$

18. E_i = {2 or 3 shows on *i*th roll}, where *i* = 1, 2, 3. Assume the E_i's are independent.

$P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2)P(E_3) = \left(\dfrac{1}{3}\right)\left(\dfrac{1}{3}\right)\left(\dfrac{1}{3}\right) = \dfrac{1}{27}$

19. Let F = {first person attends regularly} and S = {second person attends regularly}.

Then $P(F \cap S) = P(F)P(S) = \dfrac{1}{5} \cdot \dfrac{1}{5} = \dfrac{1}{25}$.

20. *P*(double on any throw) = $\dfrac{6}{36} = \dfrac{1}{6}$

Assume that the throws are independent.

P(double on all three throws) = *P*(double on 1st) · *P*(double on 2nd) · *P*(double on 3rd)

$= \dfrac{1}{6} \cdot \dfrac{1}{6} \cdot \dfrac{1}{6} = \dfrac{1}{216}$.

21. Because of replacement, assume the cards selected on the draws are independent events.

P(ace, then face card, then spade) = *P*(ace) · *P*(face card) · *P*(spade)

$= \dfrac{4}{52} \cdot \dfrac{12}{52} \cdot \dfrac{13}{52} = \dfrac{3}{676}$

22. Assume the outcomes on the rolls are independent events.

a. $P(>4, >4, >4, >4, >4, >4, >4) = \dfrac{2}{6} \cdot \dfrac{2}{6} \cdot \dfrac{2}{6} \cdot \dfrac{2}{6} \cdot \dfrac{2}{6} \cdot \dfrac{2}{6} \cdot \dfrac{2}{6} = \dfrac{1}{2187}$

b. $P(<4, <4, <4, <4, <4, <4, <4) = \dfrac{3}{6} \cdot \dfrac{3}{6} \cdot \dfrac{3}{6} \cdot \dfrac{3}{6} \cdot \dfrac{3}{6} \cdot \dfrac{3}{6} \cdot \dfrac{3}{6} = \dfrac{1}{128}$

23. a. *P*(Bill gets A $\cap$ Jim gets A $\cap$ Linda gets A)

= *P*(Bill gets A) · *P*(Jim gets A) · *P*(Linda gets A)

$= \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \dfrac{4}{5} = \dfrac{3}{10}$.

b. $P(\text{Bill no A} \cap \text{Jim no A} \cap \text{Linda no A})$
$= P(\text{Bill no A}) \cdot P(\text{Jim no A}) \cdot P(\text{Linda no A})$
$= \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{40}$

c. $P(\text{Bill no A} \cap \text{Jim no A} \cap \text{Linda gets A})$
$= P(\text{Bill no A}) \cdot P(\text{Jim no A}) \cdot P(\text{Linda gets A})$
$= \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{4}{5} = \frac{1}{10}$

24. Assume independence of rolls.
$P(\text{at least one 6}) = 1 - P(\text{no 6's}) = 1 - \left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right) = \frac{91}{216}$

25. Let A = {A survives 15 more years},
B = {B survives 15 more years}.

a. $P(A \cap B) = P(A)P(B) = \frac{2}{3} \cdot \frac{3}{5} = \frac{2}{5}$

b. $P(A' \cap B) = P(A')P(B) = \frac{1}{3} \cdot \frac{3}{5} = \frac{1}{5}$

c. $A \cap B'$ and $A' \cap B$ are mutually exclusive.
$P[(A \cap B') \cup (A' \cap B)] = P(A)P(B') + P(A')P(B) = \frac{2}{3} \cdot \frac{2}{5} + \frac{1}{3} \cdot \frac{3}{5} = \frac{7}{15}$

d. $P(\text{at least one survives}) = P(\text{exactly one survives}) + P(\text{both survive}) = \frac{7}{15} + \frac{2}{5} = \frac{13}{15}$.

e. $P(\text{neither survives}) = 1 - P(\text{at least one survives}) = 1 - \frac{13}{15} = \frac{2}{15}$.

26. Assume that drawing a particular size of paper and a particular size of envelope are independent events.
$P(\text{paper A} \cap \text{envelope A}) + P(\text{paper B} \cap \text{envelope B}) = (0.63)(0.57) + (0.37)(0.43) \approx 0.52$

27. Assume the colors selected on the draws are independent events.

a. $P(W_1 \cap G_2) = P(W_1)P(G_2) = \frac{7}{18} \cdot \frac{6}{18} = \frac{7}{54}$

b. $P[(R_1 \cap W_2) \cup (W_1 \cap R_2)] = P(R_1)P(W_2) + P(W_1)P(R_2) = \frac{5}{18} \cdot \frac{7}{18} + \frac{7}{18} \cdot \frac{5}{18} = \frac{35}{162}$

28. Assume the rolls are independent.

$P(7 \text{ on a roll}) = P\{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\} = \frac{6}{36} = \frac{1}{6}$

$P(12 \text{ on a roll}) = P\{(6, 6)\} = \frac{1}{36}$

$P(7 \text{ on one roll and 12 on the other}) = \frac{1}{6} \cdot \frac{1}{36} + \frac{1}{36} \cdot \frac{1}{6} = \frac{1}{108}$

29. Assume that the selections are independent.

$P(\text{both red} \cup \text{both white} \cup \text{both green}) = \frac{3}{19} \cdot \frac{3}{19} + \frac{7}{19} \cdot \frac{7}{19} + \frac{9}{19} \cdot \frac{9}{19} = \frac{139}{361}$

30. Assume the throws are independent. For a particular number,

$P(\text{particular number on three throws}) = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \left(\frac{1}{6}\right)^3.$

Since the particular number can be any of 6 numbers,

$P(\text{same number in 3 throws}) = 6\left(\frac{1}{6}\right)^3 = \frac{1}{36}.$

31. Assume that the draws are independent.

$P(\text{particular 1st ticket} \cap \text{particular 2nd ticket})$

$= \frac{1}{20} \cdot \frac{1}{20} = \frac{1}{400}$

$P(\text{sum is 35}) = P\{(20, 15), (19, 16), (18, 17), (17, 18), (16, 19), (15, 20)\}$

$= 6\left(\frac{1}{400}\right) = \frac{3}{200}$

32. a. $P(\{TT33\}) = P(\text{T on 1st coin})\, P(\text{T on 2nd coin})\, P(\text{3 on 1st die})\, P(\text{3 on 2nd die})$

$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{144}$

b. $P(\text{two heads, one 4 and one 6})$

$= P(\text{H on 1st coin})\, P(\text{H on 2nd coin})\, P(\text{4 on 1st die})\, P(\text{6 on 2nd die})$

$+ P(\text{H on 1st coin}) P(\text{H on 2nd coin})\, P(\text{6 on 1st die})\, P(\text{4 on 2nd die})$

$= \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{6}\right) \cdot 2 = \frac{1}{72}$

33. a. $\frac{1}{12} \cdot \frac{1}{12} \cdot \frac{1}{12} = \frac{1}{1728}$

b. To get exactly one even, there are ${}_3C_1 = 3$ ways.

$P(\text{one even and two odd}) = 3[P(\text{even 1st spin}) \cdot P(\text{odd 2nd spin}) \cdot P(\text{odd 3rd spin})]$

$= 3\left(\frac{6}{12} \cdot \frac{6}{12} \cdot \frac{6}{12}\right) = \frac{3}{8}.$

34. a. $\frac{4}{52} \cdot \frac{13}{52} \cdot \frac{2}{52} = \frac{1}{1352}$

b. $\frac{4}{52} \cdot \frac{4}{52} \cdot \frac{4}{52} = \frac{1}{2197}$

c. The queen, spade, and black ace can be drawn in any order, so there are 3! = 6 orders, thus $6 \cdot \frac{4}{52} \cdot \frac{13}{52} \cdot \frac{2}{52} = \frac{3}{676}$.

d. The ace can come first, second, or third, so $3 \cdot \frac{4}{52} \cdot \frac{48}{52} \cdot \frac{48}{52} = \frac{432}{2197}$.

35. a. The number of ways of getting exactly four correct answers out of five is ${}_5C_4 = 5$. Each of these ways has a probability of $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{1024}$. Thus

$P(\text{exactly 4 correct}) = 5 \cdot \frac{3}{1024} = \frac{15}{1024}$.

b. P(at least 4 correct) = P(exactly 4) + P(exactly 5)

$= \frac{15}{1024} + \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{64}$

c. The number of ways of getting exactly three correct answers out of five is ${}_5C_3 = 10$. Each of these ways has a probability of $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{1024}$, so

$P(\text{exactly 3 correct}) = 10 \cdot \frac{9}{1024} = \frac{45}{512}$. Thus

P(3 or more correct) $= P$(exactly 3) + P(at least 4)

$= \frac{45}{512} + \frac{1}{64} = \frac{53}{512}$.

36. a. P(none hit) = (0.5)(0.6)(0.3) = 0.09

b. P(only Linda hits) = (0.5)(0.6)(0.7) = 0.21

c. P(exactly one hits target) = P(only Bill) + P(only Jim) + P(only Linda)
= (0.5)(0.6)(0.3) + (0.5)(0.4)(0.3) + (0.5)(0.6)(0.7) = 0.36

d. P(exactly 2) = P(not Bill) + P(not Jim) + P(not Linda)
= (0.5)(0.4)(0.7) + (0.5)(0.6)(0.7) + (0.5)(0.4)(0.3) = 0.41

e. P(all hit) = (0.5)(0.4)(0.7) = 0.14

37. A wrong majority decision can occur in one of two mutually exclusive ways: exactly two wrong recommendations, or three wrong recommendations. Exactly two wrong recommendations can occur in ${}_3C_2 = 3$ mutually exclusive ways. Thus
P(wrong majority decision)
= [(0.04)(0.05)(0.9) + (0.04)(0.95)(0.1) + (0.96)(0.05)(0.1)] + (0.04)(0.05)(0.1)
= 0.0106.

Problems 8.7

1. $P(E \mid D) = \dfrac{P(E)P(D \mid E)}{P(E)P(D \mid E) + P(F)P(D \mid F)} = \dfrac{\frac{2}{5} \cdot \frac{1}{10}}{\frac{2}{5} \cdot \frac{1}{10} + \frac{3}{5} \cdot \frac{1}{5}} = \dfrac{1}{4}$

For the second part, $P(D' \mid F) = 1 - P(D \mid F) = 1 - \dfrac{1}{5} = \dfrac{4}{5}$, and

$P(D' \mid E) = 1 - P(D \mid E) = 1 - \dfrac{1}{10} = \dfrac{9}{10}$. Then

$P(F \mid D') = \dfrac{P(F)P(D' \mid F)}{P(E)P(D' \mid E) + P(F)P(D' \mid F)} = \dfrac{\frac{3}{5} \cdot \frac{4}{5}}{\frac{2}{5} \cdot \frac{9}{10} + \frac{3}{5} \cdot \frac{4}{5}} = \dfrac{4}{7}$.

2. $P(E_1 \mid S) = \dfrac{P(E_1)P(S \mid E_1)}{P(E_1)P(S \mid E_1) + P(E_2)P(S \mid E_2) + P(E_3)P(S \mid E_3)} = \dfrac{\frac{1}{5} \cdot \frac{2}{5}}{\frac{1}{5} \cdot \frac{2}{5} + \frac{3}{10} \cdot \frac{7}{10} + \frac{1}{2} \cdot \frac{1}{2}} = \dfrac{4}{27}$.

$P(E_3 \mid S') = \dfrac{P(E_3)P(S' \mid E_3)}{P(E_1)P(S' \mid E_1) + P(E_2)P(S' \mid E_2) + P(E_3)P(S' \mid E_3)} = \dfrac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{5} \cdot \frac{3}{5} + \frac{3}{10} \cdot \frac{3}{10} + \frac{1}{2} \cdot \frac{1}{2}} = \dfrac{25}{46}$.

3. D = {is Democrat},
R = {is Republican},
I = {is Independent},
V = {voted}.

$$P(D \mid V) = \frac{P(D)P(V \mid D)}{P(D)P(V \mid D) + P(R)P(V \mid R) + P(I)P(V \mid I)}$$

$$= \frac{(0.42)(0.25)}{(0.42)(0.25) + (0.33)(0.27) + (0.25)(0.15)}$$

$$= \frac{175}{386} \approx 0.453$$

4. D = {tire is domestic}
I = {tire is imported}
S = {tire is all-season}

$P(D) = \dfrac{2000}{3000} = \dfrac{2}{3}$ and $P(I) = \dfrac{1000}{3000} = \dfrac{1}{3}$.

Note: 40% = $\dfrac{2}{5}$ and 10% = $\dfrac{1}{10}$.

$$P(I \mid S) = \frac{P(I)P(S \mid I)}{P(I)P(S \mid I) + P(D)P(S \mid D)}$$

$$= \frac{\frac{1}{3} \cdot \frac{1}{10}}{\frac{1}{3} \cdot \frac{1}{10} + \frac{2}{3} \cdot \frac{2}{5}} = \frac{1}{9}$$

5. D = {has the disease}
D' = {does not have the disease}
R = {positive reaction}
N = {negative reaction} = R'

a. $$P(D \mid R) = \frac{P(D)P(R \mid D)}{P(D)P(R \mid D) + P(D')P(R \mid D')} = \frac{(0.03)(0.86)}{(0.03)(0.86) + (0.97)(0.07)} = \frac{258}{937} \approx 0.275$$

b. $$P(D \mid N) = \frac{P(D)P(N \mid D)}{P(D)P(N \mid D) + P(D')P(N \mid D')} = \frac{(0.03)(0.14)}{(0.03)(0.14) + (0.97)(0.93)} = \frac{14}{3021} \approx 0.005$$

6. I = {increase in earnings}
D = {declare a dividend}

Note: 60% = $\frac{3}{5}$ and 10% = $\frac{1}{10}$.

$$P(I \mid D) = \frac{P(I)P(D \mid I)}{P(I)P(D \mid I) + P(I')P(D \mid I')} = \frac{\frac{1}{3} \cdot \frac{3}{5}}{\frac{1}{3} \cdot \frac{3}{5} + \frac{2}{3} \cdot \frac{1}{10}} = \frac{3}{4} = 75\%$$

7. B_1 = {first bag selected}
B_2 = {second bag selected}
R = {red jelly bean drawn}

$P(B_1) = P(B_2) = \frac{1}{2}$.

$$P(B_1 \mid R) = \frac{P(B_1)P(R \mid B_1)}{P(B_1)P(R \mid B_1) + P(B_2)P(R \mid B_2)} = \frac{\frac{1}{2} \cdot \frac{4}{6}}{\frac{1}{2} \cdot \frac{4}{6} + \frac{1}{2} \cdot \frac{2}{5}} = \frac{5}{8}.$$

8. B_1 = {Bowl I selected}
B_2 = {Bowl II selected}
B_3 = {Bowl III selected}
W = {white ball selected}

$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$

$$P(B_1 \mid W) = \frac{P(B_1)P(W \mid B_1)}{P(B_1)P(W \mid B_1) + P(B_2)P(W \mid B_2) + P(B_3)P(W \mid B_3)} = \frac{\frac{1}{3} \cdot \frac{3}{5}}{\frac{1}{3} \cdot \frac{3}{5} + \frac{1}{3} \cdot \frac{3}{7} + \frac{1}{3} \cdot \frac{2}{6}} = \frac{63}{143}$$

9. A = {unit from line A}
B = {unit from line B}
D = {defective unit}.

$P(A) = \frac{300}{800} = \frac{3}{8}$

$P(B) = \frac{500}{800} = \frac{5}{8}$

$$P(A \mid D) = \frac{P(A)P(D \mid A)}{P(A)P(D \mid A) + P(B)P(D \mid B)} = \frac{\frac{3}{8} \cdot \frac{2}{100}}{\frac{3}{8} \cdot \frac{2}{100} + \frac{5}{8} \cdot \frac{5}{100}} = \frac{6}{31}$$

10. A = {unit from line A}
B = {unit from line B}
C = {unit from line C}
D = {unit from line D}
F = {defective unit}

a. $$P(A|F) = \frac{P(A)P(F|A)}{P(A)P(F|A)+P(B)P(F|B)+P(C)P(F|C)+P(D)P(F|D)}$$
$$= \frac{(0.35)(0.02)}{(0.35)(0.02)+(0.20)(0.05)+(0.30)(0.03)+(0.15)(0.04)} = \frac{7}{32}$$
Parts (b), (c), and (d) are similarly determined.

b. $\frac{10}{32} = \frac{5}{16}$

c. $\frac{9}{32}$

d. $\frac{6}{32} = \frac{3}{16}$

11. C = {call made}
T = {on time for meeting}
$$P(C|T) = \frac{P(C)P(T|C)}{P(C)P(T|C)+P(C')P(T|C')}$$
$$= \frac{(0.95)(0.9)}{(0.95)(0.9)+(0.05)(0.75)} = \frac{114}{119} \approx 0.958$$

12. J_D = {jar with dark chocolate only selected}
J_M = {jar with dark and milk chocolates selected}
D = {dark chocolate selected}
$$P(J_D) = P(J_M) = \frac{1}{2}$$
$$P(J_D|D) = \frac{P(J_D)P(D|J_D)}{P(J_D)P(D|J_D)+P(J_M)P(D|J_M)} = \frac{\frac{1}{2}\cdot\frac{50}{50}}{\frac{1}{2}\cdot\frac{50}{50}+\frac{1}{2}\cdot\frac{20}{50}} = \frac{5}{7}$$

13. W = {walking reported}
B = {bicycling reported}
R = {running reported}
C = {completed requirement}
$$P(W|C) = \frac{P(W)P(C|W)}{P(W)P(C|W)+P(B)P(C|B)+P(R)P(C|R)} = \frac{\frac{1}{2}\cdot\frac{9}{10}}{\frac{1}{2}\cdot\frac{9}{10}+\frac{1}{4}\cdot\frac{4}{5}+\frac{1}{4}\cdot\frac{2}{3}} = \frac{27}{49} \approx 0.551$$
55.1% would be expected to report walking.

14. C = {charges battery}
S = {car starts}

$$P(C' \mid S') = \frac{P(C')P(S' \mid C')}{P(C')P(S' \mid C') + P(C)P(S' \mid C)} = \frac{\frac{1}{10} \cdot \frac{4}{5}}{\frac{1}{10} \cdot \frac{4}{5} + \frac{9}{10} \cdot \frac{1}{8}} = \frac{32}{77} \approx 0.416$$

15. J = {had Japanese-made car}
E = {had European-made car}
A = {had American-made car}
B = {buy same make again}

$$P(J \mid B) = \frac{P(J)P(B \mid J)}{P(J)P(B \mid J) + P(E)P(B \mid E) + P(A)P(B \mid A)} = \frac{\frac{3}{5} \cdot \frac{85}{100}}{\frac{3}{5} \cdot \frac{85}{100} + \frac{1}{10} \cdot \frac{50}{100} + \frac{3}{10} \cdot \frac{40}{100}} = \frac{3}{4}$$

16. D = {dalhousium is present}
P = {positive test}
N = {negative test} = P'

a. $$P(D \mid P) = \frac{P(D)P(P \mid D)}{P(D)P(P \mid D) + P(D')P(P \mid D')} = \frac{(0.005)(0.80)}{(0.005)(0.80) + (0.995)(0.15)} = \frac{400}{15{,}325} \approx 0.0261$$

b. $$P(D \mid N) = \frac{P(D)P(N \mid D)}{P(D)P(N \mid D) + P(D')P(N \mid D')} = \frac{(0.005)(0.20)}{(0.005)(0.20) + (0.995)(0.85)} = \frac{100}{84{,}675} \approx 0.0012$$

17. P = {pass the exam}
A = {answer every question}

$$P(A \mid P) = \frac{P(A)P(P \mid A)}{P(A)P(P \mid A) + P(A')P(P \mid A')} = \frac{(0.75)(0.8)}{(0.75)(0.8) + (0.25)(0.50)} = \frac{24}{29} \approx 0.828$$

18. P = {predicted smoking}
S = {smoking now}

$$P(P \mid S') = \frac{P(P)P(S' \mid P)}{P(P)P(S' \mid P) + P(P')P(S' \mid P')} = \frac{(0.75)(0.7)}{(0.75)(0.7) + (0.25)(0.9)} = \frac{7}{10} = 70\%$$

19. S = {signals sent}
D = {signals detected}

$$P(S \mid D) = \frac{P(S)P(D \mid S)}{P(S)P(D \mid S) + P(S')P(D \mid S')} = \frac{\frac{2}{5} \cdot \frac{3}{5}}{\frac{2}{5} \cdot \frac{3}{5} + \frac{3}{5} \cdot \frac{1}{10}} = \frac{4}{5}$$

20. A_M = {A average at midterm}
A = {A for course}

$$P(A'_M \mid A) = \frac{P(A'_M)P(A \mid A'_M)}{P(A'_M)P(A \mid A'_M) + P(A_M)P(A \mid A_M)} = \frac{(0.4)(0.6)}{(0.4)(0.6) + (0.6)(0.7)} = \frac{4}{11} \approx 0.364$$

21. S = {movie is a success}
U = {"Two Thumbs Up"}

$$P(S \mid U) = \frac{P(S)P(U \mid S)}{P(S)P(U \mid S) + P(S')P(U \mid S')} = \frac{\frac{8}{10} \cdot \frac{70}{100}}{\frac{8}{10} \cdot \frac{70}{100} + \frac{2}{10} \cdot \frac{20}{100}} = \frac{14}{15} \approx 0.933$$

22. G_1 = {green ball drawn from Bowl 1}
R_1 = {red ball drawn from Bowl 1}
G_2 = {green ball drawn from Bowl 2}

$$P(G_1 \mid G_2) = \frac{P(G_1)P(G_2 \mid G_1)}{P(G_1)P(G_2 \mid G_1) + P(R_1)P(G_2 \mid R_1)} = \frac{\frac{5}{9} \cdot \frac{4}{8}}{\frac{5}{9} \cdot \frac{4}{8} + \frac{4}{9} \cdot \frac{3}{8}} = \frac{5}{8}$$

23. S = {is substandard request}
C = {is considered substandard request by Blackwell}

a. $P(C) = P(S)P(C \mid S) + P(S')P(C \mid S') = (0.20)(0.75) + (0.8)(0.15) = 0.27 = \frac{27}{100}$

b. $P(S \mid C) = \frac{P(S)P(C \mid S)}{P(S)P(C \mid S) + P(S')P(C \mid S')} = \frac{(0.20)(0.75)}{0.27} = \frac{0.15}{0.27} = \frac{15}{27} \approx 0.556$

c. $P(\text{Error}) = P(C' \cap S) + P(C \cap S')$
$= P(S)P(C' \mid S) + P(S')P(C \mid S')$
$= (0.20)(0.25) + (0.80)(0.15) = 0.17 = \frac{17}{100}$

24. I = {first chest selected}
II = {second chest selected}
III = {third chest selected}
G = {gold coin found}.
For the coin in the other drawer to be silver, we want the probability that the third chest was selected given that a gold coin was found.

$$P(III \mid G) = \frac{P(III)P(G \mid III)}{P(I)P(G \mid I) + P(II)P(G \mid II) + P(III)P(G \mid III)} = \frac{\frac{1}{3} \cdot \frac{1}{2}}{\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot \frac{1}{2}} = \frac{1}{3}$$

25. a. $P(L \mid E) = \frac{P(L)P(E \mid L)}{P(L)P(E \mid L) + P(M)P(E \mid M) + P(H)P(E \mid H)}$
$= \frac{(0.25)(0.49)}{(0.25)(0.49) + (0.25)(0.64) + (0.5)(0.81)} \approx 0.18$

b. $P(M \mid E) = \frac{P(M)P(E \mid M)}{P(L)P(E \mid L) + P(M)P(E \mid M) + P(H)P(E \mid H)}$
$= \frac{(0.25)(0.64)}{(0.25)(0.49) + (0.25)(0.64) + (0.5)(0.81)} \approx 0.23$

c. $P(H \mid E) = \frac{P(H)P(E \mid H)}{P(L)P(E \mid L) + P(M)P(E \mid M) + P(H)P(E \mid H)}$
$= \frac{(0.5)(0.81)}{(0.25)(0.49) + (0.25)(0.64) + (0.5)(0.81)} \approx 0.59$

d. High quality

26. **a.** (a) $P(L \mid E) = \dfrac{P(L)P(E \mid L)}{P(L)P(E \mid L) + P(M)P(E \mid M) + P(H)P(E \mid H)}$

$= \dfrac{(0.25)(0.44)}{(0.25)(0.44) + (0.25)(0.32) + (0.5)(0.18)} \approx 0.39$

(b) $P(M \mid E) = \dfrac{P(M)P(E \mid M)}{P(L)P(E \mid L) + P(M)P(E \mid M) + P(H)P(E \mid H)}$

$= \dfrac{(0.25)(0.32)}{(0.25)(0.44) + (0.25)(0.32) + (0.5)(0.18)} \approx 0.29$

(c) $P(H \mid E) = \dfrac{P(H)P(E \mid H)}{P(L)P(E \mid L) + P(M)P(E \mid M) + P(H)P(E \mid H)}$

$= \dfrac{(0.5)(0.18)}{(0.25)(0.44) + (0.25)(0.32) + (0.5)(0.18)} \approx 0.32$.

(d) Low quality

b. (a) $P(L \mid E) = \dfrac{P(L)P(E \mid L)}{P(L)P(E \mid L) + P(M)P(E \mid M) + P(H)P(E \mid H)}$

$= \dfrac{(0.25)(0.07)}{(0.25)(0.07) + (0.25)(0.04) + (0.5)(0.01)} \approx 0.54$

(b) $P(M \mid E) = \dfrac{P(M)P(E \mid M)}{P(L)P(E \mid L) + P(M)P(E \mid M) + P(H)P(E \mid H)}$

$= \dfrac{(0.25)(0.04)}{(0.25)(0.07) + (0.25)(0.04) + (0.5)(0.01)} \approx 0.31$

(c) $P(H \mid E) = \dfrac{P(H)P(E \mid H)}{P(L)P(E \mid L) + P(M)P(E \mid M) + P(H)P(E \mid H)}$

$= \dfrac{(0.5)(0.01)}{(0.25)(0.07) + (0.25)(0.04) + (0.5)(0.01)} \approx 0.15$

(d) Low quality

27. F = {fair weather}
I = {inclement weather}
W = {predict fair weather}.

$P(F \mid W) = \dfrac{P(F)P(W \mid F)}{P(F)P(W \mid F) + P(I)P(W \mid I)} = \dfrac{(0.6)(0.7)}{(0.6)(0.7) + (0.4)(0.3)} = \dfrac{7}{9} \approx 0.78$

Chapter 8 Review Problems

1. ${}_8P_3 = 8 \cdot 7 \cdot 6 = 336$

2. ${}_{20}P_1 = 20$

3. ${}_9C_7 = \dfrac{9!}{7!(9-7)!} = \dfrac{9!}{7! \cdot 2!} = \dfrac{9 \cdot 8 \cdot 7!}{7! \cdot 2 \cdot 1} = \dfrac{9 \cdot 8}{2} = 36$

4. $_{12}C_5 = \frac{12!}{5!(12-5)!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 7!} = 792$

5. For each of the first 3 characters there are 26 choices, while for each of the last 3 characters there are 10 choices. By the basic counting principle, the number of license plates that are possible is
$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$.

6. The number of choices for appetizers is 2, for the entrée it is 4, and for the dessert it is 3. By the basic counting principle, the number of complete dinners that are possible is
$2 \cdot 4 \cdot 3 = 24$.

7. Each of the five switches has 2 possible positions. By the basic counting principle, the number of different codes is
$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32$.

8. A batting order consists of nine names selected from nine names such that order is important. The number of such selections is
$_9P_9 = 9! = 362{,}880$.

9. A possibility for first, second, and third place is a selection of three of the seven teams so that order is important. Thus the number of ways the season can end is $_7P_3 = 7 \cdot 6 \cdot 5 = 210$.

10. Nine of the nine trophies can be arranged so that order is important. The first two can be placed on the top shelf, the next three on the middle shelf, and the last four on the bottom shelf. The number of such arrangements is
$_9P_9 = 9! = 362{,}880$.

11. The order of the group is not important. Thus the number of groups that can board is
$_{11}C_6 = \frac{11!}{6! \cdot 5!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 6!} = 462.$

12. There are four cards with a particular face value and there are $_4C_2$ ways of selecting two of them. Because there are 13 different face values, the number of ways of selecting two cards with the same face value is $13 \cdot {_4C_2}$. There are 12 remaining face values, so there are $12 \cdot {_4C_2}$ ways of selecting two cards having a different face value. After making these selections, there are 44 cards available with a different face value. Thus the number of 5-card hands with two cards of the same face value, another two with a different face value, and the last with yet another face value is
$13 \cdot {_4C_2} \cdot 12 \cdot {_4C_2} \cdot 44 = 13 \cdot \frac{4!}{2! \cdot 2!} \cdot 12 \cdot \frac{4!}{2!2!} \cdot 44$
$= 13 \cdot 6 \cdot 12 \cdot 6 \cdot 44 = 247{,}104.$

13. a. Three bulbs are selected from 24, and the order of selection is not important. Thus the number of possible selections is
$_{24}C_3 = \frac{24!}{3!(24-3)!} = \frac{24!}{3! \cdot 21!}$
$= \frac{24 \cdot 23 \cdot 22 \cdot 21!}{3 \cdot 2 \cdot 1 \cdot 21!} = \frac{24 \cdot 23 \cdot 22}{3 \cdot 2 \cdot 1} = 2024$.

b. Only one bulb is defective and that bulb must be included in the selection. The other two bulbs must be selected from the 23 remaining bulbs and there are $_{23}C_2$ such selections possible. Thus the number of ways of selecting three bulbs such that one is defective is
$1 \cdot {_{23}C_2} = {_{23}C_2} = \frac{23!}{2!(23-2)!} = \frac{23!}{2! \cdot 21!}$
$\frac{23 \cdot 22 \cdot 21!}{2 \cdot 1 \cdot 21!} = \frac{23 \cdot 22}{2 \cdot 1} = 253$.

14. To score 90, exactly nine questions must be correct; to score 100, all ten questions must be correct. If exactly nine questions are answered correctly, there are three ways of answering the tenth question incorrectly. But the number of ways of selecting nine of ten items is $_{10}C_9$. Thus the number of ways to score 90 is $3 \cdot {_{10}C_9}$. The number of ways to answer all ten questions correctly is $_{10}C_{10}$, or more simply, 1. Thus the number of ways to score 90 or better is
$3 \cdot {_{10}C_9} + 1 = 3 \cdot \frac{10!}{9! \cdot 1!} + 1$
$= 3 \cdot 10 + 1 = 31.$

15. In the word MISSISSIPPI, there are 11 letters with repetition: 1 M, 4 I's, 4 S's, and 2 P's. Thus the number of distinguishable permutations is
$\frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!} = 34{,}650$.

16. Nine flags must be arranged: two are red (type 1), three are green (type 2) and four are white (type 3). Thus the number of distinguishable permutations is $\frac{9!}{2!\cdot 3!\cdot 4!}=1260.$

17. Of the nine professors, four go to Dalhousie University (Cell A), three go to St. Mary's (Cell B), and two are not assigned (Cell C). The number of possible assignments is
$\frac{9!}{4!\cdot 3!\cdot 2!}=1260.$

18. Two of the three vans can be selected in ${}_3C_2$ ways. After two vans are chosen, the operator must assign 14 people so that 7 go to one van (cell 1) and 7 go to the other van (cell 2). This can be done in $\frac{14!}{7!\cdot 7!}$ ways. By the basic counting principle, the number of ways to assign the people to two vans is
${}_3C_2\cdot\frac{14!}{7!\cdot 7!}=\frac{3!}{2!\cdot 1!}\cdot\frac{14!}{7!\cdot 7!}=10,296$.

19. a. $E_1\cup E_2=\{1, 2, 3, 4, 5, 6, 7\}$

b. $E_1\cap E_2=\{4, 5, 6\}$

c. $E_1'\cup E_2=\{7,8\}\cup\{4,5,6,7\}=\{4,5,6,7,8\}$

d. The intersection of any event and its complement is $\varnothing$.

e. $(E_1\cap E_2')'=(\{1,2,3,4,5,6\}\cap\{1,2,3,8\})'$
$=\{1, 2, 3\}'=\{4, 5, 6, 7, 8\}$

f. From (b), $E_1\cap E_2\neq\varnothing$, so E_1 and E_2 are not mutually exclusive.

20. a. {1H, 2H, 3H, 4H, 5H, 6H, 1T, 2T, 3T, 4T, 5T, 6T}

b. {2H, 2T}

c. {2H, 4H, 6H}

21. a. $\{R_1R_2R_3, R_1R_2G_3, R_1G_2R_3, R_1G_2G_3, G_1R_2R_3, G_1R_2G_3, G_1G_2R_3, G_1G_2G_3\}$

b. $\{R_1R_2G_3, R_1G_2R_3, G_1R_2R_3\}$

c. $\{R_1R_2R_3, G_1G_2G_3\}$

22. $P(E_1\cup E_2)=P(E_1)+P(E_2)-P(E_1\cap E_2)$
$0.7=0.6+P(E_2)-0.2$
$P(E_2)=0.3$

23. $n(S)={}_{10}C_2=\frac{10!}{2!\cdot 8!}=\frac{10\cdot 9\cdot 8!}{2\cdot 1\cdot 8!}=\frac{10\cdot 9}{2\cdot 1}=45$
Let E be the event that box is rejected. If box is rejected, the one defective chip must be in the two-chip sample and there are nine possibilities for the other chip. Thus
$n(E)=9$
and $P(E)=\frac{n(E)}{n(S)}=\frac{9}{45}=\frac{1}{5}=0.2$.

24. Percentage of rats given drug
D = 100 – (35 + 25 + 15) = 25%.
Number of rats given C = 100(0.15) = 15.
Number of rats given D = 100(0.25) = 25.
If E = event that rat was injected with C or D, then $P(E)=\frac{n(E)}{n(S)}=\frac{15+25}{100}=0.40.$
If the experiment is repeated on a larger group of 300 rats but with the drugs given in the same proportion, then the number of rats given drug C is 300(0.15) = 45 and the number of rats given drug D is 300(0.25) = 75 and
$P(E)=\frac{n(E)}{n(S)}=\frac{45+75}{300}=0.40.$ Thus there is no effect on the previous probability.

25. Number of ways to answer exam is
$4^5=1024=n(S)$. Let
E = {exactly two questions are incorrect). The number of ways of selecting two of the five questions that are incorrect is ${}_5C_2=\frac{5!}{2!\cdot 3!}=10$.
However, there are three ways to answer a question incorrectly. Since two questions are incorrect $n(E)=10\cdot 3\cdot 3=90$. Thus
$P(E)=\frac{n(E)}{n(S)}=\frac{90}{1024}=\frac{45}{512}$.

26. a. Of the 200 cola drinkers, 35 like both A and B. Thus
$P(\text{likes both A and B})=\frac{35}{200}=\frac{7}{40}.$

b. If a person likes A but not B, then the person likes A only, and conversely. Thus

$P(\text{likes A, but not B}) = \frac{70}{200} = \frac{7}{20}$.

27. a. There are 10 jelly beans in the bag.

$n(S) = 10 \cdot 10 = 100$

$n(E_{\text{both red}}) = 4 \cdot 4 = 16$

Thus $P(E_{\text{both red}}) = \frac{n(E_{\text{both red}})}{n(S)} = \frac{16}{100} = \frac{4}{25}$.

b. $n(S) = 10 \cdot 9 = 90$

$n(E_{\text{both red}}) = 4 \cdot 3 = 12$

Thus $P(E_{\text{both red}}) = \frac{12}{90} = \frac{2}{15}$.

28. $n(S) = 6 \cdot 6 = 36$

a. $E_{2 \text{ or } 7} = \{(1, 1), (1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$

$P(E_{2 \text{ or } 7}) = \frac{n(E_{2 \text{ or } 7})}{n(S)} = \frac{7}{36}$

b. $E_{\text{multiple of 3}} = E_{3,\ 6,\ 9 \text{ or } 12}$

$= \{(1, 2), (2, 1), (1, 5), (5, 1), (2, 4), (4, 2), (3, 3), (3, 6), (6, 3), (4, 5), (5, 4), (6, 6)\}$

$P(E_{\text{multiple of 3}}) = \frac{n(E_{\text{multiple of 3}})}{n(S)} = \frac{12}{36} = \frac{1}{3}$

c. $E_{\text{no less than 7}} = E_{7,\ 8,\ 9,\ 10,\ 11, \text{ or } 12} = \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3), (2, 6), (6, 2), (3, 5), (5, 3), (4, 4), (3, 6), (6, 3), (4, 5), (5, 4), (4, 6), (6, 4), (5, 5), (5, 6), (6, 5), (6, 6)\}$

$P(E_{\text{no less than 7}}) = \frac{n(E_{7,\ 8,\ 9,\ 10,\ 11, \text{ or } 12})}{n(S)} = \frac{21}{36} = \frac{7}{12}$

29. $n(S) = 52 \cdot 52 \cdot 52$.

a. There are 26 black cards in a deck. Thus $n(E_{\text{all black}}) = 26 \cdot 26 \cdot 26$ and

$P(E_{\text{all black}}) = \frac{26 \cdot 26 \cdot 26}{52 \cdot 52 \cdot 52} = \frac{1}{8}$.

b. There are 13 diamonds in a deck, none of which are black. If E = event that two cards are black and the other is a diamond, then E occurs if the diamond is the first, second, or third card. Thus

$n(E) = 13 \cdot 26 \cdot 26 + 26 \cdot 13 \cdot 26 + 26 \cdot 26 \cdot 13 = 3 \cdot 13 \cdot 26 \cdot 26$ and $P(E) = \frac{3 \cdot 13 \cdot 26 \cdot 26}{52 \cdot 52 \cdot 52} = \frac{3}{16}$.

30. $n(S) = {}_{52}C_2 = \frac{52!}{2! \cdot 50!} = 1326$

a. There are 13 hearts in a deck. Thus

$n(E_{\text{both hearts}}) = {}_{13}C_2 = \frac{13!}{2! \cdot 11!} = 78$

and $P(E_{\text{both hearts}}) = \frac{78}{1326} = \frac{1}{17}$.

b. There are four aces and two red kings, and no red king is an ace. If E = event that one card is an ace and the other is a red king, then $n(E) = 4 \cdot 2 = 8$ and

$P(E) = \frac{8}{1326} = \frac{4}{663} \approx 0.006.$

31. $\frac{P(E)}{P(E')} = \frac{\frac{3}{8}}{1-\left(\frac{3}{8}\right)} = \frac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{5}$ or 3:5

32. $\frac{P(E)}{P(E')} = \frac{0.92}{1-0.92} = \frac{0.92}{0.08} = \frac{92}{8} = \frac{23}{2}$ or 23:2

33. $P(E) = \frac{6}{6+1} = \frac{6}{7}$

34. $P(E) = \frac{3}{3+4} = \frac{3}{7}$

35. $P\left(F' \middle| H\right) = \frac{P(F' \cap H)}{P(H)} = \frac{\frac{10}{52}}{\frac{1}{4}} = \frac{10}{13}$

36. The reduced sample space consists of {(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6), (1, 6), (2, 6), (3, 6), (4, 6), (5, 6)}.
In none of these 11 points, is the sum less than 7. Thus $P(\text{sum} < 7 \mid \text{a 6 shows}) = 0$.

37. $P(S \cap M) = P(S)P(M \mid S) = (0.6)(0.7) = 0.42$

38. $P(Q \cap H \cap AC) = P(Q)P(H)P(AC)$

$= \frac{4}{52} \cdot \frac{13}{52} \cdot \frac{1}{52} = \frac{1}{2704}$

39. a. The reduced sample space consists of {(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (1, 4), (2, 4), (3, 4), (5, 4), (6, 4)}.
In two of these 11 points, the sum of the components is 7. Thus

$P(\text{sum} = 7 \mid \text{a 4 shows}) = \frac{2}{11}$.

b. Out of 36 sample points, the event {getting a total of 7 and having a 4 show} is {(4, 3), (3, 4)}. Thus the probability of this event is $\frac{2}{36} = \frac{1}{18}$.

40. The reduced sample space consists of {(3, 6), (6, 3), (4, 5), (5, 4), (4, 6), (6, 4), (5, 5), (5, 6), (6, 5), (6, 6)}. Out of these 10 points, only one has a first toss that is less than 4. Thus the conditional probability is $\frac{1}{10}$.

41. The second number must be a 1 or 2, so the reduced sample space has $6 \cdot 2 = 12$ sample points. Of these, the event {first number ≤ second number} consists of (1, 1), (1, 2), and (2, 2). Thus the conditional probability is $\frac{3}{12} = \frac{1}{4}$.

42. It does not matter whether the first two cards are drawn or are left in place. Thus, imagine that they are merely lifted high enough for the third card to be drawn. The probability that this card is a heart is $\frac{1}{4}$.

43. a. $P(L' \mid F) = \frac{n(L' \cap F)}{n(F)} = \frac{160}{480} = \frac{1}{3}$

b. $P(L) = \frac{400}{600} = \frac{2}{3}$ and

$P(L|M) = \frac{n(L \cap M)}{n(M)} = \frac{80}{120} = \frac{2}{3}$.

Since $P(L|M) = P(L)$, events L and M are independent.

44. E = {(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)}
F = {(1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4)}

a. Since $E \cap F = \{(4,4)\} \neq \varnothing$, E and F are not mutually exclusive.

b. $P(E)=\frac{6}{36}=\frac{1}{6}$ and $P(E\mid F)=\frac{P(E\cap F)}{P(F)}=\frac{\frac{1}{36}}{\frac{6}{36}}=\frac{1}{6}$.

Since $P(E) = P(E \mid F)$, events E and F are independent.

45. P = {attend public college}
M = {from middle-class family}

$P(P)=\frac{125}{175}=\frac{5}{7}$

$P(P\mid M)=\frac{n(P\cap M)}{n(M)}=\frac{55}{80}=\frac{11}{16}$

Since $P(P \mid M) \neq P(P)$, events P and M are dependent.

46. $P(E\mid F)=\frac{P(E\cap F)}{P(F)}$

so $P(E\cap F)=P(E|F)P(F)=\frac{1}{6}\cdot\frac{1}{3}=\frac{1}{18}$, thus

$P(E\cup F)=P(E)+P(F)-P(E\cap F)$

$=\frac{1}{4}+\frac{1}{3}-\frac{1}{18}=\frac{19}{36}$.

47. a. P(none take root) = (0.3)(0.3)(0.3)(0.3) = 0.0081

b. The probability that a particular two shrubs take root and the remaining two do not is (0.7)(0.7)(0.3)(0.3). The number of ways the two that take root can be chosen from the four shrubs is ${}_4C_2$. Thus

P(exactly two take root) = ${}_4C_2(0.7)^2(0.3)^2=0.2646$.

c. For at most two shrubs to take root, either none does, exactly one does, or exactly two do.
P(none) + P(exactly one) + P(exactly two)
$=0.0081+{}_4C_1(0.7)(0.3)^3+0.2646$
= 0.0081 + 0.0756 + 0.2646
= 0.3483

48. Being effective for at least three of the persons means that it is effective for exactly three of them or for all four of them. Thus
P(exactly three) + P(all four)
$={}_4C_3(0.75)(0.75)(0.75)(0.25)+(0.75)(0.75)(0.75)(0.75)$
≈ 0.738

49. $P(R_{\text{II}})=P(G_{\text{I}})P(R_{\text{II}}\mid G_{\text{I}})+P(R_{\text{I}})P(R_{\text{II}}\mid R_{\text{I}})$

$=\frac{3}{5}\cdot\frac{4}{9}+\frac{2}{5}\cdot\frac{5}{9}=\frac{22}{45}$.

50. a. $P(W)=P(B_{\text{I}})P(W\mid B_{\text{I}})+P(B_{\text{II}})P(W\mid B_{\text{II}})$

$=\frac{1}{2}\cdot\frac{2}{6}+\frac{1}{2}\cdot\frac{3}{5}=\frac{1}{6}+\frac{3}{10}=\frac{7}{15}$

b. $P(B_{\text{II}} \mid W)$

$$= \frac{P(B_{\text{II}})P(W \mid B_{\text{II}})}{P(B_{\text{I}})P(W \mid B_{\text{I}}) + P(B_{\text{II}})P(W \mid B_{\text{II}})}$$

$$= \frac{\frac{1}{2} \cdot \frac{3}{5}}{\frac{7}{15}} = \frac{9}{14}$$

51. $P(G \mid A) = \frac{P(G \cap A)}{P(A)} = \frac{0.1}{0.4} = \frac{1}{4}$

52. S = {live within the state} and F = {first time attending}.

$$P(S' \mid F') = \frac{P(S')P(F' \mid S')}{P(S)P(F' \mid S) + P(S')P(F' \mid S')}$$

$$= \frac{\frac{98}{507} \cdot \frac{27}{100}}{\frac{409}{507} \cdot \frac{60}{100} + \frac{98}{507} \cdot \frac{27}{100}} = \frac{441}{4531} \approx 0.097$$

53. a. F = {produced by first shift}
S = {produced by second shift}
D = {scratched}

$$P(D) = P(F)P(D|F) + P(S)P(D|S)$$

$$= \frac{3000}{8000} \cdot (0.01) + \frac{5000}{8000} \cdot (0.02)$$

$$= 0.00375 + 0.0125 = 0.01625$$

b. $P(F \mid D) = \frac{P(F)P(D \mid F)}{P(F)P(D \mid F) + P(S)P(D \mid S)}$

$$= \frac{0.00375}{0.01625} = \frac{3}{13} \approx 0.23$$

54. E = {passed the exam}
S = {satisfactory performance}.

$$P(E \mid S) = \frac{P(E)P(S \mid E)}{P(E)P(S \mid E) + P(E')P(S \mid E')}$$

$$= \frac{(0.35)(0.8)}{(0.35)(0.8) + (0.65)(0.3)} = \frac{0.28}{0.475} = \frac{56}{95} \approx 0.59$$

Mathematical Snapshot Chapter 8

1. Trial and error should yield a critical value of around 0.645.

2. Possible answers: One could use cellular automata to model disease spread. The rules would be similar to the fad model, since a person who recovers from a disease is generally immune for some time afterward. One could also use cellular automata to model the formation of political opinion blocks. Each cell could be in one of three of four states, and a cell could be influenced by its neighbors. Some cells could be highly subject to neighbor influence while others were relatively immune.

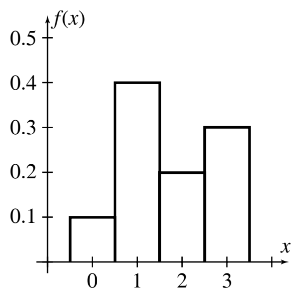
Chapter 9

Problems 9.1

1. $\mu = \sum_x x\,f(x) = 0(0.1) + 1(0.4) + 2(0.2) + 3(0.3) = 1.7$

$\text{Var}(X) = \sum_x x^2 f(x) - \mu^2 = [0^2(0.1) + 1^2(0.4) + 2^2(0.2) + 3^2(0.3)] - (1.7)^2 = 1.01$

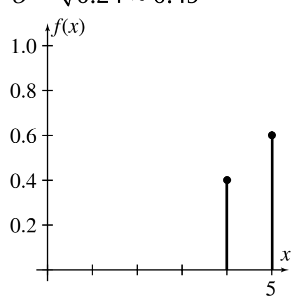
$\sigma = \sqrt{\text{Var}(X)} = \sqrt{1.01} \approx 1.00$



2. $\mu = \sum_x x\,f(x) = 4(0.4) + 5(0.6) = 4.6$

$\text{Var}(X) = [4^2(0.4) + 5^2(0.6)] - (4.6)^2 = 0.24$

$\sigma = \sqrt{0.24} \approx 0.49$



3. $\mu = \sum_x x\,f(x) = 1\left(\frac{1}{4}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{2}\right) = \frac{9}{4} = 2.25$

$\text{Var}(X) = \sum_x x^2 f(x) - \mu^2 = \left[1^2\left(\frac{1}{4}\right) + 2^2\left(\frac{1}{4}\right) + 3^2\left(\frac{1}{2}\right)\right] - \left(\frac{9}{4}\right)^2 = \frac{11}{16} = 0.6875$

$\sigma = \sqrt{\frac{11}{16}} = \frac{\sqrt{11}}{4} \approx 0.83$

4. $\mu = \sum_x x\,f(x) = 0\left(\frac{1}{7}\right) + 1\left(\frac{2}{7}\right) + 2\left(\frac{1}{7}\right) + 3\left(\frac{2}{7}\right) + 4\left(\frac{1}{7}\right) = \frac{14}{7} = 2$

$\text{Var}(X) = \left[0^2\left(\frac{1}{7}\right) + 1^2\left(\frac{2}{7}\right) + 2^2\left(\frac{1}{7}\right) + 3^2\left(\frac{2}{7}\right) + 4^2\left(\frac{1}{7}\right)\right] - 2^2 = \frac{12}{7} \approx 1.71$

$\sigma = \sqrt{\frac{12}{7}} \approx 1.31$

5. a. $P(X = 3) = 1 - [P(X = 5) + P(X = 6) + P(X = 7)] = 1 - [0.3 + 0.2 + 0.4] = 0.1$

b. $\mu = \sum_x x\,f(x) = 3(0.1) + 5(0.3) + 6(0.2) + 7(0.4) = 5.8$

c. $\sigma^2 = \sum_x x^2 f(x) - \mu^2 = [3^2(0.1) + 5^2(0.3) + 6^2(0.2) + 7^2(0.4)] - (5.8)^2 = 1.56$

6. a. $6a + 2a + 0.2 = 1 \Rightarrow a = 0.1$
Thus $P(X = 2) = 6(0.1) = 0.6$, and $P(X = 4) = 2(0.1) = 0.2$.

b. $\mu = 2(0.6) + 4(0.2) + 6(0.2) = 3.2.$

7. Distribution of *X:*

$f(0) = \frac{1}{8},\ f(1) = \frac{3}{8},\ f(2) = \frac{3}{8},\ f(3) = \frac{1}{8}$

$E(X) = \sum_x x\,f(x) = 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right) = \frac{12}{8} = \frac{3}{2} = 1.5$

$\sigma^2 = \text{Var}(X) = \sum_x x^2 f(x) - [E(x)]^2$

$= \left[0^2\left(\frac{1}{8}\right) + 1^2\left(\frac{3}{8}\right) + 2^2\left(\frac{3}{8}\right) + 3^2\left(\frac{1}{8}\right)\right] - \left(\frac{3}{2}\right)^2$

$= \frac{24}{8} - \frac{9}{4} = \frac{6}{8} = \frac{3}{4} = 0.75$

$\sigma = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \approx 0.87$

8. Distribution of *X:* $f(1) = \frac{4}{6} = \frac{2}{3},\ f(2) = \frac{2}{6} = \frac{1}{3}$

$E(X) = 1\left(\frac{2}{3}\right) + 2\left(\frac{1}{3}\right) = \frac{4}{3} \approx 1.33$

$\sigma^2 = \sum_x x^2 f(x) - [E(x)]^2 = \left[1^2\left(\frac{2}{3}\right) + 2^2\left(\frac{1}{3}\right)\right] - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \frac{2}{9} \approx 0.22$

$\sigma = \sqrt{\frac{2}{9}} \approx 0.47$

9. The number of outcomes in the sample space is $_5C_2 = 10$.
Distribution of *X:*

$$f(0) = \frac{_2C_2}{10} = \frac{1}{10},\ f(1) = \frac{_2C_1 \cdot {_3C_1}}{10} = \frac{3}{5},$$

$$f(2) = \frac{_3C_2}{10} = \frac{3}{10}$$

$$E(X) = \sum_x x\, f(x) = 0\left(\frac{1}{10}\right) + 1\left(\frac{3}{5}\right) + 2\left(\frac{3}{10}\right)$$

$$= \frac{6}{5} = 1.2$$

$$\sigma^2 = \sum_x x^2 f(x) - [E(x)]^2$$

$$= \left[0^2\left(\frac{1}{10}\right) + 1^2\left(\frac{3}{5}\right) + 2^2\left(\frac{3}{10}\right)\right] - \left(\frac{6}{5}\right)^2$$

$$= \frac{9}{5} - \frac{36}{25} = \frac{9}{25} = 0.36$$

$$\sigma = \sqrt{\frac{9}{25}} = \frac{3}{5} = 0.6$$

10. Distribution of *X:*

$$f(0) = \frac{9}{25},\ f(1) = \frac{12}{25},\ f(2) = \frac{4}{25}$$

$$E(X) = 0\left(\frac{9}{25}\right) + 1\left(\frac{12}{25}\right) + 2\left(\frac{4}{25}\right) = \frac{20}{25} = \frac{4}{5}$$

$$= 0.8$$

$$\sigma^2 = \left[0^2\left(\frac{9}{25}\right) + 1^2\left(\frac{12}{25}\right) + 2^2\left(\frac{4}{25}\right)\right] - \left(\frac{4}{5}\right)^2$$

$$= \frac{28}{25} - \frac{16}{25} = \frac{12}{25} = 0.48$$

$$\sigma = \sqrt{\frac{12}{25}} = \frac{2\sqrt{3}}{5} \approx 0.69$$

11. $f(0) = P(X = 0) = \dfrac{_2C_2}{_5C_2} = \dfrac{1}{10}$

$$f(1) = P(X = 1) = \frac{_3C_1 \cdot {_2C_1}}{_5C_2} = \frac{6}{10} = \frac{3}{5}$$

$$f(2) = P(X = 2) = \frac{_3C_2}{_5C_2} = \frac{3}{10}$$

12. $P(X = x) = \dfrac{_4C_x \cdot {_6C_{3-x}}}{_{10}C_3}$

13. a. If *X* is the gain (in dollars), then $X = -2$ or 4998.
Distribution of *X:*

$$f(-2) = \frac{7999}{8000},\ f(4998) = \frac{1}{8000}$$

$$E(x) = \sum_x xf(x)$$

$$= -2 \cdot \frac{7999}{8000} + 4998 \cdot \frac{1}{8000}$$

$$= -\frac{11{,}000}{8000} \approx -\$1.38 \text{ (a loss)}$$

b. Here $X = -4$ or 4996. Distribution of *X:*

$$f(-4) = \frac{7998}{8000},\ f(4996) = \frac{2}{8000}$$

$$E(X) = \sum_x xf(x)$$

$$= -4 \cdot \frac{7998}{8000} + 4996 \cdot \frac{2}{8000}$$

$$= -\$2.75 \text{ (a loss)}$$

14. If *X* is the gain (in dollars) per game, then $X = 10$ or -6.
Distribution of *X:*

$$f(10) = \frac{2}{8} = \frac{1}{4},\ f(-6) = \frac{6}{8} = \frac{3}{4}$$

$$E(X) = \sum_x x\, f(x) = 10 \cdot \frac{1}{4} + (-6) \cdot \frac{3}{4}$$

$$= -\$2 \text{ (a loss)}$$

15. Let X = daily earnings (in dollars).
Distribution of *X:*

$$f(200) = \frac{4}{7},\ f(-30) = \frac{3}{7}$$

$$E(X) = \sum_x x\, f(x)$$

$$= 200 \cdot \frac{4}{7} + (-30) \cdot \frac{3}{7}$$

$$= \frac{710}{7} \approx \$101.43$$

16. Let X = gain (in dollars) to the chain of a restaurant in a shopping center.
Distribution of *X:*

$f(75{,}000) = 0.65,\ f(-20{,}000) = 0.35$

$E(X) = 75{,}000(0.65) + (-20{,}000)(0.35)$
$= \$41{,}750.$

17. The probability that a person in the group is not hospitalized is
1 – (0.001 + 0.002 + 0.003 + 0.004 + 0.008) = 0.982.
Let X = gain (in dollars) to the company from a policy.
Distribution of X:
$f(10) = 0.982,\ f(-90) = 0.001,\ \ f(-190) = 0.002,\ \ f(-290) = 0.003,\ \ f(-390) = 0.004,\ f(-490) = 0.008$
$E(X) = 10(0.982) + (-90)(0.001) + (-190)(0.002) + (-290)(0.003)\ + (-390)(0.004) + (-490)(0.008)$
= \$3.00

18. $E(X)$ = 0(0.05) + 1(0.10) + 2(0.15) + 3(0.20) + 4(0.15) + 5(0.15) + 6(0.10) + 7(0.05) + 8(0.05) = 3.70

19. Let p = the annual premium (in dollars) per policy. If X = gain (in dollars) to the company from a policy, then either $X = p$ or $X = -(180{,}000 - p)$. We set $E(X) = 50$:

$$\begin{aligned} -(180{,}000 - p)(0.002) + p(0.998) &= 50 \\ -360 + 0.002p + 0.998p &= 50 \\ -360 + p &= 50 \\ p &= \$410 \end{aligned}$$

20. Let X = player's gain (in dollars) per play.
Distribution of X:

$$f(35) = \frac{1}{37},\ f(-1) = \frac{36}{37}$$

$$E(X) = 35 \cdot \frac{1}{37} + (-1) \cdot \frac{36}{37} = -\frac{1}{37} \approx -\$0.03 \text{ (a loss)}$$

21. Let X = gain (in dollars) on a play.

If 0 heads show, then $X = 0 - 1.25 = -\frac{5}{4}$.

If exactly 1 head shows, then $X = 1.00 - 1.25 = -\frac{1}{4}$.

If 2 heads show, then $X = 2.00 - 1.25 = \frac{3}{4}$.

Distribution of X:

$$f\left(-\frac{5}{4}\right) = \frac{1}{4},\ f\left(-\frac{1}{4}\right) = \frac{1}{2},\ f\left(\frac{3}{4}\right) = \frac{1}{4}$$

$$E(X) = \left(-\frac{5}{4}\right)\left(\frac{1}{4}\right) + \left(-\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) = -\frac{1}{4} = -0.25$$

Thus there is an expected loss of \$0.25 on each play.
For a fair game, let p = amount (in dollars) paid to play.
Distribution of X:

$$f(-p) = \frac{1}{4},\ f(1-p) = \frac{1}{2},\ f(2-p) = \frac{1}{4}$$

We set $E(X) = 0$:

$$\begin{aligned} (-p)\frac{1}{4} + (1-p)\frac{1}{2} + (2-p)\frac{1}{4} &= 0 \\ -\frac{p}{4} + \frac{1}{2} - \frac{p}{2} + \frac{1}{2} - \frac{p}{4} &= 0 \\ 1 - p &= 0 \\ p &= 1 \end{aligned}$$

Thus you should pay \$1 for a fair game.

Principles in Practice 9.2

1. Here $p = 0.30$, $q = 1 - p = 0.70$, and $n = 4$.

$P(X = x) = {}_nC_x p^x q^{n-x}$, $x = 0, 1, 2, 3, 4$

$P(X = 0) = {}_4C_0(0.3)^0(0.7)^4 = 0.2401$

$= \frac{2401}{10,000}$

$P(X = 1) = {}_4C_1(0.3)^1(0.7)^3 = 0.4116$

$= \frac{4116}{10,000}$

$P(X = 2) = {}_4C_2(0.3)^2(0.7)^2 = 0.2646$

$= \frac{2646}{10,000}$

$P(X = 3) = {}_4C_3(0.3)^3(0.7)^1 = 0.0756$

$= \frac{756}{10,000}$

$P(X = 4) = {}_4C_4(0.3)^4(0.7)^0 = 0.0081$

$= \frac{81}{10,000}$

Problems 9.2

1. $f(0) = {}_2C_0\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^2 = \frac{2!}{0!\cdot 2!}\cdot 1\cdot \frac{16}{25}$

$= 1\cdot 1\cdot \frac{16}{25} = \frac{16}{25}$

$f(1) = {}_2C_1\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^1 = \frac{2!}{1!\cdot 1!}\cdot \frac{1}{5}\cdot \frac{4}{5}$

$= 2\cdot \frac{1}{5}\cdot \frac{4}{5} = \frac{8}{25}$

$f(2) = {}_2C_2\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^0 = \frac{2!}{2!\cdot 0!}\cdot \frac{1}{25}\cdot 1$

$= 1\cdot \frac{1}{25}\cdot 1 = \frac{1}{25}$.

$\mu = np = 2\cdot \frac{1}{5} = \frac{2}{5}$

$\sigma = \sqrt{npq} = \sqrt{2\cdot \frac{1}{5}\cdot \frac{4}{5}}$

$= \sqrt{\frac{8}{25}} = \frac{2\sqrt{2}}{5}$

2. $f(0) = {}_3C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 = 1\cdot 1\cdot \frac{1}{8} = \frac{1}{8}$

$f(1) = {}_3C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 = 3\cdot \frac{1}{2}\cdot \frac{1}{4} = \frac{3}{8}$

$f(2) = {}_3C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 = 3\cdot \frac{1}{4}\cdot \frac{1}{2} = \frac{3}{8}$

$f(3) = {}_3C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 = 1\cdot \frac{1}{8}\cdot 1 = \frac{1}{8}$

$\mu = np = 3\cdot \frac{1}{2} = \frac{3}{2}$

$\sigma = \sqrt{npq} = \sqrt{3\cdot \frac{1}{2}\cdot \frac{1}{2}} = \frac{\sqrt{3}}{2}$

3. $f(0) = {}_3C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^3 = 1\cdot 1\cdot \frac{1}{27} = \frac{1}{27}$

$f(1) = {}_3C_1\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^2 = \frac{3!}{1!\cdot 2!}\cdot \frac{2}{3}\cdot \frac{1}{9}$

$= 3\cdot \frac{2}{3}\cdot \frac{1}{9} = \frac{2}{9}$

$f(2) = {}_3C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^1 = \frac{3!}{2!\cdot 1!}\cdot \frac{4}{9}\cdot \frac{1}{3}$

$= 3\cdot \frac{4}{9}\cdot \frac{1}{3} = \frac{4}{9}$

$f(3) = {}_3C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^0 = \frac{3!}{3!\cdot 0!}\cdot \frac{8}{27}\cdot 1$

$= 1\cdot \frac{8}{27}\cdot 1 = \frac{8}{27}$

$\mu = np = 3\cdot \frac{2}{3} = 2$; $\sigma = \sqrt{npq} = \sqrt{3\cdot \frac{2}{3}\cdot \frac{1}{3}}$

$= \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$

4. $f(0) = {}_4C_0(0.4)^0(0.6)^4 = \frac{4!}{0!\cdot 4!}\cdot 1\cdot (0.6)^4$

$= 1\cdot 1\cdot (0.6)^4 = 0.1296$

$f(1) = {}_4C_1(0.4)^1(0.6)^3 = \frac{4!}{1!\cdot 3!}(0.4)(0.6)^3$

$= 4(0.4)(0.6)^3 = 0.3456$

$f(2) = {}_4C_2(0.4)^2(0.6)^2 = \frac{4!}{2!\cdot 2!}(0.4)^2(0.6)^2$

$= 6(0.4)^2(0.6)^2 = 0.3456$

$f(3) = {}_4C_3(0.4)^3(0.6)^1 = \frac{4!}{3!\cdot 1!}(0.4)^3(0.6)$

$= 4(0.4)^3(0.6) = 0.1536$

$f(4) = {}_4C_4(0.4)^4(0.6)^0 = \frac{4!}{4!\cdot 0!}(0.4)^4 \cdot 1$

$= 1(0.4)^4 \cdot 1 = 0.0256$

$\mu = np = 4(0.4) = 1.6$

$\sigma = \sqrt{npq} = \sqrt{4(0.4)(0.6)} \approx 0.98$

5. $P(X = 5) = {}_6C_5(0.2)^5(0.8)^1$

$= 6(0.00032)(0.8) = 0.001536$

6. $P(X = 2) = {}_5C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3$

$= 10 \cdot \frac{1}{9} \cdot \frac{8}{27} = \frac{80}{243} \approx 0.3292$

7. $P(X = 2) = {}_4C_2\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^2 = 6 \cdot \frac{16}{25} \cdot \frac{1}{25}$

$= \frac{96}{625} = 0.1536$

8. $P(X = 4) = {}_7C_4(0.2)^4(0.8)^3$

$= 35(0.0016)(0.512) = 0.028672$

9. $P(X < 2) = P(X = 0) + P(X = 1)$

$= {}_5C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 + {}_5C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4$

$= 1 \cdot 1 \cdot \frac{1}{32} + 5 \cdot \frac{1}{2} \cdot \frac{1}{16} = \frac{6}{32} = \frac{3}{16}$

10. $P(X \geq 2) = 1 - [P(X = 0) + P(X = 1)]$

$= 1 - \left[{}_6C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^6 + {}_6C_1\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^5\right]$

$= 1 - \left[1 \cdot \frac{1}{729} + 6 \cdot \frac{2}{729}\right] = 1 - \frac{13}{729}$

$= \frac{716}{729} \approx 0.982$

11. Let X = number of heads that occurs.

$p = \frac{1}{2}, n = 11$

$P(X = 8) = {}_{11}C_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^3$

$= 165 \cdot \frac{1}{256} \cdot \frac{1}{8}$

$= \frac{165}{2048} \approx 0.081$

12. Let X = number of correct answers. $p = \frac{1}{4}, n = 6$

$P(X = 3) = {}_6C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^3 = 20 \cdot \frac{27}{4^6}$

$= \frac{540}{4096} \approx 0.132$

13. Let X = number of green marbles drawn. The probability of selecting a green marble on any draw is $\frac{7}{12}$, $n = 4$.

$P(X = 2) = {}_4C_2\left(\frac{7}{12}\right)^2\left(\frac{5}{12}\right)^2$

$= 6 \cdot \frac{49}{144} \cdot \frac{25}{144} = \frac{1225}{3456} \approx 0.3545$

14. Let X = number of aces selected. The probability of selecting an ace on any draw is $p = \frac{4}{52} = \frac{1}{13}$.

$n = 3$

$P(X = 2) = {}_3C_2\left(\frac{1}{13}\right)^2\left(\frac{12}{13}\right)^1 = 3 \cdot \frac{1}{169} \cdot \frac{12}{13}$

$= \frac{36}{2197} \approx 0.016$

15. Let X = number of defective switches selected. The probability that a switch is defective is $p = 0.02$, $n = 4$.

$P(X = 2) = {}_4C_2(0.02)^2(0.98)^2$

$= 6(0.0004)(0.9604) \approx 0.002$

16. $p = 0.2$, $n = 3$

$P(X = x) = {}_3C_x(0.2)^x(0.8)^{3-x}$

17. Let X = number of heads that occurs. $p=\frac{1}{4}, n=3$

a. $P(X=2)={}_3C_2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^1=3\cdot\frac{1}{16}\cdot\frac{3}{4}=\frac{9}{64}$

b. $P(X=3)={}_3C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^0=1\cdot\frac{1}{64}\cdot 1=\frac{1}{64}$

Thus

$P(X=2)+P(X=3)=\frac{9}{64}+\frac{1}{64}=\frac{10}{64}=\frac{5}{32}$

18. Let X = number of hearts selected.

$p=\frac{13}{52}=\frac{1}{4}, n=7$

a. $P(X=4)={}_7C_4\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^3=35\cdot\frac{1}{256}\cdot\frac{27}{64}=\frac{945}{16,384}\approx 0.058$

b. $P(X\geq 4)=P(X=4)+P(X=5)+P(X=6)+P(X=7)$

$$=\frac{945}{16,384}+{}_7C_5\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^2+{}_7C_6\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^1+{}_7C_7\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^0$$
$$=\frac{945}{16,384}+21\cdot\frac{1}{1024}\cdot\frac{9}{16}+7\cdot\frac{1}{4096}\cdot\frac{3}{4}+1\cdot\frac{1}{16,384}\cdot 1$$
$$=\frac{1156}{16,384}=\frac{289}{4096}\approx 0.071$$

19. Let X = number of defective in sample.

$p=\frac{1}{5},\ n=6$

$P(X\leq 1)=P(X=0)+P(X=1)$

$$={}_6C_0\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^6+{}_6C_1\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^5$$
$$=1\cdot 1\cdot\frac{4096}{15,625}+6\cdot\frac{1}{5}\cdot\frac{1024}{3125}$$
$$=\frac{10,240}{15,625}=\frac{2048}{3125}\approx 0.655$$

20. Let X = number of persons with computer.

$p=0.7, n=5$

$P(X\geq 3)=P(X=3)+P(X=4)+P(X=5)$

$={}_5C_3(0.7)^3(0.3)^2+{}_5C_4(0.7)^4(0.3)^1+{}_5C_5(0.7)^5(0.3)^0$

$=10(0.343)(0.09)+5(0.2401)(0.3)+1(0.16807)(1)$

$=0.3087+0.36015+0.16807$

$=0.83692$

21. Let X = number of hits in four at-bats.
$p = 0.300$, $n = 4$
$$P(X \geq 1) = 1 - P(X = 0) = 1 - {}_4C_0(0.300)^0(0.700)^4 = 1 - 1 \cdot 1 \cdot (0.2401) = 0.7599$$

22. Let X = number of stocks that increase in value. The probability that a stock increases in value is $p = 0.6$. Here $n = 4$. We must find
$P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X = 0) + P(X = 1)]$.
$$P(X = 0) = {}_4C_0(0.6)^0(0.4)^4 = 1 \cdot 1 \cdot (0.0256) = 0.0256$$
$$P(X = 1) = {}_4C_1(0.6)^1(0.4)^3 = 4(0.6)(0.064) = 0.1536$$
$$P(X \geq 2) = 1 - P(X < 2) = 1 - [0.0256 + 0.1536] = 1 - 0.1792 \approx 0.82$$

23. Let X = number of girls. The probability that a child is a girl is $p = \frac{1}{2}$. Here $n = 5$. We must find
$P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X = 0) + P(X = 1)]$.
$$P(X = 0) = {}_5C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 = 1 \cdot 1 \cdot \frac{1}{32} = \frac{1}{32}$$
$$P(X = 1) = {}_5C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4 = 5 \cdot \frac{1}{2} \cdot \frac{1}{16} = \frac{5}{32}$$
Thus,
$$P(X \geq 2) = 1 - [P(X = 0) + P(X = 1)] = 1 - \left[\frac{1}{32} + \frac{5}{32}\right] = 1 - \frac{3}{16} = \frac{13}{16}$$

24. $p = \frac{2}{5}, n = 50, \ q = 1 - p = 1 - \frac{2}{5} = \frac{3}{5}$
$$\sigma^2 = npq = 50 \cdot \frac{2}{5} \cdot \frac{3}{5} = 12$$

25. $\mu = 3, \sigma^2 = 2$

Since $\mu = np$, then $np = 3$. Since $\sigma^2 = npq$, then $(np)q = 2$, or $3q = 2$, so $q = \frac{2}{3}$. Thus, $p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$.

Since $np = 3$, then $n \cdot \frac{1}{3} = 3$, or $n = 9$. Thus
$$P(X = 2) = {}_9C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^7 = 36 \cdot \frac{1}{9} \cdot \frac{128}{2187} = \frac{512}{2187} \approx 0.234.$$

26. **a.** $E(X) = \mu = np = 15(0.06) = 0.9$

b. $\text{Var}(X) = \sigma^2 = npq = 15(0.06)(0.94) = 0.846$

c.
$$\begin{aligned} P(X \leq 1) &= P(X = 0) + P(X = 1) \\ &= {}_{15}C_0(0.06)^0(0.94)^{15} + {}_{15}C_1(0.06)^1(0.94)^{14} \\ &= 1 \cdot 1 \cdot (0.94)^{15} + 15(0.06)(0.94)^{14} \approx 0.77 \end{aligned}$$

Problems 9.3

1. $\begin{bmatrix} \frac{1}{2} & \frac{2}{3} \\ -\frac{3}{2} & \frac{1}{3} \end{bmatrix}$

No, since the entry at row 2 column 1 is negative.

2. $\begin{bmatrix} 0.1 & 1 \\ 0.9 & 0 \end{bmatrix}$

Yes, since all entries are nonnegative and the sum of the entries in each column is 1.

3. $\begin{bmatrix} \frac{1}{2} & \frac{1}{8} & \frac{1}{3} \\ -\frac{1}{4} & \frac{5}{8} & \frac{1}{3} \\ \frac{3}{4} & \frac{1}{4} & \frac{1}{3} \end{bmatrix}$

No, since there is a negative entry.

4. $\begin{bmatrix} 0.2 & 0.6 & 0 \\ 0.7 & 0.2 & 0 \\ 0.1 & 0.2 & 0 \end{bmatrix}$

No, since the sum of the entries in column 3 is not 1.

5. $\begin{bmatrix} 0.4 & 0 & 0.5 \\ 0.2 & 0.1 & 0.3 \\ 0.4 & 0.9 & 0.2 \end{bmatrix}$

Yes, since all entries are nonnegative and the sum of the entries in each column is 1.

6. $\begin{bmatrix} 0.5 & 0.1 & 0.3 \\ 0.4 & 0.3 & 0.3 \\ 0.6 & 0.6 & 0.4 \end{bmatrix}$

No, since the sum of the entries in column 1 is not 1.

7. $\begin{bmatrix} \frac{2}{3} & b \\ a & \frac{1}{4} \end{bmatrix}$

$\frac{2}{3} + a = 1$, so $a = \frac{1}{3}$.

$b + \frac{1}{4} = 1$, so $b = \frac{3}{4}$.

8. $\begin{bmatrix} a & b \\ \frac{5}{12} & a \end{bmatrix}$

$a + \frac{5}{12} = 1$, so $a = 1 - \frac{5}{12} = \frac{7}{12}$.

$b + a = 1$, so $b = 1 - \frac{7}{12} = \frac{5}{12}$.

9. $\begin{bmatrix} 0.4 & a & a \\ a & 0.1 & b \\ 0.3 & b & c \end{bmatrix}$

$0.4 + a + 0.3 = 1$, so $a = 0.3$.
$a + 0.1 + b = 1$, $0.3 + 0.1 + b = 1$, so $b = 0.6$.
$a + b + c = 1$, $0.3 + 0.6 + c = 1$, so $c = 0.1$.

10. $\begin{bmatrix} a & a & a \\ a & b & b \\ a & \frac{1}{4} & c \end{bmatrix}$

$a + a + a = 1, 3a = 1, a = \frac{1}{3}$

$a + b + \frac{1}{4} = 1, \frac{1}{3} + b + \frac{1}{4} = 1, b = \frac{5}{12}$

$a + b + c = 1, \frac{1}{3} + \frac{5}{12} + c = 1, c = \frac{1}{4}$

11. $\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$

Yes, all entries are nonnegative and their sum is 1.

12. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Yes, all entries are nonnegative and their sum is 1.

13. $\begin{bmatrix} 0.2 \\ 0.7 \\ 0.5 \end{bmatrix}$

No, the sum of the entries is not 1.

14. $\begin{bmatrix} 0.9 \\ -0.1 \\ 0.2 \end{bmatrix}$

No, the entry in the second row is negative.

15. $\mathbf{X}_1 = \mathbf{T}\mathbf{X}_0 = \begin{bmatrix} \frac{2}{3} & 1 \\ \frac{1}{3} & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{11}{12} \\ \frac{1}{12} \end{bmatrix}$

$\mathbf{X}_2 = \mathbf{T}\mathbf{X}_1 = \begin{bmatrix} \frac{2}{3} & 1 \\ \frac{1}{3} & 0 \end{bmatrix}\begin{bmatrix} \frac{11}{12} \\ \frac{1}{12} \end{bmatrix} = \begin{bmatrix} \frac{25}{36} \\ \frac{11}{36} \end{bmatrix}$

$\mathbf{X}_3 = \mathbf{T}\mathbf{X}_2 = \begin{bmatrix} \frac{2}{3} & 1 \\ \frac{1}{3} & 0 \end{bmatrix}\begin{bmatrix} \frac{25}{36} \\ \frac{11}{36} \end{bmatrix} = \begin{bmatrix} \frac{83}{108} \\ \frac{25}{108} \end{bmatrix}$

16. $\mathbf{X}_1 = \mathbf{T}\mathbf{X}_0 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\ \frac{5}{8} \end{bmatrix}$

$\mathbf{X}_2 = \mathbf{T}\mathbf{X}_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} \frac{3}{8} \\ \frac{5}{8} \end{bmatrix} = \begin{bmatrix} \frac{11}{32} \\ \frac{21}{32} \end{bmatrix}$

$\mathbf{X}_3 = \mathbf{T}\mathbf{X}_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} \frac{11}{32} \\ \frac{21}{32} \end{bmatrix} = \begin{bmatrix} \frac{43}{128} \\ \frac{85}{128} \end{bmatrix}$

17. $\mathbf{X}_1 = \mathbf{T}\mathbf{X}_0 = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.42 \\ 0.58 \end{bmatrix}$

$\mathbf{X}_2 = \mathbf{T}\mathbf{X}_1 = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}\begin{bmatrix} 0.42 \\ 0.58 \end{bmatrix} = \begin{bmatrix} 0.416 \\ 0.584 \end{bmatrix}$

$\mathbf{X}_3 = \mathbf{T}\mathbf{X}_2 = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}\begin{bmatrix} 0.416 \\ 0.584 \end{bmatrix} = \begin{bmatrix} 0.4168 \\ 0.5832 \end{bmatrix}$

18. $\mathbf{X}_1 = \mathbf{T}\mathbf{X}_0 = \begin{bmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix}$

$\mathbf{X}_2 = \mathbf{T}\mathbf{X}_1 = \begin{bmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{bmatrix}\begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix} = \begin{bmatrix} 0.308 \\ 0.692 \end{bmatrix}$

$\mathbf{X}_3 = \mathbf{T}\mathbf{X}_2 = \begin{bmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{bmatrix}\begin{bmatrix} 0.308 \\ 0.692 \end{bmatrix} = \begin{bmatrix} 0.6536 \\ 0.3464 \end{bmatrix}$

19. $\mathbf{X}_1 = \mathbf{T}\mathbf{X}_0 = \begin{bmatrix} 0.1 & 0 & 0.3 \\ 0.2 & 0.4 & 0.3 \\ 0.7 & 0.6 & 0.4 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 0.26 \\ 0.28 \\ 0.46 \end{bmatrix}$

$\mathbf{X}_2 = \mathbf{T}\mathbf{X}_1 = \begin{bmatrix} 0.1 & 0 & 0.3 \\ 0.2 & 0.4 & 0.3 \\ 0.7 & 0.6 & 0.4 \end{bmatrix}\begin{bmatrix} 0.26 \\ 0.28 \\ 0.46 \end{bmatrix} = \begin{bmatrix} 0.164 \\ 0.302 \\ 0.534 \end{bmatrix}$

$\mathbf{X}_3 = \mathbf{T}\mathbf{X}_2 = \begin{bmatrix} 0.1 & 0 & 0.3 \\ 0.2 & 0.4 & 0.3 \\ 0.7 & 0.6 & 0.4 \end{bmatrix}\begin{bmatrix} 0.164 \\ 0.302 \\ 0.534 \end{bmatrix} = \begin{bmatrix} 0.1766 \\ 0.3138 \\ 0.5096 \end{bmatrix}$

20. $\mathbf{X}_1 = \mathbf{TX}_0 = \begin{bmatrix} 0.4 & 0.1 & 0.2 & 0.1 \\ 0 & 0.1 & 0.3 & 0.3 \\ 0.4 & 0.7 & 0.4 & 0.4 \\ 0.2 & 0.1 & 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.3 \\ 0.4 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.17 \\ 0.21 \\ 0.49 \\ 0.13 \end{bmatrix}$

$\mathbf{X}_2 = \mathbf{TX}_1 = \begin{bmatrix} 0.4 & 0.1 & 0.2 & 0.1 \\ 0 & 0.1 & 0.3 & 0.3 \\ 0.4 & 0.7 & 0.4 & 0.4 \\ 0.2 & 0.1 & 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 0.17 \\ 0.21 \\ 0.49 \\ 0.13 \end{bmatrix} = \begin{bmatrix} 0.200 \\ 0.207 \\ 0.463 \\ 0.130 \end{bmatrix}$

$\mathbf{X}_3 = \mathbf{TX}_2 = \begin{bmatrix} 0.4 & 0.1 & 0.2 & 0.1 \\ 0 & 0.1 & 0.3 & 0.3 \\ 0.4 & 0.7 & 0.4 & 0.4 \\ 0.2 & 0.1 & 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 0.200 \\ 0.207 \\ 0.463 \\ 0.130 \end{bmatrix} = \begin{bmatrix} 0.2063 \\ 0.1986 \\ 0.4621 \\ 0.1330 \end{bmatrix}$

21. a. $\mathbf{T}^2 = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{5}{8} \end{bmatrix}$

$\mathbf{T}^3 = \mathbf{T}^2\mathbf{T} = \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{5}{8} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{7}{16} & \frac{9}{16} \\ \frac{9}{16} & \frac{7}{16} \end{bmatrix}.$

b. Entry in row 2, column 1, of $\mathbf{T}^2$ is $\frac{3}{8}$.

c. Entry in row 1, column 2 of $\mathbf{T}^3$ is $\frac{9}{16}$.

22. a. $\mathbf{T}^2 = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{4}{9} & \frac{5}{12} \\ \frac{5}{9} & \frac{7}{12} \end{bmatrix}$

$\mathbf{T}^3 = \mathbf{T}^2\mathbf{T} = \begin{bmatrix} \frac{4}{9} & \frac{5}{12} \\ \frac{5}{9} & \frac{7}{12} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{23}{54} & \frac{31}{72} \\ \frac{31}{54} & \frac{41}{72} \end{bmatrix}$

b. Entry in row 2, column 1, of $\mathbf{T}^2$ is $\frac{5}{9}$.

c. Entry in row 1, column 2 of $\mathbf{T}^3$ is $\frac{31}{72}$.

23. a. $\mathbf{T}^2 = \begin{bmatrix} 0 & 0.5 & 0.3 \\ 1 & 0.4 & 0.3 \\ 0 & 0.1 & 0.4 \end{bmatrix} \begin{bmatrix} 0 & 0.5 & 0.3 \\ 1 & 0.4 & 0.3 \\ 0 & 0.1 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.50 & 0.23 & 0.27 \\ 0.40 & 0.69 & 0.54 \\ 0.10 & 0.08 & 0.19 \end{bmatrix}$

$\mathbf{T}^3 = \mathbf{T}^2\mathbf{T} = \begin{bmatrix} 0.50 & 0.23 & 0.27 \\ 0.40 & 0.69 & 0.54 \\ 0.10 & 0.08 & 0.19 \end{bmatrix} \begin{bmatrix} 0 & 0.5 & 0.3 \\ 1 & 0.4 & 0.3 \\ 0 & 0.1 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.230 & 0.369 & 0.327 \\ 0.690 & 0.530 & 0.543 \\ 0.080 & 0.101 & 0.130 \end{bmatrix}$

b. Entry in row 2, column 1, of $\mathbf{T}^2$ is 0.40.

c. Entry in row 1, column 2 of $\mathbf{T}^3$ is 0.369.

24. a. $\mathbf{T}^2 = \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.1 \\ 0.7 & 0.8 & 0.8 \end{bmatrix} \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.1 \\ 0.7 & 0.8 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.10 & 0.10 & 0.10 \\ 0.11 & 0.11 & 0.11 \\ 0.79 & 0.79 & 0.79 \end{bmatrix}$

$\mathbf{T}^3 = \mathbf{T}^2\mathbf{T} = \begin{bmatrix} 0.10 & 0.10 & 0.10 \\ 0.11 & 0.11 & 0.11 \\ 0.79 & 0.79 & 0.79 \end{bmatrix} \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.1 \\ 0.7 & 0.8 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.10 & 0.10 & 0.10 \\ 0.11 & 0.11 & 0.11 \\ 0.79 & 0.79 & 0.79 \end{bmatrix}.$

b. Entry in row 2, column 1, of $\mathbf{T}^2$ is 0.11.

c. Entry in row 1, column 2 of $\mathbf{T}^3$ is 0.10.

25. $\mathbf{T} - \mathbf{I} = \begin{bmatrix} \frac{1}{2} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{2}{3} \\ \frac{1}{2} & -\frac{2}{3} \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -\frac{1}{2} & \frac{2}{3} & 0 \\ \frac{1}{2} & -\frac{2}{3} & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 0 & \frac{4}{7} \\ 0 & 1 & \frac{3}{7} \\ 0 & 0 & 0 \end{array}\right]$

$\mathbf{Q} = \begin{bmatrix} \frac{4}{7} \\ \frac{3}{7} \end{bmatrix}$

26. $\mathbf{T} - \mathbf{I} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -\frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{array}\right]$

$\mathbf{Q} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \end{bmatrix}$

27. $\mathbf{T} - \mathbf{I} = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{4}{5} & \frac{2}{5} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ \frac{4}{5} & -\frac{3}{5} & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{7} \\ 0 & 1 & \frac{4}{7} \\ 0 & 0 & 0 \end{array}\right]$

$\mathbf{Q} = \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix}$

28. $\mathbf{T}-\mathbf{I}=\begin{bmatrix}\frac{1}{4} & \frac{1}{3}\\ \frac{3}{4} & \frac{2}{3}\end{bmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-\frac{3}{4} & \frac{1}{3}\\ \frac{3}{4} & -\frac{1}{3}\end{bmatrix}$

$$\left[\begin{array}{cc|c}1 & 1 & 1\\ -\frac{3}{4} & \frac{1}{3} & 0\\ \frac{3}{4} & -\frac{1}{3} & 0\end{array}\right]\to\cdots\to\left[\begin{array}{cc|c}1 & 0 & \frac{4}{13}\\ 0 & 1 & \frac{9}{13}\\ 0 & 0 & 0\end{array}\right]$$

$\mathbf{Q}=\begin{bmatrix}\frac{4}{13}\\ \frac{9}{13}\end{bmatrix}$

29. $\mathbf{T}-\mathbf{I}=\begin{bmatrix}0.4 & 0.6 & 0.6\\ 0.3 & 0.3 & 0.1\\ 0.3 & 0.1 & 0.3\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-0.6 & 0.6 & 0.6\\ 0.3 & -0.7 & 0.1\\ 0.3 & 0.1 & -0.7\end{bmatrix}$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 1\\ -0.6 & 0.6 & 0.6 & 0\\ 0.3 & -0.7 & 0.1 & 0\\ 0.3 & 0.1 & -0.7 & 0\end{array}\right]\to\cdots\to\left[\begin{array}{ccc|c}1 & 0 & 0 & 0.5\\ 0 & 1 & 0 & 0.25\\ 0 & 0 & 1 & 0.25\\ 0 & 0 & 0 & 0\end{array}\right]$$

$\mathbf{Q}=\begin{bmatrix}0.5\\ 0.25\\ 0.25\end{bmatrix}$

30. $\mathbf{T}-\mathbf{I}=\begin{bmatrix}0.1 & 0.4 & 0.3\\ 0.2 & 0.2 & 0.3\\ 0.7 & 0.4 & 0.4\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-0.9 & 0.4 & 0.3\\ 0.2 & -0.8 & 0.3\\ 0.7 & 0.4 & -0.6\end{bmatrix}$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 1\\ -0.9 & 0.4 & 0.3 & 0\\ 0.2 & -0.8 & 0.3 & 0\\ 0.7 & 0.4 & -0.6 & 0\end{array}\right]\to\cdots\to\left[\begin{array}{ccc|c}1 & 0 & 0 & 0.2707\\ 0 & 1 & 0 & 0.2481\\ 0 & 0 & 1 & 0.4812\\ 0 & 0 & 0 & 0\end{array}\right]$$

$\mathbf{Q}\approx\begin{bmatrix}0.2707\\ 0.2481\\ 0.4812\end{bmatrix}$

31. a. $\mathbf{T}=\begin{array}{c}\\ \text{Flu}\\ \text{No flu}\end{array}\begin{array}{c}\begin{array}{cc}\text{Flu} & \text{No flu}\end{array}\\ \begin{bmatrix}0.1 & 0.2\\ 0.9 & 0.8\end{bmatrix}\end{array}$

b. $\mathbf{X}_0=\begin{bmatrix}\frac{120}{200}\\ \frac{80}{200}\end{bmatrix}=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}.$

If a period is 4 days, then 8 days corresponds to 2 periods, and 12 days corresponds to 3 periods. The state vector corresponding to 8 days from now is

$\mathbf{X}_2=\mathbf{T}^2\mathbf{X}_0=\begin{bmatrix}0.19 & 0.18\\ 0.81 & 0.82\end{bmatrix}\begin{bmatrix}0.6\\ 0.4\end{bmatrix}=\begin{bmatrix}0.186\\ 0.814\end{bmatrix}.$

Thus 0.186(200) $\approx$ 37 students can be expected to have the flu 8 days from now. The state vector corresponding to 12 days from now is

$$\mathbf{X}_3 = \mathbf{T}^3\mathbf{X}_0 = \begin{bmatrix} 0.181 & 0.182 \\ 0.819 & 0.818 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.1814 \\ 0.8186 \end{bmatrix}.$$

Thus 0.1814(200) $\approx$ 36 students can be expected to have the flu 12 days from now.

32. $\mathbf{T} = \begin{array}{c} \\ H \\ L \end{array}\begin{array}{c} \begin{matrix} H & L \end{matrix} \\ \begin{bmatrix} 0.55 & 0.25 \\ 0.45 & 0.75 \end{bmatrix} \end{array}$

$$\mathbf{X}_0 = \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix}$$

$$\mathbf{X}_2 = \mathbf{T}^2\mathbf{X}_0 = \begin{bmatrix} 0.415 & 0.325 \\ 0.585 & 0.675 \end{bmatrix}\begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix} = \begin{bmatrix} 0.3835 \\ 0.6165 \end{bmatrix}$$

38.35% of the members will be performing high-impact exercising.

33. a. $\mathbf{T} = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{matrix} A & B \end{matrix} \\ \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \end{array}$

b. Wednesday corresponds to step 2.

$$\mathbf{T}^2 = \begin{bmatrix} 0.61 & 0.52 \\ 0.39 & 0.48 \end{bmatrix}.$$

The probability is 0.61.

34. a. $\mathbf{X}_1 = \mathbf{T}\mathbf{X}_0$

$$= \begin{bmatrix} 0.7 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.2 \\ 0.2 & 0 & 0.6 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0.5 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.30 \\ 0.48 \\ 0.22 \end{bmatrix}$$

30% to location 1, 48% to location 2, 22% to location 3

b. $\mathbf{X}_2 = \mathbf{T}\mathbf{X}_1$

$$= \begin{bmatrix} 0.7 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.2 \\ 0.2 & 0 & 0.6 \end{bmatrix}\begin{bmatrix} 0.30 \\ 0.48 \\ 0.22 \end{bmatrix} = \begin{bmatrix} 0.350 \\ 0.458 \\ 0.192 \end{bmatrix}$$

35% to location 1, 45.8% to location 2, 19.2% to location 3

35. a. $\mathbf{T} = \begin{array}{c} \\ D \\ R \\ O \end{array}\begin{array}{c} \begin{matrix} D & R & O \end{matrix} \\ \begin{bmatrix} 0.8 & 0.1 & 0.3 \\ 0.1 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.5 \end{bmatrix} \end{array}$

b. $\mathbf{T}^2 = \begin{bmatrix} 0.68 & 0.19 & 0.41 \\ 0.18 & 0.67 & 0.29 \\ 0.14 & 0.14 & 0.30 \end{bmatrix}$

The probability is 0.19.

c. $\mathbf{X}_1 = \mathbf{T}\mathbf{X}_0$

$$= \begin{bmatrix} 0.8 & 0.1 & 0.3 \\ 0.1 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.5 \end{bmatrix}\begin{bmatrix} 0.40 \\ 0.40 \\ 0.20 \end{bmatrix} = \begin{bmatrix} 0.42 \\ 0.40 \\ 0.18 \end{bmatrix}$$

40% are expected to be Republican.

36. $\mathbf{T} = \begin{array}{c} \\ U \\ S \\ R \end{array}\begin{array}{c} \begin{matrix} U & S & R \end{matrix} \\ \begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.2 & 0.1 & 0.8 \end{bmatrix} \end{array}$

a. 15 years corresponds to step 3.

$$\mathbf{T}^3 = \begin{bmatrix} 0.412 & 0.196 & 0.196 \\ 0.219 & 0.562 & 0.219 \\ 0.369 & 0.242 & 0.585 \end{bmatrix}$$

The entry in row 3, column 2 of $\mathbf{T}^3$ is 0.242, so the probability is 0.242.

b. $\mathbf{X}_3 = \mathbf{T}^3\mathbf{X}_0$

$$= \begin{bmatrix} 0.412 & 0.196 & 0.196 \\ 0.219 & 0.562 & 0.219 \\ 0.369 & 0.242 & 0.585 \end{bmatrix} \begin{bmatrix} 0.50 \\ 0.25 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 0.304 \\ 0.30475 \\ 0.39125 \end{bmatrix}$$

The population is expected to be 30.4% urban, 30.475% suburban, 39.125% rural.

37. a. $\mathbf{T} = \begin{array}{c} \\ \text{A} \\ \text{Compet.} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \text{A} & \text{Compet.} \end{array} \\ \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \end{array}$

b. $\mathbf{X}_1 = \mathbf{T}\mathbf{X}_0 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.70 \\ 0.30 \end{bmatrix}$

$$= \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix}$$

A is expected to control 65% of the market.

c. $\mathbf{T} - \mathbf{I} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -0.2 & 0.3 & 0 \\ 0.2 & -0.3 & 0 \end{array}\right] \to \dots \to \left[\begin{array}{cc|c} 1 & 0 & 0.6 \\ 0 & 1 & 0.4 \\ 0 & 0 & 0 \end{array}\right]$$

$$\mathbf{Q} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

In the long run, A can expect to control 60% of the market.

38. a. $\mathbf{T} = \begin{array}{c} \\ \text{Fords} \\ \text{Non-fords} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \text{Fords} & \text{Non-Fords} \end{array} \\ \begin{bmatrix} 0.75 & 0.35 \\ 0.25 & 0.65 \end{bmatrix} \end{array}$

b. $\mathbf{T} - \mathbf{I} = \begin{bmatrix} 0.75 & 0.35 \\ 0.25 & 0.65 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.25 & 0.35 \\ 0.25 & -0.35 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -0.25 & 0.35 & 0 \\ 0.25 & -0.35 & 0 \end{array}\right] \to \dots \to \left[\begin{array}{cc|c} 1 & 0 & 0.5833 \\ 0 & 1 & 0.4167 \\ 0 & 0 & 0 \end{array}\right]$$

$$\mathbf{Q} \approx \begin{bmatrix} 0.5833 \\ 0.4167 \end{bmatrix}$$

In the long run, 58.33% of car purchases in the region are expected to be Fords.

39. a. $\mathbf{T} = \begin{array}{c} \\ 1 \\ 2 \end{array} \!\!\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} \frac{5}{7} & \frac{3}{7} \\ \frac{2}{7} & \frac{4}{7} \end{bmatrix} \end{array}$

b. $\mathbf{X_2} = \mathbf{T}^2\mathbf{X_0} = \begin{bmatrix} \frac{31}{49} & \frac{27}{49} \\ \frac{18}{49} & \frac{22}{49} \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{29}{49} \\ \frac{20}{49} \end{bmatrix} \approx \begin{bmatrix} 0.5918 \\ 0.4082 \end{bmatrix}$

About 59.18% in compartment 1 and 40.82% in compartment 2.

c. $\mathbf{T} - \mathbf{I} = \begin{bmatrix} \frac{5}{7} & \frac{3}{7} \\ \frac{2}{7} & \frac{4}{7} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{7} & \frac{3}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -\frac{2}{7} & \frac{3}{7} & 0 \\ \frac{2}{7} & -\frac{3}{7} & 0 \end{array}\right] \to \dots \to \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{5} \\ 0 & 1 & \frac{2}{5} \\ 0 & 0 & 0 \end{array}\right]$$

$\mathbf{Q} = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$

In the long run, there will be 60% in compartment 1 and 40% in compartment 2.

40. a. **T** =

	Works	Doesn't Work
Works	0.8	0.1
Doesn't Work	0.2	0.9

b. $\mathbf{T}^3 = \begin{bmatrix} 0.562 & 0.219 \\ 0.438 & 0.781 \end{bmatrix}$

The probability is 0.562.

c. $\mathbf{T} - \mathbf{I} = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.1 \\ 0.2 & -0.1 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -0.2 & 0.1 & 0 \\ 0.2 & -0.1 & 0 \end{array}\right] \to \dots \to \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{array}\right]$$

$\mathbf{Q} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$

In the long run, the number of machines working properly is $\left(\frac{1}{3}\right)(42) = 14$.

41. a. $\mathbf{T}-\mathbf{I}=\begin{bmatrix}\frac{3}{4} & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{2}\end{bmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-\frac{1}{4} & \frac{1}{2}\\ \frac{1}{4} & -\frac{1}{2}\end{bmatrix}$

$$\left[\begin{array}{cc|c}1 & 1 & 1\\ -\frac{1}{4} & \frac{1}{2} & 0\\ \frac{1}{4} & -\frac{1}{2} & 0\end{array}\right]\to\dots\to\left[\begin{array}{cc|c}1 & 0 & \frac{2}{3}\\ 0 & 1 & \frac{1}{3}\\ 0 & 0 & 0\end{array}\right]$$

$\mathbf{Q}=\begin{bmatrix}\frac{2}{3}\\ \frac{1}{3}\end{bmatrix}$

b. Presently, A accounts for 50% of sales and in long run A will account for $\frac{2}{3}$, or $66\frac{2}{3}\%$, of sales. Thus the percentage increase in sales above the present level is $\frac{66\frac{2}{3}-50}{50}\cdot 100\%=\frac{16\frac{2}{3}}{50}\cdot 100\%=33\frac{1}{3}\%$.

42. a. $\mathbf{T}=\begin{array}{c}\\ A\\ B\\ C\end{array}\begin{array}{c}\begin{array}{ccc}A & B & C\end{array}\\ \begin{bmatrix}0.8 & 0.2 & 0.2\\ 0.1 & 0.7 & 0.1\\ 0.1 & 0.1 & 0.7\end{bmatrix}\end{array}$

b. $\mathbf{T}^2=\begin{bmatrix}0.68 & 0.32 & 0.32\\ 0.16 & 0.52 & 0.16\\ 0.16 & 0.16 & 0.52\end{bmatrix}$

The probability is 0.52.

c. Initially 500 customers are to be considered. The probability that a customer goes to branch A is $\frac{200}{500}=0.4$; to branch B, $\frac{200}{500}=0.4$; and to branch C, $\frac{100}{500}=0.2$. Thus $\mathbf{X}_0=\begin{bmatrix}0.4\\ 0.4\\ 0.2\end{bmatrix}$.

$$\mathbf{X}_1=\mathbf{T}\mathbf{X}_0=\begin{bmatrix}0.8 & 0.2 & 0.2\\ 0.1 & 0.7 & 0.1\\ 0.1 & 0.1 & 0.7\end{bmatrix}\begin{bmatrix}0.4\\ 0.4\\ 0.2\end{bmatrix}=\begin{bmatrix}0.44\\ 0.34\\ 0.22\end{bmatrix}$$

Thus 0.44(500) = 220 customers can be expected to go to A on their next visit, 0.34(500) = 170 to B, and 0.22(500) = 110 to C.

d. $\mathbf{T}-\mathbf{I}=\begin{bmatrix}0.8 & 0.2 & 0.2\\ 0.1 & 0.7 & 0.1\\ 0.1 & 0.1 & 0.7\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-0.2 & 0.2 & 0.2\\ 0.1 & -0.3 & 0.1\\ 0.1 & 0.1 & -0.3\end{bmatrix}$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 1\\ -0.2 & 0.2 & 0.2 & 0\\ 0.1 & -0.3 & 0.1 & 0\\ 0.1 & 0.1 & -0.3 & 0\end{array}\right]\to\dots\to\left[\begin{array}{ccc|c}1 & 0 & 0 & 0.50\\ 0 & 1 & 0 & 0.25\\ 0 & 0 & 1 & 0.25\\ 0 & 0 & 0 & 0\end{array}\right]$$

In the long run, 0.50(500) = 250 can be expected to go to A, 0.25(500) = 125 to B, and 0.25(500) to C.

43. $\mathbf{T}^2=\mathbf{TT}=\begin{bmatrix}\frac{1}{2} & 1\\ \frac{1}{2} & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 1\\ \frac{1}{2} & 0\end{bmatrix}=\begin{bmatrix}\frac{3}{4} & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{2}\end{bmatrix}$

Since all entries of $\mathbf{T}^2$ are positive, **T** is regular.

44. For the matrix $\mathbf{A}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $\mathbf{A}^2=\mathbf{I}$ (the 2×2 identity matris). Thus $\mathbf{A}^n=\mathbf{I}$ if n is even, and $\mathbf{A}^n=\mathbf{A}$ if n is odd.

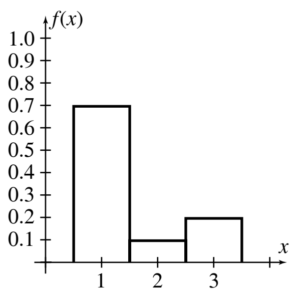
In either case there are nonpositive entries, and thus **A** is not regular.

Chapter 9 Review Problems

1. $\mu=\sum_x xf(x)=1\cdot f(1)+2\cdot f(2)+3\cdot f(3)$ = 1(0.7) + 2(0.1) + 3(0.2) = 1.5

$\text{Var}(X)=\sum_x x^2 f(x)-\mu^2=\left[1^2(0.7)+2^2(0.1)+3^2(0.2)\right]-(1.5)^2=0.65$

$\sigma=\sqrt{\text{Var}(X)}=\sqrt{0.65}\approx 0.81$



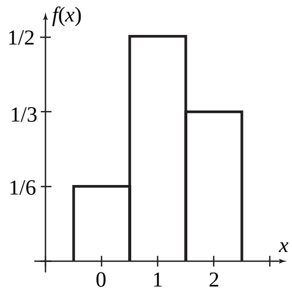
2. $\mu=\sum_x xf(x)=0\cdot\frac{1}{6}+1\cdot\frac{1}{2}+2\cdot\frac{1}{3}=\frac{7}{6}$

$\text{Var}(X)=\sum_x x^2 f(x)-\mu^2$

$=\left[0^2\cdot\frac{1}{6}+1^2\cdot\frac{1}{2}+2^2\cdot\frac{1}{3}\right]-\left(\frac{7}{6}\right)^2$

$=\frac{11}{6}-\frac{49}{36}=\frac{17}{36}$

$\sigma=\sqrt{\frac{17}{36}}=\frac{\sqrt{17}}{6}\approx 0.69$



3. a. $n(S) = 2 \cdot 6 = 12$

$E_0 = \{H1\}$, $E_1 = \{T1, H2\}$, $E_2 = \{T2, H3\}$,
$E_3 = \{T3, H4\}$, $E_4 = \{T4, H5\}$,
$E_5 = \{T5, H6\}$, $E_6 = \{T6\}$

$$f(0) = P(E_0) = \frac{n(E_0)}{n(S)} = \frac{1}{12}$$

$$f(1) = P(E_1) = \frac{n(E_1)}{n(S)} = \frac{2}{12} = \frac{1}{6}$$

Similarly, $f(2)$, $f(3)$, $f(4)$, and $f(5)$ equal $\frac{1}{6}$.

$$f(6) = P(E_6) = \frac{n(E_6)}{n(S)} = \frac{1}{12}$$

b. $E(X) = \sum_x xf(x)$

$$= 0 \cdot \frac{1}{12} + \frac{1+2+3+4+5}{6} + 6 \cdot \frac{1}{12}$$

$$= 0 + \frac{15}{6} + \frac{6}{12} = \frac{36}{12} = 3$$

4. a. $n(S) = {}_{52}C_2 = \frac{52!}{2! \cdot 50!} = \frac{52 \cdot 51}{2} = 1326$. In a deck there are 4 aces and 48 non-aces. Thus $n(E_{0\text{ aces}}) = {}_{48}C_2 = \frac{48!}{2! \cdot 46!} = \frac{48 \cdot 47}{2}$
$= 1128$.

For $E_{1\text{ ace}}$ to occur, one card is an ace and the other is non-ace. Thus
$n(E_{1\text{ ace}}) = 4 \cdot 48 = 192$.

$n(E_{2\text{ aces}}) = {}_4C_2 = \frac{4!}{2! \cdot 2!} = \frac{4 \cdot 3}{2} = 6$.

Therefore,

$$f(0) = P(E_{0\text{ aces}}) = \frac{1128}{1326} = \frac{188}{221},$$

$$f(1) = P(E_{1\text{ ace}}) = \frac{192}{1326} = \frac{32}{221},$$

$$f(2) = P(E_{2\text{ aces}}) = \frac{6}{1326} = \frac{1}{221}.$$

b. $E(X) = \sum_x xf(x) = 0 \cdot \frac{188}{221} + 1 \cdot \frac{32}{221} + 2 \cdot \frac{1}{221}$

$$= \frac{34}{221} = \frac{2}{13}$$

5. Let X = gain (in dollars) on a play. If no 10 appears, then $X = 0 - \frac{1}{4} = -\frac{1}{4}$; if exactly one 10 appears, then $X = 1 - \frac{1}{4} = \frac{3}{4}$; if two 10's appear, then $X = 2 - \frac{1}{4} = \frac{7}{4}$.

$n(S) = 52 \cdot 52$. In a deck, there are 4 10's and 48 non 10's. Thus $n(E_{\text{no }10}) = 48 \cdot 48$. The event $E_{\text{one }10}$ occurs if the first card is a 10 and the second is a non-10, or vice versa. Thus
$n(E_{\text{one }10}) = 4 \cdot 48 + 48 \cdot 4 = 2 \cdot 4 \cdot 48$.
$n(E_{\text{two }10\text{'s}}) = 4 \cdot 4$.

Dist. of X:

$$f\left(-\frac{1}{4}\right) = \frac{48 \cdot 48}{52 \cdot 52} = \frac{144}{169},$$

$$f\left(\frac{3}{4}\right) = \frac{2 \cdot 4 \cdot 48}{52 \cdot 52} = \frac{24}{169},$$

$$f\left(\frac{7}{4}\right) = \frac{4 \cdot 4}{52 \cdot 52} = \frac{1}{169}.$$

$$E(X) = -\frac{1}{4} \cdot \frac{144}{169} + \frac{3}{4} \cdot \frac{24}{169} + \frac{7}{4} \cdot \frac{1}{169}$$

$$= \frac{-144 + 72 + 7}{4 \cdot 169} = -\frac{65}{676} = -\frac{5}{52} \approx -0.10$$

There is a loss of $0.10 per play.

6. Let X = gain (in dollars) to company.
Dist. of X: $f(40{,}000) = 0.45$,
$f(-10{,}000) = 1 - 0.45 = 0.55$
$E(X) = (40{,}000)(0.45) + (-10{,}000)(0.55)$
$= 18{,}000 - 5500 = \$12{,}500$ per station

7. a. Let X = gain (in dollars) on each unit shipped. Then $P(X = -100) = 0.08$ and $P(X = 200) = 1 - 0.08 = 0.92$.
$E(X) = -100f(-100) + 200f(200)$
$= -100(0.08) + 200(0.92)$
= \$176 per unit

b. Since the expected gain per unit is \$176 and 4000 units are shipped per year, then expected annual profit is
4000(176) = \$704,000.

8. There are 41 million combinations from which to choose. Let x = gain (in dollars) per play. If the player wins, then
$x = 15{,}000{,}000 - 1.00 = 14{,}999{,}999$ and
$P(X = 14{,}999{,}999) = \dfrac{1}{41{,}000{,}000}$. If the player loses, then $X = -1.00$ and
$P(X = -1.00) = 1 - \dfrac{1}{41{,}000{,}000} = \dfrac{40{,}999{,}999}{41{,}000{,}000}$.
$E(X) = 14{,}999{,}999f(14{,}999{,}999) - 1.00f(-1.00)$
$= 14{,}999{,}999\left(\dfrac{1}{41{,}000{,}000}\right)$
$- 1.00\left(\dfrac{40{,}999{,}999}{41{,}000{,}000}\right) \approx -0.63$
There is a loss of about \$0.63 per play.

9. $f(0) = {}_4C_0(0.15)^0(0.85)^4 \approx \dfrac{4!}{0!4!}\cdot 1(0.522)$
$= 0.522$
$f(1) = {}_4C_1(0.15)^1(0.85)^3$
$\approx \dfrac{4!}{1!3!}\cdot(0.15)(0.614) = 0.368$
$f(2) = {}_4C_2(0.15)^2(0.85)^2$
$= \dfrac{4!}{2!2!}\cdot(0.0225)(0.7225) \approx 0.098$
$f(3) = {}_4C_3(0.15)^3(0.85)^1$
$= \dfrac{4!}{3!1!}\cdot(0.003375)(0.85) \approx 0.011$
$f(4) = {}_4C_4(0.15)^4(0.85)^0$
$\approx \dfrac{4!}{4!0!}\cdot(0.000506)1 = 0.0005$
$\mu = np = 4(0.15) = 0.6$
$\sigma = \sqrt{npq} = \sqrt{4(0.15)(0.85)} \approx 0.71$

10. $f(0) = {}_5C_0\left(\dfrac{1}{3}\right)^0\left(\dfrac{2}{3}\right)^5 = 1\cdot 1\cdot\dfrac{32}{243} = \dfrac{32}{243}$
$f(1) = {}_5C_1\left(\dfrac{1}{3}\right)^1\left(\dfrac{2}{3}\right)^4 = 5\cdot\dfrac{1}{3}\cdot\dfrac{16}{81} = \dfrac{80}{243}$
$f(2) = {}_5C_2\left(\dfrac{1}{3}\right)^2\left(\dfrac{2}{3}\right)^3 = 10\cdot\dfrac{1}{9}\cdot\dfrac{8}{27} = \dfrac{80}{243}$
$f(3) = {}_5C_3\left(\dfrac{1}{3}\right)^3\left(\dfrac{2}{3}\right)^2 = 10\cdot\dfrac{1}{27}\cdot\dfrac{4}{9} = \dfrac{40}{243}$
$f(4) = {}_5C_4\left(\dfrac{1}{3}\right)^4\left(\dfrac{2}{3}\right)^1 = 5\cdot\dfrac{1}{81}\cdot\dfrac{2}{3} = \dfrac{10}{243}$
$f(5) = {}_5C_5\left(\dfrac{1}{3}\right)^5\left(\dfrac{2}{3}\right)^0 = 1\cdot\dfrac{1}{243}\cdot 1 = \dfrac{1}{243}$
$\mu = np = 5\cdot\dfrac{1}{3} = \dfrac{5}{3}$
$\sigma = \sqrt{npq} = \sqrt{5\cdot\dfrac{1}{3}\cdot\dfrac{2}{3}} = \sqrt{\dfrac{10}{9}} = \dfrac{\sqrt{10}}{3} \approx 1.05$

11. $P(X \le 1) = P(X = 0) + P(X = 1)$
$= {}_5C_0\left(\dfrac{3}{4}\right)^0\left(\dfrac{1}{4}\right)^5 + {}_5C_1\left(\dfrac{3}{4}\right)^1\left(\dfrac{1}{4}\right)^4$
$= 1\cdot 1\cdot\dfrac{1}{1024} + 5\cdot\dfrac{3}{4}\cdot\dfrac{1}{256} = \dfrac{16}{1024} = \dfrac{1}{64}$

12. $P(X = 0) = {}_6C_0\left(\dfrac{2}{3}\right)^0\left(\dfrac{1}{3}\right)^6 = \dfrac{6!}{0!\cdot 6!}(1)\left(\dfrac{1}{729}\right)$
$= 1(1)\left(\dfrac{1}{729}\right) = \dfrac{1}{729}$
$P(X = 1) = {}_6C_1\left(\dfrac{2}{3}\right)^1\left(\dfrac{1}{3}\right)^5 = \dfrac{6!}{1!\cdot 5!}\left(\dfrac{2}{3}\right)\left(\dfrac{1}{243}\right)$
$= 6\left(\dfrac{2}{3}\right)\left(\dfrac{1}{243}\right) = \dfrac{12}{729}$
$P(X = 2) = {}_6C_2\left(\dfrac{2}{3}\right)^2\left(\dfrac{1}{3}\right)^4 = \dfrac{6!}{2!\cdot 4!}\left(\dfrac{4}{9}\right)\left(\dfrac{1}{81}\right)$
$= \dfrac{6\cdot 5\cdot 4!}{2\cdot 1\cdot 4!}\left(\dfrac{4}{9}\right)\left(\dfrac{1}{81}\right) = 15\left(\dfrac{4}{9}\right)\left(\dfrac{1}{81}\right) = \dfrac{60}{729}$
$P(X > 2) = 1 - P(X \le 2)$
$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$
$= 1 - \left[\dfrac{1}{729} + \dfrac{12}{729} + \dfrac{60}{729}\right] = 1 - \dfrac{73}{729} = \dfrac{656}{729}$

13. The probability that a 2 or 3 results on one roll is $\frac{2}{6} = \frac{1}{3}$. Let X = number of 2's or 3's that appear on 4 rolls. Then X is binomial with $p = \frac{1}{3}$ and $n = 4$.

$$P(X = 3) = {}_4C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^1 = 4 \cdot \frac{1}{27} \cdot \frac{2}{3} = \frac{8}{81}$$

14. Let X = number of bushes that live. Then X is binomial.

$$P(X = 0) = {}_4C_0(0.9)^0(0.1)^4 = 0.0001$$

15. Let X = number of heads that occur. Then X is binomial.

$$P(X = 0) = {}_5C_0\left(\frac{2}{5}\right)^0\left(\frac{3}{5}\right)^5 = 1 \cdot 1 \cdot \frac{243}{3125} = \frac{243}{3125}$$

$$P(X = 1) = {}_5C_1\left(\frac{2}{5}\right)^1\left(\frac{3}{5}\right)^4 = 5 \cdot \frac{2}{5} \cdot \frac{81}{625} = \frac{810}{3125}$$

$P(X \geq 2) = 1 - [P(X = 0) + P(X = 1)]$

$$= 1 - \left[\frac{243}{3125} + \frac{810}{3125}\right] = 1 - \frac{1053}{3125} = \frac{2072}{3125}$$

16. On any draw, the probability of selecting a red jelly bean is $\frac{2}{10} = \frac{1}{5}$. Let X = number of red jelly beans selected in five draws. Then X is binomial with $p = \frac{1}{5}$ and $n = 5$.

$$P(X = 0) = {}_5C_0\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^5 = 1 \cdot 1 \cdot \frac{1024}{3125}$$

$$= \frac{1024}{3125}$$

$$P(X = 1) = {}_5C_1\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^4 = 5 \cdot \frac{1}{5} \cdot \frac{256}{625}$$

$$= \frac{1280}{3125}$$

$$P(X = 2) = {}_5C_2\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^3 = 10 \cdot \frac{1}{25} \cdot \frac{64}{125}$$

$$= \frac{640}{3125}$$

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= \frac{1024}{3125} + \frac{1280}{3125} + \frac{640}{3125} = \frac{2944}{3125} = 0.94208$$

17. From column 1, $0.1 + a + 0.6 = 1$, so $a = 0.3$.
From column 2, $2a + b + b = 1$, so $2b = 1 - 2a$, or $b = \frac{1-2a}{2} = \frac{1-2(0.3)}{2} = 0.2$.
From column 3, $a + b + c = 1$, so $c = 1 - a - b$, or
$c = 1 - 0.3 - 0.2 = 0.5$.

18. From column 1, $a + a + 0.2 = 1$, so $2a = 0.8$, or $a = 0.4$.
From column 3, $b + b + a = 1$, so $2b = 1 - a$, or
$b = \frac{1-a}{2} = \frac{1-0.4}{2} = 0.3$.
From column 2, $a + b + c = 1$, so
$c = 1 - a - b = 1 - 0.4 - 0.3 = 0.3$.

19. $$\mathbf{X}_1 = \mathbf{T}\mathbf{X}_0 = \begin{bmatrix} 0.1 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0.1 \\ 0.7 & 0.3 & 0.8 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.10 \\ 0.15 \\ 0.75 \end{bmatrix}$$

$$\mathbf{X}_2 = \mathbf{T}\mathbf{X}_1 = \begin{bmatrix} 0.1 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0.1 \\ 0.7 & 0.3 & 0.8 \end{bmatrix}\begin{bmatrix} 0.10 \\ 0.15 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 0.130 \\ 0.155 \\ 0.715 \end{bmatrix}$$

$$\mathbf{X}_3 = \mathbf{T}\mathbf{X}_2 = \begin{bmatrix} 0.1 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0.1 \\ 0.7 & 0.3 & 0.8 \end{bmatrix}\begin{bmatrix} 0.130 \\ 0.155 \\ 0.715 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1310 \\ 0.1595 \\ 0.7095 \end{bmatrix}$$

20. $\mathbf{X}_1 = \mathbf{T}\mathbf{X}_0 = \begin{bmatrix} 0.4 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.5 \\ 0.4 & 0.3 & 0.4 \end{bmatrix}\begin{bmatrix} 0.1 \\ 0.3 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.13 \\ 0.50 \\ 0.37 \end{bmatrix}$

$\mathbf{X}_2 = \mathbf{T}\mathbf{X}_1 = \begin{bmatrix} 0.4 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.5 \\ 0.4 & 0.3 & 0.4 \end{bmatrix}\begin{bmatrix} 0.13 \\ 0.50 \\ 0.37 \end{bmatrix} = \begin{bmatrix} 0.139 \\ 0.511 \\ 0.350 \end{bmatrix}$

$\mathbf{X}_3 = \mathbf{T}\mathbf{X}_2 = \begin{bmatrix} 0.4 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.5 \\ 0.4 & 0.3 & 0.4 \end{bmatrix}\begin{bmatrix} 0.139 \\ 0.511 \\ 0.350 \end{bmatrix}$

$= \begin{bmatrix} 0.1417 \\ 0.5094 \\ 0.3489 \end{bmatrix}$

21. **a.** $\mathbf{T}^2 = \mathbf{T}\mathbf{T} = \begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{6}{7} & \frac{4}{7} \end{bmatrix}\begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{6}{7} & \frac{4}{7} \end{bmatrix} = \begin{bmatrix} \frac{19}{49} & \frac{15}{49} \\ \frac{30}{49} & \frac{34}{49} \end{bmatrix}$

$\mathbf{T}^3 = \mathbf{T}^2\mathbf{T} = \begin{bmatrix} \frac{19}{49} & \frac{15}{49} \\ \frac{30}{49} & \frac{34}{49} \end{bmatrix}\begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{6}{7} & \frac{4}{7} \end{bmatrix}$

$= \begin{bmatrix} \frac{109}{343} & \frac{117}{343} \\ \frac{234}{343} & \frac{226}{343} \end{bmatrix}$

b. From $\mathbf{T}^2$, entry in row 1, column 2, is $\frac{15}{49}$.

c. From $\mathbf{T}^3$, entry in row 2, column 1, is $\frac{234}{343}$.

22. **a.** $\mathbf{T}^2 = \mathbf{T}\mathbf{T}$

$= \begin{bmatrix} 0 & 0.4 & 0.3 \\ 0 & 0.3 & 0.5 \\ 1 & 0.3 & 0.2 \end{bmatrix}\begin{bmatrix} 0 & 0.4 & 0.3 \\ 0 & 0.3 & 0.5 \\ 1 & 0.3 & 0.2 \end{bmatrix}$

$= \begin{bmatrix} 0.3 & 0.21 & 0.26 \\ 0.5 & 0.24 & 0.25 \\ 0.2 & 0.55 & 0.49 \end{bmatrix}$

$\mathbf{T}^3 = \mathbf{T}^2\mathbf{T}$

$= \begin{bmatrix} 0.3 & 0.21 & 0.26 \\ 0.5 & 0.24 & 0.25 \\ 0.2 & 0.55 & 0.49 \end{bmatrix}\begin{bmatrix} 0 & 0.4 & 0.3 \\ 0 & 0.3 & 0.5 \\ 1 & 0.3 & 0.2 \end{bmatrix}$

$= \begin{bmatrix} 0.26 & 0.261 & 0.247 \\ 0.25 & 0.347 & 0.32 \\ 0.49 & 0.392 & 0.433 \end{bmatrix}$

b. From $\mathbf{T}^2$, entry in row 1, column 2, is 0.21.

c. From $\mathbf{T}^3$, entry in row 2, column 1, is 0.25.

23. $\mathbf{T} - \mathbf{I} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -\frac{2}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & -\frac{2}{3} & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]$

$\mathbf{Q} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

24. $\mathbf{T} - \mathbf{I} = \begin{bmatrix} 0.4 & 0.4 & 0.3 \\ 0.3 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -0.6 & 0.4 & 0.3 \\ 0.3 & -0.8 & 0.3 \\ 0.3 & 0.4 & -0.6 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -0.6 & 0.4 & 0.3 & 0 \\ 0.3 & -0.8 & 0.3 & 0 \\ 0.3 & 0.4 & -0.6 & 0 \end{array}\right] \to \cdots$

$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.36 \\ 0 & 1 & 0 & 0.27 \\ 0 & 0 & 1 & 0.36 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\mathbf{Q} \approx \begin{bmatrix} 0.36 \\ 0.27 \\ 0.36 \end{bmatrix}$

25. $\mathbf{T} = \begin{array}{r} \\ \text{Japanese} \\ \text{Non-Japanese} \end{array} \begin{array}{c} \text{Japanese Non-Japanese} \\ \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \end{array}$

a. $\mathbf{T}^2 = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$

$= \begin{bmatrix} 0.76 & 0.72 \\ 0.24 & 0.28 \end{bmatrix}$

From row 1, column 1, the probability that a person who currently owns a Japanese car will buy a Japanese car two cars later is 0.76. Thus 76% of people who currently own Japanese cars will own Japanese cars two cars later.

b. $\mathbf{X}_2 = \mathbf{T}^2\mathbf{X}_0 = \begin{bmatrix} 0.76 & 0.72 \\ 0.24 & 0.28 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.744 \\ 0.256 \end{bmatrix}$

Two cars from now, we expect 74.4% Japanese, 25.6% non-Japanese.

c. $\mathbf{T} - \mathbf{I} = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.6 \\ 0.2 & -0.6 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -0.2 & 0.6 & 0 \\ 0.2 & -0.6 & 0 \end{array}\right] \to \ldots \to \left[\begin{array}{cc|c} 1 & 0 & 0.75 \\ 0 & 1 & 0.25 \\ 0 & 0 & 0 \end{array}\right]$$

$\mathbf{Q} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$

In the long run, 75% Japanese cars, 25% non-Japanese cars.

26. a. $\mathbf{X}_1 = \mathbf{T}\mathbf{X}_0 = \begin{bmatrix} 0.7 & 0.4 & 0.1 \\ 0.2 & 0.5 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.3 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.49 \\ 0.27 \\ 0.24 \end{bmatrix}$

49% are expected to vote for party 1, 27% for party 2, 24% for party 3.

b. $\mathbf{T} - \mathbf{I} = \begin{bmatrix} 0.7 & 0.4 & 0.1 \\ 0.2 & 0.5 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.3 & 0.4 & 0.1 \\ 0.2 & -0.5 & 0.1 \\ 0.1 & 0.1 & -0.2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -0.3 & 0.4 & 0.1 & 0 \\ 0.2 & -0.5 & 0.1 & 0 \\ 0.1 & 0.1 & -0.2 & 0 \end{array}\right] \to \ldots \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{7} \\ 0 & 1 & 0 & \frac{5}{21} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\mathbf{Q} \approx \begin{bmatrix} 0.429 \\ 0.238 \\ 0.333 \end{bmatrix}$

In the long run, 43% will vote for party 1, 24% for party 2, and 33% for party 3.

Mathematical Snapshot Chapter 9

1. For $\mathbf{X}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, the first entry of the state vector is greater than 0.5 for $n = 7$ or greater. If $\mathbf{X}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, then $\mathbf{T}^7\mathbf{X}_0 \approx \begin{bmatrix} 0.5217 \\ 0.0000 \\ 0.4783 \\ 0.0000 \end{bmatrix}$.

2. $\mathbf{T}-\mathbf{I}=\begin{bmatrix}1 & 0.1 & 0.1 & 0.01\\ 0 & 0 & 0.9 & 0.09\\ 0 & 0.9 & 0 & 0.09\\ 0 & 0 & 0 & 0.81\end{bmatrix}-\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0.1 & 0.1 & 0.01\\ 0 & -1 & 0.9 & 0.09\\ 0 & 0.9 & -1 & 0.09\\ 0 & 0 & 0 & -0.19\end{bmatrix}$

$$\left[\begin{array}{cccc|c}1 & 1 & 1 & 1 & 1\\ 0 & 0.1 & 0.1 & 0.01 & 0\\ 0 & -1 & 0.9 & 0.09 & 0\\ 0 & 0.9 & -1 & 0.09 & 0\\ 0 & 0 & 0 & -0.19 & 0\end{array}\right]\to\dots\to\left[\begin{array}{cccc|c}1 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right].$$

3. Against Always Defect,

$$\mathbf{T}=\begin{array}{c}\\ 1\\ 2\\ 3\\ 4\end{array}\!\!\begin{array}{c}\begin{array}{cccc}1 & 2 & 3 & 4\end{array}\\ \begin{bmatrix}0 & 0 & 0 & 0\\ 1 & 0.1 & 1 & 0.1\\ 0 & 0 & 0 & 0\\ 0 & 0.9 & 0 & 0.9\end{bmatrix}\end{array}.$$

Against Always Cooperate,

$$\mathbf{T}=\begin{array}{c}\\ 1\\ 2\\ 3\\ 4\end{array}\!\!\begin{array}{c}\begin{array}{cccc}1 & 2 & 3 & 4\end{array}\\ \begin{bmatrix}1 & 0.1 & 1 & 0.1\\ 0 & 0 & 0 & 0\\ 0 & 0.9 & 0 & 0.9\\ 0 & 0 & 0 & 0\end{bmatrix}\end{array}.$$

Against regular Tit-for-tat,

$$\mathbf{T}=\begin{array}{c}\\ 1\\ 2\\ 3\\ 4\end{array}\!\!\begin{array}{c}\begin{array}{cccc}1 & 2 & 3 & 4\end{array}\\ \begin{bmatrix}1 & 0.1 & 0 & 0\\ 0 & 0 & 1 & 0.1\\ 0 & 0.9 & 0 & 0\\ 0 & 0 & 0 & 0.9\end{bmatrix}\end{array}.$$

4. With Player 2 always defecting, after one round the game is in a stable pattern of Player 1 cooperating with probability 0.1 and defecting with probability 0.9. The steady state vector in this case is $\begin{bmatrix}0\\ 0.1\\ 0\\ 0.9\end{bmatrix}$.

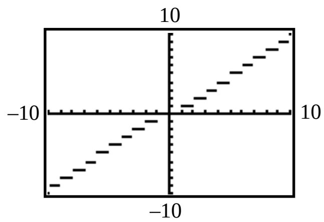
With Player 2 always cooperating, after one round the game settles into steady mutual cooperation.

With Player 2 playing standard Tit-for-tat, the probabilities gradually tilt toward mutual cooperation: $\begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix}$ is the steady state vector. In this case, it takes only one "forgiving" Tit-for-tat-er to guarantee mutual cooperation in the long run.

Chapter 10

Principles in Practice 10.1

1. The graph of the greatest integer function is shown.



$\lim\limits_{x\to a} f(x)$ does not exist when a is an integer since the limits are different depending on the side from which you approach the integer. $\lim\limits_{x\to a} f(x)$ exists for all numbers which are not integers.

2. $\lim\limits_{r\to 1} V(r) = \lim\limits_{r\to 1} \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \lim\limits_{r\to 1} r^3$

$= \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$

3. $\lim\limits_{x\to 8} R(x) = \lim\limits_{x\to 8}\left(500x - 6x^2\right)$

$= \lim\limits_{x\to 8} 500x - \lim\limits_{x\to 8} 6x^2$

$= 500 \lim\limits_{x\to 8} x - 6 \lim\limits_{x\to 8} x^2 = 500(8) - 6(8)^2$

$= 4000 - 384 = 3616$

4. $\lim\limits_{t\to 2} p = \lim\limits_{t\to 2} \frac{50\left(t^2+4t\right)}{t^2+3t+20} = \frac{\lim\limits_{t\to 2}\left[50\left(t^2+4t\right)\right]}{\lim\limits_{t\to 2}\left(t^2+3t+20\right)}$

$= \frac{50\left[2^2+4(2)\right]}{2^2+3(2)+20} = \frac{600}{30} = 20$

5. As $h \to 0$, both the numerator and denominator approach 0. For $h \neq 0$,

$\lim\limits_{h\to 0} \frac{125+2(x+h)-(125+2x)}{h}$

$= \lim\limits_{h\to 0} \frac{125+2x+2h-125-2x}{h} = \lim\limits_{h\to 0} \frac{2h}{h}$

$= \lim\limits_{h\to 0} 2 = 2$.

Problems 10.1

1. a. 1

b. 0

c. 1

2. a. 2

b. 1

c. 2

3. a. 1

b. does not exist

c. 3

4. a. –1

b. does not exist

c. 1

5. $f(-0.9) = -3.7$ $\quad f(-1.1) = -4.3$
$f(-0.99) = -3.97$ $\quad f(-1.01) = -4.03$
$f(-0.999) = -3.997$ $\quad f(-1.001) = -4.003$
estimate of limit: –4

6. $f(-3.1) = -6.1$ $\quad f(-2.9) = -5.9$
$f(-3.01) = -6.01$ $\quad f(-2.99) = -5.99$
$f(-3.001) = -6.001$ $\quad f(-2.999) = -5.999$
estimate of limit: –6

7. $f(-0.1) \approx 0.9516$ $\quad f(0.1) \approx 1.0517$
$f(-0.01) \approx 0.9950$ $\quad f(0.01) \approx 1.0050$
$f(-0.001) \approx 0.9995$ $\quad f(0.001) \approx 1.0005$
estimate of limit: 1

8. $f(-0.1) \approx 0.5132$ $\quad f(0.1) \approx 0.4881$
$f(-0.01) \approx 0.5013$ $\quad f(0.01) \approx 0.4988$
$f(-0.001) \approx 0.5001$ $\quad f(0.001) \approx 0.4999$
estimate of limit: 0.5

9. $\lim\limits_{x\to 2} 16 = 16$

10. $\lim\limits_{x\to 3} 2x = 2(3) = 6$

11. $\lim_{t\to -5}\left(t^2-5\right)=(-5)^2-5=25-5=20$

12. $\lim_{t\to 1/3}(5t-7)=5\left(\frac{1}{3}\right)-7=-\frac{16}{3}$

13. $\lim_{x\to -2}(3x^3-4x^2+2x-3)$
$=3(-2)^3-4(-2)^2+2(-2)-3$
$=-24-16-4-3$
$=-47$

14. $\lim_{r\to 9}\frac{4r-3}{11}=\frac{4(9)-3}{11}=\frac{36-3}{11}=\frac{33}{11}=3$

15. $\lim_{t\to -3}\frac{t-2}{t+5}=\frac{\lim_{t\to -3}(t-2)}{\lim_{t\to -3}(t+5)}=\frac{-3-2}{-3+5}=\frac{-5}{2}=-\frac{5}{2}$

16. $\lim_{t\to -6}\frac{x^2+6}{x-6}=\frac{\lim_{x\to -6}\left(x^2+6\right)}{\lim_{x\to -6}(x-6)}=\frac{(-6)^2+6}{(-6)-6}$
$=\frac{42}{-12}=-\frac{7}{2}$

17. $\lim_{h\to 0}\frac{h}{h^2-7h+1}=\frac{\lim_{h\to 0}h}{\lim_{h\to 0}\left(h^2-7h+1\right)}$
$=\frac{0}{0^2-7(0)+1}=0$

18. $\lim_{z\to 0}\frac{z^2-5z-4}{z^2+1}=\frac{\lim_{z\to 0}(z^2-5z-4)}{\lim_{z\to 0}(z^2+1)}$
$=\frac{0^2-5(0)-4}{0^2+1}$
$=-4$

19. $\lim_{p\to 4}\sqrt{p^2+p+5}=\sqrt{\lim_{p\to 4}\left(p^2+p+5\right)}$
$=\sqrt{4^2+4+5}=\sqrt{25}=5$

20. $\lim_{y\to 15}\sqrt{y+3}=\sqrt{\lim_{y\to 15}(y+3)}=\sqrt{15+3}=\sqrt{18}$
$=3\sqrt{2}$

21. $\lim_{x\to -2}\frac{x^2+2x}{x+2}=\lim_{x\to -2}\frac{x(x+2)}{x+2}=\lim_{x\to -2}x=-2$

22. $\lim_{x\to -1}\frac{x+1}{x+1}=\lim_{x\to -1}1=1$

23. $\lim_{x\to 2}\frac{x^2-x-2}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+1)}{x-2}$
$=\lim_{x\to 2}(x+1)=3$

24. $\lim_{t\to 0}\frac{t^3+3t^2}{t^3-4t^2}=\lim_{t\to 0}\frac{t^2(t+3)}{t^2(t-4)}=\lim_{t\to 0}\frac{t-3}{t-4}=-\frac{3}{4}$

25. $\lim_{x\to 3}\frac{x^2-x-6}{x-3}=\lim_{x\to 3}\frac{(x-3)(x+2)}{x-3}$
$=\lim_{x\to 3}(x+2)$
$=5$

26. $\lim_{t\to 2}\frac{t^2-4}{t-2}=\lim_{t\to 2}\frac{(t+2)(t-2)}{t-2}=\lim_{t\to 2}(t+2)=4$

27. $\lim_{x\to 3}\frac{x-3}{x^2-9}=\lim_{x\to 3}\frac{x-3}{(x+3)(x-3)}=\lim_{x\to 3}\frac{1}{x+3}=\frac{1}{6}$

28. $\lim_{x\to 0}\frac{x^2-2x}{x}=\lim_{x\to 0}\frac{x(x-2)}{x}=\lim_{x\to 0}(x-2)=-2$

29. $\lim_{x\to 4}\frac{x^2-9x+20}{x^2-3x-4}=\lim_{x\to 4}\frac{(x-4)(x-5)}{(x-4)(x+1)}$
$=\lim_{x\to 4}\frac{x-5}{x+1}=-\frac{1}{5}$

30. $\lim_{x\to -3}\frac{x^4-81}{x^2+8x+15}=\lim_{x\to -3}\frac{(x^2+9)(x^2-9)}{(x+3)(x+5)}$
$=\lim_{x\to -3}\frac{(x^2+9)(x+3)(x-3)}{(x+3)(x+5)}$
$=\lim_{x\to -3}\frac{(x^2+9)(x-3)}{x+5}$
$=-54$

31. $\lim_{x\to 2}\frac{3x^2-x-10}{x^2+5x-14}=\lim_{x\to 2}\frac{(3x+5)(x-2)}{(x+7)(x-2)}$
$=\lim_{x\to 2}\frac{3x+5}{x+7}=\frac{11}{9}$

32. $\lim\limits_{x\to -4}\dfrac{x^2+2x-8}{x^2+5x+4}=\lim\limits_{x\to -4}\dfrac{(x+4)(x-2)}{(x+4)(x+1)}=\lim\limits_{x\to -4}\dfrac{x-2}{x+1}=2$

33. $\lim\limits_{h\to 0}\dfrac{(2+h)^2-2^2}{h}=\lim\limits_{h\to 0}\dfrac{\left[4+4h+h^2\right]-4}{h}=\lim\limits_{h\to 0}\dfrac{4h+h^2}{h}=\lim\limits_{h\to 0}\dfrac{h(4+h)}{h}=\lim\limits_{h\to 0}(4+h)=4$

34. $\lim\limits_{x\to 0}\dfrac{(x+2)^2-4}{x}=\lim\limits_{x\to 0}\dfrac{x^2+4x}{x}=\lim\limits_{x\to 0}(x+4)=4$

35. $\lim\limits_{h\to 0}\dfrac{(x+h)^2-x^2}{h}=\lim\limits_{h\to 0}\dfrac{2xh+h^2}{h}=\lim\limits_{h\to 0}(2x+h)=2x$

36. $\lim\limits_{h\to 0}\dfrac{3(x+h)^2+7(x+h)-3x^2-7x}{h}=\lim\limits_{h\to 0}\dfrac{3x^2+6xh+3h^2+7x+7h-3x^2-7x}{h}$

$$=\lim_{h\to 0}\frac{6xh+3h^2+7h}{h}=\lim_{h\to 0}\frac{h(6x+3h+7)}{h}$$

$$=\lim_{h\to 0}(6x+3h+7)=6x+7$$

37. $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\dfrac{[7-3(x+h)]-(7-3x)}{h}=\lim\limits_{h\to 0}\dfrac{-3h}{h}=\lim\limits_{h\to 0}-3=-3$

38. $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\dfrac{[2(x+h)+3]-(2x+3)}{h}=\lim\limits_{h\to 0}\dfrac{2h}{h}=\lim\limits_{h\to 0}2=2$

39. $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\dfrac{\left[(x+h)^2-3\right]-\left(x^2-3\right)}{h}$

$$=\lim_{h\to 0}\frac{x^2+2xh+h^2-3-\left(x^2-3\right)}{h}=\lim_{h\to 0}\frac{2xh+h^2}{h}=\lim_{h\to 0}(2x+h)=2x$$

40. $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\dfrac{\left[(x+h)^2+(x+h)+1\right]-\left(x^2+x+1\right)}{h}=\lim\limits_{h\to 0}\dfrac{2xh+h^2+h}{h}=\lim\limits_{h\to 0}(2x+h+1)=2x+1$

41. $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\dfrac{[(x+h)^3-4(x+h)^2]-[x^3-4x^2]}{h}$

$$=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-4x^2-8xh-4h^2-x^3+4x^2}{h}$$

$$=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-8xh-4h^2}{h}$$

$$=\lim_{h\to 0}\frac{h(3x^2+3xh+h^2-8x-4h)}{h}$$

$$=\lim_{h\to 0}(3x^2+3xh+h^2-8x-4h)=3x^2-8x$$

42. $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{[3-(x+h)+4(x+h)^2]-(3-x+4x^2)}{h}$
$=\lim_{h\to 0}\frac{3-x-h+4x^2+8xh+4h^2-(3-x+4x^2)}{h}$
$=\lim_{h\to 0}\frac{-h+8xh+4h^2}{h}$
$=\lim_{h\to 0}\frac{h(-1+8x+4h)}{h}$
$=\lim_{h\to 0}(-1+8x+4h)$
$=-1+8x$

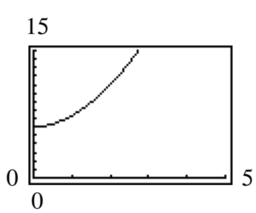
43. $\lim_{x\to 6}\frac{\sqrt{x-2}-2}{x-6}=\lim_{x\to 6}\frac{\left(\sqrt{x-2}-2\right)\left(\sqrt{x-2}+2\right)}{(x-6)\left(\sqrt{x-2}+2\right)}$
$=\lim_{x\to 6}\frac{(x-2)-4}{(x-6)\left(\sqrt{x-2}+2\right)}=\lim_{x\to 6}\frac{x-6}{(x-6)\left(\sqrt{x-2}+2\right)}$
$=\lim_{x\to 6}\frac{1}{\sqrt{x-2}+2}=\frac{1}{4}$

44. For $\lim_{x\to 3}\frac{x^2+x+c}{x^2-5x+6}=\lim_{x\to 3}\frac{x^2+x+c}{(x-3)(x-2)}$ to exist, $x-3$ must be a factor of the numerator x^2+x+c:
$x^2+x+c=(x-3)(x+r)=x^2+(r-3)x-3r$
Thus $r-3=1$, or $r=4$. So $c=-3r=-3(4)=-12$.
For $c=-12$,
$\lim_{x\to 3}\frac{x^2+x+c}{x^2-5x+6}=\lim_{x\to 3}\frac{x^2+x-12}{x^2-5x+6}=\lim_{x\to 3}\frac{(x-3)(x+4)}{(x-3)(x-2)}=\lim_{x\to 3}\frac{x+4}{x-2}=\frac{7}{1}=7$

45. **a.** $\lim_{T_c\to 0}\frac{T_h-T_c}{T_h}=\frac{T_h-0}{T_h}=\frac{T_h}{T_h}=1$

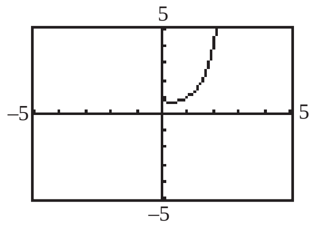
b. $\lim_{T_c\to T_h}\frac{T_h-T_c}{T_h}=\frac{T_h-T_h}{T_h}=\frac{0}{T_h}=0$

46. $\lim_{r\to 7.5\times 10^7}E=\lim_{r\to 7.5\times 10^7}-\frac{7.0\times 10^{17}}{r}=-\frac{7.0\times 10^{17}}{7.5\times 10^7}=-\frac{7.0}{7.5}\times 10^{10}\approx -9.33\times 10^9$ ft-lb

47.


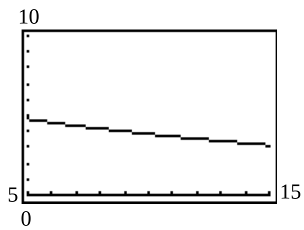
11.00

48.



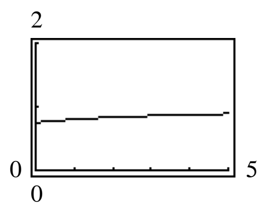
1

49.



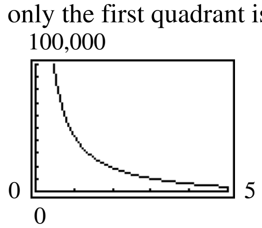
4.00

50.



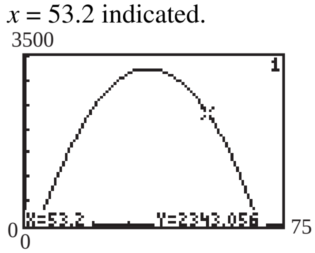
0.80

51. The graph of $C(p)$ is shown. (Negative amounts of impurities and money are not reasonable, so only the first quadrant is shown.)



As p gets closer and closer to 0, the values of $C(p)$ increase without bound, so $\lim_{p\to 0} C(p)$ does not exist.

52. The graph of $P(x)$ is shown with the value $x = 53.2$ indicated.

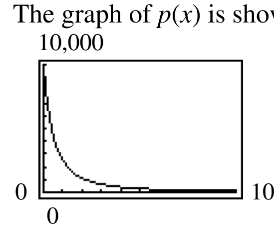


$$\lim_{x\to 53.2} P(x) = 2343.056$$

$$\lim_{x\to 53.2} P(x) = 224(53.2) - 3.1(53.2)^2 - 800$$
$$= 2343.056$$

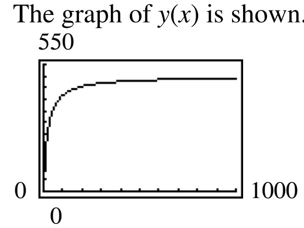
Principles in Practice 10.2

1. The graph of $p(x)$ is shown.



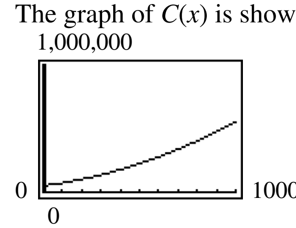
From the graph, it is apparent that $\lim_{x\to\infty} p(x) = 0$. The graph starts out high and quickly drops down toward zero. According to this function, a low price corresponds to a high demand and a high price corresponds to a low demand.

2. The graph of $y(x)$ is shown.



From the graph, it is apparent that $\lim_{x\to\infty} y(x) = 500$. The greatest yearly sales that the company can expect is \$500,000, even with unlimited spending on advertising.

3. The graph of $C(x)$ is shown.



From the graph it is apparent that $\lim_{x\to\infty} C(x) = \infty$. This indicates that the cost increases without bound the more units that you make.

4. $\lim_{x\to 1} f(x)$ does not exist since

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 100 = 100$ while

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 175 = 175$.

$\lim_{x\to 2.5} f(x) = 250$ since

$\lim_{x\to 2.5^-} f(x) = \lim_{x\to 2.5^+} f(x) = 250$

Problems 10.2

1. a. 2

b. 3

c. does not exist

d. $-\infty$

e. ∞

f. ∞

g. ∞

h. 0

i. 1

j. 1

k. 1

2. a. 0

b. $-\infty$

c. does not exist

d. ∞

e. 2

f. 1

g. 1

3. $\lim_{x\to 3^+} (x-2)$

As $x \to 3^+$, then $x-2 \to 1$.

4. $\lim_{x\to -1^+} (1-x^2) = 0$

5. $\lim_{x\to -\infty} 5x$

As *x* becomes very negative, so does 5*x*. Thus $\lim_{x\to -\infty} 5x = -\infty$.

6. $\lim_{x\to -\infty} 19 = 19$

7. $\lim_{x\to 0^-} \frac{6x}{x^4} = \lim_{x\to 0^-} \frac{6}{x^3} = -\infty$ since x^3 is negative and close to 0 for $x \to 0^-$.

8. $\lim_{x\to 2} \frac{7}{x-1} = \frac{\lim_{x\to 2} 7}{\lim_{x\to 2}(x-1)} = \frac{7}{1} = 7$

9. $\lim_{x\to -\infty} x^2 = \infty$ since x^2 is positive for $x \to -\infty$.

10. $\lim_{t\to \infty} (t-1)^3 = \infty$

11. $\lim_{h\to 0^+} \sqrt{h} = 0$ since $\sqrt{h}$ is close to 0 when *h* is positive and close to 0.

12. $\lim_{h\to 5^-} \sqrt{5-h} = 0$

13. $\lim_{x\to -2^-} \frac{-3}{x+2} = \infty$

14. $\lim_{x\to 0^-} 2^{1/2} = 2^{1/2}$

15. $\lim_{x\to 1^+} \left(4\sqrt{x-1}\right)$. As $x \to 1^+$, then *x* – 1 approaches 0 through positive values. So $\sqrt{x-1} \to 0$. Thus $\lim_{x\to 1^+} \left(4\sqrt{x-1}\right) = 4\cdot \lim_{x\to 1^+} \sqrt{x-1} = 4\cdot 0 = 0$.

16. $\lim_{x\to 2^+} \left(x\sqrt{x^2-4}\right) = 0$

17. $\lim_{x\to \infty} \sqrt{x+10}$

As *x* becomes very large, so does *x* + 10. Because square roots of very large numbers are very large, $\lim_{x\to \infty} \sqrt{x+10} = \infty$.

18. $\lim_{x\to -\infty} -\sqrt{1-10x}$

As *x* becomes very negative, 1 – 10*x* becomes very positive. Because square roots of very large numbers are very large, $\lim_{x\to -\infty} -\sqrt{1-10x} = -\infty.$

19. $\lim_{x\to\infty} \frac{3}{\sqrt{x}} = 3 \lim_{x\to\infty} \frac{1}{x^{\frac{1}{2}}} = 3\cdot 0 = 0$

20. $\lim_{x\to\infty} \frac{-6}{5x\sqrt[3]{x}} = -\frac{6}{5} \lim_{x\to\infty} \frac{1}{x^{4/3}} = -\frac{6}{5}\cdot 0 = 0$

21. $\lim_{x\to\infty} \frac{x+8}{x-3} = \lim_{x\to\infty} \frac{x}{x} = \lim_{x\to\infty} 1 = 1$

22. $\lim_{x\to\infty} \frac{2x-4}{3-2x} = \lim_{x\to\infty} \frac{2x}{-2x} = \lim_{x\to\infty} (-1) = -1$

23. $\lim_{x\to-\infty} \frac{x^2-1}{x^3+4x-3} = \lim_{x\to-\infty} \frac{x^2}{x^3} = \lim_{x\to-\infty} \frac{1}{x} = 0$

24. $\lim_{r\to\infty} \frac{r^3}{r^2+1} = \lim_{r\to\infty} \frac{r^3}{r^2} = \lim_{r\to\infty} r = \infty$

25. $\lim_{t\to\infty} \frac{3t^3+2t^2+9t-1}{5t^2-5} = \lim_{t\to\infty} \frac{3t^3}{5t^2}$
$= \lim_{t\to\infty} \frac{3t}{5}$
$= \frac{3}{5} \lim_{t\to\infty} t$
$= \infty$

26. $\lim_{x\to\infty} \frac{5x}{3x^7-x^3+4} = \lim_{x\to\infty} \frac{5x}{3x^7}$
$= \lim_{x\to\infty} \frac{5}{3x^6} = \frac{5}{3}\cdot \lim_{x\to\infty} \frac{1}{x^6} = \frac{5}{3}\cdot 0 = 0$

27. $\lim_{x\to\infty} \frac{7}{2x+1} = \lim_{x\to\infty} \frac{7}{2x} = \frac{7}{2}\cdot \lim_{x\to\infty} \frac{1}{x} = \frac{7}{2}\cdot 0 = 0$

28. $\lim_{x\to-\infty} \frac{2}{(4x-1)^3} = \lim_{x\to-\infty} \frac{2}{4^3x^3}$
$= \frac{2}{4^3}\cdot \lim_{x\to-\infty} \frac{1}{x^3} = \frac{2}{4^3}\cdot 0 = 0$

29. $\lim_{x\to\infty} \frac{3-4x-2x^3}{5x^3-8x+1} = \lim_{x\to\infty} \frac{-2x^3}{5x^3}$
$= \lim_{x\to\infty} \frac{-2}{5} = -\frac{2}{5}$

30. $\lim_{x\to-\infty} \frac{3-2x-2x^3}{7-5x^3+2x^2} = \lim_{x\to-\infty} \frac{-2x^3}{-5x^3}$
$= \lim_{x\to-\infty} \frac{2}{5} = \frac{2}{5}$

31. $\lim_{x\to 3^-} \frac{x+3}{x^2-9} = \lim_{x\to 3^-} \frac{x+3}{(x+3)(x-3)}$
$= \lim_{x\to 3^-} \frac{1}{x-3} = -\infty$

32. As $x \to -3^-$, then $3x \to -9$ and $9-x^2 \to 0$ through negative values. Thus
$\lim_{x\to-3^-} \frac{3x}{9-x^2} = \infty.$

33. $\lim_{w\to\infty} \frac{2w^2-3w+4}{5w^2+7w-1} = \lim_{w\to\infty} \frac{2w^2}{5w^2} = \lim_{w\to\infty} \frac{2}{5} = \frac{2}{5}$

34. $\lim_{x\to\infty} \frac{4-3x^3}{x^3-1} = \lim_{x\to\infty} \frac{-3x^3}{x^3} = \lim_{x\to\infty} (-3) = -3$

35. $\lim_{x\to\infty} \frac{6-4x^2+x^3}{4+5x-7x^2} = \lim_{x\to\infty} \frac{x^3}{-7x^2} = \lim_{x\to\infty} \frac{x}{-7} = -\infty$

36. $\lim_{x\to-\infty} \frac{3x-x^3}{x^3+x+1} = \lim_{x\to-\infty} \frac{-x^3}{x^3} = \lim_{x\to-\infty} (-1) = -1$

37. $\lim_{x\to-3^-} \frac{5x^2+14x-3}{x^2+3x} = \lim_{x\to-3^-} \frac{(5x-1)(x+3)}{x(x+3)}$
$= \lim_{x\to-3^-} \frac{5x-1}{x}$
$= \frac{-16}{-3}$
$= \frac{16}{3}$

38. $\lim_{t\to 3} \frac{t^2-4t+3}{t^2-2t-3} = \lim_{t\to 3} \frac{(t-1)(t-3)}{(t+1)(t-3)}$
$= \lim_{t\to 3} \frac{t-1}{t+1} = \frac{2}{4} = \frac{1}{2}$

39. $\lim_{x\to 1} \frac{x^2-3x+1}{x^2+1} = \frac{\lim_{x\to 1}\left(x^2-3x+1\right)}{\lim_{x\to 1}\left(x^2+1\right)} = \frac{-1}{2} = -\frac{1}{2}$

40. $\lim\limits_{x\to -1} \dfrac{3x^3 - x^2}{2x+1} = \dfrac{\lim\limits_{x\to -1}\left(3x^3 - x^2\right)}{\lim\limits_{x\to -1}(2x+1)} = \dfrac{-4}{-1} = 4$

41. As $x \to 1^+$, then $\dfrac{1}{x-1} \to \infty$. Thus

$\lim\limits_{x\to 1^+}\left[1+\dfrac{1}{x-1}\right] = \infty$

42. $\lim\limits_{x\to -\infty} -\dfrac{x^5 + 2x^3 - 1}{x^5 - 4x^2} = \lim\limits_{x\to -\infty} -\dfrac{x^5}{x^5}$
$= \lim\limits_{x\to -\infty}(-1) = -1$

43. $\lim\limits_{x\to -7^-} \dfrac{x^2+1}{\sqrt{x^2-49}}$. As $x \to -7^-$, then
$x^2 + 1 \to 50$ and $\sqrt{x^2-49}$
approaches 0 through positive values. Thus
$\dfrac{x^2+1}{\sqrt{x^2-49}} \to \infty$.

44. As $x \to -2^+$, then $x \to -2$ and $\sqrt{16-x^4} \to 0$ through positive values. Thus,

$\lim\limits_{x\to -2^+} \dfrac{x}{\sqrt{16-x^4}} = -\infty.$

45. As $x \to 0^+$, $x + x^2$ approaches 0 through positive values. Thus $\dfrac{5}{x+x^2} \to \infty$.

46. As $x \to \infty$, then $\dfrac{1}{x} \to 0$. Thus

$\lim\limits_{x\to\infty}\left(x+\dfrac{1}{x}\right) = \infty$.

47. $\lim\limits_{x\to 1^-} \dfrac{x}{x-1} = -\infty$

$\lim\limits_{x\to 1^+} \dfrac{x}{x-1} = \infty$

Answer: does not exist

48. $\lim\limits_{x\to 1/2^-} \dfrac{1}{2x-1} = -\infty$

$\lim\limits_{x\to 1/2^+} \dfrac{1}{2x-1} = \infty$

Answer: does not exist

49. As $x \to 1^+$, then $1 - x \to 0$ through negative values. Thus, $\lim\limits_{x\to 1^+} \dfrac{-5}{1-x} = \infty.$

50. $\lim\limits_{x\to 3^+}\left(-\dfrac{7}{x-3}\right) = -\infty$

$\lim\limits_{x\to 3^-}\left(-\dfrac{7}{x-3}\right) = +\infty$

Answer: does not exist.

51. $\lim\limits_{x\to 0^+} |x| = \lim\limits_{x\to 0^+} x = 0$

$\lim\limits_{x\to 0^-} |x| = \lim\limits_{x\to 0^-} (-x) = 0$

Thus, $\lim\limits_{x\to 0} |x| = 0.$

52. $\lim\limits_{x\to 0^+} \left|\dfrac{1}{x}\right| = \lim\limits_{x\to 0^+} \dfrac{1}{x} = \infty$

$\lim\limits_{x\to 0^-} \left|\dfrac{1}{x}\right| = \lim\limits_{x\to 0^-} \left(-\dfrac{1}{x}\right) = \infty$

Thus, $\lim\limits_{x\to 0} \left|\dfrac{1}{x}\right| = \infty.$

53. $\lim\limits_{x\to -\infty} \dfrac{x+1}{x} = \lim\limits_{x\to -\infty} \dfrac{x}{x} = \lim\limits_{x\to -\infty} 1 = 1$

54. $\lim\limits_{x\to\infty}\left[\dfrac{3}{x} - \dfrac{2x^2}{x^2+1}\right] = \lim\limits_{x\to\infty} \dfrac{3x^2 + 3 - 2x^3}{x^3 + x}$
$= \lim\limits_{x\to\infty} \dfrac{-2x^3}{x^3} = \lim\limits_{x\to\infty}(-2) = -2$

55. $f(x) = \begin{cases} 2 & \text{if } x \le 2 \\ 1 & \text{if } x > 2 \end{cases}$

a. $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} 1 = 1$

b. $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} 2 = 2$

c. $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$, so $\lim_{x\to 2} f(x)$ does not exist.

d. $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 1 = 1$

e. $\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} 2 = 2$

56. $f(x) = \begin{cases} x & \text{if } x \le 2 \\ -2+4x-x^2 & \text{if } x > 2 \end{cases}$

a. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (-2+4x-x^2) = 2$

b. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x = 2$

c. $\lim_{x\to 2} f(x) = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = 2$

d. $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (-2+4x-x^2) = -\infty$

e. $\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} x = -\infty$

57. $g(x) = \begin{cases} x & \text{if } x < 0 \\ -x & \text{if } x > 0 \end{cases}$

a. $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (-x) = 0$

b. $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} x = 0$

c. $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^-} g(x) = 0$, so $\lim_{x\to 0} g(x) = 0$

d. $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} (-x) = -\infty$

e. $\lim_{x\to -\infty} g(x) = \lim_{x\to -\infty} x = -\infty$

58. $g(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x & \text{if } x > 0 \end{cases}$

a. $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} x = 0$

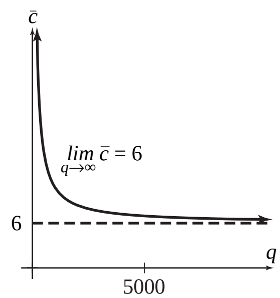
b. $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} x^2 = 0$

c. $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^-} g(x) = 0$, so $\lim_{x\to 0} g(x) = 0$

d. $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} x = \infty$

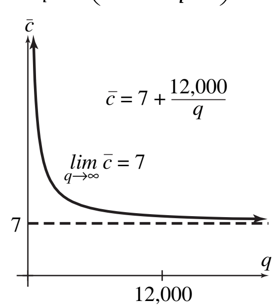
e. $\lim_{x\to -\infty} g(x) = \lim_{x\to -\infty} x^2 = \infty$

59. $\lim_{q\to -\infty} \bar{c} = \lim_{q\to\infty} \left(\frac{5000}{q} + 6\right) = 0 + 6 = 6$



60. $\lim_{q\to\infty} \bar{c} = \lim_{q\to\infty} \frac{7q+12{,}000}{q}$

$= \lim_{q\to\infty} \left(7 + \frac{12{,}000}{q}\right) = 7 + 0 = 7$



61. $\lim_{t\to\infty} \left(50{,}000 - \frac{2000}{t+1}\right) = 50{,}000 - 0 = 50{,}000$

62. $\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$

$= \lim_{x\to\infty}\frac{\left(\sqrt{x^2+x}-x\right)\left(\sqrt{x^2+x}+x\right)}{\sqrt{x^2+x}+x}$

$= \lim_{x\to\infty}\frac{\left(x^2+x\right)-x^2}{\sqrt{x^2+x}+x} = \lim_{x\to\infty}\frac{x}{\sqrt{x^2\left(1+\frac{1}{x}\right)}+x}$

$= \lim_{x\to\infty}\frac{x}{x\sqrt{1+\frac{1}{x}}+x}$

$= \lim_{x\to\infty}\frac{x}{x\left(\sqrt{1+\frac{1}{x}}+1\right)} = \lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1}$

$= \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$

63. $\lim_{x\to\infty} y = \lim_{x\to\infty}\frac{900x}{10+45x} = \lim_{x\to\infty}\frac{900x}{45x}$
$= \lim_{x\to\infty} 20 = 20$

64. $f(x) = \begin{cases}\sqrt{2-x} & \text{if } x<2\\ x^3+k(x+1) & \text{if } x\ge 2\end{cases}$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\sqrt{2-x} = 0$

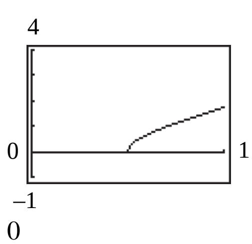
$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\left[x^3+k(x+1)\right] = 8+3k$

If $\lim_{x\to 2} f(x)$ exists, then 8 + 3k = 0. So $k = -\frac{8}{3}$.

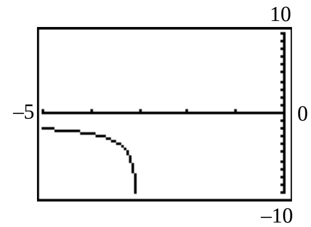
65. 1, 0.5, 0.525, 0.631, 0.912, 0.986, 0.998; conclude limit is 1.

66. 0.368, 0.135, 0.00674, 0.0000454, 3.72×10^{-44}, can't do last two. Conclude that the limit is 0.

67.

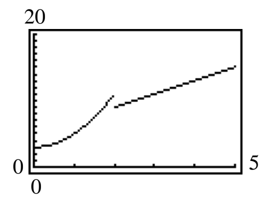


68.



$-\infty$

69.



a. 11

b. 9

c. does not exist

Problems 10.3

1. $f(x) = x^3-5x;\ x=2$

(i) f is defined at x = 2: $f(2) = -2$

(ii) $\lim_{x\to 2} f(x) = \lim_{x\to 2}(x^3-5x) = 2^3-5(2) = -2,$ which exists.

(iii) $\lim_{x\to 2} f(x) = -2 = f(2)$
Thus f is continuous at x = 2.

2. $f(x) = \frac{x-3}{5x};\ x=-3$

(i) f is defined at $x = -3$: $f(-3) = \frac{-6}{-15} = \frac{2}{5}$

(ii) $\lim_{x\to -3} f(x) = \lim_{x\to -3}\frac{x-3}{5x} = \frac{2}{5},$ which exists

(iii) $\lim_{x\to -3} f(x) = \frac{2}{5} = f(-3)$

Thus f is continuous at $x = -3$.

3. $g(x)=\sqrt{2-3x}$; $x = 0$

(i) g is defined at $x = 0$; $g(0)=\sqrt{2}$.

(ii) $\lim_{x\to 0} g(x) = \lim_{x\to 0} \sqrt{2-3x} = \sqrt{2}$, which exists

(iii) $\lim_{x\to 0} g(x) = \sqrt{2} = g(0)$

Thus g is continuous at $x = 0$.

4. $f(x)=\frac{x}{8}$; $x = 2$

(i) f is defined at $x = 2$; $f(2)=\frac{2}{8}=\frac{1}{4}$.

(ii) $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x}{8} = \frac{2}{8} = \frac{1}{4}$, which exists.

(iii) $\lim_{x\to 2} f(x) = \frac{1}{4} = f(2)$.

Thus f is continuous at $x = 2$.

5. $h(x)=\frac{x-4}{x+4}$; $x = 4$

(i) h is defined at $x = 4$, $h(4) = 0$.

(ii) $\lim_{x\to 4} h(x) = \lim_{x\to 4} \frac{x-4}{x+4} = \frac{0}{8} = 0$, which exists.

(iii) $\lim_{x\to 4} h(x) = 0 = h(4)$

Thus h is continuous at $x = 4$.

6. $f(x)=\sqrt[3]{x}$; $x = -1$

(i) f is defined at $x = -1$; $f(-1) = -1$.

(ii) $\lim_{x\to -1} f(x) = \lim_{x\to -1} \sqrt[3]{x} = \sqrt[3]{-1} = -1$, which exists.

(iii) $\lim_{x\to -1} f(x) = -1 = f(-1)$

Thus f is continuous at $x = -1$.

7. Continuous at –2 and 0 because f is a rational function and at neither point is the denominator zero.

8. Continuous at 2 and –2 because f is a polynomial function (which is continuous everywhere).

9. Discontinuous at 3 and –3 because at both points the denominator of this rational function is 0.

10. Continuous at 2 and –2 because f is a rational function and at neither point is the denominator zero.

11. $f(x)=\begin{cases} x+2 & \text{if } x \geq 2 \\ x^2 & \text{if } x < 2 \end{cases}$

f is defined at $x = 2$ and $x = 0$; $f(2) = 4$, $f(0) = 0$. Because $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x+2) = 4$ and $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = 4$, we have $\lim_{x\to 2} f(x) = 4$. In addition, $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 = 0$. Since $\lim_{x\to 2} f(x) = 4 = f(2)$ and $\lim_{x\to 0} f(x) = 0 = f(0)$, f is continuous at both 2 and 0.

Answer: Continuous at 2 and 0.

12. $f(x)=\begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Because $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = \infty$, $\lim_{x\to 0} f(x)$ does not exist. Thus f is discontinuous at $x = 0$.

At $x = -1$, f is defined; $f(-1) = -1$.

$\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{1}{x} = -1$. Since $\lim_{x\to -1} f(x) = -1 = f(-1)$, f is continuous at $x = -1$.

Answer: Discontinuous at 0, continuous at –1.

13. f is a polynomial function.

14. f is a polynomial function $\left[f(x)=\frac{2}{5}+\frac{3}{5}x-\frac{1}{5}x^2 \right]$.

15. f is a rational function and the denominator is never zero.

16. f is a polynomial function $\left[f(x)=x-x^2 \right]$.

17. None, because f is a polynomial function.

18. None, because h is a polynomial function.

19. The denominator of this rational function is zero only when $x = -4$. Thus f is discontinuous only at $x = -4$.

20. The denominator of this rational function is zero only when $x = \pm 2$. Thus f is discontinuous only at $x = \pm 2$.

21. None, because g is a polynomial function.
$\left[g(x) = \frac{8}{15}x^6 - \frac{12}{5}x^4 + \frac{18}{5}x^2 - \frac{9}{5}\right]$

22. None, because f is a polynomial function.

23. $x^2 + 2x - 15 = 0$, $(x + 5)(x - 3) = 0$, $x = -5$ or 3. Discontinuous at -5 and 3.

24. $x^2 + x = 0$, $x(x + 1) = 0$, $x = 0$ or -1. Discontinuous at 0 and -1.

25. $x^3 - x = 0$, $x\left(x^2 - 1\right) = 0$, $x(x + 1)(x - 1) = 0$, $x = 0, \pm 1$. Discontinuous at $0, \pm 1$.

26. Discontinuous at $x = \frac{3}{2}$, for which the denominator is zero.

27. $x^2 + 1 = 0$ has no real roots, so no discontinuity exists.

28. $x^4 - 1 = 0$, $\left(x^2 + 1\right)\left(x^2 - 1\right) = 0$, $\left(x^2 + 1\right)(x + 1)(x - 1) = 0$, $x = \pm 1$. Discontinuous at ± 1.

29. $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$

For $x < 0$, $f(x) = -1$, which is a polynomial and hence continuous. For $x > 0$, $f(x) = 1$, which is a polynomial and hence continuous. Because $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (-1) = -1$ and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} 1 = 1$, $\lim\limits_{x \to 0} f(x)$ does not exist.
Thus f is discontinuous at $x = 0$.

30. $f(x) = \begin{cases} 2x + 1 & \text{if } x \geq -1 \\ 1 & \text{if } x < -1 \end{cases}$

For $x < -1$, $f(x) = 1$, which is a polynomial and hence continuous. For $x > -1$, $f(x) = 2x + 1$, which is a polynomial and hence continuous. Because $\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^-} 1 = 1$ and $\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} (2x + 1) = -1$, $\lim\limits_{x \to -1} f(x)$ does not exist.
Thus f is discontinuous at $x = -1$.

31. $f(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ x - 1 & \text{if } x > 1 \end{cases}$

For $x < 1$, $f(x) = 0$, which is a polynomial and hence continuous. For $x > 1$, $f(x) = x - 1$, which is a polynomial and hence continuous. For $x = 1$, f is defined [$f(1) = 0$]. Because $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} 0 = 0$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x - 1) = 0$, then $\lim\limits_{x \to 1} f(x) = 0$.
Since $\lim\limits_{x \to 1} f(x) = 0 = f(0)$, f is continuous at $x = 1$.
f has no discontinuities.

32. $f(x) = \begin{cases} x - 3 & \text{if } x > 2 \\ 3 - 2x & \text{if } x < 2 \end{cases}$

For $x < 2$, $f(x) = 3 - 2x$, which is a polynomial and hence continuous. For $x > 2$, $f(x) = x - 3$, which is a polynomial and hence continuous. Because f is not defined at $x = 2$, it is discontinuous there.

33. $f(x) = \begin{cases} x^2 + 1 & \text{if } x > 2 \\ 8x & \text{if } x < 2 \end{cases}$

For $x < 2$, $f(x) = 8x$, which is a polynomial and hence continuous. For $x > 2$, $f(x) = x^2 + 1$, which is a polynomial and hence continuous. Because f is not defined at $x = 2$, it is discontinuous there.

34. $f(x) = \begin{cases} \frac{16}{x^2} & \text{if } x \geq 2 \\ 3x - 2 & \text{if } x < 2 \end{cases}$

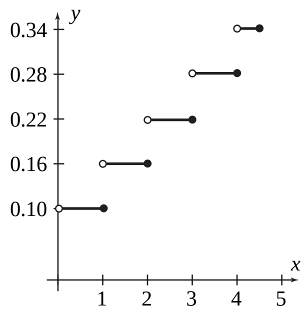
For $x < 2$, $f(x) = 3x - 2$, which is a polynomial and hence continuous. For $x > 2$, $f(x) = \frac{16}{x^2}$, which is continuous because $x > 2$ means that the denominator is never zero.

For $x = 2$, f is defined [$f(2) = 4$]. Because $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(3x-2) = 4$ and

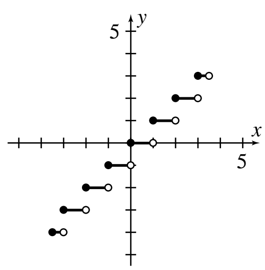
$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{16}{x^2} = 4$, then $\lim_{x\to 2} f(x) = 4$.

Since $\lim_{x\to 2} f(x) = 4 = f(2)$, f is continuous at $x = 2$.

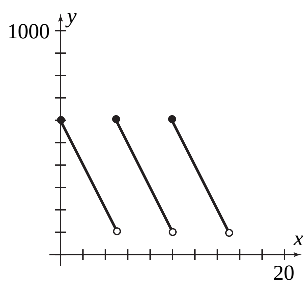
f has no discontinuities.

35. 

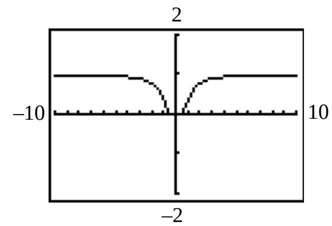
Discontinuous at 1, 2, 3, 4.

36. 

For $-3.5 \le x \le 3.5$, discontinuities at –3, –2, –1, 0, 1, 2, 3.

37. 

f is continuous at 2.
f is discontinuous at 5.
f is discontinuous at 10.

38. 

Answer: Yes

Principles in Practice 10.4

1. We need to solve $V(x) > 0$. The zeros of $V(x)$ occur when $x = 0$, $8 - 2x = 0$, and $10 - 2x = 0$, or $x = 0$, 4, and 5. These zeros determine the intervals $(-\infty, 0)$ $(0, 4)$, $(4, 5)$, and $(5, \infty)$. Using $x = -1$, 1, 4.5, and 6 for test points, we find the sign of $V(x)$:
$V(-1) = (-)(+)(+) = -$, so $V(x) < 0$ on $(-\infty, 0)$;
$V(1) = (+)(+)(+) = +$, so $V(x) > 0$ on $(0, 4)$;
$V(4.5) = (+)(-)(+) = -$, so $V(x) < 0$ on $(4, 5)$;
$V(6) = (+)(-)(-) = +$, so $V(x) > 0$ on $(5, \infty)$.
The volume is positive when $0 < x < 4$ or $5 < x$. However, $x > 5$ is unrealistic (as is $x < 0$) since the longest side of the piece of metal has length $2(5) = 10$ inches. Thus, the volume is positive when $0 < x < 4$.

Problems 10.4

1. $x^2 - 3x - 4 > 0$
$f(x) = x^2 - 3x - 4 = (x+1)(x-4)$ has zeros –1 and 4. By considering the intervals $(-\infty, -1)$, $(-1, 4)$, and $(4, \infty)$, we find $f(x) > 0$ on $(-\infty, -1)$ and $(4, \infty)$.
Answer: $(-\infty, -1)$, $(4, \infty)$

2. $x^2 - 8x + 15 > 0$
$f(x) = x^2 - 8x + 15 = (x-3)(x-5)$ has zeros 3 and 5. By considering the intervals $(-\infty, 3)$, $(3, 5)$, and $(5, \infty)$, we find $f(x) > 0$ on $(-\infty, 3)$ and $(5, \infty)$.
Answer: $(-\infty, 3)$, $(5, \infty)$

3. $x^2 - 3x - 10 \le 0$
$f(x) = (x + 2)(x - 5)$ has zeros –2 and 5. By considering the intervals $(-\infty, -2)$, $(-2, 5)$, and $(5, \infty)$, we find $f(x) < 0$ on $(-2, 5)$.
Answer: $[-2, 5]$

4. $14-5x-x^2 \le 0$, or equivalently,
$x^2+5x-14 \ge 0$
$f(x)=x^2+5x-14=(x+7)(x-2)$ has zeros –7 and 2. By considering the intervals $(-\infty, -7)$, $(-7, 2)$, and $(2, \infty)$, we find $f(x) \ge 0$ on $(-\infty, -7)$ and $(2, \infty)$.
Answer: $(-\infty, -7]$, $[2, \infty)$

5. $2x^2+11x+14<0$
$f(x)=2x^2+11x+14=(2x+7)(x+2)$ has zeros $-\frac{7}{2}$ and –2. By considering the intervals $\left(-\infty, -\frac{7}{2}\right)$, $\left(-\frac{7}{2}, -2\right)$, and $(-2, \infty)$, we find $f(x) < 0$ on $\left(-\frac{7}{2}, -2\right)$.
Answer: $\left(-\frac{7}{2}, -2\right)$

6. $x^2-4<0$. $f(x)=x^2-4=(x+2)(x-2)$ has zeros ±2. By considering the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$, we find $f(x) < 0$ on $(-2, 2)$.
Answer: $(-2, 2)$

7. $x^2+4<0$. Since x^2+4 is always positive, the inequality $x^2+4<0$ has no solution.
Answer: no solution

8. $2x^2-x-2 \le 0$. $f(x)=2x^2-x-2$ has zeros $\frac{1 \pm \sqrt{17}}{4}$. By considering the intervals $\left(-\infty, \frac{1-\sqrt{17}}{4}\right)$, $\left(\frac{1-\sqrt{17}}{4}, \frac{1+\sqrt{17}}{4}\right)$, and $\left(\frac{1+\sqrt{17}}{4}, \infty\right)$, we find $f(x) < 0$ on $\left(\frac{1-\sqrt{17}}{4}, \frac{1+\sqrt{17}}{4}\right)$.
Answer: $\left[\frac{1-\sqrt{17}}{4}, \frac{1+\sqrt{17}}{4}\right]$

9. $(x + 2)(x - 3)(x + 6) \le 0$
$f(x) = (x + 2)(x - 3)(x + 6)$ has zeros –2, 3, and –6. By considering the intervals $(-\infty, -6)$, $(-6, -2)$, $(-2, 3)$, and $(3, \infty)$, we find $f(x) < 0$ on $(-\infty, -6)$ and $(-2, 3)$.
Answer: $(-\infty, -6]$, $[-2, 3]$

10. $(x + 5)(x + 2)(x - 7) \le 0$
$f(x) = (x + 5)(x + 2)(x - 7)$ has zeros –5, –2 and 7. By considering the intervals $(-\infty, -5)$, $(-5, -2)$, $(-2, 7)$ and $(7, \infty)$, we find $f(x) < 0$ on $(-\infty, -5)$ and $(-2, 7)$.
Answer: $(-\infty, -5]$, $[-2, 7]$

11. $-x(x - 5)(x + 4) > 0$, or equivalently, $x(x - 5)(x + 4) < 0$.
$f(x) = x(x - 5)(x + 4)$ has zeros, 0, 5, and –4. By considering the intervals $(-\infty, -4)$, $(-4, 0)$, $(0, 5)$, and $(5, \infty)$, we find $f(x) < 0$ on $(-\infty, -4)$ and $(0, 5)$.
Answer: $(-\infty, -4)$, $(0, 5)$

12. $(x+2)^2 > 0$
$f(x)=(x+2)^2$ has –2 as zero. By considering the intervals $(-\infty, -2)$ and $(-2, \infty)$, we find $f(x) > 0$ on both intervals.
Answer: $(-\infty, -2)$, $(-2, \infty)$

13. $x^3+4x \ge 0$
$f(x)=x\left(x^2+4\right)$ has 0 as the only (real) zero.
By considering the intervals $(-\infty, 0)$ and $(0, \infty)$, we find $f(x) > 0$ on $(0, \infty)$.
Answer: $[0, \infty)$

14. $(x+2)^2\left(x^2-1\right)<0$
$f(x)=(x+2)^2\left(x^2-1\right)=(x+2)^2(x+1)(x-1)$
has zeros –2, –1, and 1. By considering the intervals $(-\infty, -2)$, $(-2, -1)$, $(-1, 1)$, and $(1, \infty)$, we find $f(x) < 0$ on $(-1, 1)$.
Answer: $(-1, 1)$

15. $x^3+8x^2+15x \le 0$
$f(x) = x(x + 3)(x + 5)$ has zeros 0, –3, and –5. By considering the intervals $(-\infty, -5)$, $(-5, -3)$, $(-3, 0)$, and $(0, \infty)$, we find $f(x) < 0$ on $(-\infty, -5)$ and $(-3, 0)$.
Answer: $(-\infty, -5]$, $[-3, 0]$

16. $x^3+6x^2+9x<0$

$f(x)=x(x^2+6x+9)=x(x+3)^2$ has zeros −3 and 0. By considering the intervals (∞, −3), (−3, 0) and (0, ∞), we find $f(x) < 0$ on (−∞, −3) and (−3, 0).

Answer: (−∞, −3), (−3, 0)

17. $\frac{x}{x^2-9}<0$

$f(x)=\frac{x}{x^2-9}$ is discontinuous when $x = \pm 3$; f has 0 as a zero. By considering the intervals (−∞, −3), (−3, 0), (0, 3), and (3, ∞), we find $f(x) < 0$ on (−∞, −3) and (0, 3).

Answer: (−∞, −3), (0, 3)

18. $\frac{x^2-1}{x}<0$

$f(x)=\frac{x^2-1}{x}$ is discontinuous at $x = 0$; f has zeros at ±1. By considering the intervals (−∞, −1), (−1, 0), (0, 1), and (1, ∞), we find $f(x) < 0$ on (−∞, −1) and (0, 1).

Answer: (−∞, −1), (0, 1)

19. $\frac{4}{x-1}\geq 0$

$f(x)=\frac{4}{x-1}$ is discontinuous when $x = 1$, and $f(x) = 0$ has no root. By considering the intervals (−∞, 1) and (1, ∞), we find $f(x) > 0$ on (1, ∞). Note also that $f(x) \neq 0$ for any x.

Answer: (1, ∞)

20. $\frac{3}{x^2-5x+6}>0$

$f(x)=\frac{3}{(x-2)(x-3)}$ is never zero, but is discontinuous when $x = 2, 3$. By considering the intervals (−∞, 2), (2, 3),and (3, ∞), we find $f(x) > 0$ on (−∞, 2) and (3, ∞).

Answer: (−∞, 2), (3, ∞)

21. $\frac{x^2-x-6}{x^2+4x-5}\geq 0$

$f(x)=\frac{x^2-x-6}{x^2+4x-5}=\frac{(x-3)(x+2)}{(x+5)(x-1)}$ is discontinuous at $x = -5$ and $x = 1$; f has zeros 3 and −2. By considering the intervals (−∞, −5), (−5, −2), (−2, 1), (1, 3), and (3, ∞), we find $f(x) > 0$ on (−∞, −5), (−2, 1), and (3, ∞).

Answer: (−∞, −5), [−2, 1), [3, ∞)

22. $\frac{x^2+4x-5}{x^2+3x+2}\leq 0$

$f(x)=\frac{x^2+4x-5}{x^2+3x+2}=\frac{(x+5)(x-1)}{(x+2)(x+1)}$ is discontinuous at $x = -1$ and −2; f has zeros −5 and 1. By considering the intervals (−∞, −5), (−5, −2), (−2, −1), (−1, 1), and (1, ∞), we find $f(x) < 0$ on (−5, −2) and (−1, 1).

Answer: [−5, −2), (−1, 1]

23. $\frac{3}{x^2+6x+5}\leq 0$

$f(x)=\frac{3}{x^2+6x+5}=\frac{3}{(x+5)(x+1)}$ is never zero, but is discontinuous at $x = -5$ and $x = -1$. By considering the intervals (−∞, −5), (−5, −1), and (−1, ∞), we find that $f(x) < 0$ on (−5, −1).

Answer: (−5, −1)

24. $\frac{2x+1}{x^2}\leq 0$

$f(x)=\frac{2x+1}{x^2}$ is discontinuous at $x = 0$ and f has $-\frac{1}{2}$ as a zero. By considering the intervals $\left(-\infty, -\frac{1}{2}\right)$, $\left(-\frac{1}{2}, 0\right)$, and (0, ∞), we find $f(x) < 0$ on $\left(-\infty, -\frac{1}{2}\right)$.

Answer: $\left(-\infty, -\frac{1}{2}\right]$

25. $x^2+2x\geq 2$, or equivalently, $x^2+2x-2\geq 0$.

$f(x)=x^2+2x-2$ has zeros $-1\pm\sqrt{3}$. By considering the intervals $\left(-\infty, -1-\sqrt{3}\right)$ $\left(-1-\sqrt{3}, -1+\sqrt{3}\right)$, and $\left(-1+\sqrt{3}, \infty\right)$, we find $f(x) > 0$ on $\left(-\infty, -1-\sqrt{3}\right)$ and $\left(-1+\sqrt{3}, \infty\right)$.

Answer: $\left(-\infty, -1-\sqrt{3}\right], \left[-1+\sqrt{3}, \infty\right)$

26. $x^4 - 16 \geq 0$.

$f(x) = \left(x^2 + 4\right)(x+2)(x-2)$ has (real) zeros –2 and 2. By considering the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$, we find $f(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$.

Answer: $(-\infty, -2], [2, \infty)$

27. Revenue = (no. of units)(price per unit). We want $q(28 - 0.2q) \geq 750$

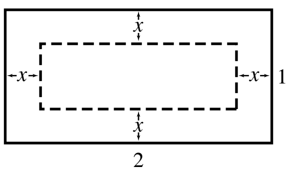
$$0.2q^2 - 28q + 750 \leq 0$$
$$q^2 - 140q + 3750 \leq 0$$

Using the quadratic formula,

$q^2 - 140q + 3750 = 0$ when $q \approx 36.09, 103.91$.

Thus $q^2 - 140q + 3750 \leq 0$ when $36.09 \leq q \leq 103.91$, so sales revenue will be at least \$750 when between 37 and 103 units, inclusive, are sold.

28.

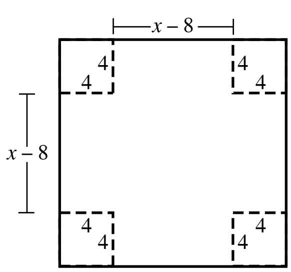


$$(2 - 2x)(1 - 2x) \geq \frac{21}{16}$$
$$64x^2 - 96x + 11 \geq 0$$
$$(8x - 11)(8x - 1) \geq 0$$

Solving gives $x \leq \frac{1}{8}$ or $x \geq \frac{11}{8}$. From the diagram, clearly, x cannot exceed $\frac{1}{2}$. Thus $x \leq \frac{1}{8}$.

Answer: $\frac{1}{8}$ mi

29.



If x is the length of a side of the piece of aluminum, then the box will be 4 by $x - 8$ by $x - 8$.

$$4(x-8)^2 \geq 324$$
$$(x-8)^2 \geq 81$$
$$x^2 - 16x - 17 \geq 0$$
$$(x - 17)(x + 1) \geq 0$$

Solving gives $x \leq -1$ or $x \geq 17$. Since x must be positive, we have $x \geq 17$.

Answer: 17 in. by 17 in.

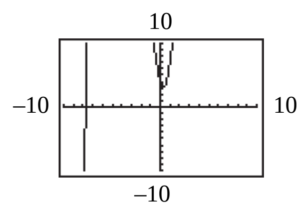
30. Let n = no. of persons over the 50 that attend. Then each of $50 + n$ persons will pay $50 - 0.50n$. We want

$$(50 + n)(50 - 0.50n) \geq 50(50)$$
$$25n - \frac{1}{2}n^2 \geq 0$$
$$n\left(25 - \frac{1}{2}n\right) \geq 0$$

Solving gives $0 \leq n \leq 50$. Thus the size of the group is between 50 and 100 inclusive.

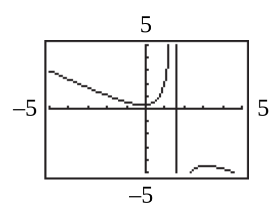
Answer: $50 \leq$ size of group ≤ 100

31.



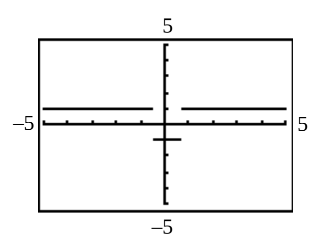
$(-\infty, -7.72]$

32.



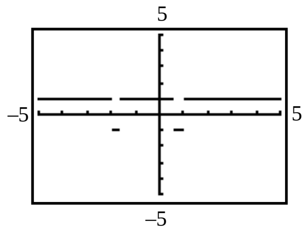
$(-\infty, 1.51)$

33.



$(-\infty, -0.5), (0.667, \infty)$

34.



(−2, −1.62), (0.62, 1)

Chapter 10 Review Problems

1. $\lim\limits_{x\to -1}\left(2x^2+6x-1\right)=2(-1)^2+6(-1)-1=-5$

2. $\lim\limits_{x\to 0}\dfrac{2x^2-3x+1}{2x^2-2}=\dfrac{\lim\limits_{x\to 0}\left(2x^2-3x+1\right)}{\lim\limits_{x\to 0}\left(2x^2-2\right)}$

$=\dfrac{1}{-2}=-\dfrac{1}{2}$

3. $\lim\limits_{x\to 3}\dfrac{x^2-9}{x^2-3x}=\lim\limits_{x\to 3}\dfrac{(x+3)(x-3)}{x(x-3)}$

$=\lim\limits_{x\to 3}\dfrac{x+3}{x}=\dfrac{6}{3}=2$

4. $\lim\limits_{x\to -4}\dfrac{2x+3}{x^2-4}=\dfrac{\lim\limits_{x\to -4}(2x+3)}{\lim\limits_{x\to -4}(x^2-4)}=\dfrac{-5}{12}=-\dfrac{5}{12}$

5. $\lim\limits_{h\to 0}(x+h)=x+0=x$

6. $\lim\limits_{x\to 2}\dfrac{x^2-4}{x^2-3x+2}=\lim\limits_{x\to 2}\dfrac{(x+2)(x-2)}{(x-2)(x-1)}$

$=\lim\limits_{x\to 2}\dfrac{x+2}{x-1}=\dfrac{4}{1}=4$

7. $\lim\limits_{x\to -4}\dfrac{x^3+4x^2}{x^2+2x-8}=\lim\limits_{x\to -4}\dfrac{x^2(x+4)}{(x+4)(x-2)}$

$\lim\limits_{x\to -4}\dfrac{x^2}{x-2}=\dfrac{16}{-6}=-\dfrac{8}{3}$

8. $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{x^2+4x-5}=\lim\limits_{x\to 1}\dfrac{(x-1)(x+2)}{(x-1)(x+5)}$

$=\lim\limits_{x\to 1}\dfrac{x+2}{x+5}=\dfrac{3}{6}=\dfrac{1}{2}$

9. As $x\to\infty,\ x+1\to\infty$. Thus $\lim\limits_{x\to\infty}\dfrac{2}{x+1}=0$.

10. $\lim\limits_{x\to\infty}\dfrac{x^2+1}{2x^2}=\lim\limits_{x\to\infty}\dfrac{x^2}{2x^2}=\lim\limits_{x\to\infty}\dfrac{1}{2}=\dfrac{1}{2}$

11. $\lim\limits_{x\to\infty}\dfrac{2x+5}{7x-4}=\lim\limits_{x\to\infty}\dfrac{2x}{7x}=\lim\limits_{x\to\infty}\dfrac{2}{7}=\dfrac{2}{7}$

12. $\lim\limits_{x\to -\infty}\dfrac{1}{x^4}=0$

13. $\lim\limits_{t\to 3^-}\dfrac{2t-3}{t-3}=-\infty$ and $\lim\limits_{t\to 3^+}\dfrac{2t-3}{t-3}=\infty$. Thus $\lim\limits_{t\to 3}\dfrac{2t-3}{t-3}$ does not exist.

14. $\lim\limits_{x\to -\infty}\dfrac{x^6}{x^5}=\lim\limits_{x\to -\infty}x=-\infty$

15. $\lim\limits_{x\to -\infty}\dfrac{x+3}{1-x}=\lim\limits_{x\to -\infty}\dfrac{x}{-x}=\lim\limits_{x\to -\infty}(-1)=-1$

16. $\lim\limits_{x\to 4}\sqrt[3]{64}=\lim\limits_{x\to 4}4=4$

17. $\lim\limits_{x\to\infty}\dfrac{x^2-1}{(3x+2)^2}=\lim\limits_{x\to\infty}\dfrac{x^2-1}{9x^2+12x+4}$

$=\lim\limits_{x\to\infty}\dfrac{x^2}{9x^2}=\lim\limits_{x\to\infty}\dfrac{1}{9}=\dfrac{1}{9}$

18. $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{x-1}=\lim\limits_{x\to 1}\dfrac{(x+2)(x-1)}{x-1}$

$=\lim\limits_{x\to 1}(x+2)=3$

19. $\lim\limits_{x\to 3^-}\dfrac{x+3}{x^2-9}=\lim\limits_{x\to 3^-}\dfrac{x+3}{(x+3)(x-3)}$

$=\lim\limits_{x\to 3^-}\dfrac{1}{x-3}=-\infty$

20. $\lim\limits_{x\to 2}\dfrac{2-x}{x-2}=\lim\limits_{x\to 2}\left[-\dfrac{x-2}{x-2}\right]=\lim\limits_{x\to 2}(-1)=-1$

21. As x becomes large, so does $3x$. Because the square roots of large numbers are also large, $\lim\limits_{x\to\infty}\sqrt{3x}=\infty$.

22. As $y \to 5^+$, $y - 5$ approaches 0 through positive values. Thus $\lim_{y\to 5^+} \sqrt{y-5} = 0$.

23. $\lim_{x\to\infty} \frac{x^{100} + \frac{1}{x^3}}{\pi - x^{97}} = \lim_{x\to\infty} \frac{x^3\left(x^{100} + \frac{1}{x^3}\right)}{x^3\left(\pi - x^{97}\right)}$

$= \lim_{x\to\infty} \frac{x^{103}+1}{\pi x^3 - x^{100}} = \lim_{x\to\infty} \frac{x^{103}}{-x^{100}}$

$= \lim_{x\to\infty} \left(-x^3\right) = -\infty$

24. $\lim_{x\to-\infty} \frac{ex^2 - x^4}{31x - 2x^3} = \lim_{x\to-\infty} \frac{-x^4}{-2x^3}$

$= \lim_{x\to-\infty} \frac{x}{2} = -\infty$

25. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x = 1$

Thus $\lim_{x\to 1} f(x) = 1$.

26. $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x+5) = 8$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 6 = 6$

Because $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$,

$\lim_{x\to 3} f(x)$ does not exist.

27. $\lim_{x\to 4^+} \frac{\sqrt{x^2-16}}{4-x} = \lim_{x\to 4^+} \frac{\sqrt{x-4}\sqrt{x+4}}{-(x-4)}$

$= \lim_{x\to 4^+} -\frac{\sqrt{x+4}}{\sqrt{x-4}}$

As $x \to 4^+$, $\sqrt{x-4}$ approaches 0 through positive values and $\sqrt{x+4} \to \sqrt{8}$. Thus

$-\frac{\sqrt{x+4}}{\sqrt{x-4}} \to -\infty$.

Answer: $-\infty$

28. $\lim_{x\to 5^+} \frac{x^2 - 3x - 10}{\sqrt{x-5}} = \lim_{x\to 5^+} \frac{(x-5)(x+2)}{\sqrt{x-5}}$

$= \lim_{x\to 5^+} \sqrt{x-5}(x+2)$

$= 0 \cdot 7$

$= 0$

29. $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{[8(x+h)-2]-[8x-2]}{h}$

$= \lim_{h\to 0} \frac{8h}{h} = \lim_{h\to 0} 8 = 8$

30. $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{\left[2(x+h)^2 - 3\right] - \left[2x^2 - 3\right]}{h}$

$= \lim_{h\to 0} \frac{4xh + 2h^2}{h}$

$= \lim_{h\to 0} \frac{h(4x+2h)}{h} = \lim_{h\to 0} (4x+2h) = 4x$

31. $y = 23\left(1 - \frac{1}{1+2x}\right)$

Considering $\frac{1}{1+2x}$, we have

$\lim_{x\to\infty} \frac{1}{1+2x} = \frac{1}{2} \cdot \lim_{x\to\infty} \frac{1}{x} = \frac{1}{2} \cdot 0 = 0$. Thus

$\lim_{x\to\infty} y = \lim_{x\to\infty} \left[23\left(1 - \frac{1}{1+2x}\right)\right] = 23(1-0) = 23$

Answer: 23

32. $\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{10x}{1+0.1x} = \lim_{x\to\infty} \frac{10x}{0.1x}$

$= \lim_{x\to\infty} 100 = 100$

Answer: 100

33. $f(x) = x + 5$; $x = 7$

(i) f is defined at $x = 7$; $f(7) = 12$

(ii) $\lim_{x\to 7} f(x) = \lim_{x\to 7} (x+5) = 7 + 5 = 12$, which exists

(iii) $\lim_{x\to 7} f(x) = 12 = f(7)$

Thus f is continuous at $x = 7$.

34. $\frac{x-5}{x^2+2}$; $x = 5$

(i) f is defined at $x = 5$; $f(5) = 0$

(ii) $\lim_{x\to 5} f(x) = \lim_{x\to 5} \frac{x-5}{x^2+2} = \frac{0}{27} = 0$, which exists

(iii) $\lim_{x\to 5} f(x) = 0 = f(5)$

Thus f is continuous at $x = 5$.

35. Since $f(x) = \frac{1}{5}x^2$ is polynomial function, it is continuous everywhere.

36. Since $f(x) = x^2 - 2$ is a polynomial function, it is continuous everywhere.

37. $f(x) = \frac{x^2}{x+3}$ is a rational function and the denominator is zero at $x = -3$. Thus f is discontinuous at $x = -3$.

38. $f(x) = \frac{0}{x^3}$ is a rational function and the denominator is zero at $x = 0$. Thus f is discontinuous at $x = 0$.

39. Since $f(x) = \frac{x-1}{2x^2+3}$ is a rational function whose denominator is never zero, f is continuous everywhere.

40. Since $f(x) = (2-3x)^3$ is a polynomial function, it is continuous everywhere.

41. $f(x) = \frac{4-x^2}{x^2+3x-4} = \frac{4-x^2}{(x+4)(x-1)}$ is a rational function and the denominator is zero only when $x = -4$ or $x = 1$, so f is discontinuous at $x = -4, 1$.

42. $f(x) = \frac{2x+6}{x^3+x} = \frac{2x+6}{x\left(x^2+1\right)}$ is a rational function and the denominator is zero only when $x = 0$, so f is discontinuous at $x = 0$.

43. $f(x) = \begin{cases} x+4 & \text{if } x > -2 \\ 3x+6 & \text{if } x \le -2 \end{cases}$

For $x < -2$, $f(x) = 3x + 6$, which is a polynomial and hence continuous. For $x > -2$, $f(x) = x + 4$, which is a polynomial and hence continuous. Because $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} (3x+6) = 0$ and $\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} (x+4) = 2$, $\lim_{x\to -2} f(x)$ does not exist. Thus f is discontinuous at $x = -2$.

44. $f(x) = \begin{cases} \frac{1}{x} & \text{if } x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$

If $x < 1$, then $f(x) = \frac{1}{x}$, which is a rational function whose denominator is zero when $x = 0$. Thus f is discontinuous at $x = 0$. If $x > 1$, then $f(x) = 1$, which a polynomial function and hence continuous. At $x = 1$, f is defined [$f(x) = 1$].

Because $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{x} = 1$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 1 = 1$, then $\lim_{x\to 1} f(x) = 1$.

Since $\lim_{x\to 1} f(x) = 1 = f(1)$, f is continuous at $x = 1$.

f is discontinuous at $x = 0$.

45. $x^2 + 4x - 12 > 0$

$f(x) = x^2 + 4x - 12 = (x+6)(x-2)$ has zeros -6 and 2. By considering the intervals $(-\infty, -6)$, $(-6, 2)$, and $(2, \infty)$, we find $f(x) > 0$ on $(-\infty, -6)$ and $(2, \infty)$.

Answer: $(-\infty, -6)$, $(2, \infty)$

46. $3x^2 - 3x - 6 \le 0$

$f(x) = 3x^2 - 3x - 6 = 3(x-2)(x+1)$ has zeros -1 and 2. By considering the intervals $(-\infty, -1)$, $(-1, 2)$, and $(2, \infty)$, we find $f(x) < 0$ on $(-1, 2)$.

Answer: $[-1, 2]$

47. $x^5 \le 7x^4$, $x^5 - 7x^4 \le 0$

$f(x) = x^5 - 7x^4 = x^4(x-7)$ has zeros 0 and 7. By considering the intervals $(-\infty, 0)$, $(0, 7)$, and $(7, \infty)$, we find $f(x) < 0$ on $(-\infty, 0)$ and $(0, 7)$.

Answer: $(-\infty, 7]$

48. $x^3 + 8x^2 + 15x \geq 0$

$f(x) = x^3 + 8x^2 + 15x = x(x+5)(x+3)$ has zeros 0, –5, and –3. By considering the intervals (–∞, –5), (–5, –3), (–3, 0), and (0, ∞), we find $f(x) > 0$ and (–5, –3) and (0, ∞).
Answer: [–5, –3], [0, ∞)

49. $\dfrac{x+5}{x^2-1} < 0$

$f(x) = \dfrac{x+5}{(x+1)(x-1)}$ is discontinuous when $x = \pm 1$, and f has –5 as a zero. By considering the intervals (–∞, –5), (–5, –1), (–1, 1), and (1, ∞), we find $f(x) < 0$ on (–∞, –5) and (–1, 1).
Answer: (–∞, –5), (–1, 1)

50. $\dfrac{x(x+5)(x+8)}{3} < 0$

$f(x) = \dfrac{x(x+5)(x+8)}{3}$ has zeros 0, –5, and –8. By considering the intervals (–∞, –8), (–8, –5), (–5, 0), and (0, ∞), we find $f(x) < 0$ on (–∞, –8) and (–5, 0).
Answer: (–∞, –8), (–5, 0)

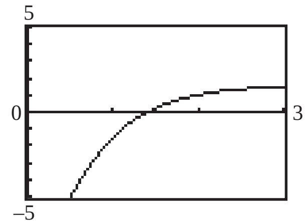
51. $\dfrac{x^2+3x}{x^2+2x-8} \geq 0$

$f(x) = \dfrac{x^2+3x}{x^2+2x-8} = \dfrac{x(x+3)}{(x+4)(x-2)}$ is discontinuous when $x = -4, 2$ and has zeros $x = -3, 0$. By considering the intervals (–∞, –4), (–4, –3), (–3, 0), (0, 2), and (2, ∞) we find $f(x) > 0$ on (–∞, –4), (–3, 0), and (2, ∞).
Answer: (–∞, –4), [–3, 0], (2, ∞)

52. $\dfrac{x^2-9}{x^2-16} \leq 0$

$f(x) = \dfrac{x^2-9}{x^2-16} = \dfrac{(x+3)(x-3)}{(x+4)(x-4)}$ is discontinuous when $x = \pm 4$ and has zeros $x = \pm 3$. By considering the intervals (–∞, –4), (–4, –3), (–3, 3), (3, 4), and (4, ∞) we find $f(x) < 0$ on (–4, –3) and (3, 4).
Answer: (–4, –3], [3, 4)

53.



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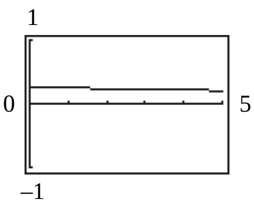
54.



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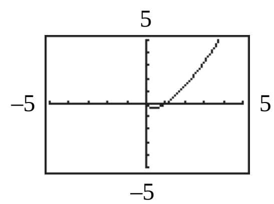

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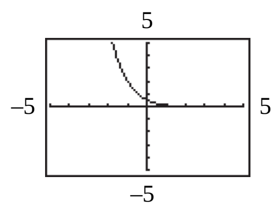
56.



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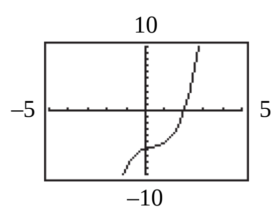
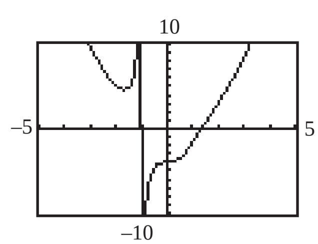


[2.00, ∞)

58.



(–1, 1.32]

Mathematical Snapshot Chapter 10

1. $D = 8432e^{-rt}$
A year from now, $t = 1$ and $D = 8000$. Thus
$8000 = 8432e^{-r}$
$e^{-r} = \frac{8000}{8432}$
$-r = \ln\frac{8000}{8432}$
$r = -\ln\frac{8000}{8432} \approx 0.053$
The rate is 5.3%.

2. $D = 8432e^{-0.06t}$
We want to find t when $D = \frac{8432}{2}$.
$\frac{8432}{2} = 8432e^{-0.06t}$
$\frac{1}{2} = e^{-0.06t}$
$-0.06t = \ln\frac{1}{2}$
$t = \frac{\ln\frac{1}{2}}{-0.06} = \frac{\ln 2}{0.06} \approx 12$
It would take about 12 years.

3. An exponential model assumes a fixed repayment rate. In reality, the repayment rate is constantly changing as a result of changing fiscal policy and other factors.

Chapter 11

Principles in Practice 11.1

1. $\frac{dH}{dt} = \frac{d}{dt}\left(6+40t-16t^2\right)$

$= \lim_{h\to 0} \frac{H(t+h)-H(t)}{h}$

$= \lim_{h\to 0} \frac{\left[6+40(t+h)-16(t+h)^2\right]-\left(6+40t-16t^2\right)}{h}$

$= \lim_{h\to 0} \frac{6+40t+40h-16t^2-32th-16h^2-6-40t+16t^2}{h}$

$= \lim_{h\to 0} \frac{40h-32th-16h^2}{h} = \lim_{h\to 0}(40-32t-16h)$

$= 40 - 32t$

$\frac{dH}{dt} = 40-32t$

Problems 11.1

1. a. $f(x) = x^3+3$, $P = (-2, -5)$

To begin, if $x = -3$, then $m_{PQ} = \frac{[(-3)^3+3]-(-5)}{-3-(-2)} = 19$. If $x = -2.5$, then $m_{PQ} = \frac{[(-2.5)^3+3]-(-5)}{-2.5-(-2)} = 15.25$.

Continuing in this manner, we complete the table:

x-value of Q	–3	–2.5	–2.2	–2.1	–2.01	–2.001
m_{PQ}	19	15.25	13.24	12.61	12.0601	12.0060

b. We estimate that $m_{\tan}$ at P is 12.

2. a. $f(x) = e^{2x}$, $P = (0, 1)$

To begin, if $x = 1$, then $m_{PQ} = \frac{e^{2(1)}-1}{1-0} \approx 6.3891$. If $x = 0.5$, then $m_{PQ} = \frac{e^{2(0.5)}-1}{0.5-0} \approx 3.4366$.

Continuing in this manner, we complete the table:

x-value of Q	1	0.5	0.2	0.1	0.01	0.001
m_{PQ}	6.3891	3.4366	2.4591	2.2140	2.0201	2.0020

b. We estimate that $m_{\tan}$ at P is 2.

3. $f(x) = x$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)-x}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$

4. $f(x) = 4x - 1$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{[4(x+h)-1]-[4x-1]}{h}$$

$$= \lim_{h\to 0}\frac{4h}{h} = \lim_{h\to 0} 4 = 4$$

5. $y = 3x + 5$. Let $y = f(x)$.

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{[3(x+h)+5]-[3x+5]}{h}$$

$$= \lim_{h\to 0}\frac{3h}{h} = \lim_{h\to 0} 3 = 3$$

6. $y = -5x$. Let $y = f(x)$.

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{[-5(x+h)]-[-5x]}{h}$$

$$= \lim_{h\to 0}\frac{-5h}{h} = \lim_{h\to 0}(-5) = -5$$

7. Let $f(x) = 5 - 4x$.

$$\frac{d}{dx}(5-4x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{[5-4(x+h)]-[5-4x]}{h}$$

$$= \lim_{h\to 0}\frac{-4h}{h} = \lim_{h\to 0}(-4) = -4$$

8. Let $f(x) = 1 - \frac{x}{2}$

$$\frac{d}{dx}\left(1-\frac{x}{2}\right) = \lim_{h\to 0}\frac{\left[1-\frac{x+h}{2}\right]-\left[1-\frac{x}{2}\right]}{h}$$

$$= \lim_{h\to 0}\frac{-\frac{h}{2}}{h} = \lim_{h\to 0}\left(-\frac{1}{2}\right) = -\frac{1}{2}$$

9. $f(x) = 3$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{3-3}{h} = \lim_{h\to 0}\frac{0}{h} = \lim_{h\to 0} 0 = 0$$

10. $f(x) = 7.01$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{7.01-7.01}{h}$$

$$= \lim_{h\to 0}\frac{0}{h} = \lim_{h\to 0} 0 = 0$$

11. Let $f(x) = x^2 + 4x - 8$.

$$\frac{d}{dx}\left(x^2+4x-8\right)$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\left[(x+h)^2+4(x+h)-8\right]-\left[x^2+4x-8\right]}{h}$$

$$= \lim_{h\to 0}\frac{x^2+2xh+h^2+4x+4h-8-x^2-4x+8}{h}$$

$$= \lim_{h\to 0}\frac{2xh+h^2+4h}{h}$$

$$= \lim_{h\to 0}(2x+h+4) = 2x+0+4 = 2x+4$$

12. $y = f(x) = x^2 + 5x + 1$

$$y' = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{[(x+h)^2+5(x+h)+1]-[x^2+5x+1]}{h}$$

$$= \lim_{h\to 0}\frac{x^2+2xh+h^2+5x+5h+1-x^2-5x-1}{h}$$

$$= \lim_{h\to 0}\frac{2xh+h^2+5h}{h}$$

$$= \lim_{h\to 0}(2x+h+5) = 2x+0+5 = 2x+5$$

13. $p = f(q) = 3q^2 + 2q + 1$

$$\frac{dp}{dq} = \lim_{h\to 0}\frac{f(q+h)-f(q)}{h}$$

$$= \lim_{h\to 0}\frac{\left[3(q+h)^2+2(q+h)+1\right]-\left[3q^2+2q+1\right]}{h}$$

$$= \lim_{h\to 0}\frac{6qh+3h^2+2h}{h}$$

$$= \lim_{h\to 0}(6q+3h+2) = 6q+0+2 = 6q+2$$

14. Let $f(x) = x^2 - x - 3$.

$$\frac{d}{dx}\left(x^2 - x - 3\right)$$
$$= \lim \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\left[(x+h)^2 - (x+h) - 3\right] - \left[x^2 - x - 3\right]}{h}$$
$$= \lim_{h\to 0} \frac{2xh + h^2 - h}{h} = \lim_{h\to 0}(2x + h - 1) = 2x - 1$$

15. $y = f(x) = \frac{6}{x}$

$$y' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{6}{x+h} - \frac{6}{x}}{h}$$

Multiplying the numerator and denominator by $x(x+h)$ gives

$$y' = \lim_{h\to 0} \frac{6x - 6(x+h)}{hx(x+h)} = \lim_{h\to 0} \frac{-6h}{hx(x+h)}$$
$$= \lim_{h\to 0}\left[-\frac{6}{x(x+h)}\right] = -\frac{6}{x(x+0)} = -\frac{6}{x^2}$$

16. $C = f(q) = 7 + 2q - 3q^2$

$$\frac{dC}{dq} = \lim_{h\to 0} \frac{f(q+h) - f(q)}{h}$$
$$= \lim_{h\to 0} \frac{\left[7 + 2(q+h) - 3(q+h)^2\right] - \left[7 + 2q - 3q^2\right]}{h}$$
$$= \lim_{h\to 0} \frac{2h - 6qh - 3h^2}{h} = \lim_{h\to 0}(2 - 6q - 3h)$$
$$= 2 - 6q$$

17. $f(x) = \sqrt{x+2}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h}$$

Rationalizing the numerator gives

$$\frac{\sqrt{x+h+2} - \sqrt{x+2}}{h}$$
$$= \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} \cdot \frac{\sqrt{x+h+2} + \sqrt{x+2}}{\sqrt{x+h+2} + \sqrt{x+2}}$$
$$= \frac{(x+h+2) - (x+2)}{h\left(\sqrt{x+h+2} + \sqrt{x+2}\right)} = \frac{1}{\sqrt{x+h+2} + \sqrt{x+2}}$$

Thus $f'(x) = \lim_{h\to 0} \frac{1}{\sqrt{x+h+2} + \sqrt{x+2}} = \frac{1}{2\sqrt{x+2}}$

18. $H(x) = \frac{3}{x-2}$

$$H'(x) = \lim_{h\to 0} \frac{H(x+h) - H(x)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{3}{x+h-2} - \frac{3}{x-2}}{h}$$

Multiplying the numerator and denominator by $(x + h - 2)(x - 2)$ gives

$$H'(x) = \lim_{h\to 0} \frac{3(x-2) - 3(x+h-2)}{h(x+h-2)(x-2)}$$
$$= \lim_{h\to 0} \frac{-3h}{h(x+h-2)(x-2)}$$
$$= \lim_{h\to 0} \frac{-3}{(x+h-2)(x-2)} = -\frac{3}{(x-2)^2}$$

19. $y = f(x) = x^2 + 4$

$$y' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\left[(x+h)^2 + 4\right] - \left[x^2 + 4\right]}{h}$$
$$= \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0}(2x + h) = 2x + 0 = 2x$$

The slope at (–2, 8) is $y'(-2) = 2(-2) = -4$.

20. $y = f(x) = 1 - x^2$

$$y' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\left[1 - (x+h)^2\right] - \left[1 - x^2\right]}{h}$$
$$= \lim_{h\to 0} \frac{-2xh - h^2}{h}$$
$$= \lim_{h\to 0}(-2x - h) = -2x$$

The slope at (1, 0) is $y'(1) = -2(1) = -2$.

21. $y = f(x) = 4x^2 - 5$

$$y' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left[4(x+h)^2 - 5\right] - \left[4x^2 - 5\right]}{h}$$

$$= \lim_{h\to 0} \frac{8xh + 4h^2}{h} = \lim_{h\to 0}(8x + 4h) = 8x$$

The slope when $x = 0$ is $y'(0) = 8(0) = 0$.

22. As shown in Example 5, $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$.

If $x = 1$, the slope is $y'(1) = \frac{1}{2}$.

23. $y = x + 4$

$$y' = \lim_{h\to 0} \frac{[(x+h)+4] - [x+4]}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

If $x = 3$, then $y' = 1$. The tangent line at the point (3, 7) is $y - 7 = 1(x - 3)$, or $y = x + 4$.

24. $y = 3x^2 - 4$

$$y' = \lim_{h\to 0} \frac{[3(x+h)^2 - 4] - [3x^2 - 4]}{h}$$

$$= \lim_{h\to 0} \frac{6xh + 3h^2}{h} = \lim_{h\to 0}(6x + 3h) = 6x$$

If $x = 1$, then $y' = 6(1) = 6$.

The tangent line at (1, –1) is $y + 1 = 6(x - 1)$ or $y = 6x - 7$.

25. $y = x^2 + 2x + 3$

$$y' = \lim_{h\to 0} \frac{\left[(x+h)^2 + 2(x+h) + 3\right] - \left[x^2 + 2x + 3\right]}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2 + 2h}{h}$$

$$= \lim_{h\to 0}(2x + h + 2) = 2x + 2$$

If $x = 1$, then $y' = 2(1) + 2 = 4$. The tangent line at the point (1, 6) is $y - 6 = 4(x - 1)$, or $y = 4x + 2$.

26. $y = (x-7)^2 = x^2 - 14x + 49$

$$y' = \lim_{h\to 0} \frac{\left[(x+h)^2 - 14(x+h) + 49\right] - \left[x^2 - 14x + 49\right]}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2 - 14h}{h} = \lim_{h\to 0}(2x + h - 14) = 2x - 14$$

If $x = 6$, then $y' = 2(6) - 14 = -2$. The tangent line at (6, 1) is $y - 1 = -2(x - 6)$, or $y = -2x + 13$.

27. $y = \frac{3}{x-1}$

$$y' = \lim_{h\to 0} \frac{\frac{3}{(x+h)-1} - \frac{3}{x-1}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3(x-1) - 3(x+h-1)}{(x+h-1)(x-1)}}{h}$$

$$= \lim_{h\to 0} \frac{-3h}{h(x+h-1)(x-1)} = \lim_{h\to 0} \frac{-3}{(x+h-1)(x-1)}$$

$$= -\frac{3}{(x-1)^2}$$

If $x = 2$, then $y' = -\frac{3}{1} = -3$. The tangent line at (2, 3) is $y - 3 = -3(x - 2)$, or $y = -3x + 9$.

28. $y = \frac{5}{1-3x}$

$$y' = \lim_{h\to 0} \frac{\frac{5}{1-3(x+h)} - \frac{5}{1-3x}}{h}$$

$$= \lim_{h\to 0} \frac{5(1-3x) - 5[1-3(x+h)]}{h[1-3(x+h)](1-3x)}$$

$$= \lim_{h\to 0} \frac{15h}{h[1-3(x+h)](1-3x)}$$

$$= \lim_{h\to 0} \frac{15}{[1-3(x+h)](1-3x)}$$

$$= \frac{15}{(1-3x)^2}$$

If $x = 2$, then $y' = \frac{15}{25} = \frac{3}{5}$. The tangent line at (2, –1) is $y + 1 = \frac{3}{5}(x - 2)$, or $y = \frac{3}{5}x - \frac{11}{5}$.

29. $r = \left(\frac{\eta}{1+\eta}\right)\left(r_L - \frac{dC}{dD}\right)$

$(1+\eta)r = \eta\left(r_L - \frac{dC}{dD}\right)$

$r + \eta r = \eta\left(r_L - \frac{dC}{dD}\right)$

$r = \eta\left(r_L - \frac{dC}{dD}\right) - \eta r$

$r = \eta\left(r_L - \frac{dC}{dD} - r\right)$

$\eta = \frac{r}{r_L - r - \frac{dC}{dD}}$

30. 1.565, 1.470

31. –3.000, 13.445

32. 0.680, 1820.369

33. –5.120, 0.038

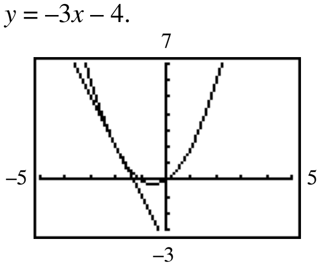
34. $y = f(x) = x^2 + x$

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

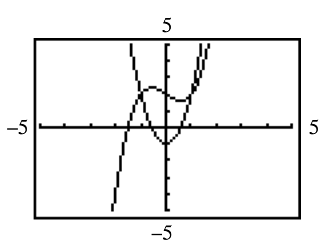
$= \lim_{h\to 0} \frac{\left[(x+h)^2 + (x+h)\right] - \left[x^2 + x\right]}{h}$

$= \lim_{h\to 0} \frac{2xh + h^2 + h}{h} = \lim_{h\to 0}(2x + h + 1) = 2x + 1$

If $x = -2$, then $f'(x) = -3$. The tangent line at the point (–2, 2) is $y - 2 = -3(x + 2)$, or $y = -3x - 4$.



35.



For the *x*-values of the points where the tangent to the graph of *f* is horizontal, the corresponding values of $f'(x)$ are 0. This is expected because the slope of a horizontal line is zero and the derivative gives the slope of the tangent line.

36. $n = 4$: $$\begin{aligned}(z-x)\sum_{i=0}^{3} x^i z^{3-i} &= (z-x)(z^3 + xz^2 + x^2z + x^3)\\ &= z^4 - xz^3 + xz^3 - x^2z^2 + x^2z^2 - x^3z + x^3z - x^4\\ &= z^4 - x^4\end{aligned}$$

$n = 3$: $$\begin{aligned}(z-x)\sum_{i=0}^{2} x^i z^{2-i} &= (z-x)(z^2 + xz + x^2)\\ &= z^3 - xz^2 + xz^2 - x^2z + x^2z - x^3\\ &= z^3 - x^3\end{aligned}$$

$n = 2$: $$(z-x)\sum_{i=0}^{1} x^i z^{1-i} = (z-x)(z+x) = z^2 - x^2$$

$$f(x) = 2x^4 + x^3 - 3x^2$$

$$\begin{aligned}f'(x) &= \lim_{z\to x} \frac{f(z) - f(x)}{z - x}\\ &= \lim_{z\to x} \frac{2z^4 + z^3 - 3z^2 - (2x^4 + x^3 - 3x^2)}{z - x}\\ &= \lim_{z\to x} \frac{2(z^4 - x^4) + (z^3 - x^3) - 3(z^2 - x^2)}{z - x}\\ &= \lim_{z\to x} \frac{2(z-x)(z^3 + xz^2 + x^2z + x^3) + (z-x)(z^2 + xz + x^2) - 3(z-x)(z+x)}{z - x}\\ &= \lim_{z\to x} [2(z^3 + xz^2 + x^2z + x^3) + (z^2 + xz + x^2) - 3(z+x)]\\ &= 2(4x^3) + (3x^2) - 3(2x)\\ &= 8x^3 + 3x^2 - 6x\end{aligned}$$

37. $n = 5$: $$\begin{aligned}(z-x)\sum_{i=0}^{4} x^i z^{4-i} &= (z-x)(z^4 + xz^3 + x^2z^2 + x^3z + x^4)\\ &= z^5 - xz^4 + xz^4 - x^2z^3 + x^2z^3 - x^3z^2 + x^3z^2 - x^4z + x^4z - x^5\\ &= z^5 - x^5\end{aligned}$$

$n = 3$: $$\begin{aligned}(z-x)\sum_{i=0}^{2} x^i z^{2-i} &= (z-x)(z^2 + xz + x^2)\\ &= z^3 - xz^2 + xz^2 - x^2z + x^2z - x^3\\ &= z^3 - x^3\end{aligned}$$

$$f(x) = 4x^5 - 3x^3$$

$$f'(x) = \lim_{z\to x} \frac{f(z)-f(x)}{z-x}$$
$$= \lim_{z\to x} \frac{4z^5-3z^3-(4x^5-3x^3)}{z-x}$$
$$= \lim_{z\to x} \frac{4(z^5-x^5)-3(z^3-x^3)}{z-x}$$
$$= \lim_{z\to x} \frac{4(z-x)(z^4+xz^3+x^2z^2+x^3z+x^4)-3(z-x)(z^2+xz+x^2)}{z-x}$$
$$= \lim_{z\to x} [4(z^4+xz^3+x^2z^2+x^3z+x^4)-3(z^2+xz+x^2)]$$
$$= 4(5x^4)-3(3x^2)$$
$$= 20x^4-9x^2$$

Principles in Practice 11.2

1. $r'(q) = \frac{d}{dq}(50q-0.3q^2)$
$= \frac{d}{dq}(50q) - \frac{d}{dq}\left(0.3q^2\right)$
$= 50\frac{d}{dq}(q) - 0.3\frac{d}{dq}\left(q^2\right)$
$= 50(1) - 0.3(2q) = 50 - 0.6q$
The marginal revenue is $r'(q) = 50 - 0.6q$.

Problems 11.2

1. $f(x) = 5$ is a constant function, so $f'(x) = 0$

2. $f(x) = \left(\frac{6}{7}\right)^{2/3}$ is a constant function, so $f'(x) = 0$

3. $y = x^6,\ y' = 6x^{6-1} = 6x^5$

4. $f'(x) = 21x^{21-1} = 21x^{20}$

5. $y = x^{80},\ \frac{dy}{dx} = 80x^{80-1} = 80x^{79}$

6. $y = x^{5.3},\ y' = 5.3x^{5.3-1} = 5.3x^{4.3}$

7. $f(x) = 9x^2,\ f'(x) = 9\left(2x^{2-1}\right) = 18x$

8. $y' = 4\left(3x^{3-1}\right) = 12x^2$

9. $g(w) = 8w^7,\ g'(w) = 8(7w^{7-1}) = 56w^6$

10. $v'(x) = ex^{e-1}$

11. $y = \frac{2}{3}x^4,\ y' = \frac{2}{3}\left(4x^{4-1}\right) = \frac{8}{3}x^3$

12. $f'(p) = \sqrt{3}\left(4p^{4-1}\right) = 4\sqrt{3}p^3$

13. $f(t) = \frac{t^7}{25},\ f'(t) = \frac{1}{25}(7t^{7-1}) = \frac{7}{25}t^6$

14. $y' = \frac{1}{7}\left(7x^{7-1}\right) = x^6$

15. $f(x) = x+3,\ f'(x) = 1+0 = 1$

16. $f'(x) = 3(1) - 0 = 3$

17. $f'(x) = 4(2x) - 2(1) + 0 = 8x - 2$

18. $F'(x) = 5(2x) - 9(1) = 10x - 9$

19. $g'(p) = 4p^{4-1} - 3\left(3p^{3-1}\right) - 0 = 4p^3 - 9p^2$

20. $f'(t) = -13(2t) + 14(1) + 0 = -26t + 14$

21. $y' = 3x^{3-1} - \left(\frac{1}{2}x^{\frac{1}{2}-1}\right) = 3x^2 - \frac{1}{2\sqrt{x}}$

22. $y' = -8\left(4x^{4-1}\right) + 0 = -32x^3$

23. $y' = -13\left(3x^{3-1}\right) + 14(2x) - 2(1) + 0$
$= -39x^2 + 28x - 2$

24. $V'(r) = 8r^{8-1} - 7\left(6r^{6-1}\right) + 3(2r) + 0 = 8r^7 - 42r^5 + 6r$

25. $f'(x) = 2\left(0 - 4x^{4-1}\right) = -8x^3$

26. $\phi'(t) = 5(3t^{3-1} - 0) = 15t^2$

27. $g(x) = \frac{1}{3}\left(13 - x^4\right),$
$g'(x) = \frac{1}{3}\left(0 - 4x^{4-1}\right) = -\frac{4}{3}x^3$

28. $f(x) = \frac{5}{2}\left(x^4 - 6\right),\ f'(x) = \frac{5}{2}\left(4x^{4-1} - 0\right) = 10x^3$

29. $h(x) = 4x^4 + x^3 - \frac{9}{2}x^2 + 8x$
$h'(x) = 4\left(4x^{4-1}\right) + 3x^{3-1} - \frac{9}{2}(2x) + 8(1)$
$= 16x^3 + 3x^2 - 9x + 8$

30. $k'(x) = -2(2x) + \frac{5}{3}(1) + 0 = -4x + \frac{5}{3}$

31. $f(x) = \frac{3}{10}x^4 + \frac{7}{3}x^3$
$f'(x) = \frac{3}{10}\left(4x^3\right) + \frac{7}{3}\left(3x^2\right) = \frac{6}{5}x^3 + 7x^2$

32. $p'(x) = \frac{1}{7}\left(7x^6\right) + \frac{2}{3}(1) = x^6 + \frac{2}{3}$

33. $f'(x) = \frac{3}{5}x^{\frac{3}{5}-1} = \frac{3}{5}x^{-2/5}$

34. $f'(x) = 2\left(-\frac{14}{5}\right)x^{\left(-\frac{14}{5}\right)-1} = -\frac{28}{5}x^{-\frac{19}{5}}$

35. $y' = \frac{3}{4}x^{\left(\frac{3}{4}\right)-1} + 2\left(\frac{5}{3}x^{\left(\frac{5}{3}\right)-1}\right) = \frac{3}{4}x^{-\frac{1}{4}} + \frac{10}{3}x^{\frac{2}{3}}$

36. $y' = 5(3x^2) - \left(-\frac{2}{5}\right)x^{-\frac{7}{5}} = 15x^2 + \frac{2}{5}x^{-\frac{7}{5}}$

37. $f(x) = 11\sqrt{x} = 11x^{\frac{1}{2}},$
$f'(x) = 11\left(\frac{1}{2}\right)x^{\left(\frac{1}{2}\right)-1} = \frac{11}{2}x^{-\frac{1}{2}} = \frac{11}{2\sqrt{x}}$

38. $y = x^{7/2},\ y' = \frac{7}{2}x^{\frac{7}{2}-1} = \frac{7}{2}x^{5/2}$

39. $f(r) = 6r^{\frac{1}{3}},\ f'(r) = 6\left(\frac{1}{3}r^{-\frac{2}{3}}\right) = 2r^{-\frac{2}{3}}$

40. $y = 4x^{\frac{1}{4}},\ y' = 4\left(\frac{1}{4}x^{-\frac{3}{4}}\right) = x^{-\frac{3}{4}}$

41. $f(x) = x^{-4},\ f'(x) = -4x^{-4-1} = -4x^{-5}$

42. $f'(s) = 2(-3s^{-4}) = -6s^{-4}$

43. $f(x) = x^{-3} + x^{-5} - 2x^{-6}$,
$f'(x) = -3x^{-3-1} + \left(-5x^{-5-1}\right) - 2\left(-6x^{-6-1}\right)$
$= -3x^{-4} - 5x^{-6} + 12x^{-7}$

44. $f'(x) = 100\left(-3x^{-4}\right) + 10\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$
$= -300x^{-4} + 5x^{-\frac{1}{2}}$

45. $y = \frac{1}{x} = x^{-1}$
$\frac{dy}{dx} = -1 \cdot x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$

46. $f(x) = 2x^{-3}$
$f'(x) = 2(-3x^{-4}) = -6x^{-4}$

47. $y = \frac{8}{x^5} = 8x^{-5}$
$y' = 8\left(-5x^{-6}\right) = -40x^{-6}$

48. $y = \frac{1}{4x^5} = \frac{1}{4}x^{-5}$
$y' = \frac{1}{4}\left(-5x^{-6}\right) = -\frac{5}{4}x^{-6}$

49. $g(x) = \frac{4}{3x^3} = \frac{4}{3}x^{-3}$
$g'(x) = \frac{4}{3}\left(-3x^{-4}\right) = -4x^{-4}$

50. $y = \frac{1}{x^2} = x^{-2},\ y' = -2x^{-3}$

51. $f(t) = \frac{1}{2}\left(\frac{1}{t}\right) = \frac{1}{2}t^{-1}$
$f'(t) = \frac{1}{2}\left(-1 \cdot t^{-2}\right) = -\frac{1}{2}t^{-2}$

52. $g(x) = \frac{7}{9}x^{-1}$
$g'(x) = \frac{7}{9}(-1x^{-2}) = -\frac{7}{9}x^{-2}$

53. $f(x) = \frac{1}{7}x + 7x^{-1}$
$f'(x) = \frac{1}{7}(1) + 7\left(-1x^{-2}\right) = \frac{1}{7} - 7x^{-2}$

54. $\Phi(x) = \frac{1}{3}x^3 - 3x^{-3}$,
$\Phi'(x) = \frac{1}{3}(3x^2) - 3(-3x^{-4}) = x^2 + 9x^{-4}$

55. $f(x) = -9x^{1/3} + 5x^{-2/5}$,
$f'(x) = -9\left(\frac{1}{3}x^{-\frac{2}{3}}\right) + 5\left(-\frac{2}{5}x^{-\frac{7}{5}}\right) = -3x^{-\frac{2}{3}} - 2x^{-\frac{7}{5}}$

56. $f'(z) = 3\left(\frac{1}{4}z^{-\frac{3}{4}}\right) - 0 - 8\left(-\frac{3}{4}z^{-\frac{7}{4}}\right) = \frac{3}{4}z^{-\frac{3}{4}} + 6z^{-\frac{7}{4}}$

57. $q(x) = \frac{1}{\sqrt[3]{8}\sqrt[3]{x^2}} = \frac{1}{2x^{2/3}} = \frac{1}{2}x^{-2/3}$
$q'(x) = \frac{1}{2}\left(-\frac{2}{3}x^{-5/3}\right) = -\frac{1}{3}x^{-5/3}$

58. $f(x) = \frac{3}{\sqrt[4]{x^3}} = 3x^{-\frac{3}{4}}$
$f'(x) = 3\left(-\frac{3}{4}x^{-\frac{7}{4}}\right) = -\frac{9}{4}x^{-\frac{7}{4}}$

59. $y = \frac{2}{x^{\frac{1}{2}}} = 2x^{-\frac{1}{2}}$
$y' = 2\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) = -x^{-\frac{3}{2}}$

60. $y = \frac{1}{2}x^{-\frac{1}{2}}$
$y' = -\frac{1}{4}x^{-\frac{3}{2}}$

61. $y = x^2\sqrt{x} = x^2\left(x^{\frac{1}{2}}\right) = x^{2+\left(\frac{1}{2}\right)} = x^{\frac{5}{2}}$
$y' = \frac{5}{2}x^{\frac{3}{2}}$

62. $f(x) = (8x^5),\ f'(x) = 40x^4$

63. $f(x) = x\left(3x^2 - 10x + 7\right) = 3x^3 - 10x^2 + 7x$

$f'(x) = 9x^2 - 20x + 7$

64. $f(x) = 3x^9 - 5x^5 + 4x^3$

$f'(x) = 27x^8 - 25x^4 + 12x^2$

$= x^2\left(27x^6 - 25x^2 + 12\right)$

65. $f(x) = x^3(3x)^2 = x^3\left(9x^2\right) = 9x^5$

$f'(x) = 45x^4$

66. $s(x) = \sqrt[3]{x}\left(\sqrt[4]{x} - 6x + 3\right)$

$= x^{1/3}(x^{1/4} - 6x + 3)$

$= x^{7/12} - 6x^{4/3} + 3x^{1/3}$

$s'(x) = \frac{7}{12}x^{-5/12} - 8x^{1/3} + x^{-2/3}$

67. $v(x) = x^{-\frac{2}{3}}(x+5) = x^{\frac{1}{3}} + 5x^{-\frac{2}{3}}$

$v'(x) = \frac{1}{3}x^{-\frac{2}{3}} - \frac{10}{3}x^{-\frac{5}{3}} = \frac{1}{3}x^{-\frac{5}{3}}(x - 10)$

68. $f(x) = x^{\frac{3}{5}}\left(x^2 + 7x + 11\right) = x^{\frac{13}{5}} + 7x^{\frac{8}{5}} + 11x^{\frac{3}{5}}$

$f'(x) = \frac{13}{5}x^{\frac{8}{5}} + \frac{56}{5}x^{\frac{3}{5}} + \frac{33}{5}x^{-\frac{2}{5}}$

$= \frac{1}{5}x^{-\frac{2}{5}}\left(13x^2 + 56x + 33\right)$

69. $f(q) = \frac{3q^2 + 4q - 2}{q} = \frac{3q^2}{q} + \frac{4q}{q} - \frac{2}{q^2}$

$= 3q + 4 - 2q^{-1}$

$f'(q) = 3(1) + 0 - 2(-q^{-2}) = 3 + 2q^{-2} = 3 + \frac{2}{q^2}$

70. $f(w) = \frac{w-5}{w^5} = w^{-4} - 5w^{-5}$

$f'(w) = -4w^{-5} + 25w^{-6} = -w^{-6}(4w - 25)$

71. $f(x) = (x+1)(x+3) = x^2 + 4x + 3$

$f'(x) = 2x + 4 = 2(x+2)$

72. $f(x) = x^2(x-2)(x+4) = x^4 + 2x^3 - 8x^2$

$f'(x) = 4x^3 + 6x^2 - 16x = 2x\left(2x^2 + 3x - 8\right)$

73. $w(x) = \frac{x^2 + x^3}{x^2} = \frac{x^2}{x^2} + \frac{x^3}{x^2} = 1 + x$

$w'(x) = 0 + 1 = 1$

74. $f(x) = \frac{7x^3 + x}{6\sqrt{x}}$

$= \frac{1}{6}\left(\frac{7x^3}{x^{1/2}} + \frac{x}{x^{1/2}}\right)$

$= \frac{1}{6}(7x^{5/2} + x^{1/2})$

$f'(x) = \frac{1}{6}\left(\frac{35}{2}x^{3/2} + \frac{1}{2}x^{-1/2}\right)$

$= \frac{1}{12}x^{1/2}(35x + x^{-1})$

75. $y' = 6x + 4$

$y'\big|_{x=0} = 4$

$y'\big|_{x=2} = 16$

$y'\big|_{x=-3} = -14$

76. $y' = -6 - 6x^2$

$y'\big|_{x=0} = -6$

$y'\big|_{x=3/2} = -\frac{39}{2}$

$y'\big|_{x=-3} = -60$

77. y is a constant, so $y' = 0$ for all x.

78. $y' = 3 - 2x^{-1/2} = 3 - \frac{2}{\sqrt{x}}$

$y'\big|_{x=4} = 2$

$y'\big|_{x=9} = \frac{7}{3}$

$y'\big|_{x=25} = \frac{13}{5}$

79. $y = 4x^2 + 5x + 6$

$y' = 8x + 5$

$y'\big|_{x=1} = 13$

An equation of the tangent line is $y - 15 = 13(x - 1)$, or $y = 13x + 2$.

80. $y = \frac{1}{5}\left(1 - x^2\right)$

$y' = \frac{1}{5}(-2x)$

$y'\big|_{x=4} = -\frac{8}{5}$

An equation of the tangent line is

$y + 3 = -\frac{8}{5}(x - 4)$, or $y = -\frac{8}{5}x + \frac{17}{5}$.

81. $y = \frac{1}{x^3} = x^{-3}$

$y' = -3x^{-4} = -\frac{3}{x^4}$

$y'\big|_{x=2} = -\frac{3}{16}$

An equation of the tangent line is

$y - \frac{1}{8} = -\frac{3}{16}(x - 2)$, or $y = -\frac{3}{16}x + \frac{1}{2}$.

82. $y = -\sqrt[3]{x} = -x^{\frac{1}{3}}$

$y' = -\frac{1}{3}x^{-\frac{2}{3}} = -\frac{1}{3x^{\frac{2}{3}}}$

$y'\big|_{x=8} = -\frac{1}{3\left(8^{\frac{2}{3}}\right)} = -\frac{1}{3 \cdot 4} = -\frac{1}{12}$

An equation of the tangent line is

$y + 2 = -\frac{1}{12}(x - 8)$, or $y = -\frac{1}{12}x - \frac{4}{3}$.

83. $y = 3 + x - 5x^2 + x^4$

$y' = 1 - 10x + 4x^3$.

When $x = 0$, then $y = 3$ and $y' = 1$. Thus an equation of the tangent line is $y - 3 = 1(x - 0)$, or $y = x + 3$.

84. $y = \frac{\sqrt{x}\left(2 - x^2\right)}{x} = x^{-\frac{1}{2}}\left(2 - x^2\right) = 2x^{-\frac{1}{2}} - x^{\frac{3}{2}}$.

$y' = -x^{-\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}}$

$y'\big|_{x=4} = -\frac{1}{8} - 3 = -\frac{25}{8}$

When $x = 4$, then $y = -7$. The tangent line is

$y + 7 = -\frac{25}{8}(x - 4)$, or $y = -\frac{25}{8}x + \frac{11}{2}$.

85. $y = \frac{5}{2}x^2 - x^3$

$y' = 5x - 3x^2$

A horizontal tangent line has slope 0, so we set $5x - 3x^2 = 0$. Then $x(5 - 3x) = 0$, $x = 0$ or $x = \frac{5}{3}$.

If $x = 0$, then $y = 0$. If $x = \frac{5}{3}$, $y = \frac{125}{54}$. This gives the points (0, 0) and $\left(\frac{5}{3}, \frac{125}{54}\right)$.

86. $y = \frac{x^5}{5} - x + 1$

$y' = x^4 - 1$

A horizontal tangent line has slope 0, so we set $x^4 - 1 = 0$. Then $x^4 = 1$, so $x = 1$ or $x = -1$. If $x = 1$, then $y = \frac{1}{5}$; if $x = -1$, then $y = \frac{9}{5}$. This gives the points $\left(1, \frac{1}{5}\right)$ and $\left(-1, \frac{9}{5}\right)$.

87. $y = x^2 - 5x + 3$

$y' = 2x - 5$

Setting $2x - 5 = 1$ gives $2x = 6$, $x = 3$. When $x = 3$, then $y = -3$. This gives the point (3, –3).

88. $y = x^4 - 31x + 11$

$y' = 4x^3 - 31$

If $4x^3 - 31 = 1$, then $x^3 = 8$, $x = 2$. When $x = 2$, then $y = -35$. This gives the point (2, –35).

89. $f(x)=\sqrt{x}+\frac{1}{\sqrt{x}}=x^{\frac{1}{2}}+x^{-\frac{1}{2}}$

$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}=\frac{1}{2\sqrt{x}}-\frac{1}{2x\sqrt{x}}=\frac{x-1}{2x\sqrt{x}}$

Thus $\frac{x-1}{2x\sqrt{x}}-f'(x)=\frac{x-1}{2x\sqrt{x}}-\frac{x-1}{2x\sqrt{x}}=0$.

90. $z=(1+b)w_p-bw_c$

$\frac{dz}{dw_c}=(1+b)\frac{dw_p}{dw_c}-b$

Rewriting the right side and factoring out 1 + *b* gives $\frac{dz}{dw_c}=(1+b)\frac{dw_p}{dw_c}-\frac{b(1+b)}{1+b}$,

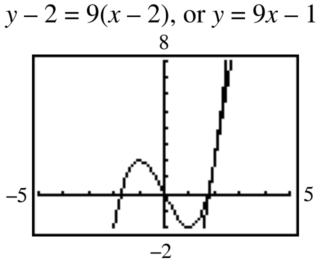
$\frac{dz}{dw_c}=(1+b)\left[\frac{dw_p}{dw_c}-\frac{b}{1+b}\right]$.

91. $y=x^3-3x$

$y'(x)=3x^2-3$

$y'|_{x=2}=3\left(2^2\right)-3=9$

The tangent line at (2, 2) is given by $y-2=9(x-2)$, or $y=9x-16$.



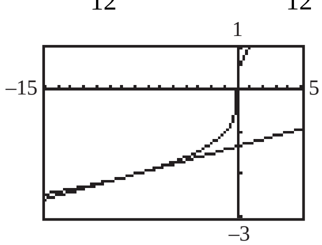
92. $y=\sqrt[3]{x}=x^{1/3}$

$y'(x)=\frac{1}{3}x^{-2/3}=\frac{1}{3\sqrt[3]{x^2}}$

$y'|_{x=-8}=\frac{1}{12}$

The tangent line at (–8, –2) is given by

$y+2=\frac{1}{12}(x+8)$, or $y=\frac{1}{12}x-\frac{4}{3}$.



Principles in Practice 11.3

1. Here $\frac{dP}{dp}=5$ and $\Delta p=25.5-25=0.5$.

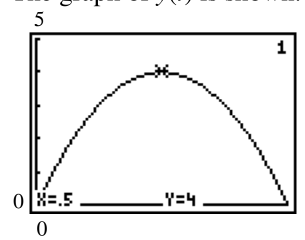
$\Delta P\approx\frac{dP}{dp}\Delta p=5(0.5)=2.5$

The profit increases by 2.5 units when the price is changed from 25 to 25.5 per unit.

2. $\frac{dy}{dt}=\frac{d}{dt}\left(16t-16t^2\right)=16-16(2t)=16-32t$

$\left.\frac{dy}{dt}\right|_{t=0.5}=16-32(0.5)=16-16=0$

The graph of *y*(*t*) is shown.



When *t* = 0.5, the object is at the peak of its flight.

3. $V'(r)=\frac{4}{3}\pi\left(3r^2\right)+4\pi(2r)=4\pi r^2+8\pi r$

When *r* = 2, $V'(r)=4\pi(2)^2+8\pi(2)=32\pi$ and

$V(r)=\frac{4}{3}\pi(2)^3+4\pi(2)^2=\frac{32\pi}{3}+16\pi=\frac{80}{3}\pi$.

The relative rate of change of the volume when *r* = 2 is $\frac{V'(2)}{V(2)}=\frac{32\pi}{\frac{80}{3}\pi}=\frac{6}{5}=1.2$. Multiplying 1.2 by 100 gives the percentage rate of change: (1.2)(100) = 120%.

Problems 11.3

1. $s=f(t)=2t^2+3t$

If $\Delta t=1$, then over [1, 2] we have

$\frac{\Delta s}{\Delta t}=\frac{f(2)-f(1)}{2-1}=\frac{14-5}{1}=9$.

If $\Delta t=0.5$, then over [1, 1.5] we have

$\frac{\Delta s}{\Delta t}=\frac{f(1.5)-f(1)}{1.5-1}=\frac{9-5}{0.5}=8$.

Continuing this way, we obtain the following table:

Δt	1	0.5	0.2	0.1	0.01	0.001
$\frac{\Delta s}{\Delta t}$	9	8	7.4	7.2	7.02	7.002

We estimate the velocity when $t = 1$ to be 7 m/s. With differentiation we get $v = \frac{ds}{dt} = 4t + 3$,

$\left.\frac{ds}{dt}\right|_{t=1} = 4(1) + 3 = 7$ m/s.

2. $y = f(x) = \sqrt{2x+5}$.
If $\Delta x = 1$, then over [3, 4] we have
$\frac{\Delta y}{\Delta x} = \frac{f(4) - f(3)}{\Delta x} = \frac{\sqrt{13} - \sqrt{11}}{1} \approx 0.2889$
If $\Delta x = 0.5$, then over [3, 3.5] we have
$\frac{\Delta y}{\Delta x} = \frac{f(3.5) - f(3)}{\Delta x} = \frac{\sqrt{12} - \sqrt{11}}{0.5} \approx 0.2950$
Continuing in this way we obtain the following table:

Δx	1	0.5	0.2	0.1	0.01	0.001
$\frac{\Delta y}{\Delta x}$	0.2889	0.2950	0.2988	0.3002	0.3014	0.3015

We estimate the rate of change to be 0.3015.
$\left(\text{Note: The actual rate of change is } \frac{1}{\sqrt{11}} \approx 0.3015.\right)$

3. $s = f(t) = 2t^2 - 4t$

a. When $t = 7$, then $s = 2(7^2) - 4(7) = 70$ m.

b. $\frac{\Delta s}{\Delta t} = \frac{f(7.5) - f(7)}{0.5} = \frac{[2(7.5)^2 - 4(7.5)] - 70}{0.5} = 25$ m/s

c. $v = \frac{ds}{dt} = 4t - 4$. If $t = 7$, then $v = 4(7) - 4 = 24$ m/s

4. $s = f(t) = \frac{1}{2}t + 1$

a. When $t = 2$, $s = \frac{1}{2}(2) + 1 = 2$ m.

b. $\frac{\Delta s}{\Delta t} = \frac{f(2.1) - f(1)}{0.1} = \frac{\left[\frac{1}{2}(2.1) + 1\right] - 2}{0.1} = 0.5$ m/s

c. $v = \frac{ds}{dt} = \frac{1}{2}$. If $t = 2$, then $v = \frac{1}{2}$ m/s

5. $s = f(t) = 2t^3 + 6$

a. When $t = 1$, $s = 2(1)^3 + 6 = 8$ m.

b. $\frac{\Delta s}{\Delta t} = \frac{f(1.02) - f(1)}{0.02}$
$= \frac{\left[2(1.02)^3 + 6\right] - 8}{0.02}$
$= 6.1208$ m/s

c. $v = \frac{ds}{dt} = 6t^2$. If $t = 1$, then
$v = 6(1)^2 = 6$ m/s

6. $s = f(t) = -3t^2 + 2t + 1$

a. When $t = 1$, $s = -3\left(1^2\right) + 2(1) + 1 = 0$ m.

b. $\frac{\Delta s}{\Delta t} = \frac{f(1.25) - f(1)}{0.25}$
$= \frac{\left[-3(1.25)^2 + 2(1.25) + 1\right] - 0}{0.25} = -4.75$ m/s

c. $v = \frac{ds}{dt} = -6t + 2$. If $t = 1$, $v = -4$ m/s

7. $s = f(t) = t^4 - 2t^3 + t$

a. When $t = 2$, $s = 2^4 - 2\left(2^3\right) + 2 = 2$ m.

b. $\frac{\Delta s}{\Delta t} = \frac{f(2.1) - f(2)}{0.1}$
$= \frac{\left[(2.1)^4 - 2(2.1)^3 + 2.1\right] - 2}{0.1} = 10.261$ m/s

c. $v = \frac{ds}{dt} = 4t^3 - 6t^2 + 1$. If $t = 2$, then
$v = 4\left(2^3\right) - 6\left(2^2\right) + 1 = 9$ m/s

8. $s = f(t) = 3t^4 - t^{7/2}$

a. When $t = 0$, $s = 3 \cdot 0^4 = 0^{7/2} = 0$.

b. $\frac{\Delta s}{\Delta t} = \frac{f\left(\frac{1}{4}\right) - f(0)}{\frac{1}{4}} = \frac{\left[3 \cdot \left(\frac{1}{4}\right)^4 - \left(\frac{1}{4}\right)^{7/2}\right] - 0}{\frac{1}{4}}$
$= \frac{1}{64}$ m/s

c. $v = \frac{ds}{dt} = 12t^3 - \frac{7}{2}t^{5/2}$. If $t = 0$, then
$v = 12(0)^3 - \frac{7}{2}(0)^{5/2} = 0$ m/s.

9. $\frac{dy}{dx} = \frac{25}{2}x^{\frac{3}{2}}$. If $x = 9$, $\frac{dy}{dx} = \frac{25}{2}(27) = 337.50$.

10. $\frac{dA}{dr} = 2\pi r$. If $r = 3$, $\frac{dA}{dr} = 2\pi(3) = 6\pi$.

11. $\frac{dT}{dT_e} = 0 + 0.27(1 - 0) = 0.27$

12. $\frac{dV}{dr} = 4\pi r^2$
When $r = 6.3 \times 10^{-4}$,
$\frac{dV}{dr} = 4\pi[6.3 \times 10^{-4}]^2 = 158.76\pi \times 10^{-8}$
$\approx 4.988 \times 10^{-6}$.

13. $c = 500 + 10q$, $\frac{dc}{dq} = 10$. When $q = 100$,
$\frac{dc}{dq} = 10$.

14. $c = 5000 + 6q$, $\frac{dc}{dq} = 6$. When $q = 36$, $\frac{dc}{dq} = 6$.

15. $\frac{dc}{dq} = 0.1(2q) + 3 = 0.2q + 3$. When $q = 5$,
$\frac{dc}{dq} = 0.2(5) + 3 = 4$.

16. $\frac{dc}{dq} = 0.2q + 3$. When $q = 3$, $\frac{dc}{dq} = 3.6$.

17. $\frac{dc}{dq} = 2q + 50$. Evaluating when $q = 15$, 16 and 17 gives 80, 82 and 84, respectively.

18. $\frac{dc}{dq} = 0.12q^2 - q + 4.4$

Evaluating when $q = 5$, 25, and 1000 gives 2.4, 54.4 and 119,004.4, respectively.

19. $\bar{c} = 0.01q + 5 + \frac{500}{q}$

$c = \bar{c}q = 0.01q^2 + 5q + 500$

$\frac{dc}{dq} = 0.02q + 5$

$\left.\frac{dc}{dq}\right|_{q=50} = 6$

$\left.\frac{dc}{dq}\right|_{q=100} = 7$

20. $\bar{c} = 2 + \frac{1000}{q}$

$c = \bar{c}q = 2q + 1000$

$\frac{dc}{dq} = 2$ for all q

21. $c = \bar{c}q = 0.00002q^3 - 0.01q^2 + 6q + 20,000$

$\frac{dc}{dq} = 0.00006q^2 - 0.02q + 6$

If $q = 100$, then $\frac{dc}{dq} = 4.6$. If $q = 500$, then $\frac{dc}{dq} = 11$.

22. $c = \bar{c}q = 0.002q^3 - 0.5q^2 + 60q + 7000$

$\frac{dc}{dq} = 0.006q^2 - q + 60$

If $q = 15$, then $\frac{dc}{dq} = 46.35$. If $q = 25$, then $\frac{dc}{dq} = 38.75$.

23. $r = 0.8q$

$\frac{dr}{dq} = 0.8$ for all q.

24. $r = q\left(15 - \frac{1}{30}q\right) = 15q - \frac{1}{30}q^2$

$\frac{dr}{dq} = 15 - \frac{1}{15}q$

For $q = 5$, $\frac{dr}{dq} = \frac{44}{3}$; for $q = 15$, $\frac{dr}{dq} = 14$; for $q = 150$, $\frac{dr}{dq} = 5$.

25. $r = 250q + 45q^2 - q^3$

$\frac{dr}{dq} = 250 + 90q - 3q^2$. Evaluating when $q = 5$, 10 and 25 gives 625, 850 and 625, respectively.

26. $r = 60q - 0.2q^2$

$\frac{dr}{dq} = 60 - 0.4q$

Evaluating when $q = 10$ and 20 gives 56 and 52, respectively.

27. $\frac{dc}{dq} = 6.750 - 0.000328(2q) = 6.750 - 0.000656q$

$\left.\frac{dc}{dq}\right|_{q=2000} = 6.750 - 0.000656(2000) = 5.438$

$\bar{c} = \frac{c}{q} = \frac{-10,484.69}{q} + 6.750 - 0.000328q$

$\bar{c}(2000) = \frac{-10,484.69}{2000} + 6.750 - 0.000328(2000)$
$= 0.851655$

28. $\frac{dc}{dq} = -0.79 + 0.04284q - 0.0003q^2$

$\left.\frac{dc}{dq}\right|_{q=70} = 0.7388$

29. $PR^{0.93} = 5,000,000$

$P = 5,000,000R^{-0.93}$

$\frac{dP}{dR} = -4,650,000R^{-1.93}$

30. $\frac{dv}{dt} = -10,500$ for all t.

31. a. $\frac{dy}{dx} = -1.5 - x$

$\left.\frac{dy}{dx}\right|_{x=6} = -1.5 - 6 = -7.5$

b. Setting $-1.5 - x = -6$ gives $x = 4.5$.

32. $c = f(q) = 0.4q^2 + 4q + 5$

$\frac{dc}{dq} = 0.8q + 4$

If $q = 2$, then $\frac{dc}{dq} = 5.6$. Over the interval [2, 3],

$\frac{\Delta c}{\Delta q} = \frac{f(3) - f(2)}{3-2} = \frac{20.6 - 14.6}{1} = 6$.

33. a. $y' = 1$

b. $\frac{y'}{y} = \frac{1}{x+4}$

c. $y'(5) = 1$

d. $\frac{1}{5+4} = \frac{1}{9} \approx 0.111$

e. 11.1%

34. a. $y' = -3$

b. $\frac{y'}{y} = \frac{-3}{7-3x} = \frac{3}{3x-7}$

c. $y'(6) = -3$

d. $\frac{3}{3(6)-7} = \frac{3}{11} \approx 0.2727$

e. 27.27%

35. a. $y' = 6x$

b. $\frac{y'}{y} = \frac{6x}{3x^2 + 7}$

c. $y'(2) = 6(2) = 12$

d. $\frac{12}{12+7} = \frac{12}{19} \approx 0.632$

e. 63.2%

36. a. $y' = -9x^2$

b. $\frac{y'}{y} = \frac{-9x^2}{5-3x^3}$

c. $y'(1) = -9$

d. $\frac{-9}{5-3} = -\frac{9}{2} = -4.5$

e. –450%

37. a. $y' = -3x^2$

b. $\frac{y'}{y} = \frac{-3x^2}{8-x^3}$

c. $y'(1) = -3$

d. $\frac{-3}{8-1} = -\frac{3}{7} \approx -0.429$

e. –42.9%

38. a. $y' = 2x + 3$

b . $\frac{y'}{y} = \frac{2x+3}{x^2 + 3x - 4}$

c. $y'(-1) = 2(-1) + 3 = 1$

d. $\frac{1}{1-3-4} = -\frac{1}{6} \approx -0.167$

e. –16.7%

39. $c = 0.3q^2 + 3.5q + 9$

$\frac{dc}{dq} = 0.6q + 3.5$

If $q = 10$, then $\frac{dc}{dq} = 0.6(10) + 3.5 = 9.5$. If $q = 10$, then $c = 74$ and

$\frac{\frac{dc}{dq}}{c}(100) = \frac{9.5}{74}(100) \approx 12.8\%$.

40. $y = \frac{100}{x} = 100x^{-1}$

$\frac{dy}{dx} = -100x^{-2} = -\frac{100}{x^2}$

If $x = 10$, $\frac{dy}{dx} = -\frac{100}{100} = -1$ and $\frac{y'}{y}(100) = \frac{-1}{10}(100) = -10\%$.

41. **a.** $\frac{dr}{dq} = 30 - 0.6q$

b. If $q = 10$, $\frac{r'}{r} = \frac{30-6}{300-30} = \frac{24}{270} = \frac{4}{45} \approx 0.09$.

c. 9%

42. **a.** $\frac{dq}{dr} = 10 - 0.4q$

b. If $q = 25$, $\frac{r'}{r} = \frac{10-0.4(25)}{10(25)-0.2(25)^2} = 0.$

c. 0%

43. $\frac{W'}{W} = \frac{0.864t^{-0.568}}{2t^{0.432}} = \frac{0.432}{t}$

44. **a.** $\frac{R_1'}{R_1} = \frac{\frac{1.3I^{0.3}}{1855.24}}{\frac{I^{1.3}}{1855.24}} = \frac{1.3}{I}$

$\frac{R_2'}{R_2} = \frac{\frac{1.3I^{0.3}}{1101.29}}{\frac{I^{1.3}}{1101.29}} = \frac{1.3}{I}$

b. They are equal.

c. $\frac{f'x}{f(x)} = \frac{nC_1x^{n-1}}{C_1x^n} = \frac{n}{x}$

$\frac{g'(x)}{g(x)} = \frac{nC_2x^{n-1}}{C_2x^n} = \frac{n}{x}$

The rates are equal.

45. The cost of $q = 20$ bikes is $q\overline{c} = 20(150) = \$3000$. The marginal cost, \$125, is the approximate cost of one additional bike. Thus the approximate cost of producing 21 bikes is \$3000 + \$125 = \$3125.

46. The relative rate of change of c is $\frac{\frac{dc}{dq}}{c}$, which is given to be $\frac{1}{q}$: $\frac{\frac{dc}{dq}}{c}=\frac{1}{q}$. Thus $\frac{dc}{dq}=\frac{c}{q}=\overline{c}$, and the marginal cost function $\left(\frac{dc}{dq}\right)$ and the average cost function $(\overline{c})$ are equal.

47. \$5.07 per unit

48. 11,275 people per year

Principles in Practice 11.4

1. $\frac{dR}{dx}=(2-0.15x)\frac{d}{dx}(225+20x)+(225+20x)\frac{d}{dx}(2-0.15x)$
$= (2-0.15x)(20) + (225+20x)(-0.15)$
$= 40 - 3x - 33.75 - 3x = 6.25 - 6x$
$\frac{dR}{dx}=6.25-6x$

2. $T(x)=x^2-\frac{1}{3}x^3$
$T'(x)=2x-x^2$
When the dosage is 1 milligram the sensitivity is $T'(1)=2(1)-1^2=1$.

Problems 11.4

1. $f'(x)=(4x+1)(6)+(6x+3)(4) = 24x + 6 + 24x + 12 = 48x + 18 = 6(8x + 3)$

2. $f'(x)=(3x-1)(7)+(7x+2)(3)=42x-1$

3. $s'(t)=(5-3t)(3t^2-4t)+(t^3-2t^2)(-3)=15t^2-20t-9t^3+12t^2-3t^3+6t^2=-12t^3+33t^2-20t$

4. $Q'(x)=(3+x)(10x)+(5x^2-2)(1)=15x^2+30x-2$

5. $f'(r)=\left(3r^2-4\right)(2r-5)+\left(r^2-5r+1\right)(6r)=6r^3-15r^2-8r+20+6r^3-30r^2+6r=12r^3-45r^2-2r+20$

6. $C'(I)=\left(2I^2-3\right)(6I-4)+\left(3I^2-4I+1\right)(4I)=12I^3-8I^2-18I+12+12I^3-16I^2+4I=2\left(12I^3-12I^2-7I+6\right)$

7. Without the product rule we have
$f(x)=x^2\left(2x^2-5\right)=2x^4-5x^2$
$f'(x)=8x^3-10x$

8. Without the product rule we have
$f(x)=3x^3\left(x^2-2x+2\right)=3x^5-6x^4+6x^3$
$f'(x)=15x^4-24x^3+18x^2$

9. $y' = \left(x^2+3x-2\right)(4x-1)+\left(2x^2-x-3\right)(2x+3)$

$= \left(4x^3+12x^2-8x-x^2-3x+2\right)+\left(4x^3-2x^2-6x+6x^2-3x-9\right)$

$= 8x^3+15x^2-20x-7$

10. $\phi'(x) = (3-5x+2x^2)(1-8x)+(2+x-4x^2)(-5+4x)$

$= 3-5x+2x^2-24x+40x^2-16x^3-10-5x+20x^2+8x+4x^2-16x^3$

$= -32x^3+66x^2-26x-7$

11. $f'(w) = (w^2+3w-7)(6w^2)+(2w^3-4)(2w+3)$

$= 6w^4+18w^3-42w^2+4w^4+6w^3-8w-12$

$= 10w^4+24w^3-42w^2-8w-12$

12. $f'(x) = \left(3x-x^2\right)(-1-2x)+\left(3-x-x^2\right)(3-2x)$

$= -3x-5x^2+2x^3+9-3x-3x^2-6x+2x^2+2x^3$

$= 4x^3-6x^2-12x+9$

13. $y' = \left(x^2-1\right)\left(9x^2-6\right)+\left(3x^3-6x+5\right)(2x)-4(8x+2)$

$= 9x^4-15x^2+6+6x^4-12x^2+10x-32x-8$

$= 15x^4-27x^2-22x-2$

14. $h'(x) = 4\left(5x^4\right)+3\left[\left(8x^2-5\right)(2)+(2x+2)(16x)\right]$

$= 20x^4+3(16x^2-10+32x^2+32x)$

$= 20x^4+144x^2+96x-30$

15. $F'(p) = \frac{3}{2}\left[(5p^{1/2}-2)(3)+(3p-1)\left(5\cdot\frac{1}{2}p^{-1/2}\right)\right]$

$= \frac{3}{2}\left[15p^{1/2}-6+\frac{15}{2}p^{1/2}-\frac{5}{2}p^{-1/2}\right]$

$= \frac{3}{4}[45p^{1/2}-12-5p^{-1/2}]$

16. $g'(x) = (x^{1/2}+5x-2)\left(\frac{1}{3}x^{-2/3}-\frac{3}{2}x^{-1/2}\right)+(x^{1/3}-3x^{1/2})\left(\frac{1}{2}x^{-1/2}+5\right)$

$= \frac{1}{3}x^{-1/6}+\frac{5}{3}x^{1/3}-\frac{2}{3}x^{-2/3}-\frac{3}{2}-\frac{15}{2}x^{1/2}+3x^{-1/2}+\frac{1}{2}x^{-1/6}+5x^{1/3}-\frac{3}{2}-15x^{1/2}$

$= \frac{1}{6}(-135x^{1/2}+40x^{1/3}+5x^{-1/6}+18x^{-1/2}-4x^{-2/3}-18)$

17. $y = 7\cdot\frac{2}{3}$ is a constant function, so $y' = 0$.

18. $y = x^3-6x^2+11x-6$

$y' = 3x^2-12x+11$

19. $y = 6x^3 + 47x^2 + 31x - 28$

$y' = 18x^2 + 94x + 31$

20. $\frac{dy}{dx} = \frac{(4x+1)(2) - (2x-3)(4)}{(4x+1)^2} = \frac{8x + 2 - 8x + 12}{(4x+1)^2}$

$= \frac{14}{(4x+1)^2}$

21. $f'(x) = \frac{(x-1)(5) - (5x)(1)}{(x-1)^2} = \frac{5x - 5 - 5x}{(x-1)^2}$

$= -\frac{5}{(x-1)^2}$

22. $H'(x) = \frac{(5-x)(-5) - (-5x)(-1)}{(5-x)^2}$

$= \frac{-25 + 5x - 5x}{(5-x)^2} = -\frac{25}{(5-x)^2}$

23. $f(x) = \frac{-13}{3x^5} = -\frac{13}{3}x^{-5}$

$f'(x) = -\frac{13}{3}(-5x^{-6}) = \frac{65}{3x^6}$

24. $f(x) = \frac{5}{7}\left(x^2 - 2\right)$

$f'(x) = \frac{5}{7}(2x) = \frac{10}{7}x$

25. $y' = \frac{(x-1)(1) - (x+2)(1)}{(x-1)^2}$

$= \frac{x - 1 - x - 2}{(x-1)^2}$

$= -\frac{3}{(x-1)^2}$

26. $h'(w) = \frac{(w-3)(6w+5) - \left(3w^2 + 5w - 1\right)(1)}{(w-3)^2}$

$= \frac{6w^2 - 13w - 15 - 3w^2 - 5w + 1}{(w-3)^2}$

$= \frac{3w^2 - 18w - 14}{(w-3)^2}$

27. $h'(z) = \frac{\left(z^2-4\right)(-2)-(6-2z)(2z)}{\left(z^2-4\right)^2}$

$= \frac{-2z^2+8-12z+4z^2}{\left(z^2-4\right)^2} = \frac{2z^2-12z+8}{\left(z^2-4\right)^2}$

$= \frac{2\left(z^2-6z+4\right)}{\left(z^2-4\right)^2}$

28. $z' = \frac{(3x^2+5x+3)(4x+5)-(2x^2+5x-2)(6x+5)}{(3x^2+5x+3)^2}$

$= \frac{12x^3+35x^2+37x+15-(12x^3+40x^2+13x-10)}{(3x^2+5x+3)^2}$

$= \frac{-5x^2+24x+25}{(3x^2+5x+3)^2}$

29. $y' = \frac{\left(x^2-5x\right)(16x-2)-\left(8x^2-2x+1\right)(2x-5)}{\left(x^2-5x\right)^2}$

$= \frac{16x^3-82x^2+10x-\left(16x^3-44x^2+12x-5\right)}{\left(x^2-5x\right)^2} = \frac{-38x^2-2x+5}{\left(x^2-5x\right)^2}$

30. $f'(x) = \frac{\left(x^2+1\right)\left(3x^2-2x\right)-\left(x^3-x^2+1\right)(2x)}{\left(x^2+1\right)^2}$

$= \frac{3x^4-2x^3+3x^2-2x-2x^4+2x^3-2x}{(x^2+1)^2}$

$= \frac{x\left(x^3+3x-4\right)}{\left(x^2+1\right)^2}$

31. $y' = \frac{\left(2x^2-3x+2\right)(2x-4)-\left(x^2-4x+3\right)(4x-3)}{\left(2x^2-3x+2\right)^2}$

$= \frac{4x^3-14x^2+16x-8-\left(4x^3-19x^2+24x-9\right)}{\left(2x^2-3x+2\right)^2}$

$= \frac{5x^2-8x+1}{\left(2x^2-3x+2\right)^2}$

32. The quotient rule can be used, or we can write

$$F(z)=\frac{z^4+4}{3z}=\frac{1}{3}\left(z^3+4z^{-1}\right),$$

so $F'(z)=\frac{1}{3}\left(3z^2-4z^{-2}\right)=\frac{3z^4-4}{3z^2}$.

33. $$g'(x)=\frac{\left(x^{100}+7\right)(0)-(1)\left(100x^{99}\right)}{\left(x^{100}+7\right)^2}=-\frac{100x^{99}}{\left(x^{100}+7\right)^2}$$

34. $$y=\frac{-9}{2x^5}=-\frac{9}{2}x^{-5}$$

$$y'=\frac{45}{2}x^{-6}$$

35. $$u(v)=\frac{v^3-8}{v}=\frac{v^3}{v}-\frac{8}{v}=v^2-8v^{-1}$$

$$u'(v)=2v+8v^{-2}=2\left(v+\frac{4}{v^2}\right)=\frac{2(v^3+4)}{v^2}$$

36. $$y=\frac{x-5}{8\sqrt{x}}=\frac{1}{8}\left(x^{\frac{1}{2}}-5x^{-\frac{1}{2}}\right)$$

$$y'=\frac{1}{8}\left(\frac{1}{2}x^{-\frac{1}{2}}+\frac{5}{2}x^{-\frac{3}{2}}\right)=\frac{1}{16}\left(\frac{1}{x^{\frac{1}{2}}}+\frac{5}{x^{\frac{3}{2}}}\right)=\frac{x+5}{16x^{\frac{3}{2}}}$$

37. $$y=\frac{3x^2-x-1}{\sqrt[3]{x}}=\frac{3x^2-x-1}{x^{\frac{1}{3}}}=3x^{\frac{5}{3}}-x^{\frac{2}{3}}-x^{-\frac{1}{3}}$$

$$y'=5x^{\frac{2}{3}}-\frac{2}{3}x^{-\frac{1}{3}}+\frac{1}{3}x^{-\frac{4}{3}}=5x^{\frac{2}{3}}-\frac{2}{3x^{\frac{1}{3}}}+\frac{1}{3x^{\frac{4}{3}}}$$

$$=\frac{15x^2-2x+1}{3x^{\frac{4}{3}}}$$

38. $$y'=\frac{\left(2x^{2.1}+1\right)\left(0.3x^{-0.7}\right)-\left(x^{0.3}-2\right)\left(4.2x^{1.1}\right)}{\left(2x^{2.1}+1\right)^2}$$

$$=\frac{0.6x^{1.4}+0.3x^{-0.7}-4.2x^{1.4}+8.4x^{1.1}}{(2x^{2.1}+1)^2}$$

$$=\frac{0.3\left(1+28x^{1.8}-12x^{2.1}\right)}{x^{0.7}\left(2x^{2.1}+1\right)^2}$$

39. $$y'=-\frac{(x-8)(0)-(4)(1)}{(x-8)^2}+\frac{(3x+1)(2)-(2x)(3)}{(3x+1)^2}$$

$$=\frac{4}{(x-8)^2}+\frac{2}{(3x+1)^2}$$

40. $$q'(x)=6x^2+\frac{(3x-5)(5)-(5x+1)(3)}{(3x-5)^2}+6x^{-4}$$

$$=6x^2-\frac{28}{(3x-5)^2}+6x^{-4}$$

41. $$y'=\frac{[(x+2)(x-4)](1)-(x-5)(2x-2)}{[(x+2)(x-4)]^2}$$

$$=\frac{x^2-2x-8-\left(2x^2-12x+10\right)}{[(x+2)(x-4)]^2}$$

$$=\frac{-\left(x^2-10x+18\right)}{[(x+2)(x-4)]^2}$$

42. $$y=\frac{(9x-1)(3x+2)}{4-5x}=\frac{27x^2+15x-2}{4-5x}$$

$$y'=\frac{(4-5x)(54x+15)-\left(27x^2+15x-2\right)(-5)}{(4-5x)^2}$$

$$=\frac{-270x^2+141x+60+135x^2+75x-10}{(4-5x)^2}$$

$$=-\frac{135x-216x-50}{(4-5x)^2}$$

43. $s'(t) = \dfrac{\left[\left(t^2-1\right)\left(t^3+7\right)\right](2t+3)-\left(t^2+3t\right)\left(5t^4-3t^2+14t\right)}{\left[\left(t^2-1\right)\left(t^3+7\right)\right]^2}$

$= \dfrac{-3t^6-12t^5+t^4+6t^3-21t^2-14t-21}{\left[\left(t^2-1\right)\left(t^3+7\right)\right]^2}$

44. $f(s) = \dfrac{17}{5s^3-10s^2+4s}$

$f'(s) = \dfrac{0-17\left[15s^2-20s+4\right]}{\left(5s^3-10s^2+4s\right)^2} = -\dfrac{17\left(15s^2-20s+4\right)}{\left(5s^3-10s^2+4s\right)^2}$

45. $y = 3x - \dfrac{\frac{2}{x}-\frac{3}{x-1}}{x-2} = 3x - \dfrac{\frac{2(x-1)-3x}{x(x-1)}}{x-2}$

$= 3x + \dfrac{x+2}{x(x-1)(x-2)} = 3x + \dfrac{x+2}{x^3-3x^2+2x}$

$y' = 3 + \dfrac{(x^3-3x^2+2x)(1)-(x+2)(3x^2-6x+2)}{[x(x-1)(x-2)]^2}$

$= 3 - \dfrac{2x^3+3x^2-12x+4}{[x(x-1)(x-2)]^2}$

46. $y = 3-12x^3 + \dfrac{1-\frac{5}{x^2+2}}{x^2+5} = 3-12x^3 + \dfrac{\frac{x^2+2-5}{x^2+2}}{x^2+5} = 3-12x^3 + \dfrac{x^2-3}{x^4+7x^2+10}$

$y' = -36x^2 + \dfrac{(x^4+7x^2+10)(2x)-(x^2-3)(4x^3+14x)}{(x^4+7x^2+10)^2} = -36x^2 + \dfrac{-2x^5+12x^3+62x}{[(x^2+2)(x^2+5)]^2}$

47. $f'(x) = \dfrac{(a-x)(1)-(a+x)(-1)}{(a-x)^2} = \dfrac{2a}{(a-x)^2}$

48. Simplifying, $f(x) = \dfrac{x^{-1}+a^{-1}}{x^{-1}-a^{-1}} \cdot \dfrac{ax}{ax} = \dfrac{a+x}{a-x}$

$f'(x) = \dfrac{(a-x)(1)-(a+x)(-1)}{(a-x)^2} = \dfrac{2a}{(a-x)^2}$

49. $y = \left(4x^2+2x-5\right)\left(x^3+7x+4\right)$

$y' = \left(4x^2+2x-5\right)\left(3x^2+7\right)+\left(x^3+7x+4\right)(8x+2)$

$y'(-1) = (-3)(10)+(-4)(-6) = -6$

50. $y = \frac{x^3}{x^4+1}$

$y' = \frac{(x^4+1)(3x^2)-(x^3)(4x^3)}{(x^4+1)^2}$

$y'(-1) = \frac{(2)(3)-(-1)(-4)}{(2)^2} = \frac{1}{2}$

51. $y = \frac{6}{x-1}$

$y' = \frac{(x-1)(0)-(6)(1)}{(x-1)^2} = -\frac{6}{(x-1)^2}$

$y'(3) = -\frac{6}{2^2} = -\frac{3}{2}$

The tangent line is $y-3 = -\frac{3}{2}(x-3)$, or $y = -\frac{3}{2}x + \frac{15}{2}$.

52. $y = \frac{x+5}{x^2} = x^{-1} + 5x^{-2}$

$y' = -x^{-2} - 10x^{-3} = -\frac{1}{x^2} - \frac{10}{x^3}$

$y'(1) = -1 - 10 = -11$

The tangent line is $y - 6 = -11(x - 1)$ or $y = -11x + 17$.

53. $y = (2x+3)\left[2\left(x^4 - 5x^2 + 4\right)\right]$

$y' = (2x+3)\left[2\left(4x^3 - 10x\right)\right] + \left[2\left(x^4 - 5x^2 + 4\right)\right](2)$

$y'(0) = (3)(0) + [2(4)](2) = 16$

The tangent line is $y - 24 = 16(x - 0)$, or $y = 16x + 24$.

54. $y = \frac{x+1}{x^2(x-4)} = \frac{x+1}{x^3-4x^2}$

$y' = \frac{\left(x^3-4x^2\right)(1)-(x+1)\left(3x^2-8x\right)}{\left(x^3-4x^2\right)^2}$

$y'(2) = \frac{(-8)(1)-(3)(-4)}{(-8)^2} = \frac{4}{64} = \frac{1}{16}$

The tangent line is $y + \frac{3}{8} = \frac{1}{16}(x-2)$, or $y = \frac{1}{16}x - \frac{1}{2}$.

55. $y = \frac{x}{2x-6}$

$y' = \frac{(2x-6)(1) - x(2)}{(2x-6)^2} = \frac{-6}{(2x-6)^2}$

If $x = 1$, then $y = \frac{1}{2-6} = -\frac{1}{4}$ and

$y' = \frac{-6}{(-4)^2} = \frac{-6}{16} = -\frac{3}{8}$.

Thus $\frac{y'}{y} = \frac{-\frac{3}{8}}{-\frac{1}{4}} = \frac{3}{2} = 1.5$.

56. $y = \frac{1-x}{1+x}$

$y' = \frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2} = -\frac{2}{(1+x)^2}$

When $x = 5$, then $\frac{y'}{y} = \frac{-\frac{1}{18}}{-\frac{2}{3}} = \frac{1}{12}$.

57. $s = \frac{2}{t^3+1}$. When $t = 1$, then $s = 1$ m.

$v = \frac{ds}{dt} = \frac{(t^3+1)(0) - 2(3t^2)}{(t^3+1)^2} = -\frac{6t^2}{(t^3+1)^2}$

If $t = 1$, then $v = -\frac{6}{4} = -1.5$ m/s.

58. $s = \frac{t+3}{t^2+7}$

$v = \frac{ds}{dt} = \frac{(t^2+7)(1) - (t+3)(2t)}{(t^2+7)^2}$

$= \frac{7 - 6t - t^2}{(t^2+7)^2} = \frac{(7+t)(1-t)}{(t^2+7)^2}$

$v = 0$ when $t = -7$ or $t = 1$. Since t is positive, we choose $t = 1$.

59. $p = 50 - 0.01q$

$r = pq = 50q - 0.01q^2$

$\frac{dr}{dq} = 50 - 0.02q$

60. $p = \frac{500}{q}$

$r = pq = 500$

$\frac{dr}{dq} = 0$

61. $p = \frac{108}{q+2} - 3$

$r = pq = \frac{108q}{q+2} - 3q$

$\frac{dr}{dq} = \frac{(q+2)(108) - (108q)(1)}{(q+2)^2} - 3$

$= \frac{216}{(q+2)^2} - 3$

62. $p = \frac{q+750}{q+50}$

$r = pq = \frac{q^2 + 750q}{q+50}$

$\frac{dr}{dq} = \frac{(q+50)(2q+750) - \left(q^2 + 750q\right)(1)}{(q+50)^2}$

$= \frac{q^2 + 100q + 37{,}500}{(q+50)^2}$

63. $\frac{dC}{dI} = 0.672$

64. $\frac{dC}{dI} = 0.712$

65. $C = 3 + I^{1/2} + 2I^{1/3}$

$\frac{dC}{dI} = 0 + \frac{1}{2}I^{-1/2} + \frac{2}{3}I^{-2/3} = \frac{1}{2\sqrt{I}} + \frac{2}{3\sqrt[3]{I^2}}$

When $I = 1$, then $\frac{dC}{dI} = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$.

$\frac{dS}{dI} = 1 - \frac{dC}{dI} = 1 - \frac{1}{2\sqrt{I}} - \frac{2}{3\sqrt[3]{I^2}}$

When $I = 1$, then $1 - \frac{dC}{dI} = 1 - \frac{7}{6} = -\frac{1}{6}$.

66. $\frac{dC}{dI} = \frac{3}{4} - \frac{1}{6\sqrt{I}}$

$\left.\frac{dC}{dI}\right|_{I=25} = \frac{43}{60}$, so $\left.\frac{dS}{dI}\right|_{I=25} = 1 - \frac{43}{60} = \frac{17}{60}$

67. $\frac{dC}{dI} = \frac{\left(\sqrt{I}+4\right)\left(\frac{8}{\sqrt{I}} + 1.2\sqrt{I} - 0.2\right) - \left(16\sqrt{I} + 0.8\sqrt{I^3} - 0.2I\right)\left(\frac{1}{2\sqrt{I}}\right)}{\left(\sqrt{I}+4\right)^2}$

$\left.\frac{dC}{dI}\right|_{I=36} \approx 0.615$, so $\frac{dS}{dI} \approx 1 - 0.615 = 0.385$ when $I = 36$.

68. $\frac{dC}{dI} = \frac{\left(\sqrt{I}+5\right)\left(\frac{10}{\sqrt{I}} + 0.75\sqrt{I} - 0.4\right) - \left(20\sqrt{I} + 0.5\sqrt{I^3} - 0.4I\right)\left(\frac{1}{2\sqrt{I}}\right)}{\left(\sqrt{I}+5\right)^2}$

$\left.\frac{dC}{dI}\right|_{I=100} \approx 0.393$, so $\frac{dS}{dI} \approx 1 - 0.393 = 0.607$ when $I = 100$.

69. Simplifying gives $C = 10 + 0.7I - 0.2I^{\frac{1}{2}}$

a. $\frac{dC}{dI} = 0.7 - 0.1I^{-\frac{1}{2}} = 0.7 - \frac{0.1}{\sqrt{I}}$

$\frac{dS}{dI} = 1 - \frac{dC}{dI} = 0.3 + \frac{0.1}{\sqrt{I}}$

$\left.\frac{dS}{dI}\right|_{I=25} = 0.3 + \frac{0.1}{5} = 0.32$

b. $\frac{\frac{dC}{dI}}{C}$ when $I = 25$ is $\frac{0.7 - \frac{0.1}{5}}{10 + 0.7(25) - 0.2(5)} \approx 0.026$

70. Simplifying S gives

$S = \frac{I - 2\sqrt{I} - 8}{\sqrt{I}+2} = \frac{\left(\sqrt{I}+2\right)\left(\sqrt{I}-4\right)}{\sqrt{I}+2} = \sqrt{I} - 4$

Thus $\frac{dS}{dI} = \frac{1}{2}I^{-1/2} = \frac{1}{2\sqrt{I}}$.

$\left.\frac{dS}{dI}\right|_{I=150} = \frac{1}{2 \cdot \sqrt{150}} \approx 0.04082$ and $\left.\frac{dC}{dI}\right|_{I=150} \approx 1 - 0.04082 \approx 0.9592$.

71. $\frac{dc}{dq} = 6 \cdot \frac{(q+2)(2q) - q^2(1)}{(q+2)^2} = 6 \cdot \frac{q^2 + 4q}{(q+2)^2} = \frac{6q(q+4)}{(q+2)^2}$

72. We assume that $\frac{d}{dq}(\overline{c}) = 0$. Thus $0 = \frac{d\overline{c}}{dq} = \frac{d}{dq}\left(\frac{c}{q}\right) = \frac{q \cdot \frac{dc}{dq} - c(1)}{q^2}$.

This implies that $q \cdot \frac{dc}{dq} - c = 0$, $q \cdot \frac{dc}{dq} = c$, $\frac{dc}{dq} = \frac{c}{q} = \overline{c}$, so the marginal cost function $\left(\frac{dc}{dq}\right)$ and the average cost function $(\overline{c})$ are equal.

73. $y = \frac{900x}{10 + 45x}$

$\frac{dy}{dx} = \frac{(10 + 45x)(900) - (900x)(45)}{(10 + 45x)^2}$

$\left.\frac{dy}{dx}\right|_{x=2} = \frac{(100)(900) - (1800)(45)}{(100)^2} = \frac{9}{10}$

74. $\text{RT} = \frac{0.05V}{A + xV}$

$\frac{d}{dV}(\text{RT}) = \frac{(A + xV)(0.05) - (0.05V)(x)}{(A + xV)^2}$

$= \frac{0.05A}{(A + xV)^2}$

Both numerator and denominator are always positive, so $\frac{d}{dV}(\text{RT}) > 0$. This rate of change means that if V increases by one unit, RT increases.

75. $y = \frac{0.7355x}{1 + 0.02744x}$

$\frac{dy}{dx} = \frac{(1 + 0.02744x)(0.7355) - (0.7355x)(0.02744)}{(1 + 0.02744x)^2}$

$= \frac{0.7355}{(1 + 0.02744x)^2}$

76. $f(x) = \frac{a(1 + x) - b(2 + n)x}{a(2 + n)(1 + x) - b(2 + n)x}$

For convenience let $c = 2 + n$.

Then $f(x) = \frac{a(1 + x) - bcx}{ac(1 + x) - bcx} = \frac{1}{c} \cdot \frac{a(1 + x) - bcx}{a(1 + x) - bx}$.

$f'(x) = \frac{1}{c} \cdot \frac{[a(1 + x) - bx](a - bc) - [a(1 + x) - bcx](a - b)}{[a(1 + x) - bx]^2}$

$= \frac{1}{c} \cdot \frac{-abc + ab}{[a(1 + x) - bx]^2} = \frac{1}{c} \cdot \frac{(-c + 1)ab}{[a(1 + x) - bx]^2}$

$= \frac{1}{2 + n} \cdot \frac{[-1(2 + n) + 1]ab}{[a(1 + x) - bx]^2} = \frac{-(1 + n)ab}{[a(1 + x) - bx]^2 (2 + n)}$

$g(x) = \frac{A + Bx}{C + Dx}$

$$g'(x) = \frac{(C+Dx)(B)-(A+Bx)(D)}{(C+Dx)^2} = \frac{CB+BDx+AD-BDx}{(C+Dx)^2} = \frac{BC-AD}{(C+Dx)^2}$$

Thus, $g'(x)$ has the form given. When $g'(x)$ is defined $\left(\text{for } x \neq \frac{C}{D}\right)$, its sign is constant.

77. $\frac{d\overline{c}}{dq} = \frac{d}{dq}\left(\frac{c}{q}\right) = \frac{q \cdot \frac{dc}{dq} - c(1)}{q^2}$. When $q = 20$ we have $\frac{\frac{d\overline{c}}{dq}}{\overline{c}} = \frac{\frac{q \cdot \frac{dc}{dq} - c}{q^2}}{\overline{c}} = \frac{\frac{20(125)-20(150)}{(20)^2}}{150} = -\frac{1}{120}$

78. $\frac{dy}{dx} = (3)(2x-1)(x-4) + (3x+1)(2)(x-4) + (3x+1)(2x-1)(1)$
$= 18x^2 - 50x + 3$

Principles in Practice 11.5

1. By the chain rule,
$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{d}{dx}\left(4x^2\right) \cdot \frac{d}{dt}(6t) = (8x)(6) = 48x$.
Since $x = 6t$, $\frac{dy}{dt} = 48(6t) = 288t$.

Problems 11.5

1. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (2u-2)(2x-1) = \left[2\left(x^2-x\right)-2\right](2x-1) = \left(2x^2-2x-2\right)(2x-1) = 4x^3 - 6x^2 - 2x + 2$

2. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left(6u^2-8\right)\left(7-3x^2\right) = 2\left(3x^6 - 42x^4 + 147x^2 - 4\right)\left(7-3x^2\right)$

3. $\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{dx} = \left(-\frac{2}{w^3}\right)(-1) = \frac{2}{w^3} = \frac{2}{(2-x)^3}$

4. $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{4}z^{-3/4}(5x^4 - 4x^3) = \frac{5x^4 - 4x^3}{4\left(\sqrt[4]{(x^5 - x^4 + 3)^3}\right)}$

5. $\frac{dw}{dt} = \frac{dw}{du} \cdot \frac{du}{dt} = (3u^2)\left[\frac{(t+1)-(t-1)}{(t+1)^2}\right] = 3u^2\left[\frac{2}{(t+1)^2}\right]$. If $t = 1$, then $u = \frac{1-1}{1+1} = 0$, so $\left.\frac{dw}{dt}\right|_{t=1} = 3(0)^2\left[\frac{2}{4}\right] = 0$.

6. $\frac{dz}{ds} = \frac{dz}{du} \cdot \frac{du}{ds} = \left(2u + \frac{1}{2\sqrt{u}}\right)(4s)$. If $s = -1$, then $u = 1$, so $\left.\frac{dz}{ds}\right|_{s=-1} = \left(\frac{5}{2}\right)(-4) = -10$

7. $\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{dx} = (6w-8)(4x)$. If $x = 0$, then $\frac{dy}{dx} = 0$.

8. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left(9u^2 - 2u + 7\right)(5)$. If $x = 1$, then $u = 3$, so $\left.\frac{dy}{dx}\right|_{x=1} = (82)(5) = 410$

9. $y' = 6(3x+2)^5 \cdot \frac{d}{dx}(3x+2)$
$= 6(3x+2)^5(3) = 18(3x+2)^5$

10. $y' = 4\left(x^2-4\right)^3 \cdot \frac{d}{dx}\left(x^2-4\right)$
$= 4\left(x^2-4\right)^3(2x) = 8x\left(x^2-4\right)^3$

11. $y' = 5(3+2x^3)^4 \cdot \frac{d}{dx}(3+2x^3)$
$= 5(3+2x^3)^4(6x^2)$
$= 30x^2(3+2x^3)$

12. $y' = 4(x^2+x)^3 \frac{d}{dx}(x^2+x)$
$= 4(x^2+x)^3(2x+1)$
$= 4(2x+1)(x^2+x)^3$

13. $y' = 2 \cdot 100\left(x^3 - 8x^2 + x\right)^{99} \cdot \frac{d}{dx}\left(x^3 - 8x^2 + x\right)$
$= 200\left(x^3 - 8x^2 + x\right)^{99}\left(3x^2 - 16x + 1\right)$
$= 200\left(3x^2 - 16x + 1\right)\left(x^3 - 8x^2 + x\right)^{99}$

14. $y = \frac{\left(2x^2+1\right)^4}{2} = \frac{1}{2}\left(2x^2+1\right)^4$
$y' = \frac{1}{2} \cdot 4\left(2x^2+1\right)^3 \frac{d}{dx}(2x^2+1)$
$= 2(2x^2+1)^3(4x) = 8x\left(2x^2+1\right)^3$

15. $y' = -3\left(x^2-2\right)^{-4} \cdot \frac{d}{dx}\left(x^2-2\right)$
$= -3\left(x^2-2\right)^{-4}(2x) = -6x\left(x^2-2\right)^{-4}$

16. $y' = -12(2x^3-8x)^{-13} \cdot \frac{d}{dx}(2x^3-8x)$
$= -12(6x^2-8)(2x^3-8x)^{-13}$

17. $y' = 2\left(-\frac{5}{7}\right)(x^2+5x-2)^{-12/7} \cdot \frac{d}{dx}(x^2+5x-2)$
$= -\frac{10}{7}(2x+5)(x^2+5x-2)^{-12/7}$

18. $y' = 4\left(-\frac{3}{2}\right)\left(7x - x^4\right)^{-\frac{5}{2}}\left(7 - 4x^3\right)$
$= -6\left(7 - 4x^3\right)\left(7x - x^4\right)^{-\frac{5}{2}}$

19. $y = \sqrt{5x^2 - x} = \left(5x^2 - x\right)^{\frac{1}{2}}$
$y' = \frac{1}{2}\left(5x^2 - x\right)^{-\frac{1}{2}}(10x-1)$
$= \frac{1}{2}(10x-1)\left(5x^2 - x\right)^{-\frac{1}{2}}$

20. $y = \sqrt{3x^2 - 7} = \left(3x^2 - 7\right)^{\frac{1}{2}}$
$y' = \frac{1}{2}\left(3x^2 - 7\right)^{-\frac{1}{2}}(6x) = 3x\left(3x^2 - 7\right)^{-\frac{1}{2}}$

21. $y = \sqrt[4]{2x-1} = (2x-1)^{\frac{1}{4}}$
$y' = \frac{1}{4}(2x-1)^{-\frac{3}{4}}(2) = \frac{1}{2}(2x-1)^{-\frac{3}{4}}$

22. $y = \sqrt[3]{8x^2 - 1} = \left(8x^2 - 1\right)^{\frac{1}{3}}$
$y' = \frac{1}{3}\left(8x^2 - 1\right)^{-\frac{2}{3}}(16x) = \frac{16}{3}x\left(8x^2 - 1\right)^{-\frac{2}{3}}$

23. $y = 2\sqrt[5]{\left(x^3+1\right)^2} = 2\left(x^3+1\right)^{\frac{2}{5}}$

$y' = 2\left(\frac{2}{5}\right)\left(x^3+1\right)^{-\frac{3}{5}}\left(3x^2\right) = \frac{12}{5}x^2\left(x^3+1\right)^{-\frac{3}{5}}$

24. $y = 7\sqrt[3]{(x^5-3)^5} = 7(x^5-3)^{5/3}$

$y' = 7\cdot\frac{5}{3}(x^5-3)^{2/3}(5x^4)$

$= \frac{175}{3}x^4(x^5-3)^{2/3}$

25. $y = \frac{6}{2x^2-x+1} = 6\left(2x^2-x+1\right)^{-1}$

$y' = 6(-1)\left(2x^2-x+1\right)^{-2}(4x-1)$

$= -6(4x-1)\left(2x^2-x+1\right)^{-2}$

26. $y = \frac{3}{x^4+2} = 3\left(x^4+2\right)^{-1}$

$y' = 3(-1)\left(x^4+2\right)^{-2}\left(4x^3\right) = -12x^3\left(x^4+2\right)^{-2}$

27. $y = \frac{1}{\left(x^2-3x\right)^2} = \left(x^2-3x\right)^{-2}$

$y' = -2\left(x^2-3x\right)^{-3}(2x-3)$

$= -2(2x-3)\left(x^2-3x\right)^{-3}$

28. $y = \frac{1}{(2+x)^4} = (2+x)^{-4}$

$y' = -4(2+x)^{-5}(1) = -4(2+x)^{-5}$

29. $y = \frac{4}{\sqrt{9x^2+1}} = 4(9x^2+1)^{-1/2}$

$y' = 4\left(-\frac{1}{2}\right)(9x^2+1)^{-3/2}(18x)$

$= -36x(9x^2+1)^{-3/2}$

30. $y = \frac{3}{\left(3x^2-x\right)^{\frac{2}{3}}} = 3\left(3x^2-x\right)^{-\frac{2}{3}}$

$y' = 3\left(-\frac{2}{3}\right)\left(3x^2-x\right)^{-\frac{5}{3}}(6x-1)$

$= -2(6x-1)\left(3x^2-x\right)^{-\frac{5}{3}}$

31. $y = \sqrt[3]{7x} + \sqrt[3]{7}x = (7x)^{\frac{1}{3}} + \sqrt[3]{7}x$

$y' = \frac{1}{3}(7x)^{-\frac{2}{3}}(7) + \sqrt[3]{7}(1) = \frac{7}{3}(7x)^{-\frac{2}{3}} + \sqrt[3]{7}$

32. $y = \sqrt{2x} + \frac{1}{\sqrt{2x}} = (2x)^{\frac{1}{2}} + (2x)^{-\frac{1}{2}}$

$y' = \left(\frac{1}{2}\right)(2x)^{-\frac{1}{2}}(2) + \left(-\frac{1}{2}\right)(2x)^{-\frac{3}{2}}(2)$

$= (2x)^{-\frac{1}{2}} - (2x)^{-\frac{3}{2}}$

33. $y' = x^2\left[5(x-4)^4(1)\right] + (x-4)^5(2x)$

$= x(x-4)^4[5x+2(x-4)]$

$= x(x-4)^4(7x-8)$

34. $y' = x\left[4(x+4)^3(1)\right] + (x+4)^4(1)$

$= (x+4)^3(4x+x+4) = (x+4)^3(5x+4)$

35. $y = 4x^2\sqrt{5x+1} = 4x^2(5x+1)^{\frac{1}{2}}$

$y' = 4x^2\left(\frac{1}{2}(5x+1)^{-\frac{1}{2}}(5)\right) + \sqrt{5x+1}(8x)$

$= 10x^2(5x+1)^{-\frac{1}{2}} + 8x\sqrt{5x+1}$

36. $y = 4x^3\sqrt{1-x^2} = 4x^3(1-x^2)^{\frac{1}{2}}$

$y' = 4x^3\left[\left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x)\right] + \sqrt{1-x^2}(12x^2)$

$= -\frac{4x^4}{\sqrt{1-x^2}} + 12x^2\sqrt{1-x^2}$

37. $y' = \left(x^2+2x-1\right)^3(5) + (5x)\left[3\left(x^2+2x-1\right)^2(2x+2)\right]$

$= 5\left(x^2+2x-1\right)^2\left[\left(x^2+2x-1\right) + 3x(2x+2)\right]$

$= 5\left(x^2+2x-1\right)^2\left(7x^2+8x-1\right)$

38. $y' = x^2\left[4\left(x^3-1\right)^3\left(3x^2\right)\right] + \left(x^3-1\right)^4(2x)$

$= 2x\left(x^3-1\right)^3\left[6x^3+\left(x^3-1\right)\right] = 2x\left(7x^3-1\right)\left(x^3-1\right)^3$

39. $y' = (8x-1)^3\left[4(2x+1)^3(2)\right] + (2x+1)^4\left[3(8x-1)^2(8)\right]$

$= 8(8x-1)^2(2x+1)^3[(8x-1)+3(2x+1)]$

$= 8(8x-1)^2(2x+1)^3(14x+2)$

$= 16(8x-1)^2(2x+1)^3(7x+1)$

40. $y' = (3x+2)^5[2(4x-5)(4)] + (4x-5)^2[5(3x+2)^4(3)]$

$= (3x+2)^4(4x-5)[8(3x+2)+15(4x-5)]$

$= (3x+2)^4(4x-5)(84x-59)$

41. $y' = 12\left(\frac{x-3}{x+2}\right)^{11}\left[\frac{(x+2)(1)-(x-3)(1)}{(x+2)^2}\right]$

$= 12\left(\frac{x-3}{x+2}\right)^{11}\left[\frac{5}{(x+2)^2}\right]$

$= \frac{60(x-3)^{11}}{(x+2)^{13}}$

42. $y' = 4\left(\frac{2x}{x+2}\right)^3\left[\frac{(x+2)(2)-2x(1)}{(x+2)^2}\right] = \frac{128x^3}{(x+2)^5}$

43. $y' = \frac{1}{2}\left(\frac{x-2}{x+3}\right)^{-\frac{1}{2}}\left[\frac{(x+3)(1)-(x-2)(1)}{(x+3)^2}\right]$

$= \frac{5}{2(x+3)^2}\left(\frac{x-2}{x+3}\right)^{-\frac{1}{2}} = \frac{5}{2(x+3)^2}\sqrt{\frac{x+3}{x-2}}$

44. $y' = \frac{1}{3}\left(\frac{8x^2-3}{x^2+2}\right)^{-\frac{2}{3}}\left[\frac{\left(x^2+2\right)(16x)-\left(8x^2-3\right)(2x)}{\left(x^2+2\right)^2}\right]$

$= \frac{1}{3}\left(\frac{8x^2-3}{x^2+2}\right)^{-\frac{2}{3}}\frac{38x}{\left(x^2+2\right)^2}$

$= \frac{38x}{3\left(8x^2-3\right)^{\frac{2}{3}}\left(x^2+2\right)^{\frac{4}{3}}}$

45. $y' = \frac{\left(x^2+4\right)^3(2)-(2x-5)\left[3\left(x^2+4\right)^2(2x)\right]}{\left(x^2+4\right)^6}$

$= \frac{\left(x^2+4\right)^2\left\{\left(x^2+4\right)(2)-(2x-5)[3(2x)]\right\}}{\left(x^2+4\right)^6}$

$= \frac{2x^2+8-12x^2+30x}{\left(x^2+4\right)^4} = \frac{-10x^2+30x+8}{\left(x^2+4\right)^4}$

$= \frac{-2\left(5x^2-15x-4\right)}{\left(x^2+4\right)^4}$

46. $y' = \frac{(3x^2+7)[4(4x-2)^3(4)]-(4x-2)^4(6x)}{(3x^2+7)^2}$

$= \frac{(4x-2)^3[16(3x^2+7)-6x(4x-2)]}{(3x^2+7)^2}$

$= \frac{(4x-2)^3(24x^2+12x+112)}{(3x^2+7)^2}$

47. $y' = \frac{(3x-1)^3\left[5(8x-1)^4(8)\right]-(8x-1)^5\left[3(3x-1)^2(3)\right]}{(3x-1)^6}$

$= \frac{(3x-1)^2(8x-1)^4[(3x-1)(40)-(8x-1)(9)]}{(3x-1)^6}$

$= \frac{(8x-1)^4(48x-31)}{(3x-1)^4}$

48. $y = \sqrt[3]{(x-2)^2(x+2)} = [(x-2)^2(x+2)]^{1/3}$

$$y' = \frac{1}{3}[(x-2)^2(x+2)]^{-2/3}[(1)(x-2)^2 + 2(x-2)(x+2)]$$
$$= \frac{1}{3}[(x-2)^2(x+2)]^{-2/3}(x-2)[x-2+2(x+2)]$$
$$= \frac{1}{3}[(x-2)^2(x+2)]^{-2/3}(x-2)(3x+2)$$
$$= \frac{1}{3}(x-2)^{-1/3}(x+2)^{-2/3}(3x+2)$$

49. $y = 6\left(5x^2+2\right)\sqrt{x^4+5} = 6\left[\left(5x^2+2\right)\left(x^4+5\right)^{\frac{1}{2}}\right]$

$$y' = 6\left[\left(5x^2+2\right)\cdot\frac{1}{2}\left(x^4+5\right)^{-\frac{1}{2}}\left(4x^3\right)+\left(x^4+5\right)^{\frac{1}{2}}(10x)\right]$$
$$= 6\left[\left(5x^2+2\right)\left(x^4+5\right)^{-\frac{1}{2}}\left(2x^3\right)+\left(x^4+5\right)^{\frac{1}{2}}(10x)\right]$$
$$= 12x\left[\left(5x^2+2\right)\left(x^4+5\right)^{-\frac{1}{2}}\left(x^2\right)+\left(x^4+5\right)^{\frac{1}{2}}(5)\right]$$

Factoring out $\left(x^4+5\right)^{-\frac{1}{2}}$ gives

$$y' = 12x\left(x^4+5\right)^{-\frac{1}{2}}\left[\left(5x^2+2\right)\left(x^2\right)+\left(x^4+5\right)(5)\right]$$
$$= 12x\left(x^4+5\right)^{-\frac{1}{2}}\left(10x^4+2x^2+25\right)$$

50. $y' = 3-4\left[x(2)(7x+1)(7)+(7x+1)^2(1)\right]$

$$= 3-4\left[147x^2+28x+1\right] = -588x^2-112x-1$$

51. $y' = 8+\dfrac{(t+4)(1)-(t-1)(1)}{(t+4)^2}-2\left(\dfrac{8t-7}{4}\right)\left(\dfrac{1}{4}\cdot 8\right)$

$$= 8+\frac{5}{(t+4)^2}-(8t-7) = 15-8t+\frac{5}{(t+4)^2}$$

52. $y = \frac{(2x^3+6)(7x-5)}{(2x+4)^2} = \frac{14x^4-10x^3+42x-30}{(2x+4)^2}$

$$y' = \frac{(2x+4)^2(56x^3-30x^2+42)-(14x^4-10x^3+42x-30)[2(2x+4)(2)]}{(2x+4)^4}$$
$$= \frac{(2x+4)[(2x+4)(56x^3-30x^2+42)-4(14x^4-10x^3+42x-30)]}{(2x+4)^4}$$
$$= \frac{112x^4-60x^3+84x+224x^3-120x^2+168-56x^4-40x^3-168x+120}{(2x+4)^3}$$
$$= \frac{4(14x^4+51x^3-30x^2-21x+72)}{(2x+4)^3}$$

53. $y' = \frac{(x^3-5)^5[(2x+1)^3(2)(x+3)(1)+(x+3)^2(3)(2x+1)^2(2)]-(2x+1)^3(x+3)^2[5(x^3-5)^4(3x^2)]}{(x^3-5)^{10}}$

54. $y' = \frac{(9x-3)\left[\sqrt{x+2}(2)\left(4x^2-1\right)(8x)+\left(4x^2-1\right)^2\left(\frac{1}{2}\right)(x+2)^{-\frac{1}{2}}\right]-\sqrt{x+2}\left(4x^2-1\right)^2(9)}{(9x-3)^2}$

55. $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \left[3(5u+6)^2(5)\right]\left[4\left(x^2+1\right)^3(2x)\right]$

When $x = 0$, then $\frac{dy}{dx} = 0$.

56. $\frac{dz}{dt} = \frac{dz}{dy}\cdot\frac{dy}{dx}\cdot\frac{dx}{dt} = (4y-4)(6)(2)$

When $t = 1$, then $x = 2$ and $y = 7$. Thus $\left.\frac{dz}{dt}\right|_{t=1} = (24)(6)(2) = 288$.

57. $y' = 3\left(x^2-7x-8\right)^2(2x-7)$

If $x = 8$, then slope $= y' = 3(64-56-8)^2(16-7) = 0$.

58. $y = (x+1)^{\frac{1}{2}}$

$y' = \frac{1}{2}(x+1)^{-\frac{1}{2}}$

If $x = 8$, $y' = \frac{1}{6}$.

59. $y = \left(x^2 - 8\right)^{\frac{2}{3}}$

$y' = \frac{2}{3}\left(x^2 - 8\right)^{-\frac{1}{3}}(2x) = \frac{4x}{3\left(x^2 - 8\right)^{\frac{1}{3}}}$

If $x = 3$, then $y' = \frac{12}{3(1)} = 4$. Thus the tangent line is $y - 1 = 4(x - 3)$, or $y = 4x - 11$.

60. $y' = 3(x+3)^2(1) = 3(x+3)^2$

If $x = -1$, $y' = 3(2)^2 = 12$.

The tangent line is $y - 8 = 12(x + 1)$ or $y = 12x + 20$.

61. $y' = \frac{(x+1)\left(\frac{1}{2}\right)(7x+2)^{-\frac{1}{2}}(7) - \sqrt{7x+2}(1)}{(x+1)^2}$

$= \frac{(x+1)\left(\frac{7}{2}\right)\frac{1}{\sqrt{7x+2}} - \sqrt{7x+2}}{(x+1)^2}$

If $x = 1$, then $y' = \frac{2\left(\frac{7}{2}\right)\left(\frac{1}{3}\right) - 3}{4} = -\frac{1}{6}$. The tangent line is $y - \frac{3}{2} = -\frac{1}{6}(x-1)$, or $y = -\frac{1}{6}x + \frac{5}{3}$.

62. $y = -3\left(3x^2 + 1\right)^{-3}$

$y' = -3(-3)\left(3x^2 + 1\right)^{-4}(6x)$

If $x = 0$, then $y' = 0$. The tangent line is $y + 3 = 0(x - 0)$, or $y = -3$.

63. $y = \left(x^2 + 9\right)^3$ and $y' = 6x\left(x^2 + 9\right)^2$. When $x = 4$, then $y = (25)^3$ and $y' = 6(4)(25)^2$, so

$\frac{y'}{y}(100) = \frac{6(4)(25)^2}{(25)^3}(100) = \frac{24}{25}(100) = 96\%$

64. $y = \frac{1}{(x^2 - 1)^3}$ and $y' = -\frac{6x}{(x^2 - 1)^4}$

When $x = 2$, $y = \frac{1}{27}$ and $y' = -\frac{12}{3^4} = -\frac{4}{27}$, so

$\left(\frac{y'}{y}\right)(100) = -\frac{4}{27} \cdot 27(100) = -400\%$

65. $q = 5m$, $p = -0.4q + 50$; $m = 6$

$\frac{dr}{dm} = \frac{dr}{dq} \cdot \frac{dq}{dm}$

$r = pq = -0.4q^2 + 50q$, $\frac{dr}{dq} = -0.8q + 50$, . For $m = 6$, then $q = 30$, so $\left.\frac{dr}{dq}\right|_{m=6} = -24 + 50 = 26.$

Also, $\frac{dq}{dm} = 5.$ Thus $\left.\frac{dr}{dm}\right|_{m=6} = (26)(5) = 130.$

66. $q = \frac{1}{20}\left(200m - m^2\right)$

$p = -0.1q + 70$; $m = 40$

$\frac{dr}{dm} = \frac{dr}{dq} \cdot \frac{dq}{dm}$

$r = pq = -0.1q^2 + 70q$, so $\frac{dr}{dq} = -0.2q + 70$. If $m = 40$, then $q = 320$, so

$\left.\frac{dr}{dq}\right|_{m=40} = -64 + 70 = 6$.

$\frac{dq}{dm} = \frac{1}{20}(200 - 2m)$. When $m = 40$, $\frac{dq}{dm} = 6$.

Thus $\left.\frac{dr}{dm}\right|_{m=40} = (6)(6) = 36$.

67. $q = \frac{10m^2}{\sqrt{m^2 + 9}}$

$p = \frac{525}{q+3}$; $m = 4$

$\frac{dr}{dm} = \frac{dr}{dq} \cdot \frac{dq}{dm}$

$r = pq = \frac{525q}{q+3}$, so

$\frac{dr}{dq} = 525 \cdot \frac{(q+3)(1) - q(1)}{(q+3)^2} = \frac{1575}{(q+3)^2}$.

If $m = 4$, then $q = 32$, so $\left.\frac{dr}{dq}\right|_{m=4} = \frac{1575}{1225} = \frac{9}{7}$.

$$\frac{dq}{dm} = \frac{\left(m^2+9\right)^{\frac{1}{2}}(20m) - 10m^2 \cdot \frac{1}{2}\left(m^2+9\right)^{-\frac{1}{2}}(2m)}{m^2+9}$$

$$= \frac{\left(m^2+9\right)^{-\frac{1}{2}}\left[20m\left(m^2+9\right) - 10m^3\right]}{m^2+9}$$

$$= \frac{10m^3+180m}{\left(m^2+9\right)^{\frac{3}{2}}}$$

When $m = 4$, then

$\frac{dq}{dm} = \frac{10(64)+180(4)}{(25)^{\frac{3}{2}}} = \frac{1360}{125} = \frac{272}{25}$. Thus

$\left.\frac{dr}{dm}\right|_{m=4} = \frac{9}{7} \cdot \frac{272}{25} \approx 13.99$.

68. $q = \frac{100m}{\sqrt{m^2+19}}$

$p = \frac{4500}{q+10}$; $m = 9$

$\frac{dr}{dm} = \frac{dr}{dq} \cdot \frac{dq}{dm}$

$r = pq = \frac{4500q}{q+10}$, so $\frac{dr}{dq} = \frac{45,000}{(q+10)^2}$.

If $m = 9$, then $q = 90$, so $\left.\frac{dr}{dq}\right|_{m=9} = \frac{9}{2}$.

$\frac{dq}{dm} = \frac{1900}{\left(m^2+19\right)^{\frac{3}{2}}}$. When $m = 9$, then $\frac{dq}{dm} = \frac{19}{10}$.

Thus $\left.\frac{dr}{dm}\right|_{m=9} = \frac{9}{2} \cdot \frac{19}{10} = 8.55$.

69. a. $\frac{dp}{dq} = 0 - \frac{1}{2}\left(q^2+20\right)^{-\frac{1}{2}}(2q) = \frac{-q}{\sqrt{q^2+20}}$

b. $$\frac{\frac{dp}{dq}}{p} = \frac{\frac{-q}{\sqrt{q^2+20}}}{100-\sqrt{q^2+20}}$$

$$= -\frac{q}{\sqrt{q^2+20}\left(100-\sqrt{q^2+20}\right)}$$

$$= -\frac{q}{100\sqrt{q^2+20} - q^2 - 20}$$

c. $r = pq = 100q - q\sqrt{q^2+20}$

$\frac{dr}{dq}$

$$= 100 - \left[q \cdot \frac{1}{2}\left(q^2+20\right)^{-\frac{1}{2}}(2q) + \sqrt{q^2+20}(1)\right]$$

$$= 100 - \frac{q^2}{\sqrt{q^2+20}} - \sqrt{q^2+20}$$

70. $p = \frac{k}{q}$; $q = f(m)$

$\frac{dr}{dm} = \frac{dr}{dq} \cdot \frac{dq}{dm}$

$r = pq = k$, so $\frac{dr}{dq} = 0$. Thus $\frac{dr}{dm} = 0 \cdot \frac{dq}{dm} = 0$.

71. $\frac{dc}{dp} = \frac{dc}{dq} \cdot \frac{dq}{dp} = (12+0.4q)(-1.5)$

When $p = 85$, then $q = 772.5$, so

$\left.\frac{dc}{dp}\right|_{p=85} = -481.5.$

72. $f(t) = 1 - \left(\frac{250}{250+t}\right)^3$

$$f'(t) = -3\left(\frac{250}{250+t}\right)^2\left[-\frac{250}{(250+t)^2}\right]$$

$$f'(100) = -3\left(\frac{250}{350}\right)^2\left[-\frac{250}{350^2}\right]$$

$$= -3\left(\frac{25}{49}\right)\left(-\frac{1}{490}\right)$$

$$= \frac{15}{4802}.$$

Thus when t increases from 100 to 101, the proportion discharged increases by approximately $\frac{15}{4802}$.

73. $\frac{dc}{dq} = \frac{\left(q^2+3\right)^{\frac{1}{2}}(10q) - \left(5q^2\right)\left[\frac{1}{2}\left(q^2+3\right)^{-\frac{1}{2}}(2q)\right]}{q^2+3}$

Multiplying numerator and denominator by $\left(q^2+3\right)^{\frac{1}{2}}$ gives

$\frac{dc}{dq} = \frac{\left(q^2+3\right)(10q) - 5q^2(q)}{\left(q^2+3\right)^{\frac{3}{2}}} = \frac{5q^3+30q}{\left(q^2+3\right)^{\frac{3}{2}}} = \frac{5q\left(q^2+6\right)}{\left(q^2+3\right)^{\frac{3}{2}}}$.

74. a. $\frac{dS}{dE} = 680E - 4360$. If $E = 16$, $\frac{dS}{dE} = 6520$.

b. Solving $680E - 4360 = 5000$ gives $680E = 9360, E \approx 13.8$.

75. $\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} = \left(4\pi r^2\right)\left[10^{-8}(2t)+10^{-7}\right]$. When $t = 10$, then $r = 10^{-8}\left(10^2\right)+10^{-7}(10) = 10^{-6}+10^{-6} = 2(10)^{-6}$.
Thus
$\left.\frac{dV}{dt}\right|_{t=10} = 4\pi\left[2(10)^{-6}\right]^2\left[10^{-8}(2)(10)+10^{-7}\right] = 4\pi\left[4(10)^{-12}\right]\left[3\left(10^{-7}\right)\right] = 48\pi(10)^{-19}$

76. a. $\frac{dp}{dI} = \frac{1}{2}(2\rho VI)^{-\frac{1}{2}}(2\rho V) = \rho V(2\rho VI)^{-\frac{1}{2}}$

b. $\frac{\frac{dp}{dI}}{p} = \frac{\rho V(2\rho VI)^{-\frac{1}{2}}}{(2\rho VI)^{\frac{1}{2}}} = \frac{1}{2I}$

77. a. $\frac{d}{dx}(I_x) = -0.001416x^3 + 0.01356x^3 + 1.696x - 34.9$

If $x = 65$, $\frac{d}{dx}(I_x) = -256.238$.

b. If $x = 65$, $\frac{\frac{d}{dx}(I_x)}{I_x} \approx \frac{-256.238}{16,236.484} \approx -0.01578$

If $x = 65$, the percentage rate of change is $\frac{\frac{d}{dx}(I_x)}{I_x}\cdot = \frac{-25,623.8}{16,236.484} = -1.578\%$.

78. $(P + a)(v + b) = k$

$v + b = \frac{k}{P+a}$

$v = \frac{k}{P+a} - b$

$v = k(P+a)^{-1} - b$

$\frac{dv}{dP} = k(-1)(P+a)^{-2} = -\frac{k}{(P+a)^2}$

79. By the chain rule, $\frac{dc}{dp} = \frac{dc}{dq} \cdot \frac{dq}{dp}$. We are given that $q = \frac{100}{p} = 100p^{-1}$, so $\frac{dq}{dp} = -100p^{-2} = \frac{-100}{p^2}$. Thus $\frac{dc}{dp} = \frac{dc}{dq}\left[\frac{-100}{p^2}\right]$. When $q = 200$, then $p = \frac{100}{200} = \frac{1}{2}$ and we are given that $\frac{dc}{dq} = 0.01$. Therefore $\frac{dc}{dp} = 0.01\left[\frac{-100}{\left(\frac{1}{2}\right)^2}\right] = -4$.

80. a. When $m = 12$, then $q = 3000$, so $r = 1500$.
Thus $p = \frac{r}{q} = \frac{1500}{3000} = \frac{1}{2} = \$0.50$.

b. $\frac{dr}{dq} = \frac{\sqrt{1000+3q}(50) - 50q\left(\frac{1}{2}\right)(1000+3q)^{-\frac{1}{2}}(3)}{1000+3q}$

$\left.\frac{dr}{dq}\right|_{q=3000} = \frac{2750}{10,000} = \frac{11}{40}$

c. $\frac{dr}{dm} = \frac{dr}{dq} \cdot \frac{dq}{dm}$. From part (b) we know $\frac{dr}{dq}$. Now,

$\frac{dq}{dm} = (2m)\left(\frac{3}{2}\right)(2m+1)^{\frac{1}{2}}(2) + (2m+1)^{\frac{3}{2}}(2)$, so $\left.\frac{dq}{dm}\right|_{m=12} = 610$.

Thus $\left.\frac{dr}{dm}\right|_{m=12} = \frac{11}{40} \cdot 610 = \frac{671}{4}$.

81. $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = f'(x)g'(t)$. We are given that $g(2) = 3$, so $x = 3$ when $t = 2$. Thus $\left.\frac{dy}{dt}\right|_{t=2} = \left.\frac{dy}{dx}\right|_{x=g(2)} \cdot \left.\frac{dx}{dt}\right|_{t=2} = f'(3)g'(2) = 10(4) = 40$.

82. a. $\lim_{q\to\infty} \bar{c} = \lim_{q\to\infty}\left(\frac{324}{\sqrt{q^2+35}} + \frac{5}{q} + \frac{19}{18}\right) = 0 + 0 + \frac{19}{18} = \frac{19}{18}$

b. $c = \bar{c}q = \frac{324q}{\sqrt{q^2+35}} + 5 + \frac{19}{18}q$

$\frac{dc}{dq} = \frac{\sqrt{q^2+35}(324) - 324q\left(\frac{1}{2}\right)\left(q^2+35\right)^{-\frac{1}{2}}(2q)}{q^2+35} + \frac{19}{18}$

$\left.\frac{dc}{dq}\right|_{q=17} = 3$

c. From part (b) the increase in cost of the additional unit is approximately \$300. Since the corresponding revenue increases by \$275, the move should not be made.

83. 86,111.37

84. 5.25

Chapter 11 Review Problems

1. $f(x) = 2 - x^2$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\left[2-(x+h)^2\right]-\left(2-x^2\right)}{h}$$
$$= \lim_{h\to 0}\frac{\left[2-x^2-2hx-h^2\right]-\left(2-x^2\right)}{h} = \lim_{h\to 0}\frac{-2hx-h^2}{h}$$
$$= \lim_{h\to 0}\frac{-h(2x+h)}{h} = \lim_{h\to 0} -(2x+h) = -2x$$

2. $f(x) = 2x^2 - 3x + 1$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{\left[2(x+h)^2-3(x+h)+1\right]-\left(2x^2-3x+1\right)}{h}$$
$$= \lim_{h\to 0}\frac{\left[2x^2+4hx+2h^2-3x-3h+1\right]-\left(2x^2-3x+1\right)}{h}$$
$$= \lim_{h\to 0}\frac{4hx+2h^2-3h}{h} = \lim_{h\to 0}\frac{h(4x+2h-3)}{h}$$
$$= \lim_{h\to 0}(4x+2h-3) = 4x-3$$

3. $f(x) = \sqrt{3x}$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{3(x+h)}-\sqrt{3x}}{h}$$
$$= \lim_{h\to 0}\frac{\sqrt{3(x+h)}-\sqrt{3x}}{h}\cdot\frac{\sqrt{3(x+h)}+\sqrt{3x}}{\sqrt{3(x+h)}+\sqrt{3x}}$$
$$= \lim_{h\to 0}\frac{3(x+h)-3x}{h\left(\sqrt{3(x+h)}+\sqrt{3x}\right)} = \lim_{h\to 0}\frac{3h}{h\left(\sqrt{3(x+h)}+\sqrt{3x}\right)}$$
$$= \lim_{h\to 0}\frac{3}{\sqrt{3(x+h)}+\sqrt{3x}}$$
$$= \frac{3}{\sqrt{3x}+\sqrt{3x}} = \frac{3}{2\sqrt{3x}} = \frac{\sqrt{3}}{2\sqrt{x}}$$

4. $f(x)=\frac{2}{1+4x}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{2}{1+4(x+h)}-\frac{2}{1+4x}}{h}$$

$$=\lim_{h\to 0}\frac{2(1+4x)-2[1+4(x+h)]}{h[1+4(x+h)](1+4x)}$$

$$=\lim_{h\to 0}\frac{-8h}{h[1+4(x+h)](1+4x)}$$

$$=\lim_{h\to 0}\frac{-8}{[1+4(x+h)](1+4x)}=\frac{-8}{[1+4(x)](1+4x)}$$

$$=-\frac{8}{(1+4x)^2}$$

5. y is a constant function, so $y'=0$.

6. $y'=e(1)x^{1-1}=ex^0=e$

7. $y'=7\left(4x^3\right)-6\left(3x^2\right)+5(2x)+0$

$$=28x^3-18x^2+10x=2x\left(14x^2-9x+5\right)$$

8. $y'=4(2x+0)-7(1)=8x-7$

9. $f(s)=s^2\left(s^2+2\right)=s^4+2s^2$

$$f'(s)=4s^3+2(2s)=4s^3+4s=4s\left(s^2+1\right)$$

10. $y=(x+3)^{\frac{1}{2}}$

$$y'=\frac{1}{2}(x+3)^{-\frac{1}{2}}(1)=\frac{1}{2}(x+3)^{-\frac{1}{2}}$$

11. $y=\frac{1}{5}\left(x^2+1\right)$

$$y'=\frac{1}{5}(2x)=\frac{2x}{5}$$

12. $y=-\frac{2}{2x^2}=-x^{-2}$, so $y'=-1(-2)x^{-3}=2x^{-3}$.

13. $y'=(x^3+7x^2)(3x^2-2x)+(x^3-x^2+5)(3x^2+14x)$

$$=3x^5+19x^4-14x^3+3x^5+11x^4-14x^3+15x^2+70x$$

$$=6x^5+30x^4-28x^3+15x^2+70x$$

14. $y'=\left(x^2+1\right)^{100}(1)+(x-6)(100)\left(x^2+1\right)^{99}(2x)=(x^2+1)^{99}[x^2+1+200x(x-6)]=\left(x^2+1\right)^{99}\left(201x^2-1200x+1\right)$

15. $f'(x) = 100\left(2x^2 + 4x\right)^{99}(4x+4) = 400(x+1)[(2x)(x+2)]^{99}$

16. $f(w) = w\sqrt{w} + w^2 = w^{\frac{3}{2}} + w^2 \quad f'(w) = \frac{3}{2}w^{\frac{1}{2}} + 2w$

17. $y = 3(2x+1)^{-1}$

$y' = 3(-1)(2x+1)^{-2}(2) = -\frac{6}{(2x+1)^2}$

18. $y = \frac{5x^2 - 8x}{2x} = \frac{5}{2}x - 4$

$y' = \frac{5}{2}$

19. $y' = (8+2x)\left[(4)\left(x^2+1\right)^3(2x)\right] + \left(x^2+1\right)^4(2)$

$= 2\left(x^2+1\right)^3\left[4x(8+2x) + \left(x^2+1\right)\right]$

$= 2\left(x^2+1\right)^3\left(32x + 8x^2 + x^2 + 1\right)$

$= 2\left(x^2+1\right)^3\left(9x^2 + 32x + 1\right)$

20. $g'(z) = \left(\frac{3}{5}\right)(2z)^{-\frac{2}{5}}(2) + 0 = \frac{6}{5}(2z)^{-\frac{2}{5}}$

21. $f'(z) = \frac{\left(z^2+4\right)(2z) - \left(z^2-1\right)(2z)}{\left(z^2+4\right)^2} = \frac{10z}{\left(z^2+4\right)^2}$

22. $y' = \frac{(x+2)^2(1) - (x-5)(2)(x+2)}{(x+2)^4} = \frac{12-x}{(x+2)^3}$

23. $y = (4x-1)^{\frac{1}{3}}$

$y' = \frac{1}{3}(4x-1)^{-\frac{2}{3}}(4) = \frac{4}{3}(4x-1)^{-\frac{2}{3}}$

24. *f* is a constant function, so $f'(x) = 0$.

25. $y = (1-x^2)^{-\frac{1}{2}}$

$y' = \left(-\frac{1}{2}\right)(1-x^2)^{-\frac{3}{2}}(-2x) = x(1-x^2)^{-\frac{3}{2}}$

26. $y = \dfrac{x^2 + x}{2x^2 + 3}$

$$y' = \frac{\left(2x^2+3\right)(2x+1) - \left(x^2+x\right)(4x)}{\left(2x^2+3\right)^2} = \frac{-2x^2+6x+3}{\left(2x^2+3\right)^2}$$

27. $h'(x) = (x-6)^4\left[3\left(x+5\right)^2\right] + (x+5)^3\left[4(x-6)^3\right]$

$= (x-6)^3(x+5)^2[3(x-6)+4(x+5)]$

$= (x-6)^3(x+5)^2(7x+2)$

28. $y' = \dfrac{x(5)(x+3)^4 - (x+3)^5(1)}{x^2} = \dfrac{(x+3)^4(4x-3)}{x^2}$

29. $y' = \dfrac{(x+6)(5) - (5x-4)(1)}{(x+6)^2} = \dfrac{34}{(x+6)^2}$

30. $f(x) = 5x^3\sqrt{3+2x^4} = 5x^3(3+2x^4)^{1/2}$

$f'(x) = (3+2x^4)^{1/2}(15x^2) + 5x^3\left[\dfrac{1}{2}(3+2x^4)^{-1/2}(8x^3)\right]$

$= 15x^2(3+2x^4)^{1/2} + 20x^6(3+2x^4)^{-1/2}$

31. $y' = 2\left(-\dfrac{3}{8}\right)x^{-\frac{11}{8}} + \left(-\dfrac{3}{8}\right)(2x)^{-\frac{11}{8}}(2) = -\dfrac{3}{4}x^{-\frac{11}{8}} - \dfrac{3}{4}\left(2^{-\frac{11}{8}}\right)x^{-\frac{11}{8}}$

$= -\dfrac{3}{4}x^{-\frac{11}{8}}\left(1+2^{-\frac{11}{8}}\right) = -\dfrac{3}{4}\left(1+2^{-\frac{11}{8}}\right)x^{-\frac{11}{8}}$

32. $y' = \dfrac{1}{2}\left(\dfrac{x}{2}\right)^{-\frac{1}{2}}\left(\dfrac{1}{2}\right) + \dfrac{1}{2}\left(\dfrac{2}{x}\right)^{-\frac{1}{2}}\left(-2x^{-2}\right) = \dfrac{1}{4}\left(\dfrac{2}{x}\right)^{\frac{1}{2}} - \dfrac{1}{x^2}\left(\dfrac{2}{x}\right)^{-\frac{1}{2}} = \left(\dfrac{2}{x}\right)^{-\frac{1}{2}}\left[\dfrac{1}{4}\left(\dfrac{2}{x}\right) - \dfrac{1}{x^2}\right]$

$= \left(\dfrac{x}{2}\right)^{\frac{1}{2}}\left[\dfrac{1}{2x} - \dfrac{1}{x^2}\right] = \dfrac{x-2}{2x^2}\sqrt{\dfrac{x}{2}}$

33. $y' = \dfrac{\left(x^2+5\right)^{\frac{1}{2}}(2x) - \left(x^2+6\right)\left(\frac{1}{2}\right)\left(x^2+5\right)^{-\frac{1}{2}}(2x)}{x^2+5}$

Multiplying the numerator and denominator by $\left(x^2+5\right)^{\frac{1}{2}}$ gives

$$y' = \frac{\left(x^2+5\right)(2x) - x\left(x^2+6\right)}{\left(x^2+5\right)^{\frac{3}{2}}} = \frac{x^3+4x}{\left(x^2+5\right)^{\frac{3}{2}}} = \frac{x\left(x^2+4\right)}{\left(x^2+5\right)^{\frac{3}{2}}}$$

34. $y = \left(7-3x^2\right)^{\frac{2}{3}}$

$y' = \frac{2}{3}\left(7-3x^2\right)^{-\frac{1}{3}}(-6x) = -4x\left(7-3x^2\right)^{-\frac{1}{3}}$

35. $y' = \frac{3}{5}\left(x^3+6x^2+9\right)^{-\frac{2}{5}}\left(3x^2+12x\right)$

$= \frac{3}{5}\left(x^3+6x^2+9\right)^{-\frac{2}{5}}(3x)(x+4)$

$= \frac{9}{5}x(x+4)\left(x^3+6x^2+9\right)^{-\frac{2}{5}}$

36. $z' = 0.4[x^2(-3)(x+1)^{-4}(1)+(x+1)^{-3}(2x)]+0$

$= 0.4(x+1)^{-4}[-3x^2+(x+1)(2x)]$

$= 0.4(x+1)^{-4}(-x^2+2x)$

37. $g(z) = -z(z-1)^2 = -z^3+2z^2-z$

$g'(z) = -3z^2+4z-1$

38. $g(z) = -\frac{3}{4}\left(z^5+2z-5\right)^{-4}$

$g'(z) = -\frac{3}{4}(-4)\left(z^5+2z-5\right)^{-5}\left(5z^4+2\right)$

$= \frac{3\left(5z^4+2\right)}{\left(z^5+2z-5\right)^5}$

39. $y = x^2-6x+4$

$y' = 2x-6$

When $x = 1$, then $y = -1$ and $y' = -4$. An equation of the tangent line is $y-(-1) = -4(x-1)$, or $y = -4x+3$.

40. $y = -2x^3+6x+1$

$y' = -6x^2+6$

When $x = 2$, then $y = -3$ and $y' = -18$. An equation of the tangent line is $y-(-3) = -18(x-2)$, or $y = -18x+33$.

41. $y = x^{\frac{1}{3}}$

$y' = \frac{1}{3}x^{-\frac{2}{3}}$

When $x = 8$, then $y = 2$ and $y' = \frac{1}{12}$. An equation of the tangent line is $y-2 = \frac{1}{12}(x-8)$, or $y = \frac{1}{12}x+\frac{4}{3}$.

42. $y = \frac{x^2}{x-12}$

$y' = \frac{(x-12)(2x)-x^2(1)}{(x-12)^2} = \frac{x^2-24x}{(x-12)^2}$

When $x = 13$, then $y = 169$ and $y' = -143$. An equation of the tangent line is $y - 169 = -143(x-13)$ or $y = -143x+2028$.

43. $f(x) = 4x^2+2x+8$

$f'(x) = 8x+2$

$f(1) = 14$ and $f'(1) = 10$. The relative rate of change is $\frac{f'(1)}{f(1)} = \frac{10}{14} = \frac{5}{7} \approx 0.714$, so the percentage rate of change is 71.4%.

44. $f(x) = \frac{x}{x+4}$

$f'(x) = \frac{(x+4)(1)-x(1)}{(x+4)^2} = \frac{4}{(x+4)^2}$

$f(1) = \frac{1}{5}$ and $f'(1) = \frac{4}{25}$. The relative rate of change is $\frac{f'(1)}{f(1)} = \frac{4}{5} = 0.8$, so the percentage rate of change is 80%.

45. $r = q(20-0.1q) = 20q-0.1q^2$

$\frac{dr}{dq} = 20-0.2q$

46. $\frac{dc}{dq} = 0.0003q^2-0.04q+3$

$\left.\frac{dc}{dq}\right|_{q=100} = 2$

47. $\frac{dC}{dI} = 0.6 - 0.25\left(\frac{1}{2}\right)I^{-\frac{1}{2}} = 0.6 - \frac{1}{8\sqrt{I}}$

$\left.\frac{dC}{dI}\right|_{I=16} \approx 0.569$

Thus the marginal propensity to consume is 0.569, so the marginal propensity to save is 1 − 0.569 = 0.431.

48. $\frac{dp}{dq} = \frac{(q+5)(1)-(q+12)(1)}{(q+5)^2} = -\frac{7}{(q+5)^2}$

49. Since $p = -0.1q + 500$, then

$r = pq = -0.1q^2 + 500q$. Thus $\frac{dr}{dq} = 500 - 0.2q$.

50. Since $\bar{c} = 0.03q + 1.2 + \frac{3}{q}$, then

$c = q\bar{c} = 0.03q^2 + 1.2q + 3$. Thus

$\frac{dc}{dq} = 0.06q + 1.2$, so $\left.\frac{dc}{dq}\right|_{q=100} = 7.2$.

51. $\frac{dc}{dq} = 0.125 + 0.00878q$

$\left.\frac{dc}{dq}\right|_{q=70} = 0.7396$

52. $q = 50m - m^2$

$p = -0.01q + 9$; $m = 10$

$\frac{dr}{dm} = \frac{dr}{dq}\cdot\frac{dq}{dm}$

$r = pq = -0.01q^2 + 9q$, so $\frac{dr}{dq} = -0.02q + 9$.

If $m = 10$, then $q = 400$, so $\left.\frac{dr}{dq}\right|_{m=10} = -8 + 9 = 1$.

$\frac{dq}{dm} = 50 - 2m$. When $m = 10$, $\frac{dq}{dm} = 30$.

Thus $\left.\frac{dr}{dm}\right|_{m=10} = (1)(30) = 30$.

53. $\frac{dy}{dx} = 42x^2 - 34x - 16$

$\left.\frac{dy}{dx}\right|_{x=2} = 84$ eggs/mm

54. $y = 12 - \frac{12}{1+3x}$

$\frac{dy}{dx} = -12(-1)(1+3x)^{-2}(3) = \frac{36}{(1+3x)^2}$

Setting $\frac{36}{(1+3x)^2} = \frac{1}{3}$ gives $(1+3x)^2 = 108$,

$1 + 3x = \pm 6\sqrt{3}$, $x = \frac{-1 \pm 6\sqrt{3}}{3}$, $x \approx 3.13$ or $x \approx -3.80$.

Because we must have $x \geq 0$, then $x \approx 3.13$.

55. a. $\frac{dt}{dT}$ when $T = 38$ is

$\left.\frac{d}{dT}\left[\frac{4}{3}T - \frac{175}{4}\right]\right|_{T=38} = \left.\frac{4}{3}\right|_{T=38} = \frac{4}{3}$.

b. $\frac{dt}{dT}$ when $T = 35$ is

$\left.\frac{d}{dT}\left[\frac{1}{24}T + \frac{11}{4}\right]\right|_{T=35} = \left.\frac{1}{24}\right|_{T=35} = \frac{1}{24}$.

56. $s = 9\left(2t^2 + 3\right)^{-1}$

$v = \frac{ds}{dt} = -9\left(2t^2+3\right)^{-2}(4t) = \frac{-36t}{\left(2t^2+3\right)^2}$

If $t = 1$, then $v = -\frac{36}{25}$ m/s.

57. $V' = \frac{1}{2}\pi d^2$. If $d = 4$ ft, then $V' = 8\pi\frac{\text{ft}^3}{\text{ft}}$.

58. $v = 128 - 32t$. Set $128 - 32t = 64$ to get $t = 2$.

59. $c = \bar{c}q = 2q^2 + \frac{10,000}{q} = 2q^2 + 10,000q^{-1}$

$\frac{dc}{dq} = 4q - 10,000q^{-2} = 4q - \frac{10,000}{q^2}$

60. $y = \frac{(x^3+2)\sqrt{x+1}}{x^4+2x} = \frac{(x^3+2)\sqrt{x+1}}{x(x^3+2)} = \frac{\sqrt{x+1}}{x}$

$$\frac{dy}{dx} = \frac{x\left(\frac{1}{2}(x+1)^{-\frac{1}{2}}(1)\right) - \sqrt{x+1}(1)}{x^2}$$

$\left.\frac{dy}{dx}\right|_{x=1} = -\frac{3}{4}\sqrt{2}$ and $y = \sqrt{2}$ when $x = 1$. An equation of the tangent line is

$y - \sqrt{2} = -\frac{3}{4}\sqrt{2}(x-1)$ or $y = -\frac{3}{4}\sqrt{2}x + \frac{7}{4}\sqrt{2}.$

61. a. $q = 10\sqrt{m^2+4900} - 700$

$p = \sqrt{19{,}300-8q};\ m = 240$

$\frac{dr}{dm} = \frac{dr}{dq}\cdot\frac{dq}{dm}.$

$r = pq = q\sqrt{19{,}300-8q}$, so

$\frac{dr}{dq} = q\left(\frac{1}{2}\right)(19{,}300-8q)^{-\frac{1}{2}}(-8) + \sqrt{19{,}300-8q}(1).$

If $m = 240$, then $q = 1800$, so

$\left.\frac{dr}{dq}\right|_{m=240} = -\frac{230}{7} \approx -32.86.$

$\frac{dq}{dm} = 10\cdot\frac{1}{2}\left(m^2+4900\right)^{-\frac{1}{2}}(2m).$

$\left.\frac{dq}{dm}\right|_{m=240} = 9.6.$ Thus

$\left.\frac{dr}{dm}\right|_{m=240} \approx (-32.86)(9.6) = -315.456$

b. $\left.\frac{\frac{dr}{dm}}{r}\right|_{m=240} = \left.\frac{-315.456}{r}\right|_{q=1800}$

$= \frac{-315.456}{1800\sqrt{4900}}$

$= -0.0025$

c. No. Since $\frac{dr}{dm} < 0$, there would be no additional revenue generated to offset the cost of \$400.

62. 21.094

63. 0.305

64. \$5.05

65. –0.32

Mathematical Snapshot Chapter 11

1. In Problems 63 and 64 of Sec. 11.4, the slope is ≈ 0.7. In Fig. 11.15 the slope is above 0.9. More is spent; less is saved.

2. In the lowest quintile, the average family spends more than it earns, thus accumulating debt.

3. The slope of the family consumption curve is $\frac{112{,}040}{\sqrt{1.9667\times10^{10}+224{,}080x}}$, which for x = 25,000 equals about 0.705. You would expect the family to spend \$705 and save \$295.

4. For x = 90,000, the slope of the consumption curve is 0.561. You would expect the family to spend \$561 and save \$439.

5. Answers may vary.

Chapter 12

Principles in Practice 12.1

1. $\frac{dq}{dp} = \frac{d}{dp}\left[25 + 2\ln\left(3p^2 + 4\right)\right]$

$= 0 + 2\frac{d}{dp}\left[\ln\left(3p^2 + 4\right)\right]$

$= 2\left(\frac{1}{3p^2 + 4}\right)\frac{d}{dp}\left(3p^2 + 4\right) = \frac{2}{3p^2 + 4}(6p)$

$= \frac{12p}{3p^2 + 4}$

2. With $I_0 = 1$, $R(I) = \log I$.

$\frac{dR}{dI} = \frac{d}{dI}[\log I] = \frac{d}{dI}\left[\frac{\ln I}{\ln 10}\right]$

$= \frac{1}{\ln 10} \cdot \frac{1}{I} = \frac{1}{I \ln 10}$

Problems 12.1

1. $\frac{dy}{dx} = 4 \cdot \frac{d}{dx}(\ln x) = 4 \cdot \frac{1}{x} = \frac{4}{x}$

2. $\frac{dy}{dx} = \frac{5}{9}\left(\frac{1}{x}\right) = \frac{5}{9x}$

3. $\frac{dy}{dx} = \frac{1}{3x-7}(3) = \frac{3}{3x-7}$

4. $\frac{dy}{dx} = \frac{1}{5x-6}(5) = \frac{5}{5x-6}$

5. $y = \ln x^2 = 2\ln x$

$\frac{dy}{dx} = 2 \cdot \frac{1}{x} = \frac{2}{x}$

6. $\frac{dy}{dx} = \frac{1}{3x^2 + 2x + 1}(6x + 2) = \frac{6x + 2}{3x^2 + 2x + 1}$

7. $\frac{dy}{dx} = \frac{1}{1 - x^2}(-2x) = -\frac{2x}{1 - x^2}$

8. $\frac{dy}{dx} = \frac{1}{-x^2+6x}(-2x+6) = \frac{-2x+6}{-x^2+6x}$
$= \frac{-2(x-3)}{-x(x-6)} = \frac{2(x-3)}{x(x-6)}$

9. $f'(X) = \frac{1}{4X^6+2X^3}(24X^5+6X^2)$
$= \frac{24X^5+6X^2}{4X^6+2X^3}$
$= \frac{6X^2(4X^3+1)}{2X^3(2X^3+1)}$
$= \frac{3(4X^3+1)}{X(2X^3+1)}$

10. $f'(r) = \frac{1}{2r^4-3r^2+2r+1}\left(8r^3-6r+2\right)$
$= \frac{8r^3-6r+2}{2r^4-3r^2+2r+1}$
$= \frac{2\left(4r^3-3r+1\right)}{2r^4-3r^2+2r+1}$

11. $f'(t) = t\left(\frac{1}{t}\right) + (\ln t)(1) = 1 + \ln t$

12. $\frac{dy}{dx} = x^2\left(\frac{1}{x}\right) + (\ln x)(2x) = x + 2x\ln x$
$= x(1+2\ln x)$

13. $\frac{dy}{dx} = x^3\left[\frac{1}{2x+5}(2)\right] + \ln(2x+5)\cdot 3x^2$
$= \frac{2x^3}{2x+5} + 3x^2\ln(2x+5)$

14. $\frac{dy}{dx} = (ax+b)^3\left[\frac{1}{(ax+b)}(a)\right] + [\ln(ax+b)]3(ax+b)^2(a)$
$= a(ax+b)^2 + 3a(ax+b)^2\ln(ax+b)$
$= a(ax+b)^2[1+3\ln(ax+b)]$

15. $y = \log_3(8x-1) = \frac{\ln(8x-1)}{\ln 3}$

$\frac{dy}{dx} = \frac{1}{\ln 3} \cdot \frac{d}{dx}[\ln(8x-1)]$

$= \frac{1}{\ln 3} \cdot \frac{1}{8x-1}(8) = \frac{8}{(8x-1)(\ln 3)}$

16. $f(w) = \log\left(w^2+w\right) = \log_{10}\left(w^2+w\right)$

$= \frac{\ln\left(w^2+w\right)}{\ln 10}$

$f'(w) = \frac{1}{\ln 10} \cdot \frac{1}{w^2+w}(2w+1)$

$= \frac{2w+1}{(\ln 10)\left(w^2+w\right)}$

17. $y = x^2 + \log_2\left(x^2+4\right) = x^2 + \frac{\ln\left(x^2+4\right)}{\ln 2}$

$\frac{dy}{dx} = 2x + \frac{1}{\ln 2}\left[\frac{1}{x^2+4}(2x)\right]$

$= 2x\left[1 + \frac{1}{(\ln 2)\left(x^2+4\right)}\right]$

18. $y = x^2 \log_2 x = x^2 \cdot \frac{\ln x}{\ln 2} = \frac{1}{\ln 2}(x^2 \ln x)$

$\frac{dy}{dx} = \frac{1}{\ln 2}\left[x^2\left(\frac{1}{x}\right) + \ln x(2x)\right]$

$= \frac{x}{\ln 2}(1 + 2\ln x)$

19. $f'(z) = \frac{z\left(\frac{1}{z}\right) - (\ln z)(1)}{z^2} = \frac{1-\ln z}{z^2}$

20. $\frac{dy}{dx} = \frac{(\ln x)(2x) - x^2\left(\frac{1}{x}\right)}{(\ln x)^2}$

$= \frac{2x\ln x - x}{\ln^2 x} = \frac{x[2\ln x - 1]}{\ln^2 x}$

21. $\frac{dy}{dx} = \frac{(\ln x)^2(2x) - (x^2+3)2(\ln x)\frac{1}{x}}{(\ln x)^4}$

$= \frac{2x^2\ln x - 2(x^2+3)}{x(\ln x)^3}$

22. $y = \ln x^{100} = 100\ln x$

$\frac{dy}{dx} = 100 \cdot \frac{1}{x} = \frac{100}{x}$

23. $y = \ln\left(x^2+4x+5\right)^3 = 3\ln\left(x^2+4x+5\right)$

$\frac{dy}{dx} = 3 \cdot \frac{1}{x^2+4x+5}(2x+4)$

$= \frac{3(2x+4)}{x^2+4x+5} = \frac{6(x+2)}{x^2+4x+5}$

24. $y = 6\ln\sqrt[3]{x} = 6 \cdot \frac{1}{3}\ln x = 2\ln x$

$\frac{dy}{dx} = 2 \cdot \frac{1}{x} = \frac{2}{x}$

25. $y = 9\ln\sqrt{1+x^2} = \frac{9}{2}\ln\left(1+x^2\right)$

$\frac{dy}{dx} = \frac{9}{2} \cdot \frac{1}{1+x^2}(2x) = \frac{9x}{1+x^2}$

26. $f(t) = \ln\left(\frac{t^5}{1+3t^2+t^4}\right) = 5\ln t - \ln(1+3t^2+t^4)$

$f'(t) = 5\left(\frac{1}{t}\right) - \frac{1}{1+3t^2+t^4}(6t+4t^3)$

$= \frac{5(1+3t^2+t^4) - t(6t+4t^3)}{t(1+3t^2+t^4)}$

$= \frac{t^4+9t^2+5}{t(1+3t^2+t^4)}$

27. $f(l) = \ln\left(\frac{1+l}{1-l}\right) = \ln(1+l) - \ln(1-l)$

$f'(l) = \frac{1}{1+l} - \frac{1}{1-l}(-1)$

$= \frac{(1-l)+(1+l)}{(1+l)(1-l)} = \frac{2}{1-l^2}$

28. $y = \ln\left(\frac{2x+3}{3x-4}\right) = \ln(2x+3) - \ln(3x-4)$

$$\frac{dy}{dx} = \frac{2}{2x+3} - \frac{3}{3x-4}$$
$$= \frac{2(3x-4)-3(2x+3)}{(2x+3)(3x-4)} = -\frac{17}{(2x+3)(3x-4)}$$

29. $y = \ln\sqrt[4]{\frac{1+x^2}{1-x^2}} = \frac{1}{4}\left[\ln\left(1+x^2\right) - \ln\left(1-x^2\right)\right]$

$$\frac{dy}{dx} = \frac{1}{4}\left[\frac{2x}{1+x^2} - \frac{-2x}{1-x^2}\right]$$
$$= \frac{1}{4}\left[\frac{2x\left(1-x^2\right)+2x\left(1+x^2\right)}{\left(1+x^2\right)\left(1-x^2\right)}\right] = \frac{x}{1-x^4}$$

30. $y = \ln\sqrt[3]{\frac{x^3-1}{x^3+1}} = \frac{1}{3}[\ln(x^3-1) - \ln(x^3+1)]$

$$\frac{dy}{dx} = \frac{1}{3}\left[\frac{3x^2}{x^3-1} - \frac{3x^2}{x^3+1}\right]$$
$$= \frac{1}{3}\left[\frac{3x^2(x^3+1)-3x^2(x^3-1)}{(x^3-1)(x^3+1)}\right]$$
$$= \frac{2x^2}{x^6-1}$$

31. $y = \ln\left[\left(x^2+2\right)^2\left(x^3+x-1\right)\right]$

$$= 2\ln\left(x^2+2\right) + \ln\left(x^3+x-1\right)$$
$$\frac{dy}{dx} = 2\cdot\frac{1}{x^2+2}(2x) + \frac{1}{x^3+x-1}\left(3x^2+1\right)$$
$$= \frac{4x}{x^2+2} + \frac{3x^2+1}{x^3+x-1}$$

32. $y = \ln\left[(5x+2)^4(8x-3)^6\right]$

$$= 4\ln(5x+2) + 6\ln(8x-3)$$
$$\frac{dy}{dx} = 4\cdot\frac{1}{5x+2}(5) + 6\cdot\frac{1}{8x-3}(8)$$
$$= \frac{20}{5x+2} + \frac{48}{8x-3}$$

33. $y = 13\ln\left(x^2\sqrt[3]{5x+2}\right)$

$$= 13\ln x^2 + 13\ln(5x+2)^{1/3}$$
$$= 26\ln x + \frac{13}{3}\ln(5x+2)$$
$$\frac{dy}{dx} = 26\left(\frac{1}{x}\right) + \frac{13}{3}\cdot\frac{1}{5x+2}(5) = \frac{26}{x} + \frac{65}{3(5x+2)}$$

34. $y = 6\ln\frac{x}{\sqrt{2x+1}} = 6\ln x - 6\ln(2x+1)^{\frac{1}{2}}$

$$= 6\ln x - 3\ln(2x+1)$$
$$\frac{dy}{dx} = \frac{6}{x} - 3\cdot\frac{1}{2x+1}(2) = \frac{6}{x} - \frac{6}{2x+1}$$

35. $\frac{dy}{dx} = \left(x^2+1\right)\left[\frac{1}{2x+1}(2)\right] + \ln(2x+1)\cdot(2x)$

$$= \frac{2\left(x^2+1\right)}{2x+1} + 2x\ln(2x+1)$$

36. $\frac{dy}{dx} = (ax+b)\left[\frac{1}{ax}(a)\right] + \ln(ax)\cdot(a)$

$$= \frac{ax+b}{x} + a\ln(ax)$$

37. $y = \ln x^3 + \ln^3 x = 3\ln x + (\ln x)^3$

$$\frac{dy}{dx} = 3\cdot\frac{1}{x} + 3(\ln x)^2\cdot\frac{1}{x} = \frac{3}{x} + \frac{3(\ln x)^2}{x}$$
$$= \frac{3\left(1+\ln^2 x\right)}{x}$$

38. $\frac{dy}{dx} = (\ln 2)x^{(\ln 2)-1}$

39. $y = \ln^4(ax) = [\ln(ax)]^4$

$$\frac{dy}{dx} = 4[\ln(ax)]^3\left(\frac{1}{ax}\cdot a\right) = \frac{4\ln^3(ax)}{x}$$

40. $y = \ln^2(2x+11) = [\ln(2x+11)]^2$

$$\frac{dy}{dx} = 2[\ln(2x+11)]\cdot\frac{1}{2x+11}(2) = \frac{4\ln(2x+11)}{2x+11}$$

41. $y = x\ln\sqrt{x-1} = \frac{1}{2}x\ln(x-1)$

$\frac{dy}{dx} = \frac{1}{2}\left[x\left(\frac{1}{x-1}\right) + \ln(x-1)\cdot(1)\right]$

$= \frac{x}{2(x-1)} + \ln\sqrt{x-1}$

42. $y = \ln\left(x^3\sqrt[4]{2x+1}\right) = 3\ln x + \frac{1}{4}\ln(2x+1)$

$\frac{dy}{dx} = 3\cdot\frac{1}{x} + \frac{1}{4}\cdot\frac{1}{2x+1}(2) = \frac{3}{x} + \frac{1}{2(2x+1)}$

43. $y = \sqrt{4+3\ln x} = (4+3\ln x)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(4+3\ln x)^{-\frac{1}{2}}\cdot\frac{3}{x} = \frac{3}{2x\sqrt{4+3\ln x}}$

44. $\frac{dy}{dx} = \frac{1}{x+\sqrt{1+x^2}}\left[1+\frac{1}{2}\left(1+x^2\right)^{-\frac{1}{2}}(2x)\right]$

$= \frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} = \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}\left(x+\sqrt{1+x^2}\right)}$

$= \frac{1}{\sqrt{1+x^2}}$

45. $y = \ln(x^2-3x-3)$

$y' = \frac{2x-3}{x^2-3x-3}$

The slope of the tangent line at $x = 4$ is

$y'(4) = \frac{8-3}{16-12-3} = 5.$ Also, if $x = 4$, then

$y = \ln(16-12-3) = \ln 1 = 0$. Thus an equation of the tangent line is $y - 0 = 5(x-4)$, or $y = 5x - 20$.

46. $y = x[\ln(x) - 1]$

$y' = x\left(\frac{1}{x}\right) + [\ln(x)-1](1) = \ln x$

When $x = e$, $y = 0$ and $y' = 1$. The equation of the tangent line is $y - 0 = 1(x - e)$, or $y = x - e$.

47. $y = \frac{x}{\ln x}$

$y' = \frac{(\ln x)(1) - x\left(\frac{1}{x}\right)}{\ln^2 x} = \frac{\ln x - 1}{\ln^2 x}$

When $x = 3$ the slope is $y'(3) = \frac{(\ln 3)-1}{\ln^2 3}$.

48. $p = \frac{25}{\ln(q+2)}$, so $r = pq = \frac{25q}{\ln(q+2)}$. Thus the marginal revenue is

$\frac{dr}{dq} = 25\cdot\frac{\ln(q+2)(1) - q\left(\frac{1}{q+2}\right)}{\ln^2(q+2)}$

$= 25\cdot\frac{(q+2)\ln(q+2)-q}{(q+2)\ln^2(q+2)}$.

49. $c = 25\ln(q+1) + 12$

$\frac{dc}{dq} = \frac{25}{q+1}$, so $\left.\frac{dc}{dq}\right|_{q=6} = \frac{25}{7}$.

50. $\overline{c} = \frac{500}{\ln(q+20)}$

$c = \overline{c}q = \frac{500q}{\ln(q+20)}$

$\frac{dc}{dq} = 500\cdot\frac{[\ln(q+20)](1) - q\left(\frac{1}{q+20}\right)}{[\ln(q+20)]^2}$

$\left.\frac{dc}{dq}\right|_{q=50} = 500\cdot\frac{\ln 70 - \frac{50}{70}}{(\ln 70)^2} \approx \$97.90$

51. $\frac{dq}{dp} = \frac{d}{dp}[25+10\ln(2p+1)]$

$= 0 + 10\frac{d}{dp}[\ln(2p+1)] = 10\left(\frac{1}{2p+1}\right)\frac{d}{dp}[2p+1]$

$= \frac{10}{2p+1}(2) = \frac{20}{2p+1}$

52. With $I_0 = 17$, $L(I) = 10\log\frac{I}{17}$.

$$\frac{dL}{dI} = \frac{d}{dI}\left[10\log\frac{I}{17}\right] = 10\frac{d}{dI}[\log I - \log 17]$$
$$= 10\frac{d}{dI}\left[\frac{\ln I}{\ln 10} - \log 17\right] = 10\left[\frac{1}{\ln 10}\cdot\frac{1}{I} - 0\right]$$
$$= \frac{10}{I\ln 10}$$

53. $A = 6\ln\left(\frac{T}{a-T} - a\right)$. Rate of change of A with respect to T:

$$\frac{dA}{dT} = 6\cdot\frac{1}{\frac{T}{a-T} - a}\left[\frac{(a-T)(1) - T(-1)}{(a-T)^2}\right]$$
$$= 6\cdot\frac{1}{\frac{T-a(a-T)}{a-T}}\left[\frac{a}{(a-T)^2}\right]$$
$$= 6\cdot\frac{a-T}{T-a^2+aT}\cdot\frac{a}{(a-T)^2}$$
$$= \frac{6a}{\left(T-a^2+aT\right)(a-T)}$$

54. If $y = \ln f(x)$, then $\frac{dy}{dx} = \frac{1}{f(x)}f'(x) = \frac{f'(x)}{f(x)}$, which is the relative rate of change of $y = f(x)$ with respect to x.

55. $\frac{d}{dx}\left(\log_b u\right) = \frac{d}{dx}\left(\frac{\ln u}{\ln b}\right)$

$$= \frac{1}{\ln b}\cdot\frac{d}{dx}(\ln u) = \frac{1}{\ln b}\left(\frac{1}{u}\cdot\frac{du}{dx}\right)$$
$$= \left(\log_b e\right)\left(\frac{1}{u}\cdot\frac{du}{dx}\right) = \frac{1}{u}\left(\log_b e\right)\frac{du}{dx}$$

56. $f'(x) = x^2(1+3\ln x)$
$f'(x) = 0$ for $x \approx 0.72$

57. Note that $f(x)$ is defined for all $x \neq 0$.

$$f'(x) = \frac{x^2\cdot\frac{1}{x^2}(2x) - \ln(x^2)\cdot 2x}{x^4} = \frac{2-2\ln(x^2)}{x^3}$$

$f'(x) = 0$ for $x \approx -1.65, 1.65$

Principles in Practice 12.2

1. The rate of change of temperature with respect to time is $\frac{dT}{dt}$. $T(t)$ has the form Ce^u where C is a constant and $u = kt$.

$$\frac{dT}{dt} = \frac{d}{dt}\left[Ce^{kt}\right] = C\frac{d}{dt}\left[e^{kt}\right]$$
$$= C\left(e^{kt}\right)\frac{d}{dt}[kt] = Ce^{kt}(k) = Cke^{kt}$$

Problems 12.2

1. $y' = 5\cdot\frac{d}{dx}(e^x) = 5e^x$

2. $y' = \frac{2e^x}{5}$

3. $y' = e^{2x^2+3}(4x) = 4xe^{2x^2+3}$

4. $y' = e^{2x^2+5}(4x) = 4xe^{2x^2+5}$

5. $y' = e^{9-5x}\cdot\frac{d}{dx}\left(9-5x\right) = e^{9-5x}(-5) = -5e^{9-5x}$

6. $f'(q) = e^{-q^3+6q-1}\left(-3q^2+6\right)$
$= -3\left(q^2-2\right)e^{-q^3+6q-1}$

7. $f'(r) = e^{3r^2+4r+4}(6r+4) = 2(3r+2)e^{3r^2+4r+4}$

8. $y' = e^{x^2+6x^3+1}(2x+18x^2)$
$= 2x(1+9x)e^{x^2+6x^3+1}$

9. $y' = x\left(e^x\right) + e^x(1) = e^x(x+1)$

10. $y' = 3x^4\left[e^{-x}(-1)\right] + e^{-x}(12x^3) = 3x^3e^{-x}(4-x)$

11. $y' = x^2\left[e^{-x^2}(-2x)\right] + e^{-x^2}(2x)$
$= 2xe^{-x^2}\left(1-x^2\right)$

12. $y' = x\left[e^{3x}(3)\right] + e^{3x}(1) = e^{3x}(3x+1)$

13. $y = \frac{1}{3}\left(e^x + e^{-x}\right)$

$y' = \frac{1}{3}\left[e^x + e^{-x}(-1)\right] = \frac{e^x - e^{-x}}{3}$

14. $\frac{dy}{dx} = \frac{(e^x + e^{-x})[e^x - e^{-x}(-1)] - (e^x - e^{-x})[e^x + e^{-x}(-1)]}{(e^x + e^{-x})^2}$

$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}$

15. $\frac{d}{dx}\left(5^{2x^3}\right) = \frac{d}{dx}\left[e^{(\ln 5)2x^3}\right]$

$= e^{(\ln 5)2x^3}[(\ln 5)6x^2]$

$= (6x^2)5^{2x^3} \ln 5$

16. $y = 2^x x^2 = e^{(\ln 2)x} x^2$

$y' = e^{(\ln 2)x}(2x) + x^2\left[e^{(\ln 2)x}(\ln 2)\right]$

$= 2x\left(2^x\right) + x^2\left(2^x\right)(\ln 2) = x\left(2^x\right)(2 + x \ln 2)$

17. $f'(w) = \frac{w^2\left[e^{2w}(2)\right] - e^{2w}[2w]}{w^4}$

$= \frac{2e^{2w}(w-1)}{w^3}$

18. $y' = e^{x-\sqrt{x}}\left(1 - \frac{1}{2}x^{-\frac{1}{2}}\right) = e^{x-\sqrt{x}}\left(1 - \frac{1}{2\sqrt{x}}\right)$

19. $y' = e^{1+\sqrt{x}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right) = \frac{e^{1+\sqrt{x}}}{2\sqrt{x}}$

20. $y' = 3(e^{2x} + 1)^2(e^{2x}(2) + 0) = 6e^{2x}(e^{2x} + 1)^2$

21. $y = x^5 - 5^x = x^5 - e^{(\ln 5)x}$

$y' = 5x^4 - e^{(\ln 5)x}(\ln 5) = 5x^4 - 5^x \ln 5$

22. $f(z) = e^{-1/z^2} = e^{-z^{-2}}$

$f'(z) = e^{-1/z^2}[-(-2z^{-3})] = \frac{2}{z^3}e^{-1/z^2}$

23. $\frac{dy}{dx} = \frac{\left(e^x+1\right)\left[e^x\right]-\left(e^x-1\right)\left[e^x\right]}{\left(e^x+1\right)^2}$

$= \frac{2e^x}{\left(e^x+1\right)^2}$

24. $y' = e^{2x}[1]+(x+6)\left[e^{2x}(2)\right] = e^{2x}(2x+13)$

25. $y = \ln e^x = x$ so $y' = 1$.

26. $y' = e^{-x}\cdot\frac{1}{x}+(\ln x)\left(-e^{-x}\right) = e^{-x}\left(\frac{1}{x}-\ln x\right)$

27. $y' = e^{x^2 \ln x^2}\left[x^2\cdot\frac{1}{x^2}(2x)+(\ln x^2)(2x)\right]$

$= 2xe^{x^2\ln x^2}(1+\ln x^2)$

28. $y = \ln e^{4x+1} = 4x+1$, so $\frac{dy}{dx} = 4$.

29. $f(x) = ee^x e^{x^2} = e^{1+x+x^2}$

$f'(x) = e^{1+x+x^2}(1+2x) = (1+2x)e^{1+x+x^2}$

$f'(-1) = [1+2(-1)]e^{1+(-1)+(-1)^2} = -e$

30. $f(x) = 5^{x^2\ln x} = \left(e^{\ln 5}\right)^{x^2\ln x} = e^{(\ln 5)x^2\ln x}$

$f'(x) = e^{(\ln 5)x^2\ln x}\left\{(\ln 5)\left[x^2\cdot\frac{1}{x}+(\ln x)(2x)\right]\right\}$

$= e^{(\ln 5)x^2\ln x}(\ln 5)[x+2x\ln x]$

$f'(1) = e^0(\ln 5)[1+0] = \ln 5$

31. $y = e^x$, $y' = e^x$. When $x = -2$, then $y = e^{-2}$ and $y' = e^{-2}$. Thus an equation of the tangent line is $y-e^{-2} = e^{-2}(x+2)$, or $y = e^{-2}x+3e^{-2}$.

32. $y' = e^x$

When $x = 1$, $y = e$ and $y' = e$. Thus an equation of the tangent line is $y - e = e(x - 1)$ or $y = ex$.

33. $\frac{dp}{dq} = 15e^{-0.001q}(-0.001) = -0.015e^{-0.001q}$

$\left.\frac{dp}{dq}\right|_{q=500} = -0.015e^{-0.5}$

34. $\frac{dp}{dq} = 9e^{-5q/750}\left(-\frac{5}{750}\right) = -0.06e^{-5q/750}$

$\left.\frac{dp}{dq}\right|_{q=300} = -0.06e^{-2}$

35. $\overline{c} = \frac{7000e^{\frac{q}{700}}}{q}$, so $c = \overline{c}q = 7000e^{\frac{q}{700}}$. The marginal cost function is $\frac{dc}{dq} = 7000e^{\frac{q}{700}}\left(\frac{1}{700}\right)$ $= 10e^{\frac{q}{700}}$. Thus $\left.\frac{dc}{dq}\right|_{q=350} = 10e^{0.5}$ and $\left.\frac{dc}{dq}\right|_{q=700} = 10e$.

36. $\overline{c} = \frac{850}{q}+4000\frac{e^{\frac{2q+6}{800}}}{q}$

$c = \overline{c}q = 850+4000e^{\frac{2q+6}{800}} = 850+4000e^{\frac{q+3}{400}}$

The marginal cost function is $\frac{dc}{dq} = 10e^{\frac{q+3}{400}}$.

$\left.\frac{dc}{dq}\right|_{q=97} = 10e^{0.25}$ and $\left.\frac{dc}{dq}\right|_{q=197} = 10e^{0.5}$.

37. $w = e^{x^3-4x}+x\ln(x-1)$ and $x = \frac{t+1}{t-1}$

By the chain rule, $\frac{dw}{dt} = \frac{dw}{dx}\cdot\frac{dx}{dt}$

$= \left[e^{x^3-4x}\left(3x^2-4\right)+x\left(\frac{1}{x-1}\right)+[\ln(x-1)(1)]\right]$

$\cdot\left[\frac{(t-1)(1)-(t+1)(1)}{(t-1)^2}\right]$

$= \left[\left(3x^2-4\right)e^{x^3-4x}+\frac{x}{x-1}+\ln(x-1)\right]\left[\frac{-2}{(t-1)^2}\right].$

When $t = 3$, then $x = \frac{3+1}{3-1} = \frac{4}{2} = 2$ and

$$\frac{dw}{dt} = [8+2+0]\left[-\frac{1}{2}\right] = -5\,.$$

38. $f'(x) = x^3$ and $u = e^x$. Let $y = f(u)$. Then

$\frac{d}{dx}[f(u)] = \frac{dy}{dx}$ and by the chain rule

$$\frac{d}{dx}[f(u)] = \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = f'(u)\frac{du}{dx} = u^3\cdot e^x$$
$$= (e^x)^3\cdot e^x = e^{3x}\cdot e^x = e^{4x}$$

39. $\frac{d}{dx}\left(c^x - x^c\right) = \frac{d}{dx}\left[\left(e^{\ln c}\right)^x - x^c\right]$

$$= \frac{d}{dx}\left[e^{(\ln c)x} - x^c\right]$$
$$= (\ln c)e^{(\ln c)x} - cx^{c-1} = (\ln c)c^x - cx^{c-1}$$
$$\left.\frac{d}{dx}(c^x - x^c)\right|_{x=1} = (\ln c)c - c$$

If this is zero, $(\ln c)c - c = 0$, or $c[\ln(c) - 1] = 0$. Since $c > 0$, we must have $\ln(c) - 1 = 0$, $\ln c = 1$, or $c = e$.

40. $f(x) = 10^{-x} + \ln(8+x) + 0.01e^{x-2}$

$$= e^{(\ln 10)(-x)} + \ln(8+x) + 0.01e^{x-2}$$
$$f'(x) = e^{(\ln 10)(-x)}(-\ln 10) + \frac{1}{8+x} + 0.01e^{x-2}$$
$$= -(\ln 10)10^{-x} + \frac{1}{8+x} + 0.01e^{x-2}$$
$$\frac{f'(2)}{f(2)} = \frac{-(\ln 10)10^{-2} + \frac{1}{10} + 0.01}{10^{-2} + \ln(10) + 0.01} \approx 0.0374$$

41. $q = 500\left(1 - e^{-0.2t}\right)$

$$\frac{dq}{dt} = 500\left(-e^{-0.2t}\right)(-0.2) = 100e^{-0.2t}$$

Thus $\left.\frac{dq}{dt}\right|_{t=10} = 100e^{-2}$.

42. $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

$$f'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}(-x)$$
$$f'(1) = \frac{1}{\sqrt{2\pi}}e^{-1/2}(-1) \approx -0.242$$

43. $P = 1.92e^{0.0176t}$

$$\frac{dP}{dt} = 1.92e^{0.0176t}(0.0176) = P(0.0176)$$
$$= 0.0176P = kP \text{ for } k = 0.0176.$$

44. $Y = k\alpha^{\beta^t} = ke^{(\ln\alpha)\beta^t}$

$$Y' = ke^{(\ln\alpha)\beta^t}(\ln\alpha)\frac{d}{dt}[\beta^t]$$
$$= k\alpha^{\beta^t}(\ln\alpha)\frac{d}{dt}[e^{(\ln\beta)t}]$$
$$= k\alpha^{\beta^t}(\ln\alpha)e^{(\ln\beta)t}(\ln\beta)$$
$$= k\alpha^{\beta^t}(\beta^t\ln\alpha)\ln\beta$$

45. Since $S = Pe^{rt}$, then $\frac{dS}{dt} = Pe^{rt}r = rPe^{rt}$. Thus

$$\frac{\frac{dS}{dt}}{S} = \frac{rPe^{rt}}{Pe^{rt}} = r.$$

46. $y = K\left(1 - e^{-ax}\right)$

$$\frac{dy}{dx} = K\left[-e^{-ax}(-a)\right] = aKe^{-ax}$$

Solving the original equation for e^{-ax} gives

$e^{-ax} = -\frac{y}{K} + 1$. Thus substitution,

$\frac{dy}{dx} = aK\left(-\frac{y}{K} + 1\right) = a(-y + K) = a\,(K - y)$, as was to be shown.

47. $N = 10^A 10^{-bM} = 10^{A-bM} = e^{(\ln 10)(A-bM)}$

$\frac{dN}{dM} = e^{(\ln 10)(A-bM)}(\ln 10)(-b)$, so

$$\frac{dN}{dM} = 10^{A-bM}(\ln 10)(-b) = -b\left(10^{A-bM}\right)\ln 10$$

48. $p = 0.89\left[0.01 + 0.99(0.85)^t\right]$

a. $\frac{dP}{dt} = 0.89\left[0.99(0.85)^t \ln(0.85)\right]$

$= 0.8811(0.85)^t \ln(0.85)$

This represents the rate of change of proportion of correct recalls with respect to length of recall interval.

b. If $t = 2$, then

$\frac{dp}{dt} = 0.8811(0.85)^2 \ln(0.85) \approx -0.10$

49. $C(t) = C_0 e^{-\left(\frac{r}{V}\right)t}$

$\frac{dC}{dt} = C_0 e^{-\left(\frac{r}{V}\right)t}\left(-\frac{r}{V}\right)$

$= [C(t)]\left(-\frac{r}{V}\right) = -\left(\frac{r}{V}\right)C(t)$

50. $C(t) = \frac{R}{r}\left[1 - e^{-\left(\frac{r}{V}\right)t}\right]$

a. $C(0) = \frac{R}{r}\left[1 - e^0\right] = \frac{R}{r}[1-1] = 0$

b. $\frac{dC}{dt} = \frac{R}{r}\left[\frac{r}{V}e^{-\left(\frac{r}{V}\right)t}\right] = \frac{R}{V}e^{-\left(\frac{r}{V}\right)t}$

$= \frac{R}{V}\left[1 - \left(1 - e^{-\left(\frac{r}{V}\right)t}\right)\right]$

$= \frac{R}{V}\left[1 - \frac{r}{R}\cdot\frac{R}{r}\left(1 - e^{-\left(\frac{r}{V}\right)t}\right)\right]$

$= \frac{R}{V}\left[1 - \frac{r}{R}C(t)\right] = \frac{R}{V} - \frac{r}{V}C(t)$

51. $f(t) = 1 - e^{-0.008t}$

$f'(t) = 0.008e^{-0.008t}$

$f'(100) = 0.008e^{-0.8} \approx 0.0036$

52. $S = \ln\frac{5}{3+e^{-I}} = \ln 5 - \ln(3 + e^{-I})$

a. Recall that $\frac{dC}{dI} = 1 - \frac{dS}{dI}$.

$\frac{dS}{dI} = -\frac{1}{3+e^{-I}}(e^{-I})(-1) = \frac{e^{-I}}{3+e^{-I}}$

Thus $\frac{dC}{dI} = 1 - \frac{dS}{dI} = 1 - \frac{e^{-I}}{3+e^{-I}} = \frac{3}{3+e^{-I}}$.

b. If $\frac{dS}{dI} = \frac{1}{8}$, then $\frac{e^{-I}}{3+e^{-I}} = \frac{1}{8}$.

$\frac{(e^I)e^{-I}}{(e^I)(3+e^{-I})} = \frac{1}{8}$

$\frac{1}{3e^I + 1} = \frac{1}{8}$

$3e^I + 1 = 8$

$e^I = \frac{7}{3}$

$I = \ln\frac{7}{3} \approx \$0.847$ billion

$= \$(0.847)(1000)$ million

$= \$847$ million

53. $f'(x) = (6x^2 + 2x - 3)e^{2x^3 + x^2 - 3x}$

$f'(x) = 0$ for $x \approx -0.89, 0.56$

54. $f'(x) = 1 - e^{-x}$

$f'(x) = 0$ gives $e^x = 1$ or $x = 0$.

Problems 12.3

1. $\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{\frac{p}{q}}{-2}$.

When $q = 5$ then $p = 40 - 2(5) = 30$, so

$\eta = \frac{\frac{30}{5}}{-2} = -3$

Because $|\eta| > 1$, demand is elastic.

2. $\eta = \frac{\frac{p}{q}}{-0.04} = \frac{\frac{6}{100}}{-0.04} = -1.5$

Because $|\eta| > 1$, demand is elastic.

3. $p = \frac{3500}{q} = 3500q^{-1}$

$\frac{dp}{dq} = -3500q^{-2} = -\frac{3500}{q^2}$

$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{\frac{p}{q}}{-\frac{3500}{q^2}} = \frac{\frac{(3500/q)}{q}}{-\frac{3500}{q^2}} = -1$

Because $|\eta| = 1$, demand has unit elasticity.

4. $\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{\frac{p}{q}}{\frac{-1000}{q^3}} = \frac{\frac{(500/q^2)}{q}}{-\frac{1000}{q^3}} = -\frac{1}{2}$, inelastic

5. $\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{\frac{p}{q}}{-\frac{500}{(q+2)^2}} = \frac{\frac{[500/(q+2)]}{q}}{-\frac{500}{(q+2)^2}} = -\frac{q+2}{q}$

When $q = 104$, then $\eta = -\frac{106}{104} = -\frac{53}{52}$. Because $|\eta| > 1$, demand is elastic.

6. $\eta = \frac{\frac{p}{q}}{-\frac{1600}{(2q+1)^2}} = \frac{\frac{800/(2q+1)}{q}}{-\frac{1600}{(2q+1)^2}} = -\frac{2q+1}{2q}$

When $q = 24$, $\eta = -\frac{49}{48}$, elastic

7. $\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{\frac{p}{q}}{\frac{-e^{\frac{q}{100}}}{100}}$

When $q = 100$, then $p = 150 - e$ and

$\eta = \frac{\frac{150-e}{100}}{-\frac{e}{100}} = -\left(\frac{150}{e} - 1\right)$. Because $|\eta| > 1$, demand is elastic.

8. $\eta = \frac{\frac{p}{q}}{\frac{-e^{-\frac{q}{200}}}{2}} = \frac{\frac{100e^{-\frac{q}{200}}}{q}}{\frac{-e^{-\frac{q}{200}}}{2}} = -\frac{200}{q}$

When $q = 200$, $\eta = -\frac{200}{200} = -1$, so demand has unit elasticity.

9. $q = 1200 - 150p$

$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{p}{q} \cdot \frac{dq}{dp} = \frac{p}{q}(-150)$

If $p = 4$, then $q = 1200 - 150(4) = 600$, so

$\eta = \frac{4}{600}(-150) = -1$. Since $|\eta| = 1$, demand has unit elasticity.

10. $q = 100 - p$

When $p = 50$, then $q = 50$.

$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{p}{q} \cdot \frac{dq}{dp}$

$\frac{dq}{dp} = -1$, so $\eta = \frac{50}{50}(-1) = -1$, unit elasticity.

11. $q = \sqrt{500 - p}$

$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{p}{q} \cdot \frac{dq}{dp}$

$\frac{dq}{dp} = \frac{1}{2}(500 - p)^{-\frac{1}{2}}(-1) = \frac{-1}{2\sqrt{500 - p}} = -\frac{1}{2q}$

$\eta = \frac{p}{q}\left(-\frac{1}{2q}\right) = -\frac{p}{2q^2}$

If $p = 400$, then $q = \sqrt{500 - 400} = 10$, so

$\eta = -\frac{400}{200} = -2$. $|\eta| > 1$, so demand is elastic.

12. $q = \sqrt{2500 - p^2}$

$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{p}{q} \cdot \frac{dq}{dp}$

$\frac{dq}{dp} = \frac{1}{2}\left(2500 - p^2\right)^{-\frac{1}{2}}(-2p)$

$= \frac{-p}{\sqrt{2500 - p^2}} = -\frac{p}{q}$

$\eta = \frac{p}{q}\left(-\frac{p}{q}\right) = -\frac{p^2}{q^2}$

If $p = 20$, then $q = \sqrt{2100}$, so we have

$\eta = -\frac{400}{2100} = -\frac{4}{21}$, inelastic.

13. $q = \frac{(p-100)^2}{2}$

$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{p}{q} \cdot \frac{dq}{dp}$

$\frac{dq}{dp} = \frac{1}{2}(2)(p-100)(1) = p-100$, so

$\eta = \frac{p}{q}(p-100)$. If $p = 20$, then

$q = \frac{(20-100)^2}{2} = 3200$. Thus

$\eta = \frac{20}{3200}(20-100) = -\frac{1}{2}$. Demand is inelastic.

14. $q = p^2 - 50p + 850$

$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{p}{q} \cdot \frac{dq}{dp}$

$\frac{dq}{dp} = 2p - 50$, so $\eta = \frac{p}{q}(2p-50)$.

If $p = 20$, then $q = 250$, and

$\eta = \frac{20}{250}(40-50) = -\frac{200}{250} = -\frac{4}{5}$, inelastic.

15. $p = 13 - 0.05q$

$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = -\frac{p}{0.05q}$

p	q	η	demand
10	60	$-\frac{10}{3}$	elastic
3	200	$-\frac{3}{10}$	inelastic
6.50	130	-1	unit elasticity

16. a. $p = 36 - 0.25q$

$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{36-0.25q}{-0.25q}$

Setting $\frac{36-0.25q}{-0.25q} = -1$ yields $q = 72$.

b. $p = 300 - q^2$

$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{300-q^2}{-2q^2} = -1$ yields $q = \pm 10$.

Since $q > 0$, we must have $q = 10$.

17. $q = 500 - 40p + p^2$

$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{p}{q} \cdot \frac{dq}{dp}$

$\frac{dq}{dp} = -40 + 2p$, so $\eta = \frac{p}{q}(2p-40)$.

When $p = 15$, then $q = 500 - 40(15) + 15^2 = 125$, so $\eta\big|_{p=15} = \frac{15}{125}(30-40) = -\frac{6}{5} = -1.2$. Now, (% change in price) $\cdot$ (η) = % change in demand. Thus if the price of 15 increases $\frac{1}{2}\%$, then the change in demand is approximately $\left(\frac{1}{2}\%\right)(-1.2) = -0.6\%$. Thus demand decreases approximately 0.6%.

18. $\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{p}{q} \cdot \frac{dq}{dp}$

$q = \sqrt{2500 - p^2}$

$\frac{dq}{dp} = \frac{-p}{\sqrt{2500-p^2}} = \frac{-p}{q}$, so

$\eta = \frac{p}{q}\left(\frac{-p}{q}\right) = -\frac{p^2}{q^2}$.

Now, if $p = 30$, then $q = \sqrt{2500 - 30^2} = 40$, so $\eta\big|_{p=30} = -\frac{(30)^2}{(40)^2} = -\frac{9}{16}$. If the price of 30 decreases to 28.5, that is, it changes by $\frac{-1.5}{30} = -5\%$, then demand would change by approximately $-5\left(-\frac{9}{16}\right)\%$, or 2.8%. (That is, demand increases by 2.8%.)

19. $p = 500 - 2q$

$$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{\frac{500-2q}{q}}{-2} = \frac{q-250}{q}$$

If demand is elastic, then $\eta = \frac{q-250}{q} < -1.$ For $q > 0$, we have $q - 250 < -q$, $2q < 250$, so $q < 125$. Thus, if $0 < q < 125$, demand is elastic.

If demand is inelastic, then $\eta = \frac{q-250}{q} > -1.$

For $q > 0$, the inequality implies $q > 125$. Thus if $125 < q < 250$, then demand is inelastic.

Since Total Revenue $= r = pq = 500q - 2q^2$, then $r' = 500 - 4q = 4(125 - q)$. If $0 < q < 125$, then $r' > 0$, so r is increasing. If $125 < q < 250$, then $r' < 0$, so r is decreasing.

20. $p = 50 - 3q$

$$r = pq = 50q - 3q^2$$

$$\frac{dr}{dq} = 50 - 6q$$

$$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{\frac{50-3q}{q}}{-3} = \frac{3q-50}{3q}$$

$$p\left(1+\frac{1}{\eta}\right) = (50-3q)\left(1+\frac{3q}{3q-50}\right) = (50-3q)\left(\frac{3q-50+3q}{3q-50}\right) = 50-6q = \frac{dr}{dq}$$

21. $p = \frac{1000}{q^2}$

$$r = pq = \frac{1000}{q}$$

$$\frac{dr}{dq} = -1000q^{-2} = -\frac{1000}{q^2}$$

$$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{\frac{1000}{q^3}}{-\frac{2000}{q^3}} = -\frac{1}{2}$$

$$p\left(1+\frac{1}{\eta}\right) = \frac{1000}{q^2}(1-2) = -\frac{1000}{q^2} = \frac{dr}{dq}$$

22. $p = mq + b$

Note: $q = \frac{p-b}{m}$

a. $\lim_{p\to b^-} \eta = \lim_{p\to b^-} \frac{\frac{p}{q}}{\frac{dp}{dq}} = \lim_{p\to b^-} \frac{\frac{p}{(p-b)/m}}{m}$

$$= \lim_{p\to b^-} \frac{p}{p-b} = -\infty$$

b. $\eta = \frac{p}{p-b}$

Thus if $p = 0$, then $\eta = 0$.

23. a. $p = \frac{a}{\sqrt{b+cq^2}} = a(b+cq^2)^{-1/2}$

$$\frac{dp}{dq} = -\frac{1}{2}a(b+cq^2)^{-3/2}(2cq) = -acq(b+cq^2)^{-3/2}$$

$$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{a(b+cq^2)^{-1/2}}{-acq^2(b+cq^2)^{-3/2}} = -\frac{b+cq^2}{cq^2}$$

Thus η does not depend on a.

b. $\eta = -\frac{b+cq^2}{cq^2} = -\left(\frac{b}{cq^2}+1\right)$

If $b, c > 0$, then $\frac{b}{cq^2}+1>1$ so $|\eta| > 1$ and demand is elastic.

c. If $|\eta| = 1$, then $\left|\frac{b}{cq^2}+1\right| = 1,$ which can only occur if $b = 0$.

24. $\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{p}{q}\cdot\frac{dq}{dp}$

We differentiate implicitly for $\frac{dq}{dp}$.

$$q^2(1+p)^2 = p$$

$$q^2\cdot 2(1+p)(1) + \left(1+p^2\right)\left(2q\frac{dq}{dp}\right) = 1$$

$$2q^2(1+p) + 2q(1+p)^2\frac{dq}{dp} = 1$$

Thus $\dfrac{dq}{dp}=\dfrac{1-2q^2(1+p)}{2q(1+p)^2}$

Hence $\eta=\dfrac{q^2(1+p)^2}{q}\cdot\dfrac{1-2q^2(1+p)}{2q(1+p)^2}=\dfrac{1-2q^2(1+p)}{2}$

If $p = 9$, we find q from the given equation:

$q^2(1+9)^2=9$

$q^2=\dfrac{9}{100}$

$q=\dfrac{3}{10}$ since $q > 0$. Thus $\eta\big|_{p=9}=\dfrac{1-2\left(\frac{3}{10}\right)^2(1+9)}{2}=-0.4$

25. a. $q=\dfrac{60}{p}+\ln\left(65-p^3\right)$

$\eta=\dfrac{\frac{p}{q}}{\frac{dp}{dq}}=\dfrac{p}{q}\dfrac{dq}{dp}=\dfrac{p}{q}\left[-\dfrac{60}{p^2}-\dfrac{3p^2}{65-p^3}\right]$

If $p = 4$, then $q=\dfrac{60}{4}+\ln 1=15$, so $\eta=\dfrac{4}{15}\left[-\dfrac{60}{16}-\dfrac{3(16)}{65-64}\right]=-\dfrac{207}{15}\approx -13.8$, and demand is elastic.

b. The percentage change in q is (–2)(–13.8) = 27.6%, so q increases by approximately 27.6%.

c. Lowering the price increases revenue because demand is elastic.

26. a. $p=50\left[(151-q)^{0.02\sqrt{q+19}}\right]$

$\ln p=\ln 50+0.02\sqrt{q+19}\,\ln(151-q)$

$\dfrac{1}{p}\dfrac{dp}{dq}=0+0.02\left[\dfrac{\sqrt{q+19}}{151-q}(-1)+\ln(151-q)\cdot\dfrac{1}{2\sqrt{q+19}}\right]$

When $q = 150$, then $p = 50$, so $\dfrac{dp}{dq}\bigg|_{q=150}=0.02(50)\left[-\dfrac{13}{1}+\dfrac{0}{26}\right]=-13$

b. $\eta\big|_{q=150}=\dfrac{\frac{p}{q}}{\frac{dp}{dq}}\Bigg|_{q=150}=\dfrac{\frac{50}{150}}{-13}\approx -0.0256$

Thus demand is inelastic.

c. (elasticity)(% change in price) = % change in demand

$(-0.0256)(\text{% change in price})=\dfrac{-10}{150}\cdot 100$

$\text{% change in price}=-\dfrac{100}{15}\left(\dfrac{1}{-0.0256}\right)=260\%$

Thus price per unit of \$50 changes by 2.6(50) = \$130, so it is approximately 50 + 130 = \$180.

d. The manufacturer should increase the price because demand is inelastic.

27. The percentage change in price is $\frac{-5}{80}\cdot 100 = -\frac{25}{4}\%$ and the percentage change in quantity is $\frac{50}{500}\cdot 100 = 10\%$.

Thus, since (elasticity)(% change in price) $\approx$ % change in demand,

$$(\text{elasticity})\left(-\frac{25}{4}\right) \approx 10.$$

$$\text{elasticity} \approx -\frac{40}{25} = -\frac{8}{5} = -1.6$$

To estimate $\frac{dr}{dq}$ when $p = 80$, we have

$$\frac{dr}{dq} = p\left(1+\frac{1}{\eta}\right) = 80\left(1+\frac{1}{-\frac{8}{5}}\right) = 30.$$

28. $\eta = \dfrac{\frac{p}{q}}{\frac{dp}{dq}} = \dfrac{2000-q^2}{-2q^2} = \dfrac{1}{2} - \dfrac{1000}{q^2}$

For $5 \le q \le 40$, $|\eta| = \frac{1000}{q^2} - \frac{1}{2}$ and $|\eta|' = -\frac{2000}{q^3}$. Since $|\eta|' < 0$, $|\eta|$ is decreasing on [5, 40] and thus $|\eta|$ is maximum at $q = 5$ and a minimum at $q = 40$.

29. $\dfrac{dp}{dq} = 200(-1)(q+5)^{-2} = \dfrac{-200}{(q+5)^2}$

Thus $\eta = \dfrac{\frac{p}{q}}{\frac{dp}{dq}} = \dfrac{\frac{200}{q(q+5)}}{-\frac{200}{(q+5)^2}} = -\dfrac{q+5}{q}$.

For $5 \le q \le 95$, $|\eta| = \frac{q+5}{q} = 1+\frac{5}{q}$ and $|\eta|' = -\frac{5}{q^2}$.

Since $|\eta|' < 0$, $|\eta|$ is decreasing on [5, 95], and thus $|\eta|$ is maximum at $q = 5$ and minimum at $q = 95$.

Principles in Practice 12.4

1. Assume that P is a function of t and differentiate both sides of $\ln\left(\frac{P}{1-P}\right) = 0.5t$ with respect to t.

$$\frac{d}{dt}\left[\ln\left(\frac{P}{1-P}\right)\right] = \frac{d}{dt}[0.5t]$$

$$\left(\frac{1}{\frac{P}{1-P}}\right)\frac{d}{dt}\left[\frac{P}{1-P}\right] = 0.5$$

$$\frac{1-P}{P}\cdot\frac{(1)(1-P)-P(-1)}{(1-P)^2}\cdot\frac{dP}{dt} = 0.5$$

$$\frac{1-P+P}{P(1-P)}\cdot\frac{dP}{dt} = 0.5$$

$$\frac{dP}{dt} = 0.5P(1-P)$$

2. $\frac{dV}{dt} = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right] = \frac{4}{3}\pi\left(3r^2\right)\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$

When $\frac{dr}{dt} = 5$ and $r = 12$,

$\frac{dV}{dt} = 4\pi(12)^2(5) = 2880\pi$. The balloon is increasing at the rate of 2880π cubic inches/minute.

3. The hypotenuse is the length of the ladder, so $x^2 + y^2 = 100$. Differentiate both sides of the equation with respect to *t*.

$\frac{d}{dt}\left[x^2 + y^2\right] = \frac{d}{dt}[100]$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

When $y = 8$, we can find *x* by using the Pythagorean theorem.

$x^2 + 8^2 = 100$

$x^2 = 100 - 64 = 36$

$x = 6$

When $x = 6$, $y = 8$, and $\frac{dx}{dt} = 3$, we have

$2(6)(3) + 2(8)\frac{dy}{dt} = 0$

$36 + 16\frac{dy}{dt} = 0$

$\frac{dy}{dt} = -\frac{36}{16} = -\frac{9}{4}$

$\frac{dy}{dt} = -\frac{9}{4}$, thus the top of the ladder is sliding down the wall at the rate of $\frac{9}{4}$ feet/sec.

Problems 12.4

1. $2x + 8yy' = 0$

$x + 4yy' = 0$

$4yy' = -x$

$y' = -\frac{x}{4y}$

2. $6x + 12yy' = 0$

$y' = -\frac{x}{2y}$

3. $6y^2y' - 14x = 0$

$y' = \frac{14x}{6y^2} = \frac{7x}{3y^2}$

4. $4x - 6yy' = 0$

$y' = \frac{2x}{3y}$

5. $x^{1/3} + y^{1/3} = 3$

$\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}y' = 0$

$y^{-2/3}y' = -x^{-2/3}$

$y' = -\frac{x^{-2/3}}{y^{-2/3}}$

$= -\frac{y^{2/3}}{x^{2/3}}$

$= -\frac{\sqrt[3]{y^2}}{\sqrt[3]{x^2}}$

$= -\sqrt[3]{\frac{y^2}{x^2}}$

6. $\left(\frac{1}{5}\right)x^{-\frac{4}{5}} + \left(\frac{1}{5}\right)y^{-\frac{4}{5}}y' = 0$

$y' = -\frac{y^{\frac{4}{5}}}{x^{\frac{4}{5}}} = -\left(\frac{y}{x}\right)^{\frac{4}{5}}$

7. $\left(\frac{3}{4}\right)x^{-\frac{1}{4}} + \left(\frac{3}{4}\right)y^{-\frac{1}{4}}y' = 0$

$y' = -\frac{y^{\frac{1}{4}}}{x^{\frac{1}{4}}}$

8. $3y^2y' = 4$

$y' = \frac{4}{3y^2}$

9. By the product rule $xy' + y(1) = 0$, $xy' = -y$,

$y' = -\frac{y}{x}$

10. $2x + xy' + y(1) - 4yy' = 0$

$$xy' - 4yy' = -2x - y$$

$$y' = \frac{-2x - y}{x - 4y} = \frac{2x + y}{-x + 4y}$$

11. $xy' + y(1) - y' - 11 = 0$

$$y'(x-1) = 11 - y$$

$$y' = \frac{11 - y}{x - 1}$$

12. $3x^2 - 3y^2y' = 3x^2y' + 6xy - 3x(2yy') - 3y^2$

$$y'(-3y^2 - 3x^2 + 6xy) = 6xy - 3y^2 - 3x^2$$

$$y' = 1$$

13. $6x^2 + 3y^2y' - 12(xy' + y) = 0$

$$3y^2y' - 12xy' = 12y - 6x^2$$

$$y'\left(3y^2 - 12x\right) = 12y - 6x^2$$

$$y'\left(y^2 - 4x\right) = 4y - 2x^2$$

$$y' = \frac{4y - 2x^2}{y^2 - 4x}$$

14. $6x^2 + (3x)y' + y(3) + 3y^2y' = 0$

$$y'\left(3x + 3y^2\right) = -6x^2 - 3y$$

$$y'\left(x + y^2\right) = -2x^2 - y$$

$$y' = -\frac{2x^2 + y}{x + y^2}$$

15. $x = \sqrt{y} + \sqrt[4]{y} = y^{1/2} + y^{1/4}$

$$1 = \frac{1}{2}y^{-1/2}y' + \frac{1}{4}y^{-3/4}y'$$

$$= y'\left(\frac{1}{2y^{1/2}} + \frac{1}{4y^{3/4}}\right) = y'\left(\frac{2y^{1/4} + 1}{4y^{3/4}}\right)$$

$$y' = \frac{4y^{3/4}}{2y^{1/4} + 1}$$

16. $x^3\left(3y^2y'\right) + y^3\left(3x^2\right) + 1 = 0$

$$y' = -\frac{1 + 3x^2y^3}{3x^3y^2}$$

17. $5x^3(4y^3y') + 15x^2y^4 - 1 + 2yy' = 0$

$$y'(20x^3y^3 + 2y) = 1 - 15x^2y^4$$

$$y' = \frac{1 - 15x^2y^4}{20x^3y^3 + 2y}$$

18. $2yy' + y' = \frac{1}{x}$

$$(2y+1)y' = \frac{1}{x}$$

$$y' = \frac{1}{x(2y+1)}$$

19. $y\left(\frac{1}{x}\right) + (\ln x)y' = x\left(e^y y'\right) + e^y(1)$

$$\left[\ln(x) - xe^y\right]y' = e^y - \frac{y}{x}$$

$$\left[\ln(x) - xe^y\right]y' = \frac{xe^y - y}{x}$$

$$y' = \frac{xe^y - y}{x\left[\ln(x) - xe^y\right]}$$

20. $\frac{xy' + y(1)}{xy} + 1 = 0$

$$xy' + y + xy = 0$$

$$xy' = -y(x+1)$$

$$y' = -\frac{y(x+1)}{x}$$

21. $\left[x\left(e^y y'\right) + e^y(1)\right] + y' = 0$

$$xe^y y' + e^y + y' = 0$$

$$\left(xe^y + 1\right)y' = -e^y$$

$$y' = -\frac{e^y}{xe^y + 1}$$

22. $8x + 18yy' = 0$

$$8x = -18yy'$$

$$y' = -\frac{8x}{18y} = -\frac{4x}{9y}$$

23. $2\left(1 + e^{3x}\right)\left(3e^{3x}\right) = \frac{1}{x+y}(1 + y')$

$$6e^{3x}\left(1 + e^{3x}\right)(x+y) = 1 + y'$$

$$y' = 6e^{3x}\left(1 + e^{3x}\right)(x+y) - 1$$

24. $e^{x+y}(1 + y') = \frac{1}{x+y}(1 + y')$

$$e^{x+y} + y'e^{x+y} = \frac{1}{x+y} + \frac{y'}{x+y}$$

$$y'\left(e^{x+y} - \frac{1}{x+y}\right) = \frac{1}{x+y} - e^{x+y}$$

$$y' = -1$$

25. $1 + [xy' + y(1)] + 2yy' = 0$

$$xy' + 2yy' = -1 - y$$

$$(x + 2y)y' = -(1 + y)$$

$$y' = -\frac{1+y}{x+2y}$$

At the point (1, 2), $y' = -\frac{1+2}{1+4} = -\frac{3}{5}.$

26. $x\left(\frac{1}{2\sqrt{y+1}} \cdot y'\right) + \sqrt{y+1}(1)$

$$= y\left(\frac{1}{2\sqrt{x+1}}\right) + \sqrt{x+1}(y')$$

$$\frac{x}{2\sqrt{y+1}} \cdot y' - \sqrt{x+1} \cdot y' = \frac{y}{2\sqrt{x+1}} - \sqrt{y+1}$$

$$\left(\frac{x}{2\sqrt{y+1}} - \sqrt{x+1}\right)y' = \frac{y}{2\sqrt{x+1}} - \sqrt{y+1}$$

$$y' = \frac{\frac{y}{2\sqrt{x+1}} - \sqrt{y+1}}{\frac{x}{2\sqrt{y+1}} - \sqrt{x+1}}$$

At (3, 3), $\frac{dy}{dx} = \frac{\frac{3}{4} - 2}{\frac{3}{4} - 2} = 1.$

27. $8x + 18yy' = 0$

$y' = -\frac{8x}{18y} = -\frac{4x}{9y}$

Thus at $\left(0, \frac{1}{3}\right)$, $y' = 0$; at (x_0, y_0),

$y' = -\frac{4x_0}{9y_0}$.

28. $2(x^2 + y^2)(2x + 2yy') = 8yy'$

$(x^2 + y^2)(x + yy') = 2yy'$

$x^3 + x^2yy' + xy^2 + y^3y' = 2yy'$

$(x^2y + y^3 - 2y)y' = -x^3 - xy^2$

$y' = \frac{-x(x^2 + y^2)}{y(x^2 + y^2 - 2)}$

At (0, 2), $y' = 0$.

29. $3x^2 + xy' + y + 2y' = 0$

$y' = -\frac{3x^2 + y}{x + 2y}$

At (–1, 1), $y' = -4$ and the tangent line is given by $y - 1 = -4[x - (-1)]$, or $y = -4x - 3$.

30. $2yy' + [xy' + y(1)] - 2x = 0$

$y' = \frac{2x - y}{2y + x}$

At (4, 3), $y' = \frac{1}{2}$ and the tangent line is given by

$y - 3 = \frac{1}{2}(x - 4)$, or $y = \frac{1}{2}x + 1$.

31. $p = 100 - q^2$

$\frac{d}{dp}(p) = \frac{d}{dp}\left(100 - q^2\right)$

$1 = -2q \cdot \frac{dq}{dp}$

$\frac{dq}{dp} = -\frac{1}{2q}$

32. $p = 400 - \sqrt{q}$

$\frac{d}{dp}(p) = \frac{d}{dp}\left(400 - \sqrt{q}\right)$

$1 = -\frac{1}{2\sqrt{q}} \cdot \frac{dq}{dp}$

$\frac{dq}{dp} = -2\sqrt{q}$

33. $p = \frac{20}{(q+5)^2}$

$\frac{d}{dp}(p) = \frac{d}{dp}\left[\frac{20}{(q+5)^2}\right]$

$\frac{d}{dp}(p) = \frac{d}{dp}\left[20(q+5)^{-2}\right]$

$1 = -\frac{40}{(q+5)^3} \cdot \frac{dq}{dp}$

$\frac{dq}{dp} = -\frac{(q+5)^3}{40}$

34. $p = \frac{10}{q^2 + 3}$

$\frac{d}{dp}(p) = \frac{d}{dp}\left[\frac{10}{q^2 + 3}\right]$

$1 = -\frac{20q}{(q^2 + 3)^2} \cdot \frac{dq}{dp}$

$\frac{dq}{dp} = -\frac{(q^2 + 3)^2}{20q}$

From the original equation, we have $q^2 + 3 = \frac{10}{p}$. Thus we can write $\frac{dq}{dp}$ as

$\frac{dq}{dp} = -\frac{\left(\frac{10}{p}\right)^2}{20q} = -\frac{5}{qp^2}$.

35. $\ln \frac{I}{I_0} = -\lambda t$

$\ln I - \ln I_0 = -\lambda t$

$\frac{1}{I}\frac{dI}{dt} = -\lambda$

$\frac{dI}{dt} = -\lambda I$

36. $1.5M = \log\left(\dfrac{E}{2.5\times 10^{11}}\right)$

$1.5M = \log E - \log\left(2.5\times 10^{11}\right)$

$\dfrac{d}{dM}(1.5M) = \dfrac{d}{dM}\left[\log E - \log\left(2.5\times 10^{11}\right)\right]$

$\dfrac{d}{dM}(1.5M) = \dfrac{d}{dM}\left[\dfrac{\ln E}{\ln 10} - \log\left(2.5\times 10^{11}\right)\right]$

$1.5 = \dfrac{1}{\ln 10}\left(\dfrac{1}{E}\cdot\dfrac{dE}{dM}\right)$

$\dfrac{dE}{dM} = 1.5E\ln 10$

$\dfrac{d}{dE}(1.5M) = \dfrac{d}{dE}\left[\dfrac{\ln E}{\ln 10} - \log(2.5\times 10^{11})\right]$

$1.5\dfrac{dM}{dE} = \dfrac{1}{\ln 10}\cdot\dfrac{1}{E}$

$\dfrac{dM}{dE} = \dfrac{1}{1.5E\ln 10}$

37. $v = f\lambda$. Differentiating implicitly with respect to λ:

$0 = f(1) + \lambda\dfrac{df}{d\lambda},\ \dfrac{df}{d\lambda} = -\dfrac{f}{\lambda}.$

Solving $v = f\lambda$ for f and differentiating: $f = \dfrac{v}{\lambda}$,

so $\dfrac{df}{d\lambda} = -\dfrac{v}{\lambda^2} = -\dfrac{f\lambda}{\lambda^2} = -\dfrac{f}{\lambda}$, which is the same as before.

38. $(P + a)(v + b) = k$

$\dfrac{d}{dP}[(P+a)(v+b)] = \dfrac{d}{dP}(k)$

$(P+a)\dfrac{dv}{dP} + (v+b)(1) = 0$

$\dfrac{dv}{dP} = -\dfrac{v+b}{P+a}$. From the original equation,

$v + b = \dfrac{k}{(P+a)}$. Thus we can write $\dfrac{dv}{dP}$ as

$\dfrac{dv}{dP} = -\dfrac{k}{(P+a)^2}$.

39. $S^2 + \dfrac{1}{4}I^2 = SI + I$. Differentiating implicitly with respect to I:

$2S\dfrac{dS}{dI} + \dfrac{1}{2}I = \left[S(1) + I\dfrac{dS}{dI}\right] + 1,$

$2S\dfrac{dS}{dI} - I\dfrac{dS}{dI} = S + 1 - \dfrac{I}{2},$

$(2S - I)\dfrac{dS}{dI} = \dfrac{2S+2-I}{2},\ \dfrac{dS}{dI} = \dfrac{2S+2-I}{2(2S-I)}.$

Marginal propensity to consume $= \dfrac{dC}{dI} = 1 - \dfrac{dS}{dI}.$

Thus $\dfrac{dC}{dI} = 1 - \dfrac{2S+2-I}{2(2S-I)}$. When $I = 16$ and $S = 12$, $\dfrac{dC}{dI} = 1 - \dfrac{24+2-16}{2(24-16)} = 1 - \dfrac{10}{16} = \dfrac{6}{16} = \dfrac{3}{8}.$

40. $\ln\dfrac{f(t)}{1-f(t)} + \sigma\dfrac{1}{1-f(t)} = C_1 + C_2 t.$ Thus

$\ln f(t) - \ln[1-f(t)] + \sigma[1-f(t)]^{-1} = C_1 + C_2 t,$

$\dfrac{f'(t)}{f(t)} + \dfrac{f'(t)}{1-f(t)} + \dfrac{\sigma f'(t)}{[1-f(t)]^2} = C_2$

$f'(t)\left[\dfrac{1}{f(t)} + \dfrac{1}{1-f(t)} + \dfrac{\sigma}{[1-f(t)]^2}\right] = C_2$

$f'(t)\left[\dfrac{[1-f(t)]^2 + f(t)[1-f(t)] + \sigma f(t)}{f(t)[1-f(t)]^2}\right] = C_2$

$f'(t)\left[\dfrac{[1-f(t)][1-f(t)+f(t)] + \sigma f(t)}{f(t)[1-f(t)]^2}\right] = C_2$

$f'(t)\left[\dfrac{[1-f(t)] + \sigma f(t)}{f(t)[1-f(t)]^2}\right] = C_2$

Thus $f'(t) = \dfrac{C_2 f(t)[1-f(t)]^2}{\sigma f(t) + [1-f(t)]}$

Problems 12.5

1. $y=(x+1)^2(x-2)\left(x^2+3\right)$. Take natural logarithms of both sides,

$\ln y=\ln\left[(x+1)^2(x-2)\left(x^2+3\right)\right]$.

Using properties of logarithms on the right side gives

$\ln y=2\ln(x+1)+\ln(x-2)+\ln\left(x^2+3\right)$.

Differentiating both sides with respect to *x*,

$$\frac{y'}{y}=\frac{2}{x+1}+\frac{1}{x-2}+\frac{2x}{x^2+3}.$$

Solving for y',

$$y'=y\left[\frac{2}{x+1}+\frac{1}{x-2}+\frac{2x}{x^2+3}\right].$$

Expressing y' in terms of *x*,

$$y'=(x+1)^2(x-2)\left(x^2+3\right)\left[\frac{2}{x+1}+\frac{1}{x-2}+\frac{2x}{x^2+3}\right]$$

2. $\ln y=\ln\left[(3x+4)(8x-1)^2\left(3x^2+1\right)^4\right]$

$=\ln(3x+4)+2\ln(8x-1)+4\ln\left(3x^2+1\right)$

$$\frac{y'}{y}=\frac{3}{3x+4}+2\cdot\frac{8}{8x-1}+4\cdot\frac{6x}{3x^2+1}$$

$$y'=y\left[\frac{3}{3x+4}+\frac{16}{8x-1}+\frac{24x}{3x^2+1}\right]$$

$$=(3x+4)(8x-1)^2\left(3x^2+1\right)^4\left[\frac{3}{3x+4}+\frac{16}{8x-1}+\frac{24x}{3x^2+1}\right]$$

3. $\ln y=\ln\left[\left(3x^3-1\right)^2(2x+5)^3\right]$

$=2\ln\left(3x^3-1\right)+3\ln(2x+5)$

$$\frac{y'}{y}=2\cdot\frac{9x^2}{3x^3-1}+3\cdot\frac{2}{2x+5}$$

$$y'=y\left[\frac{18x^2}{3x^3-1}+\frac{6}{2x+5}\right]$$

$$y'=\left(3x^3-1\right)^2(2x+5)^3\left[\frac{18x^2}{3x^3-1}+\frac{6}{2x+5}\right]$$

4. $y = (2x^2+1)\sqrt{8x^2-1}$

$$\ln y = \ln\left[(2x^2+1)\sqrt{8x^2-1}\right]$$

$$= \ln(2x^2+1) + \frac{1}{2}\ln(8x^2-1)$$

$$\frac{y'}{y} = \frac{4x}{2x^2+1} + \frac{1}{2}\cdot\frac{16x}{8x^2-1}$$

$$y' = y\left[\frac{4x}{2x^2+1} + \frac{8x}{8x^2-1}\right]$$

$$= (2x^2+1)\sqrt{8x^2-1}\left[\frac{4x}{2x^2+1} + \frac{8x}{8x^2-1}\right]$$

5. $y = \sqrt{x+1}\sqrt{x^2-2}\sqrt{x+4}$

$$\ln y = \ln\left(\sqrt{x+1}\sqrt{x^2-2}\sqrt{x+4}\right)$$

$$\ln y = \frac{1}{2}\ln(x+1) + \frac{1}{2}\ln\left(x^2-2\right) + \frac{1}{2}\ln(x+4)$$

$$\frac{y'}{y} = \frac{1}{2}\left[\frac{1}{x+1} + \frac{2x}{x^2-2} + \frac{1}{x+4}\right]$$

$$y' = \frac{y}{2}\left[\frac{1}{x+1} + \frac{2x}{x^2-2} + \frac{1}{x+4}\right]$$

$$= \frac{\sqrt{x+1}\sqrt{x^2-2}\sqrt{x+4}}{2}\left[\frac{1}{x+1} + \frac{2x}{x^2-2} + \frac{1}{x+4}\right]$$

6. $\ln y = \ln\left[(2x+1)\sqrt{x^3+2}\sqrt[3]{2x+5}\right]$

$$= \ln(2x+1) + \frac{1}{2}\ln(x^3+2) + \frac{1}{3}\ln(2x+5)$$

$$\frac{y'}{y} = \frac{2}{2x+1} + \frac{1}{2}\cdot\frac{3x^2}{x^3+2} + \frac{1}{3}\cdot\frac{2}{2x+5}$$

$$y' = y\left[\frac{2}{2x+1} + \frac{3x^2}{2(x^3+2)} + \frac{2}{3(2x+5)}\right]$$

$$= (2x+1)\sqrt{x^3+2}\sqrt[3]{2x+5}\left[\frac{2}{2x+1} + \frac{3x^2}{2(x^3+2)} + \frac{2}{3(2x+5)}\right]$$

7. $\ln y = \ln \frac{\sqrt{1-x^2}}{1-2x} = \frac{1}{2}\ln\left(1-x^2\right) - \ln(1-2x)$

$$\frac{y'}{y} = \frac{1}{2}\cdot\frac{-2x}{1-x^2} - \frac{-2}{1-2x}$$

$$y' = y\left[-\frac{x}{1-x^2} + \frac{2}{1-2x}\right]$$

$$y' = \frac{\sqrt{1-x^2}}{1-2x}\left[\frac{x}{x^2-1} + \frac{2}{1-2x}\right]$$

8. $\ln y = \ln\sqrt{\frac{x^2+5}{x+9}} = \frac{1}{2}\left[\ln\left(x^2+5\right) - \ln(x+9)\right]$

$$\frac{y'}{y} = \frac{1}{2}\left[\frac{2x}{x^2+5} - \frac{1}{x+9}\right]$$

$$y' = \frac{y}{2}\left[\frac{2x}{x^2+5} - \frac{1}{x+9}\right]$$

$$y' = \frac{1}{2}\sqrt{\frac{x^2+5}{x+9}}\left[\frac{2x}{x^2+5} - \frac{1}{x+9}\right]$$

9. $y = \frac{\left(2x^2+2\right)^2}{(x+1)^2(3x+2)}$

$$\ln y = \ln\left[\frac{\left(2x^2+2\right)^2}{(x+1)^2(3x+2)}\right]$$

$$= 2\ln\left(2x^2+2\right) - 2\ln(x+1) - \ln(3x+2)$$

$$\frac{y'}{y} = 2\cdot\frac{4x}{2x^2+2} - 2\cdot\frac{1}{x+1} - \frac{3}{3x+2}$$

$$y' = y\left[\frac{8x}{2x^2+2} - \frac{2}{x+1} - \frac{3}{3x+2}\right]$$

$$= \frac{\left(2x^2+2\right)^2}{(x+1)^2(3x+2)}\left[\frac{4x}{x^2+1} - \frac{2}{x+1} - \frac{3}{3x+2}\right]$$

10. $\ln y = \ln\frac{x\left(1+x^2\right)^2}{\sqrt{2+x^2}}$

$$= \ln x + 2\ln\left(1+x^2\right) - \frac{1}{2}\ln\left(2+x^2\right)$$

$$\frac{y'}{y} = \frac{1}{x} + 2\cdot\frac{2x}{1+x^2} - \frac{1}{2}\cdot\frac{2x}{2+x^2}$$

$$y' = y\left[\frac{1}{x} + \frac{4x}{1+x^2} - \frac{x}{2+x^2}\right]$$

$$y' = \frac{x\left(1+x^2\right)^2}{\sqrt{2+x^2}}\left[\frac{1}{x} + \frac{4x}{1+x^2} - \frac{x}{2+x^2}\right]$$

11. $y = \sqrt{\frac{(x+3)(x-2)}{2x-1}}$

$$\ln y = \ln\sqrt{\frac{(x+3)(x-2)}{2x-1}}$$

$$= \frac{1}{2}\ln(x+3) + \frac{1}{2}\ln(x-2) - \frac{1}{2}\ln(2x-1)$$

$$\frac{y'}{y} = \frac{1}{2}\cdot\frac{1}{x+3} + \frac{1}{2}\cdot\frac{1}{x-2} - \frac{1}{2}\cdot\frac{2}{2x-1}$$

$$y' = \frac{y}{2}\left[\frac{1}{x+3} + \frac{1}{x-2} - \frac{2}{2x-1}\right]$$

$$= \frac{1}{2}\sqrt{\frac{(x+3)(x-2)}{2x-1}}\left[\frac{1}{x+3} + \frac{1}{x-2} - \frac{2}{2x-1}\right]$$

12. $\ln y = \ln\sqrt[3]{\frac{6\left(x^3+1\right)^2}{x^6e^{-4x}}}$

$$= \frac{1}{3}\left[\ln(6) + 2\ln\left(x^3+1\right) - 6\ln(x) - (-4x)\ln e\right]$$

$$= \frac{1}{3}\left[\ln(6) + 2\ln\left(x^3+1\right) - 6\ln(x) + 4x\right]$$

$$\frac{y'}{y} = \frac{1}{3}\left[2\cdot\frac{3x^2}{x^3+1} - \frac{6}{x} + 4\right]$$

$$y' = \frac{y}{3}\left[\frac{6x^2}{x^3+1} - \frac{6}{x} + 4\right]$$

$$y = \frac{1}{3}\sqrt[3]{\frac{6\left(x^3+1\right)^2}{x^6e^{-4x}}}\left[\frac{6x^2}{x^3+1} - \frac{6}{x} + 4\right]$$

13. $y = x^{x^2+1}$, thus $\ln y = \ln x^{x^2+1} = (x^2+1)\ln x.$

$$\frac{y'}{y} = (x^2+1)\cdot\frac{1}{x} + (\ln x)(2x)$$

$$y' = y\left(\frac{x^2+1}{x} + 2x\ln x\right)$$

$$= x^{x^2+1}\left(\frac{x^2+1}{x} + 2x\ln x\right)$$

14. $y = (2x)^{\sqrt{x}}$. Thus

$$\ln y = \ln(2x)^{\sqrt{x}} = \sqrt{x}[\ln 2 + \ln x].$$

$$\frac{y'}{y} = \sqrt{x}\left[\frac{1}{x}\right] + [\ln 2 + \ln x]\cdot\frac{1}{2\sqrt{x}}$$

$$y' = y\left[\frac{1}{\sqrt{x}} + \frac{\ln(2x)}{2\sqrt{x}}\right]$$

$$y' = (2x)^{\sqrt{x}}\left[\frac{2+\ln(2x)}{2\sqrt{x}}\right]$$

15. $y = x^{\frac{1}{x}}$. Thus $\ln y = \frac{1}{x}\ln x = \frac{\ln x}{x}.$

$$\frac{y'}{y} = \frac{x\left(\frac{1}{x}\right) - (\ln x)(1)}{x^2}$$

$$y' = y\left[\frac{1-\ln x}{x^2}\right]$$

$$y' = \frac{x^{\frac{1}{x}}(1-\ln x)}{x^2}$$

16. $y = \left(\frac{3}{x^2}\right)^x$. Thus

$$\ln y = x\ln\left(\frac{3}{x^2}\right) = x[\ln 3 - 2\ln x].$$

$$\frac{y'}{y} = x\left(-\frac{2}{x}\right) + (\ln 3 - 2\ln x)(1)$$

$$= -2 + \ln\left(\frac{3}{x^2}\right)$$

$$y' = y\left[-2 + \ln\left(\frac{3}{x^2}\right)\right] = \left(\frac{3}{x^2}\right)^x\left[-2 + \ln\left(\frac{3}{x^2}\right)\right]$$

17. $y = (3x+1)^{2x}$. Thus

$$\ln y = \ln\left[(3x+1)^{2x}\right] = 2x\ln(3x+1)$$

$$\frac{y'}{y} = 2\left\{x\left(\frac{3}{3x+1}\right) + [\ln(3x+1)](1)\right\}$$

$$y' = 2y\left[\frac{3x}{3x+1} + \ln(3x+1)\right]$$

$$= 2(3x+1)^{2x}\left[\frac{3x}{3x+1} + \ln(3x+1)\right]$$

18. $y = (x^2+1)^{x+1}$, thus

$$\ln y = \ln(x^2+1)^{x+1} = (x+1)\ln(x^2+1).$$

$$\frac{y'}{y} = x+1\cdot\frac{2x}{x^2+1} + \ln(x^2+1)\cdot 1$$

$$y' = y\left[\frac{2x(x+1)}{x^2+1} + \ln(x^2+1)\right]$$

$$= (x^2+1)^{x+1}\left[\frac{2x(x+1)}{x^2+1} + \ln(x^2+1)\right]$$

19. $y = 4e^x x^{3x}$. Thus

$$\ln y = \ln 4 + \ln\left(e^x x^{3x}\right) = \ln 4 + \ln e^x + \ln x^{3x}$$

$$= \ln 4 + x + 3x\ln x.$$

$$\frac{y'}{y} = 1 + 3\left[x\left(\frac{1}{x}\right) + (\ln x)(1)\right]$$

$$y' = y(4 + 3\ln x)$$

$$y' = 4e^x x^{3x}(4 + 3\ln x)$$

20. $y = (\ln x)^{e^x}$. Thus $\ln y = e^x\ln(\ln x).$

$$\frac{y'}{y} = e^x\left[\frac{1}{x\ln x}\right] + [\ln(\ln x)]e^x$$

$$y' = y\left[\frac{1}{x\ln x} + \ln(\ln x)\right]e^x$$

$$= (\ln x)^{e^x}\left[\frac{1}{x\ln x} + \ln(\ln x)\right]e^x$$

21. $y=(4x-3)^{2x+1}$

$\ln y=\ln(4x-3)^{2x+1}=(2x+1)\ln(4x-3)$

$\frac{y'}{y}=(2x+1)\left[\frac{4}{4x-3}\right]+[\ln(4x-3)](2)$

$y'=y\left[\frac{4(2x+1)}{4x-3}+2\ln(4x-3)\right]$

When $x=1$, then $\frac{dy}{dx}=1\left[\frac{12}{1}+2\ln(1)\right]=12.$

22. $y=(\ln x)^{\ln x}$

$\ln y=\ln(\ln x)^{\ln x}=(\ln x)\ln(\ln x)$

$\frac{y'}{y}=(\ln x)\left[\frac{1}{\ln x}\cdot\frac{1}{x}\right]+[\ln(\ln x)]\left(\frac{1}{x}\right)$

$y'=y\left[\frac{1}{x}+\frac{\ln(\ln x)}{x}\right]$

$y'=(\ln x)^{\ln x}\left[\frac{1+\ln(\ln x)}{x}\right]$

When $x=e$, $\frac{dy}{dx}=1^1\left[\frac{1+\ln(1)}{e}\right]=e^{-1}.$

23. $y=(x+1)(x+2)^2(x+3)^2$

$\ln y=\ln(x+1)+2\ln(x+2)+2\ln(x+3)$

$\frac{y'}{y}=\frac{1}{x+1}+\frac{2}{x+2}+\frac{2}{x+3}$

$y'=y\left[\frac{1}{x+1}+\frac{2}{x+2}+\frac{2}{x+3}\right]$

When $x=0$, then $y=36$ and $y'=96$. Thus an equation of the tangent line is $y-36=96(x-0)$, or $y=96x+36$.

24. $y=x^x$

$\ln y=x\ln x$

$\frac{y'}{y}=x\cdot\frac{1}{x}+(\ln x)(1)=1+\ln x$

$y'=y(1+\ln x)=x^x(1+\ln x)$

When $x=1$, then $y=1$ and $y'=1^1(1+\ln 1)=1(1+0)=1$. An equation of the tangent line is $y-1=1(x-1)$ or $y=x$.

25. $y=e^x(x^2+1)^x$

$\ln y=\ln e^x+\ln(x^2+1)^x$

$=x+x\ln\left(x^2+1\right)$

$\frac{y'}{y}=1+\left[x\left(\frac{2x}{x^2+1}\right)\right]+\left[\ln\left(x^2+1\right)(1)\right]$

$y'=y\left[1+\frac{2x^2}{x^2+1}+\ln\left(x^2+1\right)\right]$

When $x=1$, then $y=2e$ and $y'=2e[1+1+\ln(2)]=2e(2+\ln 2)$. Thus an equation of the tangent line is $y-2e=2e(2+\ln 2)(x-1)$, or $y=(4e+2e\ln 2)x-2e-2e\ln 2$.

26. $y=x^x$

$\ln y=x\ln x$

$\frac{y'}{y}=x\cdot\frac{1}{x}+(\ln x)(1)=1+\ln x$

When $x=1$, $\frac{y'}{y}=1+\ln 1=1+0=1.$

27. $y=(3x)^{-2x}$

$\ln y=-2x\ln(3x)$

$\frac{y'}{y}=-2\left\{x\left[\frac{1}{3x}(3)\right]+[\ln(3x)](1)\right\}$

$=-2[1+\ln(3x)]$

$\frac{y'}{y}\cdot 100$ gives the percentage rate of change.

Thus $-2[1+\ln(3x)](100)=60$

$1+\ln(3x)=-0.3$

$\ln(3x)=-1.3$

$3x=e^{-1.3}$

$x=\frac{1}{3e^{1.3}}$

28. $y=[f(x)]^{g(x)}$

$\ln y=g(x)\ln[f(x)]$

$\frac{y'}{y}=g(x)\left(\frac{1}{f(x)}\cdot f'(x)\right)+\ln[f(x)]g'(x)$

$y'=y\left(f'(x)\frac{g(x)}{f(x)}+g'(x)\ln[f(x)]\right)$

$y'=[f(x)]^{g(x)}\left(f'(x)\frac{g(x)}{f(x)}+g'(x)\ln[f(x)]\right)$

29. $\frac{r'}{r}\cdot 100\% = \frac{p'}{p}\cdot 100\% + \frac{q'}{q}\cdot 100\%$

$= (1+\eta)\frac{p'}{p}100\%$

where $\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{p}{q}\cdot\frac{dq}{dp}.$

$\eta = \frac{p}{500-40p+p^2}\cdot(-40+2p)$

When $p = 15$, then $\eta = -1.2$ and a $\frac{1}{2}\%$ increase in price will result in a $(1-1.2)\left(\frac{1}{2}\%\right) = -0.1\%$ change in revenue, which is a 0.1% decrease in revenue.

30. $\frac{r'}{r}\cdot 100\% = \frac{p'}{p}\cdot 100\% + \frac{q'}{q}\cdot 100\%$

$= (1+\eta)\frac{p'}{p}100\%$

where $\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{p}{q}\cdot\frac{dq}{dp}.$

$\eta = \frac{p}{500-40p+p^2}\cdot(-40+2p)$

When $p = 15$, then $\eta = -1.2$ and a 10% decrease in price will result in a $(1 - 1.2)(-10\%) = 2\%$ change in revenue, which is a 2% increase in revenue.

Principles in Practice 12.6

1. Let $f(x) = 20x - 0.01x^2 - 850 + 3\ln x$, then $f'(x) = 20 - 0.02x + \frac{3}{x}$. $f(10) \approx -644$ and $f(50) \approx 137$, so we use 50 to be the first approximation, x_1, to find the break-even quantity between 10 and 50.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{20x_n - 0.01x_n^2 - 850 + 3\ln x_n}{20 - 0.02x_n + 3x_n^{-1}}$$

$$= x_n - \frac{20x_n^2 - 0.01x_n^3 - 850x_n + 3x_n \ln x_n}{20x_n - 0.02x_n^2 + 3}$$

$$= \frac{20x_n^2 - 0.02x_n^3 + 3x_n - \left(20x_n^2 - 0.01x_n^3 - 850x_n + 3x_n \ln x_n\right)}{20x_n - 0.02x_n^2 + 3}$$

$$= \frac{-0.01x_n^3 + 853x_n - 3x_n \ln x_n}{20x_n - 0.02x_n^2 + 3}$$

$$x_2 = 50 - \frac{f(50)}{f'(50)} \approx 42.82602$$

$$x_3 = 42.82602 - \frac{f(42.82602)}{f'(42.82602)} \approx 42.85459$$

$x_4 = 42.85459 - \frac{f(42.85459)}{f'(42.85459)} \approx 42.85459$

Since the values of x_3 and x_4 differ by less than 0.0001, we take the first break-even quantity to be $x \approx 42.85459$ or 43 televisions.

$f(1900) \approx 1073$ and $f(2000) \approx -827$, so we use 2000 to be the first approximation, x_1, for the break-even quantity between 1900 and 2000.

$x_2 = 2000 - \frac{f(2000)}{f'(2000)} \approx 1958.63703$

$x_3 = 1958.63703 - \frac{f(1958.63703)}{f'(1958.63703)} \approx 1957.74457$

$x_4 = 1957.74457 - \frac{f(1957.74457)}{f'(1957.74457)} \approx 1957.74415$

$x_5 = 1957.74415 - \frac{f(1957.74415)}{f'(1957.74415)} \approx 1957.74415$

Since the values of x_4 and x_5 differ by less than 0.0001, we take the second break-even quantity to be $x \approx 1957.74415$ or 1958 televisions.

Problems 12.6

1. We want a root of $f(x) = x^3 - 4x + 1 = 0.$ We see that $f(0) = 1$ and $f(1) = -2$ have opposite signs, so there must be a root between 0 and 1. Moreover, $f(0)$ is closer to 0 than is $f(1)$, so we select $x_1 = 0$ as our initial estimate. Since $f'(x) = 3x^2 - 4,$ the recursion formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 4x_n + 1}{3x_n^2 - 4}.$$

Simplifying gives $x_{n+1} = \frac{2x_n^3 - 1}{3x_n^2 - 4}$. Thus we obtain:

n	x_n	x_{n+1}
1	0.00000	0.25000
2	0.25000	0.25410
3	0.25410	0.25410

Because $|x_4 - x_3| < 0.0001,$ the root is approximately $x_4 = 0.25410.$

2. Let $f(x) = x^3 + 2x^2 - 1.$ $f\left(\frac{1}{2}\right) = -\frac{3}{8}$ and

$f(1) = 2$ (note the sign change). Since $f\left(\frac{1}{2}\right)$ is closer to 0 than is $f(1)$, we select $x_1 = \frac{1}{2}.$ We have

$f'(x) = 3x^2 + 4x,$ so the recursion formula is $x_{n+1} = x_n - \frac{x_n^3 + 2x_n^2 - 1}{3x_n^2 + 4x_n}$

n	x_n	x_{n+1}
1	0.50000	0.63636
2	0.63636	0.61838
3	0.61838	0.61803
4	0.61803	0.61803

Because $|x_5 - x_4| < 0.0001,$ the root is approximately $x_5 = 0.61803.$
(Note that $f'(0) = 0,$ so we cannot use 0 for x_1.)

3. Let $f(x) = x^3 - x - 1.$ We have $f(1) = -1$ and $f(2) = 5$ (note the sign change). Since $f(1)$ is closer to 0 than is $f(2)$, we choose $x_1 = 1.$ We have $f'(x) = 3x^2 - 1,$ so the recursion formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}$$
$$= \frac{2x_n^3 + 1}{3x_n^2 - 1}$$

n	x_n	x_{n+1}
1	1.00000	1.50000
2	1.50000	1.34783
3	1.34783	1.32520
4	1.32520	1.32472
5	1.32472	1.32472

Since $|x_6 - x_5| < 0.0001,$ the root is approximately $x_6 = 1.32472.$

4. Let $f(x) = x^3 - 9x + 6.$ We have $f(2.5) = -0.875$ and $f(3) = 6.$ Since $f(2.5)$ is closer to 0 than is $f(3)$, we choose $x_1 = 2.5.$ We have

$f'(x) = 3x^2 - 9$, so $x_{n+1} = x_n - \dfrac{x_n^3 - 9x_n + 6}{3x_n^2 - 9}.$

n	x_n	x_{n+1}
1	2.50000	2.58974
2	2.58974	2.58425
3	2.58425	2.58423
4	2.58423	2.58423

Since $|x_5 - x_4| < 0.0001,$ the root is approximately $x_5 = 2.58423.$

5. Let $f(x) = x^3 + x + 1.$ We have $f(-1) = -1$ and $f(0) = 1$ (note the sign change). Choose $x_1 = -1.$ Since $f'(x) = 3x^2 + 1,$ the recursion formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n + 1}{3x_n^2 + 1}$$
$$= \frac{2x_n^3 - 1}{3x_n^2 + 1}$$

n	x_n	x_{n+1}
1	–1	–0.75000
2	–0.75000	–0.68605
3	–0.68605	–0.68234
4	–0.68234	–0.68233

Because $|x_5 - x_4| < 0.0001,$ the root is approximately $x_5 = -0.68233.$

6. $x^3 = 2x + 5,$ so use $f(x) = x^3 - 2x - 5 = 0.$ We have $f(2) = -1$ and $f(3) = 16$, so $f(2)$ is closer to 0 than is $f(3)$. We choose $x_1 = 2.$ Since $f'(x) = 3x^2 - 2,$ the recursion formula is

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2} = \frac{2x_n^3 + 5}{3x_n^2 - 2}$$

n	x_n	x_{n+1}
1	2.00000	2.10000
2	2.10000	2.09457
3	2.09457	2.09455

Because $|x_4 - x_3| < 0.0001,$ the root is approximately $x_4 = 2.09455.$

7. $x^4 = 3x - 1$, so use $f(x) = x^4 - 3x + 1 = 0$. Since $f(0) = 1$ and $f(1) = -1$ (note the sign change), $f(0)$ and $f(1)$ are equally close to 0. We shall choose $x_1 = 0$. Since $f'(x) = 4x^3 - 3$, the recursion formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 3x_n + 1}{4x_n^3 - 3} = \frac{3x_n^4 - 1}{4x_n^3 - 3}$$

n	x_n	x_{n+1}
1	0.00000	0.33333
2	0.33333	0.33766
3	0.33766	0.33767

Because $|x_4 - x_3| < 0.0001$, the root is approximately $x_4 = 0.33767$.

8. Let $f(x) = x^4 + 4x - 1$. Since $f(-2) = 7$ and $f(-1) = -4$, $f(-1)$ is closer to 0 than is $f(-2)$. However, $f'(-1) = 0$, so we shall choose $x_1 = -2$. Since $f'(x) = 4x^3 + 4$, the recursion formula is

$$x_{n+1} = x_n - \frac{x_n^4 + 4x_n - 1}{4x_n^3 + 4} = \frac{3x_n^4 + 1}{4x_n^3 + 4}$$

n	x_n	x_{n+1}
1	–2.00000	–1.75000
2	–1.75000	–1.67092
3	–1.67092	–1.66332
4	–1.66332	–1.66325

Because $|x_5 - x_4| < 0.0001$, the root is approximately $x_5 = -1.66325$.

9. Let $f(x) = x^4 - 2x^3 + x^2 - 3$. $f(1) = -3$ and $f(2) = 1$ (note the sign change), so $f(2)$ is closer to 0 than is $f(1)$. We choose $x_1 = 2$. Since $f'(x) = 4x^3 - 6x^2 + 2x$, the recursion formula is

$$x_{n+1} = x_n - \frac{f(x)}{f'(x_n)} = x_n - \frac{x_n^4 - 2x_n^3 + x_n^2 - 3}{4x_n^3 - 6x_n^2 + 2x_n}$$

n	x_n	x_{n+1}
1	2.00000	1.91667
2	1.91667	1.90794
3	1.90794	1.90785

Because $|x_4 - x_3| < 0.0001$, the root is approximately $x_4 = 1.90785$.

10. Let $f(x) = x^4 - x^3 + x - 2$. $f(1) = -1$ and $f(2) = 8$, so $f(1)$ is closer to 0 than is $f(2)$. We choose $x_1 = 1$. Since $f'(x) = 4x^3 - 3x^2 + 1$, the recursion formula is

$$x_{n+1} = x_n - \frac{x_n^4 - x_n^3 + x_n - 2}{4x_n^3 - 3x_n^2 + 1}$$

n	x_n	x_{n+1}
1	1.00000	1.50000
2	1.50000	1.34677
3	1.34677	1.31040
4	1.31040	1.30858
5	1.30858	1.30857

Because $|x_6 - x_5| < 0.0001$, the root is approximately $x_6 = 1.30857$.

11. The desired number is x, where $x^3 = 71$, or $x^3 - 71 = 0$. Thus we want to find a root of $f(x) = x^3 - 71 = 0$. Since $4^3 = 64$, the solution should be close to 4, so we choose $x_1 = 4$ as our initial estimate. We have $f'(x) = 3x^2$, so the recursion formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 71}{3x_n^2} = \frac{2x_n^3 + 71}{3x_n^2}$$

n	x_n	x_{n+1}
1	4	4.146
2	4.146	4.141
4	4.141	4.141

Thus to three decimal places, $\sqrt[3]{71} = 4.141$.

12. The desired number is x, where $x^4 = 19$, or $x^4 - 19 = 0$. Thus we want to find a root of $f(x) = x^4 - 19$. Since $2^4 = 16$, the solution should be close to 2, so we choose $x_1 = 2$ as our initial estimate. We have $f'(x) = 4x^3$, so the recursion formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 19}{4x_n^3}$$
$$= \frac{3x_n^4 + 19}{4x_n^3}$$

n	x_n	x_{n+1}
1	2	2.09
2	2.09	2.09

Thus to two decimal places, $\sqrt[4]{19} = 2.09$.

13. We want real solutions to $e^x = x + 5$. Thus we want to find roots of $f(x) = e^x - x - 5 = 0$. A rough sketch of the exponential function $y = e^x$ and the line $y = x + 5$ shows that there are two intersection points: one when x is near –5, and the other when x is near 3. Thus we must find two roots. Since $f'(x) = e^x - 1$, the recursion formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{x_n} - x_n - 5}{e^{x_n} - 1}$$

If $x_1 = -5$, we obtain

n	x_n	x_{n+1}
1	–5	–4.99
2	–4.99	–4.99

If $x_1 = 3$, we obtain:

n	x_n	x_{n+1}
1	3	2.37
2	2.37	2.03
3	2.03	1.94
4	1.94	1.94

Thus the solutions are –4.99 and 1.94.

14. We must solve $\ln x = 5 - x$. That is, we must determine all roots of $f(x) = \ln(x) + x - 5 = 0$. A rough sketch shows that the graph of the logarithmic function $y = \ln x$ intersects the line $y = 5 - x$ at one point, where x is between 3 and 4. We choose $x_1 = 3$. Since $f'(x) = \frac{1}{x} + 1$, the recursion formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\ln(x_n) + x_n - 5}{\frac{1}{x_n} + 1}$$

n	x_n	x_{n+1}
1	3	3.676
2	2.676	3.693
3	3.693	3.693

Thus the solution is approximately 3.693.

15. The break-even quantity is the value of q when total revenue and total cost are equal: $r = c$, or $r - c = 0$. Thus we must find a root of $3q - \left(250 + 2q - 0.1q^3\right) = 0$, or $f(q) = q - 250 + 0.1q^3 = 0$, so $f'(q) = 1 + 0.3q^2$. The recursion formula is

$$q_{n+1} = q_n - \frac{f(q_n)}{f'(q_n)} = q_n - \frac{q_n - 250 + 0.1q_n{}^3}{1 + 0.3q_n{}^2}$$

We choose $q_1 = 13$, as suggested.

n	q_n	q_{n+1}
1	13	13.33
2	13.33	13.33

Thus $q \approx 13.33$.

16. a. The break-even quantity is the value of q when total cost = total revenue: $c = r$, $c - r = 0$. Thus we solve $40+3q+\frac{q^2}{1000}+\frac{1}{q}=7q.$ Multiplying both sides by q and simplifying, we see that the problem is equivalent to solving

$$f(q)=\frac{q^3}{1000}-4q^2+40q+1=0.$$

b. Since $f'(q)=\frac{3q^2}{1000}-8q+40,$ the recursion formula is

$$q_{n+1}=q_n-\frac{f(q_n)}{f'(q_n)}$$

$$=q_n-\frac{\frac{q_n^3}{1000}-4q_n^2+40q_n+1}{\frac{3q_n^2}{1000}-8q_n+40}$$

We select $q_1 = 10$ as suggested.

n	q_n	q_{n+1}
1	10	10.05
2	10.05	10.05

Thus $q \approx 10.05$.

17. The equilibrium quantity is the value of q for which supply and demand are equal, that is, it is a root of $2q+5=\frac{100}{q^2+1}$, or of $f(q)=2q+5-\frac{100}{q^2+1}=0.$ Since $f'(q)=2+\frac{200q}{(q^2+1)^2}$, the recursion formula is

$$q_{n+1}=q_n-\frac{f(q_n)}{f'(q_n)}=q_n-\frac{2q_n+5-\frac{100}{q_n^2+1}}{2+\frac{200q_n}{(q_n^2+1)^2}}$$

A rough sketch shows that the graph of the supply equation intersects the graph of the demand equation when q is near 3. Thus we select $q_1 = 3$.

n	q_n	q_{n+1}
1	3	2.875
2	2.875	2.880
3	2.880	2.880

Thus $q \approx 2.880$.

18. In the same manner as problem 17, we must find a root of $f(q)=0.2q^3+1.5q-8=0,$ so $f'(q)=0.6q^2+1.5.$ The recursion formula is

$$q_{n+1}=q_n-\frac{f(q_n)}{f'(q_n)}=q_n-\frac{0.2q_n^3+1.5q_n-8}{0.6q_n^2+1.5}$$

We select $q_1 = 5$ as suggested.

n	q_n	q_{n+1}
1	5	3.54
2	3.54	2.85
3	2.85	2.71
4	2.71	2.70
5	2.70	2.70

Thus $q = 2.70$, so $p = 10 - 2.70 = 7.30$ (from the demand equation).

19. For a critical value of $f(x)=\frac{x^3}{3}-x^2-5x+1,$ we want a root of $f'(x)=x^2-2x-5=0.$ Since $\frac{d}{dx}[f'(x)]=2x-2,$ the recursion formula is

$$x_{n+1}=x_n-\frac{x_n^2-2x_n-5}{2x_n-2}.$$

For the given interval [3, 4], note that $f'(3)=-2$ and $f'(4)=3$ have opposite signs. Thus there is a root x between 3 and 4. Since 3 is closer to 0, we shall select $x_1 = 3$.

n	x_n	x_{n+1}
1	3.0	3.5
2	3.5	3.45
3	3.45	3.45

Thus $x \approx 3.45$.

Principles in Practice 12.7

1. $\frac{dh}{dt} = 0 - 16(2t) = -32t$ ft/sec

$\frac{d^2h}{dt^2} = \frac{d}{dt}[-32t] = -32$ feet/sec^2

The acceleration of the rock at time t is -32 feet/sec^2 or 32 feet/sec^2 downward.

2. The rate of change of the marginal cost function with respect to x is $c''(q)$.

$c'(q) = 14q + 11$

$c'' = 14$

When $x = 3$, the rate of change of the marginal cost function is 14 dollars/unit^2.

Problems 12.7

1. $y' = 12x^2 - 24x + 6$

$y'' = 24x - 24$

$y''' = 24$

2. $y' = 5x^4 - 8x^3 + 14x$

$y'' = 20x^3 - 24x^2 + 14$

$y''' = 60x^2 - 48x$

3. $\frac{dy}{dx} = -1$

$\frac{d^2y}{dx^2} = 0$

4. $\frac{dy}{dx} = -1 - 2x$

$\frac{d^2y}{dx^2} = -2$

5. $y' = 3x^2 + e^x$

$y'' = 6x + e^x$

$y''' = 6 + e^x$

$y^{(4)} = e^x$

6. $\frac{dF}{dq} = \frac{1}{q+1}$

$\frac{d^2F}{dq^2} = -\frac{1}{(q+1)^2}$

$\frac{d^3F}{dq^3} = \frac{2}{(q+1)^3}$

7. $f(x) = x^2 \ln x$

$f'(x) = x^2\left(\frac{1}{x}\right) + (\ln x)(2x) = x(1 + 2\ln x)$

$f''(x) = x\left(\frac{2}{x}\right) + (1 + 2\ln x)(1) = 3 + 2\ln x$

8. $y = \frac{1}{x} = x^{-1}$

$y' = -x^{-2}$

$y'' = 2x^{-3}$

$y''' = -6x^{-4} = -\frac{6}{x^4}$

9. $f(q) = \frac{1}{2q^4} = \frac{1}{2}q^{-4}$

$f'(q) = -2q^{-5}$

$f''(q) = 10q^{-6}$

$f'''(q) = -60q^{-7} = -\frac{60}{q^7}$

10. $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4x^{\frac{3}{2}}}$

11. $f(r) = \sqrt{9-r} = (9-r)^{\frac{1}{2}}$

$f'(r) = -\frac{1}{2}(9-r)^{-\frac{1}{2}}$

$f''(r) = -\frac{1}{4}(9-r)^{-\frac{3}{2}} = -\frac{1}{4(9-r)^{\frac{3}{2}}}$

12. $y = e^{-4x^2}$

$y' = -8xe^{-4x^2}$

$y'' = -8\left[x\left(-8xe^{-4x^2}\right) + e^{-4x^2}(1)\right]$

$= 8e^{-4x^2}\left(8x^2 - 1\right)$

13. $y = \frac{1}{2x+3} = (2x+3)^{-1}$

$\frac{dy}{dx} = -2(2x+3)^{-2}$

$\frac{d^2y}{dx^2} = 8(2x+3)^{-3} = \frac{8}{(2x+3)^3}$

14. $y = (3x+7)^5$

$y' = 15(3x+7)^4$

$y'' = 180(3x+7)^3$

15. $y = \frac{x+1}{x-1}$

$y' = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2}$

$= -\frac{2}{(x-1)^2} = -2(x-1)^{-2}$

$y'' = 4(x-1)^{-3} = \frac{4}{(x-1)^3}$

16. $y = 2x^{\frac{1}{2}} + (2x)^{\frac{1}{2}}$

$y' = x^{-\frac{1}{2}} + \frac{1}{2}(2x)^{-\frac{1}{2}}(2) = x^{-\frac{1}{2}} + (2x)^{-\frac{1}{2}}$

$y'' = -\frac{1}{2}x^{-\frac{3}{2}} - \frac{1}{2}(2x)^{-\frac{3}{2}}(2) = -\left[\frac{1}{2x^{\frac{3}{2}}} + \frac{1}{(2x)^{\frac{3}{2}}}\right]$

17. $y = \ln[x(x+6)] = \ln(x) + \ln(x+6)$

$y' = \frac{1}{x} + \frac{1}{x+6} = x^{-1} + (x+6)^{-1}$

$y'' = -x^{-2} + (-1)(x+6)^{-2} = -\left[\frac{1}{x^2} + \frac{1}{(x+6)^2}\right]$

18. $y = \ln\frac{(2x+5)(5x-2)}{x+1}$

$= \ln(2x+5) + \ln(5x-2) - \ln(x+1)$

$y' = \frac{2}{2x+5} + \frac{5}{5x-2} - \frac{1}{x+1}$

$y'' = -\frac{4}{(2x+5)^2} - \frac{25}{(5x-2)^2} + \frac{1}{(x+1)^2}$

19. $f(z) = z^2e^z$

$f'(z) = z^2\left(e^z\right) + e^z(2z) = \left(ze^z\right)(z+2)$

$f''(z) = \left(ze^z\right)(1) + (z+2)\left[ze^z + e^z(1)\right]$

$= e^z\left(z^2 + 4z + 2\right)$

20. $y = \frac{x}{e^x}$

$\frac{dy}{dx} = \frac{e^x(1) - x\left(e^x\right)}{\left(e^x\right)^2} = \frac{1-x}{e^x}$

$\frac{d^2y}{dx^2} = \frac{e^x(-1) - (1-x)e^x}{\left(e^x\right)^2} = \frac{x-2}{e^x}$

21. $y = e^{2x} + e^{3x}$

$\frac{dy}{dx} = 2e^{2x} + 3e^{3x}$

$\frac{d^2y}{dx^2} = 4e^{2x} + 9e^{3x}$

$\frac{d^3y}{dx^3} = 8e^{2x} + 27e^{3x}$

$\frac{d^4y}{dx^4} = 16e^{2x} + 81e^{3x}$

$\frac{d^5y}{dx^5} = 32e^{2x} + 243e^{3x}$

$\left.\frac{d^5y}{dx^5}\right|_{x=0} = 32e^0 + 243e^0 = 32 + 243 = 275$

22. $y = e^{2\ln\left(x^3+1\right)} = e^{\ln\left(x^3+1\right)^2} = \left(x^3+1\right)^2$

$y' = 6x^2\left(x^3+1\right) = 6x^5 + 6x^2$

$y'' = 30x^4 + 12x$

When $x = 1$, then $y'' = 30 + 12 = 42.$

23. $x^2 + 4y^2 - 16 = 0$

$2x + 8yy' = 0$

$8yy' = -2x$

$$y' = -\frac{x}{4y}$$

$$y'' = -\frac{4y(1) - x(4y')}{16y^2}$$

$$= -\frac{4y - 4x\left(-\frac{x}{4y}\right)}{16y^2} = -\frac{4y^2 + x^2}{16y^3}$$

$$= -\frac{16}{16y^3} = -\frac{1}{y^3}$$

24. $x^2 - y^2 = 16$

$2x - 2yy' = 0$

$$y' = \frac{x}{y}$$

$$y'' = \frac{y(1) - x(y')}{y^2} = \frac{y - x\left(\frac{x}{y}\right)}{y^2}$$

$$= \frac{y^2 - x^2}{y^3} = \frac{-16}{y^3} = -\frac{16}{y^3}$$

25. $y^2 = 4x$

$2yy' = 4$

$$y' = \frac{2}{y} = 2y^{-1}$$

$$y'' = -2y^{-2}y' = -2y^{-2}\left(2y^{-1}\right) = -\frac{4}{y^3}$$

26. $9x^2 + 16y^2 = 25$

$18x + 32yy' = 0$

$$y' = -\frac{9x}{16y}$$

$$y'' = -\frac{9}{16}\cdot\frac{y(1) - xy'}{y^2} = -\frac{9}{16}\cdot\frac{y - x\left(-\frac{9x}{16y}\right)}{y^2}$$

$$= -\frac{9}{16}\cdot\frac{16y^2 + 9x^2}{16y^3} = -\frac{225}{256y^3}$$

27. $\sqrt{x} + 4\sqrt{y} = 4$

$x^{\frac{1}{2}} + 4y^{\frac{1}{2}} = 4$

$$\frac{1}{2}x^{-\frac{1}{2}} + 2y^{-\frac{1}{2}}y' = 0$$

$$2y^{-\frac{1}{2}}y' = -\frac{1}{2}x^{-\frac{1}{2}}$$

$$y' = -\frac{1}{2}\cdot\frac{x^{-\frac{1}{2}}}{2y^{-\frac{1}{2}}} = -\frac{1}{4}\cdot\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

$$y'' = -\frac{1}{4}\left[\frac{x^{\frac{1}{2}}\left(\frac{1}{2}y^{-\frac{1}{2}}y'\right) - y^{\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)}{x}\right]$$

$$= -\frac{1}{8}\left[\frac{\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}\left(-\frac{y^{\frac{1}{2}}}{4x^{\frac{1}{2}}}\right) - \frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}}{x}\right] = -\frac{1}{8}\left[\frac{-\frac{1}{4} - \frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}}{x}\right]$$

$$= \frac{1}{8}\left[\frac{\frac{1}{4} + \frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}}{x}\right] = \frac{1}{8}\left[\frac{x^{\frac{1}{2}} + 4y^{\frac{1}{2}}}{4x^{\frac{3}{2}}}\right]$$

$$= \frac{1}{8}\left[\frac{4}{4x^{\frac{3}{2}}}\right] = \frac{1}{8x^{\frac{3}{2}}}$$

28. $y^2 - 6xy = 4$

$2yy' - 6[xy' + y(1)] = 0$

$2yy' - 6xy' = 6y$

$(2y - 6x)y' = 6y$

$$y' = \frac{6y}{2y-6x} = \frac{3y}{y-3x}$$

$$y'' = 3\cdot\frac{(y-3x)y' - y(y'-3)}{(y-3x)^2} = 9\cdot\frac{y - xy'}{(y-3x)^2}$$

$$= 9\cdot\frac{y - x\left[\frac{3y}{y-3x}\right]}{(y-3x)^2} = 9\cdot\frac{y(y-3x)-3xy}{(y-3x)^3}$$

$$= 9\cdot\frac{y^2-6xy}{(y-3x)^3} = 9\cdot\frac{4}{(y-3x)^3} = \frac{36}{(y-3x)^3}$$

29. $xy + y - x = 4$

$xy' + y(1) + y' - 1 = 0$

$xy' + y' = 1 - y$

$(x+1)y' = 1 - y$

$$y' = \frac{1-y}{1+x}$$

$$y'' = \frac{(1+x)(-y') - (1-y)(1)}{(1+x)^2}$$

$$= \frac{(1+x)\left[-\frac{(1-y)}{(1+x)}\right] - (1-y)}{(1+x)^2}$$

$$= \frac{-(1-y)-(1-y)}{(1+x)^2} = \frac{-2(1-y)}{(1+x)^2} = \frac{2(y-1)}{(1+x)^2}$$

30. $x^2 + 2xy + y^2 = 1$

$2x + 2y + 2xy' + 2yy' = 0$

$(x+y)y' = -(x+y)$

$y' = -1$

$y'' = 0$

31. $y = e^{x+y}$

$y' = e^{x+y}(1+y')$

$y' - e^{x+y}y' = e^{x+y}$

$y'\left(1 - e^{x+y}\right) = e^{x+y}$

$$y' = \frac{e^{x+y}}{1-e^{x+y}}$$

$$y' = \frac{y}{1-y}$$

$$y'' = \frac{(1-y)y' - y(-y')}{(1-y)^2} = \frac{y'}{(1-y)^2}$$

$$= \frac{\frac{y}{1-y}}{(1-y)^2} = \frac{y}{(1-y)^3}$$

32. $e^x - e^y = x^2 + y^2$

$e^x - e^y y' = 2x + 2yy'$

$$y' = \frac{e^x - 2x}{e^y + 2y}$$

$$y'' = \frac{\left(e^y+2y\right)\left(e^x-2\right) - \left(e^x-2x\right)\left(e^y y' + 2y'\right)}{\left(e^y+2y\right)^2}$$

$$= \frac{\left(e^y+2y\right)\left(e^x-2\right) - \left(e^x-2x\right)\left(e^y+2\right)y'}{\left(e^y+2y\right)^2}$$

$$= \frac{\left(e^y+2y\right)^2\left(e^x-2\right) - \left(e^x-2x\right)^2\left(e^y+2\right)}{\left(e^y+2y\right)^2}$$

33. $x^2 + 3x + y^2 = 4y$

$2x + 3 + 2yy' = 4y'$

$2yy' - 4y' = -2x - 3$

$$y' = -\frac{2x+3}{2y-4} = \frac{2x+3}{4-2y}$$

$$y'' = \frac{(4-2y)(2)-(2x+3)(-2y')}{(4-2y)^2}$$

$$= \frac{2(4-2y)+2(2x+3)\left(\frac{2x+3}{4-2y}\right)}{(4-2y)^2}$$

$$= \frac{2(4-2y)^2+2(2x+3)^2}{(4-2y)^3}$$

When $x = 0$ and $y = 0$, then $\frac{d^2y}{dx^2} = \frac{2(4)^2+2(3)^2}{4^3} = \frac{25}{32}$.

34. $f(x) = (3x-5)e^{-2x}$

$f'(x) = (3x-5)\left[-2e^{-2x}\right] + e^{-2x}[3]$. Thus,

$f'(x) = e^{-2x}[-2(3x-5)+3] = (13-6x)e^{-2x}$

$f''(x) = (13-6x)\left[-2e^{-2x}\right] + e^{-2x}[-6]$

$= 2e^{-2x}[-(13-6x)-3]$

$= 4(3x-8)e^{-2x}$

$f''(x)+4f'(x)+4f(x)$

$= 4(3x-8)e^{-2x} + 4\left[(13-6x)e^{-2x}\right] + 4\left[(3x-5)e^{-2x}\right] = [4(3x-8)+4(13-6x)+4(3x-5)]e^{-2x}$

$= [0]e^{-2x} = 0$, as was to be shown.

35. $f(x) = (5x-3)^4$

$f'(x) = 20(5x-3)^3$

$f''(x) = 300(5x-3)^2$

36. $f(x) = 6x^{\frac{1}{2}} + \frac{x^{-\frac{1}{2}}}{6}$

$f'(x) = 3x^{-\frac{1}{2}} - \frac{x^{-\frac{3}{2}}}{12}$

$f''(x) = -\frac{3}{2}x^{-\frac{3}{2}} + \frac{x^{-\frac{5}{2}}}{8}$

$f'''(x) = \frac{9}{4}x^{-\frac{5}{2}} - \frac{5x^{-\frac{7}{2}}}{16}$

37. $\frac{dc}{dq} = 0.6q + 2$

$\frac{d^2c}{dq^2} = 0.6$

$\left.\frac{d^2c}{dq^2}\right|_{q=100} = 0.6$

38. $r = pq = 400q - 40q^2 - q^3$

$\frac{dr}{dq} = 400 - 80q - 3q^2$

$\frac{d^2r}{dq^2} = -80 - 6q$

When $q = 4$, $\frac{d^2r}{dq^2} = -104$.

39. $f(x) = x^4 - 6x^2 + 5x - 6$

$f'(x) = 4x^3 - 12x + 5$

$f''(x) = 12x^2 - 12 = 12(x+1)(x-1)$

Clearly $f''(x) = 0$ when $x = \pm 1$.

40. $e^y = y^2e^x$

a. $e^y y' = y^2\left(e^x\right) + e^x(2yy')$

$\left(e^y - 2ye^x\right)y' = y^2e^x$

$y' = \frac{y^2e^x}{e^y - 2ye^x} = \frac{y^2\left(\frac{e^y}{y^2}\right)}{e^y - 2y\left(\frac{e^y}{y^2}\right)}$

$= \frac{e^y}{e^y - \frac{2e^y}{y}} = \frac{1}{1-\frac{2}{y}} = \frac{y}{y-2}$

b. $y'' = \frac{(y-2)(y') - y(y')}{(y-2)^2} = \frac{-2y'}{(y-2)^2}$

$= \frac{-2\left(\frac{y}{y-2}\right)}{(y-2)^2} = -\frac{2y}{(y-2)^3} = \frac{2y}{(2-y)^3}$

41. $f'(x) = 6e^x - 3x^2 - 30x$

$f''(x) = 6\left(e^x - x - 5\right)$

$f''(x) = 0$ when $x \approx -4.99$ or 1.94.

42. $f(x) = \frac{x^5}{20} + \frac{x^4}{12} + \frac{5x^3}{6} + \frac{x^2}{2}$

$f'(x) = \frac{x^4}{4} + \frac{x^3}{3} + \frac{5x^2}{2} + x$

$f''(x) = x^3 + x^2 + 5x + 1$

$f''(x) = 0$ when $x \approx -0.21$.

Chapter 12 Review Problems

1. $y' = 3e^x + 0 + e^{x^2}(2x) + (e^2)x^{e^2-1}$

$= 3e^x + 2xe^{x^2} + e^2x^{e^2-1}$

2. $f'(w) = \left(we^w + e^w\right) + 2w = we^w + e^w + 2w$

3. $f'(r) = \frac{1}{3r^2 + 7r + 1}(6r+7) = \frac{6r+7}{3r^2+7r+1}$

4. $y = e^{\ln x} = x$. Thus $y' = 1$.

5. $y = e^{x^2+4x+5}$

$y' = e^{x^2+4x+5}(2x+4) = 2(x+2)e^{x^2+4x+5}$

6. $f(t) = \log_6\sqrt{t^2+1} = \frac{1}{2}\log_6\left(t^2+1\right)$

$= \frac{1}{2}\cdot\frac{\ln\left(t^2+1\right)}{\ln 6}$. Thus

$f'(t) = \frac{1}{2}\left(\frac{1}{\ln 6}\cdot\frac{1}{t^2+1}\cdot[2t]\right) = \frac{t}{(\ln 6)\left(t^2+1\right)}$.

7. $y' = e^x(2x) + \left(x^2+2\right)e^x = e^x\left(x^2+2x+2\right)$

8. $y = 3^{5x^3} = e^{(\ln 3)5x^3}$

$y' = e^{(\ln 3)5x^3}(\ln 3)(15x^2) = 15x^2\ln 3\left(3^{5x^3}\right)$

9. $y = \sqrt{(x-6)(x+5)(9-x)}$

$\ln y = \ln\sqrt{(x-6)(x+5)(9-x)}$

$= \frac{1}{2}[\ln(x-6) + \ln(x+5) + \ln(9-x)]$

$\frac{y'}{y} = \frac{1}{2}\left[\frac{1}{x-6} + \frac{1}{x+5} + \frac{-1}{9-x}\right]$

$y' = \frac{y}{2}\left[\frac{1}{x-6}+\frac{1}{x+5}-\frac{1}{9-x}\right]$

$= \frac{\sqrt{(x-6)(x+5)(9-x)}}{2}\left[\frac{1}{x-6}+\frac{1}{x+5}+\frac{1}{x-9}\right]$

10. $f'(t) = e^{1/t}(-1\cdot t^{-2}) = -\frac{e^{1/t}}{t^2}$

11. $y' = \frac{e^x\left(\frac{1}{x}\right)-(\ln x)\left(e^x\right)}{e^{2x}}$

$= \frac{e^x - xe^x \ln x}{xe^{2x}} = \frac{1-x\ln x}{xe^x}$

12. $y' = \frac{x^2\left(e^x - e^{-x}\right)-\left(e^x+e^{-x}\right)(2x)}{x^4}$

$= \frac{x^2e^x - x^2e^{-x} - 2xe^x - 2xe^{-x}}{x^4}$

$= \frac{e^x(x-2)-e^{-x}(x+2)}{x^3}$

13. $f(q) = \ln\left[(q+1)^2(q+2)^3\right]$

$= 2\ln(q+1)+3\ln(q+2)$

$f'(q) = \frac{2}{q+1}+\frac{3}{q+2}$

14. $y = (x+2)^3(x+1)^4(x-2)^2$

$\ln y = 3\ln(x+2) + 4\ln(x+1) + 2\ln(x-2)$

$\frac{y'}{y} = \frac{3}{x+2}+\frac{4}{x+1}+\frac{2}{x-2}$

$y' = y\left[\frac{3}{x+2}+\frac{4}{x+1}+\frac{2}{x-2}\right]$

$= (x+2)^3(x+1)^4(x-2)^2\left[\frac{3}{x+2}+\frac{4}{x+1}+\frac{2}{x-2}\right]$

15. $y = e^{(2x^2+2x-5)(\ln 2)}$

$y' = e^{(2x^2+2x-5)(\ln 2)}(4x+2)(\ln 2)$

$= (4x+2)(\ln 2)2^{2x^2+2x-5}$

16. y is a constant, so $y' = 0$.

17. $y = \frac{4e^{3x}}{xe^{x-1}} = \frac{4e^{2x+1}}{x}$

$y' = 4\cdot\frac{x\left[e^{2x+1}(2)\right]-e^{2x+1}[1]}{x^2} = \frac{4e^{2x+1}(2x-1)}{x^2}$

18. $y' = \frac{(\ln x)e^x - e^x\left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{e^x\left(\ln x - \frac{1}{x}\right)}{\ln^2 x}$

$= \frac{e^x(x\ln x - 1)}{x\ln^2 x}$

19. $y = \log_2(8x+5)^2 = 2\log_2(8x+5)$

$= 2\cdot\frac{\ln(8x+5)}{\ln 2}$

$y' = 2\cdot\frac{1}{\ln 2}\cdot\frac{8}{8x+5} = \frac{16}{(8x+5)\ln 2}$

20. $y = \ln\left(\frac{5}{x^2}\right) = \ln 5 - 2\ln x$

$y' = 0 - 2\cdot\frac{1}{x} = -\frac{2}{x}$

21. $f(l) = \ln\left(1+l+l^2+l^3\right)$

$f'(l) = \frac{1}{1+l+l^2+l^3}\left[1+2l+3l^2\right]$

$= \frac{1+2l+3l^2}{1+l+l^2+l^3}$

22. $y = (x^2)^{x^2}$

$\ln y = x^2\ln x^2 = 2x^2\ln x$

$\frac{y'}{y} = 2x^2\left(\frac{1}{x}\right)+(\ln x)(4x)$

$y' = 2xy(1+2\ln x)$

$y' = 2x(x^2)^{x^2}(1+2\ln x)$

23. $y = (x+1)^{x+1}$

$\ln y = (x+1)\ln(x+1)$

$\frac{y'}{y} = (x+1)\frac{1}{x+1}+\ln(x+1)[1] = 1+\ln(x+1)$

$y' = y[1+\ln(x+1)] = (x+1)^{x+1}[1+\ln(x+1)]$

24. $y' = \dfrac{\left(1-e^x\right)e^x - \left(1+e^x\right)\left(-e^x\right)}{\left(1-e^x\right)^2} = \dfrac{2e^x}{\left(1-e^x\right)^2}$

25. $\phi(t) = \ln\left(t\sqrt{4-t^2}\right) = \ln t + \dfrac{1}{2}\ln(4-t^2)$

$\phi'(t) = \dfrac{1}{t} + \dfrac{1}{2}\cdot\dfrac{1}{4-t^2}\cdot(-2t) = \dfrac{1}{t} - \dfrac{t}{4-t^2}$

26. $y = (x+3)^{\ln x}$

$\ln y = [\ln x]\ \ln(x + 3)$

$\dfrac{y'}{y} = (\ln x)\dfrac{1}{x+3} + \ln(x+3)\cdot\dfrac{1}{x}$

$y' = y\left[\dfrac{\ln x}{x+3} + \dfrac{\ln(x+3)}{x}\right]$

$= (x+3)^{\ln x}\left[\dfrac{\ln x}{x+3} + \dfrac{\ln(x+3)}{x}\right]$

27. $y = \dfrac{(x^2+1)^{1/2}(x^2+2)^{1/3}}{(2x^3+6x)^{2/5}}$

$\ln y = \dfrac{1}{2}\ln(x^2+1) + \dfrac{1}{3}\ln(x^2+2) - \dfrac{2}{5}\ln(2x^3+6x)$

$\dfrac{y'}{y} = \dfrac{1}{2}\left(\dfrac{1}{x^2+1}\right)(2x) + \dfrac{1}{3}\left(\dfrac{1}{x^2+2}\right)(2x) - \dfrac{2}{5}\left(\dfrac{1}{2x^3+6x}\right)(6x^2+6)$

$y' = y\left[\dfrac{x}{x^2+1} + \dfrac{2x}{3(x^2+2)} - \dfrac{6(x^2+1)}{5(x^3+3x)}\right]$

$= \dfrac{(x^2+1)^{1/2}(x^2+2)^{1/3}}{(2x^3+6x)^{2/5}}\left[\dfrac{x}{x^2+1} + \dfrac{2x}{3(x^2+2)} - \dfrac{6(x^2+1)}{5(x^3+3x)}\right]$

28. $y' = \dfrac{\sqrt{x}\left(\frac{1}{x}\right) - \ln x\left(\frac{1}{2}\right)x^{-\frac{1}{2}}}{x} = \dfrac{2-\ln x}{2x^{\frac{3}{2}}}$

29. $y = \left(x^x\right)^x = x^{x^2}$

$\ln y = \ln x^{x^2} = x^2 \ln x$

$\dfrac{y'}{y} = x^2\left(\dfrac{1}{x}\right) + (\ln x)(2x) = x + 2x\ln x$

$y' = y(x + 2x\ln x) = \left(x^x\right)^x (x + 2x\ln x)$

30. $y = x^{\left(x^x\right)}$

$\ln y = \ln x^{\left(x^x\right)} = x^x \ln x$

$\frac{y'}{y} = x^x\left(\frac{1}{x}\right) + (\ln x)\frac{d}{dx}\left(x^x\right)$

Note: If $v = x^x$, then $\ln v = \ln x^x = x \ln x$;

$\frac{v'}{v} = x\left(\frac{1}{x}\right) + (\ln x)(1) = 1 + \ln x$

$v' = \frac{d}{dx}\left(x^x\right) = v(1 + \ln x) = x^x(1 + \ln x)$

Thus $\frac{y'}{y} = x^x\left(\frac{1}{x}\right) + (\ln x)\left[x^x(1 + \ln x)\right]$

$= x^x\left[\frac{1}{x} + (1 + \ln x)\ln x\right]$

$y' = yx^x\left[\frac{1}{x} + (1 + \ln x)\ln x\right]$

$= x^{\left(x^x\right)}x^x\left[\frac{1}{x} + (1 + \ln x)\ln x\right]$

31. $y = (x+1)\ln x^2 = 2(x+1)\ln x$

$y' = 2\left[(x+1)\left(\frac{1}{x}\right) + (\ln x)(1)\right] = 2\left[\frac{x+1}{x} + \ln x\right]$

When $x = 1$, then $y' = 2\left[\frac{2}{1} + \ln 1\right] = 4.$

32. $y = \frac{e^{x^2+1}}{\sqrt{x^2+1}}$

$\ln y = \ln\left(e^{x^2+1}\right) - \frac{1}{2}\ln(x^2+1) = x^2 + 1 - \frac{1}{2}\ln(x^2+1)$

$\frac{y'}{y} = 2x - \frac{1}{2}\cdot\frac{1}{x^2+1}(2x) = x\left[2 - \frac{1}{x^2+1}\right]$

$y' = yx\left[2 - \frac{1}{x^2+1}\right]$

$y' = \frac{e^{x^2+1}}{\sqrt{x^2+1}}x\left[2 - \frac{1}{x^2+1}\right]$

When $x = 1$, then $y' = \frac{e^{1+1}}{\sqrt{1+1}}(1)\left[2 - \frac{1}{1+1}\right] = \frac{3e^2\sqrt{2}}{4}.$

33. $y = e^{e+x\ln\left(\frac{1}{x}\right)} = e^{e-x\ln x}$

$y' = e^{e-x\ln x}\left(-\left[x\left(\frac{1}{x}\right)+(\ln x)(1)\right]\right)$

$= -(1+\ln x)e^{e-x\ln x}$

When $x = e$, then $y' = -(1+\ln e)e^{e-e\ln e} = -(2)e^0 = -2.$

34. $y = \left[\dfrac{2^{5x}(x^2-3x+5)^{1/3}}{(x^2-3x+7)^3}\right]^{-1}$

$\ln y = -1\left[5x\ln 2+\frac{1}{3}\ln(x^2-3x+5)-3\ln(x^2-3x+7)\right]$

$\dfrac{y'}{y} = -\left[5\ln 2+\frac{1}{3}\cdot\frac{2x-3}{x^2-3x+5}-3\cdot\frac{2x-3}{x^2-3x+7}\right]$

$y' = -y\left[5\ln 2+\dfrac{2x-3}{3(x^2-3x+5)}-\dfrac{3(2x-3)}{x^2-3x+7}\right]$

$y' = (-1)\left[\dfrac{2^{5x}(x^2-3x+5)^{1/3}}{(x^2-3x+7)^3}\right]^{-1}\left[5\ln 2+\dfrac{2x-3}{3(x^2-3x+5)}-\dfrac{3(2x-3)}{x^2-3x+7}\right]$

When $x = 0$, then $y' = -\dfrac{343}{5^{1/3}}\left[5\ln 2-\frac{1}{5}+\frac{9}{7}\right] = -343(\ln 2)5^{2/3}-\dfrac{1862}{5^{4/3}}.$

35. $y = 3e^x$

$y' = 3e^x$

If $x = \ln 2$, then $y = 3e^{\ln 2} = 6$ and $y' = 3e^{\ln 2} = 6$.

An equation of the tangent line is $y - 6 = 6(x - \ln 2)$, $y = 6x + 6 - 6\ln 2$, $y = 6x + 6(1 - \ln 2)$. Alternatively, since $6\ln 2 = \ln 2^6 = \ln 64$, the tangent line can be written as $y = 6x + 6 - \ln 64$.

36. $y = x + x^2\ln x$

$y' = 1+\left[x^2\left(\frac{1}{x}\right)+(\ln x)(2x)\right] = 1+x+2x\ln x$

When $x = 1$, then $y = 1 + 1(0) = 1$ and $y' = 1+1+2(0) = 2$. Thus an equation of the tangent line is $y - 1 = 2(x - 1)$, or $y = 2x - 1$.

37. $y = x\left(2^{2-x^2}\right)$. To find y' we shall use logarithmic differentiation.

$\ln y = \ln\left[x\left(2^{2-x^2}\right)\right] = \ln x+\left(2-x^2\right)\ln 2$

$\dfrac{y'}{y} = \dfrac{1}{x}+(-2x)\ln 2$

$y' = y\left[\dfrac{1}{x}-2(\ln 2)x\right]$

When $x = 1$, then $y = 2$ and $y' = 2(1-2\ln 2)$. The equation of the tangent line is
$y - 2 = 2(1 - 2 \ln 2)(x - 1)$. The y-intercept of the tangent line corresponds to the point where $x = 0$:
$y - 2 = 2(1 - 2 \ln 2)(-1) = -2 + 4 \ln 2$
Thus $y = 4 \ln 2$ and the y-intercept is $(0, 4 \ln 2)$.

38. $w = 2^{x+1} + \ln(1+x^2) = e^{(\ln 2)(x+1)} + \ln(1+x^2)$

$x = \log_2(t^2+1) - e^{(t-1)^2} = \dfrac{\ln(t^2+1)}{\ln 2} - e^{(t-1)^2}$

$$\frac{dw}{dt} = \frac{dw}{dx}\cdot\frac{dx}{dt}$$
$$= \left(2^{x+1}(\ln 2) + \frac{2x}{1+x^2}\right)\left(\frac{2t}{(\ln 2)(t^2+1)} - e^{(t-1)^2}[2(t-1)]\right)$$

When $t = 1$, $x = \log_2(1+1) - e^{(1-1)^2} = 1 - 1 = 0$, $w = 2^1 + \ln 1 = 2 + 0 = 2$, and $\dfrac{dw}{dt} = (2^1 \ln 2 + 0)\left(\dfrac{2}{2\ln 2} - 1(0)\right) = 2.$

39. $y = e^{x^2-2x+1}$

$y' = e^{x^2-2x+1}[2x-2] = (2x-2)e^{x^2-2x+1}$

$y'' = 2(x-1)e^{x^2-2x+1}(2x-2) + 2e^{x^2-2x+1}$

$= 2e^{x^2-2x+1}(2(x-1)^2+1)$

At (1, 1), $y'' = 2e^0(2(0)+1) = 2.$

40. $y = x^2e^x$

$y' = x^2e^x + e^x(2x) = e^x\left(x^2+2x\right)$

$y'' = e^x(2x+2) + \left(x^2+2x\right)e^x = e^x\left(x^2+4x+2\right)$

$y''' = e^x(2x+4) + \left(x^2+4x+2\right)e^x = e^x\left(x^2+6x+6\right)$

At $(1, e)$, $y''' = e(1+6+6) = 13e$

41. $y = \ln(2x)$

$y' = \dfrac{1}{2x}(2) = x^{-1}$

$y'' = -1\cdot x^{-2} = -x^{-2}$

$y''' = -(-2)x^{-3} = \dfrac{2}{x^3}$

At $(1, \ln 2)$, $y''' = \dfrac{2}{1^3} = 2$

42. $y = x \ln x$

$$y' = x \cdot \frac{1}{x} + (\ln x)(1) = 1 + \ln x$$

$$y'' = 0 + \frac{1}{x} = \frac{1}{x}$$

At (1, 0), $y'' = \frac{1}{1} = 1$

43. $2xy + y^2 = 10$

$$2(xy' + y) + 2yy' = 0$$
$$2xy' + 2yy' = -2y$$
$$(x + y)y' = -y$$
$$y' = -\frac{y}{x + y}$$

44. $3x^2y^3 + 3x^3y^2y' = 0$

$$y'(3x^3y^2) = -3x^2y^3$$
$$y' = \frac{-3x^2y^3}{3x^3y^2} = -\frac{y}{x}$$

45. $\ln\left(xy^2\right) = xy$

$$\ln x + 2\ln y = xy$$
$$\frac{1}{x} + \frac{2}{y}y' = xy' + y$$
$$y + 2xy' = x^2yy' + xy^2$$
$$2xy' - x^2yy' = xy^2 - y$$
$$\left(2x - x^2y\right)y' = xy^2 - y$$
$$y' = \frac{xy^2 - y}{2x - x^2y}$$

46. $y^2e^{y\ln x} = e^2$

$$y^2\left[e^{y\ln x}\left(y \cdot \frac{1}{x} + (\ln x)y'\right)\right] + e^{y\ln x}\left[2yy'\right] = 0$$
$$y^2(\ln x)y' + 2yy' = -\frac{y^3}{x}$$
$$y'[y(2 + y\ln x)] = -\frac{y^3}{x}$$
$$y' = -\frac{y^2}{x(2 + y\ln x)}$$

47. $x + xy + y = 5$

$$1 + xy' + y(1) + y' = 0$$
$$(x + 1)y' = -1 - y$$
$$y' = -\frac{1 + y}{x + 1}$$
$$y'' = -\frac{(x + 1)y' - (1 + y)}{(x + 1)^2}$$

At (2, 1), $y' = -\frac{1 + 1}{2 + 1} = -\frac{2}{3}$ and

$$y'' = -\frac{3\left(-\frac{2}{3}\right) - 2}{9} = \frac{4}{9}$$

48. $xy + y^2 = 2$

$$x(y') + y(1) + 2yy' = 0$$
$$y' = -\frac{y}{x + 2y}$$
$$y'' = -\frac{(x + 2y)y' - y(1 + 2y')}{(x + 2y)^2}$$

At (–1, 2), $y' = -\frac{2}{3}$ and

$$y'' = -\frac{3\left(-\frac{2}{3}\right) - 2\left(-\frac{1}{3}\right)}{9} = \frac{4}{27}$$

49. $e^y = (y + 1)e^x$

$$e^yy' = (y + 1)e^x + e^x(y')$$
$$e^yy' - e^xy' = (y + 1)e^x$$
$$\left(e^y - e^x\right)y' = (y + 1)e^x$$
$$y' = \frac{(y + 1)e^x}{e^y - e^x} = \frac{(y + 1)\left(\frac{e^y}{y+1}\right)}{e^y - \left(\frac{e^y}{y+1}\right)} = \frac{e^y}{e^y - \frac{e^y}{y+1}}$$
$$= \frac{1}{1 - \frac{1}{y+1}} = \frac{y + 1}{y}$$
$$y'' = \frac{y(y') - (y + 1)(y')}{y^2} = \frac{-y'}{y^2} = -\frac{\frac{y+1}{y}}{y^2} = -\frac{y + 1}{y^3}$$

50. $x^{1/2}+y^{1/2}=1$

$$\frac{1}{2}x^{-1/2}+\frac{1}{2}y^{-1/2}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

$$\frac{d^2y}{dx^2}=-\frac{\left(\sqrt{x}\right)\left(\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}\right)-\left(\sqrt{y}\right)\left(\frac{1}{2\sqrt{x}}\right)}{\left(\sqrt{x}\right)^2}$$

$$=\frac{\frac{1}{2}+\frac{\sqrt{y}}{2\sqrt{x}}}{x}=\frac{\sqrt{x}+\sqrt{y}}{2x\sqrt{x}}=\frac{1}{2x\sqrt{x}}$$

51. $f'(t)$

$$=\left[-0.8e^{-0.01t}(-0.01)-0.2e^{-0.0002t}(-0.0002)\right]$$

$$=0.008e^{-0.01t}+0.00004e^{-0.0002t}$$

52. $\log N = A - bM$

$$\frac{d}{dM}(\log N)=\frac{d}{dM}(A-bM)$$

$$\frac{d}{dM}\left(\frac{\ln N}{\ln 10}\right)=\frac{d}{dM}(A-bM)$$

$$\frac{1}{\ln 10}\cdot\frac{1}{N}\frac{dN}{dM}=-b$$

$$(\log e)\frac{1}{N}\frac{dN}{dM}=-b$$

$$-\frac{dN}{dM}=\frac{bN}{\log e}$$

$$\log\left(-\frac{dN}{dM}\right)=\log\left(\frac{b}{\log e}\cdot N\right)$$

$$=\log\left(\frac{b}{\log e}\right)+\log N$$

$$=\log\left(\frac{b}{q}\right)+(A-bM)=A+\log\left(\frac{b}{q}\right)-bM$$

53. $f'(x)=(12x^3+6x^2-25)e^{3x^4+2x^3-25x}$

$f'(x)=0$ when $x\approx 1.13$.

54. $f(x)=\frac{x^5}{10}+\frac{x^4}{6}+\frac{2x^3}{3}+x^2+1$

$$f'(x)=\frac{x^4}{2}+\frac{2x^3}{3}+2x^2+2x$$

$$f''(x)=2x^3+2x^2+4x+2$$

$f''(x)=0$ when $x\approx -0.57$.

55. $p=\frac{500}{q}$

$$\eta=\frac{\frac{p}{q}}{\frac{dp}{dq}}=\frac{\frac{500/q}{q}}{-\frac{500}{q^2}}=-1$$

Since $|\eta|=1$, demand has unit elasticity when $q = 200$.

56. $p=900-q^2$

$$\eta=\frac{\frac{p}{q}}{\frac{dp}{dq}}=\frac{\frac{900-q^2}{q}}{-2q}=-\frac{900-q^2}{2q^2}$$

When $q = 10$, then $\eta=-4$. Since $|\eta|>1$, demand is elastic.

57. $p = 18 - 0.02q$

$$\eta=\frac{\frac{p}{q}}{\frac{dp}{dq}}=\frac{\frac{18-0.02q}{q}}{-0.02}=-\frac{18-0.02q}{0.02q}$$

When $q = 600$, then $\eta=-0.5$. Because $|\eta|<1$, demand is inelastic.

58. $p=20-2\sqrt{q}$

$$\eta=\frac{\frac{p}{q}}{\frac{dp}{dq}}=\frac{\frac{p}{q}}{-\frac{1}{\sqrt{q}}}=\frac{-p}{\sqrt{q}}=\frac{-p}{10-\frac{p}{2}}=\frac{2p}{p-20}$$

a. When $p = 8$, then $\eta=\frac{2(8)}{8-20}=-\frac{4}{3}$.

b. $\eta=\frac{2p}{p-20}$

If $p>\frac{20}{3}$, then $\eta<-1$, so $|\eta|>1$ and demand is elastic.

59. $\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{p}{q} \cdot \frac{dq}{dp}$

$q = \sqrt{2500 - p^2}$

$\frac{dq}{dp} = \frac{-p}{\sqrt{2500 - p^2}} = \frac{-p}{q}$, so

$\eta = \frac{p}{q}\left(\frac{-p}{q}\right) = -\frac{p^2}{q^2}$. Now, if $p = 30$, then

$q = \sqrt{2500 - 30^2} = 40$, so

$\eta\big|_{p=30} = -\frac{(30)^2}{(40)^2} = -\frac{9}{16}$

If the price of 30 decreases $\frac{2}{3}$%, then demand would change by approximately $\left(-\frac{2}{3}\right)\left(-\frac{9}{16}\right)$%, or $\frac{3}{8}$%. (That is, demand increases by approximately $\frac{3}{8}$%.)

60. a. $\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{p}{q} \cdot \frac{dq}{dp}$

$q = \sqrt{100 - p}$, where $0 < p < 100$.

$\frac{dq}{dp} = \frac{-1}{2\sqrt{100 - p}}$. Thus

$\eta = \frac{p}{\sqrt{100 - p}} \cdot \frac{-1}{2\sqrt{100 - p}}$

$= \frac{-p}{2(100 - p)} = \frac{p}{2p - 200}$

For elastic demand we want $\frac{p}{2p - 200} < -1$.

Noting that the denominator is negative for $0 < p < 100$, we multiply both sides of the inequality by $2p - 200$ and reverse the direction of the inequality

$p > -2p + 200,\ 3p > 200,\ p > \frac{200}{3}$

Thus $\frac{200}{3} < p < 100$ for elastic demand.

b. $\eta\big|_{p=40} = \frac{40}{80 - 200} = -\frac{1}{3}$

% change in q $\approx$ (% change in price) (η)

$= 5\left(-\frac{1}{3}\right)\% = -\frac{5}{3}\% = -1.67\%$. Thus demand decreases by approximately 1.67%.

61. We want a root of $f(x) = x^3 - 2x - 2 = 0$. We have $f(1) = -3$ and $f(2) = 2$ (note the sign change). Since $f(2)$ is closer to 0 than is $f(1)$, we choose $x_1 = 2$. We have $f'(x) = 3x^2 - 2$, so the recursion formula is

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n - 2}{3x_n^2 - 2}$

$= \frac{2x_n^3 + 2}{3x_n^2 - 2}$

n	x_n	x_{n+1}
1	2.00000	1.80000
2	1.80000	1.76995
3	1.76995	1.76929
4	1.76929	1.76929

Because $|x_5 - x_4| < 0.0001$, the root is approximately $x_5 = 1.7693$.

62. We want real solutions of $e^x = 3x$. Thus we want to find roots of $f(x) = e^x - 3x = 0$. A rough sketch of the exponential function $y = e^x$ and the line $y = 3x$ shows that there are two intersection points: one when x is near 0.5, and the other when x is near 1.5. Thus we must find two roots. Since $f'(x) = e^x - 3$, the recursion formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{x_n} - 3x_n}{e^{x_n} - 3}$

If $x_1 = 0.5$, we obtain

n	x_n	x_{n+1}
1	0.5	0.610
2	0.610	0.619
3	0.619	0.619

If $x_1 = 1.5$, we obtain

n	x_n	x_{n+1}
1	1.5	1.512
2	1.512	1.512

Thus the solutions are 0.619 and 1.512.

Mathematical Snapshot Chapter 12

1. $F = 25$, $D = 3400$, $V = 36.5$, $R = 0.05$.

$$q = \sqrt{\frac{2FD}{RV}} = \sqrt{\frac{2(25)(3400)}{(0.05)(36.5)}} \approx 305.2$$

The economic order quantity is 305 units.

2. If the number of units maintained as a safety margin is denoted by *m*, then the amount in stock at any time is increased by *m* units. The average inventory level is thus increased by *m* units, to $m + \frac{q}{2}$ units. The carrying cost is then

$$C(q) = \frac{FD}{q} + RV\left(m + \frac{q}{2}\right)$$
$$= \frac{FD}{q} + \frac{RVq}{2} + RVm$$

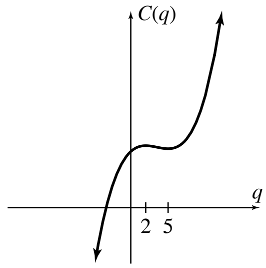
Since $\frac{d}{dq}(RVm) = 0$, the maintenance of a safetly margin does not affect the calculation of the economic order quantity.

3. Answers may vary.

Chapter 13

Principles in Practice 13.1

1. The graph of $c(q) = 2q^3 - 21q^2 + 60q + 500$ is shown.



There looks to be a relative maximum at $q = 2$ and a relative minimum at $q = 5$.

$c'(q) = 6q^2 - 42q + 60 = 6(q^2 - 7q + 10)$
$= 6(q - 5)(q - 2)$

$c'(q) = 0$ when $q = 2$ or $q = 5$. If $q < 2$, then $c'(q) = 6(-)(-) = +$, so $c(q)$ is increasing. If $2 < q < 5$, then $c'(q) = 6(-)(+) = -$, so $c(q)$ is decreasing. If $5 < q$, then $c'(q) = 6(+)(+) = +$, so $c(q)$ is increasing. When $q = 2$, there is a relative maximum, since $c'(q)$ changes from + to –. The relative maximum value is $2(2)^3 - 21(2)^2 + 60(2) + 500 = 552$. When $q = 5$, there is a relative minimum, since $c'(q)$ changes from – to +. The relative minimum value is $2(5)^3 - 21(5)^2 + 60(5) + 500 = 525$.

2. First, find $C'(t)$, with $C(t) = \dfrac{0.14t}{(t+2)^2}$.

$$C'(t) = \frac{0.14(t+2)^2 - 0.14t(2)(t+2)}{(t+2)^4}$$
$$= \frac{0.14(t+2) - 0.28t}{(t+2)^3} = \frac{0.28 - 0.14t}{(t+2)^3}$$
$$= \frac{0.14(2-t)}{(t+2)^3}$$

$C'(t) = 0$ when $t = 2$ and is undefined when $t = -2$. However, since t denotes the number of hours after an injection, negative values of t are not reasonable. If $0 \le t < 2$, $C'(t) = \frac{+}{+} = +$, so $C(t)$ is increasing. If $2 < t$, $C'(t) = \frac{-}{+} = -$, so $C(t)$ is decreasing. When $t = 2$, there is a relative maximum, since $C'(t)$ changes from + to –. The drug is at its greatest concentration 2 hours after the injection.

Problems 13.1

1. Decreasing on $(-\infty, -1)$ and $(3, \infty)$; increasing on $(-1, 3)$; relative minimum $(-1, -1)$; relative maximum $(3, 4)$.

2. Decreasing on $(-\infty, -1)$ and $(0, 1)$; increasing on $(-1, 0)$ and $(1, \infty)$; relative minima $(-1, -1)$ and $(1, -1)$; relative maximum $(0, 0)$.

3. Decreasing on $(-\infty, -2)$ and $(0, 2)$; increasing on $(-2, 0)$ and $(2, \infty)$; relative minima $(-2, 1)$ and $(2, 1)$; no relative maximum.

4. Decreasing on $(-\infty, 0)$ and $(0, \infty)$; never increasing; no relative maximum; no relative minimum.

In the following problems, we denote the critical value by CV.

5. $f'(x) = (x+3)(x-1)(x-2)$
$f'(x) = 0$ when $x = -3, 1, 2$
CV: $x = -3, 1, 2$

− + − +
−3 1 2

Increasing on $(-3, 1)$ and $(2, \infty)$; decreasing on $(-\infty, -3)$ and $(1, 2)$; relative maximum when $x = 1$; relative minima when $x = -3, 2$.

6. $f'(x) = 2x(x-1)^3$
CV: $x = 0, 1$

+ − +
0 1

Increasing on $(-\infty, 0)$ and $(1, \infty)$; decreasing on $(0, 1)$; relative maximum when $x = 0$; relative minimum when $x = 1$.

7. $f'(x) = (x+1)(x-3)^2$
CV: $x = -1, 3$

− + +
−1 3

Decreasing on $(-\infty, -1)$; increasing on $(-1, 3)$ and $(3, \infty)$; relative minimum when $x = -1$.

8. $f'(x) = \dfrac{x(x+2)}{x^2+1}$

CV: $x = 0, -2$

+ − +
−2 0

Increasing on (–∞, –2) and (0, ∞); decreasing on (–2, 0); relative maximum when $x = -2$; relative minimum when $x = 0$.

9. $y = 2x^3 + 1$

$y' = 6x^2$

CV: $x = 0$

+ +
0

Increasing on (–∞, 0); increasing on (0, ∞); no relative maximum or minimum

10. $y = x^2 + 4x + 3$

$y' = 2x + 4 = 2(x+2)$

CV: $x = -2$

− +
−2

Decreasing on (–∞, –2); increasing on (–2, ∞); relative minimum when $x = -2$.

11. $y = x - x^2 + 2$

$y' = 1 - 2x$

CV: $x = \dfrac{1}{2}$

+ −
$\frac{1}{2}$

Increasing on $\left(-\infty, \dfrac{1}{2}\right)$; decreasing on $\left(\dfrac{1}{2}, \infty\right)$;

relative maximum when $x = \dfrac{1}{2}$.

12. $y = x^3 - \dfrac{5}{2}x^2 - 2x + 6$

$y' = 3x^2 - 5x - 2 = (3x+1)(x-2)$

CV: $x = -\dfrac{1}{3}, 2$

+ − +
$-\frac{1}{3}$ 2

Increasing on $\left(-\infty, -\dfrac{1}{3}\right)$ and (2, ∞); decreasing on $\left(-\dfrac{1}{3}, 2\right)$; relative maximum when $x = -\dfrac{1}{3}$; relative minimum when $x = 2$.

13. $y = -\dfrac{x^3}{3} - 2x^2 + 5x - 2$

$y' = -x^2 - 4x + 5 = -\left(x^2 + 4x - 5\right)$

$= -(x+5)(x-1)$

CV: $x = -5, 1$

− + −
−5 1

Decreasing on (–∞, –5) and (1, ∞); increasing on (–5, 1); relative minimum when $x = -5$; relative maximum when $x = 1$.

14. $y = \dfrac{x^4}{4} + x^3$

$y' = x^3 + 3x^2 = x^2(x+3)$

CV: $x = -3, 0$

− + +
−3 0

Decreasing on (–∞, –3); increasing on (–3, 0) and (0, ∞); relative minimum at $x = -3$.

15. $y = x^4 - 2x^2$

$y' = 4x^3 - 4x = 4x\left(x^2 - 1\right) = 4x(x+1)(x-1)$

CV: $x = 0, \pm 1$

− + − +
−1 0 1

Decreasing on (–∞, –1) and (0, 1); increasing on (–1, 0) and (1, ∞); relative maximum when $x = 0$; relative minima when $x = \pm 1$.

16. $y = -3 + 12x - x^3$

$y' = 12 - 3x^2 = 3\left(4 - x^2\right) = 3(2+x)(2-x)$

CV: $x = \pm 2$

− + −
−2 2

Decreasing on (–∞, –2) and (2, ∞); increasing on (–2, 2); relative minimum when $x = -2$; relative maximum when $x = 2$.

17. $y = x^3 - \dfrac{7}{2}x^2 + 2x - 5$

$y' = 3x^2 - 7x + 2 = (3x-1)(x-2)$

CV: $x = \dfrac{1}{3}, 2$

\+ − +
$\frac{1}{3}$ 2

Increasing on $\left(-\infty, \frac{1}{3}\right)$ and $(2, \infty)$; decreasing on $\left(\frac{1}{3}, 2\right)$; relative maximum when $x = \frac{1}{3}$, relative minimum when $x = 2$.

18. $y = x^3 - 6x^2 + 12x - 6$

$y' = 3x^2 - 12x + 12 = 3\left(x^2 - 4x + 4\right) = 3(x-2)^2$

CV: $x = 2$

\+ +
2

Increasing on $(-\infty, 2)$ and $(2, \infty)$; no relative maximum or relative minimum.

19. $y = 2x^3 - \frac{11}{2}x^2 - 10x + 2$

$y' = 6x^2 - 11x - 10 = (2x-5)(3x+2)$

CV: $x = -\frac{2}{3}, \frac{5}{2}$

\+ − +
$-\frac{2}{3}$ $\frac{5}{2}$

Increasing on $\left(-\infty, -\frac{2}{3}\right)$ and $\left(\frac{5}{2}, \infty\right)$; decreasing on $\left(-\frac{2}{3}, \frac{5}{2}\right)$; relative maximum when $x = -\frac{2}{3}$; relative minimum when $x = \frac{5}{2}$.

20. $y = -5x^3 + x^2 + x - 7$

$y' = -15x^2 + 2x + 1 = -(5x+1)(3x-1)$

CV: $-\frac{1}{5}, \frac{1}{3}$

− + −
$-\frac{1}{5}$ $\frac{1}{3}$

Decreasing on $\left(-\infty, -\frac{1}{5}\right)$ and $\left(\frac{1}{3}, \infty\right)$; increasing on $\left(-\frac{1}{5}, \frac{1}{3}\right)$; relative minimum when $x = -\frac{1}{5}$; relative maximum when $x = \frac{1}{3}$.

21. $y = \frac{x^3}{3} - 5x^2 + 22x + 1$

$y' = x^2 - 10x + 22$

By the quadratic formula, $y' = 0$ when

$x = \frac{10 \pm \sqrt{(-10)^2 - 4(1)(22)}}{2(1)}$ or $x = 5 \pm \sqrt{3}$.

CV: $x = 5 \pm \sqrt{3}$

\+ − +
$5-\sqrt{3}$ $5+\sqrt{3}$

Increasing on $\left(-\infty, 5-\sqrt{3}\right)$; decreasing on $\left(5-\sqrt{3}, 5+\sqrt{3}\right)$; increasing on $\left(5+\sqrt{3}, \infty\right)$; relative maximum at $x = 5-\sqrt{3}$; relative minimum at $x = 5+\sqrt{3}$.

22. $y = \frac{9}{5}x^5 - \frac{47}{3}x^3 + 10x$

$y' = 9x^4 - 47x^2 + 10 = \left(9x^2 - 2\right)\left(x^2 - 5\right)$

$= \left(3x - \sqrt{2}\right)\left(3x + \sqrt{2}\right)\left(x - \sqrt{5}\right)\left(x + \sqrt{5}\right)$

CV: $x = \pm\frac{\sqrt{2}}{3}, \pm\sqrt{5}$

\+ − + − +
$-\sqrt{5}$ $-\frac{\sqrt{2}}{3}$ $\frac{\sqrt{2}}{3}$ $\sqrt{5}$

Increasing on $\left(-\infty, -\sqrt{5}\right)$, $\left(-\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\right)$, and $\left(\sqrt{5}, \infty\right)$; decreasing on $\left(-\sqrt{5}, -\frac{\sqrt{2}}{3}\right)$ and $\left(\frac{\sqrt{2}}{3}, \sqrt{5}\right)$; relative maxima when $x = -\sqrt{5}$, $\frac{\sqrt{2}}{3}$; relative minima when $x = -\frac{\sqrt{2}}{3}, \sqrt{5}$.

23. $y = 3x^5 - 5x^3$

$y' = 15x^4 - 15x^2 = 15x^2(x+1)(x-1)$

CV: $x = 0, \pm 1$

\+ − − +
−1 0 1

Increasing on $(-\infty, -1)$ and $(1, \infty)$; decreasing on $(-1, 0)$ and $(0, 1)$; relative maximum when $x = -1$; relative minimum when $x = 1$.

24. $y = 3x - \frac{x^6}{2}$

$y' = 3 - 3x^5 = 3(1 - x^5)$

$= 3(1 - x)(x^4 + x^3 + x^2 + x + 1)$

CV: $x = 1$

$+ \quad -$; critical value: 1

Increasing on $(-\infty, 1)$; decreasing on $(1, \infty)$; relative maximum when $x = 1$.

25. $y = -x^5 - 5x^4 + 200$

$y' = -5x^4 - 20x^3 = -5x^3(x + 4)$

CV: $x = 0, -4$

$- \quad + \quad -$; critical values: -4, 0

Decreasing on $(-\infty, -4)$ and $(0, \infty)$; increasing on $(-4, 0)$; relative minimum when $x = -4$; relative maximum when $x = 0$.

26. $y = \frac{3x^4}{2} - 4x^3 + 17$

$y' = 6x^3 - 12x^2 = 6x^2(x - 2)$

CV: $x = 0, 2$

$- \quad - \quad +$; critical values: 0, 2

Decreasing on $(-\infty, 0)$ and $(0, 2)$; increasing on $(2, \infty)$; relative minimum at $x = 2$.

27. $y = 8x^4 - x^8$

$y' = 32x^3 - 8x^7 = 8x^3\left(4 - x^4\right)$

$= 8x^3\left(2 + x^2\right)\left(2 - x^2\right)$

$= 8x^3\left(2 + x^2\right)\left(\sqrt{2} - x\right)\left(\sqrt{2} + x\right)$

CV: $x = 0, \pm\sqrt{2}$

$+ \quad - \quad + \quad -$; critical values: $-\sqrt{2}$, 0, $\sqrt{2}$

Increasing on $\left(-\infty, -\sqrt{2}\right)$ and $\left(0, \sqrt{2}\right)$; decreasing on $\left(-\sqrt{2}, 0\right)$ and $\left(\sqrt{2}, \infty\right)$; relative maxima when $x = \pm\sqrt{2}$, relative minimum when $x = 0$.

28. $y = \frac{4}{5}x^5 - \frac{13}{3}x^3 + 3x + 4$

$y' = 4x^4 - 13x^2 + 3 = \left(4x^2 - 1\right)\left(x^2 - 3\right)$

$= (2x - 1)(2x + 1)\left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right)$

CV: $x = \pm\frac{1}{2}, \pm\sqrt{3}$

$+ \quad - \quad + \quad - \quad +$; critical values: $-\sqrt{3}$, $-\frac{1}{2}$, $\frac{1}{2}$, $\sqrt{3}$

Increasing on $\left(-\infty, -\sqrt{3}\right)$, $\left(-\frac{1}{2}, \frac{1}{2}\right)$, $\left(\sqrt{3}, \infty\right)$; decreasing on $\left(-\sqrt{3}, -\frac{1}{2}\right)$ and $\left(\frac{1}{2}, \sqrt{3}\right)$; relative maxima when $x = -\sqrt{3}, \frac{1}{2}$; relative minima when $x = -\frac{1}{2}, \sqrt{3}$.

29. $y = (x^2 - 1)^4$

$y' = 8x(x^2 - 1)^3 = 8x(x + 1)^3(x - 1)^3$

CV: 0, -1, 1

$- \quad + \quad - \quad +$; critical values: -1, 0, 1

Increasing on $(-1, 0)$ and $(1, \infty)$; decreasing on $(-\infty, -1)$ and $(0, 1)$; relative maximum when $x = 0$; relative minima when $x = \pm 1$.

30. $y = \sqrt[3]{x}(x - 2)$

$y' = \frac{2(2x - 1)}{3x^{\frac{2}{3}}}$

CV: $x = 0, \frac{1}{2}$

$- \quad - \quad +$; critical values: 0, $\frac{1}{2}$

Decreasing on $(-\infty, 0)$ and $\left(0, \frac{1}{2}\right)$; increasing on $\left(\frac{1}{2}, \infty\right)$; relative minimum when $x = \frac{1}{2}$; no relative maximum.

31. $y = \frac{5}{x-1} = 5(x-1)^{-1}$

$y' = -5(x-1)^{-2} = -\frac{5}{(x-1)^2}$

CV: None, but $x = 1$ must be included in the sign chart because it is a point of discontinuity of y.

Sign chart: $-$ | $\boxed{1}$ | $-$

Decreasing on $(-\infty, 1)$ and $(1, \infty)$; no relative extremum.

32. $y = \frac{3}{x} = 3x^{-1}$

$y' = -3x^{-2} = -\frac{3}{x^2}$

CV: None, but $x = 0$ must be included in the sign chart because it is a point of discontinuity of y.

Sign chart: $-$ | $\boxed{0}$ | $-$

Decreasing on $(-\infty, 0)$ and $(0, \infty)$; no relative extremum.

33. $y = \frac{10}{\sqrt{x}} = 10x^{-\frac{1}{2}}$. [Note: $x > 0$]

$y' = -5x^{-\frac{3}{2}} = -\frac{5}{\sqrt{x^3}} < 0$ for $x > 0$.

Decreasing on $(0, \infty)$; no relative extremum.

34. $y = \frac{3x}{2x+5}$

$y' = \frac{3(2x+5)-(3x)(2)}{(2x+5)^2} = \frac{15}{(2x+5)^2}$

CV: None but $x = -\frac{5}{2}$ must be included in the sign chart because it is a point of discontinuity of y.

Sign chart: $+$ | $\boxed{-\frac{5}{2}}$ | $+$

Increasing on $\left(-\infty, -\frac{5}{2}\right)$ and $\left(-\frac{5}{2}, \infty\right)$; no relative extremum

35. $y = \frac{x^2}{2-x}$

$y' = \frac{(2-x)(2x)-x^2(-1)}{(2-x)^2} = \frac{x(4-x)}{(2-x)^2}$

CV: $x = 0, 4$, but $x = 2$ must be included in the sign chart because it is a point of discontinuity of y.

Sign chart: $-$ | 0 | $+$ | $\boxed{2}$ | $+$ | 4 | $-$

Decreasing on $(-\infty, 0)$ and $(4, \infty)$; increasing on $(0, 2)$ and $(2, 4)$; relative minimum when $x = 0$; relative maximum when $x = 4$.

36. $y = 4x^2 + \frac{1}{x}$

$y' = 8x - \frac{1}{x^2} = \frac{(2x-1)(4x^2+2x+1)}{x^2}$

CV: $x = \frac{1}{2}$, but $x = 0$ must be included in the sign chart because it is a point of discontinuity of y.

Sign chart: $-$ | $\boxed{0}$ | $-$ | $\frac{1}{2}$ | $+$

Increasing on $\left(\frac{1}{2}, \infty\right)$; decreasing on $(-\infty, 0)$ and $\left(0, \frac{1}{2}\right)$; relative minimum when $x = \frac{1}{2}$.

37. $y = \frac{x^2-3}{x+2}$

$y' = \frac{(x+2)(2x)-\left(x^2-3\right)(1)}{(x+2)^2}$

$= \frac{x^2+4x+3}{(x+2)^2} = \frac{(x+1)(x+3)}{(x+2)^2}$

CV: $x = -3, -1$, but $x = -2$ must be included in the sign chart because it is a point of discontinuity of y.

Sign chart: $+$ | -3 | $-$ | $\boxed{-2}$ | $-$ | -1 | $+$

Increasing on $(-\infty, -3)$ and $(-1, \infty)$; decreasing on $(-3, -2)$ and $(-2, -1)$; relative maximum when $x = -3$; relative minimum when $x = -1$.

38. $y = \frac{2x^2}{4x^2-25}$

$y' = \frac{(4x^2-25)(4x)-(2x^2)(8x)}{(4x^2-25)^2}$

$= -\frac{100x}{(4x^2-25)^2} = -\frac{100x}{(2x-5)^2(2x+5)^2}$

CV: $x = 0$, but $x = \pm\frac{5}{2}$ must be included in the sign chart because they are points of discontinuity of y.

+ + − −
$-\frac{5}{2}$ 0 $\frac{5}{2}$

Increasing on $\left(-\infty, -\frac{5}{2}\right)$ and $\left(-\frac{5}{2}, 0\right)$; decreasing on $\left(0, \frac{5}{2}\right)$ and $\left(\frac{5}{2}, \infty\right)$; relative maximum at $x = 0$.

39. $y = \frac{5x+2}{x^2+1}$

$y' = \frac{\left(x^2+1\right)(5)-(5x+2)(2x)}{\left(x^2+1\right)^2} = \frac{-5x^2-4x+5}{\left(x^2+1\right)^2}$

$y' = 0$ when $-5x^2-4x+5=0$; by the quadratic formula, $x = \frac{-2\pm\sqrt{29}}{5}$

CV: $x = \frac{-2\pm\sqrt{29}}{5}$

− + −
$\frac{-2-\sqrt{29}}{5}$ $\frac{-2+\sqrt{29}}{5}$

Decreasing on $\left(-\infty, \frac{-2-\sqrt{29}}{5}\right)$ and $\left(\frac{-2+\sqrt{29}}{5}, \infty\right)$; increasing on $\left(\frac{-2-\sqrt{29}}{5}, \frac{-2+\sqrt{29}}{5}\right)$; relative minimum when $x = \frac{-2-\sqrt{29}}{5}$; relative maximum when $x = \frac{-2+\sqrt{29}}{5}$.

40. $y = \sqrt[3]{x^3-9x}$

$y' = \frac{1}{3}\left(x^3-9x\right)^{-\frac{2}{3}}\left(3x^2-9\right) = \frac{\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)}{[x(x+3)(x-3)]^{\frac{2}{3}}}$

CV: $x = \pm\sqrt{3}, 0, \pm 3$

+ + − − + +
−3 $-\sqrt{3}$ 0 $\sqrt{3}$ 3

Increasing on (−∞, −3), $\left(-3, -\sqrt{3}\right)$, $\left(\sqrt{3}, 3\right)$, and (3, ∞); decreasing on $\left(-\sqrt{3}, 0\right)$ and $\left(0, \sqrt{3}\right)$; relative maximum when $x = -\sqrt{3}$; relative minimum when $x = \sqrt{3}$.

41. $y = (x-1)^{2/3}$

$y' = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3\sqrt[3]{x-1}}$

CV: $x = 1$

− +
1

Increasing on (1, ∞); decreasing on (−∞, 1); relative minimum when $x = 1$.

42. $y = x^2(x+3)^4$

$y' = x^2(4)(x+3)^3 + (x+3)^4(2x)$

$= 2x(x+3)^3[2x+(x+3)]$

$= 2x(x+3)^3(3x+3) = 6x(x+3)^3(x+1)$

CV: $x = 0, -3, -1$

− + − +
−3 −1 0

Increasing on (−3, −1) and (0, ∞); decreasing on (−∞, −3) and (−1, 0); relative maximum when $x = -1$; relative minima when $x = -3$ and $x = 0$.

43. $y = x^3(x-6)^4$

$y' = x^3\left[4(x-6)^3\right] + (x-6)^4\left(3x^2\right)$

$= x^2(x-6)^3[4x+3(x-6)]$

$= x^2(x-6)^3(7x-18)$

CV: $x = 0, 6, \frac{18}{7}$

+ + − +
0 $\frac{18}{7}$ 6

Increasing on (−∞, 0), $\left(0, \frac{18}{7}\right)$, and (6, ∞); decreasing on $\left(\frac{18}{7}, 6\right)$; relative maximum when $x = \frac{18}{7}$; relative minimum when $x = 6$.

44. $y = x(1-x)^{\frac{2}{5}}$

$y' = x\left[-\frac{2}{5}(1-x)^{-\frac{3}{5}}\right] + (1-x)^{\frac{2}{5}}(1)$

$= (1-x)^{-\frac{3}{5}}\left[-\frac{2}{5}x + (1-x)\right] = -(1-x)^{-\frac{3}{5}}\left(\frac{7}{5}x - 1\right)$

CV: $x = \frac{5}{7}, 1$

$+ \quad - \quad +$

$\frac{5}{7} \quad 1$

Increasing on $\left(-\infty, \frac{5}{7}\right)$ and $(1, \infty)$; decreasing on $\left(\frac{5}{7}, 1\right)$; relative maximum when $x = \frac{5}{7}$; relative minimum when $x = 1$.

45. $y = e^{-\pi x} + \pi$

$y' = -\pi e^{-\pi x} < 0$ for all x. Thus decreasing on $(-\infty, \infty)$; no relative extremum.

46. $y = x \ln x$. (Note: $x > 0$.)

$y' = 1 + \ln x$

$y' = 0$ when $1 + \ln x = 0$, $\ln x = -1$, or

$x = e^{-1} = \frac{1}{e}$

CV: $x = \frac{1}{e}$

$- \quad +$

$0 \quad \frac{1}{e}$

Decreasing on $\left(0, \frac{1}{e}\right)$; increasing on $\left(\frac{1}{e}, \infty\right)$; relative minimum when $x = \frac{1}{e}$.

47. $y = x^2 - 9\ln x$. [Note: $x > 0$.]

$y' = 2x - \frac{9}{x} = \frac{2x^2 - 9}{x}$

CV: $x = \frac{3\sqrt{2}}{2}$

$- \quad +$

$0 \quad \frac{3\sqrt{2}}{5}$

Decreasing on $\left(0, \frac{3\sqrt{2}}{2}\right)$; increasing on $\left(\frac{3\sqrt{2}}{2}, \infty\right)$; relative minimum when $x = \frac{3\sqrt{2}}{2}$.

48. $y = x^{-1}e^x$

$y' = x^{-1}e^x - x^{-2}e^x = e^x\left(\frac{1}{x} - \frac{1}{x^2}\right) = e^x\left(\frac{x-1}{x^2}\right)$

CV: $x = 1$, but $x = 0$ must also be included in the sign chart because it is a point of discontinuity of y.

$- \quad - \quad +$

$\boxed{0} \quad 1$

Increasing on $(1, \infty)$; decreasing on $(-\infty, 0)$ and $(0, 1)$; relative minimum when $x = 1$.

49. $y = e^x + e^{-x}$

$y' = e^x - e^{-x}$

Setting $y' = 0$ gives $e^x - e^{-x} = 0$, $e^x = e^{-x}$, $x = -x$, $x = 0$

CV: $x = 0$

$- \quad +$

0

Decreasing on $(-\infty, 0)$; increasing on $(0, \infty)$; relative minimum when $x = 0$.

50. $y = e^{-x^2/2}$

$y' = -xe^{-x^2/2}$

CV: $x = 0$

$+ \quad -$

0

Increasing on $(-\infty, 0)$; decreasing on $(0, \infty)$; relative maximum at $x = 0$

51. $y = x \ln x - x$. [Note: $x > 0$.]

$y' = \left[x \cdot \frac{1}{x} + (\ln x)(1)\right] - 1 = \ln x$

CV: $x = 1$

$- \quad +$

$0 \quad 1$

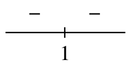
Decreasing on $(0, 1)$; increasing on $(1, \infty)$; relative minimum when $x = 1$; no relative maximum.

52. $y = \left(x^2+1\right)e^{-x}$

$y' = \left(x^2+1\right)\left(-e^{-x}\right) + e^{-x}(2x)$

$= -e^{-x}\left[\left(x^2+1\right) - 2x\right] = -e^{-x}(x-1)^2$

CV: $x = 1$



Decreasing on (–∞, 1) and (1, ∞); never increasing; no relative extremum.

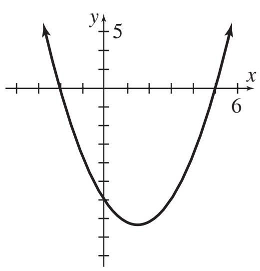
53. $y = x^2 - 3x - 10 = (x+2)(x-5)$

Intercepts (–2, 0), (5, 0), (0, –10)

$y' = 2x - 3$

CV: $x = \frac{3}{2}$

Decreasing on $\left(-\infty, \frac{3}{2}\right)$; increasing on $\left(\frac{3}{2}, \infty\right)$; relative minimum when $x = \frac{3}{2}$.



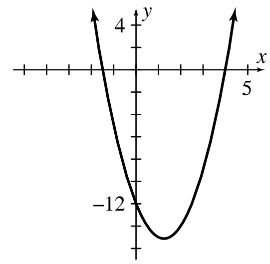
54. $y = 2x^2 - 5x - 12 = (2x+3)(x-4)$

Intercepts $\left(-\frac{3}{2}, 0\right)$, (4, 0), (0, –12)

$y' = 4x - 5 = 4\left(x - \frac{5}{4}\right)$

CV: $x = \frac{5}{4}$

Decreasing on $\left(-\infty, \frac{5}{4}\right)$; increasing on $\left(\frac{5}{4}, \infty\right)$; relative minimum when $x = \frac{5}{4}$.



55. $y = 3x - x^3 = x\left(\sqrt{3}+x\right)\left(\sqrt{3}-x\right)$

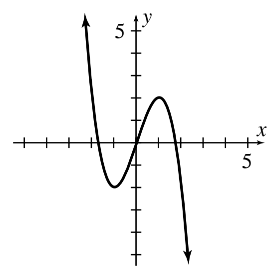
Intercepts: (0, 0), $\left(\pm\sqrt{3}, 0\right)$

Symmetric about origin.

$y' = 3 - 3x^2 = 3(1+x)(1-x)$

CV: $x = \pm 1$

Decreasing on (–∞, –1) and (1, ∞); increasing on (–1, 1); relative minimum when $x = -1$; relative maximum when $x = 1$.



56. $y = x^4 - 16 = \left(x^2+4\right)(x+2)(x-2)$

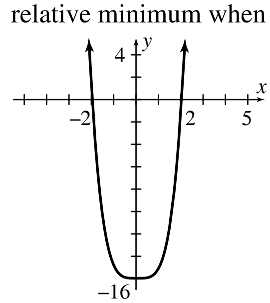
Intercepts (±2, 0), (0, –16)

Symmetric about *y*-axis.

$y' = 4x^3$

CV: $x = 0$

Decreasing on (–∞, 0); increasing on (0, ∞); relative minimum when $x = 0$.



57. $y = 2x^3 - 9x^2 + 12x = x\left(2x^2 - 9x + 12\right)$

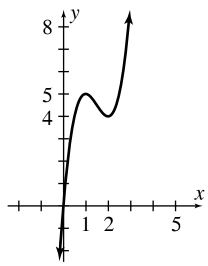
Note that $2x^2 - 9x + 12 = 0$ has no real roots.
The only intercept is (0, 0).

$y' = 6x^2 - 18x + 12 = 6\left(x^2 - 3x + 2\right)$

$= 6(x - 2)(x - 1)$

CV: $x = 1, 2$

Increasing on $(-\infty, 1)$ and $(2, \infty)$; decreasing on $(1, 2)$; relative maximum when $x = 1$; relative minimum when $x = 2$.



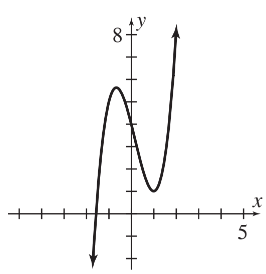
58. $y = 2x^3 - x^2 - 4x + 4$

The x-intercept is not convenient to find.
y-intercept is (0, 4).

$y' = 6x^2 - 2x - 4 = 2(3x + 2)(x - 1)$

CV: $x = -\frac{2}{3}, 1$

Increasing on $\left(-\infty, -\frac{2}{3}\right)$ and $(1, \infty)$; decreasing on $\left(-\frac{2}{3}, 1\right)$; relative maximum when $x = -\frac{2}{3}$; relative minimum when $x = 1$.



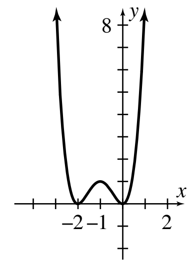
59. $y = x^4 + 4x^3 + 4x^2 = x^2(x + 2)^2$

Intercepts (0, 0), (–2, 0)

$y' = 4x^3 + 12x^2 + 8x = 4x(x + 1)(x + 2)$

CV: $x = 0, -1, -2$

Increasing on $(-2, -1)$ and $(0, \infty)$; decreasing on $(-\infty, -2)$ and $(-1, 0)$; relative maximum when $x = -1$; relative minima when $x = -2$ or $x = 0$.



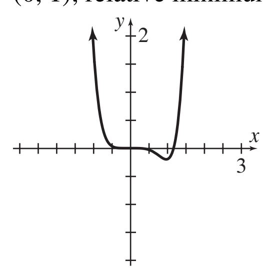
60. $y = x^6 - \frac{6}{5}x^5 = x^5\left(x - \frac{6}{5}\right)$

Intercepts (0, 0), $\left(\frac{6}{5}, 0\right)$

$y' = 6x^5 - 6x^4 = 6x^4(x - 1)$

CV: $x = 0, 1$

Increasing on $(1, \infty)$; decreasing on $(-\infty, 0)$ and $(0, 1)$; relative minimum when $x = 1$.



61. $y = (x - 1)^2(x + 2)^2$

Intercepts: (1, 0), (–2, 0), (0, 4)

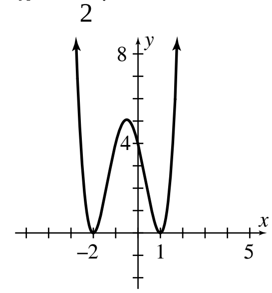
$y' = (x - 1)^2 \cdot 2(x + 2) + (x + 2)^2 \cdot 2(x - 1)$

$= 2(x - 1)(x + 2)[(x - 1) + (x + 2)]$

$= 2(x - 1)(x + 2)(2x + 1)$

CV: $x = 1, -2, -\frac{1}{2}$

Decreasing on $(-\infty, -2)$ and $\left(-\frac{1}{2}, 1\right)$; increasing on $\left(-2, -\frac{1}{2}\right)$ and $(1, \infty)$; relative minima when $x = -2$ or $x = 1$; relative maximum when $x = -\frac{1}{2}$.



62. $y = \sqrt{x}(x^2 - x - 2) = \sqrt{x}(x-2)(x+1)$

[Note $x \ge 0$.]

Intercepts (0, 0), (2, 0)

$y = x^{5/2} - x^{3/2} - 2x^{1/2}$

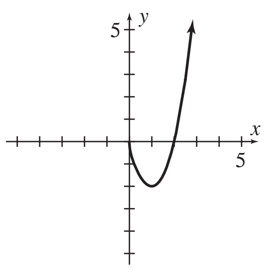
$y' = \frac{5}{2}x^{3/2} - \frac{3}{2}x^{1/2} - 2 \cdot \frac{1}{2}x^{-1/2}$

$= \frac{1}{2\sqrt{x}}(5x^2 - 3x - 2)$

$= \frac{1}{2\sqrt{x}}(5x+2)(x-1)$

CV: $x = 0, 1$ $(x \ge 0)$

Decreasing on (0, 1); increasing on (1, ∞); relative minimum when $x = 1$.



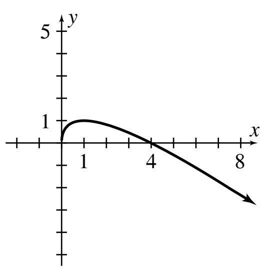
63. $y = 2\sqrt{x} - x = \sqrt{x}\left(2 - \sqrt{x}\right)$. [Note: $x \ge 0$.]

Intercepts (0, 0), (4, 0)

$y' = \frac{1}{\sqrt{x}} - 1 = \frac{1 - \sqrt{x}}{\sqrt{x}}$

CV: $x = 0, 1$

Increasing on (0, 1); decreasing on (1, ∞); relative maximum when $x = 1$.



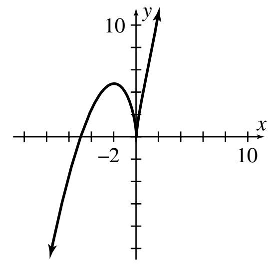
64. $y = x^{\frac{5}{3}} + 5x^{\frac{2}{3}} = x^{\frac{2}{3}}(x+5)$

Intercepts (0, 0), (–5, 0)

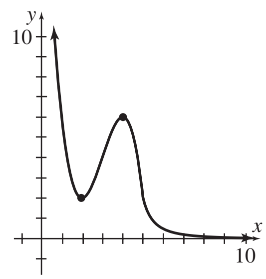
$y' = \frac{5}{3}x^{\frac{2}{3}} + \frac{10}{3x^{\frac{1}{3}}} = \frac{5(x+2)}{3x^{\frac{1}{3}}}$

CV: $x = 0, -2$

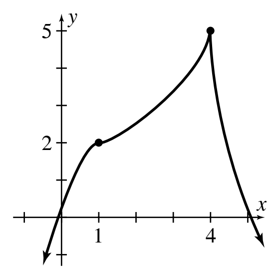
Increasing on (–∞, –2) and (0, ∞); decreasing on (–2, 0); relative maximum when $x = -2$; relative minimum when $x = 0$.



65.



66.



67. $c_f = 25{,}000$

$\bar{c}_f = \frac{c_f}{q} = \frac{25{,}000}{q}$

$\frac{d}{dq}\left(\bar{c}_f\right) = -\frac{25{,}000}{q^2} < 0$ for $q > 0$, so $\bar{c}_f$ is a decreasing function for $q > 0$.

68. $c = 3q - 3q^2 + q^3$

Marginal cost is given by $\frac{dc}{dq} = 3 - 6q + 3q^2$.

Thus $\frac{dc}{dq}$ is increasing when $\frac{d\left[\frac{dc}{dq}\right]}{dq} < 0$, that is, when $-6 + 6q > 0$. Hence $q > 1$.

69. $p = 400 - 2q$
Revenue is given by
$r = pq = (400 - 2q)q$
$= 400q - 2q^2$
Marginal revenue is $r' = 400 - 4q$. Marginal revenue is increasing when its derivative is positive. But $(r')' = -4 < 0$. Thus marginal revenue is never increasing.

70. $c = \sqrt{q}$

Marginal cost $= \frac{dc}{dq} = \frac{1}{2\sqrt{q}}$. Since

$\frac{d\left[\frac{dc}{dq}\right]}{dq} = -\frac{1}{4\sqrt{q^3}} < 0$ for $q > 0$, marginal cost is decreasing for $q > 0$.

Average cost $= \bar{c} = \frac{c}{q} = \frac{1}{\sqrt{q}}$. Since

$\frac{d\bar{c}}{dq} = -\frac{1}{2\sqrt{q^3}} < 0$ for $q > 0$, average cost is decreasing for $q > 0$.

71. $r = 240q + 57q^2 - q^3$
$r' = 240 + 114q - 3q^2 = 3(40 - q)(2 + q)$
Since $q \geq 0$, we have $q = 40$ as the only CV. Since r is increasing on (0, 40) and decreasing on (40, ∞), r is a maximum when output is 40.

72. $z = (1+b)w_p - bw_c$, w_p is function of w_c, and $b > 0$.

a. $\frac{dz}{dw_c} = (1+b)\frac{dw_p}{dw_c} - b(1)$
$= (1+b)\left[\frac{dw_p}{dw_c} - \frac{b}{1+b}\right]$ (factoring)

b. If $\frac{dw_p}{dw_c} < \frac{b}{b+1}$, then $\frac{dw_p}{dw_c} - \frac{b}{b+1} < 0$.
Because $b > 0$, then $1 + b > 0$. Thus from part (a), $\frac{dz}{dw_c} < 0$ so z is a decreasing function of w_c.

73. $E = 0.71\left(1 - \frac{T_c}{T_h}\right)$

$\frac{dE}{dT_h} = 0.71\left(\frac{T_c}{T_h^2}\right) > 0$, so as T_h increases, E increases.

74. $r = 2F + \left(1 - \frac{a}{b}\right)p - p^2 + \frac{a^2}{b}$

$\frac{dr}{dp} = \left(1 - \frac{a}{b}\right) - 2p = \frac{b-a}{b} - 2p = 2\left(\frac{b-a}{2b} - p\right)$

Setting $\frac{dr}{dp} = 0$ gives the critical value $p = \frac{b-a}{2b}$. If $p < \frac{b-a}{2b}$, then $\frac{dr}{dp} > 0$ and r is increasing. If $p > \frac{b-a}{2b}$, then $\frac{dr}{dp} < 0$ and r is decreasing. Thus revenue is maximum for $p = \frac{b-a}{2b}$.

75. $C(k) = 100\left[100 + 9k + \frac{144}{k}\right]$, $1 \leq k \leq 100$

a. $C(1) = 25{,}300$

b. $C'(k) = 100\left[9 - \frac{144}{k^2}\right] = 100\left[\frac{9k^2 - 144}{k^2}\right]$
$= 100\left[\frac{9(k+4)(k-4)}{k^2}\right]$
Since $k \geq 1$, the only critical value is $k = 4$. If $1 \leq k < 4$, then $C'(k) < 0$ and C is decreasing. If $4 < k \leq 100$, then $C'(k) > 0$ and C is increasing. Thus C has an absolute minimum for $k = 4$.

c. $C(4) = 17{,}200$

76. $P = \dfrac{100}{1+100,000e^{-0.36h}}$

$\dfrac{dP}{dh} = \dfrac{d}{dh}\left[100\left(1+100,000e^{-0.36h}\right)^{-1}\right]$

$= \dfrac{3,600,000}{e^{0.36h}\left(1+100,000e^{-0.36h}\right)^2}$

Since $\dfrac{dP}{dh} > 0$, *P* is an increasing function of *h*.

77. Relative minimum: (–3.83, 0.69)

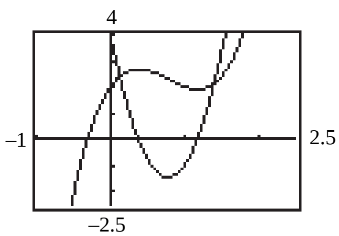
78. Relative minimum: (1.26, –5.74)

79. Relative maximum: (2.74, 3.74); relative minimum: (–2.74, –3.74)

80. Relative maximum: (0.05, 3.05)

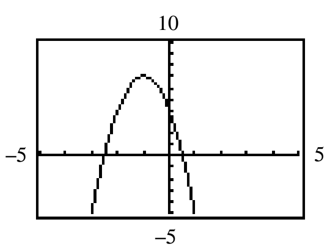
81. Relative minima: 0, 1.50, 2.00; relative maxima: 0.57, 1.77

82. *f* has relative extrema when $x \approx 0.38, 1.18$; $f'(x) = 0$ when $x \approx 0.38, 1.18$.



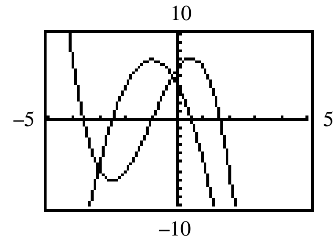
83. a. $f'(x) = 4 - 6x - 3x^2$

b.



c. $f'(x) > 0$ on (–2.53, 0.53); $f'(x) < 0$ on (–∞, –2.53), (0.53, ∞), *f* is inc. on (–2.53, 0.53); *f* is dec. on (–∞, –2.53), (0.53, ∞).

d.



84. $f'(x) = 4x^3 - 2x - 2(x+2)$

$= 4x^3 - 4x - 4$

CV: $x \approx 1.32$

Problems 13.2

1. $f(x) = x^2 - 2x + 3$ and *f* is continuous over [0, 3].

$f'(x) = 2x - 2 = 2(x-1)$

The only critical value on (0, 3) is $x = 1$. We evaluate *f* at this point and at the endpoints: $f(0) = 3$, $f(1) = 2$, and $f(3) = 6$.

Absolute maximum: $f(3) = 6$;

absolute minimum: $f(1) = 2$

2. $f(x) = -2x^2 - 6x + 5$ and *f* is continuous over [–3, 2].

$f'(x) = -4x - 6 = -4\left(x + \dfrac{3}{2}\right)$

The only critical value on (–3, 2) is $x = -\dfrac{3}{2}$. We have $f(-3) = 5$, $f\left(-\dfrac{3}{2}\right) = \dfrac{19}{2}$, and $f(2) = -15$.

Absolute maximum: $f\left(-\dfrac{3}{2}\right) = \dfrac{19}{2}$;

absolute minimum: $f(2) = -15$

3. $f(x) = \dfrac{1}{3}x^3 + \dfrac{1}{2}x^2 - 2x + 1$ and *f* is continuous over [–1, 0].

$f'(x) = x^2 + x - 2 = (x+2)(x-1)$

There are no critical values on (–1, 0), so we only have to evaluate *f* at the endpoints:

$f(-1) = \dfrac{19}{6}$ and $f(0) = 1$.

Absolute maximum: $f(-1) = \dfrac{19}{6}$

Absolute minimum: $f(0) = 1$

4. $f(x)=\frac{1}{4}x^4-\frac{3}{2}x^2$ and *f* is continuous over [0, 1].
$f'(x)=x^3-3x=x\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)$
There are no critical values on (0, 1), so we only have to evaluate *f* at the end points: *f*(0) = 0 and
$f(1)=-\frac{5}{4}$
Absolute maximum: *f*(0) = 0;
absolute minimum: $f(1)=-\frac{5}{4}$

5. $f(x)=4x^3+3x^2-18x+3$ and *f* is continuous over $\left[\frac{1}{2}, 3\right]$.
$f'(x)=12x^2+6x-18=6\left(2x^2+x-3\right)$
$=6(2x+3)(x-1)$
The only critical value on $\left(\frac{1}{2}, 3\right)$ is *x* = 1. We evaluate *f* at this point and the endpoints:
$f\left(\frac{1}{2}\right)=-\frac{19}{4}$; *f*(1) = –8, *f*(3) = 84.
Absolute maximum: *f*(3) = 84;
absolute minimum: *f*(1) = –8

6. $f(x)=x^{\frac{2}{3}}$ and *f* is continuous over [–8, 8].
$f'(x)=\frac{2}{3}x^{-\frac{1}{3}}$.
The only critical value on (–8, 8) is *x* = 0. We have *f*(–8) = 4, *f*(0) = 0, and *f*(8) = 4. Thus there is an absolute maximum when *x* = –8 or *x* = 8, and an absolute minimum when *x* = 0.
Absolute maximum: *f*(–8) = *f*(8) = 4;
absolute minimum: *f*(0) = 0

7. $f(x)=-3x^5+5x^3$ and *f* is continuous over [–2, 0].
$f'(x)=-15x^4+15x^2=15x^2\left(1-x^2\right)$
$=15x^2(1+x)(1-x)$
The only critical value on (–2, 0) is *x* = –1. We have *f*(–2) = 56, *f*(–1) = –2, and *f*(0) = 0.
Absolute maximum: *f*(–2) = 56;
absolute minimum: *f*(–1) = –2.

8. $f(x)=\frac{7}{3}x^3+2x^2-3x+1$ and *f* is continuous over [0, 3].
$f'(x)=7x^2+4x-3=(7x-3)(x+1)$
The only critical value on (0, 3) is $x=\frac{3}{7}$. We have *f*(0) = 1, $f\left(\frac{3}{7}\right)=\frac{13}{49}$, and *f*(3) = 73.
Absolute maximum: *f*(3) = 73;
absolute minimum: $f\left(\frac{3}{7}\right)=\frac{13}{49}$

9. $f(x)=3x^4-x^6$ and *f* is continuous over [–1, 2].
$f'(x)=12x^3-6x^5=6x^3\left(2-x^2\right)$
$=6x^3\left(\sqrt{2}-x\right)\left(\sqrt{2}+x\right)$
The only critical values on (–1, 2) are *x* = 0, $\sqrt{2}$. We have *f*(–1) = 2, *f*(0) = 0, $f\left(\sqrt{2}\right)=4$, and *f*(2) = –16.
Absolute maximum: $f\left(\sqrt{2}\right)=4$;
absolute minimum: *f*(2) = –16

10. $f(x)=\frac{1}{4}x^4-\frac{1}{2}x^2+3$ and *f* is continuous over [–2, 3].
$f'(x)=x^3-x=x(x^2-1)=x(x-1)(x+1)$
The critical values of *f* on (–2, 3) are *x* = –1, 0, 1. We have *f*(–2) = 5, $f(-1)=\frac{11}{4}$, *f*(0) = 3, $f(1)=\frac{11}{4}$ and $f(3)=\frac{75}{4}$.
Absolute maximum: $f(3)=\frac{75}{4}$
Absolute minimum: $f(-1)=f(1)=\frac{11}{4}$

11. $f(x)=x^4-9x^2+2$ and *f* is continuous over [–1, 3].
$f'(x)=4x^3-18x=2x\left(2x^2-9\right)$
$=2x\left(\sqrt{2}x-3\right)\left(\sqrt{2}x+3\right)$
The only critical values on (–1, 3) are *x* = 0 and $x=\frac{3}{\sqrt{2}}=\frac{3\sqrt{2}}{2}$. We have *f*(–1) = –6, *f*(0) = 2,

$f\left(\frac{3\sqrt{2}}{2}\right) = -\frac{73}{4}$, and $f(3) = 2$.

Absolute maximum: $f(0) = f(3) = 2$;

absolute minimum: $f\left(\frac{3\sqrt{2}}{2}\right) = -\frac{73}{4}$

12. $f(x) = \frac{x}{x^2+1}$ and f is continuous over [0, 2].

$f'(x) = \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

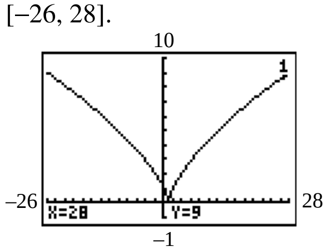
$= \frac{(1+x)(1-x)}{(x^2+1)^2}$

The only critical value on (0, 2) is $x = 1$. We have $f(0) = 0$, $f(1) = \frac{1}{2}$, and $f(2) = \frac{2}{5}$.

Absolute maximum: $f(1) = \frac{1}{2}$;

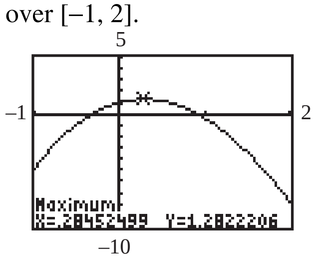
absolute minimum: $f(0) = 0$

13. $f(x) = (x-1)^{\frac{2}{3}}$ and f is continuous over [–26, 28].



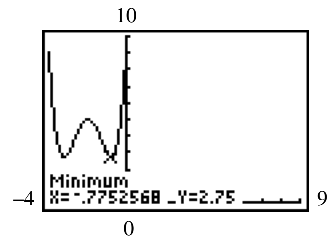
Absolute maximum: $f(-26) = f(28) = 9$;
absolute minimum: $f(1) = 0$

14. $f(x) = 0.2x^3 - 3.6x^2 + 2x + 1$ and f is continuous over [–1, 2].



Absolute maximum $f(0.28) \approx 1.28$; absolute minimum $f(2) = -7.8$

15.



a. –3.22, –0.78

b. 2.75

c. 9

d. 14,283

Problems 13.3

1. $f(x) = 2x^4 + 3x^3 + 2x - 3$

$f''(x) = 6x(4x+3)$

$f''(x)$ is 0 when $x = 0, -\frac{3}{4}$. Sign chart for f'':

$+ \quad - \quad +$ at $-\frac{3}{4}$, 0

Concave up on $\left(-\infty, -\frac{3}{4}\right)$ and $(0, \infty)$; concave down on $\left(-\frac{3}{4}, 0\right)$. Inflection points when $x = -\frac{3}{4}, 0.$.

2. $f(x) = \frac{x^5}{20} + \frac{x^4}{4} - 2x^2$

$f''(x) = (x-1)(x+2)^2$

$f''(x)$ is 0 when $x = 1, -2$. Sign chart for f'':

$- \quad - \quad +$ at -2, 1

Concave down on $(-\infty, -2)$ and $(-2, 1)$; concave up on $(1, \infty)$. Inflection point when $x = 1$.

3. $f(x) = \frac{2+x-x^2}{x^2-2x+1}$

$f''(x) = \frac{2(7-x)}{(x-1)^4}$

$f''(x)$ is 0 when $x = 7$. Although f'' is not defined when $x = 1$, f is not continuous at $x = 1$. So there is no inflection point when $x = 1$, but $x = 1$ must be considered in concavity analysis.

Sign chart for f'':

$$\begin{array}{ccccc} + & & + & & - \\ \hline & \boxed{1} & & 7 & \end{array}$$

Concave up on (–∞, 1) and (1, 7); concave down on (7, ∞). Inflection point when $x = 7$.

4. $f(x) = \frac{x^2}{(x-1)^2}$

$f''(x) = \frac{2(2x+1)}{(x-1)^4}$

$f''(x) = 0$ when $x = -\frac{1}{2}.$ Although f'' is not defined when $x = 1$, f is not continuous at $x = 1$. So there is no inflection point when $x = 1$, but $x = 1$ must be considered in concavity analysis. Sign chart of f'':

$$\begin{array}{ccccc} - & & + & & + \\ \hline & -\frac{1}{2} & & \boxed{1} & \end{array}$$

Concave up on $\left(-\frac{1}{2}, 1\right)$ and (1, ∞); concave down on $\left(-\infty, -\frac{1}{2}\right).$

Inflection point when $x = \frac{1}{2}$

5. $f(x) = \frac{x^2+1}{x^2-2}$

$f''(x) = \frac{6\left(3x^2+2\right)}{\left(x^2-2\right)^3} = \frac{6\left(3x^2+2\right)}{\left[\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)\right]^3}$

$f''(x)$ is never 0. Although f'' is not defined when $x = \pm\sqrt{2}$, f is not continuous at $x = \pm\sqrt{2}$. So there is no inflection point when $x = \pm\sqrt{2}$, but $x = \pm\sqrt{2}$ must be considered in concavity analysis. Sign chart of f'':

$$\begin{array}{ccccc} + & & - & & + \\ \hline & \boxed{-\sqrt{2}} & & \boxed{\sqrt{2}} & \end{array}$$

Concave up on $\left(-\infty, -\sqrt{2}\right)$ and $\left(\sqrt{2}, \infty\right)$; concave down on $\left(-\sqrt{2}, \sqrt{2}\right)$. No inflection point.

6. $f(x) = x\sqrt{4-x^2}$

$f''(x) = \frac{2x\left(x^2-6\right)}{\left(4-x^2\right)^{\frac{3}{2}}}$

Note that the domain of f is [–2, 2]. $f''(x)$ is 0 only when $x = 0$; f'' is not defined when $x = \pm 2$, which are the endpoints of the domain of f. The only possible point of inflection occurs when $x = 0$. Sign chart for f'':

$$\begin{array}{ccccccc} & & + & & - & & \\ \hline & -2 & & 0 & & 2 & \end{array}$$

Concave up on (–2, 0); concave down on (0, 2). Inflection point when $x = 0$.

7. $y = -2x^2 + 4x$

$y' = -4x + 4$

$y'' = -4 < 0$ for all x, so the graph is concave down for all x, that is, on (–∞, ∞).

8. $y = -74x^2 + 19x - 37$

$y' = -148x + 19$

$y'' = -148 < 0$ for all x. Thus the graph is concave down on (–∞, ∞).

9. $y = 4x^3 + 12x^2 - 12x$

$y' = 12x^2 + 24x - 12$

$y'' = 24x + 24 = 24(x+1)$

Possible inflection point when $x = -1$. Concave down on (–∞, –1): concave up on (–1, ∞); inflection point when $x = -1$.

10. $y = x^3 - 6x^2 + 9x + 1$

$y' = 3x^2 - 12x + 9$

$y'' = 6x - 12 = 6(x-2)$

Possible inflection point when $x = 2$. Concave down on (–∞, 2); concave up on (2, ∞); inflection point when $x = 2$.

11. $y = 2x^3 - 5x^2 + 5x - 2$

$y' = 6x^2 - 10x + 5$

$y'' = 12x - 10 = 12\left(x - \frac{5}{6}\right)$

Possible inflection point when $x = \frac{5}{6}.$ Concave down on $\left(-\infty, \frac{5}{6}\right)$; concave up on $\left(\frac{5}{6}, \infty\right)$; inflection point when $x = \frac{5}{6}$

12. $y = x^4 - 8x^2 - 6$

$y' = 4x^3 - 16x$

$y'' = 12x^2 - 16 = 12\left(x^2 - \frac{4}{3}\right)$

$= 12\left(x - \frac{2\sqrt{3}}{3}\right)\left(x + \frac{2\sqrt{3}}{3}\right)$

Possible inflection points $x = \pm\frac{2\sqrt{3}}{3}$. Concave up on $\left(-\infty, -\frac{2\sqrt{3}}{3}\right)$ and $\left(\frac{2\sqrt{3}}{3}, \infty\right)$; concave down on $\left(-\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right)$; inflection points when $x = \pm\frac{2\sqrt{3}}{3}$.

13. $y = 2x^4 - 48x^2 + 7x + 3$

$y' = 8x^3 - 96x + 7$

$y'' = 24x^2 - 96 = 24(x^2 - 4) = 24(x + 2)(x - 2)$

Possible inflection points when $x = \pm 2$. Concave up on $(-\infty, -2)$ and $(2, \infty)$; concave down on $(-2, 2)$; inflection points when $x = \pm 2$.

14. $y = -\frac{x^4}{4} + \frac{9x^2}{2} + 2x$

$y' = -x^3 + 9x + 2$

$y'' = -3x^2 + 9 = -3(x^2 - 3)$

$= -3\left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right)$

Possible inflection points when $x = \pm\sqrt{3}$. Concave down on $\left(-\infty, -\sqrt{3}\right)$ and $\left(\sqrt{3}, \infty\right)$; concave up on $\left(-\sqrt{3}, \sqrt{3}\right)$; inflection points when $x = \pm\sqrt{3}$.

15. $y = 2x^{\frac{1}{5}}$

$y' = \frac{2}{5}x^{-\frac{4}{5}}$

$y'' = -\frac{8}{25}x^{-\frac{9}{5}} = -\frac{8}{25x^{\frac{9}{5}}}$

y'' is not defined when $x = 0$ and y is continuous there. Thus there is a possible inflection point when $x = 0$. Concave up on $(-\infty, 0)$; concave down on $(0, \infty)$; inflection point when $x = 0$.

16. $y = \frac{7}{x^3} = 7x^{-3}$

$y' = -21x^{-4}$

$y'' = 84x^{-5} = \frac{84}{x^5}$

Although y'' is not defined when $x = 0$, y is not continuous there. Thus there is no possible inflection point. However, $x = 0$ must be considered in concavity analysis. Concave down on $(-\infty, 0)$; concave up on $(0, \infty)$; no inflection point

17. $y = \frac{x^4}{2} + \frac{19x^3}{6} - \frac{7x^2}{2} + x + 5$

$y' = 2x^3 + \frac{19}{2}x^2 - 7x + 1$

$y'' = 6x^2 + 19x - 7 = (3x - 1)(2x + 7)$

Possible inflection points when $x = -\frac{7}{2}, \frac{1}{3}$.

Concave up on $\left(-\infty, -\frac{7}{2}\right)$ and $\left(\frac{1}{3}, \infty\right)$; concave down on $\left(-\frac{7}{2}, \frac{1}{3}\right)$; inflection points when $x = -\frac{7}{2}, \frac{1}{3}$.

18. $y = -\frac{5}{2}x^4 - \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x - \frac{2}{5}$

$y' = -10x^3 - \frac{1}{2}x^2 + x + \frac{1}{3}$

$y'' = -30x^2 - x + 1 = -(5x + 1)(6x - 1)$

Possible inflection points when $x = -\frac{1}{5}, \frac{1}{6}$.

Concave down on $\left(-\infty, -\frac{1}{5}\right)$ and $\left(\frac{1}{6}, \infty\right)$;

concave up on $\left(-\frac{1}{5}, \frac{1}{6}\right)$; inflection points when $x = -\frac{1}{5}, \frac{1}{6}$.

19. $y = \frac{1}{20}x^5 - \frac{1}{4}x^4 + \frac{1}{6}x^3 - \frac{1}{2}x - \frac{2}{3}$

$y' = \frac{1}{4}x^4 - x^3 + \frac{1}{2}x^2 - \frac{1}{2}$

$y'' = x^3 - 3x^2 + x = x\left(x^2 - 3x + 1\right)$

y'' is 0 when $x = 0$ or $x^2 - 3x + 1 = 0$. Using the quadratic formula to solve $x^2 - 3x + 1 = 0$ gives $x = \frac{3 \pm \sqrt{5}}{2}$. Thus possible inflection points occur when $x = 0, \frac{3 \pm \sqrt{5}}{2}$. Concave down on $(-\infty, 0)$ and $\left(\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right)$; concave up on $\left(0, \frac{3-\sqrt{5}}{2}\right)$ and $\left(\frac{3+\sqrt{5}}{2}, \infty\right)$; inflection points when $x = 0, \frac{3 \pm \sqrt{5}}{2}$.

20. $y = \frac{1}{10}x^5 - 3x^3 + 17x + 43$

$y' = \frac{1}{2}x^4 - 9x^2 + 17$

$y'' = 2x^3 - 18x = 2x(x^2 - 9)$

$= 2x(x+3)(x-3)$

Possible inflection points when $x = 0, \pm 3$. Concave down on $(-\infty, -3)$ and $(0, 3)$; concave up on $(-3, 0)$ and $(3, \infty)$; inflection points when $x = 0, \pm 3$.

21. $y = \frac{1}{30}x^6 - \frac{7}{12}x^4 + 5x^2 + 2x - 1$

$y' = \frac{1}{5}x^5 - \frac{7}{3}x^3 + 10x + 2$

$y'' = x^4 - 7x^2 + 10 = \left(x^2 - 2\right)\left(x^2 - 5\right)$

$= \left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right)\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)$

Possible inflection points when $x = \pm\sqrt{2}, \pm\sqrt{5}$. Concave up on $\left(-\infty, -\sqrt{5}\right)$, $\left(-\sqrt{2}, \sqrt{2}\right)$, and $\left(\sqrt{5}, \infty\right)$; concave down on $\left(-\sqrt{5}, -\sqrt{2}\right)$ and $\left(\sqrt{2}, \sqrt{5}\right)$; inflection points when $x = \pm\sqrt{5}, \pm\sqrt{2}$.

22. $y = x^6 - 3x^4$

$y' = 6x^5 - 12x^3$

$y'' = 30x^4 - 36x^2 = 30x^2\left(x^2 - \frac{6}{5}\right)$

$= 30x^2\left(x - \sqrt{\frac{6}{5}}\right)\left(x + \sqrt{\frac{6}{5}}\right)$

Possible inflection points when $x = 0, \pm\sqrt{\frac{6}{5}}$.

Concave up on $\left(-\infty, -\sqrt{\frac{6}{5}}\right)$ and $\left(\sqrt{\frac{6}{5}}, \infty\right)$; concave down on $\left(-\sqrt{\frac{6}{5}}, 0\right)$ and $\left(0, \sqrt{\frac{6}{5}}\right)$.

Inflection points when $x = \pm\sqrt{\frac{6}{5}}$.

23. $y = \frac{x+1}{x-1}$

$y' = \frac{-2}{(x-1)^2}$

$y'' = \frac{4}{(x-1)^3}$

No possible inflection point, but we consider $x = 1$ in the concavity analysis. Concave down on $(-\infty, 1)$; concave up on $(1, \infty)$.

24. $y = 1 - \frac{1}{x^2}$

$y' = \frac{2}{x^3}$

$y'' = -\frac{6}{x^4}$

No possible inflection point, but we must consider $x = 0$ in the concavity analysis. Concave down on $(-\infty, 0)$ and $(0, \infty)$.

25. $y = \dfrac{x^2}{x^2+1}$

$y' = \dfrac{(x^2+1)(2x) - x^2(2x)}{(x^2+1)^2} = \dfrac{2x}{(x^2+1)^2}$

$y'' = \dfrac{(x^2+1)^2(2) - 2x(2)(x^2+1)(2x)}{(x^2+1)^4}$

$= \dfrac{(x^2+1)(2) - 8x^2}{(x^2+1)^3}$

$= \dfrac{2(1-3x^2)}{(x^2+1)^3} = \dfrac{2(1+\sqrt{3}x)(1-\sqrt{3}x)}{(x^2+1)^3}$

Possible inflection points when $x = \pm\dfrac{1}{\sqrt{3}}$.

Concave down on $\left(-\infty, -\dfrac{1}{\sqrt{3}}\right)$ and $\left(\dfrac{1}{\sqrt{3}}, \infty\right)$; concave up on $\left(-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$; inflection points when $x = \pm\dfrac{1}{\sqrt{3}}$.

26. $y = \dfrac{4x^2}{x+3}$

$y' = \dfrac{(x+3)(8x) - 4x^2(1)}{(x+3)^2} = \dfrac{4(x^2+6x)}{(x+3)^2}$

$y'' = \dfrac{(x+3)^2(4)(2x+6) - 4(x^2+6x)(2)(x+3)}{(x+3)^4}$

$= \dfrac{72}{(x+3)^3}$

No possible inflection point, but we must include $x = -3$ in the concavity analysis. Concave down on $(-\infty, -3)$; concave up on $(-3, \infty)$.

27. $y = \dfrac{21x+40}{6(x+3)^2}$

$y' = \dfrac{1}{6} \cdot \dfrac{(x+3)^2(21) - (21x+40)[2(x+3)]}{(x+3)^4}$

$= \dfrac{1}{6} \cdot \dfrac{(x+3)(21) - (21x+40)(2)}{(x+3)^3}$

$= \dfrac{1}{6} \cdot \dfrac{-21x-17}{(x+3)^3} = -\dfrac{1}{6} \cdot \dfrac{21x+17}{(x+3)^3}$

$y'' = -\dfrac{1}{6} \cdot \dfrac{(x+3)^3(21) - (21x+17)\left[3(x+3)^2\right]}{(x+3)^6}$

$= -\dfrac{1}{6} \cdot \dfrac{(x+3)(21) - (21x+17)(3)}{(x+3)^4}$

$= -\dfrac{1}{6} \cdot \dfrac{-42x+12}{(x+3)^4} = \dfrac{7x-2}{(x+3)^4}$

Possible inflection point when $x = \dfrac{2}{7}$ ($x = -3$ must be considered in concavity analysis).

Concave down on $(-\infty, -3)$ and $\left(-3, \dfrac{2}{7}\right)$; concave up on $\left(\dfrac{2}{7}, \infty\right)$; inflection point when $x = \dfrac{2}{7}$.

28. $y = 3(x^2-2)^2$

$y' = 12x(x^2-2) = 12(x^3-2x)$

$y'' = 12(3x^2-2) = 36\left(x^2 - \dfrac{2}{3}\right)$

$= 36\left(x - \dfrac{\sqrt{6}}{3}\right)\left(x + \dfrac{\sqrt{6}}{3}\right)$

Possible inflection points when $x = \pm\dfrac{\sqrt{6}}{3}$.

Concave up on $\left(-\infty, -\dfrac{\sqrt{6}}{3}\right)$ and $\left(\dfrac{\sqrt{6}}{3}, 0\right)$; concave down on $\left(-\dfrac{\sqrt{6}}{3}, \dfrac{\sqrt{6}}{3}\right)$; inflection points when $x = \pm\dfrac{\sqrt{6}}{3}$.

29. $y = 5e^x$

$y' = 5e^x$

$y'' = 5e^x$

Thus $y'' > 0$ for all x. Concave up on $(-\infty, \infty)$.

30. $y = e^x - e^{-x}$

$y' = e^x + e^{-x}$

$y'' = e^x - e^{-x}$

Setting $y'' = 0$ gives $e^x = e^{-x}$ or, equivalently, $x = 0$. Concave down on $(-\infty, 0)$; concave up on $(0, \infty)$; inflection point when $x = 0$.

31. $y = 3xe^x$

$y' = 3xe^x + 3e^x = 3e^x(x+1)$

$y'' = 3e^x(1) + 3(x+1)e^x = 3e^x(x+2)$

$y'' = 0$ if $x = -2$. Concave down on $(-\infty, -2)$; concave up on $(-2, \infty)$; inflection point when $x = -2$.

32. $y = xe^{x^2}$

$y' = 2x^2e^{x^2} + e^{x^2} = e^{x^2}(2x^2+1)$

$y'' = e^{x^2}(4x) + 2x(2x^2+1)e^{x^2} = e^{x^2}(4x^3+6x)$

$= 2xe^{x^2}(2x^2+3)$

$y'' = 0$ when $x = 0$. Concave down on $(-\infty, 0)$; concave up on $(0, \infty)$; inflection point when $x = 0$.

33. $y = \frac{\ln x}{2x}$. (Note: $x > 0$.)

$$y' = \frac{2x \cdot \frac{1}{x} - (\ln x)(2)}{4x^2} = \frac{1-\ln x}{2x^2}$$

$$y'' = \frac{2x^2\left(-\frac{1}{x}\right) - (1-\ln x)(4x)}{4x^4}$$

$$= \frac{-2x - (1-\ln x)(4x)}{4x^4}$$

$$= \frac{-1-(1-\ln x)(2)}{2x^3} = \frac{2\ln(x) - 3}{2x^3}$$

y'' is 0 if $2\ln(x) - 3 = 0$, $\ln x = \frac{3}{2}$, $x = e^{\frac{3}{2}}$.

Concave down on $\left(0, e^{\frac{3}{2}}\right)$; concave up on $\left(e^{\frac{3}{2}}, \infty\right)$; inflection point when $x = e^{\frac{3}{2}}$.

34. $y = \frac{x^2+1}{3e^x}$

$$y' = \frac{3e^x(2x) - \left(x^2+1\right)3e^x}{9e^{2x}} = \frac{2x - \left(x^2+1\right)}{3e^x}$$

$$= \frac{2x - x^2 - 1}{3e^x}$$

$$y'' = \frac{3e^x(2-2x) - \left(2x - x^2 - 1\right)3e^x}{9e^{2x}}$$

$$= \frac{(2-2x) - \left(2x - x^2 - 1\right)}{3e^x}$$

$$= \frac{x^2 - 4x + 3}{3e^x} = \frac{(x-1)(x-3)}{3e^x}$$

Possible inflection points when $x = 1, 3$. Concave up on $(-\infty, 1)$ and $(3, \infty)$; concave down on $(1, 3)$; inflection point when $x = 1, 3$.

35. $y = x^2 - x - 6 = (x-3)(x+2)$

Intercepts: $(0, -6)$, $(3, 0)$ and $(-2, 0)$

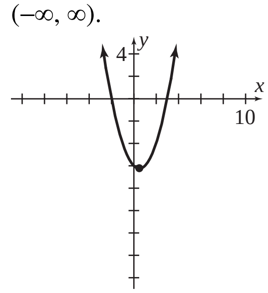
$y' = 2x - 1 = 2\left(x - \frac{1}{2}\right)$

CV: $x = \frac{1}{2}$

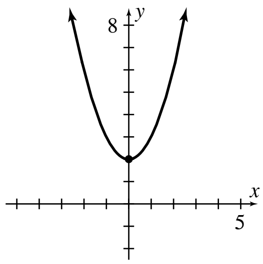
Decreasing on $\left(-\infty, \frac{1}{2}\right)$; increasing on $\left(\frac{1}{2}, \infty\right)$; relative minimum at $\left(\frac{1}{2}, -\frac{25}{4}\right)$.

$y'' = 2$

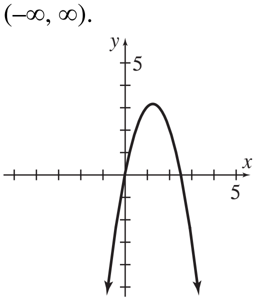
No possible inflection point. Concave up on $(-\infty, \infty)$.



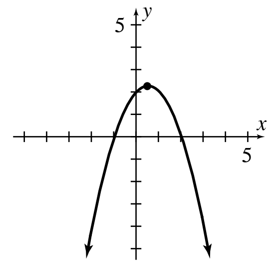
36. $y = x^2 + 2$
Intercept (0, 2)
$y' = 2x$
CV: $x = 0$
Decreasing on $(-\infty, 0)$; increasing on $(0, \infty)$; relative minimum at (0, 2).
$y'' = 2$
No possible inflection point. Concave up on $(-\infty, \infty)$. Symmetric about the *y*-axis.



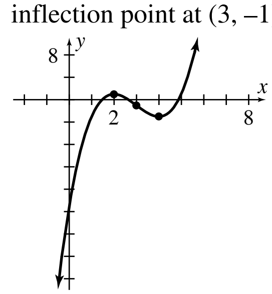
37. $y = 5x - 2x^2 = x(5 - 2x)$
Intercepts (0, 0) and $\left(\frac{5}{2}, 0\right)$
$y' = 5 - 4x$
CV: $x = \frac{5}{4}$
Increasing on $\left(-\infty, \frac{5}{4}\right)$; decreasing on $\left(\frac{5}{4}, \infty\right)$; relative maximum at $\left(\frac{5}{4}, \frac{25}{8}\right)$.
$y'' = -4$
No possible inflection point. Concave down on $(-\infty, \infty)$.



38. $y = x - x^2 + 2 = -(x - 2)(x + 1)$
Intercepts (2, 0), (–1, 0), and (0, 2)
$y' = 1 - 2x$
CV: $x = \frac{1}{2}$
Increasing on $\left(-\infty, \frac{1}{2}\right)$; decreasing on $\left(\frac{1}{2}, \infty\right)$; relative maximum at $\left(\frac{1}{2}, \frac{9}{4}\right)$
$y'' = -2$
No possible inflection point. Concave down on $(-\infty, \infty)$.



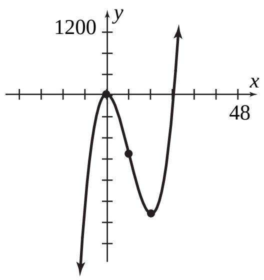
39. $y = x^3 - 9x^2 + 24x - 19$
The *x*-intercepts are not convenient to find; the *y*-intercept is (0, –19).
$y' = 3x^2 - 18x + 24 = 3(x - 2)(x - 4)$
CV: $x = 2$, $x = 4$
Increasing on $(-\infty, 2)$ and $(4, \infty)$; decreasing on (2, 4); relative maximum at (2, 1); relative minimum at (4, –3).
$y'' = 6x - 18 = 6(x - 3)$
Possible inflection point when $x = 3$. Concave down on $(-\infty, 3)$; concave up on $(3, \infty)$; inflection point at (3, –1).



40. $y = x^3 - 25x^2 = x^2(x - 25)$
Intercepts: (0, 0) and (25, 0)
$y' = 3x^2 - 50x = 3x\left(x - \frac{50}{3}\right)$
CV: $x = 0, \frac{50}{3}$
Increasing on $(-\infty, 0)$ and $\left(\frac{50}{3}, \infty\right)$; decreasing on $\left(0, \frac{50}{3}\right)$; relative maximum at (0, 0); relative minimum at $\left(\frac{50}{3}, -\frac{62{,}500}{27}\right)$.

$y'' = 6x - 50 = 6\left(x - \frac{25}{3}\right)$

Possible inflection point when $x = \frac{25}{3}$. Concave down on $\left(-\infty, \frac{25}{3}\right)$; concave up on $\left(\frac{25}{3}, \infty\right)$; inflection point at $\left(\frac{25}{3}, -\frac{31,250}{27}\right)$.



41. $y = \frac{x^3}{3} - 4x = \frac{x^3 - 12x}{3}$

$= \frac{1}{3}x\left(x + 2\sqrt{3}\right)\left(x - 2\sqrt{3}\right)$

Intercepts (0, 0) and $\left(\pm 2\sqrt{3}, 0\right)$

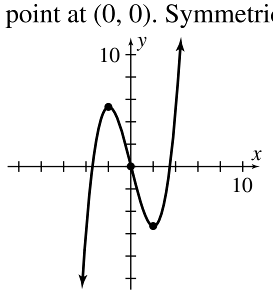
$y' = x^2 - 4 = (x+2)(x-2)$

CV: $x = \pm 2$

Increasing on (–∞, –2) and (2, ∞); decreasing on (–2, 2); relative maximum at $\left(-2, \frac{16}{3}\right)$; relative minimum at $\left(2, -\frac{16}{3}\right)$.

$y'' = 2x$

Possible inflection point when $x = 0$. Concave down on (–∞, 0); concave up on (0, ∞); inflection point at (0, 0). Symmetric about the origin.



42. $y = x^3 - 6x^2 + 9x = x(x-3)^2$

Intercepts (0, 0) and (3, 0)

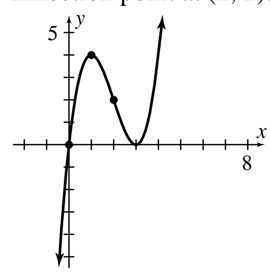
$y' = 3x^2 - 12x + 9 = 3(x-1)(x-3)$

CV: $x = 1$ and $x = 3$

Increasing on (–∞, 1) and (3, ∞); decreasing on (1, 3); relative maximum at (1, 4); relative minimum at (3, 0).

$y'' = 6x - 12 = 6(x-2)$

Possible inflection point when $x = 2$. Concave down on (–∞, 2); concave up on (2, ∞); inflection point at (2, 2).



43. $y = x^3 - 3x^2 + 3x - 3$

Intercept (0, –3)

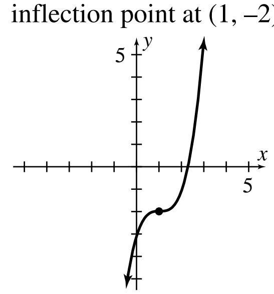
$y' = 3x^2 - 6x + 3 = 3(x-1)^2$

CV: $x = 1$

Increasing on (–∞, 1) and (1, ∞); no relative maximum or minimum

$y'' = 6(x-1)$

Possible inflection point when $x = 1$. Concave down on (–∞, 1); concave up on (1, ∞); inflection point at (1, –2).



44. $y = 2x^3 + \frac{5}{2}x^2 + 2x = x\left(2x^2 + \frac{5}{2}x + 2\right)$

Intercept (0, 0)

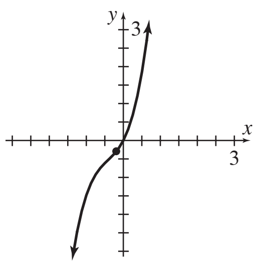
$y' = 6x^2 + 5x + 2$

CV: none

Increasing on (–∞, ∞).

$y'' = 12x + 5 = 12\left[x + \frac{5}{12}\right]$

Possible inflection point at $x = -\frac{5}{12}$. Concave down on $\left(-\infty, -\frac{5}{12}\right)$; concave up on $\left(-\frac{5}{12}, \infty\right)$; inflection point at $\left(-\frac{5}{12}, -\frac{235}{432}\right)$.



45. $y = 4x^3 - 3x^4 = x^3(4-3x)$

Intercepts (0, 0), $\left(\frac{4}{3}, 0\right)$

$y' = 12x^2 - 12x^3 = 12x^2(1-x)$

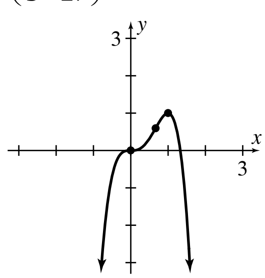
CV: $x = 0$ and $x = 1$

Increasing on $(-\infty, 0)$ and $(0, 1)$; decreasing on $(1, \infty)$; relative maximum at (1, 1).

$y'' = 24x - 36x^2 = 12x(2-3x)$

Possible inflection points at $x = 0$ and $x = \frac{2}{3}$.

Concave down on $(-\infty, 0)$ and $\left(\frac{2}{3}, \infty\right)$; concave up on $\left(0, \frac{2}{3}\right)$; inflection points at (0, 0) and $\left(\frac{2}{3}, \frac{16}{27}\right)$



46. $y = -x^3 + 2x^2 - x + 4$

Intercept (0, 4)

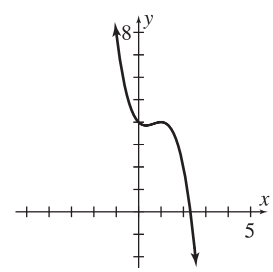
$y' = -3x^2 + 4x - 1 = -(3x-1)(x-1)$

CV: $x = \frac{1}{3}, 1$

Decreasing on $\left(-\infty, \frac{1}{3}\right)$ and $(1, \infty)$; increasing on $\left(\frac{1}{3}, 1\right)$; relative minimum at $\left(\frac{1}{3}, \frac{104}{27}\right)$; relative maximum at (1, 4)

$y'' = -6x + 4 = -6\left(x - \frac{2}{3}\right)$

Possible inflection point when $x = \frac{2}{3}$. Concave up on $\left(-\infty, \frac{2}{3}\right)$; concave down on $\left(\frac{2}{3}, \infty\right)$; inflection point at $\left(\frac{2}{3}, \frac{106}{27}\right)$



47. $y = -2 + 12x - x^3$

Intercept (0, –2)

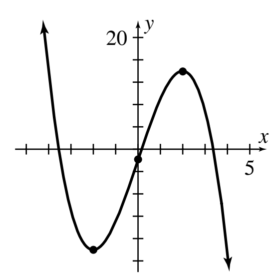
$y' = 12 - 3x^2 = 3(2+x)(2-x)$

CV: $x = \pm 2$

Decreasing on $(-\infty, -2)$ and $(2, \infty)$; increasing on $(-2, 2)$; relative minimum at (–2, –18); relative maximum at (2, 14).

$y'' = -6x$

Possible inflection point when $x = 0$. Concave up on $(-\infty, 0)$; concave down on $(0, \infty)$; inflection point at (0, –2).



48. $y = (3+2x)^3$

Intercepts (0, 27), $\left(-\frac{3}{2}, 0\right)$

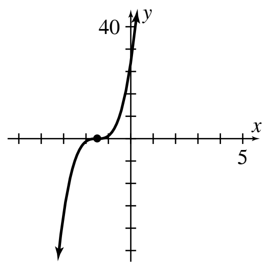
$y' = 6(3+2x)^2$

CV: $x = -\frac{3}{2}$

Increasing on $\left(-\infty, -\frac{3}{2}\right)$ and $\left(-\frac{3}{2}, \infty\right)$; no relative maximum or minimum.

$y'' = 24(3+2x)$

Possible inflection point at $x = -\frac{3}{2}$.Concave down on $\left(-\infty, -\frac{3}{2}\right)$; concave up on $\left(-\frac{3}{2}, \infty\right)$; inflection point at $\left(-\frac{3}{2}, 0\right)$.



49. $y = 2x^3 - 6x^2 + 6x - 2 = 2(x-1)^3$

Intercepts (0, –2), (1, 0)

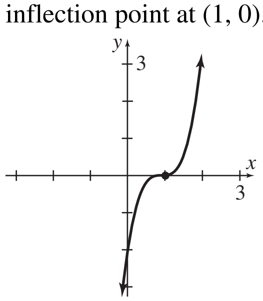
$y' = 6(x-1)^2$

CV: $x = 1$

Increasing on (–∞, 1) and (1, ∞); no relative maximum or minimum.

$y'' = 12(x-1)$

Possible inflection point when $x = 1$. Concave down on (–∞, 1); concave up on (1, ∞); inflection point at (1, 0).



50. $y = \frac{x^5}{100} - \frac{x^4}{20} = \frac{x^4}{100}(x-5)$

Intercepts (0, 0), (5, 0)

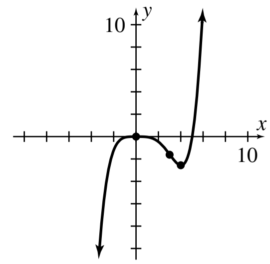
$y' = \frac{x^4}{20} - \frac{x^3}{5} = \frac{x^3}{20}(x-4)$

CV: $x = 0$ and $x = 4$

Increasing on (–∞, 0) and (4, ∞); decreasing on (0, 4); relative maximum at (0, 0); relative minimum at (4, –2.56).

$y'' = \frac{x^3}{5} - \frac{3x^2}{5} = \frac{x^2}{5}(x-3)$

Possible inflection points when $x = 0$ and $x = 3$. Concave down on (–∞, 0) and (0, 3); concave up on (3, ∞); inflection point at (3, –1.62).



51. $y = 5x - x^5 = x\left(5 - x^4\right)$

$= x\left(\sqrt{5} + x^2\right)\left(\sqrt{5} - x^2\right)$

$= x\left(\sqrt{5} + x^2\right)\left(\sqrt[4]{5} + x\right)\left(\sqrt[4]{5} - x\right)$

Intercepts (0, 0) and $\left(\pm\sqrt[4]{5}, 0\right)$

Symmetric about the origin.

$y' = 5 - 5x^4 = 5\left(1 - x^4\right) = 5\left(1 - x^2\right)\left(1 + x^2\right)$

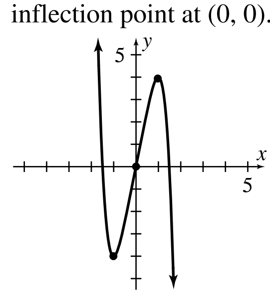
$= 5(1-x)(1+x)\left(1+x^2\right)$

CV: $x = \pm 1$

Decreasing on (–∞, –1) and (1, ∞); increasing on (–1, 1); relative minimum at (–1, –4); relative maximum at (1, 4).

$y'' = -20x^3$

Possible inflection point when $x = 0$. Concave up on (–∞, 0); concave down on (0, ∞); inflection point at (0, 0).



52. $y = x^2(x-1)^2$

Intercepts: (0, 0), (1, 0)

$y' = x^2[2(x-1)(1)] + 2x(x-1)^2$

$= 2x(x-1)(2x-1)$

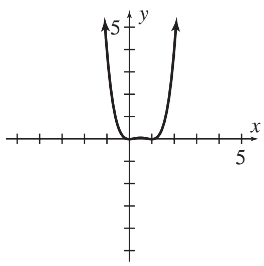
$= 4x^3 - 6x^2 + 2x$

CV: $x = 0$, 1 and $x = \frac{1}{2}$

Decreasing on $(-\infty, 0)$ and $\left(\frac{1}{2}, 1\right)$; increasing on $\left(0, \frac{1}{2}\right)$ and $(1, \infty)$; relative minima at (0, 0) and (1, 0); relative maximum at $\left(\frac{1}{2}, \frac{1}{16}\right)$

$y'' = 12x^2 - 12x + 2 = 2(6x^2 - 6x + 1)$

From the quadratic formula, there are possible inflection points when $x = \frac{3 \pm \sqrt{3}}{6}$. Concave up on $\left(-\infty, \frac{3-\sqrt{3}}{6}\right)$ and $\left(\frac{3+\sqrt{3}}{6}, \infty\right)$; concave down on $\left(\frac{3-\sqrt{3}}{6}, \frac{3+\sqrt{3}}{6}\right)$; inflection points at $\left(\frac{3-\sqrt{3}}{6}, \frac{1}{36}\right)$ and $\left(\frac{3+\sqrt{3}}{6}, \frac{1}{36}\right)$.



53. $y = 3x^4 - 4x^3 + 1$

Intercepts (0, 1) and (1, 0) [the latter is found by inspection of the equation]. No symmetry.

$y' = 12x^3 - 12x^2 = 12x^2(x-1)$

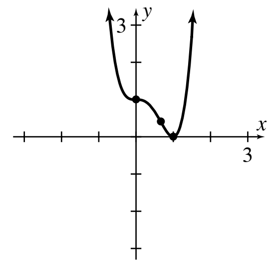
CV: $x = 0$ and $x = 1$

Decreasing on $(-\infty, 0)$ and (0, 1); increasing on $(1, \infty)$; relative minimum at (1, 0).

$y'' = 36x^2 - 24x = 12x(3x-2)$

Possible inflection points at $x = 0$ and $x = \frac{2}{3}$.

Concave up on $(-\infty, 0)$ and $\left(\frac{2}{3}, \infty\right)$; concave down on $\left(0, \frac{2}{3}\right)$; inflection points at (0, 1) and $\left(\frac{2}{3}, \frac{11}{27}\right)$.



54. $y = 3x^5 - 5x^3 = 3x^3\left[x^2 - \frac{5}{3}\right]$

$= 3x^3\left(x + \sqrt{\frac{5}{3}}\right)\left(x - \sqrt{\frac{5}{3}}\right)$

Intercepts (0, 0) and $\left(\pm\sqrt{\frac{5}{3}}, 0\right)$

Symmetric about the origin.

$y' = 15x^4 - 15x^2 = 15x^2(x+1)(x-1)$

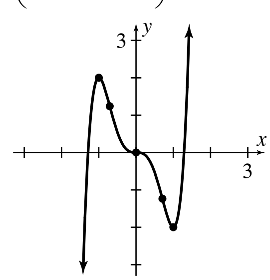
CV: $x = 0$ and $x = \pm 1$

Increasing on $(-\infty, -1)$ and $(1, \infty)$; decreasing on $(-1, 0)$ and (0, 1); relative maximum at $(-1, 2)$; relative minimum at $(1, -2)$.

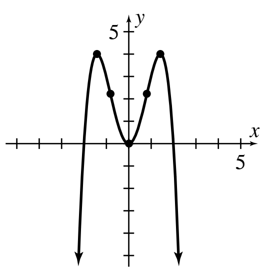
$y'' = 60x^3 - 30x = 60x\left[x + \frac{\sqrt{2}}{2}\right]\left[x - \frac{\sqrt{2}}{2}\right]$

Possible inflection points at $x = 0$ and $x = \pm\frac{\sqrt{2}}{2}$.

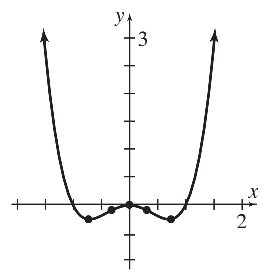
Concave down on $\left(-\infty, -\frac{\sqrt{2}}{2}\right)$ and $\left(0, \frac{\sqrt{2}}{2}\right)$; concave up on $\left(-\frac{\sqrt{2}}{2}, 0\right)$ and $\left(\frac{\sqrt{2}}{2}, \infty\right)$; inflection points at $\left(\frac{\sqrt{2}}{2}, -\frac{7\sqrt{2}}{8}\right)$, (0, 0), and $\left(-\frac{\sqrt{2}}{2}, \frac{7\sqrt{2}}{8}\right)$.



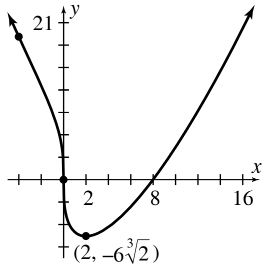
55. $y = 4x^2 - x^4 = x^2(2+x)(2-x)$
Intercepts (0, 0) and (±2, 0)
Symmetric about the *y*-axis.
$y' = 8x - 4x^3 = 4x\left(2 - x^2\right)$
$= 4x\left(\sqrt{2}+x\right)\left(\sqrt{2}-x\right)$
CV: $x = 0, \pm\sqrt{2}$
Increasing on $\left(-\infty, -\sqrt{2}\right)$ and $\left(0, \sqrt{2}\right)$; decreasing on $\left(-\sqrt{2}, 0\right)$ and $\left(\sqrt{2}, \infty\right)$; relative maxima at $\left(\pm\sqrt{2}, 4\right)$; relative minimum at (0, 0).
$y'' = 8 - 12x^2 = 12\left[\frac{2}{3} - x^2\right]$
$= 12\left(\sqrt{\frac{2}{3}} - x\right)\left(\sqrt{\frac{2}{3}} + x\right)$
Possible inflection points when $x = \pm\sqrt{\frac{2}{3}}$.
Concave down on $\left(-\infty, -\sqrt{\frac{2}{3}}\right)$ and $\left(\sqrt{\frac{2}{3}}, \infty\right)$; concave up on $\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$; inflection points at $\left(\pm\sqrt{\frac{2}{3}}, \frac{20}{9}\right)$.



56. $y = x^4 - x^2 = x^2(x+1)(x-1)$
Intercepts (0, 0), (–1, 0), and (1, 0)
Symmetric about the *y*-axis.
$y' = 4x^3 - 2x = 2x\left(\sqrt{2}x+1\right)\left(\sqrt{2}x-1\right)$
CV: $x = 0, \pm\frac{1}{\sqrt{2}}$
Decreasing on $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(0, \frac{1}{\sqrt{2}}\right)$; increasing on $\left(-\frac{1}{\sqrt{2}}, 0\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$; relative maximum at (0, 0); relative minima at $\left(\pm\frac{1}{\sqrt{2}}, -\frac{1}{4}\right)$
$y'' = 12x^2 - 2 = 12\left(x^2 - \frac{1}{6}\right)$
$= 12\left[x + \frac{1}{\sqrt{6}}\right]\left[x - \frac{1}{\sqrt{6}}\right]$
Possible inflection points when $x = \pm\frac{1}{\sqrt{6}}$.
Concave up on $\left(-\infty, -\frac{1}{\sqrt{6}}\right)$ and $\left(\frac{1}{\sqrt{6}}, \infty\right)$; concave down on $\left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$; inflection points at $\left(\pm\frac{1}{\sqrt{6}}, -\frac{5}{36}\right)$.



57. $y = x^{1/3}(x-8) = x^{4/3} - 8x^{1/3}$
Intercepts (0, 0) and (8, 0)
$y' = \frac{4}{3}x^{1/3} - \frac{8}{3}x^{-2/3}$
$= \frac{4}{3}\left[x^{1/3} - \frac{2}{x^{2/3}}\right] = \frac{4(x-2)}{3x^{2/3}}$
CV: $x = 0, 2$
Decreasing on (–∞, 0) and (0, 2); increasing on (2, ∞); relative minimum at $\left(2, -6\sqrt[3]{2}\right) \approx \left(2, -7.56\right)$.
$y'' = \frac{4}{9}x^{-2/3} + \frac{16}{9}x^{-5/3}$
$= \frac{4}{9}\left[\frac{1}{x^{2/3}} + \frac{4}{x^{5/3}}\right] = \frac{4(x+4)}{9x^{5/3}}$
Possible inflection points when $x = -4, 0$.
Concave up on (–∞, –4) and (0, ∞); concave down on (–4, 0); inflection points at $\left(-4, 12\sqrt[3]{4}\right)$ and (0, 0). Observe that at the origin the tangent line exists but it is vertical.



58. $y = (x-1)^2(x+2)^2$

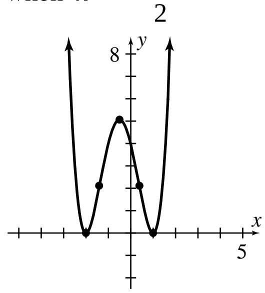
Intercepts (0, 4), (1, 0), (–2, 0)

$y' = (x-1)^2[2(x+2)] + (x+2)^2[2(x-1)]$

$= 2(x - 1)(x + 2)(2x + 1)$

CV: $x = -2, -\frac{1}{2}, 1$

Decreasing on $(-\infty, -2)$ and $\left(-\frac{1}{2}, 1\right)$; increasing on $\left(-2, -\frac{1}{2}\right)$ and $(1, \infty)$; relative maximum at $\left(-\frac{1}{2}, \frac{81}{16}\right)$; relative minima at (–2, 0) and (1, 0); $y' = 2(2x^3 + 3x^2 - 3x - 2)$, so $y'' = 6\left(2x^2 + 2x - 1\right)$. Setting $y'' = 0$ and using the quadratic formula gives possible inflection points at $x = \frac{-1 \pm \sqrt{3}}{2}$. Concave up on $\left(-\infty, \frac{-1-\sqrt{3}}{2}\right)$ and $\left(\frac{-1+\sqrt{3}}{2}, \infty\right)$; concave down on $\left(\frac{-1-\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2}\right)$; inflection points when $x = \frac{-1 \pm \sqrt{3}}{2}$



59. $y = 4x^{1/3} + x^{4/3} = x^{1/3}(4 + x)$

Intercepts (0, 0) and (–4, 0)

$y' = \frac{4}{3}x^{-2/3} + \frac{4}{3}x^{1/3} = \frac{4}{3}\left[\frac{1}{x^{2/3}} + x^{1/3}\right]$

$= \frac{4(1+x)}{3x^{2/3}}$

CV: $x = 0, -1$

Decreasing on $(-\infty, -1)$; increasing on (–1, 0) and $(0, \infty)$; rel. min at (–1, –3)

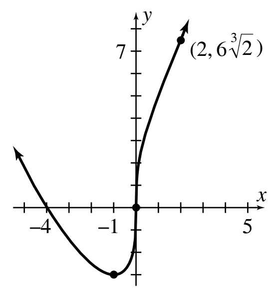
$y'' = -\frac{8}{9}x^{-5/3} + \frac{4}{9}x^{-2/3} = \frac{4}{9}\left[\frac{1}{x^{2/3}} - \frac{2}{x^{5/3}}\right]$

$= \frac{4(x-2)}{9x^{5/3}}$

Possible inflection points when $x = 0, 2$.

Concave up on $(-\infty, 0)$ and $(2, \infty)$; concave down on (0, 2); inflection point at (0, 0) and $\left(2, 6\sqrt[3]{2}\right)$.

Observe that at the origin the tangent line exists but it is vertical.



60. $y = (x+1)\sqrt{x+4}$ [Note: $x \geq -4$]

Intercepts (0, 2), (–1, 0) and (–4, 0)

$y' = (x+1)\cdot\frac{1}{2\sqrt{x+4}} + \sqrt{x+4}(1)$

$= \frac{1}{2\sqrt{x+4}}[(x+1) + 2(x+4)]$

$= \frac{3(x+3)}{2\sqrt{x+4}}$

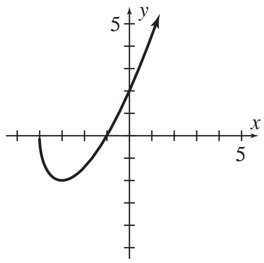
CV: $x = -3, -4$

Decreasing on (–4, –3); increasing on $(-3, \infty)$; relative minimum at (–3, –2)

$y'' = \frac{3}{2}\cdot\frac{\sqrt{x+4}(1) - (x+3)\cdot\frac{1}{2\sqrt{x+4}}}{\left(\sqrt{x+4}\right)^2}$

$= \frac{3}{4}\cdot\frac{2(x+4) - (x+3)}{(x+4)^{3/2}} = \frac{3(x+5)}{4(x+4)^{3/2}}$

No possible inflection point. Concave up on $(-4, \infty)$.



61. $y = 6x^{2/3} - \frac{x}{2} = 6x^{2/3}\left(1 - \frac{x^{1/3}}{12}\right)$

Intercepts (0, 0) and (1728, 0)

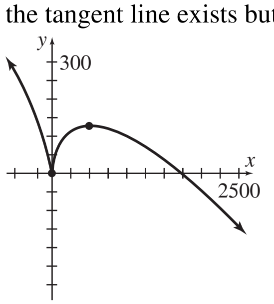
$y' = 4x^{-1/3} - \frac{1}{2} = \frac{1}{2}\left(\frac{8}{\sqrt[3]{x}} - 1\right) = \frac{1}{2}\left(\frac{8 - \sqrt[3]{x}}{\sqrt[3]{x}}\right)$

CV: $x = 0, 512$

Increasing on (0, 512); decreasing on $(-\infty, 0)$ and $(512, \infty)$; relative maximum at (512, 128); relative minimum at (0, 0).

$y'' = -\frac{4}{3}x^{-4/3} = -\frac{4}{3x^{4/3}}$

Possible inflection point at $x = 0$. Concave down on $(-\infty, 0)$ and $(0, \infty)$. Observe that at the origin the tangent line exists but it is vertical.



62. $y = 5x^{2/3} - x^{5/3} = x^{2/3}(5 - x)$

Intercepts (0, 0) and (5, 0)

$y' = \frac{10}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{5}{3}\left[\frac{2}{x^{1/3}} - x^{2/3}\right]$

$= \frac{5(2 - x)}{3x^{1/3}}$

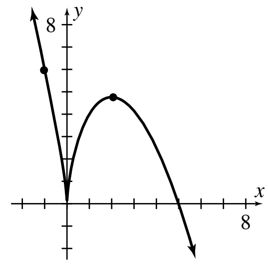
CV: $x = 0, 2$

Increasing on (0, 2); decreasing on $(-\infty, 0)$ and $(2, \infty)$; relative minimum at (0, 0); relative maximum at $\left(2, 3\sqrt[3]{4}\right) \approx (2, 4.76)$

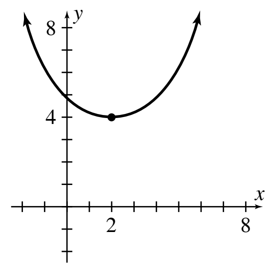
$y'' = -\frac{10}{9}x^{-4/3} - \frac{10}{9}x^{-1/3} = -\frac{10(1 + x)}{9x^{4/3}}$

Possible inflection point when $x = 0, -1$. Concave up on $(-\infty, -1)$; concave down on $(-1, 0)$, and $(0, \infty)$; inflection point at $(-1, 6)$.

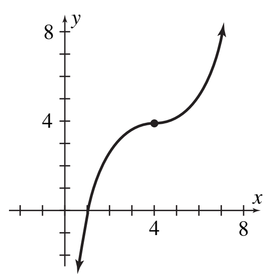
Observe that at the origin the tangent line exists but it is vertical.



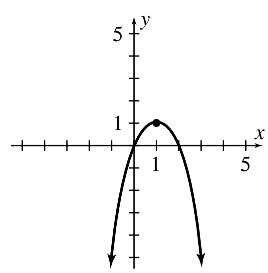
63.



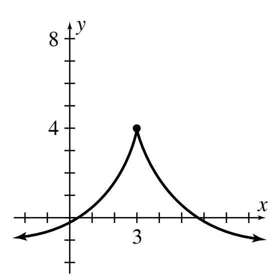
64.



65.



66.



67. $p = \frac{100}{q+2}$

$\frac{dp}{dq} = -\frac{100}{(q+2)^2} < 0$ for $q > 0$, so p is decreasing. Since $\frac{d^2p}{dq^2} = \frac{200}{(q+2)^3} > 0$ for $q > 0$, the demand curve is concave up.

68. $c = q^2 + 2q + 1$

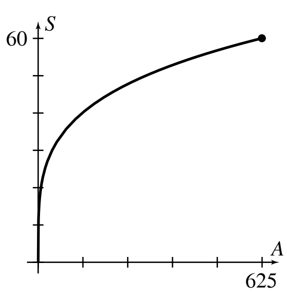
$\overline{c} = \frac{c}{q} = q + 2 + \frac{1}{q}$

$\overline{c}' = 1 - \frac{1}{q^2}$

$\overline{c}'' = \frac{2}{q^3}$

Since $\overline{c}'' > 0$ for $q > 0$, the graph of the average cost function is concave up for $q > 0$.

69. $S = f(A) = 12\sqrt[4]{A}$, $0 \le A \le 625$. For the given values of A we have $S' = 3A^{-\frac{3}{4}} > 0$ and $S'' = -\left(\frac{9}{4}\right)A^{-\frac{7}{4}} < 0$. Thus y is increasing and concave down.



70. $g(x) = e^{\frac{U_0}{A}} e^{-\frac{x^2}{2A}}$, $A > 0$, $x \ge 0$ (since x represents quantity).

$g'(x) = -\frac{e^{\frac{U_0}{A}}}{A}\left[xe^{-\frac{x^2}{2A}}\right]$

$g''(x) = -\frac{e^{\frac{U_0}{A}}}{A}\left[x \cdot e^{-\frac{x^2}{2A}}\left(-\frac{x}{A}\right) + e^{-\frac{x^2}{2A}}\right]$

$= \frac{e^{\frac{U_0}{A}}}{A^2} \cdot e^{-\frac{x^2}{2A}}\left(x^2 - A\right)$

$= \frac{e^{\frac{U_0}{A}}}{A^2} \cdot e^{-\frac{x^2}{2A}}\left(x + \sqrt{A}\right)\left(x - \sqrt{A}\right)$

If $0 \le x < \sqrt{A}$, then $g''(x) < 0$, so the graph is concave down. If $x > \sqrt{A}$, then $g''(x) > 0$, so the graph is concave up.

71. $y = 12.5 + 5.8(0.42)^x$

$y' = 5.8(0.42)^x \ln(0.42)$

Since $\ln(0.42) < 0$, we have $y' < 0$, so the function is decreasing.

$y'' = 5.8(0.42)^x \ln^2(0.42) > 0$, so the function is concave up.

72. $H = 1.00\left[1 - e^{-(0.0464t+0.0670}\right]$

$\frac{dH}{dt} = 0.0464e^{-(0.0464t+0.0670)} > 0$, so H is increasing.

$\frac{d^2H}{dt^2} = -(0.0464)^2 e^{-(0.0464t+0.0670)} < 0$, so H is concave down.

73. $n = f(r) = 0.1\ln(r) + \frac{7}{r} - 0.8$, $1 \le r \le 10$

a. $\frac{dn}{dr} = \frac{0.1}{r} - \frac{7}{r^2} = \frac{0.1r - 7}{r^2} = \frac{0.1(r - 70)}{r^2} < 0$

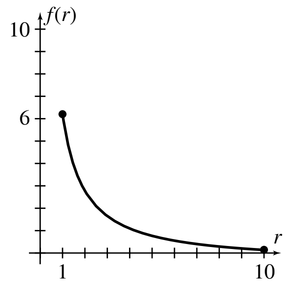
for $1 \le r \le 10$. Thus the graph of f is always falling. Also,

$\frac{d^2n}{dr^2} = -\frac{0.1}{r^2} + \frac{14}{r^3} = \frac{14 - 0.1r}{r^3}$

$= \frac{0.1(140 - r)}{r^3} > 0$

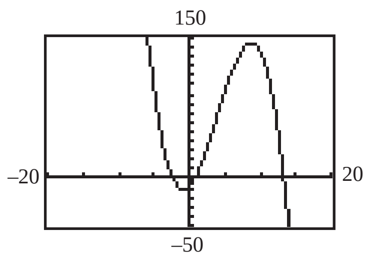
for $1 \le r \le 10$. Thus the graph is concave up.

b.



c. $\left.\dfrac{dn}{dr}\right|_{r=5} = -0.26$, so the rate of decrease is 0.26.

74.

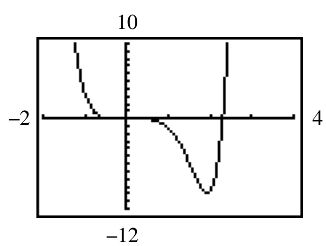


a. One relative maximum point

b. One relative minimum point

c. One inflection point

75.



Two inflection points

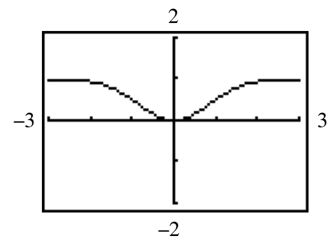
$y = x^5(x-a) = x^6 - ax^5$

$y' = 6x^5 - 5ax^4$

$y'' = 30x^4 - 20ax^3 = 10x^3(3x-2a)$

Possible inflection points when $x = 0$ and $x = \dfrac{2a}{3}$. If $a > 0$, y is concave up on $(-\infty, 0)$ and $\left(\dfrac{2a}{3}, \infty\right)$; concave down on $\left(0, \dfrac{2a}{3}\right)$. If $a < 0$, y is concave up on $\left(-\infty, \dfrac{2a}{3}\right)$ and $(0, \infty)$; concave downon $\left(\dfrac{2a}{3}, 0\right)$. In either case, y has two points of inflection, when $x = 0$ and $x = \dfrac{2a}{3}$.

76.

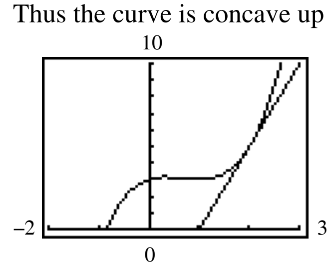


Two inflection points

77. $y = x^3 - 2x^2 + x + 3$

$y' = 3x^2 - 4x + 1$

When $x = 2$, then $y = 5$ and $y' = 5$. Thus an equation of the tangent line at $x = 2$ is $y - 5 = 5(x - 2)$, or $y = 5x - 5$. Graphing the curve and the tangent line indicates that the curve lies above the tangent line around $x = 2$. Thus the curve is concave up at $x = 2$.



78. $f(x) = 2x^3 + 3x^2 - 6x + 1$

$f'(x) = 6x^2 + 6x - 6$

$f''(x) = 12x + 6$

The relative minimum of f' occurs at a value of x for which $(f'(x))' = f''(x) = 0$. Around this value of x, $(f'(x))'$ goes from – to +. Since $(f'(x))' = f''(x)$, the concavity of f must change from concave down to concave up.

79. $f(x) = x^6 + 3x^5 - 4x^4 + 2x^2 + 1$

$f'(x) = 6x^5 + 15x^4 - 16x^3 + 4x$

$f''(x) = 30x^4 + 60x^3 - 48x^2 + 4$

Inflection points of f when $x \approx -2.61, -0.26$.

80. $f(x) = \frac{x+1}{x^2+1}$

$f'(x) = -\frac{x^2+2x-1}{(x^2+1)^2}$

$f''(x) = \frac{2(x^3+3x^2-3x-1)}{(x^2+1)^3}$

Inflection points of *f* when $x \approx -3.73, -0.27, 1.00$.

Problems 13.4

1. $y = x^2 - 5x + 6$

$y' = 2x - 5$

CV: $x = \frac{5}{2}$

$y'' = 2$

$y''\left(\frac{5}{2}\right) = 2 > 0$

Thus there is a relative minimum when $x = \frac{5}{2}$.

Because there is only one relative extremum and *f* is continuous, the relative minimum is an absolute minimum.

2. $y = 5x^2 + 20x + 2$

$y' = 10x + 20$

CV: $x = -2$

$y'' = 10$

$y''(-2) = 10 > 0$

Thus there is a relative minimum when $x = -2$. Because there is only one relative extremum and *f* is continuous, the relative minimum is an absolute minimum.

3. $y = -4x^2 + 2x - 8$

$y' = -8x + 2$

CV: $x = \frac{1}{4}$

$y'' = -8$

$y''\left(\frac{1}{4}\right) = -8 < 0$

Thus there is a relative maximum when $x = \frac{1}{4}$.

Because there is only one relative extremum and *f* is continuous, the relative maximum is an absolute maximum.

4. $y = 3x^2 - 5x + 6$

$y' = 6x - 5$

CV: $x = \frac{5}{6}$

$y'' = 6$

$y''\left(\frac{5}{6}\right) = 6 > 0$

Thus there is a relative minimum when $x = \frac{5}{6}$.

Because there is only one relative extremum and *f* is continuous, the relative minimum is an absolute minimum.

5. $y = \frac{1}{3}x^3 + 2x^2 - 5x + 1$

$y' = x^2 + 4x - 5 = (x+5)(x-1)$

CV: $x = -5, 1$

$y'' = 2x + 4$

$y''(-5) = -6 < 0 \Rightarrow$ relative maximum when $x = -5$

$y''(1) = 6 > 0 \Rightarrow$ relative minimum when $x = 1$

6. $y = x^3 - 12x + 1$

$y' = 3x^2 - 12 = 3(x+2)(x-2)$

CV: $x = \pm 2$

$y'' = 6x$

$y''(-2) = -12 < 0 \Rightarrow$ relative maximum when $x = -2$

$y''(2) = 12 > 0 \Rightarrow$ relative minimum when $x = 2$

7. $y = -x^3 + 3x^2 + 1$

$y' = -3x^2 + 6x = -3x(x-2)$

CV: $x = 0, 2$

$y'' = -6x + 6$

$y''(0) = 6 > 0 \Rightarrow$ relative minimum when $x = 0$

$y''(2) = -6 < 0 \Rightarrow$ relative maximum when $x = 2$

8. $y = x^4 - 2x^2 + 4$

$y' = 4x^3 - 4x = 4x(x+1)(x-1)$

CV: $= 0, \pm 1$

$y'' = 12x^2 - 4$

$y''(0) = -4 < 0 \Rightarrow$ relative maximum when $x = 0$

$y''(1) = 8 > 0 \Rightarrow$ relative minimum when $x = 1$

$y''(-1) = 8 > 0 \Rightarrow$ relative minimum when $x = -1$

9. $y = 7 - 2x^4$

$y' = -8x^3$

CV: $x = 0$

$y'' = -24x^2$

Since $y''(0) = 0$, the second-derivative test fails. Using the first-derivative test, we see that f increases for $x < 0$ and f decreases for $x > 0$, so there is a relative maximum when $x = 0$.

10. $y = -2x^7$

$y' = -14x^6$

CV: $x = 0$

$y'' = -84x^5$

Since $y''(0) = 0$, the second-derivative test fails. However, using the first-derivative test, we see that f decreases for $x < 0$ and for $x > 0$, so there is neither a relative maximum nor a relative minimum when $x = 0$.

11. $y = 81x^5 - 5x$

$y' = 81 \cdot 5x^4 - 5 = 5\left(81x^4 - 1\right)$

$= 5\left(9x^2 - 1\right)\left(9x^2 + 1\right)$

$= 5(3x+1)(3x-1)\left(9x^2 + 1\right)$

CV: $x = \pm\frac{1}{3}$

$y'' = 81 \cdot 5 \cdot 4x^3$

$y''\left(-\frac{1}{3}\right) = -60 < 0 \Rightarrow$ relative maximum when $x = -\frac{1}{3}$

$y''\left(\frac{1}{3}\right) = 60 > 0 \Rightarrow$ relative minimum when $x = \frac{1}{3}$

12. $y = \frac{55}{3}x^3 - x^2 - 21x - 3$

$y' = 55x^2 - 2x - 21 = (5x+3)(11x-7)$

CV: $x = -\frac{3}{5}, \frac{7}{11}$

$y'' = 110x - 2$

$y''\left(-\frac{3}{5}\right) = -68 < 0 \Rightarrow$ relative maximum when $x = -\frac{3}{5}$

$y''\left(\frac{7}{11}\right) = 68 > 0 \Rightarrow$ relative minimum when $x = \frac{11}{7}$

13. $y = \left(x^2 + 7x + 10\right)^2$

$y' = 2\left(x^2 + 7x + 10\right)(2x+7)$

$= 2(x+2)(x+5)(2x+7)$

CV: $x = -2, -5, -\frac{7}{2}$

$y'' = 2\left[\left(x^2 + 7x + 10\right)(2) + (2x+7)(2x+7)\right]$

$y''(-5) = 18 > 0 \Rightarrow$ relative minimum when $x = -5$

$y''\left(-\frac{7}{2}\right) = -9 < 0 \Rightarrow$ relative maximum when $x = -\frac{7}{2}$

$y''(-2) = 18 > 0 \Rightarrow$ relative minimum when $x = -2$

14. $y = -x^3 + 3x^2 + 9x - 2$

$y' = -3x^2 + 6x + 9 = -3(x^2 - 2x - 3)$

$= -3(x+1)(x-3)$

CV: $x = -1, 3$

$y'' = -6x + 6$

$y''(-1) = 12 > 0 \Rightarrow$ relative minimum when $x = -1$

$y''(3) = -12 < 0 \Rightarrow$ relative maximum when $x = 3$

Problems 13.5

1. $y = f(x) = \frac{x}{x-1}$

When $x = 1$ the denominator is zero but the numerator is not zero. Thus $x = 1$ is a vertical asymptote.

$\lim_{x\to\infty} \frac{x}{x-1} = \lim_{x\to\infty} \frac{x}{x} = \lim_{x\to\infty} 1 = 1$.

Similarly $\lim_{x\to-\infty} f(x) = 1$. Thus the line $y = 1$ is a horizontal asymptote.

2. $y = f(x) = \frac{x+1}{x}$

When $x = 0$ the denominator is zero but the numerator is not. Thus $x = 0$ is a vertical asymptote. $\lim_{x\to\infty} \frac{x+1}{x} = \lim_{x\to\infty} \frac{x}{x} = \lim_{x\to\infty} 1 = 1$.

Similarly $\lim_{x\to-\infty} f(x) = 1$. Thus $y = 1$ is a horizontal asymptote.

3. $f(x) = \frac{x+2}{3x-5}$

When $x = \frac{5}{3}$ the denominator is zero but the numerator is not. Thus $x = \frac{5}{3}$ is a vertical asymptote. $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{3x} = \lim_{x\to\infty} \frac{1}{3} = \frac{1}{3}$.

Similarly $\lim_{x\to-\infty} f(x) = \frac{1}{3}$. Thus $y = \frac{1}{3}$ is a horizontal asymptote.

4. $y = f(x) = \frac{2x+1}{2x+1}$

Observe that both the numerator and denominator are zero for $x = -\frac{1}{2}$. For $x \neq -\frac{1}{2}$, we have $f(x) = 1$. Thus f is a constant function for $x \neq -\frac{1}{2}$. Hence there are no vertical or horizontal asymptotes.

5. $y = f(x) = \frac{4}{x}$

When $x = 0$ the denominator is zero but the numerator is not zero, so $x = 0$ is a vertical asymptote.

$\lim_{x\to\infty} \left(\frac{4}{x}\right) = 0.$ Similarly, $\lim_{x\to-\infty} \left(\frac{4}{x}\right) = 0,$ so $y = 0$ is a horizontal asymptote.

6. $y = f(x) = 1 - \frac{2}{x^2} = \frac{x^2-2}{x^2}$

When $x = 0$ the denominator is zero but the numerator is not. Thus $x = 0$ is a vertical asymptote. $\lim_{x\to\infty} \left(1 - \frac{2}{x^2}\right) = 1 - 0 = 1.$ Similarly $\lim_{x\to-\infty} f(x) = 1,$ so $y = 1$ is a horizontal asymptote.

7. $y = f(x) = \frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)}$

Vertical asymptotes are $x = 1$ and $x = -1$.

$\lim_{x\to\infty} \frac{1}{x^2-1} = \lim_{x\to\infty} \frac{1}{x^2} = 0$. Similarly, $\lim_{x\to-\infty} f(x) = 0$. Thus $y = 0$ is a horizontal asymptote.

8. $y = f(x) = \frac{x}{x^2-4} = \frac{x}{(x-2)(x+2)}$

Vertical asymptotes: $x = 2$, $x = -2$.

$\lim_{x\to\infty} \frac{x}{x^2-4} = \lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = 0$. Similarly, $\lim_{x\to-\infty} f(x) = 0$. Thus $y = 0$ is a horizontal asymptote.

9. $y = f(x) = x^2 - 5x + 5$ is a polynomial function, so there are no horizontal or vertical asymptotes.

10. $y = f(x) = \frac{x^4}{x^3-4} = \frac{x^4}{x^3 - \left(\sqrt[3]{4}\right)^3} = \frac{x^4}{x^3 - (2^{2/3})^3}$

$= \frac{x^4}{(x - 2^{2/3})(x^2 + 2^{2/3}x + 2^{4/3})}$

Vertical asymptote: $x = 2^{2/3}$.

$\frac{x^4}{x^3-4} = x + \frac{4x}{x^3-4}$ so the line $y = x$ is an oblique asymptote.

11. $f(x) = \frac{2x^2}{x^2+x-6} = \frac{2x^2}{(x+3)(x-2)}$

Vertical asymptotes are $x = -3$ and $x = 2$.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2}{x^2} = \lim_{x\to\infty} 2 = 2$, and

$\lim_{x\to-\infty} f(x) = 2$. Thus $y = 2$ is a horizontal asymptote.

12. $f(x) = \frac{x^3}{5}$ is a polynomial function, so there are no horizontal or vertical asymptotes.

13. $y = \frac{2x^2+3x+1}{x^2-5} = \frac{2x^2+3x+1}{\left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right)}$

Vertical asymptotes are $x = -\sqrt{5}$ and $x = \sqrt{5}$.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2}{x^2} = \lim_{x\to\infty} 2 = 2$

Similarly, $\lim_{x\to-\infty} = 2$. Thus $y = 2$ is a horizontal asymptote.

14. $y = f(x) = \frac{2x^3+1}{3x(2x-1)(4x-3)}$

Vertical asymptotes are $x = 0$, $x = \frac{1}{2}$, and

$x = \frac{3}{4}$. $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^3}{24x^3} = \lim_{x\to\infty} \frac{1}{12} = \frac{1}{12}$.

Similarly, $\lim_{x\to-\infty} f(x) = \frac{1}{12}$. Thus $y = \frac{1}{12}$ is a horizontal asymptote.

15. $y = f(x) = \frac{2}{x-3} + 5 = \frac{5x-13}{x-3}$

From the denominator, $x = 3$ is a vertical asymptote.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{5x}{x} = \lim_{x\to\infty} 5 = 5$, and

$\lim_{x\to-\infty} f(x) = 5$. Thus, $y = 5$ is a horizontal asymptote.

16. $f(x) = \frac{x^2-1}{2x^2-9x+4} = \frac{x^2-1}{(2x-1)(x-4)}$

Vertical asymptotes are $x = \frac{1}{2}$ and $x = 4$.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{2x^2} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$,and

$\lim_{x\to-\infty} f(x) = \frac{1}{2}$. Thus $y = \frac{1}{2}$ is a horizontal asymptote.

17. $f(x) = \frac{3-x^4}{x^3+x^2} = \frac{3-x^4}{x^2(x+1)}$

Vertical asymptotes are $x = 0$ and $x = -1$.

$\frac{3-x^4}{x^3+x^2} = -x+1+\frac{3-x^2}{x^3+x^2}$ so the line $y = -x + 1$ is an oblique asyptote.

18. $y = f(x) = \frac{x^2+4x^3+6x^4}{3x^2}$

Observe that both the numerator and the denominator are zero when $x = 0$. For $x \neq 0$, we have

$f(x) = \frac{x^2}{3x^2}(1+4x+6x^2) = \frac{1}{3}(1+4x+6x^2)$.

Thus f is a polynomial function for $x \neq 0$. Hence there are neither horizontal nor vertical asymptotes.

19. $y = f(x) = \frac{x^2-3x-4}{1+4x+4x^2} = \frac{x^2-3x-4}{(1+2x)^2}$

From the denominator, $x = -\frac{1}{2}$ is a vertical asymptote.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{4x^2} = \lim_{x\to\infty} \frac{1}{4} = \frac{1}{4}$, and

$\lim_{x\to-\infty} f(x) = \frac{1}{4}$, so $y = \frac{1}{4}$ is a horizontal asymptote.

20. $y = f(x) = \frac{x^4+1}{1-x^4} = \frac{x^4+1}{\left(1+x^2\right)(1-x)(1+x)}$

From the denominator, vertical asymptotes are $x = 1$ and $x = -1$.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^4}{-x^4} = \lim_{x\to\infty} -1 = -1$, and $\lim_{x\to-\infty} f(x) = -1$. Thus $y = -1$ is a horizontal asymptote.

21. $y = f(x) = \frac{9x^2-16}{2(3x+4)^2} = \frac{(3x+4)(3x-4)}{2(3x+4)^2}$

When $x = -\frac{4}{3}$, both the numerator and denominator are zero. Since

$\lim_{x\to-4/3^+} f(x) = \lim_{x\to-4/3^+} \frac{3x-4}{2(3x+4)} = -\infty$, the

line $x = -\frac{4}{3}$ is a vertical asymptote.

$\lim_{x\to\infty} \frac{9x^2-16}{2(3x+4)^2} = \lim_{x\to\infty} \frac{9x^2}{18x^2} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$.

Similarly, $\lim_{x\to-\infty} f(x) = \frac{1}{2}$. Thus $y = \frac{1}{2}$ is a horizontal asymptote.

22. $y = f(x) = \frac{2}{5} + \frac{2x}{12x^2+5x-2} = \frac{24x^2+20x-4}{5(12x^2+5x-2)}$

$= \frac{4(x+1)(6x-1)}{5(3x+2)(4x-1)}$

When $x = -\frac{2}{3}$ or $x = \frac{1}{4}$, the denominator is 0, but the numerator is not. Thus, vertical asymptotes are $x = -\frac{2}{3}$ and

$x = \frac{1}{4}$. $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{24x^2}{60x^2} = \lim_{x\to\infty} \frac{2}{5} = \frac{2}{5}$.

Similarly, $\lim_{x\to-\infty} f(x) = \frac{2}{5}$. Thus, $y = \frac{2}{5}$ is a horizontal asymptote.

23. $y = f(x) = 2e^{x+2} + 4$

We have $\lim_{x\to\infty} f(x) = +\infty$ and

$\lim_{x\to-\infty} f(x) = 2 \cdot \lim_{x\to-\infty} e^x + \lim_{x\to-\infty} 4$

$= 2(0) + 4 = 4$

Thus $y = 4$ is a horizontal asymptote. There is no vertical asymptote because $f(x)$ neither increases nor decreases without bound around any fixed value of x.

24. $f(x) = 12e^{-x}$

$\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = +\infty$. Thus $y = 0$ is a horizontal asymptote. There is no vertical asymptote because $f(x)$ neither increases nor decreases without bound around any fixed value of x.

25. $y = \frac{3}{x}$

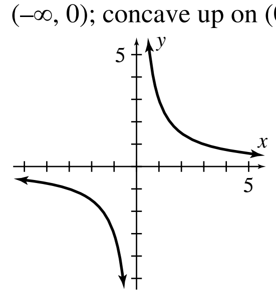
Symmetric about the origin. Vertical asymptote is $x = 0$. $\lim_{x\to\infty} \frac{3}{x} = 0 = \lim_{x\to-\infty} \frac{3}{x}$, so $y = 0$ is a horizontal asymptote.

$y' = -\frac{3}{x^2}$

CV: None, however $x = 0$ must be included in the inc.-dec. analysis. Decreasing on $(-\infty, 0)$ and $(0, \infty)$.

$y'' = \frac{6}{x^3}$

No possible inflection point, but we include $x = 0$ in the concavity analysis. Concave down on $(-\infty, 0)$; concave up on $(0, \infty)$.



26. $y = \frac{2}{2x-3}$

Intercept: $\left(0, -\frac{2}{3}\right)$

Vertical asymptote is $x = \frac{3}{2}$.

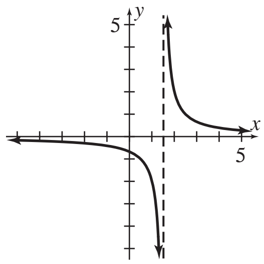
$\lim_{x\to\infty} y = 0 = \lim_{x\to-\infty} y$, so $y = 0$ is a horizontal asymptote.

$y' = -\frac{4}{(2x-3)^2}$

CV: None, but $x=\frac{3}{2}$ must be considered in the inc. dec. analysis. Decreasing on $\left(-\infty, \frac{3}{2}\right)$ and $\left(\frac{3}{2}, \infty\right)$.

$$y''=\frac{16}{(2x-3)^3}$$

No possible inflection point, but $x=\frac{3}{2}$ must be considered in the concavity analysis. Concave down on $\left(-\infty, \frac{3}{2}\right)$; concave up on $\left(\frac{3}{2}, \infty\right)$.



27. $y=\frac{x}{x-1}$

Intercept (0, 0)

Vertical asymptote is $x = 1$

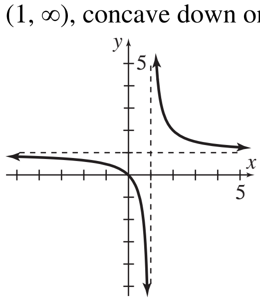
$\lim_{x\to\infty} y = 1 = \lim_{x\to-\infty} y$, so $y = 1$ is a horizontal asymptote.

$$y'=\frac{(x-1)(1)-x(1)}{(x-1)^2}=-\frac{1}{(x-1)^2}$$

CV: None, but $x = 1$ must be included in the inc.-dec. analysis. Decreasing on $(-\infty, 1)$ and $(1, \infty)$.

$$y''=\frac{2}{(x-1)^3}$$

No possible inflection point, but $x = 1$ must be included in concavity analysis. Concave up on $(1, \infty)$, concave down on $(-\infty, 1)$.



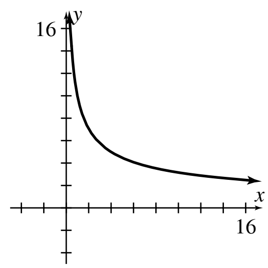
28. $y=\frac{10}{\sqrt{x}}$ (Note: $x > 0$)

$\lim_{x\to\infty} y = 0$, so $y = 0$ is a horizontal asymptote.

$\lim_{x\to 0^+} y = +\infty$, so the line $x = 0$ is a vertical asymptote.

$y'=-\frac{5}{\sqrt{x^3}}<0$ for $x > 0$. Decreasing on $(0, \infty)$.

$y''=\frac{15}{2\sqrt{x^5}}>0$ for $x > 0$. Concave up on $(0, \infty)$.



29. $y=x^2+\frac{1}{x^2}=\frac{x^4+1}{x^2}$

$x \neq 0$, so there is no y-intercept. Setting $y=0 \Rightarrow$ no x-intercept. Replacing x by $-x$ yields symmetry about the y-axis. Setting $x^2=0$ gives $x = 0$ as the only vertical asymptote. Because the degree of the numerator is greater than the degree of the denominator, no horizontal asymptote exists.

$$y=x^2+x^{-2}$$

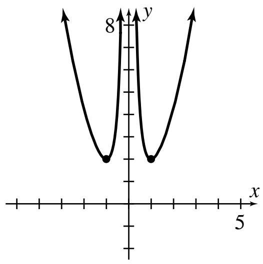
$$y'=2x-2x^{-3}=2x-\frac{2}{x^3}=\frac{2x^4-2}{x^3}=\frac{2\left(x^4-1\right)}{x^3}$$

$$=\frac{2\left(x^2+1\right)(x+1)(x-1)}{x^3}.$$

CV: $x = \pm 1$, but $x = 0$ must be included in the inc.-dec. analysis. Decreasing on $(-\infty, -1)$ and $(0, 1)$; increasing on $(-1, 0)$ and $(1, \infty)$; relative minima at $(-1, 2)$ and $(1, 2)$,

$$y''=2+\frac{6}{x^4}>0$$

for all $x \neq 0$. Concave up on $(-\infty, 0)$ and $(0, \infty)$.



30. $y = \dfrac{3x^2 - 5x - 1}{x - 2}$

Intercept: $\left(0, \dfrac{1}{2}\right)$

Vertical asymptote is $x = 2$.

$\dfrac{3x^2 - 5x - 1}{x - 2} = 3x + 1 + \dfrac{1}{x - 2}$ so $y = 3x + 1$ is an oblique asymptote.

$$y' = \frac{(x-2)(6x-5) - (3x^2 - 5x - 1)(1)}{(x-2)^2}$$

$$= \frac{3x^2 - 12x + 11}{(x-2)^2}$$

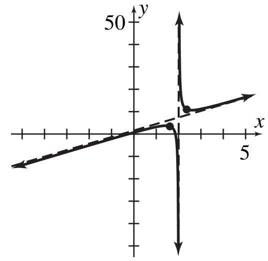
From the quadratic formula, CV: $x = \dfrac{6 \pm \sqrt{3}}{3}$, but $x = 2$ must be included in the inc.-dec. analysis. Increasing on $\left(-\infty, \dfrac{6-\sqrt{3}}{3}\right)$ and $\left(\dfrac{6+\sqrt{3}}{3}, \infty\right)$; decreasing on $\left(\dfrac{6-\sqrt{3}}{3}, 2\right)$ and $\left(2, \dfrac{6+\sqrt{3}}{3}\right)$; relative maximum at $\left(\dfrac{6-\sqrt{3}}{3}, 7-2\sqrt{3}\right)$; relative minimum at $\left(\dfrac{6+\sqrt{3}}{3}, 7+2\sqrt{3}\right)$.

$$y'' = \frac{(x-2)^2(6x-12) - (3x^2 - 12x + 11)2(x-2)}{(x-2)^4}$$

$$= \frac{(x-2)(6x-12) - 2(3x^2 - 12x + 11)}{(x-2)^3}$$

$$= \frac{2}{(x-2)^3}$$

No possible inflection point, but $x = 2$ must be included in the concavity analysis. Concave down on $(-\infty, 2)$; concave up on $(2, \infty)$



31. $y = \dfrac{1}{x^2 - 1} = \dfrac{1}{(x+1)(x-1)}$

Intercept $(0, -1)$

Symmetric about the y-axis.

Vertical asymptotes are $x = -1$ and $x = 1$.

$\lim\limits_{x\to\infty} \dfrac{1}{x^2 - 1} = 0 = \lim\limits_{x\to-\infty} \dfrac{1}{x^2 - 1}$, so $y = 0$ is a horizontal asymptote.

$$y' = -\frac{2x}{\left(x^2 - 1\right)^2}$$

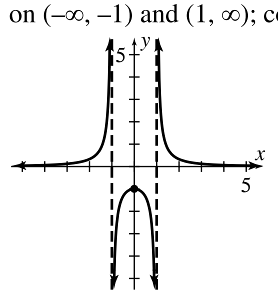
CV: $x = 0$, but $x = \pm 1$ must be included in the inc.-dec. analysis. Increasing on $(-\infty, -1)$ and $(-1, 0)$; decreasing on $(0, 1)$ and $(1, \infty)$; relative maximum at $(0, -1)$.

$$y'' = -2 \cdot \frac{\left(x^2-1\right)^2(1) - x\left[4x\left(x^2-1\right)\right]}{\left(x^2-1\right)^4}$$

$$= -2 \cdot \frac{\left(x^2-1\right)\left[\left(x^2-1\right) - 4x^2\right]}{\left(x^2-1\right)^4}$$

$$= \frac{2\left(3x^2+1\right)}{\left(x^2-1\right)^3} = \frac{2\left(3x^2+1\right)}{[(x+1)(x-1)]^3}$$

No possible inflection point, but $x = \pm 1$ must be considered in the concavity analysis. Concave up on $(-\infty, -1)$ and $(1, \infty)$; concave down on $(-1, 1)$.



32. $y = \frac{1}{x^2+1}$

Intercept (0, 1)

Symmetric about the y-axis.

$\lim_{x\to\infty} \frac{1}{x^2+1} = 0 = \lim_{x\to-\infty} \frac{1}{x^2+1}$, so $y = 0$ is a horizontal asymptote.

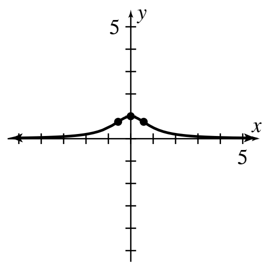
$y' = \frac{-2x}{\left(x^2+1\right)^2}$

CV: $x = 0$

Increasing on $(-\infty, 0)$; decreasing on $(0, \infty)$; relative maximum at (0, 1)

$y'' = \frac{2\left(3x^2-1\right)}{\left(x^2+1\right)^3}$

Possible inflection points at $x = \pm\frac{1}{\sqrt{3}}$. Concave up on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$; concave down on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$; inflection points at $\left(\pm\frac{1}{\sqrt{3}}, \frac{3}{4}\right)$



33. $y = \frac{1+x}{1-x}$

Intercepts: (0, 1) and (–1, 0).

$x = 1$ is the only vertical asymptote. Since

$\lim_{x\to\infty} \frac{1+x}{1-x} = \lim_{x\to\infty} \frac{x}{-x} = \lim_{x\to\infty} -1 = -1$

$= \lim_{x\to-\infty} \frac{1+x}{1-x}$

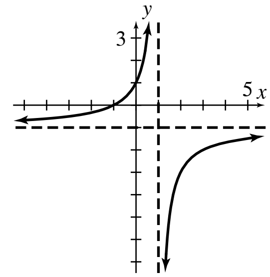
the only horizontal asymptote is $y = -1$.

$y' = \frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2} = \frac{2}{(1-x)^2}$

No critical values, but $x = 1$ must be considered in the ind.-dec. analysis. Increasing on $(-\infty, 1)$ and $(1, \infty)$.

$y'' = \frac{4}{(1-x)^3}$

No possible inflection point, but $x = 1$ must be included in the concavity analysis. Concave up on $(-\infty, 1)$; concave down on $(1, \infty)$.



34. $y = \frac{1+x}{x^2}$

Intercept is (–1, 0)

Vertical asymptote is $x = 0$.

$\lim_{x\to\infty} \frac{1+x}{x^2} = \lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = 0$

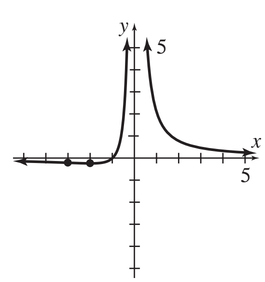
$= \lim_{x\to-\infty} \frac{1+x}{x^2}$, so $y = 0$ is the only horizontal asymptote.

$y' = -\frac{x+2}{x^3}$

CV: $x = -2$, but $x = 0$ must be included in the inc-dec. analysis. Increasing on (–2, 0); decreasing on $(-\infty, -2)$ and $(0, \infty)$; relative minimum at $\left(-2, -\frac{1}{4}\right)$.

$y'' = \frac{2(3+x)}{x^4}$

Possible inflection point when $x = 3$, but $x = 0$ must be included in the concavity analysis. Concave up on (–3, 0) and $(0, \infty)$; concave down on $(-\infty, -3)$; inflection point at $\left(-3, -\frac{2}{9}\right)$.



35. $y = \dfrac{x^2}{7x+4}$

Intercept: (0, 0)

Vertical asymptote is $x = -\dfrac{4}{7}$.

$\dfrac{x^2}{7x+4} = \dfrac{1}{7}x - \dfrac{4}{49} + \dfrac{16}{49(7x+4)}$ so $y = \dfrac{1}{7}x - \dfrac{4}{49}$ is an oblique asymptote.

$$y' = \frac{(7x+4)(2x) - x^2(7)}{(7x+4)^2}$$

$$= \frac{7x^2 + 8x}{(7x+4)^2} = \frac{x(7x+8)}{(7x+4)^2}$$

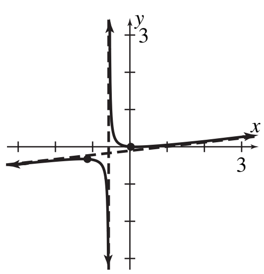
CV: $x = 0, -\dfrac{8}{7}$, but $x = -\dfrac{4}{7}$ must be included in the inc.-dec. analysis. Increasing on $\left(-\infty, -\dfrac{8}{7}\right)$ and $(0, \infty)$; decreasing on $\left(-\dfrac{8}{7}, -\dfrac{4}{7}\right)$ and $\left(-\dfrac{4}{7}, 0\right)$; relative maximum at $\left(-\dfrac{8}{7}, -\dfrac{16}{49}\right)$; relative minimum at (0, 0).

$$y'' = \frac{\left(7x^2+4\right)^2(14x+8) - \left(7x^2+8x\right)[14(7x+4)]}{(7x+4)^4}$$

$$= \frac{(7x+4)\left[(7x+4)(14x+8) - 14\left(7x^2+8x\right)\right]}{(7x+4)^4}$$

$$= \frac{32}{(7x+4)^3}$$

No possible inflection point but $x = -\dfrac{4}{7}$ must be included in concavity analysis. Concave down on $\left(-\infty, -\dfrac{4}{7}\right)$; concave up on $\left(-\dfrac{4}{7}, \infty\right)$.



36. $y = \dfrac{x^3+1}{x}$

Intercept: (–1, 0)

Vertical asymptote is $x = 0$. Because the degree of the numerator is greater than the degree of the denominator, no horizontal asymptote exists.

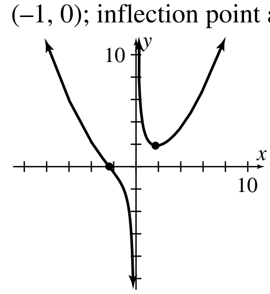
Since $y = x^2 + x^{-1}$,

$$y' = 2x - x^{-2} = 2x - \frac{1}{x^2} = \frac{2x^3 - 1}{x^2}.$$

CV: $x = \sqrt[3]{\dfrac{1}{2}}$, but $x = 0$ must be included in inc.-dec. analysis. Decreasing on $(-\infty, 0)$ and $\left(0, \sqrt[3]{\dfrac{1}{2}}\right)$; increasing on $\left(\sqrt[3]{\dfrac{1}{2}}, \infty\right)$; relative minimum at $\left(\sqrt[3]{\dfrac{1}{2}}, 3\sqrt[3]{\dfrac{1}{4}}\right)$.

$$y'' = 2 + 2x^{-3} = 2 + \frac{2}{x^3} = \frac{2\left(x^3+1\right)}{x^3}$$

Possible inflection point when $x = -1$, but $x = 0$ must be included in concavity analysis. Concave up on $(-\infty, -1)$ and $(0, \infty)$; concave down on $(-1, 0)$; inflection point at (–1, 0).



37. $y = \dfrac{9}{9x^2 - 6x - 8} = \dfrac{9}{(3x+2)(3x-4)}$

Intercept: $\left(0, -\dfrac{9}{8}\right)$

Vertical asymptotes: $x = -\dfrac{2}{3}, x = \dfrac{4}{3}$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{9}{9x^2} = \lim_{x\to\infty} \frac{1}{x^2} = 0 = \lim_{x\to-\infty} y$$

Thus $y = 0$ is a horizontal asymptote. Since

$y = 9\left(9x^2 - 6x - 8\right)^{-1}$,

$$y' = 9(-1)\left(9x^2 - 6x - 8\right)^{-2}(18x - 6)$$
$$= -\frac{54(3x-1)}{[(3x+2)(3x-4)]^2}$$

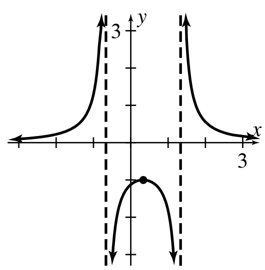
CV: $x = \frac{1}{3}$, but $x = -\frac{2}{3}$ and $x = \frac{4}{3}$ must be included in inc.-dec. analysis.

Increasing on $\left(-\infty, -\frac{2}{3}\right)$ and $\left(-\frac{2}{3}, \frac{1}{3}\right)$; decreasing on $\left(\frac{1}{3}, \frac{4}{3}\right)$ and $\left(\frac{4}{3}, \infty\right)$;

relative maximum at $\left(\frac{1}{3}, -1\right)$. Finding y'' gives:

$$y'' = -54 \cdot \frac{\left(9x^2 - 6x - 8\right)^2(3) - (3x-1)\left[2\left(9x^2 - 6x - 8\right)(18x - 6)\right]}{\left(9x^2 - 6x - 8\right)^4}$$
$$= -54 \cdot \frac{3\left(9x^2 - 6x - 8\right)\left[\left(9x^2 - 6x - 8\right) - 4(3x-1)(3x-1)\right]}{\left(9x^2 - 6x - 8\right)^4}$$
$$= \frac{-162\left(-27x^2 + 18x - 12\right)}{\left(9x^2 - 6x - 8\right)^3} = \frac{486\left(9x^2 - 6x + 4\right)}{[(3x+2)(3x-4)]^3}$$

Since $9x^2 - 6x + 4 = 0$ has no real roots, y'' is never zero. No possible inflection points, but $x = -\frac{2}{3}$ and $x = \frac{4}{3}$ must be included in concavity analysis. Concave up on $\left(-\infty, -\frac{2}{3}\right)$ and $\left(\frac{4}{3}, \infty\right)$; concave down on $\left(-\frac{2}{3}, \frac{4}{3}\right)$.



38. $y = \frac{8x^2 + 3x + 1}{2x^2}$

$8x^2 + 3x + 1$ is never 0 and x cannot be zero. Thus no intercepts. Vertical asymptote is $x = 0$.

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{8x^2}{2x^2} = \lim_{x\to\infty} 4 = 4 = \lim_{x\to-\infty} y$$

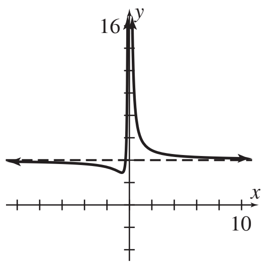
Thus $y = 4$ is a horizontal asymptote. Since $y = 4 + \frac{3}{2}x^{-1} + \frac{1}{2}x^{-2}$, we have

$$y' = -\frac{3}{2}x^{-2} - x^{-3} = -\frac{1}{2}x^{-3}(3x+2) = -\frac{3x+2}{2x^3}$$

CV: $x = -\frac{2}{3}$, but $x = 0$ must be included in the inc. dec. analysis. Decreasing on $\left(-\infty, -\frac{2}{3}\right)$ and $(0, \infty)$; increasing on $\left(-\frac{2}{3}, 0\right)$; relative minimum at $\left(-\frac{2}{3}, \frac{23}{8}\right)$.

$y'' = 3x^{-3} + 3x^{-4} = \frac{3}{x^4}(x+1)$.

Possible inflection point when $x = -1$, but $x = 0$ must be included in the concavity analysis. Concave down on $(-\infty, -1)$; concave up on $(-1, 0)$ and $(0, \infty)$; inflection point at $(-1, 3)$.



39. $y = \frac{3x+1}{(3x-2)^2}$

Intercepts: $\left(-\frac{1}{3}, 0\right), \left(0, \frac{1}{4}\right)$

Vertical asymptote is $x = \frac{2}{3}$.

$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{3x}{9x^2} = \lim_{x\to\infty} \frac{1}{3x} = 0 = \lim_{x\to-\infty} y$

Thus $y = 0$ is a horizontal asymptote.

$$y' = \frac{(3x-2)^2(3) - (3x+1)(2)(3x-2)(3)}{(3x-2)^4}$$
$$= \frac{3(3x-2)[(3x-2) - 2(3x+1)]}{(3x-2)^4}$$
$$= -\frac{3(3x+4)}{(3x-2)^3}$$

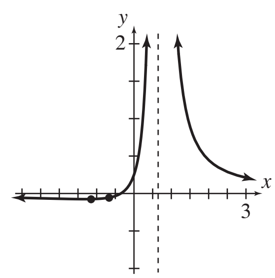
CV: $x = -\frac{4}{3}$, but $x = \frac{2}{3}$ must be included in inc.-dec. analysis.

Decreasing on $\left(-\infty, -\frac{4}{3}\right)$ and $\left(\frac{2}{3}, \infty\right)$; increasing on $\left(-\frac{4}{3}, \frac{2}{3}\right)$; relative minimum at $\left(-\frac{4}{3}, -\frac{1}{12}\right)$.

$$y'' = -3 \cdot \frac{(3x-2)^3(3) - (3x+4)(3)(3x-2)^2(3)}{(3x-2)^6}$$
$$= -3 \cdot \frac{3(3x-2)^2[(3x-2) - 3(3x+4)]}{(3x-2)^6}$$
$$= -3 \cdot \frac{3(-6x-14)}{(3x-2)^4} = \frac{18(3x+7)}{(3x-2)^4}$$

Possible inflection point when $x = -\frac{7}{3}$, but $x = \frac{2}{3}$ must be included in concavity analysis.

Concave down on $\left(-\infty, -\frac{7}{3}\right)$; concave up on $\left(-\frac{7}{3}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}, \infty\right)$; inflection point at $\left(-\frac{7}{3}, -\frac{2}{27}\right)$.



40. $y = \frac{3x+1}{(6x+5)^2}$

Intercepts: $\left(-\frac{1}{3}, 0\right), \left(0, \frac{1}{25}\right)$

Vertical asymptote is $x = -\frac{5}{6}$.

$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{3x}{36x^2} = \lim_{x\to\infty} \frac{1}{12x} = 0 = \lim_{x\to-\infty} y$

Thus $y = 0$ is horizontal asymptote.

$$y' = \frac{(6x+5)^2(3) - (3x+1)[12(6x+5)]}{(6x+5)^4}$$
$$= \frac{3(6x+5)[(6x+5) - 4(3x+1)]}{(6x+5)^4}$$
$$= \frac{3(-6x+1)}{(6x+5)^3} = \frac{-3(6x-1)}{(6x+5)^3}$$

CV: $x=\frac{1}{6}$, but $x=-\frac{5}{6}$ must be included in inc.-dec. analysis. Decreasing on $\left(-\infty, -\frac{5}{6}\right)$ and $\left(\frac{1}{6}, \infty\right)$; increasing on $\left(-\frac{5}{6}, \frac{1}{6}\right)$; relative maximum at $\left(\frac{1}{6}, \frac{1}{24}\right)$. Finding y'' gives:

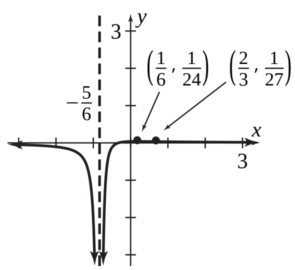
$$y''=-3\cdot\frac{(6x+5)^3(6)-(6x-1)\left[18(6x+5)^2\right]}{(6x+5)^6}$$

$$=-3\cdot\frac{6(6x+5)^2[(6x+5)-3(6x-1)]}{(6x+5)^6}$$

$$=-18\cdot\frac{-12x+8}{(6x+5)^4}$$

$$=72\cdot\frac{3x-2}{(6x+5)^4}$$

Possible inflection point when $x=\frac{2}{3}$, but $x=-\frac{5}{6}$ must be included in concavity analysis. Concave down on $\left(-\infty, -\frac{5}{6}\right)$ and $\left(-\frac{5}{6}, \frac{2}{3}\right)$; concave up on $\left(\frac{2}{3}, \infty\right)$; inflection point at $\left(\frac{2}{3}, \frac{1}{27}\right)$.



41. $y=\frac{x^2-1}{x^3}=\frac{(x+1)(x-1)}{x^3}$
Intercepts are (–1, 0) and (1, 0).
Symmetric about the origin.
Vertical asymptote $x = 0$.

$$\lim_{x\to\infty}\frac{x^2-1}{x^3}=\lim_{x\to\infty}\frac{x^2}{x^3}=\lim_{x\to\infty}\frac{1}{x}$$

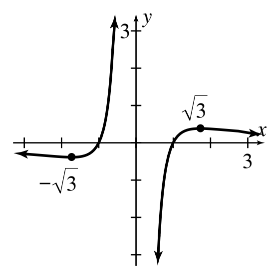
$=0=\lim_{x\to-\infty}\frac{1-x}{x^2}$, so $y = 0$ is the only horizontal asymptote. Since $y=x^{-1}-x^{-3}$, then

$$y'=-x^{-2}+3x^{-4}=x^{-4}\left(-x^2+3\right)=\frac{3-x^2}{x^4}$$

CV: $x=\pm\sqrt{3}$, but $x = 0$ must be included in the inc.-dec. analysis. Increasing on $\left(-\sqrt{3}, 0\right)$ and $\left(0, \sqrt{3}\right)$; decreasing on $\left(-\infty, -\sqrt{3}\right)$ and $\left(\sqrt{3}, \infty\right)$; relative maximum at $\left(\sqrt{3}, \frac{2\sqrt{3}}{9}\right)$; relative minimum at $\left(-\sqrt{3}, -\frac{2\sqrt{3}}{9}\right)$.

$$y''=2x^{-3}-12x^{-5}=2x^{-5}\left(x^2-6\right)=\frac{2\left(x^2-6\right)}{x^5}$$

Possible inflection points when $x=\pm\sqrt{6}$, but $x = 0$ must be included in the concavity analysis. Concave down on $\left(-\infty, -\sqrt{6}\right)$ and $\left(0, \sqrt{6}\right)$; concave up on $\left(-\sqrt{6}, 0\right)$ and $\left(\sqrt{6}, \infty\right)$; inflection points at $\left(\sqrt{6}, \frac{5\sqrt{6}}{36}\right)$ and $\left(-\sqrt{6}, \frac{-5\sqrt{6}}{36}\right)$.



42. $y = \dfrac{3x}{(x-2)^2}$

Intercept (0, 0)
Vertical asymptote at $x = 2$

$\lim\limits_{x\to\infty} \dfrac{3x}{x^2-4x+4} = \lim\limits_{x\to\infty} \dfrac{3x}{x^2} = \lim\limits_{x\to\infty} \dfrac{3}{x} = 0$ and

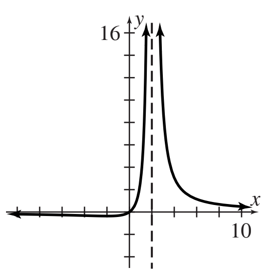
$\lim\limits_{x\to-\infty} \dfrac{3x}{x^2-4x+4} = 0,$ so $y = 0$ is the only horizontal asymptote.

$y' = \dfrac{-3(x+2)}{(x-2)^3}$

CV: $x = -2$, but $x = 2$ must be included in the inc.-dec. analysis. Decreasing on $(-\infty, -2)$ and $(2, \infty)$; increasing on $(-2, 2)$; relative maximum at $\left(-2, -\dfrac{3}{8}\right)$

$y'' = \dfrac{6(x+4)}{(x-2)^4}$

Possible inflection point when $x = -4$, but $x = 2$ must be included in the concavity analysis. Concave down on $(-\infty, -4)$; concave up on $(-4, 2)$ and $(2, \infty)$; inflection point at $\left(-4, -\dfrac{1}{3}\right)$.



43. $y = x + \dfrac{1}{x+1} = \dfrac{x^2+x+1}{x+1}$

Intercept: (0, 1). $x = -1$ is the only vertical asymptote. $y = x$ is an oblique asymptote.

$y' = \dfrac{(x+1)(2x+1) - \left(x^2+x+1\right)}{(x+1)^2}$

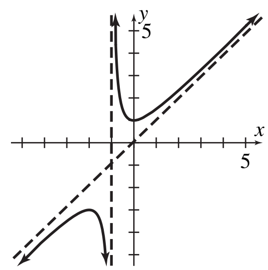
$= \dfrac{x^2+2x}{(x+1)^2} = \dfrac{x(x+2)}{(x+1)^2}$

CV: 0 and -2, but $x = -1$ must be included in the inc.-dec. analysis. Increasing on $(-\infty, -2)$ and $(0, \infty)$; decreasing on $(-2, -1)$ and $(-1, 0)$; relative maximum at $(-2, -3)$; relative minimum at (0, 1).

$y'' = \dfrac{(x+1)^2(2x+2) - \left(x^2+2x\right)[2(x+1)]}{(x+1)^4}$

$= \dfrac{(x+1)(2x+2) - \left(x^2+2x\right)[2]}{(x+1)^3} = \dfrac{2}{(x+1)^3}$

No possible inflection point, but $x = -1$ must be included in the concavity analysis. Concave down on $(-\infty, -1)$; concave up on $(-1, \infty)$.



44. $y = \dfrac{3x^4+1}{x^3}$

No intercepts
Symmetric about the origin.

Vertical asymptote is $x = 0$. $\dfrac{3x^4+1}{x^3} = 3x + \dfrac{1}{x^3}$ so $y = 3x$ is an oblique asymptote.

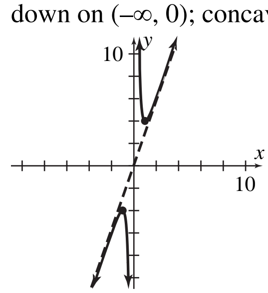
Since $y = 3x + x^{-3}$,

$y' = 3 - 3x^{-4} = 3 - \dfrac{3}{x^4} = \dfrac{3\left(x^2+1\right)(x+1)(x-1)}{x^4}$

CV: ±1, but $x = 0$ must be considered in the inc.-dec. analysis. Increasing on $(-\infty, -1)$ and $(1, \infty)$; decreasing on $(-1, 0)$ and $(0, 1)$; relative maximum at $(-1, -4)$; relative minimum at (1, 4).

$y'' = \dfrac{12}{x^5}$

No possible inflection point, but $x = 0$ must be included in the concavity analysis. Concave down on $(-\infty, 0)$; concave up on $(0, \infty)$.



45. $y = \frac{-3x^2 + 2x - 5}{3x^2 - 2x - 1} = \frac{-3x^2 + 2x - 5}{(3x+1)(x-1)}$

Note that $-3x^2 + 2x - 5$ is never zero.
Intercept: (0, 5)

Vertical asymptotes are $x = -\frac{1}{3}$ and $x = 1$.

$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{-3x^2}{3x^2} = \lim_{x\to\infty} -1 = -1 = \lim_{x\to-\infty} y$

Thus $y = -1$ is horizontal asymptote.

$$y' = \frac{\left(3x^2 - 2x - 1\right)(-6x+2) - \left(-3x^2 + 2x - 5\right)(6x-2)}{\left(3x^2 - 2x - 1\right)^2}$$

$$= \frac{2(3x-1)\left[\left(3x^2 - 2x - 1\right)(-1) - \left(-3x^2 + 2x - 5\right)\right]}{\left(3x^2 - 2x - 1\right)^2}$$

$$= \frac{12(3x-1)}{\left(3x^2 - 2x - 1\right)^2} = \frac{12(3x-1)}{[(3x+1)(x-1)]^2}$$

CV: $x = \frac{1}{3}$, but $x = -\frac{1}{3}$ and $x = 1$ must be included in inc.-dec. analysis.

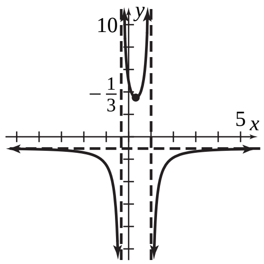
Decreasing on $\left(-\infty, -\frac{1}{3}\right)$ and $\left(-\frac{1}{3}, \frac{1}{3}\right)$; increasing on $\left(\frac{1}{3}, 1\right)$ and $(1, \infty)$; relative minimum at $\left(\frac{1}{3}, \frac{7}{2}\right)$.

$$y'' = 12 \cdot \frac{\left(3x^2 - 2x - 1\right)^2 (3) - (3x-1)\left[2\left(3x^2 - 2x - 1\right)(6x-2)\right]}{\left(3x^2 - 2x - 1\right)^4}$$

$$= 12 \cdot \frac{\left(3x^2 - 2x - 1\right)\left[3\left(3x^2 - 2x - 1\right) - 2(3x-1)(6x-2)\right]}{\left(3x^2 - 2x - 1\right)^4}$$

$$= 12 \cdot \frac{-27x^2 + 18x - 7}{\left(3x^2 - 2x - 1\right)^3} = \frac{-12\left(27x^2 - 18x + 7\right)}{[(3x+1)(x-1)]^3}$$

Since $27x^2 - 18x + 7$ is never zero, there is no possible inflection point, but $x = -\frac{1}{3}$ and $x = 1$ must be included in concavity analysis. Concave down on $\left(-\infty, -\frac{1}{3}\right)$ and $(1, \infty)$; concave up on $\left(-\frac{1}{3}, 1\right)$.



46. $y = 3x + 2 + \frac{1}{3x+2} = \frac{(3x+2)^2 + 1}{3x+2}$

$= \frac{9x^2 + 12x + 5}{3x+2}$

Note that $9x^2 + 12x + 5$ is never zero.

Intercept: $\left(0, \frac{5}{2}\right)$

Vertical asymptote is $x = -\frac{2}{3}$; oblique asymptote is $y = 3x + 2$.

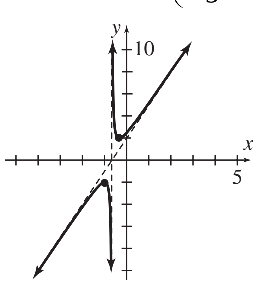
$y' = 3 - \frac{3}{(3x+2)^2} = 3 \cdot \frac{(3x+2)^2 - 1}{(3x+2)^2}$

$= 3 \cdot \frac{9x^2 + 12x + 3}{(3x+2)^2} = 9 \cdot \frac{(3x+1)(x+1)}{(3x+2)^2}$

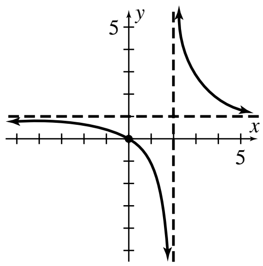
CV: $x = -\frac{1}{3}$ and $x = -1$, but $x = -\frac{2}{3}$ must be included in inc.-dec. analysis. Increasing on $(-\infty, -1)$ and $\left(-\frac{1}{3}, \infty\right)$; decreasing on $\left(-1, -\frac{2}{3}\right)$ and $\left(-\frac{2}{3}, -\frac{1}{3}\right)$; relative maximum at $(-1, -2)$; relative minimum at $\left(-\frac{1}{3}, 2\right)$.

$y'' = -3(-2)(3x+2)^{-3}(3) = \frac{18}{(3x+2)^3}$

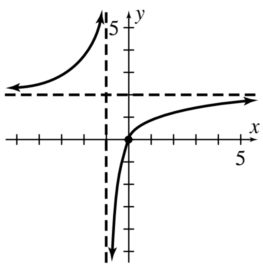
No possible inflection point, but $x = -\frac{2}{3}$ must be included in concavity analysis. Concave down on $\left(-\infty, -\frac{2}{3}\right)$; concave up on $\left(-\frac{2}{3}, \infty\right)$.



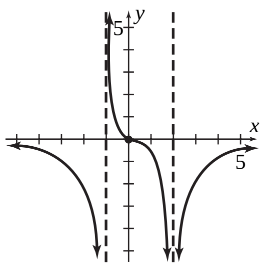
47.



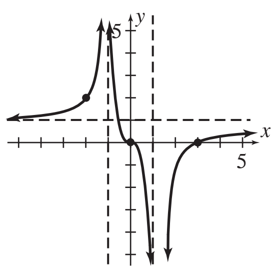
48.



49.



50.



51. When $x=-\frac{a}{b}$, then $a + bx = 0$ so $x=-\frac{a}{b}$ is a vertical asymptote.

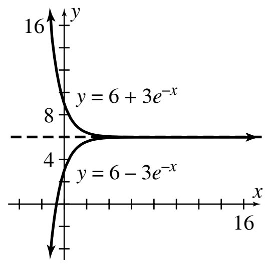
$\lim_{x\to\infty}\frac{x}{a+bx}=\lim_{x\to\infty}\frac{x}{bx}=\lim_{x\to\infty}\frac{1}{b}=\frac{1}{b}$

Thus $y=\frac{1}{b}$ is a horizontal asymptote.

52. For $y=6-3e^{-x}$ we have

$\lim_{x\to\infty}\left(6-3e^{-x}\right)=\lim_{x\to\infty}\left(6-\frac{3}{e^x}\right)=6-3(0)=6$

Thus the line $y = 6$ is a horizontal asymptote for the graph of $y=6-3e^{-x}$. For $y=6+3e^{-x}$, we obtain $\lim_{x\to\infty}\left(6+3e^{-x}\right)=6+3(0)=6$, so the line $y = 6$ is also a horizontal asymptote for the graph of $y=6+3e^{-x}$.

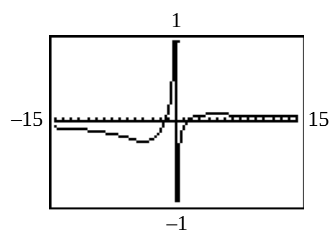


53. $\lim_{t\to\infty}\left(150-76e^{-t}\right)=\lim_{t\to\infty}\left(150-\frac{76}{e^t}\right)$

$= 150 - 0 = 150$

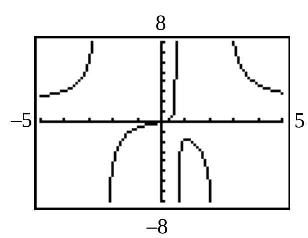
Thus $y = 150$ is a horizontal asymptote.

54.



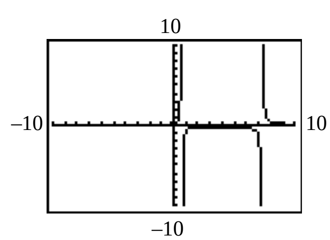
$x \approx -0.08$, $y = 0$

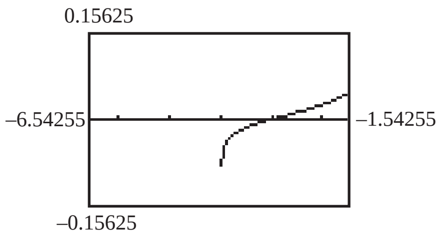
55.



$x \approx \pm 2.45$, $x \approx 0.67$, $y = 2$

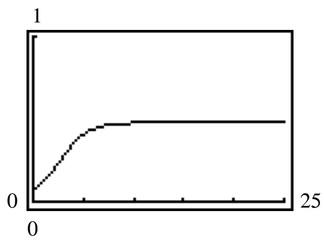
56.





In the standard window, two vertical asymptotes of the form $x = k$, where $k > 0$, are apparent ($x \approx 0.68$ and $x \approx 7.32$). By zooming around $x = -4$, another vertical asymptote is apparent ($x = -4$). Thus three vertical asymptotes exist.

57.



From the graph, it appears that $\lim\limits_{x\to\infty} y \approx 0.48$.

Thus a horizontal asymptote is $y \approx 0.48$. Algebraically, we have

$$\lim_{x\to\infty} \frac{0.34e^{0.7x}}{4.2+0.71e^{0.7x}} = \lim_{x\to\infty} \frac{\frac{0.34e^{0.7x}}{e^{0.7x}}}{\frac{4.2+0.71e^{0.7x}}{e^{0.7x}}}$$

$$= \lim_{x\to\infty} \frac{0.34}{\frac{4.2}{e^{0.7x}}+0.71} = \frac{0.34}{0+0.71} \approx 0.48$$

Problems 13.6

1. Let the numbers be x and $82 - x$. Then if $P = x(82-x) = 82x - x^2$, we have $P' = 82 - 2x$. Setting $P' = 0 \Rightarrow x = 41$. Since $P'' = -2 < 0$, there is a maximum when $x = 41$. Because $82 - x = 41$, the required numbers are 41 and 41.

2. Let the numbers be x and $20 - x$, where $0 \le x \le 20$. Let

$P = (2x)(20-x)^2 = 2x^3 - 80x^2 + 800x$.

Setting $\frac{dP}{dx} = 0$ gives

$P' = 6x^2 - 160x + 800 = 2(3x-20)(x-20) = 0$,

from which $x = \frac{20}{3}$ or $x = 20$. $P' > 0$ on $\left(0, \frac{20}{3}\right)$ and $P' < 0$ on $\left(\frac{20}{3}, 20\right)$. Thus P has a relative and absolute maximum when $x = \frac{20}{3}$.

The other number is $20 - x = \frac{40}{3}$.

3. We are given that $15x + 9(2y) = 9000$, or $y = \frac{9000-15x}{18}$. We want to maximize area A, where $A = xy$.

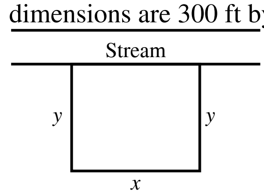
$A = xy = x\left(\frac{9000-15x}{18}\right) = \frac{1}{18}\left(9000x - 15x^2\right)$

$A' = \frac{1}{18}(9000 - 30x)$

Setting $A' = 0 \Rightarrow x = 300$. Since

$A''(300) = \frac{1}{18}(-30) < 0$, we have a maximum at

$x = 300$. Thus $y = \frac{9000-15(300)}{18} = 250$. The dimensions are 300 ft by 250 ft.

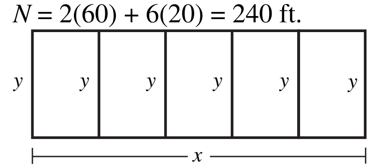


4. We are given that $xy = 1200$, or $y = \frac{1200}{x}$, and want to minimize $N = 2x + 6y$. We have

$N = 2x + 6y = 2x + 6\left(\frac{1200}{x}\right)$, $x > 0$

$N' = 2 - \frac{7200}{x^2}$

Setting $N' = 0$ yields $x^2 = 3600$, so $x = 60$. We have $N'' = \frac{14,400}{x^3}$, so $N''(60) > 0$ and we have a minimum. If $x = 60$, then $y = 20$. Thus $N = 2(60) + 6(20) = 240$ ft.



5. $c = 0.05q^2 + 5q + 500$

Avg. cost per unit $= \bar{c} = \frac{c}{q} = 0.05q + 5 + \frac{500}{q}$

$\bar{c}' = 0.05 - \frac{500}{q^2}$. Setting $\bar{c}' = 0$ yields

$0.05 = \frac{500}{q^2}$, $q^2 = 10{,}000$, $q = \pm 100$. We exclude $q = -100$ because q represents the number of units. Since $\bar{c}'' = \frac{1000}{q^3} > 0$ for $q > 0$, $\bar{c}$ is an absolute minimum when $q = 100$ units.

6. $C = 0.12s - 0.0012s^2 + 0.08$, where $0 \le s \le 60$. Setting $\frac{dC}{ds} = 0$ gives $0.12 - 0.0024s = 0 \Rightarrow s = 50$. Since $\frac{d^2C}{ds^2} = -0.0024 < 0$, a maximum occurs when $s = 50$. Thus a minimum can occur only at an endpoint of the domain. If $s = 0$, then $C = 0.08$; if $s = 60$, then $C = 2.96$. Thus the minimum cost of \$0.08 per hour occurs for $s = 0$ mi/h and might be due to depreciation, insurance, and so on.

7. $p = -5q + 30$
Since total revenue = (price)(quantity),
$r = pq = (-5q + 30)q = -5q^2 + 30q$
Setting $r' = -10q + 30 = 0 \Rightarrow q = 3$. Since $r'' = -10 < 0$, r is maximum at $q = 3$ units, for which the corresponding price is $p = -5(3) + 30 = \$15$.

8. $q = Ae^{-Bp}$

Revenue $= r = pq = pAe^{-Bp}$

$$\begin{aligned} r' &= A[e^{-Bp}(1) + pe^{-Bp}(-B)] \\ &= A(1 - Bp)e^{-Bp} \\ &= AB\left(\frac{1}{B} - p\right)e^{-Bp} \end{aligned}$$

Critical value: $p = \frac{1}{B}$

If $p < \frac{1}{B}$, then $r' > 0$ and r is increasing. If $p > \frac{1}{B}$, then $r' < 0$ and r is decreasing. Thus revenue is maximum when $p = \frac{1}{B}$. The answer does not depend on A because A represents the initial value of q, so it doesn't change q over time.

9. $f(p) = 160 - p - \frac{900}{p + 10}$, where $0 \le p \le 100$.

a. Setting $f'(p) = 0$ gives $-1 + \frac{900}{(p + 10)^2} = 0$,
$\frac{900}{(p + 10)^2} = 1$, $(p + 10)^2 = 900$,
$p + 10 = \pm 30$, from which $p = 20$.
Since $f''(p) = \frac{-1800}{(p + 10)^3} < 0$ for $p = 20$, we have an absolute maximum of $f(20) = 110$ grams.

b. $f(0) = 70$ and $f(100) = 51\frac{9}{11}$, so we have an absolute minimum of $f(100) = 51\frac{9}{11}$ grams.

10. $R = D^2\left(\frac{C}{2} - \frac{D}{3}\right) = \frac{CD^2}{2} - \frac{D^3}{3}$

The rate of change of R is $\frac{dR}{dD} = CD - D^2$. This is the function to be maximized. Setting $\frac{d}{dD}\left(\frac{dR}{dD}\right) = C - 2D = 0$ gives $D = \frac{C}{2}$. Since $\frac{d^2}{dD^2}\left(\frac{dR}{dD}\right) = -2 < 0$, the maximum rate of change occurs when $D = \frac{C}{2}$.

11. $p = 85 - 0.05q$
$c = 600 + 35q$
Profit = Total Revenue – Total Cost
$P = pq - c = (85 - 0.05q)q - (600 + 35q)$
$= -(0.05q^2 - 50q + 600)$
Setting $P' = -(0.1q - 50) = 0$ yields $q = 500$.
Since $P''(500) = -0.1 < 0$, P is a maximum when $q = 500$ units. This corresponds to a price of $p = 85 - 0.05(500) = \$60$ and a profit of $P = \$11{,}900$.

12. Cost per unit = \$3

$p = \frac{10}{\sqrt{q}}$

Profit = Total Revenue – Total Cost

$P = pq - c$

$P = \left(\frac{10}{\sqrt{q}}\right)q - (3q) = 10\sqrt{q} - 3q$

Setting $P' = \frac{5}{\sqrt{q}} - 3 = 0$ yields $q = \frac{25}{9}$.

Moreover, we have $P'' = -\frac{5}{2}q^{-\frac{3}{2}} < 0$ for $q > 0$, so P is maximum when $q = \frac{25}{9}$. The corresponding price is $p = \$6$.

13. $p = 42 - 4q$

$\bar{c} = 2 + \frac{80}{q}$

Total Cost $= c = \bar{c}q = 2q + 80$

Profit = Total Revenue – Total Cost

$P = pq - c = (42 - 4q)q - (2q + 80)$

$= -\left(4q^2 - 40q + 80\right)$

$P' = -(8q - 40)$

Setting $P' = -(8q - 40) = 0$ gives $q = 5$. We find that $P'' = -8 < 0$, so P has a maximum value when $q = 5$. The corresponding price p is $42 - 4(5) = \$22$.

14. $p = \frac{40}{\sqrt{q}}$

$\bar{c} = \frac{1}{3} + \frac{2000}{q}$

Total cost $= c = \bar{c}q = \frac{q}{3} + 2000$

Profit = Total Revenue – Total Cost

$P = pq - c = 40\sqrt{q} - \frac{q}{3} - 2000$

Setting $P' = \frac{20}{\sqrt{q}} - \frac{1}{3} = 0$ yields $q = 3600$.

Since $P'' = -10q^{-3/2} < 0$ for $q > 0$, it follows that P is a maximum when $q = 3600$. The corresponding price is $p = \frac{40}{60} \approx \$0.67$. Since $MR = \frac{20}{\sqrt{q}}$ and $MC = \frac{1}{3}$, then for $q = 3600$ we have $MR = \frac{20}{60} = \frac{1}{3} = MC$.

15. $p = q^2 - 100q + 3200$ on [0, 120]

$\bar{c} = \frac{2}{3}q^2 - 40q + \frac{10{,}000}{q}$

Profit = Total Revenue – Total Cost

Since total revenue $r = pq$ and total cost $= c = \bar{c}q$,

$P = pq - \bar{c}q$

$= q^3 - 100q^2 + 3200q - \left(\frac{2}{3}q^3 - 40q^2 + 10{,}000\right)$

$= \frac{1}{3}q^3 - 60q^2 + 3200q - 10{,}000$

$P' = q^2 - 120q + 3200 = (q - 40)(q - 80)$

Setting $P' = 0$ gives $q = 40$ or 80. Evaluating profit at $q = 0$, 40, 80, and 120 gives

$P(0) = -10{,}000$

$P(40) = \frac{130{,}000}{3} = 43{,}333\frac{1}{3}$

$P(80) = \frac{98{,}000}{3} = 32{,}666\frac{2}{3}$

$P(120) = 86{,}000$

Thus the profit maximizing output is $q = 120$ units, and the corresponding maximum profit is \$86,000.

16. a. $c = \bar{c}q = 2q^3 - 42q^2 + 228q + 210$

$\frac{dc}{dq} = 6q^2 - 84q + 228 = 6(q^2 - 14q + 38)$

Using the quadratic formula to solve $\frac{dc}{dq} = 0$ gives $q = 7 - \sqrt{11} \approx 3.68$ or $q = 7 + \sqrt{11} \approx 10.32$. Evaluating c at $q = 3$, $7 - \sqrt{11}$, $7 + \sqrt{11}$, and 12 gives 570, $434 + 44\sqrt{11} \approx 579.93$, $434 - 44\sqrt{11} \approx 288.07$, and 354, respectively. Thus the minimum cost is when $q = 7 + \sqrt{11} \approx 10.32$.

$c(10) = 290$ and $c(11) = 298$, so production should be fixed at $q = 10$ for a minimum cost of \$290.

b. $c(7) = 434$, so the minimum cost still occurs when $q = 7+\sqrt{11} \approx 10.32$ and production should again be fixed at 10 units.

17. Total fixed costs = \$1200, material-labor costs/unit = \$2, and the demand equation is $p = \frac{100}{\sqrt{q}}$.
Profit = Total Revenue – Total Cost
$P = pq - c$

$$P = \frac{100}{\sqrt{q}} \cdot q - (2q+1200)$$
$$= 100\sqrt{q} - 2q - 1200$$
$$= 2\left(50\sqrt{q} - q - 600\right)$$

Setting $P' = 2\left(\frac{25}{\sqrt{q}} - 1\right) = 0$ yields $q = 625$. We see that $P'' = -25q^{-\frac{3}{2}} < 0$ for $q > 0$, so P is maximum when $q = 625$. When $q = 625$, $\text{MR} = \frac{50}{\sqrt{625}} = 2 = \text{MC}$. When $q = 625$, then $p = \$4$.

18. Let x = number of \$10 per month increases so the monthly rate is $400 + 10x$ and the number of rented apartments is $100 - 2x$. Monthly revenue r is given by
r = (rent/apt.) (no. of apt. rented)
$r = (400 + 10x)(100 - 2x)$
$r' = (400+10x)(-2)+(100-2x)(10)$
$= 200 - 40x = 40(5 - x)$
Setting $r' = 0$ yields $x = 5$. Since $r'' = -40 < 0$, then r is maximum when $x = 5$. This results in a monthly rate for an apartment of $400 + 10(5) = \$450$.

19. If x = number of \$0.50 decreases, where $0 \le x \le 36$, then the monthly fee for each subscriber is $18 - 0.50x$, and the total number of subscribers is $4800 + 150x$. Let r be the total (monthly) revenue.
revenue = (monthly rate)(number of subscribers)
$r = (18 - 0.50x)(4800 + 150x)$
$r' = (18-0.50x)(150)+(4800+150x)(-0.50)$
$= 300 - 150x = 150(2 - x)$
Setting $r' = 0$ yields $x = 2$.
Evaluating r when $x = 0$, 2, and 36, we find that r is a maximum when $x = 2$. This corresponds to a monthly fee of $18 - 0.50(2) = \$17$ and a monthly revenue r of \$86,700.

20. Note that as the number of units produced and sold increases from 0 to 600, the profit increases from 0 to (600)(400) = \$24,000. Let q = number of units produced and sold beyond 600. Then the total profit P is given by

$$P = (600)(40)+(40-0.05q)q$$
$$= 24{,}000 + 40q - 0.05q^2$$
$$P' = 40 - 0.10q$$

Setting $P' = 0$ yields $q = 400$. Since $P'' = -0.10 < 0$, P is a maximum when $q = 400$, that is, the total number of units = 600 + 400 = 1000.

21. See the figure in the text. Given that $x^2y = 32$, we want to minimize $S = 4(xy) + x^2$. Since $y = \frac{32}{x^2}$, where $x > 0$, we have $S = 4x\left(\frac{32}{x^2}\right) + x^2 = \frac{128}{x} + x^2$, from which $S' = -\frac{128}{x^2} + 2x$. Setting $S' = 0$ gives $2x^3 = 128$, $x^3 = 64$, $x = 4$. Since $S'' = \frac{256}{x^3} + 2$, we get $S''(4) > 0$, so $x = 4$ gives a minimum. If $x = 4$, then $y = \frac{32}{16} = 2$. The dimensions are 4 ft × 4 ft × 2 ft.

22. See the figure in the text. We want to maximize $V = x^2y$ given that $4xy + x^2 = 192$, or

$$y = \frac{192 - x^2}{4x}$$
$$V = x^2\left(\frac{192-x^2}{4x}\right) = \frac{1}{4}\left(192x - x^3\right), x > 0$$
$$V' = \frac{1}{4}\left(192 - 3x^2\right) = \frac{3}{4}\left(64 - x^2\right)$$

Setting $V' = 0$ gives $x = 8$. Since $V'' = \left(\frac{3}{4}\right)(-2x)$, then $V''(8) < 0$, so $x = 8$ gives a maximum. If $x = 8$, then $y = 4$.
The dimensions are 8 ft × 8 ft × 4 ft.
The volume is $8^2(4) = 256 \text{ ft}^3$.

23. $V = x(L-2x)^2$

$= L^2x - 4Lx^2 + 4x^3$

where $0 < x < \frac{L}{2}.$

$V' = L^2 - 8Lx + 12x^2$

$= 12x^2 - 8Lx + L^2$

$= (2x - L)(6x - L)$

For $0 < x < \frac{L}{2},$ setting $V' = 0$ gives $x = \frac{L}{6}.$

Since $V' > 0$ on $\left(0, \frac{L}{6}\right)$ and $V' < 0$ on $\left(\frac{L}{6}, \frac{L}{2}\right),$ V is maximum when $x = \frac{L}{6}.$ Thus the length of the side of the square must be $\frac{L}{6}$ in., which results in a volume of

$\frac{L}{6}\left(L - \frac{L}{3}\right)^2 = \frac{2L^3}{27}$ in^3.

24. Since $xy = 240$, then $y = \frac{240}{x},$ $x > 0$. We want to minimize A where

$A = (x+10)(y+6) = (x+10)\left(\frac{240}{x} + 6\right)$

$= 300 + 6x + \frac{2400}{x}$

$A' = 6 - \frac{2400}{x^2}$

Setting $A' = 0$ gives $x = 20$. Since

$A'' = \frac{4800}{x^3} > 0$ for $x = 20$, we have a minimum.

Thus $y = 12$, so the dimensions are 20 + 10 by 12 + 6, that is, 30 in. × 18 in.

25. See the figure in the text.

$V = K = \pi r^2 h$ (1)

$S = 2\pi rh + \pi r^2$ (2)

From Equation (1) $h = \frac{K}{\pi r^2}$. Thus Equation (2) becomes

$S = \frac{2K}{r} + \pi r^2$

$\frac{dS}{dr} = -\frac{2K}{r^2} + 2\pi r = \frac{2\left(\pi r^3 - K\right)}{r^2}.$

If $S' = 0$, then $\pi r^3 - K = 0$, $\pi r^3 = K$, $r = \sqrt[3]{\frac{K}{\pi}}$. Thus

$h = \frac{K}{\pi\left(\frac{K}{\pi}\right)^{\frac{2}{3}}} = \left(\frac{K}{\pi}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{K}{\pi}}.$

Note that since $S'' = 2\pi + \frac{4K}{r^3} > 0$ for $r > 0$, we have a minimum.

26. See the figure in the text.

$S = K = 2\pi rh + \pi r^2$ (1)

$V = \pi r^2 h$ (2)

From Equation (1), $h = \frac{K - \pi r^2}{2\pi r}$. Thus Equation (2) becomes

$V = \frac{Kr - \pi r^3}{2}$

$\frac{dV}{dr} = \frac{K - 3\pi r^2}{2}.$

Setting $V' = 0$ gives $r = \sqrt{\frac{K}{3\pi}}$. Thus

$h = \frac{K - \pi\frac{K}{3\pi}}{2\pi\sqrt{\frac{K}{3\pi}}} = \frac{\frac{2}{3}K}{2\pi\sqrt{\frac{K}{3\pi}}}$

$= \frac{\frac{2}{3}K}{2\pi\sqrt{\frac{K}{3\pi}}} \cdot \frac{\sqrt{\frac{K}{3\pi}}}{\sqrt{\frac{K}{3\pi}}} = \sqrt{\frac{K}{3\pi}}$

Note that since $V'' = -3\pi r < 0$ for $r > 0$, we have a maximum.

27. $p = 600 - 2q$

$c = 0.2q^2 + 28q + 200$

Profit = Total Revenue – Total Cost

$P = pq - c$

$P = (600 - 2q)q - \left(0.2q^2 + 28q + 200\right)$

$= -\left(2.2q^2 - 572q + 200\right)$

$P' = -(4.4q - 572)$

Setting $P' = 0$ yields $q = 130$. Since $P'' = -4.4 < 0$, P is maximum when $q = 130$ units. The corresponding price is $p = 600 - 2(130) = \$340$, and the profit is $P = \$36,980$. If a tax of \$22/unit is imposed on the manufacturer, then the cost equation is

$c_1 = 0.2q^2 + 28q + 200 + 22q$

$= 0.2q^2 + 50q + 200$.

The demand equation remains the same. Thus

$P_1 = pq - c_1$

$= (600 - 2q)q - \left(0.2q^2 + 50q + 200\right)$

$= -\left(2.2q^2 - 550q + 200\right)$

$P'_1 = -(4.4q - 550)$

Setting $P'_1 = 0$ yields $q = 125$. Since $P_1'' = -4.4 < 0$, P_1 is maximum when $q = 125$ units. The corresponding price is $p = \$350$ and the profit is $P_1 = \$34{,}175$.

28. Original data: $p = 600 - 2q$, $c = 0.2q^2 + 28q + 200$. Revenue, both before and after the license fee, is given by $r = pq = 600q - 2q^2$. After the license fee, the cost equation is $c_1 = c + 1000 = 0.2q^2 + 28q + 1200$ and the profit is

$P_1 = r - c_1$

$= \left(600q - 2q^2\right) - \left(0.2q^2 + 28q + 1200\right)$

As in Problem 27, we find that P_1 has a maximum when $q = 130$ units, which gives $p = \$340$. Thus the profit-maximizing price and output remain the same. Since

Profit

$= r - c_1 = r - (c + 1000) = (r - c) - 1000$, when $q = 130$ we have

Profit = 36,980 – 1000 (from Problem 27)

= \$35,980

29. Let q = number of units in a production run. Since inventory is depleted at a uniform rate, assume that the average inventory is $\frac{q}{2}$. The value of average inventory is $10\left(\frac{q}{2}\right)$, and carrying costs are $0.128\left[10\left(\frac{q}{2}\right)\right]$. The number of production runs per year is $\frac{1000}{q}$, and total set-up costs are $40\left(\frac{1000}{q}\right)$. We want to minimize the sum C of carrying costs and set-up costs.

$C = 0.128\left[10\left(\frac{q}{2}\right)\right] + 40\left(\frac{1000}{q}\right)$

$= 0.64q + \frac{40{,}000}{q}$

$C' = 0.64 - \frac{40{,}000}{q^2}$

Setting $C' = 0$ yields $q^2 = \frac{40{,}000}{0.64} = 62{,}500$, $q = 250$ (since $q > 0$). Since $C'' = \frac{80{,}000}{q^3} > 0$, C is minimum when $q = 250$. Thus the economic lot size is 250/lot (4 lots).

30. $c = 0.004q^3 + 20q + 5000$

$p = 450 - 4q$

Profit = Total Revenue – Total Cost

$P = pq - c$

$= (450 - 4q)q - \left(0.004q^3 + 20q + 5000\right)$

$P = -\left(0.004q^3 + 4q^2 - 430q + 5000\right)$

$P' = -\left(0.012q^2 + 8q - 430\right)$

$= -2\left(0.006q^2 + 4q - 215\right)$

Setting $P' = 0$ yields

$0.006q^2 + 4q - 215 = 0$

$q = \frac{-4 \pm \sqrt{21.16}}{0.012} = \frac{-4 \pm 4.6}{0.012}$

Since $q \geq 0$, choose $q = \frac{-4 + 4.6}{0.012} = 50$. Since P is increasing on [0, 50) and decreasing on $(50, \infty)$, P is maximum when $q = 50$ units.

31. Let x = number of people over the 30.

Note: $0 \leq x \leq 10$.

Revenue = r

= (number attending)(charge/person)

= $(30 + x)(50 - 1.25x)$

$= 1500 + 12.5x - 1.25x^2$

$r' = 12.5 - 2.5x$

Setting $r' = 0$ yields $x = 5$. Since $r'' = -2.5 < 0$, r is maximum when $x = 5$, that is, when 35 attend.

32. Let N = horsepower of motor.
(Total annual cost) = C = (Annual cost to lease) + (Annual operating cost)

$$C = (200 + 0.40N) + 80{,}000\left(\frac{0.008}{N}\right)$$

$$= 200 + 0.04N + \frac{640}{N}$$

$$C' = 0.4 - \frac{640}{N^2}$$

Setting $C' = 0$ yields $N^2 = 1600$, so $N = 40$ (since $N > 0$). Since $C'' = \frac{1280}{N^3} > 0$ for $N > 0$, C is a minimum when $N = 40$ horsepower.

33. The cost per mile of operating the truck is $0.165 + \frac{s}{200}$. Driver's salary is $18/hr. The number of hours for 700 mi trip is $\frac{700}{s}$. Driver's salary for trip is $18\left(\frac{700}{s}\right)$, or $\frac{12{,}600}{s}$. The cost of operating the truck for the trip is

$$700\left[0.165 + \frac{s}{200}\right].$$

Total cost of trip is

$$C = \frac{12{,}600}{s} + 700\left(0.165 + \frac{s}{200}\right)$$

Setting $C' = -\frac{12{,}600}{s^2} + \frac{7}{2} = 0$ yields $s^2 = 3600$, or $s = 60$ (since $s > 0$). Since $C'' = \frac{25{,}200}{s^3} > 0$ for $s > 0$, C is a minimum when $s = 60$ mi/h.

34. Let q = level of production.

$$\text{Average Cost} = \bar{c} = \frac{\text{Total Cost}}{q}$$

For $0 \le q \le 5000$, we have

$$\bar{c} = \frac{30q + 10q + 20{,}000}{q} = 40 + \frac{20{,}000}{q}.$$

Note that total cost for 5000 units is 220,000.
For
$q > 5000$,

$$\bar{c} = \frac{(\text{cost for first 5000}) + \left(\begin{matrix}\text{cost for those}\\ \text{units beyond 5000}\end{matrix}\right)}{q}$$

$$= \frac{220{,}000 + [45(q - 5000) + 10(q - 5000)]}{q}$$

$$\bar{c} = 55 - \frac{55{,}000}{q}$$

If $0 < q \le 5000$, then $\bar{c}' = -\frac{20{,}000}{q^2} < 0$ and thus $\bar{c}$ is decreasing. If $q > 5000$, then $\bar{c}' = \frac{55{,}000}{q^2} > 0$ and thus $\bar{c}$ is increasing.
Hence c is minimum when q = 5000 units.

35. Profit P is given by
P = Total revenue – Total cost
= Total revenue – (salaries + fixed cost)
= $50q$ – ($1000m$ + 3000)

$$= 50\left(m^3 - 15m^2 + 92m\right) - 1000m - 3000$$

$$= 50\left(m^3 - 15m^2 + 72m - 60\right), \text{ where } 0 \le m \le 8$$

$$P' = 50\left(3m^2 - 30m + 72\right)$$

$$= 150\left(m^2 - 10m + 24\right) = 150(m - 4)(m - 6)$$

Setting $P' = 0$ gives the critical values 4 and 6. We now evaluate P at these critical values and also at the endpoints 0 and 8.
$P(0) = -3000$
$P(4) = 2600$
$P(6) = 2400$
$P(8) = 3400$
Thus Ms. Jones should hire 8 salespeople to obtain a maximum weekly profit of $3400.

36. Profit P is given by
P = Total revenue – Total cost = pq – Total cost
$= 400q - 50q^2 -$ Total cost. (q in hundreds)

$$\frac{dP}{dq} = 400 - 100q - \frac{d}{dq}(\text{Total cost})$$
$$= 400 - 100q - \text{Marginal cost}$$
$$= 400 - 100q - \frac{800}{q+5}$$
$$= \frac{400(q+5) - 100q(q+5) - 800}{q+5}$$
$$= \frac{-100q^2 - 100q + 1200}{q+5}$$
$$= \frac{-100(q+4)(q-3)}{q+5}$$

Setting $P' = 0$ gives the critical value 3 (since $q > 0$). We find that $P' > 0$ for $0 < q < 3$, and $P' < 0$ for $q > 3$. Thus there is a maximum profit when q = 3000 jackets.

37. x = tons of chemical A ($x \le 4$),
$y = \frac{24-6x}{5-x} =$ tons of chemical B, profit on A = \$2000/ton, and profit on B = \$1000/ton.

$$\text{Total Profit} = P_T = 2000x + 1000\left(\frac{24-6x}{5-x}\right)$$
$$= 2000\left[x + \frac{12-3x}{5-x}\right]$$
$$P'_T = 2000\left[1 + \frac{(5-x)(-3) - (12-3x)(-1)}{(5-x)^2}\right]$$
$$= 2000\left[1 - \frac{3}{(5-x)^2}\right]$$
$$= 2000\left[\frac{x^2 - 10x + 22}{(5-x)^2}\right]$$

Setting $P'_T = 0$ yields (by the quadratic formula)

$$x = \frac{10 \pm 2\sqrt{3}}{2} = 5 \pm \sqrt{3}$$

Because $x \le 4$, choose $x = 5 - \sqrt{3}$. Since P_T is increasing on $\left[0, 5-\sqrt{3}\right)$ and decreasing on $\left(5-\sqrt{3}, 4\right]$, P_T is a maximum for $x = 5 - \sqrt{3}$ tons. If profit on A is P/ton and profit on B is $\frac{P}{2}$/ton, then

$$P_T = Px + \frac{P}{2}\left(\frac{24-6x}{5-x}\right) = P\left[x + \frac{12-3x}{5-x}\right]$$
$$P'_T = P\left[\frac{x^2 - 10x + 22}{(5-x)^2}\right]$$

Setting $P'_T = 0$ and using an argument similar to that above, we find that P_T is a maximum when $x = 5 - \sqrt{3}$ tons.

38. x = number of floors. Let R = rate of return.

$$R = \frac{\text{Total Revenue}}{\text{Total Cost}}$$
$$= \frac{60{,}000x}{(10x)[120{,}000 + 3000(x-1)] + 1{,}440{,}000}$$
$$= \frac{2x}{x^2 + 39x + 48}$$
$$R' = 2 \cdot \frac{48 - x^2}{(x^2 + 39x + 48)^2}$$

$R' = 0$ when $x = \sqrt{48} = 4\sqrt{3}$ ($x \ge 0$). Since R is increasing on $\left(0, 4\sqrt{3}\right)$ and decreasing on $\left(4\sqrt{3}, \infty\right)$, R is a maximum when $x = 4\sqrt{3} \approx 6.93$. The number of floors in the building must be an integer, so we evaluate R when x = 6 and x = 7: $R(6) \approx 0.0377$; $R(7) \approx 0.0378$. Thus 7 floors should be built to maximize the rate of return.

39. $P(j) = Aj\frac{L^4}{V} + B\frac{V^3L^2}{1+j}$

$$\frac{dP}{dj} = \frac{AL^4}{V} - \frac{BV^3L^2}{(1+j)^2} = 0$$

Solving for $(1+j)^2$ gives $(1+j)^2 = \frac{BV^4}{AL^2}$

40. a. $\frac{d}{dv}\left(-2at_r + v - \frac{2al}{v}\right) = 1 + \frac{2al}{v^2} = 0$ when $v = \sqrt{-2al}$. Note that $\frac{d^2}{dv^2}\left(-2at_r + v - \frac{2al}{v}\right) = \frac{-4al}{v^3} > 0$ for

$a < 0$, $l > 0$, and $v > 0$. Thus $-2at_r + v - \frac{2al}{v}$ is a minimum for $v = \sqrt{-2al}$.

b. $v = \sqrt{-2(-19.6)(20)} = \sqrt{784} = 28$ ft/s.

c. $N = \dfrac{-2(-19.6)}{(-2)(-19.6)(0.5) + 28 - \frac{2(-19.6)(20)}{28}}$

≈ 0.5 cars/s = 0.5(3600) cars/h = 1800 cars/h

d. When $v = \sqrt{-2al}$, then

$$N = N(l) = \frac{-2a}{-2at_r + \sqrt{-2al} + \frac{-2al}{\sqrt{-2al}}}$$

$$= \frac{-2a}{-2at_r + 2\sqrt{-2al}} = \frac{a}{at_r - \sqrt{-2al}}$$

The relative change in N when l is reduced from 20 ft to 15 ft is $\dfrac{N(15) - N(20)}{N(20)}$.

With $a = -19.6$ ft/s^2 and $t_r = 0.5$ s, then

$$N(20) = \frac{-19.6}{(-19.6)(0.5) - \sqrt{-2(-19.6)(20)}}$$

≈ 0.5185

$$N(15) = \frac{-19.6}{(-19.6)(0.5) - \sqrt{-2(-19.6)(15)}}$$

≈ 0.5756

The relative change is

$$\frac{N(15) - N(20)}{N(20)} \approx \frac{0.5756 - 0.5158}{0.5158} \approx 0.1101$$

41. $\overline{c} = \dfrac{c}{q} = 3q + 50 - 18\ln(q) + \dfrac{120}{q}$, $q > 0$

$$\frac{d\overline{c}}{dq} = 3 - \frac{18}{q} - \frac{120}{q^2} = \frac{3q^2 - 18q - 120}{q^2}$$

$$= \frac{3\left(q^2 - 6q - 40\right)}{q^2}$$

$$= \frac{3(q-10)(q+4)}{q^2}$$

Critical value is $q = 10$ since $q \geq 0$.

Since $\dfrac{d\overline{c}}{dq} < 0$ for $0 < q < 10$, and $\dfrac{d\overline{c}}{dq} > 0$ for $q > 10$, we have a minimum when $q = 10$ cases.

This minimum average cost is

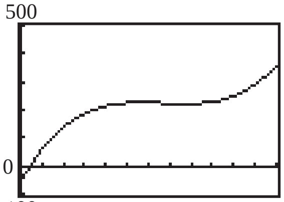
$3(10) + 50 - 18 \ln 10 + 12 \approx \$50.55$.

42. The profit function is given by

$$P = TR - TC = q^3 - 20q^2 + 160q - (30q + 50)$$

$$= q^3 - 20q^2 + 130q - 50$$

where P is in thousands of dollars, q is in tons, and $0 \le q \le 12$. From the graph, the maximum profit occurs when $q = 12$ tons. The corresponding maximum profit is \$358,000 and the selling price per ton is \$64,000.



Chapter 13 Review Problems

1. $y = \dfrac{3x^2}{x^2 - 16} = \dfrac{3x^2}{(x+4)(x-4)}$

When $x = \pm 4$ the denominator is zero and the numerator is not zero. Thus $x = 4$ and $x = -4$ are vertical asymptotes.

$$\lim_{x\to\infty} \frac{3x^2}{x^2 - 16} = \lim_{x\to\infty} \frac{3x^2}{x^2} = \lim_{x\to\infty} 3 = 3$$

Similarly, $\lim_{x\to-\infty} y = 3$. Thus $y = 3$ is the only horizontal asymptote.

2. $y = \dfrac{x+3}{9x - 3x^2} = \dfrac{x+3}{3x(3-x)}$

When $x = 0$ or $x = 3$, the denominator is zero and the numerator is not zero. Thus $x = 0$ and $x = 3$ are vertical asymptotes.

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{x}{-3x^2} = -\frac{1}{3} \lim_{x\to\infty} \frac{1}{x} = 0$$

Similarly, $\lim_{x\to-\infty} y = 0$. Thus $y = 0$ is the only horizontal asymptote.

3. $y = \dfrac{5x^2 - 3}{(3x+2)^2} = \dfrac{5x^2 - 3}{9x^2 + 12x + 4}$

When $x = -\dfrac{2}{3}$, the denominator is zero and the numerator is not zero. Thus $x = -\dfrac{2}{3}$ is a vertical asymptote.

$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{5x^2}{9x^2} = \lim_{x\to\infty} \frac{5}{9} = \frac{5}{9}$

Similarly, $\lim_{x\to-\infty} y = \frac{5}{9}$. Thus $y = \frac{5}{9}$ is the only horizontal asymptote.

4. $y = \frac{4x+1}{3x-5} - \frac{3x+1}{2x-11} = \frac{-x^2-30x-6}{(3x-5)(2x-11)}$

When $x = \frac{5}{3}$ or $x = \frac{11}{2}$, the denominator is zero and the numerator is not zero. Thus $x = \frac{5}{3}$ and $x = \frac{11}{2}$ are vertical asymptotes.

$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{-x^2}{6x^2} = \lim_{x\to\infty}\left(-\frac{1}{6}\right) = -\frac{1}{6}$

Similarly, $\lim_{x\to-\infty} y = -\frac{1}{6}$. Thus $y = -\frac{1}{6}$ is the only horizontal asymptote.

5. $f(x) = \frac{5x^2}{3-x^2}$

$f'(x) = \frac{(3-x^2)(10x)-5x^2(-2x)}{(3-x^2)^2} = \frac{10x(3-x^2+x^2)}{(3-x^2)^2} = \frac{30x}{(3-x^2)^2}$

Thus $x = 0$ is the only critical value.

Note: Although $f'\left(\pm\sqrt{3}\right)$ is not defined, $\pm\sqrt{3}$ are not critical values because $\pm\sqrt{3}$ are not in the domain of *f*.

6. $f(x) = 8(x-1)^2(x+6)^4$

$$\begin{aligned} f'(x) &= 8(2)(x-1)(x+6)^4 + 8(x-1)^2(4)(x+6)^3 \\ &= 16(x-1)(x+6)^3[x+6+2(x-1)] \\ &= 16(x-1)(x+6)^3(3x+4) \end{aligned}$$

Thus $x = 1$, $x = -6$, and $x = -\frac{4}{3}$ are the critical values.

7. $f(x) = \frac{\sqrt[3]{x+1}}{3-4x}$

$$f'(x) = \frac{(3-4x)\left[\frac{1}{3}(x+1)^{-\frac{2}{3}}\right] - (x+1)^{\frac{1}{3}}(-4)}{(3-4x)^2} = \frac{\frac{1}{3}(x+1)^{-\frac{2}{3}}[(3-4x)+12(x+1)]}{(3-4x)^2} = \frac{8x+15}{3(x+1)^{\frac{2}{3}}(3-4x)^2}$$

$f'(x)$ is zero when $x = -\frac{15}{8}$; $f'(x)$ is not defined when $x = -1$ or $x = \frac{3}{4}$. However $\frac{3}{4}$ is not in the domain of *f*.

Thus $x = -\frac{15}{8}$ and $x = -1$ are critical values.

8. $f(x) = \frac{13xe^{-\frac{5x}{6}}}{6x+5}$

$$f'(x) = 13 \cdot \frac{(6x+5)\left[x\left(-\frac{5}{6}e^{-\frac{5x}{6}}\right) + e^{-\frac{5x}{6}}(1)\right] - xe^{-\frac{5x}{6}}(6)}{(6x+5)^2}$$

$$= \frac{13}{6} \cdot \frac{-e^{-\frac{5x}{6}}\{(6x+5)[5x-6]+36x\}}{(6x+5)^2} = \frac{13}{6} \cdot \frac{-\left\{30x^2+25x-30\right\}}{e^{\frac{5x}{6}}(6x+5)^2}$$

$$= \frac{13}{6} \cdot \frac{-5\left(6x^2+5x-6\right)}{e^{\frac{5x}{6}}(6x+5)^2} = \frac{-65(2x+3)(3x-2)}{6e^{\frac{5x}{6}}(6x+5)^2}$$

$f'(x)$ is zero when $x = -\frac{3}{2}$ or $x = \frac{2}{3}$. Although $f'(x)$ is not defined when $x = -\frac{5}{6}$, $-\frac{5}{6}$ is not in the domain of *f*. Thus $x = -\frac{3}{2}$ and $x = \frac{2}{3}$ are the only critical values.

9. $f(x) = -\frac{5}{3}x^3 + 15x^2 + 35x + 10$

$$f'(x) = -5x^2 + 30x + 35$$
$$= -5(x^2 - 6x - 7) = -5(x-7)(x+1)$$

CV: $x = -1$ and $x = 7$. Decreasing on $(-\infty, -1)$ and $(7, \infty)$; increasing on $(-1, 7)$

10. $f(x) = \frac{2x^2}{(x+1)^2}$

$$f'(x) = \frac{4x(x+1)^2 - 2x^2(2)(x+1)}{(x+1)^4} = \frac{4x}{(x+1)^3}$$

CV: $x = 0$, but $x = -1$ is also considered in the inc.-dec. analysis. Increasing on $(-\infty, -1)$ and $(0, \infty)$; decreasing on $(-1, 0)$.

11. $f(x) = \frac{6x^4}{x^2-3}$

$$f'(x) = 6 \cdot \frac{\left(x^2-3\right)\left(4x^3\right) - x^4(2x)}{\left(x^2-3\right)^2}$$

$$= \frac{12x^3\left[2\left(x^2-3\right) - x^2\right]}{\left(x^2-3\right)^2} = \frac{12x^3\left(x^2-6\right)}{\left(x^2-3\right)^2}$$

$$= \frac{12x^3\left(x+\sqrt{6}\right)\left(x-\sqrt{6}\right)}{\left[\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)\right]^2}$$

CV: $x = 0$, $\pm\sqrt{6}$, but $x = \pm\sqrt{3}$ must also be considered in the inc.-dec. analysis. Decreasing on $\left(-\infty, -\sqrt{6}\right)$, $\left(0, \sqrt{3}\right)$, and $\left(\sqrt{3}, \sqrt{6}\right)$; increasing on $\left(-\sqrt{6}, -\sqrt{3}\right)$, $\left(-\sqrt{3}, 0\right)$ and $\left(\sqrt{6}, \infty\right)$.

12. $f(x) = 4\sqrt[3]{5x^3 - 7x}$

$f'(x) = 4 \cdot \frac{1}{3}(5x^3 - 7x)^{-2/3}(15x^2 - 7)$

$= \frac{4(15x^2 - 7)}{3(5x^3 - 7x)^{2/3}}$

$= \frac{4\left(\sqrt{15}x + \sqrt{7}\right)\left(\sqrt{15}x - \sqrt{7}\right)}{3[x(5x^2 - 7)]^{2/3}}$

$= \frac{4\left(\sqrt{15}x + \sqrt{7}\right)\left(\sqrt{15}x - \sqrt{7}\right)}{3\left[x\left(\sqrt{5}x + \sqrt{7}\right)\left(\sqrt{5}x - \sqrt{7}\right)\right]^{2/3}}$

CV: $x = \pm\sqrt{\frac{7}{15}}, 0, \pm\sqrt{\frac{7}{5}}$

Increasing on $\left(-\infty, -\sqrt{\frac{7}{5}}\right)$, $\left(-\sqrt{\frac{7}{5}}, -\sqrt{\frac{7}{15}}\right)$, $\left(\sqrt{\frac{7}{15}}, \sqrt{\frac{7}{5}}\right)$, and $\left(\sqrt{\frac{7}{5}}, \infty\right)$; decreasing on $\left(-\sqrt{\frac{7}{15}}, 0\right)$ and $\left(0, \sqrt{\frac{7}{15}}\right)$.

13. $f(x) = x^4 - x^3 - 14$

$f'(x) = 4x^3 - 3x^2$

$f''(x) = 12x^2 - 6x = 6x(2x - 1)$

$f''(x) = 0$ when $x = 0$ or $x = \frac{1}{2}$. Concave up on $(-\infty, 0)$ and $\left(\frac{1}{2}, \infty\right)$; concave down on $\left(0, \frac{1}{2}\right)$.

14. $f(x) = \frac{x-2}{x+2}$

$f'(x) = \frac{(x+2)(1) - (x-2)(1)}{(x+2)^2} = \frac{4}{(x+2)^2}$

$f''(x) = -\frac{8}{(x+2)^3}$

$f''(x)$ is not defined when $x = -2$. Concave up on $(-\infty, -2)$; concave down on $(-2, \infty)$

15. $f(x) = \frac{1}{2x-1} = (2x-1)^{-1}$

$f'(x) = -2(2x-1)^{-2}$

$f'' = 8(2x-1)^3 = \frac{8}{(2x-1)^3}$

$f''(x)$ is not defined when $x = \frac{1}{2}$. Concave down on $\left(-\infty, \frac{1}{2}\right)$; concave up on $\left(\frac{1}{2}, \infty\right)$.

16. $f(x) = x^3 + 2x^2 - 5x + 2$

$f'(x) = 3x^2 + 4x - 5$

$f''(x) = 6x + 4 = 2(3x + 2)$

$f''(x) = 0$ when $x = -\frac{2}{3}$. Concave down on $\left(-\infty, -\frac{2}{3}\right)$; concave up on $\left(-\frac{2}{3}, \infty\right)$.

17. $f(x) = (2x+1)^3(3x+2)$

$f'(x) = (2x+1)^3(3) + (3x+2)[3(2x+1)^2(2)]$

$= 3(2x+1)^2(2x + 1 + 6x + 4)$

$= 3(2x+1)^2(8x+5)$

$f''(x) = 3\{(2x+1)^2(8) + (8x+5)[2(2x+1)(2)]\}$

$= 12(2x+1)[2(2x+1) + 8x + 5]$

$= 12(2x+1)(12x+7)$

$f''(x) = 0$ when $x = -\frac{1}{2}$ or $x = -\frac{7}{12}$. Concave up on $\left(-\infty, -\frac{7}{12}\right)$ and $\left(-\frac{1}{2}, \infty\right)$; concave down on $\left(-\frac{7}{12}, -\frac{1}{2}\right)$.

18. $f(x) = \left(x^2 - x - 1\right)^2$

$f'(x) = 2\left(x^2 - x - 1\right)(2x - 1)$

$= 2\left(2x^3 - 3x^2 - x + 1\right)$

$f''(x) = 2\left(6x^2 - 6x - 1\right)$

$f''(x) = 0$ when $6x^2 - 6x - 1 = 0$; by the quadratic formula $x = \frac{1}{2} \pm \frac{\sqrt{15}}{6}$. Concave up on $\left(-\infty, \frac{1}{2} - \frac{\sqrt{15}}{6}\right)$ and $\left(\frac{1}{2} + \frac{\sqrt{15}}{6}, \infty\right)$; concave down on $\left(\frac{1}{2} - \frac{\sqrt{15}}{6}, \frac{1}{2} + \frac{\sqrt{15}}{6}\right)$.

19. $f(x) = 2x^3 - 9x^2 + 12x + 7$

$f'(x) = 6x^2 - 18x + 12 = 6\left(x^2 - 3x + 2\right)$

$= 6(x - 1)(x - 2)$

CV: $x = 1$ and $x = 2$

Increasing on $(-\infty, 1)$ and $(2, \infty)$; decreasing on $(1, 2)$. Relative maximum when $x = 1$; relative minimum when $x = 2$.

20. $f(x) = \frac{2x+1}{x^2}$

$f'(x) = \frac{x^2(2) - (2x+1)(2x)}{x^4}$

$= \frac{2x[x - (2x+1)]}{x^4} = \frac{2(-x-1)}{x^3} = \frac{-2(x+1)}{x^3}$

CV: $x = -1$, but $x = 0$ must be considered in inc.-dec. analysis. Decreasing on $(-\infty, -1)$ and $(0, \infty)$; increasing on $(-1, 0)$. Relative minimum when $x = -1$.

21. $f(x) = \frac{x^{10}}{10} + \frac{x^5}{5}$

$f'(x) = x^9 + x^4 = x^4(x^5 + 1)$

CV: $x = 0$ and $x = -1$

Decreasing on $(-\infty, -1)$; increasing on $(-1, 0)$ and $(0, \infty)$; relative minimum when $x = -1$

22. $f(x) = \frac{x^2}{x^2 - 4}$

$f'(x) = \frac{\left(x^2 - 4\right)(2x) - x^2(2x)}{\left(x^2 - 4\right)^2}$

$= \frac{2x\left[\left(x^2 - 4\right) - x^2\right]}{\left(x^2 - 4\right)^2} = \frac{-8x}{\left(x^2 - 4\right)^2}$

$= -\frac{8x}{[(x+2)(x-2)]^2}$

CV: $x = 0$, but $x \pm 2$ must be considered in inc.-dec. analysis. Increasing on $(-\infty, -2)$ and $(-2, 0)$; decreasing on $(0, 2)$ and $(2, \infty)$. Relative maximum when $x = 0$.

23. $f(x) = x^{\frac{2}{3}}(x + 1) = x^{\frac{5}{3}} + x^{\frac{2}{3}}$

$f'(x) = \frac{5}{3}x^{\frac{2}{3}} + \frac{2}{3}x^{-\frac{1}{3}} = \frac{1}{3}x^{-\frac{1}{3}}(5x + 2) = \frac{5x+2}{3x^{\frac{1}{3}}}$

CV: $x = 0$ and $x = -\frac{2}{5}$

Increasing on $\left(-\infty, -\frac{2}{5}\right)$ and $(0, \infty)$; decreasing on $\left(-\frac{2}{5}, 0\right)$. Relative maximum when $x = -\frac{2}{5}$; relative minimum when $x = 0$.

24. $f(x) = x^3(x - 2)^4$

$f'(x) = x^3[4(x-2)^3(1)] + (x-2)^4(3x^2)$

$= x^2(x-2)^3[4x + 3(x-2)]$

$= x^2(x-2)^3(7x - 6)$

CV: $x = 0, 2, \frac{6}{7}$

Increasing on $(-\infty, 0)$, $\left(0, \frac{6}{7}\right)$, and $(2, \infty)$; decreasing on $\left(\frac{6}{7}, 2\right)$. Relative maximum when $x = \frac{6}{7}$; relative minimum when $x = 2$.

25. $y = x^5 - 5x^4 + 3x$

$y' = 5x^4 - 20x^3 + 3$

$y'' = 20x^3 - 60x^2 = 20x^2(x - 3)$

Possible inflection points occur when $x = 0$ or $x = 3$. Concave down on $(-\infty, 0)$ and $(0, 3)$; concave up on $(3, \infty)$. Concavity changes at $x = 3$, so there is an inflection point when $x = 3$.

26. $y = \frac{x^2 + 2}{5x} = \frac{1}{5}x + \frac{2}{5}x^{-1}$

$y' = \frac{1}{5}(1 - 2x^{-2})$

$y'' = \frac{4}{5}x^{-3} = \frac{4}{5x^3}$

y'' is never zero. Although y'' is not defined when $x = 0$, y is not continuous there. Thus there is no inflection point.

27. $y = 4(3x-5)\left(x^4+2\right) = 12x^5 - 20x^4 + 24x - 40$

$y' = 60x^4 - 80x^3 + 24$

$y'' = 240x^3 - 240x^2 = 240x^2(x-1)$

Possible inflection points occur when $x = 0$ or $x = 1$. Concave down on $(-\infty, 0)$ and $(0, 1)$; concave up on $(1, \infty)$. Inflection point when $x = 1$.

28. $y = x^2 + 2\ln(-x)$ (Note: $x < 0$)

$y' = 2x + \frac{2}{x}$

$y'' = 2 - \frac{2}{x^2} = \frac{2x^2-2}{x^2} = \frac{2(x+1)(x-1)}{x^2}$

Possible inflection point occurs when $x = -1$. Concave up on $(-\infty, -1)$; concave down on $(-1, 0)$. Inflection point when $x = -1$.

29. $y = \frac{x^3}{e^x} = x^3e^{-x}$

$y' = x^3(-e^{-x}) + e^{-x}(3x^2) = -e^{-x}(x^3 - 3x^2)$

$y'' = -e^{-x}(3x^2 - 6x) - (x^3 - 3x^2)(-e^{-x})$

$= e^{-x}(x^3 - 6x^2 + 6x)$

$= xe^{-x}(x^2 - 6x + 6)$

y'' is defined for all x and y'' is zero only when $x = 0$ or $x^2 - 6x + 6 = 0$. Using the quadratic formula on the second equation, the possible points of inflection occur when $x = 0,\ 3 \pm \sqrt{3}$. Concave up on $\left(0, 3-\sqrt{3}\right)$ and $\left(3+\sqrt{3}, \infty\right)$; concave down on $(-\infty, 0)$ and $\left(3-\sqrt{3}, 3+\sqrt{3}\right)$. Inflection points when $x = 0,\ 3 \pm \sqrt{3}$.

30. $y = 6\left(x^2-4\right)^3$

$y' = 36x\left(x^2-4\right)^2$

$y'' = 36\left\{x\left[4x\left(x^2-4\right)\right] + \left(x^2-4\right)^2(1)\right\}$

$= 36\left(x^2-4\right)\left[4x^2 + \left(x^2-4\right)\right] = 36\left(x^2-4\right)\left(5x^2-4\right)$

$= 36(x+2)(x-2)\left(\sqrt{5}x+2\right)\left(\sqrt{5}x-2\right)$

Possible inflections points occur when $x = \pm 2$ or $x = \pm\frac{2}{\sqrt{5}} = \pm\frac{2\sqrt{5}}{5}$. Concave up on $(-\infty, -2)$, $\left(-\frac{2\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$, and $(2, \infty)$; concave down on $\left(-2, -\frac{2\sqrt{5}}{5}\right)$, and $\left(\frac{2\sqrt{5}}{5}, 2\right)$. Inflection points when $x = \pm 2, \pm\frac{2\sqrt{5}}{5}$.

31. $f(x) = 3x^4 - 4x^3$ and f is continuous on $[0, 2]$.

$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$

The only critical value on $(0, 2)$ is $x = 1$. Evaluating f at this value and at the endpoints gives $f(0) = 0$, $f(1) = -1$, and $f(2) = 16$. Absolute maximum: $f(2) = 16$; absolute minimum: $f(1) = -1$.

32. $f(x) = 2x^3 - 15x^2 + 36x$ and f is continuous on $[0, 3]$.

$f'(x) = 6x^2 - 30x + 36 = 6(x-2)(x-3)$

The only critical value on $(0, 3)$ is $x = 2$. Evaluating f at this value and at the endpoints gives $f(0) = 0$, $f(2) = 28$, $f(3) = 27$. Absolute maximum: $f(2) = 28$; absolute minimum: $f(0) = 0$.

33. $f(x) = \frac{x}{(5x-6)^2}$ and f is continuous on $[-2, 0]$.

$f'(x) = \frac{(5x-6)^2(1) - x[10(5x-6)]}{(5x-6)^4}$

$= \frac{(5x-6)[(5x-6)-10x]}{(5x-6)^4} = \frac{-5x-6}{(5x-6)^3}$

$= -\frac{5x+6}{(5x-6)^3}$

The only critical value on $(-2, 0)$ is $x = -\frac{6}{5}$. Evaluating f at this value and at the endpoints gives $f(-2) = -\frac{1}{128}$, $f\left(-\frac{6}{5}\right) = -\frac{1}{120}$ and $f(0) = 0$. Absolute maximum: $f(0) = 0$; absolute minimum: $f\left(-\frac{6}{5}\right) = -\frac{1}{120}$.

34. $f(x) = (x+1)^2(x-1)^{2/3}$ and *f* is continuous on [2, 3].

$$f'(x) = (x+1)^2\left[\frac{2}{3}(x-1)^{-1/3}\right] + (x-1)^{2/3}[2(x+1)]$$
$$= \frac{2}{3}(x+1)(x-1)^{-1/3}[(x+1)+3(x-1)]$$
$$= \frac{4}{3}(x+1)(x-1)^{-1/3}(2x-1) = \frac{4(x+1)(2x-1)}{3(x-1)^{1/3}}$$

There are no critical values on [2, 3]. Evaluating *f* at the endpoints gives $f(2) = 9$ and $f(3) = 16(2^{2/3}) \approx 25.4$.

Absolute maximum $f(3) = 16(2^{2/3}) \approx 25.4$; absolute minimum: $f(2) = 9$

35. $f(x) = \left(x^2+1\right)e^{-x}$

a. $f'(x) = \left(x^2+1\right)\left(-e^{-x}\right) + e^{-x}(2x)$

$$= -e^{-x}\left[\left(x^2+1\right) - 2x\right] = -e^{-x}\left(x^2 - 2x + 1\right)$$
$$= -e^{-x}(x-1)^2$$

CV: $x = 1$

Decreasing on $(-\infty, 1)$ and $(1, \infty)$. No relative extrema.

b. $f''(x) = -\left\{e^{-x}[2(x-1)] + (x-1)^2\left(-e^{-x}\right)\right\}$

$$= e^{-x}(x-1)[-2+(x-1)]$$
$$= e^{-x}(x-1)(x-3)$$

Possible inflection points when $x = 1, 3$. Concave up on $(-\infty, 1)$ and $(3, \infty)$; concave down on $(1, 3)$.

Inflection points at $(1, f(1)) = \left(1, 2e^{-1}\right)$ and $(3, f(3)) = \left(3, 10e^{-3}\right)$.

36. Let $y = f(x) = \dfrac{x}{x^2-1}$.

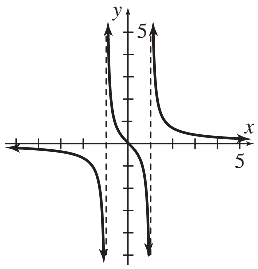
a. Replacing *x* by $-x$ and *y* by $-y$ yields $-y = \dfrac{-x}{(-x)^2-1}$, or $y = \dfrac{x}{x^2-1}$, which is the original equation. Thus the graph is symmetric about the origin. No other symmetry exists.

b. Since $f'(x) = -\dfrac{x^2+1}{(x^2-1)^2} = -\dfrac{x^2+1}{[(x+1)(x-1)]^2}$, there are no critical values. $f'(x) < 0$ for all *x*, so *f*(*x*) is decreasing on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.

c. From (b), There are no relative extrema.

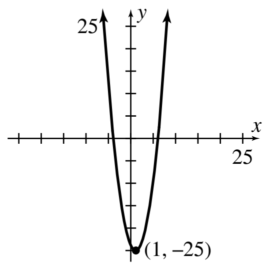
d. $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{x}{x^2} = \lim\limits_{x\to-\infty} \dfrac{1}{x} = 0$. Similarly, $\lim\limits_{x\to\infty} f(x) = 0$. Thus the line $y = 0$ is a horizontal asymptote to the graph of *f*.

e.

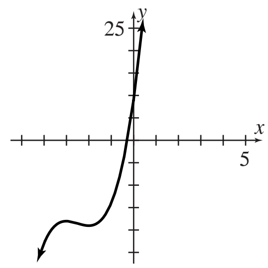


f. From the graph it is clear that no absolute extrema exist.

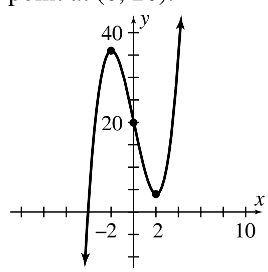
37. $y = x^2 - 2x - 24 = (x+4)(x-6)$
Intercepts: (–4, 0), (6, 0), (0, –24)
No symmetry. No asymptotes.
$y' = 2x - 2 = 2(x-1)$
CV: $x = 1$
Increasing on $(1, \infty)$; decreasing on $(-\infty, 1)$; relative minimum at
(1, –25).
$y'' = 2$
No possible inflection point. Concave up on $(-\infty, \infty)$.



38. $y = 2x^3 + 15x^2 + 36x + 9$
Intercept: (0, 9)
No symmetry; no asymptotes
$y' = 6x^2 + 30x + 36 = 6(x^2 + 5x + 6)$
$= 6(x+3)(x+2)$
CV: $x = -3, -2$
Increasing on $(-\infty, -3)$ and $(-2, \infty)$; decreasing on $(-3, -2)$; relative maximum at (–3, –18); relative minimum at (–2, –19)
$y'' = 12x + 30 = 6(2x+5)$
Possible inflection point when $x = -\frac{5}{2}$.
Concave down on $\left(-\infty, -\frac{5}{2}\right)$; concave up on $\left(-\frac{5}{2}, \infty\right)$; inflection point at $\left(-\frac{5}{2}, -\frac{37}{2}\right)$



39. $y = x^3 - 12x + 20$
Intercept: (0, 20)
No symmetry; no asymptotes
$y' = 3x^2 - 12$
$= 3\left(x^2 - 4\right) = 3(x+2)(x-2)$
CV: $x = \pm 2$
Increasing on $(-\infty, -2)$ and $(2, \infty)$; decreasing on $(-2, 2)$; relative maximum at (–2, 36); relative minimum at (2, 4).
$y'' = 6x$
Possible inflection point when $x = 0$. Concave up on $(0, \infty)$; concave down on $(-\infty, 0)$; inflection point at (0, 20).

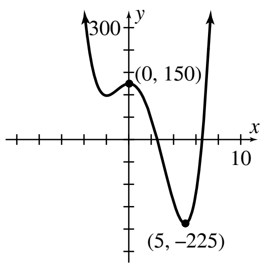


40. $y = x^4 - 4x^3 - 20x^2 + 150$
Intercept: (0, 150)
No symmetry. No asymptotes.
$y' = 4x^3 - 12x^2 - 40x = 4x(x^2 - 3x - 10)$
$= 4x(x+2)(x-5)$
CV: $x = 0, -2, 5$. Increasing on (–2, 0) and $(5, \infty)$; decreasing on $(-\infty, -2)$ and (0, 5); relative maximum at (0, 150); relative minima at (–2, 118) and (5, –225).
$y'' = 12x^2 - 24x - 40 = 4\left(3x^2 - 6x - 10\right)$
Possible inflection points when $x = 1 \pm \frac{\sqrt{39}}{3}$.
Concave up on $\left(-\infty, 1 - \frac{\sqrt{39}}{3}\right)$ and $\left(1 + \frac{\sqrt{39}}{3}, \infty\right)$; concave down on

$\left(1-\frac{\sqrt{39}}{3}, 1+\frac{\sqrt{39}}{3}\right)$; inflection points at

$\left(1-\frac{\sqrt{39}}{3}, \frac{298}{9}+16\sqrt{39}\right) \approx (-1.08, 133.03)$ and

$\left(1+\frac{\sqrt{39}}{3}, \frac{298}{9}-16\sqrt{39}\right) \approx (3.08, -66.81)$.



41. $y = x^3 - x = x\left(x^2-1\right) = x(x+1)(x-1)$

Intercepts (0, 0), (–1, 0), and (1, 0)
Symmetric about the origin. No asymptotes.

$y' = 3x^2 - 1 = \left(\sqrt{3}x+1\right)\left(\sqrt{3}x-1\right)$

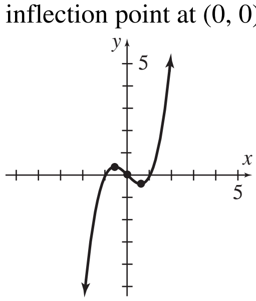
CV: $\pm\frac{\sqrt{3}}{3}$

Increasing on $\left(-\infty, -\frac{\sqrt{3}}{3}\right)$ and $\left(\frac{\sqrt{3}}{3}, \infty\right)$;

decreasing on $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$.

$y'' = 6x$

Possible inflection point when $x = 0$. Concave down on $(-\infty, 0)$; concave up on $(0, \infty)$; inflection point at (0, 0).



42. $y = \frac{x+2}{x-3}$

Intercepts: $\left(0, -\frac{2}{3}\right)$, (–2, 0)

Vertical asymptote is $x = 3$.

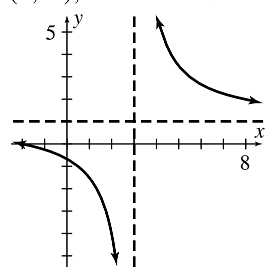
$\lim_{x\to\infty} \frac{x+2}{x-3} = 1 = \lim_{x\to-\infty} \frac{x+2}{x-3}$, so $y = 1$ is a horizontal asymptote.

$y' = -\frac{5}{(x-3)^2}$

CV: None, but $x = 3$ must be considered in the inc.-dec. analysis. Decreasing on $(-\infty, 3)$ and $(3, \infty)$.

$y'' = \frac{10}{(x-3)^3}$

No possible inflection point, but $x = 3$ must be considered in concavity analysis. Concave up on $(3, \infty)$; concave down on $(-\infty, 3)$.



43. $y = f(x) = \frac{100(x+5)}{x^2}$

Intercept: (–5, 0)
No symmetry.
$x = 0$ is the only vertical asymptote.

$\lim_{x\to\infty} y = 100 \lim_{x\to\infty} \frac{x}{x^2} = 100 \lim_{x\to\infty} \frac{1}{x} = 0$, and

$\lim_{x\to-\infty} y = 0$, so $y = 0$ is the only horizontal asymptote.

$y = 100\left[x^{-1} + 5x^{-2}\right]$

$y' = 100\left[-x^{-2} - 10x^{-3}\right] = -100\left[\frac{1}{x^2} + \frac{10}{x^3}\right]$

$= \frac{-100(x+10)}{x^3}$

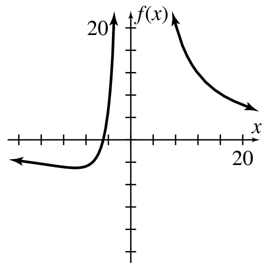
CV: $x = -10$ but $x = 0$ must be included in inc.-dec. analysis. Increasing on (–10, 0); decreasing on $(-\infty, -10)$ and $(0, \infty)$; relative minimum at (–10, –5).

$y'' = 100\left[2x^{-3} + 30x^{-4}\right] = 200\left[\frac{1}{x^3} + \frac{15}{x^4}\right]$

$= \frac{200(x+15)}{x^4}$

Possible inflection point when $x = -15$, but $x = 0$ must also be considered in concavity analysis. Concave up on (–15, 0) and $(0, \infty)$; concave down on $(-\infty, -15)$; inflection point at

$\left(-15, -\frac{40}{9}\right)$



44. $y = \frac{x^2-4}{x^2-1} = \frac{(x+2)(x-2)}{(x+1)(x-1)}$

Intercepts: (0, 4), (2, 0), (–2, 0)
Symmetric about the *y*-axis. Vertical asymptotes are $x = 1$ and $x = -1$.

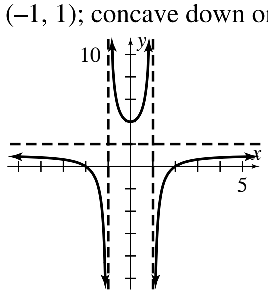
$\lim\limits_{x\to-\infty} y = \lim\limits_{x\to-\infty} \frac{x^2}{x^2} = 1 = \lim\limits_{x\to\infty} y$, so $y = 1$ is the only horizontal asymptote.

$y' = \frac{6x}{\left(x^2-1\right)^2}$

CV: $x = 0$ but $x = \pm 1$ must also be considered in inc.-dec. analysis. Increasing on (0, 1) and (1, ∞); decreasing on (–∞, –1) and (–1, 0); relative minimum at (0, 4).

$y'' = \frac{-6\left(3x^2+1\right)}{\left(x^2-1\right)^3}$

No possible inflection point, but $x = \pm 1$ must be considered in concavity analysis. Concave up on (–1, 1); concave down on (–∞, –1) and (1, ∞).



45. $y = \frac{2x}{(3x-1)^3}$

Intercept: (0, 0)
No symmetry

Vertical asymptote is $x = \frac{1}{3}$.

$\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} \frac{2x}{27x^3} = \frac{2}{27}\lim\limits_{x\to\infty} \frac{1}{x^2} = 0$
$= \lim\limits_{x\to-\infty} y,$

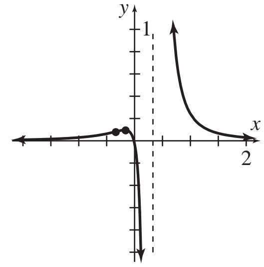
so $y = 0$ is a horizontal asymptote.

$y' = \frac{(3x-1)^3(2) - 2x[3(3x-1)^2(3)]}{(3x-1)^6}$
$= \frac{2(3x-1)^2[(3x-1)-9x]}{(3x-1)^6}$
$= \frac{2(-6x-1)}{(3x-1)^4} = \frac{-2(6x+1)}{(3x-1)^4}$

CV: $x = -\frac{1}{6}$, but $x = \frac{1}{3}$ must be considered in inc.-dec. analysis. Increasing on $\left(-\infty, -\frac{1}{6}\right)$; decreasing on $\left(-\frac{1}{6}, \frac{1}{3}\right)$ and $\left(\frac{1}{3}, \infty\right)$; relative maximum at $\left(-\frac{1}{6}, \frac{8}{81}\right)$.

$y'' = -2\cdot\frac{(3x-1)^4(6) - (6x+1)[4(3x-1)^3(3)]}{(3x-1)^8}$
$= -2\cdot\frac{6(3x-1)^3[(3x-1)-2(6x+1)]}{(3x-1)^8}$
$= \frac{-12(-9x-3)}{(3x-1)^5} = \frac{36(3x+1)}{(3x-1)^5}$

Possible inflection point when $x = -\frac{1}{3}$, but $x = \frac{1}{3}$ must be considered in concavity analysis. Concave up on $\left(-\infty, -\frac{1}{3}\right)$ and $\left(\frac{1}{3}, \infty\right)$; concave down on $\left(-\frac{1}{3}, \frac{1}{3}\right)$; inflection point at $\left(-\frac{1}{3}, \frac{1}{12}\right)$.



46. $y = 6x^{\frac{1}{3}}(2x-1)$

Intercepts: (0, 0), $\left(\frac{1}{2}, 0\right)$

No symmetry. No vertical asymptote.

As $x \to \infty$, both $6x^{1/3}$ and $2x-1 \to \infty$. As $x \to -\infty$, both $6x^{\frac{1}{3}}$ and $2x-1 \to -\infty$. Thus $\lim\limits_{x\to\infty} y = \infty = \lim\limits_{x\to-\infty} y$. So no horizontal asymptote exists. Since $y = 6\left(2x^{4/3} - x^{1/3}\right)$,

$$y' = 6\left(\frac{8}{3}x^{1/3} - \frac{1}{3}x^{-2/3}\right) = 2x^{-2/3}(8x-1)$$

$$= \frac{2(8x-1)}{x^{2/3}}$$

CV: $x = 0, \frac{1}{8}$

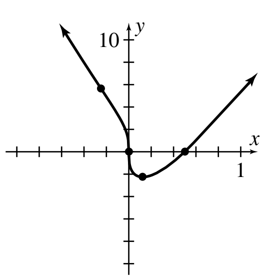
Decreasing on (–∞, 0) and $\left(0, \frac{1}{8}\right)$; increasing on $\left(\frac{1}{8}, \infty\right)$; relative minimum at $\left(\frac{1}{8}, -\frac{9}{4}\right)$.

$$y'' = 2\left(\frac{8}{3}x^{-2/3} + \frac{2}{3}x^{-5/3}\right) = \frac{4}{3}x^{-5/3}(4x+1)$$

$$= \frac{4(4x+1)}{3x^{5/3}}$$

Possible inflection points when $x = -\frac{1}{4}, 0$.

Concave up on $\left(-\infty, -\frac{1}{4}\right)$ and (0, ∞); concave down on $\left(-\frac{1}{4}, 0\right)$; inflection points at $\left(-\frac{1}{4}, \frac{9\sqrt[3]{2}}{2}\right)$ and (0, 0).



47. $f(x) = \frac{e^x + e^{-x}}{2}$

Intercept: (0, 1)

Symmetric about the *y*-axis. No asymptotes.

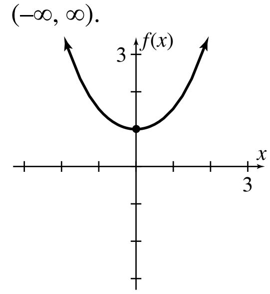
$$f'(x) = \frac{e^x - e^{-x}}{2}$$

Setting $f'(x) = 0 \Rightarrow e^x = e^{-x} \Rightarrow x = -x \Rightarrow x = 0$

CV: $x = 0$

Increasing on (0, ∞); decreasing on (–∞, 0); relative minimum at (0, 1). Finding $f''(x)$ gives:

$f''(x) = \frac{e^x + e^{-x}}{2}$. $f''(x) > 0$ for all *x*. No possible inflection point. Concave up on (–∞, ∞).



48. $y = f(x) = 1 - \ln(x^3) = 1 - 3\ln x$

$y = 0 \Rightarrow \ln\left(x^3\right) = 1 \Rightarrow x^3 = e \Rightarrow x = e^{1/3}$, so the *x*-intercept is $\left(e^{1/3}, 0\right)$. Since $x \neq 0$, there is no *y*-intercept. No symmetry. Since $\lim\limits_{x\to 0^+} y = \infty$, $x = 0$ is a vertical asymptote. No horizontal asymptote.

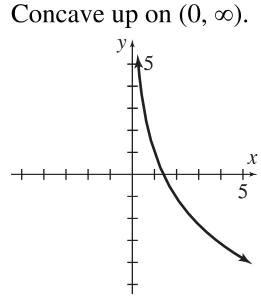
$$f'(x) = -\frac{3}{x}$$

CV: None. Decreasing on (0, ∞).

$$f''(x) = \frac{3}{x^2}$$

No possible inflection points.

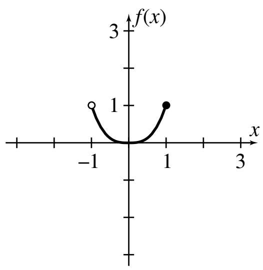
Concave up on (0, ∞).



49. a. False. $f'(x_0) = 0$ only indicates the possibility of a relative extremum at x_0, For example, if $f(x) = x^3$, then $f'(x) = 3x^2$ and $f'(0) = 0$. However there is no relative extremum at $x = 0$.

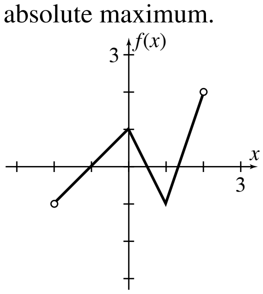
b. False. For example, let $x_1 = -1$ and $x_2 = 1$. Then $x_1 < x_2$ and $f(x_1) = -1 < f(x_2) = 1$.

c. True. The absolute minimum is $f(0) = 0$ and the absolute maximum is $f(1) = 1$.



d. False. If concavity does not change around x_0, then $(x_0, f(x_0))$ is not an inflection point. For example, consider $f(x) = x^4$. If $x_0 = 0$, then $f''(x_0) = 0$, but $(x_0, f(x_0))$ is not an inflection point. See graph in part (c).

e. False. Consider the function f whose graph is shown. On (–2, 2) it has exactly one relative maximum [at the point (0, 1)] but no absolute maximum.



50. Let $y = f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

a. Replacing x by $-x$ yields the original equation. Thus the graph is symmetric about the y-axis. No other symmetry exists.

b. $f'(x) = \frac{-xe^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$

f' is defined for all x; $f'(x) = 0$ only when $x = 0$. Thus $x = 0$ is a critical value. If $x < 0$, then $f'(x) > 0$; if $x > 0$ then $f'(x) < 0$. Thus f is increasing on $(-\infty, 0)$ and is decreasing on $(0, \infty)$.

c. From (b), f has a relative maximum when $x = 0$. The coordinates of this relative maximum are $\left(0, \frac{1}{\sqrt{2\pi}}\right)$.

d. $\lim_{x\to-\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}}(0) = 0$

$\lim_{x\to\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}}(0) = 0$

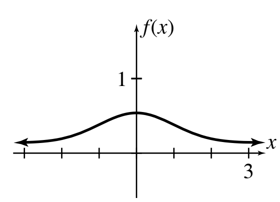
e. $f''(x) = -\frac{1}{\sqrt{2\pi}}\left[xe^{-\frac{x^2}{2}}(-x) + e^{-\frac{x^2}{2}}(1)\right]$

$= \frac{e^{-\frac{x^2}{2}}(x^2 - 1)}{\sqrt{2\pi}} = \frac{e^{-\frac{x^2}{2}}(x+1)(x-1)}{\sqrt{2\pi}}$

f'' is defined for all x; $f''(x) = 0$ when $x = \pm 1$. f is concave up on $(-\infty, -1)$ and $(1, \infty)$; f is concave down on $(-1, 1)$.

f. From (e), f changes concavity at $x = \pm 1$. Also f is continuous there. Thus f has inflection points at $x = \pm 1$; the coordinates are $\left(-1, \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi}}\right)$ and $\left(1, \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi}}\right)$

g.



h. Absolute maximum: $f(0) = \frac{1}{\sqrt{2\pi}}$

No absolute minimum.

51. $c = q^3 - 6q^2 + 12q + 18$

Marginal cost $= \frac{dc}{dq} = 3q^2 - 12q + 12$. Marginal cost is increasing when its derivative, which is $\frac{d^2c}{dq^2}$, is positive.

$$\frac{d^2c}{dq^2} = 6q - 12 = 6(q-2)$$

$\frac{d^2c}{dq^2} > 0$ for $q > 2$. Thus marginal cost is increasing for $q > 2$.

52. $r = 320q^{3/2} - 2q^2$

Marginal revenue $= \frac{dr}{dq} = 480q^{1/2} - 4q$.

Marginal revenue is increasing when its derivative, which is $\frac{d^2r}{dq^2}$ is positive.

$$\frac{d^2r}{dq^2} = 240q^{-1/2} - 4 = \frac{240}{\sqrt{q}} - 4$$

$$\frac{d^2r}{dq^2} = 0 \Rightarrow \frac{240}{\sqrt{q}} - 4 = 0 \Rightarrow 240 = 4\sqrt{q}$$

$$\Rightarrow \sqrt{q} = 60 \Rightarrow q = 3600$$

$\frac{d^2r}{dq^2} > 0$ for $0 < q < 3600$. Thus marginal revenue is increasing on (0, 3600).

53. $p = 200 - \frac{\sqrt{q}}{5}$, $q > 0$. The revenue function r is given by

$$r = pq = \left(200 - \frac{\sqrt{q}}{5}\right)q = 200q - \frac{q^{\frac{3}{2}}}{5}.$$

$$r' = 200 - \frac{3}{10}q^{\frac{1}{2}}$$

$$r'' = -\frac{3}{20}q^{-\frac{1}{2}} = -\frac{3}{20\sqrt{q}}$$

Since $r'' < 0$ for $q > 0$, the graph of the revenue function is concave down for $q > 0$.

54. a. $R(0) = 0\%$

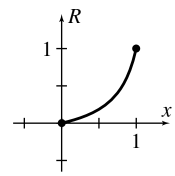
b. $R(0.5) \approx 18.5\%$

c. $R(1) = 100\%$

$$\frac{dR}{dx} = \frac{4.4}{(4.4-3.4x)^2} > 0 \text{ for } 0 \le x \le 1$$

$$\frac{d^2R}{dx^2} = \frac{29.92}{(4.4-3.4x)^3} > 0 \text{ for } 0 \le x \le 1.$$

We obtain the following graph:



55. $f(t) = At^3 + Bt^2 + Ct + D$

$f'(t) = 3At^2 + 2Bt + C$

$f''(t) = 6At + 2B$, which gives an inflection point when $6At + 2B = 0$, that is for $a = -\frac{B}{3A}$.

This value of a must be such that $f'(a) = 0$.

$$3A\left(-\frac{B}{3A}\right)^2 + 2B\left(-\frac{B}{3A}\right) + C = 0$$

$$\frac{1}{3}\left(\frac{B^2}{A}\right) - \frac{2}{3}\left(\frac{B^2}{A}\right) + C = 0$$

$$C = \frac{1}{3}\left(\frac{B^2}{A}\right)$$

$3AC = B^2$,

which was to be shown.

56. a. Let $a = p + q$. Then S becomes

$$S = \frac{ma^2}{p}\left[\frac{e^{-at}}{\left(\frac{q}{p}e^{-at} + 1\right)^2}\right]$$

$$\frac{dS}{dt}=\frac{ma^2}{p}\left[\frac{\left(\frac{q}{p}e^{-at}+1\right)^2\left(-ae^{-at}\right)-e^{-at}\left[2\left(\frac{q}{p}e^{-at}+1\right)\left(\frac{-aq}{p}e^{-at}\right)\right]}{\left(\frac{q}{p}e^{-at}+1\right)^4}\right]$$

$$=\frac{ma^2}{p}\left(ae^{-at}\right)\left(\frac{q}{p}e^{-at}+1\right)\frac{-\left(\frac{q}{p}e^{-at}+1\right)+2\frac{q}{p}e^{-at}}{\left(\frac{q}{p}e^{-at}+1\right)^4}$$

$$=\frac{ma^3}{p}e^{-at}\;\frac{\frac{q}{p}e^{-at}a-1}{\left(\frac{q}{p}e^{-at}+1\right)^3}$$

$$=\frac{\frac{m}{p}(p+q)^3e^{-(p+q)t}\left[\frac{q}{p}e^{-(p+q)t}-1\right]}{\left(\frac{q}{p}e^{-(p+q)t}+1\right)^3}$$

b. $\frac{dS}{dt}=0$ when

$$\frac{m}{p}(p+q)^3e^{-(p+q)t}\left[\frac{q}{p}e^{-(q+p)t}-1\right]=0$$

Since m, $p + q$, and $e^{-(p+q)t}$ are nonzero, we must have

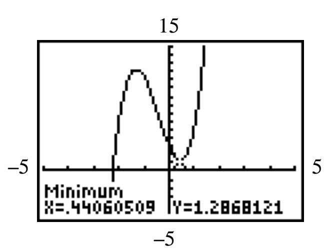
$$\frac{q}{p}e^{-(p+q)t}-1=0$$

$$e^{-(p+q)t}=\frac{p}{q}$$

$$-(p+q)t=\ln\left(\frac{p}{q}\right)$$

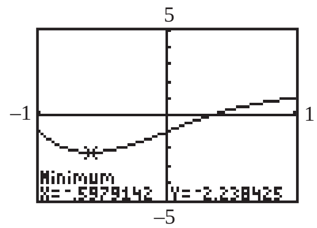
$$t=-\frac{\ln\left(\frac{p}{q}\right)}{p+q}=\frac{\ln\left(\frac{q}{p}\right)}{p+q}$$

57.



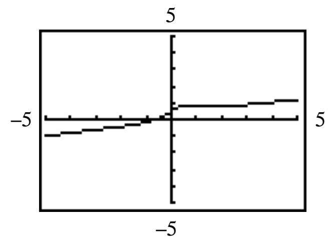
Relative maximum (–1.32, 12.28); relative minimum (0.44, 1.29)

58.



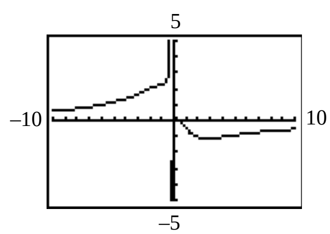
Maximum: (1, 1); minimum: (–0.60, –2.24)

59.



The *x*-value of the inflection point of *f* corresponds to the *x*-intercept of f''. Thus the *x*-value of the inflection point is $x \approx -0.60$.

60.



Horizontal asymptote $y = 0$;
vertical asymptote $x \approx -0.25$

61. $q = 80m^2 - 0.1m^4$

$$\frac{dq}{dm} = 160m - 0.4m^3 = 0.4m\left(400 - m^2\right)$$

$= 0.4m(20+m)(20-m)$. Setting $\frac{dq}{dm} = 0$ yields $m = 0$ or $m = 20$ (for $m \geq 0$). We find that *q* is increasing on (0, 20) and decreasing on (20, ∞), so *q* is maximum at $m = 20$.

62. $p = 100e^{-0.1q}$

Total revenue $= r = pq = 100qe^{-0.1q}$

$$r' = 100[e^{-0.1q}(1) + q(-0.1)e^{-0.1q}]$$
$$= 10e^{-0.1q}(10 - q)$$

$r' = 0$ when $q = 10$. Since *r* is increasing when $q < 10$ and decreasing when $q > 10$, revenue is maximized when $q = 10$.

63. $p = \sqrt{500 - q}$, where $100 \le q \le 200$.

Total revenue $= r = pq = q\sqrt{500-q}$

$$r' = q\left(\frac{1}{2}\right)(500-q)^{-1/2}(-1) + \sqrt{500-q}(1)$$
$$= \frac{1}{2}(500-q)^{-1/2}[-q + 2(500-q)]$$
$$= \frac{1000-3q}{2\sqrt{500-q}}$$
$$= \frac{3\left(\frac{1000}{3} - q\right)}{2\sqrt{500-q}}$$

No critical values on (100, 200). $r(100) = 2000$; $r(200) \approx 3464$, so 200 units should be produced for maximum revenue.

64. $c = 0.01q^2 + 5q + 100$

Avg. cost $\bar{c} = \frac{c}{q} = 0.01q + 5 + \frac{100}{q}$

$$\frac{d\bar{c}}{dq} = 0.01 - \frac{100}{q^2} = \frac{1}{100} - \frac{100}{q^2} = \frac{q^2 - 100^2}{100q^2}$$
$$= \frac{(q-100)(q+100)}{100q^2}$$

We find that $\bar{c}$ is decreasing on (0, 100) and increasing on (100, ∞), so average cost is minimum when $q = 100$.

65. $p = 500 - 3q$

$$\bar{c} = q + 200 + \frac{1000}{q}$$

Total Cost $= c = \bar{c}q = q^2 + 200q + 1000$

Profit = Total Revenue – Total Cost

$$P = pq - c = (500-3q)q - (q^2 + 200q + 1000)$$
$$= -4(q^2 - 75q + 250)$$
$$P' = -4(2q - 75)$$

Setting $P' = 0$ yields $q = 37.5$. Since $P'' = -8 < 0$, *P* is maximum when $q = 37.5$. In reality, whole units are likely. Since $P(37) = P(38) = 4624$, the maximum profit is \$4624.

66. $V = (10 - 2x)(16 - 2x)x$

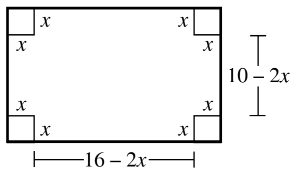
$$= 4\left(x^3 - 13x^2 + 40x\right)$$

Note: $0 < x < 5$.

$V' = 4\left(3x^2 - 26x + 40\right)$
$= 4(x - 2)(3x - 20)$

Setting $V' = 0$ gives $x = 2$ or $x = \frac{20}{3}$. On (0, 5), $x = 2$ is the only critical value. At $x = 2$ in.,

$V'' = 4(6x - 26) = 4(12 - 26) = -56 < 0$, so V is maximum at $x = 2$ in.

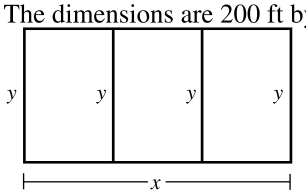


67. $2x + 4y = 800$; thus $x = 400 - 2y$
Area $= A = xy = (400 - 2y)y$
$= 400y - 2y^2$

$\frac{dA}{dy} = 400 - 4y = 4(100 - y)$

Setting $\frac{dA}{dy} = 0$ gives $y = 100$. Since $\frac{d^2A}{dy^2} = -4 < 0$, A is maximum when $y = 100$. When $y = 100$, then $x = 200$.

The dimensions are 200 ft by 100 ft.



68. $xy = 500$, so $y = \frac{500}{x}$
Printed area $= A = (x - 8)(y - 10)$

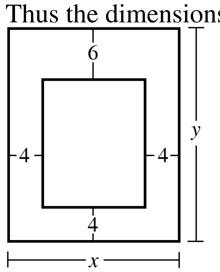
$= (x - 8)\left(\frac{500}{x} - 10\right)$

$= 580 - 10x - \frac{4000}{x}$, $x > 0$

$A' = -10 + \frac{4000}{x^2}$

Setting $A' = 0$ gives $x = 20$. When $x = 20$, $A'' = -\frac{8000}{x^3} < 0$, so A is maximum. When $x = 20$, then $y = \frac{500}{20} = 25$.

Thus the dimensions are 20 in. by 25 in.



69. **a.** $c = 2q^3 - 9q^2 + 12q + 20$, where $\frac{3}{4} \le q \le 6$.

$\frac{dc}{dq} = 6q^2 - 18q + 12 = 6\left(q^2 - 3q + 2\right)$

$= 6(q - 1)(q - 2)$

Setting $\frac{dc}{dq} = 0$ gives $q = 1$ or 2. Evaluating c at these critical values and the endpoints:

$c\left(\frac{3}{4}\right) = \frac{793}{32} \approx 24.78$, $c(1) = 25$, $c(2) = 24$, $c(6) = 200$. Thus a minimum occurs at $q = 2$, which corresponds to 200 stands and a total cost of \$24,000. This gives an average cost per stand of $\frac{24{,}000}{200} = \$120$.

b. There are no critical values of c in $3 \le q \le 6$, so we only evaluate c at the endpoints: $c(3) = 29$, $c(6) = 200$. Thus a minimum occurs at $q = 3$, which gives 300 stands.

70. $N = \frac{12{,}100 + 110t + 100t^2}{121 + t^2}$, where $t \ge 0$.

$$N' = \frac{(121 + t^2)(110 + 200t) - (12{,}100 + 110t + 100t^2)(2t)}{(121 + t^2)^2}$$

$$= \frac{110(121 - t^2)}{121 + t^2}$$

Setting $N' = 0$ gives $t = 11$, from which $N = 105$. Since $N' > 0$ for $0 \le t < 11$ and $N' < 0$ for $t > 11$, there is an absolute maximum when $t = 11$.

Mathematical Snapshot Chapter 13

1. Figure 13.74 does not readily show how long it takes for the population to reach its final size. Figure 13.75 shows that this takes about 45 days.

2. The population declines until it stabilizes between 311 and 312 (as can be verified by inspecting the final state of the variable P). This is consistent with the fact that $\frac{dP}{dt} < 0$ for all $P \ge 312$.

3. Even if the graph starts out exactly coinciding with the ideal curve, a line segment tangent to the curve at one end must (in general) lie slightly off the curve at the other end. This introduces errors that accumulate over successive iterations. The amount of cumulative error could be reduced by taking smaller time steps, such as 1 month instead of 1 year, and correspondingly drawing shorter line segments.

Chapter 14

Problems 14.1

1. $y = 5x - 7$

$$dy = \frac{d}{dx}(5x-7)dx = 5\ dx$$

2. $dy = y'dx = 0\ dx = 0$

3. $$d[f(x)] = f'(x)dx = \frac{1}{2}(x^4-9)^{-\frac{1}{2}}(4x^3)dx$$
$$= \frac{2x^3}{\sqrt{x^4-9}}dx$$

4. $d[f(x)] = f'(x)dx$
$$= 3(8x-5)\left(4x^2-5x+2\right)^2 dx$$

5. $u = x^{-2}$
$$du = \frac{d}{dx}\left(x^{-2}\right)dx = -2x^{-3}dx = -\frac{2}{x^3}dx$$

6. $u = x^{-1/2}$
$$du = u'dx = -\frac{1}{2}x^{-3/2}dx$$

7. $$dp = \frac{d}{dx}\left[\ln\left(x^2+7\right)\right]dx = \frac{1}{x^2+7}(2x)dx$$
$$= \frac{2x}{x^2+7}dx$$

8. $$dp = \frac{d}{dx}\left(e^{x^3+2x-5}\right)dx = (3x^2+2)e^{x^3+2x-5}dx$$

9. $dy = y'dx$
$$= \left[(9x+3)e^{2x^2+3}(4x) + e^{2x^2+3}(9)\right]dx$$
$$= 3e^{2x^2+3}[(3x+1)(4x)+3]dx$$
$$= 3e^{2x^2+3}\left(12x^2+4x+3\right)dx$$

10. $$y = \ln\sqrt{x^2+12} = \frac{1}{2}\ln(x^2+12)$$
$$dy = \frac{1}{2}\cdot\frac{1}{x^2+12}(2x)dx = \frac{x}{x^2+12}dx$$

11. $\Delta y = [4 - 7(3.02)] - [4 - 7(3)] = -0.14$
$dy = -7\ dx = -7(0.02) = -0.14$

12. $\Delta y = \left[5(-1.02)^2 - 5(-1)^2\right] = 0.202$
$dy = 10x\ dx = 10(-1)(-0.02) = 0.2$

13. Δy
$= [2(-1.9)^2 + 5(-1.9) - 7] - [2(-2)^2 + 5(-2) - 7]$
$= -0.28$
$dy = (4x + 5)dx = [4(-2) + 5](0.1) = -0.3$

14. $\Delta y = [3(-1.03)+2]^2 - [3(-1)+2]^2 = 0.1881$
$dy = 6(3x + 2)\ dx = 6[3(-1) + 2](-0.03) = 0.18$

15. $$\Delta y = \sqrt{32-(3.95)^2} - \sqrt{32-(4^2)} \approx 0.049$$
$$dy = \frac{-x}{\sqrt{32-x^2}}dx = \frac{-4}{\sqrt{16}}(-0.05) = 0.050$$

16. $\Delta y = \ln 4.9 - \ln 5 \approx -0.0202$
$$dy = \frac{1}{-x}(-1)dx = \frac{1}{x}dx = \frac{1}{-5}(0.1) = -0.02$$

17. a. $$f(x) = \frac{x+5}{x+1}$$
$$f'(x) = \frac{(x+1)(1)-(x+5)(1)}{(x+1)^2} = \frac{-4}{(x+1)^2}$$
$$f'(1) = \frac{-4}{4} = -1$$

b. We use $f(x + dx) \approx f(x) + dy$ with $x = 1$, $dx = 0.1$.
$f(1.1) = f(1+0.1) \approx f(1) + f'(1)dx$
$$= \frac{6}{2} + (-1)(0.1) = 2.9$$

18. a. $y = f(x) = x^{3x}$
Using logarithmic differentiation,
$\ln y = 3x \ln x$,
$$\frac{1}{y}\cdot\frac{dy}{dx} = 3x\left(\frac{1}{x}\right) + (\ln x)(3) = 3(1+\ln x)$$
$$\frac{dy}{dx} = y[3(1+\ln x)] = 3x^{3x}(1+\ln x)$$
$$f'(1) = 3(1)(1+0) = 3$$

b. We use $f(x + dx) \approx f(x) + dy$ with $x = 1$, $dx = -0.02$

$$f(0.98) = f(1-0.02) \approx f(1) + f'(1)dx$$
$$= 1^3 + (3)(-0.02) = 0.94$$

19. Let $y = f(x) = \sqrt{x}$

$$f(x+dx) \approx f(x) + dy = \sqrt{x} + \frac{1}{2\sqrt{x}}dx$$

If $x = 289$ and $dx = -1$, then

$$\sqrt{288} = f(289-1)$$
$$\approx \sqrt{289} + \frac{1}{2\sqrt{289}}(-1)$$
$$= \frac{577}{34}$$
$$\approx 16.97$$

20. Let $y = f(x) = \sqrt{x}$

$$f(x+dx) \approx f(x) + dy = \sqrt{x} + \frac{1}{2\sqrt{x}}dx$$

If $x = 121$ and $dx = 1$, then

$$\sqrt{122} = f(121+1) \approx \sqrt{121} + \frac{1}{2\sqrt{121}}(1)$$
$$= 11\frac{1}{22}.$$

21. Let $y = f(x) = \sqrt[3]{x}$

$$f(x+dx) \approx f(x) + dy = \sqrt[3]{x} + \frac{1}{3x^{\frac{2}{3}}}dx$$

If $x = 64$ and $dx = 1.5$, then

$$\sqrt[3]{65.5} = f(64+1.5) \approx \sqrt[3]{64} + \frac{1}{3\left(\sqrt[3]{64}\right)^2}(1.5)$$
$$= 4 + \frac{1.5}{3 \cdot 4^2} = 4\frac{1}{32}$$

22. Let $y = f(x) = \sqrt[4]{x}$.

$$f(x+dx) = f(x) + dy = \sqrt[4]{x} + \frac{1}{4x^{\frac{3}{4}}}dx$$

If $x = 16$ and $dx = 0.3$, then

$$\sqrt[4]{16.3} = f(16+0.3) \approx \sqrt[4]{16} + \frac{1}{4\left(\sqrt[4]{16}\right)^3}(0.3)$$
$$= 2 + \frac{0.3}{2^3} = 2\frac{3}{320}$$

23. Let $y = f(x) = \ln x$

$$f(x+dx) \approx f(x) + dy = \ln(x) + \frac{1}{x}dx$$

If $x = 1$ and $dx = -0.03$, then

$$\ln(0.97) = f(1+(-0.03))$$
$$\approx \ln(1) + \frac{1}{1}(-0.03) = -0.03$$

24. Let $y = f(x) = \ln x$

$$f(x+dx) \approx f(x) + dy = \ln(x) + \frac{1}{x}dx$$

If $x = 1$ and $dx = 0.01$, then

$$\ln 1.01 = f(1+0.01) \approx \ln(1) + \frac{1}{1}(0.01) = 0.01$$

25. Let $y = f(x) = e^x$

$$f(x+dx) \approx f(x) + dy = e^x + e^x dx$$

If $x = 0$ and $dx = 0.001$, then

$$e^{0.001} = f(0+0.001) \approx e^0 + e^0(0.001) = 1.001$$

26. Let $y = f(x) = e^x$

$$f(x+dx) \approx f(x) + dy = e^x + e^x dx$$

If $x = 0$ and $dx = -0.01$, then

$$e^{-0.01} = f(0+(-0.01)) \approx e^0 + e^0(-0.01) = 0.99$$

27. $\frac{dy}{dx} = 2$, so $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{2}$

28. $\frac{dy}{dx} = 10x + 3$, so $\frac{dx}{dy} = \frac{1}{10x+3}$

29. $\frac{dq}{dp} = 6p\left(p^2+5\right)^2$, so $\frac{dp}{dq} = \frac{1}{6p\left(p^2+5\right)^2}$

30. $\frac{dq}{dp} = \frac{1}{2\sqrt{p+5}}$, so $\frac{dp}{dq} = 2\sqrt{p+5}$

31. $q = p^{-1}$, $\frac{dq}{dp} = -1p^{-2} = \frac{-1}{p^2}$, so $\frac{dp}{dq} = -p^2$

32. $\frac{dq}{dp} = -2e^{4-2p}$, so $\frac{dp}{dq} = -\frac{1}{2e^{4-2p}} = -\frac{1}{2}e^{2p-4}$

33. $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{14x-6}$

If $x = 3$, $\frac{dx}{dy} = \frac{1}{36}$

34. $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\frac{2}{x}} = \frac{x}{2}$

If $x = 3$, $\frac{dx}{dy} = \frac{3}{2}$.

35. $p = \frac{500}{q+2}$

$\frac{dp}{dq} = \frac{-500}{(q+2)^2}$

$\frac{dq}{dp} = -\frac{(q+2)^2}{500}$

$\left.\frac{dq}{dp}\right|_{q=18} = -\left.\frac{(q+2)^2}{500}\right|_{q=18} = -\frac{4}{5}$

36. $p = 50 - \sqrt{q}$

$\frac{dp}{dq} = -\frac{1}{2\sqrt{q}}$

$\frac{dq}{dp} = -2\sqrt{q}$

$\left.\frac{dq}{dp}\right|_{q=100} = -2\sqrt{q}\Big|_{q=100} = -20$

37. $P = 397q - 2.3q^2 - 400$, q changes from 90 to 91.

$\Delta P \approx dP = P'dq = (397 - 4.6q)dq$

Choosing $q = 90$ and $dq = 1$,

$\Delta P \approx [397 - 4.6(90)](1) = -17$.

True change is

$P(91) - P(90) = 16{,}680.7 - 16{,}700 = -19.3$.

38. $r = 250q + 45q^2 - q^3$, q increases from 40 to 41.

$\Delta r \approx dr = r'dq = \left(250 + 90q - 3q^2\right)dq$

Choosing $q = 40$ and $dq = 1$,

$\Delta r \approx (-950)(1) = -950$

True change is

$r(41) - r(40) = 16{,}974 - 18{,}000 = -1026$

39. $p = \frac{10}{\sqrt{q}}$. We approximate p when $q = 24$.

$p(q + dq) \approx p + dp = \frac{10}{\sqrt{q}} - \frac{5}{\sqrt{q^3}}dq$

If $q = 25$ and $dq = -1$, then

$p(24) = p(25 + (-1)) \approx \frac{10}{\sqrt{25}} - \frac{5}{\sqrt{(25)^3}}(-1)$

$= 2 + \frac{1}{25} = \frac{51}{25} = 2.04$

40. $p = \frac{200}{\sqrt{q+8}}$

We approximate p when $q = 40$.

$p(q + dq) \approx p + dp = \frac{200}{\sqrt{q+8}} - \frac{100}{(q+8)^{\frac{3}{2}}}dq$

If $q = 41$ and $dq = 1$, then

$p(40) = p(41 - 1) \approx \frac{200}{\sqrt{49}} - \frac{100}{(49)^{\frac{3}{2}}}(1)$

$= \frac{200}{7} - \frac{100}{343} = \frac{9700}{343} \approx 28.28$

41. $c = \frac{q^4}{2} + 3q + 400$

If $q = 10$ and $dq = 2$,

$\frac{dc}{c} = \frac{\left(2q^3 + 3\right)dq}{\frac{q^4}{2} + 3q + 400} = \frac{(2003)(2)}{5430} \approx 0.7$

42. $S = 20\sqrt{I}$, I decreases from 45 to $44\frac{1}{2}$.

$\Delta S \approx dS = S'dI = \frac{10}{\sqrt{I}}dI$

Choosing $I = 45$ and $dI = -\frac{1}{2}$, then

$\Delta S \approx \frac{10}{\sqrt{45}}\left(-\frac{1}{2}\right) \approx -0.745$.

43. $V = \frac{4}{3}\pi r^3$

$\Delta V \approx dV = V'dr = 4\pi r^2 dr$

$dr = \left(6.6 \times 10^{-4}\right) - \left(6.5 \times 10^{-4}\right)$

$= 0.1 \times 10^{-4} = 10^{-5}$

$$\Delta V \approx 4\pi\left(6.5\times10^{-4}\right)^2\left(10^{-5}\right) = \left(1.69\times10^{-11}\right)\pi \text{ cm}^3 .$$

44. $(P + a)(v + b) = k$

$$P = \frac{k}{v+b} - a$$

$$dP = -k(v+b)^{-2}\,dv$$

45. **a.** We substitute $q = 40$ and $p = 20$

$$2 + \frac{40^2}{200} = \frac{4000}{20^2}$$
$$2 + 8 = 10$$
$$10 = 10$$

b. We differentiate implicitly with respect to p.

$$0 + \frac{1}{200}\left(2q\frac{dq}{dp}\right) = -\frac{8000}{p^3}$$

From part (a) $q = 40$ when $p = 20$. Substituting gives

$$\frac{1}{200}\left(2\cdot 40\frac{dq}{dp}\right) = -\frac{8000}{20^3}$$
$$\frac{dq}{dp} = -2.5$$

c. $q(p+dp) \approx q(p) + dq = q(p) + q'(p)dp$

$$\begin{aligned} q(19.20) &= q(20 + (-0.8)) \\ &\approx q(20) + q'(20)dp \\ &= 40 + (-2.5)(-0.8) \\ &= 42 \text{ units} \end{aligned}$$

46. **a.** $\text{Profit} = TR - TC = pq - \bar{c}q$

$$P = \frac{1}{2}q^3 - 66q^2 + 7000q - \left(500q - q^2 + \frac{80,000}{2}\right) = \frac{1}{2}q^3 - 65q^2 + 6500q - 40,000$$

If $q = 100$, then $P = \frac{1}{2}(100)^3 - 65(100)^2 + 6500(100) - 40,000 = 460,000$

b. We use $P(q + dq) \approx P(q) + dP$ with $q = 100$ and $dq = -2$.

$$\begin{aligned} P(98) &= P(100 + (-2)) \\ &\approx P(100) + \left(\frac{3}{2}q^2 - 130q + 6500\right)dq \\ &= 460,000 + \left[\frac{3}{2}(100)^2 - 130(100) + 6500\right](-2) \\ &= \$443,000 \end{aligned}$$

Principles in Practice 14.2

1. $\int 28.3\, dq = 28.3q + C$
   The form of the cost function is $28.3q + C$.

2. $\int 0.12t^2 dt = 0.12\frac{t^3}{3} + C = 0.04t^3 + C$
   The form of the revenue function is $R(t) = 0.04t^3 + C$.

3. Let $S(t)$ = the number of subscribers t months after the competition entered the market, then $S'(t) = -\frac{480}{t^3}$.
   $S(t) = \int -\frac{480}{t^3} dt = -480\int t^{-3} dt$
   $= -480\left(\frac{t^{-2}}{-2}\right) + C = 240t^{-2} + C = \frac{240}{t^2} + C$
   The number of subscribers is $S(t) = \frac{240}{t^2} + C$.

4. $\int\left(500 + 300\sqrt{t}\right) dt = \int\left(500 + 300t^{\frac{1}{2}}\right) dt$
   $= 500t + \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C = 500t + \frac{2}{3}t^{\frac{3}{2}} + C$
   The population is $N(t) = 500t + \frac{2}{3}t^{\frac{3}{2}} + C$

5. The amount of money saved is $\int \frac{dS}{dt} dt$.
   $\int\left(2.1t^2 - 65.4t + 491.6\right) dt$
   $= 2.1\left(\frac{t^3}{3}\right) - 65.4\left(\frac{t^2}{2}\right) + 491.6t + C$
   $= 0.7t^3 - 32.7t^2 + 491.6t + C$
   The amount of money saved is
   $S(t) = 0.7t^3 - 32.7t^2 + 491.6t + C$

Problems 14.2

1. $\int 7\, dx = 7x + C$

2. $\int \frac{1}{2x} dx = \frac{1}{2}\int \frac{1}{x} dx = \frac{1}{2}\ln|x| + C$

3. $\int x^8 dx = \frac{x^{8+1}}{8+1} + C = \frac{x^9}{9} + C$

4. $\int 5x^{24}\, dx = 5\int x^{24}\, dx = 5\cdot\frac{x^{24+1}}{24+1} + C$
   $= 5\cdot\frac{x^{25}}{25} + C = \frac{x^{25}}{5} + C$

5. $\int 5x^{-7} dx = 5\int x^{-7} dx = 5\cdot\frac{x^{-7+1}}{-7+1} + C$
   $= 5\cdot\frac{x^{-6}}{-6} + C = -\frac{5}{6x^6} + C$

6. $\int \frac{z^{-3}}{3} dz = \frac{1}{3}\int z^{-3} dz = \frac{1}{3}\cdot\frac{z^{-3+1}}{-3+1} + C$
   $= \frac{1}{3}\cdot\frac{z^{-2}}{-2} + C = -\frac{1}{6z^2} + C$

7. $\int \frac{2}{x^{10}} dx = 2\int x^{-10} dx = 2\cdot\frac{x^{-10+1}}{-10+1} + C$
   $= \frac{2x^{-9}}{-9} + C = -\frac{2}{9x^9} + C$

8. $\int \frac{7}{x^4} dx = 7\int x^{-4} dx = 7\cdot\frac{x^{-4+1}}{-4+1} + C = \frac{7x^{-3}}{-3} + C$
   $= -\frac{7}{3x^3} + C$

9. $\int \frac{1}{t^{7/4}} dt = \int t^{-7/4} dt = \frac{t^{-7/4+1}}{-\frac{7}{4}+1} + C = \frac{t^{-3/4}}{-\frac{3}{4}} + C$
   $= -\frac{4}{3t^{3/4}} + C$

10. $\int \frac{7}{2x^{\frac{9}{4}}} dx = \frac{7}{2}\int x^{-\frac{9}{4}} dx = \frac{7}{2}\cdot\frac{x^{\frac{-9}{4}+1}}{\frac{-9}{4}+1} + C$
    $= \frac{7}{2}\cdot\frac{x^{-\frac{5}{4}}}{-\frac{5}{4}} + C$
    $= -\frac{14}{5x^{\frac{5}{4}}} + C$

11. $\int (4+t)dt = \int 4\,dt + \int t\,dt = 4t + \frac{t^{t+1}}{1+1} + C$
$= 4t + \frac{t^2}{2} + C$

12. $\int \left(r^3 + 2r\right) dr = \int r^3 + 2\int r\ dr$
$= \frac{r^{3+1}}{3+1} + 2\cdot\frac{r^{1+1}}{1+1} + C$
$= \frac{r^4}{4} + r^2 + C$

13. $\int \left(y^5 - 5y\right) dy = \int y^5 dy - \int 5y\ dy$
$= \frac{y^{5+1}}{5+1} - 5\cdot\frac{y^{1+1}}{1+1} + C$
$= \frac{y^6}{6} - 5\cdot\frac{y^2}{2} + C = \frac{y^6}{6} - \frac{5y^2}{2} + C$

14. $\int \left(5 - 2w - 6w^2\right) dw$
$= \int 5\ dw - 2\int w\ dw - 6\int w^2 dw$
$= 5w - 2\cdot\frac{w^2}{2} - 6\cdot\frac{w^3}{3} + C$
$= 5w - w^2 - 2w^3 + C$

15. $\int \left(3t^2 - 4t + 5\right) dt = 3\int t^2 dt - 4\int t\ dt + \int 5\ dt$
$= 3\cdot\frac{t^3}{3} - 4\cdot\frac{t^2}{2} + 5t + C = t^3 - 2t^2 + 5t + C$

16. $\int (1 + t^2 + t^4 + t^6)dt$
$= \int 1dt + \int t^2\ dt + \int t^4\ dt + \int t^6\ dt$
$= t + \frac{t^3}{3} + \frac{t^5}{5} + \frac{t^7}{7} + C$

17. Since 7 + *e* is a constant,
$\int (7+e)dx = (7+e)x + C$.

18. $\int \left(5 - 2^{-1}\right) dx = \int \left(5 - \frac{1}{2}\right) dx = \int \frac{9}{2} dx = \frac{9}{2}x + C$

19. $\int \left(\frac{x}{7} - \frac{3}{4}x^4\right) dx = \frac{1}{7}\int x\ dx - \frac{3}{4}\int x^4 dx$
$= \frac{1}{7}\cdot\frac{x^2}{2} - \frac{3}{4}\cdot\frac{x^5}{5} + C$
$= \frac{x^2}{14} - \frac{3x^5}{20} + C$

20. $\int \left(\frac{2x^2}{7} - \frac{8}{3}x^4\right) dx = \frac{2}{7}\int x^2 dx - \frac{8}{3}\int x^4 dx$
$= \frac{2}{7}\cdot\frac{x^3}{3} - \frac{8}{3}\cdot\frac{x^5}{5} + C$
$= \frac{2x^3}{21} - \frac{8x^5}{15} + C$

21. $\int \pi e^x dx = \pi\int e^x dx = \pi e^x + C$

22. $\int \left(\frac{e^x}{3} + 2x\right) dx = \frac{1}{3}\int e^x dx + 2\int x\ dx$
$= \frac{1}{3}e^x + 2\cdot\frac{x^2}{2} + C$
$= \frac{e^x}{3} + x^2 + C$

23. $\int \left(x^{8.3} - 9x^6 + 3x^{-4} + x^{-3}\right) dx$
$= \frac{x^{9.3}}{9.3} - 9\cdot\frac{x^7}{7} + 3\cdot\frac{x^{-3}}{-3} + \frac{x^{-2}}{-2} + C$
$= \frac{x^{9.3}}{9.3} - \frac{9x^7}{7} - \frac{1}{x^3} - \frac{1}{2x^2} + C$

24. $\int (0.7y^3 + 10 + 2y^{-3})dy$
$= 0.7\cdot\frac{y^4}{4} + 10y + 2\cdot\frac{y^{-2}}{-2} + C$
$= 0.175y^4 + 10y - \frac{1}{y^2} + C$

25. $\int \frac{-2\sqrt{x}}{3} dx = -\frac{2}{3}\int x^{\frac{1}{2}} dx = -\frac{2}{3}\cdot\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$
$= -\frac{2}{3}\cdot\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = -\frac{4x^{\frac{3}{2}}}{9} + C$

26. $\int dz = \int 1\ dz = 1\cdot z + C = z + C$

27. $\int \frac{1}{4\sqrt[8]{x^2}}\,dx = \frac{1}{4}\int x^{-\frac{1}{4}}dx = \frac{1}{4}\cdot\frac{x^{-\frac{1}{4}+1}}{-\frac{1}{4}+1}+C$

$= \frac{1}{4}\cdot\frac{x^{\frac{3}{4}}}{\frac{3}{4}}+C = \frac{x^{\frac{3}{4}}}{3}+C$

28. $\int \frac{-4}{(3x)^3}\,dx = \int \frac{-4}{27x^3}\,dx = -\frac{4}{27}\int x^{-3}\,dx$

$= -\frac{4}{27}\cdot\frac{x^{-3+1}}{-3+1}+C$

$= -\frac{4}{27}\cdot\frac{x^{-2}}{-2}+C = \frac{2}{27x^2}+C$

29. $\int\left(\frac{x^3}{3}-\frac{3}{x^3}\right)dx = \frac{1}{3}\int x^3dx - 3\int x^{-3}dx$

$= \frac{1}{3}\cdot\frac{x^{3+1}}{3+1} - 3\cdot\frac{x^{-3+1}}{-3+1}+C$

$= \frac{1}{3}\cdot\frac{x^4}{4} - 3\cdot\frac{x^{-2}}{-2}+C = \frac{x^4}{12}+\frac{3}{2x^2}+C$

30. $\int\left(\frac{1}{2x^3}-\frac{1}{x^4}\right)dx = \frac{1}{2}\int x^{-3}dx - \int x^{-4}dx$

$= \frac{1}{2}\cdot\frac{x^{-2}}{-2} - \frac{x^{-3}}{-3}+C$

$= -\frac{1}{4x^2}+\frac{1}{3x^3}+C$

31. $\int\left(\frac{3w^2}{2}-\frac{2}{3w^2}\right)dw = \frac{3}{2}\int w^2dw - \frac{2}{3}\int w^{-2}dw$

$= \frac{3}{2}\cdot\frac{w^3}{3} - \frac{2}{3}\cdot\frac{w^{-1}}{-1}+C = \frac{w^3}{2}+\frac{2}{3w}+C$

32. $\int \frac{4}{e^{-s}}\,ds = 4\int e^s ds = 4e^s + C$

33. $\int \frac{3u-4}{5}\,du = \frac{1}{5}\int (3u-4)du = \frac{1}{5}\left(3\int u\ du - 4\int du\right)$

$= \frac{1}{5}\left(3\frac{u^2}{2}-4u\right)+C = \frac{3}{10}u^2 - \frac{4}{5}u + C$

$= \frac{1}{7}\left(2\int z\ dz - \int 5\ dz\right)$

$= \frac{1}{7}\left(2\cdot\frac{z^2}{2}-5z\right)+C = \frac{1}{7}\left(z^2-5z\right)+C$

34. $\int \frac{1}{12}\left(\frac{1}{3}e^x\right)dx = \int \frac{1}{36}e^x dx$

$= \frac{1}{36}\int e^x dx = \frac{1}{36}e^x + C$

35. $\int (u^e + e^u)du = \int u^e\ du + \int e^u\ du$

$= \frac{u^{e+1}}{e+1} + e^u + C$

36. $\int\left(3y^3 - 2y^2 + \frac{e^y}{6}\right)dy$

$= 3\int y^3dy - 2\int y^2dy + \frac{1}{6}\int e^y dy$

$= 3\cdot\frac{y^4}{4} - 2\cdot\frac{y^3}{3} + \frac{1}{6}\cdot e^y + C$

$= \frac{3y^4}{4} - \frac{2y^3}{3} + \frac{e^y}{6} + C$

37. $\int\left(2\sqrt{x} - 3\sqrt[4]{x}\right)dx = \int\left(2x^{\frac{1}{2}} - 3x^{\frac{1}{4}}\right)dx$

$= 2\int x^{\frac{1}{2}}dx - 3\int x^{\frac{1}{4}}dx$

$= 2\cdot\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 3\cdot\frac{x^{\frac{5}{4}}}{\frac{5}{4}} + C = \frac{4x^{\frac{3}{2}}}{3} - \frac{12x^{\frac{5}{4}}}{5} + C$

38. $\int 0\ dt = 0\cdot t + C = C$

39. $\int\left(-\frac{\sqrt[3]{x^2}}{5}-\frac{7}{2\sqrt{x}}+6x\right)dx$

$=\int\left(-\frac{x^{\frac{2}{3}}}{5}-\frac{7x^{-\frac{1}{2}}}{2}+6x\right)dx$

$=-\frac{1}{5}\int x^{\frac{2}{3}}dx-\frac{7}{2}\int x^{-\frac{1}{2}}dx+6\int x\,dx$

$=-\frac{1}{5}\cdot\frac{x^{\frac{5}{3}}}{\frac{5}{3}}-\frac{7}{2}\cdot\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+6\cdot\frac{x^2}{2}+C$

$=-\frac{3x^{\frac{5}{3}}}{25}-7x^{\frac{1}{2}}+3x^2+C$

40. $\int\left(\sqrt[3]{u}+\frac{1}{\sqrt{u}}\right)du=\int\left(u^{\frac{1}{3}}+u^{-\frac{1}{2}}\right)du$

$=\int u^{\frac{1}{3}}du+\int u^{-\frac{1}{2}}du$

$=\frac{u^{\frac{4}{3}}}{\frac{4}{3}}+\frac{u^{\frac{1}{2}}}{\frac{1}{2}}+C$

$=\frac{3u^{\frac{4}{3}}}{4}+2u^{\frac{1}{2}}+C$

41. $\int\left(x^2+5\right)(x-3)dx=\int\left(x^3-3x^2+5x-15\right)dx$

$=\frac{x^4}{4}-3\cdot\frac{x^3}{3}+5\cdot\frac{x^2}{2}-15x+C$

$=\frac{x^4}{4}-x^3+\frac{5x^2}{2}-15x+C$

42. $\int x^4\left(x^3+8x^2+7\right)dx=\int\left(x^7+8x^6+7x^4\right)dx$

$=\frac{x^8}{8}+8\cdot\frac{x^7}{7}+7\cdot\frac{x^5}{5}+C$

$=\frac{x^8}{8}+\frac{8x^7}{7}+\frac{7x^5}{5}+C$

43. $\int\sqrt{x}(x+3)dx=\int\left(x^{\frac{3}{2}}+3x^{\frac{1}{2}}\right)dx$

$=\frac{x^{\frac{5}{2}}}{\frac{5}{2}}+3\cdot\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+C$

$=\frac{2x^{\frac{5}{2}}}{5}+2x^{\frac{3}{2}}+C$

44. $\int(z+2)^2\,dz=\int\left(z^2+4z+4\right)dz$

$=\frac{z^3}{3}+4\cdot\frac{z^2}{2}+4z+C$

$=\frac{z^3}{3}+2z^2+4z+C$

45. $\int(3u+2)^3\,du=\int(27u^3+54u^2+36u+8)du$

$=27\cdot\frac{u^4}{4}+54\cdot\frac{u^3}{3}+36\cdot\frac{u^2}{2}+8u+C$

$=\frac{27}{4}u^4+18u^3+18u^2+8u+C$

46. $\int\left(\frac{2}{\sqrt[5]{x}}-1\right)^2dx=\int\left(2x^{-\frac{1}{5}}-1\right)^2dx$

$=\int\left(4x^{-\frac{2}{5}}-4x^{-\frac{1}{5}}+1\right)dx$

$=4\cdot\frac{x^{\frac{3}{5}}}{\frac{3}{5}}-4\cdot\frac{x^{\frac{4}{5}}}{\frac{4}{5}}+x+C$

$=\frac{20x^{\frac{3}{5}}}{3}-5x^{\frac{4}{5}}+x+C$

47. $\int v^{-2}\left(2v^4+3v^2-2v^{-3}\right)dv$

$=\int\left(2v^2+3-2v^{-5}\right)dv$

$=2\cdot\frac{v^3}{3}+3v-2\cdot\frac{v^{-4}}{-4}+C$

$=\frac{2v^3}{3}+3v+\frac{1}{2v^4}+C$

48. $\int\left[6e^u-u^3\left(\sqrt{u}+1\right)\right]du=\int\left[6e^u-u^{\frac{7}{2}}-u^3\right]du$

$=6\cdot e^u-\frac{u^{\frac{9}{2}}}{\frac{9}{2}}-\frac{u^4}{4}+C$

$=6e^u-\frac{2u^{\frac{9}{2}}}{9}-\frac{u^4}{4}+C$

49. $\int \frac{z^4+10z^3}{2z^2}dz = \frac{1}{2}\int\left(\frac{z^4}{z^2}+\frac{10z^3}{z^2}\right)dz$

$= \frac{1}{2}\int\left(z^2+10z\right)dz$

$= \frac{1}{2}\left(\frac{z^3}{3}+10\cdot\frac{z^2}{2}\right)+C$

$= \frac{z^3}{6}+\frac{5z^2}{2}+C$

50. $\int \frac{x^4-5x^2+2x}{5x^2}dx = \frac{1}{5}\int\left(x^2-5+\frac{2}{x}\right)dx$

$= \frac{1}{5}\left(\frac{x^3}{3}-5x+2\ln|x|\right)+C$

51. $\int \frac{e^x+e^{2x}}{e^x}dx = \int\left(\frac{e^x}{e^x}+\frac{e^{2x}}{e^x}\right)dx$

$= \int\left(1+e^x\right)dx$

$= x+e^x+C$

52. $\int \frac{(x^3+1)^2}{x^2}dx = \int \frac{x^6+2x^3+1}{x^2}dx$

$= \int (x^4+2x+x^{-2})dx$

$= \frac{x^5}{5}+2\cdot\frac{x^2}{2}+\frac{x^{-1}}{-1}+C$

$= \frac{x^5}{5}+x^2-\frac{1}{x}+C$

53. No, $F(x) - G(x)$ might be a nonzero constant.

54. **a.** $F(x) = \frac{d}{dx}\left(xe^x\right) = xe^x+e^x(1) = e^x(x+1)$

b. There is only one.

55. Because an antiderivative of the derivative of a function is the function itself, we have

$\int \frac{d}{dx}\left[\frac{1}{\sqrt{x^2+1}}\right]dx = \frac{1}{\sqrt{x^2+1}}+C.$

Principles in Practice 14.3

1. $N(t) = \int \frac{dN}{dt}dt = \int\left(800+200e^t\right)dt$

$= 800t+200e^t+C$

Since $N(5) = 40{,}000$, we have

$40{,}000 = 800(5)+200e^5+C$, so

$C = 40{,}000-\left(4000+200e^5\right)$

$= 36{,}000-200e^5 \approx 6317.37$

$N(t) = 800t+200e^t+6317.37$

2. Since $y'' = \frac{d}{dt}(y') = 84t+24$

$y' = \int (84t+24)dt = 84\left(\frac{t^2}{2}\right)+24t+C_1$

$= 42t^2+24t+C_1$

Since $y'(8) = 2891$, we have

$2891 = 42(8)^2+24(8)+C_1 = 2880+C_1$, so

$C_1 = 2891-2880 = 11$, and $y' = 42t^2+24t+11$.

$y(t) = \int y'dt = \int\left(42t^2+24t+11\right)dt$

$= 42\left(\frac{t^3}{3}\right)+24\left(\frac{t^2}{2}\right)+11t+C_2$

$= 14t^3+12t^2+11t+C_2$

Since $y(2) = 185$, we have

$185 = 14(2)^3+12(2)^2+11(2)+C_2$

$= 182+C_2$, so $C_2 = 185-182 = 3$.

$y(t) = 14t^3+12t^2+11t+3$

Problems 14.3

1. $\frac{dy}{dx} = 3x - 4$

$y = \int (3x-4)dx = \frac{3x^2}{2} - 4x + C$

Using $y(-1) = \frac{13}{2}$ gives

$\frac{13}{2} = \frac{3(-1)^2}{2} - 4(-1) + C$

$\frac{13}{2} = \frac{11}{2} + C$

Thus $C = 1$, so $y = \frac{3x^2}{2} - 4x + 1$.

2. $\frac{dy}{dx} = x^2 - x$

$y = \int \left(x^2 - x\right)dx = \frac{x^3}{3} - \frac{x^2}{2} + C$

Using $y(3) = \frac{19}{2}$ gives $\frac{19}{2} = \frac{3^3}{3} - \frac{3^2}{2} + C$

$\frac{19}{2} = \frac{9}{2} + C$

Thus, $C = 5$, so $y = \frac{x^3}{3} - \frac{x^2}{2} + 5$.

3. $y' = \frac{5}{\sqrt{x}}$

$y = \int \frac{5}{\sqrt{x}}dx = \int 5x^{-\frac{1}{2}}dx = 5 \cdot \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = 10\sqrt{x} + C$

$y(9) = 50$ implies $50 = 10\sqrt{9} + C$, $50 = 30 + C$,
$C = 20$.
Thus $y = 10\sqrt{x} + 20$.
$y(16) = 10 \cdot 4 + 20 = 60$

4. $y' = -x^2 + 2x$

$y = \int \left(-x^2 + 2x\right)dx = -\frac{x^3}{3} + x^2 + C$

$y(2) = 1$ implies $1 = -\frac{8}{3} + 4 + C$, so $C = -\frac{1}{3}$.

Thus $y = -\frac{x^3}{3} + x^2 - \frac{1}{3}$.

$y(1) = -\frac{1}{3} + 1 - \frac{1}{3} = \frac{1}{3}$

5. $y'' = -3x^2 + 4x$

$y' = \int (-3x^2 + 4x)dx = -x^3 + 2x^2 + C_1$

$y'(1) = 2$ implies $2 = -1 + 2 + C_1$, so $C_1 = 1$.

$y = \int (-x^3 + 2x^2 + 1)dx = -\frac{x^4}{4} + \frac{2x^3}{3} + x + C_2$

$y(1) = 3$ implies $3 = -\frac{1}{4} + \frac{2}{3} + 1 + C_2$, so

$C_2 = \frac{19}{12}$. Thus $y = -\frac{x^4}{4} + \frac{2x^3}{3} + x + \frac{19}{12}$.

6. $y'' = x + 1$

$y' = \int (x+1)dx = \frac{x^2}{2} + x + C_1$

$y'(0) = 0$ implies $0 = 0 + 0 + C_1$, so $C_1 = 0$.

$y = \int \left[\frac{x^2}{2} + x\right]dx = \frac{x^3}{6} + \frac{x^2}{2} + C_2$.

$y(0) = 5$ implies $5 = 0 + 0 + C_2$, so $C_2 = 5$. Thus

$y = \frac{x^3}{6} + \frac{x^2}{2} + 5$.

7. $y''' = 2x$

$y'' = \int 2x\ dx = x^2 + C_1$

$y''(-1) = 3$ implies that $3 = 1 + C_1$, so $C_1 = 2$.

$y' = \int \left(x^2 + 2\right)dx = \frac{x^3}{3} + 2x + C_2$

$y'(3) = 10$ implies $10 = 9 + 6 + C_2$, so $C_2 = -5$.

$y = \int \left(\frac{x^3}{3} + 2x - 5\right)dx = \frac{x^4}{12} + x^2 - 5x + C_3$.

$y(0) = 13$ implies that $13 = 0 + 0 - 0 + C_3$, so

$C_3 = 13$. Therefore $y = \frac{x^4}{12} + x^2 - 5x + 13$.

8. $y''' = e^x + 1$

$y'' = \int \left(e^x + 1\right)dx = e^x + x + C_1$

$y''(0) = 1$ implies $1 = 1 + 0 + C_1$, so $C_1 = 0$.

$y' = \int \left(e^x + x\right)dx = e^x + \frac{x^2}{2} + C_2$

$y'(0) = 2$ implies $2 = 1 + 0 + C_2$, so $C_2 = 1$.

$y = \int \left[e^x + \frac{x^2}{2} + 1\right]dx = e^x + \frac{x^3}{6} + x + C_3$

$y(0) = 3$ implies that $3 = 1 + 0 + 0 + C_3$, so $C_3 = 2$. Thus $y = e^x + \frac{x^3}{6} + x + 2$.

9. $\frac{dr}{dq} = 0.7$

$r = \int 0.7dq = 0.7q + C$

If $q = 0$, r must be 0, so $0 = 0 + C$, $C = 0$. Thus $r = 0.7q$. Since $r = pq$, we have

$p = \frac{r}{q} = \frac{0.7q}{q} = 0.7$. The demand function is $p = 0.7$.

10. $\frac{dr}{dq} = 10 - \frac{1}{16}q$

$r = \int \left[10 - \frac{1}{16}q\right] dq = 10q - \frac{1}{32}q^2 + C$

When $q = 0$, then $r = 0$, so $C = 0$ and $r = 10q - \frac{1}{32}q^2$. Since $r = pq$, then

$p = \frac{r}{q} = 10 - \frac{1}{32}q$. The demand function is

$p = 10 - \frac{1}{32}q$.

11. $\frac{dr}{dq} = 275 - q - 0.3q^2$

Thus $r = \int \left(275 - q - 0.3q^2\right) dq$

$= 275q - 0.5q^2 - 0.1q^3 + C$. When $q = 0$, r must be 0, so $C = 0$ and $r = 275q - 0.5q^2 - 0.1q^3$.

Since $r = pq$, then $p = \frac{r}{q} = 275 - 0.5q - 0.1q^2$.

Thus the demand function is

$p = 275 - 0.5q - 0.1q^2$.

12. $\frac{dr}{dq} = 5000 - 3(2q + 2q^3)$, so

$r = \int (5000 - 6q - 6q^3) dq$

$= 5000q - 3q^2 - \frac{3q^4}{2} + C$

When $q = 0$, then $r = 0$, so $C = 0$ and

$r = 5000q - 3q^2 - \frac{3q^4}{2}$. Since $r = pq$, then

$p = \frac{r}{q} = 5000 - 3q - \frac{3q^3}{2}$. Therefore the demand function is $p = 5000 - 3q - \frac{3q^3}{2}$.

13. $\frac{dc}{dq} = 1.35$

$c = \int 1.35dq = 1.35q + C$

When $q = 0$, then $c = 200$, so $200 = 0 + C$, or $C = 200$. Thus $c = 1.35q + 200$.

14. $\frac{dc}{dq} = 2q + 75$

$c = \int (2q + 75)dq = q^2 + 75q + C$

When $q = 0$, then $c = 2000$, so $C = 2000$. Thus the cost function is $c = q^2 + 75q + 2000$.

15. $\frac{dc}{dq} = 0.08q^2 - 1.6q + 6.5$

$c = \int \left(0.08q^2 - 1.6q + 6.5\right) dq$

$\frac{0.08}{3}q^3 - 0.8q^2 + 6.5q + C$. If $q = 0$, then $c = 8000$, from which $C = 8000$. Hence

$c = \frac{0.08}{3}q^3 - 0.8q^2 + 6.5q + 8000$. If $q = 25$,

substituting gives $c(25) = 8079\frac{1}{6}$ or \$8079.17.

16. $\frac{dc}{dq} = 0.000204q^2 - 0.046q + 6$

$c = \int (0.000204q^2 - 0.046q + 6)dq$

$= 0.000068q^3 - 0.023q^2 + 6q + C$

When $q = 0$, then $c = 15{,}000$, from which $C = 15{,}000$. The cost function is

$c = 0.000068q^3 - 0.023q^2 + 6q + 15{,}000$. When $q = 200$, substitution gives $c(200) = 15{,}824$.

17. $G = \int \left[-\frac{P}{25} + 2\right] dP = -\frac{P^2}{50} + 2P + C$

When $P = 10$, then $G = 38$, so $38 = -2 + 20 + C$, from which $C = 20$. Thus

$G = -\frac{1}{50}P^2 + 2P + 20$.

18. $\frac{dy}{dx} = -1.5 - x$

$y = \int(-1.5 - x)dx = -1.5x - \frac{x^2}{2} + C$

When $x = 1$, then $y = 57.3$, so

$57.3 = -1.5 - 0.5 + C$, or $C = 59.3$. Thus $y = -1.5x - 0.5x^2 + 59.3$.

19. $v = \int -\frac{(P_1 - P_2)r}{2l\eta}dr = -\frac{(P_1 - P_2)r^2}{4l\eta} + C$

Since $v = 0$ when $r = R$, then $0 = -\frac{(P_1 - P_2)R^2}{4l\eta} + C$, so $C = \frac{(P_1 - P_2)R^2}{4l\eta}$. Thus

$v = -\frac{(P_1 - P_2)r^2}{4l\eta} + \frac{(P_1 - P_2)R^2}{4l\eta} = \frac{(P_1 - P_2)(R^2 - r^2)}{4l\eta}$.

20. $\frac{dr}{dq} = 100 - 3q^2$

$r = \int\left(100 - 3q^2\right)dq = 100q - q^3 + C$

When $q = 0$, then $r = 0$, so $C = 0$ and $r = 100q - q^3$. Since $r = pq$, then $p = \frac{r}{q} = 100 - q^2$.

$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{\frac{p}{q}}{-2q} = -\frac{p}{2q^2}$

When $q = 5$, then $p = 75$, so $\eta = \frac{-75}{2(25)} = -\frac{3}{2}$.

21. $\frac{dc}{dq} = 0.003q^2 - 0.4q + 40$

$c = \int\left(0.003q^2 - 0.4q + 40\right)dq = 0.001q^3 - 0.2q^2 + 40q + C$

When $q = 0$, then $c = 5000$, so

$5000 = 0 - 0 + 0 + C$, or $C = 5000$. Thus $c = 0.001q^3 - 0.2q^2 + 40q + 5000$. When $q = 100$, then $c = 8000$. Since Avg. Cost $= \overline{c} = \frac{\text{Total Cost}}{\text{Quantity}} = \frac{c}{q}$, when $q = 100$, we have $\overline{c} = \frac{8000}{100} = \$80$. (Observe that knowing $\frac{dc}{dq} = 27.50$ when $q = 50$ is not relevant to the problem.)

22. $f''(x) = 30x^4 + 12x$

$f'(x) = \int(30x^4 + 12x)dx = 6x^5 + 6x^2 + C_1$

$f'(1) = 10$, so $10 = 6 + 6 + C_1$ and $C_1 = -2$.

$f'(x) = 6x^5 + 6x^2 - 2$

$f(x) = \int(6x^5 + 6x^2 - 2)dx = x^6 + 2x^3 - 2x + C_2$

Thus

$f(965.335245) - f(-965.335245)$

$= [(965.335245)^6 + 2(965.335245)^3 - 2(965.335245) + C_2]$

$- [(-965.335245)^6 + 2(-965.335245)^3 - 2(-965.335245) + C_2]$

$= 3{,}598{,}280{,}000$

Principles in Practice 14.4

1. Using the values given, $\frac{dT}{dt} = -0.5(70-60)e^{-0.5t} = -5e^{-0.5t}$

$T(t) = \int \frac{dT}{dt}dt = \int -5e^{-0.5t}dt = 10e^{-0.5t} + C$

2. The number of words memorized is $v(t)$.

$v(t) = \int \frac{35}{t+1}dt = 35\ln|t+1| + C$.

Problems 14.4

1. Let $u = x+5 \Rightarrow du = 1dx = dx$

$\int (x+5)^7[dx] = \int u^7 du = \frac{u^8}{8} + C = \frac{(x+5)^8}{8} + C$

2. $\int 15(x+2)^4 dx = 15\int (x+2)^4[dx] = 15 \cdot \frac{(x+2)^5}{5} + C = 3(x+2)^5 + C$

3. Let $u = x^2 + 3 \Rightarrow du = 2x\ dx$

$\int 2x\left(x^2+3\right)^5 dx = \int \left(x^2+3\right)^5 [2x\ dx] = \int u^5 du = \frac{u^6}{6} + C$

$= \frac{\left(x^2+3\right)^6}{6} + C$

4. Let $u = x^3 + 5x^2 + 6 \Rightarrow du = (3x^2 + 10x)dx$.

$\int \left(3x^2+10x\right)\left(x^3+5x^2+6\right)dx$

$= \int \left(x^3+5x^2+6\right)^1 \left[\left(3x^2+10x\right)dx\right]$

$= \int u\ du = \frac{u^2}{2} + C$

$= \frac{\left(x^3+5x^2+6\right)^2}{2} + C$

5. Let $u = y^3 + 3y^2 + 1 \Rightarrow du = \left(3y^2 + 6y\right)dy$

$$\int\left(3y^2+6y\right)\left(y^3+3y^2+1\right)^{\frac{2}{3}}dy$$
$$=\int\left(y^3+3y^2+1\right)^{\frac{2}{3}}\left[\left(3y^2+6y\right)dy\right]$$
$$=\int u^{\frac{2}{3}}du=\frac{u^{\frac{5}{3}}}{\frac{5}{3}}+C$$
$$=\frac{3}{5}\left(y^3+3y^2+1\right)^{\frac{5}{3}}+C$$

6. $$\int(15t^2-6t+1)(5t^3-3t^2+t)^{17}dt$$
$$=\int(5t^3-3t^2+t)^{17}[(15t^2-6t+1)dt]$$
$$=\frac{(5t^3-3t^2+t)^{18}}{18}+C$$

7. Let $u = 3x-1 \Rightarrow du = 3\ dx$

$$\int\frac{5}{(3x-1)^3}dx=\frac{5}{3}\int\frac{1}{(3x-1)^3}[3\ dx]$$
$$=\frac{5}{3}\int\frac{1}{u^3}du=\frac{5}{3}\int u^{-3}du$$
$$=\frac{5}{3}\cdot\frac{u^{-2}}{-2}+C=-\frac{5(3x-1)^{-2}}{6}+C$$

8. $$\int\frac{4x}{\left(2x^2-7\right)^{10}}dx=\int\left(2x^2-7\right)^{-10}[4x\ dx]$$
$$=-\frac{\left(2x^2-7\right)^{-9}}{9}+C$$

9. Let $u = 2x-1 \Rightarrow du = 2\ dx$.

$$\int\sqrt{2x-1}dx=\int(2x-1)^{\frac{1}{2}}dx$$
$$=\frac{1}{2}\int(2x-1)^{\frac{1}{2}}[2\ dx]$$
$$=\frac{1}{2}\int u^{\frac{1}{2}}du=\frac{1}{2}\cdot\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+C=\frac{1}{3}(2x-1)^{\frac{3}{2}}+C$$

10. Let $u = x-5 \Rightarrow du = dx$.

$$\int\frac{1}{\sqrt{x-5}}dx=\int(x-5)^{-\frac{1}{2}}[dx]$$
$$\int u^{-1/2}du=\frac{u^{1/2}}{\frac{1}{2}}+C$$
$$=\frac{(x-5)^{\frac{1}{2}}}{\frac{1}{2}}+C=2\sqrt{x-5}+C$$

11. Let $u = 7x-6 \Rightarrow du = 7\ dx$

$$\int(7x-6)^4dx=\frac{1}{7}\int(7x-6)^4[7\ dx]$$
$$=\frac{1}{7}\int u^4du=\frac{1}{7}\cdot\frac{u^5}{5}+C$$
$$=\frac{(7x-6)^5}{35}+C$$

12. $$\int x^2\left(3x^3+7\right)^3dx=\frac{1}{9}\int\left(3x^3+7\right)^3\left[9x^2dx\right]$$
$$=\frac{1}{9}\cdot\frac{\left(3x^3+7\right)^4}{4}+C$$
$$=\frac{\left(3x^3+7\right)^4}{36}+C$$

13. Let $v = 5u^2-9 \Rightarrow dv = 10u\,du$

$$\int u(5u^2-9)^{14}du=\frac{1}{10}\int(5u^2-9)^{14}[10u\,du]$$
$$\frac{1}{10}\int v^{14}dv=\frac{1}{10}\cdot\frac{v^{15}}{15}+C=\frac{(5u^2-9)^{15}}{150}+C$$

14. $$\int 9x\sqrt{1+2x^2}dx=\frac{9}{4}\int\left(1+2x^2\right)^{\frac{1}{2}}[4x\ dx]$$
$$=\frac{9}{4}\cdot\frac{\left(1+2x^2\right)^{\frac{3}{2}}}{\frac{3}{2}}+C$$
$$=\frac{3\left(1+2x^2\right)^{\frac{3}{2}}}{2}+C$$

15. Let $u = 27 + x^5 \Rightarrow du = 5x^4 dx$

$$\int 4x^4\left(27+x^5\right)^{\frac{1}{3}} dx = \frac{4}{5}\int \left(27+x^5\right)^{\frac{1}{3}}\left[5x^4 dx\right]$$

$$= \frac{4}{5}\int u^{\frac{1}{3}} du = \frac{4}{5}\cdot\frac{u^{\frac{4}{3}}}{\frac{4}{3}} + C$$

$$= \frac{3}{5}\left(27+x^5\right)^{\frac{4}{3}} + C$$

16. Let $u = 4 - 5x \Rightarrow du = -5dx$.

$$\int (4-5x)^9 dx = -\frac{1}{5}\int (4-5x)^9[-5\ dx]$$

$$= -\frac{1}{5}\int u^9 du = -\frac{1}{5}\cdot\frac{u^{10}}{10} + C = -\frac{1}{50}(4-5x)^{10} + C$$

17. Let $u = 3x \Rightarrow du = 3\ dx$

$$\int 3e^{3x} dx = \int e^{3x}[3\ dx]$$

$$= \int e^u du = e^u + C = e^{3x} + C$$

18. $\int 5e^{3t+7} dt = \frac{5}{3}\int e^{3t+7}[3\,dt] = \frac{5}{3}e^{3t+7} + C$

19. Let $u = t^2 + t \Rightarrow du = (2t+1)dt$

$$\int (2t+1)e^{t^2+t} dt = \int e^{t^2+t}[(2t+1)\ dt]$$

$$= \int e^u du = e^u + C = e^{t^2+t} + C$$

20. $\int -3w^2 e^{-w^3} dw = \int e^{-w^3}\left[-3w^2 dw\right] = e^{-w^3} + C$

21. Let $u = 7x^2 \Rightarrow du = 14x\,dx$

$$\int xe^{7x^2} dx = \frac{1}{14}\int e^{7x^2}[14x\ dx] = \frac{1}{14}\int e^u du$$

$$= \frac{1}{14}e^u + C = \frac{1}{14}e^{7x^2} + C$$

22. $\int x^3 e^{4x^4} dx = \frac{1}{16}\int e^{4x^4}\left[16x^3 dx\right]$

$$= \frac{1}{16}\cdot e^{4x^4} + C = \frac{e^{4x^4}}{16} + C$$

23. Let $u = -3x \Rightarrow du = -3dx$.

$$\int 4e^{-3x} dx = -\frac{4}{3}\int e^{-3x}[-3\ dx]$$

$$= -\frac{4}{3}\int e^u du = -\frac{4}{3}e^u + C = -\frac{4}{3}e^{-3x} + C$$

24. $\int x^4 e^{-6x^5} dx = -\frac{1}{30}\int e^{-6x^5}\left[-30x^4 dx\right]$

$$= -\frac{1}{30}e^{-6x^5} + C$$

25. Let $u = x + 5 \Rightarrow du = dx$

$$\int \frac{1}{x+5}[dx] = \int \frac{1}{u} du = \ln|u| + C = \ln|x+5| + C$$

26. $\int \frac{12x^2+4x+2}{x+x^2+2x^3} dx$

$$= \int \frac{2}{x+x^2+2x^3}[(1+2x+6x^2)dx]$$

$$= 2\ln\left|x+x^2+2x^3\right| + C$$

$$= \ln[(x+x^2+2x^3)^2] + C$$

27. Let $u = x^3 + x^4 \Rightarrow du = (3x^2 + 4x^3)dx$

$$\int \frac{3x^2+4x^3}{x^3+x^4} dx = \int \frac{1}{x^3+x^4}\left[\left(3x^2+4x^3\right)dx\right]$$

$$= \int \frac{1}{u} du = \ln|u| + C$$

$$= \ln\left|x^3+x^4\right| + C$$

28. Let $u = 1 - 3x^2 + 2x^3 \Rightarrow du = (-6x + 6x^2)dx.$

$$\int \frac{6x^2-6x}{1-3x^2+2x^3} dx$$

$$= \int \frac{1}{1-3x^2+2x^3}[(-6x+6x^2)dx$$

$$= \int \frac{1}{u} du = \ln|u| + C = \ln\left|1-3x^2+2x^3\right| + C$$

29. Let $u = z^2 - 6 \Rightarrow du = 2z\,dz$

$$\int \frac{6z}{(z^2-6)^5} = 3\int (z^2-6)^{-5}[2z\,dz]$$

$$= 3\int u^{-5} du = 3\frac{u^{-4}}{-4} + C = -\frac{3}{4}(z^2-6)^{-4} + C$$

30. $\int \frac{3}{(5v-1)^4} dv = \frac{3}{5}\int (5v-1)^{-4}[5\,dv]$

$$= \frac{3}{5}\cdot\frac{(5v-1)^{-3}}{-3} + C$$

$$= -\frac{1}{5}(5v-1)^{-3} + C$$

31. $\int \frac{4}{x}dx = 4\int \frac{1}{x}dx = 4\ln|x| + C$

32. $\int \frac{3}{1+2y}dy = 3\cdot\frac{1}{2}\int \frac{1}{1+2y}[2\ dy]$
$= \frac{3}{2}\ln|1+2y| + C$

33. Let $u = s^3 + 5 \Rightarrow du = 3s^2 ds$
$\int \frac{s^2}{s^3+5}ds = \frac{1}{3}\int \frac{1}{s^3+5}\left[3s^2 ds\right]$
$= \frac{1}{3}\int \frac{1}{u}du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln\left|s^3+5\right| + C$

34. $\int \frac{2x^2}{3-4x^3}dx = 2\left(-\frac{1}{12}\right)\int \frac{1}{3-4x^3}\left[-12x^2 dx\right]$
$= -\frac{1}{6}\ln\left|3-4x^3\right| + C$

35. Let $u = 4 - 2x \Rightarrow du = -2\ dx$
$\int \frac{5}{4-2x}dx = -\frac{5}{2}\int \frac{1}{4-2x}[-2\,dx]$
$= -\frac{5}{2}\int \frac{1}{u}du = -\frac{5}{2}\ln|u| + C = -\frac{5}{2}\ln|4-2x| + C$

36. $\int \frac{7t}{5t^2-6}dt = 7\cdot\frac{1}{10}\int \frac{1}{5t^2-6}[10t\ dt]$
$= \frac{7}{10}\ln\left|5t^2-6\right| + C$

37. $\int \sqrt{5x}\,dx = \sqrt{5}\int x^{1/2}dx = \sqrt{5}\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$
$= \frac{2\sqrt{5}}{3}x^{\frac{3}{2}} + C$

38. $\int \frac{1}{(3x)^6}dx = \frac{1}{3}\int (3x)^{-6}[3\,dx]$
$= \frac{1}{3}\cdot\frac{(3x)^{-5}}{-5} + C$
$= -\frac{1}{15}(3x)^{-5} + C$

39. Let $u = x^2 - 4 \Rightarrow du = 2x\,dx$
$\int \frac{x}{\sqrt{x^2-4}}dx = \frac{1}{2}\int \left(x^2-4\right)^{-\frac{1}{2}}[2x\ dx]$
$= \frac{1}{2}\int u^{-\frac{1}{2}}du\ = \frac{1}{2}\cdot\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$
$= \sqrt{x^2-4} + C$

40. Let $u = 1 - 3x \Rightarrow du = -3\ dx$.
$\int \frac{9}{1-3x}dx = -3\int \frac{1}{1-3x}[-3\,dx]$
$= -3\int \frac{1}{u}du = -3\ln|u| + C = -3\ln|1-3x| + C$

41. Let $u = y^4 + 1 \Rightarrow du = 4y^3 dy$
$\int 2y^3 e^{y^4+1}dy = 2\int y^3 e^{y^4+1}dy$
$= 2\cdot\frac{1}{4}\int e^{y^4+1}\left[4y^3 dy\right]$
$= \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C$
$= \frac{1}{2}e^{y^4+1} + C$

42. $\int 2\sqrt{2x-1}dx = \int (2x-1)^{\frac{1}{2}}[2\,dx]$
$= \frac{(2x-1)^{\frac{3}{2}}}{\frac{3}{2}} + C$
$= \frac{2}{3}(2x-1)^{\frac{3}{2}} + C$

43. Let $u = -2v^3 + 1 \Rightarrow du = -6v^2 dv$
$\int v^2 e^{-2v^3+1}dv = -\frac{1}{6}\int e^{-2v^3+1}\left[-6v^2 dv\right]$
$= -\frac{1}{6}\int e^u du = -\frac{1}{6}e^u + C$
$= -\frac{1}{6}e^{-2v^3+1} + C$

44. $\int \frac{x^2}{\sqrt[3]{2x^3+9}}dx = \frac{1}{6}\int \left(2x^3+9\right)^{-\frac{1}{3}}\left[6x^2dx\right]$

$= \frac{1}{6}\cdot\frac{\left(2x^3+9\right)^{\frac{2}{3}}}{\frac{2}{3}}+C$

$= \frac{1}{4}\left(2x^3+9\right)^{\frac{2}{3}}+C$

45. $\int\left(e^{-5x}+2e^x\right)dx = \int e^{-5x}dx + 2\int e^x dx$

$= -\frac{1}{5}\int e^{-5x}[-5\ dx]+2\int e^x dx$

$= -\frac{1}{5}e^{-5x}+2e^x+C$

46. $\int 4\sqrt[3]{y+1}dy = 4\int (y+1)^{\frac{1}{3}}[dy]$

$= 4\cdot\frac{(y+1)^{\frac{4}{3}}}{\frac{4}{3}}+C = 3(y+1)^{\frac{4}{3}}+C$

47. $\int (8x+10)(7-2x^2-5x)^3dx$

$= -2\int (7-2x^2-5x)^3[(-4x-5)dx]$

$= -2\cdot\frac{(7-2x^2-5x)^4}{4}+C$

$= -\frac{1}{2}(7-2x^2-5x)^4+C$

48. $\int 2ye^{3y^2}dy = 2\cdot\frac{1}{6}\int e^{3y^2}[6y\ dy] = \frac{1}{3}e^{3y^2}+C$

49. $\int \frac{x^2+2}{x^3+6x}dx = \frac{1}{3}\int \frac{1}{x^3+6x}\left[\left(3x^2+6\right)dx\right]$

$= \frac{1}{3}\ln\left|x^3+6x\right|+C$

50. $\int (e^x+2e^{-3x}-e^{5x})dx$

$= \int e^x\ dx - \frac{2}{3}\int e^{-3x}[(-3)dx] - \frac{1}{5}\int e^{5x}[5\ dx]$

$= e^x - \frac{2}{3}e^{-3x} - \frac{1}{5}e^{5x}+C$

51. $\int \frac{16s-4}{3-2s+4s^2}dx = 2\int \frac{1}{3-2s+4s^2}[(8s-2)ds]$

$= 2\ln\left|3-2s+4s^2\right|+C$

52. $\int (6t^2+4t)(t^3+t^2+1)^6dt$

$= 2\int (t^3+t^2+1)^6[(3t^2+2t)dt]$

$= 2\cdot\frac{(t^3+t^2+1)^7}{7}+C$

$= \frac{2}{7}(t^3+t^2+1)^7+C$

53. $\int x\left(2x^2+1\right)^{-1}dx = \int \frac{x}{2x^2+1}dx$

$= \frac{1}{4}\int \frac{1}{2x^2+1}[4x\ dx]$

$= \frac{1}{4}\ln\left(2x^2+1\right)+C$

54. $\int (8w^5+w^2-2)(6w-w^3-4w^6)^{-4}dw$

$= -\frac{1}{3}\int (6w-w^3-4w^6)^{-4}[(6-3w^2-24w^5)dw]$

$= -\frac{1}{3}\cdot\frac{(6w-w^3-4w^6)^{-3}}{-3}+C$

$= \frac{1}{9}(6w-w^3-4w^6)^{-3}+C$

55. $\int -\left(x^2-2x^5\right)\left(x^3-x^6\right)^{-10}dx$

$= -\frac{1}{3}\int\left(x^3-x^6\right)^{-10}\left[\left(3x^2-6x^5\right)dx\right]$

$= -\frac{1}{3}\cdot\frac{\left(x^3-x^6\right)^{-9}}{-9}+C = \frac{1}{27}\left(x^3-x^6\right)^{-9}+C$

56. $\int \frac{3}{5}(v-2)e^{2-4v+v^2}dv$

$= \frac{3}{5}\cdot\frac{1}{2}\int e^{2-4v+v^2}[(2v-4)\ dv]$

$= \frac{3}{10}e^{2-4v+v^2}+C$

57. $\int\left(2x^3+x\right)\left(x^4+x^2\right)dx$

$= \frac{1}{2}\int\left(x^4+x^2\right)^1\left[\left(4x^3+2x\right)dx\right]$

$= \frac{1}{2}\cdot\frac{\left(x^4+x^2\right)^2}{2}+C = \frac{1}{4}\left(x^4+x^2\right)^2+C$

58. $\int \left(e^{3.1}\right)^2 dx = \int e^{6.2} dx = e^{6.2}x + C$, because $e^{6.2}$ is a constant.

59. $\int \frac{7+14x}{(4-x-x^2)^5} dx$

$= -7\int (4-x-x^2)^{-5}[(-1-2x)dx]$

$= -7\frac{(4-x-x^2)^{-4}}{-4} + C$

$= \frac{7}{4}(4-x-x^2)^{-4} + C$

60. $\int \left(e^x - e^{-x}\right)^2 dx = \int \left(e^{2x} - 2 + e^{-2x}\right) dx$

$= \frac{1}{2}\int e^{2x}[2\ dx] - \int 2\ dx + \left(-\frac{1}{2}\right)\int e^{-2x}[-2\ dx]$

$= \frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x} + C$

$= \frac{1}{2}\left(e^{2x} - e^{-2x}\right) - 2x + C$

61. $u = 4x^3 + 3x^2 - 4$

$du = \left(12x^2 + 6x\right)dx = 6x(2x+1)dx$

$\int x(2x+1)e^{4x^3+3x^2-4}dx$

$= \frac{1}{6}\int e^{4x^3+3x^2-4}[6x(2x+1)dx]$

$= \frac{1}{6}\int e^u du = \frac{1}{6}e^u + C = \frac{1}{6}e^{4x^3+3x^2-4} + C$

62. $\int (u^3 - ue^{6-3u^2})du = \frac{u^4}{4} + \frac{1}{6}\int e^{6-3u^2}[-6u\,du]$

$= \frac{1}{4}u^4 + \frac{1}{6}e^{6-3u^2} + C$

63. $\int x\sqrt{\left(8-5x^2\right)^3}dx = -\frac{1}{10}\int \left(8-5x^2\right)^{\frac{3}{2}}[-10x\ dx]$

$= -\frac{1}{10}\cdot\frac{\left(8-5x^2\right)^{\frac{5}{2}}}{\frac{5}{2}} + C = -\frac{1}{25}\left(8-5x^2\right)^{\frac{5}{2}} + C$

64. $\int e^{-\frac{x}{7}}dx = -7\int e^{-\frac{x}{7}}\left[-\frac{1}{7}dx\right] = -7e^{-\frac{x}{7}} + C$

65. $\int \left(\sqrt{2x} - \frac{1}{\sqrt{2x}}\right)dx = \int \sqrt{2x}dx - \int \frac{1}{\sqrt{2x}}dx$

$= \frac{1}{2}\int (2x)^{\frac{1}{2}}[2\ dx] - \frac{1}{2}\int (2x)^{-\frac{1}{2}}[2\ dx]$

$= \frac{1}{2}\cdot\frac{(2x)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{2}\cdot\frac{(2x)^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{(2x)^{\frac{3}{2}}}{3} - \sqrt{2x} + C$

$= \frac{2\sqrt{2}}{3}x^{\frac{3}{2}} - \sqrt{2}x^{\frac{1}{2}} + C$

66. $\int 3\frac{x^4}{e^{x^5}}dx = 3\int x^4 e^{-x^5}dx = -\frac{3}{5}\int e^{-x^5}[-5x^4\ dx]$

$= -\frac{3}{5}e^{-x^5} + C$

67. $\int \left(x^2+1\right)^2 dx = \int \left(x^4 + 2x^2 + 1\right)dx$

$= \frac{x^5}{5} + \frac{2x^3}{3} + x + C$

68. $\int \left[x\left(x^2-16\right)^2 - \frac{1}{2x+5}\right]dx$

$= \frac{1}{2}\int \left(x^2-16\right)^2[2x\ dx] - \frac{1}{2}\int \frac{1}{2x+5}[2\ dx]$

$= \frac{1}{2}\cdot\frac{\left(x^2-16\right)^3}{3} - \frac{1}{2}\ln|2x+5| + C$

$= \frac{1}{6}\left(x^2-16\right)^3 - \frac{1}{2}\ln|2x+5| + C$

69. $\int \left[\frac{x}{x^2+1} + \frac{x^5}{(x^6+1)^2}\right]dx$

$= \int \frac{x}{x^2+1}dx + \int \frac{x^5}{\left(x^6+1\right)^2}dx$

$= \frac{1}{2}\int \frac{1}{x^2+1}[2x\ dx] + \frac{1}{6}\int \left(x^6+1\right)^{-2}\left[6x^5dx\right]$

$= \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{6}\cdot\frac{\left(x^6+1\right)^{-1}}{-1} + C$

$= \frac{1}{2}\ln\left(x^2+1\right) - \frac{1}{6\left(x^6+1\right)} + C$

70. $\int\left[\frac{3}{x-1}+\frac{1}{(x-1)^2}\right]dx = \int\frac{3}{x-1}[dx]+\int(x-1)^{-2}[dx]$

$= 3\ln|x-1|+\frac{(x-1)^{-1}}{-1}+C = 3\ln|x-1|-\frac{1}{x-1}+C$

71. $\int\left[\frac{2}{4x+1}-(4x^2-8x^5)(x^3-x^6)^{-8}\right]dx$

$=\frac{1}{2}\int\frac{1}{4x+1}[4\,dx]-\frac{4}{3}\int(x^3-x^6)^{-8}[(3x^2-6x^5)dx]$

$=\frac{1}{2}\ln|4x+1|-\frac{4}{3}\cdot\frac{(x^3-x^6)^{-7}}{-7}+C$

$=\frac{1}{2}\ln|4x+1|+\frac{4}{21}(x^3-x^6)^{-7}+C$

72. $\int\left(r^3+5\right)^2 dr = \int\left(r^6+10r^3+25\right)dr = \frac{1}{7}r^7+\frac{5}{2}r^4+25r+C$

73. $\int\left[\sqrt{3x+1}-\frac{x}{x^2+3}\right]dx = \int(3x+1)^{\frac{1}{2}}dx-\int\frac{x}{x^2+3}dx = \frac{1}{3}\int(3x+1)^{\frac{1}{2}}[3\ dx]-\frac{1}{2}\int\frac{1}{x^2+3}[2x\ dx]$

$=\frac{1}{3}\cdot\frac{(3x+1)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{1}{2}\ln\left(x^2+3\right)+C = \frac{2}{9}(3x+1)^{\frac{3}{2}}-\ln\sqrt{x^2+3}+C$

74. $\int\left[\frac{x}{3x^2+5}-\frac{x^2}{(x^3+1)^3}\right]dx = \frac{1}{6}\int\frac{1}{3x^2+5}[6x\,dx]-\frac{1}{3}\int(x^3+1)^{-3}[3x^2\ dx]$

$=\frac{1}{6}\ln\left|3x^2+5\right|-\frac{1}{3}\cdot\frac{(x^3+1)^{-2}}{-2}+C = \frac{1}{6}\ln\left|3x^2+5\right|+\frac{1}{6}(x^3+1)^{-2}+C$

75. Let $u=\sqrt{x} \Rightarrow du = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{1}{2\sqrt{x}}dx$.

$\int\frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\int e^{\sqrt{x}}\left[\frac{1}{2\sqrt{x}}dx\right]$

$= 2\int e^u du = 2e^u+C = 2e^{\sqrt{x}}+C$

76. $\int\left(e^5-3^e\right)dx = \left(e^5-3^e\right)x+C,$ because e^5-3^e is a constant.

77. $\int\frac{1+e^{2x}}{4e^x}dx = \frac{1}{4}\int\left(\frac{1}{e^x}+\frac{e^{2x}}{e^x}\right)dx$

$=\frac{1}{4}\int\left(e^{-x}+e^x\right)dx$

$=-\frac{1}{4}\int e^{-x}[-1\ dx]+\frac{1}{4}\int e^x dx$

$=-\frac{1}{4}e^{-x}+\frac{1}{4}e^x+C$

78. $\int \frac{2}{t^2}\sqrt{\frac{1}{t}+9}\,dt = -2\int\left(\frac{1}{t}+9\right)^{\frac{1}{2}}\left[-\frac{1}{t^2}dt\right]$

$= -2\frac{\left(\frac{1}{t}+9\right)^{\frac{3}{2}}}{\frac{3}{2}}+C$

$= -\frac{4}{3}\left(\frac{1}{t}+9\right)^{\frac{3}{2}}+C$

79. Let $u = \ln\left(x^2+2x\right) \Rightarrow du = \frac{1}{x^2+2x}(2x+2)dx$

$\int \frac{x+1}{x^2+2x}\ln\left(x^2+2x\right)dx$

$= \frac{1}{2}\int \ln\left(x^2+2x\right)\left[\frac{2x+2}{x^2+2x}dx\right]$

$= \frac{1}{2}\int u\ du = \frac{1}{2}\cdot\frac{u^2}{2}+C = \frac{1}{4}\ln^2\left(x^2+2x\right)+C$

80. Let $u = \sqrt[3]{8x^4} = 2x^{\frac{4}{3}} \Rightarrow du = \frac{8}{3}x^{\frac{1}{3}}dx$

$\int \sqrt[3]{x}e^{\sqrt[3]{8x^4}}\,dx = \frac{3}{8}\int e^{2x^{\frac{4}{3}}}\left[\frac{8}{3}x^{\frac{1}{3}}dx\right] = \frac{3}{8}\int e^u du$

$= \frac{3}{8}e^u + C = \frac{3}{8}e^{\sqrt[3]{8x^4}} + C$

81. $y = \int (3-2x)^2 dx = -\frac{1}{2}\int (3-2x)^2[-2\ dx]$

$= -\frac{1}{2}\cdot\frac{(3-2x)^3}{3}+C = -\frac{1}{6}(3-2x)^3+C$

$y(0) = 1$ implies $1 = -\frac{1}{6}(27)+C$, so $C = \frac{11}{2}$.

Thus $y = -\frac{1}{6}(3-2x)^3+\frac{11}{2}$.

82. $y = \frac{1}{2}\int \frac{1}{x^2+6}[2x\ dx] = \frac{1}{2}\ln\left(x^2+6\right)+C$

$y(1) = 0$ implies $0 = \frac{1}{2}\ln(7)+C$, so $C = -\frac{1}{2}\ln 7$.

Thus $y = \frac{1}{2}\left[\ln\left(x^2+6\right)-\ln 7\right]$, or

$y = \ln\sqrt{\frac{x^2+6}{7}}$

83. $y'' = \frac{1}{x^2}$

$y' = \int x^{-2}dx = -x^{-1}+C_1$

$y'(-2) = 3$ implies $3 = \frac{1}{2}+C_1$, so $C_1 = \frac{5}{2}$. Thus

$y' = -x^{-1}+\frac{5}{2}$.

$y = \int\left(-x^{-1}+\frac{5}{2}\right)dx = -\int \frac{1}{x}dx + \int \frac{5}{2}dx$

$= -\ln|x| + \frac{5}{2}x + C_2$

$y(1) = 2$ implies that $2 = 0+\frac{5}{2}+C_2$, so

$C_2 = -\frac{1}{2}$. Thus

$y = -\ln|x|+\frac{5}{2}x-\frac{1}{2} = \ln\left|\frac{1}{x}\right|+\frac{5}{2}x-\frac{1}{2}$.

84. $y'' = (x+1)^{3/2}$

$y' = \int (x+1)^{\frac{3}{2}}dx = \frac{2}{5}(x+1)^{\frac{5}{2}}+C_1$

$y'(3) = 0 \Rightarrow 0 = \frac{2}{5}\cdot 32 + C_1 \Rightarrow C_1 = -\frac{64}{5}$, so

$y' = \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{64}{5}$

$y = \int\left[\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{64}{5}\right]dx$

$= \frac{2}{5}\cdot\frac{(x+1)^{\frac{7}{2}}}{\frac{7}{2}} - \frac{64}{5}x + C_2$

$= \frac{4}{35}(x+1)^{\frac{7}{2}} - \frac{64}{5}x + C_2$

$y(3) = 0$ implies $0 = \frac{4}{35}\cdot 128 - \frac{64}{5}(3) + C_2$, so

$C_2 = \frac{832}{35}$. Thus $y = \frac{4}{35}(x+1)^{\frac{7}{2}} - \frac{64}{5}x + \frac{832}{35}$.

85. $V(t)=\int \frac{dV}{dt}dt=\int 8e^{0.05t}dt$

$=\frac{8}{0.05}\int e^{0.05t}[0.05\,dt]$

$=160e^{0.05t}+C$

The house cost $350,000 to build, so $V(0) = 350$.

$350=160e^0+C=160+C$

$190 = C$

$V(t)=160e^{0.05t}+190$

86. $l(t)=\int \frac{dl}{dt}dt=\int \frac{12}{2t+50}dt$

$=6\ln|2t+50|+C$

Since the expected life span was 63 years in 1940, $l(0) = 63$.

$63=6\ln|50|+C$

$C = 63 - 6\ln 50 \approx 39.53$

$l(t)=6\ln|2t+50|+39.53$

$l(58)=6\ln|166|+39.53\approx 70.20$

The expected life span for people born in 1998 (58 years after 1940) is about 70 years.

87. Note that $r > 0$.

$C=\int\left[\frac{Rr}{2K}+\frac{B_1}{r}\right]dr=\int \frac{Rr}{2K}dr+\int \frac{B_1}{r}dr$

$=\frac{R}{2K}\int r\ dr+B_1\int \frac{1}{r}dr$

$=\frac{R}{2K}\cdot\frac{r^2}{2}+B_1\ln|r|+B_2$

Thus we obtain $C=\frac{Rr^2}{4K}+B_1\ln|r|+B_2$.

88. $f(x)=\int (e^{3x+2}-3x)dx=\frac{1}{3}e^{3x+2}-\frac{3}{2}x^2+C$

$f\left(\frac{1}{3}\right)=2$ implies $2=\frac{1}{3}e^3-\frac{1}{6}+C,$ so

$C=\frac{13}{6}-\frac{1}{3}e^3.$ Thus,

$f(x)=\frac{1}{3}e^{3x+2}-\frac{3}{2}x^2+\frac{13}{6}-\frac{1}{3}e^3,$

$f(2)=\frac{1}{3}e^8-6+\frac{13}{6}-\frac{1}{3}e^3$

$=\frac{1}{6}(2e^8-2e^3-23)\approx 983.12$

Problems 14.5

1. $\int \frac{2x^6+8x^4-4x}{2x^2}dx$

$=\int\left(\frac{2x^6}{2x^2}+\frac{8x^4}{2x^2}-\frac{4x}{2x^2}\right)dx$

$=\int x^4\,dx+4\int x^2\,dx-2\int \frac{1}{x}dx$

$=\frac{x^5}{5}+\frac{4}{3}x^3-2\ln|x|+C$

2. $\int \frac{9x^2+5}{3x}dx=\int\left(3x+\frac{5}{3x}\right)dx$

$=\frac{3}{2}x^2+\frac{5}{3}\ln|x|+C$

3. $\int\left(3x^2+2\right)\sqrt{2x^3+4x+1}dx$

$=\frac{1}{2}\int\left(2x^3+4x+1\right)^{\frac{1}{2}}\left[\left(6x^2+4\right)dx\right]$

$=\frac{1}{2}\cdot\frac{\left(2x^3+4x+1\right)^{\frac{3}{2}}}{\frac{3}{2}}+C$

$=\frac{1}{3}\left(2x^3+4x+1\right)^{\frac{3}{2}}+C$

4. $\int \frac{x}{\sqrt[4]{x^2+1}}dx=\frac{1}{2}\int\left(x^2+1\right)^{-\frac{1}{4}}[2x\ dx]$

$=\frac{1}{2}\cdot\frac{\left(x^2+1\right)^{\frac{3}{4}}}{\frac{3}{4}}+C$

$=\frac{2}{3}\left(x^2+1\right)^{\frac{3}{4}}+C$

5. $\int \frac{9}{\sqrt{2-3x}}dx=9\int (2-3x)^{-1/2}dx$

$=9\left(-\frac{1}{3}\right)\int (2-3x)^{-1/2}[-3\,dx]$

$=-3\frac{(2-3x)^{1/2}}{\frac{1}{2}}+C=-6\sqrt{2-3x}+C$

6. $\int \frac{2xe^{x^2}}{e^{x^2}-2}dx=\int \frac{1}{e^{x^2}-2}\left[2xe^{x^2}dx\right]$

$=\ln\left|e^{x^2}-2\right|+C$

7. $\int 4^{7x}\,dx = \int \left(e^{\ln 4}\right)^{7x} dx = \int e^{(\ln 4)(7x)} dx$

$= \frac{1}{7\ln 4}\int e^{(\ln 4)(7x)}[7\ln 4\ dx]$

$= \frac{1}{7\ln 4}\cdot e^{(\ln 4)(7x)} + C$

$= \frac{1}{7\ln 4}\left(e^{\ln 4}\right)^{7x} + C = \frac{4^{7x}}{7\ln 4} + C$

8. $\int 5^t\,dt = \int (e^{\ln 5})^t\,dt = \int e^{(\ln 5)t}\,dt$

$= \frac{1}{\ln 5}\int e^{(\ln 5)t}[\ln 5\,dt] = \frac{1}{\ln 5}\cdot e^{(\ln 5)t} + C$

$= \frac{5^t}{\ln 5} + C$

9. $\int 2x\left(7 - e^{\frac{x^2}{4}}\right)dx = \int\left(14x - 2xe^{\frac{x^2}{4}}\right)dx$

$= 14\int x\ dx - 2\int xe^{\frac{x^2}{4}}\,dx$

$= 14\int x\ dx - 2\cdot 2\int e^{\frac{x^2}{4}}\left[\frac{1}{2}x\ dx\right]$

$= 14\cdot\frac{x^2}{2} - 4\cdot e^{\frac{x^2}{4}} + C = 7x^2 - 4e^{\frac{x^2}{4}} + C$

10. $\int\left(e^x + x^e + ex + \frac{e}{x}\right)dx$

$= \int e^x dx + \int x^e dx + e\int x\ dx + e\int \frac{1}{x}dx$

$= e^x + \frac{x^{e+1}}{e+1} + \frac{ex^2}{2} + e\ln|x| + C$

11. By long division, $\frac{6x^2 - 11x + 5}{3x-1} = 2x - 3 + \frac{2}{3x-1}$.

Thus $\int \frac{6x^2 - 11x + 5}{3x-1}dx = \int\left(2x - 3 + \frac{2}{3x-1}\right)dx$

$= 2\int x\ dx - \int 3\ dx + 2\cdot\frac{1}{3}\int\frac{1}{3x-1}[3\ dx]$

$= x^2 - 3x + \frac{2}{3}\ln|3x-1| + C$

12. $\int \frac{(3x+2)(x-4)}{x-3}dx = \int\frac{3x^2 - 10x - 8}{x-3}dx$

$= \int\left(3x - 1 - \frac{11}{x-3}\right)dx = \frac{3}{2}x^2 - x - 11\ln|x-3| + C$

13. $\int \frac{5e^{2x}}{7e^{2x}+4}dx = \frac{5}{14}\int\frac{1}{7e^{2x}+4}[7e^{2x}(2)dx]$

$= \frac{5}{14}\ln(7e^{2x}+4) + C$

14. $\int 6\left(e^{4-3x}\right)^2 dx = -\int e^{8-6x}[-6\ dx]$

$= -e^{8-6x} + C = -\left(e^{4-3x}\right)^2 + C$

15. $\int \frac{e^{\frac{7}{x}}}{x^2}dx = \int e^{\frac{7}{x}}\cdot\frac{1}{x^2}dx = -\frac{1}{7}\int e^{\frac{7}{x}}\left[-\frac{7}{x^2}dx\right]$

$= -\frac{1}{7}e^{\frac{7}{x}} + C$

16. By using long division on the integrand,

$\int \frac{2x^4 - 6x^3 + x - 2}{x-2}dx$

$= \int\left(2x^3 - 2x^2 - 4x - 7 - \frac{16}{x-2}\right)dx$

$= \frac{1}{2}x^4 - \frac{2}{3}x^3 - 2x^2 - 7x - 16\ln|x-2| + C$.

17. By using long division on the integrand,

$\int\frac{5x^3}{x^2+9}dx = \int\left(5x - \frac{45x}{x^2+9}\right)dx$

$= \int 5x\ dx - \frac{45}{2}\int\frac{1}{x^2+9}[2x\ dx]$

$= \frac{5}{2}x^2 - \frac{45}{2}\ln(x^2+9) + C$

Note that since $x^2 + 9 > 0$ for all values of x, the absolute value bars are not needed.

18. By using long division on the integrand,

$\int\frac{5-4x^2}{3+2x}dx = \int\left(-2x + 3 - \frac{4}{3+2x}\right)dx$

$= \int(-2x+3)dx - 2\int\frac{1}{3+2x}[2\ dx]$

$= -x^2 + 3x - 2\ln|3+2x| + C$

19. $\int\frac{\left(\sqrt{x}+2\right)^2}{3\sqrt{x}}dx = \frac{2}{3}\int\left(\sqrt{x}+2\right)^2\left[\frac{1}{2\sqrt{x}}dx\right]$

$= \frac{2}{3}\cdot\frac{\left(\sqrt{x}+2\right)^3}{3} + C = \frac{2}{9}\left(\sqrt{x}+2\right)^3 + C$

20. $\int \frac{5e^s}{1+3e^s} ds = \frac{5}{3}\int \frac{1}{1+3e^s}[3e^s\, ds]$
$= \frac{5}{3}\ln(1+3e^s)+C$

21. $\int \frac{5\left(x^{\frac{1}{3}}+2\right)^4}{\sqrt[3]{x^2}} dx = 3\int 5\left(x^{\frac{1}{3}}+2\right)^4 \left[\frac{1}{3}x^{-\frac{2}{3}}dx\right]$
$= 3\left(x^{\frac{1}{3}}+2\right)^5 + C$

22. $\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx = 2\int \left(1+x^{\frac{1}{2}}\right)^{\frac{1}{2}} \left[\frac{1}{2}x^{-\frac{1}{2}}dx\right]$
$= 2\cdot \frac{\left(1+x^{\frac{1}{2}}\right)^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{4}{3}\left(1+\sqrt{x}\right)^{\frac{3}{2}} + C$

23. $\int \frac{\ln x}{x} dx = \int (\ln x)\left[\frac{1}{x}dx\right] = \frac{(\ln x)^2}{2} + C$
$= \frac{1}{2}\left(\ln^2 x\right) + C$

24. $\int \sqrt{t}\left(3-t\sqrt{t}\right)^{0.6} dt = -\frac{2}{3}\int (3-t^{3/2})^{0.6}\left[-\frac{3}{2}t^{1/2}dt\right]$
$= -\frac{2}{3}\cdot\frac{(3-t^{3/2})^{1.6}}{1.6} + C = -\frac{5}{12}\left(3-t\sqrt{t}\right)^{1.6} + C$

25. $\int \frac{\ln^2(r+1)}{r+1} dr = \int [\ln(r+1)]^2\left(\frac{1}{r+1}dr\right)$
$= \frac{1}{3}\ln^3(r+1) + C$

26. $\int \frac{9x^5-6x^4-ex^3}{7x^2} dx = \int\left(\frac{9}{7}x^3 - \frac{6}{7}x^2 - \frac{e}{7}x\right)dx$
$= \frac{9}{28}x^4 - \frac{2}{7}x^3 - \frac{e}{14}x^2 + C$

27. $\int \frac{3^{\ln x}}{x} dx = \int \frac{\left(e^{\ln 3}\right)^{\ln x}}{x} dx$
$= \frac{1}{\ln 3}\int e^{(\ln 3)\ln x}\left[\frac{\ln 3}{x}dx\right]$
$= \frac{1}{\ln 3}\cdot e^{(\ln 3)\ln x} + C$
$= \frac{1}{\ln 3}\left(e^{\ln 3}\right)^{\ln x} + C = \frac{3^{\ln x}}{\ln 3} + C$

28. $\int \frac{4}{x\ln\left(2x^2\right)} dx = 2\int \frac{1}{\ln\left(2x^2\right)}\left[\frac{2}{x}dx\right]$
$= 2\ln\left|\ln\left(2x^2\right)\right| + C$

29. $\int x^2\sqrt{e^{x^3+1}}dx = \int x^2(e^{x^3+1})^{1/2}dx$
$= \frac{2}{3}\int e^{\frac{x^3+1}{2}}\left[\frac{3}{2}x^2dx\right] = \frac{2}{3}e^{\frac{x^3+1}{2}}dx$

30. By using long division on the integrand,
$\int \frac{x+3}{x+6}dx = \int\left(1-\frac{3}{x+6}\right)dx = x - 3\ln|x+6| + C$

31. $\int \frac{8}{(x+3)\ln(x+3)}dx = 8\int \frac{1}{\ln(x+3)}\left[\frac{1}{x+3}dx\right]$
$= 8\ln|\ln(x+3)| + C$

32. $\int\left(e^{e^2} + x^e - 2x\right)dx = e^{e^2}x + \frac{1}{e+1}x^{e+1} - x^2 + C$

33. By using long division on the integrand,
$\int \frac{x^3+x^2-x-3}{x^2-3}dx = \int\left(x+1+\frac{2x}{x^2-3}\right)dx$
$= \int (x+1)dx + \int \frac{1}{x^2-3}[2x\ dx]$
$= \frac{x^2}{2} + x + \ln\left|x^2-3\right| + C$

34. $\int \frac{4x\ln\sqrt{1+x^2}}{1+x^2}dx = \int \frac{4x\cdot\frac{1}{2}\ln\left(1+x^2\right)}{1+x^2}dx$
$= \int \ln\left(1+x^2\right)\left[\frac{2x}{1+x^2}dx\right] = \frac{\ln^2\left(1+x^2\right)}{2} + C$

35. $\int \frac{6x^2\sqrt{\ln(x^3+1)^2}}{x^3+1}dx$

$= \int [2\ln(x^3+1)]^{1/2}\left[\frac{6x^2}{x^3+1}dx\right]$

$= \frac{[2\ln(x^3+1)]^{3/2}}{\frac{3}{2}} + C = \frac{2}{3}\ln^{3/2}(x^3+1)^2 + C$

36. $\int 3\left(x^2+2\right)^{-\frac{1}{2}} xe^{\sqrt{x^2+2}}dx$

$= 3\int e^{\left(x^2+2\right)^{\frac{1}{2}}}\left[x\left(x^2+2\right)^{-\frac{1}{2}}dx\right] = 3e^{\sqrt{x^2+2}} + C$

37. $\int\left(\frac{x^3-1}{\sqrt{x^4-4x}} - \ln 7\right)dx$

$= \frac{1}{4}\int (x^4-4x)^{\frac{1}{2}}[(4x^3-4)dx] - \ln 7\int dx$

$= \frac{1}{6}(x^4-4x)^{\frac{3}{2}} - (\ln 7)x + C$

38. $\int \frac{x-x^{-2}}{x^2+2x^{-1}}dx = \frac{1}{2}\int\frac{1}{x^2+2x^{-1}}\left[\left(2x-2x^{-2}\right)\right]dx$

$= \frac{1}{2}\ln\left|x^2+2x^{-1}\right| + C$

39. $\int \frac{2x^4-8x^3-6x^2+4}{x^3}dx$

$= \int\left(2x-8-\frac{6}{x}+\frac{4}{x^3}\right)dx$

$= 2\int x\ dx - \int 8\ dx - 6\int\frac{1}{x}dx + 4\int x^{-3}dx$

$= 2\cdot\frac{x^2}{2} - 8x - 6\ln|x| + 4\cdot\frac{x^{-2}}{-2} + C$

$= x^2 - 8x - 6\ln|x| - \frac{2}{x^2} + C$

40. $\int\frac{e^x+e^{-x}}{e^x-e^{-x}}dx = \int\frac{1}{e^x-e^{-x}}\left[\left(e^x+e^{-x}\right)dx\right]$

$= \ln\left|e^x-e^{-x}\right| + C$

41. By using long division on the integrand,

$\int\frac{x}{x+1}dx = \int\left(1-\frac{1}{x+1}\right)dx = x - \ln|x+1| + C$

42. $\int\frac{2x}{\left(x^2+1\right)\ln\left(x^2+1\right)}dx$

$= \int\frac{1}{\ln\left(x^2+1\right)}\left[\frac{2x}{x^2+1}dx\right]$

$= \ln\left|\ln\left(x^2+1\right)\right| + C$

43. $\int\frac{xe^{x^2}}{\sqrt{e^{x^2}+2}}dx = \frac{1}{2}\int\left(e^{x^2}+2\right)^{-\frac{1}{2}}\left[2xe^{x^2}dx\right]$

$= \frac{1}{2}\cdot\frac{\left(e^{x^2}+2\right)^{\frac{1}{2}}}{\frac{1}{2}} + C = \sqrt{e^{x^2}+2} + C$

44. $\int\frac{5}{(3x+1)[1+\ln(3x+1)]^2}dx$

$= \frac{5}{3}\int[1+\ln(3x+1)]^{-2}\left[\frac{1}{3x+1}\cdot 3\,dx\right]$

$= -\frac{5}{3[1+\ln(3x+1)]} + C$

45. $\int\frac{\left(e^{-x}+6\right)^2}{e^x}dx = -\int\left(e^{-x}+6\right)^2\left[-e^{-x}dx\right]$

$= -\frac{\left(e^{-x}+6\right)^3}{3} + C$

46. $\int\left[\frac{1}{8x+1} - \frac{1}{e^x\left(8+e^{-x}\right)^2}\right]dx$

$= \frac{1}{8}\int\frac{1}{8x+1}[8\ dx] - (-1)\int\left(8+e^{-x}\right)^{-2}\left[-e^{-x}dx\right]$

$= \frac{1}{8}\ln|8x+1| + \frac{\left(8+e^{-x}\right)^{-1}}{-1} + C$

$= \frac{1}{8}\ln|8x+1| - \frac{1}{8+e^{-x}} + C$

47. $\int\left(x^3+ex\right)\sqrt{x^2+e}\,dx=\int x\left(x^2+e\right)\left(x^2+e\right)^{\frac{1}{2}}dx$

$=\frac{1}{2}\int\left(x^2+e\right)^{\frac{3}{2}}[2x\ dx]=\frac{1}{2}\cdot\frac{\left(x^2+e\right)^{\frac{5}{2}}}{\frac{5}{2}}+C$

$=\frac{1}{5}\left(x^2+e\right)^{\frac{5}{2}}+C$

48. $\int 3^{x\ln x}(1+\ln x)dx=\int\left(e^{\ln 3}\right)^{x\ln x}(1+\ln x)dx$

$=\frac{1}{\ln 3}\int e^{(\ln 3)x\ln x}[(\ln 3)(1+\ln x)dx]$

$=\frac{1}{\ln 3}\cdot e^{(\ln 3)x\ln x}+C=\frac{1}{\ln 3}\left(e^{\ln 3}\right)^{x\ln x}+C$

$=\frac{3^{x\ln x}}{\ln 3}+C$

49. $\int\sqrt{x}\sqrt{(8x)^{\frac{3}{2}}+3}dx=\int\left(8^{\frac{3}{2}}x^{\frac{3}{2}}+3\right)^{\frac{1}{2}}\cdot x^{\frac{1}{2}}dx$

$=\frac{2}{3\cdot 8^{\frac{3}{2}}}\int\left(8^{\frac{3}{2}}x^{\frac{3}{2}}+3\right)^{\frac{1}{2}}\left[8^{\frac{3}{2}}\cdot\frac{3}{2}\cdot x^{\frac{1}{2}}dx\right]$

$=\frac{2}{3\cdot 16\sqrt{2}}\cdot\frac{\left(8^{\frac{3}{2}}x^{\frac{3}{2}}+3\right)^{\frac{3}{2}}}{\frac{3}{2}}+C$

$=\frac{1}{36\sqrt{2}}\left[(8x)^{\frac{3}{2}}+3\right]^{\frac{3}{2}}+C$

50. $\int\frac{2}{x(\ln x)^{2/3}}dx=2\int(\ln x)^{-\frac{2}{3}}\left[\frac{1}{x}dx\right]$

$=6(\ln x)^{\frac{1}{3}}+C$

51. $\int\frac{\sqrt{s}}{e^{\sqrt{s^3}}}ds=-\frac{2}{3}\int e^{-s^{\frac{3}{2}}}\left[-\frac{3}{2}s^{\frac{1}{2}}ds\right]$

$=-\frac{2}{3}e^{-\sqrt{s^3}}+C$

52. $\int\frac{\ln^3 x}{3x}dx=\frac{1}{3}\int(\ln x)^3\left[\frac{1}{x}dx\right]$

$=\frac{1}{3}\cdot\frac{(\ln x)^4}{4}+C=\frac{1}{12}\ln^4 x+C$

53. $e^{\ln(x^2+1)}$ is simply x^2+1. Thus

$\int e^{\ln(x^2+1)}dx=\int(x^2+1)dx=\frac{1}{3}x^3+x+C$

54. $\int dx=\int 1\ dx=x+C$

55. $\int\frac{\ln\left(xe^x\right)}{x}dx=\int\frac{\ln x+\ln e^x}{x}dx$

$=\int\frac{\ln x+x}{x}dx=\int\left(\frac{\ln x}{x}+1\right)dx$

$=\int(\ln x)\left[\frac{1}{x}dx\right]+\int 1\ dx=\frac{\ln^2 x}{2}+x+C$

56. $\int e^{f(x)+\ln(f'(x))}dx=\int e^{f(x)}\cdot e^{\ln(f'(x))}dx$

$=\int e^{f(x)}[f'(x)\,dx]$

$=e^{f(x)}+C$

57. $\frac{dr}{dq}=\frac{200}{(q+2)^2}$

$r=\int 200(q+2)^{-2}dq=200\cdot\frac{(q+2)^{-1}}{-1}+C$

$=-\frac{200}{q+2}+C$

When $q=0$, then $r=0$, so $0=-100+C$, or $C=100$. Hence $r=-\frac{200}{q+2}+100=\frac{100q}{q+2}$. Since $r=pq$, then $p=\frac{r}{q}=\frac{100}{q+2}$.

The demand function is $p=\frac{100}{q+2}$.

58. $\frac{dr}{dq}=\frac{900}{(2q+3)^3}$

$r=\int 900(2q+3)^{-3}dq$

$=900\cdot\frac{1}{2}\int(2q+3)^{-3}[2\ dq]$

$=450\cdot\frac{(2q+3)^{-2}}{-2}+C=-\frac{225}{(2q+3)^2}+C$

When $q=0$, then $r=0$, so $0=-25+C$ or $C=25$. Hence $r=-\frac{225}{(2q+3)^2}+25$. Since

$r = pq$, then $p = \frac{r}{q} = \frac{25}{q} - \frac{225}{q(2q+3)^2}$

The demand function is $p = \frac{25}{q}\left[1 - \frac{9}{(2q+3)^2}\right]$.

59. $\frac{dc}{dq} = \frac{20}{q+5}$

$c = \int \frac{20}{q+5}\,dq = 20\int \frac{1}{q+5}\,dq = 20\ln|q+5| + C$

When $q = 0$, then $c = 2000$, so
$2000 = 20\ln(5) + C$, or $C = 2000 - 20\ln 5$.
Hence $c = 20\ln|q+5| + 2000 - 20\ln 5$

$= 20\left(\ln|q+5| - \ln 5\right) + 2000 = 20\ln\left|\frac{q+5}{5}\right| + 2000$

The cost function is $c = 20\ln\left|\frac{q+5}{5}\right| + 2000.$

60. $\frac{dc}{dq} = 3e^{0.002q}$

$c = \int 3e^{0.002q}\,dq = 3\cdot\frac{1}{0.002}\int e^{0.002q}[0.002\,dq]$

$= 1500e^{0.002q} + C$

When $q = 0$, then $c = 2000$, so $2000 = 1500 + C$, or $C = 500$.

The cost function is $c = 1500e^{0.002q} + 500.$

61. $\frac{dC}{dI} = \frac{1}{\sqrt{I}}$

$C = \int I^{-\frac{1}{2}}\,dI = \frac{I^{\frac{1}{2}}}{\frac{1}{2}} + C_1 = 2\sqrt{I} + C_1$

$C(9) = 8$ implies that $8 = 2\cdot 3 + C_1$, or $C_1 = 2$.

Thus $C = 2\sqrt{I} + 2 = 2\left(\sqrt{I} + 1\right)$.

The consumption function is $C = 2\left(\sqrt{I} + 1\right).$

62. $\frac{dC}{dI} = \frac{1}{2} - \frac{1}{2\sqrt{2I}}$

$C = \int\left(\frac{1}{2} - \frac{(2I)^{-\frac{1}{2}}}{2}\right)dI$

$= \frac{1}{2}\int dI - \frac{1}{4}\int (2I)^{-\frac{1}{2}}[2\,dI]$

$= \frac{1}{2}\cdot I - \frac{1}{4}\cdot\frac{(2I)^{\frac{1}{2}}}{\frac{1}{2}} + C_1$

$= \frac{I}{2} - \frac{\sqrt{2I}}{2} + C_1$

$C(2) = \frac{3}{4}$ implies $\frac{3}{4} = 1 - \frac{\sqrt{4}}{2} + C_1$, so $C_1 = \frac{3}{4}$.

The consumption function is $C = \frac{I}{2} - \frac{\sqrt{2I}}{2} + \frac{3}{4}$.

63. $\frac{dC}{dI} = \frac{3}{4} - \frac{1}{6\sqrt{I}}$

$C = \int\left[\frac{3}{4} - \frac{I^{-\frac{1}{2}}}{6}\right]dI = \int\frac{3}{4}\,dI - \frac{1}{6}\int I^{-\frac{1}{2}}\,dI$

$= \frac{3}{4}I - \frac{1}{6}\cdot\frac{I^{\frac{1}{2}}}{\frac{1}{2}} + C_1 = \frac{3}{4}I - \frac{\sqrt{I}}{3} + C_1$

Thus $C = \frac{3}{4}I - \frac{\sqrt{I}}{3} + C_1$.

$C(25) = 23$ implies that $23 = \frac{3}{4}\cdot 25 - \frac{5}{3} + C_1$, so

$C_1 = \frac{71}{12}$.

The consumption function is
$C = \frac{3}{4}I - \frac{1}{3}\sqrt{I} + \frac{71}{12}$.

64. $\frac{dc}{dq} = 10 - \frac{100}{q+10}$

$c = \int\left(10 - \frac{100}{q+10}\right)dq = 10q - 100\ln|q+10| + C$

Avg. cost $= \frac{c}{q} = 10 - 100\frac{\ln|q+10|}{q} + \frac{C}{q}$

When $q = 100$, then avg. cost = 50, so

$50 = 10 - 100\frac{\ln(110)}{100} + \frac{C}{100}$, or

$C = 100(40 + \ln(110))$. Thus

$c = 10q - 100\ln|q+10| + 100(40 + \ln(110))$

Evaluating c when $q = 0$ gives fixed cost:
$c(0) = -100 \ln(10) + 100(40 + \ln(110)) \approx 4240$.
The fixed cost is \$4240.

65. $\frac{dc}{dq} = \frac{100q^2 - 3998q + 60}{q^2 - 40q + 1}$

a. $\left.\frac{dc}{dq}\right|_{q=40} = \frac{100(40)^2 - 3998(40) + 60}{(40)^2 - 40(40) + 1}$
= \$140 per unit

b. To find c, we integrate $\frac{dc}{dq}$ by using long division:

$$c = \int \frac{100q^2 - 3998q + 60}{q^2 - 40q + 1}\,dq$$

$$= \int \left(100 + \frac{2q - 40}{q^2 - 40q + 1}\right) dq$$

$$= \int 100\,dq + \int \frac{1}{q^2 - 40q + 1}[(2q - 40)dq]$$

Thus $c = 100q + \ln\left|q^2 - 40q + 1\right| + C$. When $q = 0$, then $c = 10{,}000$, so
$10{,}000 = 0 + \ln(1) + C$, so $C = 10{,}000$.
Hence $c = 100q + \ln\left|q^2 - 40q + 1\right| + 10{,}000$.
When $q = 40$, then
$c = 4000 + \ln(1) + 10{,}000 = \$14{,}000$.

c. If $c = f(q)$, then

$$f(q + dq) \approx f(q) + dc = f(q) + \frac{dc}{dq}dq$$

Letting $q = 40$ and $dq = 2$, we have

$$f(42) = f(40 + 2) \approx f(40) + \left.\frac{dc}{dq}\right|_{q=40} \cdot (2)$$

$= 14{,}000 + 140(2) = \$14{,}280$

66. $\frac{dc}{dq} = \frac{9}{10}\sqrt{q}\sqrt{0.04q^{\frac{3}{2}} + 4}$

a. $\left.\frac{dc}{dq}\right|_{q=25} = \frac{9}{10}\sqrt{25}\sqrt{9} = \frac{9}{10} \cdot 5 \cdot 3 = \frac{27}{2}$
= \$13.50 per unit

b. $c = \int \frac{9}{10}\sqrt{q}\sqrt{0.04q^{\frac{3}{2}} + 4dq}$

$$= \frac{0.9}{0.06}\int \left(0.04q^{\frac{3}{2}} + 4\right)^{\frac{1}{2}} \left[0.06q^{\frac{1}{2}}dq\right]$$

$$= \frac{0.9}{0.06} \cdot \frac{\left(0.04q^{\frac{3}{2}} + 4\right)^{\frac{3}{2}}}{\frac{3}{2}} + C$$

Thus $c = 10\left(0.04q^{\frac{3}{2}} + 4\right)^{\frac{3}{2}} + C$. When $q = 0$, then $c = 360$, so $360 = 10(4)^{\frac{3}{2}} + C$, or $C = 280$. Hence $c = 10\left(0.04q^{\frac{3}{2}} + 4\right)^{\frac{3}{2}} + 280$.

When $q = 25$, then $c = 10(9)^{\frac{3}{2}} + 280 = \$550$.

c. If $c = f(q)$, then $f(q + dq) \approx f(q) + dc$
$= f(q) + \frac{dc}{dq}dq$. Letting $q = 25$ and $dq = -2$, we have

$$f(23) = f(25 - 2) \approx f(25) + \left.\frac{dc}{dq}\right|_{q=25} \cdot (-2)$$

$= 550 + 13.50(-2) = \$523$

67. $\frac{dV}{dt} = \frac{8t^3}{\sqrt{0.2t^4 + 8000}}$

$$V = \int \frac{8t^3}{\sqrt{0.2t^4 + 8000}}\,dt$$

$$= 10\int \left(0.2t^4 + 8000\right)^{-\frac{1}{2}} \left[0.8t^3\right] dt$$

$$= 10\frac{\left(0.2t^4 + 8000\right)^{\frac{1}{2}}}{\frac{1}{2}} + C$$

Thus $V = 20\sqrt{0.2t^4 + 8000} + C$. If $t = 0$, then $V = 500$, so $500 = 20\sqrt{8000} + C$,
$500 = 20\sqrt{1600 \cdot 5} + C$, $500 = 800\sqrt{5} + C$, or
$C = 500 - 800\sqrt{5}$. Hence
$V = 20\sqrt{0.2t^4 + 8000} + 500 - 800\sqrt{5}$.
When $t = 10$, then
$V = 20\sqrt{10{,}000} + 500 - 800\sqrt{5}$
$= 20(100) + 500 - 800\sqrt{5} \approx \$711$ per acre.

68. $\frac{dr}{dq} = \frac{a}{e^q + b} = \frac{ae^{-q}}{(e^q + b)e^{-q}} = \frac{ae^{-q}}{1 + be^{-q}}$

$r = \int \frac{ae^{-q}}{1+be^{-q}} dq = \left(-\frac{1}{b}\right) a \int \frac{1}{1+be^{-q}} [-be^{-q} dq]$

$= -\frac{a}{b} \ln(1 + be^{-q}) + C$

Now $r = 0$ when $q = 0$, so $0 = -\frac{a}{b}\ln(1+b) + C$,

or $C = \frac{a}{b}\ln(1+b)$. Hence

$r = -\frac{a}{b}\ln(1+be^{-q}) + \frac{a}{b}\ln(1+b)$

$= \frac{a}{b}\ln\frac{1+b}{1+be^{-q}}$

$p = \frac{r}{q} = \frac{a}{bq}\ln\frac{1+b}{1+be^{-q}}$

69. $S = \int \frac{dS}{dI} dI = \int \frac{5}{(I+2)^2} dI$

$= 5\int (I+2)^{-2} dI = 5 \cdot \frac{(I+2)^{-1}}{-1} + C_1$

Thus $S = -\frac{5}{I+2} + C_1$. If C is the total national consumption (in billions of dollars), then $C + S = I$, or $C = I - S$. Hence $C = I + \frac{5}{I+2} - C_1$.

When $I = 8$, then $C = 7.5$, so $7.5 = 8 + \frac{1}{2} - C_1$, or

$C_1 = 1$. Thus $S = 1 - \frac{5}{I+2}$. If $S = 0$, then

$0 = 1 - \frac{5}{I+2} \Rightarrow \frac{5}{I+2} = 1 \Rightarrow 5 = I + 2 \Rightarrow I = 3$

70. a. If C is total national consumption (in billions of dollars), then

$\frac{dC}{dI} = 1 - \frac{dS}{dI} = 1 - \left(\frac{1}{2} - \frac{1.8}{\sqrt[3]{3I^2}}\right)$. Thus

$\left.\frac{dC}{dI}\right|_{I=81} = 1 - \left(\frac{1}{2} - \frac{1.8}{\sqrt[3]{3(81)^2}}\right)$

$= 1 - \left(\frac{1}{2} - \frac{1.8}{27}\right) = \frac{17}{30}$.

b. $S = \int \frac{dS}{dI} dI = \int \left(\frac{1}{2} - \frac{1.8}{\sqrt[3]{3I^2}}\right) dI$

$= \int \left(\frac{1}{2} - \frac{1.8}{\sqrt[3]{3}} I^{-\frac{2}{3}}\right) dI$

$= \frac{I}{2} - \frac{1.8}{\sqrt[3]{3}} \cdot \frac{I^{\frac{1}{3}}}{\frac{1}{3}} + C_1 = \frac{I}{2} - \frac{5.4}{\sqrt[3]{3}} \cdot \sqrt[3]{I} + C_1$

Thus $S = \frac{I}{2} - 5.4\sqrt[3]{\frac{I}{3}} + C_1$. When $I = 24$,

then $S = 3$, so $3 = 12 - 5.4\sqrt[3]{8} + C_1$, or

$C_1 = 1.8$. Thus $S = \frac{I}{2} - 5.4\sqrt[3]{\frac{I}{3}} + 1.8$. If C is the total national consumption (in billions of dollars), then $C + S = I$, or

$C = I - S = I - \left(\frac{I}{2} - 5.4\sqrt[3]{\frac{I}{3}} + 1.8\right)$.

Therefore, $C = \frac{I}{2} + 5.4\sqrt[3]{\frac{I}{3}} - 1.8$.

c. From (b), when $I = 81$, then

$C = \frac{81}{2} + 5.4\sqrt[3]{\frac{81}{3}} - 1.8 = 40.5 + 16.2 - 1.8$

$= 54.9$

Thus consumption is \$54.9 billion when income is \$81 billion.

d. If $C = f(I)$, then

$f(I + dI) \approx f(I) + dC = f(I) + \frac{dC}{dI} dI$. Let

$I = 81$ and $dI = -3$. Then

$f(78) = f(81-3) \approx f(81) + \left.\frac{dC}{dI}\right|_{I=81} (-3)$

$= 54.9 + \frac{17}{30}(-3) = 53.2$. Thus when income is \$78 billion, then consumption is approximately \$53.2 billion.

Principles in Practice 14.6

1. Divide the interval [0, 10] into n subintervals of equal length Δx, so $\Delta x = \frac{10}{n}$. The endpoints of the subintervals are 0, $\frac{10}{n}$, $2\left(\frac{10}{n}\right)$, $3\left(\frac{10}{n}\right)$, ... , $(n-1)\left(\frac{10}{n}\right)$, and $n\left(\frac{10}{n}\right) = 10$. Letting S_n denote the sum of the areas of the rectangles corresponding to right-hand endpoints, we have

$$S_n = \frac{10}{n}R\left(\frac{10}{n}\right) + \frac{10}{n}R\left[2\left(\frac{10}{n}\right)\right] + \ldots + \frac{10}{n}R\left[n\left(\frac{10}{n}\right)\right]$$

$$= \frac{10}{n}\left[\left\{600 - 0.5\left(\frac{10}{n}\right)\right\} + \left\{600 - 0.5(2)\left(\frac{10}{n}\right)\right\} + \ldots + \left\{600 - 0.5(n)\left(\frac{10}{n}\right)\right\}\right]$$

$$= \frac{10}{n}\left[600n - 0.5\left(\frac{10}{n}\right)\{1 + 2 + \ldots + n\}\right]$$

$$= \frac{10}{n}\left[600n - 0.5\left(\frac{10}{n}\right)\frac{n(n+1)}{2}\right]$$

$$= \frac{10}{n}[600n - 2.5(n+1)]$$

$$= 6000 - 25\left(\frac{n+1}{n}\right)$$

Now take the limit of S_n as $n \to \infty$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[6000 - 25\left(\frac{n+1}{n}\right)\right] = \lim_{n\to\infty}\left[6000 - 25\left(1 + \frac{1}{n}\right)\right] = 6000 - 25 = 5975$$

The total revenue for selling 10 units is \$5975.

Problems 14.6

1. $f(x) = x$, $y = 0$, $x = 1$

S_3, $\Delta x = \frac{1}{3}$

$$S_3 = \frac{1}{3}f\left(\frac{1}{3}\right) + \frac{1}{3}f\left(\frac{2}{3}\right) + \frac{1}{3}f\left(\frac{3}{3}\right) = \frac{1}{3}\left[\frac{1}{3} + \frac{2}{3} + \frac{3}{3}\right] = \frac{1}{3}\cdot\frac{6}{3} = \frac{2}{3}$$

The area is approximately $\frac{2}{3}$ sq unit.

2. $f(x) = 3x$, $y = 0$, $x = 1$

S_5, $\Delta x = \frac{1}{5}$

$$S_5 = \frac{1}{5}f\left(\frac{1}{5}\right) + \frac{1}{5}f\left(\frac{2}{5}\right) + \frac{1}{5}f\left(\frac{3}{5}\right) + \frac{1}{5}f\left(\frac{4}{5}\right) + \frac{1}{5}f\left(\frac{5}{5}\right) = \frac{1}{5}\left[\frac{3}{5} + \frac{6}{5} + \frac{9}{5} + \frac{12}{5} + \frac{15}{5}\right] = \frac{1}{5}\cdot\frac{45}{5} = \frac{9}{5}$$

The area is approximately $\frac{9}{5}$ sq units.

3. $f(x)=x^2,\ y=0,\ x=1$

$S_4,\ \Delta x=\frac{1}{4}$

$$S_4=\frac{1}{4}f\left(\frac{1}{4}\right)+\frac{1}{4}f\left(\frac{2}{4}\right)+\frac{1}{4}f\left(\frac{3}{4}\right)+\frac{1}{4}f\left(\frac{4}{4}\right)$$
$$=\frac{1}{4}\left[\frac{1}{16}+\frac{4}{16}+\frac{9}{16}+\frac{16}{16}\right]$$
$$=\frac{1}{4}\cdot\frac{30}{16}=\frac{15}{32}$$

The area is approximately $\frac{15}{32}$ sq units.

4. $f(x)=x^2+1,\ y=0,\ x=0,\ x=1$

$S_2,\ \Delta x=\frac{1}{2}$

$$S_2=\frac{1}{2}f\left(\frac{1}{2}\right)+\frac{1}{2}f\left(\frac{2}{2}\right)=\frac{1}{2}\left[\frac{5}{4}+\frac{8}{4}\right]=\frac{1}{2}\cdot\frac{13}{4}=\frac{13}{8}$$

The area is approximately $\frac{13}{8}$ sq units.

5. $f(x)=4x;\ [0,1]$

$\Delta x=\frac{1}{n}$

$$S_n=\frac{1}{n}f\left(\frac{1}{n}\right)+\ldots+\frac{1}{n}f\left(n\cdot\frac{1}{n}\right)$$
$$=\frac{1}{n}\left[f\left(\frac{1}{n}\right)+\ldots+f\left(n\cdot\frac{1}{n}\right)\right]$$
$$=\frac{1}{n}\left[4\cdot\frac{1}{n}+\ldots+4\cdot\frac{n}{n}\right]$$
$$=\frac{4}{n^2}[1+\ldots+n]=\frac{4}{n^2}\cdot\frac{n(n+1)}{2}$$
$$=\frac{2(n+1)}{n}$$

6. $f(x)=3x+2;\ [0,3]$

$\Delta x=\frac{3}{n}$

$$S_n=\frac{3}{n}f\left(\frac{3}{n}\right)+\ldots+\frac{3}{n}f\left(n\cdot\frac{3}{n}\right)$$
$$=\frac{3}{n}\left[f\left(\frac{3}{n}\right)+\ldots+f\left(n\cdot\frac{3}{n}\right)\right]$$
$$=\frac{3}{n}\left\{\left[3\left(\frac{3}{n}\right)+2\right]+\ldots+\left[3\left(n\cdot\frac{3}{n}\right)+2\right]\right\}$$
$$=\frac{3}{n}\left\{\frac{9}{n}(1+\ldots+n)+2n\right\}$$
$$=\frac{3}{n}\left\{\frac{9}{n}\cdot\frac{n(n+1)}{2}+2n\right\}=\frac{27(n+1)}{2n}+6$$

7. a. $$S_n=\frac{1}{n}\left[\left(\frac{1}{n}+1\right)+\left(\frac{2}{n}+1\right)+\ldots+\left(\frac{n}{n}+1\right)\right]$$
$$=\frac{1}{n}\left[\frac{1}{n}(1+2+\ldots+n)+n\right]$$
$$=\frac{1}{n}\left[\frac{1}{n}\cdot\frac{n(n+1)}{2}+n\right]$$
$$=\frac{n+1}{2n}+1$$

b. $$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left[\frac{n+1}{2n}+1\right]$$
$$=\lim_{n\to\infty}\left[\frac{1}{2}+\frac{1}{2n}+1\right]$$
$$=\frac{1}{2}+0+1=\frac{3}{2}$$

8. a. $$S_n=\frac{2}{n}\left[\left(\frac{2}{n}\right)^2+\left(2\cdot\frac{2}{n}\right)^2+\ldots+\left(n\cdot\frac{2}{n}\right)^2\right]$$
$$=\frac{2}{n}\cdot\frac{2^2}{n^2}\left[1^2+2^2+\ldots+n^2\right]$$
$$=\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$
$$=\frac{4(n+1)(2n+1)}{3n^2}$$

b. $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{4(n+1)(2n+1)}{3n^2} = \lim_{n\to\infty} \frac{4\left[2n^2+3n+1\right]}{3n^2}$

$= \lim_{n\to\infty}\left[\frac{4}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\right] = \frac{4}{3}(2+0+0) = \frac{8}{3}$

9. $f(x) = x$, $y = 0$, $x = 1$

$\Delta x = \frac{1}{n}$

$S_n = \frac{1}{n} f\left(\frac{1}{n}\right) + \ldots + \frac{1}{n} f\left(n\cdot\frac{1}{n}\right) = \frac{1}{n}\left[\frac{1}{n}+\ldots+\frac{n}{n}\right] = \frac{1}{n^2}[1+\ldots+n] = \frac{1}{n^2}\cdot\frac{n(n+1)}{2}$

$= \frac{1}{2}\cdot\frac{n+1}{n} = \frac{1}{2}\left[1+\frac{1}{n}\right]$

$\lim_{n\to\infty} S_n = \frac{1}{2}$

The area is $\frac{1}{2}$ sq unit.

10. $f(x) = 3x$, $y = 0$, $x = 1$

$\Delta x = \frac{1}{n}$

$S_n = \frac{1}{n} f\left(\frac{1}{n}\right) + \ldots + \frac{1}{n} f\left(n\cdot\frac{1}{n}\right) = \frac{1}{n}\left[3\cdot\frac{1}{n}+\ldots+3\cdot\frac{n}{n}\right]$

$= \frac{3}{n^2}[1+\ldots+n] = \frac{3}{n^2}\cdot\frac{n(n+1)}{2} = \frac{3}{2}\cdot\frac{n+1}{n} = \frac{3}{2}\left[1+\frac{1}{n}\right]$

$\lim_{n\to\infty} S_n = \frac{3}{2}$

The area is $\frac{3}{2}$ sq units.

11. $f(x) = x^2$, $y = 0$, $x = 1$

$\Delta x = \frac{1}{n}$

$S_n = \frac{1}{n} f\left(\frac{1}{n}\right) + \ldots + \frac{1}{n} f\left(n\cdot\frac{1}{n}\right) = \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \ldots + \left(n\cdot\frac{1}{n}\right)^2\right]$

$= \frac{1}{n^3}\left[1^2+\ldots+n^2\right] = \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$

$= \frac{1}{6}\cdot\frac{2n^2+3n+1}{n^2} = \frac{1}{6}\left[2+\frac{3}{n}+\frac{1}{n^2}\right]$

$\lim_{n\to\infty} S_n = \frac{1}{3}$

The area is $\frac{1}{3}$ sq unit.

12. $y = x^2,\ y = 0, x = 1, x = 2$

$\Delta x = \dfrac{1}{n}$

$$S_n = \frac{1}{n} f\left(1+\frac{1}{n}\right) + \frac{1}{n} f\left(1+2\cdot\frac{1}{n}\right) + \cdots \frac{1}{n} f\left(1+n\cdot\frac{1}{n}\right)$$

$$= \frac{1}{n}\left\{\left[1+\frac{1}{n}\right]^2 + \cdots + \left[1+n\cdot\frac{1}{n}\right]^2\right\}$$

$$= \frac{1}{n}\left\{\left[1+2\cdot\frac{1}{n}+\frac{1}{n^2}\right] + \cdots + \left[1+2n\cdot\frac{1}{n}+n^2\cdot\frac{1}{n^2}\right]\right\}$$

$$= \frac{1}{n}\left\{n + \frac{2}{n}(1+\cdots+n) + \frac{1}{n^2}(1^2+\cdots+n^2)\right\}$$

$$= 1 + \frac{2}{n^2}\cdot\frac{n(n+1)}{2} + \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= 1 + \frac{n+1}{n} + \frac{1}{6}\cdot\frac{2n^2+3n+1}{n^2}$$

$$= 1 + \left[1+\frac{1}{n}\right] + \frac{1}{6}\left[2+\frac{3}{n}+\frac{1}{n^2}\right]$$

$$\lim_{n\to\infty} S_n = 1+1+\frac{1}{3} = \frac{7}{3}$$

The area is $\dfrac{7}{3}$ sq units.

13. $f(x) = 3x^2,\ y = 0,\ x = 1$

$\Delta x = \dfrac{1}{n}$

$$S_n = \frac{1}{n} f\left(\frac{1}{n}\right) + \ldots + \frac{1}{n} f\left(n\cdot\frac{1}{n}\right) = \frac{1}{n}\left[3\left(\frac{1}{n}\right)^2 + \ldots + 3\left(n\cdot\frac{1}{n}\right)^2\right]$$

$$= \frac{3}{n^3}\left[1^2+\ldots+n^2\right] = \frac{3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{1}{2}\cdot\frac{2n^2+3n+1}{n^2} = \frac{1}{2}\left[2+\frac{3}{n}+\frac{1}{n^2}\right]$$

$$\lim_{n\to\infty} S_n = 1$$

The area is 1 sq unit.

14. $f(x) = 9 - x^2,\ y = 0,\ x = 0$

$$\Delta x = \frac{3}{n}$$

$$S_n = \frac{3}{n} f\left(\frac{3}{n}\right) + \ldots + \frac{3}{n} f\left(n \cdot \frac{3}{n}\right)$$

$$= \frac{3}{n}\left\{\left[9 - \left(\frac{3}{n}\right)^2\right] + \ldots + \left[9 - \left(n \cdot \frac{3}{n}\right)^2\right]\right\}$$

$$= \frac{3}{n}\left\{9n - \left(\frac{3}{n}\right)^2 \left[1^2 + \ldots + n^2\right]\right\}$$

$$= 27 - \left[\frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\right]$$

$$= 27 - \frac{9}{2}\left[\frac{2n^2 + 3n + 1}{n^2}\right] = 27 - \frac{9}{2}\left[2 + \frac{3}{n} + \frac{1}{n^2}\right]$$

$$\lim_{n\to\infty} S_n = 27 - 9 = 18$$

The area is 18 sq units.

15. $\int_1^3 5x\,dx$

Let $f(x) = 5x$.

$$\Delta x = \frac{2}{n}$$

$$S_n = \frac{2}{n} f\left(1 + \frac{2}{n}\right) + \cdots + \frac{2}{n} f\left(1 + n \cdot \frac{2}{n}\right)$$

$$= \frac{2}{n}\left[5\left(1 + \frac{2}{n}\right) + \cdots + 5\left(1 + n \cdot \frac{2}{n}\right)\right]$$

$$= \frac{10}{n}\left[(1 + \cdots + 1) + \frac{2}{n}(1 + \cdots + n)\right]$$

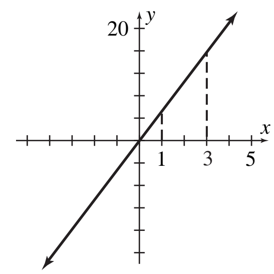
$$= \frac{10}{n}\left[n + \frac{2}{n} \cdot \frac{n(n+1)}{2}\right]$$

$$= \frac{10}{n}[n + n + 1]$$

$$= \frac{10}{n}(2n + 1)$$

$$= 20 + \frac{10}{n}$$

$$\int_1^3 5x\,dx = \lim_{n\to\infty} S_n = 20$$



16. $\int_0^4 9\,dx$

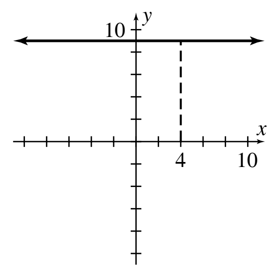
Let $f(x) = 9$

$$\Delta x = \frac{4}{n}$$

$$S_n = \frac{4}{n} f\left(\frac{4}{n}\right) + \ldots + \frac{4}{n} f\left(n \cdot \frac{4}{n}\right)$$

$$= \frac{4}{n}[9 + \ldots + 9] = \frac{4}{n} \cdot 9n = 36$$

$$\int_0^4 9\,dx = \lim_{n\to\infty} S_n = 36$$



17. $\int_0^3 -4x\,dx$

Let $f(x) = -4x$.

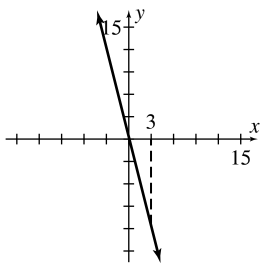
$$\Delta x = \frac{3}{n}$$

$$S_n = \frac{3}{n} f\left(\frac{3}{n}\right) + \ldots + \frac{3}{n} f\left(n \cdot \frac{3}{n}\right)$$

$$= \frac{3}{n}\left[-4\left(\frac{3}{n}\right) - \ldots - 4\left(n \cdot \frac{3}{n}\right)\right] = -\frac{36}{n^2}[1 + \ldots + n]$$

$$= -\frac{36}{n^2} \cdot \frac{n(n+1)}{2} = -18 \cdot \frac{n+1}{n} = -18\left[1 + \frac{1}{n}\right]$$

$$\int_0^3 -4x\,dx = \lim_{n\to\infty} S_n = -18$$



18. $\int_1^4 (2x+1)dx$

Let $f(x) = 2x + 1$.

$$\Delta x = \frac{4-1}{n} = \frac{3}{n}$$

$$S_n = \frac{3}{n} f\left(1+\frac{3}{n}\right)+\cdots+\frac{3}{n} f\left(1+n\cdot\frac{3}{n}\right)$$

$$= \frac{3}{n}\left\{\left[2\left(1+\frac{3}{n}\right)+1\right]+\cdots+\left[2\left(1+n\cdot\frac{3}{n}\right)+1\right]\right\}$$

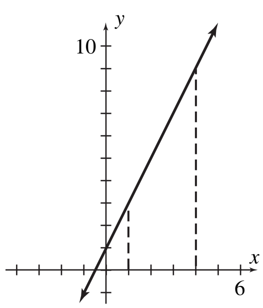
$$= \frac{3}{n}\left\{2n+\frac{6}{n}(1+2+\cdots+n)+n\right\}$$

$$= 6+\frac{18}{n^2}\cdot\frac{n(n+1)}{2}+3$$

$$= 9+9\cdot\frac{n+1}{n}$$

$$= 9+9\left(1+\frac{1}{n}\right)$$

$$\int_1^4 (2x+1)dx = \lim_{n\to\infty} S_n = 9+9 = 18$$



19. $\int_0^1 \left(x^2+x\right)dx$

Let $f(x) = x^2 + x$.

$$\Delta x = \frac{1}{n}$$

$$S_n = \frac{1}{n} f\left(\frac{1}{n}\right)+\ldots+\frac{1}{n} f\left(n\cdot\frac{1}{n}\right)$$

$$= \frac{1}{n}\left\{\left[\left(\frac{1}{n}\right)^2+\frac{1}{n}\right]+\ldots+\left[\left(n\cdot\frac{1}{n}\right)^2+n\cdot\frac{1}{n}\right]\right\}$$

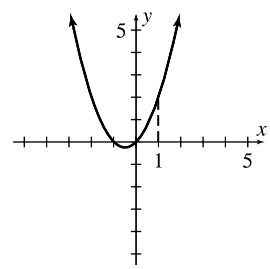
$$= \frac{1}{n}\left\{\left(\frac{1}{n}\right)^2\left[1^2+\ldots+n^2\right]+\frac{1}{n}[1+\ldots+n]\right\}$$

$$= \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{1}{n^2}\cdot\frac{n(n+1)}{2}$$

$$= \frac{1}{6}\cdot\frac{2n^2+3n+1}{n^2}+\frac{1}{2}\cdot\frac{n+1}{n}$$

$$= \frac{1}{6}\left[2+\frac{3}{n}+\frac{1}{n^2}\right]+\frac{1}{2}\left[1+\frac{1}{n}\right]$$

$$\int_0^1 \left(x^2+x\right)dx = \lim_{n\to\infty} S_n = \frac{1}{3}+\frac{1}{2} = \frac{5}{6}$$



20. $\int_1^2 (x+2)dx$

Let $f(x) = x + 2$.

$$\Delta x = \frac{1}{n}$$

$$S_n = \frac{1}{n} f\left(1+\frac{1}{n}\right)+\ldots+\frac{1}{n} f\left(1+n\cdot\frac{1}{n}\right)$$

$$= \frac{1}{n}\left\{\left[\left(1+\frac{1}{n}\right)+2\right]+\ldots+\left[\left(1+n\cdot\frac{1}{n}\right)+2\right]\right\}$$

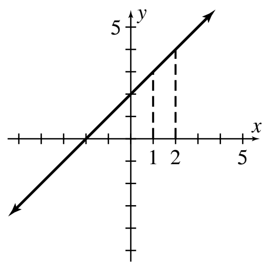
$$= \frac{1}{n}\left\{n+\frac{1}{n}(1+\cdots+n)+2n\right\}$$

$$= 1+\frac{1}{n^2}\cdot\frac{n(n+1)}{2}+2$$

$$= 3+\frac{1}{2}\cdot\frac{n+1}{n}$$

$$= 3+\frac{1}{2}\left[1+\frac{1}{n}\right]$$

$$\int_1^2 (x+2)dx = \lim_{n\to\infty} S_n = 3+\frac{1}{2} = \frac{7}{2}$$



21. $\int_2^3 \sqrt{x^2+1}\,dx$ is simply a real number. Thus
$D_x\left[\int_2^3 \sqrt{x^2+1}dx\right] = D_x$ (real number) = 0.

22. $f(x) = \begin{cases} 2 & \text{if } 0 \le x < 1 \\ 4-2x & \text{if } 1 \le x < 2 \\ 5x-10 & \text{if } 2 \le x \le 3 \end{cases}$

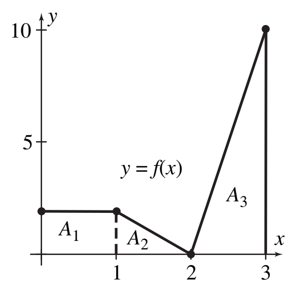
f is continuous and $f(x) \ge 0$ on [0, 3]. Thus $\int_0^3 f(x)\,dx$ gies the area *A* bounded by $y = f(x)$, $y = 0$, $x = 0$ and $x = 3$. From the diagram, this area is composed of three subareas, A_1, A_2 and A_3, and $A = A_1 + A_2 + A_3$.

A_1 = area of rectangle = (1)(2) = 2 sq unit

$A_2 = \text{area of triangle} = \frac{1}{2}(1)(2) = 1$ sq unit

$A_3 = \text{area of triangle} = \frac{1}{2}(1)(10) = 5$ sq unit

Since $A = A_1 + A_2 + A_3 = 2+1+5 = 8$ sq units, we have $\int_0^3 f(x)\,dx = 8$.



23. $f(x) = \begin{cases} 1 & \text{if } x \le 1 \\ 2-x & \text{if } 1 \le x \le 2 \\ -1+\frac{x}{2} & \text{if } x > 2 \end{cases}$

f is continuous and $f(x) \ge 0$ on [–1, 3]. Thus $\int_{-1}^3 f(x)dx$ gives the area *A* bounded by $y = f(x)$, $y = 0$, $x = -1$, and $x = 3$. From the diagram, this area is composed of three subareas, A_1, A_2, and A_3 and $A = A_1 + A_2 + A_3$.

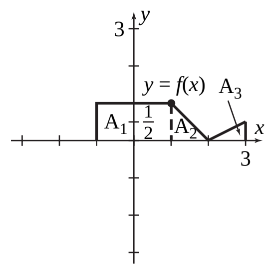
A_1 = area of rectangle = (2)(1) = 2 sq units

$A_2 = \text{area of triangle} = \frac{1}{2}(1)(1) = \frac{1}{2}$ sq unit

$A_3 = \text{area of triangle} = \frac{1}{2}(1)\left(\frac{1}{2}\right) = \frac{1}{4}$ sq unit

Since

$A = A_1 + A_2 + A_3 = 2 + \frac{1}{2} + \frac{1}{4} = \frac{11}{4}$ sq units, we have $\int_{-1}^3 f(x)dx = \frac{11}{4}$.



24. 44.6 sq units

25. 14.77 sq units

26. 1.7 sq units

27. 2.4

28. 0.7

29. –25.5

30. 0.39

Principles in Practice 14.7

1. $\int_3^6 10{,}000e^{0.02t}dt = \left(10{,}000\frac{e^{0.02t}}{0.02}\right)\Bigg|_3^6$

$= \left(500{,}000e^{0.02t}\right)\Big|_3^6$

$= 500{,}000\left(e^{0.02(6)} - e^{0.02(3)}\right)$

$= 500{,}000\left(e^{0.12} - e^{0.06}\right) \approx 32{,}830$

The total income for the chain between the third and sixth years was about \$32,830.

2. The total cost for the first 5 years is $M(5)$ or

$$M(5)-M(0)=\int_0^5 M'(x)dx$$

$$\int_0^5 \left(90x^2+5000\right)dx=\left(90\frac{x^3}{3}+5000x\right)\Bigg|_0^5$$

$$=\left(30x^3+5000x\right)\Big|_0^5=30(5)^3+5000(5)-0$$

= 3750 + 25,000 = 28,750

The total cost for the first 5 years is \$28,750.

Problems 14.7

1. $\int_0^3 5\,dx=5x\Big|_0^3=5(3)-5(0)=15-0=15$

2. $\int_2^4 (1-e)dx=(1-e)x\Big|_2^4$

$=4(1-e)-2(1-e)=2(1-e)$

3. $\int_1^2 5x\,dx=5\cdot\frac{x^2}{2}\Big|_1^2=10-\frac{5}{2}=\frac{15}{2}$

4. $\int_2^8 -5x\,dx=-5\cdot\frac{x^2}{2}\Big|_2^8=-160-(-10)=-150$

5. $\int_{-3}^1 (2x-3)dx=\left(x^2-3x\right)\Big|_{-3}^1=-2-18=-20$

6. $\int_{-1}^1 (4-9y)=\left(4y-\frac{9y^2}{2}\right)\Bigg|_{-1}^1=-\frac{1}{2}-\left(-\frac{17}{2}\right)$

$=\frac{16}{2}=8$

7. $\int_2^3 \left(y^2-2y+1\right)dy=\int_2^3 (y-1)^2\,dy=\frac{1}{3}(y-1)^3\Big|_2^3$

$=\frac{8}{3}-\frac{1}{3}=\frac{7}{3}$

8. $\int_4^1 (2t-3t^2)dt=(t^2-t^3)\Big|_4^1=0-(-48)=48$

9. $\int_{-2}^{-1}\left(3w^2-w-1\right)dw=\left(w^3-\frac{w^2}{2}-w\right)\Bigg|_{-2}^{-1}$

$=-\frac{1}{2}-(-8)=\frac{15}{2}$

10. $\int_8^9 dt=\int_8^9 1\,dt=t\Big|_8^9=9-8=1$

11. $\int_1^3 3t^{-3}\,dt=-\frac{3}{2}\cdot t^{-2}\Big|_1^3=-\frac{1}{6}-\left(-\frac{3}{2}\right)=\frac{4}{3}$

12. $\int_1^2 \frac{x^{-2}}{2}dx=-\frac{x^{-1}}{2}\Bigg|_1^2=-\frac{1}{2x}\Bigg|_1^2$

$=-\frac{1}{4}-\left(-\frac{1}{2}\right)=\frac{1}{4}$

13. $\int_{-8}^8 \sqrt[3]{x^4}\,dx=\int_{-8}^8 x^{4/3}dx$

$=\frac{3x^{7/3}}{7}\Bigg|_{-8}^8$

$=\frac{3\cdot 128}{7}-\frac{3(-128)}{7}$

$=\frac{768}{7}$

14. $\int_{1/2}^{3/2}\left(x^2+x+1\right)dx=\left(\frac{x^3}{3}+\frac{x^2}{2}+x\right)\Bigg|_{1/2}^{3/2}$

$=\frac{15}{4}-\frac{2}{3}=\frac{37}{12}$

15. $\int_{1/2}^3 \frac{1}{x^2}dx=-\frac{1}{x}\Bigg|_{1/2}^3=-\frac{1}{3}-(-2)=\frac{5}{3}$

16. $\int_9^{36}\left(\sqrt{x}-2\right)dx=\left(\frac{2}{3}x^{\frac{3}{2}}-2x\right)\Bigg|_9^{36}=72-0=72$

17. $\int_{-1}^1 (z+1)^5\,dz=\frac{(z+1)^6}{6}\Bigg|_{-1}^1=\frac{32}{3}-0=\frac{32}{3}$

18. $\int_1^8 \left(x^{\frac{1}{3}} - x^{-\frac{1}{3}}\right) dx = \left(\frac{3x^{\frac{4}{3}}}{4} - \frac{3x^{\frac{2}{3}}}{2}\right)\Bigg|_1^8$

$= 6 - \left(-\frac{3}{4}\right) = \frac{27}{4}$

19. $\int_0^1 2x^2\left(x^3-1\right)^3 dx = \frac{2}{3}\int_0^1 \left(x^3-1\right)^3\left[3x^2 dx\right]$

$= \frac{1}{6}\left(x^3-1\right)^4\Big|_0^1 = 0 - \frac{1}{6} = -\frac{1}{6}$

20. $\int_2^3 (x+2)^3 dx = \frac{(x+2)^4}{4}\Bigg|_2^3 = \frac{625}{4} - 64 = \frac{369}{4}$

21. $\int_1^8 \frac{4}{y} dy = 4\ln|y|\Big|_1^8 = 4(\ln 8 - \ln 1)$

$=4(\ln 8 - 0) = 4 \ln 8$

22. $\int_{-e^e}^{-1} \frac{6}{x} dx = 6\ln|x|\Big|_{-e^e}^{-1} = 6\ln 1 - 6\ln e^e$

$= 0 - 6e = -6e$

23. $\int_0^1 e^5 dx = e^5 x\Big|_0^1 = e^5 - 0 = e^5$

24. $\int_2^{e+1} \frac{1}{x-1} dx = \ln|x-1|\Big|_2^{e+1} = \ln e - \ln 1 = 1 - 0 = 1$

25. $\int_0^1 5x^2 e^{x^3} dx = \frac{5}{3}\int_0^1 e^{x^3}[3x^2 dx] = \frac{5}{3}e^{x^3}\Big|_0^1$

$= \frac{5}{3}(e^1 - e^0) = \frac{5}{3}(e-1)$

26. $\int_0^1 \left(3x^2+4x\right)\left(x^3+2x^2\right)^4 dx$

$= \int_0^1 \left(x^3+2x^2\right)^4\left[\left(3x^2+4x\right)dx\right]$

$= \frac{\left(x^3+2x^2\right)^5}{5}\Bigg|_0^1 = \frac{243}{5} - 0 = \frac{243}{5}$

27. $\int_4^5 \frac{2}{(x-3)^3} dx = 2\int_4^5 (x-3)^{-3} dx = 2\cdot\frac{(x-3)^{-2}}{-2}\Bigg|_4^5$

$= -\frac{1}{(x-3)^2}\Bigg|_4^5 = -\frac{1}{4} - (-1) = \frac{3}{4}$

28. $\int_{-1/3}^{20/3} \sqrt{3x+5}\, dx = \frac{1}{3}\int_{-1/3}^{20/3} (3x+5)^{\frac{1}{2}}[3\,dx]$

$= \frac{2}{9}(3x+5)^{\frac{3}{2}}\Big|_{-\frac{1}{3}}^{\frac{20}{3}}$

$= \frac{2}{9}(125-8) = 26$

29. $\int_{1/3}^2 \sqrt{10-3p}\,dp = -\frac{1}{3}\int_{1/3}^2 (10-3p)^{\frac{1}{2}}[-3\ dp]$

$= -\frac{2}{9}(10-3p)^{\frac{3}{2}}\Big|_{1/3}^2 = -\frac{2}{9}(8-27) = \frac{38}{9}$

30. $\int_{-1}^1 q\sqrt{q^2+3}\,dq = \frac{1}{2}\int_{-1}^1 \left(q^2+3\right)^{\frac{1}{2}}[2q\ dq]$

$= \frac{\left(q^2+3\right)^{\frac{3}{2}}}{3}\Bigg|_{-1}^1 = \frac{8}{3} - \frac{8}{3} = 0$

31. $\int_0^1 x^2\sqrt[3]{7x^3+1}\,dx = \frac{1}{21}\int_0^1 \left(7x^3+1\right)^{\frac{1}{3}}\left[21x^2 dx\right]$

$= \frac{1}{21}\cdot\frac{\left(7x^3+1\right)^{\frac{4}{3}}}{\frac{4}{3}}\Bigg|_0^1 = \frac{\left(7x^3+1\right)^{\frac{4}{3}}}{28}\Bigg|_0^1$

$= \frac{16}{28} - \frac{1}{28} = \frac{15}{28}$

32. $\int_0^{\sqrt{7}} \left(3x - \frac{x}{(x^2+2)^{4/3}}\right) dx$

$= \int_0^{\sqrt{7}} 3x\,dx - \frac{1}{2}\int_0^{\sqrt{7}} (x^2+2)^{-4/3}[2x\,dx]$

$= \frac{3}{2}x^2\Big|_0^{\sqrt{7}} - \frac{1}{2}\cdot\frac{(x^2+2)^{-1/3}}{-\frac{1}{3}}\Bigg|_0^{\sqrt{7}}$

$= \frac{3}{2}(7-0) + \frac{3}{2}[9^{-1/3} - 2^{-1/3}]$

$= \frac{3}{2}\left(7 + \frac{1}{\sqrt[3]{9}} - \frac{1}{\sqrt[3]{2}}\right)$

33. $\int_0^1 \frac{2x^3+x}{x^2+x^4+1}dx$

$=\frac{1}{2}\int_0^1 \frac{1}{x^4+x^2+1}\left[\left(4x^3+2x\right)dx\right]$

$=\frac{1}{2}\ln\left(x^4+x^2+1\right)\Big|_0^1=\frac{1}{2}[\ln 3-\ln 1]=\frac{1}{2}\ln 3$

34. $\int_a^b (m+ny)dy=\left(my+\frac{ny^2}{2}\right)\Bigg|_a^b=my\Big|_a^b+\frac{n}{2}y^2\Big|_a^b$

$=m(b-a)+\frac{n}{2}\left(b^2-a^2\right)$

35. $\int_0^1 \frac{e^x-e^{-x}}{2}dx=\frac{1}{2}(e^x+e^{-x})\Big|_0^1$

$=\frac{1}{2}[(e+e^{-1})+(1+1)]$

$=\frac{1}{2}\left(e+\frac{1}{e}+2\right)$

36. $\int_{-2}^1 8|x|\,dx=8\left(\int_{-2}^0 -x\ dx+\int_0^1 x\ dx\right)$

$=8\left(-\frac{x^2}{2}\Bigg|_{-2}^0+\frac{x^2}{2}\Bigg|_0^1\right)=8\left\{[0-(-2)]+\left(\frac{1}{2}-0\right)\right\}$

37. $\int_\pi^e 3(x^{-2}+x^{-3}-x^{-4})dx=3\left(\frac{x^{-1}}{-1}+\frac{x^{-2}}{-2}-\frac{x^{-3}}{-3}\right)\Bigg|_\pi^e$

$=3\left(-\frac{1}{e}-\frac{1}{2e^2}+\frac{1}{3e^3}\right)-3\left(-\frac{1}{\pi}-\frac{1}{2\pi^2}+\frac{1}{3\pi^3}\right)$

$=\frac{1}{e^3}-\frac{3}{e}-\frac{3}{2e^2}+\frac{3}{\pi}+\frac{3}{2\pi^2}-\frac{1}{\pi^3}$

38. $\int_1^2\left(6\sqrt{x}-\frac{1}{\sqrt{2x}}\right)dx$

$=6\int_1^2 x^{\frac{1}{2}}dx-\frac{1}{2}\int_1^2 (2x)^{-\frac{1}{2}}[2\,dx]$

$=\left[4x^{\frac{3}{2}}-(2x)^{\frac{1}{2}}\right]\Bigg|_1^2=\left(8\sqrt{2}-2\right)-\left(4-\sqrt{2}\right)$

$=9\sqrt{2}-6$

39. $\int_1^3 (x+1)e^{x^2+2x}dx=\frac{1}{2}\int_1^3 e^{x^2+2x}[(2x+2)dx]$

$=\frac{1}{2}e^{x^2+2x}\Big|_1^3=\frac{1}{2}\left(e^{15}-e^3\right)=\frac{e^3}{2}\left(e^{12}-1\right)$

40. $\int_1^{95}\frac{x}{\ln e^x}dx=\int_1^{95}\frac{x}{x}dx=\int_1^{95}1\,dx=x\Big|_1^{95}$

$=95-1=94$

41. Using long division on the integrand

$\int_0^2 \frac{x^6+6x^4+x^3+8x^2+x+5}{x^3+5x+1}dx$

$=\int_0^2\left[x^3+x+\frac{3x^2+5}{x^3+5x+1}\right]dx$

$=\left[\frac{x^4}{4}+\frac{x^2}{2}+\ln\left|x^3+5x+1\right|\right]\Bigg|_0^2$

$=(6+\ln 19)-0=6+\ln 19$

42. $\int_{-1}^1 \frac{2}{1+e^x}dx=\int_{-1}^1 \frac{2}{1+e^x}\cdot\frac{e^{-x}}{e^{-x}}dx$

$=2\int_{-1}^1 \frac{e^{-x}}{e^{-x}+1}dx$

$=-2\int_{-1}^1 \frac{1}{e^{-x}+1}\left[-e^{-x}dx\right]=-2\ln\left(e^{-x}+1\right)\Big|_{-1}^1$

$=-2\ln\left(e^{-1}+1\right)+2\ln(e+1)=2\ln\frac{e+1}{\frac{1}{e}+1}$

$=2\ln\frac{e^2+e}{1+e}=2\ln\frac{e(e+1)}{1+e}=2\ln e=2$

43. $\int_0^2 f(x)dx=\int_0^{1/2}4x^2dx+\int_{1/2}^2\ 2x\ dx$

$=\frac{4x^3}{3}\Bigg|_0^{1/2}+x^2\Bigg|_{1/2}^2=\left(\frac{1}{6}-0\right)+\left(4-\frac{1}{4}\right)=\frac{47}{12}$

44. $\left(\int_1^3 x\,dx\right)^3-\int_1^3 x^3dx=\left(\frac{x^2}{2}\Bigg|_1^3\right)^3-\frac{x^4}{4}\Bigg|_1^3$

$=\left(\frac{9}{2}-\frac{1}{2}\right)^3-\left(\frac{81}{4}-\frac{1}{4}\right)$

$=4^3-20$

$=44$

45. $f(x)=\int_1^x 3\frac{1}{t^2}\,dt=-\frac{3}{t}\Big|_1^x=-\frac{3}{x}+3=3-\frac{3}{x}$

$\int_e^1 f(x)dx=\int_e^1\left(3-\frac{3}{x}\right)dx=\left(3x-3\ln|x|\right)\Big|_e^1$

$=(3-0)-(3e-3)=6-3e$

46. $\int_7^7 e^{x^2}dx+\int_0^{\sqrt{2}}\frac{1}{3\sqrt{2}}dx=0+\frac{1}{3\sqrt{2}}\int_0^{\sqrt{2}}1\,dx$

$=\frac{1}{3\sqrt{2}}x\Big|_0^{\sqrt{2}}$

$=\frac{1}{3\sqrt{2}}\left(\sqrt{2}-0\right)$

$=\frac{1}{3}$

47. $\int_1^3 f(x)dx=\int_1^2 f(x)dx+\int_2^3 f(x)dx$, so

$\int_1^2 f(x)dx=\int_1^3 f(x)dx-\int_2^3 f(x)dx$

$=\int_1^3 f(x)dx+\int_3^2 f(x)dx=4+3=7$

48. $\int_1^4 f(x)dx=\int_1^2 f(x)dx+\int_2^4 f(x)dx$

$\int_1^4 f(x)dx=\int_1^3 f(x)dx-\int_2^3 f(x)dx+\int_2^4 f(x)dx$

$6=2-\int_2^3 f(x)dx+5$, so $\int_2^3 f(x)\,dx=7-6=1$.

49. $\int_2^3 e^{x^3}dx$ is a constant, so $\frac{d}{dx}\int_2^3 e^{x^3}dx=0$. Thus

$\int_2^3\left(\frac{d}{dx}\int_2^3 e^{x^3}dx\right)dx=\int_2^3 0\,dx=C\Big|_2^3=C-C=0$

50. $f(x)=\int_e^x\frac{e^t-e^{-t}}{e^t+e^{-t}}dt$

$=\int_e^x\frac{1}{e^t+e^{-t}}[(e^t-e^{-t})\,dt]$

$=\ln\left|e^t+e^{-t}\right|\Big|_e^x$

$=\ln(e^x+e^{-x})-\ln(e^e-e^{-e})$

$f'(x)=\frac{1}{e^x+e^{-x}}[e^x+e^{-x}(-1)]-0$

$=\frac{e^x-e^{-x}}{e^x+e^{-x}}$

51. $\int_0^T\alpha^{\frac{5}{2}}dt=\alpha^{\frac{5}{2}}t\Big|_0^T=\alpha^{\frac{5}{2}}T-0=\alpha^{\frac{5}{2}}T$

52. $\mu=\int_0^1(x\cdot 1)dx=\int_0^1 x\,dx=\frac{x^2}{2}\Big|_0^1=\frac{1}{2}-0=\frac{1}{2}$

$\sigma^2=\int_0^1(x-\mu)^2 f(x)dx=\int_0^1\left(x-\frac{1}{2}\right)^2dx$

$=\frac{1}{3}\left(x-\frac{1}{2}\right)^3\Big|_0^1=\frac{1}{3}\left[\frac{1}{8}-\left(-\frac{1}{8}\right)\right]=\frac{1}{12}$. Thus

$\mu=\frac{1}{2}$; $\sigma^2=\frac{1}{12}$

53. The total number receiving between a and b dollars equals the number $N(a)$ receiving a or more dollars minus the number $N(b)$ receiving b or more dollars. Thus

$N(a)-N(b)=\int_b^a -Ax^{-B}dx$.

54. $\int_0^{10^{-4}}x^{-\frac{1}{2}}dx=\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_0^{10^{-4}}=2\sqrt{x}\Big|_0^{10^{-4}}=2\sqrt{10^{-4}}-0$

$=2\left(10^{-2}\right)=0.02$

55. $\int_0^5 2000e^{-0.06t}dt=2000\cdot\frac{1}{-0.06}\int_0^5 e^{-0.06t}[-0.06\,dt]$

$=-\frac{2000}{0.06}e^{-0.06t}\Big|_0^5=-\frac{2000}{0.06}\left(e^{-0.03}-1\right)$

$\approx$ \$8639

56. $\int_0^t \left(e^{-a\tau} - e^{-b\tau}\right) d\tau$

$= \frac{1}{-a}\int_0^t e^{-a\tau}[-a\ d\tau] - \frac{1}{-b}\int_0^t e^{-b\tau}[-b\ d\tau]$

$= \left(-\frac{e^{-a\tau}}{a} + \frac{e^{-b\tau}}{b}\right)\Bigg|_0^t$

$= \left(-\frac{e^{-at}}{a} + \frac{e^{-bt}}{b}\right) - \left(-\frac{1}{a} + \frac{1}{b}\right)$

$= \frac{1-e^{-at}}{a} - \frac{1-e^{-bt}}{b}$

57. $\int_{36}^{64} 10,000\sqrt{100-t}\,dt$

$= (-1)(10,000)\int_{36}^{64}(100-t)^{\frac{1}{2}}[(-1)\ dt]$

$= -\frac{2}{3}(10,000)(100-t)^{\frac{3}{2}}\Big|_{36}^{64}$

$= -\frac{2}{3}(10,000)[216-512]$

$\approx$ 1,973,333

58. $\int_0^t 3000e^{0.05\tau}d\tau = 3000\cdot\frac{1}{0.05}\int_0^t e^{0.05\tau}[0.05\ d\tau]$

$= 60,000e^{0.05\tau}\Big|_0^t = 60,000\left(e^{0.05t}-1\right)$

59. $\int_{65}^{75}(0.2q+8)dq = \left(0.1q^2+8q\right)\Big|_{65}^{75}$

$= 1162.5 - 942.5 = \$220$

60. $\int_{90}^{180}(0.004q^2 - 0.5q + 50)\,dq$

$= \frac{0.004}{3}q^3 - 0.25q^2 + 50q\Big|_{90}^{180}$

$= 8676 - 3447$

$= \$5229$

61. $\int_{500}^{800}\frac{2000}{\sqrt{300q}}dq = \int_{500}^{800}\frac{2000}{10\sqrt{3q}}dq$

$= \frac{200}{\sqrt{3}}\int_{500}^{800}q^{-1/2}dq = \frac{200}{\sqrt{3}}\cdot\frac{q^{1/2}}{\frac{1}{2}}\Bigg|_{500}^{800}$

$= \frac{400}{\sqrt{3}}\sqrt{q}\Big|_{500}^{800} = \frac{400}{\sqrt{3}}\left(\sqrt{800}-\sqrt{500}\right) \approx \$1367.99$

62. $\int_{10}^{20}\left(250+90q-3q^2\right)dq = \left(250q+45q^2-q^3\right)\Big|_{10}^{20}$

$= 15,000 - 6000 = \$9000$

63. $\int_0^{12}(8t+10)dt = \left(4t^2+10t\right)\Big|_0^{12} = 696 - 0 = 696$

$\int_6^{12}(8t+10)dt = \left(4t^2+10t\right)\Big|_6^{12} = 696 - 204 = 492$

64. $\int_0^{700}\frac{81\times10^6}{(300+t)^4}dt = \left(81\times10^6\right)\int_0^{700}(300+t)^{-4}dt$

$= \left(81\times10^6\right)\frac{(300+t)^{-3}}{-3}\Bigg|_0^{700}$

$= -\left(27\times10^6\right)\frac{1}{(300+t)^3}\Bigg|_0^{700}$

$= -\left(27\times10^6\right)\left(\frac{1}{1000^3} - \frac{1}{300^3}\right)$

$= -\left(27\times10^6\right)\left(\frac{1}{10^9} - \frac{1}{27\cdot10^6}\right)$

$= -\frac{27}{10^3} + 1 = -\frac{27}{1000} + 1 = \frac{973}{1000} = 0.973$

65. $G = \int_{-R}^{R} i\ dx = ix\Big|_{-R}^{R} = iR - (-iR) = 2Ri$

66. $E = \int_{-R}^{R} \frac{i}{2}\left[e^{-k(R-x)} + e^{-k(R+x)}\right] dx$

$= \frac{i}{2}\left[\frac{1}{k}\int_{-R}^{R} e^{-k(R-x)}[k\ dx] + \frac{1}{-k}\int_{-R}^{R} e^{-k(R+x)}[-k\ dx]\right]$

$= \frac{i}{2k}\left[\int_{-R}^{R} e^{-k(R-x)}[k\ dx] - \int_{-R}^{R} e^{-k(R+x)}[-k\ dx]\right]$

$= \frac{i}{2k}\left[e^{-k(R-x)} - e^{-k(R+x)}\right]\Big|_{-R}^{R}$

$= \frac{i}{2k}\left[\left(1 - e^{-k(2R)}\right) - \left(e^{-k(2R)} - 1\right)\right]$

$= \frac{i}{2k}\left[2 - 2e^{-2kR}\right] = \frac{i}{k}\left(1 - e^{-2kR}\right)$

67. $A = \dfrac{\int_0^R (m+x)[1-(m+x)]dx}{\int_0^R [1-(m+x)]dx} = \dfrac{\int_0^R \left(m + x - m^2 - 2mx - x^2\right)dx}{\int_0^R (1-m-x)dx}$

$= \dfrac{\left[mx + \frac{x^2}{2} - m^2x - mx^2 - \frac{x^3}{3}\right]\Big|_0^R}{\left[x - mx - \frac{x^2}{2}\right]\Big|_0^R}$

$= \dfrac{\left[mR + \frac{R^2}{2} - m^2R - mR^2 - \frac{R^3}{3}\right] - 0}{\left[R - mR - \frac{R^2}{2}\right] - 0}$

$= \dfrac{R\left[m + \frac{R}{2} - m^2 - mR - \frac{R^2}{3}\right]}{R\left[1 - m - \frac{R}{2}\right]} = \dfrac{m + \frac{R}{2} - m^2 - mR - \frac{R^2}{3}}{1 - m - \frac{R}{2}}$

68. $\int_{2.5}^{3.5} (1 + 2x + 3x^2)dx = (x + x^2 + x^3)\Big|_{2.5}^{3.5}$

$= 58.625 - 24.375$

$= 34.25$

69. $\int_0^4 \frac{1}{(4x+4)^2}\,dx = \frac{1}{4}\int_0^4 (4x+4)^{-2}[4\ dx] = \frac{1}{4}\cdot\frac{(4x+4)^{-1}}{-1}\Big|_0^4$

$= -\frac{1}{4}\cdot\frac{1}{4x+4}\Big|_0^4 = -\frac{1}{16}\cdot\frac{1}{x+1}\Big|_0^4 = -\frac{1}{16}\left(\frac{1}{5} - 1\right) = \frac{1}{20} = 0.05$

70. $\int_0^1 e^{3t}\,dt = \frac{1}{3}\int_0^1 e^{3t}[3\ dt] = \frac{e^{3t}}{3}\Big|_0^1 = \frac{1}{3}\left(e^3 - 1\right) \approx 6.36$

71. 3.52

72. 0.23

73. 14.34

74. 3.64

Principles in Practice 14.8

1. In this case, $f(t)=\frac{60}{\sqrt{t^2+9}}$, $n = 5$, $a = 0$, and $b = 5$. Thus $h=\frac{b-a}{n}=\frac{5-0}{5}=1$. The terms to be added are

$f(0)=\frac{60}{\sqrt{0^2+9}}=\frac{60}{3}=20$

$2f(1)=\frac{2(60)}{\sqrt{1^2+9}}=\frac{120}{\sqrt{10}}\approx 37.9473$

$2f(2)=\frac{2(60)}{\sqrt{2^2+9}}=\frac{120}{\sqrt{13}}\approx 33.2820$

$2f(3)=\frac{2(60)}{\sqrt{3^2+9}}=\frac{120}{\sqrt{18}}\approx 28.2843$

$2f(4)=\frac{2(60)}{\sqrt{4^2+9}}=\frac{120}{5}=24$

$f(5)=\frac{60}{\sqrt{5^2+9}}=\frac{60}{\sqrt{34}}\approx 10.2899$

The sum of the above terms is 153.8035. The estimate of the radius after 5 seconds is

$\int_0^5 \frac{60}{\sqrt{t^2+9}}\,dt \approx \frac{1}{2}(153.8035)\approx 76.90$ feet.

2. In this case, $f(t)=0.3e^{0.2t^2}$, $n = 8$, $a = 0$, and $b = 4$. Thus, $h=\frac{b-a}{n}=\frac{4}{8}=0.5$. The terms to be added are

$f(0)=0.3e^0=0.3$

$4f(0.5)=4(0.3)e^{0.05}\approx 1.2615$

$2f(1)=2(0.3)e^{0.2}\approx 0.7328$

$4f(1.5)=4(0.3)e^{0.45}\approx 1.8820$

$2f(2)=2(0.3)e^{0.8}\approx 1.3353$

$4f(2.5)=4(0.3)e^{1.25}\approx 4.1884$

$2f(3)=2(0.3)e^{1.8}\approx 3.6298$

$4f(3.5)=4(0.3)e^{2.45}\approx 13.9060$

$f(4)=0.3e^{3.2}\approx 7.3598$

The sum of the above terms is 34.5956. The estimate of the amount the culture grew over the first four hours is

$\int_0^4 0.3e^{0.2t^2}\,dt \approx \frac{0.5}{3}(34.5956)\approx 5.77$ grams.

Problems 14.8

1. $f(x)=\frac{170}{1+x^2}$, $n = 6$, $a = –2$, $b = 4$. Trapezoidal

$h=\frac{b-a}{n}=\frac{4-(-2)}{6}=\frac{6}{6}=1$

$$\begin{aligned} f(-2)&=34= & 34\\ 2f(-1)&=2(85)= & 170\\ 2f(0)&=2(170)= & 340\\ 2f(1)&=2(85)= & 170\\ 2f(2)&=2(34)= & 68\\ 2f(3)&=2(17)= & 34\\ f(4)&=10= & \underline{10}\\ && 826 \end{aligned}$$

$\int_{-2}^4 \frac{170}{1+x^2}\,dx \approx \frac{1}{2}(826)=413$

2. $f(x)=\frac{170}{1+x^2}$, $n = 6$, $a = –2$, $b = 4$

Simpson's

$h=\frac{b-a}{n}=\frac{4-(-2)}{6}=\frac{6}{6}=1$

$$\begin{aligned} f(-2)&=34= & 34\\ 4f(-1)&=4(85)= & 340\\ 2f(0)&=2(170)= & 340\\ 4f(1)&=4(85)= & 340\\ 2f(2)&=2(34)= & 68\\ 4f(3)&=4(17)= & 68\\ f(4)&=10= & \underline{10}\\ && 1200 \end{aligned}$$

$\int_{-2}^4 \frac{170}{1+x^2}\,dx \approx \frac{1}{3}(1200)=400$

3. $f(x)=x^2$, $n=5$, $a=0$, $b=1$

Trapezoidal

$$h=\frac{b-a}{n}=\frac{1-0}{5}=\frac{1}{5}=0.2$$

$$\begin{aligned} f(0) &= 0.0000 \\ 2f(0.2) &= 0.0800 \\ 2f(0.4) &= 0.3200 \\ 2f(0.6) &= 0.7200 \\ 2f(0.8) &= 1.2800 \\ f(1) &= \underline{1.0000} \\ & 3.4000 \end{aligned}$$

$$\int_0^1 x^2dx \approx \frac{0.2}{2}(3.4000)=0.340$$

Actual value: $\int_0^1 x^2dx=\left.\frac{x^3}{3}\right|_0^1=\frac{1}{3}\approx 0.333$

4. $f(x)=x^2$, $n=4$, $a=0$, $b=1$

Simpson's

$$h=\frac{b-a}{n}=\frac{1-0}{4}=0.25$$

$$\begin{aligned} f(0) &= 0.0000 \\ 4f(0.25) &= 0.2500 \\ 2f(0.50) &= 0.5000 \\ 4f(0.75) &= 2.2500 \\ f(1) &= \underline{1.0000} \\ & 4.0000 \end{aligned}$$

$$\int_0^1 x^2dx \approx \frac{0.25}{3}(4.0000)=\frac{1}{3}\approx 0.333$$

Actual value: $\int_0^1 x^2dx=\left.\frac{x^3}{3}\right|_0^1=\frac{1}{3}\approx 0.333$

5. $f(x)=\frac{1}{x^2}$, $n=4$, $a=1$, $b=4$

Simpson's

$$h=\frac{b-a}{n}=\frac{4-1}{4}=0.75$$

$$\begin{aligned} f(1) &= 1.0000 \\ 4f(1.75) &= 1.3061 \\ 2f(2.50) &= 0.3200 \\ 4f(3.25) &= 0.3787 \\ f(4) &= \underline{0.0625} \\ & 3.0673 \end{aligned}$$

$$\int_1^4 \frac{1}{x^2}dx \approx \frac{0.75}{3}(3.0673)\approx 0.767$$

Actual value:

$$\int_1^4 \frac{1}{x^2}dx=-\left.\frac{1}{x}\right|_1^4=-\frac{1}{4}-(-1)=0.750$$

6. $f(x)=\frac{1}{x}$, $n=6$, $a=1$, $b=4$

Trapezoidal

$$h=\frac{b-a}{n}=\frac{4-1}{6}=0.5$$

$$\begin{aligned} f(1) &= 1.0000 \\ 2f(1.5) &= 1.3333 \\ 2f(2) &= 1.0000 \\ 2f(2.5) &= 0.8000 \\ 2f(3) &= 0.6667 \\ 2f(3.5) &= 0.5714 \\ f(4) &= \underline{0.2500} \\ & 5.6214 \end{aligned}$$

$$\int_1^4 \frac{1}{x}dx \approx \frac{0.5}{2}(5.6214)\approx 1.405$$

Actual value:

$$\int_1^4 \frac{1}{x}dx=\ln|x|\Big|_1^4=\ln 4-\ln 1=\ln 4-0=\ln 4$$

$\approx$ 1.386

7. $f(x)=\frac{x}{x+1}$, $n=4$, $a=0$, $b=2$

Trapezoidal

$$h=\frac{b-a}{n}=\frac{2-0}{4}=0.5$$

$$\begin{aligned} f(0) &= 0.0000 \\ 2f(0.5) &= 0.6667 \\ 2f(1) &= 1.0000 \\ 2f(1.5) &= 1.2000 \\ f(2) &= \underline{0.6667} \\ & 3.5334 \end{aligned}$$

Thus

$$\int_0^2 \frac{x}{x+1}dx \approx \frac{0.5}{2}(3.5334)\approx 0.883$$

8. $f(x)=\frac{1}{x+x^2}$, $n=4$, $a=2$, $b=4$

Simpson's

$$h=\frac{b-a}{n}=\frac{4-2}{4}=0.5$$

$$\begin{aligned} f(2) &= 0.1667 \\ 4f(2.5) &= 0.4571 \\ 2f(3) &= 0.1667 \\ 4f(3.5) &= 0.2540 \\ f(4) &= \underline{0.0500} \\ & \quad 1.0945 \end{aligned}$$

$$\int_2^4 \frac{dx}{x+x^2} \approx \frac{0.5}{3}(1.0945) \approx 0.182$$

9. $\int_{45}^{70} l(t)dt,$ males, $n = 5$, $a = 45$, $b = 70$

$$h = \frac{70-45}{5} = 5$$

$$\begin{aligned} l(45) &= 93,717 \\ 2l(50) &= 183,232 \\ 2l(55) &= 177,292 \\ 2l(60) &= 168,376 \\ 2l(65) &= 155,094 \\ l(70) &= \underline{68,375} \\ & \quad 846,086 \end{aligned}$$

$$\int_{45}^{70} l(t)dt \approx \frac{5}{2}(846,086) = 2,115,215$$

10. $\int_{35}^{55} l(t)dt$, females, $n = 4$, $a = 35$, $b = 55$

$$h = \frac{55-35}{4} = 5$$

$$\begin{aligned} l(35) &= 97,964 \\ 2l(40) &= 194,796 \\ 2l(45) &= 193,164 \\ 2l(50) &= 190,784 \\ l(55) &= \underline{93,562} \\ & \quad 770,270 \end{aligned}$$

$$\int_{35}^{55} l(t)dt \approx \frac{5}{2}(770,270) = 1,925,675$$

11. $a = 1$, $b = 5$, $h = 1$

$$\begin{aligned} f(1) &= 0.4 &&= 0.4 \\ 4f(2) &= 4(0.6) &&= 2.4 \\ 2f(3) &= 2(1.2) &&= 2.4 \\ 4f(4) &= 4(0.8) &&= 3.2 \\ f(5) &= 0.5 &&= \underline{0.5} \\ & && \quad 8.9 \end{aligned}$$

$$\int_1^5 f(x)dx \approx \frac{1}{3}(8.9) \approx 3.0$$

The area is about 3.0 square units.

12. $a = 2$, $b = 5$, $h = 0.5$

$$\begin{aligned} f(2) &= 0 \\ 4f(2.5) &= 24 \\ 2f(3) &= 20 \\ 4f(3.5) &= 44 \\ 2f(4) &= 28 \\ 4f(4.5) &= 60 \\ f(5) &= \underline{16} \\ & \quad 192 \end{aligned}$$

$$\int_2^5 f(x)\,dx \approx \frac{0.5}{3}(192) = 32$$

The area is about 32 square units.

13. $\int_1^3 f(x)dx$, $n = 4$, $a = 1$, $b = 3$

$$h = \frac{3-1}{4} = 0.5$$

$$\begin{aligned} f(1) &= 1 &&= 1 \\ 4f(1.5) &= 4(2) &&= 8 \\ 2f(2) &= 2(2) &&= 4 \\ 4f(2.5) &= 4(0.5) &&= 2 \\ f(3) &= 1 &&= \underline{1} \\ & && \quad 16 \end{aligned}$$

$$\int_1^3 f(x)dx \approx \frac{0.5}{3}(16) = \frac{8}{3}$$

14. $f(x) = \dfrac{2}{\sqrt{1+x}}$, $a = 1$, $b = 3$, $n = 4$

$$h = \frac{3-1}{4} = 0.5$$

Simpson's

$$\begin{aligned} f(1) &\approx 1.4142 \\ 4f(1.5) &\approx 5.0596 \\ 2f(2) &\approx 2.3094 \\ 4f(2.5) &\approx 4.2762 \\ f(3) &= \underline{1.0000} \\ & \quad 14.0594 \end{aligned}$$

$$\int_1^3 \frac{2}{\sqrt{1+x}}dx \approx \frac{0.5}{3}(14.0594) \approx 2.343$$

For the actual value, we have

$$\int_1^3 \frac{2}{\sqrt{1+x}}dx = 2\int_1^3 (1+x)^{-1/2}dx$$

$$= 2[2(1+x)^{1/2}]\Big|_1^3 = 4\left(2-\sqrt{2}\right) \approx 2.343$$

15. $f(x)=\sqrt{1-x^2}$, $a = 0$, $b = 1$, $n = 4$

$h=\frac{1-0}{4}=0.25$

Simpson's

$$\begin{aligned} f(0) &= 1.0000 \\ 4f(0.25) &= 3.8730 \\ 2f(0.50) &= 1.7321 \\ 4f(0.75) &= 2.6458 \\ f(1) &= \underline{0.0000} \\ & 9.2509 \end{aligned}$$

$$\int_0^1 \sqrt{1-x^2}\,dx \approx \frac{0.25}{3}(9.2509) \approx 0.771$$

16. $\int_0^{80} \frac{dr}{dq}\,dq = r(80)-r(0) = r(80)$

[since $r(0) = 0$]

Using Simpson's rule with $h = 10$ and

$f(q)=\frac{dr}{dq}$:

$$\begin{aligned} f(0) &= 10 &&= 10 \\ 4f(10) &= 4(9) &&= 36 \\ 2f(20) &= 2(8.5) &&= 17 \\ 4f(30) &= 4(8) &&= 32 \\ 2f(40) &= 2(8.5) &&= 17 \\ 4f(50) &= 4(7.5) &&= 30 \\ 2f(60) &= 2(7) &&= 14 \\ 4f(70) &= 4(6.5) &&= 26 \\ f(80) &= 7 &&= \underline{7} \\ & && 189 \end{aligned}$$

$$\int_0^{80} \frac{dr}{dq}\,dq \approx \frac{10}{3}(189) = 630$$

The total revenue is about \$630.

17. Let $f(x)$ = distance from near to far shore at point x on highway. Then area $\approx \int_0^4 f(x)dx$. Using Simpson's rule with $h = 0.5$:

$$\begin{aligned} f(0) &= 0.5-0.5 &&= 0 &&= 0 \\ 4f(0.5) &= 4(2.3-0.3) &&= 4(2) &&= 8 \\ 2f(1) &= 2(2.2-0.7) &&= 2(1.5) &&= 3 \\ 4f(1.5) &= 4(3-1) &&= 4(2) &&= 8 \\ 2f(2) &= 2(2.5-0.5) &&= 2(2) &&= 4 \\ 4f(2.5) &= 4(2.2-0.2) &&= 4(2) &&= 8 \\ 2f(3) &= 2(1.5-0.5) &&= 2(1) &&= 2 \\ 4f(3.5) &= 4(1.3)-0.8) &&= 4(0.5) &&= 2 \\ f(4) &= 1-1 &&= 0 &&= \underline{0} \\ & && && 35 \end{aligned}$$

$$\text{Area } \approx \int_0^4 f(x)dx \approx \frac{0.5}{3}(35) = \frac{35}{6}\text{ km}^2$$

18. a. $MC=\frac{dc}{dq}$

$\int_0^{100} \frac{dc}{dq}\,dq$

$= c(100)-c(0)$

$=$ (total cost of 100 units) $-$ (fixed costs)

$=$ total variable costs of 100 units

Using the trapezoidal rule with $h = 20$ and

$f(q)=\frac{dc}{dq}$ to estimate the integral:

$$\begin{aligned} f(0) &= 260 \\ 2f(20) &= 500 \\ 2f(40) &= 480 \\ 2f(60) &= 400 \\ 2f(80) &= 480 \\ f(100) &= \underline{250} \\ & 2370 \end{aligned}$$

$$\int_0^{100} \frac{dc}{dq}\,dq \approx \frac{20}{2}(2370) = \$23,700$$

b. $MR=\frac{dr}{dq}$

$\int_0^{100} \frac{dr}{dq}\,dq = r(100)-r(0) = r(100)$

[since $r(0) = 0$]

$=$ total revenue from sale of 100 units

Using the trapezoidal rule with $h = 20$ and

$g(q)=\frac{dr}{dq}$ to estimate the integral:

$$\begin{array}{rcr} g(0) & = & 410 \\ 2g(20) & = & 700 \\ 2g(40) & = & 600 \\ 2g(60) & = & 500 \\ 2g(80) & = & 540 \\ g(100) & = & 250 \\ \hline & & 3000 \end{array}$$

$$\int_0^{100} \frac{dr}{dq}\,dq \approx \frac{20}{2}(3000) = \$30{,}000$$

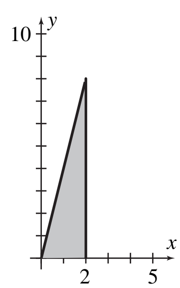
c. At $q = 100$: total revenue = 30,000
total cost = (total var. costs) + (fixed costs)
= 23,700 + 2000 = 25,700
Thus maximum profit
= (total revenue) − (total costs)
= 30,000 − 25,700 = $4300.

Problems 14.9

In Problems 1–34, answers are assumed to be expressed in square units.

1. $y = 4x$, $x = 2$

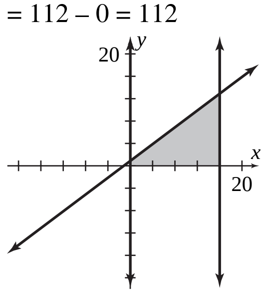
Area $= \int_0^2 4x\,dx = 2x^2\Big|_0^2 = 8 - 0 = 8$



2. $y = \frac{3}{4}x + 1$, $x = 0$, $x = 16$

Area $= \int_0^{16}\left(\frac{3}{4}x + 1\right)dx = \left(\frac{3x^2}{8} + x\right)\Bigg|_0^{16}$

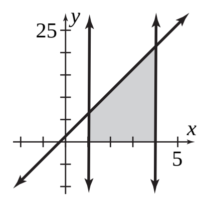
$= 112 - 0 = 112$



3. $y = 5x + 2$, $x = 1$, $x = 4$

Area $= \int_1^4 (5x + 2)dx$

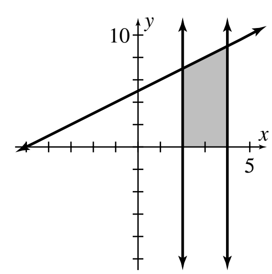
$= \left(\frac{5x^2}{2} + 2x\right)\Bigg|_1^4 = 48 - \frac{9}{2} = \frac{87}{2}$



4. $y = x + 5$, $x = 2$, $x = 4$

Area $= \int_2^4 (x + 5)dx = \left(\frac{x^2}{2} + 5x\right)\Bigg|_2^4$

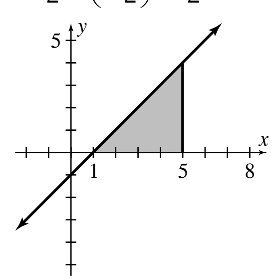
$= 28 - 12 = 16$



5. $y = x - 1$, $x = 5$

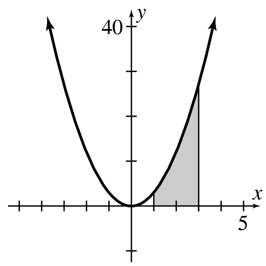
Area $= \int_1^5 (x - 1)dx = \left(\frac{x^2}{2} - x\right)\Bigg|_1^5$

$= \frac{15}{2} - \left(-\frac{1}{2}\right) = \frac{16}{2} = 8$



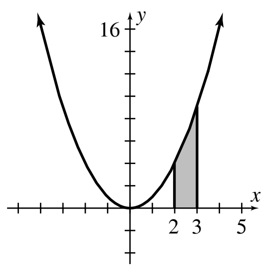
6. $y = 3x^2,\ x = 1, x = 3$

$\text{Area} = \int_1^3 3x^2\,dx = x^3\Big|_1^3 = 27 - 1 = 26$



7. $y = x^2,\ x = 2,\ x = 3$

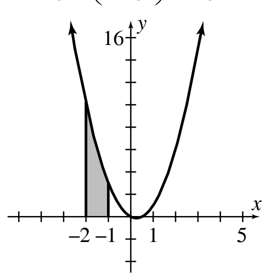
$\text{Area} = \int_2^3 x^2 dx = \frac{x^3}{3}\Big|_2^3 = 9 - \frac{8}{3} = \frac{19}{3}$



8. $y = 2x^2 - x$, $x = -2, x = -1$

$\text{Area} = \int_{-2}^{-1}\left(2x^2 - x\right)dx = \left(\frac{2x^3}{3} - \frac{x^2}{2}\right)\Big|_{-2}^{-1}$

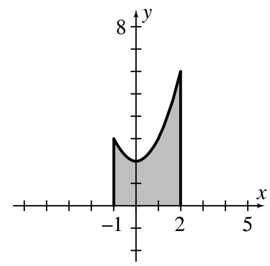
$= -\frac{7}{6} - \left(-\frac{44}{6}\right) = \frac{37}{6}$



9. $y = x^2 + 2$, $x = -1, x = 2$

$\text{Area} = \int_{-1}^{2}\left(x^2 + 2\right)dx = \left(\frac{x^3}{3} + 2x\right)\Big|_{-1}^{2}$

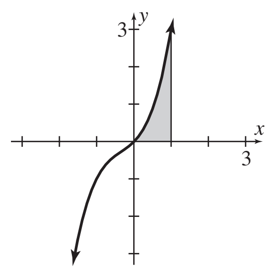
$= \frac{20}{3} - \left(-\frac{7}{3}\right) = \frac{27}{3} = 9$



10. $y = x + x^2 + x^3,\ x = 1$

$\text{Area} = \int_0^1 (x + x^2 + x^3)dx = \left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}\right)\Big|_0^1$

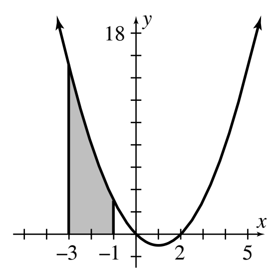
$= \frac{13}{12} - 0 = \frac{13}{12}$



11. $y = x^2 - 2x$, $x = -3, x = -1$

$\text{Area} = \int_{-3}^{-1}\left(x^2 - 2x\right)dx = \left(\frac{x^3}{3} - x^2\right)\Big|_{-3}^{-1}$

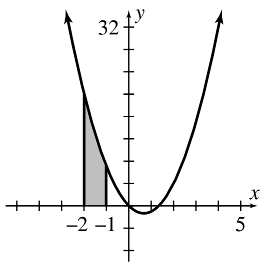
$= -\frac{4}{3} - (-18) = \frac{50}{3}$



12. $y = 3x^2 - 4x$, $x = -2$, $x = -1$

Area $= \int_{-2}^{-1} \left(3x^2 - 4x\right) dx = \left(x^3 - 2x^2\right)\Big|_{-2}^{-1}$

$= -3 - (-16) = 13$

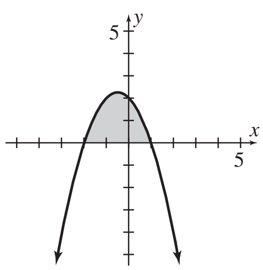


13. $y = 2 - x - x^2$

Area $= \int_{-2}^{1} (2 - x - x^2)dx = \left(2x - \frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_{-2}^{1}$

$= \frac{7}{6} - \left(-\frac{10}{3}\right)$

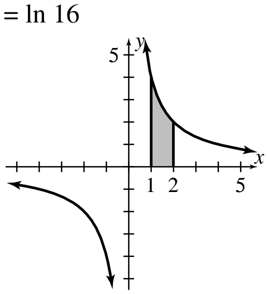
$= \frac{9}{2}$



14. $y = \frac{4}{x}$, $x = 1$, $x = 2$

Area $= \int_{1}^{2} \frac{4}{x} dx = 4 \ln|x|\Big|_{1}^{2} = 4 \ln(2) - 0 = 4 \ln 2$

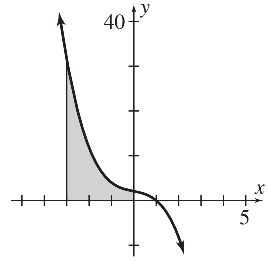
$= \ln 16$



15. $y = 2 - x - x^3$, $x = -3$, $x = 0$

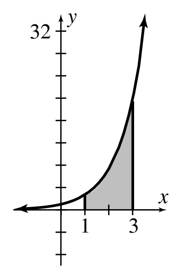
Area $= \int_{-3}^{0} \left(2 - x - x^3\right) dx = \left(2x - \frac{x^2}{2} - \frac{x^4}{4}\right)\Bigg|_{-3}^{0}$

$= 0 - \left(-\frac{123}{4}\right) = \frac{123}{4}$



16. $y = e^x$, $x = 1$, $x = 3$

Area $= \int_{1}^{3} e^x dx = e^x\Big|_{1}^{3} = e^3 - e$

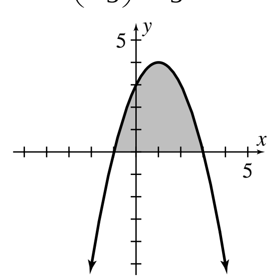


17. $A = 3 + 2x - x^2$

Area $= \int_{-1}^{3} \left(3 + 2x - x^2\right) dx$

$= \left(3x + x^2 - \frac{x^3}{3}\right)\Bigg|_{-1}^{3}$

$= 9 - \left(-\frac{5}{3}\right) = \frac{32}{3}$

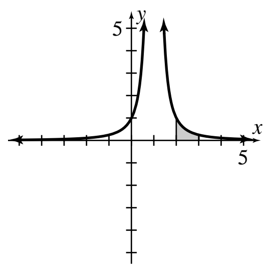


18. $y = \dfrac{1}{(x-1)^2},\ x = 2, x = 3$

$$\text{Area} = \int_2^3 \frac{1}{(x-1)^2}\,dx = \int_2^3 (x-1)^{-2}\,dx$$

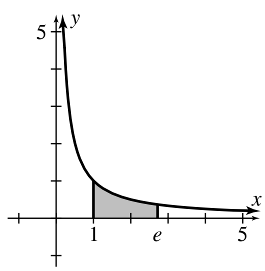
$$= \left.\frac{(x-1)^{-1}}{-1}\right|_2^3 = \left.\left(-\frac{1}{x-1}\right)\right|_2^3$$

$$= -\frac{1}{2} - (-1) = \frac{1}{2}$$



19. $y = \dfrac{1}{x}$, $x = 1$, $x = e$

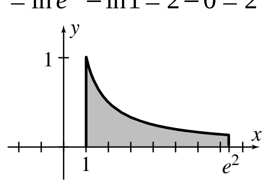
$$\text{Area} = \int_1^e \frac{1}{x}\,dx = \ln|x|\Big|_1^e = \ln e - \ln 1 = 1 - 0 = 1$$



20. $y = \dfrac{1}{x},\ x = 1,\ x = e^2$

$$\text{Area} = \int_1^{e^2} \frac{1}{x}\,dx = \ln|x|\Big|_1^{e^2}$$

$$= \ln e^2 - \ln 1 = 2 - 0 = 2$$

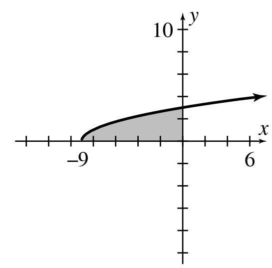


21. $y = \sqrt{x+9}$, $x = -9$, $x = 0$

$$\text{Area} = \int_{-9}^0 \sqrt{x+9}\,dx = \int_{-9}^0 (x+9)^{\frac{1}{2}}\,dx$$

$$= \left.\frac{(x+9)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{-9}^0 = \left.\frac{2(x+9)^{\frac{3}{2}}}{3}\right|_{-9}^0$$

$$= 18 - 0 = 18$$

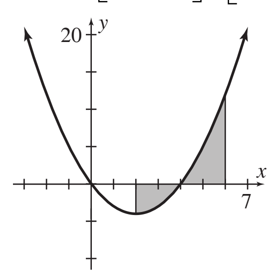


22. $y = x^2 - 4x$, $x = 2$, $x = 6$

$$\text{Area} = \int_2^4 -(x^2 - 4x)dx + \int_4^6 (x^2 - 4x)dx$$

$$= \left.\left(-\frac{x^3}{3} + 2x^2\right)\right|_2^4 + \left.\left(\frac{x^3}{3} - 2x^2\right)\right|_4^6$$

$$= \left[\frac{32}{3} - \frac{16}{3}\right] + \left[0 - \left(-\frac{32}{3}\right)\right] = 16$$

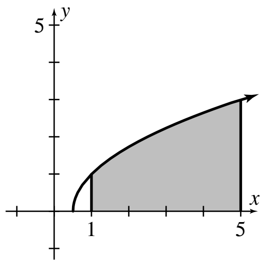


23. $y = \sqrt{2x-1}$, $x = 1$, $x = 5$

$$\text{Area} = \int_1^5 \sqrt{2x-1}\,dx$$

$$= \frac{1}{2}\int_1^5 (2x-1)^{\frac{1}{2}}[2\ dx]$$

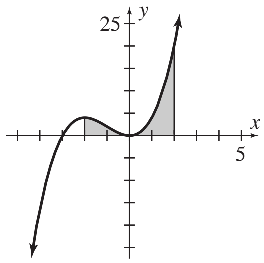
$$= \left.\frac{(2x-1)^{\frac{3}{2}}}{3}\right|_1^5 = 9 - \frac{1}{3} = \frac{26}{3}$$



24. $y = x^3 + 3x^2$, $x = -2$, $x = 2$

$$\text{Area} = \int_{-2}^{2}\left(x^3 + 3x^2\right)dx = \left(\frac{x^4}{4} + x^3\right)\Bigg|_{-2}^{2}$$

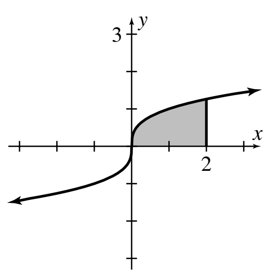
$= 12 - (-4) = 16$



25. $y = \sqrt[3]{x}$, $x = 2$

$$\text{Area} = \int_0^2 \sqrt[3]{x}\,dx = \int_0^2 x^{\frac{1}{3}}\,dx = \frac{3x^{\frac{4}{3}}}{4}\Bigg|_0^2 = \frac{3(2)^{\frac{4}{3}}}{4} - 0$$

$$= \frac{3\left(2\sqrt[3]{2}\right)}{4} = \frac{3}{2}\sqrt[3]{2}$$



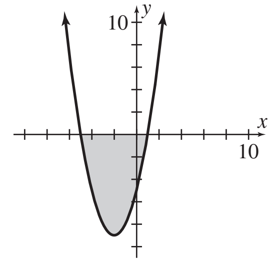
26. $y = x^2 + 4x - 5$, $x = -5$, $x = 1$

$$\text{Area} = \int_{-5}^{1} -(x^2 + 4x - 5)\,dx$$

$$= -\left(\frac{x^3}{3} + 2x^2 - 5x\right)\Bigg|_{-5}^{1}$$

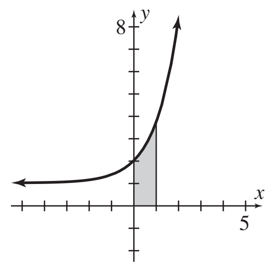
$$= -\left(-\frac{8}{3} - \frac{100}{3}\right)$$

$= 36$



27. $y = e^x + 1$, $x = 0$, $x = 1$

$$\text{Area} = \int_0^1 (e^x + 1)dx = (e^x + x)\Big|_0^1 = (e^1 + 1) - 1 = e$$

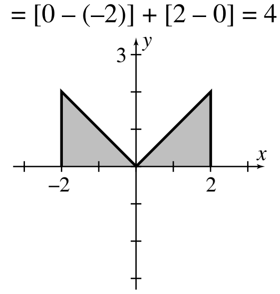


28. $y = |x|$, $x = -2$, $x = 2$

$$\text{Area} = \int_{-2}^{2} |x|\,dx = \int_{-2}^{0} (-x)dx + \int_0^2 x\,dx$$

$$= -\frac{x^2}{2}\Bigg|_{-2}^{0} + \frac{x^2}{2}\Bigg|_0^2$$

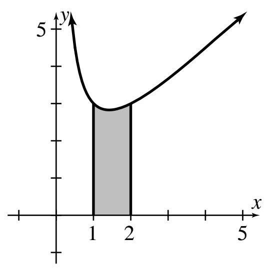
$= [0 - (-2)] + [2 - 0] = 4$



29. $y = x + \frac{2}{x}$, $x = 1$, $x = 2$

$$\text{Area } = \int_1^2 \left(x + \frac{2}{x}\right) dx = \left(\frac{x^2}{2} + 2\ln|x|\right)\bigg|_1^2$$

$$= (2 + 2\ln 2) - \frac{1}{2} = \frac{3}{2} + 2\ln 2 = \frac{3}{2} + \ln 4$$



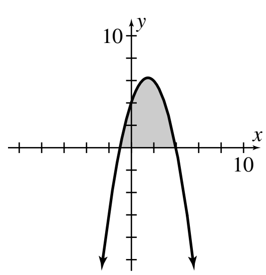
30. $y = 4 + 3x - x^2$

$$\text{Area} = \int_{-1}^4 (4 + 3x - x^2) dx$$

$$= \left(4x + \frac{3x^2}{2} - \frac{x^3}{3}\right)\bigg|_{-1}^4$$

$$= \frac{56}{3} - \left(-\frac{13}{6}\right)$$

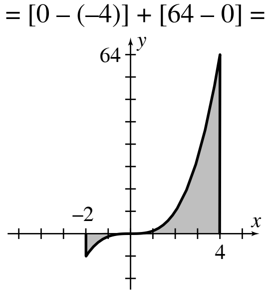
$$= \frac{125}{6}$$



31. $y = x^3$, $x = -2$, $x = 4$

$$\text{Area } = \int_{-2}^0 -x^3 dx + \int_0^4 x^3 dx = -\frac{x^4}{4}\bigg|_{-2}^0 + \frac{x^4}{4}\bigg|_0^4$$

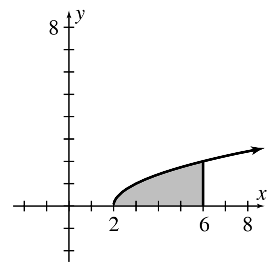
$= [0 - (-4)] + [64 - 0] = 68$



32. $y = \sqrt{x-2}$, $x = 2$, $x = 6$

$$\text{Area } = \int_2^6 \sqrt{x-2}\, dx = \frac{2(x-2)^{\frac{3}{2}}}{3}\bigg|_2^6$$

$$= \frac{16}{3} - 0 = \frac{16}{3}$$

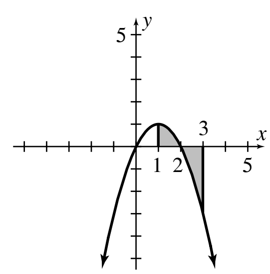


33. $y = 2x - x^2$, $x = 1$, $x = 3$

$$\text{Area } = \int_1^2 \left(2x - x^2\right) dx + \int_2^3 -\left(2x - x^2\right) dx$$

$$= \left(x^2 - \frac{x^3}{3}\right)\bigg|_1^2 - \left(x^2 - \frac{x^3}{3}\right)\bigg|_2^3$$

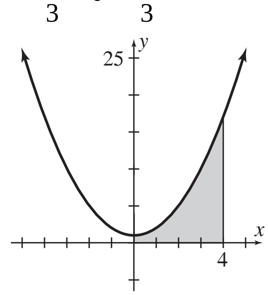
$$= \left[\frac{4}{3} - \frac{2}{3}\right] - \left[0 - \frac{4}{3}\right] = \frac{6}{3} = 2$$



34. $y = x^2 + 1$, $x = 0$, $x = 4$

$$\text{Area } = \int_0^4 \left(x^2 + 1\right) dx = \left(\frac{x^3}{3} + x\right)\bigg|_0^4$$

$$= \frac{76}{3} - 0 = \frac{76}{3}$$

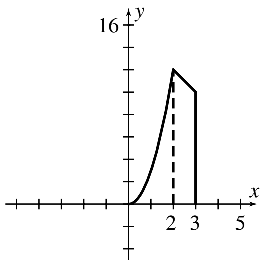


35. $f(x) = \begin{cases} 3x^2 & \text{if } 0 \le x < 2 \\ 16-2x & \text{if } x \ge 2 \end{cases}$

Area $= \int_0^3 f(x)dx = \int_0^2 3x^2 dx + \int_2^3 (16-2x)dx$

$= x^3\Big|_0^2 + \left(16x - x^2\right)\Big|_2^3$

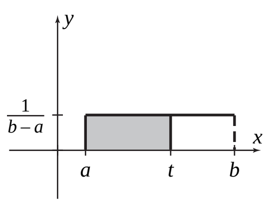
= [8 – 0] + [39 – 28] = 19 sq units



36. $y = \frac{1}{b-a}$

Area $= \int_a^t \frac{1}{b-a}dx = \frac{x}{b-a}\Big|_a^t$

$= \frac{t}{b-a} - \frac{a}{b-a} = \frac{t-a}{b-a}$ sq units



37. a. $P(0 \le x \le 1) = \int_0^1 \frac{1}{8}x\,dx = \frac{x^2}{16}\Big|_0^1 = \frac{1}{16} - 0$

$= \frac{1}{16}$

b. $P(2 \le x \le 4) = \int_2^4 \frac{1}{8}x\,dx = \frac{x^2}{16}\Big|_2^4 = 1 - \frac{1}{4} = \frac{3}{4}$

c. $P(x \ge 3) = \int_3^4 \frac{1}{8}x\,dx = \frac{x^2}{16}\Big|_3^4 = 1 - \frac{9}{16} = \frac{7}{16}$

38. a. $P(1 \le x \le 2) = \int_1^2 \frac{1}{3}(1-x)^2\,dx$

$= \frac{1}{3}(-1)\int_1^2 (1-x)^2[-dx] = -\frac{1}{3}\cdot\frac{(1-x)^3}{3}\Big|_1^2$

$= -\frac{1}{9}(-1-0) = \frac{1}{9}$

b. $P\left(1 \le x \le \frac{5}{2}\right) = \int_1^{5/2} \frac{1}{3}(1-x)^2\,dx$

$= -\frac{1}{9}(1-x)^3\Big|_1^{5/2} = -\frac{1}{9}\left(-\frac{27}{8} - 0\right) = \frac{3}{8}$

c. $P(x \le 1) = \int_0^1 \frac{1}{3}(1-x)^2\,dx = -\frac{1}{9}(1-x)^3\Big|_0^1$

$= -\frac{1}{9}(0-1) = \frac{1}{9}$

d. $\int_0^3 f(x)dx = \int_0^1 f(x)dx + \int_1^3 f(x)dx$

$1 = \frac{1}{9} + P(x \ge 1)$

Thus, $P(x \ge 1) = \frac{8}{9}$.

39. a. $P(3 \le x \le 7) = \int_3^7 \frac{1}{x}dx = \ln|x|\Big|_3^7$

$= \ln 7 - \ln 3 = \ln\frac{7}{3}$

b. $P(x \le 5) = \int_e^5 \frac{1}{x}dx = \ln|x|\Big|_e^5$

$= \ln(5) - \ln e = \ln(5) - 1$

c. $P(x \ge 4) = \int_4^{e^2} \frac{1}{x}dx = \ln|x|\Big|_4^{e^2}$

$= \ln e^2 - \ln 4 = 2 - \ln 4$

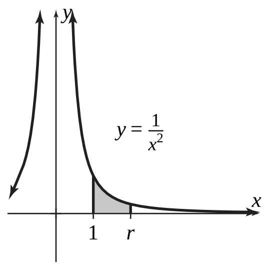
d. $P\left(e \le x \le e^2\right) = \int_e^{e^2} \frac{1}{x}dx$

$= \ln|x|\Big|_e^{e^2} = \ln e^2 - \ln e$

= 2 – 1 = 1

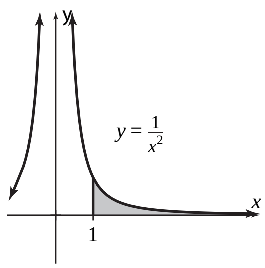
40. a. $\int_1^r \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^r = -\frac{1}{r} + 1 = 1 - \frac{1}{r}$

b.



c. $\lim\limits_{r\to\infty} \int_1^r \frac{1}{x^2}\,dx = \lim\limits_{r\to\infty}\left(1 - \frac{1}{r}\right)$ [from part (a)]
$= 1 - 0 = 1$

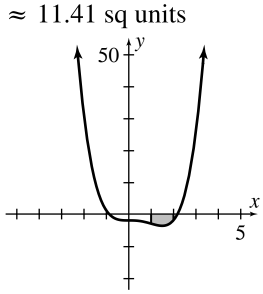
d.



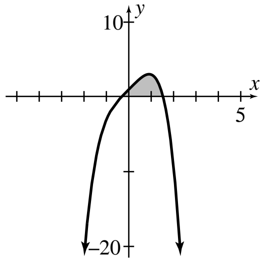
41. 1.89 sq units

42. 7.18 sq units

43. The x-intercept on [1, 3] is $A \approx 2.190327947$
Area $= \int_1^A -\left(x^4 - 2x^3 - 2\right)dx + \int_A^3 \left(x^4 - 2x^3 - 2\right)dx$
≈ 11.41 sq units



44. The x-intercepts are $A \approx -0.3294085282$ and $B \approx 1.539613346$
Area $= \int_A^B \left(1 + 3x - x^4\right)dx \approx 3.53$ sq units



Problems 14.10

1. Area $= \int_a^b (y_{\text{UPPER}} - y_{\text{LOWER}})\,dx = \int_{-2}^{3} \left[(x+6) - x^2\right] dx$

2. Area $= \int_a^b (y_{\text{UPPER}} - y_{\text{LOWER}})\,dx = \int_0^2 \left(2x - x^2\right) dx$

3. Intersection points:
$x^2 - x = 2x,\ x^2 - 3x = 0,\ x(x-3) = 0 \Rightarrow x = 0$ or $x = 3$
Area $= \int_0^3 (y_{\text{UPPER}} - y_{\text{LOWER}})\,dx + \int_3^4 (y_{\text{UPPER}} - y_{\text{LOWER}})\,dx$
$= \int_0^3 \left[2x - \left(x^2 - x\right)\right] dx + \int_3^4 \left[\left(x^2 - x\right) - 2x\right] dx$

4. Intersection points: $x(x-3)^2 = 2x$, $x(x-3)^2 - 2x = 0$, $x\left[(x-3)^2 - 2\right] = 0$, $x(x^2 - 6x + 7) = 0 \Rightarrow x = 0$, $3 \pm \sqrt{2}$
(from the quadratic formula)
Area $= \int_0^{3-\sqrt{2}} (y_{\text{UPPER}} - y_{\text{LOWER}})\,dx + \int_{3-\sqrt{2}}^{3+\sqrt{2}} (y_{\text{UPPER}} - y_{\text{LOWER}})\,dx$
$= \int_0^{3-\sqrt{2}} \left[x(x-3)^2 - 2x\right] dx + \int_{3-\sqrt{2}}^{3+\sqrt{2}} \left[2x - x(x-3)^2\right] dx$

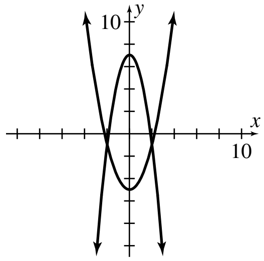
5. The graphs of $y = 1 - x^2$ and $y = x - 1$ intersect when $1 - x^2 = x - 1$, $0 = x^2 + x - 2$, $0 = (x-1)(x+2) \Rightarrow x = 1$ or $x = -2$. When $x = 1$, then $y = 0$. We use horizontal elements, where y ranges from 0 to 1. Solving $y = x - 1$ for x gives $x = y + 1$, and solving $y = 1 - x^2$ for x gives $x^2 = 1 - y$, $x = \pm\sqrt{1-y}$. We must choose $x = \sqrt{1-y}$ because x is not negative over the given region.
Area $= \int_0^1 (x_{\text{RIGHT}} - x_{\text{LEFT}})\,dy = \int_0^1 \left[(y+1) - \sqrt{1-y}\right] dy$

6. The graphs of $y = 2x$ and $y = -2x - 8$ intersect when $2x = -2x - 8$, $4x = -8$, $x = -2$. When $x = -2$, then $y = -4$. We use horizontal elements, where y ranges from -4 to 4. Solving $y = 2x$ for x gives $x = \frac{y}{2}$; solving $y = -2x - 8$ for x gives $2x = -y - 8$, $x = \frac{-y-8}{2}$.
Area $= \int_{-4}^{4} (x_{\text{RIGHT}} - x_{\text{LEFT}})\,dy = \int_{-4}^{4} \left[\frac{y}{2} - \left(\frac{-y-8}{2}\right)\right] dy$

7. The graphs of $y = x^2 - 5$ and $y = 7 - 2x^2$ intersect when $x^2 - 5 = 7 - 2x^2$, $3x^2 = 12$, $x^2 = 4$, so $x = \pm\sqrt{4} = \pm 2$. We use vertical elements.

$$\text{Area} = \int_1^2 (y_{\text{UPPER}} - y_{\text{LOWER}})\,dx$$

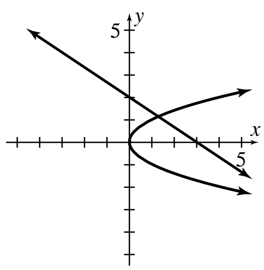
$$= \int_1^2 [(7 - 2x^2) - (x^2 - 5)]\,dx$$



8. The curves $y^2 = x$ and $2y = 3 - x$ (or $x = 3 - 2y$) intersect when $y^2 = 3 - 2y$, $y^2 + 2y - 3 = 0$, $(y + 3)(y - 1) = 0 \Rightarrow y = -3$ or 1. We use horizontal elements.

$$\text{Area} = \int_0^1 (x_{\text{RIGHT}} - x_{\text{LEFT}})\,dy$$

$$= \int_0^1 [(3 - 2y) - y^2]\,dy$$



In Problems 9–34, the answers are assumed to be expressed in square units.

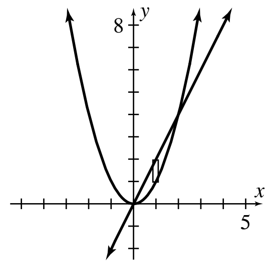
9. $y = x^2$, $y = 2x$

Region appears below.

Intersection: $x^2 = 2x$, $x^2 - 2x = 0$, $x(x - 2) = 0$, so $x = 0$ or 2.

$$\text{Area} = \int_0^2 \left(2x - x^2\right) dx = \left(x^2 - \frac{x^3}{3}\right)\Bigg|_0^2$$

$$= \left(4 - \frac{8}{3}\right) - 0 = \frac{4}{3}$$



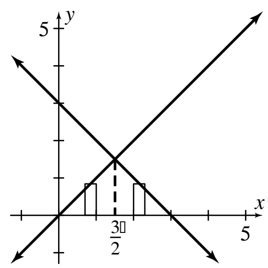
10. $y = x$, $y = -x + 3$, $y = 0$. Region appears below.

Intersection: $x = -x + 3$, $2x = 3$, $x = \frac{3}{2}$

$$\text{Area} = \int_0^{3/2} x\,dx + \int_{3/2}^3 (-x + 3)\,dx$$

$$= \frac{x^2}{2}\Bigg|_0^{3/2} + \left(-\frac{x^2}{2} + 3x\right)\Bigg|_{3/2}^3$$

$$= \left[\frac{9}{8} - 0\right] + \left[\left(-\frac{9}{2} + 9\right) - \left(-\frac{9}{8} + \frac{9}{2}\right)\right] = \frac{9}{4}$$

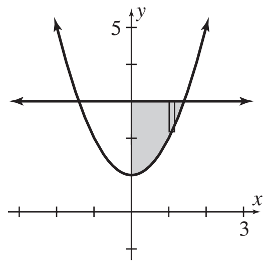


11. $y = x^2 + 1$, $x \geq 0$, $x = 0$, $y = 3$. Region appears below.

Intersection: $x^2 + 1 = 3$, so $x = \pm\sqrt{2}$

$$\text{Area} = \int_0^{\sqrt{2}} [3 - (x^2 + 1)]\,dx = \int_0^{\sqrt{2}} (2 - x^2)\,dx$$

$$= \left(2x - \frac{x^3}{3}\right)\Bigg|_0^{\sqrt{2}} = \frac{4\sqrt{2}}{3} - 0 = \frac{4\sqrt{2}}{3}$$



12. $y = x^2 + 1$, $y = x + 3$. Region appears below.

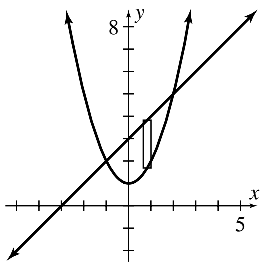
Intersection: $x^2 + 1 = x + 3$, $x^2 - x - 2 = 0$,
$(x + 1)(x - 2) = 0$, so $x = -1, 2$

$$\text{Area } = \int_{-1}^{2}\left[(x+3)-\left(x^2+1\right)\right]dx$$

$$= \int_{-1}^{2}\left(x+2-x^2\right)dx$$

$$= \left(\frac{x^2}{2}+2x-\frac{x^3}{3}\right)\Bigg|_{-1}^{2}$$

$$= \left(2+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)=\frac{9}{2}$$



13. $y = 10 - x^2$, $y = 4$. Region appears below.

Intersection: $10 - x^2 = 4$, $x^2 = 6$, so $x = \pm\sqrt{6}$

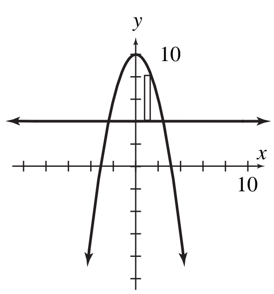
$$\text{Area} = \int_{-\sqrt{6}}^{\sqrt{6}}[(10-x^2)-4]\,dx$$

$$= \int_{-\sqrt{6}}^{\sqrt{6}}(6-x^2)\,dx$$

$$= \left(6x-\frac{x^3}{3}\right)\Bigg|_{-\sqrt{6}}^{\sqrt{6}}$$

$$= \left(6\sqrt{6}-\frac{6\sqrt{6}}{3}\right)-\left(-6\sqrt{6}+\frac{6\sqrt{6}}{3}\right)$$

$$= 8\sqrt{6}$$

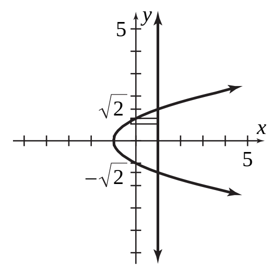


14. $y^2 = x + 1$, $x = 1$. Region appears below.

Intersection: $y^2 = 2$, $y = \pm\sqrt{2}$

$$\text{Area } = \int_{-\sqrt{2}}^{\sqrt{2}}\left[1-\left(y^2-1\right)\right]dy = \left(2y-\frac{y^3}{3}\right)\Bigg|_{-\sqrt{2}}^{\sqrt{2}}$$

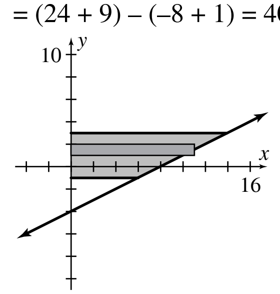
$$= \left(2\sqrt{2}-\frac{2\sqrt{2}}{3}\right)-\left(-2\sqrt{2}+\frac{2\sqrt{2}}{3}\right)=\frac{8\sqrt{2}}{3}$$



15. $x = 8 + 2y$, $x = 0$, $y = -1$, $y = 3$. Region appears below.

$$\text{Area } = \int_{-1}^{3}(8+2y)dy = \left(8y+y^2\right)\Big|_{-1}^{3}$$

$= (24 + 9) - (-8 + 1) = 40$



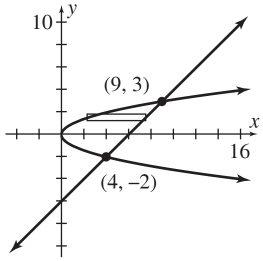
16. $y = x - 6$, $y^2 = x$. Region appears below.

Intersection: $y^2 = y + 6$, $y^2 - y - 6 = 0$,
$(y + 2)(y - 3) = 0$, so $y = -2, 3$.

$$\text{Area } = \int_{-2}^{3}\left[(y+6)-(y^2)\right]dy$$

$$= \left(\frac{y^2}{2}+6y-\frac{y^3}{3}\right)\Bigg|_{-2}^{3}$$

$$= \left(\frac{9}{2}+18-9\right)-\left(2-12+\frac{8}{3}\right)=\frac{125}{6}$$



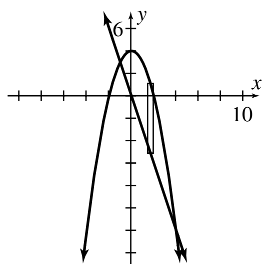
17. $y = 4 - x^2$, $y = -3x$. Region appears below.

Intersection: $-3x = 4 - x^2$, $x^2 - 3x - 4 = 0$, $(x + 1)(x - 4) = 0$, so $x = -1$ or 4.

Area $\int_{-1}^{4}\left[\left(4 - x^2\right) - (-3x)\right]dx$

$$= \left(4x - \frac{x^3}{3} + \frac{3x^2}{2}\right)\Bigg|_{-1}^{4}$$

$$= \left(16 - \frac{64}{3} + 24\right) - \left(-4 + \frac{1}{3} + \frac{3}{2}\right) = \frac{125}{6}$$

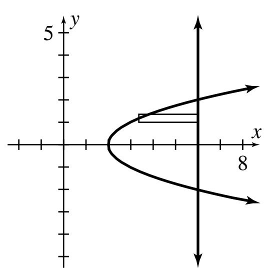


18. $x = y^2 + 2$, $x = 6$. Region appears below.

Intersection: $y^2 + 2 = 6$, $y^2 = 4$, $y = \pm 2$

Area $= \int_{-2}^{2}\left[6 - \left(y^2 + 2\right)\right]dy = \int_{-2}^{2}\left(4 - y^2\right)dy$

$$= \left(4y - \frac{y^3}{3}\right)\Bigg|_{-2}^{2} = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = \frac{32}{3}$$



19. $y^2 = 4x$, $y = 2x - 4$. Region appears below.

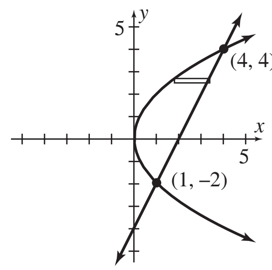
Intersection: $y^2 = 4\left(\frac{y}{2} + 2\right)$, $y^2 - 2y - 8 = 0$, $(y + 2)(y - 4) = 0$, so $y = -2$ or 4.

Area $= \int_{-2}^{4}\left[\left(\frac{y}{2} + 2\right) - \frac{y^2}{4}\right]dy$

$$= \left(\frac{y^2}{4} + 2y - \frac{y^3}{12}\right)\Bigg|_{-2}^{4}$$

$$= \left(4 + 8 - \frac{16}{3}\right) - \left(1 - 4 + \frac{2}{3}\right)$$

$$= 9$$



20. $y = x^3$, $y = x + 6$, $x = 0$

Region appears below.

Intersection: $x^3 = x + 6$, $x^3 - x - 6 = 0$,

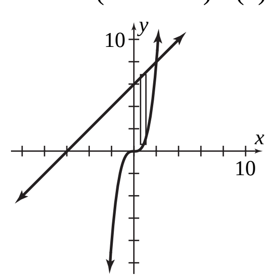
$(x - 2)(x^2 + 2x + 3) = 0 \Rightarrow x = 2$

$x^3 = 0 \Rightarrow x = 0$

Area $= \int_{0}^{2}[(x + 6) - x^3]\,dx$

$$= \left(\frac{x^2}{2} + 6x - \frac{x^4}{4}\right)\Bigg|_{0}^{2}$$

$$= (2 + 12 - 4) - (0) = 10$$



21. $2y = 4x - x^2$, $2y = x - 4$. Region appears below.

Intersection: $x - 4 = 4x - x^2$, $x^2 - 3x - 4 = 0$, $(x + 1)(x - 4) = 0$, so $x = -1$ or 4. Note that the y-values of the curves are given by $y = \frac{4x - x^2}{2}$ and $y = \frac{x-4}{2}$.

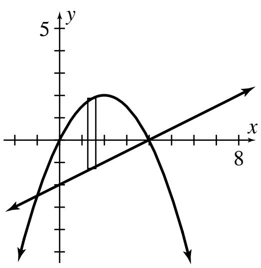
$$\text{Area} = \int_{-1}^{4}\left[\left(\frac{4x - x^2}{2}\right) - \left(\frac{x-4}{2}\right)\right]dx$$

$$= \int_{-1}^{4}\left(\frac{3}{2}x - \frac{x^2}{2} + 2\right)dx$$

$$= \left(\frac{3x^2}{4} - \frac{x^3}{6} + 2x\right)\Bigg|_{-1}^{4}$$

$$= \left(12 - \frac{64}{6} + 8\right) - \left(\frac{3}{4} + \frac{1}{6} - 2\right)$$

$$= \frac{125}{12}$$

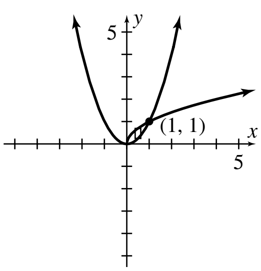


22. $y = \sqrt{x}$, $y = x^2$. Region appears below.

Intersection: $x^2 = \sqrt{x}$, $x^4 = x$, $x^4 - x = 0$, $x\left(x^3 - 1\right) = 0$, so $x = 0, 1$.

$$\text{Area} = \int_0^1\left(\sqrt{x} - x^2\right)dx = \left(\frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3}\right)\Bigg|_0^1$$

$$= \left(\frac{2}{3} - \frac{1}{3}\right) - 0 = \frac{1}{3}$$



23. $y^2 = 3x$, $3x - 2y = 15$ $\left(\text{or } x = \frac{2y+15}{3}\right)$. Region appears below.

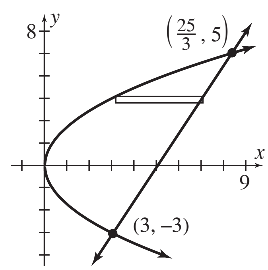
Intersection: $y^2 = 3\left(\frac{2y+15}{3}\right)$, $y^2 - 2y - 15 = 0$, $(y + 3)(y - 5) = 0$, so $y = -3$ or 5.

$$\text{Area} = \int_{-3}^{5}\left[\left(\frac{2}{3}y + 5\right) - \frac{y^2}{3}\right]dy$$

$$= \left(\frac{1}{3}y^2 + 5y - \frac{y^3}{9}\right)\Bigg|_{-3}^{5}$$

$$= \left(\frac{25}{3} + 25 - \frac{125}{9}\right) - (3 - 15 + 3)$$

$$= \frac{256}{9}$$

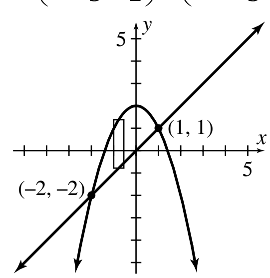


24. $y = 2 - x^2$, $y = x$. Region appears below.

Intersection: $x = 2 - x^2$, $x^2 + x - 2 = 0$, $(x+2)(x-1) = 0 \Rightarrow x = -2$ or 1.

$$\text{Area } \int_{-2}^{1}\left[\left(2 - x^2\right) - x\right]dx = \left(2x - \frac{x^3}{3} - \frac{x^2}{2}\right)\Bigg|_{-2}^{1}$$

$$= \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - 2\right) = \frac{9}{2}$$

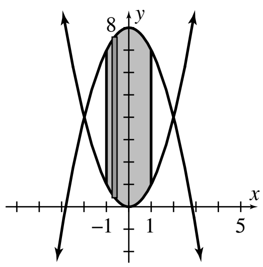


25. $y = 8 - x^2$, $y = x^2$, $x = -1$, $x = 1$. Region appears below.

Intersection: $x^2 = 8 - x^2$, $2x^2 = 8$, $x^2 = 4$, so $x = \pm 2$.

Area $= \int_{-1}^{1}\left[\left(8 - x^2\right) - x^2\right]dx = \int_{-1}^{1}\left(8 - 2x^2\right)dx$

$= \left(8x - \frac{2x^3}{3}\right)\Bigg|_{-1}^{1} = \left(8 - \frac{2}{3}\right) - \left(-8 + \frac{2}{3}\right) = \frac{44}{3}$



26. $y^2 = 6 - x$, $3y = x + 12$. Region appears below.

$y^2 = 6 - (3y - 12)$, $y^2 + 3y - 18 = 0$,

$(y + 6)(y - 3) = 0$, so $y = -6, 3$

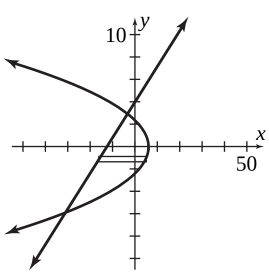
Area $= \int_{-6}^{3}[(6 - y^2) - (3y - 12)]\,dy$

$= \int_{-6}^{3}(18 - y^2 - 3y)\,dy$

$= \left(18y - \frac{y^3}{3} - \frac{3y^2}{2}\right)\Bigg|_{-6}^{3}$

$= \left(54 - 9 - \frac{27}{2}\right) - (-108 + 72 - 54)$

$= \frac{243}{2}$

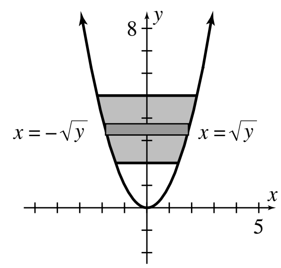


27. $y = x^2$, $y = 2$, $y = 5$. Region appears below.

Area

$= \int_{2}^{5}\left[\sqrt{y} - \left(-\sqrt{y}\right)\right]dy = \int_{2}^{5}2\sqrt{y}\,dy = 2 \cdot \frac{y^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_{2}^{5}$

$= \frac{4y^{\frac{3}{2}}}{3}\Bigg|_{2}^{5} = \frac{4 \cdot 5\sqrt{5}}{3} - \frac{4 \cdot 2\sqrt{2}}{3} = \frac{4}{3}\left(5\sqrt{5} - 2\sqrt{2}\right)$



28. $y = x^3 + x$, $y = 0$ (x-axis), $x = -1$, $x = 2$

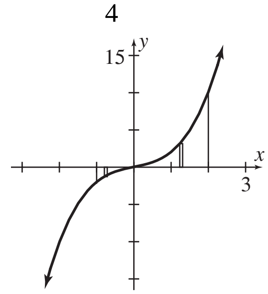
Region appears below.

Area $= \int_{-1}^{0} -(x^3 + x)dx + \int_{0}^{2}(x^3 + x)dx$

$= \left(-\frac{x^4}{4} - \frac{x^2}{2}\right)\Bigg|_{-1}^{0} + \left(\frac{x^4}{4} + \frac{x^2}{2}\right)\Bigg|_{0}^{2}$

$= \left[0 - \left(-\frac{1}{4} - \frac{1}{2}\right)\right] + [(4 + 2) - 0]$

$= \frac{27}{4}$



29. $y = x^3 - 1$, $y = x - 1$. Region appears below. Intersection: $x^3 - 1 = x - 1$, $x^3 - x = 0$, $x\left(x^2 - 1\right) = 0$,

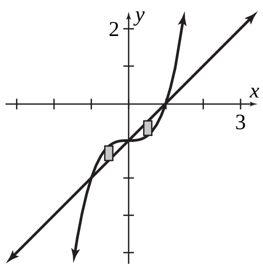
$x(x + 1)(x - 1) = 0$, so $x = 0$ or $x = \pm 1$.

Area $= \int_{-1}^{0} [x^3 - 1 - (x - 1)]\,dx + \int_{0}^{1} [x - 1 - (x^3 - 1)]\,dx$

$= \int_{-1}^{0} \left(x^3 - x\right) dx + \int_{0}^{1} \left(x - x^3\right) dx$

$= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_{-1}^{0} + \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_{0}^{1}$

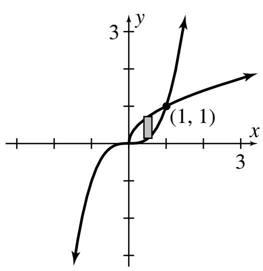
$= \left[0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right] + \left[\left(\frac{1}{2} - \frac{1}{4}\right) - 0\right] = \frac{1}{2}$



30. $y = x^3$, $y = \sqrt{x}$. Region appears below. Intersection: $x^3 = \sqrt{x}$, $x^6 = x$, $x^6 - x = 0$, $x\left(x^5 - 1\right) = 0$, $x = 0, 1$

Area $= \int_{0}^{1} \left(\sqrt{x} - x^3\right) dx = \left(\frac{2x^{\frac{3}{2}}}{3} - \frac{x^4}{4}\right)\Big|_{0}^{1}$

$= \left(\frac{2}{3} - \frac{1}{4}\right) - 0 = \frac{5}{12}$

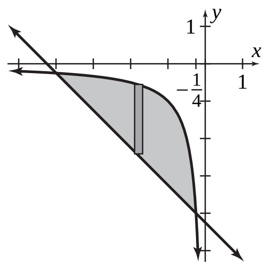


31. $4x + 4y + 17 = 0$, $y = \frac{1}{x}$. Region appears below. Intersection: $\frac{-17 - 4x}{4} = \frac{1}{x}$, $-17x - 4x^2 = 4$, $4x^2 + 17x + 4 = 0$,

$(4x + 1)(x + 4) = 0$, so $x = -\frac{1}{4}$ or -4.

$$\text{Area } = \int_{-4}^{-1/4}\left[\frac{1}{x}-\left(\frac{-17-4x}{4}\right)\right]dx = \left(\ln|x|+\frac{17}{4}x+\frac{x^2}{2}\right)\Bigg|_{-4}^{-1/4}$$

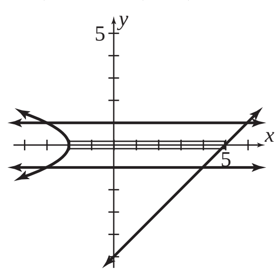
$$=\left(\ln\frac{1}{4}-\frac{17}{16}+\frac{1}{32}\right)-(\ln 4-17+8)=\frac{255}{32}-4\ln 2$$



32. $y^2 = -x-2,\ x-y=5,\ y=-1,\ y=1.$
Region appears below.
Intersection: $y^2 = -x-2$ intersects $y = \pm 1$ when $x = -3$; $x - y = 5$ intersects $y = 1$ when $x = 6$; $x - y = 5$ intersects $y = -1$ when $x = 4$

$$\text{Area} = \int_{-1}^{1}[(y+5)-(-y^2-2)]\,dy = \int_{-1}^{1}(y+7+y^2)\,dy = \left(\frac{y^2}{2}+7y+\frac{y^3}{3}\right)\Bigg|_{-1}^{1}$$

$$=\left(\frac{1}{2}+7+\frac{1}{3}\right)-\left(\frac{1}{2}-7-\frac{1}{3}\right)=\frac{44}{3}$$

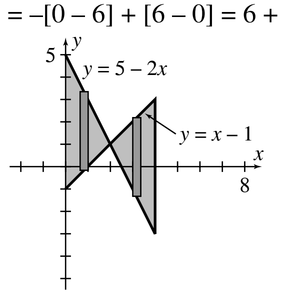


33. $y = x - 1$, $y = 5 - 2x$. Region appears below.
Intersection: $x - 1 = 5 - 2x$, $3x = 6$, so $x = 2$.

$$\text{Area } = \int_0^2 [(5-2x)-(x-1)]dx + \int_2^4[(x-1)-(5-2x)]dx = \int_0^2(6-3x)dx+\int_2^4(3x-6)dx$$

$$=-\frac{1}{3}\int_0^2(6-3x)[-3\ dx]+\frac{1}{3}\int_2^4(3x-6)[3\ dx] = -\frac{(6-3x)^2}{6}\Bigg|_0^2+\frac{(3x-6)^2}{6}\Bigg|_2^4$$

$= -[0 - 6] + [6 - 0] = 6 + 6 = 12$



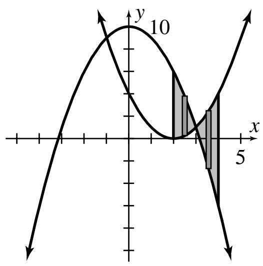
34. $y = x^2 - 4x + 4,\ y = 10 - x^2$. Region appears below.

Intersection: $x^2 - 4x + 4 = 10 - x^2$, $2x^2 - 4x - 6 = 0$, $x^2 - 2x - 3 = 0$, $(x - 3)(x + 1) = 0$, so $x = 3, -1$.

$$\text{Area} = \int_2^3 \left[\left(10 - x^2\right) - \left(x^2 - 4x + 4\right)\right] dx + \int_3^4 \left[\left(x^2 - 4x + 4\right) - \left(10 - x^2\right)\right] dx$$

$$= \int_2^3 \left(6 + 4x - 2x^2\right) dx + \int_3^4 \left(2x^2 - 4x - 6\right) dx = 2\left\{\int_2^3 \left(3 + 2x - x^2\right) dx + \int_3^4 \left(x^2 - 2x - 3\right) dx\right\}$$

$$= 2\left\{\left(3x + x^2 - \frac{x^3}{3}\right)\Bigg|_2^3 + \left(\frac{x^3}{3} - x^2 - 3x\right)\Bigg|_3^4\right\} = 2\left\{\left[9 - \frac{22}{3}\right] + \left[-\frac{20}{3} - (-9)\right]\right\} = 2\{4\} = 8$$



35. $$\frac{\text{Area between curve and diag.}}{\text{Area under diagonal}} = \frac{\int_0^1 \left[x - \left(\frac{14}{15}x^2 + \frac{1}{15}x\right)\right] dx}{\int_0^1 x\, dx}$$

$$\text{Numerator} = \int_0^1 \left[\frac{14}{15}x - \frac{14}{15}x^2\right] dx = \frac{14}{15}\int_0^1 \left(x - x^2\right) dx = \frac{14}{15}\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{14}{15}\left[\left(\frac{1}{2} - \frac{1}{3}\right) - 0\right] = \frac{14}{15}\cdot\frac{1}{6} = \frac{7}{45}$$

$$\text{Denominator} = \int_0^1 x\, dx = \frac{x^2}{2}\Bigg|_0^1 = \frac{1}{2}$$

$$\text{Coefficient of inequality} = \frac{\frac{7}{45}}{\frac{1}{2}} = \frac{14}{45}$$

36. $$\frac{\text{Area between curve and diag.}}{\text{Area under diagonal}} = \frac{\int_0^1 \left[x - \left(\frac{11}{12}x^2 + \frac{1}{12}x\right)\right] dx}{\int_0^1 x\, dx}$$

$$\text{Numerator} = \frac{11}{12}\int_0^1 \left(x - x^2\right) dx = \frac{11}{12}\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{11}{12}\left[\left(\frac{1}{2} - \frac{1}{3}\right) - 0\right] = \frac{11}{12}\cdot\frac{1}{6} = \frac{11}{72}$$

Denominator $= \frac{1}{2}$ (see Problem 35).

$$\text{Coefficient of inequality} = \frac{\frac{11}{72}}{\frac{1}{2}} = \frac{11}{36}$$

37. $y^2 = 3x,\ y = mx$

Intersection: $(mx)^2 = 3x,\ m^2x^2 = 3x$

$m^2x^2 - 3x = 0,\ x(m^2x - 3) = 0,\ x = 0$ or

$x = \frac{3}{m^2}.$

If $x = 0$, then $y = 0$; if $x = \frac{3}{m^2}$, then $y = \frac{3}{m}.$

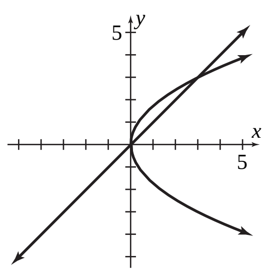
With horizontal elements,

$$\text{Area} = \int_0^{3/m}\left(\frac{y}{m} - \frac{y^2}{3}\right)dy = \left(\frac{y^2}{2m} - \frac{y^3}{9}\right)\Bigg|_0^{\frac{3}{m}}$$

$$= \frac{9}{2m^3} - \frac{3}{m^3} = \frac{3}{2m^3} \text{ square units}$$

Note: With vertical elements,

$$\text{Area} = \int_0^{3/m^2}\left(\sqrt{3}\sqrt{x} - mx\right)dx.$$



38. a. $y = x^2 - 1$, $y = 2x + 2$

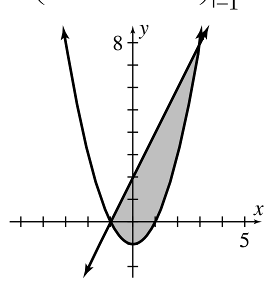
Intersection: $x^2 - 1 = 2x + 2$,

$x^2 - 2x - 3 = 0$, $(x - 3)(x + 1)$, so $x = 3$ and -1. The area is

$$\int_{-1}^{3}\left[2x + 2 - \left(x^2 - 1\right)\right]dx$$

$$= \int_{-1}^{3}\left(-x^2 + 2x + 3\right)dx$$

$$= \left(-\frac{x^3}{3} + x^2 + 3x\right)\Bigg|_{-1}^{3} = \frac{32}{3}$$



b. The area below is $\int_{-1}^{1}\left(1 - x^2\right)dx = \frac{4}{3}$. Thus the area above is $\frac{32}{3} - \frac{4}{3} = \frac{28}{3}$. Hence the percentage above the x-axis is

$$\frac{\frac{28}{3}}{\frac{32}{3}} \cdot 100 = 87.5\%$$

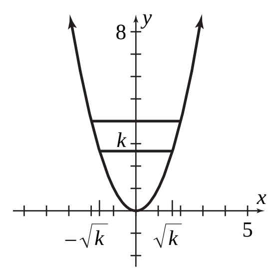
39. $y = x^2$ and $y = k$ intersect when $x^2 = k,\ x = \pm\sqrt{k}$. Equating areas gives

$$\int_{-\sqrt{k}}^{\sqrt{k}}\left(k - x^2\right)dx = \frac{1}{2}\int_{-2}^{2}\left(4 - x^2\right)dx$$

$$\left(kx - \frac{x^3}{3}\right)\Bigg|_{-\sqrt{k}}^{\sqrt{k}} = \frac{1}{2}\left(4x - \frac{x^3}{3}\right)\Bigg|_{-2}^{2}$$

$$\frac{4}{3}k^{\frac{3}{2}} = \frac{16}{3}$$

$$k^{\frac{3}{2}} = 4 \Rightarrow k = 4^{\frac{2}{3}} = \left(2^2\right)^{\frac{2}{3}} = 2^{\frac{4}{3}} \approx 2.52$$



40. 0.23 sq units

41. 4.76 sq units

42. Two integrals are involved.
Answer: 36.65 sq units

43. Two integrals are involved.
Answer: 7.26 sq units

44. Three integrals are involved.
Answer: 358.18 sq units

Problems 14.11

1. $\left.\begin{array}{l} D: p = 22 - 0.8q \\ S: p = 6 + 1.2q \end{array}\right\}$

Equilibrium pt. $= (q_0, p_0) = (8, 15.6)$

$$\text{CS} = \int_0^{q_0} [f(q) - p_0]\,dq$$

$$= \int_0^8 [(22 - 0.8q) - 15.6] = \int_0^8 (6.4 - 0.8q)dq$$

$$= \left(6.4q - 0.4q^2\right)\Big|_0^8 = (51.2 - 25.6) - 0 = 25.6$$

$$\text{PS} = \int_0^{q_0} [p_0 - g(q)]\,dq$$

$$= \int_0^8 [15.6 - (6 + 1.2q)]dq = \int_0^8 (9.6 - 1.2q)dq$$

$$= \left(9.6q - 0.6q^2\right)\Big|_0^8 = (76.8 - 38.4) - 0 = 38.4$$

2. $\left.\begin{array}{l} D: p = 2200 - q^2 \\ S: p = 400 + q^2 \end{array}\right\}$

Equilibrium point $= (q_0, p_0) = (30, 1300)$

$$\text{CS} = \int_0^{30} [(2200 - q^2) - 1300]$$

$$= \int_0^{30} (900 - q^2)\,dq = \left(900q - \frac{q^3}{3}\right)\Bigg|_0^{30}$$

$$= (27{,}000 - 9000) - 0 = 18{,}000$$

$$\text{PS} = \int_0^{30} [1300 - (400 + q^2)]\,dq$$

$$= \int_0^{30} (900 - q^2)\,dq$$

$$= \left(900q - \frac{q^3}{3}\right)\Bigg|_0^{30}$$

$$= (27{,}000 - 9000) - 0$$

$$= 18{,}000$$

3. $\left.\begin{array}{l} D: p = \frac{50}{q+5} \\ S: p = \frac{q}{10} + 4.5 \end{array}\right\}$

Equilibrium pt. $= (q_0, p_0) = (5, 5)$

$$\text{CS} = \int_0^{q_0} [f(q) - p_0]\,dq$$

$$= \int_0^5 \left[\frac{50}{q+5} - 5\right] dq = \left(50\ln|q+5| - 5q\right)\Big|_0^5$$

$= [50 \ln(10) - 25] - [50 \ln(5)]$
$= 50[\ln(10) - \ln(5)] - 25 = 50 \ln(2) - 25$

$$\text{PS} = \int_0^{q_0} [p_0 - g(q)]\,dq$$

$$= \int_0^5 \left[5 - \left(\frac{q}{10} + 4.5\right)\right] dq = \int_0^5 \left(0.5 - \frac{q}{10}\right) dq$$

$$= \left(0.5q - \frac{q^2}{20}\right)\Bigg|_0^5 = (2.5 - 1.25) - 0 = 1.25$$

4. $\left.\begin{array}{l} D: p = 400 - q^2 \\ S: p = 20q + 100 \end{array}\right\}$

Equilibrium pt. $(q_0, p_0) = (10, 300)$

$$\text{CS} = \int_0^{10} \left[\left(400 - q^2\right) - 300\right] dq$$

$$= \int_0^{10} \left(100 - q^2\right) dq$$

$$= \left(100q - \frac{q^3}{3}\right)\Bigg|_0^{10} = \left(1000 - \frac{1000}{3}\right) - 0 = \frac{2000}{3}$$

$$\text{PS} = \int_0^{10} [300 - (20q + 100)]dq$$

$$= \int_0^{10} (200 - 20q)dq$$

$$= \left(200q - 10q^2\right)\Big|_0^{10} = (2000 - 1000) - 0 = 1000$$

5. $\left.\begin{array}{l} D: q = 100(10 - 2p) \\ S: q = 50(2p - 1) \end{array}\right\}$

Equilibrium pt. $= (q_0, p_0) = (300, 3.5)$

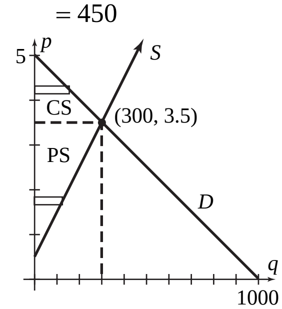
We use horizontal strips and integrate with respect to p.

$$\text{CS} = \int_{3.5}^{5} 100(10 - 2p)dp = 100(10p - p^2)\Big|_{3.5}^{5}$$

$= 100[(50 - 25) - (35 - 12.25)]$
$= 225$

$$\text{PS} = \int_{0.5}^{3.5} 50(2p - 1)dp = 50(p^2 - p)\Big|_{0.5}^{3.5}$$

$= 50[(12.25 - 3.5) - (0.25 - 0.5)]$
$= 450$



6. $\left.\begin{array}{l} D: q=\sqrt{100-p} \\ S: q=\frac{p}{2}-10 \end{array}\right\}$

Equilibrium pt. $=\left(q_0,\ p_0\right)=(8,\ 36)$

Integrating with respect to p,

$$\text{CS}=\int_{36}^{100}\sqrt{100-p}\,dp$$

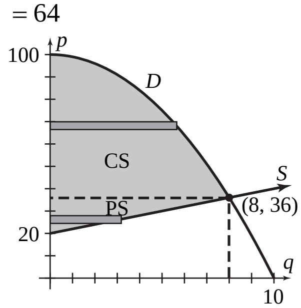
$$=-\frac{2}{3}(100-p)^{\frac{3}{2}}\Big|_{36}^{100}$$

$$=0-\left(-\frac{2}{3}\cdot 512\right)=\frac{1024}{3}$$

$$\text{PS}=\int_{20}^{36}\left[\frac{p}{2}-10\right]dp$$

$$=\left(\frac{p^2}{4}-10p\right)\Big|_{20}^{36}=(324-360)-(100-200)$$

$$=64$$



7. We integrate with respect to p. From the demand equation, when $q = 0$, then $p = 100$.

$$\text{CS}=\int_{84}^{100}10\sqrt{100-p}\,dp$$

$$=\int_{84}^{100}-10(100-p)^{\frac{1}{2}}[-dp]$$

$$=-\frac{20}{3}(100-p)^{\frac{3}{2}}\Big|_{84}^{100}$$

$$=-\frac{20}{3}\left[0-(16)^{\frac{3}{2}}\right]=-\frac{20}{3}(-64)$$

$$=426\frac{2}{3}\approx \$426.67$$

8. At equilibrium, $p=\dfrac{400-p^2}{60}+5$,

$60p=400-p^2+300$, $p^2+60p-700=0$,

$(p+70)(p-10)=0 \Rightarrow p=10$ and

$q=400-10^2=300$, so equilibrium pt. is

$\left(q_0,\ p_0\right)=(300,\ 10)$.

$$\text{PS}=\int_0^{300}\left[10-\left(\frac{q}{60}+5\right)\right]dq$$

$$=\left(5q-\frac{q^2}{120}\right)\Bigg|_0^{300}=(1500-750)-0=750$$

For CS we integrate with respect to p. From the demand equation, $q=0 \Rightarrow p=20$.

$$\text{CS}=\int_{10}^{20}\left(400-p^2\right)dp=\left(400p-\frac{p^3}{3}\right)\Bigg|_{10}^{20}$$

$$=\left(8000-\frac{8000}{3}\right)-\left(4000-\frac{1000}{3}\right)=1666\frac{2}{3}$$

9. At equilibrium,

$2^{11-q}=2^{q+1} \Rightarrow 11-q=q+1 \Rightarrow q=5$, so

$p=2^{11-5}=64$

$$\text{CS}=\int_0^5\left(2^{11-q}-64\right)dq=\left(-\frac{2^{11-q}}{\ln 2}-64q\right)\Bigg|_0^5$$

$$=-\frac{64}{\ln 2}-320-\left(-\frac{2^{11}}{\ln 2}-0\right)$$

$\approx$ 2542.307 hundred $\approx$ \$254,000

10. a. (10 + 10)(30 + 20) = 1000, (20)(50) = 1000, 1000 = 1000

30 − 4(10) + 10 = 0, 30 − 40 + 10 = 0, 0 = 0

b. $(p + 10)(q + 20) = 1000$, $p+10=\dfrac{1000}{q+20}$,

$$p=\frac{1000}{q+20}-10$$

$$\text{CS}=\int_0^{30}\left[\left(\frac{1000}{q+20}-10\right)-10\right]dq$$

$$=[1000\ln(q+20)-20q]\Big|_0^{30}$$

$$=1000\ln(50)-600-[1000\ln(20)]$$

$$=1000\ln\left(\frac{50}{20}\right)-600$$

$$=1000\ln\left(\frac{5}{2}\right)-600$$

11. CS $\approx$ 1197; PS $\approx$ 477

12. Let $p = f(q)$.

$$PS = \int_0^{40} [80 - f(q)]\,dq$$
$$= \int_0^{40} 80\,dq - \int_0^{40} f(q)\,dq$$
$$= 3200 - \int_0^{40} f(q)\,dq$$

Use the trapezoid rule with $h = 10$ to estimate $\int_0^{40} f(q)\,dq$:

$f(0)$	=	25	=	25
$2f(10)$	=	2(49)	=	98
$2f(20)$	=	2(59)	=	118
$2f(30)$	=	2(71)	=	142
$f(40)$	=	80	=	80
				463

$$\int_0^{40} f(q)\,dq \approx \frac{10}{2}(463) = 2315$$

Thus PS = 3200 − 2315 = $885.

Chapter 14 Review Problems

1. $\int \left(x^3 + 2x - 7\right) dx = \frac{x^4}{4} + 2 \cdot \frac{x^2}{2} - 7x + C$
$= \frac{x^4}{4} + x^2 - 7x + C$

2. $\int dx = \int 1\,dx = 1 \cdot x + C = x + C$

3. $\int_0^8 \left(\sqrt{2x} + 2x\right) dx = \int_0^8 \left(\sqrt{2}x^{\frac{1}{2}} + 2x\right) dx$
$= \left(\sqrt{2} \cdot \frac{2x^{\frac{3}{2}}}{3} + x^2\right)\Bigg|_0^8$
$= \left(\sqrt{2} \cdot \frac{2}{3}\left(2\sqrt{2}\right)^3 + 64\right) - 0$
$= \frac{64}{3} + 64 = \frac{256}{3}$

4. $\int \frac{4}{5-3x}\,dx = 4\left(-\frac{1}{3}\right)\int \frac{1}{5-3x}[-3\,dx]$
$= -\frac{4}{3}\ln|5-3x| + C$

5. $\int \frac{6}{(x+5)^3}\,dx = 6\int (x+5)^{-3}\,dx$
$= \frac{6(x+5)^{-2}}{-2} + C$
$= -3(x+5)^{-2} + C$

6. $\int_3^9 (y-6)^{301}\,dy = \frac{(y-6)^{302}}{302}\Bigg|_3^9$
$= \frac{3^{302}}{302} - \frac{(-3)^{302}}{302} = 0$

7. $\int \frac{6x^2 - 12}{x^3 - 6x + 1}\,dx = 2\int \frac{1}{x^3 - 6x + 1}\left[\left(3x^2 - 6\right)dx\right]$
$= 2\ln\left|x^3 - 6x + 1\right| + C$

8. $\int_0^2 xe^{4-x^2}\,dx = -\frac{1}{2}\int_0^2 e^{4-x^2}[-2x\,dx]$
$= -\frac{1}{2}e^{4-x^2}\Big|_0^2$
$= -\frac{1}{2}\left(e^0 - e^4\right) = \frac{1}{2}\left(e^4 - 1\right)$

9. $\int_0^1 \sqrt[3]{3t+8}\,dt = \frac{1}{3}\int_0^1 (3t+8)^{\frac{1}{3}}[3\,dt]$
$= \frac{1}{3} \cdot \frac{(3t+8)^{\frac{4}{3}}}{\frac{4}{3}}\Bigg|_0^1$
$= \frac{(3t+8)^{\frac{4}{3}}}{4}\Bigg|_0^1 = \frac{11\sqrt[3]{11}}{4} - 4$

10. $\int \frac{4-2x}{7}\,dx = \int \left(\frac{4}{7} - \frac{2}{7}x\right) dx = \frac{4}{7}x - \frac{1}{7}x^2 + C$

11. $\int y(y+1)^2\,dy = \int \left(y^3 + 2y^2 + y\right) dy$
$= \frac{y^4}{4} + \frac{2y^3}{3} + \frac{y^2}{2} + C$

12. $\int_0^1 10^{-8}\,dx = 10^{-8}x\Big|_0^1 = 10^{-8} - 0 = 10^{-8}$

13. $\int \frac{\sqrt[5]{t}-\sqrt[3]{t}}{\sqrt{t}}dt = \int\left(\frac{t^{1/5}}{t^{1/2}}-\frac{t^{1/3}}{t^{1/2}}\right)dt$

$= \int (t^{-3/10} - t^{-1/6})dt = \frac{t^{7/10}}{\frac{7}{10}} - \frac{t^{5/6}}{\frac{5}{6}} + C$

$= \frac{10}{7}t^{7/10} - \frac{6}{5}t^{5/6} + C$

14. $\int \frac{(0.5x-0.1)^4}{0.4}dx$

$= \frac{1}{0.4}\cdot\frac{1}{0.5}\int (0.5x-0.1)^4[0.5\ dx]$

$= \frac{1}{0.2}\cdot\frac{(0.5x-0.1)^5}{5} + C = (0.5x-0.1)^5 + C$

15. $\int_1^3 \frac{2t^2}{3+2t^3}dt = \frac{1}{3}\int_1^3 \frac{1}{3+2t^3}[6t^2\ dt]$

$= \frac{1}{3}\ln(3+2t^3)\Big|_1^3$

$= \frac{1}{3}[\ln(57) - \ln(5)] = \frac{1}{3}\ln\left(\frac{57}{5}\right)$

16. $\int \frac{4x^2-x}{x}dx = \int (4x-1)dx = 2x^2 - x + C$

17. $\int x^2\sqrt{3x^3+2}dx = \frac{1}{9}\int\left(3x^3+2\right)^{\frac{1}{2}}\left[9x^2dx\right]$

$= \frac{1}{9}\cdot\frac{\left(3x^3+2\right)^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{27}\left(3x^3+2\right)^{\frac{3}{2}} + C$

18. $\int (8x^3+4x)(x^4+x^2)^{5/2}dx$

$= 2\int (x^4+x^2)^{5/2}[(4x^3+2x)dx]$

$= 2\cdot\frac{(x^4+x^2)^{7/2}}{\frac{7}{2}} + C = \frac{4}{7}(x^4+x^2)^{7/2} + C$

19. $\int\left(e^{2y}-e^{-2y}\right)dy$

$= \frac{1}{2}\int e^{2y}[2\ dy] - \left(-\frac{1}{2}\right)\int e^{-2y}[-2\ dy]$

$= \frac{1}{2}e^{2y} + \frac{1}{2}e^{-2y} + C = \frac{1}{2}\left(e^{2y}+e^{-2y}\right) + C$

20. $\int \frac{8x}{3\sqrt[3]{7-2x^2}}dx = \frac{8}{3}\left(-\frac{1}{4}\right)\int\left(7-2x^2\right)^{-\frac{1}{3}}[-4x\ dx]$

$= -\frac{2}{3}\cdot\frac{3\left(7-2x^2\right)^{\frac{2}{3}}}{2} + C = -\left(7-2x^2\right)^{\frac{2}{3}} + C$

21. $\int\left(\frac{1}{x}+\frac{2}{x^2}\right)dx = \int\frac{1}{x}dx + 2\int x^{-2}dx$

$= \ln|x| + 2\cdot\frac{x^{-1}}{-1} + C$

$= \ln|x| - \frac{2}{x} + C$

22. $\int_0^2 \frac{3e^{3x}}{1+e^{3x}}dx = \int_0^2 \frac{1}{1+e^{3x}}[3e^{3x}\ dx]$

$= \ln(1+e^{3x})\Big|_0^2$

$= \ln(1+e^6) - \ln(1+1)$

$= \ln\left(\frac{1+e^6}{2}\right)$

23. $\int_{-2}^1 10\left(y^4-y+1\right)dy = 10\left(\frac{y^5}{5}-\frac{y^2}{2}+y\right)\Bigg|_{-2}^1$

$= 10\left(\frac{1}{5}-\frac{1}{2}+1\right) - 10\left(-\frac{32}{5}-2-2\right) = 111$

24. $\int_7^{70} dx = x\Big|_7^{70} = 70 - 7 = 63$

25. $\int_1^2 5x\sqrt{5-x^2}dx = -\frac{5}{2}\int_1^2 (5-x^2)^{1/2}[-2x\,dx]$

$= -\frac{5}{2}\cdot\frac{(5-x^2)^{3/2}}{\frac{3}{2}}\Bigg|_1^2 = -\frac{5}{3}(5-x^2)^{3/2}\Big|_1^2$

$= -\frac{5}{3}(1^{3/2} - 4^{3/2}) = -\frac{5}{3}(1-8) = \frac{35}{3}$

26. $\int_0^1 (2x+1)\left(x^2+x\right)^4 dx = \int_0^1 \left(x^2+x\right)^4 [(2x+1)\ dx] = \left.\frac{\left(x^2+x\right)^5}{5}\right|_0^1 = \frac{2^5}{5} - 0 = \frac{32}{5}$

27. $\int_0^1 \left[2x - \frac{1}{(x+1)^{\frac{2}{3}}}\right] dx = 2\int_0^1 x\ dx - \int_0^1 (x+1)^{-\frac{2}{3}}[dx] = \left[2\cdot\frac{x^2}{2} - \frac{(x+1)^{\frac{1}{3}}}{\frac{1}{3}}\right]\Bigg|_0^1 = \left[x^2 - 3(x+1)^{\frac{1}{3}}\right]\Big|_0^1$

$= \left[1 - 3\sqrt[3]{2}\right] - [0-3] = 4 - 3\sqrt[3]{2}$

28. $\int_3^{27} 3\left(\sqrt{3x} - 2x + 1\right) dx = 3\int_3^{27}\left(\sqrt{3}x^{\frac{1}{2}} - 2x + 1\right) dx = 3\left(\sqrt{3}\cdot\frac{2}{3}x^{\frac{3}{2}} - x^2 + x\right)\Bigg|_3^{27}$

$= 3\left[\left(\sqrt{3}\cdot\frac{2}{3}\left(3\sqrt{3}\right)^3 - 729 + 27\right) - \left(\sqrt{3}\cdot\frac{2}{3}\left(\sqrt{3}\right)^3 - 9 + 3\right)\right] = 3(-540) = -1620$

29. $\int \frac{\sqrt{t}-3}{t^2} dt = \int \left[\frac{t^{\frac{1}{2}}}{t^2} - \frac{3}{t^2}\right] dt = \int \left(t^{-\frac{3}{2}} - 3t^{-2}\right) dt = \frac{t^{-\frac{1}{2}}}{-\frac{1}{2}} - 3\cdot\frac{t^{-1}}{-1} + C = -2t^{-\frac{1}{2}} + 3t^{-1} + C = \frac{3}{t} - \frac{2}{\sqrt{t}} + C$

30. $\int \frac{3z^3}{z-1} dz = 3\int \left(z^2 + z + 1 + \frac{1}{z-1}\right) dz$

$= 3\left(\frac{z^3}{3} + \frac{z^2}{2} + z + \ln|z-1|\right) + C$

31. $\int_{-1}^{0} \frac{x^2+4x-1}{x+2} dx = \int_{-1}^{0}\left(x + 2 - \frac{5}{x+2}\right) dx = \left(\frac{x^2}{2} + 2x - 5\ln|x+2|\right)\Bigg|_{-1}^{0}$

$= (-5\ln 2) - \left(\frac{1}{2} - 2 - 0\right) = \frac{3}{2} - 5\ln 2$

32. $\int \frac{\left(x^2+4\right)^2}{x^2} dx = \int \frac{x^4 + 8x^2 + 16}{x^2} dx$

$= \int \left(x^2 + 8 + 16x^{-2}\right) dx$

$= \frac{x^3}{3} + 8x + 16\frac{x^{-1}}{-1} + C = \frac{x^3}{3} + 8x - \frac{16}{x} + C$

33. $\int 9\sqrt{x}\sqrt{x^{\frac{3}{2}}+1}\, dx = 9\cdot\frac{2}{3}\int \left(x^{\frac{3}{2}}+1\right)^{\frac{1}{2}} \left[\frac{3}{2}x^{\frac{1}{2}} dx\right]$

$= 6\cdot\frac{\left(x^{\frac{3}{2}}+1\right)^{\frac{3}{2}}}{\frac{3}{2}} + C = 4\left(x^{\frac{3}{2}}+1\right)^{\frac{3}{2}} + C$

34. $\int \frac{e^{\sqrt{5x}}}{\sqrt{3x}}\,dx = \frac{1}{\sqrt{3}}\int \frac{e^{\sqrt{5}x^{\frac{1}{2}}}}{x^{\frac{1}{2}}}\,dx$

$= \frac{2}{\sqrt{3}\cdot\sqrt{5}} 1\int e^{\sqrt{5}x^{\frac{1}{2}}}\left[\frac{\sqrt{5}}{2}x^{-\frac{1}{2}}\,dx\right]$

$= \frac{2}{\sqrt{15}}\left(e^{\sqrt{5}x^{\frac{1}{2}}}\right) + C$

$= \frac{2}{\sqrt{15}}e^{\sqrt{5x}} + C$

35. $\int_1^e \frac{e^{\ln x}}{x^2}\,dx = \int_1^e \frac{x}{x^2}\,dx = \int_1^e \frac{1}{x}\,dx = \ln|x|\Big|_1^e$

= ln *e* – ln 1
= 1 – 0 = 1

36. $\int \frac{6x^2+4}{e^{x^3+2x}}\,dx = -2\int e^{-\left(x^3+2x\right)}\left[-\left(3x^2+2\right)dx\right]$

$= -2e^{-\left(x^3+2x\right)} + C = \frac{-2}{e^{x^3+2x}} + C$

37. $\int \frac{(1+e^{2x})^3}{e^{-2x}}\,dx = \frac{1}{2}\int (1+e^{2x})^3[2e^{2x}dx]$

$= \frac{(1+e^{2x})^3}{8} + C$

38. $\int \frac{3}{e^{3x}\left(6+e^{-3x}\right)^2}\,dx$

$= -\int \left(6+e^{-3x}\right)^{-2}\left[-3e^{-3x}dx\right]$

$= -\frac{\left(6+e^{-3x}\right)^{-1}}{-1} + C = \frac{1}{6+e^{-3x}} + C$

39. $\int 3\sqrt{10^{3x}}\,dx = 3\int e^{\frac{3x}{2}\ln 10}\,dx$

$= 3\cdot\frac{2}{3\ln 10}\int e^{\frac{3x}{2}\ln 10}\left[\frac{3\ln 10}{2}\,dx\right]$

$= \frac{2}{\ln 10}e^{\frac{3x}{2}\ln 10} + C = \frac{2}{\ln 10}10^{\frac{3x}{2}} + C$

$= \frac{2\sqrt{10^{3x}}}{\ln 10} + C$

40. $\int \frac{5x^3+15x^2+37x+3}{x^2+3x+7}\,dx$

$= \int\left(\frac{5x^3+15x^2+35x}{x^2+3x+7} + \frac{2x+3}{x^2+3x+7}\right)dx$

$= \int 5x\,dx + \int \frac{1}{x^2+3x+7}[(2x+3)\,dx]$

$= \frac{5x^2}{2} + \ln\left|x^2+3x+7\right| + C$

41. $y = \int\left(e^{2x}+3\right)dx = \int e^{2x}dx + \int 3\,dx$

$= \frac{1}{2}\int e^{2x}[2\,dx] + \int 3\,dx$

$= \frac{1}{2}e^{2x} + 3x + C$

$y(0) = -\frac{1}{2}$ implies that $-\frac{1}{2} = \frac{1}{2} + 0 + C$, so $C = -1$. Thus $y = \frac{1}{2}e^{2x} + 3x - 1$

42. $y = \int \frac{x+5}{x}\,dx = \int\left(1+\frac{5}{x}\right)dx = x + 5\ln|x| + C$

$y(1) = 3$ implies $3 = 1 + 0 + C$, so $C = 2$. Thus $y = x + 5\ln|x| + 2$

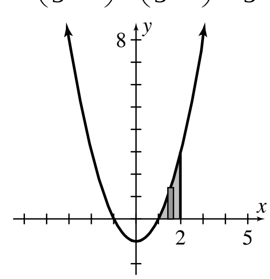
In Problems 43–58, answers are assumed to be expressed in square units.

43. $y = x^2 - 1$, $x = 2$, $y \geq 0$. Region appears below.

Area $= \int_1^2\left(x^2-1\right)dx$

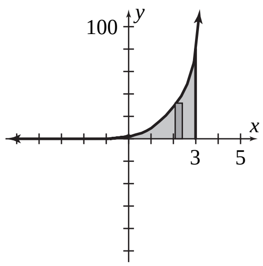
$= \left(\frac{x^3}{3} - x\right)\Big|_1^2$

$= \left(\frac{8}{3}-2\right) - \left(\frac{1}{3}-1\right) = \frac{4}{3}$



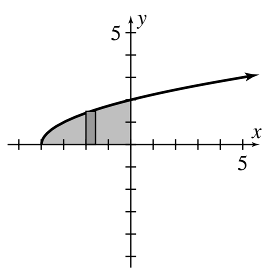
44. $y = 4e^x$, $x = 0$, $x = 3$. Region appears below.

Area $= \int_0^3 4e^x dx = 4e^x \Big|_0^3 = 4(e^3 - 1)$



45. $y = \sqrt{x+4}$, $x = 0$. Region appears below.

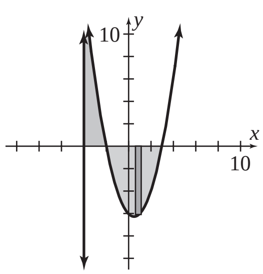
Area $= \int_{-4}^0 \sqrt{x+4}dx = \int_{-4}^0 (x+4)^{\frac{1}{2}}[dx] = \frac{(x+4)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_{-4}^0 = \frac{2(x+4)^{\frac{3}{2}}}{3}\Bigg|_{-4}^0 = \frac{16}{3} - 0 = \frac{16}{3}$



46. $y = x^2 - x - 6$, $x = -4$, $x = 3$. Region appears below.

Area $= \int_{-4}^{-2} (x^2 - x - 6)\,dx + \int_{-2}^3 -(x^2 - x - 6)\,dx = \left(\frac{x^3}{3} - \frac{x^2}{2} - 6x\right)\Bigg|_{-4}^{-2} - \left(\frac{x^3}{3} - \frac{x^2}{2} - 6x\right)\Bigg|_{-2}^{3}$

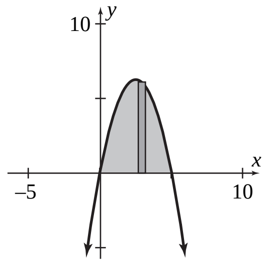
$= \left[\left(-\frac{8}{3} - 2 + 12\right) - \left(-\frac{64}{3} - 8 + 24\right)\right] - \left[\left(9 - \frac{9}{2} - 18\right) - \left(-\frac{8}{3} - 2 + 12\right)\right] = \frac{67}{2}$



47. $y = 5x - x^2$. Region appears below.

$$\text{Area } = \int_0^5 \left(5x - x^2\right)dx = \left(\frac{5x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^5$$

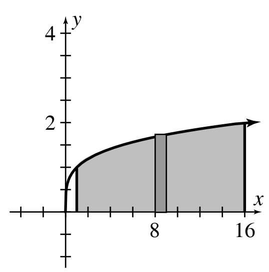
$$= \left(\frac{125}{2} - \frac{125}{3}\right) - 0 = \frac{125}{6}$$



48. $y = \sqrt[4]{x}$, $x = 1$, $x = 16$. Region appears below.

$$\text{Area } = \int_1^{16} \sqrt[4]{x}\,dx = \int_1^{16} x^{\frac{1}{4}}\,dx = \frac{4x^{\frac{5}{4}}}{5}\Bigg|_1^{16}$$

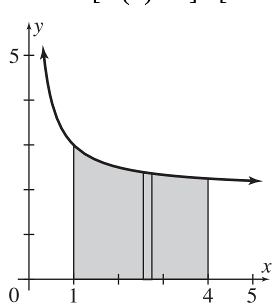
$$= \frac{128}{5} - \frac{4}{5} = \frac{124}{5}$$



49. $y = \frac{1}{x} + 2,\ x = 1, x = 4$. Region appears below.

$$\text{Area } = \int_1^4 \left(\frac{1}{x} + 2\right)dx = \left(\ln|x| + 2x\right)\Big|_1^4$$

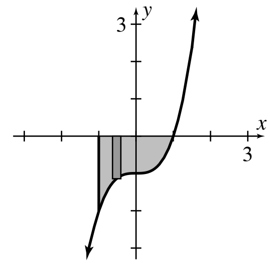
$$= [\ln(4) + 8] - [0 + 2] = 6 + \ln 4$$



50. $y = x^3 - 1$, $x = -1$. Region appears below.

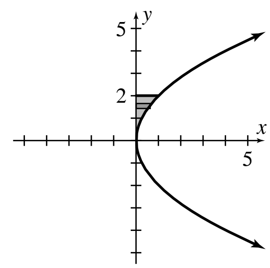
$$\text{Area } = \int_{-1}^1 -\left(x^3 - 1\right)dx = -\left(\frac{x^4}{4} - x\right)\Bigg|_{-1}^1$$

$$= -\left(-\frac{3}{4}\right) + \left(\frac{5}{4}\right) = 2$$



51. $y^2 = 4x$, $x = 0$, $y = 2$. Region appears below.

$$\text{Area } = \int_0^2 \frac{y^2}{4}\,dy = \frac{y^3}{12}\Bigg|_0^2 = \frac{8}{12} - 0 = \frac{2}{3}$$



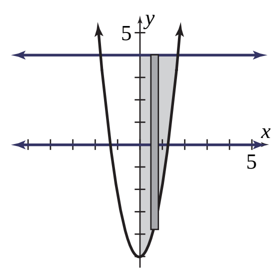
52. $y = 3x^2 - 5, x = 0, y = 4$. Region appears below.

$3x^2 - 5 = 4,\ 3x^2 = 9,\ x^2 = 3,$ so $x = \pm\sqrt{3}.$

$$\text{Area} = \int_0^{\sqrt{3}} [4 - (3x^2 - 5)]\,dx$$

$$= \int_0^{\sqrt{3}} [9 - 3x^2]\,dx = (9x - x^3)\Big|_0^{\sqrt{3}}$$

$$= \left(9\sqrt{3} - 3\sqrt{3}\right) - 0 = 6\sqrt{3}$$



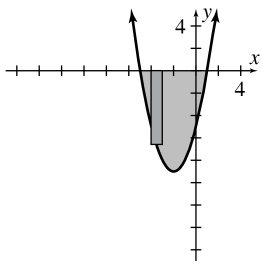
53. $y = x^2 + 4x - 5$, $y = 0$. Region appears below.

$x^2 + 4x - 5 = 0$, $(x + 5)(x - 1) = 0$, so $x = -5, 1$.

$$\text{Area } = \int_{-5}^{1} -\left(x^2 + 4x - 5\right) dx$$

$$= -\left(\frac{x^3}{3} + 2x^2 - 5x\right)\Bigg|_{-5}^{1}$$

$$= -\left(\frac{1}{3} + 2 - 5\right) + \left(-\frac{125}{3} + 50 + 25\right) = 36$$



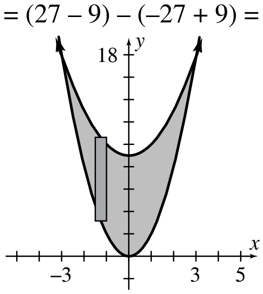
54. $y = 2x^2$, $y = x^2 + 9$. Region appears below.

$2x^2 = x^2 + 9$, $x^2 = 9$, so $x = \pm 3$

$$\text{Area } = \int_{-3}^{3}\left[\left(x^2 + 9\right) - \left(2x^2\right)\right] dx$$

$$= \int_{-3}^{3}\left(9 - x^2\right) dx = \left(9x - \frac{x^3}{3}\right)\Bigg|_{-3}^{3}$$

$= (27 - 9) - (-27 + 9) = 36$



55. $y = x^2 - x$, $y = 10 - x^2$. Region appears below.

$x^2 - x = 10 - x^2$, $2x^2 - x - 10 = 0$,

$(x + 2)(2x - 5) = 0$, so $x = -2$ or $\frac{5}{2}$.

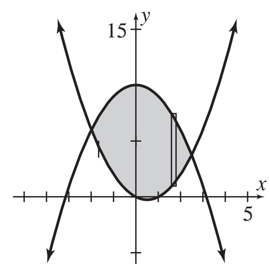
$$\text{Area} = \int_{-2}^{5/2}[(10 - x^2) - (x^2 - x)]dx$$

$$= \int_{-2}^{5/2}(10 + x - 2x^2)dx$$

$$= \left(10x + \frac{x^2}{2} - \frac{2x^3}{3}\right)\Bigg|_{-2}^{5/2}$$

$$= \left(25 + \frac{25}{8} - \frac{125}{12}\right) - \left(-20 + 2 + \frac{16}{3}\right)$$

$$= \frac{243}{8}$$

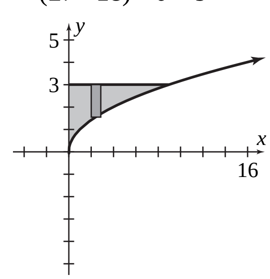


56. $y = \sqrt{x}$, $x = 0$, $y = 3$. Region appears below.

$\sqrt{x} = 3$, so $x = 9$.

$$\text{Area } = \int_{0}^{9}\left(3 - \sqrt{x}\right) dx = \left(3x - \frac{2x^{\frac{3}{2}}}{3}\right)\Bigg|_{0}^{9}$$

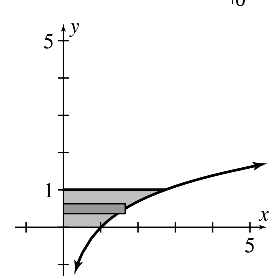
$= (27 - 18) - 0 = 9$



57. $y = \ln x$, $x = 0$, $y = 0$, $y = 1$. Region appears below.

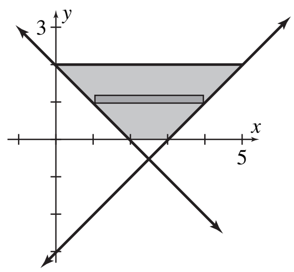
$y = \ln x \Rightarrow x = e^y$

$$\text{Area } = \int_{0}^{1} e^y dy = e^y\Big|_{0}^{1} = e - 1$$



58. $y = 2 - x$, $y = x - 3$, $y = 0$, $y = 2$. Region appears below.

$$\text{Area} = \int_0^2 [(y+3)-(2-y)]\,dy$$
$$= \int_0^2 (2y+1)\,dy = (y^2+y)\Big|_0^2$$
$$= (4+2) - 0 = 6$$



59. $r = \int\left(100 - \frac{3}{2}\sqrt{2q}\right)dq = \int 100\,dq - \frac{3}{2}\sqrt{2}\int q^{\frac{1}{2}}\,dq$

$$= 100q - \frac{3}{2}\sqrt{2}\cdot\frac{q^{\frac{3}{2}}}{\frac{3}{2}} + C = 100q - \sqrt{2}q^{\frac{3}{2}} + C$$

When $q = 0$, then $r = 0$. Thus $0 = 0 - 0 + C$, so $C = 0$. Hence $r = 100q - \sqrt{2}q^{\frac{3}{2}}$. Since $r = pq$, then $p = \frac{r}{q} = 100 - \sqrt{2}q^{\frac{1}{2}} = 100 - \sqrt{2q}$. Thus $p = 100 - \sqrt{2q}$.

60. $c = \int\left(q^2 + 7q + 6\right)dq = \frac{q^3}{3} + \frac{7}{2}q^2 + 6q + C$

When $q = 0$, then $c = 2500$. Thus $2500 = 0 + 0 + 0 + C$, so $C = 2500$. Hence $c = \frac{q^3}{3} + \frac{7}{2}q^2 + 6q + 2500$. When $q = 6$, then $c = \$2734$.

61. $\int_{15}^{25}(250 - q - 0.2q^2)\,dq$

$$= \left(250q - \frac{q^2}{2} - \frac{0.2q^3}{3}\right)\Bigg|_{15}^{25}$$
$$= \left(6250 - 312.5 - \frac{3125}{3}\right) - (3750 - 112.5 - 225)$$
$$\approx \$1483.33$$

62. $\int_{10}^{33}\frac{1000}{\sqrt{3q+70}}\,dq = 1000\cdot\frac{1}{3}\int_{10}^{33}(3q+70)^{-\frac{1}{2}}[3\,dq]$

$$= \frac{1000}{3}\cdot\frac{(3q+70)^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_{10}^{33}$$
$$= \frac{2000}{3}\sqrt{3q+70}\Big|_{10}^{33}$$
$$= \frac{2000}{3}[13-10] = \$2000$$

63. $\int_0^{100} 0.008e^{-0.008t}\,dt = -\int_0^{100} e^{0.008t}[-0.008\;dt]$

$$= -e^{-0.008t}\Big|_0^{100} = -e^{-0.8} + 1 \approx 0.5507$$

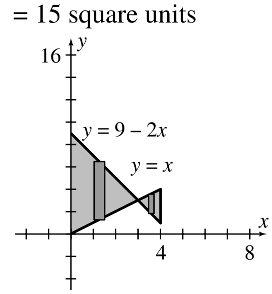
64. $\int_0^5 4000e^{0.05t}\,dt = 4000\cdot\frac{1}{0.05}\int_0^5 e^{0.05t}[0.05\;dt]$

$$= \frac{4000}{0.05}e^{0.05t}\Big|_0^5 = \frac{4000}{0.05}\left[e^{0.25} - 1\right] \approx \$22{,}722$$

65. $y = 9 - 2x$, $y = x$; from $x = 0$ to $x = 4$. Region appears below. Intersection: $x = 9 - 2x$, $3x = 9$, so $x = 3$.

$$\text{Area} = \int_0^3 [(9-2x)-x]\,dx + \int_3^4 [x-(9-2x)]\,dx$$
$$= \int_0^3 (9-3x)\,dx + \int_3^4 (3x-9)\,dx$$
$$= \left(9x - \frac{3x^2}{2}\right)\Bigg|_0^3 + \left(\frac{3x^2}{2} - 9x\right)\Bigg|_3^4$$
$$= \left[\left(27 - \frac{27}{2}\right) - 0\right] + \left[(24-36) - \left(\frac{27}{2} - 27\right)\right]$$

= 15 square units



66. $y = 2x^2,\ y = 2 - 5x$; from $x = -1$ to $x = \frac{1}{3}$.

Region appears below.

$2x^2 = 2 - 5x,\ 2x^2 + 5x - 2 = 0,$

$x = \frac{-5 \pm \sqrt{41}}{4}$ (from the quadratic formula),

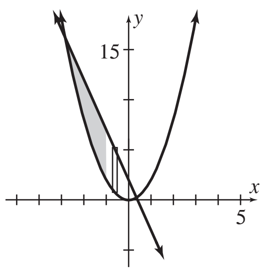
$x \approx -2.85$ or 0.35.

$$\text{Area} = \int_{-1}^{1/3}[(2-5x) - 2x^2]dx$$

$$= \left(2x - \frac{5x^2}{2} - \frac{2x^3}{3}\right)\Bigg|_{-1}^{1/3}$$

$$= \left(\frac{2}{3} - \frac{5}{18} - \frac{2}{81}\right) - \left(-2 - \frac{5}{2} + \frac{2}{3}\right)$$

$$= \frac{340}{81} \text{ square units}$$



67. $\left.\begin{array}{l} D:\ p = 0.01q^2 - 1.1q + 30 \\ S:\ p = 0.01q^2 + 8 \end{array}\right\}$

Equilibrium pt. $= (q_0,\ p_0) = (20, 12)$

$$\text{CS} = \int_0^{q_0}[f(q) - p_0]dq$$

$$= \int_0^{20}\left[\left(0.01q^2 - 1.1q + 30\right) - 12\right]dq$$

$$= \int_0^{20}\left(0.01q^2 - 1.1q + 18\right)dq$$

$$= \left(\frac{0.01q^3}{3} - \frac{1.1q^2}{2} + 18q\right)\Bigg|_0^{20}$$

$$= \left(\frac{80}{3} - 220 + 360\right) - 0 = 166\frac{2}{3}$$

$$\text{PS} = \int_0^{q_0}[p_0 - g(q)]dq = \int_0^{20}\left[12 - \left(0.01q^2 + 8\right)\right]dq$$

$$= \int_0^{20}\left(4 - 0.01q^2\right)dq = \left(4q - \frac{0.01q^3}{3}\right)\Bigg|_0^{20}$$

$$= \left(80 - \frac{80}{3}\right) - 0 = 53\frac{1}{3}$$

68. $\left.\begin{array}{l} D:\ p = (q-5)^2 \\ S:\ p = q^2 + q + 3 \end{array}\right\}$

Equilibrium pt. $= (q_0,\ p_0) = (2, 9)$

$$\text{CS} = \int_0^2\left[(q-5)^2 - 9\right]dq = \left[\frac{(q-5)^3}{3} - 9q\right]\Bigg|_0^2$$

$$= \left(\frac{-27}{3} - 18\right) - \left(-\frac{125}{3} - 0\right) = \$14\frac{2}{3} \text{ thousand}$$

$\approx$ \$14,667

69. $$\int_{q_0}^{q_n}\frac{dq}{q - \hat{q}} = -(u+v)\int_0^n dt$$

$$\ln|q - \hat{q}|\Big|_{q_0}^{q_n} = -(u+v)t\Big|_0^n$$

$$\ln|q_n - \hat{q}| - \ln|q_0 - \hat{q}| = -(u+v)n$$

$$\ln|q_0 - \hat{q}| - \ln|q_n - \hat{q}| = (u+v)n$$

$$\ln\left|\frac{q_0 - \hat{q}}{q_n - \hat{q}}\right| = (u+v)n$$

$$n = \frac{1}{u+v}\ln\left|\frac{q_0 - \hat{q}}{q_n - \hat{q}}\right|$$

as was to be shown.

70. $$Q = \int_0^R 2\pi rv\ dr = 2\pi\int r\cdot\frac{(P_1 - P_2)\left(R^2 - r^2\right)}{4\eta l}dr$$

$$= \frac{\pi(P_1 - P_2)}{2\eta l}\int_0^R r\left(R^2 - r^2\right)dr$$

$$= \frac{\pi(P_1 - P_2)}{2\eta l}\int_0^R\left(R^2 r - r^3\right)dr$$

$$= \frac{\pi(P_1 - P_2)}{2\eta l}\left(\frac{R^2 r^2}{2} - \frac{r^4}{4}\right)\Bigg|_0^R$$

$$= \frac{\pi(P_1 - P_2)}{2\eta l}\left[\left(\frac{R^4}{2} - \frac{R^4}{4}\right) - 0\right]$$

$$= \frac{\pi(P_1 - P_2)}{2\eta l}\left(\frac{R^4}{4}\right) = \frac{\pi R^4(P_1 - P_2)}{8\eta l}$$

As was to be shown.

71. Case 1. $r \neq -1$

$$g(x) = \frac{1}{k}\int_1^{1/x} ku^r\,du = \int_1^{1/x} u^r\,du = \left.\frac{u^{r+1}}{r+1}\right|_1^{1/x}$$

$$= \frac{1}{r+1}\left(x^{-r-1} - 1\right)$$

$$g'(x) = \frac{1}{r+1}\left[-(r+1)x^{-r-2}\right] = -\frac{1}{x^{r+2}}$$

Case 2. $r = -1$

$$g(x) = \frac{1}{k}\int_1^{1/x} ku^{-1}du = \int_1^{1/x}\frac{1}{u}du$$

$$= \ln|u|\Big|_1^{1/x} = \ln\left(\frac{1}{x}\right) - 0 = -\ln x$$

$$g'(x) = -\frac{1}{x} = -\frac{1}{x^{r+2}}$$

72. Two integrals are needed.
Answer: 101.75 sq units

73. Two integrals are involved.
Answer: 15.08 sq units

74. Two integrals are needed.
Answer: 32.75

75. CS $\approx$ 1148; PS $\approx$ 251

Mathematical Snapshot Chapter 14

1. a. $\int_0^5 f(t)dt = \int_0^5 (100-2t)dt = \left(100t - t^2\right)\Big|_0^5$

$= (500 - 25) - 0 = 475$

b. $\int_{20}^{25} f(t)dt = \int_{20}^{25}(100-2t)dt = (100t - t^2)\Big|_{20}^{25}$

$= (2500-625)-(2000-400) = 275$

2. a. Total revenue $= \int_0^R (m+st)f(t)dt$

$$= \int_0^{80} (50+0.2t)\cdot(40-0.5t)dt$$

$$= \int_0^{80}\left(2000 - 17t - 0.1t^2\right)dt$$

$$= \left(2000t - \frac{17}{2}t^2 - \frac{1}{30}t^3\right)\Bigg|_0^{80}$$

$$= 160{,}000 - 54{,}400 - \frac{51{,}200}{3} \approx \$88{,}533.33$$

b. Total number of units sold

$$= \int_0^R f(t)dt = \int_0^{80}(40-0.5t)dt$$

$$= \left(40t - 0.25t^2\right)\Big|_0^{80} = 3200 - 1600 = 1600$$

c. Average delivered price

$$= \frac{\text{total revenue}}{\text{total number of units sold}}$$

$$\approx \frac{88{,}533.33}{1600} \approx \$55.33$$

3. a. Total revenue

$$= \int_0^R (m+st)f(t)dt = \int_0^{30}(100+t)(900-t^2)dt$$

$$= \int_0^{30}\left(90{,}000 + 900t - 100t^2 - t^3\right)dt$$

$$= 90{,}000t + 450t^2 - \frac{100}{3}t^3 - \frac{1}{4}t^4\Big|_0^{30}$$

= 2,700,000 + 405,000 – 900,000 – 202,500
= \$2,002,500

b. Total number of units sold

$$= \int_0^R f(t)dt = \int_0^{30}\left(900 - t^2\right)dt$$

$$= \left(900t - \frac{1}{3}t^3\right)\Bigg|_0^{30} = 27{,}000 - 9000 = 18{,}000$$

c. Average delivered price

$$= \frac{\text{total revenue}}{\text{total number of units sold}}$$

$$= \frac{2{,}002{,}500}{18{,}000} = \$111.25$$

4. Answers may vary.

Chapter 15

Principles in Practice 15.1

1. $S(t) = \int -4te^{0.1t}\,dt$

Let $u = -4t$ and $dv = e^{0.1t}dt$, so $du = -4\,dt$, and

$v = \int e^{0.1t}dt = \frac{1}{0.1}e^{0.1t} = 10e^{0.1t}$.

$\int -4te^{0.1t}dt = (-4t)\left(10e^{0.1t}\right) - \int\left(10e^{0.1t}\right)(-4)dt$

$= -40te^{0.1t} + \int 40e^{0.1t}dt$

$= -40te^{0.1t} + 40\frac{e^{0.1t}}{0.1} + C$

$= -40te^{0.1t} + 400e^{0.1t} + C$

$S(t) = -40te^{0.1t} + 400e^{0.1t} + C$ and $S(0) = 5000$

$5000 = 0 + 400e^0 + C$

$C = 4600$

$S(t) = -40te^{0.1t} + 400e^{0.1t} + 4600$

2. $P(t) = \int 0.1t(\ln t)^2\,dt$

Let $u = (\ln t)^2$ and $dv = 0.1t\,dt$, so

$du = 2(\ln t)\left(\frac{1}{t}\right)dt = \frac{2\ln t}{t}dt$ and

$v = \int 0.1t\,dt = 0.1\frac{t^2}{2} = 0.05t^2$

$\int 0.1t(\ln t)^2\,dt$

$= 0.05t^2(\ln t)^2 - \int\left(0.05t^2\right)\left(\frac{2\ln t}{t}\right)dt$

$= 0.05(t\ln t)^2 - \int 0.1t\ln t\,dt$

For $\int 0.1t\ln t\,dt$, let $u = \ln t$ and $dv = 0.1t\,dt$, so

$du = \frac{1}{t}dt$ and $v = 0.05t^2$.

$\int 0.1t\ln t\,dt = 0.05t^2\ln t - \int\left(0.05t^2\right)\left(\frac{1}{t}\right)dt$

$= 0.05t^2\ln t - \int 0.05t\,dt$

$= 0.05t^2\ln t - 0.05\frac{t^2}{2} + C$

$= 0.05t^2\ln t - 0.025t^2 + C$

Thus,

$P(t) = 0.05(t\ln t)^2 - \left(0.05t^2\ln t - 0.025t^2\right) + C$

$= 0.05(t\ln t)^2 - 0.05t^2\ln t + 0.025t^2 + C$

Problems 15.1

1. $\int f(x)dx = uv - \int v\,du$

$= x\cdot\frac{2}{3}(x+5)^{\frac{3}{2}} - \int\frac{2}{3}(x+5)^{\frac{3}{2}}dx$

$= \frac{2}{3}x(x+5)^{\frac{3}{2}} - \frac{2}{3}\cdot\frac{2}{5}(x+5)^{\frac{5}{2}} + C$

$= \frac{2}{3}x(x+5)^{\frac{3}{2}} - \frac{4}{15}(x+5)^{\frac{5}{2}} + C$

2. $\int xe^{3x+1}dx$

If $u = x$ and $dv = e^{3x+1}dx,$ then $du = dx$ and

$v = \frac{1}{3}e^{3x+1}$.

$\int xe^{3x+1}dx = \frac{x}{3}e^{3x+1} - \int\frac{1}{3}e^{3x+1}dx$

$= \frac{x}{3}e^{3x+1} - \frac{1}{3}\cdot\frac{1}{3}\int e^{3x+1}[3\,dx]$

$= \frac{x}{3}e^{3x+1} - \frac{1}{9}e^{3x+1} + C$

$= \frac{1}{9}e^{3x+1}(3x-1) + C$

3. $\int xe^{-x}dx$

Letting $u = x$, $dv = e^{-x}dx$, then $du = dx$,

$v = -e^{-x}$.

$\int xe^{-x}dx = -xe^{-x} - \int -e^{-x}dx$

$= -xe^{-x} - \int e^{-x}[-dx] = -xe^{-x} - e^{-x} + C$

$= -e^{-x}(x+1) + C$

4. $\int xe^{-5x}\,dx$

Letting $u = x$, $dv = e^{-5x}\,dx$, then $du = dx$, $v = -\frac{1}{5}e^{-5x}$.

$$\int xe^{-5x}\,dx = -\frac{xe^{-5x}}{5} - \int -\frac{1}{5}e^{-5x}\,dx$$
$$= -\frac{xe^{-5x}}{5} + \frac{e^{-5x}}{5(-5)} + C$$
$$= -\frac{e^{-5x}}{5}\left(x + \frac{1}{5}\right) + C$$

5. $\int y^3 \ln y\,dy$

Letting $u = \ln y$, $dv = y^3dy$, then $du = \left(\frac{1}{y}\right)dy$, $v = \frac{y^4}{4}$

$$\int y^3 \ln y\,dy = \frac{y^4 \ln y}{4} - \int \frac{y^4}{4}\left(\frac{1}{y}dy\right)$$
$$= \frac{y^4 \ln y}{4} - \int \frac{y^3}{4}dy = \frac{y^4 \ln y}{4} - \frac{y^4}{16} + C$$
$$= \frac{y^4}{4}\left[\ln(y) - \frac{1}{4}\right] + C$$

6. $\int x^2 \ln x\,dx$

Letting $u = \ln x$, $dv = x^2dx$, then $du = \frac{1}{x}dx$, $v = \frac{x^3}{3}$

$$\int x^2 \ln x\,dx = \frac{x^3 \ln x}{3} - \int \frac{x^3}{3}\left(\frac{1}{x}dx\right)$$
$$= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C = \frac{x^3}{3}\left[\ln(x) - \frac{1}{3}\right] + C$$

7. $\int \ln(4x)\,dx$

Letting $u = \ln(4x)$, $dv = dx$, then $du = \left(\frac{1}{x}\right)dx$, $v = x$.

$$\int \ln(4x)dx = x\ln(4x) - \int x\left(\frac{1}{x}dx\right)$$
$$= x\ln(4x) - \int dx = x\ln(4x) - x + C$$
$$= x[\ln(4x) - 1] + C$$

8. $\int \left(\frac{t}{e^t}\right)dt$

Letting $u = t$, $dv = e^{-t}dt$, then $du = dt$, $v = -e^{-t}$

$$\int \left(\frac{t}{e^t}\right)dt = -te^{-t} - \int -e^{-t}dt$$
$$= -te^{-t} - e^{-t} + C = -e^{-t}(t+1) + C$$

9. $\int 3x\sqrt{2x+3}\,dx$

Letting $u = 3x$, $dv = \sqrt{2x+3}\,dx$, then $du = 3dx$, $v = \frac{1}{3}(2x+3)^{3/2}$.

$$\int 3x\sqrt{2x+3}\,dx$$
$$= 3x \cdot \frac{1}{3}(2x+3)^{3/2} - \int \frac{1}{3}(2x+3)^{3/2} \cdot 3\,dx$$
$$= x(2x+3)^{3/2} - \frac{1}{5}(2x+3)^{5/2} + C$$
$$= \frac{1}{5}(2x+3)^{3/2}[5x - (2x+3)] + C$$
$$= \frac{1}{5}(2x+3)^{3/2}(3x-3) + C$$
$$= \frac{3}{5}(2x+3)^{3/2}(x-1) + C$$

10. $\int \frac{12x}{\sqrt{1+4x}}dx$

Letting $u = 12x$, $dv = (1+4x)^{-\frac{1}{2}}dx$, then $du = 12dx$, $v = \frac{1}{2}(1+4x)^{\frac{1}{2}}$

$$\int \frac{12x}{\sqrt{1+4x}}dx = 12x \cdot \frac{\sqrt{1+4x}}{2} - \int \frac{(1+4x)^{\frac{1}{2}}}{2} \cdot 12dx$$
$$= 6x\sqrt{1+4x} - (1+4x)^{\frac{3}{2}} + C$$
$$= \sqrt{4x-1}[6x - (1+4x)] + C$$
$$= (2x-1)\sqrt{4x+1} + C$$

11. $\int \frac{x}{(5x+2)^3}\,dx$

Letting $u = x$, $dv = (5x+3)^{-3}\,dx$, then $du = dx$ and $v = -\frac{1}{10}(5x+3)^{-2}$.

$$\int \frac{x}{(5x+2)^3}\,dx$$
$$= -\frac{x}{10(5x+3)^2} - \int -\frac{1}{10}(5x+3)^{-2}\,dx$$
$$= -\frac{x}{10(5x+3)^2} + \frac{1}{10}\cdot\frac{(5x+3)^{-1}}{5(-1)} + C$$
$$= -\frac{x}{10(5x+3)^2} - \frac{1}{50(5x+3)} + C$$

12. $\int \frac{\ln(x+1)}{2(x+1)}\,dx = \frac{1}{2}\int \ln(x+1)\left[\frac{1}{x+1}\,dx\right]$

(Form: $\int u^n du$)

$$\int \frac{\ln(x+1)}{2(x+1)}\,dx = \frac{\ln(x+1)^2}{4} + C$$

13. $\int \frac{\ln x}{x^2}\,dx$

Letting $u = \ln x$, $dv = x^{-2}dx$, then $du = \frac{1}{x}dx$, $v = -x^{-1}$.

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} - \int -x^{-1}\left(\frac{1}{x}dx\right)$$
$$= -\frac{\ln x}{x} + \int x^{-2}dx = -\frac{\ln x}{x} - \frac{1}{x} + C$$
$$= -\frac{1}{x}(1 + \ln x) + C$$

14. $\int \frac{3x+5}{e^{2x}}\,dx$

Letting $u = 3x + 5$, $dv = e^{-2x}dx$, then $du = 3dx$ and $v = -\frac{1}{2}e^{-2x}$.

$$\int \frac{3x+5}{e^{2x}}\,dx = -\frac{3x+5}{2e^{2x}} - \int -\frac{1}{2}e^{-2x}\cdot 3\,dx$$
$$= -\frac{3x+5}{2e^{2x}} + \frac{3}{2}\int e^{-2x}dx$$
$$= -\frac{3x+5}{2e^{2x}} + \frac{3}{2}\left(-\frac{1}{2}e^{-2x}\right) + C$$
$$= -\frac{1}{4e^{2x}}[2(3x+5)+3] + C$$
$$= -\frac{1}{4e^{2x}}(6x+13) + C$$

15. $\int_1^2 4xe^{2x}dx$

Letting $u = 4x$, $dv = e^{2x}dx$, then $du = 4dx$, $v = \frac{1}{2}e^{2x}$

$$\int_1^2 4xe^{2x}dx = \left[2xe^{2x} - \int 2e^{2x}dx\right]\Bigg|_1^2$$
$$= \left[2xe^{2x} - e^{2x}\right]\Bigg|_1^2 = e^{2x}(2x-1)\Big|_1^2$$
$$= e^4(3) - e^2(1) = e^2\left(3e^2 - 1\right)$$

16. $\int_1^2 2xe^{-3x}\,dx$

Letting $u = 2x$, $dv = e^{-3x}\,dx$, then $du = 2\,dx$ and $v = -\frac{1}{3}e^{-3x}$.

$$\int_1^2 2xe^{-3x}\,dx$$
$$=\left[-\frac{2xe^{-3x}}{3}-\int -\frac{2}{3}e^{-3x}\,dx\right]\Bigg|_1^2$$
$$=\left[-\frac{2xe^{-3x}}{3}+\frac{2}{3}\cdot\frac{e^{-3x}}{-3}\right]\Bigg|_1^2$$
$$=\left[-\frac{2xe^{-3x}}{3}-\frac{2e^{-3x}}{9}\right]\Bigg|_1^2$$
$$=\left[-\frac{2e^{-3x}}{3}\left(x+\frac{1}{3}\right)\right]\Bigg|_1^2$$
$$=\left[-\frac{2e^{-6}}{3}\left(2+\frac{1}{3}\right)\right]-\left[-\frac{2e^{-3}}{3}\left(1+\frac{1}{3}\right)\right]$$
$$=-\frac{2e^{-6}}{3}\left[\frac{7}{3}-e^{3}\left(\frac{4}{3}\right)\right]$$
$$=-\frac{2}{9e^{6}}[7-4e^{3}]$$

17. $\int_0^1 xe^{-x^2}\,dx=-\frac{1}{2}\int_0^1 e^{-x^2}(-2x\,dx)$ (Form: $\int e^u\,du$)

$$=-\frac{1}{2}e^{-x^2}\Big|_0^1=-\frac{1}{2}\left(e^{-1}-1\right)=\frac{1}{2}\left(1-e^{-1}\right)$$

18. $\int \frac{3x^3}{\sqrt{4-x^2}}\,dx$

Letting $u=3x^2$, $dv=x\left(4-x^2\right)^{-\frac{1}{2}}dx$, then $du=6x\,dx$, $v=-\left(4-x^2\right)^{\frac{1}{2}}$.

$$\int \frac{3x^3}{\sqrt{4-x^2}}\,dx$$
$$=-3x^2\left(4-x^2\right)^{\frac{1}{2}}-\int -\left(4-x^2\right)^{\frac{1}{2}}(6x\,dx)$$
$$=-3x^2\left(4-x^2\right)^{\frac{1}{2}}-2\left(4-x^2\right)^{\frac{3}{2}}+C$$
$$=-\sqrt{4-x^2}\left[3x^2+2\left(4-x^2\right)\right]+C$$
$$=-\left(x^2+8\right)\sqrt{4-x^2}+C$$

19. $\int_1^2 \frac{3x}{\sqrt{4-x}}\,dx$

Letting $u=3x$, $dv=(4-x)^{-\frac{1}{2}}dx$, then $du=3dx$, $v=-2(4-x)^{\frac{1}{2}}$.

$$\int_1^2 \frac{3x}{\sqrt{4-x}}\,dx$$
$$=\left[-6x(4-x)^{\frac{1}{2}}-\int -2(4-x)^{\frac{1}{2}}(3\,dx)\right]\Bigg|_1^2$$
$$=\left[-6x(4-x)^{\frac{1}{2}}-4(4-x)^{\frac{3}{2}}\right]\Bigg|_1^2$$
$$=\left\{-2\sqrt{4-x}[3x+2(4-x)]\right\}\Big|_1^2$$
$$=\left\{-2\sqrt{4-x}(x+8)\right\}\Big|_1^2=-2\left(10\sqrt{2}-9\sqrt{3}\right)$$
$$=2\left(9\sqrt{3}-10\sqrt{2}\right)$$

20. $\int (\ln x)^2\,dx$

Letting $u=(\ln x)^2$, $dv=dx$, then $du=\left[\frac{2\ln x}{x}\right]dx$, $v=x$.

$$\int (\ln x)^2\,dx=x(\ln x)^2-\int x\left[\frac{2\ln x}{x}dx\right]$$
$$=x(\ln x)^2-2\int \ln(x)dx$$

For $\int \ln(x)dx$, let $u=\ln x$, $dv=dx$. Then $du\left(\frac{1}{x}\right)dx$, $v=x$, so

$$\int \ln(x)dx=x\ln x-\int x\left(\frac{1}{x}dx\right)=x[\ln(x)-1]+C_1.$$

Thus $\int (\ln x)^2\,dx=x\left[(\ln x)^2-2\ln(x)+2\right]+C$.

21. $\int 3(2x-2)\ln(x-2)dx$

Letting $u = 3\ln(x-2)$, $dv = (2x-2)dx$, then $du = \frac{3}{x-2}dx$ and $v = x^2 - 2x = x(x-2)$.

$\int 3(2x-2)\ln(x-2)dx$

$= 3x(x-2)\ln(x-2) - \int x(x-2)\cdot\frac{3}{x-2}dx$

$= 3x(x-2)\ln(x-2) - \int 3x\,dx$

$= 3x(x-2)\ln(x-2) - \frac{3}{2}x^2 + C$

22. $\int \frac{xe^x}{(x+1)^2}dx$

Letting $u = xe^x$, $dv = (x+1)^{-2}dx$, then $du = (x+1)e^x dx$, $v = -(x+1)^{-1}$.

$\int \frac{xe^x}{(x+1)^2}dx = -\frac{xe^x}{x+1} + \int e^x dx$

$= -\frac{xe^x}{x+1} + e^x + C$

$= e^x\left(1 - \frac{x}{x+1}\right) = e^x\left(\frac{x+1-x}{x+1}\right) + C = \frac{e^x}{x+1} + C$

23. $\int x^2 e^x dx$

Letting $u = x^2$, $dv = e^x dx$, then $du = 2x\,dx$ and $v = e^x$.

$\int x^2 e^x dx = x^2 e^x - \int e^x(2x\ dx)$

$= x^2 e^x - 2\int xe^x dx$

For $\int xe^x dx$, let $u = x$, $dv = e^x dx$. Then $du = dx$, $v = e^x$ and

$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C_1$

$= e^x(x-1) + C_1$.

Thus $\int x^2 e^x dx = x^2 e^x - 2\left[e^x(x-1)\right] + C$

$= e^x\left(x^2 - 2x + 2\right) + C$.

24. $\int_e^3 \sqrt[3]{x}\ln(x^5)\,dx = 5\int_e^3 x^{\frac{1}{3}}\ln x\,dx$

Letting $u = \ln x$ and $dv = x^{\frac{1}{3}}dx$, then $du = \frac{1}{x}dx$ and $v = \frac{3}{4}x^{\frac{4}{3}}$.

$5\int_e^3 x^{\frac{1}{3}}\ln x\,dx$

$= 5\left(\left[\ln x\cdot\frac{3}{4}x^{\frac{4}{3}} - \int\frac{3}{4}x^{\frac{4}{3}}\cdot\frac{1}{x}dx\right]\Bigg|_e^3\right)$

$= 5\left(\left[\frac{3\ln x}{4}x^{\frac{4}{3}} - \frac{3}{4}\int x^{\frac{1}{3}}dx\right]\Bigg|_e^3\right)$

$= 5\left(\left[\frac{3\ln x}{4}x^{\frac{4}{3}} - \frac{9}{16}x^{\frac{4}{3}}\right]\Bigg|_e^3\right)$

$= 5\left(\left[\frac{3\ln 3}{4}3^{\frac{4}{3}} - \frac{9}{16}3^{\frac{4}{3}}\right] - \left[\frac{3\ln e}{4}e^{\frac{4}{3}} - \frac{9}{16}e^{\frac{4}{3}}\right]\right)$

$= 5\left(\frac{3}{4}3^{\frac{4}{3}}\left[\ln 3 - \frac{3}{4}\right] - \frac{3}{4}e^{\frac{4}{3}}\left[\frac{1}{4}\right]\right)$

25. $\int\left(x - e^{-x}\right)^2 dx = \int\left(x^2 - 2xe^{-x} + e^{-2x}\right)dx$

$= \frac{x^3}{3} - \frac{e^{-2x}}{2} - 2\int xe^{-x}dx$

Using Problem 3 for $\int xe^{-x}dx$,

$\int\left(x - e^{-x}\right)^2 dx = \frac{x^3}{3} - \frac{e^{-2x}}{2} + 2e^{-x}(x+1) + C$

26. $\int x^2 e^{3x}dx$

Letting $u = x^2$, $dv = e^{3x}dx$, then $du = 2x\,dx$ and $v = \frac{1}{3}e^{3x}$.

$\int x^2 e^{3x}dx = \frac{1}{3}x^2 e^{3x} - \int\frac{1}{3}e^{3x}\cdot 2x\,dx$

$= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int xe^{3x}dx$

For $\int xe^{3x}dx$, let $u = x$, $dv = e^{3x}dx$, then $du = dx$, $v = \frac{1}{3}e^{3x}$, and

$\int xe^{3x}dx = \frac{1}{3}xe^{3x} - \int\frac{1}{3}e^{3x}dx$

$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C_1$.

Thus,

$$\int x^2e^{3x}dx = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left(\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right) + C$$
$$= \frac{1}{27}e^{3x}(9x^2 - 6x - 2) + C$$

27. $\int x^3e^{x^2}\,dx$

Letting $u = x^2$, $dv = xe^{x^2}dx$, then $du = 2x\,dx$, $v = \left(\frac{1}{2}\right)e^{x^2}$.

$$\int x^3e^{x^2}dx = \frac{x^2e^{x^2}}{2} - \int \frac{e^{x^2}}{2}(2x\,dx)$$
$$= \frac{x^2e^{x^2}}{2} - \frac{e^{x^2}}{2} + C = \frac{e^{x^2}}{2}\left(x^2 - 1\right) + C$$

28. $\int x^5e^{x^2}\,dx$

Letting $u = x^4$ and $dv = xe^{x^2}dx$, then $du = 4x^3\,dx$ and $v = \frac{1}{2}e^{x^2}$.

$$\int x^5e^{x^2}dx = \frac{x^4}{2}e^{x^2} - \int \frac{1}{2}e^{x^2}\cdot 4x^3\,dx$$
$$= \frac{x^4}{2}e^{x^2} - 2\int x^3e^{x^2}dx$$

Using Problem 27 for $\int x^3e^{x^2}dx$,

$$\int x^5e^{x^2}dx = \frac{x^4}{2}e^{x^2} - 2\cdot\left[\frac{1}{2}e^{x^2}(x^2 - 1)\right] + C$$
$$= \frac{x^4}{2}e^{x^2} - e^{x^2}(x^2 - 1) + C$$
$$= \frac{1}{2}e^{x^2}(x^4 - 2x^2 + 2) + C$$

29. $\int \left(2^x + x\right)^2 dx = \int \left(2^{2x} + 2x2^x + x^2\right)dx$

$$= \int 2^{2x}dx + \int x2^{x+1}dx + \int x^2dx$$

For $\int x2^{x+1}dx$, let $u = x$, $dv = 2^{x+1}dx$. Then $du = dx$, $v = \frac{1}{\ln 2}\cdot 2^{x+1}$ and

$$\int x2^{x+1}dx = \frac{x}{\ln 2}\cdot 2^{x+1} - \frac{1}{\ln 2}\int 2^{x+1}dx$$
$$= \frac{x}{\ln 2}\cdot 2^{x+1} - \frac{1}{\ln^2 2}\cdot 2^{x+1} + C_1. \text{ Thus}$$

$$\int \left(2^x + x\right)^2 dx = \int 2^{2x}dx + \int x2^{x+1}dx + \int x^2dx$$
$$= \frac{1}{2}\int 2^{2x}[2\,dx] + \int x2^{x+1}dx + \int x^2dx$$
$$= \frac{1}{2\ln 2}\cdot 2^{2x} + \frac{x}{\ln 2}\cdot 2^{x+1} - \frac{1}{\ln^2 2}\cdot 2^{x+1} + \frac{x^3}{3} + C$$
$$= \frac{1}{\ln 2}\cdot 2^{2x-1} + \frac{x}{\ln 2}\cdot 2^{x+1} - \frac{1}{\ln^2 2}\cdot 2^{x+1} + \frac{x^3}{3} + C$$

30. $\frac{d}{dx}\left[\ln\left(x + \sqrt{x^2+1}\right)\right]$

$$= \frac{1}{x + \sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$
$$= \frac{1}{x + \sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}}$$

For $\int \ln\left(x + \sqrt{x^2+1}\right)dx$, let $u = \ln\left(x + \sqrt{x^2+1}\right)$, $dv = dx$. Then $du = \frac{1}{\sqrt{x^2+1}}dx$, $v = x$, and

$$\int \ln\left(x + \sqrt{x^2+1}\right)dx$$
$$= x\ln\left(x + \sqrt{x^2+1}\right) - \int x\left(x^2+1\right)^{-\frac{1}{2}}dx$$
$$= x\ln\left(x + \sqrt{x^2+1}\right) - \frac{1}{2}\int \left(x^2+1\right)^{-\frac{1}{2}}[2x\,dx]$$
$$= x\ln\left(x + \sqrt{x^2+1}\right) - \sqrt{x^2+1} + C$$

31. Area $= \int_1^{e^3}(\ln x)dx$. Letting $u = \ln x$, $dv = dx$, then $du = \left(\frac{1}{x}\right)dx$, $v = x$.

$$\int_1^{e^3}(\ln x)dx = \left[(x\ln x) - \int x\cdot\frac{1}{x}dx\right]\Bigg|_1^{e^3}$$
$$= \left[(x\ln x) - \int dx\right]\Bigg|_1^{e^3} = [x\ln(x) - x]\Big|_1^{e^3}$$
$$= \left[e^3\cdot 3 - e^3\right] - [1\cdot 0 - 1] = 2e^3 + 1$$

The area is $(2e^3 + 1)$ sq units.

32. Area $= \int_0^1 x^2 e^x \, dx.$

Letting $u = x^2$, $dv = e^x dx$, then $du = 2x\, dx$ and $v = e^x$.

$\int x^2 e^x = x^2 e^x - 2\int xe^x \, dx$

For $\int xe^x \, dx$, let $u = x$ and $dv = e^x \, dx$, then $du = dx$ and $v = e^x$.

$\int xe^x = xe^x - \int e^x \, dx = xe^x - e^x = e^x(x-1).$

Thus $\int_0^1 x^2 e^x \, dx = (x^2 e^x - 2[e^x(x-1)])\Big|_0^1$

$= (e^x[x^2 - 2x + 2])\Big|_0^1$

$= e - 2$

The area is $(e - 2)$ sq units.

33. Area $= \int_1^2 x^2 \ln x \, dx.$

Letting $u = \ln x$, $dv = x^2 dx$, then $du = \frac{1}{x} dx$, $v = \frac{x^3}{3}$.

$$\int_1^2 x^2 \ln x \, dx = \left(\frac{x^3}{3}\ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} dx\right)\Bigg|_1^2$$

$$= \left(\frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx\right)\Bigg|_1^2$$

$$= \left(\frac{x^3}{3}\ln x - \frac{1}{9}x^3\right)\Bigg|_1^2$$

$$= \left(\frac{8}{3}\ln 2 - \frac{8}{9}\right) - \left(0 - \frac{1}{9}\right)$$

$$= \frac{8}{3}\ln(2) - \frac{7}{9}$$

The area is $\left(\frac{8}{3}\ln(2) - \frac{7}{9}\right)$ sq units.

34. $p = 10(q+10)e^{-(0.1q+1)}$

When $q = 20$, then $p = 300e^{-3}$.

$$\text{CS} = \int_0^{20} \left[10(q+10)e^{-(0.1q+1)} - 300e^{-3}\right] dq$$

$$= 10\underbrace{\int_0^{20} (q+10)e^{-(01.q+1)} dq}_{I_1} - 300e^{-3} \underbrace{\int_0^{20} dq}_{I_2}$$

For I_1, let $u = q + 10$, $dv = e^{-(0.1q+1)} dq$. Then $du = dq$, $v = -10e^{-(0.1q+1)}$, and

$$I_1 = \left[-10(q+10)e^{-(0.1q+1)} + 10\int e^{-(0.1q+1)} dq\right]\Bigg|_0^{20}$$

$$= \left[-10(q+10)e^{-(0.1q+1)} - 100e^{-(0.1q+1)}\right]\Bigg|_0^{20}$$

$$= -10e^{-(0.1q+1)}[(q+10)+10]\Big|_0^{20}$$

$$= -10e^{-(0.1q+1)}(q+20)\Big|_0^{20} = -400e^{-3} + 200e^{-1}$$

$I_2 = q\Big|_0^{20} = 20 - 0 = 20$

Thus

$\text{CS} = 10I_1 - 300e^{-3}I_2$

$= 10\left(-400e^{-3} + 200e^{-1}\right) - 300e^{-3}(20)$

$= -10{,}000e^{-3} + 2000e^{-1} \approx 237.89$

CS ≈ \$237.89

35. a. Consider $\int p \; dq$. Letting $u = p$, $dv = dq$, then $du = \frac{dp}{dq} dq$, $v = q$. Thus

$$\int p \; dq = pq - \int q \frac{dp}{dq} dq = r - \int q \frac{dp}{dq} dq$$

(since $r = pq$).

b. From (a), $r = \int p \; dq + \int q \frac{dp}{dq} dq$.

Combining the integrals gives

$$r = \int \left(p + q\frac{dp}{dq}\right) dq.$$

c. From (b), $\frac{dr}{dq} = p + q\frac{dp}{dq}$. Thus

$$\int_0^{q_0} \left(p + q\frac{dp}{dq}\right) dq$$

$$= \int_0^{q_0} \frac{dr}{dq} dq = r\left(q_0\right) - r(0) = r\left(q_0\right)$$

[since $r(0) = 0$].

36. $\int f(x)e^x dx$

Letting $u = f(x)$, $dv = e^x dx$, then $du = f'(x)dx$, $v = e^x$. Using integration by parts,

$\int f(x)e^x dx = f(x)e^x - \int f'(x)e^x dx$. Thus

$\int f(x)e^x dx + \int f'(x)e^x dx = f(x)e^x + C$

37. f and its inverse f^{-1} satisfy the equation $f(f^{-1}(x)) = x$. Differentiating this equation using the Chain Rule we get: $f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$. Thus $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$. Now to evaluate $\int f^{-1}(x)\,dx$ we will use integration by parts, letting $u = f^{-1}(x)$ and $dv = dx$. Then $du = \frac{1}{f'(f^{-1}(x))}dx$ and $v = x$.

So $\int f^{-1}(x)\,dx = xf^{-1}(x) - \int \frac{x}{f'(f^{-1}(x))}dx.$

To evaluate $\int \frac{x}{f'(f^{-1}(x))}dx$ we will use the fact that $x = f(f^{-1}(x))$ and $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$.

Hence

$$\int \frac{x}{f'(f^{-1}(x))}dx = \int f(f^{-1}(x)) \cdot (f^{-1})'(x)\,dx = F(f^{-1}(x))$$

since $F' = f$. Finally,

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - F(f^{-1}(x)) + C.$$

Principles in Practice 15.2

1. $r(q) = \int r'(q)dq = \int \frac{5(q+4)}{q^2+4q+3}dq$

Express $\frac{5(q+4)}{q^2+4q+3}$ as a sum of partial fractions.

$$\frac{5(q+4)}{q^2+4q+3} = \frac{5(q+4)}{(q+1)(q+3)} = \frac{A}{q+1} + \frac{B}{q+3}$$

$5(q + 4) = A(q + 3) + B(q + 1)$

When $q = -3$, we get $5(1) = -2B$, so $B = -\frac{5}{2}$.

When $q = -1$, we get $5(3) = A(2)$, so $A = \frac{15}{2}$.

$$r(q) = \int \frac{5(q+4)}{q^2+4q+3}dx$$

$$= \int \frac{\frac{15}{2}}{q+1}dq - \int \frac{\frac{5}{2}}{q+3}dq$$

$$= \frac{15}{2}\ln|q+1| - \frac{5}{2}\ln|q+3| + C$$

$$= \frac{5}{2}\ln\left|\frac{(q+1)^3}{q+3}\right| + C$$

Since $r(0) = 0$, $0 = \frac{5}{2}\ln\left|\frac{1}{3}\right| + C$ so $C = \frac{5}{2}\ln 3$ and

$$r(q) = \frac{5}{2}\ln\left|\frac{3(q+1)^3}{q+3}\right|.$$

2. $V(t) = \int V'(t)dt = \int \frac{300t^3}{t^2+6}dt$

Since the degree of the numerator is greater than the degree of the denominator, we first divide $300t^3$ by t^2+6 to reduce the fraction.

$$\frac{300t^3}{t^2+6} = \frac{300t^3+1800t-1800t}{t^2+6}$$

$$= \frac{300t\left(t^2+6\right)-1800t}{t^2+6} = 300t - \frac{1800t}{t^2+6}$$

t^2+6 is irreducible. To integrate $\frac{1800t}{t^2+6}$, let $u = t^2+6$, so $du = 2t\,dt$

$$\int \frac{300t^3}{t^2+6}dt = \int 300t\;dt - \int \frac{1800t}{t^2+6}dt$$

$$= 150t^2 - 900\ln\left|t^2+6\right| + C$$

$$V(t) = 150t^2 - 900\ln\left(t^2+6\right) + C$$

Problems 15.2

1. $\frac{10x}{x^2+7x+6} = \frac{10x}{(x+6)(x+1)} = \frac{A}{x+6} + \frac{B}{x+1}$

$10x = A(x+1) + B(x+6)$

If $x = -1$, then $-10 = 5B$, or $B = -2$. If $x = -6$, then $-60 = -5A$, or $A = 12$.

Answer $\frac{12}{x+6} - \frac{2}{x+1}$

2. $\frac{x+5}{x^2-1}=\frac{x+5}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$

$x + 5 = A(x + 1) + B(x - 1)$

If $x = -1$, then $4 = -2B$, or $B = -2$. If $x = 1$, then $6 = 2A$, or $A = 3$.

Answer $\frac{3}{x-1}-\frac{2}{x+1}$

3. $\frac{2x^2}{x^2+5x+6}=2+\frac{-10x-12}{x^2+5x+6}$

$\frac{-10x-12}{x^2+5x+6}=\frac{-10x-12}{(x+2)(x+3)}=\frac{A}{x+2}+\frac{B}{x+3}$

$-10x - 12 = A(x + 3) + B(x + 2)$

If $x = -3$, then $18 = -B$, or $B = -18$.

If $x = -2$, then $8 = A$.

Answer: $2+\frac{8}{x+2}-\frac{18}{x+3}$

4. $\frac{2x^2-15}{x^2+5x}=2+\frac{-10x-15}{x^2+5x}$ (by long division).

$\frac{-10x-15}{x^2+5x}=\frac{-10x-15}{x(x+5)}=\frac{A}{x}+\frac{B}{x+5}$

$-10x - 15 = A(x + 5) + Bx$. If $x = 0$, then $-15 = 5A$, or $A = -3$. If $x = -5$, then $35 = -5B$, or $B = -7$.

Answer: $2-\frac{3}{x}-\frac{7}{x+5}$

5. $f(x)=\frac{x+4}{x^2+4x+4}=\frac{x+4}{(x+2)^2}=\frac{A}{x+2}+\frac{B}{(x+2)^2}$

$x + 4 = A(x + 2) + B$

If $x = -2$, then $2 = B$. If $x = 0$, then $4 = 2A + B$, $2A = 4 - B = 4 - 2 = 2$, or $A = 1$.

Answer: $\frac{1}{x+2}+\frac{2}{(x+2)^2}$

6. $\frac{2x+3}{x^2(x-1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}$

$2x+3=Ax(x-1)+B(x-1)+Cx^2$

If $x = 0$, then $3 = -B$, or $B = -3$. If $x = 1$, then $5 = C$. If $x = -1$, then $1 = 2A - 2B + C$, $1 = 2A + 6 + 5$, or $A = -5$.

Answer: $-\frac{5}{x}-\frac{3}{x^2}+\frac{5}{x-1}$

7. $\frac{x^2+3}{x^3+x}=\frac{x^2+3}{x\left(x^2+1\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$

$x^2+3=A\left(x^2+1\right)+(Bx+C)x$

$x^2+3=(A+B)x^2+Cx+A$

Thus $A + B = 1$, $C = 0$, $A = 3$. This gives $A = 3$, $B = -2$, $C = 0$.

Answer: $\frac{3}{x}-\frac{2x}{x^2+1}$

8. $\frac{3x^2+5}{\left(x^2+4\right)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{\left(x^2+4\right)^2}$

$3x^2+5=(Ax+B)\left(x^2+4\right)+(Cx+D)$

$3x^2+5=Ax^3+Bx^2+(4A+C)x+(4B+D)$

Thus $A = 0$, $B = 3$, $4A + C = 0$, $4B + D = 5$. This gives $A = 0$, $B = 3$, $C = 0$, $D = -7$.

Answer: $\frac{3}{x^2+4}-\frac{7}{\left(x^2+4\right)^2}$

9. $\frac{5x-2}{x^2-x}=\frac{5x-2}{x(x-1)}=\frac{A}{x}+\frac{B}{x-1}$

$5x - 2 = A(x - 1) + Bx$

If $x = 1$, then $3 = B$. If $x = 0$, then $-2 = -A$, or $A = 2$.

$$\int\frac{5x-2}{x^2-x}dx=\int\left(\frac{2}{x}+\frac{3}{x-1}\right)dx$$

$$=2\ln|x|+3\ln|x-1|+C=\ln\left|x^2(x-1)^3\right|+C$$

10. $\frac{7x+6}{x^2+3x}=\frac{7x+6}{x(x+3)}=\frac{A}{x}+\frac{B}{x+3}$

$7x + 6 = A(x + 3) + Bx$

If $x = -3$, then $-15 = -3B$, or $B = 5$.

If $x = 0$, then $6 = 3A$, or $A = 2$.

$$\int\frac{7x+6}{x^2+3x}dx=\int\left(\frac{2}{x}+\frac{5}{x+3}\right)dx$$

$$=2\ln|x|+5\ln|x+3|+C$$

$$=\ln\left|x^2(x+3)^5\right|+C$$

11. $\dfrac{x+10}{x^2-x-2}=\dfrac{x+10}{(x+1)(x-2)}=\dfrac{A}{x+1}+\dfrac{B}{x-2}$

$x + 10 = A(x - 2) + B(x + 1)$

If $x = 2$, then $12 = 3B$, or $B = 4$. If $x = -1$, then $9 = -3A$, or $A = -3$.

$$\int \frac{x+10}{x^2-x-2}dx=\int\left(\frac{-3}{x+1}+\frac{4}{x-2}\right)dx$$

$$=-3\ln|x+1|+4\ln|x-2|+C=\ln\left|\frac{(x-2)^4}{(x+1)^3}\right|+C$$

12. $\dfrac{2x-1}{x^2-x-12}=\dfrac{2x-1}{(x-4)(x+3)}=\dfrac{A}{x-4}+\dfrac{B}{x+3}$

$2x - 1 = A(x + 3) + B(x - 4)$

If $x = -3$, then $-7 = -7B$, or $B = 1$. If $x = 4$, then $7 = 7A$, or $A = 1$.

$$\int \frac{2x-1}{x^2-x-12}dx=\int\left(\frac{1}{x-4}+\frac{1}{x+3}\right)dx$$

$$=\ln|x-4|+\ln|x+3|+C=\ln|(x-4)(x+3)|+C$$

13. $\dfrac{3x^3-3x+4}{4x^2-4}=\dfrac{1}{4}\cdot\dfrac{3x^3-3x+4}{x^2-1}$

$$=\frac{1}{4}\left(3x+\frac{4}{x^2-1}\right)$$

$$\frac{4}{x^2-1}=\frac{4}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$$

$4 = A(x + 1) + B(x - 1)$

If $x = -1$, then $4 = -2B$, or $B = -2$. If $x = 1$, then $4 = 2A$, or $A = 2$.

$$\int \frac{3x^3-3x+4}{4x^2-4}dx=\frac{1}{4}\int\left(3x+\frac{2}{x-1}+\frac{-2}{x+1}\right)dx$$

$$=\left(\frac{1}{4}\right)\left[\frac{3x^2}{2}+2\ln|x-1|-2\ln|x+1|\right]+C$$

$$=\left(\frac{1}{4}\right)\left[\frac{3x^2}{2}+\ln\left|\frac{x-1}{x+1}\right|^2\right]+C$$

14. $\dfrac{7(4-x^2)}{(x-4)(x-2)(x+3)}=\dfrac{7(2+x)(2-x)}{(x-4)(x-2)(x+3)}$

$$=\frac{-7(x+2)}{(x-4)(x+3)}=\frac{A}{x-4}+\frac{B}{x+3}$$

$-7(x + 2) = A(x + 3) + B(x - 4)$

If $x = -3$, then $7 = -7B$, or $B = -1$. If $x = 4$, then $-42 = 7A$, or $A = -6$.

$$\int \frac{7(4-x^2)}{(x-4)(x-2)(x+3)}dx=\int\left(\frac{-6}{x-4}+\frac{-1}{x+3}\right)dx$$

$$=-6\ln|x-4|-\ln|x+3|+C$$

$$=-\ln\left|(x-4)^6(x+3)\right|+C$$

15. $\dfrac{3x-4}{x^3-x^2-2x}=\dfrac{3x-4}{x(x+1)(x-2)}=\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x-2}$

$3x-4=A(x+1)(x-2)+Bx(x-2)+Cx(x+1)$

If $x = 0$, then $-4 = -2A$, or $A = 2$.

If $x = -1$, then $-7 = 3B$, or $B=-\frac{7}{3}$.

If $x = 2$, then $2 = 6C$, or $C=\frac{1}{3}$.

$$\int\frac{3x-4}{x^3-x^2-2x}dx=\int\left(\frac{2}{x}+\frac{-\frac{7}{3}}{x+1}+\frac{\frac{1}{3}}{x-2}\right)dx$$
$$=2\ln|x|-\frac{7}{3}\ln|x+1|+\frac{1}{3}\ln|x-2|+C$$
$$=\ln\left|\frac{x^2\sqrt[3]{x-2}}{\sqrt[3]{(x+1)^7}}\right|+C$$

16. $\dfrac{4-x}{x^4-x^2}=\dfrac{4-x}{x^2(x+1)(x-1)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x+1}+\dfrac{D}{x-1}$

$4-x=Ax(x+1)(x-1)+B(x+1)(x-1)+Cx^2(x-1)+Dx^2(x+1)$

If $x = 0$, then $4 = -B$, or $B = -4$. If $x = -1$, then $5 = -2C$, or $C=-\frac{5}{2}$. If $x = 1$, then $3 = 2D$, or $D=\frac{3}{2}$. If $x = 2$, then $2 = 6A + 3B + 4C + 12D$, $2 = 6A - 12 - 10 + 18$, or $2 = 6A - 4$, so $A = 1$.

$$\int\frac{4-x}{x^4-x^2}dx=\int\left(\frac{1}{x}-\frac{4}{x^2}+\frac{-\frac{5}{2}}{x+1}+\frac{\frac{3}{2}}{x-1}\right)dx$$
$$=\ln|x|+\frac{4}{x}-\frac{5}{2}\ln|x+1|+\frac{3}{2}\ln|x-1|+C$$
$$=\frac{4}{x}+\frac{1}{2}\ln\left|\frac{x^2(x-1)^3}{(x+1)^5}\right|+C$$

17. $\displaystyle\int\frac{2(3x^5+4x^3-x)}{x^6+2x^4-x^2-2}dx=\int\frac{1}{x^6+2x^4-x^2-2}\left[\left(6x^5+8x^3-2x\right)dx\right]$

$\left(\text{Form: }\int\left(\frac{1}{u}\right)du\right)$ (Partial fractions not required.)

Answer: $\ln\left|x^6+2x^4-x^2-2\right|+C$

18. $\dfrac{x^4-2x^3+6x^2-11x+2}{x^3-3x^2+2x}=x+1+\dfrac{7x^2-13x+2}{x^3-3x^2+2x}$

$\dfrac{7x^2-13x+2}{x^3-3x^2+2x}=\dfrac{7x^2-13x+2}{x(x-1)(x-2)}=\dfrac{A}{x}+\dfrac{B}{x-1}+\dfrac{C}{x-2}$

$7x^2-13x+2=A(x-1)(x-2)+Bx(x-2)+Cx(x-1)$

If $x = 0$, then $2 = 2A$, or $A = 1$. If $x = 1$, then $-4 = -B$, or $B = 4$. If $x = 2$, then $4 = 2C$, or $C = 2$.

$$\int \frac{x^4-2x^3+6x^2-11x+2}{x^3-3x^2+2x} = \int\left(x+1+\frac{1}{x}+\frac{4}{x-1}+\frac{2}{x-2}\right)dx$$
$$= \frac{x^2}{2}+x+\ln|x|+4\ln|x-1|+2\ln|x-2|+C$$
$$= \frac{x^2}{2}+x+\ln\left|x(x-1)^4(x-2)^2\right|+C$$

19. $\dfrac{2x^2-5x-2}{(x-2)^2(x-1)} = \dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$

$2x^2-5x-2 = A(x-2)^2+B(x-1)(x-2)+C(x-1)$

If $x = 1$, then $-5 = A$. If $x = 2$, then $-4 = C$.
If $x = 0$, then $-2 = 4A + 2B - C$, $-2 = -20 + 2B + 4$, or $B = 7$.

$$\int \frac{2x^2-5x-2}{(x-2)^2(x-1)}dx = \int\left[\frac{-5}{x-1}+\frac{7}{x-2}+\frac{-4}{(x-2)^2}\right]dx$$
$$= -5\ln|x-1|+7\ln|x-2|+\frac{4}{x-2}+C = \frac{4}{x-2}+\ln\left|\frac{(x-2)^7}{(x-1)^5}\right|+C$$

20. $\dfrac{-3x^3+2x-3}{x^2(x^2-1)} = \dfrac{-3x^3+2x-3}{x^2(x+1)(x-1)} = \dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x+1}+\dfrac{D}{x-1}$

$-3x^3+2x-3 = Ax(x+1)(x-1)+B(x+1)(x-1)+Cx^2(x-1)+Dx^2(x+1)$

If $x = 0$, then $-3 = -B$, or $B = 3$. If $x = -1$, then $-2 = -2C$, or $C = 1$. If $x = 1$, then $-4 = 2D$, or $D = -2$. If $x = 2$, then $-23 = 6A + 3B + 4C + 12D$, $-23 = 6A + 9 + 4 - 24$, or $A = -2$.

$$\int \frac{-3x^3+2x-3}{x^2(x^2-1)}dx = \int\left(\frac{-2}{x}+\frac{3}{x^2}+\frac{1}{x+1}+\frac{-2}{x-1}\right)dx$$
$$= -2\ln|x|-\frac{3}{x}+\ln|x+1|-2\ln|x-1|+C = -\frac{3}{x}+\ln\left|\frac{x+1}{x^2(x-1)^2}\right|+C$$

21. $\dfrac{2(x^2+8)}{x^3+4x} = \dfrac{2x^2+16}{x\left(x^2+4\right)} = \dfrac{A}{x}+\dfrac{Bx+C}{x^2+4}$

$2x^2+16 = A\left(x^2+4\right)+(Bx+C)x$

$2x^2+16 = (A+B)x^2+Cx+4A$

Thus $A + B = 2$, $C = 0$, $4A = 16$. This gives $A = 4$, $B = -2$, $C = 0$.

$$\int \frac{2(x^2+8)}{x^3+4x}dx = \int\left(\frac{4}{x}+\frac{-2x}{x^2+4}\right)dx = 4\int\frac{1}{x}dx-\int\frac{1}{x^2+4}[2x\ dx] = 4\ln|x|-\ln\left(x^2+4\right)+C = \ln\left[\frac{x^4}{x^2+4}\right]+C$$

22. $\dfrac{4x^3-3x^2+2x-3}{(x^2+3)(x+1)(x-2)} = \dfrac{Ax+B}{x^2+3}+\dfrac{C}{x+1}+\dfrac{D}{x-2}$

$4x^3-3x^2+2x-3 = (Ax+B)(x+1)(x-2)+C(x^2+3)(x-2)+D(x^2+3)(x+1)$

If $x = -1$, then $-12 = -12C$, or $C = 1$.
If $x = 2$, then $21 = 21D$, or $D = 1$.
If $x = 0$, then $-3 = -2B - 6C + 3D$, $-3 = -2B - 6 + 3$, $0 = -2B$, or $B = 0$.

If $x = 1$, then $0 = -2(A + B) - 4C + 8D$, $0 = -2A - 4 + 8$, $-4 = -2A$, or $A = 2$.

$$\int \frac{4x^3-3x^2+2x-3}{(x^2+3)(x+1)(x-2)}dx = \int\left(\frac{2x}{x^2+3}+\frac{1}{x+1}+\frac{1}{x-2}\right)dx$$
$$= \ln(x^2+3)+\ln|x+1|+\ln|x-2|+C$$
$$= \ln\left|(x^2+3)(x+1)(x-2)\right|+C$$

23. $\dfrac{-x^3+8x^2-9x+2}{\left(x^2+1\right)(x-3)^2} = \dfrac{Ax+B}{x^2+1}+\dfrac{C}{x-3}+\dfrac{D}{(x-3)^2}$

$$-x^3+8x^2-9x+2 = (Ax+B)(x-3)^2+C(x-3)\left(x^2+1\right)+D\left(x^2+1\right)$$
$$= (Ax+B)\left(x^2-6x+9\right)+C\left(x^3-3x^2+x-3\right)+D\left(x^2+1\right)$$
$$= (A+C)x^3+(B-6A-3C+D)x^2+(9A-6B+C)x+(9B-3C+D)$$

Thus $A + C = -1$, $B - 6A - 3C + D = 8$, $9A - 6B + C = -9$, $9B - 3C + D = 2$. This gives $A = -1$, $B = 0$, $C = 0$, $D = 2$.

$$\int \frac{-x^3+8x^2-9x+2}{\left(x^2+1\right)(x-3)^2}dx = \int\left(\frac{-x}{x^2+1}+\frac{0}{x-3}+\frac{2}{(x-3)^2}\right)dx = -\frac{1}{2}\ln\left(x^2+1\right)-\frac{2}{x-3}+C$$

24. $\dfrac{5x^4+9x^2+3}{x(x^2+1)^2} = \dfrac{A}{x}+\dfrac{Bx+C}{x^2+1}+\dfrac{Dx+E}{(x^2+1)^2}$

$$5x^4+9x^2+3 = A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x$$
$$= A(x^4+2x^2+1)+(Bx+C)(x^3+x)+Dx^2+Ex$$
$$= (A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A$$

Thus, $A + B = 5$, $C = 0$, $2A + B + D = 9$, $C + E = 0$, and $A = 3$. This gives $A = 3$, $B = 2$, $C = 0$, $D = 1$, and $E = 0$.

$$\int \frac{5x^4+9x^2+3}{x(x^2+1)^2}dx = \int\left(\frac{3}{x}+\frac{2x}{x^2+1}+\frac{x}{(x^2+1)^2}\right)dx$$
$$= 3\ln|x|+\ln\left|x^2+1\right|-\frac{1}{2(x^2+1)}+C$$
$$= \ln\left|x^3(x^2+1)\right|-\frac{1}{2(x^2+1)}+C$$

25. $\dfrac{14x^3+24x}{\left(x^2+1\right)\left(x^2+2\right)} = \dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+2}$

$$14x^3+24x = \left(x^2+2\right)(Ax+B)+\left(x^2+1\right)(Cx+D) = (A+C)x^3+(B+D)x^2+(2A+C)x+(2B+D)$$

Thus $A + C = 14$, $B + D = 0$, $2A + C = 24$, $2B + D = 0$.
This gives $A = 10$, $B = 0$, $C = 4$, $D = 0$.

$$\int \frac{14x^3+24x}{\left(x^2+1\right)\left(x^2+2\right)}dx = \int\left(\frac{10x}{x^2+1}+\frac{4x}{x^2+2}\right)dx$$

$$= 5\int \frac{1}{x^2+1}[2\ dx] + 2\int \frac{1}{x^2+2}[2\ dx]$$

$$= 5\ln\left(x^2+1\right) + 2\ln\left(x^2+2\right) + C$$

$$= \ln\left[\left(x^2+1\right)^5\left(x^2+2\right)^2\right] + C$$

26. $\frac{12x^3+20x^2+28x+4}{3(x^2+2x+3)(x^2+1)} = \frac{1}{3}\left(\frac{Ax+B}{x^2+2x+3}+\frac{Cx+D}{x^2+1}\right)$

$$12x^3+20x^2+28x+4 = (Ax+B)(x^2+1)+(x^2+2x+3)(Cx+D)$$

$$= (A+C)x^3+(B+D+2C)x^2+(A+2D+3C)x+(B+3D)$$

Thus, $A + C = 12$, $B + D + 2C = 20$, $A + 2D + 3C = 28$, $B + 3D = 4$. This gives $A = 4$, $B = 4$, $C = 8$, $D = 0$.

$$\int \frac{12x^3+20x^2+28x+4}{3\left(x^2+2x+3\right)\left(x^2+1\right)}dx = \frac{1}{3}\int\left(\frac{4x+4}{x^2+2x+3}+\frac{8x}{x^2+1}\right)dx$$

$$= \frac{1}{3}\left[2\ln\left(x^2+2x+3\right)+4\ln\left(x^2+1\right)\right]+C$$

$$= \ln\left[\left(x^2+2x+3\right)^{\frac{2}{3}}\left(x^2+1\right)^{\frac{4}{3}}\right]+C$$

27. $\frac{3x^3+8x}{(x^2+2)^2} = \frac{Ax+B}{x^2+2}+\frac{Cx+D}{(x^2+2)^2}$

$$3x^3+8x = (Ax+B)(x^2+2)+Cx+D$$

$$= Ax^3+Bx^2+(2A+C)x+(2B+D)$$

Thus, $A = 3$, $B = 0$, $2A + C = 8$, $2B + D = 0$.
This gives $A = 3$, $B = 0$, $C = 2$, $D = 0$.

$$\int \frac{3x^3+8x}{(x^2+2)^2}dx = \int\left(\frac{3x}{x^2+2}+\frac{2x}{(x^2+2)^2}\right)dx = \frac{3}{2}\ln(x^2+2)-\frac{1}{x^2+2}+C$$

28. $\int \frac{3x^2-8x+4}{x^3-4x^2+4x-6}dx = \int \frac{1}{x^3-4x^2+4x-6}\left[\left(3x^2-8x+4\right)dx\right]$

$\left(\text{Form: } \int\left(\frac{1}{u}\right)du\right)$ (Partial fractions not required.)

Answer: $\ln\left|x^3-4x^2+4x-6\right|+C$

29. $\frac{2-2x}{x^2+7x+12}=\frac{2-2x}{(x+3)(x+4)}=\frac{A}{x+3}+\frac{B}{x+4}$

$2 - 2x = A(x + 4) + B(x + 3)$

If $x = -4$, then $10 = -B$, or $B = -10$. If $x = -3$, then $8 = A$.

$$\int_0^1 \frac{2-2x}{x^2+7x+12}dx=\int_0^1\left(\frac{8}{x+3}+\frac{-10}{x+4}\right)dx$$

$$=\left[8\ln|x+3|-10\ln|x+4|\right]\Big|_0^1$$

$= 8 \ln 4 - 10 \ln 5 - (8 \ln 3 - 10 \ln 4)$

$= 18 \ln(4) - 10 \ln(5) - 8 \ln(3)$

30. $\frac{3x^2+15x+13}{x^2+4x+3}=3+\frac{3x+4}{x^2+4x+3}$

$$=3+\frac{3x+4}{(x+1)(x+3)}$$

$\frac{3x+4}{(x+1)(x+3)}=\frac{A}{x+1}+\frac{B}{x+3}$

$3x + 4 = A(x + 3) + B(x + 1)$

If $x = -1$, then $1 = 2A$, or $A=\frac{1}{2}$. If $x = -3$, then

$-5 = -2B$, or $B=\frac{5}{2}$.

$$\int_1^2 \frac{3x^2+15x+13}{x^2+4x+3}dx$$

$$=\int_1^2\left(3+\frac{1}{2}\cdot\frac{1}{x+1}+\frac{5}{2}\cdot\frac{1}{x+3}\right)dx$$

$$=\left(3x+\frac{1}{2}\ln|x+1|+\frac{5}{2}\ln|x+3|\right)\Big|_1^2$$

$$=6+\frac{1}{2}\ln 3+\frac{5}{2}\ln 5-\left(3+\frac{1}{2}\ln 2+\frac{5}{2}\ln 4\right)$$

$$=3+\frac{1}{2}\ln 3+\frac{5}{2}\ln 5-\frac{1}{2}\ln 2-\frac{5}{2}\ln 4$$

31. Note that $\frac{6\left(x^2+1\right)}{(x+2)^2}\ge 0$ on [0, 1].

Area $=\int_0^1 \frac{6(x^2+1)}{(x+2)^2}dx$

$\frac{6(x^2+1)}{(x+2)^2}=6+\frac{-24x-18}{(x+2)^2}$ (by long division)

$\frac{-24x-18}{(x+2)^2}=\frac{A}{x+2}+\frac{B}{(x+2)^2}$

$-24x - 18 = A(x + 2) + B$

If $x = -2$, then $30 = B$. If $x = 0$, then $-18 = 2A + B$, $-18 = 2A + 30$, or $A = -24$.

$$\int_0^1 \frac{6\left(x^2+1\right)}{(x+2)^2}dx$$

$$=\int_0^1\left[6+\frac{-24}{x+2}+\frac{30}{(x+2)^2}\right]dx$$

$$=\left[6x-24\ln|x+2|-\frac{30}{x+2}\right]\Big|_0^1$$

$$=6-24\ln 3-10-(-24\ln 2-15)$$

$$=11+24\ln\frac{2}{3}$$

The area is $11+24\ln\frac{2}{3}$ sq units.

32. $\text{CS}=\int_0^{10}\left[\frac{200(q+3)}{q^2+7q+6}-\frac{325}{22}\right]dq$

$\frac{200(q+3)}{q^2+7q+6}=\frac{200(q+3)}{(q+6)(q+1)}=\frac{A}{q+6}+\frac{B}{q+1}$

$200(q + 3) = A(q + 1) + B(q + 6)$

If $q = -1$, then $400 = 5B$, or $B = 80$. If $q = -6$, then $-600 = -5A$, or $A = 120$.

$$\text{CS}=\int_0^{10}\left[\frac{120}{q+6}+\frac{80}{q+1}-\frac{325}{22}\right]dq$$

$$=\left[120\ln|q+6|-80\ln|q+1|-\frac{325}{22}q\right]\Big|_0^{10}$$

$$=\left[120\ln(16)+80\ln(11)-\frac{3250}{22}\right]-[120\ln(6)]$$

$$=120\ln\frac{8}{3}+80\ln(11)-\frac{1625}{11}\approx \$161.80$$

Problems 15.3

1. Let $u = x$, $a^2 = 9$. Then $du = dx$.

$$\int\frac{dx}{(9-x^2)^{3/2}}=\frac{x}{9\sqrt{9-x^2}}+C$$

2. Let $u = 2x$, $a^2 = 25$. Then $du = 2dx$.

$$\int \frac{dx}{\left(25-4x^2\right)^{\frac{3}{2}}} = \frac{1}{2}\int \frac{(2dx)}{\left[25-(2x)^2\right]^{\frac{3}{2}}}$$

$$= \frac{1}{2}\left[\frac{(2x)}{25\sqrt{25-(2x)^2}}\right]+C$$

$$= \frac{x}{25\sqrt{25-4x^2}}+C$$

3. Let $u = 4x$, $a^2 = 3$. Then $du = 4\,dx$.

$$\int \frac{dx}{x^2\sqrt{16x^2+3}} = 4\int \frac{(4\,dx)}{(4x)^2\sqrt{(4x)^2+3}}$$

$$= 4\left[-\frac{\sqrt{(4x)^2+3}}{3(4x)}\right]+C$$

$$= -\frac{\sqrt{16x^2+3}}{3x}+C$$

4. Let $u = x^2$, $a^2 = 9$. Then $du = 2x\,dx$.

$$\int \frac{3\,dx}{x^3\sqrt{x^4-9}} = \frac{3}{2}\int \frac{(2x\,dx)}{(x^2)^2\sqrt{(x^2)^2-9}}$$

$$= \frac{3}{2}\left[-\frac{-\sqrt{(x^2)^2-9}}{9x^2}+C\right]$$

$$= \frac{\sqrt{x^4-9}}{6x^2}+C$$

5. Formula 5 with $u = x$, $a = 6$, $b = 7$. Then $du = dx$.

$$\int \frac{dx}{x(6+7x)} = \frac{1}{6}\ln\left|\frac{x}{6+7x}\right|+C$$

6. Formula 8 with $u = x$, $a = 2$, $b = 5$. Then $du = dx$.

$$\int \frac{3x^2\,dx}{(2+5x)^2} = 3\left[\int \frac{x^2\,dx}{(2+5x)^2}\right]$$

$$= 3\left[\frac{x}{25}-\frac{4}{125(2+5x)}-\frac{4}{125}\ln|2+5x|\right]+C$$

7. Formula 28 with $u = x$, $a = 3$. Then $du = dx$.

$$\int \frac{dx}{x\sqrt{x^2+9}} = \frac{1}{3}\ln\left|\frac{\sqrt{x^2+9}-3}{x}\right|+C$$

8. Formula 32 with $u = x$, $a^2 = 7$. Then $du = dx$.

$$\int \frac{dx}{(x^2+7)^{3/2}} = \frac{x}{7\sqrt{x^2+7}}+C$$

9. Formula 12 with $u = x$, $a = 2$, $b = 3$, $c = 4$, $k = 5$. Then $du = dx$.

$$\int \frac{x\,dx}{(2+3x)(4+5x)}$$

$$= \frac{1}{2}\left[\frac{4}{5}\ln|4+5x|-\frac{2}{3}\ln|2+3x|\right]+C$$

10. Formula 37 with $u = 5x$, $a = 2$. Then $du = 5\,dx$.

$$\int 2^{5x}\,dx = \frac{1}{5}\int 2^{5x}(5\,dx) = \frac{1}{5}\cdot\frac{2^{5x}}{\ln 2}+C$$

11. Formula 45 with $u = x$, $a = 5$, $b = 2$, $c = 3$. Then $du = dx$. $\int \frac{dx}{5+2e^{3x}} = \frac{1}{15}\left(3x-\ln\left|5x+2e^{3x}\right|\right)+C$

12. Formula 14 with $u = x$, $a = 1$, $b = 1$. Then $du = dx$.

$$\int x^2\sqrt{1+x}\,dx = \frac{2\left(8-12x+15x^2\right)(1+x)^{\frac{3}{2}}}{105}+C$$

13. Formula 9 with $u = x$, $a = 5$, $b = 2$. Then $du = dx$.

$$\int \frac{7\,dx}{x(5+2x)^2} = 7\left[\int \frac{dx}{x(5+2x)^2}\right]$$

$$= 7\left[\frac{1}{5(5+2x)}+\frac{1}{25}\ln\left|\frac{x}{5+2x}\right|\right]+C$$

14. Formula 20 with $u = \sqrt{11}x$, $a = \sqrt{5}$. Then $du = \sqrt{11}\,dx$.

$$\int \frac{dx}{x\sqrt{5-11x^2}} = \int \frac{\sqrt{11}dx}{\left(\sqrt{11}x\right)\sqrt{\left(\sqrt{5}\right)^2-\left(\sqrt{11}x\right)^2}}$$

$$= -\frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}+\sqrt{5-11x^2}}{\sqrt{11}x}\right|+C$$

15. Formula 3 with $u = x$, $a = 2$, $b = 1$. Then $du = dx$.

$$\int_0^1 \frac{x\,dx}{2+x} = \left(x - 2\ln|2+x|\right)\Big|_0^1 = 1 - 2\ln 3 + 2\ln 2 = 1 - \ln 9 + \ln 4 = 1 + \ln\left(\frac{4}{9}\right)$$

16. Formula 4 with $u = x$, $a = 3$, $b = 7$. Then $du = dx$.

$$\int \frac{2x^2 dx}{3+7x} = 2\int \frac{x^2 dx}{3+7x} = 2\left(\frac{x^2}{14} - \frac{3x}{49} + \frac{9}{343}\ln|3+7x|\right) + C$$

17. Formula 23 with $u = x$, $a^2 = 3$. Then $du = dx$.

$$\int \sqrt{x^2-3}\,dx = \frac{1}{2}\left(x\sqrt{x^2-3} - 3\ln\left|x + \sqrt{x^2-3}\right|\right) + C$$

18. Formula 11 with $u = x$, $a = 1$, $b = 5$, $c = 3$, $k = 2$. Then $du = dx$.

$$\int \frac{dx}{(1+5x)(2x+3)} = \frac{1}{13}\ln\left|\frac{1+5x}{2x+3}\right| + C$$

19. Formula 38 with $u = x$, $a = 12$. Then $du = dx$.

$$\int_0^{1/12} xe^{12x}dx = \frac{e^{12x}}{144}(12x-1)\Bigg|_0^{1/12} = \frac{1}{144}[e(0) - 1(-1)] = \frac{1}{144}$$

20. Formula 46 with $u = 3x$, $a = 2$, $b = 5$.
Then $du = 3\,dx$.

$$\int \sqrt{\frac{2+3x}{5+3x}}dx = \frac{1}{3}\int \sqrt{\frac{2+3x}{5+3x}}(3\,dx) = \frac{1}{3}\left[\sqrt{(2+3x)(5+3x)} - 3\ln\left(\sqrt{2+3x} + \sqrt{5+3x}\right)\right] + C$$

21. Formula 39 with $u = x$, $n = 2$, $a = 1$. Then $du = dx$.

$$\int x^2e^x dx = x^2e^x - 2\int xe^x dx$$

Applying Formula 38 on $\int xe^x dx$ with $u = x$, $a = 1$ (so $du = dx$) gives $\int xe^x dx = e^x(x-1) + C_1$. Thus

$$\int x^2e^x dx = x^2e^x - 2\left[e^x(x-1)\right] + C = e^x\left[x^2 - 2(x-1)\right] + C = e^x\left(x^2 - 2x + 2\right) + C$$

22. Formula 6 with $u = x$, $a = 1$, $b = 1$. Then $du = dx$.

$$\int_1^2 \frac{4\,dx}{x^2(1+x)} = 4\int_1^2 \frac{dx}{x^2(1+x)} = 4\left(-\frac{1}{x} + \ln\left|\frac{1+x}{x}\right|\right)\Bigg|_1^2 = 4\left(-\frac{1}{2} + \ln\frac{3}{2}\right) - 4(-1 + \ln 2) = 2 + 4\ln\frac{3}{4}$$

23. Formula 26 with $u = \sqrt{5}x$, $a^2 = 1$. Then $du = \sqrt{5}\,dx$.

$$\int \frac{\sqrt{5x^2+1}}{2x^2}dx = \frac{5}{2\sqrt{5}}\int \frac{\sqrt{5x^2+1}}{5x^2}\left(\sqrt{5}\,dx\right)$$

$$= \frac{\sqrt{5}}{2}\left(-\frac{\sqrt{5x^2+1}}{\sqrt{5}x} + \ln\left|\sqrt{5}x + \sqrt{5x^2+1}\right|\right) + C$$

24. Formula 17 with $u = x$, $a = 2$, $b = -1$. Then $du = dx$.

$$\int \frac{dx}{x\sqrt{2-x}} = \frac{1}{\sqrt{2}} \ln\left|\frac{\sqrt{2-x}-\sqrt{2}}{\sqrt{2-x}+\sqrt{2}}\right| + C$$

25. Formula 7 with $u = x$, $a = 1$, $b = 3$. Then $du = dx$.

$$\int \frac{x\,dx}{(1+3x)^2} = \frac{1}{9}\left(\ln|1+3x| + \frac{1}{1+3x}\right) + C$$

26. Formula 47 with $u = 3x$, $a = 5$, $b = 6$. Then $du = 3\,dx$.

$$\int \frac{3\,dx}{\sqrt{(5+3x)(6+3x)}} = \ln\left|\frac{11}{2} + 3x + \sqrt{(5+3x)(6+3x)}\right| + C$$

27. Formula 34 with $u = \sqrt{5}x,\ a = \sqrt{7}$. Then $du = \sqrt{5}dx$

$$\int \frac{dx}{7-5x^2} = \frac{1}{\sqrt{5}} \int \frac{1}{\left(\sqrt{7}\right)^2 - \left(\sqrt{5}x\right)^2}\left(\sqrt{5}dx\right) = \frac{1}{\sqrt{5}}\left(\frac{1}{2\sqrt{7}} \ln\left|\frac{\sqrt{7}+\sqrt{5}x}{\sqrt{7}-\sqrt{5}x}\right|\right) + C$$

28. Formula 24 with $u = \sqrt{3}x,\ a^2 = 6$. Then $du = \sqrt{3}dx$.

$$\int 7x^2\sqrt{3x^2-6}\,dx = \frac{7}{\left(\sqrt{3}\right)^3} \int \left(\sqrt{3}x\right)^2 \sqrt{\left(\sqrt{3}x\right)^2 - 6}\left(\sqrt{3}\,dx\right)$$

$$= \frac{7}{3\sqrt{3}}\left[\frac{\sqrt{3}x}{8}(6x^2-6)\sqrt{3x^2-6} - \frac{36}{8}\ln\left|\sqrt{3}x + \sqrt{3x^2-6}\right|\right] + C$$

29. Formula 42 with $u = 3x$, $n = 5$. Then $du = 3\,dx$.

$$\int 36x^5 \ln(3x)dx = 36\int x^5 \ln(3x)dx = \frac{36}{3^6}\int (3x)^5 \ln(3x)(3\,dx)$$

$$= \frac{4}{81}\left[\frac{(3x)^6 \ln(3x)}{6} - \frac{(3x)^6}{36}\right] + C = x^6[6\ln(3x) - 1] + C$$

30. Formula 10 with $u = x$, $a = 3$, $b = 2$. Then $du = dx$.

$$\int \frac{5\,dx}{x^2(3+2x)^2} = 5\left[\int \frac{dx}{x^2(3+2x)^2}\right] = 5\left[-\frac{3+4x}{9x(3+2x)} + \frac{4}{27}\ln\left|\frac{3+2x}{x}\right|\right] + C$$

31. Formula 13 with $u = x$, $a = 1$, $b = 3$. Then
$du = dx$.

$$\int 270x\sqrt{1+3x}dx = 270\int x\sqrt{1+3x}\,dx = 270\left[\frac{2(9x-2)(1+3x)^{\frac{3}{2}}}{15\cdot 9}\right] + C$$

$$= 4(9x-2)(1+3x)^{\frac{3}{2}} + C$$

32. Formula 42 with $u = x$, $n = 2$. Then $du = dx$.

$$\int 9x^2 \ln x\,dx = 9\int x^2 \ln x\,dx = 9\left(\frac{x^3 \ln x}{3} - \frac{x^3}{9}\right) + C = 3x^3(\ln x) - x^3 + C$$

33. Formula 27 with $u = 2x$, $a^2 = 13$. Then $du = 2\,dx$.

$$\int \frac{dx}{\sqrt{4x^2-13}} = \frac{1}{2}\int \frac{1}{\sqrt{(2x)^2-13}}(2\,dx)$$
$$= \frac{1}{2}\ln\left|2x+\sqrt{4x^2-13}\right|+C$$

34. Formula 44 with $u = 2x$. Then $du = 2\,dx$.

$$\int \frac{dx}{x\ln(2x)} = \int \frac{(2\,dx)}{(2x)\ln(2x)}$$
$$= \ln\left|\ln(2x)\right|+C$$

35. Formula 21 with $u = 3x$, $a^2 = 16$. Then $du = 3\,dx$.

$$\int \frac{2\,dx}{x^2\sqrt{16-9x^2}} = 2(3)\int \frac{(3\,dx)}{(3x)^2\sqrt{16-(3x)^2}}$$
$$= 6\left(-\frac{\sqrt{16-9x^2}}{16(3x)}\right)+C$$
$$= -\frac{\sqrt{16-9x^2}}{8x}+C$$

36. Formula 22 with $u = \sqrt{3}x$, $a = \sqrt{2}$. Then $du = \sqrt{3}dx$.

$$\int \frac{\sqrt{2-3x^2}}{x}dx = \int \frac{\sqrt{\left(\sqrt{2}\right)^2-\left(\sqrt{3}x\right)^2}}{\sqrt{3}x}\left(\sqrt{3}dx\right)$$
$$= \sqrt{2-3x^2}-\sqrt{2}\ln\left|\frac{\sqrt{2}+\sqrt{2-3x^2}}{\sqrt{3}x}\right|+C$$

37. Formula 45 with $u = \sqrt{x}$, $a = \pi$, $b = 7$, $c = 4$.

Then $du = \frac{1}{2\sqrt{x}}dx$

$$\int \frac{dx}{\sqrt{x}\left(\pi+7e^{4\sqrt{x}}\right)} = 2\int \frac{1}{\pi+7e^{4\sqrt{x}}}\left(\frac{1}{2\sqrt{x}}dx\right)$$
$$= 2\left[\frac{1}{4\pi}\left(4\sqrt{x}-\ln\left|\pi+7e^{4\sqrt{x}}\right|\right)\right]+C$$
$$= \frac{1}{2\pi}\left(4\sqrt{x}-\ln\left|\pi+7e^{4\sqrt{x}}\right|\right)+C$$

38. Formula 2 with $u = x^3$, $a = 1$, $b = 2$. Then $du = 3x^2\,dx$.

$$\int_0^1 \frac{3x^2\,dx}{1+2x^3} = \frac{1}{2}\ln\left|1+2x^3\right|\Big|_0^1$$
$$= \frac{1}{2}\ln|3|-\frac{1}{2}\ln|1| = \ln\sqrt{3}$$

39. Can be put in the form $\int \frac{1}{u}du$.

$$\int \frac{x\,dx}{x^2+1} = \frac{1}{2}\int \frac{1}{x^2+1}(2x\,dx)$$
$$= \frac{1}{2}\ln\left(x^2+1\right)+C$$

40. Can be put in the form $\int e^u du$.

$$\int 3x\sqrt{x}e^{x^{5/2}}dx = 3\cdot\frac{2}{5}\int e^{x^{5/2}}\left[\frac{5}{2}x^{3/2}dx\right]$$
$$= \frac{6}{5}e^{x^{5/2}}+C$$

41. Can be put in the form $\int u^n du$.

$$\int 6x\sqrt{2x^2+1}dx = \frac{3}{2}\int\left(2x^2+1\right)^{\frac{1}{2}}(4x\,dx)$$
$$= \frac{3}{2}\cdot\frac{\left(2x^2+1\right)^{\frac{3}{2}}}{\frac{3}{2}}+C$$
$$= \left(2x^2+1\right)^{\frac{3}{2}}+C$$

42. $$\int \frac{5x^3-\sqrt{x}}{2x}dx = \int\left(\frac{5}{2}x^2-\frac{1}{2}x^{-\frac{1}{2}}\right)dx$$
$$= \frac{5}{6}x^3-\sqrt{x}+C$$

43. $$\int \frac{1}{x^2-5x+6}dx = \int \frac{1}{(x-3)(x-2)}dx$$

Formula 11 with $u = x$, $a = -3$, $b = 1$, $c = -2$, and $k = 1$. Then $du = dx$.

$$\int \frac{1}{x^2-5x+6}dx = \int \frac{1}{(x-3)(x-2)}dx$$
$$= \ln\left|\frac{x-3}{x-2}\right|+C$$

44. Can be put in the form $\int u^n du$.

$$\int \frac{e^{2x}}{\sqrt{e^{2x}+3}}dx = \frac{1}{2}\int \left(e^{2x}+3\right)^{-\frac{1}{2}}(2e^{2x}dx)$$
$$= \sqrt{e^{2x}+3}+C$$

45. Formula 42 with $u = x$ and $n = 3$. Then $du = dx$.

$$\int x^3 \ln x\, dx = \frac{x^4}{4}\left[\ln(x) - \frac{1}{4}\right] + C$$

46. Formula 38 with $u = x$ and $a = -1$. Then $du = dx$.

$$\int_0^3 xe^{-x}dx = e^{-x}(-x-1)\Big|_0^3 = e^{-3}(-4) - 1(-1)$$
$$= 1 - 4e^{-3}$$

47. Formula 38 with $u = x^2$ and $a = 3$. Then $du = 2x\, dx$.

$$\int 4x^3e^{3x^2}dx = 2\int x^2e^{3x^2}[2x\, dx]$$
$$= 2\left[\frac{e^{3x^2}}{9}(3x^2-1)\right] + C$$
$$= \frac{2}{9}e^{3x^2}(3x^2-1) + C$$

48. Formula 14 with $u = x$, $a = 3$ and $b = 2$. Then $du = dx$.

$$\int_1^2 35x^2\sqrt{3+2x}dx = 35\int_1^2 x^2\sqrt{3+2x}dx$$
$$= 35\cdot\frac{2\left(72-72x+60x^2\right)(3+2x)^{\frac{3}{2}}}{840}\Bigg|_1^2$$
$$= 98\sqrt{7} - 25\sqrt{5}$$

49. Formula 43 and then Formula 41. For Formula 43, let $u = x$, $n = 0$, and $m = 2$. Then $du = dx$.

$$\int \ln^2 x\ dx = x\ln^2 x - 2\int \ln x\ dx$$

Now we apply Formula 41 to the last integral with $u = x$ (so $du = dx$).

$$\int \ln^2 x\, dx = x(\ln x)^2 - 2x(\ln x) + 2x + C$$

50. Formula 41 with $u = x^2$. Then $du = 2x\, dx$.

$$\int_1^3 3x\ln x^2\, dx$$
$$= \frac{3}{2}\int_1^e \ln(x^2)[2x\, dx] = \frac{3}{2}[x^2\ln(x^2) - x^2]\Big|_1^e$$
$$= \frac{3}{2}[(e^2\ln(e^2) - e^2) - (1\cdot\ln 1 - 1)]$$
$$= \frac{3}{2}(e^2+1)$$

51. Formula 15 with $u = x$, $a = 4$ and $b = -1$. Then $du = dx$.

$$\int_1^2 \frac{x\ dx}{\sqrt{4-x}} = \frac{2(-x-8)\sqrt{4-x}}{3}\Bigg|_1^2$$
$$= \frac{2}{3}\left(9\sqrt{3} - 10\sqrt{2}\right)$$

52. Formula 13 with $u = x$, $a = 2$, and $b = 3$. Then $du = dx$.

$$\int_2^3 x\sqrt{2+3x}\, dx = \frac{2(9x-4)(2+3x)^{3/2}}{135}\Bigg|_2^3$$
$$= \frac{2}{135}[23(11)^{3/2} - 14(8)^{3/2}]$$
$$= \frac{2}{135}\left(253\sqrt{11} - 224\sqrt{2}\right)$$

53. Can be put in the form $\int u^n du$.

$$\int_0^1 \frac{2x\ dx}{\sqrt{8-x^2}} = -\int_0^1 \left(8-x^2\right)^{-\frac{1}{2}}(-2x\ dx)$$
$$= -\frac{\left(8-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_0^1$$
$$= -2\left(8-x^2\right)^{\frac{1}{2}}\Big|_0^1 = -2\left(\sqrt{7}-\sqrt{8}\right)$$
$$= -2\left(\sqrt{7} - 2\sqrt{2}\right)$$
$$= 2\left(2\sqrt{2} - \sqrt{7}\right)$$

54. Formula 39 with $u = x$, $n = 2$, $a = 3$. Then $du = dx$.

$$\int x^2e^{3x}\,dx = \frac{x^2e^{3x}}{3} - \frac{2}{3}\int xe^{3x}\,dx$$

For $\int xe^{3x}\,dx$, use Formula 38 with $u = x$ and $a = 3$. Then $du = dx$.

$$\int x^2e^{3x}\,dx = \frac{x^2e^{3x}}{3} - \frac{2}{3}\left[\frac{e^{3x}}{9}(3x-1)\right]$$

$$= \frac{e^{3x}}{27}[9x^2 - 6x + 2]$$

$$\int_0^{\ln 2} x^2e^{3x}\,dx = \left(\frac{e^{3x}}{27}[9x^2 - 6x + 2]\right)\Bigg|_0^{\ln 2}$$

$$= \frac{8}{27}[9(\ln 2)^2 - 6\ln 2 + 2] - \frac{1}{27}[2]$$

$$= \frac{2}{27}[36(\ln 2)^2 - 24\ln 2 + 7]$$

55. Integration by parts or Formula 42. For Formula 42, let $u = 2x$, $n = 1$. Then $du = 2\,dx$.

$$\int_1^2 x\ln(2x)dx = \frac{1}{4}\int_1^2 (2x)\ln(2x)[2\,dx]$$

$$= \frac{1}{4}\left[\frac{(2x)^2\ln(2x)}{2} - \frac{(2x)^2}{4}\right]\Bigg|_1^2$$

$$= 2\ln(4) - 1 - \frac{1}{2}\ln(2) + \frac{1}{4}$$

$$= 2\ln\left(2^2\right) - \frac{1}{2}\ln(2) - \frac{3}{4}$$

$$= 4\ln(2) - \frac{1}{2}\ln(2) - \frac{3}{4}$$

$$= \frac{7}{2}(\ln 2) - \frac{3}{4}$$

56. Can be put in the form $\int k\,dx$.

$$\int_1^2 dx = \int_1^2 1\,dx = x\Big|_1^2 = 2 - 1 = 1$$

57. Formula 5 with $u = q$, $a = 1$, and $b = -1$. Then $du = dq$.

$$\int_{q_0}^{q_n} \frac{dq}{q(1-q)} = \ln\left|\frac{q}{1-q}\right|\Bigg|_{q_0}^{q_n} = \ln\left|\frac{q_n}{1-q_n}\right| - \ln\left|\frac{q_0}{1-q_0}\right|$$

$$= \ln\left|\frac{q_n(1-q_0)}{q_0(1-q_n)}\right|$$

58. Formula 6 with $u = q$, $a = 1$ and $b = -1$. Then $du = dq$.

$$n = -\frac{1}{0.4}\int_{0.3}^{0.1}\frac{dq}{q^2(1-q)}$$

$$= -\frac{1}{0.4}\left[-\frac{1}{q} - \ln\left|\frac{1-q}{q}\right|\right]\Bigg|_{0.3}^{0.1}$$

$$= -\frac{1}{0.4}\left\{[-10 - \ln 9] - \left[-\frac{10}{3} - \ln\frac{7}{3}\right]\right\}$$

$$= -\frac{1}{0.4}\left(-\frac{20}{3} - \ln 9 + \ln\frac{7}{3}\right) \approx 20$$

59. a. For $\int_0^9 1000e^{-0.04t}dt$, the form $\int e^u du$ can be applied.

$$\int_0^9 1000e^{-0.04t}dt$$

$$= \frac{1000}{-0.04}\int_0^9 e^{-0.04t}(-0.04\,dt)$$

$$= -\frac{1000}{0.04}e^{-0.04t}\Big|_0^9$$

$$= -\frac{1000}{0.04}(e^{-0.36} - 1)$$

$$\approx \$7558.09$$

b. For $\int_0^{10} 500te^{-0.06t}dt$ use Formula 38 with $t = u$ and $a = -0.06$, so $du = dt$.

$$\int_0^{10} 500te^{-0.06t}dt$$

$$= 500\int_0^{10} te^{-0.06t}dt$$

$$= 500\left[\frac{e^{-0.06t}}{0.0036}(-0.06t - 1)\right]\Bigg|_0^{10}$$

$$= \frac{500}{0.0036}[e^{-0.6}(-1.6) - (-1)]$$

$$\approx \$16{,}930.75$$

60. $$\int_0^T ke^{-rt}dt = k\left(-\frac{1}{r}\right)\int_0^T e^{-rt}(-r\,dt) = \frac{-ke^{-rt}}{r}\Bigg|_0^T$$

$$= -\frac{ke^{-rT}}{r} + \frac{k}{r} = k\left(\frac{1-e^{-rT}}{r}\right)$$

61. a. $\int_0^{10} 400e^{0.06(10-t)}dt = 400\int_0^{10} e^{0.6-0.06t}dt$

$= 400\int_0^{10} e^{0.6}e^{-0.06t}dt$

$= 400e^{0.6}\int_0^{10} e^{-0.06t}dt$

$= 400e^{0.6}\left(\frac{1}{-0.06}\right)\int_0^{10} e^{-0.06t}(-0.06\ dt)$

$= \frac{400e^{0.6}}{-0.06}e^{-0.06t}\Big|_0^{10} = \frac{400e^{0.6}}{-0.06}\left[e^{-0.6}-1\right]$

$\approx \$5481$

b. Use Formula 38 with $u = t$ and $a = -0.04$, so $du = dt$.

$\int_0^5 40te^{0.04(5-t)}dt = 40\int_0^5 te^{0.2}e^{-0.04t}dt$

$= 40e^{0.2}\int_0^5 te^{-0.04t}dt$

$= 40e^{0.2}\left[\frac{e^{-0.04t}}{0.0016}(-0.04t-1)\right]\Bigg|_0^5$

$= \frac{40e^{0.2}}{0.0016}\left[e^{-0.2}(-0.2-1)-1(-1)\right] \approx \$535$

62. Use Formula 38 with $u = t$ and $a = -0.07$, so $du = dt$.

$\int_0^5 50,000te^{-0.07t}\ dt = 50,000\int_0^5 te^{-0.07t}\ dt$

$= 50,000\left[\frac{e^{-0.07t}}{0.0049}(-0.07t-1)\right]\Bigg|_0^5$

$= \frac{50,000}{0.0049}[e^{-0.35}(-1.35)-1(-1)]$

$= \$496,640$

Problems 15.4

1. $\bar{f} = \frac{1}{3-(-1)}\int_{-1}^3 x^2\ dx = \frac{1}{4}\cdot\frac{x^3}{3}\Big|_{-1}^3 = \frac{1}{4}\left(9-\frac{-1}{3}\right)$

$= \frac{7}{3}$

2. $\bar{f} = \frac{1}{2-1}\int_1^2 (3x-1)dx = \left(\frac{3x^2}{2}-x\right)\Big|_1^2 = \frac{7}{2}$

3. $\bar{f} = \frac{1}{2-(-1)}\int_{-1}^2 \left(2-3x^2\right)dx$

$= \frac{1}{3}\left(2x-x^3\right)\Big|_{-1}^2 = -1$

4. $\bar{f} = \frac{1}{3-1}\int_1^3 \left(x^2+x+1\right)dx$

$= \frac{1}{2}\left(\frac{x^3}{3}+\frac{x^2}{2}+x\right)\Big|_1^3 = \frac{22}{3}$

5. $\bar{f} = \frac{1}{3-(-3)}\int_{-3}^3 2t^5dt$

$= \frac{1}{6}\cdot\frac{t^6}{3}\Big|_{-3}^3$

$= \frac{1}{18}[3^6-(-3)^6]$

$= 0$

6. $\bar{f} = \frac{1}{4-0}\int_0^4 t\sqrt{t^2+9}dt$

$= \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\int_0^4 \sqrt{t^2+9}[2t\ dt]$

$= \frac{1}{8}\left[\frac{2\left(t^2+9\right)^{\frac{3}{2}}}{3}\right]\Bigg|_0^4 = \frac{49}{6}$

7. $\bar{f} = \frac{1}{9-1}\int_1^9 6\sqrt{x}dx = \frac{1}{8}\left(4x^{\frac{3}{2}}\right)\Big|_1^9 = 13$

8. $\bar{f} = \frac{1}{3-1}\int_1^3 \frac{5}{x^2}dx = \frac{1}{2}\cdot-\frac{5}{x}\Big|_1^3 = \frac{1}{2}\left(-\frac{5}{3}+5\right)$

$= \frac{5}{3}$

9. $\bar{P} = \frac{1}{100-0}\int_0^{100}\left(369q-2.1q^2-400\right)dq$

$= \frac{1}{100}\left(184.5q^2-0.7q^3-400q\right)\Big|_0^{100}$

$= \frac{1}{100}(1,845,000-700,000-40,000)-0$

$= 11,050$

Answer: \$11,050

10. $\bar{c} = \frac{1}{500-100}\int_{100}^{500}\left(4000+10q+0.1q^2\right)dq$

$= \frac{1}{400}\left(4000q+5q^2+\frac{0.1q^3}{3}\right)\Bigg|_{100}^{500} \approx 17,333.33$

Answer: \$17,333.33

11. $\frac{1}{2-0}\int_0^2 3000e^{0.05t}\,dt$

$= \frac{3000}{2}\cdot\frac{1}{0.05}\int_0^2 e^{0.05t}[0.05\ dt]$

$= 30,000e^{0.05t}\Big|_0^2 = 30,000\left(e^{0.1}-1\right) \approx 3155.13$

Answer: \$3155.13

12. $\bar{C} = \frac{1}{T-0}\int_0^T \frac{R}{F(t)}dt = \frac{1}{T}\int_0^T \frac{R(1+\alpha t)^2}{F_1}dt$

$\frac{R}{TF_1}\cdot\frac{1}{\alpha}\int_0^T (1+\alpha t)^2[\alpha\ dt] = \frac{R}{\alpha TF_1}\left[\frac{(1+\alpha t)^3}{3}\right]\Bigg|_0^T$

$= \frac{R}{\alpha TF_1}\left[\frac{(1+\alpha T)^3}{3}-\frac{1}{3}\right]$

$= \frac{R}{3\alpha TF_1}\left[1+3\alpha T+3\alpha^2T^2+\alpha^3T^3-1\right]$

$= \frac{R}{3\alpha TF_1}(3\alpha T)\left(1+\alpha T+\frac{1}{3}\alpha^2T^2\right)$

$= \frac{R\left(1+\alpha T+\frac{1}{3}\alpha^2T^2\right)}{F_1}$

13. Average value $= \frac{1}{q_0-0}\int\frac{dr}{dq}dq$.

$= \frac{1}{q_0}\left[r(q_0)-r(0)\right]$

But $r(0) = 0$, so avg. value $= \frac{r(q_0)}{q_0}$. Since

$r(q_0)$

= [price per unit when q_0 units are sold] $\cdot q_0$,

we have

avg. value $= \frac{\begin{bmatrix}\text{price per unit}\\ \text{when } q_0 \text{ units}\\ \text{are sold}\end{bmatrix}\cdot q_0}{q_0}$

= price per unit when $\cdot q_0$ units are sold.

14. $\bar{f} = \frac{1}{1-0}\int_0^1\frac{1}{x^2-4x+5}dx \approx 0.32$

Principles in Practice 15.5

1. Separating variables, we have

$\frac{dI}{dx} = -0.0085I$

$\frac{dI}{I} = -0.0085dx$

$\int\frac{1}{I}dI = -\int 0.0085\ dx$

$\ln|I| = -0.0085x + C_1$

To solve for I, we convert to exponential Formula

$I = e^{-0.0085x+C_1} = Ce^{-0.0085x}$. Since $I = I_0$ when $x = 0$, $I_0 = Ce^0 = C$, so

$I(x) = I_0e^{-0.0085x}$.

Problems 15.5

1. $y' = 2xy^2$

$\frac{dy}{dx} = 2xy^2$

$\frac{dy}{y^2} = 2x\ dx$

$\int y^{-2}dy = \int 2x\ dx$

$-\frac{1}{y} = x^2 + C$

$y = -\frac{1}{x^2+C}$

2. $y' = x^2y^2$

$\frac{dy}{dx} = x^2y^2$

$\frac{dy}{y^2} = x^2\ dx$

$$\int \frac{dy}{y^2} = \int x^2\,dx$$
$$-\frac{1}{y} = \frac{x^3}{3} + C_1$$
$$-\frac{1}{y} = \frac{1}{3}(x^3 + 3C_1)$$
$$\frac{1}{y} = -\frac{1}{3}(x^3 + C)$$
$$y = -\frac{3}{x^3 + C}$$

3. $\frac{dy}{dx} - 3x\sqrt{x^2+1} = 0$

$$dy = 3x\left(x^2+1\right)^{\frac{1}{2}}dx$$
$$\int dy = 3\int x\left(x^2+1\right)^{\frac{1}{2}}dx$$
$$\int dy = \frac{3}{2}\int\left(x^2+1\right)^{\frac{1}{2}}[2x\ dx]$$
$$y = \frac{3}{2}\cdot\frac{\left(x^2+1\right)^{\frac{3}{2}}}{\frac{3}{2}} + C$$
$$y = \left(x^2+1\right)^{\frac{3}{2}} + C$$

4. $\frac{dy}{dx} = \frac{x}{y}$

$$y\,dy = x\,dx$$
$$\int y\ dy = \int x\ dx$$
$$\frac{y^2}{2} = \frac{x^2}{2} + C_1$$
$$y^2 = x^2 + 2C_1$$
$$y^2 = x^2 + C$$

5. $\frac{dy}{dx} = y$, where $y > 0$.

$$\frac{dy}{y} = dx$$
$$\int \frac{dy}{y} = \int dx$$
$$\ln y = x + C_1$$

$y = e^{x+C_1} = e^{C_1}e^x = Ce^x$, where $C = e^{C_1}$. Thus

$y = Ce^x$, where $C > 0$.

6. $y' = e^x y^3$

$$\frac{dy}{dx} = e^x y^3$$
$$\frac{dy}{y^3} = e^x dx$$
$$\int \frac{dy}{y^3} = \int e^x dx$$
$$-\frac{1}{2y^2} = e^x + C$$
$$y^2 = -\frac{1}{2(e^x + C)}$$

7. $y' = \frac{y}{x}$, where $x, y > 0$.

$$\frac{dy}{dx} = \frac{y}{x}$$
$$\frac{dy}{y} = \frac{dx}{x}$$
$$\int \frac{dy}{y} = \int \frac{dx}{x}$$
$$\ln y = \ln x + C_1$$

$\ln y = \ln x + \ln C$, where $C > 0$.

$\ln y = \ln(Cx) \Rightarrow y = Cx$, where $C > 0$.

8. $\frac{dy}{dx} + xe^x = 0$

$$dy = -xe^x dx$$
$$\int dy = \int -xe^x dx$$
$$y = \int -xe^x dx$$

Using integration by parts or formula 38 gives

$$y = (1-x)e^x + C$$

9. $y' = \frac{1}{y^2}$ where $y(1) = 1$.

$$\frac{dy}{dx} = \frac{1}{y^2}$$
$$y^2 dy = dx$$
$$\int y^2\,dy = \int dx$$
$$\frac{y^3}{3} = x + C$$

Given $y(1) = 1$, we obtain $\frac{1^3}{3} = 1 + C$, so

$C = -\frac{2}{3}$. Thus $y^3 = 3\left(x - \frac{2}{3}\right) = 3x - 2$,

$y = \sqrt[3]{3x-2}$.

10. $y' = e^{x-y}$, where $y(0) = 0$

$\frac{dy}{dx} = \frac{e^x}{e^y}$

$e^y dy = e^x dx$

$\int e^y dy = \int e^x dx$

$e^y = e^x + C$

Since $y(0) = 0$, we have $e^0 = e^0 + C$, $1 = 1 + C$, $C = 0$. Thus $e^y = e^x$, so $y = x$.

11. $e^y y' - x^2 = 0$, where $y = 0$ when $x = 0$.

$e^y \frac{dy}{dx} = x^2$

$e^y dy = x^2 dx$

$\int e^y dy = \int x^2 dx$

$e^y = \frac{x^3}{3} + C$

Given that $y(0) = 0$, we have $e^0 = 0 + C$, so $1 = C \Rightarrow e^y = \frac{x^3}{3} + 1$, $e^y = \frac{x^3+3}{3}$, so

$y = \ln \frac{x^3+3}{3}$.

12. $x^2 y' + \frac{1}{y^2} = 0$, where $y(1) = 2$

$x^2 \frac{dy}{dx} = -\frac{1}{y^2}$

$y^2 dy = -\frac{dx}{x^2}$

$\int y^2 dy = -\int \frac{dx}{x^2}$

$\frac{y^3}{3} = \frac{1}{x} + C$

Now, $y(1) = 2$ implies $C = \frac{5}{3}$. Thus

$\frac{y^3}{3} = \frac{1}{x} + \frac{5}{3}$, $y^3 = \frac{3}{x} + 5$, $y = \sqrt[3]{\frac{3}{x} + 5}$.

13. $(3x^2+2)^3 y' - xy^2 = 0$, where $y(0) = \frac{3}{2}$.

$(3x^2+2)^3 \frac{dy}{dx} = xy^2$

$\frac{dy}{y^2} = \frac{x}{(3x^2+2)^3}$

$\int \frac{dy}{y^2} = \int \frac{x}{(3x^2+2)^3} dx$

$\int y^{-2} dy = \frac{1}{6} \int (3x^2+2)^{-3} [6x\, dx]$

$-\frac{1}{y} = -\frac{1}{12(3x^2+2)^2} + C$

Given that $y(0) = \frac{3}{2}$ we have

$-\frac{1}{\frac{3}{2}} = -\frac{1}{2(2)^2} + C$, $-\frac{2}{3} = -\frac{1}{48} + C$, so

$C = -\frac{31}{48}$. Thus,

$$-\frac{1}{y} = -\frac{1}{12(3x^2+2)^2} - \frac{31}{48}$$

$$= -\frac{4 + 31(3x^2+2)^2}{48(3x^2+2)^2}.$$

Hence, $y = \frac{48(3x^2+2)^2}{4 + 31(3x^2+2)^2}$.

14. $y' + x^3 y = 0$ and $y = e$ when $x = 0$.

$\frac{dy}{dx} = -x^3 y$

$\frac{dy}{y} = -x^3\, dx$

$\int \frac{dy}{y} = -\int x^3\, dx$

$\ln|y| = -\frac{x^4}{4} + C$

Given $y(0) = e$, $\ln e = 0 + C$, so $C = 1$.

Thus $\ln y = -\frac{x^4}{4} + 1$, so $y = e^{-\frac{x^4}{4}+1}$.

15. $\frac{dy}{dx}=\frac{3x\sqrt{1+y^2}}{y}$, where $y>0$ and $y(1)=\sqrt{8}$.

$$\frac{y\ dy}{\sqrt{1+y^2}}=3x\ dx$$

$$\frac{1}{2}\int\left(1+y^2\right)^{-\frac{1}{2}}[2y\ dy]=3\int x\ dx$$

$$\left(1+y^2\right)^{\frac{1}{2}}=\frac{3x^2}{2}+C$$

$$y(1)=\sqrt{8}\Rightarrow(1+8)^{\frac{1}{2}}=\frac{3}{2}+C$$

$$C=\frac{3}{2}$$

Thus

$$\left(1+y^2\right)^{\frac{1}{2}}=\frac{3x^2}{2}+\frac{3}{2}$$

$$1+y^2=\left[\frac{3x^2}{2}+\frac{3}{2}\right]^2$$

$$y^2=\left[\frac{3x^2}{2}+\frac{3}{2}\right]^2-1$$

Since $y>0$, $y=\sqrt{\left[\frac{3x^2}{2}+\frac{3}{2}\right]^2-1}$.

16. $2y\left(x^3+2x+1\right)\frac{dy}{dx}=\frac{3x^2+2}{\sqrt{y^2+9}}$, where $y(0)=0$.

$$\int 2y\sqrt{y^2+9}dy=\int\frac{3x^2+2}{x^3+2x+1}dx$$

$$\int\left(y^2+9\right)^{\frac{1}{2}}[2y\ dy]=\int\frac{1}{x^3+2x+1}\left[\left(3x^2+2\right)dx\right]$$

$$\frac{2}{3}\left(y^2+9\right)^{\frac{3}{2}}=\ln\left|x^3+2x+1\right|+C$$

Now $y(0)=0$ implies that $\frac{2}{3}(27)=\ln(1)+C$, so $C=18$. Thus

$$\frac{2}{3}\left(y^2+9\right)^{\frac{3}{2}}=\ln\left|x^3+2x+1\right|+18.$$

17. $2\frac{dy}{dx}=\frac{xe^{-y}}{\sqrt{x^2+3}}$, where $y(1)=0$.

$$e^y\ dy=\frac{1}{2}x\left(x^2+3\right)^{-\frac{1}{2}}dx$$

$$\int e^y dy=\frac{1}{2}\cdot\frac{1}{2}\int\left(x^2+3\right)^{-\frac{1}{2}}[2x\ dx]$$

$$e^y=\frac{1}{2}\left(x^2+3\right)^{\frac{1}{2}}+C$$

Now, $y(1)=0\Rightarrow e^0=\frac{1}{2}(2)+C$, so $C=0$. Thus

$$e^y=\frac{1}{2}\left(x^2+3\right)^{\frac{1}{2}}\Rightarrow y=\ln\left(\frac{1}{2}\sqrt{x^2+3}\right).$$

18. $x(y^2+1)^{3/2}dx=e^{x^2}y\,dy$, where $y(0)=0$.

$$xe^{-x^2}dx=y(y^2+1)^{-3/2}dy$$

$$\int xe^{-x^2}dx=\int y(y^2+1)^{-3/2}dy$$

$$-\frac{1}{2}\int e^{-x^2}[-2x\,dx]=\frac{1}{2}\int(y^2+1)^{-3/2}[2y\,dy]$$

$$-\frac{1}{2}e^{-x^2}=\frac{1}{2}\frac{(y^2+1)^{-1/2}}{-1/2}+C$$

$$\frac{1}{2}e^{-x^2}=(y^2+1)^{-1/2}+C$$

Now $y(0)=0$ gives $\frac{1}{2}=\frac{1}{\sqrt{1}}+C$, so $C=-\frac{1}{2}$.

Thus $\frac{1}{2}e^{-x^2}=(y^2+1)^{-1/2}-\frac{1}{2}$ or

$$e^{-x^2}=2(y^2+1)^{-1/2}-1.$$

19. $(q+1)^2\frac{dc}{dq}=cq$

$$\int\frac{1}{c}dc=\int\frac{q}{(q+1)^2}dq$$

Using partial fractions or Formula 7 for $\int\frac{q}{(q+1)^2}dq$, we obtain

$\ln c=\ln(q+1)+\frac{1}{q+1}+C$. Now, fixed cost is given to be e, which means that $c=e$ when $q=0$. This implies $1=0+1+C$, so $C=0$. Thus

$$\ln c=\ln(q+1)+\frac{1}{q+1}\Rightarrow c=e^{\ln(q+1)+\frac{1}{q+1}},$$

$$c=e^{\ln(q+1)}e^{\frac{1}{q+1}},\ \text{or}\ c=(q+1)e^{\frac{1}{q+1}}.$$

20. $\frac{dy}{dx} = xe^{x-y} = \frac{xe^x}{e^y}$

$\int e^y dy = \int xe^x dx$

Using integration by parts or formula 38 gives $e^y = e^x(x-1) + C$. Now, $f(1) = 0 \Rightarrow 1 = e(0) + C$, $1 = C$, so $e^y = e^x(x-1) + 1$, $y = \ln\left[e^x(x-1)+1\right]$. Thus $f(2) = \ln\left(e^2+1\right)$.

21. $\frac{dy}{dt} = -0.025y$

$\int \frac{1}{y} dy = -0.025 \int dt$

$\ln|y| = -0.025t + C$

Given that $y = 1000$ when $t = 0$, we have $\ln 1000 = -0 + C = C$. Thus $\ln|y| = -0.025t + \ln 1000$. To find t when money is 95% new, we note that y would be 5%(1000) = 50. Solving $\ln 50 = -0.025t + \ln 1000$ gives

$t = \frac{\ln 1000 - \ln 50}{0.025} \approx 120$ weeks.

22. $\frac{dr}{dq} = (50-4q)e^{-\frac{r}{5}}$

$\int e^{\frac{r}{5}} dr = \int (50-4q)dq$

$5e^{\frac{r}{5}} = 50q - 2q^2 + C$

Since $r = 0$ when $q = 0$, we have $5(1) = C$, $C = 5$.

$5e^{\frac{r}{5}} = 50q - 2q^2 + 5$

$e^{\frac{r}{5}} = 10q - \frac{2}{5}q^2 + 1$

$\frac{r}{5} = \ln\left|10q - \frac{2}{5}q^2 + 1\right|$

$r = 5\ln\left|10q - \frac{2}{5}q^2 + 1\right|$

Since $r = pq$, $p = \frac{1}{q}r = \frac{5}{q}\ln\left|10q - \frac{2}{5}q^2 + 1\right|$.

23. Let N be the population at time t, where $t = 0$ corresponds to 1985. Since N follows exponential growth, $N = N_0e^{kt}$. Now, $N = 40{,}000$ when $t = 0$, so $N_0 = 40{,}000$. Therefore $N = 40{,}000e^{kt}$. Since $N = 48{,}000$ when $t = 10$, we have $48{,}000 = 40{,}000e^{10k}$, $1.2 = e^{10k}$, $\ln 1.2 = 10k$, $k = \frac{\ln 1.2}{10}$

Thus

$N = 40{,}000e^{\ln(1.2)\left(\frac{t}{10}\right)}$ (*)

$N = 40{,}000e^{0.18\left(\frac{t}{10}\right)}$

$N = 40{,}000e^{0.018t}$ (First form)

From (*), we have $N = 40{,}000\left[e^{\ln 1.2}\right]^{\frac{t}{10}}$, so

$N = 40{,}000(1.2)^{\frac{t}{10}}$ (Second form)

At year 2005, $t = 20$ and so

$N = 40{,}000(1.2)^{\frac{20}{10}} = 40{,}000(1.2)^2 = 57{,}600$.

24. Exponential growth applies, so $N = N_0e^{kt}$. When $t = 0$, then $N = 50{,}000$, So $N_0 = 50{,}000$. Thus $N = 50{,}000e^{kt}$. When $t = 50$, then $N = 100{,}000$, or $100{,}000 = 50{,}000e^{50k}$ or $k = \frac{\ln 2}{50}$. Thus

$N = 50{,}000e^{\frac{t\ln 2}{50}}$ (*)

$N = 50{,}000e^{\left(\frac{0.69}{50}\right)t}$

$N = 50{,}000e^{0.0138t}$ (First form)

From (*), $N = 50{,}000\left[e^{\ln 2}\right]^{\frac{t}{50}}$, so

$N = 50{,}000(2)^{\frac{t}{50}}$. (Second form)

When $t = 100$, then

$N = 50{,}000(2)^{\frac{100}{50}} = 50{,}000(2)^2 = 200{,}000$

25. Let N be the population (in billions) at time t, where t is the number of years past 1930. N follows exponential growth, so $N = N_0e^{kt}$. When $t = 0$, then $N = 2$, so $N_0 = 2$. Thus $N = 2e^{kt}$. Since $N = 3$ when $t = 30$, then

$3 = 2e^{30k}$

$\frac{3}{2} = e^{30k}$

$30k = \ln\frac{3}{2}$

$k = \frac{\ln\frac{3}{2}}{30}$

Thus $N = 2e^{\frac{t}{30}\ln\frac{3}{2}}$.
In 2015, $t = 85$ and so
$N = 2e^{\frac{85}{30}\ln\frac{3}{2}} \approx 2e^{1.14882}$ billion.

26. Let N = population at time t and N_0 = population at $t = 0$. Then $N = N_0e^{kt}$. When $t = 100$, then $N = 3N_0$, so

$3N_0 = N_0e^{100k}$ or $k = \frac{\ln 3}{100}$.

Setting $N = 2N_0$ and solving for t gives

$2N_0 = N_0e^{\frac{t\ln 3}{100}}$

$2 = e^{\frac{t\ln 3}{100}}$

$\ln 2 = \frac{t\ln 3}{100}$

$t = \frac{100\ln 2}{\ln 3} \approx 63.$

The population will double in approximately 63 years.

27. Let N be amount of sample that remains after t seconds. Then $N = N_0e^{-\lambda t}$, where N_0 is the initial amount present. When $t = 100$, then $N = 0.3N_0$. Thus

$0.3N_0 = N_0e^{-100\lambda}$

$0.3 = e^{-100\lambda}$

$-100\lambda = \ln 0.3$

$\lambda = -\frac{\ln 0.3}{100}$

Thus $\lambda \approx 0.01204$. The half-life is

$\frac{\ln 2}{\lambda} = \frac{\ln 2}{-\frac{\ln 0.3}{100}} = -100\frac{\ln 2}{\ln 0.3} \approx 57.57$ s.

28. $N = N_0e^{-\lambda t}$

After 100 s, 70% remains.

$0.7N_0 = N_0e^{-100\lambda}$

$0.7 = e^{-100\lambda}$

$-100\lambda = \ln 0.7$

$\lambda = -\frac{\ln 0.7}{100}$

$\lambda \approx 0.0035667$

The half-life is

$\frac{\ln 2}{\lambda} = \frac{\ln 2}{-\frac{\ln 0.7}{100}} = -100\frac{\ln 2}{\ln 0.7} \approx 194.3$ s.

29. Let N be the amount of ${}^{14}C$ present in the scroll t years after it was made. Then $N = N_0e^{-\lambda t}$, where N_0 is amount of ${}^{14}C$ present when $t = 0$. We must find t when $N = 0.7N_0$.

$0.7N_0 = N_0e^{-\lambda t}$

$0.7 = e^{-\lambda t}$

$-\lambda t = \ln 0.7$

so $t = -\frac{\ln 0.7}{\lambda}$. By Equation 15 in the text,

$\lambda = \frac{\ln 2}{5730}$, so

$t = -\frac{\ln 0.7}{\frac{\ln 2}{5730}} = -\frac{5730\ln 0.7}{\ln 2} \approx 2900$ years.

30. $N = N_0e^{-\lambda t}$

$0.1N_0 = N_0e^{-\lambda t}$

$0.1 = e^{-\lambda t}$

$-\lambda t = \ln(0.1)$

$t = -\frac{\ln 0.1}{\lambda}$

By Equation 15 in the text, $\lambda = \frac{\ln 2}{5730}$, so

$t = -\frac{\ln 0.1}{\frac{\ln 2}{5730}} = -\frac{5730\ln 0.1}{\ln 2} \approx 19,000$ years.

31. $\frac{dN}{dt} = kN$

$N = Ae^{kt}$

$N_0 = Ae^{kt_0}$

$A = \frac{N_0}{e^{kt_0}}$

Thus $N = \frac{N_0}{e^{kt_0}}\left(e^{kt}\right) = N_0 e^{kt-kt_0}$, or

$N = N_0 e^{k(t-t_0)}$, where $t \ge t_0$.

32. a. From Equation 15 in the text, $140 = \frac{\ln 2}{\lambda}$.

Thus $\lambda = \frac{\ln 2}{140}$.

b. $N = N_0 e^{-\lambda t} = N_0 e^{-\frac{t \ln 2}{140}} = N_0 e^{-\frac{365 \ln 2}{140}}$

$\frac{N}{N_0} = e^{-\frac{365 \ln 2}{140}} \approx 0.164$

33. $N = N_0 e^{-\lambda t}$

When $t = 2$, then $N = 10$. Thus $10 = N_0 e^{-2\lambda}$,

$N_0 = 10e^{2\lambda}$. By Equation 15 in the text,

$6 = \frac{\ln 2}{\lambda}$

$\lambda = \frac{\ln 2}{6}$

Thus $N_0 = 10e^{2\frac{\ln 2}{6}} = 10e^{\frac{\ln 2}{3}} \approx 12.6$ units.

34. $N = N_0 e^{-\lambda t}$

We want to find t when

$N = \left(\frac{3}{5}\right)N_0$

$\left(\frac{3}{5}\right)N_0 = N_0 e^{-\lambda t}$

$\frac{3}{5} = e^{-\lambda t}$

$-\lambda t = \ln\left(\frac{3}{5}\right)$

$t = -\frac{\ln \frac{3}{5}}{\lambda}$

By Equation 15 in the text, $8 = \frac{\ln 2}{\lambda}$, $\lambda = \frac{\ln 2}{8}$.

Thus $t = -\frac{\ln \frac{3}{5}}{\frac{\ln 2}{8}} = -\frac{8 \ln \frac{3}{5}}{\ln 2} \approx 5.9$ days.

35. $\frac{dA}{dt} = 200 - 0.50A$

$\int \frac{dA}{200 - 0.50A} = \int dt$

$-\frac{1}{0.50}\ln(200 - 0.50A) = t + C_1$

$\ln(200 - 0.50A) = -0.50t - 0.50C_1$

$= -0.50t + C_2$

Thus

$200 - 0.50A = e^{-0.50t + C_2} = e^{-0.50t}e^{C_2}$

$200 - \frac{A}{2} = Ce^{-0.50t}$

Given that $A = 0$ when $t = 0$, we have $C = 200$,

so $200 - \frac{A}{2} = 200e^{-\frac{t}{2}}$

$200 - 200e^{-\frac{t}{2}} = \frac{A}{2}$

$200\left(1 - e^{-\frac{t}{2}}\right) = \frac{A}{2}$

Thus $A = 400\left(1 - e^{-\frac{t}{2}}\right)$. If $t = 1$,

$A = 400\left(1 - e^{-\frac{1}{2}}\right) \approx 157$ grams per square meter.

36. $\frac{dP}{dx} = k(150,000 - 2P)$

$\int \frac{dP}{150,000 - 2P} = \int k\,dx$

$-\frac{1}{2}\ln[150,000 - 2P] = kx + C$

Since $P(0) = 15,000$, we have

$-\frac{1}{2}\ln[150,000 - 30,000] = C$, so

$-\frac{1}{2}\ln[150,000 - 2P] = kx - \frac{1}{2}\ln[120,000]$.

Since $P(1000) = 70{,}000$,

$$-\frac{1}{2}\ln[150,000-140,000] = 1000k-\frac{1}{2}\ln[120,000]$$

$$k=\frac{1}{2}\cdot\frac{\ln[120,000]-\ln[10,000]}{1000}=\frac{\ln 12}{2000}$$

Thus

$$-\frac{1}{2}\ln[150,000-2P]=\frac{\ln 12}{2000}x-\frac{1}{2}\ln[120,000]$$

$$\ln[150,000-2P]=-\frac{\ln 12}{1000}x+\ln[120,000]$$

$$150,000-2P=e^{-\frac{\ln 12}{1000}x}e^{\ln[120,000]}$$

$$150,000-2P=120,000e^{-\frac{\ln 12}{1000}x}$$

$$P=\frac{1}{2}\left(150,000-120,000e^{-\frac{\ln 12}{1000}x}\right)$$

$$=75,000-60,000\left(12^{-\frac{x}{1000}}\right)$$

If $x = 2000$, then

$$P=75,000-60,000(12^{-2})\approx \$74,583.$$

37. a. $$\frac{dV}{dt}=kV$$

$$\int\frac{1}{V}dV=\int k\ dt$$

$$\ln V=kt+C_1$$

$$V=e^{kt}e^{C_1}$$

or $V=Ce^{kt}$. Now $t = 0$ corresponds to July 1, 1996 where $V = 0.75 \cdot 80{,}000 = 60{,}000$, so $60{,}000 = C(1)$. Thus $V=60,000e^{kt}$. Also $V = 38{,}900$ for $t=9.5$, so

$$38,900=60,000e^{9.5k}$$

$$\frac{389}{600}=e^{9.5k}$$

$$9.5k=\ln\left(\frac{389}{600}\right)$$

$$k=\frac{1}{9.5}\ln\left(\frac{389}{600}\right)$$

Thus $V=60,000e^{\frac{t}{9.5}\ln\left(\frac{389}{600}\right)}$.

b. $$14,000=60,000e^{\frac{t}{9.5}\ln\left(\frac{389}{600}\right)}$$

$$\frac{7}{30}=e^{\frac{t}{9.5}\ln\left(\frac{389}{600}\right)}$$

$$\ln\left(\frac{7}{30}\right)=\frac{t}{9.5}\ln\left(\frac{389}{600}\right)$$

$$t=\frac{9.5\ln\left(\frac{7}{30}\right)}{\ln\left(\frac{389}{600}\right)}\approx 31.903$$

This corresponds to about 31 years and 11 months after July 1, 1996 ⇒ June 2028.

Problems 15.6

1. $$N=\frac{M}{1+be^{-ct}}$$

$M = 100{,}000$

Since $N = 50{,}000$ at $t = 0$ (1995), we have

$50,000=\frac{100,000}{1+b}$, so $1+b=\frac{100,000}{50,000}=2$, or $b=1$.

Hence, $N=\frac{100,000}{1+e^{-ct}}$. If $t = 5$, then $N = 60{,}000$, so

$$60,000=\frac{100,000}{1+e^{-5c}}$$

$$1+e^{-5c}=\frac{100,000}{60,000}=\frac{5}{3}$$

$$e^{-5c}=\frac{5}{3}-1=\frac{2}{3}$$

$$e^{-c}=\left(\frac{2}{3}\right)^{1/5}$$

Hence, $N=\frac{100,000}{1+\left(\frac{2}{3}\right)^{t/5}}$. In 2005, $t = 10$, so

$$N=\frac{100,000}{1+\left(\frac{2}{3}\right)^{2}}\approx 69,200.$$

2. $$N=\frac{M}{1+be^{-ct}}$$

Since $M = 500$, and $N = 200$ when $t = 0$, we have

$$200=\frac{500}{1+b}$$

$$1+b=\frac{500}{200}=\frac{5}{2}\Rightarrow b=\frac{3}{2}.$$

Hence $N=\frac{500}{1+\frac{3}{2}e^{-ct}}$. When $t = 1$ we are given

$N = 300$. Thus

$$300 = \frac{500}{1+\frac{3}{2}e^{-c}}$$

$$1+\frac{3}{2}e^{-c} = \frac{500}{300} = \frac{5}{3}$$

$$\frac{3}{2}e^{-c} = \frac{2}{3}$$

$$e^{-c} = \frac{4}{9}$$

Hence $N = \frac{500}{1+\frac{3}{2}\left(\frac{4}{9}\right)^t}$. When $t = 2$, then

$$N = \frac{500}{1+\frac{3}{2}\left(\frac{4}{9}\right)^2} \approx 386.$$

3. $N = \frac{M}{1+be^{-ct}}$

$M = 40{,}000$, and $N = 20$ when $t = 0$, so

$$20 = \frac{40{,}000}{1+b}$$

$$1+b = \frac{40{,}000}{20} = 2000$$

$$b = 1999$$

Hence $N = \frac{40{,}000}{1+1999e^{-ct}}$.

Since $N = 100$ when $t = 1$, $100 = \frac{40{,}000}{1+1999e^{-c}}$,

$$1+1999e^{-c} = \frac{40{,}000}{100} = 400$$

$$e^{-c} = \frac{399}{1999}$$

Hence $N = \frac{40{,}000}{1+1999\left(\frac{399}{1999}\right)^t}$.

If $t = 2$, then $N = \frac{40{,}000}{1+1999\left(\frac{399}{1999}\right)^2} \approx 500$.

4. $N = \frac{M}{1+be^{-ct}}$

Since $M = 30{,}000$, and $N = 400$ when $t = 0$, we have

$$400 = \frac{30{,}000}{1+b}$$

$$1+b = \frac{30{,}000}{400} = 75$$

$$b = 74$$

Hence $N = \frac{30{,}000}{1+74e^{-ct}}$. If $t = 1$, then $N = 1200$. Thus

$$1200 = \frac{30{,}000}{1+74e^{-c}}$$

$$1+74e^{-c} = \frac{30{,}000}{1200} = 25$$

$$74e^{-c} = 24$$

$$e^{-c} = \frac{24}{74} = \frac{12}{37}$$

Hence $N = \frac{30{,}000}{1+74\left(\frac{12}{37}\right)^t}$.

5. $N = \frac{M}{1+be^{-ct}}$

$M = 100{,}000$, and since $N = 500$ when $t = 0$, we have

$$500 = \frac{100{,}000}{1+b}$$

$$1+b = \frac{100{,}000}{500} = 200$$

$$b = 199$$

Hence $N = \frac{100{,}000}{1+199e^{-ct}}$. If $t = 1$, then $N = 1000$. Thus

$$1000 = \frac{100{,}000}{1+199e^{-c}}$$

$$1+199e^{-c} = \frac{100{,}000}{1000} = 100$$

$$199e^{-c} = 99$$

$$e^{-c} = \frac{99}{199}$$

Hence $N = \frac{100{,}000}{1+199\left(\frac{99}{199}\right)^t}$. If $t = 2$, then

$$N = \frac{100{,}000}{1+199\left(\frac{99}{199}\right)^2} \approx 1990.$$

6. a.
$$\frac{dN}{dt} = N(1-N)$$
$$\frac{dN}{N(1-N)} = dt$$
$$\int \frac{dN}{N(1-N)} = \int dt$$
Using Formula 5 in the Table of Integrals, for $\int \frac{dN}{N(1-N)}$, we get $\ln\left|\frac{N}{1-N}\right| = t + C$.

Since $N(0) = \frac{1}{2}$, $\ln\left|\frac{\frac{1}{2}}{1-\frac{1}{2}}\right| = \ln 1 = 0 = C$.

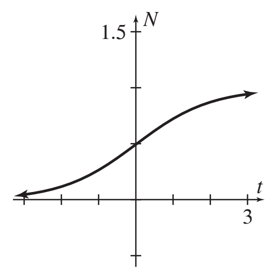
Also, $N > 0$, and since $M = 1$, $N < 1$. Thus $\ln\left(\frac{N}{1-N}\right) = t$.
$$\frac{N}{1-N} = e^t$$
$$N = (1-N)e^t$$
$$N(e^t + 1) = e^t$$
$$N = \frac{e^t}{e^t + 1} = \frac{1}{1+e^{-t}}$$

b.
$$\frac{dN}{dt} = N(1-N) = N - N^2$$
$$\frac{d^2N}{dt^2} = 1 - 2N$$
$$\frac{d^2N}{dt^2} = 0 \text{ when } N = \frac{1}{2}.$$
$1 - 2N > 0$ when $N < \frac{1}{2}$ and $1 - 2N < 0$ when $N > \frac{1}{2}$, so there is an inflection point when $N = \frac{1}{2}$.
$$\frac{1}{2} = \frac{1}{1+e^{-t}}$$
$$1 + e^{-t} = 2$$
$$e^{-t} = 1$$
$$t = 0$$
Thus the point $\left(0, \frac{1}{2}\right)$ is an inflection point on the graph.

c.
$$f(t) = \frac{1}{1+e^{-t}} - \frac{1}{2} = \frac{2-(1+e^{-t})}{2(1+e^{-t})} = \frac{1-e^{-t}}{2(1+e^{-t})} = \frac{e^t - 1}{2(e^t + 1)}$$
Replace t by $-t$ then multiply numerator and denominator by e^t.
$$\frac{e^{-t}-1}{2(e^{-t}+1)} = \frac{1-e^t}{2(1+e^t)} = -\frac{e^t - 1}{2(e^t+1)} = -f(t)$$
Thus, $f(t)$ is symmetric about the origin.

d. The graph of $N(t)$ is the graph of $f(t)$ shifted $\frac{1}{2}$ unit upward. Thus, since $f(t)$ is symmetric about (0, 0), $N(t)$ is symmetric about $\left(0, \frac{1}{2}\right)$.
$$N(t) = f(t) + \frac{1}{2}$$
$$N(-t) = f(-t) + \frac{1}{2} = -f(t) + \frac{1}{2} = -\left[f(t) + \frac{1}{2}\right] + 1 = 1 - N(t)$$

e.



7. a.
$$N = \frac{375}{1+e^{5.2-2.3t}} = \frac{375}{1+e^{5.2}e^{-2.3t}} \approx \frac{375}{1+181.27e^{-2.3t}}$$

b.
$$\lim_{t\to\infty} N = \frac{375}{1+181.27(0)} = 375$$

8. a. $N = \dfrac{0.2524}{e^{-2.128x} + 0.005125}$

$= \dfrac{\frac{0.2524}{0.005125}}{\frac{e^{-2.128x}+0.005125}{0.005125}}$

$\approx \dfrac{49.25}{\frac{e^{-2.128x}}{0.005125}+1} \approx \dfrac{49.25}{1+195.1e^{-2.128x}}$

b. If $x = 0$, then $N \approx \dfrac{49.25}{1+195.1} \approx 0.2511 \text{ cm}^2$.

9. $\dfrac{dT}{dt} = k(T-a)$ where $a = -10$.

$\dfrac{dT}{T+10} = k\,dt$

$\int \dfrac{dT}{T+10} = \int k\,dt$

Thus $\ln(T + 10) = kt + C$. At $t = 0$, we have $T = 28$, so $\ln(28 + 10) = 0 + C$, $C = \ln 38$, and $\ln(T + 10) = kt + \ln 38$.

$\ln(T + 10) - \ln 38 = kt$

Hence $\ln\left(\dfrac{T+10}{38}\right) = kt$.

If $t = 1$, then $T = 20$. Thus $\ln\left(\dfrac{20+10}{38}\right) = k \cdot 1$, so

$k = \ln\dfrac{30}{38} = \ln\dfrac{15}{19}$. Hence

$\ln\left(\dfrac{T+10}{38}\right) = \left(\ln\dfrac{15}{19}\right)t$.

If $T = 37$, then $\ln\left(\dfrac{47}{38}\right) = \left(\ln\dfrac{15}{19}\right)t$

$$t = \frac{\ln\frac{47}{38}}{\ln\frac{15}{19}} \approx -0.90 \text{ hr}$$

which corresponds to 54 minutes. Time of murder: 4:15 A.M. – 54 min = 3:21 A.M.

10. $\dfrac{dp}{dt} = kp(I-p)$

This is logistic growth, so the maximum rate of formation (growth) occurs when $p = \dfrac{I}{2}$, which is when there are equal amounts of both enzymes.

11. $\dfrac{dx}{dt} = k(200{,}000 - x)$

$\int \dfrac{dx}{200{,}000-x} = \int k\ dt$

$-\ln(200{,}000 - x) = kt + C$

$\ln(200{,}000 - x) = -kt - C$

$200{,}000 - x = e^{-kt-C} = e^{-C}e^{-kt} = Ae^{-kt}$, where $A = e^{-C}$. Thus $x = 200{,}000 - Ae^{-kt}$. If $t = 0$, then $x = 50{,}000$, so $50{,}000 = 200{,}000 - A \Rightarrow A = 150{,}000$. Thus $x = 200{,}000 - 150{,}000e^{-kt}$. If $t = 1$, then $x = 100{,}000$, so

$100{,}000 = 200{,}000 - 150{,}000e^{-k}$

$150{,}000e^{-k} = 100{,}000$

$e^{-k} = \dfrac{100{,}000}{150{,}000} = \dfrac{2}{3}$

Thus $x = 200{,}000 - 150{,}000\left(\dfrac{2}{3}\right)^t$. If $t = 3$, then

$x = 200{,}000 - 150{,}000\left(\dfrac{8}{27}\right) \approx \$155{,}555.56$.

12. $\dfrac{dN}{dt} = kN^2$

$\int \dfrac{dN}{N^2} = \int k\ dt$

$-\dfrac{1}{N} = kt + C$

If $t = 0$, then $N = N_0$. Thus $-\dfrac{1}{N_0} = C$, so

$-\dfrac{1}{N} = kt - \dfrac{1}{N_0}$

$\dfrac{1}{N} = \dfrac{-kN_0t+1}{N_0}$

$N = \dfrac{N_0}{1-kN_0t}$

As $t \to \left(\dfrac{1}{kN_0}\right)^-$, then $1 - kN_0t \to 0^+$, so $N \to \infty$.

13. $\frac{dN}{dt}=k(M-N)$

$\int \frac{dN}{M-N}=\int k\ dt$

$-\ln(M-N)=kt+C$

If $t=0$, then $N=N_0$, so $-\ln(M-N_0)=C$.

Thus we have

$-\ln(M-N)=kt-\ln(M-N_0)$

$\ln(M-N_0)-\ln(M-N)=kt$

$\ln\frac{M-N_0}{M-N}=kt$

$\ln\frac{M-N}{M-N_0}=-kt$

$\frac{M-N}{M-N_0}=e^{-kt}$

$M-N=(M-N_0)e^{-kt}$

$N=M-(M-N_0)e^{-kt}$

Principles in Practice 15.7

1. $\int_0^\infty\left(3e^{-0.1t}-3e^{-0.3t}\right)dt$

$=\lim_{r\to\infty}\int_0^r\left(3e^{-0.1t}-3e^{-0.3t}\right)dt$

$=\lim_{r\to\infty}\left(-30e^{-0.1t}+10e^{-0.3t}\right)\Big|_0^r$

$=\lim_{r\to\infty}\left[-\frac{30}{e^{-0.1r}}+\frac{10}{e^{0.3r}}-\left(-30e^0+10e^0\right)\right]$

$=\lim_{r\to\infty}\left[-\frac{30}{e^{0.1r}}+\frac{10}{e^{0.3r}}-(-20)\right]$

$= 0 + 0 + 20 = 20$

The total amount of the drug that is eliminated is approximately 20 milliliters.

Problems 15.7

1. $\int_3^\infty\frac{1}{x^3}dx=\lim_{r\to\infty}\int_3^r x^{-3}dx$

$=\lim_{r\to\infty}\frac{x^{-2}}{-2}\Big|_3^r=-\frac{1}{2}\lim_{r\to\infty}\frac{1}{x^2}\Big|_3^r$

$=-\frac{1}{2}\lim_{r\to\infty}\left(\frac{1}{r^2}-\frac{1}{9}\right)=-\frac{1}{2}\left(0-\frac{1}{9}\right)=\frac{1}{18}$

2. $\int_1^\infty\frac{1}{(3x-1)^2}dx=\lim_{r\to\infty}\frac{1}{3}\int_1^r(3x-1)^{-2}[3\,dx]$

$=\lim_{r\to\infty}\left[-\frac{1}{3(3x-1)}\right]\Big|_1^r$

$=\lim_{r\to\infty}\left[-\frac{1}{3(3r-1)}+\frac{1}{6}\right]$

$=0+\frac{1}{6}$

$=\frac{1}{6}$

3. $\int_1^\infty\frac{1}{x}dx=\lim_{r\to\infty}\int_1^r\frac{1}{x}dx=\lim_{r\to\infty}\ln|x|\Big|_1^r$

$=\lim_{r\to\infty}\left(\ln|r|-0\right)$

$=\lim_{r\to\infty}\ln|r|=\infty\Rightarrow$ diverges

4. $\int_2^\infty\frac{1}{\sqrt[3]{(x+2)^2}}dx=\lim_{r\to\infty}\int_2^r(x+2)^{-\frac{2}{3}}dx$

$=\lim_{r\to\infty}\frac{(x+2)^{\frac{1}{3}}}{\frac{1}{3}}\Big|_2^r$

$=\lim_{r\to\infty}3\left[\sqrt[3]{r+2}-\sqrt[3]{(4)}\right]$

$=\infty\Rightarrow$ diverges

5. $\int_1^\infty e^{-x}dx=\lim_{r\to\infty}-\int_1^r e^{-x}[-dx]=\lim_{r\to\infty}(-e^{-x})\Big|_1^r$

$=\lim_{r\to\infty}\left(-e^{-r}+e^{-1}\right)=\lim_{r\to\infty}\left(-\frac{1}{e^r}+\frac{1}{e}\right)$

$=0+\frac{1}{e}=\frac{1}{e}$

6. $\int_0^\infty\left(5+e^{-x}\right)dx=\lim_{r\to\infty}\int_0^r\left(5+e^{-x}\right)dx$

$=\lim_{r\to\infty}\left(5x-e^{-x}\right)\Big|_0^r=\lim_{r\to\infty}\left[\left(5r-e^{-r}\right)-(0-1)\right]$

$=\lim_{r\to\infty}\left(5r-\frac{1}{e^r}+1\right)=\infty\Rightarrow$ diverges

7. $\int_1^\infty\frac{1}{\sqrt{x}}dx=\lim_{r\to\infty}\int_1^r x^{-\frac{1}{2}}dx=\lim_{r\to\infty}2x^{\frac{1}{2}}\Big|_1^r$

$=\lim_{r\to\infty}\left(2\sqrt{r}-2\right)=\infty\Rightarrow$ diverges

8. $\int_4^\infty \frac{x}{\sqrt{\left(x^2+9\right)^3}}dx = \lim_{r\to\infty} \frac{1}{2}\int_4^r \left(x^2+9\right)^{-\frac{3}{2}}[2x\ dx]$

$= \lim_{r\to\infty}\left[-\left(x^2+9\right)^{-\frac{1}{2}}\right]\Bigg|_4^r = \lim_{r\to\infty}\left[-\frac{1}{\sqrt{r^2+9}}+\frac{1}{5}\right]$

$= 0+\frac{1}{5}=\frac{1}{5}$

9. $\int_{-\infty}^{-3}\frac{1}{(x+1)^2}dx = \lim_{r\to-\infty}\int_r^{-3}(x+1)^{-2}dx$

$= \lim_{r\to-\infty} -\frac{1}{x+1}\Bigg|_r^{-3}$

$= \lim_{r\to-\infty}\left[\frac{1}{2}+\frac{1}{r+1}\right]$

$= \frac{1}{2}+0$

$= \frac{1}{2}$

10. $\int_{-\infty}^{3}\frac{1}{\sqrt{7-x}}dx = \lim_{r\to-\infty} -\int_r^3(7-x)^{-\frac{1}{2}}[-dx]$

$= \lim_{r\to-\infty} -2(7-x)^{\frac{1}{2}}\Bigg|_r^3$

$= \lim_{r\to-\infty}\left(-4+2\sqrt{7-r}\right)=\infty \Rightarrow$ diverges

11. $\int_{-\infty}^{\infty}2xe^{-x^2}dx = \int_{-\infty}^{0}2xe^{-x^2}dx+\int_0^\infty 2xe^{-x^2}dx$

$\int_{-\infty}^{0}2xe^{-x^2}dx = \lim_{r\to-\infty} -\int_r^0 e^{-x^2}[-2x\,dx]$

$= \lim_{r\to-\infty} -e^{-x^2}\Bigg|_r^0$

$= \lim_{r\to-\infty}\left[-1+\frac{1}{e^{r^2}}\right]=-1+0=-1$

$\int_0^\infty 2xe^{-x^2}dx = \lim_{r\to\infty} -\int_0^r e^{-x^2}[-2x\,dx]$

$= \lim_{r\to\infty} -e^{x^2}\Bigg|_0^r$

$= \lim_{r\to\infty}\left[-\frac{1}{e^{r^2}}+1\right]=0+1=1$

Thus $\int_{-\infty}^{\infty}2xe^{-x^2}dx = -1+1=0.$

12. $\int_{-\infty}^{\infty}(5-3x)dx = \int_{-\infty}^{0}(5-3x)dx+\int_0^\infty (5-3x)dx$

$\int_{-\infty}^{0}(5-3x)dx = \lim_{r\to-\infty}\int_r^0(5-3x)dx$

$= \lim_{r\to-\infty}\left(5x-\frac{3}{2}x^2\right)\Bigg|_r^0$

$= \lim_{r\to-\infty}\left[(0-0)-\left(5r-\frac{3}{2}r^2\right)\right]$

$= \lim_{r\to-\infty}\left(-5r+\frac{3}{2}r^2\right)=\infty$

Thus $\int_{-\infty}^{\infty}(5-3x)dx$ diverges

13. a. $\int_{800}^\infty \frac{k}{x^2}dx=1$

$\lim_{r\to\infty} k\int_{800}^r x^{-2}dx=1$

$\lim_{r\to\infty} -\frac{k}{x}\Bigg|_{800}^r=1$

$\lim_{r\to\infty}\left(-\frac{k}{r}+\frac{k}{800}\right)=1$

$0+\frac{k}{800}=1$

$k=800$

b. $\int_{1200}^\infty \frac{800}{x^2}dx = \lim_{r\to\infty} 800\int_{1200}^r x^{-2}dx$

$= \lim_{r\to\infty} -\frac{800}{x}\Bigg|_{1200}^r$

$= \lim_{r\to\infty}\left(-\frac{800}{r}+\frac{800}{1200}\right)=0+\frac{2}{3}=\frac{2}{3}$

14. $\int_1^\infty ke^{-2x}dx=1$

$\lim_{r\to\infty} -\frac{k}{2}\int_1^r e^{-2x}[-2\,dx]=1$

$\lim_{r\to\infty} -\frac{ke^{-2x}}{2}\Bigg|_1^r=1$

$\lim_{r\to\infty}\left(-\frac{k}{2e^{2r}}+\frac{k}{2e^2}\right)=1$

$0+\frac{k}{2e^2}=1$

Thus $k=2e^2$.

15. $\int_0^{\infty} 240{,}000e^{-0.06t}\,dt$

$= \lim_{r\to\infty} \frac{240{,}000}{-0.06}\int_0^r e^{-0.06t}[-0.06\,dt]$

$= \lim_{r\to\infty} -\frac{240{,}000}{0.06}e^{-0.06t}\Big|_0^r$

$= \lim_{r\to\infty} -\frac{240{,}000}{0.06}\left(\frac{1}{e^{0.06r}}-1\right)$

$= -\frac{240{,}000}{0.06}(-1) = 4{,}000{,}000$

16. $\alpha = \int_{x_c}^{\infty} e^{-x}dx = \lim_{r\to\infty} -\int_{x_c}^{r} e^{-x}[-dx]$

$= \lim_{r\to\infty} -e^{-x}\Big|_{x_c}^{r}$

$= \lim_{r\to\infty}\left(-\frac{1}{e^r}+e^{-x_c}\right) = 0+e^{-x_c} = e^{-x_c}$

$\beta = \int_{x_c}^{\infty}\frac{1}{8}e^{-\frac{x}{8}}dx = \lim_{r\to\infty} -\int_{x_c}^{r} e^{-\frac{x}{8}}\left[-\frac{1}{8}dx\right]$

$= \lim_{r\to\infty} -e^{-\frac{x}{8}}\Big|_{x_c}^{r}$

$= \lim_{r\to\infty}\left[-\frac{1}{e^{\frac{r}{8}}}+e^{-\left(\frac{1}{8}\right)x_c}\right]$

$= 0+e^{-\left(\frac{1}{8}\right)x_c}$

$= e^{-\left(\frac{1}{8}\right)x_c}$

17. Area $= -\int_{-\infty}^{0} -e^{3x}\,dx = \lim_{r\to-\infty}\frac{1}{3}\int_r^0 e^{3x}[3\,dx]$

$= \lim_{r\to-\infty}\frac{1}{3}\cdot e^{3x}\Big|_r^0 = \lim_{r\to-\infty}\left[\frac{1}{3}-\frac{1}{3}e^r\right]$

$= \frac{1}{3}-0 = \frac{1}{3}$ sq units

18. $V = \pi_0\int_0^{\infty} e^{\theta t}e^{-\rho t}dt = \pi_0\lim_{r\to\infty}\int_0^r e^{(\theta-\rho)t}dt$

$= \pi_0\lim_{r\to\infty}\frac{1}{\theta-\rho}\int_0^r e^{(\theta-\rho)t}[(\theta-\rho)dt]$

$= \lim_{r\to\infty}\frac{\pi_0}{\theta-\rho}e^{(\theta-\rho)t}\Big|_0^r = \lim_{r\to\infty}\frac{\pi_0}{\theta-\rho}\left[e^{(\theta-\rho)r}-1\right]$

$= \frac{\pi_0}{\theta-\rho}[0-1]$ (since $\theta-\rho<0$)

$= -\frac{\pi_0}{\theta-\rho}$

Thus $V = -\frac{\pi_0}{\theta-\rho} = \frac{\pi_0}{\rho-\theta}$.

19. $\int_0^{\infty}\frac{40{,}000}{(t+2)^2}dt = \lim_{r\to\infty}\int_0^r\frac{40{,}000}{(t+2)^2}dt$

$= \lim_{r\to\infty} -\frac{40{,}000}{t+2}\Big|_0^r$

$= \lim_{r\to\infty}\left[-\frac{40{,}000}{r+2}+\frac{40{,}000}{2}\right]$

$= 0+\frac{40{,}000}{2} = 20{,}000$ increase

Chapter 15 Review Problems

1. Use Formula 42 with $u = x$ and $n = 1$. Then $du = dx$.

$\int x\ln x\,dx = \frac{x^2\ln x}{2}-\frac{x^2}{4}+C$

2. Use Formula 27 with $u = 2x$, $a^2 = 1$. Then $du = 2\,dx$.

$\int\frac{1}{\sqrt{4x^2+1}}dx = \frac{1}{2}\int\frac{1}{\sqrt{(2x)^2+1}}(2\,dx)$

$= \frac{1}{2}\ln\left|2x+\sqrt{4x^2+1}\right|+C$

3. Use Formula 23 with $u = 3x$, $a^2 = 16$. Then $du = 3\ dx$.

$$\int_0^2 \sqrt{9x^2+16}\,dx = \frac{1}{3}\int_0^2 \sqrt{(3x)^2+16}(3\ dx)$$

$$= \frac{1}{3}\left[\frac{1}{2}\left((3x)\sqrt{9x^2+16}+16\ln\left|3x+\sqrt{9x^2+16}\right|\right)\right]\Bigg|_0^2$$

$$= \left(2\sqrt{13}+\frac{8}{3}\ln\left(6+2\sqrt{13}\right)\right)-\left(0+\frac{8}{3}\ln 4\right)$$

$$= 2\sqrt{13}+\frac{8}{3}\ln\left(\frac{6+2\sqrt{13}}{4}\right)$$

$$= 2\sqrt{13}+\frac{8}{3}\ln\left(\frac{3+\sqrt{13}}{2}\right)$$

4. By long division, $\int \frac{16x}{3-4x}dx = \int\left(-4+\frac{12}{3-4x}\right)dx = -4x-3\ln|3-4x|+C$

Or, by Formula 3 with $u = x$, $a = 3$, and $b = -4$. Then $du = dx$.

$$\int \frac{16x}{3-4x}dx = 16\int \frac{x}{3-4x}dx = 16\left[\frac{x}{-4}-\frac{3}{16}\ln|3-4x|\right]+C = -4x-3\ln|3-4x|+C$$

5. $\int \frac{15x-2}{(3x+1)(x-2)}dx = \int\left(\frac{15x}{(3x+1)(x-2)}-\frac{2}{(3x+1)(x-2)}\right)dx$

For $\int \frac{15x}{(3x+1)(x-2)}dx$, use Formula 12 with $u = x$, $a = 1$, $b = 3$, $c = -2$, and $k = 1$. Then $du = dx$.

$$\int \frac{15x}{(3x+1)(x-2)}dx = 15\int \frac{x}{(3x+1)(x-2)}dx = 15\left[\frac{1}{-7}\left(-2\ln|x-2|-\frac{1}{3}\ln|3x+1|\right)\right]+C$$

For $\int \frac{2}{(3x+1)(x-2)}dx$, use Formula 11 with

$u = x$, $a = 1$, $b = 3$, $c = -2$, and $k = 1$. Then $du = dx$.

$$\int \frac{2}{(3x+1)(x-2)}dx = 2\int \frac{dx}{(3x+1)(x-2)} = 2\left(\frac{1}{-7}\ln\left|\frac{3x+1}{x-2}\right|\right)+C$$

Thus, $\int \frac{15x-2}{(3x+1)(x-2)}dx$

$$= \frac{30}{7}\ln|x-2|+\frac{5}{7}\ln|3x+1|+\frac{2}{7}\ln\left|\frac{3x+1}{x-2}\right|+C$$

$$= \frac{30}{7}\ln|x-2|+\frac{5}{7}\ln|3x+1|+\frac{2}{7}\ln|3x+1|-\frac{2}{7}\ln|x-2|+C$$

$$= 4\ln|x-2|+\ln|3x+1|+C$$

6. The integral can be put in the form $\int \frac{1}{u}\,du$ with $u = \ln x$.

$$\int_e^{e^2} \frac{1}{x\ln x}dx = \int_e^{e^2} \frac{1}{\ln x}\left[\frac{1}{x}dx\right] = \ln|\ln x|\Big|_e^{e^2}$$

$$= \ln\left|\ln e^2\right| - \ln|\ln e| = \ln 2 - \ln 1 = \ln(2) - 0 = \ln 2$$

7. Use Formula 9 with $u = x$, $a = 2$, and $b = 1$. Then $du = dx$.

$$\int \frac{dx}{x(x+2)^2} = \frac{1}{2(x+2)} + \frac{1}{4}\ln\left|\frac{x}{x+2}\right| + C$$

8. Use Formula 35 with $u = x$ and $a = 1$. Then $du = dx$.

$$\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

9. Use Formula 21 with $u = 4x$ and $a^2 = 9$. Then $du = 4\,dx$.

$$\int \frac{dx}{x^2\sqrt{9-16x^2}} = 4\int \frac{(4\,dx)}{(4x)^2\sqrt{9-(4x)^2}}$$

$$= 4\left[-\frac{\sqrt{9-16x^2}}{9(4x)}\right] + C$$

$$= -\frac{\sqrt{9-16x^2}}{9x} + C$$

10. Use Formula 42 with $u = x^2$ and $n = 1$. Then $du = 2x\,dx$.

$$\int x^3 \ln x^2 dx = \frac{1}{2}\int (x^2)\ln(x^2)[2x\,dx]$$

$$= \frac{1}{2}\left(\frac{(x^2)^2\ln x^2}{2} - \frac{(x^2)^2}{2^2}\right) + C$$

$$= \frac{1}{4}x^4\ln x^2 - \frac{1}{8}x^4 + C$$

11. Use Formula 35 with $u = x$ and $a = 3$. Then $du = dx$.

$$\int \frac{9\,dx}{x^2-9} = 9\int \frac{dx}{x^2-9} = 9\left[\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|\right] + C$$

$$= \frac{3}{2}\ln\left|\frac{x-3}{x+3}\right| + C$$

12. Use Formula 15 with $u = x$, $a = 2$, and $b = 5$. Then $du = dx$.

$$\int \frac{x}{\sqrt{2+5x}}dx = \frac{2(5x-4)\sqrt{2+5x}}{75} + C$$

13. Use Formula 38 with $u = x$ and $a = 7$. Then $du = dx$.

$$\int 49xe^{7x}\,dx = 49\int xe^{7x}\,du$$

$$= 49\left[\frac{e^{7x}}{49}(7x-1)\right] + C = e^{7x}(7x-1) + C$$

14. Use Formula 45 with $u = x$, $a = 2$, $b = 3$, and $c = 4$. Then $du = dx$.

$$\int \frac{dx}{2+3e^{4x}} = \frac{1}{8}\left[4x - \ln\left(2+3e^{4x}\right)\right] + C$$

15. The integral has the form $\int \frac{1}{u}du$.

$$\int \frac{dx}{2x\ln x^2} = \frac{1}{4}\int \frac{1}{\ln x^2}\left[\frac{2}{x}dx\right] = \frac{1}{4}\ln\left|\ln x^2\right| + C$$

16. Use Formula 5 with $u = x$, $a = 2$, and $b = 1$. Then $du = dx$.

$$\int \frac{dx}{x(2+x)} = \frac{1}{2}\ln\left|\frac{x}{2+x}\right| + C$$

17. Long division or Formula 3. For long division,

$$\int \frac{2x}{3+2x}dx = \int\left[1 - \frac{3}{3+2x}\right]dx$$

$$= x - 3\cdot\frac{1}{2}\int \frac{1}{3+2x}[2\,dx]$$

$$= x - \frac{3}{2}\ln|3+2x| + C.$$

For Formula 3, use $u = x$, $a = 3$, and $b = 2$. Then $du = dx$.

$$\int \frac{2x}{3+2x}dx = 2\int \frac{x}{3+2x}dx$$

$$= 2\left(\frac{x}{2} - \frac{3}{4}\ln|3+2x|\right) + C = x - \frac{3}{2}\ln|3+2x| + C$$

18. Use Formula 30 with $u = 2x$ and $a^2 = 9$. Then $du = 2\,dx$.

$$\int \frac{dx}{x^2\sqrt{4x^2-9}} = 2\int \frac{(2\,dx)}{(2x)^2\sqrt{(2x)^2-9}}$$

$$= 2\left(-\frac{-\sqrt{4x^2-9}}{9(2x)}\right) + C = \frac{\sqrt{4x^2-9}}{9} + C$$

19. Partial fractions

$$\frac{5x^2+2}{x^3+x} = \frac{5x^2+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$5x^2+2 = A(x^2+1)+(Bx+C)x$$
$$= (A+B)x^2 + Cx + A$$

Thus, $A + B = 5$, $C = 0$, $A = 2$. This gives $A = 2$, $B = 3$, $C = 0$.

$$\int \frac{5x^2+2}{x^3+x}dx = \int \left[\frac{2}{x} + \frac{3x}{x^2+1}\right]dx$$
$$= 2\ln|x| + \frac{3}{2}\ln\left(x^2+1\right) + C$$

20. Partial fractions

$$\frac{3x^3+5x^2+4x+3}{x^4+x^3+x^2} = \frac{3x^3+5x^2+4x+3}{x^2(x^2+x+1)}$$
$$= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+x+1}$$

$$3x^3+5x^2+4x+3$$
$$= Ax(x^2+x+1)+B(x^2+x+1)+(Cx+D)x^2$$
$$= (A+C)x^3+(A+B+D)x^2+(A+B)x+B$$

Thus, $A + C = 3$, $A + B + D = 5$, $A + B = 4$, $B = 3$.
This gives $A = 1$, $B = 3$, $C = 2$, $D = 1$.

$$\int \frac{3x^3+5x^2+4x+3}{x^4+x^3+x^2}dx$$
$$= \int \left[\frac{1}{x} + \frac{3}{x^2} + \frac{2x+1}{x^2+x+1}\right]dx$$
$$= \ln|x| - \frac{3}{x} + \ln\left(x^2+x+1\right) + C$$

21. Integration by parts

$u = \ln(x + 1)$

$$dv = (x+1)^{-\frac{1}{2}}dx$$

Then $du = \frac{1}{x+1}dx$ and $v = 2(x+1)^{\frac{1}{2}}$.

$$\int \frac{\ln(x+1)}{\sqrt{x+1}}dx$$
$$= 2(x+1)^{\frac{1}{2}}\ln(x+1) - 2\int (x+1)^{-\frac{1}{2}}dx$$
$$= 2(x+1)^{\frac{1}{2}}\ln(x+1) - 4(x+1)^{\frac{1}{2}} + C$$
$$= 2\sqrt{x+1}[\ln(x+1)-2]+C$$

22. Integration by parts

$u = x^2$

$dv = e^x dx$

Then $du = 2x\,dx$ and $v = e^x$.

$$\int x^2e^x dx = x^2e^x - \int 2xe^x dx$$

For $\int 2xe^x dx$, use integration by parts again.

$u = 2x$

$dv = e^x dx$

Then $du = 2\,dx$ and $v = e^x$.

$$\int 2xe^x dx = 2xe^x - \int 2e^x dx = 2xe^x - 2e^x + C$$
$$\int x^2e^x dx = x^2e^x - (2xe^x - 2e^x) + C$$
$$= e^x(x^2 - 2x + 2) + C$$

23. $\overline{f} = \frac{1}{4-2}\int_2^4 \left(3x^2+2x\right)dx = \frac{1}{2}\left(x^3+x^2\right)\Big|_2^4$

$$= \frac{1}{2}[(64+16)-(8+4)] = 34$$

24. $\overline{f} = \frac{1}{1-0}\int_0^1 t^2e^t\,dt$

For $\int t^2e^t\,dt$, use Formula 39 with $u = t$, $n = 2$, and $a = 1$. Then $du = dt$.

$$\int t^2e^t\,dt = t^2e^t - 2\int te^t\,dt$$

For $\int te^t\,dt$, use Formula 38 with $u = t$ and $a = 1$. Then $du = dt$.

$$\int t^2e^t\,dt = t^2e^t - 2[e^t(t-1)] + C$$
$$= e^t(t^2-2t+2)+C$$

Thus,

$$\overline{f} = \int_0^1 t^2e^t\,dt = e^t(t^2-2t+2)\Big|_0^1 = e(1) - 1(2)$$
$$= e - 2.$$

25. $y' = 3x^2y + 2xy$, $y > 0$

$$\frac{dy}{y} = \left(3x^2+2x\right)dx$$
$$\int \frac{dy}{y} = \int \left(3x^2+2x\right)dx$$

$\ln y = x^3 + x^2 + C_1$, from which $y = e^{x^3+x^2+C_1}$,

$y = Ce^{x^3+x^2}$, where $C > 0$.

26. $y' - 2xe^{x^2-y+3} = 0$, $y(0) = 3$

$\frac{dy}{dx} = 2xe^{x^2+3}e^{-y}$

$e^y dy = 2xe^{x^2+3}dx$

$\int e^y dy = \int 2xe^{x^2+3}dx$

$e^y = e^{x^2+3} + C$

$y(0) = 3$ implies $e^3 = e^3 + C$, $C = 0$. Thus $e^y = e^{x^2+3}$, or $y = x^2 + 3$.

27. $\int_1^\infty \frac{1}{x^{2.5}}dx = \lim_{r\to\infty}\int_1^r x^{-2.5}dx$

$= \lim_{r\to\infty} \frac{x^{-1.5}}{-1.5}\bigg|_1^r$

$= \lim_{r\to\infty} -\frac{2}{3x^{1.5}}\bigg|_1^r$

$= \lim_{r\to\infty}\left(-\frac{2}{3r^{1.5}} + \frac{2}{3}\right)$

$= 0 + \frac{2}{3}$

$= \frac{2}{3}$

28. $\int_{-\infty}^0 e^{2x}\,dx = \lim_{r\to-\infty}\int_r^0 e^{2x}\,dx = \lim_{r\to-\infty}\frac{e^{2x}}{2}\bigg|_r^0$

$= \lim_{r\to-\infty}\left[\frac{1}{2} - \frac{1}{2}e^{2r}\right] = \frac{1}{2} - 0 = \frac{1}{2}$

29. $\int_1^\infty \frac{1}{2x}dx = \lim_{r\to\infty}\int_1^r \frac{1}{2x}dx = \lim_{r\to\infty}\frac{1}{2}\ln|x|\bigg|_1^r$

$= \lim_{r\to\infty}\left[\frac{1}{2}\ln|r| - 0\right] = \infty \Rightarrow$ diverges

30. $\int_{-\infty}^\infty xe^{1-x^2}dx = \int_{-\infty}^0 xe^{1-x^2}dx + \int_0^\infty xe^{1-x^2}dx$

$\int_{-\infty}^0 xe^{1-x^2}dx = \lim_{r\to-\infty} -\frac{1}{2}\int_r^0 e^{1-x^2}[-2x\ dx]$

$= \lim_{r\to-\infty} -\frac{1}{2}e^{1-x^2}\bigg|_r^0$

$= \lim_{r\to-\infty}\left[-\frac{1}{2}e + \frac{1}{2}e^{1-r^2}\right] = -\frac{1}{2}e - 0 = -\frac{1}{2}e$

$\int_0^\infty xe^{1-x^2}dx = \lim_{r\to\infty} -\frac{1}{2}\int_0^r e^{1-x^2}[-2x\ dx]$

$= \lim_{r\to\infty} -\frac{1}{2}e^{1-x^2}\bigg|_0^r$

$= \lim_{r\to\infty}\left[-\frac{1}{2}e^{1-r^2} + \frac{1}{2}e\right] = 0 + \frac{1}{2}e = \frac{1}{2}e$

Thus $\int_{-\infty}^\infty xe^{1-x^2} = -\frac{1}{2}e + \frac{1}{2}e = 0$

31. $N = N_0e^{kt}$

Since $N = 100{,}000$ when $t = 0$ (1985), $N_0 = 100{,}000$. Thus $N = 100{,}000e^{kt}$.

Since $N = 120{,}000$ when $t = 15$, then

$120{,}000 = 100{,}000e^{15k}$

$1.2 = e^{15k}$

$\ln 1.2 = 15k$, or $k = \frac{\ln 1.2}{15}$. Thus

$N = 100{,}000e^{t\frac{\ln 1.2}{15}} = 100{,}000\left(e^{\ln 1.2}\right)^{\frac{t}{15}}$

$= 100{,}000(1.2)^{\frac{t}{15}}$

For the year 2015 we have $t = 30$ and

$N = 100{,}000(1.2)^{\frac{30}{15}} = 100{,}000(1.2)^2 = 144{,}000$

32. $N = N_0e^{kt}$

When $t = 0$, then $N = 40{,}000$. Thus $N_0 = 40{,}000$ and $N = 40{,}000e^{kt}$. When $t = 10$, then $N = 80{,}000$, so

$80{,}000 = 40{,}000e^{10k}$

$2 = e^{10k}$

$10\,k = \ln 2$, or $k = \frac{\ln 2}{10}$. Thus $N = 40{,}000e^{\frac{t\ln 2}{10}}$.

33. $N = N_0e^{-\lambda t}$, where N_0 is the original amount present. When $t = 100$, then $N = 0.95N_0$, so we have

$0.95N_0 = N_0e^{-100\lambda}$

$0.95 = e^{-100\lambda}$

$-100\lambda = \ln 0.95$

$\lambda = -\frac{\ln 0.95}{100} \approx 0.0005$ (decay constant). After 200 years, $N = N_0e^{-200\lambda}$. Thus

$\frac{N}{N_0} = e^{-200\lambda} = e^{-200\left[-\frac{\ln 0.95}{100}\right]} = e^{2\ln 0.95}$

$\approx 0.90 = 90\%$

34. $\frac{dq}{dt} = -kq$

$\frac{dq}{q} = -k\,dt$

$\int \frac{dq}{q} = \int -k\,dt$

$\ln q = -kt + C$

When $t = 0$, $q = q_0$, so $\ln q_0 = 0 + C = C$. Thus

$\ln q = -kt + \ln q_0$

$q = e^{-kt}e^{\ln q_0} = q_0e^{-kt}$

When $t = \frac{7}{k}$, $\frac{q}{q_0} = e^{-7} \approx 0.09\%$.

35. $N = \frac{450}{1+be^{-ct}}$

If $t = 0$, then $N = 2$. Thus $2 = \frac{450}{1+b}$,

$1+b = \frac{450}{2} = 225$, $b = 224$, so $N = \frac{450}{1+224e^{-ct}}$.

If $t = 6$, then

$N = 300 \Rightarrow 300 = \frac{450}{1+224e^{-6c}}$

$1+224e^{-6c} = \frac{450}{300} = \frac{3}{2}$

$224e^{-6c} = \frac{3}{2} - 1 = \frac{1}{2}$

$e^{-6c} = \frac{1}{448}$

$e^{6c} = 448$

$6c = \ln 448$

$c = \frac{\ln 448}{6} \approx 1.02$

Thus $N \approx \frac{450}{1+224e^{-1.02t}}$.

36. $N = \frac{2000}{1+be^{-ct}}$

When $t = 0$ (last year), then $N = 1000$. Thus

$1000 = \frac{2000}{1+b}$

$1+b = \frac{2000}{1000} = 2$

$b = 1$

So $N = \frac{2000}{1+e^{-ct}}$. When $t = 1$ then $N = 1100$. Thus

$1100 = \frac{2000}{1+e^{-c}}$

$1+e^{-c} = \frac{2000}{1100} = \frac{20}{11}$

$e^{-c} = \frac{9}{11}$

Hence $N = \frac{2000}{1+\left(\frac{9}{11}\right)^t}$. When $t = 2$, then

$N = \frac{2000}{1+\left(\frac{9}{11}\right)^2} \approx 1200$.

37. $\frac{dT}{dt} = k(T-25)$

$\frac{dT}{T-25} = k\ dt$

$\int \frac{dT}{T-25} = \int k\ dt$

$\ln(T-25) = kt + C$

If $t = 0$, then $T = 35$. Thus ln 10 = C, so $\ln(T - 25) = kt + \ln 10$, or $\ln\left(\frac{T-25}{10}\right) = kt$. If $t = 1$, then $T = 34$ and $\ln\left(\frac{9}{10}\right) = k$. Thus $\ln\left(\frac{T-25}{10}\right) = (\ln 0.9)t$. If

$T = 37$,

$\ln\frac{12}{10} = (\ln 0.9)t$

$\ln 1.2 = (\ln 0.9)t$,

$t = \frac{\ln 1.2}{\ln 0.9} \approx -1.73$

Note that 1.73 hr corresponds approximately to 1 hr 44 min. Thus

6:00 P.M. – 1 hr 44 min = 4:16 P.M.

38. Use Formula 38 with $u = t$, and $a = -0.06$, so $du = dt$.

$$\int_0^{12} 10te^{-0.06t}\,dt = 10\left[\frac{e^{-0.06t}}{0.0036}(-0.06t-1)\right]\Bigg|_0^{12}$$

$$= \frac{10}{0.0036}[e^{-0.72}(-0.72-1)-(-1)]$$

$$\approx \$452$$

39. $\int_0^{\infty} f(x)\,dx$

$$= \lim_{r\to\infty}\int_0^r (0.007e^{-0.01x} + 0.00005e^{-0.0002x})dx$$

$$= \lim_{r\to\infty}(-0.7e^{-0.01x} - 0.25e^{-0.0002x})\Big|_0^r$$

$$= \lim_{r\to\infty}\left[-\frac{0.7}{e^{0.01r}} - \frac{0.25}{e^{-0.0002r}} - (-0.7-0.25)\right]$$

$$= 0 - 0 + 0.7 + 0.25$$

$$= 0.95$$

40. $\int_{-\infty}^{t_1} A_0e^{kt}\,dt = \lim_{r\to-\infty}\int_r^{t_1} A_0e^{kt}\,dt$

$$= \lim_{r\to-\infty} A_0\cdot\frac{1}{k}\int_r^{t_1} e^{kt}(k\ dt)$$

$$= \lim_{r\to-\infty}\frac{A_0e^{kt}}{k}\Bigg|_r^{t_1} = \lim_{r\to-\infty}\frac{A_0}{k}\left(e^{kt_1} - e^{kr}\right)$$

$= \frac{A_0}{k}e^{kt_1}$ since $e^{kr} \to 0$ as $r \to -\infty$ for $k > 0$.

$$\int_{t_1}^{t_2} A_0e^{kt}\,dt = \frac{A_0e^{kt}}{k}\Bigg|_{t_1}^{t_2} = \frac{A_0}{k}\left(e^{kt_2} - e^{kt_1}\right)$$

$$= \frac{A_0}{k}e^{kt_1}\left(e^{kt_2}e^{-kt_1} - 1\right)$$

$$= \frac{A_0}{k}e^{kt_1}\left[e^{k(t_2-t_1)} - 1\right]. \qquad (1)$$

If $A_0e^{kt_2} = 2A_0e^{kt_1}$, then $e^{kt_2} = 2e^{kt_1}$, $2 = \frac{e^{kt_2}}{e^{kt_1}} = e^{k(t_2-t_1)}$. Substituting into (1) gives

$$\frac{A_0}{k}e^{kt_1}[2-1] = \frac{A_0}{k}e^{kt_1}.$$

41. a. Total revenue $= r(12) - r(0) = \int_0^{12}\frac{dr}{dq}dq$.

$f(q) = \frac{dr}{dq}$

$n = 4$, $a = 0$, $b = 12$

$h = \frac{b-a}{n} = \frac{12-0}{4} = 3$

Trapezoidal

$f(0) =$	25
$2f(3) =$	44
$2f(6) =$	36
$2f(9) =$	26
$f(12) =$	7
	138

$TR \approx \frac{3}{2}(138) = 207$

Simpson's

$f(0) =$	25
$4f(3) =$	88
$2f(6) =$	36
$4f(9) =$	52
$f(12) =$	7
	208

$TR \approx \frac{3}{3}(208) = 208$

b. Total variable cost $c(12) - c(0) = \int_0^{12}\frac{dc}{dq}dq$

$f(q) = \frac{dc}{dq}$

$a = 0$, $b = 12$

Using as few data values as possible, we choose $n = 1$ for Trapezoidal and $n = 2$ for Simpson's (n must be even).

Trapezoidal ($n = 1$)

$h = \frac{b-a}{n} = \frac{12-0}{1} = 12$

$f(0) = 15$

$f(12) = \underline{\ \ 7}$

$\quad 22$

$VC \approx \frac{12}{2}(22) = 132$

Simpson's (n = 2)

$h = \frac{b-a}{n} = \frac{12-0}{2} = 6$

$f(0) = 15$

$4f(6) = 48$

$f(12) = \underline{\ \ 7}$

$\quad 70$

$VC \approx \frac{6}{3}(70) = 140$

To each of our results we must add on the fixed cost of 25 to obtain total cost. Thus for trapezoidal we get $TC \approx 132 + 25 = 157$, and for Simpson's we have $TC \approx 140 + 25 = 165$.

c. We use the relation $P(12) = \int_0^{12}\left[\frac{dr}{dq} - \frac{dc}{dq}\right]dq - 25$. First we determine variable cost for each rule with $n = 4$ and $h = \frac{b-a}{n} = \frac{12-0}{4} = 3$.

Trapezoidal

$f(0) = 15$

$2f(3) = 28$

$2f(6) = 24$

$2f(9) = 20$

$f(12) = \underline{\ \ 7}$

$\quad 94$

$VC \approx \frac{3}{2}(94) = 141$

Simpson's

$f(0) = 15$

$4f(3) = 56$

$2f(6) = 24$

$4f(12) = 40$

$f(12) = \underline{\ \ \ 7}$

$\quad 142$

$VC \approx \frac{3}{3}(142) = 142$

Using these results and those of part (a), we have:

Trapezoidal

$P(12) \approx 207 - 141 - 25 = 41$

Simpson's

$P(12) \approx 208 - 142 - 25 = 41$

Mathematical Snapshot Chapter 15

1. $C = 2000,\ w_0 = 200$

$w_{eq} = \frac{C}{17.5} = \frac{2000}{17.5} \approx 114$

$w(t) = \frac{C}{17.5} + \left(w_0 - \frac{C}{17.5}\right)e^{-0.005t}$

$= \frac{2000}{17.5} + \left(200 - \frac{2000}{17.5}\right)e^{-0.005t}$

Letting $w(t) = 175$ and solving for t gives

$175 = \frac{2000}{17.5} + \left(200 - \frac{2000}{17.5}\right)e^{-0.005t}$

$175 - \frac{2000}{17.5} = \left(200 - \frac{2000}{17.5}\right)e^{-0.005t}$

$\frac{175 - \frac{2000}{17.5}}{200 - \frac{2000}{17.5}} = e^{-0.005t}$

$-0.005t = \ln\left[\frac{175 - \frac{2000}{17.5}}{200 - \frac{2000}{17.5}}\right]$

$t = \frac{\ln\left[\frac{175-\frac{2000}{17.5}}{200-\frac{2000}{17.5}}\right]}{-0.005} \approx 69$

Thus $w_{eq} = 114$ and $t = 69$ days.

2. $\frac{dw}{dt} = \frac{1}{3500}(C - 17.5w)$

$\frac{dw}{C - 17.5w} = \frac{1}{3500}dt$

$\int \frac{dw}{C - 17.5w} = \int \frac{1}{3500}dt$

$-\frac{1}{17.5}\ln|C - 17.5w| = \frac{1}{3500}t + C_1$

$\ln|C-17.5w| = -\frac{17.5}{3500}t - 17.5C_1 = -0.005t + C_2$

$|C-17.5w| = e^{-0.005t+C_2} = e^{C_2}e^{-0.005t} = C_3e^{-0.005t}$

Thus $C-17.5w = C_4e^{-0.005t}$, where C_4 is a constant and $C_4 = \pm C_3$. When $t = 0$, then $w = w_0$, so $C-17.5w_0 = C_4$. Thus $C-17.5w = (C-17.5w_0)e^{-0.005t}$

$-17.5w = -C + (C-17.5w_0)e^{-0.005t}$

$w = \frac{C}{17.5} + \left(-\frac{C}{17.5} + \frac{17.5}{17.5}w_0\right)e^{-0.005t}$

$w = \frac{C}{17.5} + \left(w_0 - \frac{C}{17.5}\right)e^{-0.005t}$

which is Equation 2.

3. $w(t) = \frac{C}{17.5} + \left(w_0 - \frac{C}{17.5}\right)e^{-0.005t}$

Since $\frac{C}{17.5} = w_{eq}$, we have $w(t) = w_{eq} + (w_0 - w_{eq})e^{-0.005t}$. Simplifying the equation $w(t+d) = w(t) - \frac{1}{2}\left[w(t) - w_{eq}\right]$ gives $w(t+d) = \frac{1}{2}\left[w(t) + w_{eq}\right]$. Thus

$w_{eq} + (w_0 - w_{eq})e^{-0.005(t+d)} = \frac{1}{2}\left[w_{eq} + (w_0 - w_{eq})e^{-0.005t} + w_{eq}\right]$, or

$w_{eq} + (w_0 - w_{eq})e^{-0.005(t+d)} = w_{eq} + \frac{1}{2}(w_0 - w_{eq})e^{-0.005t}$

Solving for d gives

$$e^{-0.005(t+d)} = \frac{1}{2}e^{-0.005t}$$

$$e^{-0.005t}e^{-0.005d} = \frac{1}{2}e^{-0.005t}$$

$$e^{-0.005d} = \frac{1}{2}$$

$$-0.005d = \ln\frac{1}{2} = -\ln 2$$

$$d = \frac{\ln 2}{0.005}$$

as was to be shown.

4. BMI = $\frac{w}{h^2}$, so w = BMI · h^2 with w in kilograms and h in meters. 5 feet, 8 inches equals 68 inches, or 1.7272 meters. The upper BMI limit then corresponds to a weight of $24.9(1.7272)^2 \approx 74.28$ kilograms, or about 163 pounds. So the woman would need to lose 27 pounds. On a 2200 calorie-per-day diet,

$w_{eq} = \frac{2200}{17.5} \approx 125.71$ lb and the weight function is

$w(t) = 125.71 + (190 - 125.71)e^{-0.005t} = 125.71 + 64.29e^{-0.005t}$.

The solution of the equation $163 = 125.71 + 64.29e^{-0.005t}$ is $t \approx 109$. It would take about 109 days.

5. Answers may vary.

Chapter 16

Principles in Practice 16.1

1. The uniform density function is given by
$$f(x)=\begin{cases}\frac{1}{60}, & \text{if } 0\le x\le 60\\ 0, & \text{otherwise.}\end{cases}$$
The probability of waiting between 25 and 45 minutes is
$$P(25\le X\le 45)=\int_{25}^{45}\frac{1}{60}dx=\left.\frac{x}{60}\right|_{25}^{45}$$
$$=\frac{45-25}{60}=\frac{20}{60}=\frac{1}{3}.$$

2. The exponential density function is given by $f(x)=\begin{cases}\frac{1}{10}e^{-\frac{x}{10}}, & \text{if } x\ge 0\\ 0, & \text{if } x<0.\end{cases}$

The probability that the break pads will break down after the warranty period is
$P(5 < X) = 1 - P(0 \le X \le 5)$
$$=1-\int_0^5\frac{1}{10}e^{-\frac{x}{10}}dx=1-\left(\left.-e^{-\frac{x}{10}}\right|_0^5\right)$$
$$=1-\left(-e^{-\frac{5}{10}}+e^0\right)=1+e^{-\frac{1}{2}}-1=e^{-\frac{1}{2}}\approx 0.607$$

3. The exponential density function is given by
$$f(x)=\begin{cases}0.2e^{-0.2x}, & \text{for } x\ge 0\\ 0, & \text{otherwise.}\end{cases}$$
The mean is given by $\mu=\frac{1}{k}=\frac{1}{0.2}=5.$

The standard deviation is given by
$\sigma=\frac{1}{k}=\frac{1}{0.2}=5.$

Problems 16.1

1. a. $P(1<X<2)=\int_1^2\frac{1}{6}(x+1)dx=\left.\frac{(x+1)^2}{12}\right|_1^2$
$$=\frac{9}{12}-\frac{4}{12}=\frac{5}{12}$$

b. $P(X<2.5)=\int_1^{2.5}\frac{1}{6}(x+1)dx=\left.\frac{(x+1)^2}{12}\right|_1^{2.5}$
$$=\frac{49}{48}-\frac{4}{12}=\frac{11}{16}=0.6875$$

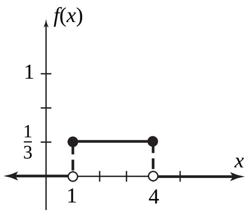
c. $P\left(X\ge\frac{3}{2}\right)=\int_{3/2}^3\frac{1}{6}(x+1)dx=\left.\frac{(x+1)^2}{12}\right|_{3/2}^3$
$$=\frac{16}{12}-\frac{25}{48}=\frac{13}{16}=0.8125$$

d. $\int_1^c\frac{1}{6}(x+1)dx=\frac{1}{2}$
$$\left.\frac{(x+1)^2}{12}\right|_1^c=\frac{1}{2}$$
$$\frac{(c+1)^2}{12}-\frac{1}{3}=\frac{1}{2}$$
$$(c+1)^2-4=6$$
$$(c+1)^2=10$$
$$c+1=\pm\sqrt{10}$$
$$c=-1\pm\sqrt{10}$$
We choose $c=-1+\sqrt{10}$ since $1<c<3$.

2. a. $P(3000<X<4000)=\int_{3000}^{4000}\frac{1000}{x^2}dx$
$$=-\left.\frac{1000}{x}\right|_{3000}^{4000}=-\frac{1}{4}-\left(-\frac{1}{3}\right)=\frac{1}{12}$$

b. $P(X>2000)=\int_{2000}^{\infty}\frac{1000}{x^2}dx$
$$=\lim_{r\to\infty}\int_{2000}^{r}\frac{1000}{x^2}dx$$
$$=\lim_{r\to\infty}-\left.\frac{1000}{x}\right|_{2000}^{r}=\lim_{r\to\infty}\left(-\frac{1000}{r}+\frac{1}{2}\right)=\frac{1}{2}$$

3. a. $f(x)=\begin{cases}\frac{1}{3}, & \text{if } 1\le x\le 4\\ 0, & \text{otherwise}\end{cases}$



b. $P\left(\frac{3}{2}<X<\frac{7}{2}\right)=\frac{\frac{7}{2}-\frac{3}{2}}{4-1}=\frac{2}{3}$

c. $P(0<X<1)=\int_0^1 0\,dx=0$

d. $P(X\le 3.5)=P(1\le X\le 3.5)$
$=\frac{3.5-1}{4-1}=\frac{2.5}{3}=\frac{5}{6}$

e. $P(X>3)=P(3<X\le 4)=\frac{4-3}{4-1}=\frac{1}{3}$

f. $P(X=2)=0$

g. $P(X<5)=P(1\le X\le 4)=\frac{4-1}{4-1}=1$

h. $\mu=\int_1^4 x\left(\frac{1}{3}\right)dx=\left.\frac{x^2}{6}\right|_1^4=\frac{16}{6}-\frac{1}{6}=\frac{5}{2}$

i. $\sigma^2=\int_1^4 x^2\left(\frac{1}{3}\right)dx-\mu^2$
$=\left.\frac{x^3}{9}\right|_1^4-\left(\frac{5}{2}\right)^2$
$=\left[\frac{64}{9}-\frac{1}{9}\right]-\frac{25}{4}$
$=7-\frac{25}{4}$
$=\frac{3}{4}$

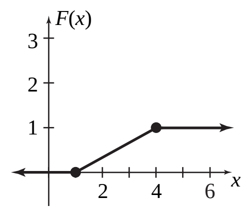
Thus, $\sigma=\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}.$

j. If $1\le x\le 4$, $F(x)=\int_1^x \frac{1}{3}dt=\left.\frac{t}{3}\right|_1^x=\frac{x-1}{3}.$

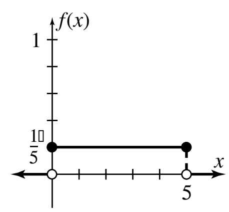
Thus $F(x)=\begin{cases}0, & \text{if } x<1\\ \frac{x-1}{3}, & \text{if } 1\le x\le 4\\ 1, & \text{if } x>4.\end{cases}$

$P(X<2)=F(2)=\frac{2-1}{3}=\frac{1}{3}$

$P(1<X<3)=F(3)-F(1)=\frac{2}{3}-0=\frac{2}{3}$



4. a. $f(x)=\begin{cases}\frac{1}{5}, & \text{if } 0\le x\le 5\\ 0, & \text{otherwise}\end{cases}$



b. $\frac{3-1}{5-0}=\frac{2}{5}$

c. $\frac{5-4.5}{5-0}=\frac{0.5}{5}=\frac{1}{10}$

d. $P(X=4)=0$

e. $P(X>2)=P(2<X\le 5)=\frac{5-2}{5-0}=\frac{3}{5}$

f. $P(X<5)=P(0\le X\le 5)=\frac{5-0}{5-0}=1$

g. $P(X>5)=\int_5^\infty 0dx=0$

h. $\mu=\int_0^5 x\left(\frac{1}{5}\right)dx=\left.\frac{x^2}{10}\right|_0^5=\frac{25}{10}-0=\frac{5}{2}$

i. $\sigma^2 = \int_0^5 x^2\left(\frac{1}{5}\right)dx - \mu^2 = \left.\frac{x^3}{15}\right|_0^5 - \left(\frac{5}{2}\right)^2$

$= \left[\frac{125}{15} - 0\right] - \frac{25}{4} = \frac{25}{3} - \frac{25}{4} = \frac{25}{12}$

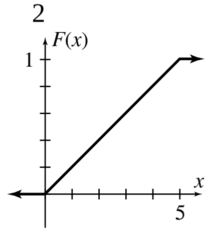
Thus $\sigma = \sqrt{\frac{25}{12}} = \frac{5}{\sqrt{12}} = \frac{5\sqrt{3}}{6}$.

j. If $0 \le x \le 5$, $F(x) = \int_0^x \frac{1}{5}dt = \left.\frac{t}{5}\right|_0^x = \frac{x}{5}$

Thus $F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{5}, & \text{if } 0 \le x \le 5 \\ 1, & \text{if } x > 5 \end{cases}$

$P(1 < X < 3.5) = F(3.5) - F(1) = \frac{3.5}{5} - \frac{1}{5}$

$= \frac{1}{2}$



5. a. $f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$

b. $\mu = \int_a^b x\left(\frac{1}{b-a}\right)dx = \left.\frac{x^2}{2(b-a)}\right|_a^b$

$= \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$

c. $\sigma^2 = \int_a^b x^2\left(\frac{1}{b-a}\right)dx - \mu^2$

$= \left.\frac{x^3}{3(b-a)}\right|_a^b - \left(\frac{a+b}{2}\right)^2$

$= \frac{b^3 - a^3}{3(b-a)} - \frac{(a+b)^2}{4}$

$= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4}$

$= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}$

Thus $\sigma = \frac{b-a}{\sqrt{12}}$.

6. a. $\int_a^b k\,dx = 1$

$kx\Big|_a^b = 1$

$k(b-a) = 1$

$k = \frac{1}{b-a}$

Thus X is uniformly distributed.

b. If $a \le x \le b$

$F(x) = \int_a^x \frac{1}{b-a}dt = \left.\frac{1}{b-a}t\right|_a^x = \frac{x-a}{b-a}$

Thus $F(x) = \begin{cases} 0, & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases}$

7. $f(x) = \begin{cases} 3e^{-3x}, & \text{if } x \ge 0 \\ 0, & \text{if } x < 0 \end{cases}$

a. $P(1 < X < 4) = \int_1^4 3e^{-3x}dx$

$= -e^{-3x}\Big|_1^4$

$= -e^{-12} + e^{-3}$

≈ 0.04978

b. $P(X < 4) = \int_0^4 3e^{-3x}dx$

$= -e^{-3x}\Big|_0^4$

$= -e^{-12} + 1$

≈ 0.99999

c. $P(X > 6) = 1 - P(X \le 6)$
$= 1 - \int_0^6 3e^{-3x}\,dx$
$= 1 - (-e^{-18} + 1)$
$= e^{-18} \approx 0.00000$

d. From the text, $\mu = \sigma = \frac{1}{k} = \frac{1}{3}$.

$P(\mu - 2\sigma < X < \mu + 2\sigma) = P\left(-\frac{1}{3} < X < 1\right)$
$= \int_0^1 3e^{-3x}\,dx$
$= -e^{-3x}\Big|_0^1$
$= -e^{-3} + 1$
≈ 0.95021

e. $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} 3e^{-3x}\,dx$
$= \lim_{r\to\infty} \int_0^r 3e^{-3x}\,dx$
$= \lim_{r\to\infty} (-e^{-3x})\Big|_0^r$
$= \lim_{r\to\infty} (-e^{-3r} + 1)$
$= 0 + 1 = 1$

f. $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$
If $x \ge 0$,
$F(x) = \int_0^x 3e^{-3t}\,dt = -e^{-3t}\Big|_0^x = -e^{-3x} + 1.$
Thus $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ -e^{-3x} + 1 & \text{if } x \ge 0 \end{cases}$

8. $f(x) = \begin{cases} 0.5e^{-0.5x}, & \text{if } x \ge 0 \\ 0, & \text{if } x < 0 \end{cases}$

a. $P(X > 4) = 1 - P(X \le 4) = 1 - \int_0^4 0.5e^{-0.5x}\,dx$
$= 1 - \left(-e^{-0.5x}\right)\Big|_0^4 = 1 - \left(-e^{-2} + 1\right) = e^{-2} \approx 0.135$

b. $P(0.5 < X < 2.6) = \int_{0.5}^{2.6} 0.5e^{-0.5x}\,dx$
$= -e^{-0.5x}\Big|_{0.5}^{2.6} = -e^{-1.3} + e^{-0.25} \approx 0.506$

c. $P(X < 5) = \int_0^5 0.5e^{-0.5x}\,dx = -e^{-0.5x}\Big|_0^5$
$= -e^{-2.5} + 1 \approx 0.918$

d. $P(X = 4) = 0$

e. $P(0 < X < c) = \frac{1}{2}$
$\int_0^c 0.5e^{-0.5x}\,dx = \frac{1}{2}$
$-e^{-0.5x}\Big|_0^c = \frac{1}{2}$
$-e^{-0.5c} + 1 = \frac{1}{2}$
$e^{-0.5c} = \frac{1}{2}$
$-0.5c = \ln\frac{1}{2}$
$-0.5c = -\ln 2$
$c = 2\ln 2$

9. a. $\int_0^4 kx\,dx = 1$
$\frac{kx^2}{2}\Big|_0^4 = 1$
$8k = 1$
$k = \frac{1}{8}$

b. $P(2 < X < 3) = \int_2^3 \frac{x}{8}\,dx = \frac{x^2}{16}\Big|_2^3 = \frac{9}{16} - \frac{4}{16}$
$= \frac{5}{16}$

c. $P(X > 2.5) = \int_{2.5}^4 \frac{x}{8}\,dx = \frac{x^2}{16}\Big|_{2.5}^4$
$= 1 - \frac{25}{64} = \frac{39}{64} \approx 0.609$

d. $P(X > 0) = P(0 \le X \le 4) = 1$

e. $\mu = \int_0^4 x\left(\frac{x}{8}\right)dx = \frac{x^3}{24}\Big|_0^4 = \frac{64}{24} - 0 = \frac{8}{3}$

f. $\sigma^2 = \int_0^4 x^2\left(\frac{x}{8}\right)dx - \mu^2 = \left.\frac{x^4}{32}\right|_0^4 - \left(\frac{8}{3}\right)^2$

$= 8 - \frac{64}{9} = \frac{8}{9}$

Thus $\sigma = \sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$

g. $P(X < c) = \frac{1}{2}$

$\int_0^c \frac{x}{8}dx = \frac{1}{2}$

$\left.\frac{x^2}{16}\right|_0^c = \frac{1}{2}$

$\frac{c^2}{16} = \frac{1}{2}$

$c^2 = 8$

$c = \pm 2\sqrt{2}$

We choose $c = 2\sqrt{2}$ since $0 < c < 4$.

h. $P(3 < X < 5) = P(3 < X < 4) = \int_3^4 \frac{x}{8}dx$

$= \left.\frac{x^2}{16}\right|_3^4 = \frac{16}{16} - \frac{9}{16} = \frac{7}{16}$

10. a. $P(2 \le X \le 4) = 1$

$\int_2^4 \left(\frac{x}{2} + k\right)dx = 1$

$\left.\left(\frac{x^2}{4} + kx\right)\right|_2^4 = 1$

$(4 + 4k) - (1 + 2k) = 1$

$2k = -2$

$k = -1$

b. $P(X \ge 2.5) = P(2.5 \le X \le 4)$

$= \int_{2.5}^4 \left(\frac{x}{2} - 1\right)dx = \left.\left(\frac{x^2}{4} - x\right)\right|_{2.5}^4$

$= \left[(4-4) - \left(\frac{25}{16} - \frac{5}{2}\right)\right] = \frac{15}{16} = 0.9375$

c. $\mu = \int_2^4 x\left(\frac{x}{2} - 1\right)dx = \int_2^4 \left(\frac{x^2}{2} - x\right)dx$

$= \left.\left(\frac{x^3}{6} - \frac{x^2}{2}\right)\right|_2^4$

$= \left[\left(\frac{32}{3} - 8\right) - \left(\frac{4}{3} - 2\right)\right] = \frac{10}{3}$

d. $P(2 < X < \mu) = \int_2^{10/3} \left(\frac{x}{2} - 1\right)dx$

$= \left.\left(\frac{x^2}{4} - x\right)\right|_2^{10/3}$

$= \left[\left(\frac{25}{9} - \frac{10}{3}\right) - (1 - 2)\right] = \frac{4}{9}$

11. $P(X \le 7) = \int_0^7 \frac{1}{10}dx = \left.\frac{x}{10}\right|_0^7 = \frac{7}{10}$

$E(X) = \int_0^{10} x\left(\frac{1}{10}\right)dx = \left.\frac{x^2}{20}\right|_0^{10} = 5 \text{ min}$

12. $P(X < 12) = \frac{12 - 11.93}{12.07 - 11.93} = \frac{0.07}{0.14} = \frac{1}{2}$

$P(X = 12) = 0$

$E(X) = \int_{11.93}^{12.07} x\left(\frac{1}{12.07 - 11.93}\right)dx$

$= \int_{11.93}^{12.07} \frac{x}{0.14}dx$

$= \left.\frac{x^2}{0.28}\right|_{11.93}^{12.07} = 12 \text{ oz}$

13. $P(X > 1) = 1 - P(X \le 1)$

$= 1 - \int_0^1 3e^{-3x}dx = 1 - \left.\left(-e^{-3x}\right)\right|_0^1$

$= 1 - \left(-e^{-3} + 1\right) \approx 0.050$

14. $P(X \le 3) = \int_0^3 \frac{2}{5} e^{-\frac{2}{5}x}\, dx = -e^{-\frac{2}{5}x}\Big|_0^3$

$= -e^{-\frac{6}{5}} + 1 \approx 0.699$

$P(X > 5) = 1 - P(X \le 5) = 1 - \int_0^5 \frac{2}{5} e^{-\frac{2}{5}x}\, dx$

$= 1 - \left(-e^{-\frac{2}{5}x}\right)\Big|_0^5 = 1 - (-e^{-2} + 1)$

≈ 0.135

Problems 16.2

1. a. $P(0 < Z < 1.7) = A(1.7) = 0.4554$

b. $P(0.43 < Z < 2.89) = A(2.89) - A(0.43)$
$= 0.4981 - 0.1664$
$= 0.3317$

c. $P(Z > -1.23) = 0.5 + A(1.23) = 0.5 + 0.3907$
$= 0.8907$

d. $P(Z \le 2.91) = 0.5 + A(2.91) = 0.5 + 0.4982$
$= 0.9982$

e. $P(-2.51 < Z \le 1.3) = A(2.51) + A(1.3)$
$= 0.4940 + 0.4032$
$= 0.8972$

f. $P(Z > 0.03) = 0.5 - A(0.03) = 0.5 - 0.0120$
$= 0.4880$

2. a. $P(-1.96 < Z < 1.96) = 2A(1.96)$
$= 2(0.4750)$
$= 0.9500$

b. $P(-2.11 < Z < -1.35) = A(2.11) - A(1.35)$
$= 0.4826 - 0.4115$
$= 0.0711$

c. $P(Z < -1.05) = 0.5 - A(1.05)$
$= 0.5 - 0.3531$
$= 0.1469$

d. $P(Z > 3\sigma) = P(Z > 3) = 0.5 - A(3)$
$= 0.5 - 0.4987$
$= 0.0013$

e. $P(|Z| > 2) = P(Z < -2) + P(Z > 2)$
$= 2[0.5 - A(2)]$
$= 2[0.5 - 0.4772]$
$= 0.0456$

f. $P\left(|Z| < \frac{1}{2}\right) = P\left(-\frac{1}{2} < Z < \frac{1}{2}\right) = 2A\left(\frac{1}{2}\right)$
$= 2(0.1915)$
$= 0.3830$

3. $P(Z < z_0) = 0.5517$
$0.5 + A(z_0) = 0.5517$
$A(z_0) = 0.0517$
$z_0 = 0.13$

4. $P(Z < z_0) = 0.0668$
$0.5 - A(-z_0) = 0.0668$
$A(-z_0) = 0.4332$
$-z_0 = 1.5$
$z_0 = -1.5$

5. $P(Z > z_0) = 0.8599$
$0.5 + A(-z_0) = 0.8599$
$A(-z_0) = 0.3599$
$-z_0 = 1.08$
$z_0 = -1.08$

6. $P(Z > z_0) = 0.4129$
$0.5 - A(z_0) = 0.4129$
$A(z_0) = 0.0871$
$z_0 = 0.22$

7. $P(-z_0 < Z < z_0) = 0.2662$
$2A(z_0) = 0.2662$
$A(z_0) = 0.1331$
$z_0 = 0.34$

8. $P(|Z| > z_0) = 0.3174$

$P(Z > z_0) = \frac{0.3174}{2} = 0.1587$

$0.5 - A(z_0) = 0.1587$
$A(z_0) = 0.3413$
$z_0 = 1.00$

9. a. $P(X<27)=P\left(Z<\frac{27-16}{4}\right)$
$=P(Z<2.75)=0.5+A(2.75)$
$=0.5+0.4970=0.9970$

b. $P(X<10)=P\left(Z<\frac{10-16}{4}\right)$
$=P(Z<-1.5)=0.5-A(1.5)$
$=0.5-0.4332=0.0668$

c. $P(10.8<X<12.4)$
$=P\left(\frac{10.8-16}{4}<Z<\frac{12.4-16}{4}\right)$
$=P(-1.3<Z<-0.9)=A(1.3)-A(0.9)$
$=0.4032-0.3159=0.0873$

10. a. $P(X>150)=P\left(Z>\frac{150-200}{40}\right)$
$=P(Z>-1.25)=0.5+A(1.25)$
$=0.5+0.3944=0.8944$

b. $P(210<X<250)$
$=P\left(\frac{210-200}{40}<Z<\frac{250-200}{40}\right)$
$=P(0.25<Z<1.25)=A(1.25)-A(0.25)$
$=0.3944-0.0987=0.2957$

11. $P(X>-2)=P\left(Z>\frac{-2-(-3)}{2}\right)$
$=P\left(Z>\frac{1}{2}\right)=0.5-A\left(\frac{1}{2}\right)$
$=0.5-0.1915=0.3085$

12. $P(X<3)=P\left(Z<\frac{3-0}{1.5}\right)=P(Z<2)$
$=0.5+A(2)=0.5+0.4772=0.9772$

13. Since $\sigma^2=100$, $\sigma=10$. Thus
$35=65-30=\mu-3\sigma$
$95=65+30=\mu+3\sigma$
Thus $P(35<X\le 95)=P(\mu-3\sigma<X\le\mu+3\sigma)$
$=0.997$ (99.7%)

14. $P(X>\mu-\sigma)=P(X>7)=P\left(Z>\frac{7-8}{1}\right)$
$=P(Z>-1)=0.5+A(1)$
$=0.5+0.3413=0.8413$

15. $P(X>54)=P(Z>z_0)=0.0401$
$0.5-A(z_0)=0.0401$
$A(z_0)=0.4599$
$z_0=1.75$
Since $\frac{54-40}{\sigma}=1.75$
$\frac{14}{\sigma}=1.75$
so $\sigma=\frac{14}{1.75}=8$.

16. Case 1. Suppose $x_0>16$. Then
$P(16<X<x_0)=P\left(0<Z<\frac{x_0-16}{2.25}\right)$
$=A\left(\frac{x_0-16}{2.25}\right)=0.4641$. Thus
$\frac{x_0-16}{2.25}=1.8$, so $x_0=20.05$.

Case 2. Suppose $x_0<16$. Then
$P(x_0<X<16)=P\left(\frac{x_0-16}{2.25}<Z<0\right)$
$=A\left(-\frac{x_0-16}{2.25}\right)=0.4641$. Thus
$-\frac{x_0-16}{2.25}=1.8$, so $x_0=11.95$.

Therefore, x_0 can be either 11.95 or 20.05.

17. Let X be score on test. Then the probability that X lies within $2\sigma=2(100)=200$ points of 500 is 0.95. Thus, 95% of those who took the test had scores between 300 and 700.

18. Let X be score on test and let x_0 be least score a person could get and yet score in about the top 20 percent. Then
$P(X\ge x_0)=P\left(Z\ge\frac{x_0-65}{10}\right)=0.20.$ Thus
$0.5-A\left(\frac{x_0-65}{10}\right)=0.20$ or $A\left(\frac{x_0-65}{10}\right)=0.30.$
Hence $\frac{x_0-65}{10}\approx 0.84.$
so $x_0\approx 73.4\approx 74.$

19. Let X be height of an adult. Then
$P(X<72)=P\left(Z<\frac{72-68}{3}\right)=P(Z<1.33)$
$= 0.5 + A(1.33) = 0.5 + 0.4082 = 0.9082$
90.82% are over 6 feet.

20. Let X be the yearly income (in dollars) of a person in the group.

a. $P(X<46{,}000)=P\left(Z<\frac{46{,}000-60{,}000}{5000}\right)$
$=P(Z<-2.8)$
$=0.5-A(2.8)$
$=0.5-0.4974$
$=0.0026$
0.0026(10,000) = 26 people have yearly incomes less than $46,000.

b. $P(X>75{,}000)=P\left(\frac{75{,}000-60{,}000}{5000}\right)$
$=P(Z>3)$
$=0.5-A(3)$
$=0.5-0.4987$
$=0.0013$
0.0013(10,000) = 13 people have yearly incomes over $75,000.

21. Let X be IQ of a child in population.

a. $P(X>125)=P\left(Z>\frac{125-100.4}{11.6}\right)$
$= P(Z > 2.12) = 0.5 - A(2.12)$
$= 0.5 - 0.4830 = 0.0170.$
Thus 1.7% of the children have IQ's greater than 125.

b. If x_0 is the value, then $P(X>x_0)=0.90$.

Thus $P\left(Z>\frac{x_0-100.4}{11.6}\right)=0.90$ or

$0.5+A\left(-\frac{x_0-100.4}{11.6}\right)=0.90$. Hence

$A\left(-\frac{x_0-100.4}{11.6}\right)=0.4$, so

$-\frac{x_0-100.4}{11.6}=1.28$ or $x_0=85.552\approx 85.6$.

22. Since $P(4<X<16)=P(\mu-3\sigma<X<\mu+3\sigma)$ $= 0.25 \neq 0.997$, X cannot be normally distributed.

Principles in Practice 16.3

1. X is the number of winners and X is binomial with $n = 60$ and $p=\frac{1}{4}$. To find $P(X = 20)$, use the normal approximation to the binomial distribution with $\mu=np=60\left(\frac{1}{4}\right)=15$

and $\sigma=\sqrt{npq}=\sqrt{60\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)}=\sqrt{\frac{45}{4}}\approx 3.35$.

Converting the correct X-values 19.5 and 20.5 to Z-values gives

$z_1=\frac{19.5-15}{\sqrt{\frac{45}{4}}}\approx 1.34$

$z_2=\frac{20.5-15}{\sqrt{\frac{45}{4}}}\approx 1.64$

Thus $P(X = 20) \approx P(1.34 \le Z \le 1.64)$
$= A(1.64) - A(1.34) = 0.4495 - 0.4099 = 0.0396$
The probability of 20 winners out of 60 contestants is 0.0396.

Problems 16.3

1. $n = 150$, $p = 0.4$, $q = 0.6$,
$\mu = np = 150(0.4) = 60$,
$\sigma=\sqrt{npq}=\sqrt{150(0.4)(0.6)}=\sqrt{36}=6$
$P(X\le 52)=P(X<52.5)$
$z=\frac{52.5-60}{6}=-1.25$
$P(X \le 52) = P(X \le 52.5)$
$\approx P(Z \le -1.25) = 0.5 - A(1.25)$
$= 0.5 - 0.3944 = 0.1056$
$P(X\ge 74)=P(X\ge 73.5)\approx P\left(Z\ge\frac{73.5-60}{6}\right)$
$= P(Z \ge 2.25)$
$= 0.5 - A(2.25)$
$= 0.5 - 0.4878 = 0.0122$

2. $n = 50$, $p = 0.3$, $q = 0.7$, $\mu = np = 50(0.3) = 15$,
$\sigma = \sqrt{npq} = \sqrt{50(0.3)(0.7)} = \sqrt{10.5} \approx 3.24$

$$P(X = 19) = P(18.5 \le X \le 19.5) \approx P\left(\frac{18.5-15}{\sqrt{10.5}} \le Z \le \frac{19.5-15}{\sqrt{10.5}}\right) = P(1.08 \le Z \le 1.39) = A(1.39) - A(1.08) = 0.4177 - 0.3599 = 0.0578$$

$$P(X \le 18) = P(X \le 18.5) \approx P\left(Z \le \frac{18.5-15}{\sqrt{10.5}}\right) = P(Z \le 1.08) = 0.5 + A(1.08) = 0.5 + 0.3599 = 0.8599$$

3. $n = 200$, $p = 0.6$, $q = 0.4$,
$\mu = np = 200(0.6) = 120$
$\sigma = \sqrt{npq} = \sqrt{200(0.6)(0.4)} = \sqrt{48} \approx 6.93$

$$P(X = 125) = P(124.5 \le X \le 125.5) \approx P\left(\frac{124.5-120}{\sqrt{48}} \le Z \le \frac{125.5-120}{\sqrt{48}}\right) = P(0.65 \le Z \le 0.79) = A(0.79) - A(0.65) = 0.2852 - 0.2422 = 0.0430$$

$$P(110 \le X \le 135) = P(109.5 \le X \le 135.5) \approx P\left(\frac{109.5-120}{\sqrt{48}} \le Z \le \frac{135.5-120}{\sqrt{48}}\right) = P(-1.52 \le Z \le 2.24) = A(1.52) + A(2.24) = 0.4357 + 0.4875 = 0.9232$$

4. $n = 25$, $p = 0.25$, $q = 0.75$,
$\mu = np = 25(0.25) = 6.25$,
$\sigma = \sqrt{npq} = \sqrt{25(0.25)(0.75)} = \sqrt{4.6875} \approx 2.17$

$$P(X \ge 7) = P(X \ge 6.5) \approx P\left(Z \ge \frac{6.5-6.25}{\sqrt{4.6875}}\right) = P(Z \ge 0.12) = 0.5 - A(0.12) = 0.5 - 0.0478 = 0.4522$$

5. Let X = no. of times 5 occurs. Then X is binomial with $n = 300$, $p = \frac{1}{6}$, $q = \frac{5}{6}$,
$\mu = np = 50$, $\sigma = \sqrt{npq} = \sqrt{\frac{125}{3}} \approx 6.45$.
$P(45 \le X \le 60) = P(44.5 \le X \le 60.5)$

$$\approx P\left(\frac{44.5-50}{\sqrt{\frac{125}{3}}} \le Z \le \frac{60.5-50}{\sqrt{\frac{125}{3}}}\right)$$

$= P(-0.85 \le Z \le 1.63) = A(0.85) + A(1.63)$
$= 0.3023 + 0.4484 = 0.7507$

6. Let X = no. of heads that occurs. Then X is binomial with $n = 200$, $p = 0.4$, $q = 0.6$,
$\mu = np = 80$, $\sigma = \sqrt{npq} = \sqrt{48} \approx 6.93$.
$P(90 \le X \le 100) = P(89.5 \le X \le 100.5)$

$$\approx P\left(\frac{89.5-80}{\sqrt{48}} \le Z \le \frac{100.5-80}{\sqrt{48}}\right)$$

$= P(1.37 \le Z \le 2.96) = A(2.96) - A(1.37)$
$= 0.4985 - 0.4147 = 0.0838$

7. Let X = no. of trucks out of service. Then X can be considered binomial with $n = 60$, $p = 0.1$,
$q = 0.9$, $\mu = np = 6$, $\sigma = \sqrt{npq} = \sqrt{5.4} \approx 2.32$

$$P(X \ge 7) = P(X \ge 6.5) \approx P\left(Z \ge \frac{6.5-6}{\sqrt{5.4}}\right)$$

$= P(Z \ge 0.22) = 0.5 - A(0.22)$
$= 0.5 - 0.0871 = 0.4129$

8. Let X = no. of defective items in sample. Then X is binomial with $n = 200$, $p = 0.05$, $q = 0.95$,
$\mu = 200(0.05) = 10$, $\sigma = \sqrt{npq} = \sqrt{9.5} \approx 3.08$
$P(X \ge 7) = P(X \ge 6.5)$

$$z = \frac{6.5-10}{\sqrt{9.5}} \approx -1.14$$

$P(X \ge 7) = P(X \ge 6.5) \approx P(Z \ge -1.14)$
$= 0.5 + A(1.14) = 0.5 + 0.3729 = 0.8729$

9. Let X = no. of correct answers. Then X is binomial and $p = 0.5$, $q = 0.5$. If $n = 25$, then $\mu = np = 25(0.5) = 12.5$,
$\sigma = \sqrt{npq} = \sqrt{25(0.5)(0.5)} = \sqrt{6.25} = 2.5$ and

$$\begin{aligned} P(X \geq 13) &= P(X \geq 12.5) \\ &\approx P\left(Z \geq \frac{12.5-12.5}{2.5}\right) \\ &= P(Z \geq 0.00) \\ &= 0.5 - A(0.00) \\ &= 0.5 - 0 \\ &= 0.5 \end{aligned}$$

If $n = 100$, then $\mu = 100(0.5) = 50$,
$\sigma = \sqrt{npq} = \sqrt{100(0.5)(0.5)} = \sqrt{25} = 5$

$$\begin{aligned} P(X \geq 60) &= P(X \geq 59.5) \\ &= P\left(Z \geq \frac{59.5-50}{5}\right) \\ &= P(Z \geq 1.9) \\ &= 0.5 - A(1.9) \\ &= 0.5 - 0.4713 \\ &= 0.0287 \end{aligned}$$

10. Let X = no. of correct answers on last 20 questions. Then X is binomial with $n = 20$, $p = 0.25$, $q = 0.75$, $\mu = np = 20(0.25) = 5$,
$\sigma = \sqrt{npq} = \sqrt{20(0.25)(0.75} = \sqrt{3.75} \approx 1.94$

$$\begin{aligned} P(X \geq 10) &= P(X \geq 9.5) \\ &\approx P\left(Z \geq \frac{9.5-5}{\sqrt{3.75}}\right) \\ &= P(Z \geq 2.32) \\ &= 0.5 - A(2.32) \\ &= 0.5 - 0.4898 \\ &= 0.0102 \end{aligned}$$

11. Let X = no. of deals consisting of three cards of one suit and two cards of another suit. Then X is binomial with $n = 100$, $p = 0.1$, $q = 0.9$,
$\mu = np = 10$, $\sigma = \sqrt{npq} = \sqrt{9} = 3$.

$$\begin{aligned} P(X \geq 16) &= P(X \geq 15.5) \\ &\approx P\left(Z \geq \frac{15.5-10}{3}\right) \\ &= P(Z \geq 1.83) \\ &= 0.5 - A(1.83) \\ &= 0.5 - 0.4664 \\ &= 0.0336 \end{aligned}$$

12. Let X = the number of subjects that choose the cola from the sponsoring company. Then X is binomial with $n = 35$, $p = 0.5$, $q = 0.5$,
$\mu = np = 17.5$, $\sigma = \sqrt{npq} = \sqrt{8.75} \approx 2.96$

$$\begin{aligned} P(X \geq 25) &= P(X \geq 24.5) \\ &= P\left(Z \geq \frac{24.5-17.5}{\sqrt{8.75}}\right) \\ &= P(Z \geq 2.37) \\ &= 0.5 - A(2.37) = 0.5 - 0.4911 \\ &= 0.0089 \end{aligned}$$

Chapter 16 Review Problems

1. a.

$$\begin{aligned} P(0 \leq X \leq 1) &= 1 \\ \int_0^1 \left(\frac{1}{3} + kx^2\right) dx &= 1 \\ \left.\left(\frac{x}{3} + \frac{kx^3}{3}\right)\right|_0^1 &= 1 \\ \frac{1}{3} + \frac{k}{3} &= 1 \\ \frac{k}{3} &= \frac{2}{3} \\ k &= 2 \end{aligned}$$

b. $P\left(\frac{1}{2} < X < \frac{3}{4}\right) = \int_{1/2}^{3/4} \left(\frac{1}{3} + 2x^2\right) dx$

$$= \left.\left(\frac{x}{3} + \frac{2x^3}{3}\right)\right|_{1/2}^{3/4} = \left.\frac{1}{3}\left(x + 2x^3\right)\right|_{1/2}^{3/4}$$

$$= \frac{1}{3}\left[\left(\frac{3}{4} + \frac{27}{32}\right) - \left(\frac{1}{2} + \frac{1}{4}\right)\right] = \frac{9}{32}$$

c. $P\left(X \geq \frac{1}{2}\right) = \int_{1/2}^{1} \left(\frac{1}{3} + 2x^2\right) dx$

$$= \left.\left(\frac{x}{3} + \frac{2x^3}{3}\right)\right|_{1/2}^{1} = \left.\frac{1}{3}\left(x + 2x^3\right)\right|_{1/2}^{1}$$

$$= \frac{1}{3}\left[(1+2) - \left(\frac{1}{2} + \frac{1}{4}\right)\right] = \frac{3}{4}$$

d. If $0 \le x \le 1$, $F(x) = \int_0^x \left(\frac{1}{3} + 2t^2\right) dt$

$$= \left(\frac{t}{3} + \frac{2t^3}{3}\right)\Bigg|_0^x = \frac{x}{3} + \frac{2x^3}{3}$$

Therefore, $F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{3} + \frac{2x^3}{3}, & \text{if } 0 \le x \le 1 \\ 1, & \text{if } x > 1 \end{cases}$

2. $f(x) = \begin{cases} \frac{1}{3}e^{-(1/3)x}, & \text{if } x \ge 0 \\ 0, & \text{if } x < 0 \end{cases}$

$$P(X > 2) = 1 - P(X \le 2) = 1 - \int_0^2 \frac{1}{3} e^{-(1/3)x} dx$$

$$= 1 - \left[-e^{-(1/3)x}\Big|_0^2\right] = 1 - \left[-e^{-2/3} + 1\right]$$

$$= e^{-2/3} \approx 0.513$$

3. a. $\mu = \int_0^5 x\left(\frac{2}{25}x\right) dx = \frac{2x^3}{75}\Bigg|_0^5 = \frac{10}{3}$

b. $\sigma^2 = \int_0^5 x^2 \left(\frac{2}{25}x\right) dx - \mu^2$

$$= \frac{x^4}{50}\Bigg|_0^5 - \left(\frac{10}{3}\right)^2$$

$$= \frac{625}{50} - \frac{100}{9} = \frac{25}{18}$$

Thus $\sigma = \sqrt{\frac{25}{18}} \approx 1.18.$

4. $P(X < 5) = \int_2^5 \frac{1}{6-2} dx = \int_2^5 \frac{1}{4} dx = \frac{x}{4}\Bigg|_2^5$

$$= \frac{5}{4} - \frac{2}{4} = \frac{3}{4}$$

5. $P(X > 22) = P\left(Z > \frac{22-20}{4}\right)$

$= P(Z > 0.5) = 0.5 - A(0.5)$
$= 0.5 - 0.1915 = 0.3085$

6. $P(X < 21) = P\left(Z < \frac{21-20}{4}\right)$

$= P(Z < 0.25) = 0.5 + A(0.25)$
$= 0.5 + 0.0987 = 0.5987$

7. $P(14 < X < 18) = P\left(\frac{14-20}{4} < Z < \frac{18-20}{4}\right)$

$= P(-1.5 < Z < -0.5)$
$= A(1.5) - A(0.5) = 0.4332 - 0.1915 = 0.2417$

8. $P(X > 10) = P\left(Z > \frac{10-20}{4}\right)$

$= P(Z > -2.5) = 0.5 + A(2.5)$
$= 0.5 + 0.4938 = 0.9938$

9. $P(X < 23) = P\left(Z < \frac{23-20}{4}\right)$

$= P(Z < 0.75) = 0.5 + A(0.75)$
$= 0.5 + 0.2734 = 0.7734$

10. $P(23 < X < 33) = P\left(\frac{23-20}{4} < Z < \frac{33-20}{4}\right)$

$= P(0.75 < Z < 3.25)$
$= A(3.25) - A(0.75)$
$= 0.4994 - 0.2734$
$= 0.2260$

11. $n = 100$, $p = 0.35$, $q = 0.65$, $\mu = np = 35$,
$\sigma = \sqrt{npq} = \sqrt{22.75} \approx 4.77$
$P(25 \le X \le 47) = P(24.5 \le X \le 47.5)$

$$\approx P\left(\frac{24.5-35}{\sqrt{22.75}} \le Z \le \frac{47.5-35}{\sqrt{22.75}}\right)$$

$= P(-2.20 \le Z \le 2.62) = A(2.20) + A(2.62)$
$= 0.4861 + 0.4956 = 0.9817$

12. $n = 100$, $p = 0.35$, $q = 0.65$, $\mu = np = 35$,
$\sigma = \sqrt{npq} = \sqrt{22.75} \approx 4.77$
$P(X = 48) = P(47.5 \le X \le 48.5)$

$$\approx P\left(\frac{47.5-35}{\sqrt{22.75}} \le Z \le \frac{48.5-35}{\sqrt{22.75}}\right)$$

$= P(2.62 \le Z \le 2.83) = A(2.83) - A(2.62)$
$= 0.4977 - 0.4956 = 0.0021$

13. Let X = height of an individual. X is normally distributed with $\mu = 68$ and $\sigma = 2$.

$P(X > 72) = P\left(Z > \frac{72-68}{2}\right)$

$= P(Z > 2) = 0.5 - A(2)$
$= 0.5 - 0.4772 = 0.0228$

14. Let X = number of heads that occurs. X is binomial with $n = 500$, $p = 0.5$, $q = 0.5$, $\mu = np = 250$, $\sigma = \sqrt{npq} = \sqrt{125} \approx 11.18$.

$$P(X \geq 215) = P(X \geq 214.5)$$
$$\approx P\left(Z \geq \frac{214.5 - 250}{\sqrt{125}}\right)$$
$$= P(Z \geq -3.18) = 0.5 + A(3.18)$$
$$= 0.5 + 0.4993 = 0.9993$$

Mathematical Snapshot Chapter 16

1. The result should correspond to the known distribution function.

2. The derivative of the logistic function should roughly coincide with the normal probability density function used to generate the values.

3. The list of earthquake magnitudes will appear to have a normal density function. This is surprising, since one would expect something more like an exponential density function, with most earthquakes being very low-magnitude. Presumably, the explanation for the normal density function is that as magnitude declines, the likelihood of a quake's being reported and thus appearing on the list also goes down.

Chapter 17

Principles in Practice 17.1

1. **a.** $c(500, 700) = 160 + 2(500) + 3(700)$
$= 160 + 1000 + 2100 = 3260$
The cost of manufacturing 500 12-ounce and 700 20-ounce mugs is $3260.

b. $c(1000, 750) = 160 + 2(1000) + 3(750)$
$= 160 + 2000 + 2250 = 4410$
The cost of manufacturing 1000 12-ounce mugs and 750 20-ounce mugs is $4410.

Problems 17.1

1. $f(1, 2) = 4(1) - (2)^2 + 3 = 4 - 4 + 3 = 3$

2. $f(2, -1) = 3(2)^2(-1) - 4(-1) = -12 + 4 = -8$

3. $g(0, 3, -1) = e^{2\cdot 0}[3(3) + (-1)] = e^0(8) = 8$

4. $g(3, 1, -2)$
$= (3)^2(1)(-2) + 3(1)^2(-2) + 3(1)(-2)^2$
$= -18 - 6 + 12 = -12$

5. $h(-3, 3, 5, 4) = \dfrac{-3(3)}{5^2 - 4^2} = \dfrac{-9}{25-16} = \dfrac{-9}{9} = -1$

6. $h(1, 5, 3, 1) = \ln(1 \cdot 1) = \ln 1 = 0$

7. $g(4, 8) = 2(4)\left(4^2 - 5\right) = 2(4)(11) = 88$

8. $g(8, 4) = 8\sqrt{4} + 10 = 8(2) + 10 = 16 + 10 = 26$

9. $F(2, 0, -1) = 3$

10. $F(1, 0, 3) = \dfrac{2(1)}{(0+1)(3)} = \dfrac{2}{3}$

11. $f(x_0 + h, y_0) = e^{(x_0+h)+y_0} = e^{x_0+h+y_0}$

12. $f(r+t, r) = (r+t)^2 r - 3r^3 = r\left(t^2 + 2rt - 2r^2\right)$

13. $f(400, 400, 80) = \dfrac{400(400)}{80} = 2000$

14. $P(3, 4) = \dfrac{4!\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^{4-3}}{3!(4-3)!} = \dfrac{4!\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)}{3!1!} = \dfrac{3}{64}$

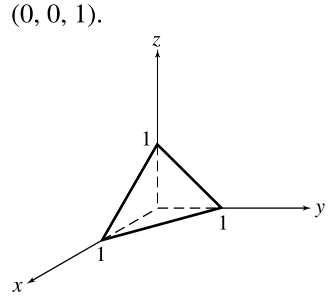
15. A plane parallel to the x,z-plane has the form y = constant. Because (0, 2, 0) lies on the plane, the equation is $y = 2$.

16. A plane parallel to the y,z-plane has the form x = constant. Because (–2, 0, 0) lies on the plane, the equation is $x = -2$.

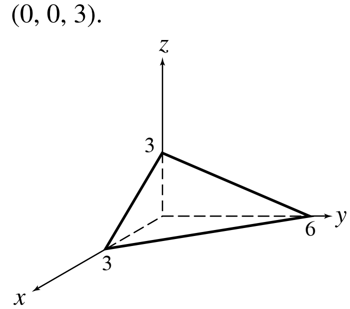
17. A plane parallel to the x,y-plane has the form z = constant. Because (2, 7, 6) lies on the plane, the equation is $z = 6$.

18. A plane parallel to the y,z-plane has the form x = constant. Because (–4, –2, 7) lies on the plane, the equation is $x = -4$.

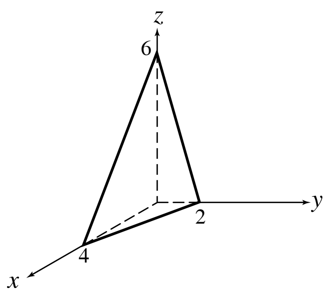
19. $x + y + z = 1$ can be put in the form $Ax + By + Cz + D = 0$, so the graph is a plane. The intercepts are (1, 0, 0), (0, 1, 0), and (0, 0, 1).



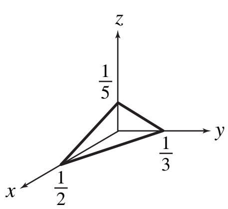
20. $2x + y + 2z = 6$ can be put in the form $Ax + By + Cz + D = 0$, so the graph is a plane. The intercepts are (3, 0, 0), (0, 6, 0), and (0, 0, 3).



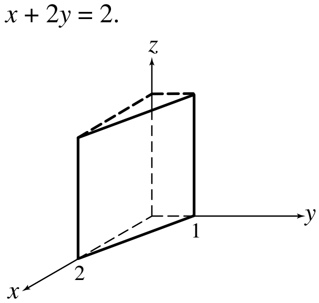
21. $3x + 6y + 2z = 12$ can be put in the form $Ax + By + Cz + D = 0$, so the graph is a plane. The intercepts are (4, 0, 0), (0, 2, 0), and (0, 0, 6).



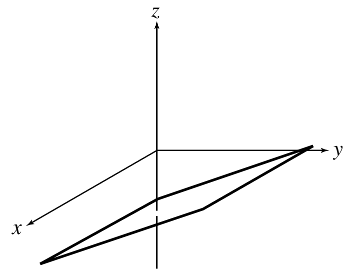
22. $2x + 3y + 5z = 1$ can be put in the form $Ax + By + Cz + D = 0$, so the graph is a plane. The intercepts are $\left(\frac{1}{2}, 0, 0\right)$, $\left(0, \frac{1}{3}, 0\right)$, and $\left(0, 0, \frac{1}{5}\right)$.



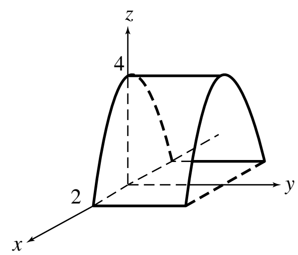
23. $x + 2y = 2$ can be put in the form $Ax + By + Cz + D = 0$, so the graph is a plane. There are only two intercepts: (2, 0, 0) and (0, 1, 0). The x,y-trace is $x + 2y = 2$, which is a line. For any fixed value of z, we obtain the line $x + 2y = 2$.



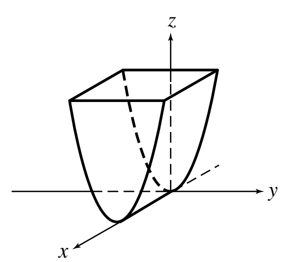
24. $y = 3z + 2$ can be put in the form $Ax + By + Cz + D = 0$, so the graph is a plane. There are only two intercepts: (0, 2, 0) and $\left(0, 0, -\frac{2}{3}\right)$. The y,z-trace is $y - 3z = 2$, which is a line. For any fixed value of x, we obtain the line $y - 3z = 2$.



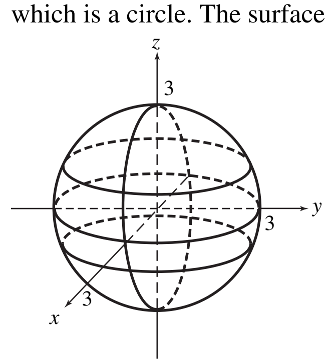
25. $z = 4 - x^2$. The x,z-trace is $z = 4 - x^2$, which is a parabola. For any fixed value of y, we obtain the parabola $z = 4 - x^2$.



26. $y = z^2$. The y,z-trace is $y = z^2$, which is a parabola. For any fixed value of x, we obtain the parabola $y = z^2$.

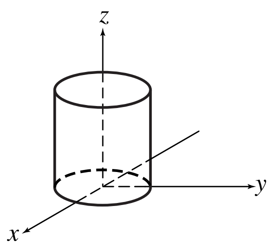


27. $x^2 + y^2 + z^2 = 9$. The x,y-trace is $x^2 + y^2 = 9$, which is a circle. The x,z-trace is $x^2 + z^2 = 9$, which is a circle. The y,z-trace is $y^2 + z^2 = 9$, which is a circle. The surface is a sphere.



28. $3x^2+2y^2=1$

The *x*,*y*-trace is $3x^2+2y^2=1$, which is an ellipse. For any fixed value of *z*, we obtain the ellipse $3x^2+2y^2=1$.



Problems 17.2

1. $f(x, y)=4x^2+3y^2-7$

$f_x(x, y)=4(2x)+0+0=8x$

$f_y(x, y)=0+3(2y)+0=6y$

2. $f(x, y)=2x^2+3xy$

$f_x(x, y)=2(2x)+3(1)y=4x+3y$

$f_y(x, y)=0+3x(1)=3x$

3. $f(x, y)=2y+1$

$f_x(x, y)=0+0=0$

$f_y(x, y)=2(1)+0=2$

4. $f(x, y)=\ln 2$

$f_x(x, y)=0$

$f_y(x,y)=0$

5. $g(x, y)=3x^4y+2xy^2-5xy+8x-9y$

$g_x(x, y)=3(4)x^3y+2(1)y^2-5(1)y+8(1)$
$=12x^3y+2y^2-5y+8$

$g_y(x, y)=3x^4(1)+2x(2)y-5x(1)-9(1)$
$=3x^4+4xy-5x-9$

6. $g(x, y)=(x+1)^2+(y-3)^3+5xy^3-2$

$g_x(x, y)=2(x+1)+0+5(1)y^3-0$
$=2(x+1)+5y^3$

$g_y(x, y)=0+3(y-3)^2+5x\left(3y^2\right)-0$
$=3(y-3)^2+15xy^2$

7. $g(p, q)=\sqrt{pq}=(pq)^{\frac{1}{2}}$

$$g_p(p, q)=\frac{1}{2}(pq)^{-\frac{1}{2}}\cdot q=\frac{q}{2\sqrt{pq}}$$

$$g_q(p, q)=\frac{1}{2}(pq)^{-\frac{1}{2}}\cdot p=\frac{p}{2\sqrt{pq}}$$

8. $g(w, z)=\sqrt[3]{w^2+z^2}=\left(w^2+z^2\right)^{\frac{1}{3}}$

$$g_w(w, z)=\frac{1}{3}\left(w^2+z^2\right)^{-\frac{2}{3}}(2w)=\frac{2w}{3\left(w^2+z^2\right)^{\frac{2}{3}}}$$

$$g_z(w, z)=\frac{1}{3}\left(w^2+z^2\right)^{-\frac{2}{3}}(2z)=\frac{2z}{3\left(w^2+z^2\right)^{\frac{2}{3}}}$$

9. $h(s, t)=\frac{s^2+4}{t-3}$

$$h_s(s, t)=\frac{1}{t-3}(2s)=\frac{2s}{t-3}$$

Rewriting $h(s, t)$ as $\left(s^2+4\right)(t-3)^{-1}$, we have

$$h_t(s, t)=\left(s^2+4\right)\left[(-1)(t-3)^{-2}(1)\right]=-\frac{s^2+4}{(t-3)^2}$$

10. $h(u, v)=\frac{8uv^2}{u^2+v^2}$

$$h_u(u, v)=8v^2\frac{\left(u^2+v^2\right)(1)-u(2u)}{\left(u^2+v^2\right)^2}=\frac{8v^2\left(v^2-u^2\right)}{\left(u^2+v^2\right)^2}$$

$$h_v(u, v)=8u\frac{\left(u^2+v^2\right)(2v)-v^2(2v)}{\left(u^2+v^2\right)^2}=\frac{16u^3v}{\left(u^2+v^2\right)^2}$$

11. $u(q_1, q_2) = \frac{1}{2}\ln(q_1+2) + \frac{1}{3}\ln(q_2+5)$

$u_{q_1}(q_1, q_2) = \frac{1}{2}\cdot\frac{1}{q_1+2} + 0 = \frac{1}{2(q_1+2)}$

$u_{q_2}(q_1, q_2) = 0 + \frac{1}{3}\cdot\frac{1}{q_2+5} = \frac{1}{3(q_2+5)}$

12. $Q(l, k) = 2l^{0.38}k^{1.79} - 3l^{1.03} + 2k^{0.13}$

$Q_l(l, k) = 2(0.38)l^{0.38-1}k^{1.79} - 3(1.03)l^{1.03-1} + 0 = 0.76l^{-0.62}k^{1.79} - 3.09l^{0.03}$

$Q_k(l, k) = 2l^{0.38}(1.79)k^{1.79-1} - 0 + 2(0.13)k^{0.13-1} = 3.58l^{0.38}k^{0.79} + 0.26k^{-0.87}$

13. $h(x, y) = \frac{x^2+3xy+y^2}{\sqrt{x^2+y^2}}$

$$h_x(x, y) = \frac{\left(x^2+y^2\right)^{\frac{1}{2}}[2x+3y] - \left(x^2+3xy+y^2\right)\left[\frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}}(2x)\right]}{\left[\left(x^2+y^2\right)^{\frac{1}{2}}\right]^2}$$

$$= \frac{\left(x^2+y^2\right)^{-\frac{1}{2}}\left[\left(x^2+y^2\right)(2x+3y) - \left(x^2+3xy+y^2\right)x\right]}{x^2+y^2}$$

$$= \frac{2x^3+3x^2y+2xy^2+3y^3-x^3-3x^2y-xy^2}{\left(x^2+y^2\right)^{\frac{3}{2}}} = \frac{x^3+xy^2+3y^3}{\left(x^2+y^2\right)^{\frac{3}{2}}}$$

$$h_y(x, y) = \frac{\left(x^2+y^2\right)^{\frac{1}{2}}[3x+2y] - \left(x^2+3xy+y^2\right)\left[\frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}}(2y)\right]}{\left[\left(x^2+y^2\right)^{\frac{1}{2}}\right]^2}$$

$$= \frac{\left(x^2+y^2\right)^{-\frac{1}{2}}\left[\left(x^2+y^2\right)(3x+2y) - \left(x^2+3xy+y^2\right)y\right]}{x^2+y^2}$$

$$= \frac{3x^3+2x^2y+3xy^2+2y^3-x^2y-3xy^2-y^3}{\left(x^2+y^2\right)^{\frac{3}{2}}} = \frac{3x^3+x^2y+y^3}{\left(x^2+y^2\right)^{\frac{3}{2}}}$$

14. $h(x,\ y)=\dfrac{\sqrt{x+9}}{x^2y+y^2x}$

$$h_x(x,\ y)=\frac{\left(x^2y+y^2x\right)\frac{1}{2}(x+9)^{-\frac{1}{2}}-(x+9)^{\frac{1}{2}}\left(2xy+y^2\right)}{\left(x^2y+y^2x\right)^2}$$

$$=\frac{\frac{1}{2}(x+9)^{-\frac{1}{2}}\left[x^2y+y^2x-2(x+9)\left(2xy+y^2\right)\right]}{\left(x^2y+y^2x\right)^2}$$

$$=\frac{y\left[x^2+xy-2(x+9)(2x+y)\right]}{2(x+9)^{\frac{1}{2}}\left[xy(x+y)\right]^2}$$

$$=\frac{y\left[x^2+xy-4x^2-36x-2xy-18y\right]}{2(x+9)^{\frac{1}{2}}x^2y^2(x+y)^2}=\frac{-\left(3x^2+xy+36x+18y\right)}{2x^2y\sqrt{x+9}(x+y)^2}$$

Since $h(x,\ y)=\sqrt{x+9}\left(x^2y+y^2x\right)^{-1}$, then

$$h_y(x,\ y)=\sqrt{x+9}(-1)\left(x^2y+y^2x\right)^{-2}\left(x^2+2xy\right)$$

$$=\frac{-\sqrt{x+9}\left(x^2+2xy\right)}{\left(x^2y+y^2x\right)^2}=\frac{-x\sqrt{x+9}(x+2y)}{x^2y^2(x+y)^2}=\frac{-\sqrt{x+9}(x+2y)}{xy^2(x+y)^2}$$

15. $z=e^{5xy}$

$$\frac{\partial z}{\partial x}=e^{5xy}(5y)=5ye^{5xy};\ \frac{\partial z}{\partial y}=e^{5xy}(5x)=5xe^{5xy}$$

16. $z=(x^2+y^2)e^{2x+3y+1}$

$$\frac{\partial z}{\partial x}=(x^2+y^2)[e^{2x+3y+1}(2)]+e^{2x+3y+1}[2x]=(2x^2+2y^2+2x)e^{2x+3y+1}$$

$$\frac{\partial z}{\partial y}=(x^2+y^2)[e^{2x+3y+1}(3)]+e^{2x+3y+1}[2y]=(3x^2+3y^2+2y)e^{2x+3y+1}$$

17. $z=5x\ln\left(x^2+y\right)$

$$\frac{\partial z}{\partial x}=5\left\{x\left[\frac{1}{x^2+y}(2x)\right]+\ln\left(x^2+y\right)[1]\right\}=5\left[\frac{2x^2}{x^2+y}+\ln\left(x^2+y\right)\right]$$

$$\frac{\partial z}{\partial y}=5x\left(\frac{1}{x^2+y}[1]\right)=\frac{5x}{x^2+y}$$

18. $z = \ln(5x^3y^2 + 2y^4)^4 = 4\ln(5x^3y^2 + 2y^4)$

$$\frac{\partial z}{\partial x} = 4 \cdot \frac{1}{5x^3y^2 + 2y^4}[5(3x^2)y^2 + 0] = \frac{60x^2y^2}{5x^3y^2 + 2y^4} = \frac{60x^2y^2}{y^2(5x^3 + 2y^2)} = \frac{60x^2}{5x^3 + 2y^2}$$

$$\frac{\partial z}{\partial y} = 4 \cdot \frac{1}{5x^3y^2 + 2y^4}[5x^3(2y) + 2(4y^3)] = \frac{4(10x^3y + 8y^3)}{5x^3y^2 + 2y^4} = \frac{8y(5x^3 + 4y^2)}{y(5x^3y + 2y^3)} = \frac{8(5x^3 + 4y^2)}{5x^3y + 2y^3}$$

19. $f(r, s) = (r + 2s)^{\frac{1}{2}}\left(r^3 - 2rs + s^2\right)$

$$f_r(r, s) = (r + 2s)^{\frac{1}{2}}\left[3r^2 - 2s\right] + \left(r^3 - 2rs + s^2\right)\left[\frac{1}{2}(r + 2s)^{-\frac{1}{2}}(1)\right]$$

$$= \sqrt{r + 2s}\left(3r^2 - 2s\right) + \frac{r^3 - 2rs + s^2}{2\sqrt{r + 2s}}$$

$$f_s(r, s) = (r + 2s)^{\frac{1}{2}}\left[-2r + 2s\right] + \left(r^3 - 2rs + s^2\right)\left[\frac{1}{2}(r + 2s)^{-\frac{1}{2}}(2)\right]$$

$$= 2(s - r)\sqrt{r + 2s} + \frac{r^3 - 2rs + s^2}{\sqrt{r + 2s}}$$

20. $f(r, s) = (rs)^{\frac{1}{2}}e^{2+r}$

$$f_r(r, s) = (rs)^{\frac{1}{2}}\left[e^{2+r}(1)\right] + e^{2+r}\left[\frac{1}{2}(rs)^{-\frac{1}{2}}(s)\right] = \left[\sqrt{rs} + \frac{s}{2\sqrt{rs}}\right]e^{2+r}$$

$$f_s(r, s) = e^{2+r}\left[\frac{1}{2}(rs)^{-\frac{1}{2}}(r)\right] = \frac{re^{2+r}}{2\sqrt{rs}}$$

21. $f(r, s) = e^{3-r}\ln(7 - s)$

$$f_r(r, s) = \ln(7 - s)\left[e^{3-r}(-1)\right] = -e^{3-r}\ln(7 - s)$$

$$f_s(r, s) = e^{3-r}\left[\frac{1}{7 - s}(-1)\right] = \frac{e^{3-r}}{s - 7}$$

22. $f(r, s) = \left(5r^2 + 3s^3\right)(2r - 5s)$

$$f_r(r, s) = \left(5r^2 + 3s^3\right)[2] + (2r - 5s)[10r] = 2\left(5r^2 + 3s^3\right) + 10r(2r - 5s)$$

$$f_s(r, s) = \left(5r^2 + 3s^3\right)[-5] + (2r - 5s)\left[9s^2\right] = -5\left(5r^2 + 3s^3\right) + 9s^2(2r - 5s)$$

23. $g(x, y, z) = 2x^3y^2 + 2xy^3z + 4z^2$

$$g_x(x, y, z) = 2y^2(3x^2) + 2y^3z(1) + 0 = 6x^2y^2 + 2y^3z$$

$$g_y(x, y, z) = 2x^3(2y) + 2xz(3y^2) + 0 = 4x^3y + 6xy^2z$$

$$g_z(x, y, z) = 0 + 2xy^3(1) + 4(2z) = 2xy^3 + 8z$$

24. $g(x, y, z) = 2xy^2z^6 - 4x^2y^3z^2 + 3xyz$

$g_x(x, y, z) = 2(1)y^2z^6 - 4(2x)y^3z^2 + 3(1)yz$
$= 2y^2z^6 - 8xy^3z^2 + 3yz$

$g_y(x, y, z) = 2x(2y)z^6 - 4x^2(3y^2)z^2 + 3x(1)z$
$= 4xyz^6 - 12x^2y^2z^2 + 3xz$

$g_z(x, y, z) = 2xy^2(6z^5) - 4x^2y^3(2z) + 3xy(1)$
$= 12xy^2z^5 - 8x^2y^3z + 3xy$

25. $g(r, s, t) = e^{s+t}(r^2 + 7s^3)$

$g_r(r, s, t) = e^{s+t}[2r + 0] = 2re^{s+t}$

$g_s(r, s, t) = e^{s+t}\left[0 + 21s^2\right] + \left(r^2 + 7s^3\right)\left[e^{s+t}(1)\right]$
$= \left(7s^3 + 21s^2 + r^2\right)e^{s+t}$

$g_t(r, s, t) = \left(r^2 + 7s^3\right)\left[e^{s+t}(1)\right] = e^{s+t}\left(r^2 + 7s^3\right)$

26. $g(r, s, t, u) = rs\ln(2t + 5u)$

$g_r(r, s, t, u) = (1)s\ln(2t + 5u) = s\ln(2t + 5u)$

$g_s(r, s, t, u) = r(1)\ln(2t + 5u) = r\ln(2t + 5u)$

$g_t(r, s, t, u) = rs\left[\frac{1}{2t + 5u}(2)\right] = \frac{2rs}{2t + 5u}$

$g_u(r, s, t, u) = rs\left[\frac{1}{2t + 5u}(5)\right] = \frac{5rs}{2t + 5u}$

27. $f(x, y) = x^3y + 7x^2y^2$

$f_x(x, y) = 3x^2y + 14xy^2$

$f_x(1, -2) = 3(1)^2(-2) + 14(1)(-2)^2 = 50$

28. $z = \sqrt{2x^3 + 5xy + 2y^2}$

$\frac{\partial z}{\partial x} = \frac{6x^2 + 5y}{2\sqrt{2x^3 + 5xy + 2y^2}}$

$\left.\frac{\partial z}{\partial x}\right|_{(0, 1)} = \frac{5}{2\sqrt{2}}$

29. $g(x, y, z) = e^x\sqrt{y + 2z}$

$g_z(x, y, z) = e^x\left[\frac{1}{2}(y + 2z)^{-\frac{1}{2}}(2)\right] = \frac{e^x}{\sqrt{y + 2z}}$

$g_z(0, 6, 4) = \frac{1}{\sqrt{6 + 8}} = \frac{1}{\sqrt{14}}$

30. $g(x, y, z) = \dfrac{3x^2y^2 + 2xy + x - y}{xy - yz + xz}$

$$g_y(x, y, z) = \frac{(xy - yz + xz)(6x^2y + 2x - 1) - (3x^2y^2 + 2xy + x - y)(x - z)}{(xy - yz + xz)^2}$$

$$g_y(1, 1, 5) = \frac{(1-5+5)(6+2-1) - (3+2+1-1)(1-5)}{(1-5+5)^2} = 27$$

31. $h(r, s, t, u) = \left(s^2 + tu\right)\ln(2r + 7st)$

$$h_s(r, s, t, u) = \frac{7t\left(s^2 + tu\right)}{2r + 7st} + 2s\ln(2r + 7st)$$

$h_s(1, 0, 0, 1) = 0$

32. $h(r, s, t, u) = \dfrac{7r + 3s^2u^2}{s}$

$h_t(r, s, t, u) = 0$

$h_t(4, 3, 2, 1) = 0$

33. $f(r, s, t) = rst\left(r^2 + s^3 + t^4\right) = r^3st + rs^4t + rst^5$

$f_s(r, s, t) = r^3(1)t + r\left(4s^3\right)t + r(1)t^5 = r^3t + 4rs^3t + rt^5$

$f_s(1, -1, 2) = 2 + (-8) + 32 = 26$

34. $z = \dfrac{x^2 + y^2}{e^{x^2+y^2}} = (x^2 + y^2)e^{-(x^2+y^2)}$

$$\frac{\partial z}{\partial x} = (2x)e^{-(x^2+y^2)} + (x^2 + y^2)e^{-(x^2+y^2)}(-2x)$$

$$= 2xe^{-(x^2+y^2)}[1 - (x^2 + y^2)]$$

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=0\\y=0}} = 2(0)e^0[1 - (0)] = 0$$

By symmetry, $\dfrac{\partial z}{\partial y} = 2ye^{-(x^2+y^2)}[1 - (x^2 + y^2)]$.

$$\left.\frac{\partial z}{\partial y}\right|_{\substack{x=1\\y=1}} = 2(1)e^{-2}[1 - (2)] = -\frac{2}{e^2}$$

35. $z = xe^{x-y} + ye^{y-x}$

$$\frac{\partial z}{\partial x} = \left[xe^{x-y} + e^{x-y}\right] + \left[ye^{y-x}(-1)\right]$$

$$\frac{\partial z}{\partial y} = \left[xe^{x-y}(-1)\right] + \left[ye^{y-x} + e^{y-x}\right]$$

Thus $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = e^{x-y} + e^{y-x}$, as was to be shown.

36. $u = f(t, r, z) = \dfrac{(1+r)^{1-z}\ln(1+r)}{(1+r)^{1-z} - t}$

$$\frac{\partial u}{\partial z} = \ln(1+r)\frac{\partial}{\partial z}\left[\frac{(1+r)^{1-z}}{(1+r)^{1-z}-t}\right]$$

$$= \ln(1+r)\frac{\left[(1+r)^{1-z}-t\right]\frac{\partial}{\partial z}\left[(1+r)^{1-z}\right] - (1+r)^{1-z}\left\{\frac{\partial}{\partial z}\left[(1+r)^{1-z}\right]-0\right\}}{\left[(1+r)^{1-z}-t\right]^2}$$

$$= \ln(1+r)\frac{-t\frac{\partial}{\partial z}\left[(1+r)^{1-z}\right]}{\left[(1+r)^{1-z}-t\right]^2}$$

$$= \ln(1+r)\frac{-t\left\{(1+r)^{1-z}\ln(1+r)[-1]\right\}}{\left[(1+r)^{1-z}-t\right]^2}$$

$= \dfrac{t(1+r)^{1-z}\ln^2(1+r)}{\left[(1+r)^{1-z}-t\right]^2}$, as was to be shown.

37. $F(b, C, T, i) = \dfrac{bT}{C} + \dfrac{iC}{2}$

$$\frac{\partial F}{\partial C} = \frac{\partial}{\partial C}\left[\frac{bT}{C}\right] + \frac{\partial}{\partial C}\left[\frac{iC}{2}\right] = -\frac{bT}{C^2} + \frac{i}{2}$$

38. From $\eta = \dfrac{\frac{r}{D}}{\frac{\partial r}{\partial D}}$, we have $\dfrac{\partial r}{\partial D} = \dfrac{r}{D\eta}$. Substituting into Equation (3) gives

$$r_L = r + D\cdot\frac{r}{D\eta} + \frac{dC}{dD}$$

$$r_L = r + \frac{r}{\eta} + \frac{dC}{dD}$$

$$r_L = r\left[1 + \frac{1}{\eta}\right] + \frac{dC}{dD}$$

$$r_L = r\left[\frac{\eta+1}{\eta}\right] + \frac{dC}{dD}$$

$$r_L = r\left[\frac{1+\eta}{\eta}\right] + \frac{dC}{dD}$$

which is Equation (4).

39. $R = f(r, a, n) = \dfrac{r}{1+a\left(\frac{n-1}{2}\right)} = r\left[1+a\left(\dfrac{n-1}{2}\right)\right]^{-1}$

$$\frac{\partial R}{\partial n} = r(-1)\left[1+a\left(\frac{n-1}{2}\right)\right]^{-2}\cdot\frac{a}{2} = -\frac{ra}{2\left[1+a\left(\frac{n-1}{2}\right)\right]^2}$$

Problems 17.3

1. $c = 7x + 0.3y^2 + 2y + 900$

$$\frac{\partial c}{\partial y} = 0.6y + 2$$

When $x = 20$ and $y = 30$, then

$$\frac{\partial c}{\partial y} = 0.6(30) + 2 = 20.$$

2. $c = x\sqrt{x+y} + 5000$

$$\frac{\partial c}{\partial x} = \frac{x}{2\sqrt{x+y}} + \sqrt{x+y}$$

When $x = 40$ and $y = 60$, then

$$\frac{\partial c}{\partial x} = \frac{40}{2\sqrt{100}} + \sqrt{100} = 12.$$

3. $c = 0.03(x+y)^3 - 0.6(x+y)^2 + 9.5(x+y) + 7700$

$$\frac{\partial c}{\partial x} = 0.09(x+y)^2 - 1.2(x+y) + 9.5$$

When $x = 50$ and $y = 80$, then

$$\frac{\partial c}{\partial x} = 0.09(130)^2 - 1.2(130) + 9.5 = 1374.5.$$

4. $P = 15lk - 3l^2 + 5k^2 + 500$

$$\frac{\partial P}{\partial k} = 15l + 10k$$

$$\frac{\partial P}{\partial l} = 15k - 6l$$

5. $P = 2.314l^{0.357}k^{0.643}$

$$\frac{\partial P}{\partial l} = 2.314(0.357)l^{-0.643}k^{0.643} = 0.826098\left(\frac{k}{l}\right)^{0.643}$$

$$\frac{\partial P}{\partial k} = 2.314(0.643)l^{0.357}k^{-0.357} = 1.487902\left(\frac{l}{k}\right)^{0.357}$$

6. $P = Al^{\alpha}k^{\beta}$

a. $\dfrac{\partial P}{\partial l} = A\alpha l^{\alpha-1}k^{\beta} = \left(\dfrac{\alpha}{l}\right)Al^{\alpha}k^{\beta} = \dfrac{\alpha P}{l}$

b. $\dfrac{\partial P}{\partial k} = A\beta l^{\alpha}k^{\beta-1} = \left(\dfrac{\beta}{k}\right)Al^{\alpha}k^{\beta} = \dfrac{\beta P}{k}$

c. From parts (a) and (b),

$$l\frac{\partial P}{\partial l} + k\frac{\partial P}{\partial k} = l\left(\frac{\alpha P}{l}\right) + k\left(\frac{\beta P}{k}\right) = \alpha P + \beta P = P(\alpha + \beta) = P(1) = P$$

7. $\dfrac{\partial q_A}{\partial p_A} = -50$, $\dfrac{\partial q_A}{\partial p_B} = 2$, $\dfrac{\partial q_B}{\partial q_A} = 4$, $\dfrac{\partial q_B}{\partial p_B} = -20$

Since $\dfrac{\partial q_A}{\partial p_B} > 0$ and $\dfrac{\partial q_B}{\partial p_A} > 0$ the products are competitive.

8. $\dfrac{\partial q_A}{\partial p_A} = -1$, $\dfrac{\partial q_A}{\partial p_B} = -2$, $\dfrac{\partial q_B}{\partial p_A} = -2$, $\dfrac{\partial q_B}{\partial p_B} = -3$

Since $\dfrac{\partial q_A}{\partial p_B} < 0$ and $\dfrac{\partial q_B}{\partial p_A} < 0$ the products are complementary.

9. $q_A = 100p_A^{-1}p_B^{-\frac{1}{2}}$

$q_B = 500p_B^{-1}p_A^{-\frac{1}{3}}$

$$\frac{\partial q_A}{\partial p_A} = 100(-1)p_A^{-2}p_B^{-\frac{1}{2}} = \frac{-100}{p_A^2 p_B^{\frac{1}{2}}}$$

$$\frac{\partial q_A}{\partial p_B} = 100\left(-\frac{1}{2}\right)p_A^{-1}p_B^{-\frac{3}{2}} = \frac{-50}{p_A p_B^{\frac{3}{2}}}$$

$$\frac{\partial q_B}{\partial p_A} = 500\left(-\frac{1}{3}\right)p_B^{-1}p_A^{-\frac{4}{3}} = \frac{-500}{3p_B p_A^{\frac{4}{3}}}$$

$$\frac{\partial q_B}{\partial p_B} = 500(-1)p_B^{-2}p_A^{-\frac{1}{3}} = -\frac{500}{p_B^2 p_A^{\frac{1}{3}}}$$

Since $\dfrac{\partial q_A}{\partial p_B} < 0$ and $\dfrac{\partial q_B}{\partial p_A} < 0$, the products are complementary.

10. $\dfrac{\partial P}{\partial l} = 15.18l^{-0.54}k^{0.52}$

$\dfrac{\partial P}{\partial k} = 17.16l^{0.46}k^{-0.48}$

If $l = 1$ and $k = 1$, then $\dfrac{\partial P}{\partial l} = 15.18$ and

$\dfrac{\partial P}{\partial k} = 17.16$

11. $\frac{\partial P}{\partial B} = 0.01A^{0.27}B^{-0.99}C^{0.01}D^{0.23}E^{0.09}F^{0.27}$

$\frac{\partial P}{\partial C} = 0.01A^{0.27}B^{0.01}C^{-0.99}D^{0.23}E^{0.09}F^{0.27}$

12. $P = \frac{kl}{2k+3l}$

a. $\frac{\partial P}{\partial k} = \frac{l(2k+3l) - kl(2)}{(2k+3l)^2} = \frac{3l^2}{(2k+3l)^2}$

$\frac{\partial P}{\partial l} = \frac{k(2k+3l) - kl(3)}{(2k+3l)^2} = \frac{2k^2}{(2k+3l)^2}$

b. When $k = l$, then

$$\frac{\partial P}{\partial k} + \frac{\partial P}{\partial l} = \frac{3l^2}{(2l+3l)^2} + \frac{2l^2}{(2l+3l)^2} = \frac{5l^2}{25l^2} = \frac{1}{5}$$

13. $\frac{\partial z}{\partial x} = 4480$. If a staff manager with an M.B.A. degree had an extra year of work experience before the degree, the manager would receive \$4480 more per year in extra compensation.

14. $S_g = 7S_e^{\frac{1}{3}}S_i^{\frac{1}{2}}$

$\frac{\partial S_g}{\partial S_e} = 7\left(\frac{1}{3}\right)S_e^{-\frac{2}{3}}S_i^{\frac{1}{2}} = \left(\frac{7}{3}\right)\frac{\sqrt{S_i}}{\sqrt[3]{S_e^2}}$

$\frac{\partial S_g}{\partial S_i} = 7\left(\frac{1}{2}\right)S_e^{\frac{1}{3}}S_i^{-\frac{1}{2}} = \left(\frac{7}{2}\right)\frac{\sqrt[3]{S_e}}{\sqrt{S_i}}$

If $S_e = 125$ and $S_i = 100$, then

$\frac{\partial S_g}{\partial S_e} = \left(\frac{7}{3}\right)\frac{10}{5^2} = \frac{14}{15}$ and $\frac{\partial S_g}{\partial S_i} = \left(\frac{7}{2}\right)\frac{5}{10} = \frac{7}{4}$.

Thus if S_e increases to 126 and S_i remains at 100, then S_g increases by approximately $\frac{14}{15}$; if S_i increases to 101 and S_e remains at 125, then S_g increases by approximately $\frac{7}{4}$.

15. a. $\frac{\partial R}{\partial w} = -1.015$; $\frac{\partial R}{\partial s} = -0.846$

b. One for which $w = w_0$ and $s = s_0$ since increasing *w* by 1 while holding *s* fixed decreases the reading ease score.

16. $\omega = b^{-1}L^{-1}\sqrt{\frac{\tau}{\pi\rho}} = \frac{1}{bL}\sqrt{\frac{\tau}{\pi}}\rho^{-\frac{1}{2}} = \frac{1}{bL}\sqrt{\frac{1}{\pi\rho}}\tau^{\frac{1}{2}}$

$\frac{\partial \omega}{\partial b} = (-1)b^{-2}L^{-1}\sqrt{\frac{\tau}{\pi\rho}} = -\frac{1}{b^2L}\sqrt{\frac{\tau}{\pi\rho}}$

$\frac{\partial \omega}{\partial L} = b^{-1}(-1)L^{-2}\sqrt{\frac{\tau}{\pi\rho}} = -\frac{1}{bL^2}\sqrt{\frac{\tau}{\pi\rho}}$

$\frac{\partial \omega}{\partial \rho} = \frac{1}{bL}\sqrt{\frac{\tau}{\pi}}\left(-\frac{1}{2}\right)\rho^{-\frac{3}{2}} = -\frac{1}{2bL\rho^{\frac{3}{2}}}\sqrt{\frac{\tau}{\pi}}$

$\frac{\partial \omega}{\partial \tau} = \frac{1}{bL}\sqrt{\frac{1}{\pi\rho}}\left(\frac{1}{2}\right)\tau^{-\frac{1}{2}} = \frac{1}{2bL}\sqrt{\frac{1}{\pi\rho\tau}}$

17. $\frac{\partial g}{\partial x} = \frac{1}{V_F} > 0$ for $V_F > 0$. Thus if *x* increases and V_F and V_S are fixed, then *g* increases.

18. $q_A = e^{-(p_A+p_B)}$ and $q_B = \frac{16}{p_A^2 p_B^2} = 16p_A^{-2}p_B^{-2}$

a. $\frac{\partial q_A}{\partial p_B} = -e^{-(p_A+p_B)} < 0$

$\frac{\partial q_B}{\partial p_A} = -32p_A^{-3}p_B^{-2} < 0$

Since both are < 0, A and B are complementary.

b. Note that p_A and p_B are in units of thousands of dollars. When $p_A = 1$ and $p_A = 2$, then $\frac{\partial q_A}{\partial p_B} = -e^{-3} = -\frac{1}{e^3}$.

A decrease in the price of B of \$20 is a decrease in p_B of $\frac{20}{2000} = 0.01$. Thus the change in q_B is approximately $-\frac{1}{e^3}(-0.01) = \frac{0.01}{e^3}$. So demand increases by approximately $\frac{0.01}{e^3}$ unit.

19. a. $\frac{\partial q_A}{\partial p_A} = 10\sqrt{p_B}\left(-\frac{1}{2}p_A^{-\frac{3}{2}}\right)$

$\frac{\partial q_A}{\partial p_B} = \frac{10}{\sqrt{p_A}}\left(\frac{1}{2}p_B^{-\frac{1}{2}}\right)$

When $p_A = 9$ and $p_B = 16$, then $\frac{\partial q_A}{\partial p_A} = 10(4)\left(-\frac{1}{2}\cdot\frac{1}{27}\right) = -\frac{20}{27}$ and

$\frac{\partial q_A}{\partial p_B} = \frac{10}{3}\left(\frac{1}{2}\cdot\frac{1}{4}\right) = \frac{5}{12}$.

b. From (a), when $p_A = 9$ and $p_B = 16$, then $\frac{\partial q_A}{\partial p_B} = \frac{5}{12}$. Hence each \$1 reduction in p_B decreases q_A by approximately $\frac{5}{12}$ unit. Thus a \$2 reduction in p_B (from \$16 to \$14) decreases the demand for A by approximately $\frac{5}{12}(2) = \frac{5}{6}$ unit.

20. $c = \frac{q_A^2\left(q_B^3 + q_A\right)^{\frac{1}{2}}}{17} + q_A q_B^{\frac{1}{3}} + 600$

a. $\frac{\partial c}{\partial q_A} = \frac{1}{17}\left[q_A^2\cdot\frac{1}{2}\left(q_B^3 + q_A\right)^{-\frac{1}{2}} + \left(q_B^3 + q_A\right)^{\frac{1}{2}}(2q_A)\right] + q_B^{\frac{1}{3}}$

$= \frac{1}{17}\left[\frac{1}{2}q_A^2\left(q_B^3 + q_A\right)^{-\frac{1}{2}} + 2q_A\left(q_B^3 + q_A\right)^{\frac{1}{2}}\right] + q_B^{\frac{1}{3}}$

$\frac{\partial c}{\partial q_B} = \frac{1}{17}\left[q_A^2\cdot\frac{1}{2}\left(q_B^3 + q_A\right)^{-\frac{1}{2}}\left(3q_B^2\right)\right] + q_A\cdot\frac{1}{3}q_B^{-\frac{2}{3}}$

$= \frac{1}{17}\left[\frac{3}{2}q_A^2 q_B^2\left(q_B^3 + q_A\right)^{-\frac{1}{2}}\right] + \frac{1}{3}q_A q_B^{-\frac{2}{3}}$

b. When $q_A = 17$ and $q_B = 8$, then

$\frac{\partial c}{\partial q_A} = \frac{1}{17}\left[\frac{1}{2}(17)^2\left(\frac{1}{23}\right) + 2(17)(23)\right] + 2 = \left[\frac{1}{2}(17)\frac{1}{23} + 2(23)\right] + 2 \approx 48.37$.

c. From (b), if q_A is reduced by one unit (from 17 to 16) while q_B remains at 8, then the cost will decrease by approximately \$48.37.

21. a. $\frac{\partial R}{\partial E_r} = 2.5945 - 0.1608E_r - 0.0277I_r$

If $E_r = 18.8$ and $I_r = 10$, then $\frac{\partial R}{\partial E_r} = -0.70564$. Since $\frac{\partial R}{\partial E_r} < 0$, such a candidate should not be so advised.

b. $\frac{\partial R}{\partial N} = 0.8579 - 0.0122N$

If $\frac{\partial R}{\partial N} < 0$, then $N > 70.3 \approx 70\%$

22. $S = \frac{AT + 450}{\sqrt{A+T^2}}$. Note: *A* is expressed in hundreds of dollars.

a. $$\frac{\partial S}{\partial T} = \frac{\left(A+T^2\right)^{\frac{1}{2}}(A) - (AT+450)\left[\frac{1}{2}\left(A+T^2\right)^{-\frac{1}{2}}(2T)\right]}{\left(\sqrt{A+T^2}\right)^2}$$

$$= \frac{\left(A+T^2\right)^{-\frac{1}{2}}\left[\left(A+T^2\right)A - (AT+450)T\right]}{A+T^2} = \frac{A^2 - 450T}{\left(A+T^2\right)^{\frac{3}{2}}}$$

as was to be shown.

b. We want to find when $\frac{\partial S}{\partial T} < 0$ and $A = \frac{9000}{100} = 90$. First we find when $\frac{\partial S}{\partial T} = 0$ and $A = 90$:

$\frac{90^2 - 450T}{\left(90+T^2\right)^{\frac{3}{2}}} = 0 \Rightarrow 90^2 - 450T = 0 \Rightarrow T = \frac{90^2}{450} = 18$.

$\frac{\partial S}{\partial T} > 0$ for $T < 18$, and $\frac{\partial S}{\partial T} < 0$ for $T > 18$. Thus 18 months elapse before the sales volume begins to decrease.

23. $q_A = 1000 - 50p_A + 2p_B$

$\eta_{p_A} = \left(\frac{p_A}{q_A}\right)\frac{\partial q_A}{\partial p_A} = \left(\frac{p_A}{q_A}\right)(-50)$

$\eta_{p_B} = \left(\frac{p_B}{q_A}\right)\frac{\partial q_A}{\partial p_B} = \left(\frac{p_B}{q_A}\right)(2)$

When $p_A = 2$ and $p_B = 10$, then $q_A = 920$, from which $\eta_{p_A} = -\frac{5}{46}$ and $\eta_{p_B} = \frac{1}{46}$

24. $q_A = 60 - 3p_A - 2p_B$

$\eta_{p_A} = \left(\frac{p_A}{q_A}\right)\frac{\partial q_A}{\partial p_A} = \left(\frac{p_A}{q_A}\right)(-3)$

$\eta_{p_B} = \left(\frac{p_B}{q_A}\right)\frac{\partial q_A}{\partial p_B} = \left(\frac{p_B}{q_A}\right)(-2)$

When $p_A = 5$ and $p_B = 3$, then $q_A = 39$, from which $\eta_{p_A} = -\frac{5}{13}$ and $\eta_{p_B} = -\frac{2}{13}$.

25. $q_A = \frac{100}{p_A\sqrt{p_B}}$

$\eta_{p_A} = \left(\frac{p_A}{q_A}\right)\frac{\partial q_A}{\partial p_A} = \left(\frac{p_A}{q_A}\right)\left(\frac{-100}{p_A^2\sqrt{p_B}}\right)$

$\eta_{p_B} = \left(\frac{p_B}{q_A}\right)\frac{\partial q_A}{\partial p_B} = \left(\frac{p_B}{q_A}\right)\left(\frac{-50}{p_A\sqrt{p_B^3}}\right)$

When $p_A = 1$ and $p_B = 4$, then $q_A = 50$. This gives $\eta_{p_A} = -1$ and $\eta_{p_B} = -\frac{1}{2}$.

Problems 17.4

1. $4x + 0 + 10z\frac{\partial z}{\partial x} = 0$

$\frac{\partial z}{\partial x} = -\frac{4x}{10z} = -\frac{2x}{5z}$

2. $2z\frac{\partial z}{\partial x} - 10x + 0 = 0$

$\frac{\partial z}{\partial x} = \frac{10x}{2z} = \frac{5x}{z}$

3. $6z^2\frac{\partial z}{\partial y} - 0 - 8y = 0$

$\frac{\partial z}{\partial y} = \frac{8y}{6z^2} = \frac{4y}{3z^2}$

4. $0 + 2y + 6z^2\frac{\partial z}{\partial y} = 0$

$\frac{\partial z}{\partial y} = \frac{-2y}{6z^2} = -\frac{y}{3z^2}$

5. $x^2 - 2y - z^2 + y\left(x^2z^2\right) = 20$

$2x - 0 - 2z\frac{\partial z}{\partial x} + y\left[x^2 \cdot 2z\frac{\partial z}{\partial x} + z^2 \cdot 2x\right] = 0$

$\left(2x^2yz - 2z\right)\frac{\partial z}{\partial x} = -2x - 2xyz^2$

$\frac{\partial z}{\partial x} = \frac{-2x\left(1 + yz^2\right)}{2z\left(x^2y - 1\right)} = \frac{x\left(yz^2 + 1\right)}{z\left(1 - x^2y\right)}$

6. $3z^2\frac{\partial z}{\partial x} + 2x^2\left(2z\frac{\partial z}{\partial x}\right) + 2z^2(2x) - y = 0$

$(3z^2 + 4x^2z)\frac{\partial z}{\partial x} = y - 4xz^2$

$\frac{\partial z}{\partial x} = \frac{y - 4xz^2}{3z^2 + 4x^2z}$

7. $0 + e^y + e^z\frac{\partial z}{\partial y} = 0$

$\frac{\partial z}{\partial y} = -\frac{e^y}{e^z} = -e^{y-z}$

8. $xyz + 3y^3x^2 - \ln z^3 = 0$ so

$xyz + 3y^3x^2 - 3\ln z = 0.$

$xz + xy\frac{\partial z}{\partial y} + 9y^2x^2 - 3 \cdot \frac{1}{z}\frac{\partial z}{\partial y} = 0$

$\left(xy - \frac{3}{z}\right)\frac{\partial z}{\partial y} = -xz - 9y^2x^2$

$\left(\frac{xyz - 3}{z}\right)\frac{\partial z}{\partial y} = -xz - 9y^2x^2$

$\frac{\partial z}{\partial y} = \frac{z(-xz - 9y^2x^2)}{xyz - 3}$

9. $\frac{1}{z}\frac{\partial z}{\partial x} + 9\frac{\partial z}{\partial x} - y = 0$

$\left(\frac{1}{z} + 9\right)\frac{\partial z}{\partial x} = y$

$\left(\frac{1 + 9z}{z}\right)\frac{\partial z}{\partial x} = y$

$\frac{\partial z}{\partial x} = \frac{yz}{9 + z}$

10. $\frac{1}{x} + 0 - \frac{1}{z}\frac{\partial z}{\partial x} = 0$

$-\frac{1}{z}\frac{\partial z}{\partial x} = -\frac{1}{x}$

$\frac{\partial z}{\partial x} = \frac{z}{x}$

11. $\left(2z\frac{\partial z}{\partial y}+6x\right)\sqrt{x^3+5}=0$

$2z\frac{\partial z}{\partial y}+6x=0$

$\frac{\partial z}{\partial y}=\frac{-6x}{2z}=-\frac{3x}{z}$

12. $xz(1+y)-5=0$

$\left[x\frac{\partial z}{\partial x}+z\cdot 1\right](1+y)-0=0$

$x\frac{\partial z}{\partial x}+z=0$

$\frac{\partial z}{\partial x}=-\frac{z}{x}$

If $x = 1$, $y = 4$, $z = 1$, then $\frac{\partial z}{\partial x}=-\frac{1}{1}=-1$.

13. $3x\left(2z\frac{\partial z}{\partial x}\right)+3z^2+2y\left(2z\frac{\partial z}{\partial x}\right)-7y(4x^3)=0$

$(6xz+4yz)\frac{\partial z}{\partial x}=28x^3y-3z^2$

$\frac{\partial z}{\partial x}=\frac{28x^3y-3z^2}{6xz+4yz}$

$\left.\frac{\partial z}{\partial x}\right|_{(1,\ 0,\ 1)}=\frac{28(1)^3(0)-3(1)}{6(1)(1)-4(0)(1)}=\frac{-3}{6}=-\frac{1}{2}$

14. $e^{zx}\cdot x\frac{\partial z}{\partial y}=x\left[y\frac{\partial z}{\partial y}+z\cdot 1\right]$

$\left(xe^{zx}-xy\right)\frac{\partial z}{\partial y}=xz$

$\frac{\partial z}{\partial y}=\frac{xz}{x\left(e^{zx}-y\right)}$

$\frac{\partial z}{\partial y}=\frac{z}{e^{zx}-y}$

If $x = 1$, $y=-e^{-1}$, $z = -1$, then

$\frac{\partial z}{\partial y}=\frac{-1}{e^{-1}-\left(-e^{-1}\right)}=-\frac{e}{2}$.

15. $e^{yz}\cdot y\frac{\partial z}{\partial x}=-y\left[x\frac{\partial z}{\partial x}+z\cdot 1\right]$.

$\left(ye^{yz}+xy\right)\frac{\partial z}{\partial x}=-yz$

$\frac{\partial z}{\partial x}=-\frac{yz}{y\left(e^{yz}+x\right)}$

$\frac{\partial z}{\partial x}=-\frac{z}{e^{yz}+x}$

If $x=-\frac{e^2}{2}$, $y = 1$, $z = 2$, then

$\frac{\partial z}{\partial x}=-\frac{2}{e^2+\frac{-e^2}{2}}=-\frac{2}{\frac{e^2}{2}}=-\frac{4}{e^2}$.

16. $\frac{1}{2}\left(xz+y^2\right)^{-\frac{1}{2}}\left[x\frac{\partial z}{\partial y}+2y\right]-x=0$

$\frac{x}{2\sqrt{xz+y^2}}\frac{\partial z}{\partial y}=x-\frac{y}{\sqrt{xz+y^2}}=\frac{x\sqrt{xz+y^2}-y}{\sqrt{xz+y^2}}$

$\frac{\partial z}{\partial y}=\frac{2\left(x\sqrt{xz+y^2}-y\right)}{x}$

If $x = 2$, $y = 2$, $z = 6$, then $\frac{\partial z}{\partial y}=\frac{2(2\cdot 4-2)}{2}=6$.

17. $\frac{1}{z}\frac{\partial z}{\partial x}=4+0$

$\frac{\partial z}{\partial x}=4z$

If $x = 5$, $y = -20$, $z = 1$, then $\frac{\partial z}{\partial x}=4$.

18. $\frac{\left(s^2+t^2\right)\left[4rs^2\frac{\partial r}{\partial t}\right]-2r^2s^2[2t]}{\left(s^2+t^2\right)^2}=1$

$4rs^2\left(s^2+t^2\right)\frac{\partial r}{\partial t}=4r^2s^2t+\left(s^2+t^2\right)^2$

$\frac{\partial r}{\partial t}=\frac{4r^2s^2t+\left(s^2+t^2\right)^2}{4rs^2\left(s^2+t^2\right)}$

If $r = 1$, $s = 1$, $t = 1$, then $\frac{\partial r}{\partial t}=\frac{4+2^2}{4(2)}=1$.

19. $$\frac{(rs)\left[2t\frac{\partial t}{\partial r}\right]-\left(s^2+t^2\right)[s]}{(rs)^2}=0$$

$$2rst\frac{\partial t}{\partial r}-s\left(s^2+t^2\right)=0$$

$$2rst\frac{\partial t}{\partial r}=s\left(s^2+t^2\right)$$

$$\frac{\partial t}{\partial r}=\frac{s\left(s^2+t^2\right)}{2rst}=\frac{s^2+t^2}{2rt}$$

If $r = 1$, $s = 2$, $t = 4$, then $\frac{\partial t}{\partial r}=\frac{4+16}{2\cdot 1\cdot 4}=\frac{20}{8}=\frac{5}{2}$.

20. $$\frac{1}{x+y+z}\left(1+\frac{\partial z}{\partial x}\right)+yz+xy\frac{\partial z}{\partial x}=\frac{\partial z}{\partial x}e^{x+y+z}+ze^{x+y+z}\left(1+\frac{\partial z}{\partial x}\right)$$

When $x = 0$, $y = 1$, and $z = 0$, then

$$\frac{1}{1}\left(1+\frac{\partial z}{\partial x}\right)+(1)(0)+(0)(1)\frac{\partial z}{\partial x}=\frac{\partial z}{\partial x}e^1+0(e^1)\left(1+\frac{\partial z}{\partial x}\right)$$

$$1+\frac{\partial z}{\partial x}=e\frac{\partial z}{\partial x},\ 1=\frac{\partial z}{\partial x}(e-1),\ \frac{\partial z}{\partial x}=\frac{1}{e-1}$$

21. $c+\sqrt{c}=12+q_A\sqrt{9+q_B^2}$

a. If $q_A=6$ and $q_B=4$, then $c+\sqrt{c}=12+6(5)=42$, $\sqrt{c}=42-c$, $c=(42-c)^2=42^2-84c+c^2$, $c^2-85c+1764=0$,

$$c=\frac{85\pm\sqrt{(-85)^2-4(1)(1764)}}{2}=\frac{85\pm\sqrt{169}}{2}=\frac{85\pm 13}{2}$$

Thus $c = 49$ or $c = 36$. However $c = 49$ is extraneous but $c = 36$ is not. Thus $c = 36$.

b. Differentiating with respect to q_A:

$$\frac{\partial c}{\partial q_A}+\frac{1}{2\sqrt{c}}\cdot\frac{\partial c}{\partial q_A}=\sqrt{9+q_B^2}.$$

$$\left(1+\frac{1}{2\sqrt{c}}\right)\frac{\partial c}{\partial q_A}=\sqrt{9+q_B^2}.$$

When $q_A=6$ and $q_B=4$, then $c = 36$ and $\left(1+\frac{1}{12}\right)\frac{\partial c}{\partial q_A}=5$, $\frac{13}{12}\cdot\frac{\partial c}{\partial q_A}=5$, or

$$\frac{\partial c}{\partial q_A}=\frac{60}{13}.$$

Differentiating with respect to q_B:

$$\frac{\partial c}{\partial q_B}+\frac{1}{2\sqrt{c}}\cdot\frac{\partial c}{\partial q_B}=q_A\cdot\frac{q_B}{\sqrt{9+q_B^2}}$$

$$\left(1+\frac{1}{2\sqrt{c}}\right)\frac{\partial c}{\partial q_B}=\frac{q_Aq_B}{\sqrt{9+q_B^2}}$$

When $q_A=6$ and $q_B=4$, then $c = 36$ and $\left(1+\frac{1}{12}\right)\frac{\partial c}{\partial q_B}=\frac{24}{5}$, $\frac{13}{12}\cdot\frac{\partial c}{\partial q_B}=\frac{24}{5}$, or

$$\frac{\partial c}{\partial q_B}=\frac{288}{65}.$$

Problems 17.5

1. $f_x(x,\ y)=6(1)y^2=6y^2$
$f_{xy}(x,\ y)=6(2y)=12y$
$f_y(x,\ y)=6x(2y)=12xy$
$f_{yx}(x,\ y)=12(1)y=12y$

2. $f_x(x,\ y)=6x^2y^2+12xy^3-3y$
$f_{xx}(x,\ y)=12xy^2+12y^3$

3. $f_y(x,\ y)=3$
$f_{yy}(x,\ y)=0$
$f_{yyx}(x,\ y)=0$

4. $f_x(x, y) = \left(x^2 + xy + y^2\right)[2x + y] + \left(x^2 + xy + 1\right)[2x + y]$

$= (2x + y)\left[\left(x^2 + xy + y^2\right) + \left(x^2 + xy + 1\right)\right]$

$= (2x + y)\left(2x^2 + 2xy + y^2 + 1\right)$

$f_{xy}(x, y) = (2x + y)[2x + 2y] + \left(2x^2 + 2xy + y^2 + 1\right)[1]$

$= \left(4x^2 + 6xy + 2y^2\right) + \left(2x^2 + 2xy + y^2 + 1\right)$

$= 6x^2 + 8xy + 3y^2 + 1$

5. $f_y(x, y) = 9\left[e^{2xy}(2x)\right] = 18xe^{2xy}$

$f_{yx}(x, y) = 18\left[x\left(e^{2xy} \cdot 2y\right) + e^{2xy}(1)\right] = 18e^{2xy}(2xy + 1)$

$f_{yxy}(x, y) = 18\left[e^{2xy}(2x) + (2xy + 1)\left(e^{2xy} \cdot 2x\right)\right]$

$= 18e^{2xy}(2x)[1 + (2xy + 1)] = 18e^{2xy}(2x)[2 + 2xy]$

$= 72x(1 + xy)e^{2xy}$

6. $f_x(x, y) = \frac{1}{x^2 + y^2} \cdot 2x = \frac{2x}{x^2 + y^2}$

$f_{xx}(x, y) = \frac{\left(x^2 + y^2\right)[2] - (2x)[2x]}{\left(x^2 + y^2\right)^2} = \frac{2\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^2}$

$f_{xy}(x, y) = (2x)(-1)\left(x^2 + y^2\right)^{-2}[2y] = -\frac{4xy}{\left(x^2 + y^2\right)^2}$

7. $f(x, y) = (x + y)^2(xy) = \left(x^2 + 2xy + y^2\right)(xy) = x^3y + 2x^2y^2 + xy^3$

$f_x(x, y) = 3x^2y + 4xy^2 + y^3$

$f_y(x, y) = x^3 + 4x^2y + 3xy^2$

$f_{xx}(x, y) = 6xy + 4y^2$

$f_{yy}(x, y) = 4x^2 + 6xy$

8. $f_x(x, y, z) = 2xy^3z^4$

$f_{xz}(x, y, z) = 8xy^3z^3$

$f_z(x, y, z) = 4x^2y^3z^3$

$f_{zx}(x, y, z) = 8xy^3z^3$

9. $z = e^{\sqrt{x^2+y^2}}$

$$\frac{\partial z}{\partial y} = e^{\sqrt{x^2+y^2}} \cdot \frac{2y}{2\sqrt{x^2+y^2}} = \frac{ye^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} = \frac{zy}{\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\sqrt{x^2+y^2}\left(e^{\sqrt{x^2+y^2}} + y \cdot e^{\sqrt{x^2+y^2}} \cdot \frac{y}{\sqrt{x^2+y^2}}\right) - ye^{\sqrt{x^2+y^2}}\left(\frac{y}{\sqrt{x^2+y^2}}\right)}{\left(\sqrt{x^2+y^2}\right)^2}$$

$$= \frac{(x^2+y^2)\left(e^{\sqrt{x^2+y^2}} + e^{\sqrt{x^2+y^2}} \cdot \frac{y^2}{\sqrt{x^2+y^2}}\right) - y^2 e^{\sqrt{x^2+y^2}}}{(x^2+y^2)^{3/2}}$$

$$= e^{\sqrt{x^2+y^2}}\ \frac{x^2+y^2+y^2\sqrt{x^2+y^2}-y^2}{(x^2+y^2)^{3/2}}$$

$$= e^{\sqrt{x^2+y^2}}\ \frac{x^2+y^2\sqrt{x^2+y^2}}{(x^2+y^2)^{3/2}}$$

$$= z \cdot \frac{x^2+y^2\sqrt{x^2+y^2}}{(x^2+y^2)^{3/2}}$$

10. $\frac{\partial z}{\partial x} = \frac{1}{y} \cdot \frac{1}{x^2+5}(2x) = \frac{2x}{y\left(x^2+5\right)}$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{2x}{x^2+5}\left(-\frac{1}{y^2}\right) = -\frac{2x}{y^2\left(x^2+5\right)}$$

11. $f_y(x, y, z) = 0$

$f_{yx}(x, y, z) = 0$

$f_{yxx}(x, y, z) = 0$

Thus $f_{yxx}(4, 3, -2) = 0$.

12. $f_x(x, y, z) = z^2\left(6x - 4y^3\right)$

$f_{xy}(x, y, z) = z^2\left(-12y^2\right) = -12y^2z^2$

$f_{xyz}(x, y, z) = -24y^2z$. Thus $f_{xyz}(1, 2, 3) = -24(4)(3) = -288$.

13. $f_k(l, k) = 18l^3k^5 - 14l^2k^6$

$f_{kl}(l, k) = 54l^2k^5 - 28lk^6$

$f_{klk}(l, k) = 270l^2k^4 - 168lk^5$

Thus $f_{klk}(2, 1) = 270(4)(1) - 168(2)(1) = 744.$

14. $f_x(x, y) = 9x^2y^2 + y - 2xy^2$

$f_{xx}(x, y) = 18xy^2 - 2y^2$

$f_{xxy}(x, y) = 36xy - 4y$

$f_{xy} = 18x^2y + 1 - 4xy$

$f_{xyx} = 36xy - 4y = f_{xxy}(x, y)$

Thus $f_{xxy}(5, 1) = f_{xyx}(5, 1) = 36(5)(1) - 4 = 176.$

15. $f_x(x, y) = y^2e^x + \frac{1}{x}$

$f_{xy}(x, y) = 2ye^x$

$f_{xyy}(x, y) = 2e^x$

Thus $f_{xyy}(1, 1) = 2e$.

16. $f_x(x, y) = 3x^2 - 6y^2 + 2x$

$f_{xy}(x, y) = -12y$

Thus $f_{xy}(1, -1) = 12$.

17. $\frac{\partial c}{\partial q_B} = \frac{1}{3}\left(3q_A^2 + q_B^3 + 4\right)^{-\frac{2}{3}}\left(3q_B^2\right)$

$= q_B^2\left(3q_A^2 + q_B^3 + 4\right)^{-\frac{2}{3}}$

$\frac{\partial^2 c}{\partial q_A \partial q_B} = -\frac{2}{3}q_B^2\left(3q_A^2 + q_B^3 + 4\right)^{-\frac{5}{3}}(6q_A)$

$= -4q_Aq_B^2\left(3q_A^2 + q_B^3 + 4\right)^{-\frac{5}{3}}$

When $p_A = 25$ and $p_B = 4$, then $q_A = 10 - 25 + 16 = 1$ and $q_B = 20 + 25 - 44 = 1$, and $\frac{\partial^2 c}{\partial q_A \partial q_B} = -4(8)^{-\frac{5}{3}} = -\frac{4}{32} = -\frac{1}{8}$.

18. $f_x(x, y) = 4x^3y^4 + 9x^2y^2 - 7$

$f_{xy}(x, y) = 16x^3y^3 + 18x^2y$

$f_{xx}(x, y) = 12x^2y^4 + 18xy^2$

$f_{xyx}(x, y) = 48x^2y^3 + 36xy$

$f_{xxy}(x, y) = 48x^2y^3 + 36xy$

Thus $f_{xyx}(x, y) = f_{xxy}(xy)$.

19. $f_x(x, y) = 24x^2 + 4xy^2$

$f_y(x, y) = 4x^2y + 20y^3$

$f_{xy}(x, y) = 8xy$

$f_{yx}(x, y) = 8xy$

Thus $f_{xy}(x, y) = f_{yx}(x, y)$.

20. $f_x(x, y) = ye^{xy}$

$f_{xx}(x, y) = y^2e^{xy}$

$f_{xy}(x, y) = y(xe^{xy}) + e^{xy}(1) = e^{xy}(xy + 1)$

$f_y(x, y) = xe^{xy}$

$f_{yy}(x, y) = x^2e^{xy}$

$f_{yx}(x, y) = x(ye^{xy}) + e^{xy}(1) = e^{xy}(xy + 1)$

Thus, $f_{xx}(x, y) + f_{xy}(x, y) + f_{yx}(x, y) + f_{yy}(x, y)$

$= y^2e^{xy} + e^{xy}(xy + 1) + e^{xy}(xy + 1) + x^2e^{xy}$

$= e^{xy}(x^2 + 2xy + y^2 + 2)$

$= f(x, y)((x + y)^2 + 2)$

21. $\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}$

$\frac{\partial^2 z}{\partial x^2} = \frac{\left(x^2 + y^2\right)(2) - (2x)(2x)}{\left(x^2 + y^2\right)^2} = \frac{2\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^2}$

$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}$

$\frac{\partial^2 z}{\partial y^2} = \frac{\left(x^2 + y^2\right)(2) - (2y)(2y)}{\left(x^2 + y^2\right)^2} = \frac{2\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}$

$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^2} + \frac{2\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2} = 0$

22. $6z\frac{\partial z}{\partial x} - 6x^2 = 0$

$\frac{\partial z}{\partial x} = \frac{6x^2}{6z} = \frac{x^2}{z}$

$\frac{\partial^2 z}{\partial x^2} = \frac{z(2x) - x^2\frac{\partial z}{\partial x}}{z^2} = \frac{2xz - x^2\left(\frac{x^2}{z}\right)}{z^2} = \frac{2xz^2 - x^4}{z^3}$

23. $2z\frac{\partial z}{\partial y}+2y=0$

$\frac{\partial z}{\partial y}=-\frac{2y}{2z}=-\frac{y}{z}$

$\frac{\partial^2 z}{\partial y^2}=-\frac{z(1)-y\cdot\frac{\partial z}{\partial y}}{z^2}=-\frac{z-y\left(-\frac{y}{z}\right)}{z^2}=-\frac{z^2+y^2}{z^3}$

From the original equation, $z^2+y^2=3x^2$. Thus

$\frac{\partial^2 z}{\partial y^2}=-\frac{3x^2}{z^3}$.

24. $2z^2=x^2+2xy+xz$ (Eq. 1).

Differentiating both sides of Eq. 1 with respect to *y*:

$4z\frac{\partial z}{\partial y}=2x+x\frac{\partial z}{\partial y}$, $(4z-x)\frac{\partial z}{\partial y}=2x$,

$\frac{\partial z}{\partial y}=\frac{2x}{4z-x}$.

Differentiating both sides of Eq. 1 with respect to *x*:

$4z\frac{\partial z}{\partial x}=2x+2y+\left[x\frac{\partial z}{\partial x}+z(1)\right]$,

$(4z-x)\frac{\partial z}{\partial x}=2x+2y+z$, $\frac{\partial z}{\partial x}=\frac{2x+2y+z}{4z-x}$.

Differentiating $\frac{\partial z}{\partial y}$ with respect to *x*:

$$\frac{\partial^2 z}{\partial x\partial y}=2\cdot\frac{(4z-x)[1]-x\left[4\frac{\partial z}{\partial x}-1\right]}{(4z-x)^2}$$

$$=2\cdot\frac{(4z-x)-x\left[\frac{4(2x+2y+z)}{4z-x}-1\right]}{(4z-x)^2}$$

$$=2\cdot\frac{(4z-x)^2-x[4(2x+2y+z)-(4z-x)]}{(4z-x)^3}$$

$$=2\cdot\frac{16z^2-8xz-8x^2-8xy}{(4z-x)^3}$$

$$=16\cdot\frac{2z^2-xz-x^2-xy}{(4z-x)^3}$$

$$=16\cdot\frac{\left(x^2+2xy+xz\right)-xz-x^2-xy}{(4z-x)^3}$$

$$=\frac{16xy}{(4z-x)^3}.$$

Problems 17.6

1. $z=5x+3y$, $x=2r+3s$, $y=r-2s$

$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=(5)(2)+(3)(1)=13$

$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}=(5)(3)+(3)(-2)=9$

2. $z=2x^2+3xy+2y^2$, $x=r^2-s^2$, $y=r^2+s^2$

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$
$$=(4x+3y)(2r)+(3x+4y)(2r)$$
$$=14r(x+y)$$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$=(4x+3y)(-2s)+(3x+4y)(2s)$$
$$=-2s(x-y)$$

3. $z=e^{x+y}$, $x=t^2+3$, $y=\sqrt{t^3}$

$$\frac{dz}{dt}=\frac{\partial t}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$=e^{x+y}(2t)+e^{x+y}\left(\frac{3}{2}t^{1/2}\right)$$
$$=e^{x+y}\left(2t+\frac{3}{2}\sqrt{t}\right)$$

4. $z=\sqrt{8x+y}$, $x=t^2+3t+4$, $y=t^3+4$

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$=\frac{4}{\sqrt{8x+y}}(2t+3)+\frac{1}{2\sqrt{8x+y}}\left(3t^2\right)$$
$$=\frac{3t^2+16t+24}{2\sqrt{8x+y}}$$

5. $w = x^2z^2 + xyz + yz^2$, $x = 5t$, $y = 2t + 3$, $z = 6 - t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial t}\frac{dz}{dt}$$

$$= \left(2xz^2 + yz\right)(5) + \left(xz + z^2\right)(2) + \left(2x^2z + xy + 2yz\right)(-1)$$

$$= 5\left(2xz^2 + yz\right) + 2\left(xz + z^2\right) - \left(2x^2z + xy + 2yz\right)$$

6. $w = \ln\left(x^2 + y^2 + z^2\right)$, $x = 2 - 3t$, $y = t^2 + 3$, $z = 4 - t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= \frac{2x}{x^2 + y^2 + z^2}(-3) + \frac{2y}{x^2 + y^2 + z^2}(2t) + \frac{2z}{x^2 + y^2 + z^2}(-1)$$

$$= \frac{-2(3x - 2yt + z)}{x^2 + y^2 + z^2}$$

7. $z = \left(x^2 + xy^2\right)^3$, $x = r + s + t$, $y = 2r - 3s + 8t$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= 3\left(x^2 + xy^2\right)^2\left(2x + y^2\right)[1] + 3\left(x^2 + xy^2\right)^2(2xy)[8]$$

$$= 3\left(x^2 + xy^2\right)^2\left(2x + y^2 + 16xy\right)$$

8. $z = \sqrt{x^2 + y^2}$, $x = r^2 + s - t$, $y = r - s + t$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{x}{\sqrt{x^2 + y^2}}(2r) + \frac{y}{\sqrt{x^2 + y^2}}(1) = \frac{2xr + y}{\sqrt{x^2 + y^2}}$$

9. $w = x^2 + xyz + z^2$, $x = r^2 - s^2$, $y = rs$, $z = r^2 + s^2$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= (2x + yz)(-2s) + (xz)(r) + \left(xy + 2z\right)(2s)$$

$$= -2s(2x + yz) + r(xz) + 2s(xy + 2z)$$

10. $w = e^{xyz}$, $x = r^2s^3$, $y = \ln(r - s)$, $z = \sqrt{rs^2}$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= yze^{xyz}(2rs^3) + xze^{xyz}\left(\frac{1}{r - s}\right) + xye^{xyz}\left(\frac{s^2}{2\sqrt{rs^2}}\right)$$

$$= e^{xyz}\left(2rs^3yz + \frac{xz}{r - s} + \frac{xys^2}{2\sqrt{rs^2}}\right)$$

11. $y = x^2 - 7x + 5$, $x = 19rs + 2s^2t^2$

$$\frac{\partial y}{\partial r} = \frac{dy}{dx}\frac{\partial x}{\partial r} = (2x-7)(19s) = 19s(2x-7)$$

12. $y = 4 - x^2$, $x = 2r + 3s - 4t$

$$\frac{\partial y}{\partial t} = \frac{dy}{dx}\frac{\partial x}{\partial t} = (-2x)(-4) = 8x$$

13. $z = (4x+3y)^3$, $x = r^2s$, $y = r - 2s$; $r = 0$, $s = 1$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$= 12(4x+3y)^2(2rs) + 9(4x+3y)^2(1)$$

$$= 3(4x+3y)^2(8rs+3)$$

When $r = 0$, $s = 1$, then $x = 0$, $y = -2$, and $\frac{\partial z}{\partial r} = 324$.

14. $z = \sqrt{2x+3y}$, $x = 3t + 5$, $y = t^2 + 2t + 1$; $t = 1$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \frac{2}{2\sqrt{2x+3y}}(3) + \frac{3}{2\sqrt{2x+3y}}(2t+2)$$

$$= \frac{3(t+2)}{\sqrt{2x+3y}}$$

When $t = 1$, then $x = 8$, $y = 4$ and $\frac{dz}{dt} = \frac{9}{\sqrt{28}} = \frac{9}{2\sqrt{7}}$.

15. $w = e^{2x+3y}(x^2 + 4z^2)$, $x = rs$, $y = 2s - 3r$, and $z = r + s$; $r = 1$, $s = 0$.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= [2e^{2x+3y}(x^2+4z^2) + e^{2x+3y}(2x)](r) + 3e^{2x+3y}(x^2+4z^2)(2) + e^{2x+3y}(8z)(1)$$

When $r = 1$, $s = 0$, then $x = 0$, $y = -3$ and $z = 1$.

$$\frac{\partial w}{\partial s} = [2e^{-9}(4) + e^{-9}(0)](1) + 3e^{-9}(4)(2) + e^{-9}(8)(1)$$

$$= e^{-9}[8 + 24 + 8] = \frac{40}{e^9}$$

16. $y = \frac{x}{x-5}$, $x = 2t^2 - 3rs - r^2t$; $r = 0$, $s = 2$, $t = -1$

$$\frac{\partial y}{\partial t} = \frac{dy}{dx}\frac{\partial x}{\partial t} = \frac{-5}{(x-5)^2}\left(4t - r^2\right)$$

When $r = 0$, $s = 2$, and $t = -1$, then $x = 2$ and $\frac{\partial y}{\partial t} = \frac{20}{9}$

17. $\dfrac{\partial c}{\partial p_A} = \dfrac{\partial c}{\partial q_A}\dfrac{\partial q_A}{\partial p_A} + \dfrac{\partial c}{\partial q_B}\dfrac{\partial q_B}{\partial p_A}$

$$= \left[\frac{1}{3}\left(3q_A^2 + q_B^3 + 4\right)^{-\frac{2}{3}}(6q_A)\right](-1) + \left[\frac{1}{3}\left(3q_A^2 + q_B^3 + 4\right)^{-\frac{2}{3}}\left(3q_B^2\right)\right](1)$$

$$= \left(3q_A^2 + q_B^3 + 4\right)^{-\frac{2}{3}}\left(-2q_A + q_B^2\right)$$

$$\frac{\partial c}{\partial p_B} = \frac{\partial c}{\partial q_A}\frac{\partial q_A}{\partial p_B} + \frac{\partial c}{\partial q_B}\frac{\partial q_B}{\partial p_B}$$

$$= \left[\frac{1}{3}\left(3q_A^2 + q_B^3 + 4\right)^{-\frac{2}{3}}(6q_A)\right](2p_B) + \left[\frac{1}{3}\left(3q_A^2 + q_B^3 + 4\right)^{-\frac{2}{3}}\left(3q_B^2\right)\right](-11)$$

$$= \left(3q_A^2 + q_B^3 + 4\right)^{-\frac{2}{3}}\left(4q_A p_B - 11q_B^2\right)$$

When $p_A = 25$ and $p_B = 4$, then $q_A = 10 - 25 + 16 = 1$, $q_B = 20 + 25 - 44 = 1$, and $\dfrac{\partial c}{\partial p_A} = (8)^{-\frac{2}{3}}(-1) = -\dfrac{1}{4}$ and $\dfrac{\partial c}{\partial p_B} = (8)^{-\frac{2}{3}}(5) = \dfrac{5}{4}$.

18. a. $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$

b. Since $\dfrac{dy}{dt} = 1$, from (a), $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}$

19. a. $\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t}$

b. $w = 2x^2 \ln|3x - 5y|$, $x = s\sqrt{t^2 + 2}$ and $y = t - 3e^{2-s}$.

$$\frac{\partial w}{\partial t} = \left[4x\ln|3x - 5y| + \frac{2x^2(3)}{3x - 5y}\right]\frac{s(2t)}{2\sqrt{t^2 + 2}} + \left[\frac{2x^2}{3x - 5y}(-5)\right](1)$$

When $s = 1$ and $t = 0$, then $x = \sqrt{2}$ and $y = -3e$.

$$\frac{\partial w}{\partial t} = \left[4\sqrt{2}\ln\left|3\sqrt{2} - 5(-3e)\right| + \frac{2(2)(3)}{3\sqrt{2} - 5(-3e)}\right](0) + \left[\frac{2(2)}{3\sqrt{2} - 5(-3e)}(-5)\right]$$

$$= -\frac{20}{3\sqrt{2} + 15e}$$

20. $p = aP - whL$, where $P = f(l, k)$ and $l = Lg(h)$.

$$\frac{\partial p}{\partial L} = a\frac{\partial P}{\partial L} - wh = a\left[\frac{\partial P}{\partial l}\frac{\partial l}{\partial L} + \frac{\partial P}{\partial k}\frac{\partial k}{\partial L}\right] - wh$$

$$= a\left[\frac{\partial P}{\partial l}g(h) + \frac{\partial P}{\partial k}\cdot 0\right] - wh = a\frac{\partial P}{\partial l}g(h) - wh$$

$$\frac{\partial p}{\partial h} = a\frac{\partial P}{\partial h} - wL = a\left[\frac{\partial P}{\partial l}\frac{\partial l}{\partial h} + \frac{\partial P}{\partial k}\frac{\partial k}{\partial h}\right] - wL$$

$$= a\left[\frac{\partial P}{\partial l}Lg'(h) + \frac{\partial P}{\partial k}\cdot 0\right] - wL$$

$$= a\frac{\partial P}{\partial l}Lg'(h) - wL$$

Problems 17.7

1. $f(x, y) = x^2 + y^2 - 5x + 4y + xy$

$$\begin{cases} f_x(x, y) = 2x + y - 5 = 0 \\ f_y(x, y) = x + 2y + 4 = 0 \end{cases}$$

Solving the system gives the critical point $\left(\frac{14}{3}, -\frac{13}{3}\right)$.

2. $f(x, y) = x^2 + 4y^2 - 6x + 16y$

$$\begin{cases} f_x(x, y) = 2x - 6 = 0 \\ f_y(x, y) = 8y + 16 = 0 \end{cases}$$

Critical point: (3, –2)

3. $f(x, y) = \frac{5}{3}x^3 + \frac{2}{3}y^3 - \frac{15}{2}x^2 + y^2 - 4y + 7$

$$\begin{cases} f_x(x, y) = 5x^2 - 15x = 0 \\ f_y(x, y) = 2y^2 + 2y - 4 = 0 \end{cases}$$

Both equations are easily solved by factoring.
Critical points: (0, –2), (0, 1), (3, –2), (3, 1)

4. $f(x, y) = xy - x + y$

$f_x(x, y) = y - 1$

$f_y(x, y) = x + 1$

Critical point: (–1, 1)

5. $f(x, y, z) = 2x^2 + xy + y^2 + 100 - z(x + y - 200)$

$$\begin{cases} f_x(x, y, z) = 4x + y - z = 0 \\ f_y(x, y, z) = x + 2y - z = 0 \\ f_z(x, y, z) = -x - y + 200 = 0 \end{cases}$$

Solving the system gives the critical point (50, 150, 350).

6. $f(x, y, z, w) = x^2 + y^2 + z^2 - w(x - y + 2z - 6)$

$$\begin{cases} f_x(x, y, z, w) = 2x - w = 0 \\ f_y(x, y, z, w) = 2y + w = 0 \\ f_z(x, y, z, w) = 2z - 2w = 0 \\ f_w(x, y, z, w) = -x + y - 2z + 6 = 0 \end{cases}$$

Solving the system gives the critical point (1, –1, 2, 2).

7. $f(x, y) = x^2 + 3y^2 + 4x - 9y + 3$

$$\begin{cases} f_x(x, y) = 2x + 4 = 0 \\ f_y(x, y) = 6y - 9 = 0 \end{cases}$$

Critical point $\left(-2, \frac{3}{2}\right)$

Second-Derivative Test

$f_{xx}(x, y) = 2$, $f_{yy}(x, y) = 6$, $f_{xy}(x, y) = 0$. At $\left(-2, \frac{3}{2}\right)$, $D = (2)(6) - 0^2 = 12 > 0$ and $f_{xx}(x, y) = 2 > 0$. Thus at $\left(-2, \frac{3}{2}\right)$ there is a relative minimum.

8. $f(x, y) = -2x^2 + 8x - 3y^2 + 24y + 7$

$$\begin{cases} f_x(x, y) = -4x + 8 = 0 \\ f_y(x, y) = -6y + 24 = 0 \end{cases}$$

Critical point: (2, 4)
Second-Derivative Test

$f_{xx}(x, y) = -4$, $f_{yy}(x, y) = -6$, $f_{xy}(x, y) = 0$. At (2, 4), $D = (-4)(-6) - 0^2 = 24 > 0$ and $f_{xx}(x, y) = -4 < 0$; thus there is a relative maximum at (2, 4).

9. $f(x, y) = y - y^2 - 3x - 6x^2$

$$\begin{cases} f_x(x, y) = -3 - 12x = 0 \\ f_y(x, y) = 1 - 2y = 0 \end{cases}$$

Critical point $\left(-\frac{1}{4}, \frac{1}{2}\right)$

Second-Derivative Test

$f_{xx}(x, y) = -12$, $f_{yy}(x, y) = -2$, $f_{xy}(x, y) = 0$

At $\left(-\frac{1}{4}, \frac{1}{2}\right)$, $D = (-12)(-2) - 0^2 = 24 > 0$ and $f_{xx}(x, y) = -12 < 0$. Thus at $\left(-\frac{1}{4}, \frac{1}{2}\right)$ there is a relative maximum.

10. $f(x, y) = 2x^2 + \frac{3}{2}y^2 + 3xy - 10x - 9y + 2$

$\begin{cases} f_x(x, y) = 4x + 3y - 10 = 0 \\ f_y(x, y) = 3y + 3x - 9 = 0 \end{cases}$

Critical point: (1, 2)
Second-Derivative Test
$f_{xx}(x, y) = 4,\ f_{yy}(x, y) = 3,\ f_{xy}(x, y) = 3.$

At (1, 2), $D = (4)(3) - 3^2 = 3 > 0$ and $f_{xx}(x, y) = 4 > 0;$ thus there is a relative minimum at (1, 2).

11. $f(x, y) = x^2 + 3xy + y^2 + x + 3$

$\begin{cases} f_x(x, y) = 2x + 3y + 1 = 0 \\ f_y(x, y) = 3x + 2y = 0 \end{cases}$

Critical point: $\left(\frac{2}{5}, -\frac{3}{5}\right)$

Second-Derivative Test

$f_{xx}(x, y) = 2,\quad f_{yy} = 2,\ f_{xy} = 3.$ At $\left(\frac{2}{5}, -\frac{3}{5}\right)$, $D = (2)(2) - (3)^2 = -5 < 0,$ so there is no relative extremum at $\left(\frac{2}{5}, -\frac{3}{5}\right)$.

12. $f(x, y) = \frac{x^3}{3} + y^2 - 2x + 2y - 2xy$

$\begin{cases} f_x(x, y) = x^2 - 2 - 2y = 0 \\ f_y(x, y) = 2y + 2 - 2x = 0 \end{cases}$

Critical points: (2, 1), (0, –1)
Second-Derivative Test
$f_{xx}(x, y) = 2x,\ f_{yy}(x, y) = 2,\ f_{xy}(x, y) = -2.$

At (2, 1), $D = (4)(2) - (-2)^2 = 4 > 0$ and $f_{xx}(x, y) = 4 > 0$, so a relative minimum at (2, 1). At (0, –1), $D = (0)(2) - (-2)^2 = -4 < 0$; thus neither at (0, –1).

13. $f(x, y) = \frac{1}{3}\left(x^3 + 8y^3\right) - 2\left(x^2 + y^2\right) + 1$

$\begin{cases} f_x(x, y) = x^2 - 4x = 0 \\ f_y(x, y) = 8y^2 - 4y = 0 \end{cases}$

Critical points: (0, 0), $\left(4, \frac{1}{2}\right)$, $\left(0, \frac{1}{2}\right)$, (4, 0)

Second-Derivative Test
$f_{xx}(x, y) = 2x - 4,\ f_{yy}(x, y) = 16y - 4,$ $f_{xy}(x, y) = 0$. At (0, 0),
$D = (-4)(-4) - 0^2 = 16 > 0$ and $f_{xx}(x, y) = -4 < 0$; thus a relative maximum.

At $\left(4, \frac{1}{2}\right)$, $D = (4)(4) - 0^2 = 16 > 0$ and $f_{xx}(x, y) = 4 > 0$; thus a relative minimum.

At $\left(0, \frac{1}{2}\right)$, $D = (-4)(4) - 0^2 = -16 < 0$; thus neither.

At (4, 0), $D = (4)(-4) - 0^2 = -16 < 0$, thus neither.

14. $f(x, y) = x^2 + y^2 - xy + x^3$

$\begin{cases} f_x(x, y) = 2x - y + 3x^2 = 0 \\ f_y(x, y) = 2y - x = 0 \end{cases}$

Critical points: (0, 0), $\left(-\frac{1}{2}, -\frac{1}{4}\right)$

Second-Derivative Test
$f_{xx}(x, y) = 2 + 6x,\ f_{yy}(x, y) = 2,$ $f_{xy}(x, y) = -1$. At (0, 0),
$D = (2)(2) - (-1)^2 = 3 > 0$ and $f_{xx}(x, y) = 2 > 0$; thus relative minimum.

At $\left(-\frac{1}{2}, -\frac{1}{4}\right)$, $D = (-1)(2) - (-1)^2 = -3 < 0$; thus neither.

15. $f(l, k) = \frac{l^2}{2} + 2lk + 3k^2 - 69l - 164k + 17$

$\begin{cases} f_l(l, k) = l + 2k - 69 = 0 \\ f_k(l, k) = 2l + 6k - 164 = 0 \end{cases}$

Critical point: (43, 13)
Second-Derivative Test
$f_{ll}(l, k) = 1,\ f_{kk}(l, k) = 6,\ f_{lk}(l, k) = 2$

At (43, 13), $D = (1)(6) - 2^2 = 2 > 0$ and $f_{ll}(l, k) = 1 > 0$; thus there is a relative minimum at (43, 13).

16. $f(l, k) = l^2 + k^2 - 2lk$

$\begin{cases} f_l(l, k) = 2l - 2k \\ f_k(l, k) = 2k - 2l \end{cases}$

Critical points: (r, r) where r is any real number.
Second Derivative Test
$f_{ll}(l, k) = 2,\ f_{kk}(l, k) = 2,$ and $f_{lk}(l, k) = -2.$
At (r, r), $D = (2)(2) - (-2)^2 = 0,$ thus we cannot make a conclusion.

17. $f(p, q) = pq - \frac{1}{p} - \frac{1}{q}$

$\begin{cases} f_p(p, q) = q + \frac{1}{p^2} = 0 \\ f_q(p, q) = p + \frac{1}{q^2} = 0 \end{cases}$

Critical point: (–1, –1)
Second-Derivative Test
$f_{pp}(p, q) = -\frac{2}{p^3}$, $f_{qq}(p, q) = -\frac{2}{q^3}$,
$f_{pq}(p, q) = 1$. At (–1, –1),
$D = (2)(2) - 1^2 = 3 > 0$ and $f_{pp}(p, q) = 2 > 0$;
thus there is a relative minimum at (–1, –1).

18. $f(x, y) = (x-3)(y-3)(x+y-3)$
$= (y-3)(x^2 + xy - 6x - 3y + 9)$
$= (x-3)(xy - 3x + y^2 - 6y + 9)$

$\begin{cases} f_x(x, y) = (y-3)(2x+y-6) = 0 \\ f_y(x, y) = (x-3)(x+2y-6) = 0 \end{cases}$

Critical points: (2, 2), (3, 3), (3, 0), (0, 3)
Second-Derivative Test
$f_{xx}(x, y) = 2(y-3)$, $f_{yy}(x, y) = 2(x-3)$,
$f_{xy}(x, y) = 2x + 2y - 9$. At (2, 2),
$D = (-2)(-2) - (-1)^2 = 3 > 0$ and
$f_{xx}(x, y) = -2 < 0$; thus relative maximum.
At (3, 3), $D = (0)(0) - 3^2 = -9 < 0$; thus neither.
At (3, 0), $D = (-6)(0) - (-3)^2 = -9 < 0$; thus neither. At (0, 3), $D = (0)(-6) - (-3)^2 = -9 < 0$; thus neither.

19. $f(x, y) = (y^2 - 4)(e^x - 1)$

$\begin{cases} f_x(x, y) = e^x(y^2 - 4) = 0 & (1) \\ f_y(x, y) = 2y(e^x - 1) = 0 & (2) \end{cases}$

Critical points: (0, –2), (0, 2)
[Note that $y = 0$ does not give rise to a common solution of (1) and (2).]
Second-Derivative Test
$f_{xx}(x, y) = e^x(y^2 - 4)$, $f_{yy}(x, y) = 2(e^x - 1)$,
$f_{xy}(x, y) = 2ye^x$. At (0, –2),
$D = (0)(0) - (-4)^2 = -16 < 0$; thus neither. At (0, 2), $D = (0)(0) - (4)^2 = -16 < 0$; thus neither.

20. $f(x, y) = \ln(xy) + 2x^2 - xy - 6x$

$\begin{cases} f_x(x, y) = \frac{1}{x} + 4x - y - 6 = 0 \\ f_y(x, y) = \frac{1}{y} - x = 0 \end{cases}$

The only critical point is $\left(\frac{3}{2}, \frac{2}{3}\right)$.

$f_{xx}(x, y) = -\frac{1}{x^2} + 4$, $f_{yy}(x, y) = -\frac{1}{y^2}$,
$f_{xy}(x, y) = -1$. At $\left(\frac{3}{2}, \frac{2}{3}\right)$,
$D = \left(\frac{32}{9}\right)\left(\frac{-9}{4}\right) - (-1)^2 = -9 < 0$; thus neither.

21. $P = f(l, k)$
$= 1.08l^2 - 0.03l^3 + 1.68k^2 - 0.08k^3$

$\begin{cases} P_l = 2.16l - 0.09l^2 = 0 \\ P_k = 3.36k - 0.24k^2 = 0 \end{cases}$

Critical points: (0, 0), (0, 14), (24, 0), (24, 14)
Second-Derivative Test
$P_{ll} = 2.16 - 0.18l$, $P_{kk} = 3.36 - 0.48k$, $P_{lk} = 0$.
At (0, 0), $D = (2.16)(3.36) - 0^2 > 0$ and $P_{ll} = 2.16 > 0$; thus relative minimum.
At (0, 14), $D = (2.16)(-3.36) - 0^2 < 0$; thus no extremum. At (24, 0),
$D = (-2.16)(3.36) - 0^2 < 0$; thus no extremum.
At (24, 14), $D = (-2.16)(-3.36) - 0^2 > 0$ and $P_{ll} = -2.16 < 0$; thus $l = 24$, $k = 14$ gives a relative maximum.

22. $Q = 18c + 20d - 2c^2 - 4d^2 - cd$

$$\begin{cases} Q_c = 18 - 4c - d = 0 \\ Q_d = 20 - 8d - c = 0 \end{cases}$$

Critical point: $c = 4$, $d = 2$

$Q_{cc} = -4$, $Q_{dd} = -8$, $Q_{cd} = -1$

When $c = 4$ and $d = 2$, then $D = (-4)(-8) - (-1)^2 > 0$ and $Q_{cc} = -4 < 0$; thus relative maximum at $c = 4$, $d = 2$.

23. Profit per lb for A = $p_\text{A} - 60$.

Profit per lb for B = $p_\text{B} - 70$.

Total Profit $= P = (p_\text{A} - 60)q_\text{A} + (p_\text{B} - 70)q_\text{B}$

$P = (p_\text{A} - 60)[5(p_\text{B} - p_\text{A})] + (p_\text{B} - 70)[500 + 5(p_\text{A} - 2p_\text{B})]$

Thus

$$\begin{cases} \dfrac{\partial P}{\partial p_\text{A}} = -10(p_\text{A} - p_\text{B} + 5) = 0 \\ \dfrac{\partial P}{\partial p_\text{B}} = 10(p_\text{A} - 2p_\text{B} + 90) = 0 \end{cases}$$

Critical point: $p_\text{A} = 80$, $p_\text{B} = 85$

$\dfrac{\partial^2 P}{\partial p_\text{A}^2} = -10$, $\dfrac{\partial^2 P}{\partial p_\text{B}^2} = -20$, $\dfrac{\partial^2 P}{\partial p_\text{B} \partial p_\text{A}} = 10$. When $p_\text{A} = 80$ and $p_\text{B} = 85$, then

$D = (-10)(-20) - (10)^2 = 100 > 0$ and $\dfrac{\partial^2 P}{\partial p_\text{A}^2} = -10 < 0$; thus relative maximum at $p_\text{A} = 80$, $p_\text{B} = 85$.

24. Profit per lb for A = $p_\text{A} - a$.

Profit per lb for $B = p_\text{B} - b$.

Total Profit $= P = (p_\text{A} - a)q_\text{A} + (p_\text{B} - b)q_\text{B}$

$P = (p_\text{A} - a)[5(p_\text{B} - p_\text{A})] + (p_\text{B} - b)[500 + 5(p_\text{A} - 2p_\text{B})]$

$$\begin{cases} \dfrac{\partial P}{\partial p_\text{A}} = -5(2p_\text{A} - 2p_\text{B} + b - a) = 0 \\ \dfrac{\partial P}{\partial p_\text{B}} = 5(2p_\text{A} - 4p_\text{B} + 2b - a + 100) = 0 \end{cases}$$

Critical point: $p_\text{A} = 50 + \dfrac{a}{2}$, $p_\text{B} = 50 + \dfrac{b}{2}$

$\dfrac{\partial^2 P}{\partial p_\text{A}^2} = -10$, $\dfrac{\partial^2 P}{\partial p_\text{B}^2} = -20$, $\dfrac{\partial^2 P}{\partial p_\text{B} \partial p_\text{A}} = 10$

When $p_\text{A} = 50 + \dfrac{a}{2}$ and $p_\text{B} = 50 + \dfrac{b}{2}$, then $D = (-10)(-20) - (10)^2 = 100 > 0$ and $\dfrac{\partial^2 P}{\partial p_\text{A}^2} = -10 < 0$; thus a relative maximum at $p_\text{A} = 50 + \dfrac{a}{2}$, $p_\text{B} = 50 + \dfrac{b}{2}$.

25. $p_A = 100 - q_A$, $p_B = 84 - q_B$, $c = 600 + 4(q_A + q_B)$.

Revenue from market A $= r_A = p_A q_A = (100 - q_A)q_A$. Revenue from market B $= r_B = p_B q_B = (84 - q_B)q_B$.

Total Profit = Total Revenue – Total Cost

$P = (100 - q_A)q_A + (84 - q_B)q_B - [600 + 4(q_A + q_B)]$

$$\begin{cases} \frac{\partial P}{\partial q_A} = 96 - 2q_A = 0 \\ \frac{\partial P}{\partial q_B} = 80 - 2q_B = 0 \end{cases}$$

Critical point: $q_A = 48$, $q_B = 40$

$\frac{\partial^2 P}{\partial q_A^2} = -2$, $\frac{\partial^2 P}{\partial q_B^2} = -2$, $\frac{\partial^2 P}{\partial q_B \partial q_A} = 0$.

At $q_A = 48$ and $q_B = 40$, then $D = (-2)(-2) - 0^2 = 4 > 0$ and $\frac{\partial^2 P}{\partial q_A^2} = -2 < 0$; thus relative maximum at $q_A = 48$, $q_B = 40$. When $q_A = 48$ and $q_B = 40$, then selling prices are $p_A = 52$, $p_B = 44$, and profit = 3304.

26. $q_A = 3 - p_A + 2p_B$, $q_B = 5 + 5p_A - 2p_B$

Revenue from A = $p_A q_A$. Revenue from B = $p_B q_B$.

Total cost of producing q_A units of A and q_B units of B is $3q_A + 2q_B$.

Total Profit = Total Revenue – Total Cost

$P = p_A q_A + p_B q_B - (3q_A + 2q_B)$

$P = -p_A^2 - 2p_B^2 + 7p_A p_B - 4p_A + 3p_B - 19$

$$\begin{cases} \frac{\partial P}{\partial p_A} = -2p_A + 7p_B - 4 \\ \frac{\partial P}{\partial p_B} = 7p_A - 4p_B + 3 \end{cases}$$

Critical point: $p_A = -\frac{5}{41}$, $p_B = \frac{22}{41}$

$\frac{\partial^2 P}{\partial p_A^2} = -2$, $\frac{\partial^2 P}{\partial p_B^2} = -4$, $\frac{\partial^2 P}{\partial p_B \partial p_A} = 7$

At $p_A = -\frac{5}{41}$, $p_B = \frac{22}{41}$, then $D = (-2)(-4) - 7^2 = -41 < 0$, so this is not a relative extremum. Thus the profit cannot be maximized.

27. $c = \frac{3}{2}q_A^2 + 3q_B^2$, $p_A = 60 - q_A^2$, $p_B = 72 - 2q_B^2$

Total Profit = Total Revenue – Total Cost

$P = (p_A q_A + p_B q_B) - c$

$P = 60q_A - q_A^3 + 72q_B - 2q_B^3 - \left(\frac{3}{2}q_A^2 + 3q_B^2\right)$

$$\begin{cases} \frac{\partial P}{\partial q_A} = 60 - 3q_A - 3q_A^2 = 3(5 + q_A)(4 - q_A) \\ \frac{\partial P}{\partial q_B} = 72 - 6q_B - 6q_B^2 = 6(4 + q_B)(3 - q_B) \end{cases}$$

Since we want $q_A \geq 0$ and $q_B \geq 0$, the critical point occurs when $q_A = 4$ and $q_B = 3$.

$\frac{\partial^2 P}{\partial q_A^2} = -3 - 6q_A$, $\frac{\partial^2 P}{\partial q_B^2} = -6 - 12q_B$, $\frac{\partial^2 P}{\partial q_B \partial q_A} = 0$. When $q_A = 4$ and $q_B = 3$, then $D = (-27)(-42) - 0^2 > 0$ and $\frac{\partial^2 P}{\partial q_A^2} = -27 < 0$; thus relative maximum at $q_A = 4$, $q_B = 3$.

28. $c = 2(q_A + q_B + q_A q_B)$,
Total Profit = Total Revenue − Total Cost

$$P = (p_A q_A + p_B q_B) - c$$
$$= p_A(20 - 2p_A) + p_B(10 - p_B) - [20 - 2p_A + 10 - p_B + (20 - 2p_A)(10 - p_B)]$$
$$= -2p_A^2 - p_B^2 - 2p_A p_B + 42p_A + 31p_B + 230$$

$$\begin{cases} \frac{\partial P}{\partial p_A} = -4p_A - 2p_B + 42 \\ \frac{\partial P}{\partial p_B} = -2p_A - 2p_B + 31 \end{cases}$$

Critical point: $p_A = \frac{11}{2}$, $p_B = 10$

$\frac{\partial^2 P}{\partial p_A^2} = -4$, $\frac{\partial^2 P}{\partial p_B^2} = -2$, $\frac{\partial^2 P}{\partial p_B \partial p_A} = -2$

When $p_A = \frac{11}{2}$, $p_B = 10$, then $D = (-4)(-2) - (-2)^2 = 4 > 0$, and $\frac{\partial^2 p}{\partial p_A^2} = -4 < 0$, so the maximum profit occurs when $p_A = 5.5$ and $p_B = 10$. At these prices, $q_A = 9$, $q_B = 0$, and the total profit is 40.5.

29. Refer to the diagram in the text.
$xyz = 6$
$C = 3xy + 2[1(xz)] + 2[0.5(yz)]$
Note that $z = \frac{6}{xy}$. Thus

$$C = 3xy + 2xz + yz = 3xy + 2x\left(\frac{6}{xy}\right) + y\left(\frac{6}{xy}\right) = 3xy + \frac{12}{y} + \frac{6}{x}$$

$$\begin{cases} \frac{\partial C}{\partial x} = 3y - \frac{6}{x^2} = 0 \\ \frac{dC}{dy} = 3x - \frac{12}{y^2} = 0 \end{cases}$$

A critical point occurs at $x = 1$ and $y = 2$. Thus $z = 3$.

$\frac{\partial^2 C}{\partial x^2} = \frac{12}{x^3}$, $\frac{\partial^2 C}{\partial y^2} = \frac{24}{y^3}$, $\frac{\partial^2 C}{\partial x \partial y} = 3$.

When $x = 1$ and $y = 2$, then $d = (12)(3) - (3)^2 = 27 > 0$ and $\frac{\partial^2 C}{\partial x^2} = 12 > 0$. Thus we have a minimum. The dimensions should be 1 ft by 2 ft by 3 ft.

30. $p = 92 - q_A - q_B$, $c_A = 10q_A$, $c_B = 0.5q_B^2$
Since Profit = Total Revenue – Total Cost, then
Profit of A $= pq_A - c_A$ and
Profit of B $= pq_B - c_B$.
Thus profit P of monopoly is
$P = pq_A - c_A + pq_B - c_B$
$= p(q_A + q_B) - c_A - c_B$
$= (92 - q_A - q_B)(q_A + q_B) - 10q_A - 0.5q_B^2$
$= 82q_A + 92q_B - q_A^2 - 2q_Aq_B - 1.5q_B^2$

$$\begin{cases} \dfrac{\partial P}{\partial q_A} = 82 - 2q_A - 2q_B = 0 \\ \dfrac{\partial P}{\partial q_B} = 92 - 2q_A - 3q_B = 0 \end{cases}$$

Critical point: $q_A = 31, q_B = 10$

$$\frac{\partial^2 P}{\partial q_A^2} = -2, \frac{\partial^2 P}{\partial q_B^2} = -3, \frac{\partial^2 P}{\partial q_B \partial q_A} = -2$$

When $q_A = 31$ and $q_B = 10$, then
$D = (-2)(-3) - (-2)^2 = 2 > 0$ and
$\dfrac{\partial^2 P}{\partial q_A^2} = -2 < 0$; thus relative maximum at
$q_A = 31, q_B = 10$.

31. $y = \dfrac{3x - 7}{2}$

$$f(x, y) = -2x^2 + 5\left(\frac{3x - 7}{2}\right)^2 + 7$$

Setting the derivative equal to 0 gives
$-4x + 5(2)\left(\dfrac{3x - 7}{2}\right)\left(\dfrac{3}{2}\right) = 0$,
$-4x + \dfrac{15}{2}(3x - 7) = 0$, $-8x + 15(3x - 7) = 0$,
$37x = 105$, or $x = \dfrac{105}{37}$. The second-derivative is
$\dfrac{37}{2} > 0$, so we have a relative minimum. If
$x = \dfrac{105}{37}$, then $y = \dfrac{28}{37}$. Thus there is a relative
minimum at $\left(\dfrac{105}{37}, \dfrac{28}{37}\right)$.

32. $y = \dfrac{x - 10}{4}$

$$f(x, y) = x^2 + 4\left(\frac{x - 10}{4}\right)^2 + 6$$

Setting the derivative equal to 0 gives
$2x + 4(2)\left(\dfrac{x - 10}{4}\right)\left(\dfrac{1}{4}\right) = 0$, from which $x = 2$.
The second-derivative is $\dfrac{5}{2} > 0$, so we have a relative minimum If $x = 2$, then $y = -2$. Thus at $(2, -2)$ there is a relative minimum

33. $c = q_A^2 + 3q_B^2 + 2q_Aq_B + aq_A + bq_B + d$
We are given that $(q_A, q_B) = (3, 1)$ is a critical point.

$$\begin{cases} \dfrac{\partial c}{\partial q_A} = 2q_A + 2q_B + a = 0 \\ \dfrac{\partial c}{\partial q_B} = 6q_B + 2q_A + b = 0 \end{cases}$$

Substituting the given values for q_A and q_B into both equations gives $a = -8$ and $b = -12$. Since
$c = 15$ when $q_A = 3$ and $q_B = 1$, from the joint-cost function we have
$15 = 3^2 + 3(1^2) + 2(3)(1) + (-8)(3) + (-12) + d$,
$15 = -18 + d$, $33 = d$. Thus $a = -8$, $b = -12$, $d = 33$.

34. $D(a, b) > 0$
$f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2 > 0$
$f_{xx}(a, b)f_{yy}(a, b) > (f_{xy}(a, b))^2 \geq 0$

a. Since the product $f_{xx}(a, b)f_{yy}(a, b)$ is positive, $f_{xx}(a, b)$ and $f_{yy}(a, b)$ must have the same sign. That is $f_{xx}(a, b) < 0$ if and only if $f_{yy}(a, b) < 0$.

b. Since the product $f_{xx}(a, b)f_{yy}(a, b)$ is positive, $f_{xx}(a, b)$ and $f_{yy}(a, b)$ must have the same sign. That is $f_{xx}(a, b) > 0$ if and only if $f_{yy}(a, b) > 0$.

35. a. Profit = Total Revenue – Total Cost

$P = p_A q_A + p_B q_B -$ total cost

$= \left(35 - 2q_A^2 + q_B\right)q_A + (20 - q_B + q_A)q_B - \left(-8 - 2q_A^3 + 3q_A q_B + 30q_A + 12q_B + \frac{1}{2}q_A^2\right)$

$P = 5q_A - \frac{1}{2}q_A^2 - q_A q_B + 8q_B - q_B^2 + 8$

$$\begin{cases} \frac{\partial P}{\partial q_A} = 5 - q_A - q_B = 0 \\ \frac{\partial P}{\partial q_B} = -q_A + 8 - 2q_B = 0 \end{cases}$$

Critical point: $q_A = 2$, $q_B = 3$

$\frac{\partial^2 P}{\partial q_A^2} = -1$, $\frac{\partial^2 P}{\partial q_B^2} = -2$, $\frac{\partial^2 P}{\partial q_B \partial q_A} = -1$

At $q_A = 2$ and $q_B = 3$, then $D = (-1)(-2) - (-1)^2 = 1 > 0$ and $\frac{\partial^2 P}{\partial q_A^2} = -1 < 0$; thus there is a relative maximum profit for 2 units of A and 3 units of B.

b. Substituting $q_A = 2$ and $q_B = 3$ into the formulas for p_A, p_B, and P gives a selling price for A of 30, a selling price for B of 19, and a relative maximum profit of 25.

36. $P = 250\left[\frac{5x}{2+x} + \frac{2y}{5+y}\right] - x - y$

$$\begin{cases} \frac{\partial P}{\partial x} = \frac{2500}{(2+x)^2} - 1 \\ \frac{\partial P}{\partial y} = \frac{2500}{(5+y)^2} - 1 \end{cases}$$

Critical point: $x = 48$, $y = 45$

$\frac{\partial^2 P}{\partial x^2} = -\frac{5000}{(2+x)^3}$, $\frac{\partial^2 P}{\partial y^2} = -\frac{5000}{(5+y)^3}$, $\frac{\partial^2 P}{\partial y \partial x} = 0$

At $x = 48$ and $y = 45$, then $D = \left(-\frac{1}{25}\right)\left(-\frac{1}{25}\right) - 0^2 > 0$ and $\frac{\partial^2 P}{\partial x^2} = -\frac{1}{25} < 0$.

Thus relative maximum profit at $x = 48$, $y = 45$.

37. a. $P = 5T\left(1 - e^{-x}\right) - 20x - 0.1T^2$

b. $\frac{\partial P}{\partial T} = 5\left(1 - e^{-x}\right) - 0.2T$

$\frac{\partial P}{\partial x} = 5Te^{-x} - 20$

At the point $(T, x) = (20, \ln 5)$,

$\frac{\partial P}{\partial T} = 5\left(1 - e^{-\ln 5}\right) - 0.2(20)$

$= 5\left(1 - \frac{1}{5}\right) - 4 = 0$

$\frac{\partial P}{\partial x} = 5(20)e^{-\ln 5} - 20 = 100\left(\frac{1}{5}\right) - 20 = 0$

Thus (20, ln 5) is a critical point. In a similar fashion we verify that $\left(5, \ln\frac{5}{4}\right)$ is a critical point.

c. $\frac{\partial^2 P}{\partial T^2} = -0.2$, $\frac{\partial^2 P}{\partial x^2} = -5Te^{-x}$, $\frac{\partial^2 P}{\partial T\partial x} = 5e^{-x}$

At (20, ln 5),

$D = (-0.2)\left[-5(20)e^{-\ln 5}\right] - \left(5e^{-\ln 5}\right)^2 = 20\left(\frac{1}{5}\right) - \left[5\left(\frac{1}{5}\right)\right]^2 = 3 > 0$,

and $\frac{\partial^2 P}{\partial T^2} = -0.2 < 0$. Thus we get a relative maximum at (20, ln 5).

At $\left(5, \ln\frac{5}{4}\right)$,

$D = (-0.2)\left[-5(5)e^{-\ln\left(\frac{5}{4}\right)}\right] - \left[5e^{-\ln\left(\frac{5}{4}\right)}\right]^2 = 5\left(\frac{4}{5}\right) - \left[5\left(\frac{4}{5}\right)\right]^2 = -12 < 0$, so there is no relative extremum at $\left(5, \ln\frac{5}{4}\right)$.

Problems 17.8

1. $f(x, y) = x^2 + 4y^2 + 6,\ 2x - 8y = 20$

$F(x, y, \lambda) = x^2 + 4y^2 + 6 - \lambda(2x - 8y - 20)$

$$\begin{cases} F_x = 2x - 2\lambda = 0 & (1) \\ F_y = 8y + 8\lambda = 0 & (2) \\ F_\lambda = -2x + 8y + 20 = 0 & (3) \end{cases}$$

From (1), $x = \lambda$; from (2), $y = -\lambda$. Substituting $x = \lambda$ and $y = -\lambda$ into (3) gives $-2\lambda - 8\lambda + 20 = 0$, $-10\lambda = -20$, so $\lambda = 2$. Thus $x = 2$ and $y = -2$. Critical point of F: (2, –2, 2). Critical point of f: (2, –2).

2. $f(x, y) = -2x^2 + 5y^2 + 7$, $3x - 2y = 7$

$F(x, y, \lambda) = -2x^2 + 5y^2 + 7 - \lambda(3x - 2y - 7)$

$$\begin{cases} F_x = -4x - 3\lambda = 0 & (1) \\ F_y = 10y + 2\lambda = 0 & (2) \\ F_\lambda = -3x + 2y + 7 = 0 & (3) \end{cases}$$

From (1), $x=-\frac{3\lambda}{4}$; from (2), $y=-\frac{\lambda}{5}$.

Substituting $x=-\frac{3\lambda}{4}$ and $y=-\frac{\lambda}{5}$ into (3)

gives $\frac{9\lambda}{4}-\frac{2\lambda}{5}+7=0$, from which $\lambda=-\frac{140}{37}$.

Thus $x=\frac{105}{37}$ and $y=\frac{28}{37}$. Critical point of *F*:

$\left(\frac{105}{37}, \frac{28}{37}, -\frac{140}{37}\right)$. Critical point of *f*:

$\left(\frac{105}{37}, \frac{28}{37}\right)$.

3. $f(x, y, z)=x^2+y^2+z^2,\ 2x+y-z=9$

$F(x, y, z, \lambda)=x^2+y^2+z^2-\lambda(2x+y-z-9)$

$$\begin{cases} F_x=2x-2\lambda=0 & (1)\\ F_y=2y-\lambda=0 & (2)\\ F_z=2z+\lambda=0 & (3)\\ F_\lambda=-2x-y+z+9=0 & (4)\end{cases}$$

From (1), $x=\lambda$; from (2), $y=\frac{\lambda}{2}$; from (3),

$z=-\frac{\lambda}{2}$. Substituting into (4) gives

$-2\lambda-\frac{\lambda}{2}+\left(\frac{-\lambda}{2}\right)+9=0$, $-6\lambda+18=0$, so

$\lambda=3$. Thus $x = 3$, $y=\frac{3}{2}$, $z=-\frac{3}{2}$. Critical point

of *F*: $\left(3, \frac{3}{2}, -\frac{3}{2}, 3\right)$. Critical point of *f*:

$\left(3, \frac{3}{2}, -\frac{3}{2}\right)$.

4. $f(x, y, z) = x + y + z,\ xyz = 8$

$F(x, y, z, \lambda)=x+y+z-\lambda(xyz-8)$

$$\begin{cases} F_x=1-\lambda yz=0 & (1)\\ F_y=1-\lambda xz=0 & (2)\\ F_z=1-\lambda xy=0 & (3)\\ F_\lambda=-xzy+8=0 & (4)\end{cases}$$

From (1) and (2), $\lambda yz=\lambda xz$, so $y = x$. From (2) and (3), $\lambda xz=\lambda xy$, so $y = z$. Therefore $x = y = z$, so from (4), $x = y = z = 2$. Hence, Critical point of *f* is (2, 2, 2). Note that it is not necessary to determine λ.

5. $f(x, y, z)=2x^2+xy+y^2+z,\ x+2y+4z=3$

$F(x, y, z, \lambda)$

$=2x^2+xy+y^2+z-\lambda(x+2y+4z-3)$

$$\begin{cases} F_x=4x+y-\lambda=0\\ F_y=x+2y-2\lambda=0\\ F_z=1-4\lambda=0\\ F_\lambda=-x-2y-4z-3=0\end{cases}$$

From the third equation we have $\lambda=\frac{1}{4}$.

Substituting this value into the first two equations and then eliminating *y* gives $x = 0$ and $y=\frac{1}{4}$.

Finally, solving for *z* in the last equation gives $z=-\frac{7}{8}$.

Critical point of *F*: $\left(0, \frac{1}{4}, -\frac{7}{8}, \frac{1}{4}\right)$

Critical point of *f*: $\left(0, \frac{1}{4}, -\frac{7}{8}\right)$

6. $f(x, y, z)=xyz^2$, $x-y+z=20$ $\left(xyz^2\neq 0\right)$

$f(x, y, z, \lambda)=xyz^2-\lambda(x-y+z-20)$

$$\begin{cases} F_x=yz^2-\lambda=0 & (1)\\ F_y=xz^2+\lambda=0 & (2)\\ F_z=2xyz-\lambda=0 & (3)\\ F_\lambda=-x+y-z+20=0 & (4)\end{cases}$$

From (1) and (2), $y = –x$, From (1) and (3), $z = 2x$. Hence from (4), $x = 5$, so $y = –5$ and $z = 10$. Critical point of *f* is (5, –5, 10). Note that it is not necessary to determine λ.

7. $f(x, y, z) = xyz,\ x + 2y + 3z = 18\ (xyz \neq 0)$

$f(x, y, z, \lambda)=xyz-\lambda(x+2y+3z-18)$

$$\begin{cases} F_x=yz-\lambda=0 & (1)\\ F_y=xz-2\lambda=0 & (2)\\ F_z=xy-3\lambda=0 & (3)\\ F_\lambda=-x-2y-3z+18=0 & (4)\end{cases}$$

From (1) and (2), $y=\frac{x}{2}$. From (1) and (3),

$z=\frac{x}{3}$. Hence from (4), $x = 6$, so $y = 3$ and $z = 2$.

Critical point of *f* is (6, 3, 2). Note that it is not necessary to determine λ.

8. $f(x, y, z) = x^2 + y^2 + z^2$, $x + y + z = 3$

$F(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(x + y + z - 3)$

$$\begin{cases} F_x = 2x - \lambda = 0 & (1) \\ F_y = 2y - \lambda = 0 & (2) \\ F_z = 2z - \lambda = 0 & (3) \\ F_\lambda = -x - y - z + 3 = 0 & (4) \end{cases}$$

From (1)–(3), $x = y = z = \frac{\lambda}{2}$. Substituting into (4), $-\frac{\lambda}{2} - \frac{\lambda}{2} - \frac{\lambda}{2} + 3 = 0$, so $\lambda = 2$. Thus $x = 1, \ y = 1, \ z = 1$.

Critical point of *F*: $(1, 1, 1, 2)$. Critical point of *f*: $(1, 1, 1)$.

9. $f(x, y, z) = x^2 + 2y - z^2$, $2x - y = 0$, $y + z = 0$

Since there are two constraints, two Lagrange multipliers are used.

$F(x, y, z, \lambda_1, \lambda_2) = x^2 + 2y - z^2 - \lambda_1(2x - y) - \lambda_2(y + z)$

$$\begin{cases} F_x = 2x - 2\lambda_1 = 0 & (1) \\ F_y = 2 + \lambda_1 - \lambda_2 = 0 & (2) \\ F_z = -2z - \lambda_2 = 0 & (3) \\ F_{\lambda_1} = -2x + y = 0 & (4) \\ F_{\lambda_2} = -y - z = 0 & (5) \end{cases}$$

From (1), $x = \lambda_1$. From (3), $z = -\frac{\lambda_2}{2}$. From (4) and (5), $2x = -z$, so $\lambda_1 = \frac{\lambda_2}{4}$. Substituting $\lambda_1 = \frac{\lambda_2}{4}$ into (2) yields $\lambda_2 = \frac{8}{3}$. Thus $\lambda_1 = \frac{2}{3}$, $x = \frac{2}{3}$, and $z = -\frac{4}{3}$. From (5), $y = -z$ and hence $y = \frac{4}{3}$. Critical point of *f*: $\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right)$

10. $f(x, y, z) = x^2 + y^2 + z^2$, $x + y + z = 4$, $x - y + z = 4$

Since there are two constraints, two Lagrange multipliers are used.

$F(x, y, z, \lambda_1, \lambda_2) = x^2 + y^2 + z^2 - \lambda_1(x + y + z - 4) - \lambda_2(x - y + z - 4)$

$$\begin{cases} F_x = 2x - \lambda_1 - \lambda_2 = 0 & (1) \\ F_y = 2y - \lambda_1 + \lambda_2 = 0 & (2) \\ F_z = 2z - \lambda_1 - \lambda_2 = 0 & (3) \\ F_{\lambda_1} = -x - y - z + 4 = 0 & (4) \\ F_{\lambda_2} = -x + y - z + 4 = 0 & (5) \end{cases}$$

From (4) and (5), $y = 0$. From (1) and (3), $z = x$. Substituting into (5) gives $x = 2$. Thus $z = 2$.

Critical point of *f*: $(2, 0, 2)$

11. $f(x, y, z) = xy^2z$, $x + y + z = 1$, $x - y + z = 0$ $(xyz \neq 0)$

Since there are two constraints, two Lagrange multipliers are used.

$F(x, y, z, \lambda_1, \lambda_2) = xy^2z - \lambda_1(x + y + z - 1) - \lambda_2(x - y + z)$

$$\begin{cases} F_x = y^2z - \lambda_1 - \lambda_2 = 0 & (1) \\ F_y = 2xyz - \lambda_1 + \lambda_2 = 0 & (2) \\ F_z = xy^2 - \lambda_1 - \lambda_2 = 0 & (3) \\ F_{\lambda_1} = -x - y - z + 1 = 0 & (4) \\ F_{\lambda_2} = -x + y - z = 0 & (5) \end{cases}$$

Subtracting (3) from (1) gives $y^2z - xy^2 = 0$, so $x = z$ (since $xy^2z \neq 0$). Subtracting (5) from (4) gives $-2y + 1 = 0$, so $y = \frac{1}{2}$. Substituting $z = x$ and $y = \frac{1}{2}$ in (5) gives $-2x + \frac{1}{2} = 0$, so $x = \frac{1}{4}$. Thus, $z = \frac{1}{4}$. Critical point of *f*: $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$.

12. $f(x, y, z, w) = 3x^2 + y^2 + 2z^2 - 5w^2$, $x + 6y + 3z + 2w = 4$

$F(x, y, z, w, \lambda) = 3x^2 + y^2 + 2z^2 - 5w^2 - \lambda(x + 6y + 3z + 2w - 4)$

$$\begin{cases} F_x = 6x - \lambda = 0 \\ F_y = 2y - 6\lambda = 0 \\ F_z = 4z - 3\lambda = 0 \\ F_w = -10w - 2\lambda = 0 \\ F_\lambda = -x - 6y - 3z - 2w + 4 = 0 \end{cases}$$

Solving the first four equations for *x*, *y*, *z*, and *w* in terms of λ gives $x = \frac{\lambda}{6}$, $y = 3\lambda$, $z = \frac{3\lambda}{4}$ and $w = -\frac{\lambda}{5}$.

Substituting into the last equation gives $\lambda = \frac{240}{1201}$. Thus $x = \frac{40}{1201}$, $y = \frac{720}{1201}$, $z = \frac{180}{1201}$ and $w = -\frac{48}{1201}$.

Critical point of *F*: $\left(\frac{40}{1201}, \frac{720}{1201}, \frac{180}{1201}, -\frac{48}{1201}, \frac{240}{1201}\right)$

Critical point of *f*: $\left(\frac{40}{1201}, \frac{720}{1201}, \frac{180}{1201}, -\frac{48}{1201}\right)$

13. We minimize $c = f(q_1, q_2) = 0.1q_1^2 + 7q_1 + 15q_2 + 1000$ subject to the constraint $q_1 + q_2 = 100$.

$F(q_1, q_2, \lambda) = 0.1q_1^2 + 7q_1 + 15q_2 + 1000 - \lambda(q_1 + q_2 - 100)$

$$\begin{cases} F_{q_1} = 0.2q_1 + 7 - \lambda = 0 & (1) \\ F_{q_2} = 15 - \lambda = 0 & (2) \\ F_\lambda = -q_1 - q_2 + 100 = 0 & (3) \end{cases}$$

From (2), $\lambda = 15$. Substituting $\lambda = 15$ into (1) gives $0.2q_1 + 7 - 15 = 0$, so $q_1 = 40$. Substituting $q_1 = 40$ into (3) gives $-40 - q_2 + 100 = 0$, so $q_2 = 60$. Thus $\lambda = 15$, $q_1 = 40$, and $q = 60$. Thus plant 1 should produce 40 units and plant 2 should produce 60 units.

14. We minimize $c = 3q_1^2 + q_1q_2 + 2q_2^2$ subject to the constraint $q_1 + q_2 = 200$.

$F(q_1, q_2, \lambda) = 3q_1^2 + q_1q_2 + 2q_2^2 - \lambda(q_1 + q_2 - 200)$

$$\begin{cases} F_{q_1} = 6q_1 + q_2 - \lambda = 0 & (1) \\ F_{q_2} = q_1 + 4q_2 - \lambda = 0 & (2) \\ F_\lambda = -q_1 - q_2 + 200 = 0 & (3) \end{cases}$$

Eliminating λ from (1) and (2) yields $q_1 = \frac{3}{5}q_2$. Substituting $q_1 = \frac{3}{5}q_2$ into (3) yields $q_2 = 125$ and thus $q_1 = 75$. Thus plant 1 should produce 75 units and plant 2 should produce 125 units.

15. We maximize $f(l, k) = 12l + 20k - l^2 - 2k^2$ subject to the constraint $4l + 8k = 88$.

$F(l, k, \lambda) = 12l + 20k - l^2 - 2k^2 - \lambda(4l + 8k - 88)$

$$\begin{cases} F_l = 12 - 2l - 4\lambda = 0 & (1) \\ F_k = 20 - 4k - 8\lambda = 0 & (2) \\ F_\lambda = -4l - 8k + 88 = 0 & (3) \end{cases}$$

Eliminating λ from (1) and (2) yields $k = l - 1$. Substituting $k = l - 1$ into (3) yields $l = 8$, so $k = 7$. Therefore the greatest output is $f(8, 7) = 74$ units (when $l = 8$, $k = 7$).

16. We maximize $f(l, k) = 20l + 25k - l^2 - 3k^2$ subject to the constraint $2l + 4k = 50$.

$F(l, k, \lambda) = 20l + 25k - l^2 - 3k^2 - \lambda(2l + 4k - 50)$

$$\begin{cases} F_l = 20 - 2l - 2\lambda = 0 & (1) \\ F_k = 25 - 6k - 4\lambda = 0 & (2) \\ F_\lambda = -2l - 4k + 50 = 0 & (3) \end{cases}$$

From (1), $l = 10 - \lambda$ and from (2), $k = \frac{25}{6} - \frac{2}{3}\lambda$. Substituting these expressions for l and k into (3) yields $\lambda = -\frac{20}{7}$. Thus $l = \frac{90}{7}$ and $k = \frac{85}{14}$. Therefore the greatest output is $f\left(\frac{90}{7}, \frac{85}{14}\right) = \frac{3725}{28} \approx 133$ units (when $l = \frac{90}{7}$, $k = \frac{85}{14}$).

17. We maximize $P(x, y) = 9x^{\frac{1}{4}}y^{\frac{3}{4}} - x - y$ subject to the constraint $x + y = 60{,}000$.

$F(x, y, \lambda) = 9x^{\frac{1}{4}}y^{\frac{3}{4}} - x - y - \lambda(x + y - 60{,}000)$

$$\begin{cases} F_x = \frac{9}{4}x^{-\frac{3}{4}}y^{\frac{3}{4}} - 1 - \lambda = 0 & (1) \\ F_y = \frac{27}{4}x^{\frac{1}{4}}y^{-\frac{1}{4}} - 1 - \lambda = 0 & (2) \\ F_\lambda = -x - y + 60{,}000 = 0 & (3) \end{cases}$$

Solving (2) for λ and substituting in (1) gives $\frac{9}{4}x^{-\frac{3}{4}}y^{\frac{3}{4}} - \frac{27}{4}x^{\frac{1}{4}}y^{-\frac{1}{4}} = 0$, $\frac{9}{4}x^{-\frac{3}{4}}y^{\frac{3}{4}} = \frac{27}{4}x^{\frac{1}{4}}y^{-\frac{1}{4}}$, $y = 3x$.
Substituting for y in (3) gives $-4x + 60{,}000 = 0$, so $x = 15{,}000$, from which $y = 45{,}000$. Thus each month \$15,000 should be spent on newspaper advertising and \$45,000 on TV advertising.

18. We maximize $f(l, k) = 6l^{\frac{2}{5}}k^{\frac{3}{5}}$ subject to the constraint $25l + 69k = 25{,}875$.

$F(l, k, \lambda) = 6l^{\frac{2}{5}}k^{\frac{3}{5}} - \lambda(25l + 69k - 25{,}875)$

$$\begin{cases} F_l = \frac{12}{5}l^{-\frac{3}{5}}k^{\frac{3}{5}} - 25\lambda = 0 \\ F_k = \frac{18}{5}l^{\frac{2}{5}}k^{-\frac{2}{5}} - 69\lambda = 0 \\ F_\lambda = -25l - 69k + 25{,}875 = 0 \end{cases}$$

From the first two equations, $\frac{12}{5}l^{-\frac{3}{5}}k^{\frac{3}{5}} = 25\lambda$ and $\frac{18}{5}l^{\frac{2}{5}}k^{-\frac{2}{5}} = 69\lambda$. Thus, $\frac{\frac{12}{5}l^{-\frac{3}{5}}k^{\frac{3}{5}}}{\frac{18}{5}l^{\frac{2}{5}}k^{-\frac{2}{5}}} = \frac{25\lambda}{69\lambda} = \frac{25}{29}$, from which

$k = \frac{25}{46}l$. Substituting this for k in the third equation and solving for l gives $l = 414$ so $k = 225$.

414 units of labor and 225 units of capital should be invested.

19. We minimize $B(x, y, z) = x^2 + y^2 + 2z^2$ subject to $x + y = 20$ and $y + z = 20$.

Since there are two constraints, two Lagrange multipliers are used.

$F(x, y, z, \lambda_1, \lambda_2) = x^2 + y^2 + 2z^2 - \lambda_1(x + y - 20) - \lambda_2(y + z - 20)$

$$\begin{cases} F_x = 2x - \lambda_1 = 0 & (1) \\ F_y = 2y - \lambda_1 - \lambda_2 = 0 & (2) \\ F_z = 4z - \lambda_2 = 0 & (3) \\ F_{\lambda_1} = -x - y + 20 = 0 & (4) \\ F_{\lambda_2} = -y - z + 20 = 0 & (5) \end{cases}$$

Eliminating y from (4) and (5) gives $x = z$. From (1) and (3), $\lambda_1 = 2x$ and $\lambda_2 = 4z$. Substituting in (2) we have $2y - 2x - 4z = 0$, $2y - 2x - 4x = 0$, $2y - 6x = 0$, $y = 3x$. Substituting in (5) gives $-(3x) - x + 20 = 0$, so $x = 5$. Thus $z = 5$ and $y = 15$. Therefore, $x = 5$, $y = 15$, $z = 5$.

20. **a.** $P = TR - TC = 64q - (8l + 16k)$

$$= 64\left[\frac{65 - 4(l-4)^2 - 2(k-5)^2}{16}\right] - 8l - 16k$$

$$P = -196 - 16l^2 + 120l - 8k^2 + 64k$$

b. $P_l = -32l + 120 = 0 \Rightarrow l = \frac{15}{4}$

$P_k = -16k + 64 = 0 \Rightarrow k = 4$

Thus there is one critical point: $(l, k) = \left(\frac{15}{4}, 4\right)$

Second-Derivative Test: $P_{ll} = -32$, $P_{kk} = -16$, $P_{lk} = 0$.

Thus $D(l, k) = P_{ll}P_{kk} - [P_{lk}]^2 = (-32)(-16) - 0^2 = 512$. At $\left(\frac{15}{4}, 4\right)$, $D\left(\frac{15}{4}, 4\right) = 512 > 0$ and $P_{ll} = -32 < 0$.

Thus there is a relative maximum at $l = \frac{15}{4}$, $k = 4$. Substituting these values into the profit function gives a relative maximum profit of \$157.00.

c. $F(l, k, q, \lambda) = 64q - 8l - 16k - \lambda\left[16q - 65 + 4(l-4)^2 + 2(k-5)^2\right]$

$$\begin{cases} F_l = -8 - 8\lambda(l-4) = 0 & (1) \\ F_k = -16 - 4\lambda(k-5) = 0 & (2) \\ F_q = 64 - 16\lambda = 0 & (3) \\ F_\lambda = -16q + 65 - 4(l-4)^2 - 2(k-5)^2 = 0 & (4) \end{cases}$$

From (3), $\lambda = 4$. Substituting $\lambda = 4$ into (1) gives –8 – 32(*l* – 4) = 0, so $l = \frac{15}{4}$. Similarly, from (2) we get $k = 4$. Substituting for *l* and *k* in (4) gives $q = \frac{251}{64}$. Thus $(l, k, q) = \left(\frac{15}{4}, 4, \frac{251}{64}\right)$.

21. $U = x^3y^3$, $p_x = 2$, $p_y = 3$, $I = 48$ $\left(x^3y^3 \neq 0\right)$

We want to maximize $U = x^3y^3$ subject to 2*x* + 3*y* = 48.

$F(x, y, \lambda) = x^3y^3 - \lambda(2x + 3y - 48)$

$$\begin{cases} F_x = 3x^2y^3 - 2\lambda = 0 & (1) \\ F_y = 3x^3y^2 - 3\lambda = 0 & (2) \\ F_\lambda = -2x - 3y + 48 = 0 & (3) \end{cases}$$

From (1), $\lambda = \frac{3}{2}x^2y^3$ and from (2), $\lambda = x^3y^2$. Thus $\frac{3}{2}x^2y^3 = x^3y^2$, so $x = \frac{3}{2}y$.

Substituting this expression for *x* into (3) yields *y* = 8. Hence $x = \left(\frac{3}{2}\right)8 = 12$.

22. $U = 40x - 5x^2 + 4y - 2y^2$, $p_x = 2$, $p_y = 3$, $I = 10$

We want to maximize $U = 40x - 5x^2 + 4y - 2y^2$ subject to 2*x* + 3*y* = 10.

$F(x, y, \lambda) = 40x - 5x^2 + 4y - 2y^2 - \lambda(2x + 3y - 10)$ $\begin{cases} F_x = 40 - 10x - 2\lambda = 0 \\ F_y = 4 - 4y - 3\lambda = 0 \\ F_\lambda = -2x - 3y + 10 = 0 \end{cases}$

From the first equation, $x = 4 - \frac{\lambda}{5}$ and from the second equation $y = 1 - \frac{3\lambda}{4}$.

Substituting these values into the third equation gives $\lambda = \frac{20}{53}$. Thus $x = \frac{208}{53}$ and $y = \frac{38}{53}$.

23. $U = f(x, y, z) = xyz$

$p_x = p_y = p_z = 1$, $I = 100$

$(xyz \neq 0)$

We want to maximize $U = xyz$ subject to

$x + y + z = 100$.

$F(x, y, z, \lambda) = xyz - \lambda(x + y + z - 100)$

$$\begin{cases} F_x = yz - \lambda = 0 & (1) \\ F_y = xz - \lambda = 0 & (2) \\ F_z = xy - \lambda = 0 & (3) \\ F_\lambda = -x - y - z + 100 = 0 & (4) \end{cases}$$

From (1) and (2), $yz = xz$, so $y = x$. Similarly, from (1) and (3), $z = x$. Substituting $y = x$ and $z = x$ into (4) yields $x = \frac{100}{3}$. Thus $y = \frac{100}{3}$ and $z = \frac{100}{3}$.

24. To maximize $U = f(x, y)$ subject to the constraint $xp_x + yp_y = I$, we consider

$F(x, y, \lambda) = f(x, y) - \lambda\left(xp_x + yp_y - I\right)$.

For maximum satisfaction,

$F_x = f_x(x, y) - \lambda p_x = 0$ (1)

and

$F_y = f_y(x, y) - \lambda p_y = 0$ (2)

From (1), $\lambda = \frac{f_x(x, y)}{p_x}$ and from (2), $\lambda = \frac{f_y(x, y)}{p_y}$. Thus $\lambda = \frac{f_x(x, y)}{p_x} = \frac{f_y(x, y)}{p_y}$

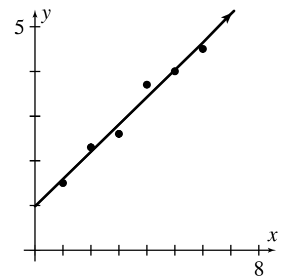
Since $f_x(x, y)$ represents change in total utility from a one unit change in X (which costs p_x), then $\frac{f_x(x, y)}{p_x}$ is the marginal utility of a dollar's worth of X. Likewise $\frac{f_y(x, y)}{p_y}$ is the marginal utility of a dollar's worth of Y. Thus maximum satisfaction is obtained when the consumer allocates the budget so that the marginal utility of a dollar's worth of X is equal to the marginal utility of a dollars worth of Y. Similarly, for $U = f(x, y, z, w)$ subject to the constraint $xp_x + yp_y + zp_z + wp_w = I$, U is maximized when

$$\lambda = \frac{f_x(x, y, z, w)}{p_x} = \frac{f_y(x, y, z, w)}{p_y} = \frac{f_z(x, y, z, w)}{p_z} = \frac{f_w(x, y, z, w)}{p_w}.$$

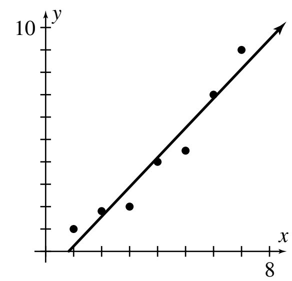
That is, U is maximized when the marginal utility of a dollar's worth of each of the products is the same.

Problems 17.9

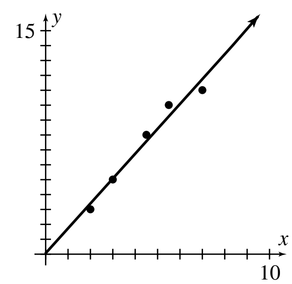
1. $n = 6$, $\Sigma x_i = 21$, $\Sigma y_i = 18.6$, $\Sigma x_i y_i = 75.7$, $\Sigma x_i^2 = 91$.
$a = 0.98$
$b = 0.61$
Thus $\hat{y} = 0.98 + 0.61x$. When $x = 3.5$, then $\hat{y} = 3.12$.



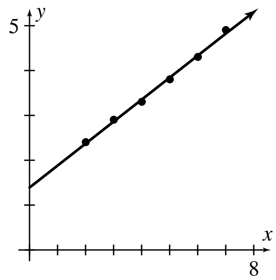
2. $n = 7$, $\Sigma x_i = 28$, $\Sigma y_i = 29.3$, $\Sigma x_i y_i = 154.1$, $\Sigma x_i^2 = 140$. $a = -1.09$, $b = 1.32$. Thus $\hat{y} = -1.09 + 1.32x$. When $x = 3.5$, then $\hat{y} = 3.53$.



3. $n = 5$, $\Sigma x_i = 22$, $\Sigma y_i = 37$, $\Sigma x_i y_i = 189$, $\Sigma x_i^2 = 112.5$. $a = 0.057$, $b = 1.67$. Thus $\hat{y} = 0.057 + 1.67x$. When $x = 3.5$, then $\hat{y} = 5.90$.



4. $n=6,\ \Sigma x_i=27,\ \Sigma y_i=21.6,\ \Sigma x_i y_i=105.8$, $\Sigma x_i^2=139$. $a=1.39$, $b=0.49$. Thus $\hat{y}=1.39+0.49x$. When $x=3.5$, then $\hat{y}=3.12$.



5. $n=6,\ \Sigma p_i=250,\ \Sigma q_i=322,\ \Sigma p_i q_i=11{,}690$,
$\Sigma p_i^2=13{,}100.$
$a=80.5$
$b=-0.643$
Thus $\hat{q}=80.5-0.643p.$

6. $n=4,\ \sum x_i=80,\ \sum y_i=23.9,\ \sum x_i y_i=498.4,$ $\sum x_i^2=1920,\ a=4.7,\ b=0.06.$
Thus $\hat{y}=4.7+0.06x.$ When $x=20$, then $\hat{y}=5.9.$

7. $n=4,\ \Sigma x_i=160,\ \Sigma y_i=420.8,\ \Sigma x_i y_i=16{,}915.2$, $\Sigma x_i^2=7040$. $a=100,\ b=0.13$. Thus $\hat{y}=100+0.13x$. When $x=40$, then $\hat{y}=105.2$.

8. $n=4,\ \sum x_i=539,\ \sum y_i=569,$
$\sum x_i y_i=76{,}736,\ \sum x_i^2=72{,}691,\ a=1.95,$
$b=1.04.$
Thus $\hat{y}=1.95+1.04x.$

9.

Year (x)	1	2	3	4	5
Production (y)	10	15	16	18	21

$n=5,\ \Sigma x_i=15,\ \Sigma y_i=80,\ \Sigma x_i y_i=265,\ \Sigma x_i^2=55.$
$a=8.5$
$b=2.5$
Thus $\hat{y}=8.5+2.5x$

10.

Year (x)	1	3	5	7
Index (y)	77	100	126	134

$n=4,\ \Sigma x_i=16,\ \Sigma y_i=437,\ \Sigma x_i y_i=1945$, $\Sigma x_i^2=84$. $a=69.85$, $b=9.85$. Thus $\hat{y}=69.85+9.85x$.

11. a.

Year (x)	1	2	3	4	5
Quantity (y)	35	31	26	24	26

$n=5,\ \Sigma x_i=15,\ \Sigma y_i=142,\ \Sigma x_i y_i=401$, $\Sigma x_i^2=55$. $a=35.9$, $b=-2.5$. Thus $\hat{y}=35.9-2.5x$.

b.

Year (x)	−2	−1	0	1	2
Quantity (y)	35	31	26	24	26

$n=5,\ \Sigma x_i=0,\ \Sigma y_i=142,\ \Sigma x_i y_i=-25$, $\Sigma x_i^2=10$. $a=\frac{\Sigma y_i}{n}=28.4$ and $b=\frac{\Sigma x_i y_i}{\Sigma x_i^2}=-2.5$. Thus $\hat{y}=28.4-2.5x$.

12.

Year (x)	−2	−1	0	1	2
Index (y)	357	380	403	434	462

$n=5,\ \Sigma x_i=0,\ \Sigma y_i=2036,\ \Sigma x_i y_i=264$, $\Sigma x_i^2=10$. $a=\frac{\Sigma y_i}{n}=407.2$ and $b=\frac{\Sigma x_i y_i}{\Sigma x_i^2}=26.4$. Thus $\hat{y}=407.2+26.4x$.

Problems 17.10

1. $\int_0^3\int_0^4 x\,dy\,dx=\int_0^3 xy\Big|_0^4 dx=\int_0^3 4x\,dx=2x^2\Big|_0^3=18$

2. $\int_1^4\int_0^3 y\,dy\,dx=\int_1^4 \frac{y^2}{2}\Big|_0^3 dx=\int_1^4 \frac{9}{2}dx=\frac{9x}{2}\Big|_1^4=\frac{27}{2}$

3. $\int_0^1\int_0^1 xy\,dx\,dy=\int_0^1 \frac{x^2y}{2}\Big|_0^1 dy=\int_0^1 \frac{y}{2}dy=\frac{y^2}{4}\Big|_0^1=\frac{1}{4}$

4. $\int_0^2\int_0^3 x^2dy\,dx=\int_0^2 x^2y\Big|_0^3 dx=\int_0^2 3x^2dx=x^3\Big|_0^2=8$

5. $\int_1^3 \int_1^2 \left(x^2 - y\right) dx\, dy = \int_1^3 \left(\frac{x^3}{3} - xy\right)\Bigg|_1^2 dy$

$= \int_1^3 \left[\left(\frac{8}{3} - 2y\right) - \left(\frac{1}{3} - y\right)\right] dy = \int_1^3 \left(\frac{7}{3} - y\right) dy$

$= \left(\frac{7}{3}y - \frac{y^2}{2}\right)\Bigg|_1^3 = \left(7 - \frac{9}{2}\right) - \left(\frac{7}{3} - \frac{1}{2}\right) = \frac{2}{3}$

6. $\int_{-2}^3 \int_0^2 (y^2 - 2xy)\, dy\, dx = \int_{-2}^3 \left(\frac{y^3}{3} - xy^2\right)\Bigg|_0^2 dx$

$= \int_{-2}^3 \left[\left(\frac{8}{3} - 4x\right) - 0\right] dx = \int_{-2}^3 \left(\frac{8}{3} - 4x\right) dx$

$= \left(\frac{8}{3}x - 2x^2\right)\Bigg|_{-2}^3 = (8 - 18) - \left(-\frac{16}{3} - 8\right)$

$= \frac{10}{3}$

7. $\int_0^1 \int_0^2 (x + y)\, dy\, dx = \int_0^1 \left(xy + \frac{y^2}{2}\right)\Bigg|_0^2 dx$

$= \int_0^1 (2x + 2)\, dx = \left(x^2 + 2x\right)\Big|_0^1 = 3$

8. $\int_0^3 \int_0^x \left(x^2 + y^2\right) dy\, dx = \int_0^3 \left(x^2 y + \frac{y^3}{3}\right)\Bigg|_0^x dx$

$= \int_0^3 \left(x^3 + \frac{x^3}{3}\right) dx = \int_0^3 \frac{4x^3}{3}\, dx = \frac{x^4}{3}\Bigg|_0^3 = 27$

9. $\int_1^4 \int_0^{5x} y\, dy\, dx = \int_1^4 \frac{y^2}{2}\Bigg|_0^{5x} dx = \int_1^4 \frac{25}{2} x^2\, dx$

$= \frac{25}{6} x^3\Bigg|_1^4 = \frac{525}{2}$

10. $\int_1^2 \int_0^{x-1} 2y\, dy\, dx = \int_1^2 y^2\Big|_0^{x-1} dx$

$= \int_1^2 (x-1)^2\, dx = \frac{(x-1)^3}{3}\Bigg|_1^2 = \frac{1}{3}$

11. $\int_0^1 \int_{3x}^{x^2} 14x^2 y\, dy\, dx = \int_0^1 \left(7x^2 y^2\right)\Big|_{3x}^{x^2} dx$

$= \int_0^1 \left(7x^6 - 63x^4\right) dx = \left(x^7 - \frac{63x^5}{5}\right)\Bigg|_0^1 = -\frac{58}{5}$

12. $\int_0^2 \int_0^{x^2} xy\, dy\, dx = \int_0^2 \frac{xy^2}{2}\Bigg|_0^{x^2} dx$

$= \int_0^2 \frac{x^5}{2}\, dx = \frac{x^6}{12}\Bigg|_0^2 = \frac{16}{3}$

13. $\int_0^3 \int_0^{\sqrt{9-x^2}} y\, dy\, dx = \int_0^3 \frac{y^2}{2}\Bigg|_0^{\sqrt{9-x^2}} dx$

$= \int_0^3 \left(\frac{9 - x^2}{2} - 0\right) dx = \frac{1}{2}\int_0^3 (9 - x^2)\, dx$

$= \frac{1}{2}\left(9x - \frac{x^3}{3}\right)\Bigg|_0^3 = \frac{1}{2}(27 - 9) - 0 = 9$

14. $\int_0^1 \int_{y^2}^{y} y\, dx\, dy = \int_0^1 xy\Big|_{y^2}^{y} dy = \int_0^1 (y^2 - y^3)\, dy$

$= \left(\frac{y^3}{3} - \frac{y^4}{4}\right)\Bigg|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

15. $\int_{-1}^1 \int_x^{1-x} 3(x + y)\, dy\, dx = \int_{-1}^1 3\left(xy + \frac{y^2}{2}\right)\Bigg|_x^{1-x} dx$

$= \int_{-1}^1 3\left[x(1-x) + \frac{(1-x)^2}{2} - \left(x^2 + \frac{x^2}{2}\right)\right] dx$

$= \int_{-1}^1 3\left[x - \frac{5x^2}{2} + \frac{(1-x)^2}{2}\right] dx$

$= 3\left[\frac{x^2}{2} - \frac{5x^3}{6} - \frac{(1-x)^3}{6}\right]\Bigg|_{-1}^1$

$= 3\left[\frac{1}{2} - \frac{5}{6} - 0\right] - 3\left[\frac{1}{2} + \frac{5}{6} - \frac{4}{3}\right] = -1$

16. $\int_0^3\int_{y^2}^{3y} 5x\,dx\,dy = \int_0^3 \frac{5x^2}{2}\bigg|_{y^2}^{3y} dy$

$= \int_0^3\left(\frac{45y^2}{2} - \frac{5y^4}{2}\right)dy$

$= \left(\frac{15y^3}{2} - \frac{y^5}{2}\right)\bigg|_0^3 = \frac{405}{2} - \frac{243}{2} = 81$

17. $\int_0^1\int_0^y e^{x+y}dx\,dy = \int_0^1 e^{x+y}\bigg|_0^y dy = \int_0^1\left(e^{2y} - e^y\right)dy$

$= \left[\frac{e^{2y}}{2} - e^y\right]\bigg|_0^1 = \frac{e^2}{2} - e - \left(\frac{1}{2} - 1\right) = \frac{e^2}{2} - e + \frac{1}{2}$

18. $\int_0^1\int_0^1 e^{y-x}dx\,dy = \int_0^1 -e^{y-x}\Big|_0^1 dy$

$= \int_0^1(-e^{y-1} + e^y)dy = (-e^{y-1} + e^y)\Big|_0^1$

$= (-e^0 + e^1) - (-e^{-1} + e^0) = -1 + e + e^{-1} - 1$

$= -2 + e + e^{-1}$

19. $\int_{-1}^0\int_{-1}^2\int_1^2 6xy^2z^3dx\,dy\,dz = \int_{-1}^0\int_{-1}^2 3x^2y^2z^3\Big|_1^2 dy\,dz$

$= \int_{-1}^0\int_{-1}^2 9y^2z^3dy\,dz = \int_{-1}^0 3y^3z^3\Big|_{-1}^2 dz$

$= \int_{-1}^0 27z^3dz = \frac{27z^4}{4}\bigg|_{-1}^0 = -\frac{27}{4}$

20. $\int_0^1\int_0^x\int_0^{x+y} x^2dz\,dy\,dx = \int_0^1\int_0^x x^2z\Big|_0^{x+y} dy\,dx$

$= \int_0^1\int_0^x [x^2(x+y) - 0]dy\,dx = \int_0^1\int_0^x (x^3 + x^2y)dy\,dx$

$= \int_0^1\left(x^3y + \frac{x^2y^2}{2}\right)\bigg|_0^x dx = \int_0^1\left[\left(x^4 + \frac{x^4}{2}\right) - 0\right]dx$

$= \int_0^1 \frac{3x^4}{2}dx = \frac{3x^5}{10}\bigg|_0^1 = \frac{3(1)^5}{10} - 0 = \frac{3}{10}$

21. $\int_0^1\int_{x^2}^x\int_0^{xy} dz\,dy\,dx = \int_0^1\int_{x^2}^x z\Big|_0^{xy} dy\,dx$

$= \int_0^1\int_{x^2}^x xy\,dy\,dx = \int_0^1 \frac{xy^2}{2}\bigg|_{x^2}^x dx$

$= \int_0^1\left[\frac{x^3}{2} - \frac{x^5}{2}\right]dx = \left[\frac{x^4}{8} - \frac{x^6}{12}\right]\bigg|_0^1 = \frac{1}{24}$

22. $\int_1^e\int_{\ln x}^x\int_0^y dz\,dy\,dx = \int_1^e\int_{\ln x}^x z\Big|_0^y dy\,dx$

$= \int_1^e\int_{\ln x}^x y\,dy\,dx = \int_1^e \frac{y^2}{2}\bigg|_{\ln x}^x dx = \int_1^e \frac{x^2}{2} - \frac{(\ln x)^2}{2}dx$

$= \left[\frac{x^3}{6} - \frac{1}{2}(x\ln^2 x - 2x\ln x + 2x)\right]\bigg|_1^e$

$= \frac{e^3}{6} - \frac{e}{2} - \left(\frac{1}{6} - 1\right) = \frac{e^3}{6} - \frac{e}{2} + \frac{5}{6}$

23. $P(0 \le x \le 2, 1 \le y \le 2) = \int_1^2\int_0^2 e^{-(x+y)}dx\,dy$

$= \int_1^2 -e^{-(x+y)}\Big|_0^2 dy = \int_1^2\left[-e^{-(2+y)} + e^{-y}\right]dy$

$= \left[e^{-(2+y)} - e^{-y}\right]\Big|_1^2 = e^{-4} - e^{-2} - e^{-3} + e^{-1}$

24. $P(3 \le x \le 4,\ 2 \le y \le 6) = \int_2^6\int_3^4 12e^{-4x-3y}dx\,dy$

$= \int_2^6\left(-3e^{-4x-3y}\right)\Big|_3^4 dy$

$= \int_2^6\left(-3^{-16-3y} + 3e^{-12-3y}\right)dy$

$= \left(e^{-16-3y} - e^{-12-3y}\right)\Big|_2^6$

$= e^{-34} - e^{-30} - e^{-22} + e^{-18}$

25. $P\left(x \ge \frac{1}{2},\ y \ge \frac{1}{3}\right) = \int_{1/3}^1\int_{1/2}^1 1\,dx\,dy$

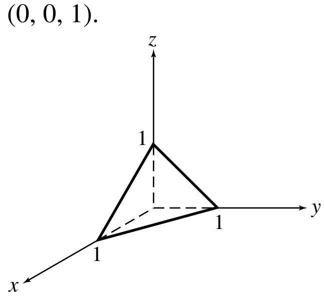
$= \int_{1/3}^1 x\Big|_{1/2}^1 dy = \int_{1/3}^1\left(1 - \frac{1}{2}\right)dy$

$= \int_{1/3}^1 \frac{1}{2}dy = \frac{1}{2}y\bigg|_{1/3}^1 = \frac{1}{2}\left(1 - \frac{1}{3}\right) = \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}$

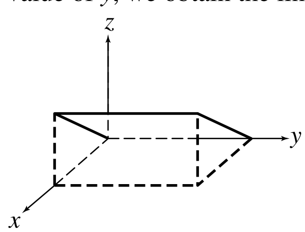
26. $\int_0^1\int_0^1 \frac{1}{8}dx\,dy = \int_0^1 \frac{x}{8}\bigg|_0^1 dy = \int_0^1 \frac{1}{8}dy = \frac{y}{8}\bigg|_0^1 = \frac{1}{8}$

Chapter 17 Review Problems

1. $x + y + z = 1$ can be put in the form $Ax + By + Cz + D = 0$, so the graph is a plane. The intercepts are (1, 0, 0), (0, 1, 0), and (0, 0, 1).

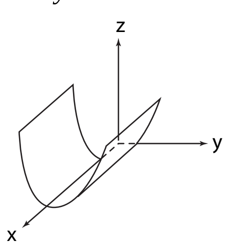


2. $z = x$ can be put in the form $Ax + By + Cz + D = 0$, so the graph is a plane. Every point on the y-axis is an intercept. The x,z-trace is $z = x$, which is a line. For any fixed value of y, we obtain the line $z = x$.

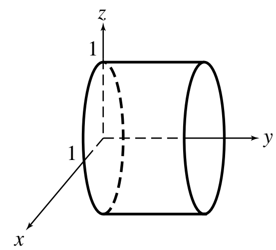


3. $z = y^2$

The y,z-trace is $z = y^2$, which is a parabola. For any fixed value of x, we obtain the curve $z = y^2$.



4. $x^2 + z^2 = 1$. The x,z-trace is $x^2 + z^2 = 1$, which is a circle. For any fixed value of y, we obtain the circle $x^2 + z^2 = 1$.



5. $f_x(x,\ y) = 4(2x) + 6(1)y + 0 - 0 = 8x + 6y$

$f_y(x,\ y) = 0 + 6x(1) + 2y - 0 = 6x + 2y$

6. $\dfrac{\partial P}{\partial l} = 3l^2 + 0 - (1)k = 3l^2 - k$

$\dfrac{\partial P}{\partial k} = 0 + 3k^2 - l(1) = 3k^2 - l$

7. $\dfrac{\partial z}{\partial x} = \dfrac{(x+y)(1) - x(1)}{(x+y)^2} = \dfrac{y}{(x+y)^2}$

Because $z = x(x+y)^{-1}$,

$\dfrac{\partial z}{\partial y} = x\left[(-1)(x+y)^{-2}(1)\right] = -\dfrac{x}{(x+y)^2}$.

8. $f_{p_B}(p_A,\ p_B) = 0 + 5(1-0) = 5$

9. $f(x,\ y) = \ln\sqrt{x^2+y^2} = \dfrac{1}{2}\ln\left(x^2+y^2\right)$

$\dfrac{\partial}{\partial y}[f(x,\ y)] = \dfrac{1}{2}\cdot\dfrac{1}{x^2+y^2}(2y) = \dfrac{y}{x^2+y^2}$

10. $w = \dfrac{x}{\sqrt{x^2+y^2}} = x(x^2+y^2)^{-\frac{1}{2}}$

$\dfrac{\partial w}{\partial y} = x\left[-\dfrac{1}{2}(x^2+y^2)^{-\frac{3}{2}}(2y)\right] = -\dfrac{xy}{(x^2+y^2)^{\frac{3}{2}}}$

11. $w_x(x,\ y,\ z) = 2xyze^{x^2yz}$

$w_{xy}(x,\ y,\ z) = 2xz\left[y\left(e^{x^2yz}\cdot x^2z\right) + e^{x^2yz}\cdot 1\right]$

$= 2xze^{x^2yz}\left(x^2yz + 1\right)$

12. $f_x(x, y) = y\left[x\left(\frac{1}{xy}\cdot y\right) + \ln(xy)\cdot 1\right] = y[1+\ln(xy)]$

$f_{xy}(x, y) = y\left[\frac{1}{xy}\cdot x\right] + [1+\ln(xy)]\cdot 1 = 1+1+\ln(xy) = 2 + \ln(xy)$

13. $\frac{\partial}{\partial z}[f(x, y, z)] = (x+y+z)(2z)+(x^2+y^2+z^2)(1) = 3z^2+2z(x+y)+x^2+y^2$

$\frac{\partial^2}{\partial z^2}[f(x, y, z)] = 6z+2(x+y) = 2x+2y+6z$

14. $z = \left(x^2-y\right)\left(y^2-2xy\right) = x^2y^2-2x^3y-y^3+2xy^2$

$\frac{\partial z}{\partial y} = 2x^2y-2x^3-3y^2+4xy$

$\frac{\partial^2 z}{\partial y^2} = 2x^2-6y+4x$

15. $w = e^{x+y+z}\ln xyz = e^{x+y+z}(\ln x+\ln y+\ln z)$

$\frac{\partial w}{\partial y} = e^{x+y+z}(\ln x+\ln y+\ln z)+e^{x+y+z}\left(\frac{1}{y}\right)$

$= e^{x+y+z}\left[\ln xyz+\frac{1}{y}\right]$

By symmetry, $\frac{\partial w}{\partial x} = e^{x+y+z}\left[\ln xyz+\frac{1}{x}\right]$.

$\frac{\partial^2 w}{\partial z\partial x} = e^{x+y+z}\left[\ln xyz+\frac{1}{x}\right]+e^{x+y+z}\left[\frac{1}{z}\right]$

$= e^{x+y+z}\left[\ln xyz+\frac{1}{x}+\frac{1}{z}\right]$.

16. $\frac{\partial P}{\partial l} = 100\left[(0.11)l^{0.11-1}\right]k^{0.89} = 11l^{-0.89}k^{0.89}$

$\frac{\partial^2 P}{\partial k\partial l} = 11l^{-0.89}\left[(0.89)k^{0.89-1}\right] = 9.79l^{-0.89}k^{-0.11}$

17. $f(x, y, z) = \frac{x+y}{xz} = \frac{1}{z}+\frac{y}{xz}$

$f_x(x, y, z) = -\frac{y}{x^2z}$

$f_{xy}(x, y, z) = -\frac{1}{x^2z}$

$f_{xyz}(x, y, z) = \frac{1}{x^2z^2}$

$f_{xyz}(2, 7, 4) = \frac{1}{2^2\cdot 4^2} = \frac{1}{64}$

18. $f_x(x, y, z) = 6e^{y^2 \ln(z+1)}$

$f_{xy}(x, y, z) = 12y\ln(z+1)e^{y^2 \ln(z+1)}$

$f_{xyz}(x, y, z) = 12y\left[\ln(z+1)\left\{e^{y^2 \ln(z+1)} \cdot \frac{y^2}{z+1}\right\} + e^{y^2 \ln(z+1)} \cdot \frac{1}{z+1}\right]$

$f_{xyz}(0, 1, 0) = 12[0+1] = 12$

19. $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = (2x+2y)\left(e^r\right) + (2x+6y)\left(\frac{1}{r+s}\right)$

$= 2(x+y)e^r + \frac{2(x+3y)}{r+s}$

$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = (2x+2y)(0) + (2x+6y)\left(\frac{1}{r+s}\right)$

$= \frac{2(x+3y)}{r+s}$

20. $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \left(\frac{1}{x} + ye^{xy} - y\right)(2s) + \left(-\frac{1}{y} + xe^{xy} - x\right)(2(r+s))$

21. $2x + 2y - 4z\frac{\partial z}{\partial x} + \left[x\frac{\partial z}{\partial x} + z(1)\right] + 0 = 0$

$(-4z + x)\frac{\partial z}{\partial x} = -(2x+2y+z)$

$\frac{\partial z}{\partial x} = \frac{-(2x+2y+z)}{-4z+x} = \frac{2x+2y+z}{4z-x}$

22. $z^2 + \ln(yz) + \ln z + x + z = 0$ or $z^2 + \ln y + \ln z + \ln z + x + z = 0,$ thus $z^2 + \ln y + 2\ln z + x + z = 0.$

$2z\frac{\partial z}{\partial y} + \frac{1}{y} + 2 \cdot \frac{1}{z}\frac{\partial z}{\partial y} + \frac{\partial z}{\partial y} = 0$

$\frac{\partial z}{\partial y}\left[2z + \frac{2}{z} + 1\right] = -\frac{1}{y}$

$\frac{\partial z}{\partial y}\left[\frac{2z^2 + 2 + z}{z}\right] = -\frac{1}{y}$

$\frac{\partial z}{\partial y} = -\frac{z}{y(2z^2 + 2 + z)}$

23. $P = 20l^{0.7}k^{0.3}$. Marginal productivity functions are given by $\frac{\partial P}{\partial l} = 20(0.7)l^{-0.3}k^{0.3}$ and $\frac{\partial P}{\partial k} = 20(0.3)l^{0.7}k^{-0.7}$. Thus $\frac{\partial P}{\partial l} = 14l^{-0.3}k^{0.3}$ and $\frac{\partial P}{\partial k} = 6l^{0.7}k^{-0.7}$.

24. $c = 3x + 0.05xy + 9y + 500$
Marginal cost with respect to x is
$\frac{\partial c}{\partial x} = 3 + 0.05y.$ When $x = 50$ and $y = 100$, then
$\frac{\partial c}{\partial x} = 8.$

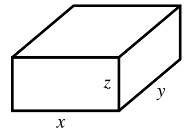
25. $q_A = 100 - p_A + 2p_B$, $q_B = 150 + 3p_A - 2p_B$.
Since $\frac{\partial q_A}{\partial p_B} = 2 > 0$ and $\frac{\partial q_B}{\partial p_A} = 3 > 0$, A and B are competitive products.

26. $\frac{\partial \alpha}{\partial P} = 0.530$; $\frac{\partial \alpha}{\partial S} = -0.027$

27. $f(x, y) = x^2 + 2y^2 - 2xy - 4y + 3$
$$\begin{cases} f_x(x, y) = 2x - 2y = 0 \\ f_y(x, y) = 4y - 2x - 4 = 0 \end{cases}$$
Critical point: (2, 2)
$f_{xx}(x, y) = 2,\ f_{yy}(x, y) = 4,\ f_{xy}(x, y) = -2$
At (2, 2), $D = (2)(4) - (-2)^2 = 4 > 0$ and $f_{xx}(x, y) = 2 > 0$; thus relative minimum at (2, 2).

28. $f(w, z) = 2w^3 + 2z^3 - 6wz + 7$
$$\begin{cases} f_w(w, z) = 6w^2 - 6z = 0 \\ f_z(w, z) = 6z^2 - 6w = 0 \end{cases}$$
Critical points: (0, 0), (1, 1)
$f_{ww}(w, z) = 12w,\ f_{zz}(w, z) = 12z,$
$f_{wz}(w, z) = -6$
At (0, 0), $D = (0)(0) - (-6)^2 = -36 < 0$; thus neither relative maximum nor relative minimum.
At (1, 1), $D = (12)(12) - (-6)^2 = 108 > 0$ and $f_{ww}(w, z) = 12 > 0$; thus relative minimum at (1, 1).

29.



$xyz = 32$ ($xyz \neq 0$). Let S be the amount of cardboard used.
$S = xy + 2yz + 2xz$
$= xy + 2y\left[\frac{32}{xy}\right] + 2x\left[\frac{32}{xy}\right]$
$= xy + \frac{64}{x} + \frac{64}{y}$
$\frac{\partial S}{\partial x} = y - \frac{64}{x^2},\ \frac{\partial S}{\partial y} = x - \frac{64}{y^2}$
The critical point occurs when $x = 4$, $y = 4$, and $z = 2$, which gives a minimum. The dimensions are 4 ft by 4 ft by 2 ft.

30. $f(x, y) = ax^2 + by^2 + cxy - 10x - 20y$
$f_x(x, y) = 2ax + cy - 10$;
$f_y(x, y) = 2by + cx - 20$
At (1, 2), $f_x(x, y) = 0$ and $f_y(x, y) = 0$.
Thus
$2a + 2c - 10 = 0$ (1)
and
$4b + c - 20 = 0$ (2)
From Eq. (1), $a = 5 - c$; from Eq. (2),
$b = \frac{20 - c}{4}$.
$f_{xx}(x, y) = 2a,\ f_{yy}(x, y) = 2b,\ f_{xy}(x, y) = c$
At (1, 2), $D = (2a)(2b) - c^2 = 0$.
Thus $4ab - c^2 = 0$, $4(5 - c)\left(\frac{20 - c}{4}\right) - c^2 = 0$,
$100 - 25c + c^2 - c^2 = 0$, $100 = 25c$, or $c = 4$.
So $a = 5 - c = 5 - 4 = 1$ and
$b = \frac{20 - c}{4} = \frac{20 - 4}{4} = 4$.
Thus $a = 1$, $b = 4$, $c = 4$.

31. Profit $= P = (p_A - 50)q_A + (p_B - 60)q_B$

$P = (p_A - 50)[250(p_B - p_A)] + (p_B - 60)[32,000 + 250(p_A - 2p_B)]$.

$\frac{\partial P}{\partial p_A} = (p_A - 50)(-250) + [250(p_B - p_A)](1) + 250(p_B - 60)$

$= -500p_A + 500p_B - 250(10) = 500(-p_A + p_B - 5)$

Also, $\frac{\partial P}{\partial p_B} = (p_A - 50)(250) + (p_B - 60)(-500) + [32,000 + 250(p_A - 2p_B)](1)$

$= 500p_A - 1000p_B + 49,500 = 500(p_A - 2p_B + 99)$

Setting $\frac{\partial P}{\partial p_A} = 0$ and $\frac{\partial P}{\partial p_B} = 0$ gives

$-p_A + p_B - 5 = 0$ (1)

and

$p_A - 2p_B + 99 = 0$ (2)

Adding Equations (1) and (2) gives $-p_B + 94 = 0$. So $p_B = 94$. From Equation (1), $p_A = p_B - 5$, so $p_A = 94 - 5 = 89$. At $p_A = 89$ and $p_B = 94$, $D = \frac{\partial^2 P}{\partial p_A^2}\frac{\partial^2 P}{\partial p_B^2} - \frac{\partial^2 P}{\partial p_B \partial p_A} = (-500)(-1000) - (500)^2 > 0$ and

$\frac{\partial^2 P}{\partial p_A^2} = -500 < 0$.

Thus there is a relative maximum profit when the price of A is 89 cents per pound and the price of B is 94 cents per pound.

32. $f(x, y, z) = xy^2z,\ x + y + z - 1 = 0$

$F(x, y, z, \lambda) = xy^2z - \lambda(x + y + z - 1)$

$$\begin{cases} F_x = y^2z - \lambda = 0 \\ F_y = 2xyz - \lambda = 0 \\ F_z = xy^2 - \lambda = 0 \\ F_\lambda = -x - y - z + 1 = 0 \end{cases}$$

From the first and third equations, we have $\lambda = y^2z = xy^2$, so $x = z$ (since $xyz \neq 0$). With $x = z$ in the second equation, we have $\lambda = 2x^2y$. Combining this with $\lambda = xy^2$, we get $y = 2x$. Substituting $y = 2x$ and $z = x$ in the fourth equation gives $-x - 2x - x + 1 = 0$ so $4x = 1$ and $x = \frac{1}{4}$. Thus $y = \frac{1}{2}$ and $z = \frac{1}{4}$. The critical point is $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$.

33. $f(x, y, z) = x^2 + y^2 + z^2$, $3x + 2y + z = 14$

$F(x, y, z, \lambda)$

$= x^2 + y^2 + z^2 - \lambda(3x + 2y + z - 14)$

$$\begin{cases} F_x = 2x - 3\lambda = 0 & (1) \\ F_y = 2y - 2\lambda = 0 & (2) \\ F_z = 2z - \lambda = 0 & (3) \\ F_\lambda = -3x - 2y - z + 14 = 0 & (4) \end{cases}$$

From (1), $x = \frac{3\lambda}{2}$; from (2), $y = \lambda$, from (3), $z = \frac{\lambda}{2}$. Substituting into (4) gives $-3\left(\frac{3\lambda}{2}\right) - 2\lambda - \frac{\lambda}{2} + 14 = 0$, from which $\lambda = 2$.

Thus $x = 3$, $y = 2$, and $z = 1$. Critical point of F: (3, 2, 1, 2), so the critical point of f is (3, 2, 1).

34. $n = 5$,

$\Sigma t_i = 104$, $\Sigma p_i = 381$, $\Sigma t_i p_i = 7482$, $\Sigma t_i^2 = 3192$

$$a = \frac{\left(\Sigma t_i^2\right)\left(\Sigma p_i\right) - \left(\Sigma t_i\right)\left(\Sigma t_i p_i\right)}{n\Sigma t_i^2 - \left(\Sigma t_i\right)^2} = 85.15$$

$$b = \frac{n\Sigma t_i p_i - \left(\Sigma t_i\right)\left(\Sigma p_i\right)}{n\Sigma t_i^2 - \left(\Sigma t_i\right)^2} = -0.43$$

Thus $\hat{p} = 85.15 - 0.43t$

35.

Year (x)	1	2	3	4	5	6
Expenditures (y)	15	22	21	26	27	34

$n = 6$, $\Sigma x_i = 21$, $\Sigma y_i = 145$, $\Sigma x_i y_i = 565$, $\Sigma x_i^2 = 91$

$$a = \frac{\left(\Sigma x_i^2\right)\left(\Sigma y_i\right) - \left(\Sigma x_i\right)\left(\Sigma x_i y_i\right)}{n\Sigma x_i^2 - \left(\Sigma x_i\right)^2} = 12.67$$

$$b = \frac{n\Sigma x_i y_i - \left(\Sigma x_i\right)\left(\Sigma y_i\right)}{n\Sigma x_i^2 - \left(\Sigma x_i\right)^2} = 3.29$$

Thus $\hat{y} = 12.67 + 3.29x$

36. $$\int_1^2 \int_0^y x^2 y^2 \, dx\, dy = \int_1^2 \left.\frac{x^3 y^2}{3}\right|_0^y dy = \int_1^2 \frac{y^5}{3} dy$$

$$= \left.\frac{y^6}{18}\right|_1^2 = \frac{64}{18} - \frac{1}{18} = \frac{63}{18} = \frac{7}{2}$$

37. $$\int_0^1 \int_0^{y^2} xy \, dx\, dy = \int_0^1 \left.\frac{x^2 y}{2}\right|_0^{y^2} dy$$

$$= \int_0^1 \left(\frac{y^4 \cdot y}{2} - 0\right) dy = \frac{1}{2}\int_0^1 y^5 dy$$

$$= \left.\frac{1}{2} \cdot \frac{y^6}{6}\right|_0^1 = \frac{1}{2} \cdot \frac{1}{6} - 0 = \frac{1}{12}$$

38. $$\int_1^4 \int_{x^2}^{2x} y \, dy\, dx = \int_1^4 \left.\frac{y^2}{2}\right|_{x^2}^{2x} dx$$

$$= \int_1^4 \left(2x^2 - \frac{x^4}{2}\right) dx$$

$$= \left.\left(\frac{2x^3}{3} - \frac{x^5}{10}\right)\right|_1^4$$

$$= \left(\frac{128}{3} - \frac{512}{5}\right) - \left(\frac{2}{3} - \frac{1}{10}\right)$$

$$= -\frac{603}{10}$$

39. $$\int_0^1 \int_{\sqrt{x}}^{x^2} 7\left(x^2 + 2xy - 3y^2\right) dy\, dx$$

$$= 7\int_0^1 \left.\left(x^2 y + xy^2 - y^3\right)\right|_{\sqrt{x}}^{x^2} dx$$

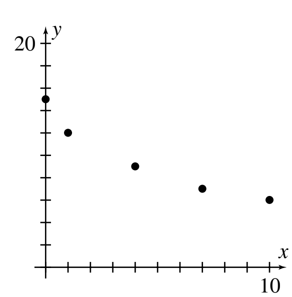
$$= 7\int_0^1 \left[\left(x^4 + x^5 - x^6\right) - \left(x^{\frac{5}{2}} + x^2 - x^{\frac{3}{2}}\right)\right] dx$$

$$= 7\left.\left[\frac{x^5}{5} + \frac{x^6}{6} - \frac{x^7}{7} - \frac{2x^{\frac{7}{2}}}{7} - \frac{x^3}{3} + \frac{2x^{\frac{5}{2}}}{5}\right]\right|_0^1$$

$$= 7\left[\frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{2}{7} - \frac{1}{3} + \frac{2}{5}\right] - 0 = \frac{1}{30}$$

Mathematical Snapshot Chapter 17

1.



$y = Ce^{ax} + 5$, $y - 5 = Ce^{ax}$, $\ln(y - 5) = ax + \ln C$

x	y	$y-5$	$\ln(y-5)$
0	15	10	2.30259
1	12	7	1.94591
4	9	4	1.38629
7	7	2	0.69315
10	6	1	0.00000

$n = 5,\ \Sigma x_i = 22,\ \Sigma \ln(y_i - 5) = 6.32794$,

$\Sigma[x_i \ln(y_i - 5)] = 12.34312,\ \Sigma x_i^2 = 166$

$$a = \frac{n\Sigma[x_i \ln(y_i-5)] - (\Sigma x_i)[\Sigma \ln(y_i-5)]}{n(\Sigma x_i^2) - (\Sigma x_i)^2}$$

$$= \frac{5(12.34312) - 22(6.32794)}{5(166) - (22)^2} \approx -0.22399$$

$$\ln C = \frac{(\Sigma x_i^2)[\Sigma \ln(y_i-5)] - (\Sigma x_i)\{\Sigma[x_i \ln(y_i-5)]\}}{n(\Sigma x_i^2) - (\Sigma x_i)^2}$$

$$= \frac{166(6.32794) - 22(12.34312)}{5(166) - (22)^2} \approx 2.25112$$

$C \approx e^{2.25112} \approx 9.50$

Thus $y = 9.50e^{-0.22399x} + 5$.

2. $y = \frac{C}{x^r}$. $\ln y = \ln \frac{C}{x^r}$, $\ln y = \ln C - \ln x^r$,
$\ln y = \ln(C) - r \ln x$.
Thus $r(\ln x) + (\ln y) - \ln(C) = 0$. Since $\ln C$ and r are constants, $\ln x$ and $\ln y$ are linearly related.

3. Newton's law of cooling: $\frac{dT}{dt} = k(T-a)$, where $a = 45$. Thus $\frac{dT}{dt} = k(T-45)$, $\frac{dT}{T-45} = k\,dt$,
$\int \frac{dT}{T-45} = \int k\,dt$, $\ln|T-45| = kt + C$. Because $T - 45 > 0$, $\ln(T-45) = kt + C$. Thus
$T - 45 = e^{kt+C}$, or
$T = e^{kt+C} + 45 = e^C e^{kt} + 45 = C_1 e^{kt} + 45$, where $C_1 = e^C$. So $T = C_1 e^{kt} + 45$. When $t = 0$, then $T = 124$. Hence $124 = C_1 + 45$, or $C_1 = 79$. Thus $T = 79e^{kt} + 45$. When $t = 128$, then $T = 64$, so
$64 = 79e^{128k} + 45$, $19 = 79e^{128k}$, $e^{128k} = \frac{19}{79}$,
$128k = \ln\frac{19}{79}$, $k = \frac{\ln\left(\frac{19}{79}\right)}{128} \approx -0.01113$. Thus
$T = 79e^{-0.01113t} + 45$.

4. Using a graphics calculator on the points displayed in Fig. 17.25 produces the same result as in the snapshot. Performing an exponential regression on the points shown in Fig. 17.24, however, does not produce the right curve, because the exponential model lacks a constant term. This difficulty can be overcome by subtracting 45 (the long-term temperature) from every temperature value, running the exponential regression, and then adding a constant term of 45 to the resulting model.