Given: Common substances

Tar	Sand
"Silly Putty"	Jello
Modeling clay	Toothpaste
Wax	Shaving cream

Some of these substances exhibit characteristics of solids and fluids under different conditions.

Find: Explain and give examples.

Solution:

Tar, wax, and Jello behave as solids at room temperature or below at ordinary pressures. At high pressures or over long periods, they exhibit fluid characteristics. At higher temperatures, all three liquefy and become viscous fluids.

Modeling clay and silly putty show fluid behavior when sheared slowly. However, they fracture under suddenly applied stress, which is a characteristic of solids.

Toothpaste behaves as a solid when at rest in the tube. When the tube is squeezed hard, toothpaste "flows" out the spout, showing fluid behavior. Shaving cream behaves similarly.

Sand act solid when in repose (a sand "pile"). However, it "flows" from a spout or down a steep incline.

42-381 50 SHEETS 5 SQUARE
42-382 100 SHEETS 5 SQUARE
42-389 200 SHEETS 5 SQUARE
NATIONAL

Given: Tank to contain 10 kg of O_2 at 14 MPa, 35°C.

Find: Tank volume and diameter if spherical.

Solution: Assume ideal gas behavior.

Basic equations: $p = \rho RT$ (p = absolute pressure)

$$\rho = \frac{m}{\forall}$$

Substituting, we obtain $p = \frac{mRT}{\forall}$, so

$$\forall = \frac{mRT}{p}$$

From Table A.6, $R = 259.8\ N \cdot m/kg \cdot K$, so

$$\forall = 10\ kg \times 259.8\ \frac{N \cdot m}{kg \cdot K} \times (273+35)\ K \times \frac{m^2}{(14 \times 10^6 + 101 \times 10^3)\ N}$$

$$\forall = 0.0567\ m^3 \longleftarrow \forall$$

For a sphere, $\forall = \frac{4}{3}\pi R^3 = \frac{1}{6}\pi D^3$, so

$$D = \left[\frac{6\forall}{\pi}\right]^{\frac{1}{3}} = \left[\frac{6}{\pi} \times 0.0567\ m\right]^{\frac{1}{3}} = 0.477\ m \longleftarrow D$$

Given: Cylindrical tank, $D = 150\,mm$, $L = 1300\,mm$, contains nitrogen at pressure $p = 204$ atmospheres (gage)

Find: the mass of nitrogen in the tank.

Solution:

Assume ideal gas behavior

Basic equations: $p = \rho RT$ (p = absolute pressure)

$$\rho = \frac{m}{V}$$

Substituting, we obtain $p = \frac{mRT}{V}$, so

$$m = \frac{pV}{RT}$$

Since $V = \frac{\pi D^2 L}{4}$, then

$$m = \frac{\pi p D^2 L}{4RT}$$

From Table A.6, $R = 296.8\ N \cdot m/kg \cdot K$

Assuming $T = 20^\circ C = 293\,K$, then

$$m = \frac{\pi}{4} \times (204+1)\,\cancel{atm} \times 101 \times 10^3 \frac{N}{m^2 \cdot \cancel{atm}} \times (0.15)^2 m^2 \times 1.30\,\cancel{m} \times \frac{kg \cdot K}{296.8\,\cancel{N \cdot m}} \times \frac{1}{293\,\cancel{K}}$$

$$m = 5.47\,kg$$

Given: Five basic conservation laws stated in Section 1-4.

Write: A word statement of each, as they apply to a system.

Solution: Assume that laws are to be written for a system.

(a) Conservation of mass - The mass of a system is constant by definition.

(b) Newton's second law of motion - The net force acting on a system is directly proportional to the product of the system mass times its acceleration.

(c) First law of thermodynamics - The change in stored energy of a system equals the net energy added to the system as heat and work.

(d) Second law of thermodynamics - The entropy of any isolated system cannot decrease during any process between equilibrium states.

(e) Principle of angular momentum - The net torque acting on a system is equal to the rate of change of angular momentum of the system.

Open-Ended Problem Statement: Consider the physics of "skipping" a stone across the water surface of a lake. Compare these mechanisms with a stone as it bounces after being thrown along a roadway.

Discussion: Observation and experience suggest two behaviors when a stone is thrown along a water surface:

(1) If the angle between the path of the stone and the water surface is steep the stone may penetrate the water surface. Some momentum of the stone will be converted to momentum of the water in the resulting splash. After penetrating the water surface, the high drag* of the water will slow the stone quickly. Then, because the stone is heavier than water it will sink.

(2) If the angle between the path of the stone and the water surface is shallow the stone may not penetrate the water surface. The splash will be smaller than if the stone penetrated the water surface. This will transfer less momentum to the water, causing less reduction in speed of the stone. The only drag force on the stone will be from friction on the water surface. The drag will be momentary, causing the stone to lose only a portion of its kinetic energy. Instead of sinking, the stone may skip off the surface and become airborne again.

When the stone is thrown with speed and angle just right, it may skip several times across the water surface. With each skip the stone loses some forward speed. After several skips the stone loses enough forward speed to penetrate the surface and sink into the water.

Observation suggests that the shape of the stone significantly affects skipping. Essentially spherical stones may be made to skip with considerable effort and skill from the thrower. Flatter, more disc-shaped stones are more likely to skip, provided they are thrown with the flat surface(s) essentially parallel to the water surface; spin may be used to stabilize the stone in flight.

By contrast, no stone can ever penetrate the pavement of a roadway. Each collision between stone and roadway will be inelastic; friction between the road surface and stone will affect the motion of the stone only slightly. Regardless of the initial angle between the path of the stone and the surface of the roadway, the stone may bounce several times, then finally it will roll to a stop.

The shape of the stone is unlikely to affect trajectory of bouncing from a roadway significantly.

* Compared to the negligible aerodynamic drag in air.

13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Open-Ended Problem Statement: The barrel of a bicycle tire pump becomes quite warm during use. Explain the mechanisms responsible for the temperature increase.

Discussion: Two phenomena are responsible for the temperature increase: (1) friction between the pump piston and barrel and (2) temperature rise of the air as it is compressed in the pump barrel.

Friction between the pump piston and barrel converts mechanical energy (force on the piston moving through a distance) into thermal energy as a result of friction. Lubricating the piston helps to provide a good seal with the pump barrel and reduces friction (and therefore force) between the piston and barrel.

Temperature of the trapped air rises as it is compressed. The compression is not adiabatic because it occurs during a finite time interval. Heat is transferred from the warm compressed air in the pump barrel to the cooler surroundings. This raises the temperature of the barrel, making its outside surface warm (or even hot!) to the touch.

13-782 500 SHEETS FILLER 5 SQUARE
42-381 500 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Air at standard conditions - $p = 29.9$ in Hg, $T = 59°F$
Uncertainty: in p is ± 0.1 in. Hg, in T is $\pm 0.5°F$
Note that 29.9 in Hg corresponds to 14.7 psia

Find: a) air density using ideal gas equation of state.
b) estimate of uncertainty in calculated value.

Solution:

$$\rho = \frac{p}{RT} = 14.7 \frac{lbf}{in^2} \times \frac{lb \cdot °R}{53.3\ ft \cdot lbf} \times \frac{1}{519°R} \times 144 \frac{in^2}{ft^2}$$

$$\rho = 0.0765\ lbm/ft^3 \quad \rho$$

The uncertainty in density is given by

$$u_\rho = \left[\left(\frac{p}{\rho} \frac{\partial \rho}{\partial p} u_p \right)^2 + \left(\frac{T}{\rho} \frac{\partial \rho}{\partial T} u_T \right)^2 \right]^{1/2}$$

$$\frac{p}{\rho} \frac{\partial \rho}{\partial p} = RT \frac{1}{RT} = \frac{RT}{RT} = 1 \; ; \quad u_p = \frac{\pm 0.1}{29.9} = \pm 0.334\%$$

$$\frac{T}{\rho} \frac{\partial \rho}{\partial T} = \frac{T}{\rho} \left(-\frac{p}{RT^2} \right) = -\frac{p}{\rho RT} = -1 \; ; \quad u_T = \frac{\pm 0.5}{460+59} = \pm 0.0963\%$$

Then

$$u_\rho = \left[(u_p)^2 + (-u_T)^2 \right]^{1/2} = \pm \left[(0.334)^2 + (-0.0963)^2 \right]$$

$$u_\rho = \pm 0.348\% \quad (\pm 2.66 \times 10^{-4}\ lbm/ft^3) \quad u_\rho$$

Given: Air at pressure, $p = 759 \pm 1$ mm Hg and temperature, $T = -20 \pm 0.5°C$.
Note that 759 mm Hg corresponds to 101 kPa.

Find: (a) air density using ideal gas equation of state
(b) estimate of uncertainty in calculated value

Solution:

$$\rho = \frac{p}{RT} = 101 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287 N \cdot m} \times \frac{1}{253 K} = 1.39 \ kg/m^3$$ ← ρ

The uncertainty in density is given by

$$u_\rho = \left[\left(\frac{p}{\rho} \frac{\partial \rho}{\partial p} u_p \right)^2 + \left(\frac{T}{\rho} \frac{\partial \rho}{\partial T} u_T \right)^2 \right]^{1/2}$$

$$\frac{p}{\rho} \frac{\partial \rho}{\partial p} = RT \frac{1}{RT} = 1 \ ; \quad u_p = \frac{\pm 1}{759} = \pm 0.132\%$$

$$\frac{T}{\rho} \frac{\partial \rho}{\partial T} = \frac{T}{\rho} \left(-\frac{p}{RT^2} \right) = -\frac{p}{\rho RT} = -1 \ ; \quad u_T = \frac{\pm 0.5}{273 - 20} = \pm 0.198\%$$

Then

$$u_\rho = \left[(u_p)^2 + (-u_T)^2 \right]^{1/2} = \pm \left[(0.132)^2 + (-0.198)^2 \right]^{1/2}$$

$$u_\rho = \pm 0.238\% \quad (\pm 3.31 \times 10^{-3} \ kg/m^3)$$ ← u_ρ

Given: Mass flow rate of water determined by collecting discharge over a timed interval is 0.3 kg/s.
Scales can be read to nearest 0.05 kg.
Stopwatch can be read to nearest 0.2 s.

Find: Estimate precision of flow rate calculation for time intervals of (a) 10 s, and (b) 1 min.

Solution: Apply methodology of uncertainty analysis, Appendix F:

Computing equations: $\dot{m} = \frac{\Delta m}{\Delta t}$

$$u_{\dot{m}} = \pm\left[\left(\frac{\Delta m}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta m}u_{\Delta m}\right)^2 + \left(\frac{\Delta t}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta t}u_{\Delta t}\right)^2\right]^{1/2}$$

Thus $\frac{\Delta m}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta m} = \Delta t\left(\frac{1}{\Delta t}\right) = 1$ and $\frac{\Delta t}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta t} = \frac{\Delta t^2}{\Delta m}\left[(-1)\frac{\Delta m}{\Delta t^2}\right] = -1$

The uncertainties are expected to be ± half the least counts of the measuring instruments.

Tabulating results:

Time Interval, Δt (s)	Error in Δt (s)	Uncertainty in Δt (percent)	Water Collected, Δm (kg)	Error in Δm (kg)	Uncertainty in Δm (percent)	Uncertainty in $\dot{m}$ (percent)
10	±0.10	±1.0	3.0	±0.025	±0.833	±1.30
60	±0.10	±0.167	18.0	±0.025	±0.139	±0.217

A time interval of about 15 seconds should be chosen to reduce the uncertainty in results to ±1 percent.

Given: Pet food can $H = 102 \pm 1$ mm (20 to 1)
$D = 73 \pm 1$ mm (20 to 1)
$m = 397 \pm 1$ g (20 to 1)

Find: Magnitude and estimated uncertainty of pet food density.

Solution: Density is $\rho = \frac{m}{\forall} = \frac{m}{\pi R^2 H} = \frac{4}{\pi}\frac{m}{D^2 H}$ or $\rho = \rho(m, D, H)$

From uncertainty analysis

$$u_\rho = \pm\left[\left(\frac{m}{\rho}\frac{\partial \rho}{\partial m} u_m\right)^2 + \left(\frac{D}{\rho}\frac{\partial \rho}{\partial D} u_D\right)^2 + \left(\frac{H}{\rho}\frac{\partial \rho}{\partial H} u_H\right)^2\right]^{\frac{1}{2}}$$

Evaluating, $\frac{m}{\rho}\frac{\partial \rho}{\partial m} = \frac{m}{\rho}\frac{4}{\pi}\frac{1}{D^2 H} = \frac{1}{\rho}\frac{4m}{\pi D^2 H} = 1$; $u_m = \frac{\pm 1}{397} = \pm 0.252\%$

$\frac{D}{\rho}\frac{\partial \rho}{\partial D} = \frac{D}{\rho}(-2)\frac{4m}{\pi D^3 H} = (-2)\frac{1}{\rho}\frac{4m}{\pi D^2 H} = -2$; $u_D = \frac{\pm 1}{73} = \pm 1.37\%$

$\frac{H}{\rho}\frac{\partial \rho}{\partial H} = \frac{H}{\rho}(-1)\frac{4m}{\pi D^2 H^2} = (-1)\frac{1}{\rho}\frac{4m}{\pi D^2 H} = -1$; $u_H = \frac{\pm 1}{102} = \pm 0.980\%$

Substituting

$$u_\rho = \pm\left\{[(1)(0.252)]^2 + [(-2)(1.37)]^2 + [(-1)(0.980)]^2\right\}^{1/2}$$

$u_\rho = \pm 2.92$ percent ← u_ρ

$$\forall = \frac{\pi}{4} D^2 H = \frac{\pi}{4} \times (73)^2 \text{ mm}^2 \times 102 \text{ mm} \times \frac{\text{m}^3}{10^9 \text{ mm}^3} = 4.27 \times 10^{-4} \text{ m}^3$$

$$\rho = \frac{m}{\forall} = \frac{397 \text{ g}}{4.27 \times 10^{-4} \text{ m}^3} \times \frac{\text{kg}}{1000 \text{ g}} = 930 \text{ kg/m}^3$$

Thus $\rho = 930 \pm 27.2$ kg/m^3 (20 to 1) ← ρ

Given: Standard American golf ball: $m = 1.62 \pm 0.01$ oz (20 to 1)

$D = 1.68 \pm 0.01$ in. (20 to 1)

Find: (a) Density and specific gravity.
(b) Estimate uncertainties in calculated values.

Solution: Density is mass per unit volume, so

$$\rho = \frac{m}{\forall} = \frac{m}{\frac{4}{3}\pi R^3} = \frac{3}{4\pi}\frac{m}{(D/2)^3} = \frac{6}{\pi}\frac{m}{D^3}$$

$$\rho = \frac{6}{\pi} \times 1.62\,oz \times \frac{1}{(1.68)^3\,in.^3} \times \frac{0.4536\,kg}{16\,oz} \times \frac{in.^3}{(0.0254)^3\,m^3} = 1130\ kg/m^3$$

and

$$SG = \frac{\rho}{\rho_{H_2O}} = 1130\,\frac{kg}{m^3} \times \frac{m^3}{1000\,kg} = 1.13$$

The uncertainty in density is given by

$$u_\rho = \pm\left[\left(\frac{m}{\rho}\frac{\partial \rho}{\partial m}u_m\right)^2 + \left(\frac{D}{\rho}\frac{\partial \rho}{\partial D}u_D\right)^2\right]^{1/2}$$

$$\frac{m}{\rho}\frac{\partial \rho}{\partial m} = \frac{m}{\rho}\frac{1}{\forall} = \frac{\forall}{\forall} = 1\ ;\ u_m = \pm\frac{0.01}{1.62} = \pm 0.617 \text{ percent}$$

$$\frac{D}{\rho}\frac{\partial \rho}{\partial D} = \frac{D}{\rho}\left(-3\,\frac{6}{\pi}\frac{m}{D^4}\right) = \frac{\pi D^4}{6m}\left(-3\,\frac{6}{\pi}\frac{m}{D^4}\right) = -3\ ;\ u_D = \pm 0.595 \text{ percent}$$

Thus $u_\rho = \pm\left[(u_m)^2 + (-3u_D)^2\right]^{1/2}$

$$= \pm\left\{(0.617)^2 + [-3(0.595)]^2\right\}^{1/2}$$

$$u_\rho = \pm 1.89 \text{ percent } (\pm 21.4\ kg/m^3)$$

$$u_{SG} = u_\rho = \pm 1.89 \text{ percent } (\pm 0.0214)$$

Finally,

$\rho = 1130 \pm 21.4\ kg/m^3$ (20 to 1) ← ρ

$SG = 1.13 \pm 0.0214$ (20 to 1) ← SG

Given: Standard British golf ball:

$m = 45.9 \pm 0.3\,g$ (20 to 1)

$D = 41.1 \pm 0.3\,mm$ (20 to 1)

Find: (a) Density and specific gravity

(b) Estimate of uncertainties in calculated values.

Solution:

Density is mass per unit volume, so

$$\rho = \frac{m}{V} = \frac{m}{\frac{4}{3}\pi R^3} = \frac{m}{\frac{4}{3}\pi (D/2)^3} = \frac{6}{\pi}\frac{m}{D^3}$$

$$\rho = \frac{6}{\pi} \times 0.0459\,kg \times \frac{1}{(0.0411)^3}\,m^3 = 1260\,kg/m^3$$

and

$$SG = \frac{\rho}{\rho_{H_2O}} = \frac{1260\,kg}{m^3} \times \frac{m^3}{1000\,kg} = 1.26$$

The uncertainty in density is given by

$$u_\rho = \pm\left[\left(\frac{\partial \rho}{\partial m} u_m\right)^2 + \left(\frac{\partial \rho}{\partial D} u_D\right)^2\right]^{1/2}$$

$$\frac{\partial \rho}{\partial m} = \frac{1}{V} = \frac{1}{\frac{\pi}{6}D^3} = 1 \;;\quad u_m = \pm \frac{0.3}{45.9} = \pm 0.654\,\%$$

$$\frac{\partial \rho}{\partial D} = \frac{\partial}{\partial D}\left(\frac{6}{\pi}\frac{m}{D^3}\right) = -3\left(\frac{6}{\pi}\frac{m}{D^4}\right) = -3$$

$$u_D = \pm \frac{0.3}{41.1} = 0.730\,\%$$

Thus

$$u_\rho = \pm\left[(u_m)^2 + (-3u_D)^2\right]^{1/2} = \pm\left\{(0.654)^2 + [-3(0.730)]^2\right\}^{1/2}$$

$$u_\rho = \pm 2.29\,\% \quad (\pm 28.9\,kg/m^3)$$

$$u_{SG} = u_\rho = \pm 2.29\,\% \quad (\pm 0.0289)$$

Summarizing

$\rho = 1260 \pm 28.9\,kg/m^3$ (20 to 1) ← ρ

$SG = 1.26 \pm 0.0289$ (20 to 1) ← SG

Given: Soda can with estimated dimensions $D = 66.0 \pm 0.5$ mm, $H = 110 \pm 0.5$ mm. Soda has SG = 1.055

Find: (a) volume of soda in the can (based on measured mass of full and empty can).
(b) estimate average depth to which the can is filled and the uncertainty in the estimate.

Solution:

Measurements on a can of coke give

$m_f = 386.5 \pm 0.50$ g, $m_e = 17.5 \pm 0.50$ g $\therefore m = m_f - m_e = 369 \pm u_m$ g

$$u_m = \pm\left[\left(\frac{m}{m_f}\frac{\partial m}{\partial m_f}u_{m_f}\right)^2 + \left(\frac{m}{m_e}\frac{\partial m}{\partial m_e}u_{m_e}\right)^2\right]^{1/2}$$

$$u_{m_f} = \pm\frac{0.5\text{ g}}{386.5\text{ g}} = \pm 0.00129, \quad u_{m_e} = \pm\frac{0.50}{17.5} = 0.0286$$

$$\therefore u_m = \pm\left\{\left[\frac{386.5}{369}(1)(0.00129)\right]^2 + \left[\frac{17.5}{369}(-1)(0.0286)\right]^2\right\}^{1/2} = 0.0019 \leftarrow ---$$

Density is mass per unit volume and $SG = \rho/\rho_{H_2O}$ so

$$\forall = \frac{m}{\rho} = \frac{m}{\rho_{H_2O}\,SG} = 369\text{ g} \times \frac{m^3}{1000\text{ kg}} \times \frac{1}{1.055} \times \frac{\text{kg}}{1000\text{ g}} = 350 \times 10^{-6}\ m^3 \leftarrow \forall$$

The reference value ρ_{H_2O} is assumed to be precise. Since SG is specified to three places beyond the decimal point, assume $u_{SG} = \pm 0.001$. Then

$$u_V = \pm\left[\left(\frac{m}{V}\frac{\partial V}{\partial m}u_m\right)^2 + \left(\frac{m}{SG}\frac{\partial V}{\partial SG}\right)^2\right]^{1/2} = \pm\left\{\left[(1)u_m\right]^2 + \left[(-1)u_{SG}\right]^2\right\}^{1/2}$$

$$u_V = \pm\left\{\left[(1)(0.0019)\right]^2 + \left[(-1)(0.001)\right]^2\right\}^{1/2} = 0.0021 \text{ or } 0.21\% \leftarrow ---$$

$$\forall = \frac{\pi}{4}D^2 L \quad \text{or} \quad L = \frac{4\forall}{\pi D^2} = \frac{4}{\pi} \times \frac{350\times10^{-6}\ m^3}{(0.066)^2\ m^2} \times \frac{10^3\text{ mm}}{m} = 102\text{ mm} \leftarrow L$$

$$u_L = \pm\left[\left(\frac{L}{\forall}\frac{\partial \forall}{\partial \forall}u_\forall\right)^2 + \left(\frac{L}{D}\frac{\partial \forall}{\partial D}u_D\right)^2\right]^{1/2}$$

$$\frac{L}{\forall}\frac{\partial \forall}{\partial \forall} = \frac{\forall}{\frac{\pi}{4}D^2 L} \cdot 1 = 1 \qquad u_D = \pm\frac{0.5\text{ mm}}{66\text{ mm}} = 0.0076$$

$$\frac{L}{D}\frac{\partial \forall}{\partial D} = D\,\frac{\forall}{\frac{\pi}{4}D^2 L}\left(\frac{-\forall}{D^3}\right) = -2$$

$$u_L = \pm\left\{\left[(1)(0.0021)\right]^2 + \left[(-2)(0.0076)\right]^2\right\}^{1/2} = 0.0153 \text{ or } 1.53\% \leftarrow u_L$$

Note: (1) printing on the can states the content as 355 ml. This suggests that the implied accuracy of the SG value may be over stated.
(2) results suggest that over seven percent of the can height is void of soda.

Given: Nominal mass flow rate of water determined by collecting discharge (in a beaker) over a timed interval is $\dot{m} = 100$ g/s
- scales have capacity of 1 kg, with least count of 1 g.
- timer has least count of 0.1 s.
- beakers with volume of 100, 500, 1000 mL are available
  - tare mass of 1000 mL beaker is 500 g.

Find: Estimate (a) time intervals, and (b) uncertainties, in measuring mass flow rate from using each of the three beakers.

Solution: To estimate time intervals assume beaker is filled to maximum volume in case of 100 and 500 mL beakers and to maximum allowable mass of water (500 g) in case of 1000 mL beaker.

Then $\dot{m} = \frac{\Delta m}{\Delta t}$ and $\Delta t = \frac{\Delta m}{\dot{m}} = \frac{\rho \Delta t}{\dot{m}}$

Tabulating results

	100 mL	500 mL	1000 mL
Δm =	100 mL	500 mL	1000 mL
Δt =	1 s	5 s	5 s

Apply the methodology of uncertainty analysis, Appendix E

Computing equation:

$$u_{\dot{m}} = \pm \left[\left(\frac{\Delta m}{\dot{m}} \frac{\partial \dot{m}}{\partial \Delta m} u_{\Delta m} \right)^2 + \left(\frac{\Delta t}{\dot{m}} \frac{\partial \dot{m}}{\partial \Delta t} u_{\Delta t} \right)^2 \right]^{1/2}$$

The uncertainties are expected to be ± half the least counts of the measuring instruments

$$\delta \Delta m = \pm 0.5 \text{ g} \qquad \delta \Delta t = 0.05 \text{ s}$$

$$\frac{\Delta m}{\dot{m}} \frac{\partial \dot{m}}{\partial \Delta m} = \Delta t \left(\frac{1}{\Delta t} \right) = 1 \quad \text{and} \quad \frac{\Delta t}{\dot{m}} \frac{\partial \dot{m}}{\partial \Delta t} = \frac{(\Delta t)^2}{\Delta m} \left[-\frac{\Delta m}{(\Delta t)^2} \right] = -1$$

$$\therefore u_{\dot{m}} = \pm \left[(u_{\Delta m})^2 + (-u_{\Delta t})^2 \right]^{1/2}$$

Tabulating results:

Beaker Volume Δt (mL)	Water collected Δm (g)	Error in Δm (g)	Uncertainty in Δm (percent)	Time Interval Δt (s)	Error in Δt (s)	Uncertainty in Δt (percent)	Uncertainty in $\dot{m}$ (percent)
100	100	± 0.50	± 0.50	1.0	± 0.05	± 5.0	± 5.03
500	500	± 0.50	± 0.10	5.0	± 0.05	± 1.0	± 1.0
1000	500	± 0.50	± 0.10	5.0	± 0.05	± 1.0	± 1.0

Since the scales have a capacity of 1 kg and the tare mass of the 1000 mL beaker is 500 g, there is no advantage in using the larger beaker. The uncertainty in $\dot{m}$ could be reduced to ± 0.50 percent by using the large beaker if a scale with greater capacity the same least count were available

Given: Lateral acceleration, $a = 0.823\,g$, measured on 200-ft diameter skid pad.

Path deviation: ± 2 ft
Vehicle speed: ± 0.5 mph
} measurement uncertainty

Find: (a) Estimate uncertainty in lateral acceleration.
(b) How could experimental procedure be improved?

Solution: Lateral acceleration is given by $a = V^2/R$.

From Appendix F, $u_a = \pm\left[(2u_V)^2 + (u_R)^2\right]^{1/2}$

From the given data,

$$V^2 = aR\,;\quad V = \sqrt{aR} = \left[0.823 \times 32.2\,\frac{ft}{s^2} \times 100\,ft\right]^{1/2} = 51.5\ ft/s$$

Then

$$u_V = \pm\frac{\delta V}{V} = \pm 0.5\,\frac{mi}{hr} \times \frac{s}{51.5\,ft} \times 5280\,\frac{ft}{mi} \times \frac{hr}{3600\,s} = \pm 0.0142$$

and

$$u_R = \pm\frac{\delta R}{R} = \pm 2\,ft \times \frac{1}{100\,ft} = \pm 0.0200$$

so

$$u_a = \pm\left[(2 \times 0.0142)^2 + (0.0200)^2\right]^{1/2} = \pm 0.0347$$

$$u_a = \pm 3.47 \text{ percent}$$

Experimental procedure could be improved by using a larger circle, assuming the absolute errors in measurement are constant.

For $D = 400$ ft, $R = 200$ ft

$$V = \sqrt{aR} = \left[0.823 \times 32.2\,\frac{ft}{s^2} \times 200\,ft\right]^{1/2} = 72.8\ ft/s = 49.6\ mph$$

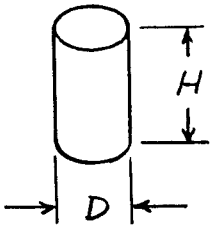
$$u_V = \pm\frac{0.5\ mph}{49.6\ mph} = \pm 0.0101\,;\quad u_R = \pm\frac{2\,ft}{200\,ft} = \pm 0.0100$$

$$u_a = \pm\left[(2 \times 0.0101)^2 + (0.0100)^2\right]^{1/2} = \pm 0.0225 \text{ or } \pm 2.3 \text{ percent}$$

Given: Dimensions of soda can:

$D = 66$ mm

$H = 110$ mm



Find: Measurement precision needed to allow volume to be estimated with an uncertainty of ± 0.5 percent or less.

Solution: Use the methods of Appendix F:

Computing equations: $\forall = \frac{\pi D^2 H}{4}$

$$u_\forall = \pm\left[\left(\frac{H}{\forall}\frac{\partial \forall}{\partial H} u_H\right)^2 + \left(\frac{D}{\forall}\frac{\partial \forall}{\partial D} u_D\right)^2\right]^{\frac{1}{2}}$$

Since $\forall = \frac{\pi D^2 H}{4}$, then $\frac{\partial \forall}{\partial H} = \frac{\pi D^2}{4}$ and $\frac{\partial \forall}{\partial D} = \frac{\pi D H}{2}$

Let $u_D = \pm \frac{\delta x}{D}$ and $u_H = \pm \frac{\delta x}{H}$, substituting,

$$u_\forall = \pm\left[\left(\frac{4H}{\pi D^2 H}\frac{\pi D^2}{4}\frac{\delta x}{H}\right)^2 + \left(\frac{4D}{\pi D^2 H}\frac{\pi D H}{2}\frac{\delta x}{D}\right)^2\right]^{\frac{1}{2}} = \pm\left[\left(\frac{\delta x}{H}\right)^2 + \left(\frac{2\delta x}{D}\right)^2\right]^{\frac{1}{2}}$$

Solving, $$u_\forall^2 = \left(\frac{\delta x}{H}\right)^2 + \left(\frac{2\delta x}{D}\right)^2 = (\delta x)^2\left[\left(\frac{1}{H}\right)^2 + \left(\frac{2}{D}\right)^2\right]$$

$$\delta x = \pm \frac{u_\forall}{\left[\left(\frac{1}{H}\right)^2 + \left(\frac{2}{D}\right)^2\right]^{\frac{1}{2}}} = \pm \frac{0.005}{\left[\left(\frac{1}{110\ mm}\right)^2 + \left(\frac{2}{66\ mm}\right)^2\right]^{\frac{1}{2}}} = \pm 0.158\ mm$$ ← δ:

Check: $u_H = \pm \frac{\delta x}{H} = \pm \frac{0.158\ mm}{110\ mm} = \pm 1.44 \times 10^{-3}$

$u_D = \pm \frac{\delta x}{D} = \pm \frac{0.158\ mm}{66\ mm} = \pm 2.39 \times 10^{-3}$

$u_\forall = \pm\left[(u_H)^2 + (2u_D)^2\right]^{\frac{1}{2}} = \pm\left[(0.00144)^2 + (0.00478)^2\right]^{\frac{1}{2}} = \pm 0.00499$ ✓✓

If δx represents half the least count, a minimum resolution of about $2\delta x \approx 0.32$ mm is needed.

Given: American golf ball, $m = 1.62 \pm 0.01$ oz, $D = 1.68$ in.

Find: Precision to which D must be measured to estimate density within uncertainty of ± 1 percent.

Solution: Apply uncertainty concepts

Definition: Density, $\rho \equiv \frac{m}{\forall}$ $\qquad \forall = \frac{4}{3}\pi R^3 = \frac{\pi D^3}{6}$

Computing equation: $u_R = \pm\left[\left(\frac{x_1}{R}\frac{\partial R}{\partial x_1} u_{x_1}\right)^2 + \cdots\right]^{1/2}$

From the definition, $\rho = \frac{m}{\pi D^3/6} = \frac{6m}{\pi D^3} = \rho(m, D)$

Thus $\frac{m}{\rho}\frac{\partial \rho}{\partial m} = 1$ and $\frac{D}{\rho}\frac{\partial \rho}{\partial D} = 3$, so

$$u_\rho = \pm\left[(1\, u_m)^2 + (3\, u_D)^2\right]^{1/2}$$

$$u_\rho^2 = u_m^2 + 9\, u_D^2$$

Solving, $u_D = \pm \frac{1}{3}\left[u_\rho^2 - u_m^2\right]^{\frac{1}{2}}$

From the data given, $u_\rho = \pm 0.0100$

$$u_m = \frac{\pm 0.01 \text{ oz}}{1.62 \text{ oz}} = \pm 0.00617$$

$$u_D = \pm \frac{1}{3}\left[(0.0100)^2 - (0.00617)^2\right]^{\frac{1}{2}} = \pm 0.00262 \text{ or } \pm 0.262\%$$

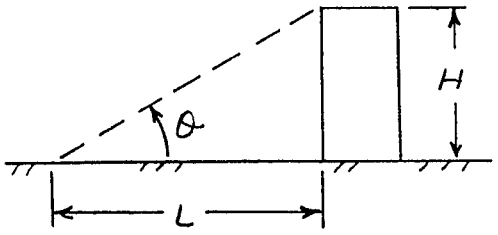
Since $u_D = \pm \frac{\delta D}{D}$, then

$$\delta D = \pm D\, u_D = \pm 1.68 \text{ in.} \times 0.00262 = \pm 0.00441 \text{ in.}$$

The ball diameter must be measured to a precision of ± 0.00441 in. (± 0.112 mm) or better to estimate density within ± 1 percent. A micrometer or caliper could be used.

Given: Height of building measured by line of sight:

$L = 100 \pm 0.5$ ft, $\theta = 30 \pm 0.2$ deg



Find: (a) Estimate of H and its uncertainty.
(b) Angle and L to minimize uncertainty in H.
(c) Evaluation and plot of optimum θ as a function of H.

Solution: Apply uncertainty concepts from Appendix F.

Computing equation: $H = H(L,\theta)$; $u_H = \pm\left[\left(\frac{L}{H}\frac{\partial H}{\partial L}u_L\right)^2 + \left(\frac{\theta}{H}\frac{\partial H}{\partial \theta}u_\theta\right)^2\right]^{\frac{1}{2}}$

From trigonometry, $\tan\theta = \frac{H}{L}$, so $H = L\tan\theta$

$$\frac{\partial H}{\partial L} = \tan\theta \,, \quad \frac{\partial H}{\partial \theta} = L(1+\tan^2\theta)$$

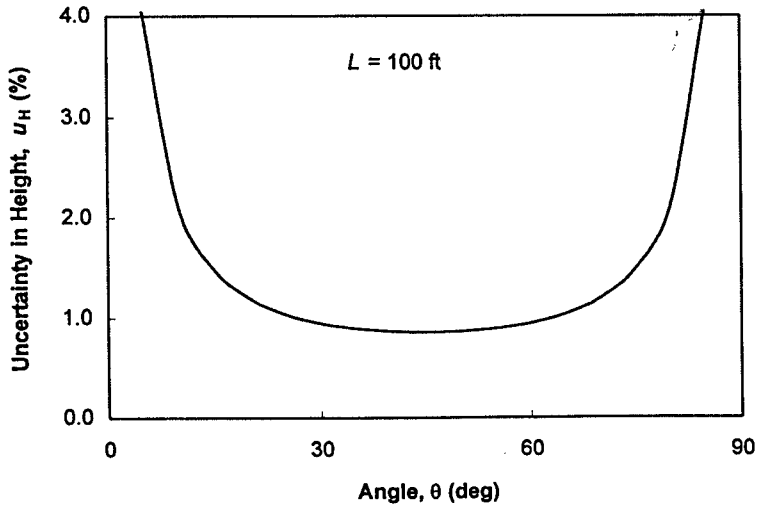
$$u_H = \pm\left[\left(\frac{L}{L\tan\theta}\tan\theta\, u_L\right)^2 + \left(\frac{\theta}{L\tan\theta}L(1+\tan^2\theta)u_\theta\right)^2\right]^{\frac{1}{2}}$$

$$u_H = \pm\left[u_L^2 + \left(\frac{\theta}{\tan\theta}(1+\tan^2\theta)\,u_\theta\right)^2\right]^{\frac{1}{2}} \qquad (1)$$

Evaluating with $\theta = \pi/6$ radians, $u_H = \pm 0.949$ percent ← u_H

Therefore $H = 57.7 \pm 0.548$ ft ← H

Equation 1 is difficult to optimize analytically. For any L it may be evaluated and the minimum u_H determined by varying θ. Plotting the results:



Plot ←

From the plot, the best measurement angle for $H = 100$ ft is $\theta = 45°$, but with a broad minimum, so $30° < \theta < 60°$ are nearly optimum. θ_{op} ←

Plots for other values of L are similar. Thus $L \approx H$ gives minimum uncertainty in building height.

Given: Piston-cylinder device to have $\forall = 1\ mm^3$

Molded plastic parts with dimensional uncertainties, $\delta = \pm 0.002$ in.

Find: (a) Estimate of uncertainty in dispensed volume that results from the dimensional uncertainties.
(b) Determine the ratio of stroke length to bore diameter that minimizes $u_\forall$; plot of the results.
(c) Is this result influenced by the magnitude of δ?

Solution: Apply uncertainty concepts from Appendix F:

Computing equation: $\forall = \frac{\pi D^2 L}{4}$; $u_\forall = \pm\left[\left(\frac{L}{\forall}\frac{\partial \forall}{\partial L}u_L\right)^2 + \left(\frac{D}{\forall}\frac{\partial \forall}{\partial D}u_D\right)^2\right]^{\frac{1}{2}}$

From $\forall$, $\frac{L}{\forall}\frac{\partial \forall}{\partial L} = 1$, and $\frac{D}{\forall}\frac{\partial \forall}{\partial D} = 2$, so $u_\forall = \pm\left[u_L^2 + (2u_D)^2\right]^{\frac{1}{2}}$

The dimensional uncertainty is $\delta = \pm 0.002\ in. \times 25.4\frac{mm}{in.} = \pm 0.0508\ mm$

Assume $D = 1\ mm$. Then $L = \frac{4\forall}{\pi D^2} = \frac{4}{\pi} \times 1\ mm^3 \times \frac{1}{(1)^2 mm^2} = 1.27\ mm$

$$\left.\begin{aligned} u_D &= \pm\frac{\delta}{D} = \pm\frac{0.0508}{1} = \pm 5.08\ \text{percent} \\ u_L &= \pm\frac{\delta}{L} = \pm\frac{0.0508}{1.27} = \pm 4.00\ \text{percent} \end{aligned}\right\} u_\forall = \pm\left[(4.00)^2 + (2(5.08))^2\right]^{\frac{1}{2}}$$

$$u_\forall = \pm 10.9\ \text{percent}$$ ← $u_\forall$

To minimize $u_\forall$, substitute in terms of D:

$$u_\forall = \pm\left[(u_L)^2 + (2u_D)^2\right] = \pm\left[\left(\frac{\delta}{L}\right)^2 + \left(2\frac{\delta}{D}\right)^2\right]^{\frac{1}{2}} = \pm\left[\left(\frac{\pi D^2 \delta}{4\forall}\right)^2 + \left(2\frac{\delta}{D}\right)^2\right]^{\frac{1}{2}}$$

This will be minimum when D is such that $\partial[\,]/\partial D = 0$, or

$$\frac{\partial[\,]}{\partial D} = \left(\frac{\pi\delta}{4\forall}\right)^2 4D^3 + (2\delta)^2\left(-2\frac{1}{D^3}\right) = 0\ ;\quad D^6 = 2\left(\frac{4\forall}{\pi}\right)^2\ ;\quad D = 2^{1/6}\left(\frac{4\forall}{\pi}\right)^{1/3}$$

Thus

$$D_{opt} = 2^{1/6}\left(\frac{4}{\pi} \times 1\ mm^3\right)^{1/3} = 1.22\ mm$$

The corresponding L is

$$L_{opt} = \frac{4\forall}{\pi D^2} = \frac{4}{\pi} \times 1\ mm^3 \times \frac{1}{(1.22)^2 mm^2} = 0.855\ mm$$

The optimum stroke-to-bore ratio is

$$(L/D)_{opt} = \frac{0.855\ mm}{1.22\ mm} = 0.701$$ (see table and plot on next page) ← L/D.

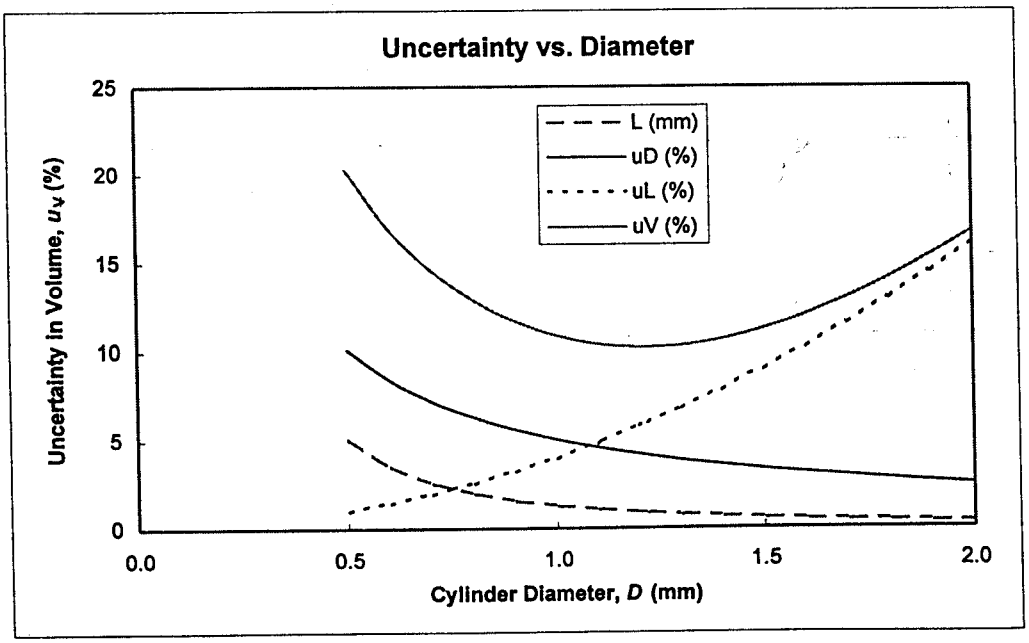
Note that δ drops out of the optimization equation. This optimum L/D is independent of the magnitude of δ. However, the magnitude of the optimum $u_\forall$ increases as δ increases.

Uncertainty in volume of cylinder:

δ = 0.002 in. 0.0508 mm

V = 1 mm^3

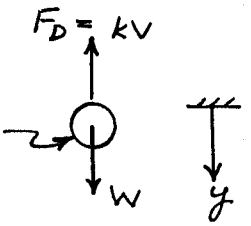
D (mm)	L (mm)	L/D (---)	u_D (%)	u_L (%)	u_V (%)
0.5	5.09	10.2	10.2	1.00	20.3
0.6	3.54	5.89	8.47	1.44	17.0
0.7	2.60	3.71	7.26	1.96	14.6
0.8	1.99	2.49	6.35	2.55	13.0
0.9	1.57	1.75	5.64	3.23	11.7
1.0	1.27	1.27	5.08	3.99	10.9
1.1	1.05	0.957	4.62	4.83	10.4
1.2	0.884	0.737	4.23	5.75	10.2
1.22	0.855	0.701	4.16	5.94	10.2
1.3	0.753	0.580	3.91	6.74	10.3
1.4	0.650	0.464	3.63	7.82	10.7
1.5	0.566	0.377	3.39	8.98	11.2
1.6	0.497	0.311	3.18	10.2	12.0
1.7	0.441	0.259	2.99	11.5	13.0
1.8	0.393	0.218	2.82	12.9	14.1
1.9	0.353	0.186	2.67	14.4	15.4
2.0	0.318	0.159	2.54	16.0	16.7
2.1	0.289	0.137	2.42	17.6	18.2
2.2	0.263	0.120	2.31	19.3	19.9
2.3	0.241	0.105	2.21	21.1	21.6
2.4	0.221	0.092	2.12	23.0	23.4
2.5	0.204	0.081	2.03	24.9	25.3



Given: Small particle accelerating from rest in a fluid.
Net weight is W, resisting force $F_D = kV$, where V is speed.

Find: Time required to reach 95 percent of terminal speed, V_t.

Solution: Consider the particle to be a system. Apply Newton's second law.



Basic equation: $\Sigma F_y = ma_y$

Assumptions: (1) W is net weight
(2) Resisting force acts opposite to V

Then

$$\Sigma F_y = W - kV = ma_y = m\frac{dV}{dt} = \frac{W}{g}\frac{dV}{dt}$$

or $$\frac{dV}{dt} = g\left(1 - \frac{k}{W}V\right)$$

Separating variables,

$$\frac{dV}{1 - \frac{k}{W}V} = g\,dt$$

Integrating, noting that velocity is zero initially,

$$\int_0^V \frac{dV}{1-\frac{k}{W}V} = -\frac{W}{k}\ln\left(1-\frac{k}{W}V\right)\Big]_0^V = \int_0^t g\,dt = gt$$

or

$$1 - \frac{k}{W}V = e^{-\frac{kgt}{W}} \quad ; \quad V = \frac{W}{k}\left[1 - e^{-\frac{kgt}{W}}\right]$$

But $V \to V_t$ as $t \to \infty$, so $V_t = \frac{W}{k}$. Therefore

$$\frac{V}{V_t} = 1 - e^{-\frac{kgt}{W}}$$

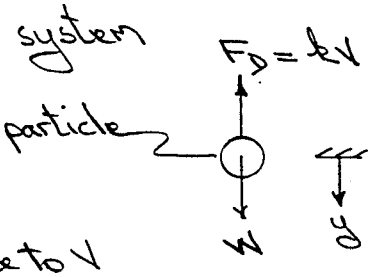
When $\frac{V}{V_t} = 0.95$, then $e^{-\frac{kgt}{W}} = 0.05$ and $\frac{kgt}{W} = 3$. Thus

$$t = 3W/gk$$

Given: Small particle accelerating from rest in a fluid. Net weight is W, resisting force is $F_D = kV$, where V is speed.

Find: Distance required to reach 95 percent of terminal speed, V_t.

Solution: Consider the particle to be a system. Apply Newton's second law.



Basic equation: $\Sigma F_y = ma_y$

Assumptions: (1) W is net weight
(2) Resisting force acts opposite to V

Then, $\Sigma F_y = W - kV = ma_y = \frac{W}{g}\frac{dV}{dt} = \frac{W}{g}V\frac{dV}{dy}$

or $1 - \frac{k}{W}V = \frac{V}{g}\frac{dV}{dy}$

At terminal speed, $a_y = 0$ and $V = V_t = \frac{W}{k}$. Then

$1 - \frac{V}{V_t} = \frac{1}{g}V\frac{dV}{dy}$

Separating variables

$$\frac{V\,dV}{1 - \frac{1}{V_t}V} = g\,dy$$

Integrating, noting that velocity is zero initially

$$y = \int_0^{0.95V_t} \frac{V\,dV}{1 - \frac{1}{V_t}V} = \left[-VV_t - V_t^2 \ln\left(1 - \frac{V}{V_t}\right)\right]_0^{0.95V_t}$$

$$y = -0.95V_t^2 - V_t^2 \ln(1 - 0.95) - V_t^2 \ln(1)$$

$$y = -V_t^2\left[0.95 + \ln 0.05\right] = 2.05V_t^2$$

$$\therefore y = \frac{2.05}{g}V_t^2 = 2.05\frac{W^2}{gk^2}$$

Given: Sky diver with $m = 75\ kg$ and $F_D = kv^2$; $k = 0.228\ \frac{N \cdot s^2}{m^2}$

Find: (a) Maximum speed in free fall
(b) Speed reached in fall of 100 m

Plot: (a) Speed $v = v(t)$ and (b) $v = v(y)$

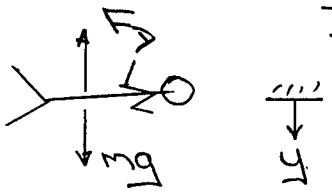
Solution:

Treat the sky diver as a system; apply Newton's 2nd law

Basic equation: $\Sigma F_y = ma_y$

Assumptions: $F_D = kv^2$ acts opposite to v

Initial conditions $v = 0$ at $t = 0$ and $y = 0$



Then $\Sigma F_y = mg - kv^2 = ma$ ..... (1)

At terminal speed, $a_y = 0$ and $v = v_t$,

so $mg - kv_t^2 = 0$. Thus

$$v_t = \sqrt{\frac{mg}{k}} = \left[75\ kg \times 9.81\ \frac{m}{s^2} \times \frac{m^2}{0.228\ N \cdot s^2} \times \frac{N \cdot s^2}{kg \cdot m}\right]^{1/2} = 56.8\ m/s \quad \underline{v_t}$$

(b) To solve for v at $y = 100$ m, we need an expression for $v(y)$.

Note that $a_y = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy} v = v \frac{dv}{dy}$

Then substituting into Eq. 1,

$$mg - kv^2 = mv \frac{dv}{dy} \quad \text{or} \quad 1 - \frac{kv^2}{mg} = \frac{v}{g}\frac{dv}{dy}$$

Separating variables and integrating

$$\int_0^v \frac{v\,dv}{1 - kv^2/mg} = \int_0^y g\,dy$$

$$-\frac{m}{2k} \ln\left(1 - \frac{kv^2}{mg}\right)\Big|_0^v = gy \quad \text{or} \quad \ln\left(1 - \frac{kv^2}{mg}\right) = -\frac{2k}{m} y$$

Thus, $1 - \frac{kv^2}{mg} = e^{-2ky/m}$ and $v^2 = \frac{mg}{k}\left(1 - e^{-2ky/m}\right)$

$$v = v_t \left[1 - e^{-2ky/m}\right]^{1/2} \quad \text{----- (2)}$$

At $y = 100$ m, $v = 56.8\ \frac{m}{s} \left[1 - e^{-2 \times 0.228\ \frac{N \cdot s^2}{m^2} \times 100\ m \times \frac{1}{75\ kg} \times \frac{kg \cdot m}{N \cdot s^2}}\right]^{1/2}$

$v = 38.3\ m/s$ $\quad \underline{v_{y=100m}}$

From Eq. 2, we can plot $v(y) = v_t\left[1 - e^{-2ky/m}\right]^{1/2}$

or $v/v_t = \left[1 - e^{-2ky/m}\right]^{1/2}$ . . . . . (2a)

To obtain an expression for $v = v(t)$ we write

$$\Sigma F_y = mg - kv^2 = ma = m\frac{dv}{dt}$$

Separating variables and integrating

$$\int_0^t dt = \int_0^v \frac{m\,dv}{mg - kv^2} = \frac{m}{k}\int_0^v \frac{dv}{\frac{mg}{k} - v^2}$$

$$t = \frac{1}{2}\sqrt{\frac{m}{kg}}\ln\left|\frac{\sqrt{\frac{mg}{k}} + v}{\sqrt{\frac{mg}{k}} - v}\right| = \frac{1}{2}\sqrt{\frac{m}{kg}}\ln\left|\frac{v_t + v}{v_t - v}\right|$$

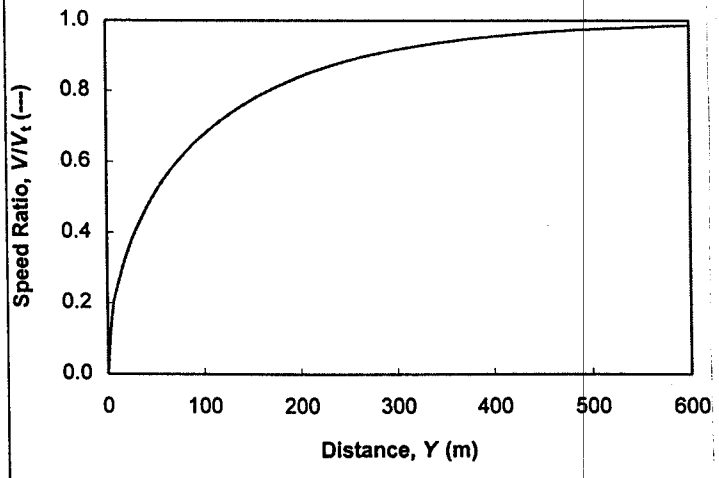
Then,

$$\frac{v_t + v}{v_t - v} = e^{2\sqrt{\frac{kg}{m}}\,t}$$, and

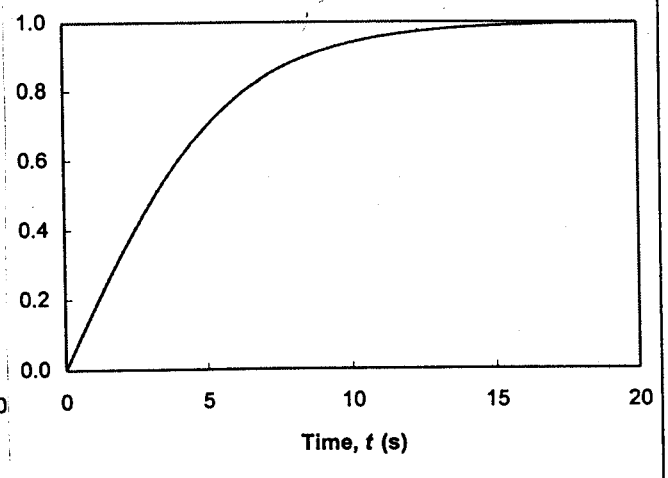
$$\frac{v}{v_t} = \frac{\left(e^{2\sqrt{\frac{kg}{m}}\,t} - 1\right)}{\left(e^{2\sqrt{\frac{kg}{m}}\,t} + 1\right)} = \tanh\left(v_t\sqrt{\frac{k}{m}}\,t\right) \quad \text{- - - - - (3)}$$

Eqs. 2a and 3 are plotted below

Eq. 2a: Speed Ratio vs. Distance



Eq. 3: Speed Ratio vs. Time



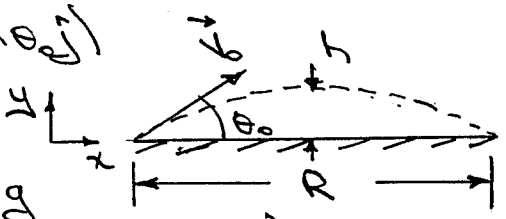
Given: Long bow at range, $R = 100$ m. Maximum height of arrow is $h = 10$ m. Neglect air resistance.

Find: Estimate of (a) speed, and (b) angle, of arrow leaving the bow.

Plot: (a) release speed, and (b) angle, as a function of h

Solution:

Let $\vec{V}_0 = u_0\hat{i} + v_0\hat{j} = V_0(\cos\theta_0\hat{i} + \sin\theta_0\hat{j})$



$\Sigma F_y = m\frac{dv}{dt} = -mg$, so

$v = v_0 - gt$, and $t_f = 2t_{v=0} = 2v_0/g$

Also, $mv\frac{dv}{dy} = -mg$, $v\,dv = -g\,dy$, $0 - \frac{v_0^2}{2} = -gh$

Thus $h = v_0^2/2g$ (1)

$\Sigma F_x = m\frac{du}{dt} = 0$, so $u = u_0 = \text{const}$, and $R = u_0 t_f = \frac{2u_0 v_0}{g}$ (2)

From (1) $v_0^2 = 2gh$ ..... (3)

(2) $u_0 = \frac{gR}{2v_0} = \frac{gR}{2\sqrt{2gh}}$ $\therefore u_0^2 = \frac{gR^2}{8h}$

Then

$$V_0^2 = u_0^2 + v_0^2 = \frac{gR^2}{8h} + 2gh \quad \text{and} \quad V_0 = \left[2gh + \frac{gR^2}{8h}\right]^{1/2} \quad \text{...(4)}$$

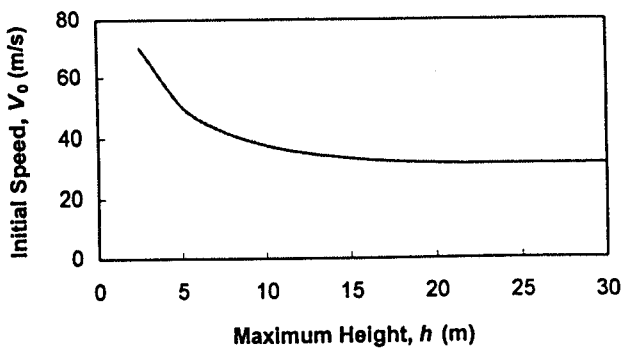
$$V_0 = \left[2\times 9.81\frac{m}{s^2}\times 10\,m + \frac{9.81}{8}\frac{m}{s^2}\times(100)^2\,m^2\times\frac{1}{10\,m}\right]^{1/2} = 37.7\,m/s \leftarrow V_0$$

From Eq. 3 $v_0 = \sqrt{2gh} = V_0\sin\theta$, $\theta = \sin^{-1}\frac{\sqrt{2gh}}{V_0}$ ....(5)

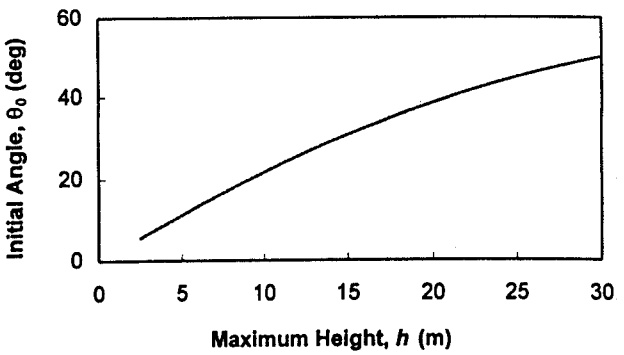
$$\theta = \sin^{-1}\left[\left(2\times 9.81\frac{m}{s}\times 10\,m\right)^{1/2}\frac{s}{37.7\,m}\right] = 21.8^\circ \leftarrow \theta_0$$

Plots of $V_0 = V_0(h)$ {Eq 4} and $\theta_0 = \theta_0(h)$ {Eq. 5} are presented below.

Eq. 4: Initial Speed vs. Max. Height



Eq. 5: Initial Angle vs. Max. Height



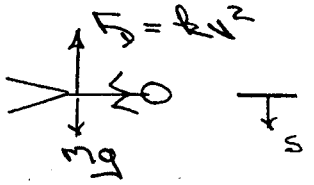
Given: Sky diver with $m = 80\ kg$ falling straight down.
Drag force is $F_D = kV^2$; $k = 0.27\ N \cdot s^2/m^2$; V is speed

Find: (a) Terminal speed, V_t, of sky diver
(b) Vertical distance to reach $V = 0.95\,V_t$
(c) Vertical distance to reach same speed if $F_D = 0$

Plot: (a) Speed $V = V(t)$, and (b) $V = V(s)$

Solution:

Consider sky diver as a system; apply Newton's 2nd law of motion.



Basic equation: $\Sigma F_s = m\frac{dV}{dt} = mV\frac{dV}{ds} = mg - F_D$ (1)

Assumptions: Initial condition $V = 0$ at $t = 0$, $s = 0$.
$F_D = kV^2$ acts opposite to V

Then $\Sigma F_y = mg - kV^2 = ma$

At terminal speed $a = 0$ and $V = V_t$. Thus.

$mg - kV_t^2 = 0$ and $V_t = \sqrt{\frac{mg}{k}}$ (2)

$$V_t = \left[80\ kg \times 9.81\ \frac{m}{s^2} \times 0.27\ \frac{N \cdot s^2}{m^2} \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 53.9\ m/s \leftarrow V_t$$

(b) To solve for s at $V = 0.95\,V_t$, we need an expression relating V and y. From Eq. (1)

$$mg - kV^2 = mV\frac{dV}{ds} \quad \text{or} \quad 1 - \frac{kV^2}{mg} = \frac{V}{g}\frac{dV}{ds}$$

Separating variables and integrating.

$$\int_0^V \frac{V\,dV}{1 - kV^2/mg} = \int_0^s g\,ds$$

$$-\frac{m}{2k}\ln\left(1 - \frac{kV^2}{mg}\right)\Big|_0^V = gs \quad \text{or} \quad s = -\frac{m}{2k}\ln\left(1 - \frac{kV^2}{mg}\right)$$

or $$s = -\frac{m}{2k}\ln\left(1 - \frac{V^2}{V_t^2}\right) \quad (3)$$

For $V = 0.95\,V_t$, then

$$s = -\frac{1}{2} \times 80\ kg \times 0.27\ \frac{N \cdot s^2}{m^2} \ln\left[1 - (0.95)^2\right] \times \frac{m^2}{N \cdot s^2} \times \frac{N \cdot m}{kg} = 345\ m \leftarrow s$$

(c) For free fall without air resistance $\Sigma F_s = mg = mV\frac{dV}{ds}$

Integrating we obtain $\frac{V^2}{2} = gs$ or

$$s = \frac{V^2}{2g}$$

13 782 500 SHEETS FILLER 5 SQUARE
42-381 50 SHEETS EYE EASE 5 SQUARE
42-382 100 SHEETS EYE EASE 5 SQUARE
42-389 200 SHEETS EYE EASE 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National Brand

Evaluating, we obtain

$$s = \frac{1}{2} \times \left[0.95(53.9)\ \frac{s}{m}\right]^2 \times 9.81\ \frac{s}{m} = 134\ m \qquad s_{F_s=0}$$

Thus air resistance significantly increases the vertical fall distance needed to reach a given speed.

From Eq. 3 we can write $v = v(s)$ as

$$\frac{v}{v_t} = \left[1 - e^{-2ks/m}\right]^{1/2} \qquad (4)$$

To obtain an expression for $v = v(t)$ we write

$$\Sigma F_s = mg - kv^2 = m\frac{dv}{dt}$$

Separating variables and integrating we have

$$\int_0^t dt = \int_0^v \frac{m\,dv}{mg - kv^2} = \frac{m}{k}\int_0^v \frac{dv}{\frac{mg}{k} - v^2}$$

Res

$$t = \frac{1}{2}\sqrt{\frac{m}{kg}}\ln\frac{\sqrt{\frac{mg}{k}} + v}{\sqrt{\frac{mg}{k}} - v} = \frac{1}{2}\sqrt{\frac{m}{kg}}\ln\frac{|v_t + v|}{|v_t - v|}$$

and

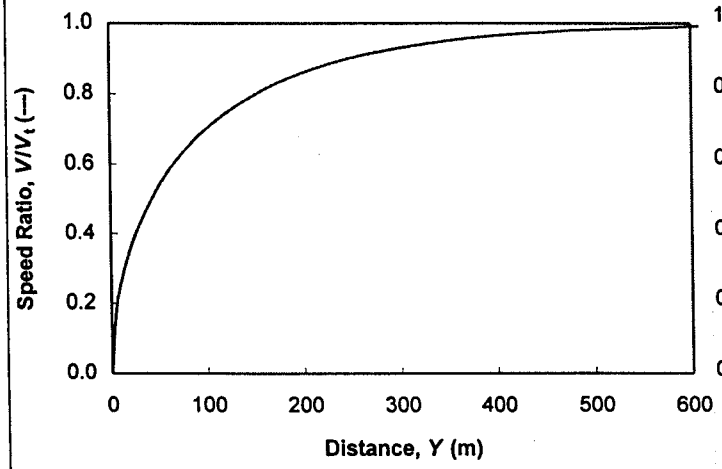
$$\frac{v_t + v}{v_t - v} = e^{2\sqrt{\frac{kg}{m}}t}$$

Finally

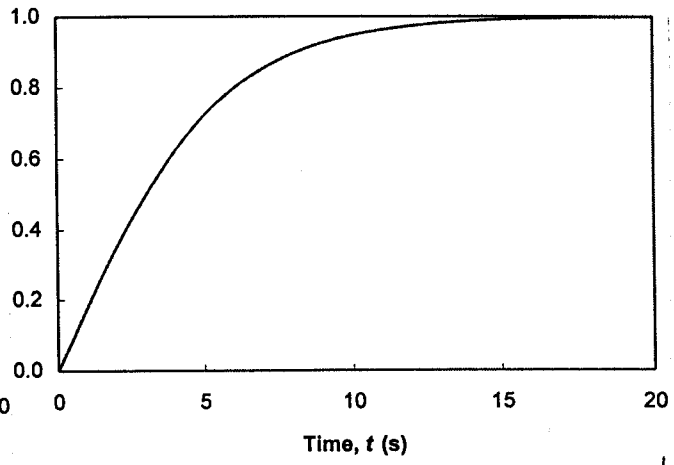
$$\frac{v}{v_t} = \frac{\left(e^{2\sqrt{\frac{kg}{m}}t} - 1\right)}{\left(e^{2\sqrt{\frac{kg}{m}}t} + 1\right)} = \tanh\left(t\sqrt{\frac{kg}{m}}\right) \qquad (5)$$

Eqs 4 and 5 are plotted below

Eq. 4: Speed Ratio vs. Distance



Eq. 5: Speed Ratio vs. Time



Given: Basic dimensions M, L, t and T.

Find: Dimensional representation of quantities below, and typical units in SI and English systems.

Solution:

(a) $\text{Power} = \frac{\text{Energy}}{\text{Time}} = \frac{\text{Force} \times \text{Distance}}{\text{Time}}$

From Newton's second law, Force = Mass x Acceleration

$\therefore \text{Power} = \frac{\text{Mass} \times \text{Acc.} \times \text{Dist.}}{\text{Time}} = \left[\frac{M \frac{L}{t^2} L}{t}\right] = \left[\frac{ML^2}{t^3}\right]$; $\frac{kg \cdot m^2}{s^3}$ or $\frac{slug \cdot ft^2}{s^3}$

(b) $\text{Pressure} = \frac{\text{Force}}{\text{Area}} = \left[\frac{F}{L^2}\right] = \left[\frac{ML}{t^2}\frac{1}{L^2}\right] = \left[\frac{M}{Lt^2}\right]$; $\frac{kg}{m \cdot s^2}$ or $\frac{slug}{ft \cdot s^2}$

(c) $\text{Modulus of elasticity} = \frac{\text{Force}}{\text{Area}} = \left[\frac{M}{Lt^2}\right]$; $\frac{kg}{m \cdot s^2}$ or $\frac{slug}{ft \cdot s^2}$

(d) $\text{Angular velocity} = \frac{\text{Radians}}{\text{Time}} = \left[\frac{1}{t}\right]$; $\frac{1}{s}$ or $\frac{1}{s}$

(e) $\text{Energy} = \text{Force} \times \text{Distance} = \left[\frac{ML}{t^2} L\right] = \left[\frac{ML^2}{t^2}\right]$; $\frac{kg \cdot m^2}{s^2}$ or $\frac{slug \cdot ft^2}{s^2}$

(f) $\text{Momentum} = \text{Mass} \times \text{Velocity} = \left[M \frac{L}{t}\right] = \left[\frac{ML}{t}\right]$; $\frac{kg \cdot m}{s}$ or $\frac{slug \cdot ft}{s}$

(g) $\text{Shear stress} = \frac{\text{Force}}{\text{Area}} = \left[\frac{M}{Lt^2}\right]$; $\frac{kg}{m \cdot s^2}$ or $\frac{slug}{ft \cdot s^2}$

(h) $\text{Specific heat} = \frac{\text{Energy}}{\text{Mass} \times \text{Temperature}} = \left[\frac{\frac{ML^2}{t^2}}{M\,T}\right] = \left[\frac{L^2}{t^2 T}\right]$; $\frac{m^2}{s^2 \cdot K}$ or $\frac{ft^2}{s^2 \cdot {}^\circ R}$

(i) $\text{Thermal expansion coefficient} = \frac{\text{Change in length / Length}}{\text{Temperature}} = \left[\frac{1}{T}\right]$;

$\frac{1}{K}$ or $\frac{1}{{}^\circ R}$

Given: Basic dimensions F, L, t and T.

Find: Dimensional representation of quantities below, and typical units in SI and English systems.

Solution:

(a) Power = $\frac{\text{Energy}}{\text{Time}} = \frac{\text{Force} \times \text{Distance}}{\text{Time}} = \left[\frac{FL}{t}\right]$; $\frac{N \cdot m}{s}$ or $\frac{lbf \cdot ft}{s}$

(b) Pressure = $\frac{\text{Force}}{\text{Area}} = \left[\frac{F}{L^2}\right]$; $\frac{N}{m^2}$ or $\frac{lbf}{ft^2}$

(c) Modulus of elasticity = $\frac{\text{Force}}{\text{Area}} = \left[\frac{F}{L^2}\right]$; $\frac{N}{m^2}$ or $\frac{lbf}{ft^2}$

(d) Angular velocity = $\frac{\text{Radians}}{\text{Time}} = \left[\frac{1}{t}\right]$; $\frac{1}{s}$ or $\frac{1}{s}$

(e) Energy = Force x Distance = $[FL]$; N·m or lbf·ft

(f) Moment of a force = Force x Distance = $[FL]$; N·m or lbf·ft

(g) Momentum = Mass x Velocity = $\left[\frac{ML}{t}\right]$

From Newton's second law, $F = ma$, so $m = \frac{F}{a}$

$\therefore$ Momentum = $\frac{\text{Force} \times \text{Velocity}}{\text{Acceleration}} = \left[\frac{F\,\frac{L}{t}}{\frac{L}{t^2}}\right] = [Ft]$; N·s or lbf·s

(h) Shear stress = $\frac{\text{Force}}{\text{Area}} = \left[\frac{F}{L^2}\right]$; $\frac{N}{m^2}$ or $\frac{lbf}{ft^2}$

(i) Strain = $\frac{\text{Change in length}}{\text{Length}} = \left[\frac{L}{L}\right] = [-]$; (-) or (-)

Given: Unit conversions

Find: Convert (a) 1 Pa to lbf/in.^{2} (pressure)
(b) 1 gallon to liters (volume)
(c) 1 N·s/m^2 to lbf·s/ft^2 (viscosity)

Solution: Use basic definitions

(a) $1\ Pa = 1\ \frac{N}{m^2} = 1\ \frac{N}{m^2} \times \frac{1\ lbf}{4.448\ N} \times (0.0254)\frac{m^2}{in.^2} = 1.45 \times 10^{-4}\ lbf/in.^2$ (a)

(b) $1\ gal = 1\ gal \times 231\ \frac{in.^3}{gal} \times (0.0254)^3 \frac{m^3}{in.^3} \times 1000\ \frac{L}{m^3} = 3.79\ L$ (b)

(c) $1\ N \cdot s/m^2 = 1\ \frac{N \cdot s}{m^2} \times \frac{1\ lbf}{4.448\ N} \times (0.3048)^2 \frac{m^2}{ft^2} = 2.09 \times 10^{-2}\ \frac{lbf \cdot s}{ft^2}$ (c)

13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-399 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Unit conversions

Find: Convert (a) 1 N/m² to lbf/in.² (pressure)
(b) 1 hp to N·m/s (power)
(c) 1 Btu/lbm to N·m/kg (specific energy)

Solution: Use basic definitions

(a) $1\ Pa \equiv 1\ \frac{N}{m^2} \times \frac{lbf}{4.448\ N} \times (0.0254)^2 \frac{m^2}{in.^2} = 1.45 \times 10^{-4}\ \frac{lbf}{in.^2}$ (a)

(b) $1\ hp \equiv 550\ \frac{ft \cdot lbf}{s} \times 0.3048\ \frac{m}{ft} \times 4448\ \frac{N}{lbf} = 746\ \frac{N \cdot m}{s}$ (b)

(c) $1\ \frac{Btu}{lbm} \equiv 778.2\ \frac{ft \cdot lbf}{lbm} \times 0.3048\ \frac{m}{ft} \times 4.448\ \frac{N}{lbf} \times \frac{lbm}{0.4536\ kg}$

$1\ \frac{Btu}{lbm} = 2330\ \frac{N \cdot m}{kg} \left(= 2330\ \frac{J}{kg}\right)$ (c)

13-782 500 SHEETS, FILLER, 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Density of mercury is $\rho = 26.3$ slug/ft³.

Acceleration of gravity on moon is $g_m = 5.47$ ft/s².

Find: (a) Specific gravity of mercury.
(b) Specific volume of mercury, in m³/kg.
(c) Specific weight on Earth.
(d) Specific weight on moon.

Solution: Apply definitions: $\gamma \equiv \rho g$, $v \equiv 1/\rho$, $SG \equiv \rho/\rho_{H_2O}$

Thus $SG = 26.3 \frac{\text{slug}}{\text{ft}^3} \times \frac{\text{ft}^3}{1.94 \text{ slug}} = 13.6$ ← SG

$$v = \frac{\text{ft}^3}{26.3 \text{ slug}} \times (0.3048)^3 \frac{\text{m}^3}{\text{ft}^3} \times \frac{\text{slug}}{32.2 \text{ lbm}} \times \frac{\text{lbm}}{0.4536 \text{ kg}} = 7.37 \times 10^{-5} \text{ m}^3/\text{kg}$$ ← v

On Earth,

$$\gamma_E = 26.3 \frac{\text{slug}}{\text{ft}^3} \times 32.2 \frac{\text{ft}}{\text{s}^2} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} = 847 \text{ lbf/ft}^3$$ ← γ

On the moon,

$$\gamma_m = 26.3 \frac{\text{slug}}{\text{ft}^3} \times 5.47 \frac{\text{ft}}{\text{s}^2} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} = 144 \text{ lbf/ft}^3$$ ← γ

{ Note that the mass-based quantities (SG and v) are independent of gravity. }

Given: Quantities as listed below.

Find: Conversion factors.

Solution:

(a) Volume flow rate: $m^3/s \rightarrow ft^3/s$

$$1 \frac{m^3}{s} = 1 \frac{m^3}{s} \times \frac{ft^3}{(0.3048)^3 m^3} = 35.3 \frac{ft^3}{s} \quad \text{(a)}$$

(b) Volume flow rate of water: $ft^3/s \rightarrow gal/min$

$$1 \frac{ft^3}{s} = 1 \frac{ft^3}{s} \times \frac{(12)^3 \frac{in.^3}{ft^3}}{231 \frac{in.^3}{gal}} \times \frac{60 \frac{s}{min}}{} = 449 \frac{gal}{min} \quad \text{(b)}$$

(c) Volume flow rate of water: $gal/hr \rightarrow L/s$

$$1 \frac{gal}{hr} = 1 \frac{gal}{hr} \times \frac{ft^3}{7.48 gal} \times \frac{(0.3048)^3 \frac{m^3}{ft^3}}{} \times \frac{hr}{3600 s} \times \frac{10^{-3} \frac{L}{m^3}}{}$$

$$1 \frac{gal}{hr} = 1.05 \times 10^{-3} L/s \quad \text{(c)}$$

(d) Volume flow rate of air: $SCFM \rightarrow m^3/hr$

$$1 \, SCFM = 1 \, SCFM \times \frac{1 ft^3}{SCFM \cdot min} \times 60 \frac{min}{hr} \times \frac{(0.3048)^3 \frac{m^3}{ft^3}}{}$$

$$1 \, SCFM = 1.70 \, m^3/hr \quad \text{(d)}$$

13 782 500 SHEETS, HILLER, 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: In European usage, 1 kgf is the force exerted on 1 kg mass in standard gravity.

Find: Convert 32 psi to units of kgf/cm².

Solution: Apply Newton's second law.

Basic equation: $F = ma$

The force exerted on 1 kg in standard gravity is

$$F = 1\ kg \times 9.81\ \frac{m}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 9.81\ N = 1\ kgf$$

Setting up a conversion from psi to kgf/cm²,

$$1\ \frac{lbf}{in.^2} = 1\ \frac{lbf}{in.^2} \times 4.448\ \frac{N}{lbf} \times \frac{in.^2}{(2.54)^2\ cm^2} \times \frac{kgf}{9.81\ N} = 0.0703\ \frac{kgf}{cm^2}$$

or

$$1 \equiv \frac{0.0703\ kgf/cm^2}{psi}$$

Thus

$$32\ psi = 32\ psi \times \frac{0.0703\ kgf/cm^2}{psi}$$

$$32\ psi = 2.25\ kgf/cm^2$$

Given: System of units based on dimensions of force, length, and time

unit of force = basketball = 21 oz
unit of length = three-pointer = 21 ft
unit of time = shot clock = 45 s

Find: (a) conversion factors to SI units
(b) relation between units of mass in the two systems.

Solution:

(a) The conversion factors are

$$1\ \text{basketball} = 1\ \text{basketball} \times \frac{21\ \text{oz}}{\text{basketball}} \times \frac{1\ \text{lbf}}{16\ \text{oz}} \times \frac{4.448\ \text{N}}{\text{lbf}} = 5.84\ \text{N} \quad [F]$$

$$1\ \text{three-pointer} = 1\ \text{three-pointer} \times \frac{21\ \text{ft}}{\text{three-pointer}} \times \frac{0.3048\ \text{m}}{\text{ft}} = 6.40\ \text{m} \quad [L]$$

$$1\ \text{shot clock} = 45\ \text{s} \quad [t]$$

(b) From Newtons second law $F = ma$

$[F] = \left[M \frac{L}{t^2}\right]$ $\therefore M = \frac{Ft^2}{L}$ and the unit of mass is

$$M = \frac{\text{basketball} \cdot (\text{shot clock})^2}{\text{three-pointer}}$$

To convert mass to SI

$$M_{bb} = \frac{\text{basketball} \cdot (\text{shot clock})^2}{\text{three-pointer}} \times \frac{5.84\ \text{N}}{\text{basketball}} \times \frac{(45)^2 \cdot \text{s}^2}{(\text{shot clock})^2} \times \frac{\text{three-pointer}}{6.40\ \text{m}} \times \frac{\text{kg} \cdot \text{m}}{\text{N} \cdot \text{s}^2}$$

$$M_{bb} = 1850\ \text{kg}$$

$$\frac{\text{mass (bball units)}}{\text{mass (SI units)}} = 1850$$ mass ratio

Given: System of units based on basic dimensions of force, velocity, and length

unit of force = baseball = 5.1 oz
unit of length = homerun = 385 ft
unit of velocity = fastball = 90 mph

Find: (a) conversion factors to SI units
(b) unit of mass in the "baseball" system of units
(c) relation between SI and "baseball" units of mass

Solution:

(a) The conversion factors are

$$1\ \text{baseball} = 1\ \text{baseball} \times \frac{5.1\ \text{oz}}{\text{baseball}} \times \frac{1\ \text{lbf}}{16\ \text{oz}} \times \frac{4.448\ \text{N}}{1\ \text{lbf}} = 1.42\ \text{N} \quad \text{F}$$

$$1\ \text{homerun} = 1\ \text{homerun} \times \frac{385\ \text{ft}}{\text{homerun}} \times \frac{0.3048\ \text{m}}{\text{ft}} = 117\ \text{m} \quad \text{L}$$

$$1\ \text{fastball} = 1\ \text{fastball} \times \frac{90\ \text{mi}}{\text{hr}\cdot\text{fastball}} \times \frac{\text{hr}}{3600\ \text{s}} \times \frac{5280\ \text{ft}}{\text{mi}} \times \frac{0.3048\ \text{m}}{\text{ft}} = 40.2\ \frac{\text{m}}{\text{s}} \quad \text{V}$$

(b) With basic dimensions of F, L, V, then time, t, and mass, M, are secondary dimensions

Since $V = \frac{L}{t}$, then $t = \frac{L}{V}$

From Newtons second law $F = ma = \frac{ML}{t^2}$

so $M = \frac{Ft^2}{L} = F\left(\frac{L}{V}\right)^2 \frac{1}{L} = \frac{FL}{V^2}$ and the unit of mass is

$$M = \frac{\text{baseball}\cdot\text{homerun}}{(\text{fastball})^2} \quad \text{M}$$

(c) To convert mass unit to SI

$$M = \frac{\text{baseball}\cdot\text{homerun}}{(\text{fastball})^2} \times \frac{1.42\ \text{N}}{\text{baseball}} \times \frac{117\ \text{m}}{\text{homerun}} \times \frac{(\text{fastball})^2}{(40.2)^2\ \text{m}^2/\text{s}^2} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}$$

$$M = 0.103\ \text{kg} \quad \text{M}$$

Given: Empty container weighing 2.90 lbf when empty, has a mass of 1.95 slug when filled with water at 90°F.

Find: (a) Weight of water in the container
(b) Container volume in ft³

Solution:

Basic equation: $F = ma$

Weight is the force of gravity on a body, $W = mg$

Then $W_t = W_{H_2O} + W_c$

$$W_{H_2O} = W_t - W_c = mg - W_c$$

$$W_{H_2O} = 1.95\ \text{slug} \times 32.2\ \frac{\text{ft}}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} - 2.90\ \text{lbf} = 59.9\ \text{lbf} \quad W_{H_2O}$$

The volume is given by

$$\forall = \frac{m_{H_2O}}{\rho} = \frac{m_{H_2O}\,g}{\rho g} = \frac{W_{H_2O}}{\rho g}$$

From Table A.7, $\rho = 1.93\ \text{slug/ft}^3$ at $T = 90°\text{F}$

$$\therefore \forall = 59.9\ \text{lbf} \times \frac{\text{ft}^3}{1.93\ \text{slug}} \times 32.2\ \frac{\text{ft}}{\text{s}^2} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2} = 0.964\ \text{ft}^3 \quad \forall$$

Given : Velocity fields listed below. (a and b are constants)

Determine: (a) Dimensions of each velocity field
(b) Whether the flow is steady or unsteady

Solution:

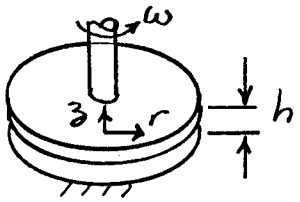
The dimensions of each velocity field will be determined relative to an xyz coordinate system.

FIELD	DIMENSIONS	STEADY or UNSTEADY?
(1) $\vec{V} = [ae^{-bx}]\hat{i}$	one-dimensional $\vec{V} = \vec{V}(x)$	steady $\vec{V} \neq \vec{V}(t)$
(2) $\vec{V} = ax^2\hat{i} + bx\hat{j}$	one-dimensional $\vec{V} = \vec{V}(x)$	steady $\vec{V} \neq \vec{V}(t)$
(3) $\vec{V} = [ax^2e^{-bt}]\hat{i}$	one-dimensional $\vec{V} = \vec{V}(x)$	unsteady $\vec{V} = \vec{V}(t)$
(4) $\vec{V} = ax\hat{i} - by\hat{j}$	two-dimensional $\vec{V} = \vec{V}(x,y)$	steady $\vec{V} \neq \vec{V}(t)$
(5) $\vec{V} = (ax+t)\hat{i} - by\hat{j}$	two-dimensional $\vec{V} = \vec{V}(x,y)$	unsteady $\vec{V} = \vec{V}(t)$
(6) $\vec{V} = ax^2\hat{i} + bxy\hat{j}$	two-dimensional $\vec{V} = \vec{V}(x,y)$	steady $\vec{V} \neq \vec{V}(t)$
(7) $\vec{V} = a(x^2+y^2)^{1/2}(1/z^3)\hat{k}$	three-dimensional $\vec{V} = \vec{V}(x,y,z)$	steady $\vec{V} \neq \vec{V}(t)$
(8) $\vec{V} = axy\hat{i} - byzt\hat{j}$	three-dimensional $\vec{V} = \vec{V}(x,y,z)$	unsteady $\vec{V} = \vec{V}(t)$

Given: Viscous liquid sheared between parallel disks.

Upper disk rotates, lower fixed.

Velocity field is $\vec{V} = \hat{e}_\theta r\omega z/h$.



Find: (a) Dimensions of velocity field.
(b) satisfy physical boundary conditions.

Solution: To find dimensions, compare to $\vec{V} = \vec{V}(x,y,z)$ form.

The given field is $\vec{V} = \vec{V}(r,z)$. Two space coordinates are included, so field is 2-D. ← 2-D

Flow must satisfy the no-slip condition:

(1) At lower disk, $\vec{V} = 0$, since stationary.

$z = 0$, so $\vec{V} = \hat{e}_\theta r\omega(0)/h = 0$ ∴ satisfied ← $z =$

(2) At upper disk, $\vec{V} = \hat{e}_\theta r\omega$, since it rotates as a solid body.

$z = h$, so $\vec{V} = \hat{e}_\theta r\omega(h)/h = \hat{e}_\theta r\omega$ ∴ satisfied ← $z =$

Given: Velocity field, $\vec{V} = ax\hat{i} - by\hat{j}$ $(a = b = 1\ sec^{-1})$

Find: Equation for the flow streamlines, and

Plot: Representative streamlines for $x \geq 0$ and $y \geq 0$

Solution:

The slope of the streamlines in the x-y plane is given by

$$\frac{dy}{dx} = \frac{v}{u}$$

For $\vec{V} = ax\hat{i} - by\hat{j}$, then $u = ax$, $v = -by$. Hence

$$\frac{dy}{dx} = \frac{v}{u} = -\frac{by}{ax}$$

To solve the differential equation, separate variables and integrate

$$\int \frac{dy}{y} = -\int \frac{b}{a}\frac{dx}{x}$$

$$\ln y = -\frac{b}{a}\ln x + \text{constant}$$

$$\ln y = \ln x^{-b/a} + \ln c \qquad \text{where constant} = \ln c$$

Then $y = c\,x^{-b/a}$ $y(x)$

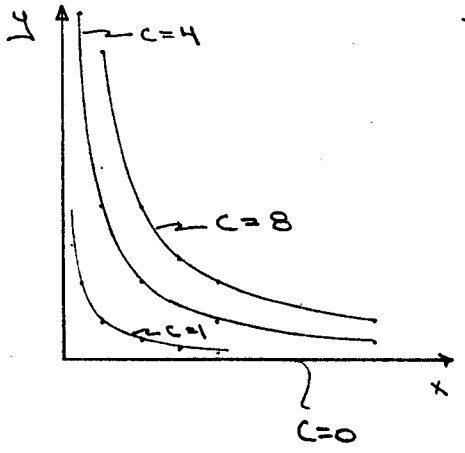
or alternately $x = \left(\frac{y}{c}\right)^{-a/b} = \left(\frac{c}{y}\right)^{a/b}$

For a given velocity field, the constants a and b are fixed. Different streamlines are obtained by assigning different values to the constant of integration, c.

Since $a = b = 1\ sec^{-1}$, then $a/b = 1$, and the streamlines are given by the equation

$$y = cx^{-1} = \frac{c}{x} \quad \text{or} \quad x = \frac{c}{y}$$

For $c = 0$ $y = 0$ for all x and $x = 0$ for all y.



The equation $y = \frac{c}{x}$ is the equation of a hyperbola.

Curves are shown for different values of c

Given: Flow in a corner may be modeled by the velocity field $\vec{V} = -ax\hat{i} + by\hat{j}$; $a = b = 2\ s^{-1}$

Find: an equation for the flow streamlines.

Plot: the streamlines in the first quadrant including one through point (1, 1)

Solution

The slope of the streamlines in the x-y plane is given by

$$\frac{dy}{dx} = \frac{v}{u}$$

For $\vec{V} = -ax\hat{i} + by\hat{j}$, then $u = -ax$ and $v = by$. Hence

$$\frac{dy}{dx} = \frac{v}{u} = -\frac{b}{a}\frac{y}{x}$$

or

$$\frac{1}{y}\frac{dy}{dx} + \frac{b}{a}\frac{1}{x}\frac{dx}{dx} = 0$$

. This can be integrated to obtain

$$\ln y + \frac{b}{a}\ln x = \text{constant} = \ln c$$

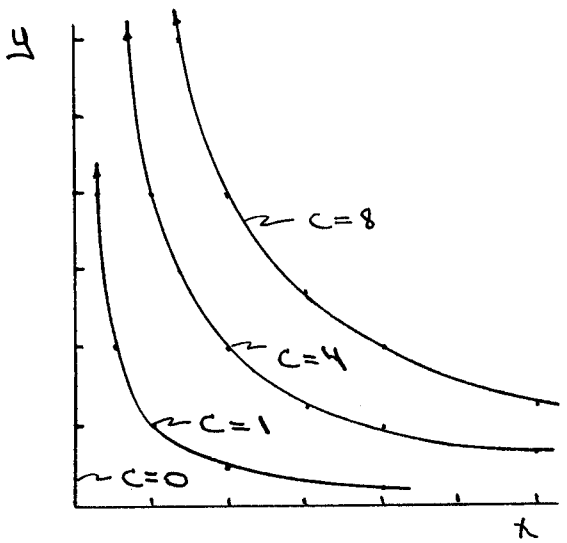
This can be simplified to give $y x^{b/a} = c$

For $a = b$, then $xy = c$

Different streamlines are obtained by assigning different values to the constant, C.

For the streamline passing through the point (1,1) the constant has the value, $c = 1$.

For $c = 0$, $y = 0$ for all x and $x = 0$ for all y.



The equation $y = c/x$ is the equation of a hyperbola

Since $\vec{V} = -ax\hat{i} + by\hat{j}$, the flow into the corner is from right to left.

Given: Flow field, $\vec{V} = (Ax+B)\hat{i} + (-Ay)\hat{j}$
where $A = 10\ ft/s/ft$, $B = 3\ ft/s$, coordinates-m.

Plot: a few streamlines including the one that passes through the point $(x,y) = (1,2)$

<u>Solution:</u> Since $\vec{V} \neq \vec{V}(z)$ and $w = 0$, flow is the same in any xy-plane.

Streamlines are parallel to the velocity $\vec{V}$ so $\left(\frac{dy}{dx}\right)_{streamline} = \frac{v}{u}$

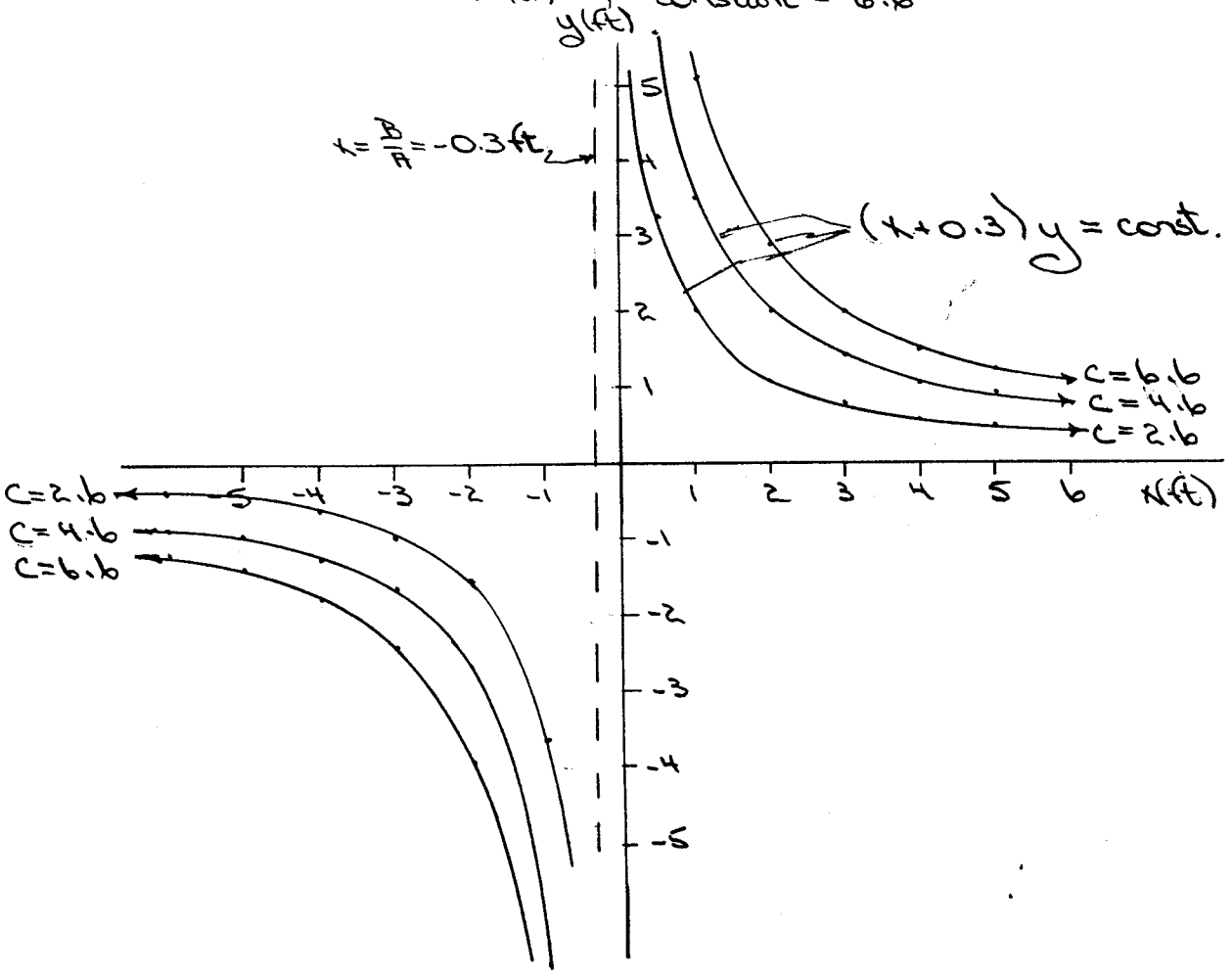
$$\therefore \frac{dy}{dx} = \frac{v}{u} = \frac{-Ay}{(Ax+B)} = -\frac{y}{x + B/A}$$

Then $$y\frac{dy}{} + \frac{dx}{x + B/A} = 0$$

Integrating

$$\ln y + \ln\left(x + \frac{B}{A}\right) = \ln C \quad \text{or} \quad \left(x + B/A\right)y = (x+0.3)y = \text{const}$$

Line through $(x,y) = (1,2)$, constant = 2.6
$= (2,2)$, constant = 4.6
$= (3,2)$, constant = 6.6



Given: Velocity field, $\vec{V} = ax^2\hat{i} + bxy\hat{j}$

$a = 2\ m^{-1}s^{-1}$, $b = -4\ m^{-1}s^{-1}$; coordinates in meters

Find: (a) Dimensions of flow field. Why?

(b) Velocity components at $(x,y,z) = (2, 1/2, 0)$

(c) Equation for streamline through this point.

Plot: several streamlines in first quadrant including the one through $(2, 1/2, 0)$

Solution:

(a) Dimensions are determined by the number of space coordinates needed to specify the velocity field.

$\vec{V} = \vec{V}(x,y)$, so field is 2-D (and steady) ← (a)

(b) At $(x,y,z) = (2, 1/2, 0)$

$u = ax^2 = \frac{2}{m\cdot s} \times (2)^2\ m^2 = 8\ m/s$

$v = bxy = -\frac{4}{m\cdot s} \times 2m \times \frac{1}{2}m = 4\ m/s$; $w = 0$ ← (b)

(c) Since $\vec{V} \neq \vec{V}(z)$ and $w = 0$, flow is the same in any xy-plane. Streamlines are parallel to $\vec{V}$, so

$$\left(\frac{dy}{dx}\right)_{streamline} = \frac{v}{u} = \frac{bxy}{ax^2} = \frac{b}{a}\frac{y}{x} \quad \text{and} \quad \frac{dy}{y} = \frac{b}{a}\frac{dx}{x}$$

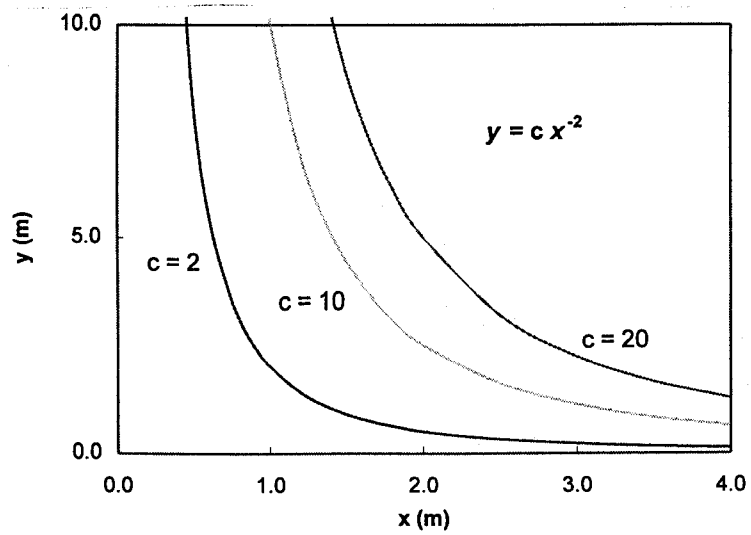
Integrating, $\ln y = \frac{b}{a}\ln x + c_1 = \frac{b}{a}\ln x + \ln c$

or $y = cx^{b/a} = cx^{-2}$ (1)

At point $(2, 1/2, 0)$, $c = x^2 y = (2)^2\ m^2 \times \frac{1}{2}m = 2m^3$

Thus $y = cx^{-2}$ where $c = 2m^3$ ← (c)

(d) Streamlines, $y = cx^{-2}$, are plotted below



Given: Velocity field, $\vec{V} = Axy\,\hat{i} + By^2\,\hat{j}$

$A = 1\ m^{-1}s^{-1}$, $B = -0.5\ m^{-1}s^{-1}$; coordinates in meters

Find: Equation for flow streamlines

Plot: several streamlines in upper half plane

Solution:

Streamlines are tangent to the velocity vector, so

$$\left.\frac{dy}{dx}\right|_{streamline} = \frac{v}{u} = \frac{By^2}{Axy} = \frac{By}{Ax} = \frac{-0.5}{1}\,\frac{m\cdot s}{m\cdot s}\,\frac{1}{10}\,\frac{y}{x} = -\frac{y}{2x}$$

Separating variables,

$$\frac{dx}{x} = -2\frac{dy}{y} \quad \text{or} \quad \frac{dx}{x} + 2\frac{dy}{y} = 0$$

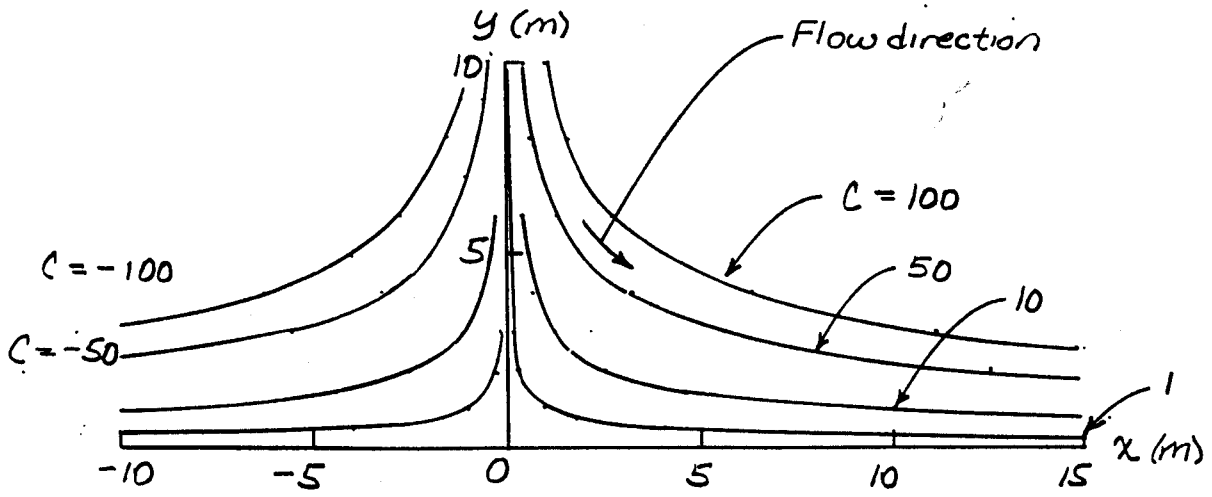
Integrating,

$$\ln x + 2\ln y = c_1 = \ln c \quad \text{or} \quad \ln x + \ln y^2 = \ln c$$

Taking antilogarithms,

$$xy^2 = c$$ ← (Equation for streamlines)

Plotting:



Given: Steady, incompressible flow in xy plane with

$$\vec{V} = \frac{A}{x}\hat{i} + \frac{Ay}{x^2}\hat{j} \quad \text{where } A = 2\ m^2/s$$

and coordinates are in meters.

Find: (a) Equation for streamline through $(x,y) = (1,3)$.
(b) Time required for a fluid particle to move from $x = 1\ m$ to $x = 3\ m$.

Solution: The velocity field is $\vec{V} = u\hat{i} + v\hat{j}$, so $u = \frac{A}{x}$, $v = \frac{Ay}{x^2}$

Computing equations: $\left(\frac{dy}{dx}\right)_{streamline} = \frac{v}{u}$; $u_p = \frac{dx}{dt}$

Substituting, $\frac{dy}{dx} = \frac{Ay}{x^2}\frac{x}{A} = \frac{y}{x}$ so $\frac{dx}{x} = \frac{dy}{y}$

Integrating, $\ln x = \ln y + C^* = \ln y + \ln c$ or $x = cy$

For point $(x,y) = (1,3)$, $c = \frac{x}{y} = \frac{1}{3}$

Thus $x = \frac{y}{3}$ is equation (a)

For a particle, $u_p = \frac{dx}{dt} = \frac{A}{x}$ or $x\,dx = A\,dt$

Integrating, $\int_{x_0}^{x} x\,dx = \frac{x^2 - x_0^2}{2} = At$ so $t = \frac{x^2 - x_0^2}{2A}$

$$t = \frac{1}{2} \times \left[(3)^2 m^2 - (1)^2 m^2\right] \times \frac{s}{2\,m^2} = 2\ s \quad \text{(b)}$$

Given: Velocity field $\vec{V} = ax\hat{i} - by\hat{j}$, where $a = b = 1\ s^{-1}$.

Find: (a) Show that particle motion is described by the parametric equations $x_p = c_1 e^{at}$ and $y_p = c_2 e^{-bt}$
(b) Obtain equation of pathline for particle located at (1,2) at $t=0$
(c) Compare pathline with streamline through same point

Solution

(a) A particle moving in the velocity field $\vec{V} = ax\hat{i} - by\hat{j}$ will have velocity components $u = ax$, $v = -by$

Thus $u_p = \frac{dx}{dt} = ax$ or $\frac{dx}{x} = a\,dt$ and $\int \frac{dx}{x} = \int a\,dt$ ..(1)

$v_p = \frac{dy}{dt} = -by$ or $\frac{dy}{y} = -b\,dt$ and $\int \frac{dy}{y} = -\int b\,dt$ ..(2)

Integrating Eqs. (1) and (2) we obtain

$\ln x = at + \ln c_1$ or $\frac{x}{c_1} = e^{at}$ and $x = c_1 e^{at}$

$\ln y = -bt + \ln c_2$ or $\frac{y}{c_2} = e^{-bt}$ and $y = c_2 e^{-bt}$ ← Q.E.D.

(b) To obtain the equation of the pathline we eliminate t from the parametric equations.

$x = c_1 e^{at}$ $\therefore \ln \frac{x}{c_1} = at$ or $t = \frac{1}{a} \ln \frac{x}{c_1}$

$y = c_2 e^{-bt}$ $\therefore \ln \frac{y}{c_2} = -bt$ or $t = -\frac{1}{b} \ln \frac{y}{c_2}$

Equating expressions for t, we obtain

$\frac{1}{a} \ln \frac{x}{c_1} = -\frac{1}{b} \ln \frac{y}{c_2}$ or $-\frac{b}{a} \ln \frac{x}{c_1} = \ln \frac{y}{c_2}$

Thus $\left(\frac{x}{c_1}\right)^{-b/a} = \frac{y}{c_2}$ or $y\left(\frac{x}{c_1}\right)^{b/a} = c_2$

At $t=0$ $x = 1 = c_1$, $y = 2 = c_2$. Since $a=b$, then the pathline of the particle is $xy = 2$. ← Pathline

(c) The streamline in the x-y plane has slope $\frac{dy}{dx} = \frac{v}{u} = -\frac{b}{a}\frac{y}{x}$

Thus $\frac{dy}{y} + \frac{b}{a}\frac{dx}{x} = 0$. This can be integrated to obtain

$\ln y + \frac{b}{a} \ln x = \text{constant} = \ln c$

Simplifying we obtain $yx^{b/a} = c$. With $b=a$, the equation of the streamline through point (1,2) is then

$xy = 2$ ← Streamline

Given: Velocity field, $\vec{V} = -ax\hat{i} + by\hat{j}$, where $a = b = 2\,s^{-1}$, and the coordinates are measured in meters.

Find: (a) Parametric equations for particle motion.
(b) Pathline equation for particle at $(x,y) = (2,1)$ at $t = 0$.
(c) Compare with streamline through same point.

Solution: Velocity field is $\vec{V} = u\hat{i} + v\hat{j}$, so $u = -ax$, $v = by$

Computing equations: $u_p = \frac{dx}{dt}$, $v_p = \frac{dy}{dt}$

$$\left.\frac{dy}{dx}\right)_{streamline} = \frac{v}{u}$$

$u_p = \frac{dx}{dt} = -ax$ so $\frac{dx}{x} = -a\,dt$ and $\ln\frac{x}{x_0} = -at$

Thus $x = x_0 e^{-at}$

$v_p = \frac{dy}{dt} = +by$ so $\frac{dy}{y} = +b\,dt$ and $\ln\frac{y}{y_0} = +bt$

Thus $y = y_0 e^{bt} = y_0 e^{at}$ Parametric (a)

Solving for e^{at}, $e^{at} = \frac{y}{y_0} = \frac{x_0}{x}$ so $xy = x_0 y_0$

For particle at $(x_0, y_0) = (2,1)$

$xy = 2\,m \times 1\,m = 2\,m^2$ Pathline (b)

The streamline is $\frac{dy}{dx} = \frac{v}{u} = \frac{by}{-ax} = \frac{a\,y}{-ax} = -\frac{y}{x}$

Separating variables and integrating,

$$\int\frac{dy}{y} + \int\frac{dx}{x} = 0 \quad \text{or} \quad \ln x + \ln y = C^* = \ln C \quad \text{or} \quad xy = C$$

For point $(x,y) = (2,1)$ the streamline is $C = 2\,m \times 1\,m = 2\,m^2$

$xy = 2\,m^2$ Streamline (c)

{ Note: as expected, pathline and streamline coincide for this steady flow. }

Given: Velocity field $\vec{V} = (ax\hat{i} - ay\hat{j})(2 + \cos\omega t)$

where $a = 3\,s^{-1}$ and $\omega = \pi\,s^{-1}$; x and y measured in m

Find: (a) Algebraic equation for streamline at $t = 0$

(b) Plot streamline through point $(x,y) = (2,4)$ at $t = 0$

(c) Will the streamline change with time? Explain.

(d) Show velocity vector at same point, time. Tangent? Explain

Solution: For a streamline, $\frac{dy}{v} = \frac{dx}{x}$. From the given field, at $t = 0$,

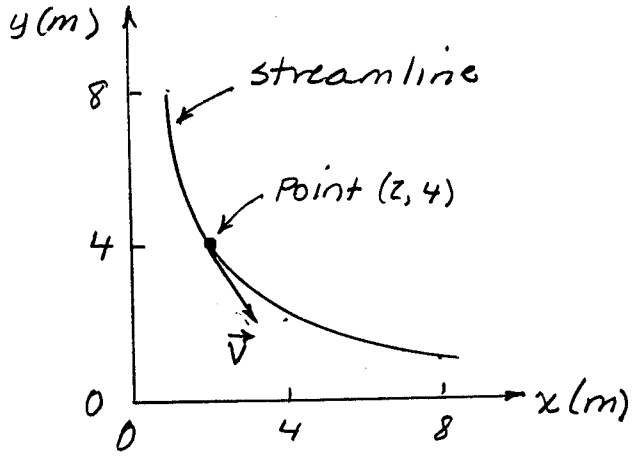
$$u = 2ax \text{ and } v = -2ay, \text{ so } \frac{dy}{v} = -\frac{dy}{2ay} = \frac{dx}{u} = \frac{dx}{2ax}$$

or

$$\frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating, $\ln x + \ln y = \ln c$ or $xy = c$ ← streamline ($t = 0$)

For point $(x,y) = (2,4)$, $xy = (2)(4) = c = 8$, or $xy = 8$ ← Thru $(x,y) = (2,4$



Streamline pattern will not change with time, since $\frac{dy}{dx} \neq f(t)$. ← Tim

At point (2,4) at $t = 0$, $u = 2ax = (2)(3\,s^{-1})(2\,m) = 12$ m/s

$$v = -2ay = -(2)(3\,s^{-1})(4\,m) = -24 \text{ m/s}$$

The velocity vector is tangent to the streamline. ← Tangent

Given: Velocity field $\vec{V} = A\hat{i} + Bt\hat{j}$, where $A = 2$ m/s, $B = 0.3$ m/s^2, and coordinates are in meters.

Find: (a) position functions for particle located at $(x_0, y_0) = 1, 1$ at time $t = 0$
(b) algebraic expression for pathline of particle of part (a)

Plot: the pathline and compare with streamline through the same point at $t = 0, 1, 2$ s.

Solution:

For a particle $u = \frac{dx}{dt}$ and $v = \frac{dy}{dt}$

Then, $u = A = dx/dt$, $\int_{x_0}^{x} dx = \int_0^t A\,dt$ and $x = x_0 + At$ (1a)

$v = Bt = dy/dt$, $\int_{y_0}^{y} dy = \int_0^t Bt\,dt$ and $y = y_0 + \frac{1}{2}Bt^2$ (1b)

Substituting values for A, B, x_0, and y_0, then

$x = 1 + 2t$ and $y = 1 + 0.15t^2$ ← x, y

(b) To determine the pathline for the particle, we eliminate t from the parametric equations of part (a). From Eq 1a, $t = (x - x_0)/A$. Substituting into Eq (1b), then

$$y - y_0 = \frac{B(x - x_0)^2}{2A^2} \quad (2)$$

Substituting numerical values,

$y = 1 + 0.0375(x - 1)^2$ ← pathline

(c) The streamline is found (at given t) from $(dy/dx)_s = v/u$

$$\left(\frac{dy}{dx}\right)_{\text{streamline}} = \frac{v}{u} = \frac{Bt}{A}$$

$\therefore y = \frac{Bt}{A}x + C$

Through point (1,1)

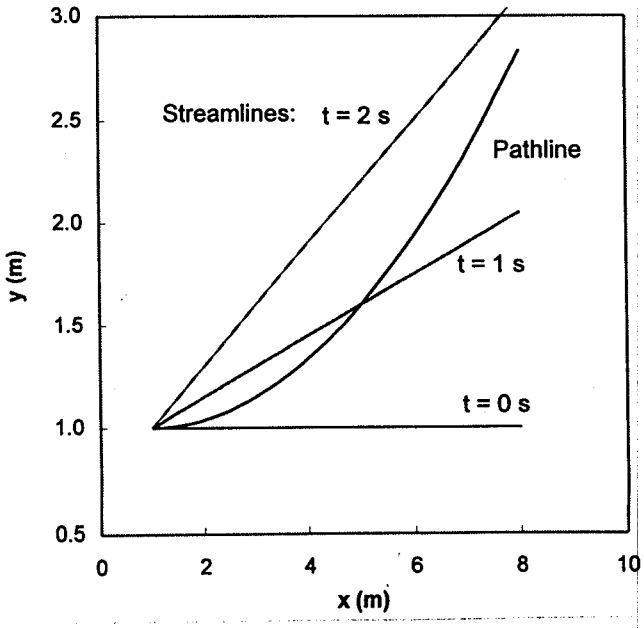
$C = 1 - \frac{0.3}{2}t = 1 - 0.15t$

$y = 1 + 0.15t(x - 1)$ ← Streamline through (1,1)

@ $t = 0$, $y = 1$

$t = 1$ s, $y = 1 + 0.15(x - 1)$

$t = 2$ s, $y = 1 + 0.30(x - 1)$



Plot shows that pathline and streamlines do not coincide for unsteady flow.

Given: Velocity field $\vec{V} = Bx(1+At)\hat{i} + Cy\hat{j}$, with $A = 0.5\ s^{-1}$, $B = C = 1\ s^{-1}$; coordinates measured in meters.

Plot: the pathline of the particle that passed through the point (1,1,0) at time t=0. Compare with the streamlines through the same point at the instants t = 0, 1, and 2s.

Solution:

For a particle, $u = \frac{dx}{dt}$ and $v = \frac{dy}{dt}$

Then

$$u = Bx(1+At) = \frac{dx}{dt}, \quad \int_{x_0}^{x} \frac{dx}{x} = \int_0^t B(1+At)\,dt$$

$$\ln \frac{x}{x_0} = B\left[t + \tfrac{1}{2}At^2\right]_0^t = B\left[t + \tfrac{1}{2}At^2\right] \quad \therefore x = x_0 e^{B\left(t+\frac{1}{2}At^2\right)}$$

$$v = Cy = \frac{dy}{dt}, \quad \int_0^t C\,dt = \int_{y_0}^{y} \frac{dy}{y} \quad \therefore y = y_0 e^{Ct}$$

The pathline may be plotted by varying t as shown below.

The streamline is found (at given t) from $\left(\frac{dy}{dx}\right)_{\text{streamline}} = \frac{v}{u}$.

Then

$$\frac{dy}{dx} = \frac{Cy}{Bx(1+At)} \quad \text{and} \quad (1+At)\frac{dy}{y} = \frac{C}{B}\frac{dx}{x}$$

and

$$(1+At)\ln y = \frac{C}{B}\ln x + \ln c_1, \quad c_1 x^{C/B} = y^{(1+At)}$$

Streamline through point (1,1,0) gives $c_1 = 1$. Then on substituting for A, B, and C we obtain

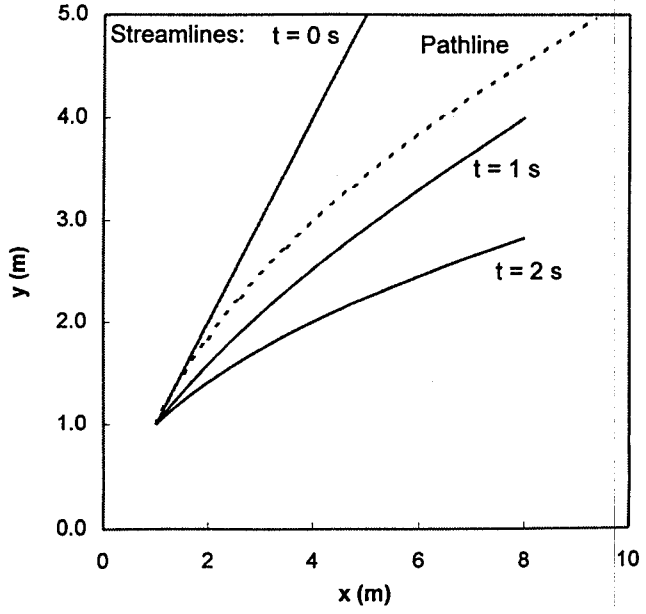
$$x = y^{(1+0.5t)}$$

At t=0, $x = y^{1.5}$

t=1s, $x = y^{2}$

t=2s, $x = y^{2.5}$

Streamline



Given: Velocity field $\vec{V} = ax(1+bt)\hat{i} + cy\hat{j}$, where $a = c = 1\,s^{-1}$, $b = 0.2\,s^{-1}$, and coordinates are measured in meters.

Plot: the pathline (over the interval, $0 \le t \le 3s$) of the particle that passed through the point $(x_0, y_0) = (1,1)$ at $t = 0$. Compare with the streamline plotted through the same point at $t = 0, 1$, and 2 s.

Solution:

For a particle, $u = dx/dt$ and $v = dy/dt$

Then $u = ax(1+bt) = \frac{dx}{dt}$, $\int_{x_0}^{x} \frac{dx}{x} = \int_0^t a(1+bt)\,dt$

$\ln \frac{x}{x_0} = a(t + \frac{1}{2}bt^2)$ and $x = x_0 e^{a(t+\frac{1}{2}bt^2)}$.

$v = cy = \frac{dy}{dt}$, $\int_0^t c\,dt = \int_{y_0}^{y} \frac{dy}{y}$ and $y = y_0 e^{ct}$ } (x,y) pathline

The pathline may be plotted by varying t as shown below

The streamline is found (at given t) from $(dy/dx)_s = \frac{v}{u}$

Then $\frac{dy}{dx} = \frac{cy}{ax(1+bt)}$ and $(1+bt)\frac{dy}{y} = \frac{c}{a}\frac{dx}{x}$

so $(1+bt)\ln y = \frac{c}{a}\ln x + \ln c_1$, $c_1 x^{c/a} = y^{(1+bt)}$

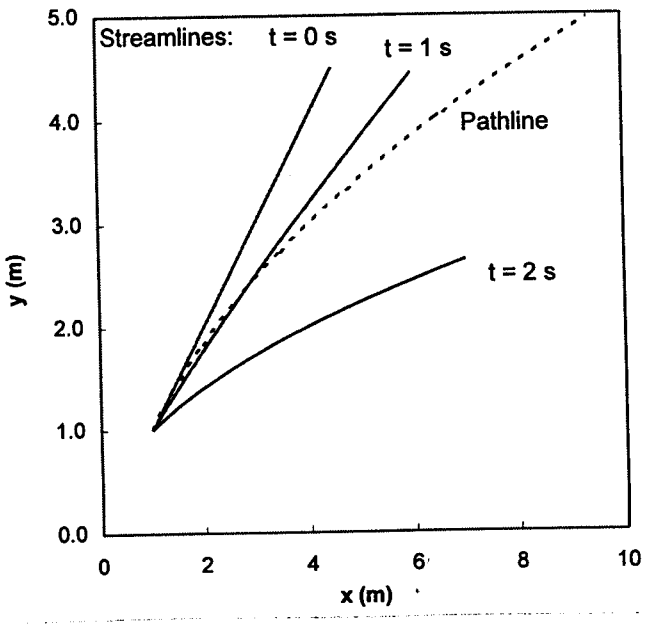
Streamline through point $(1,1)$ gives $c_1 = 1$. Then on substituting for a, b, and c we obtain

$x = y^{(1+0.2t)}$ Streamline

At $t=0$, $x = y$

$t=1s$, $x = y^{1.2}$

$t=2s$, $x = y^2$



Given: Velocity field $\vec{V} = ax\hat{i} + b\hat{j}$ where $a = 0.2\ s^{-2}$, $b = 3\ m/s$ and coordinates are measured in meters.

Plot: the pathline (during the interval $0 \le t \le 3s$) of the particle that passed through the point $(x_0, y_0) = (3, 1)$ at time $t = 0$.
Compare with the streamline plotted through the same point at $t = 1, 2$, and 3 s.

Solution:

For a particle, $u = dx/dt$ and $v = dy/dt$

Then, $u = ax = dx/dt$, $\int_{x_0}^{x} \frac{dx}{x} = \int_0^t a\,dt$

$\ln \frac{x}{x_0} = \frac{1}{2} at^2$ and $x = x_0 e^{\frac{1}{2}at^2}$ $\therefore x = 3e^{0.1t^2}$

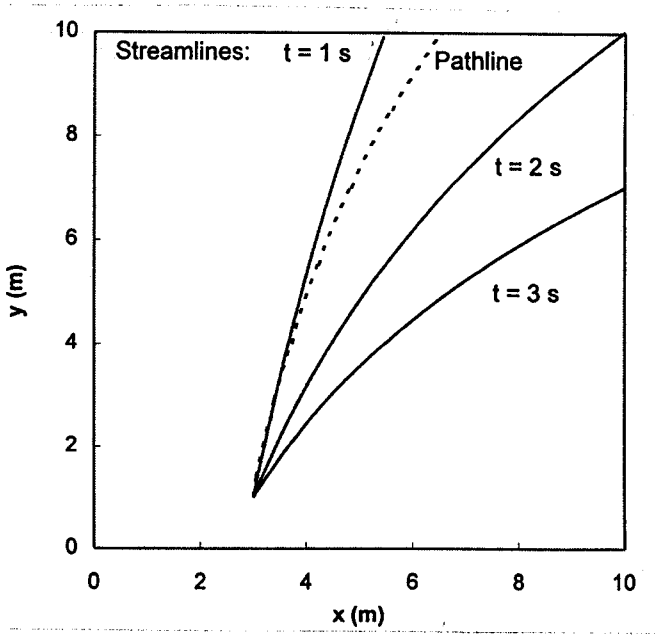
Also, $v = dy/dt = b$, $\int_{y_0}^{y} dy = \int_0^t b\,dt$, $y = y_0 + bt$ $\therefore y = 1 + 3t$ } (x, y) pathline

The pathline may be plotted by varying t as shown below.
The streamline is found (at given t) from $(dy/dx)_s = \frac{v}{u}$.

Then $\frac{dy}{dx} = \frac{b}{axt}$ and the streamline through (x_0, y_0) at time is $\int_{y_0}^{y} dy = \int_{t_0}^{t} \frac{b}{at} \frac{dx}{x}$ or $y = y_0 + \frac{b}{at} \ln \frac{x}{x_0}$

Substituting for a, b, x_0, and y_0, $y = 1 + \frac{15}{t} \ln \frac{x}{3}$ streamline

At $t = 1$, $y = 1 + 15 \ln x/3$
$t = 2$, $y = 1 + 7.5 \ln x/3$
$t = 3$, $y = 1 + 5 \ln x/3$



Given: Velocity field $\vec{V} = ax\hat{i} + by(1+ct)\hat{j}$, where $a = b = 2\,s^{-1}$, $c = 0.4\,s^{-1}$, and coordinates are measured in meters

Plot: the pathline (during the interval $0 \le t \le 1.5s$) of the particle that passed through the point $(x_0, y_0) = (1,1)$ at time $t = 0$.
Compare with the streamline plotted through the same point at $t = 0$, 1, and 1.5s.

Solution:

For a particle, $u = dx/dt$ and $v = dy/dt$

Then $u = dx/dt = ax$, $\int_{x_0}^{x} \frac{dx}{x} = \int_0^t a\,dt$, $\ln \frac{x}{x_0} = at$, $x = x_0 e^{at}$

Also $v = dy/dt = by(1+ct)$, $\int_{y_0}^{y} \frac{dy}{y} = \int_0^t b(1+ct)\,dt$

$\ln \frac{y}{y_0} = b\left(t + \frac{1}{2}ct^2\right)$ $\quad y = y_0 e^{b\left(t + \frac{1}{2}ct^2\right)}$

Substituting for a, b, c, x_0, and y_0

$x = e^{2t}$, $y = e^{(2t + 0.4t^2)}$ (x, y) pathline

The streamline is found (at given t) from $dy/dx\big|_s = v/u$.

Then $\frac{dy}{dx} = \frac{by(1+ct)}{ax}$, $\int_{y_0}^{y} \frac{dy}{y} = \int_{x_0}^{x} \frac{b(1+ct)}{a}\frac{dx}{x}$, $\ln \frac{y}{y_0} = \frac{b(1+ct)}{a}\ln\frac{x}{x_0}$

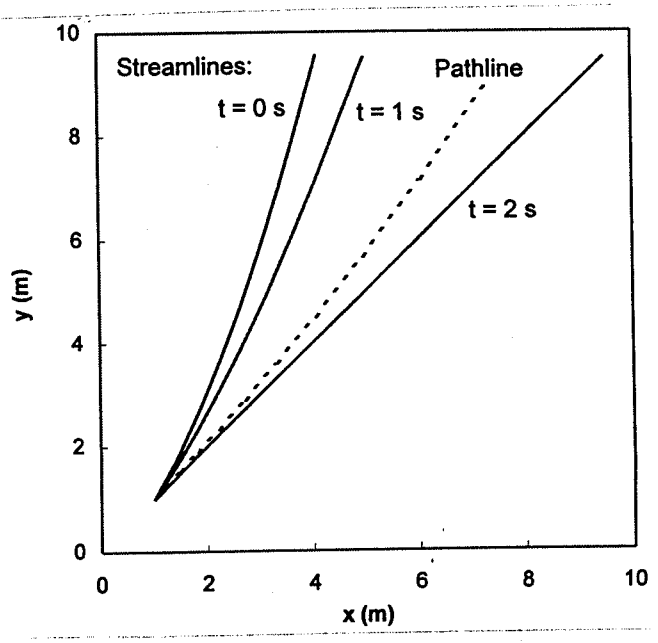
$y = y_0 \left(\frac{x}{x_0}\right)^{\frac{b(1+ct)}{a}}$. Substituting for a, b, c, x_0, and y_0.

$y = x^{(1+0.4t)}$ streamline

At $t = 0$, $y = x$

$t = 1s$, $y = x^{1.4}$

$t = 1.5s$, $y = x^{1.6}$



Given: Velocity field $\vec{V} = Bx(1+At)\hat{i} + Cy\hat{j}$, with $A = 0.5\,s^{-1}$, $B = C = 4\,s^{-1}$; coordinates measured in meters.

Plot: the streakline formed by particles that passed through point $(x_0, y_0, z_0) = (1,1,0)$ during interval from $t=0$ to $t = 3s$.
Compare with streamlines through point at $t = 0, 1$, and $2s$

Solution

Streakline at $t = 3s$ connects particles that passed through point $(1,1,0)$ at earlier times $t_0 = 0, 1$, and $2s$

For a particle, $u = \frac{dx}{dt}$ and $v = \frac{dy}{dt}$

Then $u = Bx(1+At) = \frac{dx}{dt}$, $\int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} B(1+At)\,dt$

$\therefore \ln \frac{x}{x_0} = B\left[t + \frac{1}{2}At^2\right]_{t_0}^{t} = B\left[(t-t_0) + \frac{1}{2}AB(t^2 - t_0^2)\right]$

$x = x_0 e^{B\left[(t-t_0) + \frac{1}{2}AB(t^2-t_0^2)\right]}$ (1a)

Also $v = Cy = \frac{dy}{dt}$, $\int_{t_0}^{t} C\,dt = \int_{y_0}^{y} \frac{dy}{y}$, $\therefore y = y_0 e^{C(t-t_0)}$ (1b)

The velocity vector is tangent to the streamline

$\left.\frac{dy}{dx}\right|_{streamline} = \frac{v}{u} = \frac{Cy}{Bx(1+At)}$ and $\frac{(1+At)}{B}\frac{dy}{y} = \frac{1}{x}\,dx$

Then $\frac{(1+At)}{B}\ln y = \ln x + \ln c_1$ and $c_1 x^{C/B} = y^{(1+At)}$

Streamline through point $(1,1,0)$ gives $c_1 = 1$. Then on substituting for A, B, and C we obtain

$x = y^{(1+0.5t)}$ ← Streamline

At $t=0$, $x = y$
$t = 1s$, $x = y^{1.5}$
$t = 2s$, $x = y^2$
} These streamlines through (1,1,0) are shown on the plot.

Points on the streakline have coordinates given by Eqs 1a & 1b

$x = x_0 e^{B\left[(t-t_0) + \frac{1}{2}AB(t^2-t_0^2)\right]}$ $\quad y = y_0 e^{C(t-t_0)}$

Substituting for A, B, and C

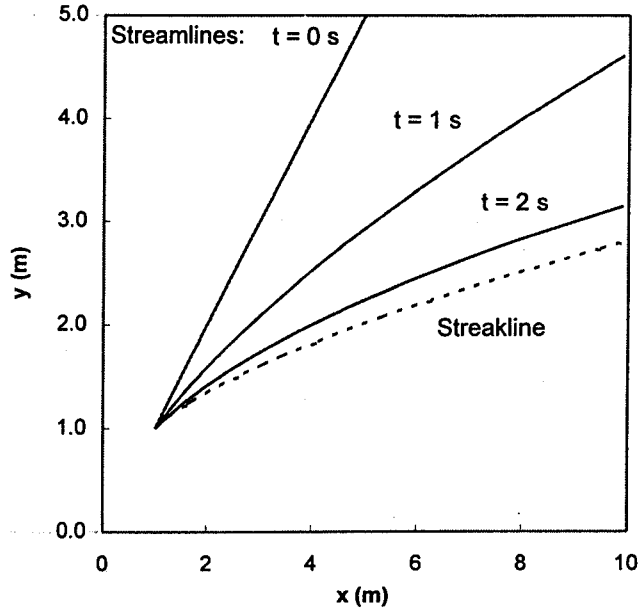
$x = x_0 e^{\left[(t-t_0) + 0.25(t^2-t_0^2)\right]}$ $\quad y = y_0 e^{(t-t_0)}$

The streakline through $(x_0, y_0) = (1,1)$ at time $t = 3s$ is obtained by substituting $x_0 = 1$, $y_0 = 1$, $t = 3s$ and varying t_0 in these equations.

Thus, $x = e^{[(3-t_o)+0.25(9-t_o^2)]}$

$y = e^{(3-t_o)}$

give points (obtained by varying t_o) on the streakline through (1,1,0) at $t = 3s$.



13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Velocity field $\vec{V} = ax(1+bt)\hat{i} + cy\hat{j}$, where $a = c = 1\,s^{-1}$, $b = 0.2\,s^{-1}$, and coordinates are measured in meters

Plot: the streakline that passes through the point $(x_0, y_0) = (1,1)$ during the interval $0 \le t \le 3s$. Compare with the streamlines plotted through the same point at $t = 0$, 1, and 2s

Solution:

Streakline at $t = 3s$ connects particles that passed through point (x_0, y_0) at earlier times $\tau = 0, 1, 2,$ and 3s.

For a particle, $u = dx/dt$ and $v = dy/dt$

Then $u = ax(1+bt) = \frac{dx}{dt}$ and $\int_{x_0}^{x} \frac{dx}{x} = \int_{\tau}^{t} a(1+bt)\,dt$

$\ln \frac{x}{x_0} = a\left(t + \frac{b}{2}t^2\right)_{\tau}^{t} = a\left[(t-\tau) + \frac{b}{2}(t^2 - \tau^2)\right]$

$x = x_0\, e^{a\left[(t-\tau) + \frac{b}{2}(t^2-\tau^2)\right]}$

Also $v = \frac{dy}{dt} = cy$, $\int_{y_0}^{y} \frac{dy}{y} = \int_{\tau}^{t} c\,dt$, $\ln \frac{y}{y_0} = c(t-\tau)$, $y = y_0\, e^{c(t-\tau)}$

Substituting for a, b, c, x_0, and y_0, gives:

$x = e^{\left[(t-\tau) + 0.1(t^2-\tau^2)\right]}$, $y = e^{(t-\tau)}$ ← (x,y) streakline

The streakline may be plotted by substituting values for τ in the range $0 \le \tau \le 3s$ as shown below.

The streamline is found (at given t) from $(dy/dx)_s = v/u$

Thus $dy/dx = \frac{cy}{ax(1+bt)}$ and $\int_{y_0}^{y} \frac{dy}{y} = \int_{x_0}^{x} \frac{c}{a(1+bt)} \frac{dx}{x}$

$\ln \frac{y}{y_0} = \frac{c}{a(1+bt)} \ln \frac{x}{x_0}$ or $y = y_0 \left[\frac{x}{x_0}\right]^{c/a(1+bt)}$

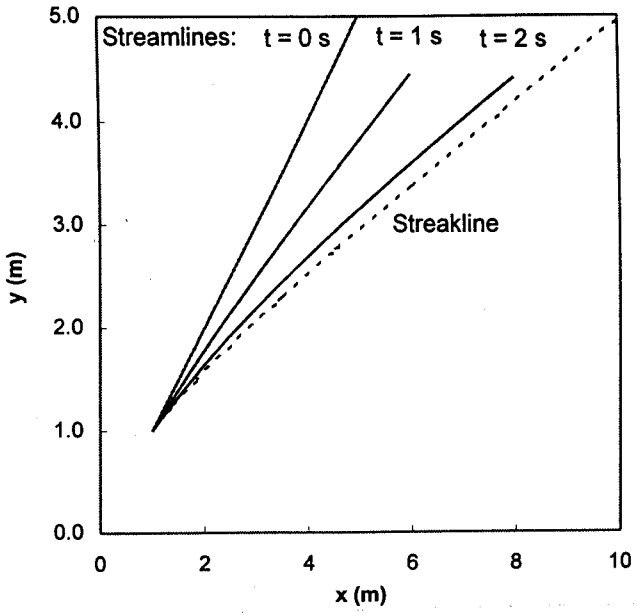
Substituting values for x_0, y_0, a, b, c, then

$y = x^{1/(1+0.2t)}$ or $x = y^{(1+0.2t)}$ ← streamline

At $t = 0$, $x = y_{1.0}$

$t = 1s$, $x = y_{1.2}$

$t = 2s$, $x = y_{1.4}$



13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Hydrogen bubbles generated at location (0,0) in an unsteady flow field given by

$u = 1$ m/s $v = 1$ m/s $0 \le t < 2$ s

$u = 0$ $v = 1.5$ m/s $2 \le t \le 4$ s

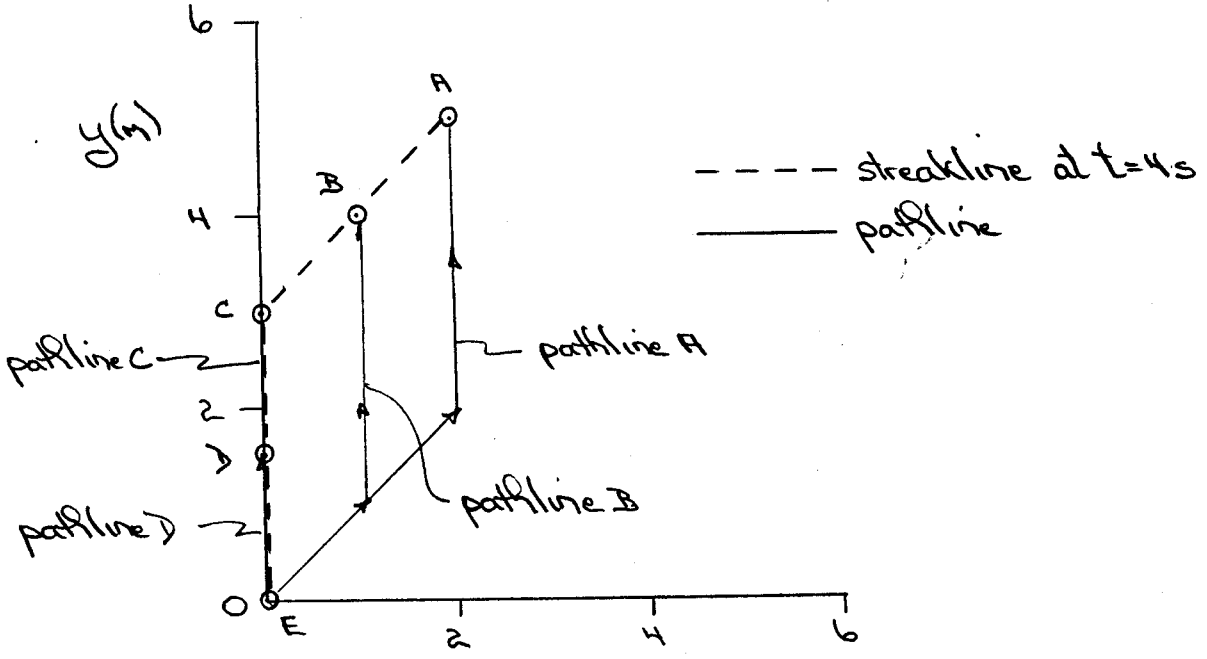
Find: (a) pathlines of bubbles leaving origin at $t = 0, 1, 2, 3, 4$ s
(b) locate bubbles at $t = 4$ s
(c) position of streakline at $t = 4$ s (show as dashed)

Solution:

Coordinates of particle at time t given by $x = \int_0^t u\,dt$, $y = \int_0^t v\,dt$

Set up table to locate particles at successive times

Bubble No	$t = 0$	Location (x,y) meters at time $t = 1$ s	$t = 2$ s	$t = 3$ s	$t = 4$ s
A	0,0	1,1	2,2	2,3.5	2,5.0
B		0,0	1,1	1,2.5	1,4.0
C			0,0	0,1.5	0,3.0
D				0,0	0,1.5
E					0,0



Given: Velocity field $\vec{V} = axt\,\hat{i} + b\,\hat{j}$, where $a = 0.3\ s^{-1}$, $b = 2\ m/s$, and coordinates are in meters.

Plot: the pathline (during the interval $0 \le t \le 3s$) of the particle that passed through the point $(x_0, y_0) = (1,2)$ at time $t = 0$
Compare with the streakline through the same point at the instant $t = 3s$.

Solution:

The pathline and streakline are based on parametric equations for a particle.

For a particle $u = dx/dt$ and $v = dy/dt$

then $u = \frac{dx}{dt} = axt$, $\int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} at\,dt$, $\ln\frac{x}{x_0} = \frac{1}{2}a(t^2 - t_0^2)$

$$x = x_0 e^{\frac{1}{2}a(t^2 - t_0^2)}$$

Also $v = dy/dt = b$, $\int_{y_0}^{y} dy = \int_{t_0}^{t} b\,dt$, $y = y_0 + b(t - t_0)$

In the above equations, x_0, y_0 are coordinates of particle at t_0.

(a) The pathline is obtained by following the particle that passed through the point $(x_0, y_0) = (1,2)$ at time $t_0 = 0$

Thus
$$\left.\begin{aligned} x &= x_0 e^{\frac{1}{2}at^2} = e^{0.15t^2} \\ y &= y_0 + bt = 2 + 2t \end{aligned}\right\} \leftarrow (x,y)\ \text{pathline}$$

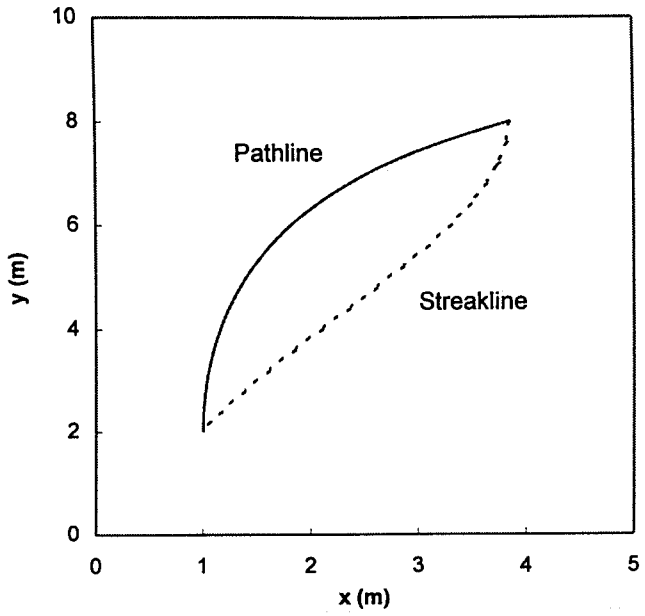
The pathline may be plotted by varying t, $(0 \le t \le 3s)$ as shown below

(b) The streakline is obtained by locating (and connecting) at time $t = 3s$, all the particles that passed through the point $(x_0, y_0) = (1,2)$ at some earlier time t_0.

Thus
$$\left.\begin{aligned} x &= x_0 e^{\frac{1}{2}a(9 - t_0^2)} = e^{0.15(9 - t_0^2)} \\ y &= y_0 + b(t - t_0) = 2 + 2(3 - t_0) \end{aligned}\right\} \leftarrow (x,y)\ \text{streakline}$$

The streakline may be plotted by varying t_0 $(0 \le t_0 \le 3s)$ as shown below.

13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-398 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand



13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Velocity field in xy plane, $\vec{V} = a\hat{i} + bx\hat{j}$, where

$a = 2\ m/s$ and $b = 1\ s^{-1}$!

Find: (a) Equation for streamline through $(x,y) = (2,5)$.

(b) At $t = 2\ s$, coordinates of particle $(0,4)$ at $t = 0$.

(c) At $t = 3\ s$, coordinates of particle $(1, 4.25)$ at $t = 1\ s$.

(d) compare pathline, streamline, streakline.

Solution: For a streamline $\frac{dx}{u} = \frac{dy}{v}$

For $\vec{V} = a\hat{i} + bx\hat{j}$, $u = a$ and $v = bx$, so $\frac{dx}{a} = \frac{dy}{bx}$ or

$$x\,dx = \frac{a}{b}\,dy$$

Integrating

$$\frac{x^2}{2} = \frac{a}{b}y + C' \quad \text{or} \quad y = \frac{b}{2a}x^2 + C$$

Evaluating C at $(x,y) = (2,5)$,

$$C = y - \frac{b}{2a}x^2 = 5m - \frac{1}{2}\times\frac{1}{s}\times\frac{s}{2m}(2m)^2 = 4\ m$$

Streamline through $(x,y) = (2,5)$ is $y = \frac{x^2}{4} + 4$ (a)

To locate particles, derive parametric equations

$$u_p = \frac{dx}{dt} = a, \quad dx = a\,dt, \quad \text{and} \quad x - x_0 = a(t - t_0)$$

$$v_p = \frac{dy}{dt} = bx, \quad dy = bx\,dt = b(x_0 + at - at_0)$$

$$y - y_0 = bx_0(t - t_0) + \frac{a}{2}(t^2 - t_0^2) - at_0(t - t_0)$$

For the particle at $(x_0, y_0) = (0,4)$ at $t = 0$,

$x = 0 + at$ so at $t = 2s$, $x = 2\frac{m}{s}\times 2s = 4\ m$

$y = 4 + \frac{at^2}{2}$ so at $t = 2s$, $y = 4 + \frac{1}{2}\times 2\frac{m}{s}\times(2)^2 s^2$

$y = 8\ m$ (b)

For the particle at $(x,y) = (1, 4.25)$ at $t = 1\,s$,

$$x = x_0 + a(t - t_0) = 1 + a(t-1)$$

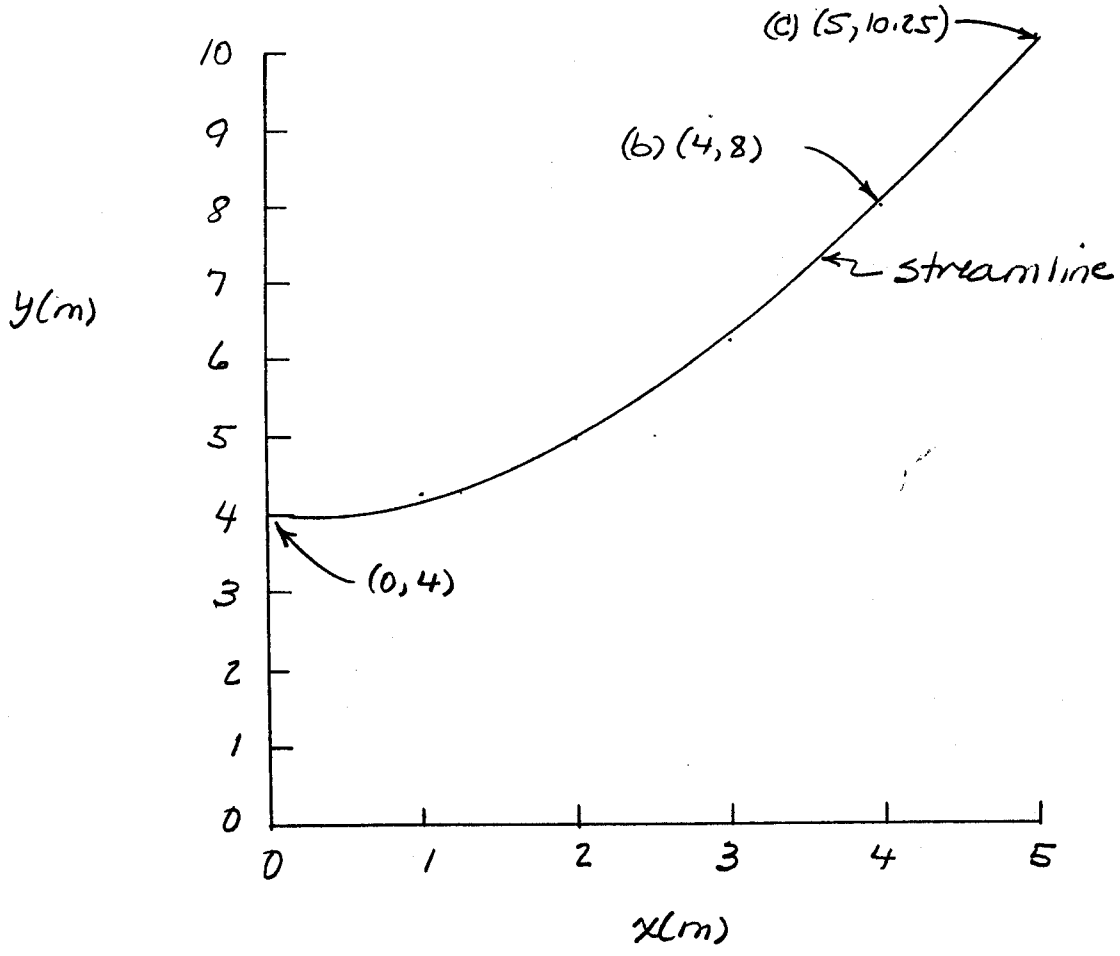
so at $t = 3\,s$, $\quad x = 1 + 2\frac{m}{s}(3-1)s = 5\,m$

$$y = y_0 + bx_0(t-t_0) + \frac{a}{2}(t^2 - t_0^2) - at_0(t-t_0)$$

$$= 4.25 + \frac{1}{s} \times 1m \times (t-1) + \frac{1}{2} \times 2\frac{m}{s}(t^2-1) - 2\frac{m}{s} \times 1s\,(t-1)$$

so at $t = 3\,s$, $\quad y = 4.25 + 2 + 8 - 4 = 10.25\,m$ (C)

All these points lie on the same streamline, as shown below:



For this steady flow, streamlines, pathlines, and streaklines coincide, as expected.

Given: Velocity field $\vec{V} = ay\hat{i} + b\hat{j}$, where $a = 1\,s^{-1}$, and $b = 2\,m/s$; coordinates are measured in meters.

Find: (a) Equation of streamline through $(x,y) = (6,6)$
(b) At $t = 1\,s$, coordinates of particle that passed through point $(x_0, y_0) = (1,4)$ at $t_0 = 0$
(c) At $t = 3\,s$, coordinates of particle that passed through point $(x_0, y_0) = (-3,0)$ at $t_0 = 1\,s$.

Solution

The velocity vector is tangent to the streamlines

$$\left(\frac{dy}{dx}\right)_{streamline} = \frac{v}{u} = \frac{b}{ay} \quad \text{or} \quad \int_6^y ay\,dy = \int_6^x b\,dx$$

Then $\frac{1}{2}ay^2\Big]_6^y = bx\Big]_6^x$, $2b(x-6) = a(y^2 - 36)$

and

$$4(x-6) = y^2 - 36 \quad \text{or} \quad x = \frac{y^2}{4} - 3 \qquad \text{Streamline}$$

(b) Follow particle that passed through (1,4) @ t=0

$$u = \frac{dx}{dt} = ay \quad \therefore \int_{x_0}^{x} dx = \int_0^t ay\,dt \quad \{\text{need } y = y(t)\}$$

$$v = \frac{dy}{dt} = b \quad \therefore \int_{y_0}^{y} dy = \int_0^t b\,dt \quad \text{and} \quad y = y_0 + bt \qquad (1a)$$

Then

$$x - x_0 = \int_{x_0}^{x} dx = \int_0^t a(y_0 + bt)\,dt = ay_0 t + \frac{1}{2}abt^2$$

$$x = x_0 + ay_0 t + \frac{1}{2}abt^2 \qquad (1b)$$

Following particle through (1,4) at t=0, then at t=1s

$$x_p = 1 + (1)(4)(1) + \frac{1}{2}(2)(1)^2 = 6 \quad \text{and} \quad y_p = 4 + 2(1) = 6 \qquad (x_p, y_p)$$

(c) Streakline. At $t = 3\,s$, locate position of particle that passed through $(x_0, y_0) = (-3,0)$ at earlier time $t_0 = 1\,s$.

For a particle

$$v = \frac{dy}{dt} = b \quad \therefore \int_{y_0}^{y} dy = \int_{t_0}^{t} b\,dt \quad \text{and} \quad y = y_0 + b(t - t_0) \qquad (2a)$$

$$u = \frac{dx}{dt} = ay \quad \therefore \int_{x_0}^{x} dx = \int_{t_0}^{t} ay\,dt = \int_{t_0}^{t} a\left[y_0 + b(t - t_0)\right]dt$$

and

$$x = x_0 + ay_0(t - t_0) + \frac{ab}{2}(t^2 - t_0^2) - abt_0(t - t_0) \qquad \text{-----(2b)}$$

Then from Eqs 2a, 2b for $t = 3\,s$ and $t_0 = 1\,s$

$$x = -3 + 0 + \frac{(1)2}{2}\left[(3)^2 - (1)^2\right] - (1)(2)(1)(3-1) = 1$$

$$y = 0 + 2(3-1) = 4 \qquad (x,y) = (1,4)$$

Since points (6,6), (1,4), and (-3,0) are all on the same streamline ($x = y^2/4 - 3$), pathlines, streaklines = streamlines coincide

Given: Velocity field $\vec{V} = a\hat{i} + b\hat{j}$, where $a = 0.4\ m/s^2$, $b = 2\ m/s$, and coordinates are measured in meters.

Find: (a) At $t = 2s$, coordinates of particle that passed through $(x_0, y_0) = (2,1)$ at $t = 0$
(b) At $t = 3s$, coordinates of the particle that passed through (x_0, y_0) at $t = 2s$

Plot: the pathline and streakline through point (2,1); compare with the streamlines through the same point at $t = 0, 1, 2s$

Solution:

The pathline and streakline are based on parametric equations for a particle.

For a particle $u = dx/dt$ and $v = dy/dt$

Thus $u = \frac{dx}{dt} = at$, $\int_{x_0}^{x} dx = \int_{t_0}^{t} at\,dt$, $x = x_0 + \frac{1}{2}a(t^2 - t_0^2)$ (1a)

$v = \frac{dy}{dt} = b$, $\int_{y_0}^{y} dy = \int_{t_0}^{t} b\,dt$, $y = y_0 + b(t - t_0)$. (1b)

In the above equations, x_0, y_0 are coordinates of the particle at time t_0

(a) The pathline is obtained by following the particle that passed through the point $(x_0, y_0) = (2,1)$ at time $t_0 = 0$

Thus $x = x_0 + \frac{1}{2}at^2 = 2 + 0.2t^2$
$y = y_0 + bt = 1 + 2t$
} ← (x,y) pathline

At $t = 2s$, particle is at $(x,y) = (2.8, 5)\ m$ ← (a)

The pathline may be plotted by varying t $(0 \le t \le 3s)$ as shown below.

(b) The streakline is obtained by locating (and connecting) at time $t = 3s$, all the particles that passed through the point $(x_0, y_0) = (2,1)$ at some earlier time t_0

Thus $x = x_0 + \frac{1}{2}a(9 - t_0^2) = 2 + 0.2(9 - t_0^2)$
$y = y_0 + b(t - t_0) = 1 + 2(3 - t_0)$
} ← (x,y) streakline

At $t = 2s$, particle is at $(x,y) = (3,3)$ ← (b)

The streakline may be plotted by varying t_0 $(0 \le t_0 \le 3s)$ as shown below.

The streamline is found (at given t) from $(dy/dx)_s = \frac{v}{u}$.

Then: $dy/dx = \frac{b}{at}$, $\int_{y_0}^{y} dy = \int_{t_0}^{t} \frac{b}{at}\,dt$, $y - y_0 = \frac{b}{at}(t - t_0)$

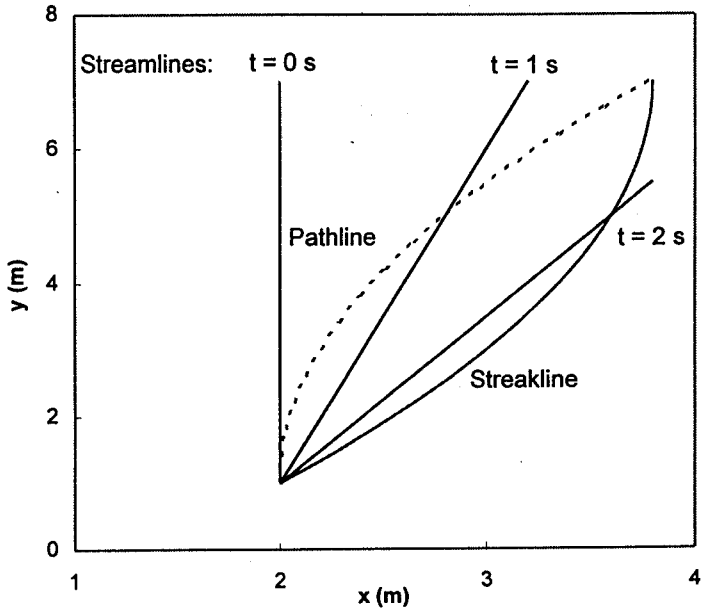
Streamline through point (2,1) gives $y - 1 = \frac{b}{at}(t-2)$

$y = 1 + \frac{5(t-2)}{t}$ or $t = 2 + \frac{t}{5}(y-1)$ ← streamline

At $t=0$, $t=2$

$t=1$, $y = 5t - 9$

$t=2$, $y = 2.5t - 4$



Given: Velocity field $\vec{V} = ay\hat{i} + bt\hat{j}$, where $a = 1\ s^{-1}$; $b = 0.5\ m/s^2$, t in s.

Find: (a) At $t = 2$ s, particle that passed (1, 2) at $t = 0$ s

(b) At $t = 3$ s, particle that passed (1, 2) at $t = 2$ s

(c) Plot pathline and streakline through (1, 2); compare with streamlines at $t = 0, 1, 2$ s.

Solution: Pathline and streakline are based on parametric equations for a particle. Thus

$$v = \frac{dy}{dt} = bt, \text{ so } dy = bt\,dt, \text{ and } y - y_0 = \frac{b}{2}(t^2 - t_0^2)$$

and $$u = \frac{dx}{dt} = ay = a\left[y_0 + \frac{b}{2}(t^2 - t_0^2)\right]$$

so $$x\Big]_{x_0}^{x} = a\left[y_0 t + \frac{b}{2}\left(\frac{t^3}{3} - t_0^2 t\right)\right]_{t_0}^{t}; \quad x = x_0 + ay_0(t - t_0) + \frac{ab}{2}\left(\frac{t^3 - t_0^3}{3} + t_0^2(t_0 - t)\right)$$

where x_0, y_0 are coordinates of particle at t_0.

For (a), $t_0 = 0$, and $(x_0, y_0) = (1, 2)$. Thus at $t = 2$ s, $y = y_0 + \frac{bt^2}{2}$

$$y = 2\,m + \frac{1}{2} \times 0.5\frac{m}{s^2} \times (2)^2 s^2 = 3.00\,m$$

$$x = 1\,m + \frac{1}{s} \times 2\,m\,(2 - 0)s + \frac{1}{2} \times \frac{1}{s} \times 0.5\frac{m}{s^2}\left(\frac{(2)^3 - 0}{3} + 0\right)s^3 = 5.67\,m$$

$t_0 = 0$ (a)

At $t = 2$ s, $(x, y) =$ (5.67, 3.00) m ←

For (b), $t_0 = 2$ s, and $(x_0, y_0) = (1, 2)$. Thus at $t = 3$ s, the particle is at

$$y(3) = 2\,m + \frac{1}{2} \times 0.5\frac{m}{s^2}\left[(3)^2 - (2)^2\right]s^2 = 3.25\,m$$

$$x(3) = 1\,m + \frac{1}{s} \times 2\,m\,(3 - 2)s + \frac{1}{2} \times \frac{1}{s} \times 0.5\frac{m}{s^2}\left(\frac{(3)^3 - (2)^3}{3} + (2)^2(2 - 3)\right)s^3 = 3.58\,m$$

At $t = 3$ s, $t_0 = 2$ s (b)

$(x, y) =$ (3.58, 3.25) ←

For (c), the streakline may be plotted at any t by varying t_0, as shown on the next page.

The streamline is found (at given t) from $\frac{dx}{u} = \frac{dy}{v}$

Substituting $u = ay$ and $v = bt$, $dx = \frac{ay}{bt}dy$ or $y^2 = \frac{2bt}{a}x + C$

Thus $C = y_0^2 - \frac{2bt}{a}x_0$

For $t = 0$, $y^2 = C$; at $(x_0, y_0) = (1, 2)$, then $C = 4$

$t = 1$, $y^2 = \frac{2b}{a}x + C$; at $(x_0, y_0) = (1, 2)$, then $C = 3$

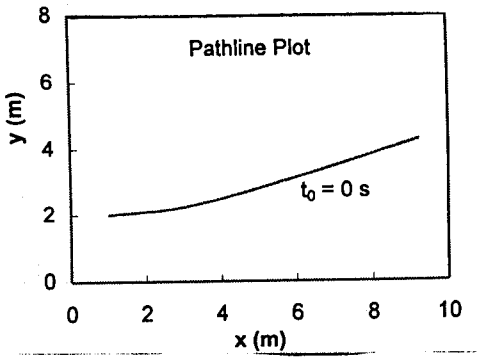
$t = 2$, $y^2 = \frac{4b}{a}x + C$; at $(x, y) = (1, 2)$, $C = 2$; for $t = 3$ s, $C = 1$ ←

(c)

Recall $\vec{V} = ay\hat{i} + bt\hat{j}$, where $a = 1\,s^{-1}$, $b = 0.5\ m/s^2$, $(x_0, y_0) = (1, 2)\ m$.

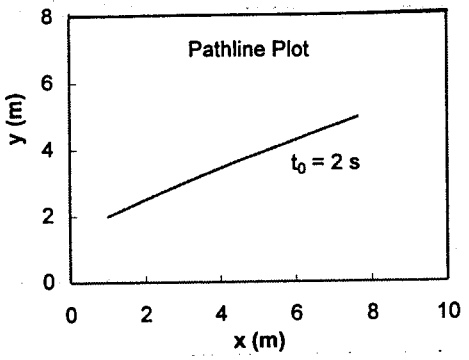
Part (a): Pathline of particle located at (x_0, y_0) at $t_0 = 0$ s:

t_0 (s)	t (s)	x (m)	y (m)
0	0	1.00	2.00
0	1	3.08	2.25
0	2	5.67	3.00
0	3	9.25	4.25



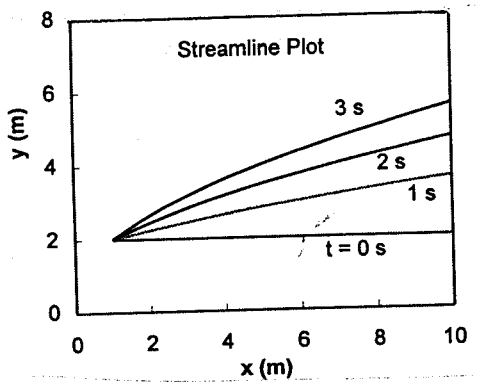
Part (b): Pathline of particle located at (x_0, y_0) at $t_0 = 2$ s:

t_0 (s)	t (s)	x (m)	y (m)
2	2	1.00	2.00
2	3	3.58	3.25
2	4	7.67	5.00



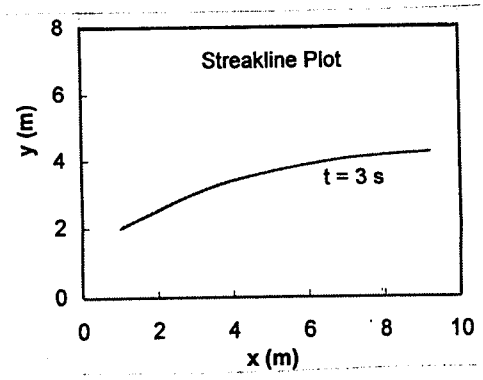
Part (c): Streamlines through point (x_0, y_0) at $t = 0, 1, 2$, and 3 s:

	t (s)	0	1	2	3
	c =	4.0	3.0	2.0	1.0
t_0 (s)	x (m)	y (m)	y (m)	y (m)	y (m)
0	1	2.00	2.00	2.00	2.00
0	2	2.00	2.24	2.45	2.65
0	3	2.00	2.45	2.83	3.16
0	4	2.00	2.65	3.16	3.61
0	5	2.00	2.83	3.46	4.00
0	6	2.00	3.00	3.74	4.36
0	7	2.00	3.16	4.00	4.69
0	8	2.00	3.32	4.24	5.00
0	9	2.00	3.46	4.47	5.29
0	10	2.00	3.61	4.69	5.57



Streakline at $t = 3$ s of particles that passed thru point (x_0, y_0):

t_0 (s)	t (s)	x (m)	y (m)
0	3	9.25	4.25
1	3	6.67	4.00
2	3	3.58	3.25
3	3	1.00	2.00



Given: Variation of air viscosity with temperature (absolute) is

$$\mu = \frac{bT^{1/2}}{1 + S/T}$$

where $b = 1.458 \times 10^{-6}$ kg/m.s.K$^{1/2}$, $S = 110.4$K

Find: Equation for calculating air viscosity in British Gravitational units as a function of absolute temperature in degrees Rankine. Check result using data from Appendix A

Solution:

Convert constants

$$b = 1.458 \times 10^{-6} \frac{kg}{m.s.K^{1/2}} \times \frac{lbm}{0.4536 kg} \times \frac{slug}{32.17 lbm} \times \frac{lb.s^2}{slug.ft} \times \frac{0.3048 m}{ft} \times \left(\frac{5K}{9°R}\right)^{1/2}$$

$$b = 2.27 \times 10^{-8} \text{ lbf.s/ft}^2.°R^{1/2}$$

$$S = 110.4K \times \frac{9°R}{5K} = 198.7°R$$

Then in British Gravitational Units

$$\mu = \frac{2.27 \times 10^{-8} T^{1/2}}{1 + 198.7/T}$$ ← μ

where units of T are °R; μ is in lbf.s/ft^2

Evaluate at T = 80°F (539.7°R)

$$\mu = \frac{2.27 \times 10^{-8} \times (539.7)^{1/2}}{1 + 198.7/539.7} = 3.855 \times 10^{-7} \text{ lbf.s/ft}^2$$

From Table A.9 (Appendix A) at T = 80°F

$\mu = 3.86 \times 10^{-7}$ lbf.s/ft^2 ✓ check.

Given: Variation of air viscosity with temperature (absolute) is

$$\mu = \frac{bT^{1/2}}{1+S/T}$$

where $b = 1.458\times10^{-6}\ \frac{kg}{m\cdot s\cdot K^{1/2}}$

$S = 110.4\ K$

Find: Equation for kinematic viscosity of air (in SI units) as a function of temperature at atmospheric pressure. Assume ideal gas behavior. Check result using data from Appendix A.

Solution:

For an ideal gas, $p = \rho RT$. From Table A.6, $R = 286.9\ N\cdot m/kg\cdot K$

The kinematic viscosity, $\nu \equiv \mu/\rho$

$$\therefore \nu = \frac{\mu}{\rho} = \frac{\mu RT}{p} = \frac{RT}{p}\frac{bT^{1/2}}{1+S/T} = \frac{Rb}{p}\frac{T^{3/2}}{1+S/T} = \frac{b'T^{3/2}}{1+S/T}$$

where $b' = \frac{Rb}{p} = 286.9\ \frac{N\cdot m}{kg\cdot K}\times 1.458\times10^{-6}\ \frac{kg}{m\cdot s\cdot K^{1/2}}\times\frac{m^2}{101.3\times10^3\ N}$

$$b' = 4.129\times10^{-9}\ m^2/s\cdot K^{3/2}$$

$$\therefore \nu = \frac{b'T^{3/2}}{1+S/T}$$

where $b' = 4.129\times10^{-9}\ m^2/s\cdot K^{3/2}$, $S = 110.4\ K$

units of T are (K); ν is in m^2/s

Evaluate at $T = 20°C = 293.2\ K$

$$\nu = \frac{4.129\times10^{-9}\ (293.2)^{3/2}}{1+110.4/293.2} = 1.506\times10^{-5}\ m^2/s$$

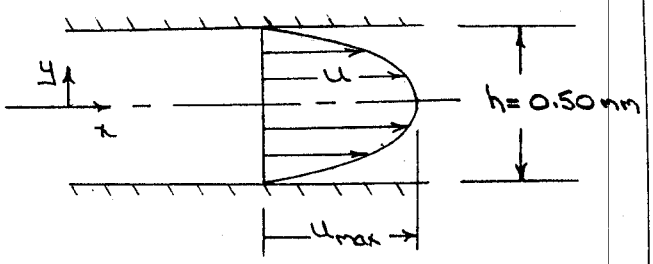
From Table A.10 (Appendix A) at $T = 20°C$

$\nu = 1.51\times10^{-5}\ m^2/s$ ✓ Check.

Given: Flow of water @ 15°C between parallel plates as shown.

$$\frac{u}{u_{max}} = \left[1 - \left(\frac{2y}{h}\right)^2\right]$$

$u_{max} = 0.30$ m/s



Find: Shear stress on upper plate (indicate direction); sketch the variation of shear stress across the channel

Solution

Basic equation $\tau_{yx} = \mu \frac{du}{dy}$

$$\frac{du}{dy} = \frac{d}{dy}\left\{u_{max}\left[1 - \left(\frac{2y}{h}\right)^2\right]\right\}$$

$$\frac{du}{dy} = u_{max}\left(-\frac{4}{h^2}\right)2y = -\frac{8 u_{max} y}{h^2}$$

At upper plate, $y = +\frac{h}{2}$, so

$$\tau_{yx}\left(@\ y = \frac{h}{2}\right) = \mu \left(\frac{du}{dy}\right)_{y=\frac{h}{2}} = -\frac{8\mu u_{max}}{h^2}\left(\frac{h}{2}\right) = -\frac{4\mu u_{max}}{h}$$

From Table A.8, for water @ 15°C, $\mu = 1.14 \times 10^{-3}$ N·s/m². Thus

$$\tau_{yx} = -\frac{4\mu u_{max}}{h} = -4 \times 1.14 \times 10^{-3} \frac{N \cdot s}{m^2} \times 0.30 \frac{m}{s} \times \frac{1}{5 \times 10^{-4}} m$$

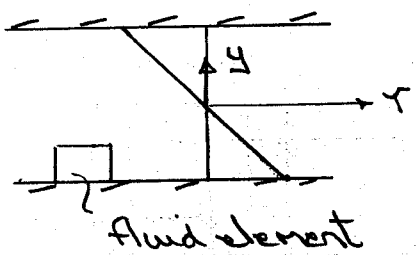
$\tau_{yx} = -2.74$ N/m² ← τ_{yx}

The upper plate is a minus y surface. Since $\tau_{yx} < 0$, the shear stress on the upper plate must act in the plus x direction.

The shear stress varies linearly with y

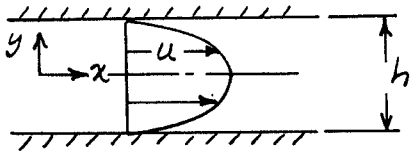
$$\tau = \mu \frac{du}{dy} = -\frac{8 u_{max}}{h^2} y$$

The shear stress on the surface of the fluid element shown (a positive y surface) is illustrated in the sketch.



Given: Laminar flow between parallel plates.

$$\frac{u}{u_{max}} = 1 - \left(\frac{2y}{h}\right)^2$$



$T = 15°C$, $u_{max} = 0.05$ m/s, $h = 5$ mm, water

Find: Force on $A = 0.3$ m² section of lower plate.

Solution: Apply definitions of Newtonian fluid, shear stress.

Basic equations: $\tau = \frac{F}{A}$, $\tau_{yx} = \mu \frac{du}{dy}$

Assumptions: (1) Newtonian fluid

From the given profile, $u = u_{max}\left[1 - \left(\frac{2y}{h}\right)^2\right]$, so $\frac{du}{dy} = u_{max}(-2)\left(\frac{2y}{h}\right)\left(\frac{2}{h}\right)$

At lower surface, $y = -h/2$ $= -\frac{8 u_{max} y}{h^2}$

$$\tau_{yx}(\text{lower}) = \mu \left.\frac{du}{dy}\right)_{y=-h/2} = \mu\left[-\frac{8 u_{max}(-h/2)}{h^2}\right] = \frac{4\mu u_{max}}{h}$$

$\tau_{yx} > 0$ and surface is positive, so to right.

$$F = \tau_{yx} A = \frac{4\mu u_{max} A}{h}$$

From Appendix A, Table A.8, $\mu = 1.14 \times 10^{-3}$ N·s/m at 15°C, so

$$F = 4 \times 1.14 \times 10^{-3} \frac{N \cdot s}{m^2} \times 0.05 \frac{m}{s} \times 0.3 m^2 \times \frac{1}{5 mm} \times 10^3 \frac{mm}{m}$$

$F = 0.0137$ N (to right)

F ←

Open-Ended Problem Statement: Explain how an ice skate interacts with the ice surface. What mechanism acts to reduce sliding friction between skate and ice?

Discussion: The normal freezing and melting temperature of ice is 0°C (32°F) at atmospheric pressure. The melting temperature of ice decreases as pressure is increased. Therefore ice can be caused to melt at a temperature below the normal melting temperature when the ice is subjected to increased pressure.

A skater is supported by relatively narrow blades with a short contact against the ice. The blade of a typical skate is less than 3 mm wide. The length of blade in contact with the ice may be just ten or so millimeters. With a 3 mm by 10 mm contact patch, a 75 kg skater is supported by a pressure between skate blade and ice on the order of tens of megaPascals (hundreds of atmospheres). Such a pressure is enough to cause ice to melt rapidly.

When pressure is applied to the ice surface by the skater, a thin surface layer of ice melts to become liquid water and the skate glides on this thin liquid film. Viscous friction is quite small, so the effective friction coefficient is much smaller than for sliding friction.

The magnitude of the viscous drag force acting on each skate blade depends on the speed of the skater, the area of contact, and the thickness of the water layer on top of the ice.

The phenomenon of static friction giving way to viscous friction is similar to the hydroplaning of a pneumatic tire caused by a layer of water on the road surface.

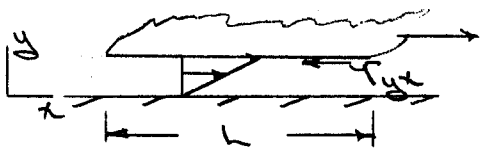
13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-398 200 SHEETS EYE-EASE® 5 SQUARE
42-389 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Skater, of weight $W = 112$ lbf, glides on one skate at speed $V = 20.5$ ft/s. Skate blade, of length $L = 11.5$ in and width $w = 0.125$ in. glides on thin film of water of height $h = 5.75 \times 10^{-5}$ in.

Find: the deceleration of the skater due to viscous shear.

Solution:

Model flow as one-dimensional shear flow



Basic equation: $\tau_{yx} = \mu \frac{du}{dy}$

Assumptions: 1. Newtonian fluid
2. Linear velocity profile
3. Neglect end effects.

From Table A.7, Appendix A, at 32°F

$\mu = 3.66 \times 10^{-5}$ lbf·s/ft²

$$\tau_{yx} = \mu \frac{du}{dy} = \mu \frac{V}{h} = 3.66 \times 10^{-5} \frac{\text{lbf}\cdot\text{s}}{\text{ft}^2} \times 20.5 \frac{\text{ft}}{\text{s}} \times 5.75 \times 10^{-5} \text{ in} \times \frac{12 \text{ in}}{\text{ft}}$$

$\tau_{yx} = 157$ lbf/ft²

$\Sigma F_x = m a_x$ $\quad \therefore \tau_{yx} A = -\frac{W}{g} a_x$

$$a_x = -\frac{\tau_{yx} A}{W} g = -\frac{\tau_{yx} L w g}{W}$$

$$= -157 \frac{\text{lbf}}{\text{ft}^2} \times 11.5 \text{ in} \times 0.125 \text{ in} \times 32.2 \frac{\text{ft}}{\text{s}^2} \times \frac{1}{112 \text{ lbf}} \times \frac{\text{ft}^2}{144 \text{ in}^2}$$

$a_x = -0.451$ ft/s² $\longleftarrow a_x$

Given: Thin film of crude oil ($SG = 0.85$, $\mu = 2.15 \times 10^{-3}$ lbf·s/ft²) with thickness $h = 0.125$ in, flows down a 30° incline. The velocity profile is given by

$$u = \frac{\rho g}{\mu}\left(hy - \frac{y^2}{2}\right)\sin\theta$$

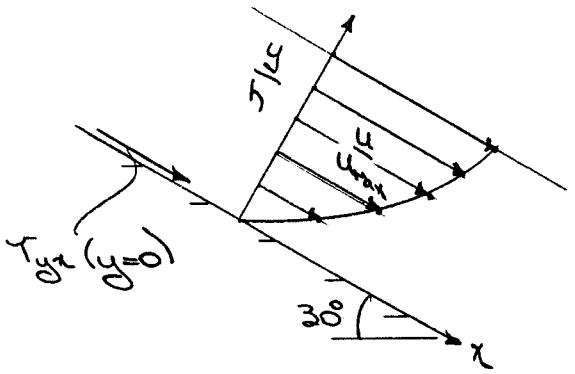
Find: (a) the magnitude and direction of the shear stress acting on the surface.
(b) Plot the velocity profile.

Solution:

To plot the profile, note that $u = u_{max}$ at $y = h$

$$u_{max} = \frac{\rho g}{\mu}\frac{h^2}{2}\sin\theta \qquad \therefore \frac{u}{u_{max}} = 2\left[\frac{y}{h} - \frac{1}{2}\left(\frac{y}{h}\right)^2\right]$$

y/h	u/u_{max}
0	0
0.2	0.36
0.4	0.64
0.6	0.84
0.8	0.96
1.0	1.00



The shear stress is given by $\tau_{yx} = \mu \frac{du}{dy}$

$$\therefore \tau_{yx} = \mu \frac{d}{dy}\left[\frac{\rho g}{\mu}\left(hy - \frac{y^2}{2}\right)\sin\theta\right] = \mu \frac{\rho g}{\mu}\sin\theta\,(h - y)$$

At the inclined surface, $y = 0$

$$\therefore \tau_{yx} = \rho g h \sin\theta = SG\,\rho_{H_2O}\, g h \sin\theta$$

$$\tau_{yx} = 0.85 \times 1.94 \frac{slug}{ft^3} \times 32.2 \frac{ft}{s^2} \times 0.125\,in \times \frac{ft}{12\,in} \times \sin 30° \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$\tau_{yx} = 0.277\ lbf/ft^2$$

The surface is a positive y surface. Since $\tau_{yx} > 0$, the stress must act in the positive x direction as shown on the sketch above.

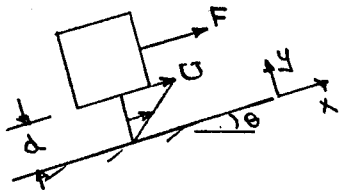
Given: Block of weight 10 lbf, 10 in. on each edge, is pulled up a plane, inclined at 15° to the horizontal, over a film of SAE 10W oil at 100°F. The speed of the block is constant at 5 ft/s and the oil film thickness is 0.001 in. Velocity profile in film is linear.

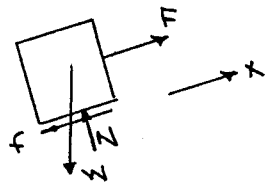
Find: Force required.

Solution:

Since the block is moving at constant velocity, V, then $\Sigma \vec{F}_{ext} = 0$

Consider the forces along the direction of motion and look at a free body diagram of the block.





Since $\Sigma F_x = 0$, then $F - f - W \sin\theta = 0$

Now the friction force, $f = \tau A$

where $\tau = \mu \frac{du}{dy}$

For small gap (linear velocity profile) $\tau = \mu \frac{V}{d}$

Hence $f = \tau A = \mu \frac{V}{d} A$

and $F - \mu \frac{V}{d} A - W \sin\theta = 0$

Thus

$$F = \mu \frac{V}{d} A + W \sin\theta$$

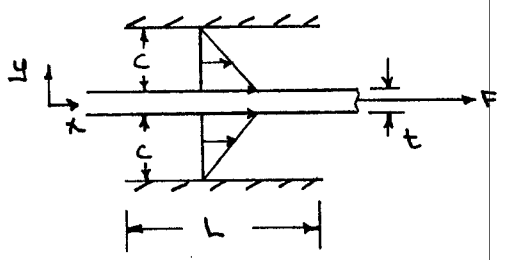
From Fig A.2, Appendix A, for SAE 10W oil @ 100°F (38°C), $\mu = 3.7 \times 10^{-2}$ N·s/m²

$$F = \mu \frac{V}{d} A + W \sin\theta$$

$$= 3.7 \times 10^{-2} \frac{N \cdot s}{m^2} \times 2.09 \times 10^{-2} \frac{lbf \cdot s}{ft^2} \cdot \frac{m^2}{N \cdot s} \times \frac{5 ft}{s} \times (10)^2 in^2 \times \frac{1}{0.001 in} \times \frac{ft}{12 in} + 10 lbf \sin 15$$

$F = 34.8$ lbf ← F

Given: Tape, of width $w = 1.00$ in is to be coated on both sides with lubricant by drawing in through narrow gap of length, L, as shown.



$c = 0.012$ in. $t = 0.015$ in, $L = 0.75$ in

Lubricant: $\mu = 0.021$ slug/ft·s completely fills gap, velocity distribution is linear

Maximum allowable force in tape is $F = 7.5$ lbf

Find: Maximum allowable tape speed.

Solution:

$$\Sigma F_x = ma_x$$

Since V_{tape} = constant, then $\Sigma F_x = 0$ and driving force is balanced by friction force, F_f

$F_f = \tau A$ where $\tau = \mu \frac{du}{dy}$

On top surface of tape, $\tau_t = \mu \frac{du}{dy} = \mu \frac{V_{c+t/2} - V_{t/2}}{(t/2+c) - t/2} = -\mu \frac{V}{c}$

negative τ on positive surface means F_f acts to left

On bottom surface of tape, $\tau_b = \mu \frac{du}{dy} = \mu \frac{V_{-(c+t/2)} - V_{-t/2}}{(-t/2-c) - (-t/2)} = \mu \frac{V}{c}$

positive τ on negative surface means F_f acts to left

Hence, $\Sigma F_x = 0 = F - F_{f_t} - F_{f_b}$

$F = F_{f_t} + F_{f_b} = |\tau_t A| + |\tau_b A|$

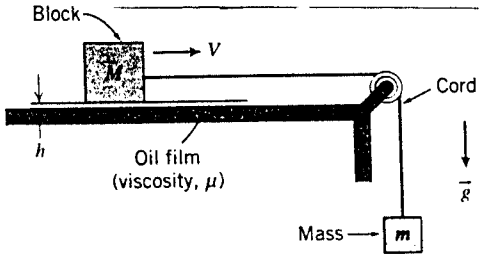
$F = \mu \frac{V}{c} A + \mu \frac{V}{c} A = 2\mu \frac{V}{c} A$

Solving for V,

$$V = \frac{Fc}{2\mu A} = 7.5 \text{ lbf} \times 0.012 \text{ in} \times \frac{1}{2} \times \frac{\text{ft·s}}{0.021 \text{ slug}} \times \frac{1}{(1.00 \text{ in})(0.75 \text{ in})} \times \frac{\text{ft·slug}}{\text{lbf·s}^2} \times \frac{12 \text{ in}}{1 \text{ ft}}$$

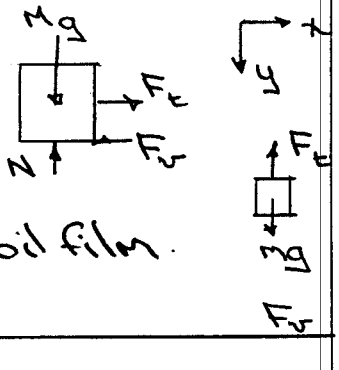
$$V = 34.3 \text{ ft/s} \longleftarrow V$$

Given: Block of mass M slides on thin film of oil of thickness h. Contact area of block is A. At time t=0, mass m is released from rest.



Find: (a) Expression for viscous force on block when moving at speed V
(b) Differential equation governing block speed as a function of time
(c) Expression for block speed V = V(t); sketch.

Solution:



Basic equations: $\tau_{yx} = \mu \frac{du}{dy}$ $\quad \Sigma \vec{F} = m\vec{a}$

Assumptions: (1) Newtonian fluid
(2) Linear velocity profile in oil film.

Then, $F_v = \tau A = \mu \frac{du}{dy} A = \mu \frac{\Delta u}{\Delta y} A = \mu \frac{V}{h} A$

For the block, $\Sigma F_x = F_t - F_v = M \frac{dV_b}{dt}$ (1)

For the falling mass $\Sigma F_y = mg - F_t = m \frac{dV_m}{dt}$, or

$$F_t = mg - m \frac{dV_m}{dt} \quad (2)$$

Since $V_b = V_m = V$, then substituting from Eq. (2) into (1) gives

$$mg - m\frac{dV}{dt} - F_v = M \frac{dV}{dt} = mg - m\frac{dV}{dt} - \mu \frac{V}{h} A$$

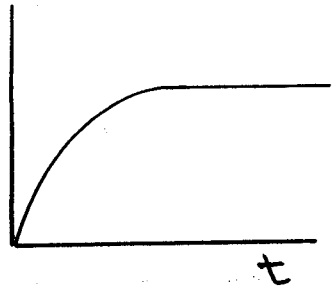
Finally,

$$mg - \mu \frac{V}{h} A = (M+m) \frac{dV}{dt}$$ Diff. Eq.

To solve we separate variables and integrate

$$t = \int_0^t dt = \int_0^V \frac{(M+m)}{mg - \mu \frac{V}{h} A} dV = -\frac{(M+m)}{\mu A / h} \ln\left(mg - \frac{\mu V A}{h}\right)\Big|_0^V$$

$$t = -\frac{(M+m)/h}{\mu A} \ln\left(1 - \frac{\mu V A}{mgh}\right)$$



Taking antilogarithms,

$$1 - \frac{\mu V A}{mgh} = e^{-\frac{\mu A t}{(M+m)h}}$$

Solving for V,

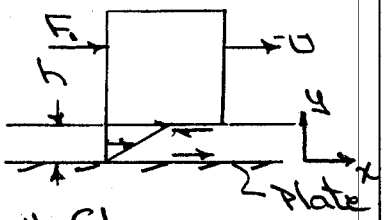
$$V = \frac{mgh}{\mu A}\left(1 - e^{-\frac{\mu A t}{(M+m)h}}\right)$$ V

The velocity increases exponentially to $V_{max} = \frac{mgh}{\mu A}$

Given: Block of mass M moves at steady speed U under influence of constant force F_1 on a thin film of oil of thickness h and viscosity μ; block is square, a mm on a side.

Find: (a) Magnitude and direction of shear stress acting on bottom of block and supporting plate.
(b) Expression for time required to lose 95% of its initial speed when force is suddenly removed.
(c) Expect shape of speed vs time curve.

Solution:



Basic equations: $\tau_{yx} = \mu \frac{du}{dy}$ $\quad \Sigma \vec{F} = m\vec{a}$

Assumptions: (1) Newtonian fluid
(2) Linear velocity profile in oil film

$$\tau_{yx} = \mu \frac{du}{dy} = \mu \frac{\Delta u}{\Delta y} = \mu \frac{U}{h} \longleftarrow \tau_{yx}$$

Bottom of block is $-y$ surface, so τ_{yx} acts to left
Plate surface is $+y$ surface, so τ_{yx} acts to right

Viscous shear force on block is $F_v = \tau A = \tau a^2 = \frac{\mu U a^2}{h}$

When F_1 is removed, block slows under action of F_v

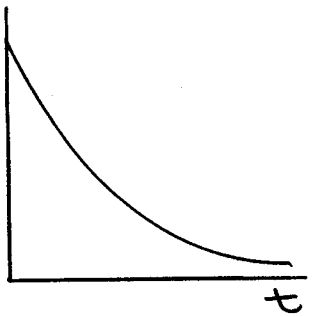
$$\Sigma F_x = m \frac{dU}{dt} = -F_v = -\frac{\mu U a^2}{h}$$

Separating variables and integrating we have

$$\int_{U_i}^{U} \frac{dU}{U} = -\int_0^t \frac{\mu a^2}{h}\, dt$$

Then $$\ln \frac{U}{U_i} = -\frac{\mu a^2}{h} t \quad \text{---- (1)}$$

and $$t = -\frac{h}{\mu a^2} \ln \frac{U}{U_i}$$



For $U - U_i = 0.05$

$$t = 3.0 \frac{h}{\mu a^2} \longleftarrow t$$

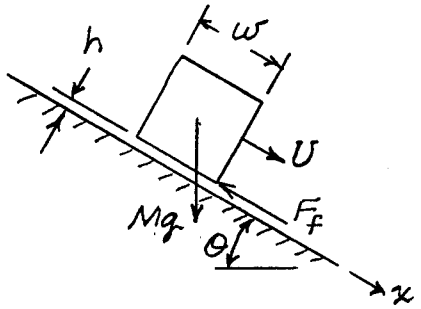
From Eq. (1) we can write

$$U = U_1 e^{-\frac{\mu a^2 t}{h}}$$

The speed thus decreases exponentially with time.

Given: Block sliding on oil film on inclined plane (SAE 30 @ 20°C).

Find: (a) Initial acceleration
(b) Expression for speed vs. time
(c) Sketch V(t).
(d) Speed when sliding without oil > or < terminal speed, why?



$M = 2\ kg,\ w = 0.2\ m,\ h = 0.02\ mm,\ \theta = 30°$

Solution: Apply Newton's second law, viscous stress.

Basic equations: $\Sigma F_x = M\frac{dU}{dt}$ $\tau = \mu\frac{dV}{dy}$ $A = w^2 = 0.04\ m^2$

Assumptions: (1) Newtonian fluid (from Fig. A.2, $\mu = 0.4\ \frac{N\cdot s}{m^2}$ at 20°C)
(2) Linear velocity profile in oil $\frac{dV}{dy} = \frac{U}{h}$

Then
$$\Sigma F_x = Mg\sin\theta - F_f = Mg\sin\theta - \mu\frac{U}{h}A = M\frac{dU}{dt}$$

so $\frac{dU}{dt} = g\sin\theta - \frac{\mu A}{Mh}U$; at $U=0$, $\frac{dU}{dt} = g\sin\theta = 4.91\ m/s^2$ ← $\frac{dU}{dt}$

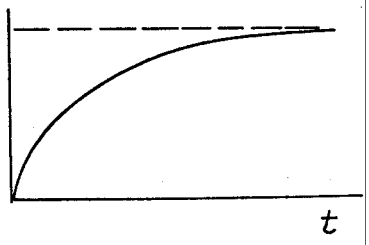
Separating variables $\frac{dU}{g\sin\theta - \frac{\mu A}{Mh}U} = dt$

Letting $a = g\sin\theta$ and $b = \frac{\mu A}{Mh}$ and integrating gives

$$\int_0^U \frac{dU}{a-bU} = -\frac{1}{b}\ln(a-bU)\Big]_0^U = t \quad \text{or} \quad \ln\left(\frac{a-bU}{a}\right) = \ln\left(1-\frac{bU}{a}\right) = -bt$$

Thus $1-\frac{bU}{a} = e^{-bt}$ and $U = \frac{a}{b}(1-e^{-bt}) = \frac{g\sin\theta\, Mh}{\mu A}\left(1-e^{-\frac{\mu A}{Mh}t}\right)$ ← $U(t)$

Sketching:



Evaluating: $a = g\sin\theta = 4.91\ m/s^2$

$$b = \frac{\mu A}{Mh} = 0.4\ \frac{N\cdot s}{m^2} \times (0.2)^2 m^2 \times \frac{1}{2\ kg} \times \frac{1}{0.02\times10^{-3}\ m} \times \frac{kg\cdot m}{N\cdot s^2}$$

$b = 400\ s^{-1}$

Thus $U = 4.91\ \frac{m}{s^2} \times \frac{s}{400}\left(1-e^{-400t}\right) = 0.0123\left(1-e^{-400t}\right)\ m/s$

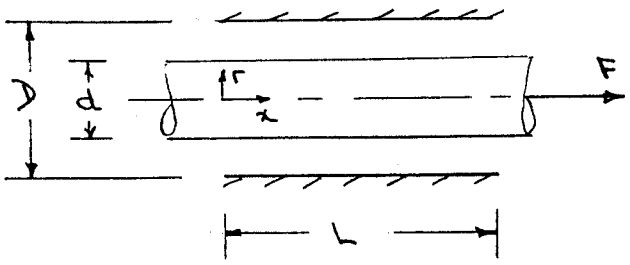
and terminal speed is $U_t = 0.0123\ m/s$

For Coulomb friction, $F_f = \mu_k N = \mu_k Mg\cos\theta$, so $Mg\sin\theta - \mu_k Mg\cos\theta = M\frac{dU}{dt}$

so $\frac{dU}{dt} = g(\sin\theta - \mu_k\cos\theta)$

When $\sin\theta > \mu_k\cos\theta$, speed increases continuously. ← μ_k
When $\sin\theta \le \mu_s\cos\theta$, block does not move.

Given: Wire, of diameter d, is to be coated with varnish by drawing it through a circular die of diameter, D, and length, L.



$d = 0.8$ mm, $D = 0.9$ mm, $L = 20$ mm

Varnish, $\mu = 20$ centipoise fills the space between wire and die
Wire is drawn through at speed, $V = 50$ m/s

Find: Force required to pull the wire

Solution:

$$\Sigma F_x = ma_x$$

Since V_{wire} = constant, applied force must be sufficient to balance friction force, F_f

$$F_f = \tau A \quad \text{where } \tau = \mu \frac{du}{dr} \quad \text{and } A = \pi d L$$

Assuming a linear velocity distribution in varnish

$$\tau_s = \mu \left(\frac{du}{dr}\right)_s = \mu \frac{V_{D/2} - V_{d/2}}{D/2 - d/2} = -\mu \frac{V}{(D-d)/2}$$

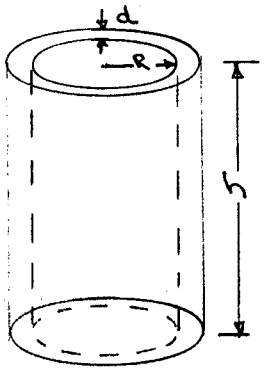
(negative stress on positive r surface must act in negative x direction)

$$F - F_f = 0$$

$$F = \tau A = \mu \frac{2V}{(D-d)} \times \pi d L$$

$$F = 20\ \text{cp} \times \frac{\text{gm}}{100\ \text{cm}\cdot\text{s}\cdot\text{cp}} \times 2\pi \times 50\ \frac{\text{m}}{\text{s}} \times 0.8\text{mm} \times 20\text{mm} \times \frac{1}{0.1\text{mm}} \times \frac{1\ \text{cm}}{10\text{mm}} \times \frac{\text{kg}}{1000\text{gm}} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$F = 1.01\ \text{N} \longleftarrow F$$



Given: Concentric cylinder viscometer
$R = 1.5$ in. $d = 0.001$ in $h = 6$ in.
Inner cylinder rotates at 250 rpm.
Gap filled with castor oil at 90F.

Determine: Torque required to rotate the inner cylinder

Solution:

The required torque must balance the resisting torque of the shear force.

The shear force is given by $F = \tau A$ where $A = 2\pi Rh$

For a Newtonian fluid $\tau = \mu \frac{du}{dy}$

For small gap (linear profile) $\tau = \mu \frac{V}{d}$

where V = tangential velocity of inner cylinder $= R\omega$

Hence $$F = \tau A = \mu \frac{R\omega}{d} 2\pi Rh = \frac{2\pi \mu R^2 \omega h}{d}$$

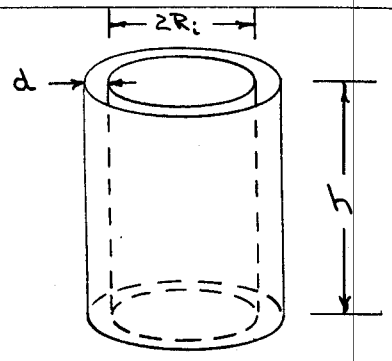
and the torque $$T = RF = \frac{2\pi \mu R^3 \omega h}{d}$$

From Fig A.2, for castor oil at 90F (32°C), $\mu = 3.80 \times 10^{-1}$ N·s/m²

Substituting numerical values.

$$T = \frac{2\pi \mu R^3 \omega h}{d} = 2\pi \times 3.80\times10^{-1} \frac{N\cdot s}{m^2} \times 2.09\times10^{-2} \frac{lbf\cdot s\cdot m^2}{ft^2\cdot N\cdot s} \times (1.5)^3 in^3 \times 250 \frac{rev}{min} \times 6 in \times \frac{1}{10^{-3} in} \times 2\pi \frac{rad}{rev} \times \frac{min}{60 s} \times \frac{ft^3}{1728 in^3}$$

$T = 15.3$ ft·lbf



Given: Concentric cylinder viscometer

$R_i = 37.5$ mm, $d = 0.02$ mm, $h = 150$ mm

Inner cylinder rotates at $\omega = 100$ rpm, under torque, $T = 0.021$ N.m

Find: Viscosity of liquid in clearance gap.

Solution

The imposed torque must balance the resisting torque of the shear force.

The shear force is given by $F = \tau A$ where $A = 2\pi R_i h$

For a Newtonian fluid $\tau = \mu \frac{du}{dy}$

Since the velocity profile is assumed to be linear, $\tau = \mu \frac{V}{d}$

where V is the tangential velocity of the inner cylinder, $V = R_i \omega$

Thus,

$$F = \tau A = \mu \frac{V}{d} 2\pi R_i h = \frac{2\pi \mu R_i^2 \omega h}{d}$$

and the torque $T = R_i F = \frac{2\pi \mu R_i^3 \omega h}{d}$

Solving for μ,

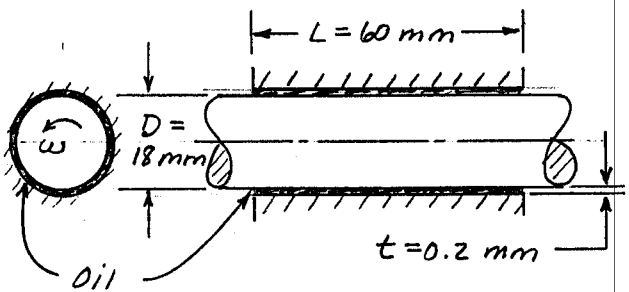
$$\mu = \frac{T d}{2\pi R_i^3 \omega h} = 0.021 \text{ N.m} \times 0.02 \text{ mm} \times \frac{1}{2\pi} \times \frac{1}{(37.5)^3} \text{ mm}^3 \times \frac{\text{min}}{100 \text{ rev}} \times \frac{1}{150 \text{ mm}}$$

$$\times \frac{\text{rev}}{2\pi \text{ rad}} \times \frac{60 \text{ s}}{\text{min}} \times (1000)^3 \frac{\text{mm}^3}{\text{m}^3}$$

$$\mu = 8.07 \times 10^{-4} \text{ N.s/m}^2$$

Given: Shaft turning inside stationary journal as shown, N = 20 rps.

Torque, $T = 0.0036\ N \cdot m$



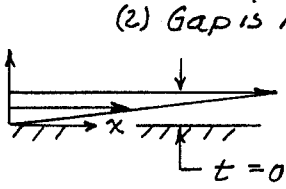
Find: Estimate viscosity of oil.

Solution: Basic equation $\tau_{yx} = \mu \frac{du}{dy}$

Assumptions: (1) Newtonian fluid

(2) Gap is narrow, so velocity profile is linear, $\frac{du}{dy} \approx \frac{\Delta u}{\Delta y}$

Then



Shear stress is

$$\tau_{yx} \approx \mu \frac{\Delta u}{\Delta y} = \mu \frac{U}{t} = \frac{\mu \omega D}{2t}$$

Neglecting end effects, torque is

$$T = FR = \tau_{yx} A R = \tau_{yx} (\pi D L) \frac{D}{2} = \frac{\mu \pi \omega D^3 L}{4t}$$

Solving for viscosity

$$\mu = \frac{4tT}{\pi \omega D^3 L}$$

$$= \frac{4}{\pi} \times 0.2\ mm \times 0.0036\ N \cdot m \times \frac{s}{20\ rev} \times \frac{1}{(18)^3\ mm^3} \times \frac{1}{60\ mm} \times \frac{rev}{2\pi\ rad} \times (1000)^3 \frac{mm^3}{m^3}$$

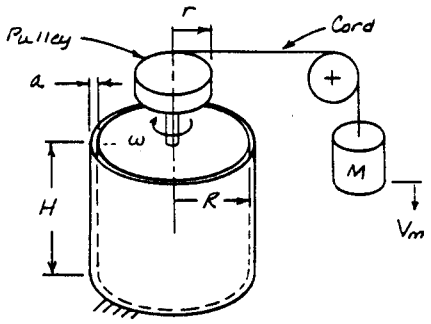
$$\mu = 0.0208\ N \cdot s / m^2$$

μ

{ From Fig. A.2, this oil appears somewhat less viscous than SAE 10W, assuming the oil is at room temperature. }

Given: Concentric-cylinder viscometer, driven by falling mass.

$M = 0.10$ kg $r = 25$ mm
$R = 50$ mm $a = 0.25$ mm
$H = 50$ mm $V_m = 40$ mm/s



After starting transient, V_m = const.

Find: (a) An algebraic expression for viscosity of the liquid, in terms of M, g, V_m, r, R, a, and H.

(b) Evaluate using the data given.

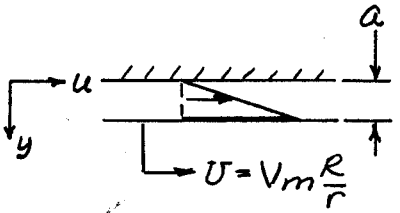
Solution: Apply Newton's law of viscosity.

Basic equations: $\tau = \mu \frac{du}{dy}$ $\Sigma M = 0$ $T = \tau A R$

Assumptions: (1) Newtonian liquid
(2) Narrow gap, so linear velocity profile
(3) Steady angular speed

Summing torques on the rotor

$$\Sigma M = Mgr - \tau A R = I\cancel{\alpha}^{=0(3)} = 0 \quad ; \quad A = 2\pi RH$$



Because $a \ll R$, treat the gap as plane. Then

$$\tau = \mu \frac{du}{dy} \approx \mu \frac{\Delta u}{\Delta y} = \mu \frac{U-0}{a-0} = \mu \frac{U}{a} = \frac{\mu V_m R}{ar}$$

Substituting,

$$Mgr - \frac{\mu V_m R}{ar} 2\pi RH\, R = Mgr - \frac{2\pi \mu V_m R^3 H}{ar} = 0$$

So

$$\mu = \frac{Mgr^2 a}{2\pi V_m R^3 H} \qquad \mu$$

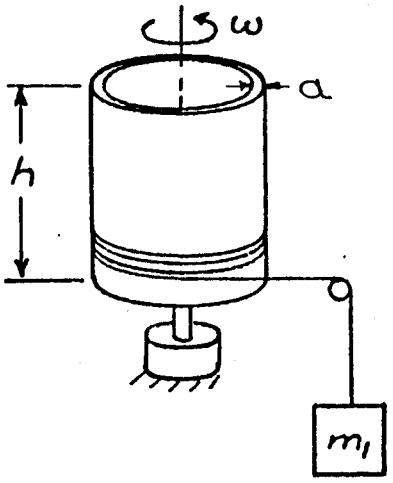
Evaluating for the given data

$$\mu = \frac{1}{2\pi} \times 0.10\ \text{kg} \times 9.81 \frac{\text{m}}{\text{s}^2} \times (0.025)^2 \text{m}^2 \times 0.0002\ \text{m} \times \frac{\text{s}}{0.040\ \text{m}}$$
$$\times \frac{1}{(0.050)^3 \text{m}^3} \times \frac{1}{0.050\ \text{m}} \times \frac{\text{N s}^2}{\text{kg} \cdot \text{m}}$$

$$\mu = 0.0781\ \text{N} \cdot \text{s} / \text{m}^2 \quad (78.1\ \text{mPa} \cdot \text{s}) \qquad \mu$$

Given: Thin outer cylinder (mass m_2, and radius R) of a concentric-cylinder viscometer is driven by the falling mass, m_1. Clearance between outer cylinder and stationary inner cylinder is a. Bearing friction, air resistance and mass of liquid in the viscometer may be neglected

Find: (a) algebraic expression for the torque due to viscous shear acting on cylinder at angular speed ω.
(b) differential equation and solution for $\omega(t)$
(c) expression for ω_{max}



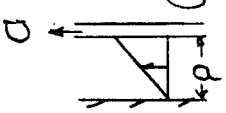
Solution:

Basic equations: $\tau = \mu \frac{du}{dy}$

$\Sigma F = ma$, $\Sigma M = I\alpha$

Assume: (1) Newtonian fluid
(2) linear velocity profile

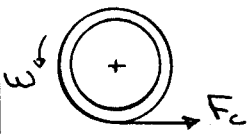
In the gap, $\tau = \mu \frac{du}{dy} = \mu \frac{U}{a} = \frac{\mu R\omega}{a}$



$T = \tau A R = \frac{\mu R\omega}{a}(2\pi Rh)R$

$T = \frac{2\pi R^3 \mu h}{a}\omega$ ← T

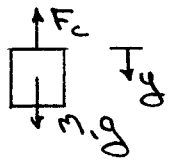
During acceleration, let the tension in the cord be F_c



For the cylinder $\Sigma M = F_c R - T = I\alpha = m_2 R^2 \frac{d\omega}{dt}$ ... (1)

For the mass $\Sigma F_y = m_1 g - F_c = ma = m_1 \frac{dV}{dt} = m_1 R \frac{d\omega}{dt}$ ... (2)

$\therefore F_c = m_1 g - m_1 R \frac{d\omega}{dt}$



Substituting into eq. (1)

$m_1 gR - \frac{2\pi R^3 \mu h}{a}\omega = (m_1 + m_2)R^2 \frac{d\omega}{dt}$.

Let $m_1 gR = b$, $-2\pi R^3 \mu h/a = c$, $(m_1 + m_2)R^2 = f$.

Then, $b + c\omega = f\frac{d\omega}{dt}$ or $\int_0^t \frac{1}{f}\,dt = \int_0^\omega \frac{d\omega}{(b+c\omega)}$

Integrating, $\frac{1}{f}t = \frac{1}{c}\ln(b+c\omega)\Big]_0^\omega = \frac{1}{c}\ln\frac{(b+c\omega)}{b} = \frac{1}{c}\ln\left(1+\frac{c}{b}\omega\right)$.

$\frac{c}{f}t = \ln\left(1+\frac{c}{b}\omega\right) \Rightarrow e^{\frac{c}{f}t} = \left(1+\frac{c}{b}\omega\right) \Rightarrow \omega = \frac{b}{c}\left(e^{\frac{c}{f}t} - 1\right)$

Substituting for b, c, and f

$\omega = \frac{m_1 gRa}{2\pi R^3 \mu h}\left(1 - e^{\frac{-2\pi R^3 \mu h}{a(m_1+m_2)R^2}t}\right) = \frac{m_1 ga}{2\pi R^2 \mu h}\left[1 - e^{-\frac{2\pi R\mu h}{a(m_1+m_2)}t}\right]$ ← ω(t)

Maximum ω occurs at $t \to \infty$

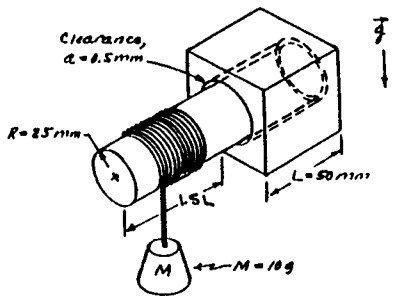
$\omega_{max} = \frac{m_1 ga}{2\pi R^2 \mu h}$ ← ω_{max}

Given: Circular aluminum shaft in journal.

Symmetric clearance gap

filled with SAE 10W-30 at 30°C.

Shaft turned by mass and cord.

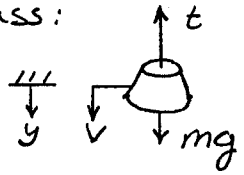


Find: (a) Develop and solve a differential equation for angular speed as a function of time.
(b) Calculate maximum angular speed.
(c) Estimate time needed to reach 95 percent of maximum speed.

Solution: Apply summation of torques and Newton's second laws.

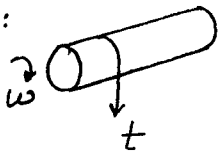
Basic equations: $\Sigma T = I\frac{d\omega}{dt}$ $\Sigma F = m\frac{dV}{dt}$ $V = R\omega$

For the mass:



$$\Sigma F_y = mg - t = m\frac{dV}{dt} = mR\frac{d\omega}{dt} \quad (1)$$

For the shaft:



$$\Sigma T = tR - T_{viscous} = I\frac{d\omega}{dt} \quad (2)$$

$$T_{viscous} = \tau RA = \mu\frac{V}{a}R\,2\pi RL = \frac{2\pi\mu\omega R^3 L}{a}$$

Assume: (1) Newtonian liquid, (2) small gap, (3) Linear profile

Then Eq. 2 becomes $$tR - \frac{2\pi\mu R^3 L}{a}\omega = I\frac{d\omega}{dt} \quad ; \quad I = \frac{1}{2}MR^2 \quad (3)$$

Multiplying Eq. 1 by R and combining with Eq. 3 gives

$$mgR - mR^2\frac{d\omega}{dt} - \frac{2\pi\mu R^3 L}{a}\omega = I\frac{d\omega}{dt} \quad \text{or} \quad mgR - \frac{2\pi\mu R^3 L}{a}\omega = (I+mR^2)\frac{d\omega}{dt} \quad (4)$$

This may be written $A - B\omega = C\frac{d\omega}{dt}$ where $A = mgR$, $B = \frac{2\pi\mu R^3 L}{a}$, $C = I + mR^2$

Separating variables $$\frac{d\omega}{A-B\omega} = \frac{dt}{C}$$

Integrating $$\int_0^\omega \frac{d\omega}{A-B\omega} = -\frac{1}{B}\ln(A-B\omega)\Big]_0^\omega = -\frac{1}{B}\ln\left(1-\frac{B\omega}{A}\right) = \int_0^t \frac{dt}{C} = \frac{t}{C}$$

Simplifying $$1 - \frac{B\omega}{A} = e^{-Bt/C} \quad \text{or} \quad \omega = \frac{A}{B}\left[1 - e^{-Bt/C}\right] \quad (5) \quad \omega(t)$$

The maximum angular speed ($t \to \infty$) is $\omega = A/B$.

$$A = mgR = 0.010\ kg \times 9.81\ \frac{m}{s^2} \times 0.025\ m \times \frac{N\cdot s^2}{kg\cdot m} = 2.45\times10^{-3}\ N\cdot m$$

$$B = \frac{2\pi\mu R^3 L}{a} = 2\pi \times 0.095\ \frac{kg}{m\cdot s} \times (0.025)^3\ m^3 \times 0.050\ m \times \frac{1}{0.0005\ m} \times \frac{N\cdot s^2}{kg\cdot m} = 9.33\times10^{-4}\ N\cdot m\cdot s$$

Evaluating, $\omega_{max} = \frac{A}{B} = 2.45\times10^{-3}\,N\cdot m \times \frac{1}{9.33\times10^{-4}\,N\cdot m\cdot sec} = 2.63\ rad/s$.

Thus

$$\omega_{max} = 2.63\,\frac{rad}{s}\times\frac{rev}{2\pi\,rad}\times\frac{60\,s}{min} = 25.1\ rpm \qquad \omega_{max}$$

From Eq. 5, $\omega = 0.95\,\omega_{max}$ when $e^{-Bt/C} = 0.05$, or $Bt/C \simeq 3$; $t \simeq \frac{3C}{B}$

$$C = I + mR^2 = \tfrac{1}{2}MR^2 + mR^2 = (\tfrac{1}{2}M + m)R^2$$

$$M = \pi R^2(1.5L + L)\rho = 2.5\pi R^2 L\, SG\, \rho_w$$

$$M = 2.5\pi \times (0.025)^2 m^2 \times 0.050\,m \times (2.64)1000\,\frac{kg}{m^3} = 0.648\ kg$$

$$C = (\tfrac{1}{2}\times 0.648\ kg + 0.010\ kg)(0.025)^2 m^2 = 2.09\times10^{-4}\ kg\cdot m^2$$

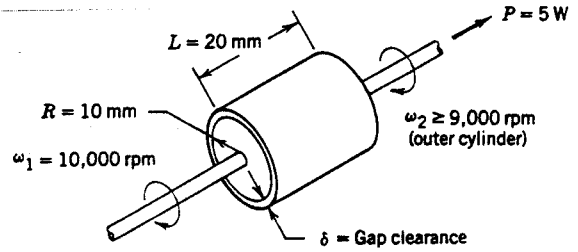
Thus

$$t_{95} = 3\times 2.09\times10^{-4}\,kg\cdot m^2 \times \frac{1}{9.33\times10^{-4}\,N\cdot m\cdot s}\times\frac{N\cdot s^2}{kg\cdot m} = 0.671\ s. \qquad t_{95}$$

{The terminal speed could have been computed from Eq. 4 by setting $d\omega/dt \to 0$, without solving the differential equation.}

Given: Coupling, fabricated of concentric cylinders as shown, must transmit power $\mathcal{P} = 5$ W. Minimum clearance gap, $\delta = 0.5$ mm is to be filled with fluid of viscosity μ. Other dimensions and properties are as indicated.

Find: viscosity of fluid.



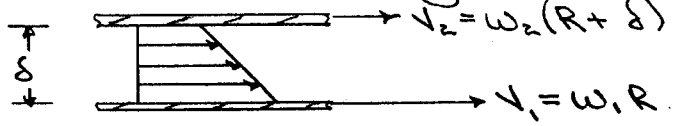
Solution:

Basic equations: $\tau_{r\theta} = \mu \dfrac{du}{dr}$

- shear force, $F = \tau A$
- torque, $T = FR$
- power, $\mathcal{P} = T\omega$

Assumptions:
(1) Newtonian fluid
(2) linear velocity profile in the gap.

Model flow in the gap



$$\tau_{r\theta} = \mu \frac{du}{dr} = \mu \frac{\Delta V}{\Delta r} = \mu \frac{[\omega_1 R - \omega_2 (R+\delta)]}{\delta}$$

$$\tau_{r\theta} \approx \mu \frac{(\omega_1 - \omega_2)R}{\delta} \quad \{\delta \ll R\}$$

For the output

$$\mathcal{P} = T\omega_2 = \omega_2 F R = \omega_2 \tau A_2 R = \omega_2 \frac{\mu(\omega_1 - \omega_2)R}{\delta} \times 2\pi R L \times R$$

$$\mathcal{P} = \frac{2\pi \mu \omega_2 (\omega_1 - \omega_2) R^3 L}{\delta}$$

Solving for the viscosity,

$$\mu = \frac{\mathcal{P}\delta}{2\pi \omega_2 (\omega_1 - \omega_2) R^3 L}$$

$$= \frac{5 \text{ W} \times 5\times10^{-4} \text{ m}}{2\pi} \times 9000 \frac{\text{rev}}{\text{min}} \times 1000 \frac{\text{rev}}{\text{min}} \times \frac{1}{(0.01)^3 \text{ m}^3} \times \frac{1}{0.02 \text{ m}}$$

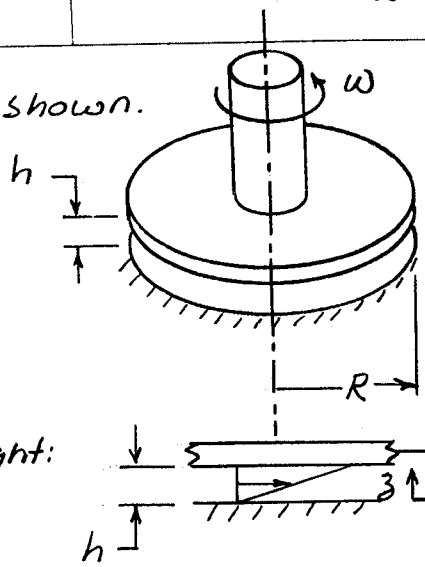
$$\times \frac{\text{N.m}}{\text{s.W}} \times (2\pi)^2 \frac{\text{rad}^2}{\text{rev}^2} \times \frac{3600 \text{ s}^2}{\text{min}^2}$$

$$\mu = 0.202 \text{ N.s/m}^2 \quad \leftarrow \mu$$

{This viscosity corresponds to SAE 30 oil at 30°C}

Problem 2.45

Given: Parallel-disk apparatus as shown.



Find: (a) Algebraic expression for shear stress at any radial location.
(b) Expression for the torque needed to turn the upper disk.

Solution: Use r, θ, z coordinates at right:

Basic equations: $\tau_{z\theta} = \mu \frac{dV_\theta}{dz}$

$$dT = r\,dF = r\,\tau_{z\theta}\,dA$$

Assumptions: (1) Newtonian fluid
(2) No-slip condition
(3) Linear velocity profile (in narrow gap)

The velocity at any radial location on the rotating disk is $V_\theta = \omega r$.

Since the velocity profile is linear, then

$$\tau_{z\theta} = \mu \frac{dV_\theta}{dz} = \mu \frac{\Delta V}{\Delta z} = \mu \frac{(\omega r - 0)}{(h - 0)} = \frac{\mu \omega r}{h}$$ ← $\tau_{z\theta}$

and

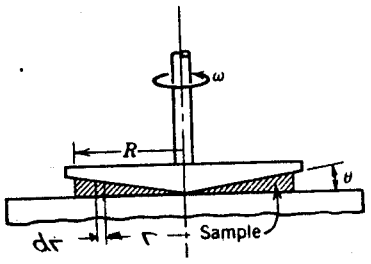
$$dT = r\,\tau_{z\theta}\,dA = r\,\mu \frac{\omega r}{h}\,2\pi r\,dr = \frac{2\pi\mu\omega r^3}{h}\,dr$$

Integrating

$$T = \int_A dT = \int_0^R \frac{2\pi\mu\omega r^3}{h}\,dr = \frac{\pi\mu\omega r^4}{2h}\Big]_0^R$$

$$T = \frac{\pi\mu\omega R^4}{2h}$$ ← T

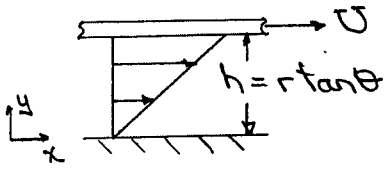
Given: Cone and plate viscometer shown.
Apex of cone just touches the plate,
θ is very small



Find: (a) Derive an expression for the shear rate in the liquid that fills the gap
(b) Evaluate the torque on the driven cone in terms of the shear stress and geometry of the system.

Solution:

Since the angle θ is very small, the average gap width is also very small.
It is reasonable to assume a linear velocity profile across the gap and to neglect end effects.



The shear (deformation) rate is given by

$$\dot{\gamma} = \frac{du}{dy} = \frac{\Delta u}{\Delta y}$$

At any radius, r,

the velocity $U = \omega r$ and

the gap width $h = r \tan\theta$

$$\therefore \dot{\gamma} = \frac{\omega r}{r \tan\theta} = \frac{\omega}{\tan\theta}$$

Since θ is very small, $\tan\theta \approx \theta$ and

$$\dot{\gamma} = \frac{\omega}{\theta} \quad \dot{\gamma}$$

Note: The shear rate is independent of r. The entire sample is subjected to the same shear rate.

The torque on the driven cone is given by

$$T = \int r\,dF \quad \text{where} \quad dF = \tau_{yx}\,dA$$

Since $\dot{\gamma}$ is a constant (for a given ω) then τ_{yx} = constant

and

$$T = \int r\,dF = \int_A r\,\tau_{yx}\,dA = \tau_{yx}\int_0^R r\,2\pi r\,dr$$

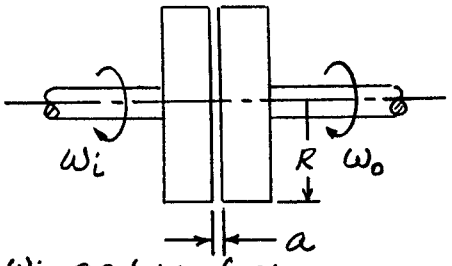
$$T = \frac{2}{3}\pi R^3 \tau_{yx} \quad T$$

Given: Viscous clutch made from pair of closely spaced disks.

Input speed, ω_i

Output speed, ω_o

Viscous oil in gap, μ



Find algebraic expressions in terms of μ, R, a, ω_i, and ω_o for:
(a) Torque transmitted, T
(b) Power transmitted
(c) Slip ratio, $s = \Delta\omega/\omega_i$, in terms of T
(d) Efficiency, η, in terms of s, ω_i, and T

Solution: Apply Newton's law of viscosity

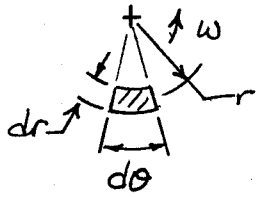
Basic equations: $\tau = \mu \frac{du}{dy}$ $\quad dF = \tau dA$ $\quad dT = r\,dF$

Assumptions: (1) Newtonian liquid
(2) Narrow gap so velocity profile is linear

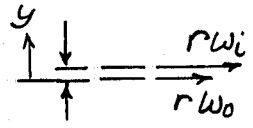
Consider a segment of plates:

$$\tau = \mu \frac{du}{dy} = \mu \frac{\Delta u}{\Delta y} = \mu \frac{r(\omega_i - \omega_o)}{a}$$

$dA = r\,dr\,d\theta$



End View



Bottom View

$$dF = \tau dA = \frac{\mu r \Delta\omega}{a} r\,dr\,d\theta = \frac{\mu \Delta\omega}{a} r^2\,dr\,d\theta \;;\; dT = r\,dF = \frac{\mu \Delta\omega}{a} r^3\,dr\,d\theta$$

Integrating

$$T = \int_0^{2\pi}\int_0^{R} dT = \frac{\mu \Delta\omega}{a}\int_0^{2\pi}\int_0^{R} r^3\,dr\,d\theta = \frac{2\pi\mu \Delta\omega}{a}\int_0^{R} r^3\,dr = \frac{\pi\mu \Delta\omega R^4}{2a} \qquad T$$

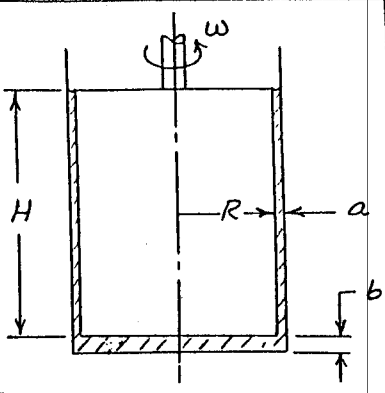
$$P_o = T\omega_o = \frac{\pi\mu\omega_o \Delta\omega R^4}{2a} \quad \text{(power transmitted)} \qquad P$$

$$s = \frac{\Delta\omega}{\omega_i} = \frac{2aT}{\pi\mu R^4 \omega_i} \qquad s$$

Efficiency is $\eta = \frac{\text{Power out}}{\text{Power in}} = \frac{T\omega_o}{T\omega_i} = \frac{\omega_o}{\omega_i}$. But $\omega_o = \omega_i - \Delta\omega$, so

$$\eta = \frac{\omega_i - \Delta\omega}{\omega_i} = 1 - \frac{\Delta\omega}{\omega_i} = 1 - s \qquad \eta$$

Given: Concentric-cylinder viscometer shown. When inner cylinder rotates at angular speed ω viscous retarding torque arises around circumference of inner cylinder and on cylinder bottom.

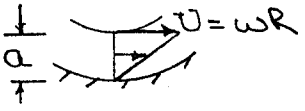


Find: (a) expression for viscous torque due to gap of width a
(b) expression for viscous torque on bottom due to gap of width b
(c) For $T_{bottom}/T_{annulus} \le 0.01$, plot b/a vs geometric variables.
(d) What are design implications?
(e) What design modifications can you recommend?

Solution: Basic equation $\tau_{yx} = \mu \frac{du}{dy}$

Assumptions: (1) linear velocity profile (2) Newtonian liquid

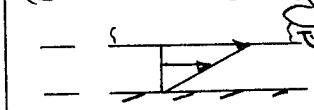
(a) in annular gap



$$\tau = \mu \frac{du}{dr} = \mu \frac{\Delta u}{\Delta r} = \mu \frac{U}{a} = \mu \frac{\omega R}{a}$$

$$\text{Torque} = RF_t = R\tau A = R\mu \frac{\omega R}{a}(2\pi R H) = \frac{2\pi\mu\omega R^3 H}{a} \quad \text{(a)}$$

(b) in bottom gap



$$\tau = \mu \frac{du}{dz} = \mu \frac{\Delta u}{\Delta z} = \mu \frac{U}{b} = \mu \frac{\omega r}{b} \quad \text{(varies with } r\text{)}$$

$$\text{Torque} = \int dT = \int r\,dF = \int r\tau\,dA = \int_0^R r\,\mu\frac{\omega r}{b}\,2\pi r\,dr$$

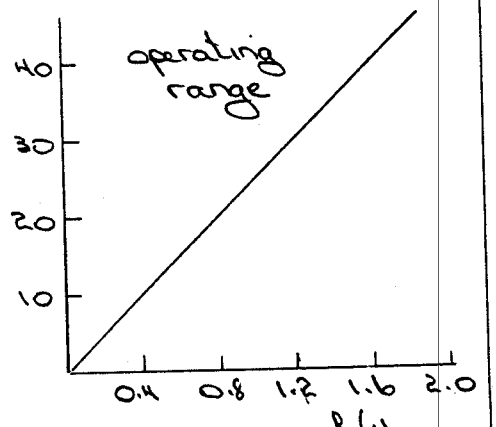
$$\text{Torque} = \frac{2\pi\mu\omega}{b}\int_0^R r^3\,dr = \frac{2\pi\mu\omega}{b}\left[\frac{r^4}{4}\right]_0^R = \frac{\pi\mu\omega}{2b}R^4 \quad \text{(b)}$$

(c) For $T_{bottom}/T_{annulus} \le \frac{1}{100}$, then:

$$\frac{T_{bot}}{T_{an}} = \frac{\pi\mu\omega}{2b}R^4 \times \frac{a}{2\pi\mu\omega R^3 H} \le \frac{1}{100}$$

$$\frac{aR}{4bH} \le \frac{1}{100}$$

or $$\frac{b}{a} \ge 25\frac{R}{H}$$



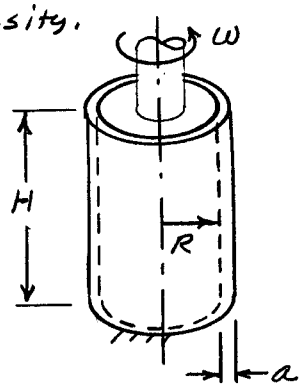
(d) The plot shows the operating range. Specific design would depend on other constraints.

For $a = 1$ mm with $R/H = 1/2$ gives $b = 12.5$ mm

(e) For a given value of R/H, the dimension b could be effectively increased by "hollowing out" the inner cylinder as shown by the dashed lines in the diagram above.

Given: Concentric-cylinder viscometer, liquid similar to water.
Goal is to obtain ±1 percent accuracy in viscosity value.

Specify: Configuration and dimensions to achieve ±1% measurement.
Parameter to be measured to compute viscosity.

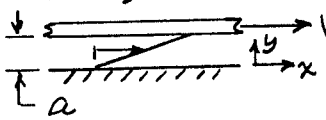


Solution: Apply definition of Newtonian fluid

Computing equation: $\tau = \mu \frac{du}{dy}$

Assumptions: (1) Steady
(2) Newtonian liquid
(3) Narrow gap, so "unroll" it
(4) Linear velocity profile in gap
(5) Neglect end effects

Flow model:



$$u = V\frac{y}{a} = \omega R\frac{y}{a}\ ;\ \frac{du}{dy} = \frac{\omega R}{a}$$

Thus $\tau = \mu\frac{du}{dy} = \mu\frac{\omega R}{a}$ and torque on rotor is $T = R\tau A$, where $A = 2\pi RH$

Consequently $T = R\mu\frac{\omega R}{a}2\pi RH = \frac{2\pi\mu\omega R^3 H}{a}$, or

$$\mu = \frac{Ta}{2\pi\omega R^3 H} \qquad \mu$$

From this equation the uncertainty in μ is (see Appendix F),

$$u_\mu = \pm\left[u_T^2 + u_a^2 + u_\omega^2 + (3u_R)^2 + u_H^2\right]^{\frac{1}{2}} = \pm\left[13\,u^2\right]^{\frac{1}{2}} = \pm 3.61\,u$$

if the uncertainty of each parameter equals u. Thus

$$u = \pm\frac{u_\mu}{3.61} = \pm\frac{1\text{ percent}}{3.61} = \pm 0.277\text{ percent} \qquad u$$

Typical dimensions for a bench-top unit might be

$H = 200$ mm, $R = 75$ mm, $a = 0.02$ mm, and $\omega = 10.5$ rad/s (100 rpm)

From Appendix A, Table A.8, water has $\mu = 1.00\times10^{-3}$ N·s/m² at T = 20°C.

The corresponding torque would be

$$T = 2\pi \times 1.00\times10^{-3}\frac{N\cdot s}{m^2}\times\frac{10.5}{s}\times(0.075)^3 m^3 \times 0.2\,m \times\frac{1}{0.00002\,m} = 0.278\ N\cdot m \qquad T$$

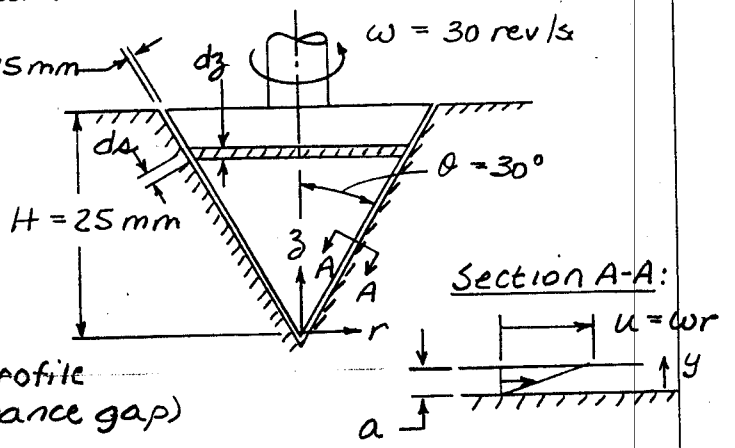
It should be possible to measure this torque quite accurately.

{Many details would need to be considered (e.g. bearings, temperature rise, etc.) to produce a workable device.}

Given: Conical pointed shaft turning in conical bearing.

Lubricant is heavy oil with viscosity of SAE 30 at 30°C.

Find: (a) Algebraic expression for shear stress at height, z.
(b) Torque that acts on shaft.



Solution: Basic equations:

$$\tau = \mu \frac{du}{dy}$$

$$dT = r\tau\, dA$$

Assumptions: (1) Newtonian fluid
(2) No-slip condition
(3) Linear velocity profile (in narrow clearance gap)

Along the conical surface, $\tan\theta = \frac{r}{z}$, so $r = z\tan\theta$

Then $\tau = \mu \frac{du}{dy} = \mu \frac{\Delta u}{\Delta y} = \mu \frac{(\omega r - 0)}{(a - 0)} = \frac{\mu\omega z \tan\theta}{a}$ τ

Consider the cross-hatched element of area: $dz = ds\cos\theta$

$$dA = 2\pi r\, ds = 2\pi r \frac{dz}{\cos\theta}$$

The viscous torque on the element of area is:

$$dT = r\tau\, dA = r \frac{\mu\omega z \tan\theta}{a} 2\pi r \frac{dz}{\cos\theta} \; ; \; r = z\tan\theta$$

$$dT = \frac{2\pi\mu\omega z^3 \tan^3\theta}{a\cos\theta} dz$$

Integrating

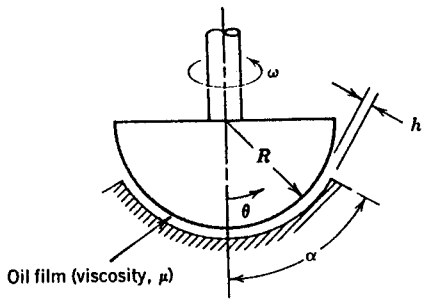
$$T = \int_A dT = \int_0^H \frac{2\pi\mu\omega\tan^3\theta}{a\cos\theta} z^3 dz = \frac{2\pi\mu\omega\tan^3\theta}{a\cos\theta} \frac{z^4}{4}\Big]_0^H$$

$$T = \frac{\pi\mu\omega\tan^3\theta\, H^4}{2a\cos\theta} \quad (\mu \approx 0.2\ N\cdot s/m^2 \text{ from Fig. A.2})$$

$$= \frac{\pi}{2} \times 0.2 \frac{N\cdot s}{m^2} \times 30 \frac{rev}{s} \times \tan^3 30° \times (0.025)^4 m^4 \times \frac{1}{0.25\times10^{-3} m} \times \frac{1}{\cos 30°} \times 2\pi \frac{rad}{rev}$$

$$T = 0.0206\ N\cdot m$$

Given: Spherical thrust bearing shown:



Find: Obtain and plot an algebraic expression for the torque on the spherical member, as a function of α.

Solution: Apply definitions

Computing equations: $\tau = \mu \frac{du}{dy}$ $\quad T = \int_A r\tau\, dA$

Assumptions: (1) Newtonian fluid, (2) Narrow gap, (3) Laminar flow

From the figure, $r = R\sin\theta$ $\quad u = \omega r = \omega R\sin\theta$

$$\tau = \mu\frac{du}{dy} = \mu\left(\frac{u-0}{h}\right) = \mu\frac{u}{h} = \frac{\mu\omega R\sin\theta}{h}$$

$$dA = 2\pi r R\, d\theta = 2\pi R^2 \sin\theta\, d\theta$$

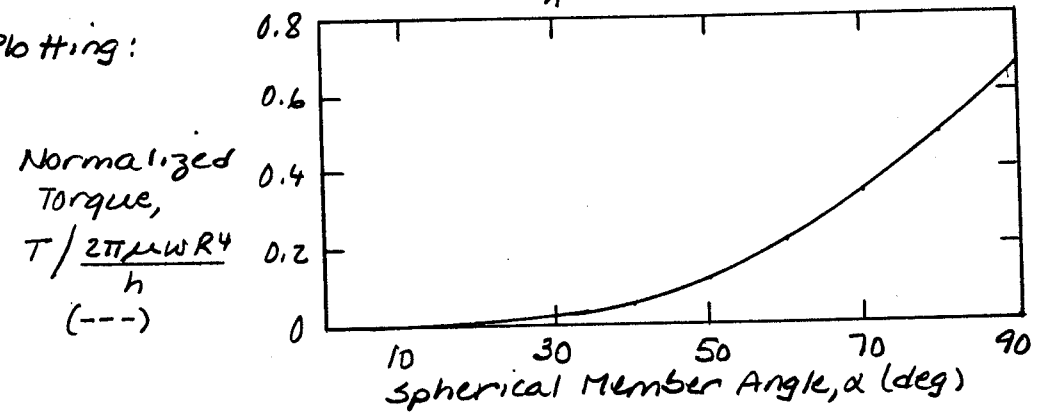
Thus

$$T = \int_0^\alpha R\sin\theta\left(\frac{\mu\omega R\sin\theta}{h}\right) 2\pi R^2 \sin\theta\, d\theta = \frac{2\pi\mu\omega R^4}{h}\int_0^\alpha \sin^3\theta\, d\theta$$

$$T = \frac{2\pi\mu\omega R^4}{h}\left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^\alpha = \frac{2\pi\mu\omega R^4}{h}\left[\frac{\cos^3\alpha}{3} - \cos\alpha + \frac{2}{3}\right] \quad \leftarrow T$$

To plot, normalize to $\left[T \Big/ \frac{2\pi\mu\omega R^4}{h}\right] = \left[\frac{\cos^3\alpha}{3} - \cos\alpha + \frac{2}{3}\right]$

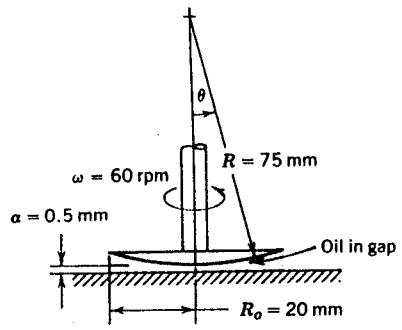
Plotting:



{Check dimensions: $\left[\frac{\mu\omega R^4}{h}\right] = \frac{Ft}{L^2} \times \frac{1}{t} \times L^4 \times \frac{1}{L} = FL$ ✓✓}

Given: Rotating bearing shown:

Narrow gap filled with viscous oil, $\mu = 1250$ cp.



Find: (a) Algebraic expression for shear stress on spherical member.
(b) Find maximum shear stress.
(c) Algebraic expression for viscous torque on spherical member.
(d) Evaluate torque.

Solution: Apply definitions

Computing equations: $\tau = \mu \frac{du}{dy}$ $\quad T = \int_A r\tau\, dA$

Assumptions: (1) Newtonian fluid, (2) Narrow gap, (3) Laminar motion

From the figure, $r = R\sin\theta$, $u = \omega r = \omega R\sin\theta$, $\frac{du}{dy} = \frac{u-0}{h} = \frac{u}{h}$

$h = a + R(1-\cos\theta)$ $\quad dA = 2\pi r\, dr = 2\pi R\sin\theta R\cos\theta\, d\theta$

Thus $\tau = \dfrac{\mu\omega R\sin\theta}{a+R(1-\cos\theta)}$ ← τ

From the table below, $\tau_{max} = 67.9$ N/m² at $\theta = 6.5°$ (not at R_o) ← τ_{max}

Torque is $T = \displaystyle\int_0^{\theta_{max}} \frac{\mu\omega R^4 \sin^2\theta\cos\theta}{a+R(1-\cos\theta)}\, d\theta$ ← T

This must be evaluated numerically or graphically. From Appendix G, 1 Poise = 0.1 kg/m·s. Thus $\mu = 1.25$ kg/m·s $= 1.25$ N·s/m²

$$\tau = 1.25\frac{N\cdot s}{m^2} \times 2\pi\frac{rad}{s} \times 0.075\, m \times \sin 6.5° \times \frac{1}{0.0005 + 0.075(1-\cos 6.5°)\, m} = 67.9\ N/m^2$$

Tabulating results of similar calculations gives:

theta (deg)	tau (N/m^2)	function (---)	torque (N-m)	sumt (N-m)
0.5	10.2	0.15	4.13E-06	4.13E-06
1.5	29.3	1.30	3.55E-05	3.96E-05
2.5	45.0	3.33	9.07E-05	1.30E-04
3.5	56.2	5.81	1.58E-04	2.89E-04
4.5	63.2	8.39	2.29E-04	5.17E-04
5.5	66.8	10.82	2.95E-04	8.12E-04
6.5	67.9	12.96	3.53E-04	1.17E-03
7.5	67.3	14.80	4.03E-04	1.57E-03
8.5	65.8	16.32	4.45E-04	2.01E-03
9.5	63.6	17.58	4.79E-04	2.49E-03
10.5	61.1	18.60	5.07E-04	3.00E-03
11.5	58.5	19.42	5.29E-04	3.53E-03
12.5	56.0	20.08	5.47E-04	4.08E-03
13.5	53.5	20.60	5.61E-04	4.64E-03
14.5	51.0	21.01	5.72E-04	5.21E-03
15.5	48.8	21.32	5.81E-04	5.79E-03

Here "function" is $\dfrac{\sin^2\theta\cos\theta}{a+R(1-\cos\theta)}$

and $\Delta\theta = 1\ deg = 0.0175$ rad

for the numerical integration.

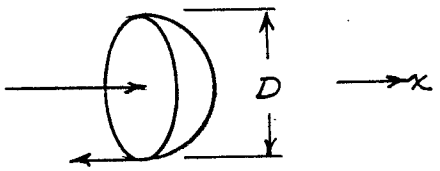
$\theta_{max} = \sin^{-1}\dfrac{R_o}{R} = \sin^{-1}\left(\dfrac{20}{75}\right) = 15.5°$

Integrated torque = 5.50E-03 N-m ← T

Given: Small gas bubbles form in soda when opened; $D = 0.1$ mm.

Find: Estimate pressure difference from inside to outside such a bubble.

Solution: Consider a free-body diagram of half a bubble:



Two forces act:

Pressure: $F_p = \Delta p \frac{\pi D^2}{4}$

Surface tension: $F_\sigma = \sigma \pi D$

Summing forces for equilibrium

$$\Sigma F_x = F_p - F_\sigma = \Delta p \frac{\pi D^2}{4} - \sigma \pi D = 0$$

so $\frac{\Delta p D}{4} - \sigma = 0$ or $\Delta p = \frac{4\sigma}{D}$

Assuming soda-gas interface is similar to water-air, then $\sigma = 72.8$ mN/m, and

$$\Delta p = 4 \times 72.8 \times 10^{-3} \frac{N}{m} \times \frac{1}{0.1 \times 10^{-3} m} = 2.91 \times 10^3 \frac{N}{m^2} = 2.91 \text{ kPa}$$ Δp

Problem 2.54

Open-Ended Problem Statement: Slowly fill a glass with water to the maximum possible level before it overflows. Observe the water level closely. Explain how it can be higher than the rim of the glass.

Discussion: Surface tension can cause the maximum water level in a glass to be higher than the rim of the glass. The same phenomenon causes an isolated drop of water to "bead up" on a smooth surface.

Surface tension between the water/air interface and the glass acts as an invisible membrane that allows trapped water to rise above the level of the rim of the glass. The mechanism can be envisioned as forces that act in the surface of the liquid above the rim of the glass. Thus the water appears to defy gravity by attaining a level higher than the rim of the glass.

To experimentally demonstrate that this phenomenon is the result of surface tension, set the liquid level nearly as far above the glass rim as you can get it, using plain water. Add a drop of liquid detergent (the detergent contains additives that reduce the surface tension of water). Watch as the excess water runs over the side of the glass.

13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 150 SHEETS EYE-EASE® 5 SQUARE
42-388 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U. S. A.
National® Brand

Problem 2.55

Open-Ended Problem Statement: Observe small water drops as they form at the tip of a medicine dropper. Comment on the stages in the development of a drop.

Discussion: The following three stages in drop formation are observed:

(1) The drop is less than a hemisphere in size. In the beginning it is not really a drop at all. Surface tension holds the impending drop in place below the tip of the medicine dropper. The surface of the drop becomes more curved and begins to approach a hemispherical shape as more water is squeezed into the drop.

(2) The drop is between a hemisphere and a sphere in size. Surface tension still holds the drop in place below the tip of the medicine dropper. As the drop grows, it first tends to become roughly spherical in shape. As the drop continues to grow it necks down at the dropper and begins to resemble the shape of a punching bag.

(3) The drop separates from the tip of the medicine dropper. When the drop reaches its ultimate size, surface tension just balances the weight of the drop and keeps it attached to the tip of the medicine dropper. Adding more water causes the drop to separate from the tip of the dropper.

After separating from the dropper, the drop returns to a spherical shape as it falls.

At first it may be difficult to predict when the drop will separate from the dropper. Practice makes it easier to identify imminent separation.

The three stages in the behavior of a drop appear to be common to all medicine droppers (we used a glass medicine dropper and a plastic medicine dropper for our experiments). However, the water drop sizes may be different for each medicine dropper, depending on its tip configuration and the material from which it is made.

13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-386 200 SHEETS EYE-EASE® 5 SQUARE
42-387 100 RECYCLED WHITE 5 SQUARE
42-389 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Problem 2.56

Open-Ended Problem Statement: Plan an experiment to measure the surface tension of a liquid similar to water. If necessary, review the NCFMF video *Surface Tension* for ideas. Which method would be most suitable for use in an undergraduate laboratory? What experimental precision could be expected?

Discussion: Two basic kinds of experiment are possible for an undergraduate laboratory:

(1) Using a clear small-diameter tube, compare the capillary rise of the unknown liquid with that of a known liquid (compare with water, because it is similar to the unknown liquid).

This method would be simple to set up and should give fairly accurate results. A vertical traversing optical microscope could be used to increase the precision of measuring the liquid height in each tube.

A drawback to this method is that the specific gravity and contact angle of the two liquids must be the same to allow the capillary rises to be compared.

The capillary rise would be largest and therefore easiest to measure accurately in a tube with the smallest practical diameter. Tubes of several diameters could be used if desired.

(2) Dip an object into a pool of test liquid and measure the vertical force required to pull the object from the liquid surface.

The object might be made rectangular (e.g., a sheet of plastic material) or circular (e.g., a metal ring). The net force* needed to pull the same object from each liquid should be proportional to the surface tension of each liquid.

This method would be simple to set up. However, the force magnitudes to be measured would be quite small.

A drawback to this method is that the contact angles of the two liquids must be the same.

The first method is probably best for undergraduate laboratory use. A quantitative estimate of experimental measurement uncertainty is impossible without knowing details of the test setup. It might be reasonable to expect results accurate to within ±10% of the true surface tension.

13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 100 RECYCLED WHITE 5 SQUARE
42-390 200 RECYCLED WHITE 5 SQUARE
42-386 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

* Net force is the total vertical force minus the weight of the object. A buoyancy correction would be necessary if part of the object were submerged in the test liquid.

Given: Water, with bulk modulus assumed constant.

Find: (a) Percent change in density at 100 atm
(b) Plot percent change vs. p/p_{atm} up to 50,000 psi.
(c) Comment on assumption of constant density.

Solution: By definition, $E_v = \frac{dp}{d\rho/\rho}$. Assume E_v = constant. Then

$$\frac{d\rho}{\rho} = \frac{dp}{E_v}$$

Integrating, from ρ_0 to ρ gives $\ln\frac{\rho}{\rho_0} = \frac{p-p_0}{E_v} = \frac{\Delta p}{E_v}$, so $\frac{\rho}{\rho_0} = e^{\Delta p/E_v}$

The relative change in density is

$$\frac{\Delta\rho}{\rho_0} = \frac{\rho-\rho_0}{\rho_0} = \frac{\rho}{\rho_0} - 1 = e^{\Delta p/E_v} - 1$$

From Table A.2, $E_v = 2.24$ GPa for water at 20°C.

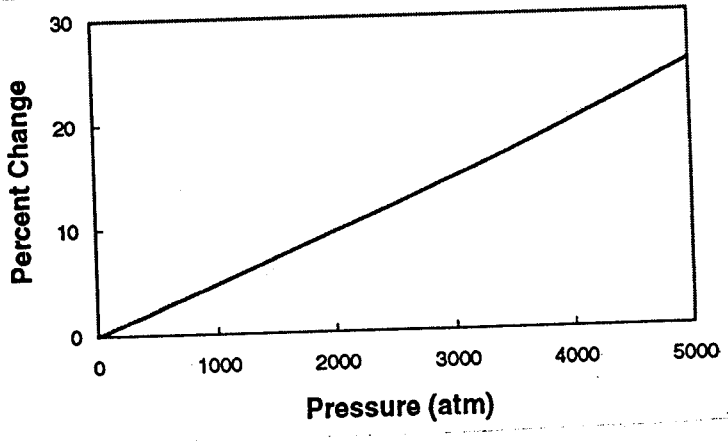
For p = 100 atm (gage), Δp = 100 atm, so

$$\frac{\Delta\rho}{\rho_0} = \exp\left(100\ \text{atm} \times \frac{1}{2.24\times10^9\ \text{Pa}} \times 101.325\times10^3\ \frac{\text{Pa}}{\text{atm}}\right) - 1 = 0.00453, \text{ or } 0.453\%$$

For Δp = 50,000 psi,

$$\frac{\Delta\rho}{\rho_0} = \exp\left(50{,}000\ \text{psi} \times \frac{1}{2.24\times10^9\ \text{Pa}} \times \frac{101.325\times10^3\ \text{Pa}}{14.696\ \text{psi}}\right) - 1 = 0.166 \text{ or } 16.6\%$$

Thus constant density is not a reasonable assumption for a cutting jet operating at 50,000 psi. Constant density (5% change) would be reasonable up to $\Delta p \approx 16{,}000$ psi.



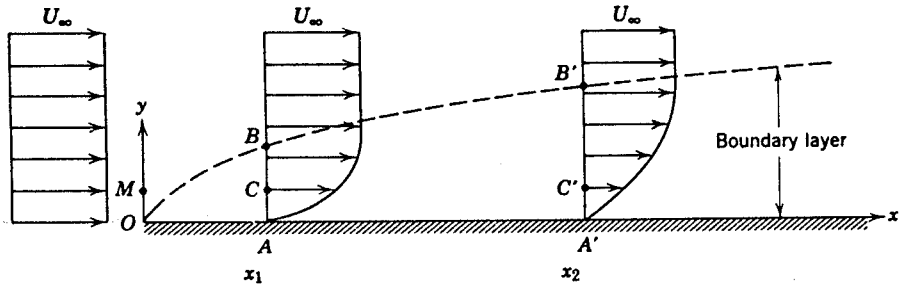
Given: Incompressible laminar viscous flow over a semi-infinite flat plate shown in Fig. 2.11.

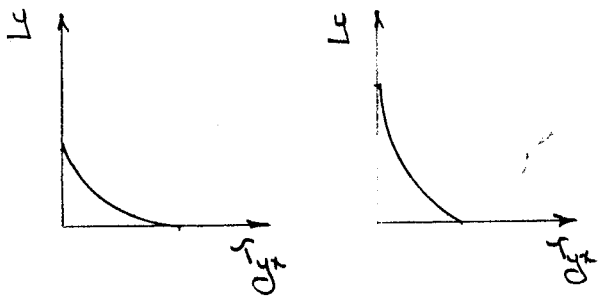
Find: (a) Sketch τ_{yx} as a function of y at locations x_1 and x_2
(b) Sketch τ_{yx} along plate surface ($y=0$) as a function of x

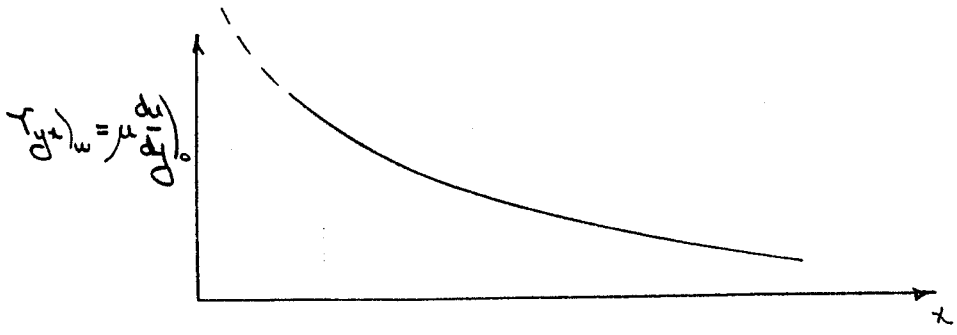
Solution: Assuming a Newtonian fluid, $\tau_{yx} = \mu \frac{du}{dy}$

Consider the stress on a positive y surface. Since velocity gradient is positive, $\tau_{yx} > 0$ and stress acts in positive x direction

At the edge of the boundary layer, $du/dy = 0$ and hence $\tau_{yx} = 0$

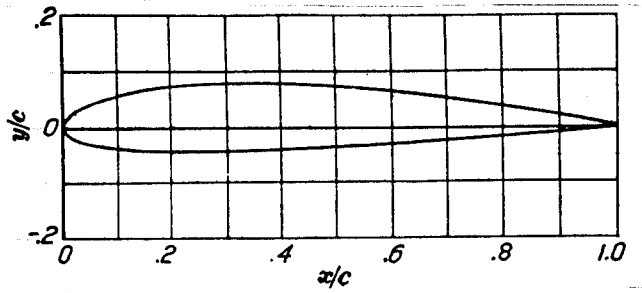






Open-Ended Problem Statement: How does an airplane wing develop lift?

Discussion: The sketch shows the cross-section of a typical airplane wing. The airfoil section is rounded at the front, curved across the top, reaches maximum thickness about a third of the way back, then tapers slowly to a fine trailing edge. The bottom of the airfoil section is relatively flat. (The discussion below also applies to a symmetric airfoil at an angle of incidence that produces lift.)



NACA 2412 Wing Section

It is both a popular expectation and an experimental fact that air flows more rapidly over the curved top surface of the airfoil section than along the relatively flat bottom. In the NCFMF video *Flow Visualization*, timelines placed in front of the airfoil indicate that fluid flows more rapidly along the top of the section than along the bottom.

In the absence of viscous effects (this is a valid assumption outside the boundary layers on the airfoil) pressure falls when flow speed increases. Thus the pressures on the top surface of the airfoil where flow speed is higher are lower than the pressures on the bottom surface where flow speed does not increase. (Actual pressure profiles measured for a lifting section are shown in the NCFMF video *Boundary Layer Control*.) The unbalanced pressures on the top and bottom surfaces of the airfoil section create a net force that tends to develop lift on the profile.

13-782 500 SHEETS FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-388 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U. S. A.
National® Brand

Given: Compressed nitrogen is shipped in cylindrical tanks of diameter, $D = 0.25\,m$ and length $L = 1.3\,m$. Gas in the tank is at $P = 20\,MPa$ and $T = 20^{\circ}C$
The maximum allowable stress in the tank wall is 210 MPa

Find: (a) mass of gas in the tank.
(b) theoretical minimum thickness of the cylinder wall

Solution

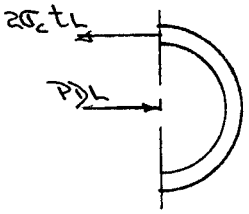
Assuming ideal gas behavior, $PV = MRT$

$$M = \frac{PV}{RT} = \frac{P}{RT}\,\frac{\pi D^2}{4}L$$

$$M = 20\times10^6\,\frac{N}{m^2}\times\frac{\pi}{4}\times(0.25)^2\,m^2\times1.3\,m\times\frac{kg\cdot K}{297\,J}\times\frac{1}{293\,K}\times\frac{J}{N\cdot m}$$

$$M = 14.7\,kg$$ ← M

To determine wall thickness, consider a section of the pipe wall



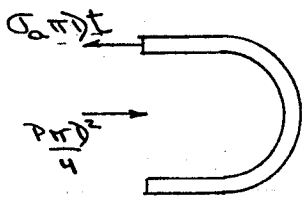
$$\Sigma F = 0 = PDL - 2\sigma_c t L$$

Then, $$t = \frac{PDL}{2\sigma_c L} = \frac{PD}{2\sigma_c}$$

$$t = 20\times10^6\,\frac{N}{m^2}\times\frac{0.25\,m}{2}\times\frac{m^2}{210\times10^6\,N}\times\frac{10^3\,mm}{m}$$

$$t = 11.9\,mm$$ ← t

For the ends of the cylinder



$$\Sigma F = 0 = P\frac{\pi D^2}{4} - \sigma_a \pi D t$$

and $$t = \frac{PD}{4\sigma_a}$$

Hence minimum thickness is determined by circumferential stress

Open-Ended Problem Statement: Pressure variations that result from altitude changes can cause ear "popping" and discomfort to airplane passengers or those driving in mountainous terrain. Each individual is affected differently in terms of the elevation change necessary to cause one "ear pop." Determine the pressure change, expressed in millimeters of water, that corresponds to the elevation difference required to cause an ear pop on a standard day at an altitude of 2000 m.

Discussion: The elevation change needed to cause an ear pop will vary for different persons. However, if we express the pressure change in millimeters of water per meter of elevation change, we can compute the result for any individual's physiology.

Basic equation: $\frac{dp}{dz} = -\rho g$

Assumptions: (1) Δz to cause "ear pop" < 100 m, so ρ_{air} = constant

Then $dp = -\rho_{air} g\, dz$ or $\Delta p = -\rho_{air} g\, \Delta z_{air}$

From Table A.3 (Standard Atmosphere) $T = 275.2$ K and $\rho/\rho_{SL} = 0.8217$ at $z = 2000$ m

$$T = 275.2\ K - 273.2 = 2^\circ C$$

$$\rho = 0.8217\,\rho_{SL} = 0.8217 \times 1.23 \frac{kg}{m^3} = 1.01\ kg/m^3$$

In terms of height change, Δh, of water column, $\Delta p = -\rho_{H_2O} g\, \Delta h_{H_2O}$

Combining, $\rho_{air} g\, \Delta z_{air} = \rho_{H_2O} g\, \Delta h_{H_2O}$

and $\Delta h_{H_2O} = \frac{\rho_{air}}{\rho_{H_2O}} \Delta z_{air}$

Assuming Δz_{air} of 1 m, $\Delta h_{H_2O} = \frac{1.01}{999} \times 1\,m = 1.01\ mm\ (H_2O)$ ← Δh

If the elevation change needed to produce one "ear pop," then the corresponding pressure change is

$$\Delta h_{H_2O} = \frac{1.01\ mm\ (H_2O)}{1\ m\ (air)} \times 75\ m\ (air) = 75\ mm\ (H_2O)$$ ← Δh (p

{ Of course we have simplified the problem by assuming ρ_{air} = constant. We would expect a significant variation among individuals, too. }

Given: Pure water on a standard day

Find: Boiling temperature at (a) 1000m, and (b) 2000m. Compare with sea level value

Solution

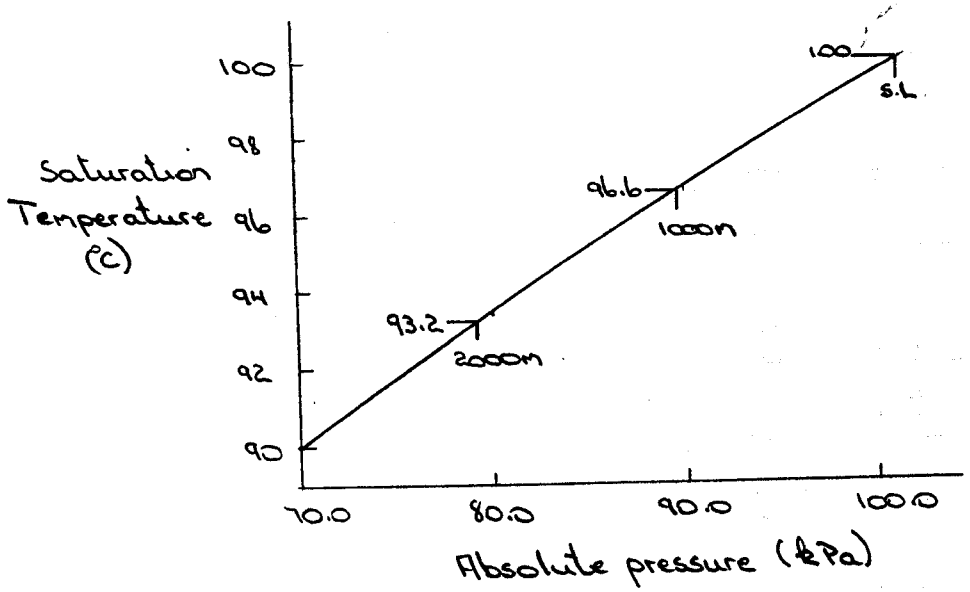
We can determine the atmospheric pressure at the given altitudes from table A.3, Appendix A

Elevation (m)	$\frac{P}{P_0}$	P (kPa)	T_{sat}^* (°C)
0	1.000	101	100
1000	0.887	89.6	96.6
2000	0.785	79.3	93.2

* T_{sat} obtained from plot of T_{sat} vs P given below

Data from Steam Tables gives T_{sat}

P (kPa)	T_{sat} (°C)
70	90.0
80	93.5
90	96.7
101.325	100.0

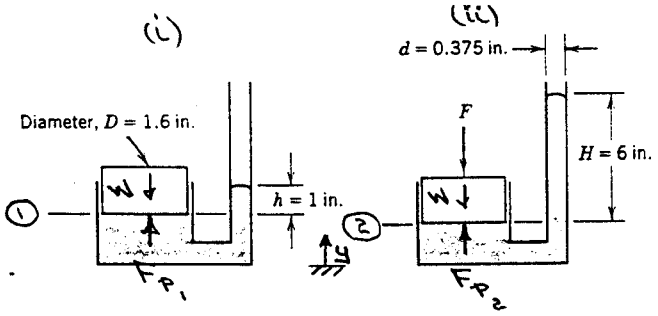


{ These data show that T_{sat} drops about 3.4°C / 1000m }

Given: The tube shown is filled with mercury at 20°C

Find: the force applied to the piston

Solution:



Basic equations: $\frac{dp}{dy} = -\rho g$

$$\vec{F} = -\int p\, d\vec{A}$$

For ρ = constant in a static fluid

$$p = p_{atm} - \rho g(y - y_0)$$

where $p = p_{atm}$ at $y = y_0$

Then

$$p_1 = p_{atm} + \rho g h \qquad \text{and} \qquad F_{p_1} = \rho g h A \text{ (gage)}$$

For fbd (i) $\Sigma F_y = 0 = F_{p_1} - W = 0$ and $W = F_{p_1} = \rho g h A$

Also $p_2 = p_{atm}$ and $\rho g H$ and $F_2 = \rho g H A$ (gage)

For fbd (ii) $\Sigma F_y = 0 = F_{p_2} - W - F = 0$

$$\therefore F = F_{p_2} - W = \rho g H A - \rho g h A = \rho g A (H - h)$$

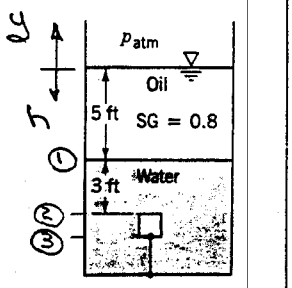
$$F = \rho_{H_2O} SG\, g \frac{\pi}{4} D^2 (H - h) \qquad \text{From Fig. A.1, App. A, SG} = 13.54$$

$$F = 1000 \frac{kg}{m^3} \times 13.54 \times 9.81 \frac{m}{s^2} \times \frac{\pi}{4} (1.6)^2 in^2 (6-1) in \times (0.0254)^3 \frac{m^3}{in^3} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F = 21.9\ N$$

Edited by Foxit Reader
Copyright(C) by Foxit Software Company,2005-2007
For Evaluation Only.

Given: Cube of solid oak, 1 ft on a side, is submerged by tether as shown.



Find: (a) the force of water on bottom surface
(b) the tension in the tether.

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $\vec{F} = -\int p\, d\vec{A}$

Assumptions: (a) static fluid
(b) SG_{oil} = constant, ρ_{H_2O} = constant

Then $\int_{p_1}^{p_3} dp = \int_{h_0}^{h_3} \rho g\, dh = \int_{h_0}^{h_1} SG_{oil}\, \rho_{H_2O} g\, dh + \int_{h_1}^{h_3} \rho_{H_2O} g\, dh$

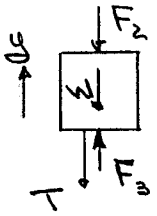
$p_3 - p_{atm} = SG_{oil}\, \rho_{H_2O} g (h_1 - h_0) + \rho_{H_2O} g (h_3 - h_1)$

$= 0.8 \times 1.94 \frac{slug}{ft^3} \times 32.2 \frac{ft}{s^2} \times 5 ft + 1.94 \frac{slug}{ft^3} \times 32.2 \frac{ft}{s^2} \times 4 ft$

$p_3 - p_{atm} = 500 \frac{slug}{ft \cdot s^2} \times \frac{lbf \cdot s^2}{ft \cdot slug} = 500\ lbf/ft^2$

Since the pressure over the bottom surface is uniform,

$\vec{F}_3 = -\int_A p\, d\vec{A} = -p\vec{A} = [p_{atm} + 500 \frac{lbf}{ft^2}] \times ft^2 \hat{j} = 2620 \hat{j}\ lbf$ ← $\vec{F}_3$



The force F_2 on the top of the cube is $F_2 = p_2 A$

The pressure on the top of the cube is

$p_2 - p_{atm} = SG_{oil}\, \rho_{H_2O} g (h_1 - h_0) + \rho_{H_2O} g (h_2 - h_1)$

The weight of the block is $W = \rho g \forall = SG_{oak}\, \rho_{H_2O} g \forall$

where $SG_{oak} = 0.77$ (Table A.1, Appendix A)

Then for the fbd of the block, $\sum F_y = 0 = F_3 - F_2 - W - T$

$T = F_3 - F_2 - W = [p_{atm} + SG_{oil}\, \rho_{H_2O} g (h_1 - h_0) + \rho_{H_2O} g (h_3 - h_1)] A$

$- [p_{atm} + SG_{oil}\, \rho_{H_2O} g (h_1 - h_0) + \rho_{H_2O} g (h_2 - h_1)] A - SG_{oak}\, \rho_{H_2O} g \forall$

$T = \rho_{H_2O} g (h_3 - h_2) A - SG_{oak}\, \rho_{H_2O} g \forall$

$= 1.94 \frac{slug}{ft^3} \times 32.2 \frac{ft}{s^2} \times 1 ft \times 1 ft^2 - 0.77 \times 1.94 \frac{slug}{ft^3} \times 32.2 \frac{ft}{s^2} \times 1 ft^3$

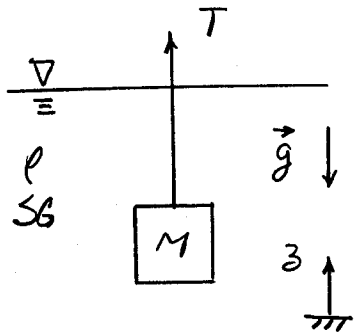
$T = 14.4 \frac{slug \cdot ft}{s^2} \times \frac{lbf \cdot s^2}{ft \cdot slug} = 14.4\ lbf$ ← T

Given: Submerged cube with 4 in. sides; $M = 0.569$ slug.

Top horizontal, 6 in. below surface; $T = 11.5$ lbf.

Find: (a) Density and (b) specific gravity of liquid.

<u>Solution</u>: Use free-body diagram, definition of SG.



Computing equations: $\Sigma \vec{F}_3 = m\vec{a}_3 = 0$

$$SG = \frac{\rho}{\rho_{H_2O}(4C)}$$

Assumptions: (1) static liquid

(2) $F_B = \rho_\ell \forall$

Then $\Sigma F_3 = T + F_B - Mg = 0$

$$T + \rho_\ell \forall g - Mg = 0$$

$$\rho_\ell = \frac{Mg - T}{\forall g} = \left[0.569 \text{ slug} \times 32.2 \frac{ft}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} - 11.5 lbf\right]\left(\frac{12}{4}\right)^3 \frac{1}{ft^3} \times \frac{s^2}{32.2 ft} \times \frac{slug \cdot ft}{lbf \cdot s^2}$$

$$\rho_\ell = 5.72 \text{ slug/ft}^3 \quad \rho_\ell$$

$$SG = \frac{\rho}{\rho_{H_2O}(4C)} = 2.95 \quad SG$$

Check: $F_B = \rho_\ell \forall g = 5.72 \frac{slug}{ft^3} \times \left(\frac{4}{12}\right)^3 ft^3 \times 32.2 \frac{ft}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} = 6.82 lbf$

$$T + F_B - Mg = 11.5 + 6.82 - 0.569(32.2) = 0 \quad \checkmark\checkmark$$

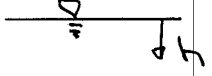
Given: Air bubble, $D = 10$ mm, released at depth $h = 30$ m below surface of sea; water at $T = 30°C$.

Find: Estimate of bubble diameter as it reaches the water surface.

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$ $\quad p = \rho RT$ $\quad \rho = \frac{m}{\forall}$

Assumptions: (1) T = constant = 30°C
(2) air behaves as ideal gas
(3) ρ_{H_2O} = constant



$h_1 = 30$ m
$h_2 = 0$

From ideal gas eq. $p = \rho RT = \frac{m}{\forall} RT$

Since m and T are constant, then $p_1 \forall_1 = p_2 \forall_2$ (1)

Also $\int_{p_1}^{p_2} dp = \int_{h_1}^{h_2} \rho g \, dh$ $\quad p_2 - p_1 = \rho g (h_2 - h_1)$

$p_1 = p_2 + \rho g h_1 = p_{atm} + \rho g h_1$

From Eq. (1) $\frac{\forall_2}{\forall_1} = \frac{p_1}{p_2} = \frac{p_{atm} + \rho g h_1}{p_{atm}} = 1 + \frac{\rho g h_1}{p_{atm}}$

From Table A.8 (Appendix A) at $T = 30°C$, $\rho = 996$ kg/m³

From Table A.2, $SG_{seawater} = 1.025$

$$\therefore \frac{\forall_2}{\forall_1} = 1 + (996)(1.025)\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 30\,m \times \frac{N \cdot s^2}{kg \cdot m} \times 1.01\times10^5 \frac{1}{N/m^2}$$

$$\frac{\forall_2}{\forall_1} = 3.975$$

Since $\forall \propto D^3$, $\left(\frac{D_2}{D_1}\right)^3 = \frac{\forall_2}{\forall_1}$

and $D_2 = D_1 \left(\frac{\forall_2}{\forall_1}\right)^{1/3} = 10\,mm\,(3.975)^{1/3} = 15.8$ mm $\longleftarrow D_2$

13.782 500 SHEETS, FILLER, 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Problem 3.8

Open-Ended Problem Statement: When filling the lawnmower with gasoline, we noticed the need to vent the gas can to obtain adequate flow into the tank. Discuss and comment on what is meant by "venting," and why this should be necessary.

Discussion: As liquid flows from an unvented gas can, the air remaining in the can must expand. This expansion reduces the air pressure in the can to below atmospheric. As gasoline continues to flow from the can, the expansion of air continues and the air pressure above the gasoline becomes lower and lower.

Sub-atmospheric pressure in the air above the gasoline in the can counteracts the gravity head that tends to cause flow from the can into the mower tank. At some gasoline level in the can, the gravity head will be canceled by the reduced air pressure. Then flow of gasoline from the can will cease.

A vent is an opening from the air space to the atmosphere. Venting the gas can allows air to flow into the tank as the liquid flows out and the air volume increases. The vent ensures that the air pressure above the gasoline surface is always atmospheric. The constant pressure above the liquid surface permits gasoline to continue to flow from the can by gravity.

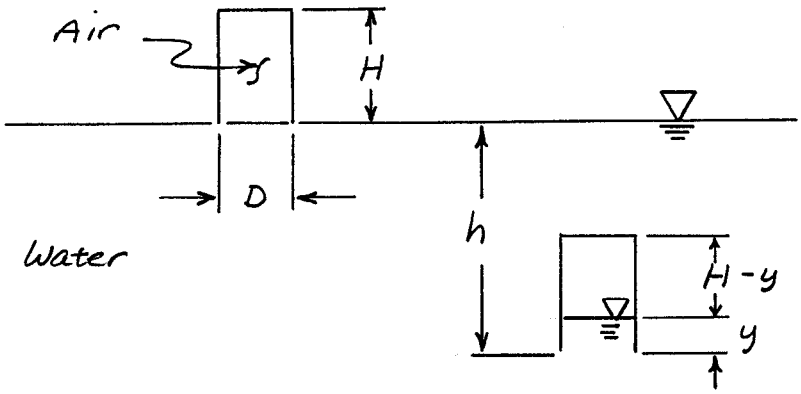
In general the air vent area may be considerably smaller than the liquid outlet area. However, in order to function properly the vent must be of sufficient size to freely pass air.

13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-388 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U. S. A.
National® Brand

Edited by Foxit Reader
Copyright(C) by Foxit Software Company,2005-2007
For Evaluation Only.

Problem 3.9

Given: Cylindrical cup lowered slowly beneath pool surface.



Find: Expression for y in terms of h and H. Plot: y/H vs. h/H.

Solution: Apply ideal gas and hydrostatic equations.

Basic equations: $p\forall = mRT$ $\quad \frac{dp}{dh} = \rho g$

Assumptions: (1) T = constant
(2) Static liquid
(3) Incompressible liquid

Using (1), $p\forall = p_a \frac{\pi D^2}{4} H = p \frac{\pi D^2}{4}(H-y)$; or $p_a H = p(H-y)$

Integrating $\frac{dp}{dh} = \rho g$ gives $p - p_a = \rho g(h-y)$ in container.

Thus

$$p_a H = [p_a + \rho g(h-y)](H-y) = p_a H - p_a y + \rho g(h-y)(H-y)$$

Expanding,

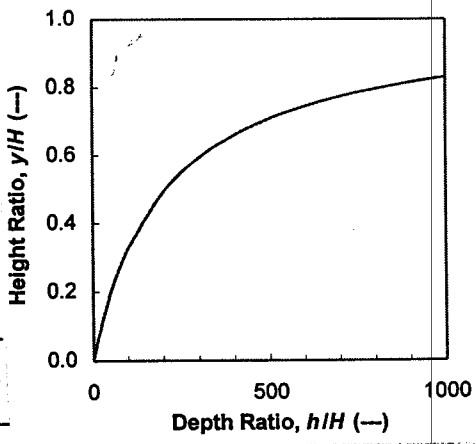
$$0 = \rho g h H - \rho g h y - \rho g y H + \rho g y^2 - p_a y$$

or

$$0 = hH - \left[(h+H) + \frac{p_a}{\rho g}\right] y + y^2$$

Using the quadratic equation

$$y = \frac{h + H + p_a/\rho g - \sqrt{\left[h + H + \frac{p_a}{\rho g}\right]^2 - 4hH}}{2}$$



(Note $y \leq H$, so the minus sign must be used.) In terms of y/H, this becomes

$$\frac{y}{H} = \frac{\frac{h}{H} + 1 + \frac{p_a}{\rho g H} - \sqrt{\left[\frac{h}{H} + 1 + \frac{p_a}{\rho g H}\right]^2 - 4\frac{h}{H}}}{2}$$

(See plot above)

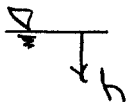
Given: Behavior of seawater to be modeled by assuming constant bulk modulus.

Find: The percent deviations in (a) density, and (b) pressure, at depth $h = 10$ km, as compared to values obtained assuming constant density.

Plot: the results over range of $0 \le h \le 10$ km

Solution

Basic equation: $\frac{dp}{dh} = \rho g$ Definition: $E_v = \frac{dp}{d\rho/\rho}$



Then, $d\rho = \rho g\, dh = \frac{d\rho}{\rho} E_v$ and $\int_{\rho_0}^{\rho} \frac{d\rho}{\rho^2} = \int_0^h \frac{g\, dh}{E_v}$

We obtain

$$-\frac{1}{\rho}\Big]_{\rho_0}^{\rho} = -\frac{1}{\rho} + \frac{1}{\rho_0} = \frac{-\rho_0 + \rho}{\rho\rho_0} = \frac{\rho g h}{E_v} \quad \text{or} \quad \rho - \rho_0 = \rho\rho_0 \frac{g h}{E_v}$$

Then

$$\rho\left(1 - \frac{\rho_0 g h}{E_v}\right) = \rho_0 \quad \text{and} \quad \frac{\rho}{\rho_0} = \frac{1}{\left(1 - \frac{\rho_0 g h}{E_v}\right)}$$

Finally, $$\frac{\Delta\rho}{\rho_0} = \frac{\rho - \rho_0}{\rho_0} = \frac{\rho}{\rho_0} - 1 = \frac{\rho_0 g h / E_v}{(1 - \rho_0 g h / E_v)} \quad \text{---- (1)}$$

To determine an expression for the percent deviation in pressure we write $\int_{p_{atm}}^{p} dp = E_v \int_{\rho_0}^{\rho} \frac{d\rho}{\rho}$

Then $p - p_{atm} = E_v \ln \rho/\rho_0$

For ρ = constant, $\int_{p_{atm}}^{p} dp = \rho_0 g \int_0^h dh$ and $p - p_{atm} = \rho_0 g h$

Then $$\frac{p - p_{\rho=c}}{p_{\rho=c}} = \frac{\Delta p}{p_{\rho=c}} = \frac{E_v \ln \rho/\rho_0 - \rho_0 g h}{\rho_0 g h} = \frac{E_v}{\rho_0 g h} \ln \frac{\rho}{\rho_0} - 1 \quad \text{---- (2)}$$

From Table A.2 for seawater SG = 1.025, $E_v = 2.42$ GN/m^2. Then

$$\frac{E_v}{\rho_0 g} = 2.42 \times 10^9 \frac{N}{m^2} \times \frac{1}{(1000)(1.025)\frac{kg}{m^3}} \times 9.81 \frac{m}{s^2} \times \frac{kg \cdot m}{N \cdot s^2} \times \frac{km}{10^3 m} = 240.7 \text{ km}$$

Substituting into eqs (1) and (2)

$$\frac{\Delta\rho}{\rho_0} = \frac{4.155 \times 10^{-3} h}{1 - 4.155 \times 10^{-3} h} \quad \text{....(1a)}$$

$$\frac{\Delta p}{p_0} = \frac{240.7}{h} \ln\left[\frac{1}{1 - 4.155 \times 10^{-3} h}\right] - 1 \quad \text{.... (2a)}$$

13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

$\frac{\Delta\rho}{\rho_0} = 0.0215$ or 2.15 % $\left.\frac{\Delta\rho}{\rho_0}\right|_{h=10\,km}$

Both $\Delta\rho/\rho_0$ and $\Delta p/p_0$ are plotted as a function of depth h (in km) below.

The computing equations are

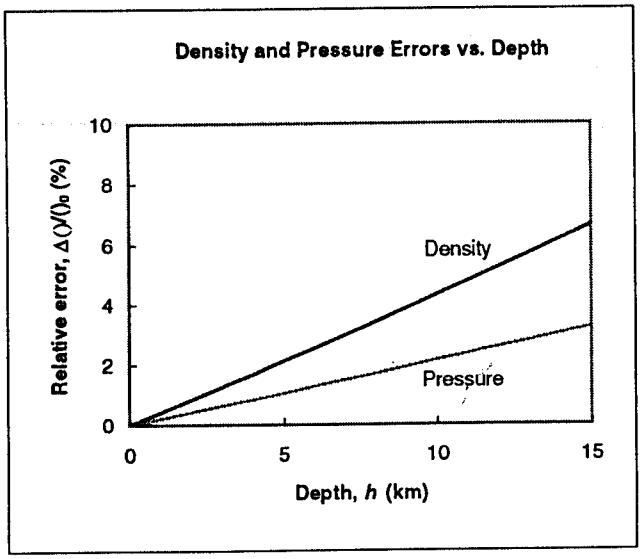
$$\frac{\Delta\rho}{\rho_0} = \frac{\rho_0 g h / E_v}{(1 - \rho_0 g h / E_v)}$$

$$\frac{\Delta p}{p_0} = \frac{E_v}{\rho_0 g h} \ln \frac{\rho}{\rho_0} - 1$$

Density and pressure variation of seawater:

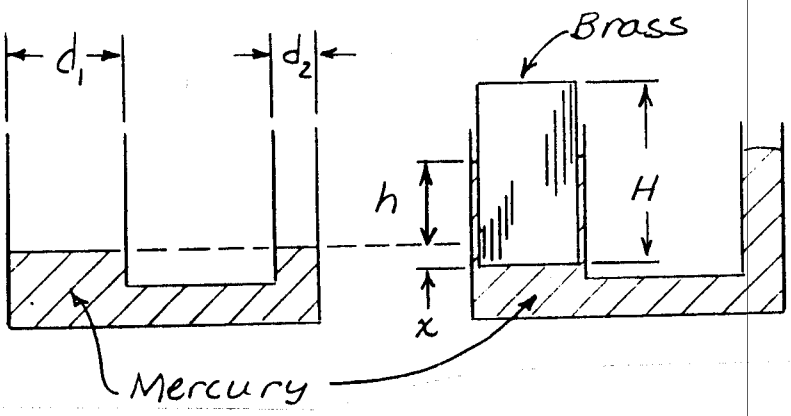
E_v = 2.42 GN/m^2 Bulk modulus of seawater

Depth, h (km)	Density Error, $\Delta\rho/\rho_0$ (—)	Pressure Error, $\Delta p/p_0$ (—)
0	0	0
1	0.417	0.219
2	0.838	0.429
3	1.26	0.639
4	1.69	0.851
5	2.12	1.06
6	2.56	1.28
7	3.00	1.49
8	3.44	1.71
9	3.88	1.93
10	4.34	2.15
11	4.79	2.37
12	5.25	2.59
13	5.71	2.81
14	6.18	3.04
15	6.65	3.26



Given: Container of mercury with vertical tubes $d_1 = 39.5$ mm and $d_2 = 12.7$ mm.

Brass cylinder with $D = 37.5$ mm and $H = 76.2$ mm is introduced into larger tube, where it floats.

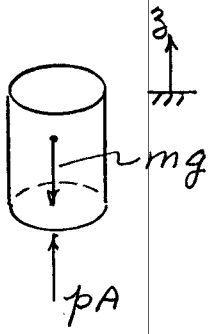


Find: (a) Pressure on bottom of cylinder.
(b) New equilibrium level, h, of mercury.

Solution: Analyze free-body diagram of cylinder, apply hydrostatics.

Computing equations: $\Sigma F_z = 0$; $\frac{dp}{dz} = -\rho g$; $\rho = SG\,\rho_{H_2O}$

Assumptions: (1) Static liquid
(2) Incompressible liquid



For the cylinder $\Sigma F_z = p\frac{\pi D^2}{4} - \rho_{brass} g \frac{\pi D^2}{4} H = 0$

Thus $p = \rho_{brass} g H = SG_{brass}\,\rho_{H_2O}\, g H$

From Table A.1, $SG_{brass} = 8.55$ at 20°C, so

$$p = 8.55 \times 1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.0762\,m \times \frac{N \cdot s^2}{kg \cdot m} = 6.39\ kPa\ (gage)$$

This pressure must be produced by a column of mercury $h+x$ in height. Thus, using SG_{Hg} from Table A.1,

$$p = \rho_{Hg} g (h+x) = SG_{Hg}\,\rho_{H_2O}\, g (h+x) = SG_{brass}\,\rho_{H_2O}\, g H$$

Thus $h + x = \frac{SG_{brass}}{SG_{Hg}} H = \frac{8.55}{13.55} H = 0.631 H$ (1)

But the volume of mercury must remain constant. Therefore

$$\frac{\pi D^2}{4} x = \frac{\pi (d_1^2 - D^2)}{4} h + \frac{\pi d_2^2}{4} h \quad \text{or} \quad x\left[\left(\frac{d_1}{D}\right)^2 - 1 + \left(\frac{d_2}{D}\right)^2\right] = 0.224\, h$$

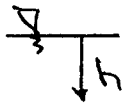
Substituting into Eq. 1,

$$h + x = h + 0.224\, h = 1.224\, h = 0.631\, H \quad \text{or} \quad h = \frac{0.631}{1.224} H = 0.516\, H$$

Given: Model behavior of seawater by assuming constant bulk modulus

Find: (a) expression density as a function of depth, h.
(b) Show that result may be written as $\rho = \rho_0 + bh$
(c) evaluate the constant b
(d) use results of (b) to obtain equation for p(h)
(e) determine percent error in predicted pressure at h = 1000 m

Solution: From Table A.2, App A, $SG_0 = 1.025$, $E_v = 2.42\ GN/m^2$



Basic equation: $\frac{dp}{dh} = \rho g$ Definition: $E_v = \frac{dp}{d\rho/\rho}$

Then, $dp = \rho g\, dh = E_v \frac{d\rho}{\rho}$ and $\frac{d\rho}{\rho^2} = \frac{g}{E_v} dh$

Integrating, $\int_{\rho_0}^{\rho} \frac{d\rho}{\rho^2} = \int_0^h \frac{g}{E_v} dh$ and $-\frac{1}{\rho}\Big]_{\rho_0}^{\rho} = \frac{gh}{E_v}$

Then, $\frac{gh}{E_v} = -\frac{1}{\rho} + \frac{1}{\rho_0} = \frac{-\rho_0 + \rho}{\rho\rho_0}$ or $\rho - \rho_0 = \rho\rho_0 \frac{gh}{E_v}$

$\therefore \rho\left(1 - \rho_0 \frac{gh}{E_v}\right) = \rho_0$ and $\frac{\rho}{\rho_0} = \frac{1}{\left\{1 - \frac{\rho_0 gh}{E_v}\right\}}$ ← p(h)

For $\frac{\rho_0 g}{E_v} \ll 1$, $\frac{\rho}{\rho_0} \approx 1 + \frac{\rho_0 g}{E_v} h$

Thus, $\rho \approx \rho_0 + \frac{\rho_0^2 g}{E_v} h = \rho_0 + bh$ where $b = \frac{\rho_0^2 g}{E_v}$ ← Q.E.D.

Since $dp = \rho g\, dh$, then an approximate expression for p(h) is

$$p - p_{atm} = \int_{p_{atm}}^{p} dp = \int_0^h (\rho_0 + bh) g\, dh = \left(\rho_0 h + \frac{bh^2}{2}\right) g$$

$$p_{approx} = p_{atm} + \left(\rho_0 h + \frac{\rho_0^2 g h^2}{E_v}\right) g = p_{atm} + \rho_0 hg\left[1 + \frac{\rho_0 gh}{E_v}\right]$$ ← p_{approx}

The exact solution for p(h) is obtained by utilizing the exact equation for $\rho(h)$. Thus.

$$p - p_{atm} = \int_{p_{atm}}^{p} dp = \int_{\rho_0}^{\rho} E_v \frac{d\rho}{\rho} = E_v \ln \frac{\rho}{\rho_0}$$

$$p = p_{atm} + E_v \ln\left\{1 - \frac{\rho_0 gh}{E_v}\right\}^{-1}$$ ← p_{exact}

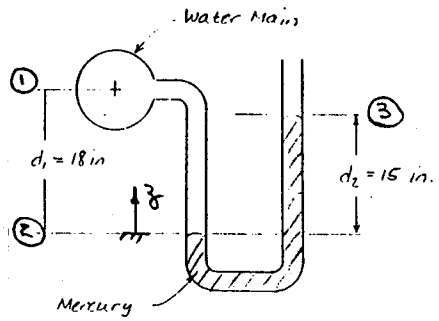
$$\frac{\rho_0 gh}{E_v} = (1.025)1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 10^3 m \times 2.42 \times 10^9 \frac{1}{N/m^2} \times \frac{N \cdot s^2}{kg \cdot m} = 4.16 \times 10^{-3}$$

Substituting numerical values, $p_{approx} = p_{atm} + 9.851$ MPa

$p_{exact} = p_{atm} + 10.076$ MPa

$$\text{error} = \frac{p_{exact} - p_{app}}{p_{exact}} = \frac{10.076 - 9.851}{10.076} = 0.0224 = 2.24\%$$ ← error

Given: Pressure in a water main is measured using the dual-fluid manometer shown.



Find: the gage pressure in the water main.

Solution:

Basic equation: $\frac{dp}{dz} = -\rho g$

Assumptions: static, incompressible liquids

For ρ = constant, integration of basic equation from reference point ① to any point gives

$$p - p_0 = -\rho g (z - z_0)$$

Then

$$p_1 - p_2 = -\rho g (z_1 - z_2) = -\rho_{H_2O} g d_1$$

and

$$p_2 - p_3 = -\rho g (z_2 - z_3) = -\rho_{Hg} g (-d_2) = \rho_{Hg} g d_2$$

Adding we obtain

$$p_1 - p_3 = -\rho_{H_2O} g d_1 + \rho_{Hg} g d_2 = g(-\rho_{H_2O} d_1 + SG_{Hg} \rho_{H_2O} d_2)$$

$$p_1 - p_3 = \rho_{H_2O} g (-d_1 + SG_{Hg} d_2)$$

From Table A.2, $SG_{Hg} = 13.6$

$$\therefore p_1 - p_3 = 1.94 \frac{slug}{ft^3} \times 32.2 \frac{ft}{sec^2} \left[-18 in + 13.6 (15 in)\right] \times \frac{ft^3}{1728 in^3} \times \frac{lbf \cdot sec^2}{ft \cdot slug}$$

$$p_1 - p_3 = 6.72 \text{ psi}$$

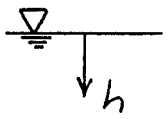
Since $p_3 = p_{atm}$, $p_1 = 6.72$ psig

Given: U-tube manometer, partially filled with water, then $\forall_{oil} = 3.25\ cm^3$ of Meriam red oil is added to the left side.

Find: Equilibrium height, H, when both legs are open to atmosphere.

Solution: Apply basic pressure-height relation.

Basic equation: $\frac{dp}{dh} = +\rho g$



Assumptions: (1) Incompressible liquid
(2) h measured down

Integration gives

$$p_2 - p_1 = \rho g (h_2 - h_1)$$

Thus

$$p_B = p_A + \rho_{oil} g L$$

$$p_D = p_C + \rho_{water} g (L - H)$$

Since $p_A = p_C = p_{atm}$, then

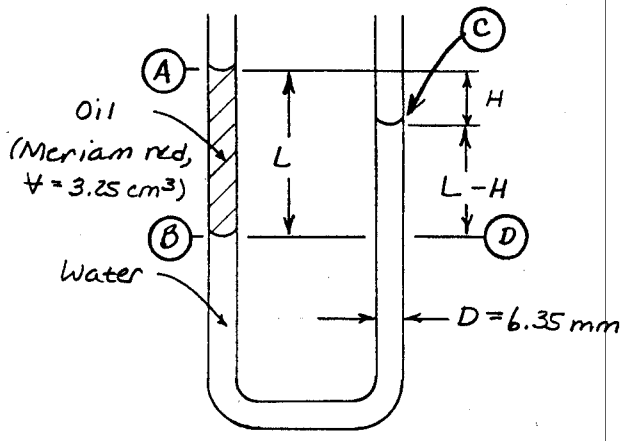
$$\rho_{oil} g L = \rho_{water} g (L - H)$$

or

$$SG_{oil} L = L - H$$

Thus

$$H = L (1 - SG_{oil})$$



From the volume of oil, $\forall = \frac{\pi D^2}{4} L$, so

$$L = \frac{4\forall}{\pi D^2} = \frac{4}{\pi} \times 3.25\ cm^3 \times \frac{1}{(6.35)^2 mm^2} \times (10)^3 \frac{mm^3}{cm^3} = 103\ mm$$

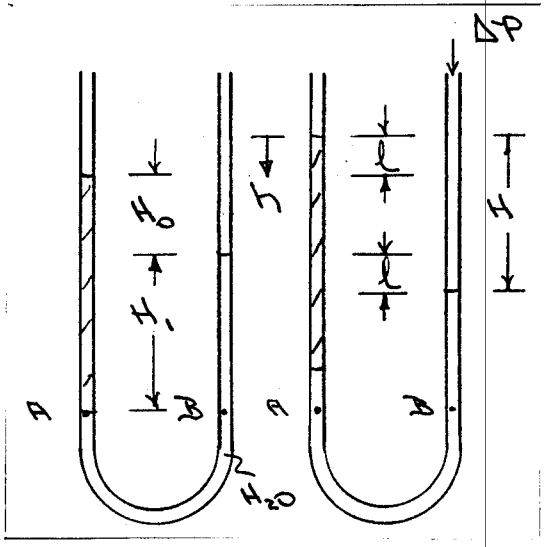
Finally, since SG = 0.827 (Table A.1, Appendix A), then

$$H = 103\ mm\ (1 - 0.827) = 17.8\ mm$$

H

Given: Two fluid manometer contains water and kerosene. With both tubes open to atmosphere, the free surface elevations differ by $H_o = 20.0$ mm

Find: Elevation difference, H, between free-surface of fluids when a gage pressure of 98.0 Pa is applied to the right tube.



Solution:

Basic equation: $\frac{dp}{dh} = \rho g$; $\Delta p = \rho g \Delta h$

Assumptions: (1) static fluid
(2) gravity is the only body force

When the gage pressure $\Delta p = 98.0$ Pa is applied to the right tube, the water in the right tube is displaced downward a distance, l; the kerosene in the left tube is displaced upward the same distance, l.

Under the applied gage pressure, Δp, the elevation difference, H, is

$$H = H_o + 2l$$

Since points A & B are at the same elevation in the same fluid $p_A = p_B$.

Initially (left diagram), $p_A = \rho_k g(H_o + H_1)$, $p_B = \rho g H_1$

and hence

$$\rho_k g(H_o + H_1) = \rho g H_1$$

or

$$H_1 = \frac{\rho_k H_o}{\rho - \rho_k} = \frac{SG_k H_o}{(1 - SG_k)}$$. From table A.2, $SG_k = 0.82$

$$\therefore H_1 = \frac{0.82}{(1-0.82)} 20\text{mm} = 91.1 \text{ mm}$$ ←

Under the applied pressure Δp (right diagram),

$p_A = \rho_k g(H_o + H_1) + \rho g l$, $p_B = \Delta p + \rho g (H_1 - l)$.

$$\therefore SG_k (H_o + H_1) + l = \frac{\Delta p}{\rho g} + (H_1 - l)$$

Solving for l.

$$l = \frac{1}{2}\left[H_1 + \frac{\Delta p}{\rho g} - SG_k(H_o + H_1)\right]$$

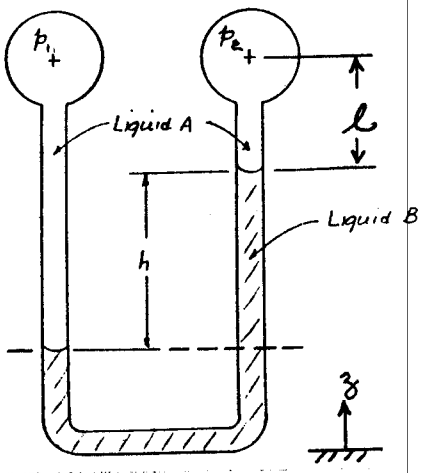
$$= \frac{1}{2}\left[91.1\text{ mm} + \frac{98\text{N}}{\text{m}^2} \times \frac{\text{m}^3}{999\text{kg}} \times \frac{\text{s}^2}{9.81\text{m}} \times \frac{\text{kg.m}}{\text{N.s}^2} \times \frac{10^3\text{mm}}{\text{m}} - 0.82(20+91.1)\text{mm}\right]$$

$$l = 5\text{mm}$$

$$H = H_o + 2l = 30\text{mm}$$ ←

Given: Manometer with two liquids as shown. $SG_A = 0.88$, $SG_B = 2.95$.

Find: Deflection, h, when $p_1 - p_2 = 870$ Pa.



Solution: Apply hydrostatics.

Basic equation: $\frac{dp}{dz} = -\rho g$ $\quad SG = \frac{\rho}{\rho_{H_2O}(4C)}$

Assumptions: (1) Static liquids
(2) Incompressible

Integrating, $p_A - p_B = -\rho(z_A - z_B)g$

$$p_A - p_B = \rho g(z_B - z_A)$$

or $\Delta p = \rho g \Delta h$

For left leg, $p_A = p_1 + (\ell + h)\rho_A g$

For right leg, $p_A = p_2 + \ell \rho_A g + h \rho_B g$

Subtracting, $p_1 - p_2 + hg(\rho_A - \rho_B) = 0$

$$p_1 - p_2 = hg(\rho_B - \rho_A)$$

Thus $$h = \frac{p_1 - p_2}{(\rho_B - \rho_A)g} = \frac{p_1 - p_2}{(SG_B - SG_A)\rho_{H_2O} g}$$

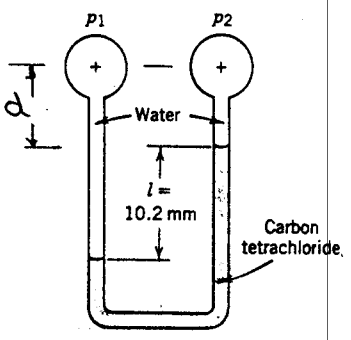
$$h = 870 \frac{N}{m^2} \times \frac{m^3}{(2.95 - 0.88)\,1000\,kg} \times \frac{s^2}{9.81\,m} \times \frac{kg \cdot m}{N \cdot s^2} = 0.0428\ m$$

$h = 42.8$ mm

h

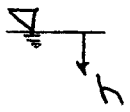
Given: Two-fluid manometer shown

Find: Pressure difference, $p_1 - p_2$



Solution:

Basic equation: $\frac{dp}{dh} = \rho g$



Assumptions: (1) static liquid
(2) incompressible
(3) g = constant

Then, $dp = \rho g\, dh$ and $\Delta p = \rho g \Delta h$

Starting at point ① and progressing to point ② we have

$$p_1 + \rho_{H_2O} g(d+l) - \rho_{ct} g l - \rho_{H_2O} g d = p_2$$

$$\therefore p_1 - p_2 = \rho_{ct} g l - \rho_{H_2O} g l = SG_{ct} \rho_{H_2O} g l - \rho_{H_2O} g l$$

$$p_1 - p_2 = \rho_{H_2O} g l (SG_{ct} - 1)$$

From Table A.2, Appendix A, $SG_{ct} = 1.595$

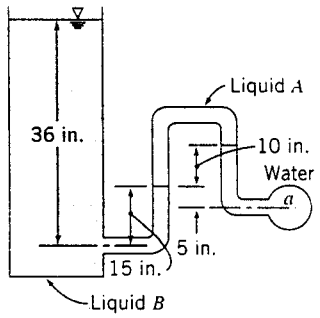
$$\therefore p_1 - p_2 = 1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 10.2 mm \times \frac{m}{10^3 mm} (1.595 - 1) \frac{N \cdot s^2}{kg \cdot m}$$

$$p_1 - p_2 = 59.5 \; N/m^2 \quad \leftarrow p_1 - p_2$$

Given: Manometer system as shown

SG Liquid A = 0.75

SG Liquid B = 1.20



Find: Gage pressure at point "a"

Solution:

Basic equation: $\frac{dP}{dz} = -\gamma$ dz

Assumptions: (1) static fluid
(2) gravity is only body force
(3) z axis direction vertically
(4) γ = constant

$dP = -\gamma\, dz$

For γ = constant, then $\Delta P = -\gamma \Delta z$, i.e. $P_j - P_i = -\gamma (z_j - z_i)$

$P_2 - P_1 = -\gamma_B (z_2 - z_1)$

$P_3 - P_2 = -\gamma_B (z_3 - z_2)$

$P_4 - P_3 = -\gamma_A (z_4 - z_3)$

$P_5 - P_4 = -\gamma_{H_2O} (z_5 - z_4)$

Summing these equations recognizing that $P_5 = P_a$ and $P_1 = P_{atm}$ then

$$P_a - P_{atm} = -\gamma_B (z_3 - z_1) - \gamma_A (z_4 - z_3) - \gamma_{H_2O} (z_5 - z_4)$$

$$= 1.20 \times 62.4 \frac{lbf}{ft^3} \times 21 in \times \frac{ft}{12 in} - 0.75 \times 62.4 \frac{lbf}{ft^3} \times \frac{10}{12} ft + 62.4 \frac{lbf}{ft^3} \times \frac{15}{12} ft$$

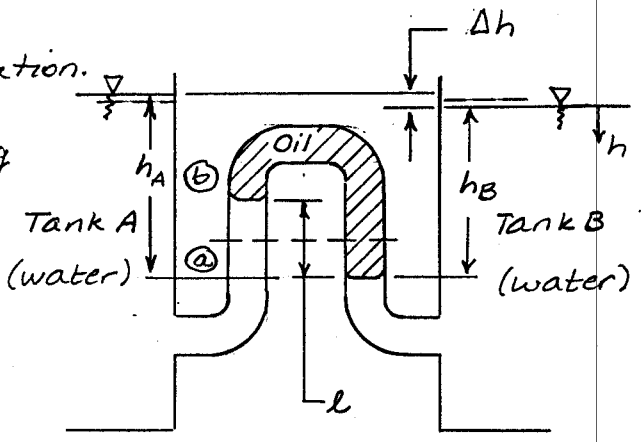
$$P_{a\,gage} = 170 \frac{lbf}{ft^2} \times \frac{ft^2}{144 in^2}$$

$$P_{a\,gage} = 1.18 \text{ psig}$$

P_a

Given: Two-fluid manometer; oil is second fluid.

Find: SG needed for 10 to 1 amplification.



Solution: Basic equation $\frac{dp}{dz} = -\rho g$

Assumptions: (1) Static liquid
(2) Incompressible

Then $dp = \rho g\, dh$

$$p = p_0 + \rho g h$$

For left leg, $p_a = p_{atm} + \rho_{H_2O} g h_A$

$$p_b = p_a - \rho_{H_2O} g \ell = p_{atm} + \rho_{H_2O} g (h_A - \ell) \qquad (1)$$

For right leg, $p_a = p_{atm} + \rho_{H_2O} g h_B$

$$p_b = p_a - SG_{oil} \rho_{H_2O} g \ell = p_{atm} + \rho_{H_2O} g (h_B - SG_{oil} \ell) \qquad (2)$$

Combining,

$$p_{atm} + \rho_{H_2O} g (h_A - \ell) = p_{atm} + \rho_{H_2O} g (h_B - SG_{oil} \ell)$$

or

$$h_A - \ell = h_B - SG_{oil} \ell \; ; \; h_A - h_B = \Delta h = \ell (1 - SG_{oil})$$

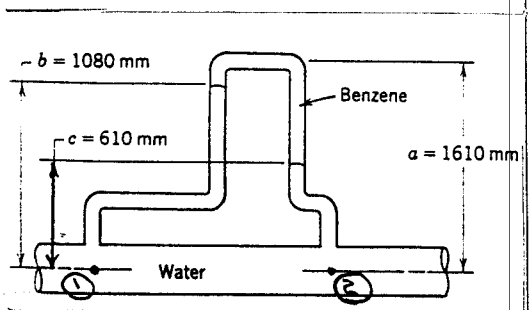
Finally

$$SG_{oil} = 1 - \frac{\Delta h}{\ell} = 1 - \frac{1}{10} = 0.900$$

SG

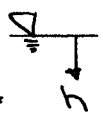
Given: Two-fluid inverted manometer

Find: Pressure difference, $p_1 - p_2$



Solution:

Basic equation: $\frac{dp}{dh} = \rho g$



Assumptions: (1) static liquid*
(2) incompressible
(3) g = constant

Then, $dp = \rho g\, dh$ and $\Delta p = \rho g \Delta h$

Starting at point (1) and progressing to point (2) we have

$$p_1 - \rho_{H_2O} g b + \rho_{Benz} g (b-c) + \rho_{H_2O} g c = p_2$$

$$\therefore p_1 - p_2 = \rho_{H_2O} g (b-c) - \rho_{Benz} g (b-c)$$

$$p_1 - p_2 = \rho_{H_2O} g (b-c) \left[1 - \frac{\rho_{Benz}}{\rho_{H_2O}}\right] = \rho_{H_2O} g (b-c) \left[1 - SG_{Benz}\right]$$

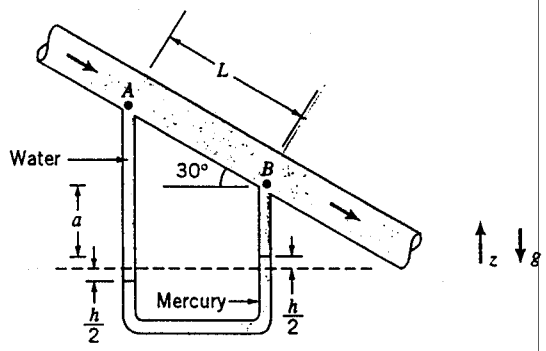
From Table A.2, Appendix A, $SG_{Benz} = 0.879$

$$\therefore p_1 - p_2 = 1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times (1080 - 610) mm \times \frac{m}{10^3 mm} \left[1 - 0.879\right] \frac{N \cdot s^2}{kg \cdot m}$$

$$p_1 - p_2 = 558\ N/m^2 \longleftarrow p_1 - p_2$$

* Note: Although water in the pipe is in motion, the streamlines are straight and the vertical pressure variation across the pipe is hydrostatic.

Given: Water flow in an inclined pipe as shown. Pressure difference, $p_A - p_B$, measured with two-fluid manometer. $L = 5$ ft, $h = 6$ in.



Find: Pressure difference, $p_A - p_B$.

Solution:

Basic equation: $\frac{dp}{dh} = \rho g$ where h is measured positive down

Assumptions: (1) static liquid
(2) incompressible
(3) g = constant

Then, $dp = \rho g\, dh$ and $\Delta p = \rho g \Delta h$

Start at p_A and progress through manometer to p_B

$$p_A + \rho_{H_2O} g L \sin 30^\circ + \cancel{\rho_{H_2O} g a} + \rho_{H_2O} g h - \rho_{Hg} g h - \cancel{\rho_{H_2O} g a} = p_B$$

$$p_A - p_B = \rho_{Hg} g h - \rho_{H_2O} g h - \rho_{H_2O} g L \sin 30^\circ$$

$$= SG_{Hg} \rho_{H_2O} g h - \rho_{H_2O} g h - \rho_{H_2O} g L \sin 30^\circ$$

$$p_A - p_B = \rho_{H_2O} g \left[h (SG_{Hg} - 1) - L \sin 30^\circ \right]$$

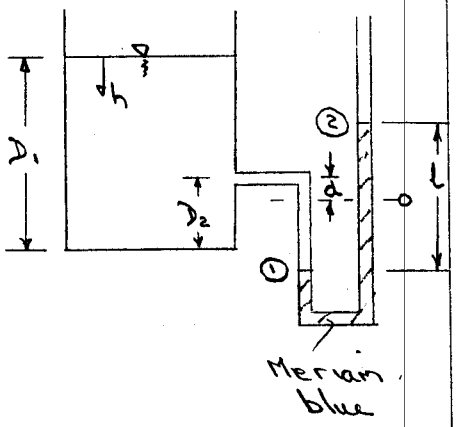
From Table A.2, $SG_{Hg} = 13.55$

Then,

$$p_A - p_B = 1.94 \frac{\text{slug}}{\text{ft}^3} \times 32.2 \frac{\text{ft}}{\text{s}^2} \left[0.5 \text{ ft} (13.55 - 1) - 5 \text{ ft} \sin 30^\circ \right] \times \frac{\text{lbf} \cdot \text{s}^2}{\text{ft} \cdot \text{slug}}$$

$$p_A - p_B = 236 \text{ lbf/ft}^2 \quad (1.64 \text{ psi}) \qquad \underline{\qquad p_A - p_B \qquad}$$

Given: A U-tube manometer is connected to the open tank filled with water as shown (manometer fluid is mercian blue)



$\gamma_1 = 2.5\,m$, $\gamma_2 = 0.7\,m$, $d = 0.2\,m$

Find: The manometer deflection, l.

Solution:

Basic equation: $\frac{dP}{dh} = \rho g$

For γ = constant $\Delta P = \rho g \Delta h$

Then, beginning at the free surface and accounting for the changes in pressure with elevation,

$$P_{atm} + (P_1 - P_{atm}) + (P_2 - P_1) = P_2 = P_{atm}$$

$$\rho_{H_2O} g\left[(\gamma_1 - \gamma_2) + d + \frac{l}{2}\right] - \rho_{mb} g\, l = 0$$

$$(\gamma_1 - \gamma_2) + d + \frac{l}{2} = \frac{\rho_{mb}}{\rho_{H_2O}} l = (S.G)_{mb}\, l$$

and

$$l = \frac{(\gamma_1 - \gamma_2) + d}{\left[(S.G)_{mb} - \frac{1}{2}\right]}$$

(From Table A.1, Appendix A, SG = 1.75.)

$$l = \frac{(2.5 - 0.7)\,m + 0.2\,m}{(1.75 - 0.5)}$$

$$l = 1.6\,m$$

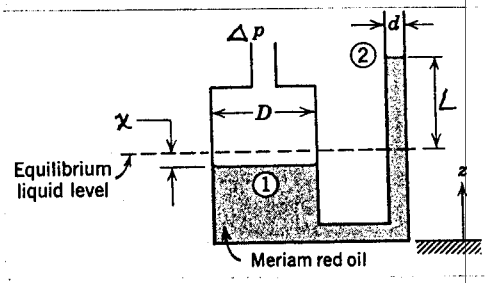
Given: Reservoir manometer with vertical tubes $D = 18$ mm and $d = 6$ mm diameter. Gage liquid is Meriam red oil.

Find: (a) Algebraic expression for deflection L in small tube when gage pressure Δp is applied to the reservoir.
(b) Evaluate L when Δp is equivalent to 25 mm H_2O (gage).

Solution: Use the diagram of Example Problem 3.2, apply hydrostatics.

Computing equations: $\frac{dp}{dh} = +\rho g$; $\Delta p = \rho g \Delta h$; $\rho = SG\,\rho_{H_2O}$

Assumptions: (1) Static liquid
(2) Incompressible liquid



Then $\Delta p = \rho_{oil}\, g\,(x + L)$

From conservation of volume,

$$\frac{\pi D^2}{4} x = \frac{\pi d^2}{4} L \; ; \; x = \left(\frac{d}{D}\right)^2 L$$

so

$$\Delta p = \rho_{water}\, g\, \Delta h = \rho_{oil}\, g \left[\left(\frac{d}{D}\right)^2 L + L\right] = \rho_{oil}\, g L \left[1 + \left(\frac{d}{D}\right)^2\right]$$

Solving for L,

$$L = \frac{\Delta p}{\rho_{oil}\, g \left[1 + (d/D)^2\right]} \quad \leftarrow L$$

Substituting $\Delta p = \rho_{water}\, g\, \Delta h$,

$$L = \frac{\rho_{water}\, g\, \Delta h}{SG_{oil}\, \rho_{water}\, g \left[1 + (d/D)^2\right]} = \frac{\Delta h}{SG_{oil}\left[1 + (d/D)^2\right]}$$

Evaluating, with $SG_{oil} = 0.827$ (Table A.1),

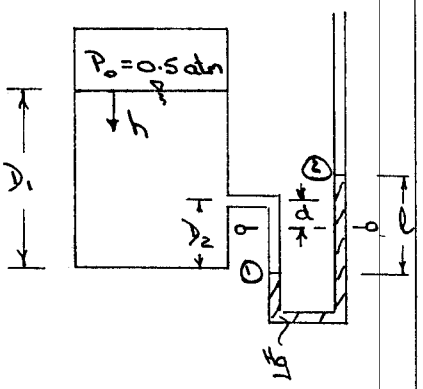
$$L = \frac{25.0 \text{ mm}}{0.827\left[1 + (6/18)^2\right]} = 27.2 \text{ mm} \quad \leftarrow L$$

{Note: $A \equiv \frac{L}{\Delta h_e} = \frac{27.2 \text{ mm}}{25.0 \text{ mm}} = 1.09$ for this manometer.}

Given: A U-tube manometer is connected to a closed tank filled with water as shown. The manometer fluid is Hg.

$D_1 = 2.5\,m$, $D_2 = 0.7\,m$, $d = 0.2\,m$

At the water surface $P_0 = 0.5$ atm (gage)



Find: The manometer deflection l.

Solution

Basic equation $\frac{dP}{dh} = \rho g$

For γ = constant $\Delta P = \rho g \Delta h$

Then, beginning at the free surface and accounting for pressure changes with elevation,

$$P_0 + (P_1 - P_0) + (P_2 - P_1) = P_2 = P_{atm}$$

$$P_0 + \rho_{H_2O} g\left[(D_1 - D_2) + d + \frac{l}{2}\right] - \rho_{Hg} g\, l = P_{atm}$$

$$\frac{P_0 - P_{atm}}{\rho_{H_2O} g} + (D_1 - D_2) + d + \frac{l}{2} = \frac{\rho_{Hg} g}{\rho_{H_2O} g} l = (S.G)_{Hg}\, l$$

and

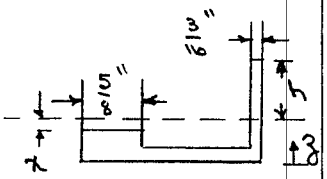
$$l = \frac{(P_0 - P_{atm})/\rho_{H_2O} g + (D_1 - D_2) + d}{(S.G)_{Hg} - 0.5}$$

$$= \frac{0.5\,atm \times 1.01\times10^5 \frac{N}{m^2\cdot atm} \times 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} + (2.5 - 0.7)m + 0.2m}{13.6 - 0.5}$$

$$l = 0.546\,m \longleftarrow l$$

Given: Reservoir manometer with dimensions shown
Manometer fluid SG = 0.827

Find: required distance between marks on vertical scale for 1 in of water ΔP



Solution:

Basic equation: $\frac{dP}{dz} = -\gamma$

Assumptions: (1) static fluid
(2) gravity is only body force
(3) z axis directed vertically

$dP = -\gamma\, dz$

For constant γ, $\Delta P = P_1 - P_2 = -\gamma(z_1 - z_2)$

Under applied pressure $\Delta P = \gamma_{oil}(x+h)$
But conditions of problem require $\Delta P = \gamma_{H_2O}\, \ell$ where $\ell = 1$ in
$\therefore \gamma_{oil}(x+h) = \gamma_{H_2O}\, \ell$

Since the volume of the oil must remain constant

$$x A_{res} = h A_{tube}$$

$$\therefore x = h \frac{A_{tube}}{A_{res}}$$

and $$\gamma_{oil}\left(h \frac{A_t}{A_r} + h\right) = \gamma_{H_2O}\, \ell$$

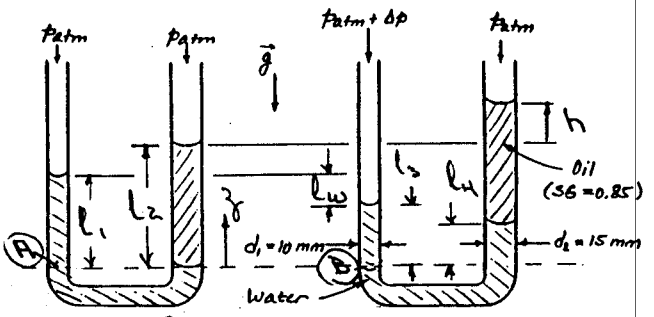
$$\therefore \frac{h}{\ell} = \frac{\gamma_{H_2O}}{\gamma_{oil}} \left(\frac{1}{\frac{A_t}{A_r} + 1} \right) = \frac{1}{SG_{oil}\left[\left(\frac{D_t}{D_r}\right)^2 + 1\right]}$$

$$\frac{h}{\ell} = \frac{1}{0.827\left[\left(\frac{\frac{3}{16}}{\frac{8}{5}}\right)^2 + 1\right]} = \frac{1}{0.827\left[(0.3)^2 + 1\right]}$$

$$\frac{h}{\ell} = 1.11$$

For $\ell = 1.0$ in as given, then $h = 1.11$ in.

Given: U-tube manometer with tubes of different diameter and two liquids, as shown.



Find: (a) the deflection, h, for $\Delta p = 250\ N/m^2$.
(b) the sensitivity of the manometer.

Plot: the manometer sensitivity as a function of d_2/d_1

Solution:

Basic equation: $\frac{dp}{dz} = -\rho g$

Assumptions: (1) static liquid (2) incompressible

Integrating the basic equation from reference state at z_0 to general state at z gives

$$p - p_0 = -\rho g(z - z_0) = \rho g(z_0 - z)$$

From the left diagram: $p_A - p_{atm} = \rho_w g l_1 = \rho_o g l_2$ ----(1)

From the right diagram $p_B - (p_{atm} + \Delta p) = \rho_w g l_3$ ----(2)

$$p_B - p_{atm} = \rho_w g l_4 + \rho_o g l_2 \quad \text{----(3)}$$

Subtracting Eq.2 from Eq.3 and then employing Eq.1 gives

$$\Delta p = \rho_w g(l_4 - l_3) + \rho_o g l_2 = \rho_w g(l_4 + l_1 - l_3)$$

Define $l_w = l_1 - l_3$. Note $l_4 = h$. Then $\Delta p = \rho_w g(h + l_w)$ ...(4)

We can relate l_w to h by recognizing the volume of water must be conserved

$$\therefore \pi \frac{d_1^2}{4} l_w = \pi \frac{d_2^2}{4} h \quad \text{and} \quad l_w = h\left(\frac{d_2}{d_1}\right)^2$$

Substituting into Eq.4 gives

$$\Delta p = \rho_w g\left[h + h\left(\frac{d_2}{d_1}\right)^2\right] = \rho_w g h\left[1 + \left(\frac{d_2}{d_1}\right)^2\right]$$

Solving for h,

$$h = \frac{\Delta p}{\rho_w g[1 + (d_2/d_1)^2]} = 250 \frac{N}{m^2} \times \frac{m^3}{999 kg} \times \frac{s^2}{9.81 m} \times \frac{1}{[1 + (15/10)^2]} \times \frac{kg \cdot m}{N \cdot s^2} \times \frac{10^3\ mm}{m}$$

$$h = 7.85\ mm \quad \text{h}$$

(b) The sensitivity of the manometer is defined as

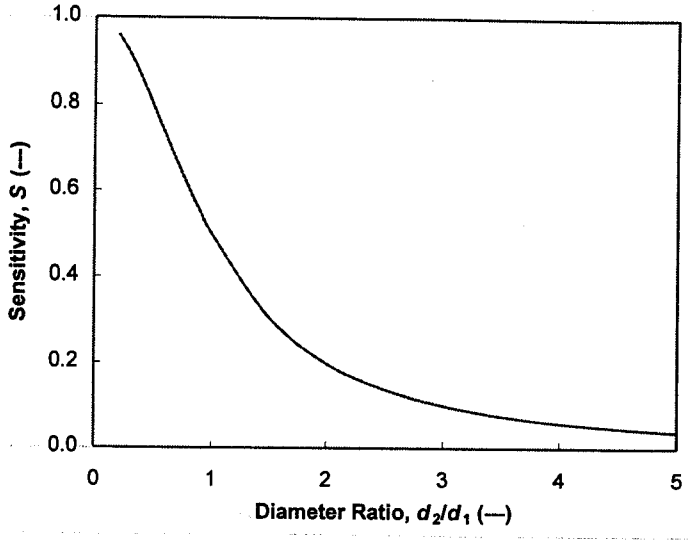
$$S = \frac{h}{\Delta h_e} = \frac{\text{actual deflection}}{\text{equivalent } \Delta h_{H_2O}} \qquad \text{where } \Delta p = \rho_{H_2O} g \Delta h_e$$

$$\therefore S = \frac{h}{\Delta h_e} = \frac{1}{[1 + (d_2/d_1)^2]} = \frac{1}{[1 + (1.5)^2]} = 0.308 \quad \text{S}$$

The design is a poor one. The sensitivity could be improved by interchanging d_2 and d_1, i.e. having $d_2/d_1 < 1.0$ as shown in the plot below.

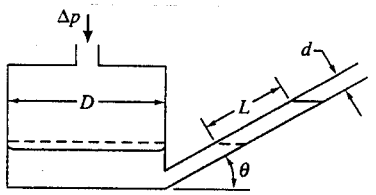
$$S = \frac{1}{[1 + (d_2/d_1)^2]}$$

The manometer sensitivity, as a function of diameter ratio d_2/d_1, is shown below.



13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Inclined manometer as shown filled with oil, SG = 0.897



Find: Angle, θ, such that applied pressure of 1 in. H_2O gage gives 5" oil deflection along incline. Also determine sensitivity

Solution:

Basic equation: $\frac{dp}{dz} = -\gamma$

Assumptions: (1) static fluid
(2) gravity is only body force
(3) z axis directed vertically

$dp = -\gamma\, dz$

For constant γ, $\Delta p = p_1 - p_2 = -\gamma(z_1 - z_2)$

Under applied pressure $\Delta p = \gamma_{oil}(L\sin\theta + x)$

where $\Delta p = 1\text{ in } H_2O = \gamma_{H_2O} h = 62.4 \frac{lbf}{ft^3} \times 1\text{ in} \times \frac{ft}{12\text{ in}} = 5.2 \frac{lbf}{ft^2}$.

Since the volume of the oil must remain constant

$$x A_{res} = L A_{tube}$$

$$\therefore x = L \frac{A_{tube}}{A_{res}}$$

and

$$\Delta p = \gamma_{oil}\left(L\sin\theta + L\frac{A_t}{A_r}\right) = \gamma_{oil}\left[L\sin\theta + L\left(\frac{d}{D}\right)^2\right]$$

Solving for $\sin\theta$,

$$\sin\theta = \frac{\Delta p}{\gamma_{oil} L} - \left(\frac{d}{D}\right)^2$$

$$= 5.2 \frac{lbf}{ft^2} \times \frac{ft^3}{0.897(62.4)lbf} \times \frac{1}{5\text{ in}} \times \frac{12\text{ in}}{ft} - \left(\frac{1}{4(3)}\right)^2$$

$$\sin\theta = 0.2161$$

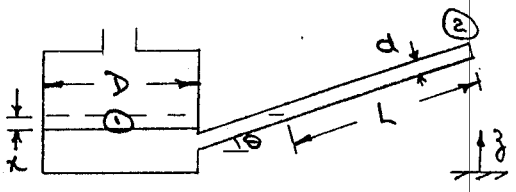
$$\theta = 12.5° \quad \leftarrow \theta$$

The manometer sensitivity, $S = \frac{L}{\Delta h_e} = \frac{5\text{ in}}{1\text{ in}} = 5$ ← S

Given: Inclined manometer as shown

$D = 96\,mm$, $d = 8\,mm$

Angle θ is such that liquid deflection is five times that of U-tube manometer under same applied pressure difference



Find: angle, θ and manometer sensitivity

Solution:

Basic equation $\frac{dP}{dz} = -\rho g$

Then $dP = -\rho g\, dz$ and for constant ρ

$$\Delta P = P_1 - P_2 = -\rho g (z_1 - z_2)$$

For the inclined manometer,

$$P_1 - P_{atm} = \rho g (L \sin\theta + x)$$

Since the volume of the oil must remain constant,

$$x A_{res} = L A_{tube}$$

$$x = L \frac{A_{tube}}{A_{res}} = L \left(\frac{d}{D}\right)^2$$

Then

$$P_1 - P_{atm} = \rho g (L\sin\theta + x) = \rho g \left(L\sin\theta + L\left(\frac{d}{D}\right)^2\right) = \rho g L \left(\sin\theta + \left(\frac{d}{D}\right)^2\right)$$

For a U-tube manometer

$$P_1 - P_{atm} = -\rho g (z_1 - z_2) = \rho g h$$

Hence,

$$\frac{(P_1 - P_{atm})_{incl}}{(P_1 - P_{atm})_{U\text{-}tube}} = \frac{\rho g L \left[\sin\theta + \left(\frac{d}{D}\right)^2\right]}{\rho g h}$$

For same applied pressure and $L/h = 5$

$$1 = 5\left[\sin\theta + \left(\frac{d}{D}\right)^2\right]$$

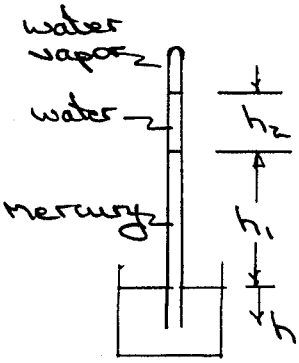
$$\theta = \sin^{-1}\left[0.2 - \left(\frac{d}{D}\right)^2\right] = \sin^{-1}\left[0.2 - \left(\frac{8}{96}\right)^2\right] = 11.1^\circ \quad \Theta$$

The manometer sensitivity $S = \frac{L}{\Delta h} = \frac{L}{\rho g h} = \frac{5}{\rho g}$

Given: Barometer with 6.5 in. of water on top of the mercury column of height 28.35 in.; Temperature $T = 70°F$

Find: (a) Barometric pressure in psia.
(b) Effect of increase in ambient temperature (to $T_a = 85°F$) on length of mercury column for same barometric pressure.

Solution:



Basic equation: $\frac{dp}{dh} = \rho g$

Assumptions: (1) static liquid
(2) incompressible
(3) g = constant

For, $dp = \rho g\, dh$ and $\Delta p = \rho g \Delta h$

Start at the free surface of the mercury ($p = p_{atm}$) and progress through the barometer to p_v (vapor pressure of the water)

$$p_{atm} - \rho_{Hg} g h_1 - \rho_{H_2O} g h_2 = p_v$$

$$p_{atm} = \rho_{Hg} g h_1 + \rho_{H_2O} g h_2 + p_v = \rho_{H_2O} SG_{Hg} h_1 + \rho_{H_2O} g h_2 + p_v$$

$$p_{atm} = \rho_{H_2O} g \left[SG_{Hg} h_1 + h_2 \right] + p_v$$

From Table A.2, $SG_{Hg} = 13.55$

Table A.7 $\rho_{H_2O} = 1.93$ slug/ft^3, $p_v = 0.363$ psia.

Evaluating,

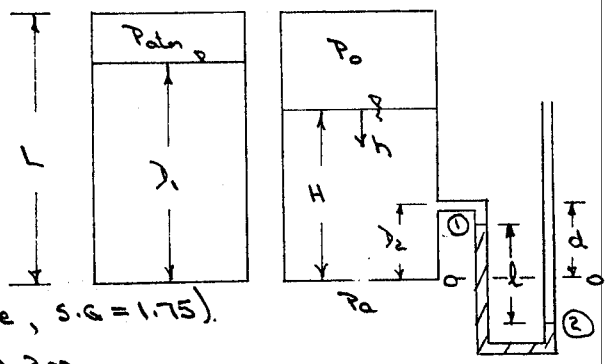
$$p_{atm} = 1.93 \frac{\text{slug}}{\text{ft}^3} \times 32.2 \frac{\text{ft}}{\text{s}^2} \left[13.55 \times 28.35 \text{ in} + 6.5 \text{ in} \right] \frac{\text{ft}}{12 \text{ in}} \times \frac{\text{ft}^2}{144 \text{ in}^2} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{ft} \cdot \text{slug}} + 0.363 \text{ psia}$$

$$p_{atm} = 14.4 \text{ psia}$$

At $T = 85°F$, the vapor pressure of water is estimated (from Table A.7) to be ≈ 0.60 psia. For the same barometric pressure the length of the mercury column would be shorter at the higher ambient temperature.

Given: Sealed tank of cross-section A and height $L = 3.0\,m$ is filled with water to a depth $h_1 = 2.5\,m$.
Water drains slowly from the tank until system attains equilibrium.
U-tube manometer is connected to tank as shown.
(manometer fluid is merian blue, S.G = 1.75).

$h_1 = 2.5\,m$, $h_2 = 0.7\,m$, $d = 0.2\,m$



Find: The manometer deflection, ℓ, under equilibrium conditions

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$ $\qquad p\forall = MRT$

For γ = constant $\Delta p = \rho g \Delta h$

To determine the surface pressure p_o under equilibrium conditions treat air above water as an ideal gas

$$\frac{p_a \forall_a}{p_o \forall_o} = \frac{MRT_a}{MRT_o} \qquad \text{Assuming } T_a = T_o, \text{ then}$$

$$p_o = \frac{\forall_a}{\forall_o} p_a = \frac{A(L-h_1)}{A(L-H)} p_a = \frac{(L-h_1)}{(L-H)} p_a$$

Under equilibrium conditions, $p_o + \rho_{H_2O} g H = p_a$

Hence, $\frac{(L-h_1)}{(L-H)} p_a + \rho_{H_2O} g H = p_a$ or $\rho_{H_2O} g H^2 - H(p_a + \rho_{H_2O} g L) + h_1 p_a = 0$

and

$$H = \frac{(p_a + \rho_{H_2O} g L) \pm \sqrt{(p_a + \rho_{H_2O} g L)^2 - 4 \rho_{H_2O} g h_1 p_a}}{2 \rho_{H_2O} g}$$

$$H = \frac{\left[1.01\times10^5 \frac{N}{m^2} + \frac{999\,kg}{m^3}\times\frac{9.81\,m}{sec^2}\times 3m \times \frac{N\cdot sec^2}{kg\cdot m}\right] \pm \sqrt{[\,]^2 - 4\times\frac{999\,kg}{m^3}\times\frac{9.81\,m}{sec^2}\times 2.5m \times 1.01\times10^5 \frac{N}{m^2}\times\frac{N\cdot sec^2}{kg\cdot m}}}{2\times 999\frac{kg}{m^3}\times\frac{9.81\,m}{sec^2}\times\frac{N\cdot sec^2}{kg\cdot m}}$$

$H = 10.9\,m$ or $2.36\,m$. From physical considerations $H = 2.36\,m$

$$p_o = \frac{(L-h_1)}{(L-H)} p_a = \frac{(3.0-2.5)}{(3.0-2.36)} \times 1.01\times10^5\ N/m^2 = 7.89\times10^4\ N/m^2$$

For the manometer, $p_o + (p_1 - p_o) + (p_2 - p_1) = p_2 = p_{atm}$

$$p_o + \rho_{H_2O} g (H - h_2 + d - \tfrac{\ell}{2}) + \rho_{Hb} g \ell = p_{atm}$$

$$\frac{p_{atm} - p_o}{\rho_{H_2O} g} - H + h_2 - d = (S.G)_{Hb}\,\ell - \frac{\ell}{2} = \ell\left[(S.G)_{Hb} - 0.5\right]$$

$$\ell = \frac{(p_{atm} - p_o)/\rho_{H_2O} g - H + h_2 - d}{(S.G)_{Hb} - 0.5} = \frac{(10.1 - 7.89)\times10^4 \frac{N}{m^2}\times\frac{m^3}{999\,kg}\times\frac{s^2}{9.81\,m}\times\frac{kg\,m}{N\cdot s^2} - 2.36m + 0.7m - 0.2m}{1.75 - 0.5}$$

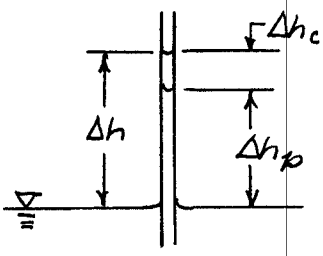
$\ell = 0.316\,m$ ←

Given: Water column standing at $\Delta h = 50$ mm in $D = 2.5$ mm glass tube.

Find: (a) Column height if surface tension were zero.
(b) Column height in $D = 1$ mm tube.

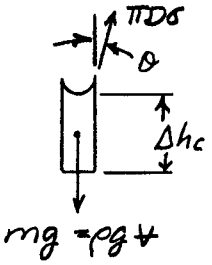
Solution: Assume column height is sum of capillary rise and rise caused by pressure difference,

$$\Delta h = \Delta h_c + \Delta h_p$$



Choose a free-body diagram of Δh_c for analysis:

$$\Sigma F_z = \pi D \sigma \cos\theta - \frac{\pi D^2}{4} \rho g \Delta h_c = 0$$



Assumptions: (1) Neglect volume under meniscus
(2) Δh_p remains constant

Then $$\Delta h_c = \frac{4\sigma}{\rho g D} \cos\theta$$

For water (Table A.4), $\sigma = 72.8$ mN/m and $\theta \approx 0$, so $\cos\theta = 1$, and

$$\Delta h_c = \frac{4\sigma}{\rho g D}$$

For the $D = 2.5$ mm tube,

$$\Delta h_c = 4 \times 72.8 \times 10^{-3} \frac{N}{m} \times \frac{m^3}{999\ kg} \times \frac{s^2}{9.81\ m} \times \frac{1}{0.0025\ m} \times \frac{kg \cdot m}{N \cdot s^2} = 0.0119\ m \text{ or } 11.9\ mm$$

Then

$$\Delta h_p = \Delta h - \Delta h_c = (50.0 - 11.9)\ mm = 38.1\ mm \quad (\sigma = 0)$$

Δh_p

For the $D = 1.0$ mm tube,

$$\Delta h_c = 4 \times 72.8 \times 10^{-3} \frac{N}{m} \times \frac{m^3}{999\ kg} \times \frac{s^2}{9.81\ m} \times \frac{1}{0.001\ m} \times \frac{kg \cdot m}{N \cdot s^2} = 0.0297\ m \text{ or } 29.7\ mm$$

So

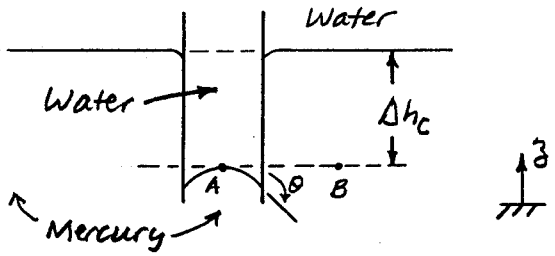
$$\Delta h = \Delta h_c + \Delta h_p = (29.7 + 38.1)\ mm = 67.8\ mm \quad (D = 1.0\ mm \text{ tube})$$

Δh

Given: Interface between two immiscible liquids of different densities.

Find: (a) Expression for difference in level caused by surface tension.
(b) Plot of height change caused by capillary action versus tube diameter for water over mercury.

Solution: Choose the small volume of water as a free-body:



Summing forces,

$$\Sigma F_z = p_A \frac{\pi D^2}{4} - \rho_{H_2O} g \Delta h_c \frac{\pi D^2}{4} + \pi D \sigma \cos\theta = 0$$

Assumptions: (1) Neglect volume between meniscus and plane AB
(2) $p_A = p_B = \rho_{Hg} g \Delta h_c$

Solving, $$\Delta h_c = -\frac{4\sigma\cos\theta}{(\rho_{Hg} - \rho_{H_2O})gD} = -\frac{4\sigma\cos\theta}{\rho_{H_2O} g D (SG_{Hg} - 1)}$$ Δh_c

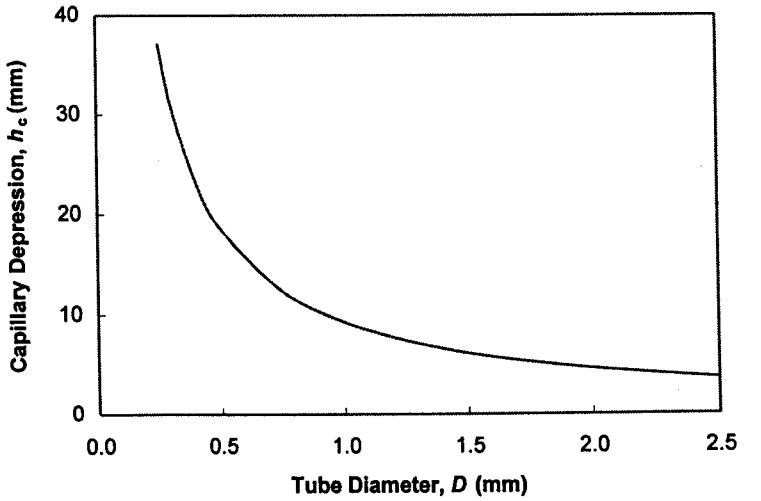
For mercury (Table A.4), $\sigma = 375$ mN/m and $\theta = 140°$, so

$$\Delta h_c (mm) = -4 \times 375 \times 10^{-3} \frac{N}{m} \times \frac{m^3}{1000 kg} \times \frac{s^2}{9.81 m} \times \frac{1}{D(mm)} \times \frac{1}{(13.6-1)} \times (1000)^2 \frac{mm^2}{m^2} \times \frac{kg \cdot m}{N \cdot s^2} \times \cos 140°$$

$$\Delta h_c (mm) = \frac{9.30}{D(mm)} \quad \text{(water over mercury)}$$ Δh_c

{The meniscus moves downward, so Δh_c is a capillary depression for this case.}

The plot is:



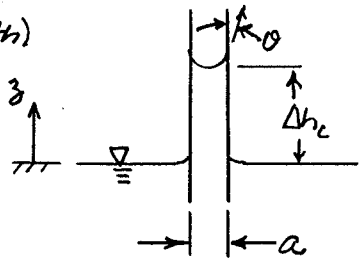
Given: Level change caused by capillary action in tube, between plates.

Find: (a) Comparison of configurations.
(b) Plots of level change for air over water vs. diameter, spacing.

Solution: Choose elements shown, analyze free-body diagrams.

(i) Parallel plates, spacing a.

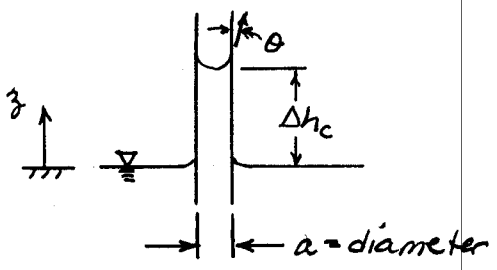
(w = width)



$$\Sigma F_z = 2w\sigma\cos\theta - wa\rho g\Delta h_c = 0$$

$$\Delta h_c = \frac{2\sigma\cos\theta}{\rho g a}$$

(ii) Circular tube, diameter a.



$$\Sigma F_z = \pi a\sigma\cos\theta - \frac{\pi D^2}{4}\rho g\,\Delta h_c = 0$$

$$\Delta h_c = \frac{4\sigma\cos\theta}{\rho g a}$$

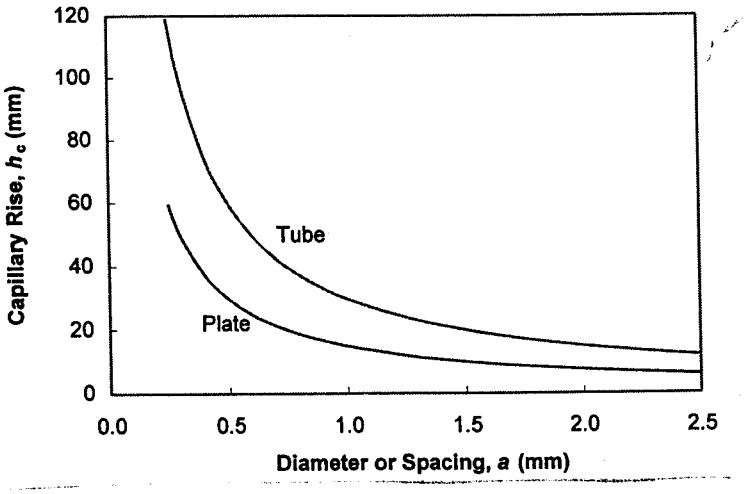
For water (Table A.4) $\sigma = 72.8$ mN/m and $\theta \approx 0°$, so $\cos\theta \approx 1$. Then

$$\Delta h_c(\text{planar}) = 2\times 72.8\times 10^{-3}\frac{N}{m}\times(1)\times\frac{m^3}{999\,kg}\times\frac{s^2}{9.81\,m}\times\frac{1}{a(mm)}\times(1000)^2\frac{mm^2}{m^2}\times\frac{kg\cdot m}{N\cdot s^2} = \frac{14.9}{a(mm)}$$

$$\Delta h_c(\text{tube}) = 2\Delta h_c(\text{planar}) = \frac{29.8}{a(mm)}$$

Δh_c

The plot is



Given: Atmosphere in which T = constant = 30°C between sea level and 5 km altitude.

Find: (a) Elevation change, Δz, corresponding to a 1% reduction in air pressure.
(b) Elevation change, Δz, necessary to effect a 15% reduction in density

Plot: p_2/p_1 and ρ_2/ρ_1 vs. z.

Solution:



Basic equations: $\frac{dp}{dz} = -\rho g$, $p = \rho RT$

Assumptions: (1) static, isothermal fluid
(2) g = constant
(3) ideal gas behavior

Then, $\frac{dp}{dz} = -\rho g = -\frac{p}{RT}g$ and $\frac{dp}{p} = -\frac{g}{RT_0}dz$

Separating variables and integrating,

$$\int_{p_1}^{p_2}\frac{dp}{p} = -\frac{g}{RT_0}\int_{z_1}^{z_2} dz \quad \text{and} \quad \ln\frac{p_2}{p_1} = -\frac{g}{RT_0}\Delta z$$

$$\Delta z = -\frac{RT_0}{g}\ln\frac{p_2}{p_1} \qquad (1)$$

For an ideal gas, $\frac{p_2}{p_1} = \frac{\rho_2 RT_0}{\rho_1 RT_0} = \frac{\rho_2}{\rho_1}$

Thus, $$\Delta z = -\frac{RT_0}{g}\ln\frac{\rho_2}{\rho_1} \qquad (2)$$

From Table A.6 $R_{air} = 287\ N\cdot m/kg\cdot K$

Evaluating, $\frac{RT_0}{g} = 287\frac{N\cdot m}{kg\cdot K}\times(273+30)K\times\frac{s^2}{9.81\ m}\times\frac{kg\cdot m}{N\cdot s^2} = 8860\ m$

For a one percent reduction in pressure, $p_2/p_1 = 0.99$. From (1)

$$\Delta z = -8860\ m\ \ln(0.99) = 89.0\ m \qquad (a)$$

For a 15% reduction in density, $\rho_2/\rho_1 = 0.85$. From (2)

$$\Delta z = -8860\ m\ \ln(0.85) = 1440\ m \qquad (b)$$

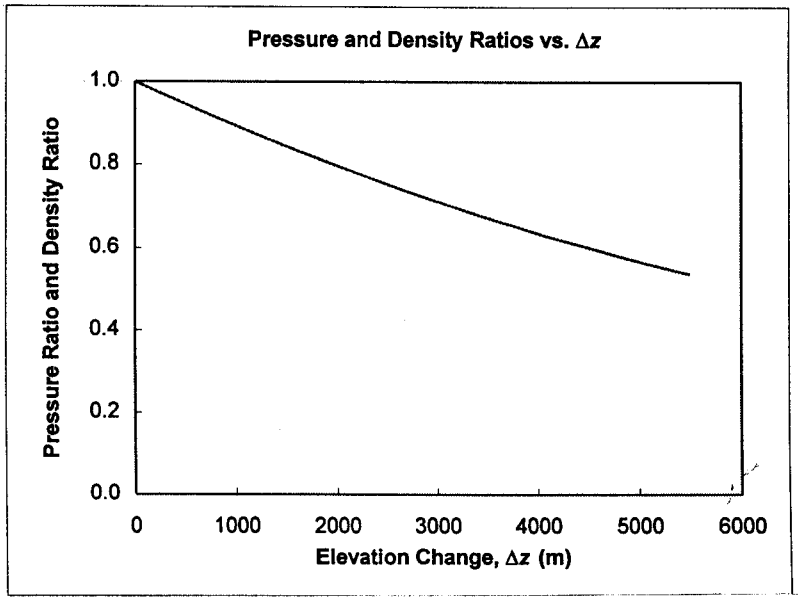
To plot p_2/p_1 and ρ_2/ρ_1, we rewrite eqs. (1) and (2) as

$$\frac{p_2}{p_1} = \frac{\rho_2}{\rho_1} = e^{-g\Delta z/RT_0}$$

The plot is presented below

Pressure and density ratio variation with altitude (T = 30°C):

Elevation Change Δz (m)	Pressure Ratio p_2/p_1 (---)	Density Ratio ρ_2/ρ_1 (---)
0	1.00	1.00
500	0.945	0.945
1000	0.893	0.893
1500	0.844	0.844
2000	0.798	0.798
2500	0.754	0.754
3000	0.713	0.713
3500	0.674	0.674
4000	0.637	0.637
4500	0.602	0.602
5000	0.569	0.569
5500	0.538	0.538



Note: Since T = constant, both ratios are the same!

Given: Martian atmosphere behaves as an ideal gas, T = constant
$M_m = 32.0$, $T = 200\ K$, $g = 3.92\ m/s^2$, $\rho_0 = 0.015\ kg/m^3$

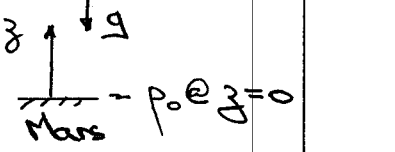
Find: Density at $z = 20\ km$.

Plot: the ratio ρ/ρ_0 (ratio of density to surface density) vs z; compare with earth's atmosphere.

Solution:

Basic equations: $\frac{dp}{dz} = -\rho g$; $p = \rho RT$; $R = R_u/M_m$

Assumptions: (1) static fluid
(2) g constant
(3) ideal gas.



Since T = constant, $dp = d(\rho RT) = RT\,d\rho$

$$\frac{dp}{dz} = RT\frac{d\rho}{dz} = -\rho g \quad \text{and} \quad \int_{\rho_0}^{\rho}\frac{d\rho}{\rho} = -\int_0^z \frac{g}{RT}\,dz$$

$$\ln\frac{\rho}{\rho_0} = -gz/RT \quad \text{and} \quad \frac{\rho}{\rho_0} = e^{-gz/RT} \qquad ----(1)$$

Evaluating

$$R = \frac{R_u}{M_m} = 8314.3\ \frac{N\cdot m}{kgmole\cdot K} \times \frac{kgmole}{32.0\ kg} = 260\ \frac{N\cdot m}{kg\cdot K}$$

$$\rho = 0.015\ \frac{kg}{m^3} \times \exp\left[-3.92\ \frac{m}{s^2} \times 20\times10^3\ m \times 260\ \frac{kg\cdot K}{N\cdot m} \times \frac{1}{200\ K} \times \frac{N\cdot s^2}{kg\cdot m}\right]$$

$$\rho = 0.00332\ kg/m^3$$

$\rho_{z=20\,km}$

For the Martian atmosphere, Eq. 1 gives $\rho/\rho_0 = e^{-0.0754\,z\,(km)}$

For the earth's atmosphere, ρ/ρ_0 is given in Table A.3

Both ρ/ρ_0 variations are plotted below.

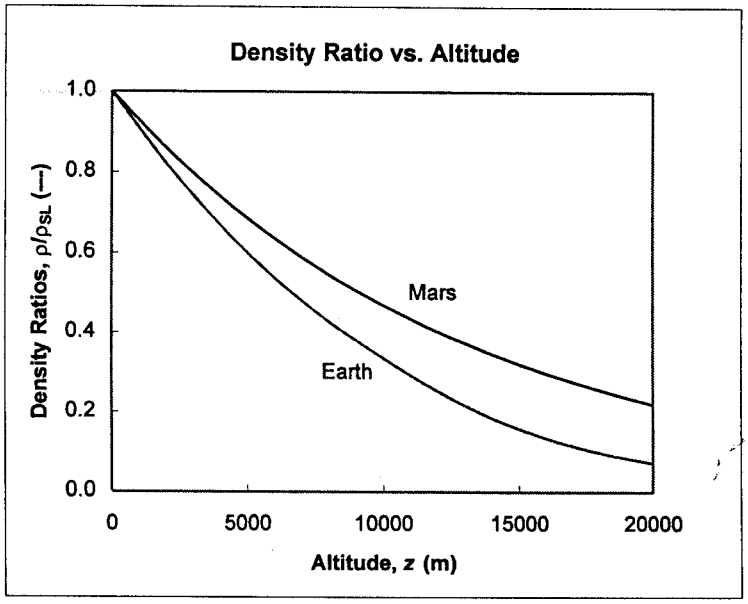
Note from the plot:

- on Mars $\rho/\rho_0 = 0.221$ at $z = 20\ km$, whereas
- on Earth, $\rho/\rho_0 = 0.073$ at $z = 20\ km$.

The difference is caused by (a) the larger gravity on Earth, and (b) temperature decrease with altitude in our atmosphere.

Density vs. elevation in Martian and Earth atmospheres:

Elevation Change Δz (m)	Density Ratio (Earth) ρ/ρ_{SL} (---)	Density Ratio (Mars) ρ/ρ_{SL} (---)
0	1.000	1.00
2000	0.8217	0.860
4000	0.6689	0.740
6000	0.5389	0.636
8000	0.4292	0.547
10000	0.3376	0.470
12000	0.2546	0.405
14000	0.1860	0.348
16000	0.1359	0.299
18000	0.09930	0.257
20000	0.07258	0.221



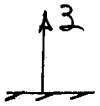
13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U. S. A.
National® Brand

Given: Atmospheric conditions at ground level ($z=0$) in Denver, Colorado are $p_0 = 83.2$ kPa, $T_0 = 25°C$. Pike's peak is at elevation $z = 2690$ m

Find: Pressure on Pike's peak assuming (a) an incompressible, and (b) an adiabatic atmosphere.

Plot: p/p_0 vs z for both cases.

Solution:



Basic equations: $dp/dz = -\rho g$; $p = \rho RT$

Assumptions: (1) static fluid, (2) g = constant, (3) ideal gas behavior

(a) For an incompressible atmosphere $\int_{p_0}^{p} dp = -\int_0^z \rho g\, dz$

$$p - p_0 = \rho g z = \frac{p_0}{RT_0} g z \quad \text{and} \quad p = p_0\left[1 - \frac{gz}{RT_0}\right] \quad \text{------(1)}$$

At $z = 2690$ m

$$p = 83.2\text{ kPa}\left[1 - \frac{9.81\text{ m}}{\text{s}^2} \times 2690\text{ m} \times \frac{\text{kg·K}}{287\text{ N·m}} \times \frac{1}{298\text{ K}} \times \frac{\text{N·s}^2}{\text{kg·m}}\right] = 57.5\text{ kPa}$$

$p_{p=c}$

(b) For an adiabatic atmosphere $p/\rho^k = \text{constant}$, $\rho = \rho_0 (p/p_0)^{1/k}$

$$\frac{dp}{dz} = -\rho g = -g\rho_0 \left(\frac{p}{p_0}\right)^{1/k} \quad \text{or} \quad \int_{p_0}^{p} \frac{dp}{p^{1/k}} = -\int_0^z \rho_0 p_0^{-1/k} g\, dz$$

For $\frac{k}{(k-1)} p^{-\frac{1}{k}+1}\Big|_{p_0}^{p} = -\rho_0 p_0^{-1/k} g z$ or $\frac{k}{(k-1)}\left[p^{(k-1)/k} - p_0^{(k-1)/k}\right] = -\rho_0 p_0^{1/k} g z$

and $\frac{k}{(k-1)} p_0^{(k-1)/k}\left[\left(\frac{p}{p_0}\right)^{(k-1)/k} - 1\right] = -\rho_0 p_0^{-1/k} g z$

$$\left(\frac{p}{p_0}\right)^{(k-1)/k} = 1 - \frac{(k-1)}{k} p_0^{-1/k - (k-1)/k} \rho_0 g z \qquad \rho_0 g z = 1 - \frac{(k-1)}{k} p_0^{-1} \rho_0 g z$$

and $\frac{p}{p_0} = \left[1 - \frac{(k-1)}{k} \frac{\rho_0}{p_0} g z\right]^{k/k-1} = \left[1 - \frac{(k-1)}{k} \frac{gz}{RT_0}\right]^{k/k-1}$ ------(2)

Evaluating at $z = 2690$ m

$$p = 83.2\text{ kPa}\left[1 - \frac{0.4}{1.4} \times \frac{9.81\text{ m}}{\text{s}^2} \times 2690\text{ m} \times \frac{\text{kg·K}}{287\text{ N·m}} \times \frac{1}{298\text{ K}} \times \frac{\text{N·s}^2}{\text{kg·m}}\right]^{\frac{1.4}{0.4}}$$

$p = 60.2$ kPa $\quad p_{adi}$

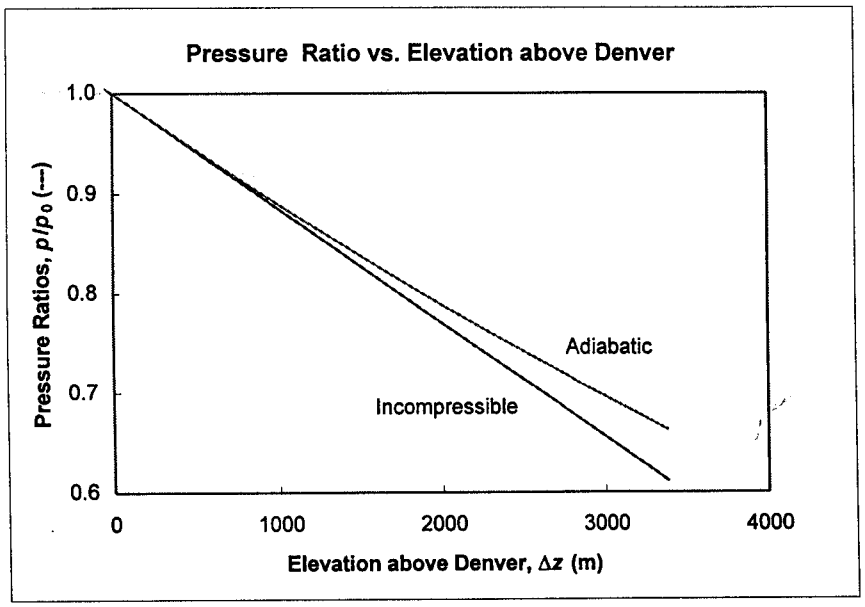
The pressure ratio p/p_0 vs z is plotted for an incompressible atmosphere (Eq. 1) and an adiabatic atmosphere (Eq. 2) below.

Incompressible case $\quad p/p_0 = [1 - 0.115z]$ (z in km).

Adiabatic case $\quad p/p_0 = [1 - 0.0328z]^{3.5}$ (z in km)

Pressure ratio vs. elevation above Denver:

Elevation z (m)	Elevation above Denver z (m)	Pressure Ratio (T = C) p/p_0 (---)	Pressure Ratio (adiabatic) p/p_0 (---)
0	-1610	1.185	1.20
500	-1110	1.127	1.13
1000	-610	1.070	1.07
1500	-110	1.013	1.01
2000	390	0.955	0.956
2500	890	0.898	0.902
3000	1390	0.841	0.849
3500	1890	0.783	0.800
4000	2390	0.726	0.752
4300	2690	0.691	0.724
4500	2890	0.669	0.706
5000	3390	0.611	0.662



13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Open-Ended Problem Statement: A hydropneumatic elevator consists of a piston-cylinder assembly to lift the elevator cab. Hydraulic oil, stored in an accumulator tank pressurized by air, is valved to the piston as needed to lift the elevator. When the elevator descends, oil is returned to the accumulator. Design the least expensive accumulator that can satisfy the system requirements. Assume the lift is 3 floors, the maximum load is 10 passengers, the maximum system pressure is 800 kPa (gage). For column bending strength, the piston diameter must be at least 150 mm. The elevator cab and piston have a combined mass of 3,000 kg, and are to be purchased. Perform the analysis needed to define, as a function of system operating pressure, the piston diameter, the accumulator volume and diameter, and the wall thickness. Discuss safety features that your company should specify for the complete elevator system. Would it be preferable to use a completely pneumatic design or a completely hydraulic design? Why?

Discussion: The design requirements are specified, except that a typical floor height is about 12 ft, making the total required lift about 36 ft.)

A spreadsheet was used to calculate the system properties for various pressures. Results are presented on the next page, followed by a sample calculation.

Total cost dropped quickly as system pressure was increased. A shallow minimum was reached in the 100-110 psig range.

The lowest-cost solution was obtained at a system pressure of about 100 psig. At this pressure, the reservoir of 140 gal required a 3.30 ft diameter pressure sphere with a 0.250 in. wall thickness. The welding cost was $311 and the material cost $433, for a total cost of $744.

Accumulator wall thickness was constrained at 0.250 in. for pressures below 100 psi; it increased for higher pressures (this caused the discontinuity in slope of the curve at 100 psig). The mass of steel became constant above 110 psig.

No allowance was made for the extra volume needed to pressurize the accumulator.

Fail-safe design is essential for an elevator to be used by the public. The control circuitry should be redundant. Failures must be easy to spot. For this reason, hydraulic actuation is good: leaks will be readily apparent. The final design must be reviewed, approved, and stamped by a professional engineer since the design involves public safety.

The terminology used in the solution is defined in Table 1.

Table 1. Symbols, definitions, and units

Symbol	Definition	Units
p	system pressure	psig
A_p	area of lift piston	$in.^2$
V_{oil}	volume of oil	gal
D_s	diameter of (spherical) accumulator	ft
t	wall thickness of spherical accumulator	in.
A_w	area of weld	$in.^2$
C_w	cost of weld	$
M_s	mass of (steel) accumulator	lbm
C_s	cost of steel	$
C_t	total cost	$

Results of the system simulation and sample calculations are presented on the next page.

Table 2. Results of system simulation

Input Data:	Cab and piston weight:	W_{cab} =	6,000	lbf
	Passenger weight:	W_{pax} =	1,500	lbf
	Total weight:	W_{tot} =	7,500	lbf
	Allowable stress:	σ =	4,000	psi
	Minimum wall thickness:	t =	0.250	in.
	Welding cost factor:	cf_w =	5.00	\$/in.^{2}
	Steel cost factor:	cf_s =	1.25	\$/pound

Results:

p (psig)	A_p (in.^{2})	V_{oil} (gal)	D_s (ft)	t (in.)	A_w (in.^{2})	C_w (\$)	M_s (lbm)	C_s (\$)	C_t (\$)
20	375	701	5.64	0.250	106.2	\$531	1012	\$1,265	\$1,796
30	250	468	4.92	0.250	92.8	\$464	772	\$965	\$1,429
40	188	351	4.47	0.250	84.3	\$422	638	\$797	\$1,218
50	150	281	4.15	0.250	78.3	\$391	549	\$687	\$1,078
60	125	234	3.91	0.250	73.7	\$368	487	\$608	\$976
70	107	200	3.71	0.250	70.0	\$350	439	\$549	\$899
80	93.8	175	3.55	0.250	66.9	\$335	402	\$502	\$837
90	83.3	156	3.41	0.250	64.4	\$322	371	\$464	\$786
100	75.0	140	3.30	0.250	62.1	\$311	346	\$433	\$743
110	68.2	128	3.19	0.263	63.4	\$317	342	\$428	\$745
120	62.5	117	3.10	0.279	65.3	\$326	342	\$428	\$754
130	57.7	108	3.02	0.294	67.1	\$335	342	\$428	\$763

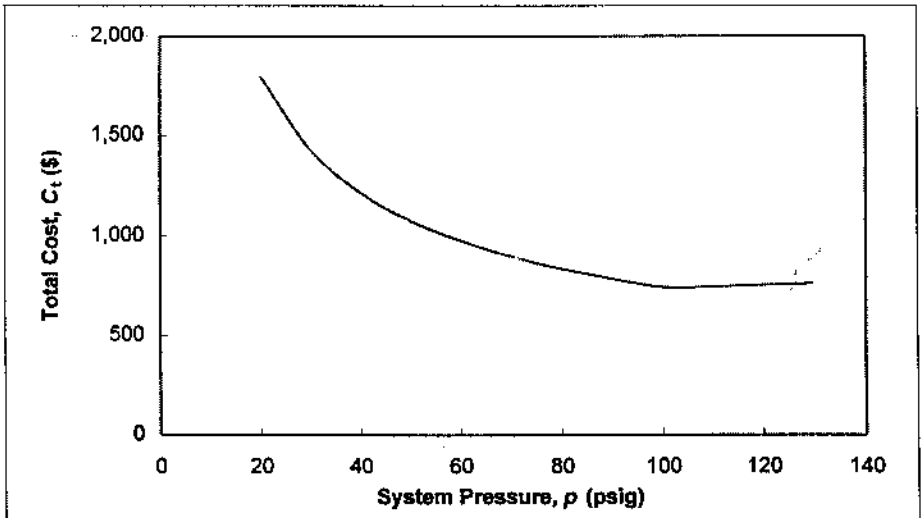


Fig. 1 Total cost versus system pressure

Sample Calculation (p = 20 psig):

$$W_t = p A_p \; ; \; A_p = \frac{W_t}{p} = 7500 \text{ lbf} \times \frac{\text{in.}^2}{20 \text{ lbf}} = 375 \text{ in.}^2$$

$$\forall_{oil} = A_p L = 375 \text{ in.}^2 \times \frac{1}{36 \text{ ft}} \times \frac{\text{ft}^2}{144 \text{ in.}^2} \times 7.48 \frac{\text{gal}}{\text{ft}^3} = 701 \text{ gal}$$

$$\forall_{oil} = \forall_s = \frac{4\pi R_s^3}{3} = \frac{\pi D_s^3}{6} \; ; \; D_s = \left(\frac{6 \forall_{oil}}{\pi}\right)^{1/3} = \left(\frac{6}{\pi} \times 701 \text{ gal} \times \frac{\text{ft}^3}{7.48 \text{ gal}}\right)^{1/3} = 5.64 \text{ ft}$$

From a force balance on the sphere:



Thus $p\frac{\pi D_s^2}{4} = \pi D_s t \sigma$, so $t = \frac{p}{\sigma}\frac{D_s}{4} = \frac{1}{4} \times 20\frac{lbf}{in.^2} \times \frac{in.^2}{4000\,lbf} \times 5.64\,ft \times 12\frac{in.}{ft} = 0.0846$

Therefore $t = t_{min} = 0.250$ in.

$$A_w = \pi D_s t = \pi \times 5.64\,ft \times 0.25\,in. \times 12\frac{in.}{ft} = 106\ in.^2$$

$$C_w = \frac{\$5.00}{in.^2} \times 106\ in.^2 = \$531$$

$$M_s = 4\pi R_s^2 t \rho_s = \pi D_s^2 t SG_s \rho_{H_2O} = \pi \times (5.64)^2\,ft^2 \times 0.25\,in. \times 7.8 \times 62.4\frac{lbm}{ft^3} \times \frac{ft}{12\,in.} = 1012\,lbm$$

$$C_s = \frac{\$1.25}{lbm} \times 1012\ lbm = \$1265$$

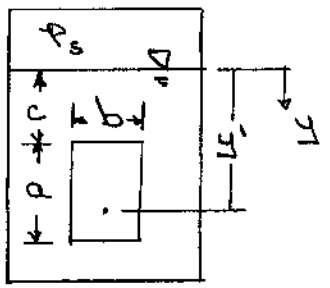
and

$$C_t = C_w + C_s = \$531 + \$1265 = \$1,796$$

C_t

13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-398 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Door located in plane vertical wall of water tank as shown
$a = 1.5$ m, $b = 1$ m, $c = 1$ m.
Atmospheric pressure acts on outer surface of door.



Find: (a) For $p_s = p_{atm}$, resultant force on door and line of action of force
(b) Resultant force and line of action if $p_s = 0.3$ atm(gage)

Plot: F/F_o and y'/y'_c over range of p_s/p_{atm}. (F_o is resultant force when $p_s = p_{atm}$; y'_c is y coordinate of centroid.)

Solution:

Basic equations: $\frac{dp}{dy} = \rho g$; $F_R = \int_A p\,dA$; $y'F_R = \int_A y p\,dA$

Assumptions: (1) static liquid
(2) incompressible liquid

Note: We will obtain a general expressions for F and y' (needed for the plot) and then simply for cases (a) & (b).

Since $dp = \rho g\,dy$ then $p = p_s + \rho g y$

Because p_{atm} acts on the outside of the door, then p_s is the surface gage pressure.

$$F_R = \int p\,dA = \int_c^{(c+a)} p\,b\,dy = \int_c^{c+a} (p_s + \rho g y)\,b\,dy = b\left[p_s y + \frac{\rho g y^2}{2}\right]_c^{c+a}$$

$$F_R = b\left[p_s a + \frac{\rho g}{2}\{(c+a)^2 - c^2\}\right] = b\left[p_s a + \frac{\rho g}{2}(a^2 + 2ac)\right] \quad \text{---------(1)}$$

$$y'F_R = \int y p\,dA \quad \text{and} \quad y' = \frac{1}{F_R}\int_c^{c+a} y(p_s + \rho g y)\,b\,dy$$

$$y' = \frac{b}{F_R}\left[\frac{p_s y^2}{2} + \frac{\rho g y^3}{3}\right]_c^{c+a}$$

$$y' = \frac{b}{F_R}\left[\frac{p_s}{2}\{(c+a)^2 - c^2\} + \frac{\rho g}{3}\{(c+a)^3 - c^3\}\right] \quad \text{---------(2)}$$

(a) For $p_s = 0$ (gage) then

from Eq. 1 $F_R = \frac{\rho g b}{2}(a^2 + 2ac)$.

$$F_R = \frac{1}{2} \times 999\ \frac{kg}{m^3} \times 9.81\ \frac{m}{s^2} \times 1\ m\left[(1.5\ m)^2 + 2(1.5\ m)(1\ m)\right] \frac{N \cdot s^2}{kg \cdot m} = 25.7\ kN \quad \underline{F_a}$$

From Eq. 2

$$y' = \frac{b}{F_{R_o}}\frac{\rho g}{3}\left[(c+a)^3 - c^3\right]$$

$$y' = \frac{1\ m}{25.7\ kN} \times \frac{999}{3}\ \frac{kg}{m^3} \times 9.81\ \frac{m}{s^2}\left[(2.5)^3 - 1\right] m^3 \frac{N \cdot s^2}{kg \cdot m} \cdot \frac{kN}{10^3\ N} = 1.86\ m \quad \underline{y'_o}$$

b) For $p_s = 0.3$ atm (gage) then

from Eq. 1 $\quad F_R = b\left[p_s a + \frac{\rho g}{2}(a^2 + 2ac)\right]$

$$F_R = 1\,m\left[0.3\,atm \times 1.01\times10^5\,\frac{N}{m^2\cdot atm}(1.5\,m) + \frac{1}{2}\times 999\,\frac{kg}{m^3}\times 9.8\,\frac{m}{s^2}\left\{(1.5)^2 + 2(1.5)(1)\right\}m^2\cdot\frac{N}{kg\cdot m}\cdot\frac{kg}{N\cdot s^2}\right]$$

$$F_R = 71.2\,kN$$

F_R

$$y_c' = \frac{1}{F_R}\left[\frac{p_s}{2}\left\{(c+a)^2 - c^2\right\} + \frac{\rho g}{3}\left\{(c+a)^3 - c^3\right\}\right]$$

$$y_c' = \frac{1\,m}{71.2\,kN}\left[\frac{1}{2}\times 0.3\,atm\times 1.01\times10^5\,\frac{N}{m^2\cdot atm}\left\{(2.5)^2 - 1\right\} + \frac{1}{3}\times 999\,kg\times 9.81\,\frac{m}{s^2}\times\left\{(2.5)^3 - 1\right\}m^3\times\frac{N\cdot s^2}{kg\cdot m}\times\frac{kN}{10^3\,N}\right]$$

$$y_c' = 1.79\,m$$

y_c'

The value of F/F_0 is obtained from Eq. 1 and $F_{R_0} = 25.7\,kN$.

$$\frac{F}{F_0} = \frac{1}{25.7\,kN}\,b\left[p_s a + \frac{\rho g}{2}(a^2 + 2ac)\right] = 0.0389\left[151.5\,p_s + 25.7\right]$$

with p_s in atm.

F/F_0

For the gate $y_c = c + \frac{a}{2} = 1.75\,m$. Then from Eq. 2

$$\frac{y_c'}{y_c} = \frac{1}{F_R(1.75)}\left[\frac{p_s}{2}\left\{(c+a)^2 - c^2\right\} + \frac{\rho g}{3}\left\{(c+a)^3 - c^3\right\}\right] = \frac{0.571}{F}\left[265\,p_s + 47.8\right]$$

with F in kN, p_s in atm.

y_c'/y_c

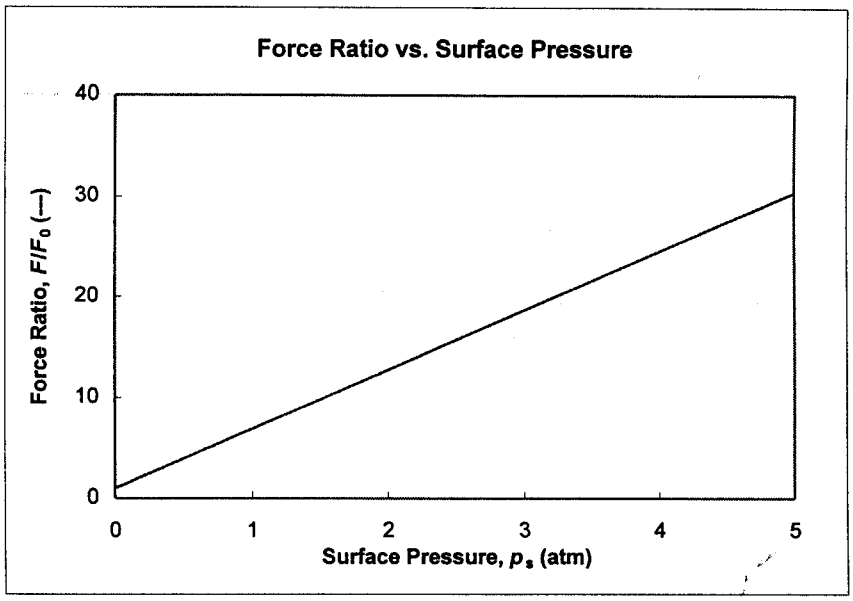
The plots are shown below

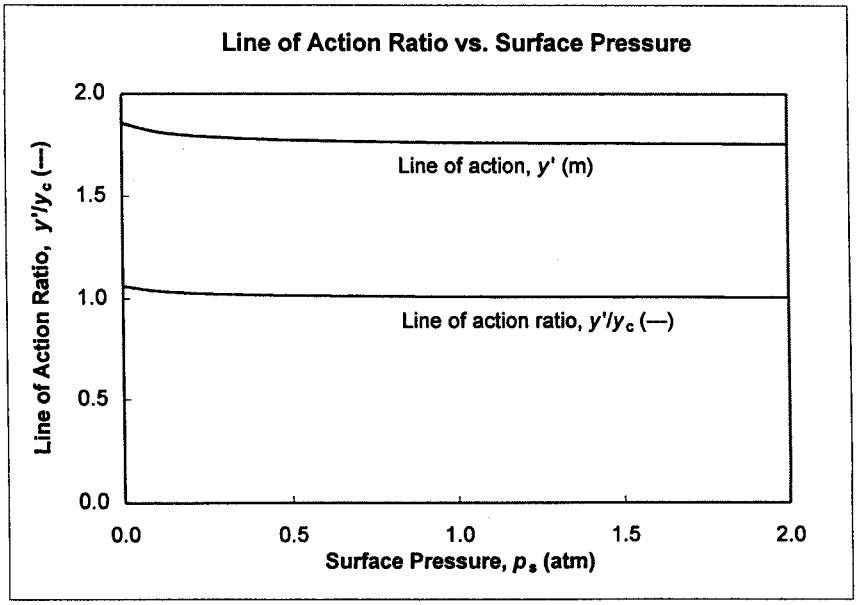
Note: The force on the gate increases linearly with increase in surface pressure.

The line of action of the resultant force is always below the centroid of the gate; y_c'/y_c approaches unity as the surface pressure is increased.

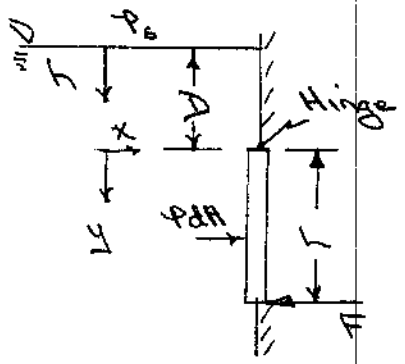
Force ratio and line of action ratio vs. surface pressure:

Surface Pressure, p_s (atm)	Force Ratio, F/F_0 (---)	Force, F_0 (kN)	Line of Action Ratio, y'/y_c (---)	Line of Action, y' (m)
0	1.00	25.7	1.0623	1.86
0.1	1.59	40.8	1.0388	1.82
0.2	2.18	56.0	1.0281	1.80
0.3	2.77	71.1	1.0219	1.79
0.5	3.95	101	1.0151	1.78
1.0	6.89	177	1.00822	1.76
2.0	12.8	329	1.00399	1.76
3.0	18.7	480		
4.0	24.6	632		
5.0	30.5	783		





Given: Door, of width $b = 1\,m$, located in plane vertical wall of water tank is hinged along upper edge.
$D = 1\,m$, $L = 1.5\,m$
Atmospheric pressure acts on outer surface of door; force F is applied at lower applied at lower edge to keep door closed.



Find: (a) Force F, if $p_s = p_{atm}$.
(b) Force F, if $p_s = 0.5$ atm.

Plot: F/F_0 over range of p_s/p_{atm}. (F_0 is force required when $p_s = p_{atm}$)

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$; $F_R = \int p\,dA$; $\Sigma M_z = 0$

Assumptions: (1) static fluid (2) ρ = constant
(3) door is in equilibrium

Since $\Sigma M_z = 0$ for equilibrium, taking moments about the hinge

$$\Sigma M_z = 0 = F L - \int y\,p\,dA = F L - \int_0^L y\,p\,b\,dy$$

and $$F = \frac{1}{L}\int_0^L y\,p\,b\,dy \qquad (1)$$

Note: We will obtain a general expression for F (needed for the plot) and then simplify for cases (a) and (b)

Since $dp = \rho g\,dh$, then $p = p_s + \rho g h$

$h = D + y$ and hence $p = p_s + \rho g (D + y)$.

Because p_{atm} acts on the outside of the door, p_s is the surface gage pressure.

From Eq (1), $$F = \frac{1}{L}\int_0^L y\left[p_{sg} + \rho g(D+y)\right] b\,dy$$

$$F = \frac{b}{L}\left[p_{sg}\frac{y^2}{2} + \rho g\left(\frac{D y^2}{2} + \frac{y^3}{3}\right)\right]_0^L$$

$$F = \frac{b}{L}\left[p_{sg}\frac{L^2}{2} + \rho g\left(\frac{D L^2}{2} + \frac{L^3}{3}\right)\right] = b\left[p_{sg}\frac{L}{2} + \rho g L\left(\frac{D}{2} + \frac{L}{3}\right)\right] \qquad (2)$$

(a) For $p_s = p_{atm}$, $p_{sg} = 0$

$$F_0 = \rho g b L\left(\frac{D}{2} + \frac{L}{3}\right) \qquad (3)$$

$$F_0 = 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 1m \times 1.5m\left(\frac{1m}{2} + \frac{1.5m}{3}\right) \times \frac{N\cdot s^2}{kg\cdot m} \times \frac{kN}{10^3 N} = 14.7\,kN$$

F_0

(b) For $p_{sg} = 0.5$ atm (50.6 kPa), from Eq. (2)

$$F = p_{s_0} bL/2 + \rho g L b (L/2 + L/3)$$

$$F = 50.6 \frac{kN}{m^2} \times 1m \times 1.5\frac{m}{2} + 14.7\ kN = 52.7\ kN \quad \leftarrow F)_{p_s = 0.5}$$

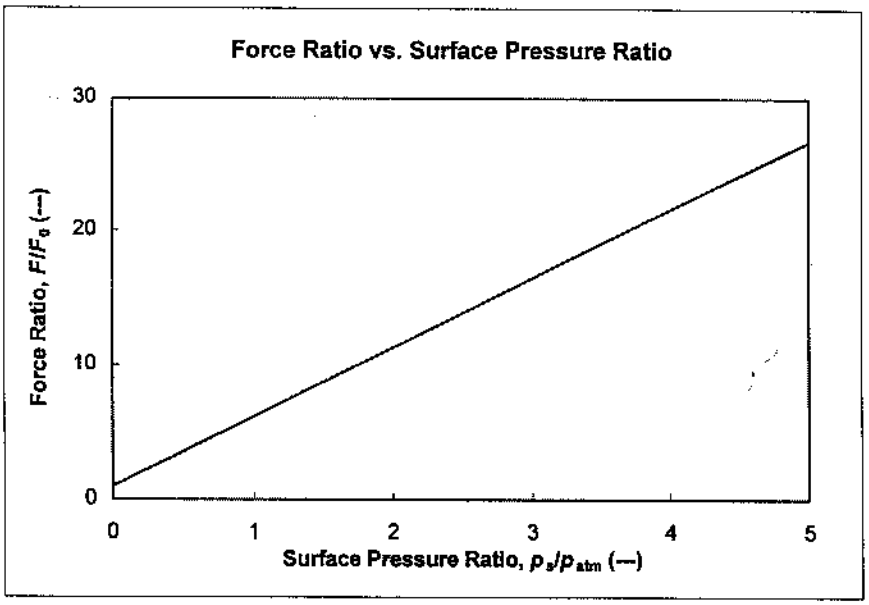
From Eqs. (2) and (3) we can write

$$\frac{F}{F_0} = \frac{b\left[p_{sg}\frac{L}{2} + \rho g L\left(\frac{L}{2} + \frac{L}{3}\right)\right]}{\rho g b L\left(\frac{L}{2} + \frac{L}{3}\right)} = 1 + \frac{p_{sg}}{2\rho g\left(\frac{L}{2} + \frac{L}{3}\right)}$$

Substituting values

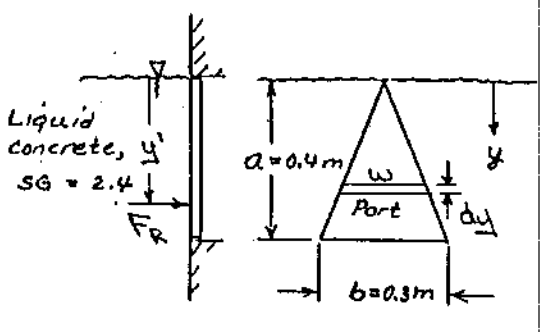
$$\frac{F}{F_0} = 1 + \frac{p_{sg}}{0.194} \quad \text{(with } p_{sg} \text{ in atmospheres)} \qquad (4)$$

F/F_0 is plotted as a function of $p_{s\,gage}/p_{atm}$.



Given: Triangular port in the side of a form containing liquid concrete, as shown

Find: (a) the resultant force that acts on the port
(b) point of application of the resultant force.



Solution:

Basic equations: $F_R = \int p\,dA$ $\frac{dp}{dy} = \rho g$ $\rho = SG\,\rho_{H_2O}$

$\Sigma M_s = y' F_R = \int y\,dF_R = \int y p\,dA$

Assumptions: (1) static fluid (2) ρ = constant
(3) p_{atm} acts at surface + on outside of port.

Under these assumptions, the pressure at any point in the liquid is given by $p = \rho g y$.

Also $dA = w\,dy$ where $\frac{w}{b} = \frac{y}{a}$ or $w = \frac{b}{a} y$

Then

$$F_R = \int p\,dA = \int_0^a \rho g y\, w\,dy = \int_0^a \rho g y \frac{b}{a} y\,dy = \int_0^a \rho g \frac{b}{a} y^2\,dy$$

$$F_R = \rho g \frac{b}{a}\left[\frac{y^3}{3}\right]_0^a = \rho g \frac{b a^2}{3} = SG\,\rho_{H_2O} g \frac{b a^2}{3}$$

$$F_R = \frac{2.4}{3} \times 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 0.3\,m \times (0.4)^2\,m^2 \times \frac{N \cdot s^2}{kg \cdot m} = 376\,N \leftarrow F_R$$

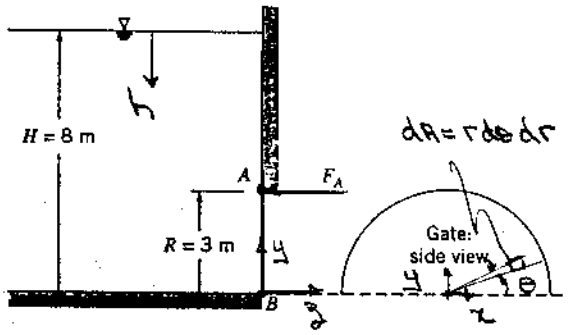
$$\Sigma M_{axis} = y' F_R = \int y p\,dA = \int_0^a y \rho g y \frac{b}{a} y\,dy = \int_0^a \rho g \frac{b}{a} y^3\,dy$$

$$y' F_R = \rho g \frac{b}{a}\left[\frac{y^4}{4}\right]_0^a = \rho g \frac{b}{a}\frac{a^4}{4} = SG\,\rho_{H_2O} g \frac{b a^3}{4}$$

$$y' = \frac{M_{axis}}{F_R} = SG\,\rho_{H_2O} g \frac{b a^3}{4} \times \frac{3}{SG\,\rho_{H_2O} g b a^2} = \frac{3a}{4} = \frac{3}{4} \times 0.4\,m = 0.3\,m \leftarrow y'$$

Given: Semicircular plane gate AB is hinged along B and held in place by horizontal force F_A.

Find: Force F_A required to hold gate in place



Solution:

Basic equations: $\frac{dp}{dh} = \rho g$; $F_R = \int p\,dA$; $\Sigma M_z = 0$

Assumptions: (1) static fluid (2) ρ = constant (3) door is in equilibrium.

Since $\Sigma M_z = 0$ for equilibrium, taking moments about the hinge B,

$$F_A = \frac{1}{R}\int y\,p\,dA$$

Since $dp = \rho g\,dh$ and $p = p_s + \rho g h$

Because the free surface is at atmospheric pressure, and atmospheric pressure acts on the outside of the gate, $p = \rho g h$

and $F_A = \frac{1}{R}\int y\,\rho g h\,dA$

For the circular gate, $dA = r\,dr\,d\theta$, $y = r\sin\theta$, $h = H - y$.

so $F_A = \frac{1}{R}\int_0^{\pi}\int_0^{R} r\sin\theta\,\rho g(H - r\sin\theta)\,r\,dr\,d\theta$

$$F_A = \frac{\rho g}{R}\int_0^{\pi}\int_0^{R}(Hr^2 - r^3\sin\theta)\sin\theta\,dr\,d\theta = \frac{\rho g}{R}\int_0^{\pi}\left[\frac{Hr^3}{3} - \frac{r^4}{4}\sin\theta\right]_0^{R}\sin\theta\,d\theta$$

$$F_A = \frac{\rho g}{R}\int_0^{\pi}\left[\frac{HR^3}{3} - \frac{R^4}{4}\sin\theta\right]\sin\theta\,d\theta = \frac{\rho g}{R}\int_0^{\pi}\left[\frac{HR^3}{3}\sin\theta - \frac{R^4}{4}\sin^2\theta\right]d\theta$$

$$= \frac{\rho g}{R}\left[\left(-\frac{HR^3}{3}\cos\theta\right)_0^{\pi} - \frac{R^4}{4}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi}\right] = \frac{\rho g}{R}\left[\frac{2HR^3}{3} - \frac{\pi R^4}{8}\right]$$

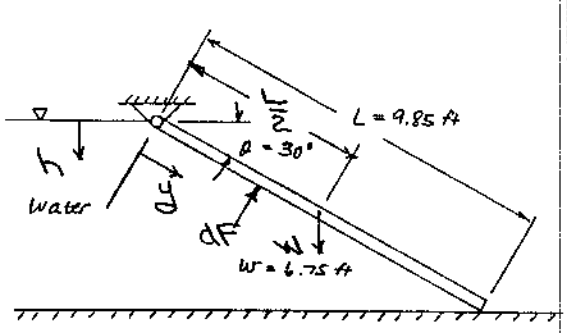
$$F_A = \rho g\left[\frac{2HR^2}{3} - \frac{\pi R^3}{8}\right]$$

$$= 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2}\left[\frac{2\times 8m\times 9m^2}{3} - \frac{\pi}{8}27m^3\right] \times \frac{N\cdot s^2}{kg\cdot m} \times \frac{kN}{10^3\,N}$$

F_A = 366 kN

Given: Plane gate of uniform thickness and width $w = 6.75$ ft holds back a depth of water as shown.

Find: the minimum weight, W, of the gate needed to insure gate remains closed.



Solution:

Basic equations: $F = \int p\,dA$ $\quad \frac{dp}{dh} = \rho g$

$\Sigma M_o = 0$ $\quad M = \int y\,dF$

Assumptions: (1) static fluid (2) ρ = constant
(3) p_{atm} acts at surface of water and along top surface of the gate.

Under these assumptions, the pressure at any point in the liquid is given by $p = \rho g h = \rho g y \sin\theta$

$$\Sigma M_o = 0 = \int y\,dF - W\frac{L}{2}\cos\theta \qquad \{\text{moment about axis through O is + cc}\}$$

Then

$$W = \frac{2}{L\cos\theta}\int y\,dF = \frac{2}{L\cos\theta}\int y\,p\,dA = \frac{2}{L\cos\theta}\int_0^L y\,\rho g y \sin\theta\, w\,dy$$

$$W = \frac{2\rho g w \tan\theta}{L}\int_0^L y^2\,dy = \frac{2\rho g w \tan\theta}{L}\left[\frac{y^3}{3}\right]_0^L$$

$$W = \frac{2}{3}\rho g w L^2 \tan\theta$$

$$W = \frac{2}{3}\times 1.94\ \frac{slug}{ft^3}\times 32.2\ \frac{ft}{s^2}\times 6.75\ ft\times(9.85)^2\ ft^2\times\tan 30^\circ\times\frac{lbf\cdot s^2}{ft\cdot slug}$$

$$W_{min} = 15{,}800\ lbf$$

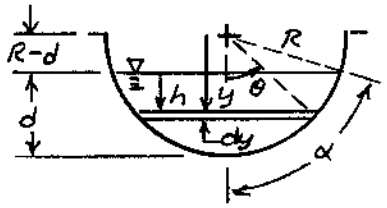
Given: Semi-cylindrical trough, partly filled with water to depth d.

Find: (a) General expressions for F_R and y' on end of trough, if open to atmosphere.
(b) Plots of results vs. d/R for $0 \le d/R \le 1$.

Solution: Apply basic equations for hydrostatics of incompressible liquid.

Computing equations: $p = \rho g h \quad F_R = \int_A p\,dA \quad y'F_R = \int_A y\,p\,dA$

Assumptions: (1) Static liquid
(2) ρ = constant



$$p = \rho g h = \rho g [y - (R-d)]$$

$$= \rho g R \left[\frac{y}{R} - \left(1 - \frac{d}{R}\right)\right] = \rho g R (\cos\theta - \cos\alpha)$$

$$h = y - (R-d)$$

$$\cos\alpha = \frac{R-d}{R} = 1 - \frac{d}{R}$$

$$w = 2R\sin\theta$$

$$dA = w\,dy = 2R\sin\theta\,dy; \quad y = R\cos\theta$$

$$dy = -R\sin\theta\,d\theta$$

$$F_R = \int_{R-d}^{R} p\,w\,dy = \int_{R-d}^{R} \rho g R (\cos\theta - \cos\alpha)\, 2R\sin\theta\, (-R\sin\theta)\,d\theta$$

The new limits are $y = R \to \theta = 0$ and $y = R-d \to \theta = \alpha$, so

$$F_R = 2\rho g R^3 \int_{\alpha}^{0} (-\sin^2\theta\cos\theta + \sin^2\theta\cos\alpha)\,d\theta = 2\rho g R^3 \int_0^{\alpha} (\sin^2\theta\cos\theta - \sin^2\theta\cos\alpha)\,d\theta$$

$$= 2\rho g R^3 \left[\frac{\sin^3\theta}{3} - \cos\alpha\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)\right]_0^{\alpha} = 2\rho g R^3 \left[\frac{\sin^3\theta}{3} - \cos\alpha\left(\frac{\theta}{2} - \frac{\sin\theta\cos\theta}{2}\right)\right]_0^{\alpha}$$

$$F_R = 2\rho g R^3 \left[\frac{\sin^3\alpha}{3} - \cos\alpha\left(\frac{\alpha}{2} - \frac{\sin\alpha\cos\alpha}{2}\right)\right] \quad \leftarrow F_R$$

$$y'F_R = \int_{R-d}^{R} y\,p\,w\,dy = \int_{R-d}^{R} R\cos\theta\,\rho g R (\cos\theta - \cos\alpha)\, 2R\sin\theta (-R\sin\theta)\,d\theta$$

$$= 2\rho g R^4 \int_0^{\alpha} \sin^2\theta\cos\theta (\cos\theta - \cos\alpha)\,d\theta = 2\rho g R^4 \int_0^{\alpha} (\sin^2\theta\cos^2\theta - \cos\alpha\sin^2\theta\cos\theta)\,d\theta$$

$$= 2\rho g R^4 \left[\frac{1}{8}\left(\theta - \frac{\sin 4\theta}{4}\right) - \cos\alpha\,\frac{\sin^3\theta}{3}\right]_0^{\alpha}$$

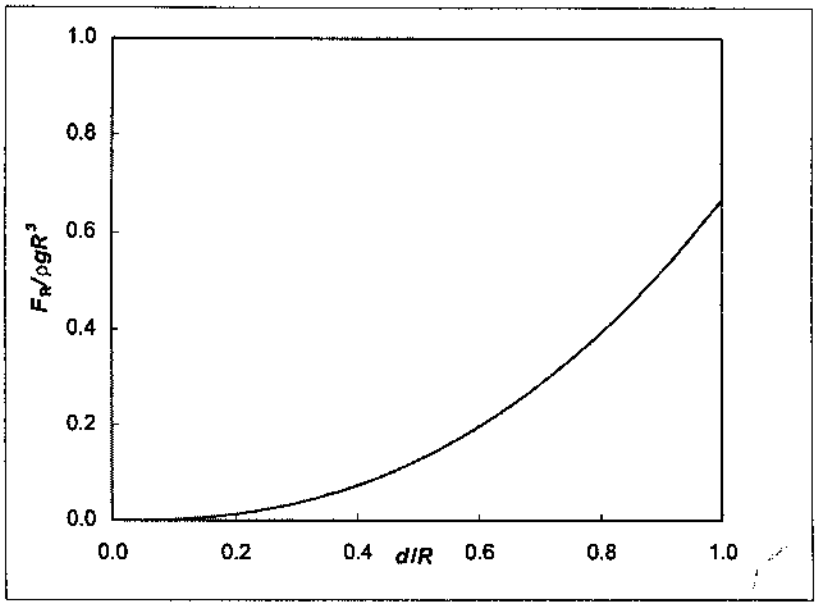
$$y'F_R = 2\rho g R^4 \left[\frac{1}{8}\left(\alpha - \frac{\sin 4\alpha}{4}\right) - \cos\alpha\,\frac{\sin^3\alpha}{3}\right] \quad \leftarrow y'F_R$$

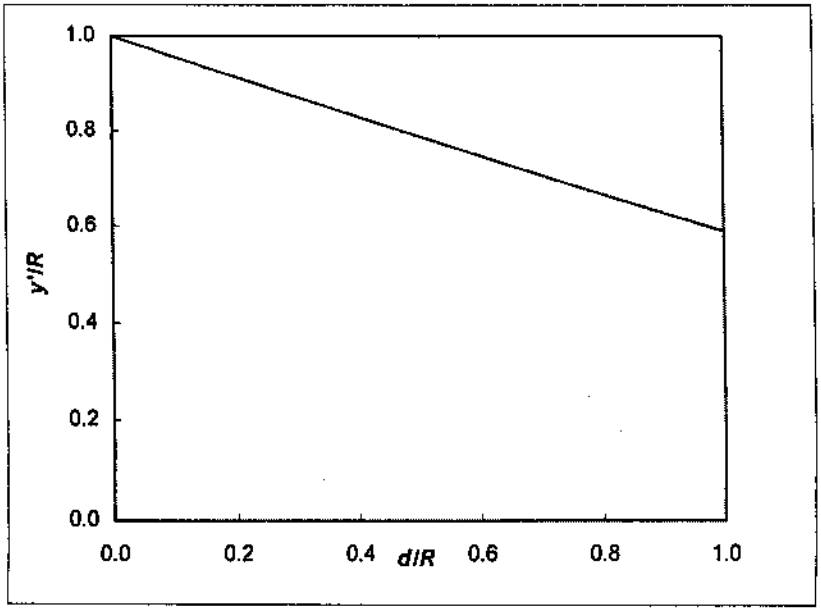
and

$$y' = \frac{y'F_R}{F_R} \quad \text{or} \quad \frac{y'}{R} = \frac{y'F_R}{R F_R} \quad \leftarrow y'$$

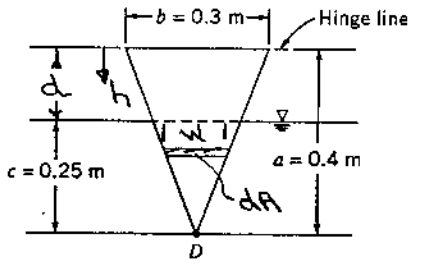
Resultant force and line of action on end of semi-cylindrical water trough:

d/R	α (rad)	α (deg)	$F_R/\rho gR^3$	$y'F_R/\rho gR^4$	y'/R
0	0.001	0.08	7.54E-16	7.54E-16	1.000
0.05	0.318	18.2	0.000419	0.000410	0.979
0.1	0.451	25.8	0.00236	0.00226	0.957
0.2	0.644	36.9	0.0132	0.0121	0.915
0.3	0.795	45.6	0.0360	0.0314	0.873
0.4	0.927	53.1	0.0730	0.0606	0.831
0.5	1.05	60.0	0.126	0.0994	0.790
0.6	1.16	66.4	0.196	0.147	0.749
0.7	1.27	72.5	0.285	0.202	0.708
0.8	1.37	78.5	0.392	0.262	0.668
0.9	1.47	84.3	0.520	0.326	0.628
1.0	1.57	90.0	0.667	0.393	0.589





Given: Window, in shape of isosceles triangle and hinged at the top is located in the vertical wall of a form that contains concrete.



Find: the minimum force applied at D needed to keep the window closed.

Plot: the results over the range of concrete depth $0 \le c \le a$

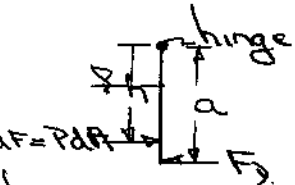
Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $F = \int p\,dA$, $\Sigma M = 0$

Assumptions: (1) static fluid (2) ρ = constant
(3) p_{atm} acts at the free surface and on the outside of the window.

For $dp = \rho g\,dh$ gives $p = \rho g(h-d)$ for $h > d$
and $p = 0$ for $h < d$
where $d = a - c$

Summing moments about the hinge



$$F_D = \frac{1}{a}\int h\,p\,dA = \frac{1}{a}\int_d^a h\,\rho g(h-d)\,w\,dh$$

From the law of similar triangles $\frac{w}{b} = \frac{a-h}{a}$; $w = \frac{b}{a}(a-h)$

$$F_D = \frac{b}{a^2}\rho g\int_d^a h(h-d)(a-h)\,dh \qquad \{\rho = SG_{concrete}\,\rho_{H_2O}\}$$

$$F_D = \frac{b}{a^2}\rho g\int_d^a \left[-h^3 + h^2(a+d) - adh\right]dh$$

$$F_D = \frac{b}{a^2}\rho g\left[-\frac{h^4}{4} + \frac{h^3}{3}(a+d) - \frac{1}{2}adh^2\right]_d^a$$

$$F_D = \frac{b}{a^2}\rho g\left[-\frac{1}{4}(a^4-d^4) + \frac{1}{3}(a^3-d^3)(a+d) - \frac{1}{2}ad(a^2-d^2)\right]$$

$$F_D = b\rho g a^2\left[-\frac{1}{4}\left(1-\frac{d^4}{a^4}\right) + \frac{1}{3}\left(1-\frac{d^3}{a^3}\right)\left(1+\frac{d}{a}\right) - \frac{1}{2}\frac{d}{a}\left(1-\frac{d^2}{a^2}\right)\right] \quad (1)$$

Evaluating with $\rho = SG_{conc}\,\rho_{H_2O}$ ($SG = 2.5$ - Table A.1)

$$b\rho g a^2 = 0.3\,m \times 2.5 \times 10^3\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times (0.4)^2\,m^2 \times \frac{N \cdot s^2}{kg \cdot m} = 1177\,N$$

For $a = 0.4\,m$, $c = 0.25\,m$, $d = a - c = 0.15\,m$, $\frac{d}{a} = 0.375$

The term [ ] in Eq. 1 has a value of 0.0280

Then for the conditions given

$$F_D = 1177\,N \times 0.0280 = 33.0\,N$$

To plot F_D vs c/a for $0 \le c \le a$, recognize

Since $d = a - c$, then $\frac{d}{a} = 1 - \frac{c}{a}$

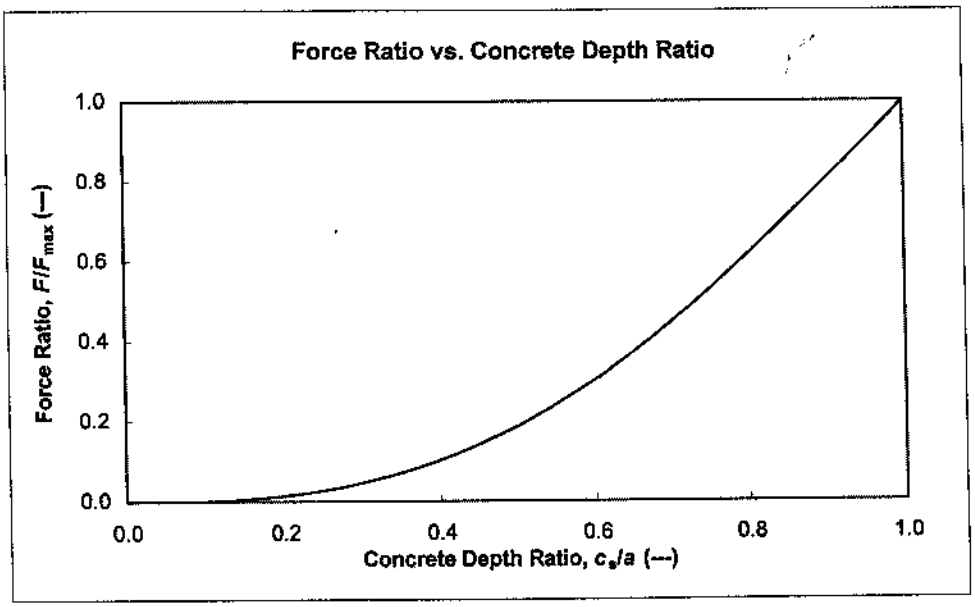
and

$$F_D = 1177\,N\left[-\frac{1}{4}\left\{1-\left(\frac{d}{a}\right)^4\right\} + \frac{1}{3}\left\{1-\left(\frac{d}{a}\right)^3\right\}\left(1+\frac{d}{a}\right) - \frac{1}{2}\frac{d}{a}\left\{1-\left(\frac{d}{a}\right)^2\right\}\right]$$

The results are plotted below.

Hinge force vs. concrete depth ratio:

Depth Ratio, c/a (---)	Depth Ratio, d/a (---)	Force Ratio, F/F_{max} (---)
0	1.0	0.0000
0.1	0.9	0.0019
0.2	0.8	0.0144
0.3	0.7	0.0459
0.4	0.6	0.102
0.5	0.5	0.187
0.6	0.4	0.302
0.625	0.375	0.336
0.7	0.3	0.446
0.8	0.2	0.614
0.9	0.1	0.802
1.0	0.0	1.000

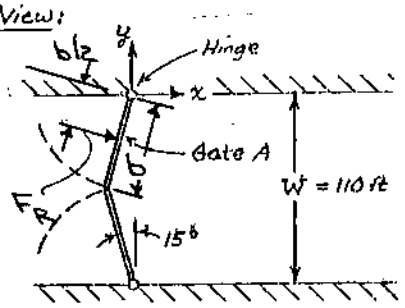


Given: Pair of plane gates close a channel of width, $W = 110$ ft; each gate is hinged at channel wall. Gate edges are forced together at the channel center by water pressure. Water depth, $D = 32$ ft. Neglect the weight of the gate.

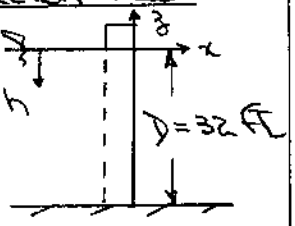
Find: (a) force exerted by water on gate A
(b) force components exerted by the gate on hinge A

Solution:

Plan View:



Elevation View



Basic equations: $\frac{dp}{dh} = \rho g$; $p = p_{atm} + \rho g h$

Assumptions: (1) static liquid
(2) gravity only body force
(3) h positive down from free surface.
(4) p_{atm} acts on both sides of gate

Then

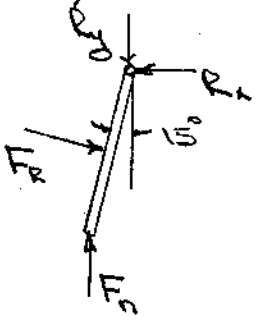
$$F_R = \int p\,dA = \int_0^{D/\sin 15^\circ} \rho g h b\,dh = \rho g \frac{W}{2\cos 15^\circ}\left[\frac{h^2}{2}\right]_0^{D/\sin 15^\circ} = \frac{\rho g W D^2}{4\sin 15^\circ \cos 15^\circ}$$

$$= \frac{1}{4\cos 15} \times 1.94 \frac{\text{slug}}{\text{ft}^3} \times 32.2 \frac{\text{ft}}{\text{s}^2} \times 110\,\text{ft} \times (32)^2\,\text{ft}^2 \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}$$

$$F_R = 1.82 \times 10^6 \text{ lbf}$$

Since the gate width, $b = \frac{W}{2\cos 15} = 56.9$ ft, is constant the line of action of F_R is located at $b/2$ from the hinge

To find the reaction forces at the hinge, consider a FBD of the gate.



Since we have neglect the weight of the gate, the reaction force at the hinge has only components R_x and R_y.

The contact force, F_n, between the pair of gates must act perpendicular to the channel walls (from symmetry conditions).

$$\Sigma M_o = 0 = F_R \frac{b}{2} - F_n b \sin 15^\circ$$

$$\therefore F_n = \frac{F_R}{2\sin 15} = \frac{1.82 \times 10^6 \text{ lbf}}{2\sin 15} = 3.52 \times 10^6 \text{ lbf}$$

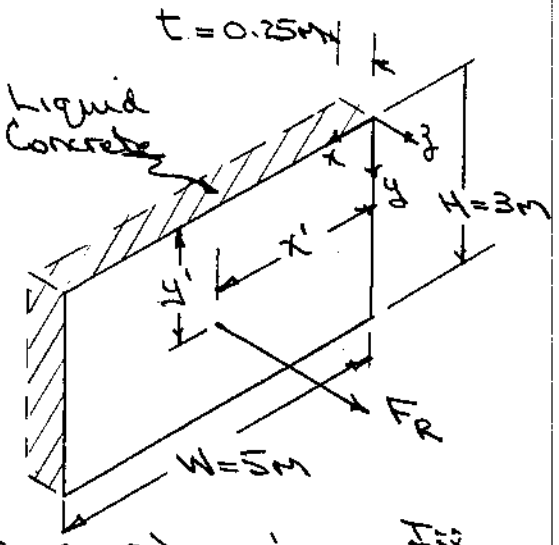
$$\Sigma F_x = F_R \cos 15 - R_x = 0 \quad \therefore R_x = F_R \cos 15 = 1.82 \times 10^6 \text{ lbf} \cos 15 = 1.76 \times 10^6 \text{ lbf}$$

$$\Sigma F_y = -R_y - F_R \sin 15 + F_n = 0 \quad \therefore R_y = F_n - F_R \sin 15 = 3.52 \times 10^6 - 1.82 \times 10^6 \sin 15$$

$$R_y = 3.04 \times 10^6 \text{ lbf}$$

The force on the hinge (from the gate) is $\vec{F}_A = (1.76\hat{i} + 3.04\hat{j}) \times 10^6$ lbf $\leftarrow \vec{F}_B$

Given: Liquid concrete poured between vertical forms as shown



Find: (a) Resultant force on form
(b) Line of application

Solution:

Basic equation: $\frac{dp}{dy} = \rho g$

Computing equations:

$F_R = p_c A$ (3.14); $y' = y_c + \frac{I_{xx}}{A y_c}$ (3.15a); $x' = x_c + \frac{I_{xy}}{A y_c}$

For the rectangular plate: $x_c = 2.5m$, $y_c = 1.5m$.

$I_{xx} = \frac{1}{12} W H^3$, $I_{xy} = 0$

Assumptions: (1) static liquid (2) incompressible liquid
(3) p_{atm} acts at free surface and on the vertical form.

Then on integrating $dp = \rho g dy$, we obtain $p = \rho g y$

$F_R = p_c A = \rho g y_c A = \rho g y_c W H = SG_{conc} \rho_{H_2O} y_c W H$

$F_R = 2.5 \times 10^3 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 1.5m \times 5m \times 3m \times \frac{N \cdot s^2}{kg \cdot m}$ {SG=2.5, Table A.1}

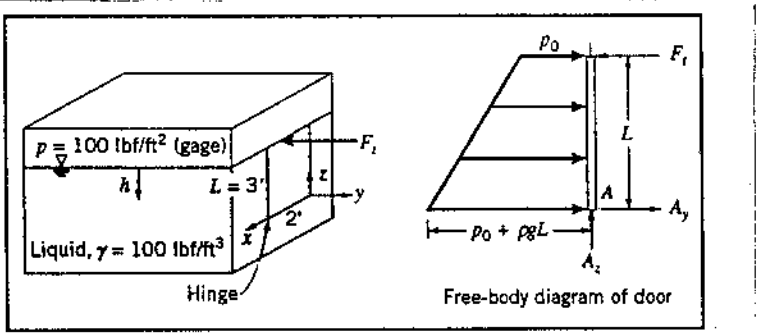
$F_R = 552 \, kN$ ← F_R

$y' = y_c + \frac{I_{xx}}{A y_c} = y_c + \frac{\frac{1}{12} W H^3}{W H y_c} = y_c + \frac{\frac{1}{12} H^2}{y_c} = 1.5m + \frac{\frac{1}{12}(3m)^2}{1.5m} = 2.0m$

$x' = x_c = 2.5m$

Line of application is through $(x', y') = (2.5, 2.0)m$ ← (x', y')

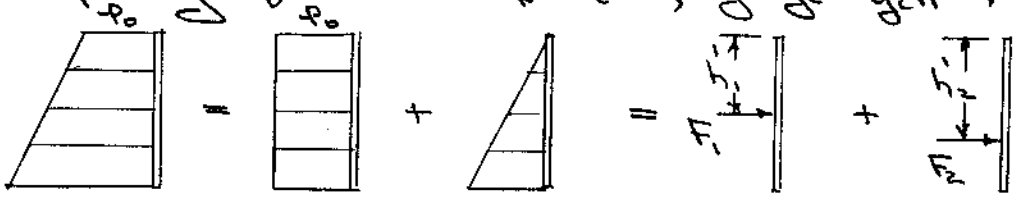
Given: Door as shown in the figure; x axis is along the hinge
From Ex. Prob 3.6, pressure in liquid is $p = p_{0_{gage}} + \rho g h$



Find: Force required to keep door shut (by considering the distributed force to be the sum of a force F_1 caused by uniform gage pressure, and force F_2 caused by the liquid)

Solution:

Computing equations: $F_R = p_c A$; $y' = y_c + \frac{I_{xx}}{y_c A}$; $I_{xx} = \frac{bL^3}{12}$



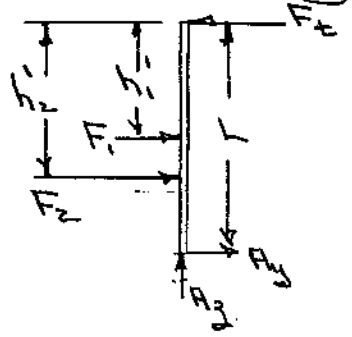
$F_1 = p_0 A = 100 \frac{lbf}{ft^2} \times 3ft \times 2ft = 600\,lb$ { applied at $(x', z') = (1.0, 1.5)$ ft

$F_2 = p_c A = \rho g h_c L b = \gamma h_c L b = 100 \frac{lb}{ft^3} \times 1.5ft \times 3ft \times 2ft = 900\,lbf$.

For the rectangular door $I_{xx} = \frac{1}{12} b L^3$

$h'_2 = h_c + \frac{I_{xx}}{A h_c} = h_c + \frac{\frac{1}{12} bL^3}{bL h_c} = h_c + \frac{1}{12}\frac{L^2}{h_c} = 1.5m + \frac{1}{12}\frac{(3m)^2}{1.5m} = 2.0m$

The free-body diagram of the door is then



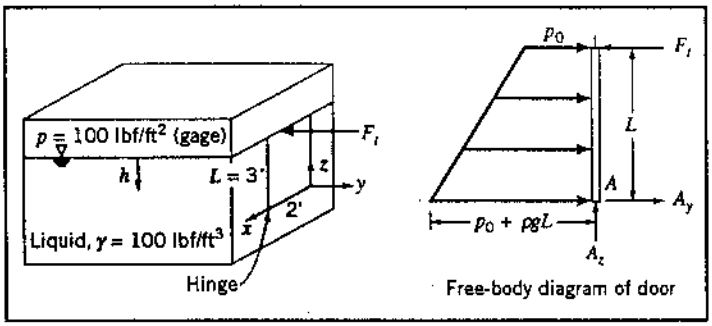
$\Sigma M_{Hx} = 0 = L F_t - F_1 (L - h'_1) - F_2 (L - h'_2)$

$F_t = F_1 \left(1 - \frac{h'_1}{L}\right) + F_2 \left(1 - \frac{h'_2}{L}\right)$

$= 600\,lb \left(1 - \frac{1.5}{3.0}\right) + 900\,lb \left(1 - \frac{2}{3}\right)$

$F_t = 600\,lb$ ← F_t

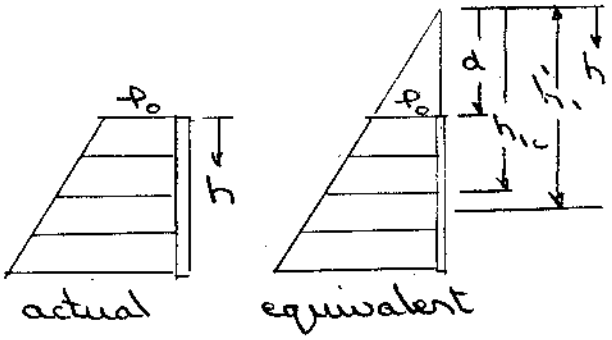
Given: Door as shown in the figure; x axis is along the hinge. From Ex. Prob. 3.6, pressure in liquid is $p = p_{gage} + \gamma h$



Find: Force required to keep door shut (by artificially increasing liquid depth so that, with new free surface at atm, p at top of door is 100 lb/ft²

Solution

Computing equations: $F_R = p_c A$; $y' = y_c + \frac{I_{xc}}{y_c A}$; $I_{xc} = \frac{bL^3}{12}$



In the equivalent system the liquid depth has been increased by an amount d, such that with $p_{surface}$ (@ $h_1 = 0$) = 0 gage p_0 (at top of gate) is 100 lb/ft²

$$p_0 = \rho g d = \gamma d$$

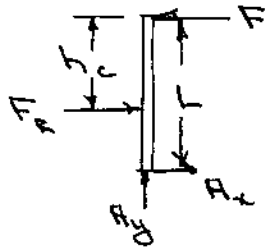
$$d = \frac{p_0}{\gamma} = \frac{100\ lbf/ft^2}{100\ lbf/ft^3} = 1\ ft$$

$$F_R = p_c A = \rho g h_{1,c} bL = \gamma h_{1,c} bL = \gamma (d + \tfrac{L}{2}) bL = 100 \tfrac{lbf}{ft^3} (1 + \tfrac{3}{2}) ft \times 2 ft \times 3 ft$$

$$F_R = 1500\ lbf$$

$$h_1' = h_{1,c} + \frac{I_{xc}}{h_{1,c} A} = h_{1,c} + \frac{bL^3}{12 h_{1,c} bL} = h_{1,c} + \frac{L^2}{12 h_{1,c}} = 2.5 ft + \frac{(3)^2 ft^2}{12 \times 2.5 ft} = 2.80 ft$$

Then, the free body diagram of the door becomes



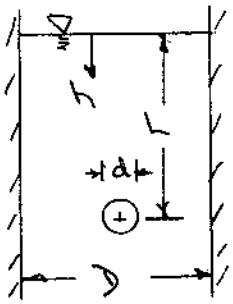
$$h_c = h_1' - d = 2.80 ft - 1.0 ft = 1.80 ft$$

$$\Sigma M_A = 0 = F_t h - F_R (L - h_c)$$

$$F_t = F_R \left(1 - \frac{h_c}{L}\right) = 1500\ lbf \left(1 - \frac{1.80}{3.00}\right)$$

$$F_t = 600\ lbf$$ ← F_t

Given: Circular access port, of diameter $d = 0.6\,m$, in side of water standpipe, of diameter $D = 7\,m$, is held in place by eight bolts evenly spaced around circumference of the port.
Center of the port is located at distance $L = 12\,m$ below the free surface of the water



Find: (a) Total force on the port
(b) Appropriate bolt diameter

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $\sigma = \frac{F}{A}$

Computing equation: $F_R = p_c A$

Assumptions: (1) static fluid
(2) incompressible
(3) force distributed uniformly over the bolts
(4) appropriate working stress for steel bolts is $\sigma = 100\,MPa$
(5) p_{atm} acts at free surface and on the outside of the port

Then on integrating $dp = \rho g\,dh$ we obtain $p = \rho g h$.

$$F_R = p_c A = \rho g h_c \pi R^2 = \rho g L \pi R^2$$

$$F_R = 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 12\,m \times \pi \times (0.3\,m)^2 \times \frac{N \cdot s^2}{kg \cdot m} = 33.3\,kN \quad \leftarrow F_R$$

$\sigma = \frac{F}{A}$ where A (total area of bolts) $= 8 \times \frac{\pi d_b^2}{4}$

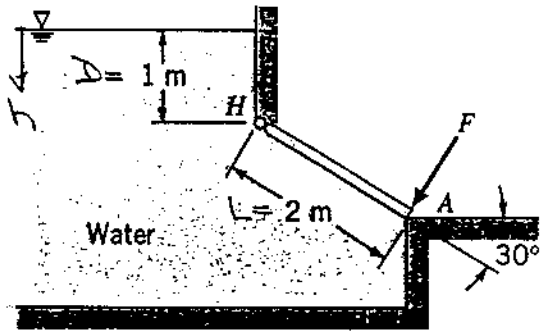
Then

$$\sigma = \frac{F}{2\pi d_b^2}$$

$$d_b = \left[\frac{F}{2\pi\sigma}\right]^{1/2} = \left[\frac{33.3 \times 10^3\,N}{2\pi} \times 10^8\,\frac{N}{m^2} \times 10^6\,\frac{mm^2}{m^2}\right]^{1/2} = 7.28\,mm \quad \leftarrow d_b$$

Given: Gate of width $b = 2$ m, hinged at H.

Find: Force F_A required to hold gate closed.



Solution:

Basic equations: $\frac{dp}{dh} = \rho g$

$\Sigma M_z = 0$

Computing equations: $F_R = p_c A$; $y' = y_c + \frac{I_{xx}}{y_c A}$; $I_{xx} = \frac{bL^3}{12}$

Assumptions: (1) static liquid (2) ρ = constant (3) p_{atm} acts at free surface and on top of the gate.

Then on integrating $dp = \rho g\, dh$, we obtain $p = \rho g h$

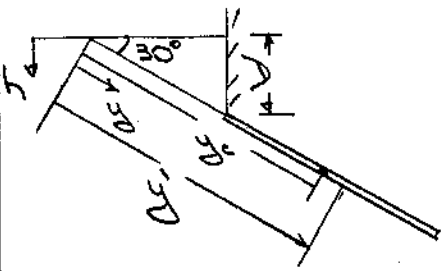
$F_R = p_c A = \rho g h_c A = \rho g h_c L b$ $\qquad h_c = D + \frac{L}{2}\sin 30° = 1\text{m} + \frac{2\text{m}}{2}\sin 30°$

$h_c = 1.5$ m

$F_R = 999 \frac{\text{kg}}{\text{m}^3} \times 9.81 \frac{\text{m}}{\text{s}^2} \times 1.5\text{m} \times 2\text{m} \times 2\text{m} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$

$F_R = 58.8$ kN

When using the computing equation to find y', we must use coordinates, with origin at the location where $p_{gage} = 0$

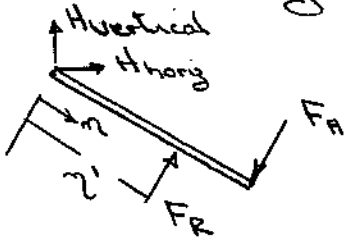


$y_c = \frac{D}{\sin 30°} + \frac{L}{2} = \frac{1\text{m}}{\sin 30°} + \frac{2\text{m}}{2} = 3.0$ m

$y' = y_c + \frac{I_{xx}}{A y_c} = y_c + \frac{bL^3}{12 y_c bL} = y_c + \frac{L^2}{12 y_c}$

$y' = 3.0\text{m} + \frac{(2\text{m})^2}{(12)3.0\text{m}} = 3.111$ m

The free body diagram of the gate is as shown



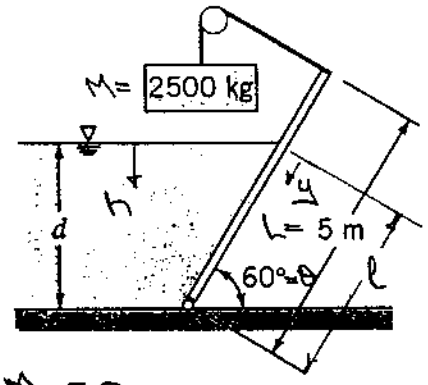
Summing moments about H

$\Sigma M_H = 0 = r' F_R - L F_A$

where $r' = y' - \frac{D}{\sin 30°} = 3.111\text{m} - \frac{D}{\sin 30°} = 1.11$ m

$F_A = \frac{1}{L} r' F_R = \frac{1.111\text{m}}{2.0\text{m}} \times 58.8\text{kN} = 32.6$ kN ← F_A

Given: Gate shown has width $b = 3\ m$; mass of gate is negligible. Gate is in equilibrium



Find: Water depth, d

Solution:

Basic equation: $\frac{dp}{dh} = \rho g$ $\quad \Sigma M_O = 0$

Computing equations: $F_R = p_c A$; $y' = y_c + \frac{I_{xx}}{y_c A}$; $I_{xx} = \frac{bL^3}{12}$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts at free surface and on underside of gate

Then on integrating $dp = \rho g\, dh$, we obtain $p = \rho g h$

$F_R = p_c A = \rho g h_c A \qquad h_c = \frac{d}{2}, \quad A = b \times \frac{d}{\sin\theta}$

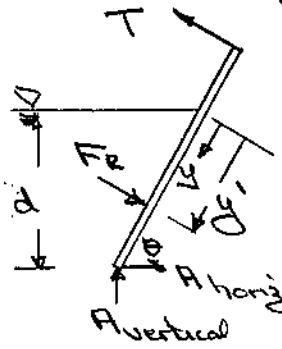
$F_R = \rho g \frac{d}{2} \frac{d}{\sin\theta} b = \frac{\rho g b d^2}{2\sin\theta}$

$y' = y_c + \frac{I_{xx}}{y_c A} = y_c + \frac{1}{12}\frac{b l^3}{y_c l b}$ where l is length of gate in contact with the water

$y' = y_c + \frac{l^2}{12 y_c} \qquad l = \frac{d}{\sin\theta}, \quad y_c = \frac{l}{2} = \frac{d}{2\sin\theta}$

$y' = \frac{d}{2\sin\theta} + \frac{1}{12}\left(\frac{d}{\sin\theta}\right)^2 \frac{2\sin\theta}{d} = \frac{d}{2\sin\theta} + \frac{d}{6\sin\theta} = \frac{2d}{3\sin\theta}$

The free body diagram of the gate is as shown.



Summing moments about A

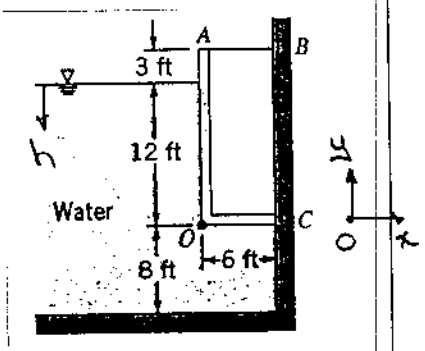
$\Sigma M_O = 0 = TL - (l - y')F_R \qquad T = Mg$

$Mgl = (l - y')F_R = \left(\frac{d}{\sin\theta} - \frac{2d}{3\sin\theta}\right)\frac{\rho g b d^2}{2\sin\theta}$

$MgL = \frac{1}{3}\frac{d}{\sin\theta} \times \frac{\rho g b d^2}{2\sin\theta} = \frac{\rho g b d^3}{6\sin^2\theta}$

$d^3 = \frac{6\sin^2\theta\, ML}{\rho b}$

$d = \left[\frac{6 \times \sin^2 60° \times 2500\ kg \times 5\ m \times 999\ kg/m^3 \times \frac{1}{3}\ m}\right]^{1/3} = 2.66\ m$ ← d



Given: Gate AOC, hinged along O, has width $b = 6$ ft; weight of gate may be neglected. Gate is sealed at C

Find: Force in bar AB

Solution:

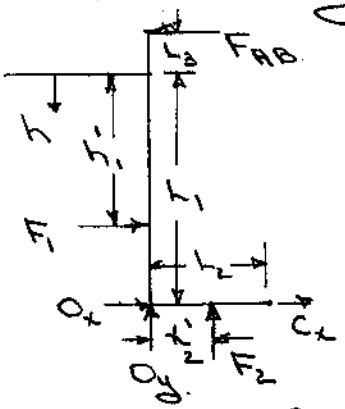
Basic equations: $\frac{dp}{dh} = \rho g$; $\Sigma M_O = 0$

Computing equations: $F_R = p_c A$; $y' = y_c + \frac{I_{ii}}{y_c A}$; $I_{ii} = \frac{bL^3}{12}$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts at free surface and on outside of gate
(4) no resisting moment in hinge along O
(5) no vertical resisting force at C

Then on integrating $dp = \rho g dh$, we obtain $p = \rho g h$

The free body diagram of the gate is as shown.



F_1 is resultant of distributed force on L_1
F_2 " " " uniform force on L_2
F_{AB} is force of bar
C_x is force from seal at C

$$F_1 = p_c A_1 = \rho g h_{c_1} b L_1$$

$$F_1 = 1.94 \frac{\text{slug}}{\text{ft}^3} \times 32.2 \frac{\text{ft}}{\text{s}^2} \times 6\text{ft} \times 6\text{ft} \times 12\text{ft} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}} = 27.0 \times 10^3 \text{ lbf}$$

$$h_1' = h_{c_1} + \frac{bL_1^3}{12 h_{c_1} b L_1} = \frac{L_1}{2} + \frac{L_1^2}{12 \times \frac{L_1}{2}} = \frac{L_1}{2} + \frac{L_1}{6} = \frac{2}{3} L_1 = \frac{2}{3} \times 12\text{ft} = 8\text{ft}$$

$$F_2 = p_c A_2 = \rho g h_{c_2} b L_2 = \rho g \frac{L_2}{2} b L_2$$

$$F_2 = 1.94 \frac{\text{slug}}{\text{ft}^3} \times 32.2 \frac{\text{ft}}{\text{s}^2} \times 12\text{ft} \times 6\text{ft} \times 6\text{ft} = 27.0 \times 10^3 \text{ lbf}$$

Since the pressure is uniform over surface ②, the force F_2 acts at the centroid of the surface, i.e. $x_2' = L_2/2 = 3$ ft

Then summing moments about O gives

$$\Sigma M_{O_z} = 0 = (L_1 + L_3) F_{AB} + x_2' F_2 - (L_1 - h_1') F_1$$

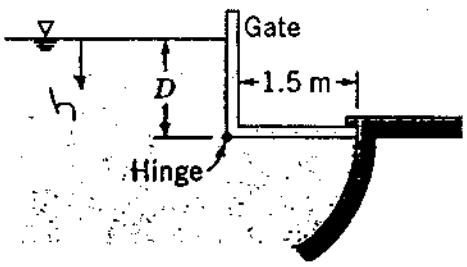
$$F_{AB} = \frac{1}{(L_1 + L_3)} \left[(L_1 - h_1') F_1 - x_2' F_2 \right] = \frac{1}{15\text{ft}} \left[(12-8)\text{ft} \times 27{,}000\text{lbf} - 3\text{ft} \times 27{,}000\text{lbf} \right]$$

F_{AB}

$$F_{AB} = 1800 \text{ lbf}$$

Thus bar AB is in compression

Given: Water rising on the left side of the gate causes it to open automatically. Neglect weight of gate.



Find: Depth, D, above the hinge at which the gate begins to open.

Solution:

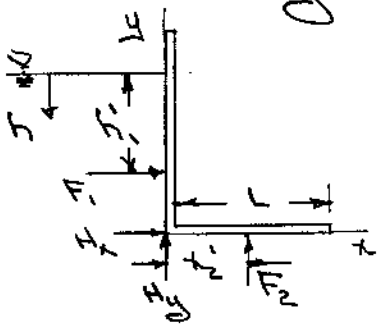
Basic equations: $\frac{dp}{dh} = \rho g$; $\Sigma M_O = 0$

Coupling equations: $F_R = p_c A$; $y' = y_c + \frac{I_{xx}}{y_c A}$; $I_{xx} = \frac{b D^3}{12}$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts at free surface and on outside of gate
(4) no resisting moment in hinge.

Then on integrating $dp = \rho g\, dh$, we obtain $p = \rho g h$

The free body diagram of the gate is as shown.



F_1 is resultant of distributed force on vertical section
F_2 " " " uniform force on horizontal "

Let width of gate be b.

$F_1 = p_{c_1} A_1 = \rho g h_{c_1} bD = \rho g \frac{D}{2} bD = \frac{1}{2}\rho g b D^2$

$h'_1 = h_{c_1} + \frac{bD^3/12}{h_{c_1} bD} = \frac{D}{2} + \frac{D}{12}\frac{2}{D} = \left(\frac{1}{2} + \frac{1}{6}\right)D = \frac{2}{3}D$

$F_2 = p_{c_2} A_2 = \rho g h_{c_2} bL = \rho g \frac{D}{2} bL$

Since the pressure is uniform over the horizontal surface, the force F_2 acts at the centroid of the surface, i.e. $h'_2 = L/2$

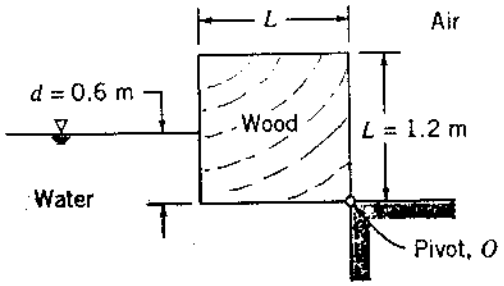
For summing moments about the hinge

$\Sigma M_{@H} = 0 = F_2 h'_2 - F_1 (D - h'_1) = \rho g \frac{D}{2} bL \frac{L}{2} - \frac{1}{2}\rho g b D^2 \left(D - \frac{2}{3}D\right)$

$\therefore \quad L^3 - \frac{D^3}{3} = 0$

$D = \sqrt{3}\, L = \sqrt{3} \times 1.5\,m = 2.60\,m$

Given: Long, square wooden block, pivoted on one edge, in equilibrium in water as shown. Friction in pivot is negligible.



Find: Specific gravity of the wood.

Solution:

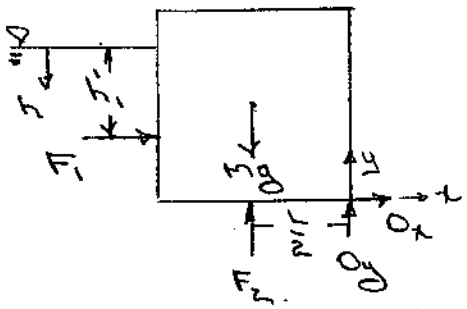
Basic equations: $\frac{dp}{dh} = \rho g$, $\Sigma M_{pt} = 0$

Computing equations: $F_R = p_c A$; $y' = y_c + \frac{I_{xx}}{y_c A}$; $I_{xx} = \frac{bd^3}{12}$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts at free surface and on outside of the block
(4) no resisting moment in hinge (given)

Then on integrating $dp = \rho g\, dh$, we obtain $p = \rho g h$

The free body diagram of the block is as shown.



F_1 is the resultant of distributed force on vertical face

F_2 is the resultant of the uniform force on the bottom face

m = mass = $\rho_{H_2O} g V = SG \rho L^2 b$ where b is the length of the block

$$F_1 = p_{c_1} A_1 = \rho g h_{c_1} db = \rho g \frac{d}{2} db = \frac{1}{2} \rho g b d^2$$

$$h'_1 = h_{c_1} + \frac{bd^3/12}{h_{c_1} bd} = \frac{d}{2} + \frac{d^2/12}{d/2} = d\left(\frac{1}{2} + \frac{1}{6}\right) = \frac{2}{3} d$$

$$F_2 = p_{c_2} A_2 = \rho g h_{c_2} bL = \rho g d b L$$

F_2 due to uniform pressure acts at centroid of surface

Then summing moments about the hinge gives

$$mg \frac{L}{2} - F_1 h'_1 - F_2 \frac{L}{2} = 0$$

$$SG \rho L^2 b g \frac{L}{2} - \frac{1}{2} \rho g b d^2 \left(d - \frac{2}{3} d\right) - \rho g d b L \frac{L}{2} = 0$$

$$SG \frac{L^3}{2} - \frac{d^3}{6} - \frac{dL^2}{2} = 0$$

$$SG = \frac{1}{3}\left(\frac{d}{L}\right)^3 + \frac{d}{L} = \frac{1}{3}\left(\frac{0.6}{1.2}\right)^3 + \frac{0.6}{1.2} = 0.542 \quad \underline{\text{SG}}$$

Problem 3.55

Open-Ended Problem Statement: Explore the cross-section shape and size required for a solid concrete gravity dam to hold back static water in a reservoir of depth D. For ease in preparing forms for concrete, assume the dam is built with plane surfaces. Neglect hydrostatic pressure under the dam. Define the dam shape that requires the minimum amount of concrete for a given water depth and obtain the dimensions of the best dam, expressed in terms of depth D. Discuss the possible effects of hydrostatic pressure beneath the dam.

Discussion: The weight of a gravity dam resists the resultant force from the static water in the reservoir that acts on the face of the dam. For simplicity, assume the reservoir water level is at the top of the dam, and that the dam does not slip with respect to the soil. Thus the potential failure mode for the dam is tipping.

Three possible profiles for the proposed gravity dam are shown. For each shape, the hydrostatic force was decomposed into a horizontal and a vertical component. Moments were summed about the lower left-hand corner of each dam to determine the minimum base dimension b needed to prevent the dam from tipping. The cross-sectional areas for the three shapes (which are proportional to the amount of concrete needed to build the dam) were $A_1 = 0.238\ D^2$, $A_2 = 0.373\ D^2$, and $A_3 = 0.228\ D^2$. Thus shape 3 requires the least concrete.

More careful study of these results suggested that a hybrid of shapes 1 and 3 might require even less concrete. The hybrid is shown as shape 4. The minimum cross-sectional area for shape 4 was $A_4 = 0.226\ D^2$ for $\alpha = 0.3$. Thus shape 4 is better than either shape 1 or shape 3; it requires about 1 percent less concrete than shape 3.

The required cross-sectional area for shape 4 is plotted as a function of α in the figure (note the exaggerated vertical scale). The minimum is quite shallow: values of α in the range $0.24 \le \alpha \le 0.36$ give essentially the same cross-sectional area.

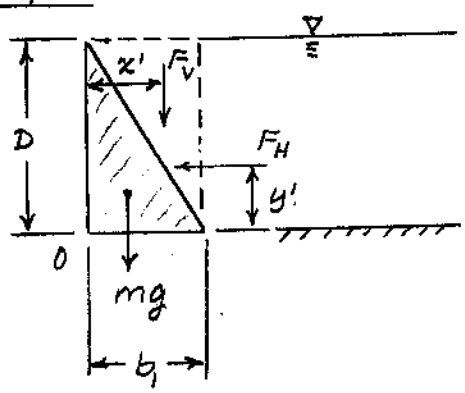
Limiting profiles for shape 4 are $\alpha = 0$ and $\alpha = 1$. Using $\alpha = 1$ produces shape 1; using $\alpha = 0$ produces shape 3. Numerical values from the hybrid analysis confirm the cross-sectional areas for these two profiles.

Hydrostatic pressure under the dam would create another force tending to tip the dam. This would have to be added to the tipping moments included in the present analysis. Thus the present analysis predicts dam base length and cross-sectional area smaller than if hydrostatic pressure were included. (Numerous civil engineering texts treat this more complete analysis. See, e.g., Streeter, Bedford, and Wiley, *Fluid Mechanics*, 9th ed. New York: McGraw-Hill, 1998.)

Use computing equations of Section 3-5.2: $F_H = p_c A$ $\quad y' = \frac{1}{3} D$

$F_V = \rho g \forall$ $\quad x' = \frac{2}{3} b$

Shape 1:



Sum moments about point 0:

$$\overset{\curvearrowleft +}{\Sigma M_0} = y' F_H - x' F_V - mg \frac{b}{3} = 0$$

$$F_H = \rho g \frac{D}{2} w D = \rho g w \frac{D^2}{2}$$

$$F_V = \rho g \frac{bD}{2} w = \rho g w \frac{bD}{2}$$

$$mg = SG \rho g w \frac{bD}{2}$$

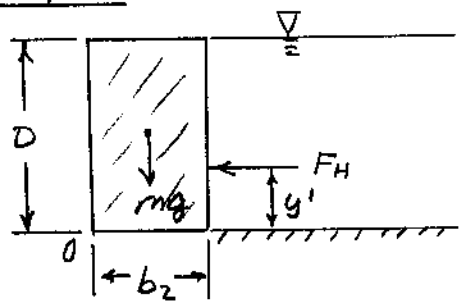
Then $\frac{1}{3} D \rho g w \frac{D^2}{2} - \frac{2}{3} b \rho g w \frac{bD}{2} - SG \rho g w \frac{bD}{2} \frac{b}{3} = 0$ [Use SG = 2.4 (Table A.1).]

Simplifying $\frac{D^3}{6} - \frac{b^2 D}{3} - SG\,\frac{b^2 D}{6} = 0$

or $D^2 = (2 + SG)\,b^2$ or $b = \sqrt{\frac{1}{2+SG}}\,D$

Thus the cross-section is $A_1 = \frac{bD}{2} = \frac{1}{2}\sqrt{\frac{1}{2+SG}}\,D^2 = 0.238\,D^2$ A_1

Shape 2:



Sum moments about point 0:

$\circlearrowleft + \Sigma M_0 = y' F_H - \frac{b}{2} mg = 0$

$F_H = \rho g \frac{D}{2} w D \qquad y' = \frac{D}{3}$

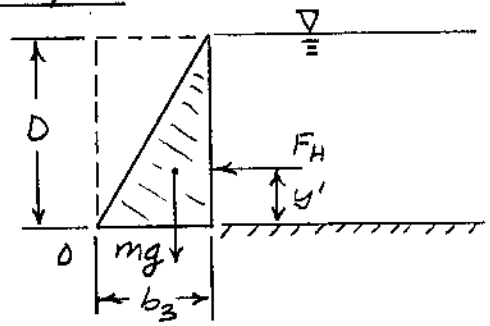
$mg = SG \rho g\, b w D$

Then $\frac{D}{3} \rho g \frac{D}{2} w D - \frac{b}{2} SG \rho g\, b w D = 0$

simplifying, $\frac{D^3}{3} - SG\, b^2 D = 0$ so $D^2 = 3SG\, b^2$ or $b = \sqrt{\frac{1}{3SG}}\,D$

Thus the cross-section is $A_2 = bD = \sqrt{\frac{1}{3SG}}\,D^2 = 0.373\,D^2$ A_2

Shape 3:



Sum moments about point 0:

$\circlearrowleft + \Sigma M_0 = y' F_H - \frac{2b}{3} mg \qquad y' = \frac{D}{3}$

$F_H = \rho g \frac{D}{2} w D$

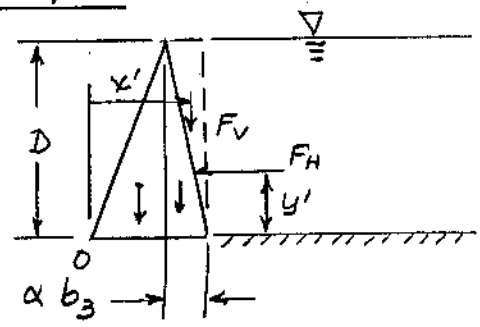
$mg = SG \rho \quad w g$

Then $\rho g w \frac{D^3}{6} - SG \rho g w \frac{bD}{2} \frac{2}{3} b = 0$

Simplifying, $\frac{D^3}{6} - SG\,\frac{b^2 D}{3} = 0$ so $D^2 = 2SG\, b^2$ or $b = \sqrt{\frac{1}{2SG}}\,D$

Thus the cross-section is $A_3 = \frac{bD}{2} = \frac{1}{2}\sqrt{\frac{1}{2SG}}\,D^2 = 0.228\,D^2$ A_3

Shape 4:



Sum moments about point 0:

$\circlearrowleft + \Sigma M_0 = y' F_H - x' F_V - x_1 m_1 g - x_2 m_2 g = 0$

$F_H = \rho g \frac{D}{2} w D; \quad y' = \frac{D}{3}$

$F_V = \rho g \frac{\alpha b_3 D}{2} w; \quad x' = (1-\alpha) b_3 + \frac{2}{3} \alpha b_3$

$m_1 = SG \rho \frac{(1-\alpha) b_3 D}{2} w; \quad x_1 = \frac{2}{3}(1-\alpha) b_3$

$m_2 = SG \rho \frac{\alpha b_3 D}{2} w; \quad x_2 = (1-\alpha) b_3 + \frac{1}{3} \alpha b_3$

Substituting

$$\overleftarrow{+}\Sigma M_0 = \frac{D}{3}\rho g\frac{D}{2}wD - \left[\left(1-\alpha\right)b_3 + \frac{2}{3}\alpha b_3\right]\rho g\alpha\frac{b_3}{2}Dw$$

$$-\frac{2}{3}(1-\alpha)b_3\,SG\rho g\frac{(1-\alpha)b_3}{2}Dw - \left[(1-\alpha)b_3 + \frac{1}{3}\alpha b_3\right]SG\rho g\frac{\alpha b_3}{2}Dw = 0$$

Simplifying, $b^2\left[(2-\alpha)SG + (3\alpha-\alpha^2)\right] = D^2$

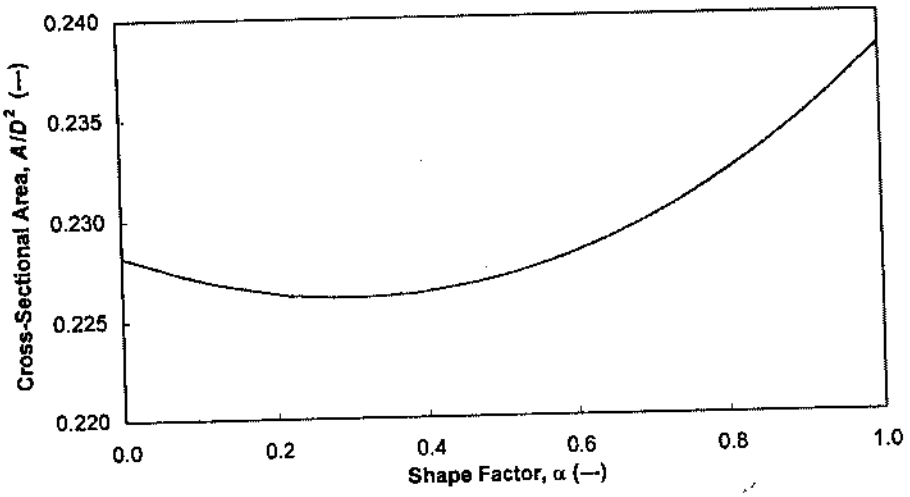
Or $b = \sqrt{\frac{1}{(2-\alpha)SG + (3\alpha-\alpha^2)}}\,D$ and $A_4 = \frac{b_4 D}{2} = \frac{1}{2}\sqrt{\quad}D^2$

The minimum* area occurs where $\alpha = 0.3$. Then

$$A_4 = \frac{1}{2}\sqrt{\frac{1}{(2-0.3)2.4 + (3(0.3)-(0.3)^2)}}\,D^2 = 0.226\,D^2$$

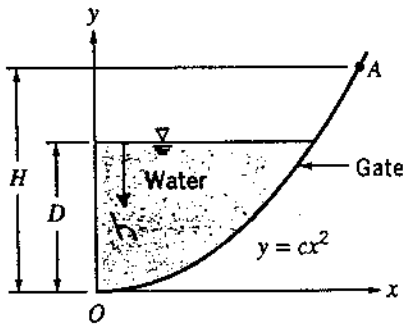
A_4

Plotting:



*Determined by setting $\frac{dA}{d\alpha} = 0$.

Given: Parabolic gate, hinged at O, has width $b = 2\,m$.
$c = 0.25\,m^{-1}$, $D = 2\,m$, $H = 3\,m$



Find: (a) Magnitude and line of action of vertical force on gate due to water
(b) Horizontal force applied at A needed for equilibrium
(c) Vertical force applied at A needed for equilibrium

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $\Sigma M_O = 0$, $F_V = \int p\,dA_y$, $x'F_V = \int x\,dF_V$

Computing equations $F_H = p_c A$, $h' = h_c + \frac{I_{xc}}{h_c A}$

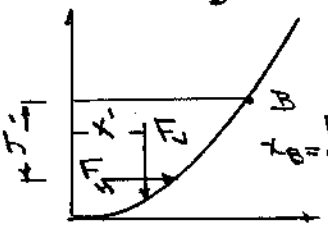
Assumptions: (1) static liquid (2) ρ = constant
(3) Patm acts on the surface of the water and along the outside surface of the gate

Then on integrating $dp = \rho g\,dh$, we obtain $p = \rho g h$

(a) $F_V = \int p\,dA_y = \int_0^{\sqrt{D/c}} \rho g h\, b\,dx = \int_0^{\sqrt{D/c}} \rho g (D-y)\, b\,dx = \int_0^{\sqrt{D/c}} \rho g (D - cx^2)\, b\,dx$

$F_V = \rho g b \left[Dx - \frac{cx^3}{3} \right]_0^{\sqrt{D/c}} = \rho g b \left[\frac{D^{3/2}}{c^{1/2}} - \frac{1}{3}\left(\frac{D}{c}\right)^{3/2} \right] = \frac{2}{3} \frac{\rho g b}{c^{1/2}} D^{3/2}$ — (1)

$F_V = \frac{2}{3} \times 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 2\,m \times (2\,m)^{3/2} \left(\frac{0.25}{1\,m}\right)^{0.5} \times \frac{N \cdot s^2}{kg \cdot m} = 73.9\,kN$ ← F_V



$x' = \frac{1}{F_V} \int x\,dF_V = \frac{1}{F_V} \int x\,p\,dA_y = \frac{1}{F_V} \int_0^{\sqrt{D/c}} x\,\rho g h\, b\,dx$

$x' = \frac{1}{F_V} \int_0^{\sqrt{D/c}} x\,\rho g (D - cx^2)\, b\,dx$

$x' = \frac{b\rho g}{F_V} \left[\frac{Dx^2}{2} - \frac{cx^4}{4} \right]_0^{\sqrt{D/c}} = \frac{b\rho g}{F_V} \left[\frac{D}{2}\frac{D}{c} - \frac{1}{4}\frac{D^2}{c^2} \right] = \frac{b\rho g}{F_V} \frac{D^2}{4c}$

Substituting for F_V from Eq. 1

$x' = \frac{b\rho g}{4c} \times \frac{3}{2} \frac{c^{1/2}}{\rho g b} \frac{1}{D^{3/2}} = \frac{3}{8} \left(\frac{D}{c}\right)^{1/2} = \frac{3}{8}\left[2\,m \times \frac{m}{0.25} \right]^{1/2} = 1.06\,m$ ← x'

In order to sum moments about point O to find the required force at A required for equilibrium, we need to find the horizontal force of the water on the gate and its line of action.

$$F_H = p_c A = \rho g h_c (bd) = \rho g b \frac{d^2}{2} \qquad \left\{ h_c = \frac{d}{2} \right\}$$

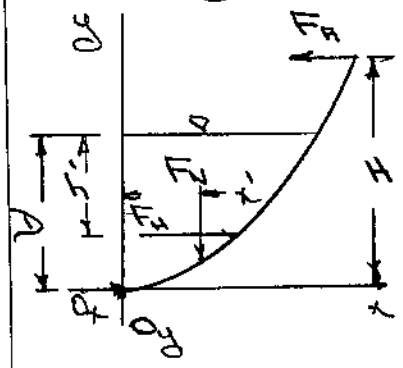
$$F_H = 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 2\,m \times \frac{(2\,m)^2}{2} \times \frac{N}{1\,kg \cdot m/s^2} = 39.2\,kN$$

$$h' = h_c + \frac{I_{xx_c}}{A h_c} = h_c + \frac{b d^3/12}{bd\, h_c} \qquad \left\{ I_{xx_c} = \frac{bd^3}{12} \text{ and } A = bd \right\}$$

$$h' = \frac{d}{2} + \frac{d^2}{12 \frac{d}{2}} \qquad \left\{ h_c = \frac{d}{2} \right\}$$

$$h' = \frac{d}{2} + \frac{d}{6} = \frac{2}{3} d$$

(b) Horizontal force applied at A for equilibrium



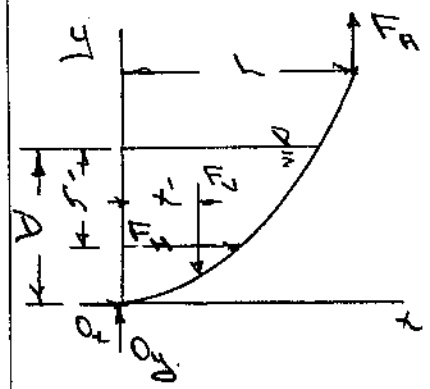
$$\Sigma M_O = 0 = F_{A_H} H - F_V x' - F_H (D - h')$$

$$F_{A_H} = \frac{1}{H} \left[F_V x' + F_H (D - h') \right]$$

$$= \frac{1}{3\,m} \left[13.9\,kN \times 1.06\,m + 39.2\,kN \times \left(2 - \frac{4}{3}\right) m \right]$$

$$F_{A_H} = 34.8\,kN \leftarrow$$

(c) Vertical force applied at A for equilibrium



$$\Sigma M_O = 0 = F_{A_V} L - F_V x' - F_H (D - h')$$

$$F_{A_V} = \frac{1}{L} \left[F_V x' + F_H (D - h') \right]$$

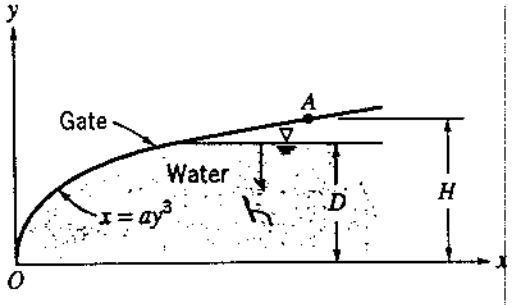
$L = x$ @ $y = H$. Since $y = cx^2$

$$L = \sqrt{\frac{H}{c}} = \left[\frac{3\,m}{0.25 \frac{1}{m}} \right]^{1/2} = 3.46\,m$$

$$F_{A_V} = \frac{1}{3.46\,m} \left[13.9\,kN \times 1.06\,m + 39.2\,kN \times \left(2 - \frac{4}{3}\right) m \right]$$

$$F_{A_V} = 30.2\,kN \leftarrow$$

Given: Gate, hinged at O, has width $b = 1.5\ m$
$a = 1.0\ m^{-2}$, $D = 1.20\ m$,
$H = 1.40\ m$



Find: (a) Magnitude and moment about O of vertical force on gate due to water.
(b) Horizontal force applied at A needed for equilibrium

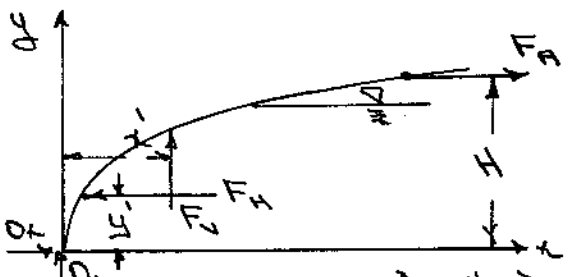
Solution

Basic equations: $\frac{dp}{dh} = \rho g$, $F_V = \int p\,dA_y$, $x'F_V = \int x\,dF_V$

$y'F_H = \int y\,dF_H$, $F_H = \int p\,dA_x$, $\Sigma M_{bz} = 0$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts on the surface of the water and along the top surface of the gate

Then on integrating $dp = \rho g\,dh$, we obtain $p = \rho g h$



$$F_V = \int p\,dA_y = \int \rho g h\, b\,dx$$

$h = D - y$ $\quad x = ay^3$ $\quad dx = 3ay^2\,dy$

$$F_V = \int_0^D \rho g (D-y)\, b\, 3ay^2\,dy$$

$$F_V = 3\rho g b a \left[\frac{D}{3}y^3 - \frac{1}{4}y^4\right]_0^D = 3\rho g b a \frac{D^4}{12} = \rho g b a \frac{D^4}{4}$$

$$F_V = 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 1.5\,m \times 1.0\,m^{-2} \times (1.20\,m)^4 \times \frac{N \cdot s^2}{kg \cdot m} = 7.62\,kN \quad \uparrow F_V$$

The moment of F_V about O is given by

$$x'F_V = \int x\,dF_V = \int x\,p\,dA_y = \int x \rho g h\, b\,dx$$

$$= \rho g b \int_0^D ay^3 (D-y)\, 3ay^2\,dy = 3\rho g b a^2 \int_0^D y^5 (D-y)\,dy$$

$$= 3\rho g b a^2 \left[\frac{D}{6}y^6 - \frac{1}{7}y^7\right]_0^D = \rho g b a^2 \frac{D^7}{14}$$

$$x'F_V = 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 1.5\,m \times (1.0)^2\,m^{-4} \times \frac{(1.20\,m)^7}{14} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$x'F_V = 3.76\,kN \cdot m \quad \text{(counterclockwise)} \quad \circlearrowleft x'F_V$$

From the free body diagram of the gate

$$\Sigma M_B = x'F_V + y'F_H - HF_A$$

$$y'F_H = \int y\,dF_H = \int y\,p\,dA_x = \int y\,\rho g h\,b\,dy = \rho g b \int_0^H y(H-y)\,dy$$

$$= \rho g b \left[\frac{Hy^2}{2} - \frac{y^3}{3}\right]_0^H = \rho g b \frac{H^3}{6}$$

$$y'F_H = \frac{1}{6} \times 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 1.5\,m \times (1.20\,m)^3 \times \frac{N \cdot s^2}{kg \cdot m} = 4.23\,kN \cdot m$$

(counterclockwise)

So

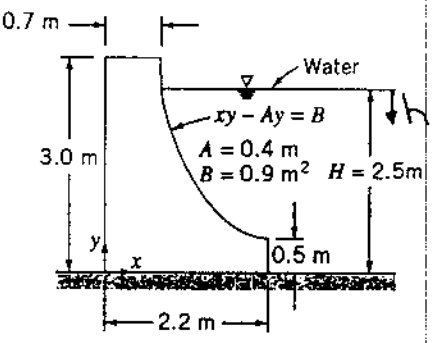
$$F_A = \frac{1}{H}\left[x'F_V + y'F_H\right] = \frac{1}{1.20\,m}\left[3.76 + 4.23\right] kN \cdot m$$

$$F_A = 5.11\,kN$$

13-782 600 SHEETS FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Dam with cross-section shown (width $b = 50m$)

Find: (a) Magnitude and line of action of vertical force on dam due to water.
(b) If it is possible for water force to overturn the dam



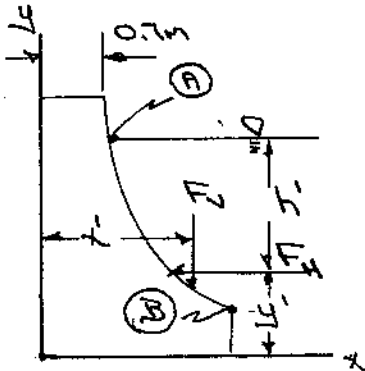
Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $F_v = \int p\,dA_y$, $x'F_v = \int x\,dF_v$, $\Sigma M_{O'} = 0$

Computing equations: $F_H = p_c A$, $h' = h_c + \frac{I_{xc}}{h_c A}$

Assumptions: (1) static fluid (2) ρ = constant
(3) p_{atm} acts on the surface of the water and on the back side of the dam.

Then on integrating $dp = \rho g\,dh$ we obtain $p = \rho g h$



$$F_V = \int p\,dA_y = \int_{x_A}^{x_B} \rho g h b\,dx = \rho g b \int_{x_A}^{x_B} (H - y)\,dx$$

$y(x - A) = B$ so $y = \left(\frac{B}{x-A}\right)$

$$F_V = \rho g b \int_{x_A}^{x_B} \left(H - \frac{B}{(x-A)}\right) dx$$

$$= \rho g b \left[Hx - B \ln(x-A) \right]_{x_A}^{x_B}$$

$$F_V = \rho g b \left[H(x_B - x_A) - B \ln \frac{(x_B - A)}{(x_A - A)} \right] \quad \text{---(1)}$$

$$F_V = 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 50m \left[2.5m(2.2 - 0.76)m - 0.9m^2 \ln \frac{(2.2 - 0.4)}{(0.76 - 0.4)} \right] \frac{N \cdot s^2}{kg \cdot m}$$

$$F_V = 1.05 \times 10^6 N \quad \leftarrow F_V$$

$$x'F_V = \int x\,dF_V = \int_{x_A}^{x_B} x \rho g b \left(H - \frac{B}{(x-A)}\right) dx = \rho g b \int_{x_A}^{x_B} \left[Hx - \frac{Bx}{(x-A)} \right] dx$$

$$x'F_V = \rho g b \left[H\frac{x^2}{2} - Bx - BA \ln(x-A) \right]_{x_A}^{x_B}$$

$$x'F_V = \rho g b \left[\frac{H}{2}(x_B^2 - x_A^2) - B(x_B - x_A) - BA \ln \frac{(x_B - A)}{(x_A - A)} \right] \quad \text{---(2)}$$

$$x' = 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 50m \left\{ \frac{2.5m}{2} \left[(2.2)^2 m^2 - (0.76)^2 m^2 - 0.9m^2 (2.2 - 0.76)m \right. \right.$$

$$\left. \left. - 0.9m^2 \times 0.4m \ln \frac{2.2 - 0.4}{0.76 - 0.4} \right] \frac{N \cdot s^2}{kg \cdot m} \right\} \times \frac{1}{1.05 \times 10^6 N}$$

$$x' = 1.61m \quad \leftarrow x'$$

From the free-body diagram of the dam we see that it is the horizontal component of the resultant force of the water that tends to overturn the dam.

Thus, neglecting the weight of the dam, the net moment tending to overturn the dam is

$$\Sigma M_{O_3} = x' F_V - y' F_H$$

$$F_H = p_c A = \rho g h_c b H = \rho g \frac{H}{2} b H = \frac{1}{2}\rho g b H^2, \qquad y' = H - h'$$

$$h' = h_c + \frac{I_{xx}}{h_c A} = \frac{H}{2} + \frac{\frac{1}{12} b H^3}{\frac{H}{2} b H} = \frac{H}{2} + \frac{H}{6} = \frac{2}{3} H$$

$$\therefore y' F_H = \left(H - \frac{2}{3}H\right) \frac{1}{2}\rho g b H^2 = \frac{1}{6}\rho g b H^3 \qquad (3)$$

The tipping moment is a maximum at $H = 3.0$ m. At $H = 3.0$ m,

$$y' F_H = \frac{1}{6}\rho g b (3\text{m})^3 = 4.50\, \rho g b$$

From Eq. (2), at these conditions

$$x' F_V = \rho g b \left\{ \frac{3.0}{2}\text{m} \left[(2.2\text{m})^2 - (0.7\text{m})^2 \right] - 0.9\text{m}^2 \left[(2.2 - 0.7)\text{m} \right] - 0.9\text{m}^2 \times 0.4\text{m} \times \ln \frac{2.2 - 0.4}{0.7 - 0.4} \right\}$$

$$x' F_V = 4.53\, \rho g b$$

Thus at $H = 3.0$ m, $\Sigma M_{O_3} = 4.50\, \rho g b - 4.53\, \rho g b = -0.03\, \rho g b$

The weight of the gate would produce a clockwise moment. Even neglecting this, the gate would not tip.

Note: The maximum net tipping moment occurs at a water depth $H = 0.5$m

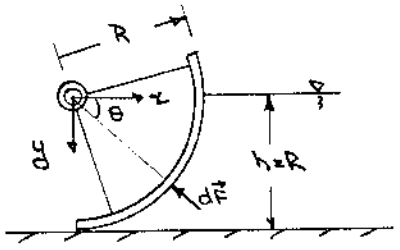
At this condition

$$y' F_H = \frac{1}{6}\rho g b H^3 = 999 \frac{\text{kg}}{\text{m}^3} \times 9.81 \frac{\text{m}}{\text{s}^2} \times 5\text{m} \times \frac{(0.5\text{m})^3}{6} \cdot \frac{\text{N}}{\text{kg}\cdot\text{m}/\text{s}^2}$$

$$y' F_H = 10.2\ \text{kN}\cdot\text{m}$$

The moment from the weight of the gate would be sufficient to prevent tipping.

Given: Gate formed in the shape of a circular arc has width of w meters. Liquid is water; depth $h = R$



Find: (a) magnitude and direction of the net vertical force component due to fluids acting on the gate
(b) line of action of vertical component of the force.

Solution.

Basic equations: $\vec{F}_R = -\int p\,d\vec{A}$ $\quad \frac{dp}{dy} = \rho g$ $\quad x' F_{R_y} = \int x\,dF$

Assumptions: (1) static fluid
(2) ρ = constant
(3) y is measured positive downward from free surface

$$F_{R_y} = \vec{F}_R \cdot \vec{j} = \int d\vec{F} \cdot \vec{j} = -\int p\,d\vec{A} \cdot \vec{j} = -\int p\,dA \sin\theta = -\int_0^{\pi/2} p \sin\theta\, w R\, d\theta$$

We can obtain an expression for p as a function of y

$$\frac{dp}{dy} = \rho g \qquad dp = \rho g\, dy \qquad \text{and} \qquad p - p_0 = \int_{p_0}^{p} dp = \int_0^y \rho g\, dy = \rho g y$$

Since atmospheric pressure acts at the free surface and on the back surface of the gate, then the appropriate expression for p is $p = \rho g y$

Along the surface of the gate,

$$y = R\sin\theta \quad \text{and hence} \quad p = \rho g R \sin\theta$$

Thus,

$$F_{R_y} = -\int_0^{\pi/2} p \sin\theta\, w R\, d\theta = -\rho g w R^2 \int_0^{\pi/2} \sin^2\theta\, d\theta = -\rho g w R^2 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$F_{R_y} = -\frac{\rho g w R^2 \pi}{4} \qquad \{F_{R_y} \text{ acts upward}\}$$

F_{R_y}

For any element of surface area, $d\vec{A}$, the force, $d\vec{F}$, acts normal to the surface. Thus each $d\vec{F}$ has a line of action through the origin. Consequently, the line of action of $\vec{F}_R$ must also be through the origin.

We can find the line of action of F_{R_y} by recognizing that the moment of F_{R_y} about an axis through the origin must be equal to the sum of the moments of dF_y about the same axis.

$$x' F_{R_y} = \int x\, dF_y = \int x(-p\,dA \sin\theta) = -\int x p\, dA \sin\theta$$

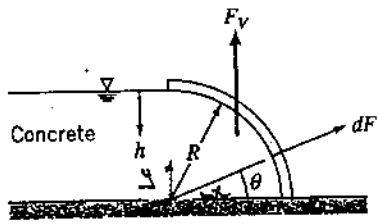
$$x' F_{R_y} = -\int_0^{\pi/2} R\cos\theta\, \rho g R \sin\theta\, w R\, d\theta \sin\theta = -\rho g w R^3 \int_0^{\pi/2} \sin^2\theta \cos\theta\, d\theta$$

$$x' = -\frac{\rho g w R^3}{F_{R_y}} \int_0^{\pi/2} \sin^2\theta \cos\theta\, d\theta = \frac{-\rho g w R^3}{-\frac{\rho g w R^2 \pi}{4}} \left[\frac{1}{3}\sin^3\theta\right]_0^{\pi/2}$$

$$x' = \frac{4R}{3\pi}$$

Given: Liquid concrete is poured into form shown; width $w = 4.25\ m$

Find: Magnitude and line of action of vertical force on form



Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $F_V = \int p\,dA_y$, $x'F_V = \int x\,dF_V$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts on the liquid surface and along the outside of the form.

Then on integrating $dp = \rho g\,dh$, we obtain $p = \rho g h$

$F_V = \int p\,dA_y = \int \rho g h\,dA \sin\theta$ $\qquad dA = w R\,d\theta$, $h = R - y = R - R\sin\theta$

$F_V = \int_0^{\pi/2} \rho g R(1-\sin\theta)\sin\theta\, w R\,d\theta = \rho g R^2 w \int_0^{\pi/2} (\sin\theta - \sin^2\theta)\,d\theta$

$F_V = \rho g R^2 w \left[-\cos\theta - \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \rho g R^2 w \left[-0 + 1 - \frac{\pi}{4} + 0 + 0 - 0\right]$

$F_V = \rho g R^2 w \left(1 - \frac{\pi}{4}\right)$ $\qquad \{\ \rho = SG\,\rho_{H_2O}$; $SG = 2.5$ (Table A.1)

$F_V = 2.5 \times 1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times (0.313\,m)^2 \times 4.25\,m \left(1 - \frac{\pi}{4}\right) \times \frac{N \cdot s^2}{kg \cdot m}$

$F_V = 2.19\ kN$ ← F_V

$x'F_V = \rho g R^2 w \int_0^{\pi/2} x(\sin\theta - \sin^2\theta)\,d\theta = \rho g R^2 w \int_0^{\pi/2} R\cos\theta(\sin\theta - \sin^2\theta)\,d\theta$

$= \rho g R^3 w \int_0^{\pi/2} (\sin\theta\cos\theta - \sin^2\theta\cos\theta)\,d\theta = \rho g R^3 w \left[\frac{\sin^2\theta}{2} - \frac{\sin^3\theta}{3}\right]_0^{\pi/2}$

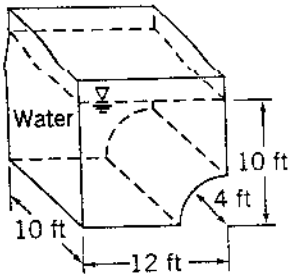
$x'F_V = \rho g R^3 w \left[\frac{1}{2} - \frac{1}{3}\right] = \frac{\rho g R^3 w}{6}$

$x' = \frac{\rho g R^3 w}{6 F_V} = \frac{\rho g R^3 w}{6} \cdot \frac{1}{\rho g R^2 w\left(1 - \frac{\pi}{4}\right)} = \frac{R}{6\left(1 - \frac{\pi}{4}\right)} = \frac{0.313\,m}{6\left(1 - \frac{\pi}{4}\right)}$

$x' = 0.243\ m$ ← x'

Given: Open tank as shown.
width of curved surface $b = 10$ ft

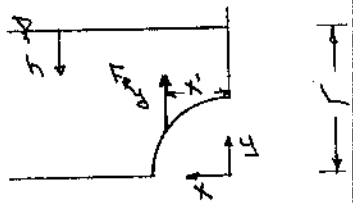
Find: (a) vertical force component, F_{R_y}, on curved surface
(b) line of action of F_{R_y}.



Solution:

Basic equations: $\vec{F}_R = -\int p\,d\vec{A}$ $\quad \frac{dp}{dh} = \gamma$ $\quad \vec{r}' \times \vec{F}_R = \int \vec{r} \times d\vec{F} = -\int \vec{r} \times p\,d\vec{A}$

Assumptions: (1) static fluid
(2) gravity is only body force
(3) γ = constant = 62.4 lbf/ft³
(4) h is measured positive downward from free surface



$$F_{R_y} = \vec{F}_R \cdot \hat{j} = -\int p\,d\vec{A} \cdot \hat{j} = -\int p\,dA_y = -\int p\,b\,dx$$

We can obtain an expression for p as a function of y

$$\frac{dp}{dh} = \gamma \qquad dp = \gamma\,dh \qquad p - p_0 = \int_{p_0}^{p} dp = \int_0^h \gamma\,dh = \gamma h$$

Since atmospheric pressure acts at the free surface and on the underside of the curved surface, then the appropriate expression for p is $p = \gamma h$

Now, $h = L - y$ $\quad \therefore p = \gamma(L-y)$

$F_{R_y} = -\int p\,b\,dx = -\int \gamma(L-y)\,b\,dx$. Along the surface $y = (R^2 - x^2)^{1/2}$ and so

$$F_{R_y} = -\gamma b \int_0^R \{L - (R^2-x^2)^{1/2}\}\,dx = -\gamma b \left[Lx - \tfrac{1}{2}\left(x\sqrt{R^2-x^2} + R^2 \arcsin\frac{x}{R}\right)\right]_0^R$$

$$= -\gamma b \left\{LR - \tfrac{1}{2}\left(R^2 \arcsin 1\right) + \tfrac{1}{2}R^2 \arcsin 0\right\} = \gamma b R \left\{L - \frac{R}{2}\arcsin 1\right\}$$

$$= -\gamma b R \left\{L - R\frac{\pi}{4}\right\}$$

$$F_{R_y} = -62.4 \frac{\text{lbf}}{\text{ft}^3} \times 10\text{ft} \times 4\text{ft} \times \left\{10\text{ft} - 4\text{ft} \times \frac{\pi}{4}\right\} = -17{,}100 \text{ lbf} \quad \text{(it acts downward)} \qquad F_R$$

$$x'\hat{i} \times F_{R_y}\hat{j} = \int x\hat{i} \times dF_{R_y}\hat{j} = \int x\hat{i} \times (-p\,dA_y\hat{j}) = -\int x\hat{i} \times p\,b\,dx\,\hat{j}$$

$$x' F_{R_y}\hat{k} = -\hat{k}\int x\,p\,b\,dx$$

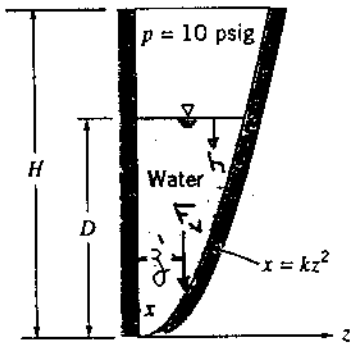
$$x' = -\frac{1}{F_{R_y}}\int_0^R x\,p\,b\,dx = -\frac{1}{F_{R_y}}\int_0^R x\,\gamma(L-y)\,b\,dx = -\frac{\gamma b}{F_{R_y}}\int_0^R x\left\{L - (R^2-x^2)^{1/2}\right\}dx$$

$$= -\frac{\gamma b}{F_{R_y}}\left[L\frac{x^2}{2} + \frac{1}{3}\sqrt{(R^2-x^2)^3}\right]_0^R = -\frac{\gamma b}{F_{R_y}}\left[L\frac{R^2}{2} - \frac{1}{3}R^3\right] = -\frac{\gamma b R^2}{F_{R_y}}\left[\frac{L}{2} - \frac{R}{3}\right]$$

$$x' = -62.4 \frac{\text{lbf}}{\text{ft}^3} \times 10\text{ft} \times (4)^2\text{ft}^2 \times \frac{1}{(-17{,}100)\text{lbf}}\left[\frac{10\text{ft}}{2} - \frac{4\text{ft}}{3}\right] \qquad x'$$

$$x' = 2.14 \text{ ft}$$

Given: Tank shown, of width $b = 2$ ft, is filled with water to depth D.
$k = 0.5\ \text{ft}^{-1}$, $H = 12$ ft
$p_s = 10$ psig.



Find: (a) Magnitude and line of action of vertical force on curved wall for $D = 8$ ft
(b) F_V and z' for $D = 4$ ft

Plot: F_V and z' vs D for $0 \le D \le H$ with $p_s = 10$ psig.

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $F_V = \int p\, dA_x$, $z' F_V = \int z\, dF_V$

Assumptions: (1) static liquid (2) ρ = constant
(2) p_{atm} acts on outside of tank

Then on integrating $dp = \rho g\, dh$, we obtain $p = p_s + \rho g h$
where p_s is the gage pressure at the free surface.

$$F_V = \int p\, dA_x = \int_0^{z_1} (p_s + \rho g h)\, b\, dz$$

Note $h = D - x$ $\quad x = kz^2$ $\quad \therefore h = D - kz^2$

$z = \left(\frac{x}{k}\right)^{1/2}$ At $x = D$, $z = z_1 = \left(\frac{D}{k}\right)^{1/2}$

$$\therefore F_V = \int_0^{z_1} \left[p_s + \rho g (D - kz^2)\right] b\, dz = b\left[p_s z + \rho g \left(Dz - k\frac{z^3}{3}\right)\right]_0^{z_1}$$

$$F_V = b\left[p_s z_1 + \rho g \left(D z_1 - \frac{k z_1^3}{3}\right)\right] = b\left[p_s \left(\frac{D}{k}\right)^{1/2} + \rho g \left\{D\left(\frac{D}{k}\right)^{1/2} - \frac{k}{3}\left(\frac{D}{k}\right)^{3/2}\right\}\right]$$

$$F_V = b\left(\frac{D}{k}\right)^{1/2}\left[p_s + \rho g\left\{D - \frac{D}{3}\right\}\right] = b\left(\frac{D}{k}\right)^{1/2}\left[p_s + \frac{2}{3}\rho g D\right] \quad \text{— (1)}$$

$$z' F_V = \int_0^{z_1} z\left[p_s + \rho g (D - kz^2)\right] b\, dz = b\int_0^{z_1} \left[p_s z + \rho g (Dz - kz^3)\right] dz$$

$$= b\left[p_s \frac{z^2}{2} + \rho g \left(\frac{Dz^2}{2} - \frac{kz^4}{4}\right)\right]_0^{z_1} = b\left[p_s \frac{z_1^2}{2} + \rho g \left(\frac{D z_1^2}{2} - \frac{k z_1^4}{4}\right)\right]$$

$$= b\left[\frac{p_s}{2}\frac{D}{k} + \rho g \left(\frac{D}{2}\frac{D}{k} - \frac{k}{4}\frac{D^2}{k^2}\right)\right]$$

$$z' F_V = \frac{bD}{2k}\left[p_s + \rho g\left(D - \frac{D}{2}\right)\right] = \frac{bD}{2k}\left[p_s + \rho g \frac{D}{2}\right]$$

$$z' = \frac{bD}{2k F_V}\left[p_s + \rho g \frac{D}{2}\right] = \frac{1}{2}\left(\frac{D}{k}\right)^{1/2} \frac{\left[p_s + \rho g \frac{D}{2}\right]}{\left[p_s + \frac{2}{3}\rho g D\right]} \quad \text{— (2)}$$

(c) Evaluating for $D = 8$ ft

$$F_V = b\left(\frac{2}{\rho}\right)^{1/2}\left[p_s + \frac{\rho g D}{2}\right]$$

$$= 2\text{ ft} \times \left(\frac{8\text{ ft}}{0.5\text{ ft}^{-1}}\right)^{1/2}\left[10\frac{\text{lbf}}{\text{in}^2} \times 144\frac{\text{in}^2}{\text{ft}^2} + \frac{1}{2} \times 1.94\frac{\text{slug}}{\text{ft}^3} \times 32.2\frac{\text{ft}}{\text{s}^2} \times 8\text{ ft} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}\right]$$

$F_V = 14{,}200$ lbf ← F_V

$$\bar{x}' = \frac{b}{2\rho F_V}\left[p_s + \frac{\rho g D^2}{2}\right]$$

$$= \frac{2\text{ ft} \times 8\text{ ft}}{2} \times \frac{\text{ft}}{0.5} \times \frac{1}{14{,}200\text{ lbf}}\left[10\frac{\text{lbf}}{\text{in}^2} \times 144\frac{\text{in}^2}{\text{ft}^2} + \frac{1}{2} \times 1.94\frac{\text{slug}}{\text{ft}^3} \times 32.2\frac{\text{ft}}{\text{s}^2} \times 8^2\text{ ft}^2 \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}\right]$$

$\bar{x}' = 1.90$ ft ← $\bar{x}'_{D=8'}$

(d) Evaluating for $D = 4$ ft

$$F_V = 2\text{ ft} \times \left(\frac{4\text{ ft}}{0.5\text{ ft}^{-1}}\right)^{1/2}\left[1440\frac{\text{lbf}}{\text{ft}^2} + \frac{1}{2} \times 1.94\frac{\text{slug}}{\text{ft}^3} \times 32.2\frac{\text{ft}}{\text{s}^2} \times 4\text{ ft} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}\right]$$

$F_V = 9{,}090$ lbf ←

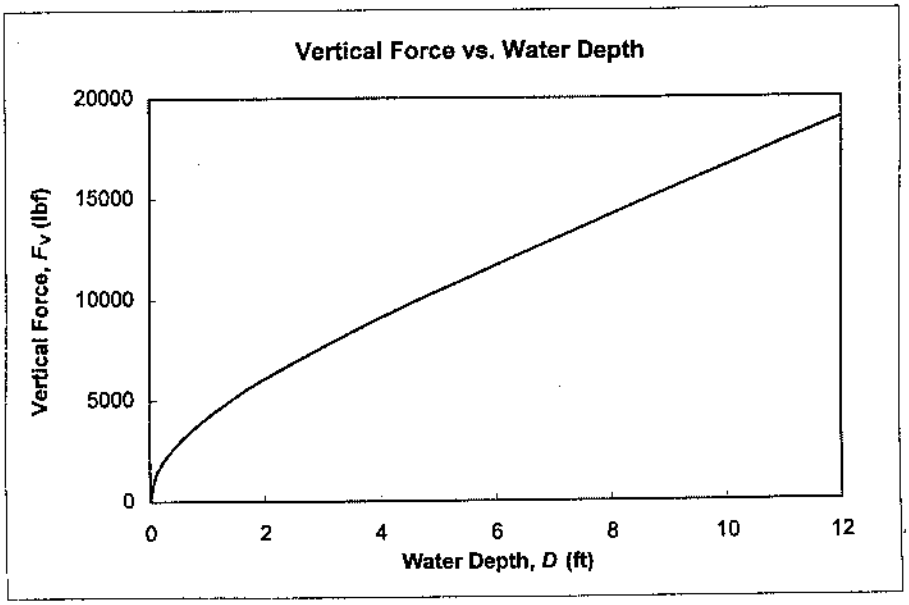
$$\bar{x}' = \frac{2\text{ ft} \times 4\text{ ft}}{2} \times \frac{\text{ft}}{0.5} \times \frac{1}{9{,}090\text{ lbf}}\left[1440\frac{\text{lbf}}{\text{ft}^2} + \frac{1}{2} \times 1.94\frac{\text{slug}}{\text{ft}^3} \times 32.2\frac{\text{ft}}{\text{s}^2} \times 4^2\text{ ft}^2 \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}\right]$$

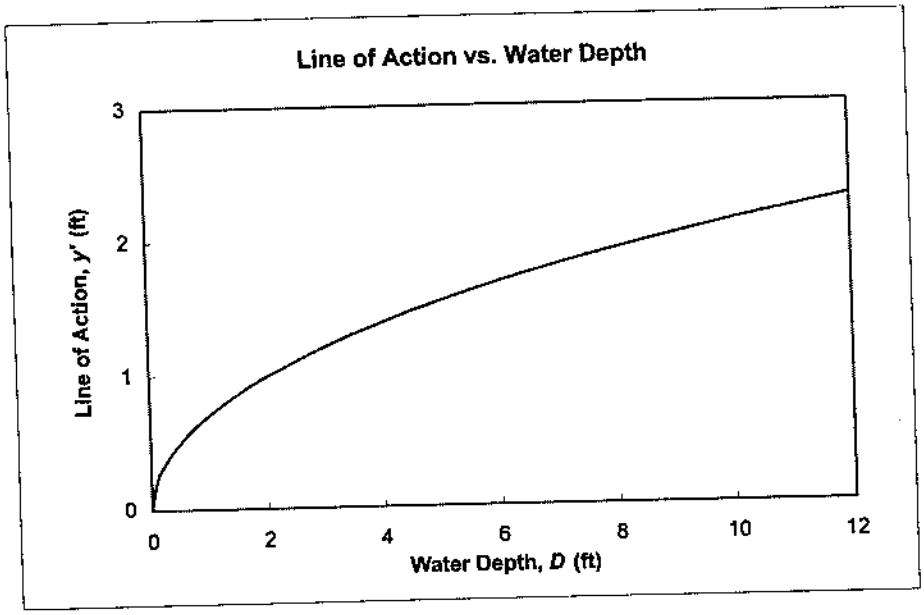
$\bar{x}' = 1.38$ ft ← $\bar{x}'_{D=4'}$

For the plot we have

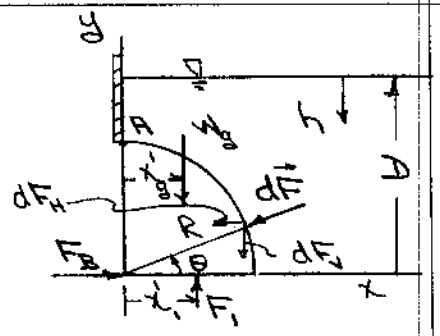
$$F_V = 2.828\,D^{1/2}\left[1440 + 41.6D\right] \qquad \{D \text{ in ft}\}$$

$$\bar{x}' = \frac{2D}{F_V}\left[1440 + 31.2D\right] \qquad \{D \text{ in ft}, F_V \text{ in lbf}\}$$




Line of Action vs. Water Depth
Line of Action, y' (ft)
Water Depth, D (ft)
0
1
2
3
0
2
4
6
8
10
12

Given: Concrete gate in the form of a quarter cylinder, hinged at A, has width $b = 2\,m$. Liquid is water. $R = 2\,m$, $D = 3\,m$



Find: Force on the stop at B.

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $\vec{F}_R = -\int p\,d\vec{A}$, $\Sigma M_{A_z} = 0$

Assumptions: (1) static liquid (2) ρ = constant

$\Sigma M_{A_z} = 0 = x_1' F_1 - W_g x_g' + R F_B - \int (R-y)\,dF_H - \int x\,dF_V$

$dF_V = dF \sin\theta = p\,dA \sin\theta$; $dF_H = dF\cos\theta = p\,dA\cos\theta$

$dp = \rho g\,dh$ and $p - p_0 = \rho g h = \rho g (D-y)$. Also $dA = b R\,d\theta$

Then $R F_B = W_g x_g' - F_1 x_1' + \int_0^{\pi/2} (R-y)\, p\, bR\cos\theta\,d\theta + \int_0^{\pi/2} x\, p\, bR\sin\theta\,d\theta$

$R F_B = W_g x_g' - F_1 x_1' + \int_0^{\pi/2} (R - R\sin\theta)\,\rho g (D-y)\, bR\cos\theta\,d\theta + \int_0^{\pi/2} R\cos\theta\,\rho g (D-y)\, bR\sin\theta\,d\theta$

$= W_g x_g' - F_1 x_1' + \rho g b R^2 \int_0^{\pi/2} (1-\sin\theta)(D - R\sin\theta)\cos\theta\,d\theta + \rho g b R^2 \int_0^{\pi/2} \sin\theta\cos\theta (D - R\sin\theta)\,d\theta$

$= W_g x_g' - F_1 x_1' + \rho g b R^2 \int_0^{\pi/2} \left[D\cos\theta - (D+R)\sin\theta\cos\theta + R\sin^2\theta\cos\theta \right] d\theta$

$+ \rho g b R^2 \int_0^{\pi/2} \left[D\sin\theta\cos\theta - R\sin^2\theta\cos\theta \right] d\theta$

$= W_g x_g' - F_1 x_1' + \rho g b R^2 \left[D\sin\theta - (D+R)\frac{1}{2}\sin^2\theta + R\,\frac{\sin^3\theta}{3} \right]_0^{\pi/2}$

$+ \rho g b R^2 \left[D\,\frac{\sin^2\theta}{2} - R\,\frac{\sin^3\theta}{3} \right]_0^{\pi/2}$

$= W_g x_g' - F_1 x_1' + \rho g b R^2 \left[D - \frac{1}{2}(D+R) + \frac{1}{2} D \right]$

$R F_B = W_g x_g' - F_1 x_1' + \rho g b R^2 \left[D - \frac{R}{2} \right]$

$F_1 = p_1 A_1 = \rho g D\, bR$ also $x_1' = \frac{D}{2}$

$W_g = \rho_g \forall_g = \rho_g g \frac{\pi R^2}{4} b = SG\,\rho g \frac{\pi R^2}{4} b$ {From Table A.1, SG = 2.4}

also $x_g' = \frac{4R}{3\pi}$

$\therefore$ $R F_B = SG\,\rho g \frac{\pi R^2}{4} b \cdot \frac{4R}{3\pi} - \rho g D\, bR \cdot \frac{D}{2} + \rho g b R^2 \left[D - \frac{R}{2} \right]$

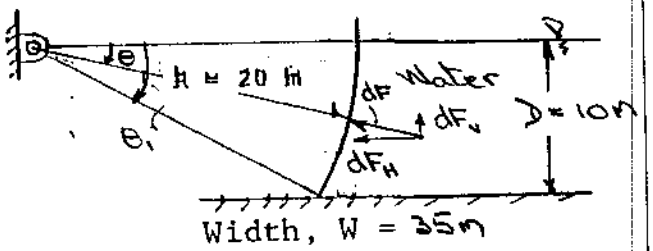
$F_B = SG_{gde}\,\rho g \frac{bR^2}{3} + \rho g b R \left(D - \frac{R}{2} \right) = \rho g b R \left[\frac{(SG_{gde})R}{3} + \left(D - \frac{R}{2} \right) \right]$

$F_B = 1000\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 2\,m \times 2\,m \left[\frac{(2.4)(2)}{3} + \frac{1}{2} \right] m \times \frac{N \cdot s^2}{kg \cdot m}$

$F_B = 82.4\ kN$ ← F_B

Given: Tainter gate as shown

Find: Force of the water acting on the gate.



Solution:

Basic equations: $dF = p\,dA$; $\frac{dp}{dh} = \rho g$

Assumptions: (1) static fluid
(2) ρ = constant
(3) p_{atm} acts at free surface and on surface of gate

For ρ = const, $\int dp = \int \rho g\,dh$ yields $p - p_{atm} = \rho g h = \rho g R \sin\theta$ (3)

$dF_H = dF\cos\theta = p\,dA\cos\theta = \rho g R \sin\theta\, W R\,d\theta \cos\theta$ $\{dA = W R\,d\theta\}$

$F_H = \int dF_H = \int_0^{\theta_1} \rho g W R^2 \sin\theta\cos\theta\,d\theta$ where $\theta_1 = \sin^{-1}\frac{10}{20} = 30°$

$F_H = \rho g W R^2 \int_0^{30°} \sin\theta\cos\theta\,d\theta = \rho g W R^2 \left[\frac{\sin^2\theta}{2}\right]_0^{30°} = \frac{\rho g W R^2}{8}$

$F_H = \frac{1}{8} \times 999 \frac{kg}{m^3} \times 9.81 \frac{m}{sec^2} \times 35m \times (20m)^2 \times \frac{N \cdot sec^2}{kg \cdot m} = 1.72 \times 10^7\ N$ ← F_H

$dF_V = dF\sin\theta = p\,dA\sin\theta = \rho g R\sin\theta\, W R\,d\theta \sin\theta$

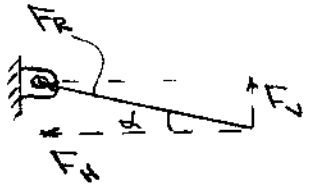
$F_V = \int dF_V = \rho g W R^2 \int_0^{30°} \sin^2\theta\,d\theta = \rho g W R^2 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/6}$

$F_V = \rho g W R^2 \left[\frac{\pi}{12} - \frac{0.866}{4}\right] = 0.0453\,\rho g W R^2$

$F_V = 0.0453 \times 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 35m \times (20m)^2 \times \frac{N \cdot s^2}{kg \cdot m} = 6.22 \times 10^6\ N$ ← F_V

Since the gate surface in contact with the water is a circular arc, all elements dF, of the force and hence the line of action of the resultant force must pass through the pivot. Thus

$F_R = \left[F_H^2 + F_V^2\right]^{1/2} = \left[(17.2\times10^6)^2 + (6.22\times10^6)^2\right]^{1/2} = 1.83 \times 10^7\ N$ ← F_R

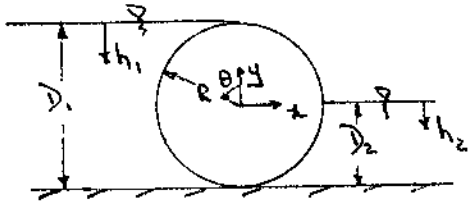


$\alpha = \tan^{-1}\frac{F_V}{F_H} = \tan^{-1}\frac{6.22}{17.2}$

$\alpha = 19.9°$ ← α

F_R passes through pivot at angle α to the horizontal

Given: Cylindrical weir of radius, $R = 1.5$ m and length, $L = 6$ m as shown
Liquid is water
$D_1 = 3$ m $D_2 = 1.5$ m



Find: Magnitude and direction of resultant force of water on the weir.

Solution:

Basic equations: $\vec{F}_R = -\int p d\vec{A}$ $\frac{dp}{dh} = \rho g$

Assumptions: (1) static fluid
(2) ρ = constant
(3) h is measured positive down from free surface

$$F_{R_x} = \int dF_x = \vec{F}_R \cdot \hat{i} = \int d\vec{F} \cdot \hat{i} = -\int p d\vec{A} \cdot \hat{i} = -\int p dA \cos(90+\theta) = \int p dA \sin\theta$$

$$F_{R_y} = \int dF_y = \vec{F}_R \cdot \hat{j} = \int d\vec{F} \cdot \hat{j} = -\int p d\vec{A} \cdot \hat{j} = -\int p dA \cos\theta$$

Since $dA = LR d\theta$,

$$F_{R_x} = \int_0^{3\pi/2} pLR \sin\theta \, d\theta \quad \text{and} \quad F_{R_y} = -\int_0^{3\pi/2} pLR \cos\theta \, d\theta$$

We can obtain an expression for p as a function of h

$$\frac{dp}{dh} = \rho g \qquad dp = \rho g dh \quad \text{and} \quad p - p_0 = \int_{p_0}^{p} dp = \int_0^h \rho g dh = \rho g h$$

Since atmospheric pressure acts over the first quadrant of the cylinder and both free surfaces, the appropriate expression for p is $p = \rho g h$.

For
$0 \le \theta \le \pi$, $h_1 = R - R\cos\theta = R(1-\cos\theta)$ and hence $p_1 = \rho g R(1-\cos\theta)$
$\pi \le \theta \le \frac{3\pi}{2}$, $h_2 = -R\cos\theta$ and hence $p_2 = -\rho g R \cos\theta$

$$F_{R_x} = \int_0^{3\pi/2} pLR\sin\theta \, d\theta = \int_0^{\pi} \rho g R(1-\cos\theta) LR \sin\theta \, d\theta + \int_{\pi}^{3\pi/2} (-\rho g R\cos\theta) LR \sin\theta \, d\theta$$

$$= \rho g R^2 L \int_0^{\pi} (1-\cos\theta)\sin\theta \, d\theta - \rho g R^2 L \int_{\pi}^{3\pi/2} \cos\theta \sin\theta \, d\theta$$

$$= \rho g R^2 L \left[-\cos\theta - \tfrac{1}{2}\sin^2\theta\right]_0^{\pi} - \rho g R^2 L \left[\tfrac{1}{2}\sin^2\theta\right]_{\pi}^{3\pi/2} = \rho g R^2 L \left[2 - \tfrac{1}{2}\right] = \tfrac{3}{2}\rho g R^2 L$$

$$F_{R_x} = \tfrac{3}{2} \times 999 \tfrac{kg}{m^3} \times 9.81 \tfrac{m}{s^2} \times (1.5)^2 m^2 \times 6m \times \tfrac{N \cdot s^2}{kg \cdot m} = 198 \text{ kN}$$

$$F_{R_y} = -\int_0^{3\pi/2} pLR\cos\theta = -\int_0^{\pi} \rho g R(1-\cos\theta) LR\cos\theta \, d\theta - \int_{\pi}^{3\pi/2} (-\rho g R\cos\theta) LR\cos\theta \, d\theta$$

$$= -\rho g R^2 L \int_0^{\pi} (1-\cos\theta)\cos\theta \, d\theta + \rho g R^2 L \int_{\pi}^{3\pi/2} \cos^2\theta \, d\theta$$

$$= -\rho g R^2 L \left[\sin\theta - \tfrac{\theta}{2} - \tfrac{\sin 2\theta}{4}\right]_0^{\pi} + \rho g R^2 L \left[\tfrac{\theta}{2} + \tfrac{\sin 2\theta}{4}\right]_{\pi}^{3\pi/2} = \rho g R^2 L \left[\tfrac{\pi}{2} + \tfrac{3\pi}{4} - \tfrac{\pi}{2}\right] = \tfrac{3\pi}{4}\rho g R^2 L$$

$$F_{R_y} = \tfrac{3\pi}{4} \times 999 \tfrac{kg}{m^3} \times 9.81 \tfrac{m}{s^2} \times (1.5)^2 m^2 \times 6m \times \tfrac{N \cdot s^2}{kg \cdot m} = 312 \text{ kN}$$

$$\vec{F}_R = \hat{i} F_{R_x} + \hat{j} F_{R_y} = 198\hat{i} + 312\hat{j} \text{ kN}$$

$$F_R = \sqrt{F_{R_x}^2 + F_{R_y}^2} = \left[(198)^2 + (312)^2\right]^{1/2} \text{kN} = 370 \text{ kN} \quad \leftarrow F_R$$

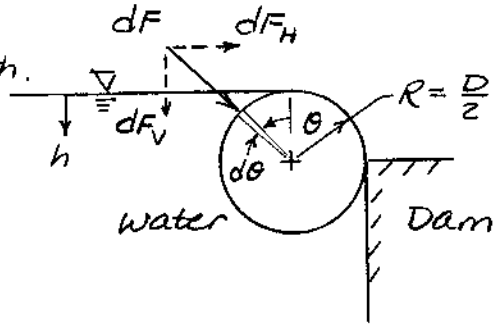
Since all elements of force $d\vec{F}$ are normal to the surface, the direction α,

$$\alpha = \tan^{-1} F_{R_y}/F_{R_x} = \tan^{-1} 312/198 = 57.6^\circ \quad \leftarrow \alpha$$

Given: Cylindrical log floating against dam.

Find: (a) Mass per unit length

(b) Contact force per unit length.



Solution: Use hydrostatic equations

Basic equations: $\frac{dp}{dh} = \rho g \qquad dF = p\,dA$

Assumptions: (1) Static liquid
(2) Incompressible
(3) Neglect p_{atm} (it acts everywhere)

Then $p - \cancel{p_0}^{(3)} = \rho g h = \rho g R(1-\cos\theta)$

$dF = p\,dA = \rho w R\,d\theta, \quad dF_H = dF\sin\theta, \quad dF_V = dF\cos\theta$

$$F_H = \int_0^{3\pi/2} \rho g R(1-\cos\theta)\, wR\sin\theta\, d\theta = \rho g w R^2\left[-\cos\theta - \frac{\sin^2\theta}{2}\right]_0^{3\pi/2} = \rho g w R^2\left[-\frac{(-1)^2}{2} - (-1)\right]$$

$$F_H = \frac{1}{2}\rho g w R^2 \qquad \frac{F_H}{w} = \frac{1}{2}\rho g R^2 \qquad \frac{F_H}{w}$$

$$F_V = \int_0^{3\pi/2} \rho g R(1-\cos\theta)\, wR\cos\theta\, d\theta = \int_0^{3\pi/2} \rho g w R^2\left(\cos\theta - \frac{1+\cos 2\theta}{2}\right) d\theta$$

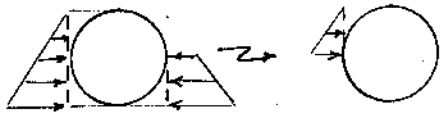
$$F_V = \rho g w R^2\left[\sin\theta - \frac{\theta + \frac{1}{2}\sin 2\theta}{2}\right]_0^{3\pi/2} = \rho g w R^2\left[-1 - \frac{3\pi}{4}\right] = -\rho g w R^2\left[1 + \frac{3\pi}{4}\right]$$

From a free-body diagram of the log

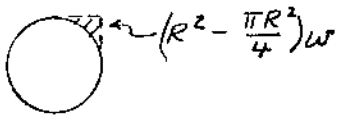
$$\Sigma F_y = -mg - F_V = 0 \qquad m = -\frac{F_V}{g} = \rho w R^2\left[1 + \frac{3\pi}{4}\right]$$

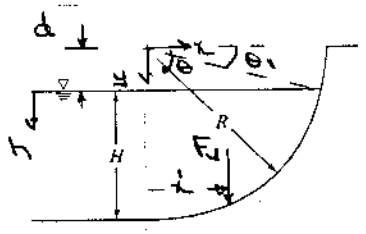
$$\frac{m}{w} = \rho R^2\left[1 + \frac{3\pi}{4}\right] \qquad \frac{m}{w}$$

Check: $F_H = p_c A = \rho g \frac{R}{2}\, wR = \frac{1}{2}\rho g w R^2$ ✓



$$F_V = -\rho g \forall - \rho g\left[R^2 - \frac{\pi R^2}{4}\right] w = -\rho g w\left[-\pi R^2 - R^2 + \frac{\pi R^2}{4}\right] = -\rho g w R^2\left[1 + \frac{3\pi}{4}\right]$$ ✓





Given: Curved surface, in shape of quarter cylinder, with radius $R = 0.3\,m$ and width $w = 1.25\,m$ is filled to depth $H = 0.24\,m$ with liquid concrete.

Find: (a) Magnitude, and (b) line of action, of the vertical force on the form from the concrete.

Plot: F_V and x' over the range of depth $0 \le H \le R$

Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $F_V = \int p\,dA_y$, $x' F_V = \int x\,dF_V$

Assumptions: (1) static liquid (2) ρ = constant (3) p_{atm} acts at surface of concrete

Then on integrating $dp = \rho g\,dh$, we obtain $p = \rho g h$

$$F_V = \int p\,dA_y = \int \rho g h\,dA \sin\theta \qquad dA = w R\,d\theta$$

From the geometry: $y = R\sin\theta$, $h = y - d$, $d = R - H$

$$F_V = \int_{\theta_1}^{\pi/2} \rho g (R\sin\theta - d)\sin\theta\, w R\,d\theta \qquad \text{where } \theta_1 = \sin^{-1}\frac{d}{R}$$

$$F_V = \rho g R w \int_{\theta_1}^{\pi/2} (R\sin^2\theta - d\sin\theta)\,d\theta = \rho g R w \left[R\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + d\cos\theta \right]_{\theta_1}^{\pi/2}$$

$$F_V = \rho g R w \left[R\left(\frac{\pi}{4} - \frac{\theta_1}{2} + \frac{\sin 2\theta_1}{4}\right) - d\cos\theta_1 \right] \qquad (1)$$

Evaluating, $\theta_1 = \sin^{-1}\frac{d}{R} = \sin^{-1}\frac{0.3 - 0.24}{0.30} = 11.5°$

$\rho = SG\,\rho_{H_2O}$ ($SG = 2.50$, Table A.1)

$$F_V = 1000 \frac{kg}{m^3} \times 2.5 \times 9.81 \frac{m}{s^2} \times 0.3m \times 1.25m \times \frac{N \cdot s^2}{kg \cdot m} \left[0.3m\left(\frac{\pi}{4} - 0.0639\frac{\pi}{2} + \frac{\sin 23°}{4}\right) - 0.06m\cos 11.5° \right]$$

$$F_V = 1.62\,kN \qquad F_V$$

$$x' F_V = \rho g R w \int_{\theta_1}^{\pi/2} x(R\sin^2\theta - d\sin\theta)\,d\theta = \rho g R^2 w \int_{\theta_1}^{\pi/2} (R\sin^2\theta\cos\theta - d\sin\theta\cos\theta)\,d\theta$$

$$= \rho g R^2 w \left[R\frac{\sin^3\theta}{3} + d\frac{\cos^2\theta}{2} \right]_{\theta_1}^{\pi/2}$$

$$x' F_V = \rho g R^2 w \left[\frac{R}{3}(1 - \sin^3\theta_1) - \frac{d}{2}\cos^2\theta_1 \right]$$

$$x' = SG\,\rho_{H_2O}\,\frac{g R^2 W}{F_V}\left[\frac{R}{3}(1-\sin^3\theta_1) - \frac{R}{2}\cos^2\theta_1\right] \quad \text{(2)}$$

$$x' = 2.5 \times 1000\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times (0.3\,m)^2 \times 1.25\,m \times \frac{1}{1.162\times10^3\,N} \times \frac{N\cdot s^2}{kg\cdot m} \times \left[\frac{0.3\,m}{3}(1-\sin^3 11.5^\circ) - \frac{0.3\,m}{2}\cos^2 11.5^\circ\right]$$

$$x' = 0.120\,m \quad \leftarrow x'$$

The computing equations for the required plots are:

$$\theta_1 = \sin^{-1}\frac{R-H}{R} + \sin^{-1}\left(1-\frac{H}{R}\right) \quad \text{(3)}$$

$$F_V = SG\,\rho_{H_2O}\,g R^2 W\left[\frac{\pi}{4} - \frac{\theta_1}{2} + \frac{\sin 2\theta_1}{4} - \left(1-\frac{H}{R}\right)\cos\theta_1\right] \quad \text{(1a)}$$

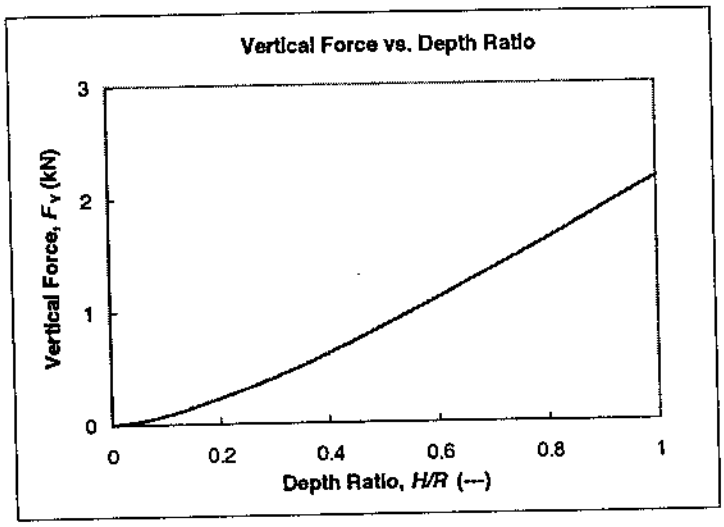
$$x' = SG\,\frac{\rho_{H_2O}\,g R^3 W}{F_V}\left[\frac{1}{3}(1-\sin^3\theta_1) - \frac{1}{2}\left(1-\frac{H}{R}\right)\cos^2\theta_1\right] \quad \text{(2a)}$$

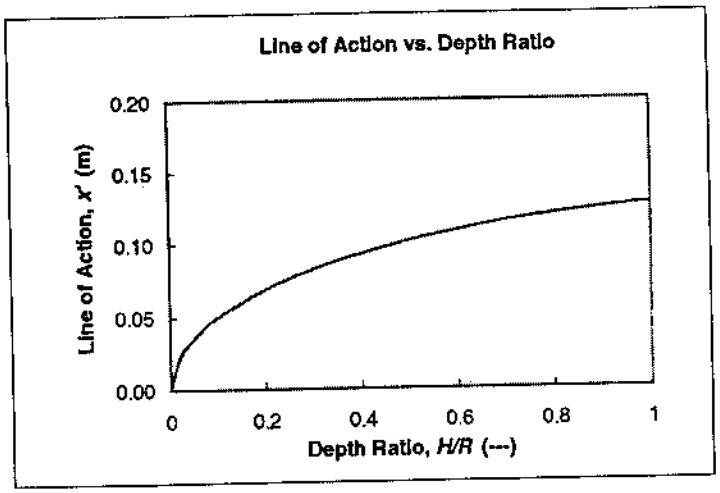
Force and line of action vs. liquid concrete depth:

Radius:	***R* =**	**0.3**	**m**
Specific gravity:	***SG* =**	**2.5**	**—**
Width:	***W* =**	**1.25**	**m**

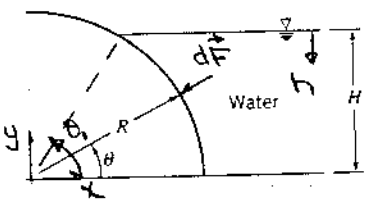
Depth Ratio, *H/R* (—)	Concrete Depth, *H* (m)	Angle, θ_1 (deg)	Vertical Force, F_V (kN)	Line of Action, *x'* (m)
0	0	90.0	0	0
0.02	0.006	78.5	0.00734	0.0224
0.05	0.015	71.8	0.0289	0.0352
0.1	0.03	64.2	0.0810	0.0494
0.2	0.06	53.1	0.226	0.0685
0.3	0.09	44.4	0.408	0.0822
0.4	0.12	36.9	0.617	0.0930
0.5	0.15	30.0	0.847	0.102
0.6	0.18	23.6	1.09	0.109
0.7	0.21	17.5	1.35	0.115
0.8	0.24	11.5	1.62	0.120
0.9	0.27	5.7	1.89	0.124
1.0	0.30	0.0	2.17	0.127

National Brand
SQUARE

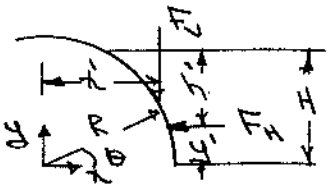

Vertical Force vs. Depth Ratio
Vertical Force, F_V (kN)
3
2
1
0
0
0.2
0.4
0.6
0.8
1
Depth Ratio, H/R (---)


Line of Action vs. Depth Ratio
Line of Action, x' (m)
0.20
0.15
0.10
0.05
0.00
0
0.2
0.4
0.6
0.8
1
Depth Ratio, H/R (---)

Given: Curved surface, in shape of quarter cylinder, with radius $R = 0.750\,m$ and width $W = 3.55\,m$; water stands to depth $H = 0.650\,m$



Find: Magnitude and line of action of:
(a) Vertical force, and
(b) horizontal force
on the curved surface.



Solution:

Basic equations: $\frac{dp}{dh} = \rho g$, $F_V = \int p\,dA_y$, $x'F_V = \int x\,dF_V$

Computing equations: $F_H = p_c A$, $h' = h_c + \frac{I_{xc}}{h_c A}$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{atm} acts at free surface of the water

Then on integrating $dp = \rho g\,dh$, we obtain $p = \rho g h$.

From the geometry $h = H - R\sin\theta$, $y = R\sin\theta$, $x = R\cos\theta$

$\theta_1 = \sin^{-1} H/R$, $dA = W R\,d\theta$

$$F_V = \int p\,dA_y = \int \rho g h\,dA \sin\theta = \int_0^{\theta_1} \rho g (H - R\sin\theta)\sin\theta\, W R\,d\theta$$

$$F_V = \rho g W R \int_0^{\theta_1} (H\sin\theta - R\sin^2\theta)\,d\theta = \rho g W R \left[-H\cos\theta - R\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)\right]_0^{\theta_1}$$

$$F_V = \rho g W R \left[H(1-\cos\theta_1) - R\left(\frac{\theta_1}{2} - \frac{\sin 2\theta_1}{4}\right)\right] \quad \text{---------- (1)}$$

Evaluating for $\theta_1 = \sin^{-1}\frac{H}{R} = \sin^{-1}\frac{0.650}{0.750} = 60^\circ\ (\pi/3)$.

$$F_V = 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 3.55\,m \times 0.75\,m \left[0.65\,m\left(1-\cos 60^\circ\right) - 0.75\,m\left(\frac{\pi}{6} - \frac{\sin 120^\circ}{4}\right)\right] \frac{N\cdot s^2}{kg\cdot m}$$

$$F_V = 2.47\,kN \quad \longleftarrow F_V$$

$$x'F_V = \rho g W R \int_0^{\theta_1} R\cos\theta (H\sin\theta - R\sin^2\theta)\,d\theta = \rho g W R^2 \int_0^{\theta_1} (H\sin\theta\cos\theta - R\sin^2\theta\cos\theta)\,d\theta$$

$$x'F_V = \rho g W R^2 \left[\frac{H}{2}\sin^2\theta - R\frac{\sin^3\theta}{3}\right]_0^{\theta_1}$$

$$x' = \frac{\rho g W R^2}{F_V}\left[\frac{H}{2}\sin^2\theta_1 - \frac{R}{3}\sin^3\theta_1\right] \quad \text{---------- (2)}$$

$$x' = 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 3.55\,m \times (0.75\,m)^2 \times \frac{1}{2.47\times10^3\,N} \left[\frac{0.650\,m}{2}\sin^2 60^\circ - \frac{0.750\,m}{3}\sin^3 60^\circ\right] \frac{N\cdot s^2}{kg\cdot m}$$

$$x' = 0.645\,m \quad \longleftarrow x'$$

$$F_H = p_c A = \rho g h_c H W = \rho g \frac{H}{2} H W = \frac{\rho g H^2 W}{2} \quad \text{---------- (3)}$$

$$F_H = \frac{1}{2} \times 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times (0.65\,m)^2 \times 3.55\,m \times \frac{N\cdot s^2}{kg\cdot m} = 7.35\,kN \quad \longleftarrow F_H$$

$$y' = y_c + \frac{I_{xc}}{y_c A} = y_c + \frac{\frac{1}{12} W H^3}{\frac{H}{2} W H} = \frac{H}{2} + \frac{1}{12}\frac{W H^3}{W H^2} = \frac{2H}{3} + \frac{H}{6} = \frac{3H}{4}$$

$$x_c - x_c' = H - y' = H - \frac{3}{4}H = \frac{1}{4}H \qquad (4)$$

$$= \frac{1}{4}H = \frac{1}{4} \times 0.650\ \text{m} = 0.217\ \text{m}$$

x_c'

The computing equations for the plot are:

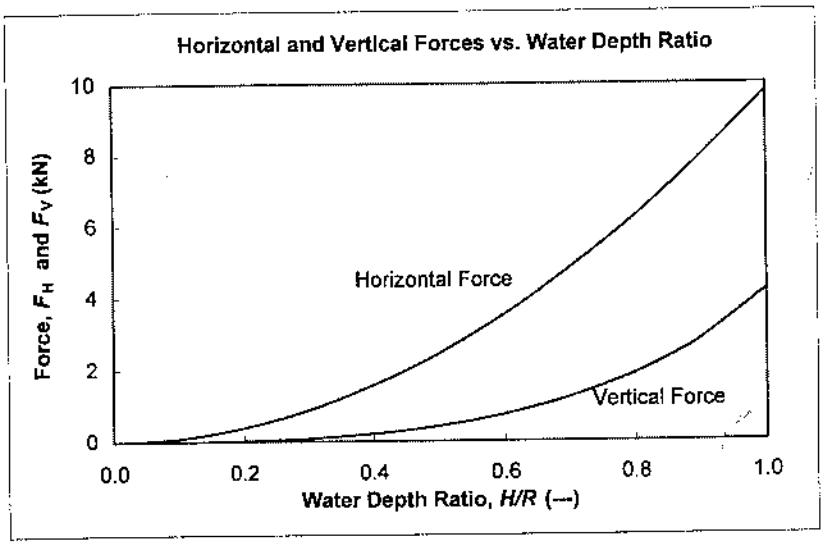
$$\theta_1 = \sin^{-1}\frac{H}{R}$$

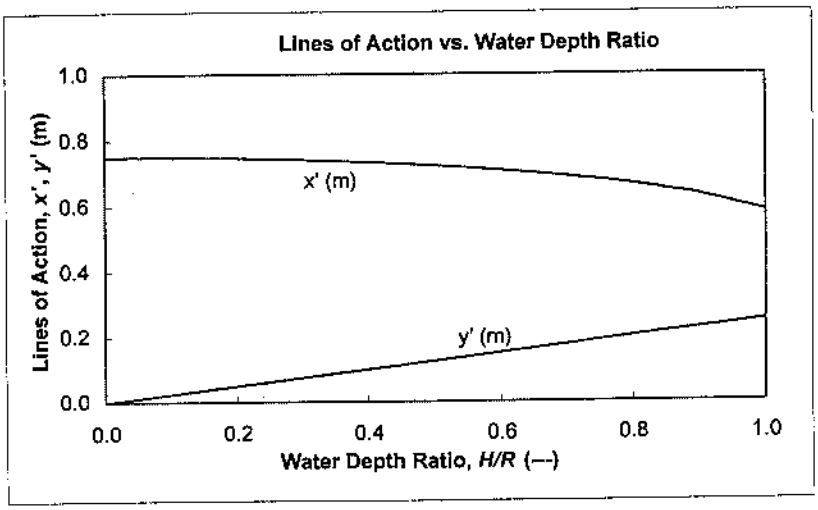
$$F_V = \rho g W R^2 \left[\frac{H}{R}(1-\cos\theta_1) - \frac{\theta_1}{2} + \frac{\sin 2\theta_1}{4}\right]$$

$$x' = \frac{\rho g W R^3 \sin^2\theta_1}{F_V}\left[\frac{1}{2}\frac{H}{R} - \frac{1}{3}\sin\theta_1\right]$$

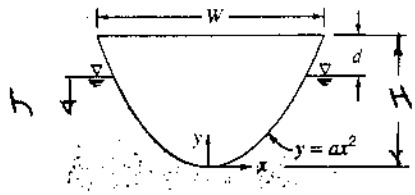
$$F_H = \frac{\rho g H^2 W}{2}$$

$$x_c' = \frac{1}{4}H$$





Given: Model cross section of canoe, by $y = ax^2$, where $a = 3.89\ m^{-1}$; coordinates are in meters. Assume constant width $W = 0.6m$ over entire length $L = 5.25\ m$.



Find: Expression relating total mass of canoe and contents to distance d; determine maximum allowable total mass without swamping the canoe.

Solution:

At any value of d the weight of the canoe and its contents is balanced by the net vertical force of the water on the canoe.

Basic equations: $\frac{dp}{dh} = \rho g$, $F_V = \int p\,dA_y$

Assumptions: (1) static liquid (2) ρ = constant
(3) p_{dn} acts at free surface of the water and on inner surface of canoe.

Then on integrating $dp = \rho g\,dh$, we obtain $p = \rho g h$

$F_V = \int p\,dA_y = \int \rho g h L\,dx$ where $h = (H-d) - y$

$y = ax^2$, At surface $y = H-d$ $\therefore x = \sqrt{\frac{H-d}{a}}$

$$F_V = 2\int_0^{\sqrt{(H-d)/a}} \rho g\left[(H-d) - ax^2\right] L\,dx = 2\rho g L\left[(H-d)x - a\frac{x^3}{3}\right]_0^{\sqrt{(H-d)/a}}$$

$$F_V = 2\rho g L\left[\frac{(H-d)^{3/2}}{\sqrt{a}} - \frac{a}{3}\frac{(H-d)^{3/2}}{a^{3/2}}\right] = \frac{2\rho g L}{\sqrt{a}}(H-d)^{3/2}\left[1 - \frac{1}{3}\right]$$

$$F_V = \frac{4}{3}\frac{\rho g L}{\sqrt{a}}(H-d)^{3/2} = Mg$$

$$\therefore M = \frac{4\rho L (H-d)^{3/2}}{3\sqrt{a}}$$

At $d = 0$, $x = W/2$, $y = H = 0.35m$

For $d = 0$, $M = \frac{4}{3} \times 999\frac{kg}{m^3} \times 5.25m \times (0.35m)^{3/2} \times \left(\frac{m}{3.89}\right)^{1/2} = 734\ kg$

This does not provide any cushion from swamping.

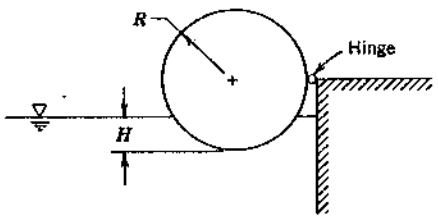
Set $d = 0.050\ m$

$$M = \frac{4}{3} \times 999\frac{kg}{m^3} \times 5.25m \times (0.30m)^{3/2} \times \left(\frac{m}{3.89}\right)^{1/2} = 583\ kg$$

The answer clearly depends on the allowed risk of swamping!

Given: Cylinder, of mass M, length L, and radius R, is hinged along it's length and immersed in an incompressible liquid to depth H.

Find: a general expression for the cylinder specific gravity as a function of $\alpha = H/R$ needed to hold the cylinder in equilibrium for $0 \le \alpha \le 1$.

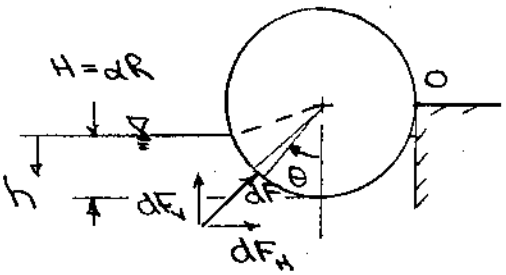


<u>Solution:</u> Apply fluid statics.

Basic eqs.: $\frac{dp}{dh} = \rho g$, $F = \int p\,dA$, $\Sigma M = 0$

Assumptions: (1) static liquid
(2) ρ = constant
$\therefore p = \rho g h$

For $0 \le \alpha \le 1$, F_H causes no net moment about O



$$dF_v = dF\cos\theta = p\,dA\cos\theta = \rho g h\, w R\, d\theta \cos\theta$$

$$h + R(1-\cos\theta) = H \quad , \quad \therefore h = H - R(1-\cos\theta).$$

$$dF_v = \rho g\left[H - R(1-\cos\theta)\right] w R\cos\theta\, d\theta = \rho g w R^2\left[\frac{H}{R} - (1-\cos\theta)\right]\cos\theta\, d\theta$$

$$dF_v = \rho g w R^2\left[(\alpha-1)\cos\theta + \cos^2\theta\right] d\theta = \rho g w R^2\left[(\alpha-1)\cos\theta + \frac{1+\cos 2\theta}{2}\right]$$

For $\alpha \le 1$, $F_H = 0$, and

$$F_v = \int_{-\theta_{max}}^{\theta_{max}} dF_v = 2\int_0^{\theta_{max}} dF_v \quad \text{where } \cos\theta_{max} = \frac{R-H}{R} = 1-\alpha$$

$$\theta_{max} = \cos^{-1}(1-\alpha)$$

$$F_v = 2\rho g w R^2 \int_0^{\theta_{max}}\left[(\alpha-1)\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right] d\theta$$

$$F_v = 2\rho g w R^2\left[(\alpha-1)\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\theta_{max}}$$

$$\sin\theta_{max} = \sqrt{1-\cos^2\theta_{max}} = \left[1-(1-\alpha)^2\right]^{1/2} = \left[1-1+2\alpha-\alpha^2\right]^{1/2} = \sqrt{\alpha(2-\alpha)}$$

$$\sin 2\theta_{max} = 2\sin\theta_{max}\cos\theta_{max} = 2\sqrt{\alpha(2-\alpha)}\,(1-\alpha)$$

Then,

$$F_v = 2\rho g w R^2\left[(\alpha-1)\sqrt{\alpha(2-\alpha)} + \frac{1}{2}\cos^{-1}(1-\alpha) + \frac{1}{2}(1-\alpha)\sqrt{\alpha(2-\alpha)}\right]$$

$$F_v = 2\rho g w R^2\left[\frac{1}{2}\cos^{-1}(1-\alpha) - \frac{1}{2}(1-\alpha)\sqrt{\alpha(2-\alpha)}\right]$$

$$F_v = \rho g w R^2\left[\cos^{-1}(1-\alpha) - (1-\alpha)\sqrt{\alpha(2-\alpha)}\right]$$

The line of action of the vertical force due to the liquid is through the centroid of the displaced liquid, i.e. through the center of the cylinder

The weight of the cylinder is given by

$$W = mg = \rho_c \forall g = SG\, \rho\, \pi R^2 w g$$

where $SG = \rho_c/\rho$ and the gravity force acts through the center of the cylinder

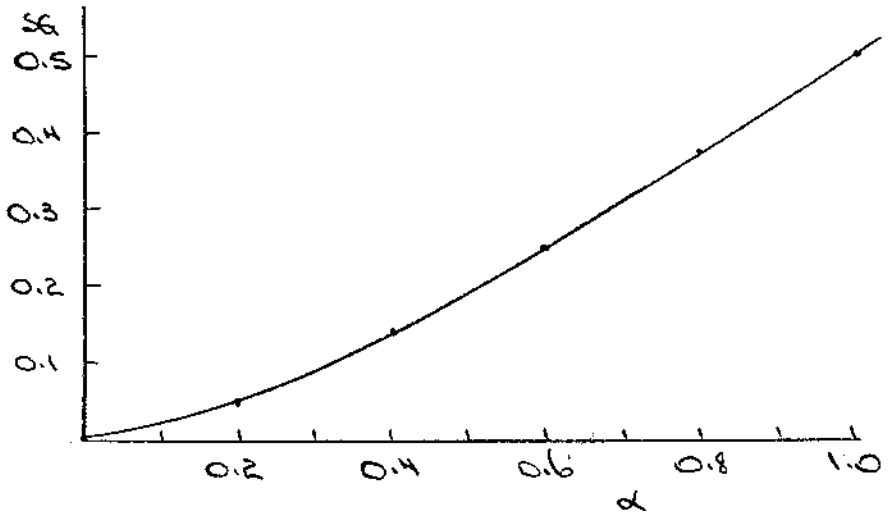
$$\Sigma M_o = WR - F_V R = 0 \quad \therefore \quad W = F_V \text{ and}$$

$$SG\, \cancel{\rho}\, \cancel{\pi R^2}\, \cancel{w}\, \cancel{g} = \cancel{\rho g w R^2} \left[\cos^{-1}(1-\alpha) - (1-\alpha)\sqrt{\alpha(2-\alpha)}\right]$$

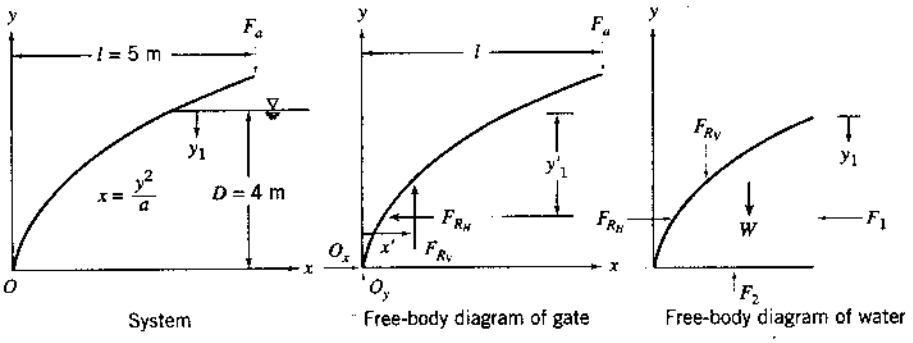
$$SG = \frac{1}{\pi}\left[\cos^{-1}(1-\alpha) + (\alpha-1)\sqrt{\alpha(2-\alpha)}\right] \qquad SG(0 \le \alpha \le 1)$$

Tabulating values

α	SG
0	0
0.2	0.052
0.4	0.142
0.6	0.252
0.8	0.374
1.0	0.500



Given: Gate of Example Problem 3.7, with gate mass, ρ'' (kg/m²).



Find: (a) F_a if $\rho'' = 70.2$ kg/m²
(b) Plot of F_a vs. ρ''
(c) Value of ρ'' for which $F_a = 0$.

Solution: Let mass per unit area of gate be ρ'' (kg/m²), and L' be the line of action of the gate wight. Then

$$L'M = \int x\,dm = \int_0^S x w \rho''\,ds, \text{ where } s \text{ is measured along the gate surface.}$$

But $ds = \sqrt{(dx)^2+(dy)^2} = \left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}} dy$; $x = \frac{y^2}{a}$; $dx = \frac{2y}{a}dy$; $\frac{dx}{dy} = \frac{2y}{a}$

Thus $ds = \left[1+\left(\frac{2y}{a}\right)^2\right]^{\frac{1}{2}} dy$. Substituting,

$$L'M = \int_0^S \frac{y^2}{a} w\rho'' \left[1+\left(\frac{2y}{a}\right)^2\right]^{\frac{1}{2}} dy = \int_0^S w\rho'' \frac{a}{4}\left(\frac{2y}{a}\right)^2 \left[1+\left(\frac{2y}{a}\right)^2\right]^{\frac{1}{2}} \frac{a}{2}\, d\left(\frac{2y}{a}\right)$$

Let $\lambda = \frac{2y}{a}$. Note at $s = 0$, $y = 0$, so $\lambda = 0$; at $s = S$, $y = \sqrt{aL}$, so $\lambda = \sqrt{4L/a}$. Thus

$$L'M = \frac{a^2}{8} w\rho'' \int_0^{\sqrt{4L/a}} \lambda^2 \left[1+\lambda^2\right]^{\frac{1}{2}} d\lambda$$

Integrating (see § 232.01, p. 56 of [1])

$$L'M = \frac{a^2 w\rho''}{8}\left[\frac{\lambda[1+\lambda^2]^{3/2}}{4} - \frac{\lambda[1+\lambda^2]^{\frac{1}{2}}}{8} - \frac{1}{8}\ln\left\{\lambda + [1+\lambda^2]^{1/2}\right\}\right]_0^{\sqrt{4L/a}}$$

$$L'M = \frac{a^2 w\rho''}{8}\left[\sqrt{\frac{4L}{a}}\,\frac{[1+4L/a]^{3/2}}{4} - \sqrt{\frac{4L}{a}}\,\frac{[1+4L/a]^{1/2}}{8} - \frac{1}{8}\ln\left\{\sqrt{\frac{4L}{a}} + \left[1+\frac{4L}{a}\right]^{1/2}\right\}\right]$$

Substituting $a = 4$ m, $L = 5$ m, and $w = 5$ m gives $4L/a = 5$, and

$$L'M = \frac{a^2 w\rho''}{8}(7.338) = 0.9173\, a^2 w\rho''$$

$$L'M = 0.9173 \times (4)^2 m^2 \times 5\,m \times 70.2\,\frac{kg}{m^2} = 5{,}151 \text{ kg·m} \quad \leftarrow L'M$$

Similarly

$$M = \int_0^S w\rho''\,ds = \int_0^S w\rho''\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}} dy = \int_0^S w\rho''\left[1+\lambda^2\right]^{\frac{1}{2}} \frac{a}{2}\,d\lambda$$

$$M = \frac{aw\rho''}{2}\int_0^{\sqrt{4L/a}} \left[1+\lambda^2\right]^{\frac{1}{2}} d\lambda$$

Integrating (see §230.01 of [1]),

$$M = \frac{aw\rho''}{2}\left[\frac{\lambda[1+\lambda^2]^{1/2}}{2} + \frac{1}{2}\ln\left\{\lambda + [1+\lambda^2]^{1/2}\right\}\right]_0^{\sqrt{4L/a}}$$

$$M = \frac{aw\rho''}{2}\left[\sqrt{\frac{4L}{a}}\,\frac{[1+\frac{4L}{a}]^{1/2}}{2} + \frac{1}{2}\ln\left\{\sqrt{\frac{4L}{a}} + [1+\frac{4L}{a}]^{1/2}\right\}\right] = 1.755\,aw\rho''$$

$$M = 1.755 \times 4\,m \times 5\,m \times 70.2\,\frac{kg}{m^2} = 2464\,kg$$ M

Finally,

$$L' = \frac{L'M}{M} = \frac{5151\,kg\cdot m}{2464\,kg} = 2.09\,m \quad \text{(note } \rho'' \text{ cancels out)}$$ L'

To determine F_a, sum moments about O using the diagram above,

$$\overset{+}{\circlearrowleft}\Sigma M_O = y'F_H + x'F_V - L'Mg - LF_a = 0$$

or $$F_a = \frac{1}{L}(y'F_H + x'F_V - L'Mg)$$

Using values from the solution to Example Problem 3.7,

$$F_a = \frac{1}{5m}\left(\frac{4}{3}m \times 392\,kN + 1.2m \times 261\,kN - 0.9173\rho''a^2wg\right)$$

$$= \frac{1}{5}\left[(523+313)10^3\,N\cdot m - 0.9173\rho'' \times (4)^2m^2 \times 5m \times 9.81\,\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m}\right]$$

$$F_a = \frac{1}{5}(8.36\times10^5 - 719.9\rho'') = 1.672\times10^5 - 144\rho''\,(kg/m^2) \qquad (1)$$

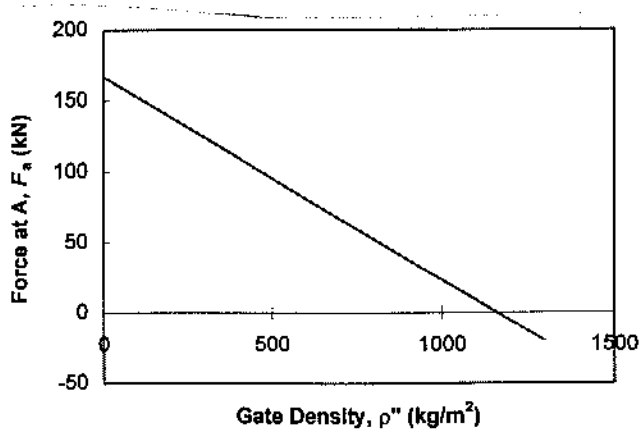
When $\rho'' = 70.2\,kg/m^2$,

$$F_a = 1.672\times10^5 - 144 \times 70.2 = 157\,kN \quad \text{(down, in direction shown)}$$ F_a

From Eq. 1, $F_a = 0$ when

$$\rho'' = \frac{1.672\times10^5}{144} = 1161\,kg/m^2$$ ρ'' when $F_a = 0$

The plot is:



Reference: [1] Dwight, Tables of Integrals, 4th ed. NY: Macmillan, 1961.

Given: Canoe, modelled as a right circular semi-cylindrical shell, floats in water of depth, d. The shell has outer radius, R = 0.35 m and length, L = 5.25m.

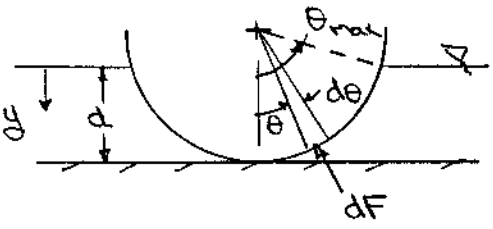
Find: (a) a general algebraic expression for the maximum total mass that can be floated, as a function of depth and
(b) evaluate for the given conditions with d = 0.245 m

Plot: the results over the range of water depth $0 \le d \le R$.

Solution:

Basic equations: $\frac{dp}{dy} = \rho g$; $p = p_{atm} + \rho g y$; $F_R = \int p\, dA$

End view of canoe



Assumptions: (1) static liquid
(2) p_{atm} acts on both inside & outside surfaces.

Geometry $y = y(\theta)$ for given d

$y = d - (R - R\cos\theta) = d - R + R\cos\theta$

$\theta_{max} = \cos^{-1}\frac{R-d}{R}$

A fbd of the canoe gives $\Sigma F_y = 0 = Mg - F_V$
where F_V is the vertical force of the water on the canoe

$$F_V = \int dF_V = \int dF\cos\theta = \int_A p\, dA\cos\theta = \int_{-\theta_{max}}^{\theta_{max}} \rho g y\, LR\, d\theta\cos\theta$$

$$F_V = 2\int_0^{\theta_{max}} \rho g LR\left[(d-R)\cos\theta + R\cos^2\theta\right] d\theta$$

$$F_V = 2\rho g LR\left[(d-R)\sin\theta + R\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\right]_0^{\theta_{max}}$$

$$F_V = 2\rho g LR\left[(d-R)\sin\theta_{max} + R\left(\frac{\theta_{max}}{2} + \frac{\sin 2\theta_{max}}{4}\right)\right]$$

where $\theta_{max} = \cos^{-1}\frac{(R-d)}{R}$.

Since $M = F_V/g$

$$M = 2\rho LR\left[(d-R)\sin\theta_{max} + R\left(\frac{\theta_{max}}{2} + \frac{\sin 2\theta_{max}}{4}\right)\right] \quad \leftarrow M(d)$$

For R = 0.35m, L = 5.25m and d = 0.245m,

$$\theta_{max} = \cos^{-1}\frac{(R-d)}{R} = \cos^{-1}\frac{(0.35-0.245)}{0.35} = \cos^{-1} 0.30 = 72.5°$$

$\theta_{max} = 0.403$ rad

$$M = 2 \times 999\ \frac{kg}{m^3} \times 5.25m \times 0.35m\left[(0.245-0.35)\sin 72.5 + 0.35\left(\frac{0.403rad}{2} + \frac{1}{4}\sin 145\right)\right] m$$

$M = 631\ kg$ ← M

The computing equations for the plot are.

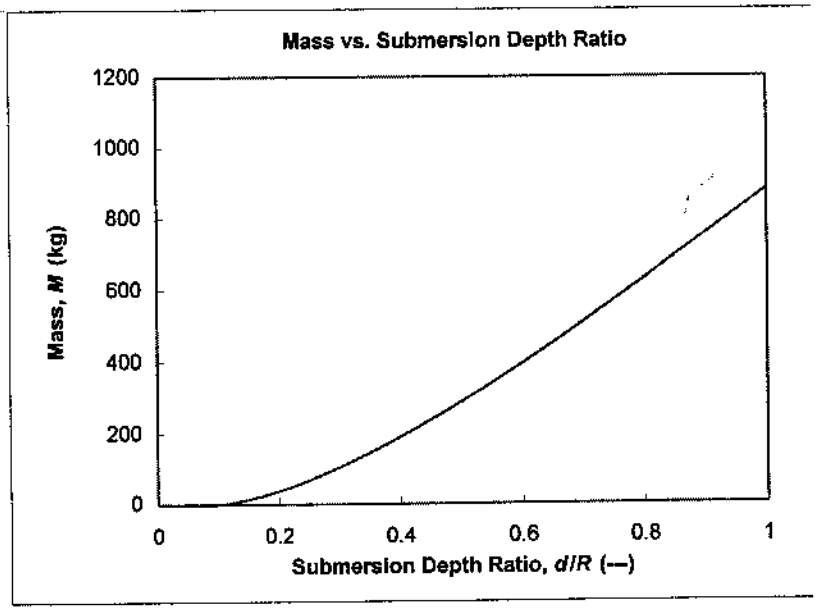
$$\theta_{max} = \cos^{-1}\left(1 - \frac{d}{R}\right)$$

$$M = 2\rho L R^2 \left[\frac{\theta_{max}}{2} + \frac{\sin 2\theta_{max}}{4} - \left(1 - \frac{d}{R}\right)\sin\theta_{max} \right]$$

Mass of canoe vs. depth of submersion ratio:

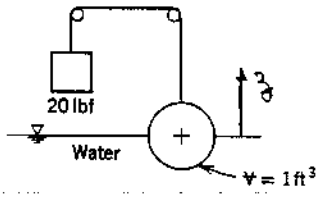
Density:	ρ =	999	kg/m^3
Length:	L =	5.25	m
Radius:	R =	0.35	m

d (m)	d/R (---)	θ_{max} (rad)	θ_{max} (deg)	Mass (kg)
0	0	0	0	0
0.035	0.10	0.45	25.8	37.7
0.070	0.20	0.64	36.9	105
0.105	0.30	0.80	45.6	190
0.140	0.40	0.93	53.1	287
0.175	0.50	1.05	60.0	395
0.210	0.60	1.16	66.4	509
0.245	0.70	1.27	72.5	630
0.280	0.80	1.37	78.5	754
0.315	0.90	1.47	84.3	881
0.350	1.00	1.57	90.0	1009



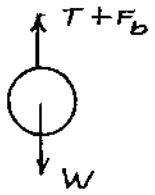
13-782 500 SHEETS, FILLER 5 SQUARE
42-381 500 SHEETS EYE-EASE® 5 SQUARE
42-391 150 SHEETS EYE-EASE® 5 SQUARE
42-396 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U. S. A.
National® Brand

Given: Sphere of $\forall = 1\ ft^3$ floating as shown.



Find: (a) Specific weight of sphere.
(b) New equilibrium position if 20-pound weight is removed.

Solution: Balance forces on sphere.



Basic equation: $\Sigma F_z = m a_z = 0$

Computing equation: $F_{buoyancy} = \rho_{H_2O} g \forall_{displaced} = \gamma \forall_{disp}$

Then $\Sigma F_z = T + F_b - W = 0$

$$\Sigma F_z = T + \gamma_{H_2O} \forall_{disp} - W = T + \gamma_{H_2O} \frac{\forall_s}{2} - \gamma_s \forall_s = 0$$

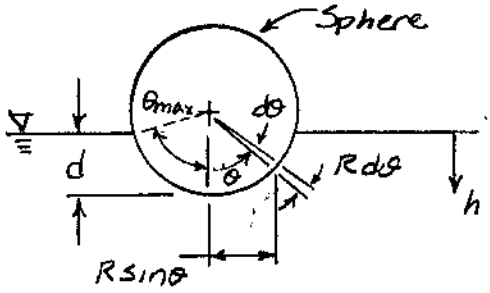
Thus $$\gamma_s = \frac{T + \gamma_{H_2O} \frac{\forall_s}{2}}{\forall_s} = \frac{T}{\forall_s} + \frac{\gamma_{H_2O}}{2}$$

$$\gamma_s = 20\ lbf \times \frac{1}{1\ ft^3} + \frac{1}{2} \times 62.4\ \frac{lbf}{ft^3} = 51.2\ lbf/ft^3$$ γ_s

To find new equilibrium position, evaluate force from water on sphere.

Basic equations: $\frac{dp}{dh} = \gamma_{H_2O}$

$dF = p\,dA$



Assumptions: (1) Static liquid
(2) Incompressible
(3) Neglect p_{atm}, because it acts everywhere

Then $dF_V = \cos\theta\, p\, dA$; $p = \gamma h$; $d = h + R(1-\cos\theta)$; $h = d - R(1-\cos\theta)$

$$dA = 2\pi R \sin\theta\, R\, d\theta = 2\pi R^2 \sin\theta\, d\theta$$

$$dF_V = \cos\theta\, \gamma \left[d - R(1-\cos\theta)\right] 2\pi R^2 \sin\theta\, d\theta = 2\pi R^3 \gamma \left[\frac{d}{R} - (1-\cos\theta)\right] \sin\theta \cos\theta\, d\theta$$

Now $$F_V = \int_A dF_V = \int_0^{\theta_{max}} 2\pi R^3 \gamma \left[\frac{d}{R} - (1-\cos\theta)\right] \sin\theta \cos\theta\, d\theta$$

At θ_{max}, $\cos\theta_{max} = \frac{R-d}{R} = 1 - \frac{d}{R}$, so

$$F_V = 2\pi R^3 \gamma \left\{\left(1 - \frac{d}{R}\right)\left[\frac{1}{2}\left(1 - \frac{d}{R}\right)^2 - \frac{1}{2}\right] - \frac{1}{3}\left(1 - \frac{d}{R}\right)^3 + \frac{1}{3}\right\}$$

$$F_V = 2\pi R^3 \gamma \left[\frac{1}{6}\left(1 - \frac{d}{R}\right)^3 - \frac{1}{2}\left(1 - \frac{d}{R}\right) + \frac{1}{3}\right]$$ F_V

But $\forall_s = \frac{4\pi R^3}{3}$, so

$F_V = F_{buoyancy} = \frac{3}{2}\left(\frac{4\pi R^3}{3}\right)\gamma_{H_2O} f\left(\frac{d}{R}\right)$; $f\left(\frac{d}{R}\right) = \frac{1}{6}\left(1-\frac{d}{R}\right)^3 - \frac{1}{2}\left(1-\frac{d}{R}\right) + \frac{1}{3}$

Because $T = 0$, $F_B = W = \gamma_S \forall_S = \frac{3}{2}\gamma_{H_2O}\forall_S f\left(\frac{d}{R}\right)$

Thus at equilibrium $f\left(\frac{d}{R}\right) = \frac{2}{3}\frac{\gamma_S}{\gamma_{H_2O}} = \frac{2}{3}SG_S = \frac{2}{3}\times\frac{51.2}{62.4} = 0.547$

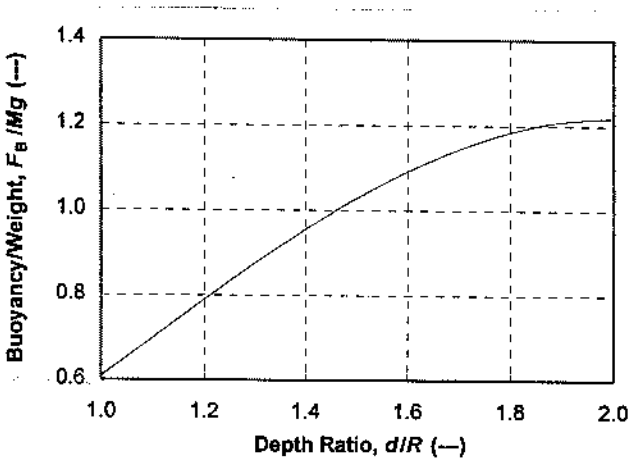
By iteration, using a spreadsheet, $f(d/R) = 0.547$ when $d/R = 1.46$ ← d/R

For $\forall = 1\,ft^3$, $\forall = \frac{4\pi R^3}{3}$ and $R = \left(\frac{3\forall}{4\pi}\right)^{1/3} = 0.620\,ft$

Thus $d = 1.46R = 1.46 \times 0.62\,ft = 0.906\,ft$ ← d

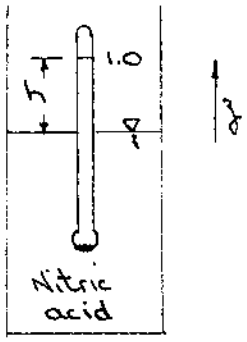
The spreadsheet results and plot are shown below.

dIR (---)	f(dIR)	FvIMg (---)
1	0.333	0.609
1.1	0.383	0.700
1.2	0.432	0.790
1.3	0.479	0.875
1.4	0.523	0.956
1.46	0.547	1.000
1.5	0.563	1.03
1.6	0.597	1.09
1.7	0.626	1.14
1.8	0.648	1.18
1.9	0.662	1.21
2	0.667	1.22



500 SHEETS, FILLER, 5 SQUARE
50 SHEETS EYE-EASE® 5 SQUARE
100 SHEETS EYE-EASE® 5 SQUARE
200 SHEETS EYE-EASE® 5 SQUARE
100 RECYCLED WHITE 5 SQUARE
200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Hydrometer, as shown, submerged in nitric acid, S.G = 1.5
When immersed in water, h = 0 and immersed volume is 15 cm³.
Stem diameter d = 6 mm.



Find: The distance, h

Solution:

Basic equation: $\Sigma \vec{F} = m\vec{a} = 0$

Computing equation: $F_{buoyancy} = \rho g \forall_d$

Assumptions: (1) static conditions
(2) ρ = constant

$$\Sigma \vec{F} = 0 = M\vec{g} + \vec{F}_{buoyancy}$$

Using the data given for water, we can calculate M

$$-Mg + F_b = 0 \qquad M = \frac{F_b}{g} = \rho_{H_2O} \forall_{H_2O}$$

When immersed in nitric acid

$$M = \rho_{n.a} \forall_{n.a} \qquad \text{where } \forall_{n.a} = \forall_{H_2O} - \frac{\pi d^2 h}{4}$$

Since the mass is the same in both cases

$$M = \rho_{H_2O} \forall_{H_2O} = \rho_{n.a}\left(\forall_{H_2O} - \frac{\pi d^2 h}{4}\right)$$

$$\frac{\pi d^2 h}{4} = \forall_{H_2O} - \frac{\rho_{H_2O}}{\rho_{n.a}} \forall_{H_2O} = \forall_{H_2O}\left(1 - \frac{1}{S.G_{n.a}}\right)$$

$$h = \frac{4 \forall_{H_2O}}{\pi d^2}\left(1 - \frac{1}{S.G_{n.a}}\right)$$

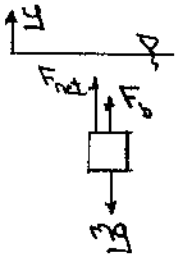
$$h = \frac{4}{\pi} \times 15\ cm^3 \times \frac{1}{6^2\ mm^2}\left(1 - \frac{1}{1.5}\right) \times \frac{1000\ mm^3}{cm^3} = 177\ mm$$

h

Given: Specific gravity of a person is to be determined from measurements of weight in air and the net weight when totally immersed in water.

Find: Expression for the specific gravity of a person from the measurements.

Solution:



For equilibrium $\Sigma F_y = 0$

$F_{net} = mg - F_b$

$F_b = \rho_{H_2O} g \forall$ $\quad F_{air} = mg$

$\therefore F_{net} = F_{air} - \rho_{H_2O} g \forall$ $\quad$ and $\forall = \dfrac{F_{air} - F_{net}}{\rho_{H_2O} g}$

$F_{air} = mg = \rho \forall g = \dfrac{\rho}{\rho_{H_2O}} (F_{air} - F_{net})$

Let $\rho^* = \rho_{H_2O}$ at 4C. Then

$F_{air} = \dfrac{\rho / \rho^*}{\rho_{H_2O} / \rho^*} (F_{air} - F_{net}) = \dfrac{SG}{SG_{H_2O}} (F_{air} - F_{net})$

Solving for SG,

$SG = SG_{H_2O} \dfrac{F_{air}}{(F_{air} - F_{net})}$

SG

Given: Experiment performed by Archimedes to identify the material content of King Hero's crown.

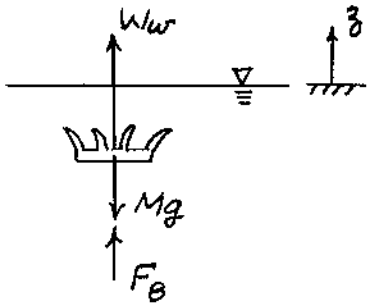
Measured weight of crown in air, W_a, and in water, W_w.

Find: Expression for specific gravity of crown as function of W_a and W_w.

Solution: Apply principle of bouyancy to free-body of crown:

Computing equation: $F_B = \rho_{H_2O} g \forall$

Assumptions: (1) Static liquid
(2) Incompressible liquid



Free-body diagram of crown in water:

$$\Sigma F_z = W_w - Mg + F_B = ma_z = 0$$

or

$$W_w - Mg + \rho_{H_2O} \forall g = 0$$

For the crown in air, $W_a = Mg$

Combining, $W_w - W_a + \rho_{H_2O} g \forall$, so $\forall = \dfrac{W_a - W_w}{\rho_{H_2O} g}$

The crown's density is $\rho_c = \dfrac{M}{\forall} = \dfrac{W_a}{g\forall} = \rho_{H_2O} \dfrac{W_a}{W_a - W_w}$

The crown's specific gravity is $SG = \dfrac{\rho_c}{\rho_{H_2O}} = \dfrac{W_a}{W_a - W_w}$ ← SG

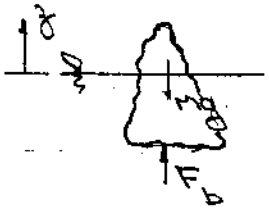
{Note: by definition, $SG = \rho / \rho_{H_2O}(4°C)$, so the measured temperature of water and data from Table A.7 or A.8 may be used to correct the density to 4°C.}

Given: Iceberg floating in sea water

Find: Quantify the statement "only the tip of an iceberg shows"

Solution:

A floating body is buoyed up by a force equal to the weight of the displaced liquid



$$\Sigma F_z = 0 = F_b - mg$$

$$F_b = \rho_s \forall_{sub} g \qquad m = \rho \forall_{total}$$

$$\therefore \rho_s \forall_{sub} g = \rho \forall_{tot} g$$

$$\therefore \forall_{sub} = \forall_{tot} \frac{\rho}{\rho_{sw}} = \forall_{tot} \frac{\rho/\rho^*}{\rho_{sw}/\rho^*}$$

where $\rho^* = \rho_{H_2O}$ at 4C.

$$\forall_{sub} = \forall_{tot} \frac{SG_{ice}}{SG_{sw}}$$

$$\forall_{not\,sub} = \forall_{tot} - \forall_{sub} = \forall_{tot}\left(1 - \frac{SG_{ice}}{SG_{sw}}\right) \quad \left\{\begin{array}{l}\text{Table A.1, } SG_{ice} = 0.917 \\ \text{Table A.2, } SG_{sw} = 1.025\end{array}\right\}$$

$$\therefore \frac{\forall_{not\,sub}}{\forall_{tot}} = 1 - \frac{SG_{ice}}{SG_s} = 1 - \frac{0.917}{1.025}$$

$$\frac{\forall_{not\,sub}}{\forall_{tot}} = 0.105 \qquad (10\% \text{ shows})$$

Given: Hydrogen bubble, with diameter $d = 0.025$ mm, rise slowly when immersed in water. The drag force on a bubble is given by $F_D = 3\pi\mu V d$, where V is bubble speed relative to the water

Find: (a) the buoyancy force on a hydrogen bubble immersed in water.
(b) estimate of terminal speed of bubble rising in water

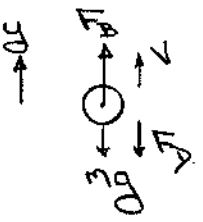
Solution:

Basic equations: $F_B = \rho g \forall$, $\Sigma \vec{F} = m\vec{a}$

For a sphere, $\forall = \frac{\pi}{6} d^3$

$$\therefore F_B = \rho g \forall = \frac{\rho g \pi d^3}{6} = \frac{\pi}{6} \times 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times (0.025 \times 10^{-3})^3 m^3 \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_B = 8.02 \times 10^{-8} \text{ N} \longleftarrow F_B$$



$$\Sigma F_y = F_B - mg - F_D = ma_y$$

At terminal speed, $a_y = 0$. Hence

$$F_D = 3\pi\mu V d = F_B - mg$$

and $$V = \frac{F_B - mg}{3\pi\mu d}$$

At $T = 20°C$, from Table A.8 (Appendix A) $\mu = 1.0 \times 10^{-3}$ N·s/m²

Treat hydrogen as an ideal gas. Assume $T = 20$ C, $p = 1.1$ atm

$$mg = \rho \forall g = \frac{p}{RT} \forall g = \frac{p}{RT} \frac{\pi d^3}{6} g$$ { From Table A.6, $R = 4124 \frac{J}{kg \cdot K}$ }

$$mg = 1.1 \text{ atm} \times 1.01 \times 10^5 \frac{N/m^2}{atm} \times \frac{kg \cdot K}{4124 J} \times \frac{1}{293 K} \times \frac{\pi}{6} \times (0.025 \times 10^{-3})^3 \times 9.81 \frac{m}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$mg = 7.38 \times 10^{-15} \text{ N}$$

$$\therefore V = \frac{(8.02 \times 10^{-8} - 7.38 \times 10^{-15}) N}{3\pi} \times \frac{1 \times 10^3 \text{ m}^2/\text{s}}{\text{N} \cdot \text{s}} \times 0.025 \times 10^{-3} \text{ m}$$

$$V = 3.40 \times 10^{-4} \text{ m/s} \quad \text{or} \quad 0.341 \text{ mm/s} \longleftarrow V$$

(As noted by Prof. Kline in the movie, "Flow Visualization," bubbles rise slowly!).

Given: Balloons with hot air, helium, and hydrogen. Claim lift per cubic foot of 0.018, 0.066, and 0.071 lbf/ft³ for respective gases, with air heated to 150°F over ambient.

Find: (a) Evaluate claims
(b) Compare air at 250°F above ambient.

Solution: Assume ambient conditions are STP, $\rho_{gas} = \rho_{air}$, and apply ideal gas equation of state.

(Use data from Table A.6.)

Basic equations: $Lift = \rho_{air} g\forall - \rho_{gas} g\forall$, $p = \rho RT$

Then

$$Lift/\forall = g(\rho_a - \rho_g) = \rho_{ag}\left(1 - \frac{\rho_g}{\rho_a}\right) = \rho_{ag}\left(1 - \frac{R_a T_a}{R_g T_g}\right); \quad \rho_{ag} = 0.0765 \frac{lbf}{ft^3}$$

For helium

$$\frac{L}{\forall} = 0.0765 \frac{lbf}{ft^3}\left[1 - \frac{53.33 \frac{ft \cdot lbf}{lbm \cdot R} \times (460+59)R \times \frac{lbm \cdot R}{386.1\ ft \cdot lbf} \times \frac{1}{(460+59F)}}{}\right]$$

$$\frac{L}{\forall} = 0.0659\ lbf/ft^3 \quad \text{(rounds to 0.066)}$$ ← He

For hydrogen

$$\frac{L}{\forall} = 0.0765 \frac{lbf}{ft^3}\left(1 - \frac{53.33}{766.5}\right) = 0.0712\ lbf/ft^3 \text{ (rounds to 0.071)}$$ ← H_2

For air at 150°F above ambient,

$$\frac{L}{\forall} = 0.0765 \frac{lbf}{ft^3}\left[1 - \frac{53.33(460+59)}{53.33(460+59+150)}\right] = 0.0172\ lbf/ft^3$$ ← Air $\Delta T = 150$

For air at 250°F above ambient,

$$\frac{L}{\forall} = 0.0765 \frac{lbf}{ft^3}\left[1 - \frac{53.33(460+59)}{53.33(460+59+250)}\right] = 0.0249\ lbf/ft^3$$ ← Air $\Delta T = 250$ F

Agreement with claims is good.

Air at $\Delta T = 250°F$ gives 45 percent more lift than at $\Delta T = 150°F$.

{ Hot air balloon needs 40.2 ft³/lbf of lift at $\Delta T = 250°F$! }

Open-Ended Problem Statement: Gas bubbles are released from the regulator of a submerged Scuba diver. What happens to the bubbles as they rise through the seawater?

Discussion: Air bubbles released by a submerged diver should be close to ambient pressure at the depth where the diver is swimming. The bubbles are small compared to the depth of submersion, so each bubble is exposed to essentially constant pressure. Therefore the released bubbles are nearly spherical in shape.

The air bubbles are buoyant in water, so they begin to rise toward the surface. The bubbles are quite light, so they reach terminal speed quickly. At low speeds the spherical shape should be maintained. At higher speeds the bubble shape may be distorted.

As the bubbles rise through the water toward the surface, the hydrostatic pressure decreases. Therefore the bubbles expand as they rise. As the bubbles grow larger, one would expect the tendency for distorted bubble shape to be exaggerated.

13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-384 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-394 200 RECYCLED WHITE 5 SQUARE
42-398 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

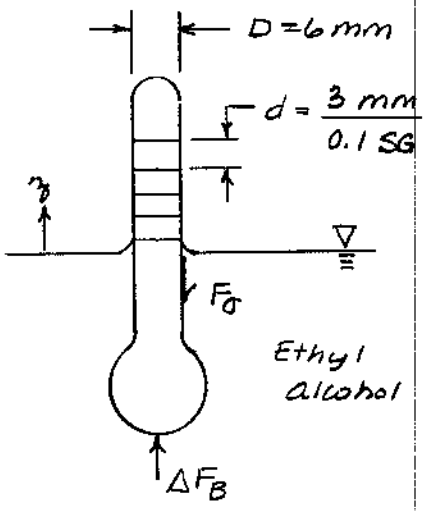
Given: Glass hydrometer used to measure SG of liquids.

Stem has $D = 6$ mm; distance between marks on stem is $d = 3$ mm per 0.1 SG

Hydrometer floats in ethyl alcohol (assume contact angle is 0).

Find: Magnitude of error introduced by surface tension.

Solution: Consider a free-body diagram of the floating hydrometer



Surface tension will cause the hydrometer to sink Δh lower into the liquid. Thus for this change,

$$\Sigma F_z = \Delta F_B - F_\sigma = ma_z = 0$$

Computing equation: $\Delta F_B = \rho g \Delta \forall$

Assumptions: (1) Static liquid (3) $\theta \approx 0$
(2) Incompressible liquid

Then $\Delta \forall = \frac{\pi D^2}{4} \Delta h$ and $\Delta F_B = \rho g \frac{\pi D^2}{4} \Delta h$

and $F_\sigma = \pi D \sigma \cos\theta = \pi D \sigma$

Combining $\rho g \frac{\pi D^2}{4} \Delta h = \pi D \sigma$ or $\Delta h = \frac{4\sigma}{\rho g D} = \frac{4\sigma}{SG \rho_{H_2O} g D}$

From Table A.2, SG = 0.789 and from Table A.4, $\sigma = 22.3$ mN/m for ethanol, so

$$\Delta h = \frac{4}{0.789} \times 22.3 \times 10^{-3} \frac{N}{m} \times \frac{m^3}{1000\,kg} \times \frac{s^2}{9.81\,m} \times \frac{1}{0.006\,m} \times \frac{kg \cdot m}{N \cdot s^2} = 1.92 \times 10^{-3}\,m$$

Thus the change in SG will be

$$\Delta SG = 1.92 \times 10^{-3}\,m \times \frac{0.1\ SG}{3\,mm} \times 1000 \frac{mm}{m} = 0.0640$$

ΔS

{ From the diagram, surface tension acts to cause the hydrometer to float lower in the liquid. Therefore surface tension results in an indicated SG smaller than the actual SG. }

Given: Glass hydrometer used to measure SG of liquids.

Stem has $D = 6$ mm; distance between marks on stem is $d = 3$ mm per 0.1 SG

Hydrometer floats in ethyl alcohol (assume contact angle is 0°).

Find: Magnitude of error introduced by surface tension.

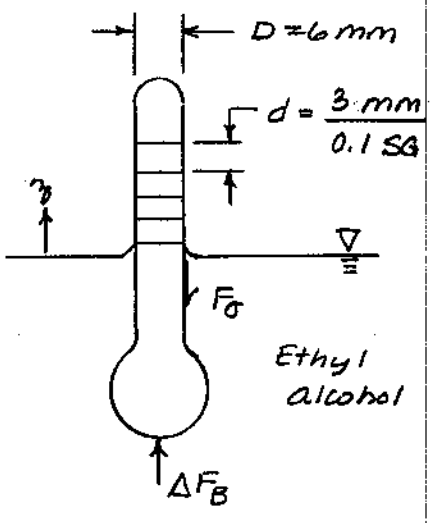
Solution: Consider a free-body diagram of the floating hydrometer

Surface tension will cause the hydrometer to sink Δh lower into the liquid. Thus for this change,

$$\Sigma F_z = \Delta F_B - F_\sigma = ma_z = 0$$

Computing equation: $\Delta F_B = \rho g \Delta \forall$

Assumptions: (1) Static liquid (3) $\theta \approx 0$
(2) Incompressible liquid



Then $\Delta \forall = \frac{\pi D^2}{4} \Delta h$ and $\Delta F_B = \rho g \frac{\pi D^2}{4} \Delta h$

and $F_\sigma = \pi D \sigma \cos\theta = \pi D \sigma$

Combining $\rho g \frac{\pi D^2}{4} \Delta h = \pi D \sigma$ or $\Delta h = \frac{4\sigma}{\rho g D} = \frac{4\sigma}{SG \rho_{H_2O} g D}$

From Table A.2, SG = 0.789 and from Table A.4, $\sigma = 22.3$ mN/m for ethanol, so

$$\Delta h = \frac{4}{0.789} \times 22.3 \times 10^{-3} \frac{N}{m} \times \frac{m^3}{1000\,kg} \times \frac{s^2}{9.81\,m} \times \frac{1}{0.006\,m} \times \frac{kg \cdot m}{N \cdot s^2} = 1.92 \times 10^{-3}\,m$$

Thus the change in SG will be

$$\Delta SG = 1.92 \times 10^{-3}\,m \times \frac{0.1\,SG}{3\,mm} \times 1000 \frac{mm}{m} = 0.0640$$

ΔSG

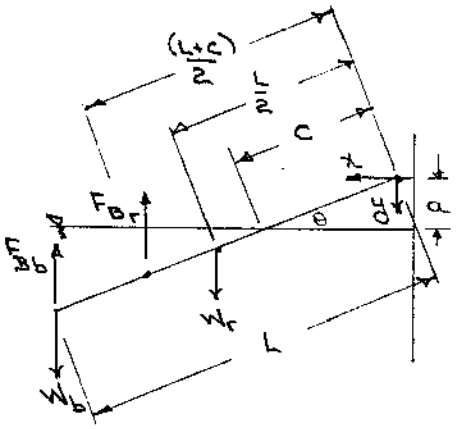
{From the diagram, surface tension acts to cause the hydrometer to float lower in the liquid. Therefore surface tension results in an indicated SG smaller than the actual SG.}

Given: Weight as shown in water on rod:

$L = 10\,ft$, $A = 3\,in^2$, $W_r = 3\,lbf$

$a = 1\,ft$, $W_b = 67\,lbf$.

Find: θ for equilibrium condition



Solution:

Basic equations: $\sum \vec{M} = 0$ for equilibrium

Moment of force $= \vec{r} \times \vec{F}$

Compiling equation: $\vec{F}_B = -\gamma \forall_{displaced} \hat{k}$

Let $()_r$ refer to rod

$()_b$ refer to block

Summing moments about the hinge

$$(W_b - F_{B_b})\hat{j} \times L(\hat{i}\cos\theta + \hat{j}\sin\theta) + (-F_{B_r})\hat{j} \times \frac{(L+c)}{2}(\hat{i}\cos\theta + \hat{j}\sin\theta) + W_r \hat{j} \times \frac{L}{2}(\hat{i}\cos\theta + \hat{j}\sin\theta)$$

$$\{-W_b L\cos\theta + F_{B_b} L\cos\theta + F_{B_r}\frac{(L+c)}{2}\cos\theta - W_r \frac{L}{2}\cos\theta\}\hat{k} = 0$$

$$-W_b L + F_{B_b} L + F_{B_r}\frac{(L+c)}{2} - W_r \frac{L}{2} = 0$$

$$F_{B_r} = \gamma \forall_{dis} = \gamma A \ell = \gamma A (L-c)$$

$$\therefore -W_b L + F_{B_b} L + \gamma A (L-c)\frac{(L+c)}{2} - W_r \frac{L}{2} = 0$$

$$-2W_b L + 2F_{B_b} L + \gamma A (L^2 - c^2) - W_r L = 0$$

$$\therefore \gamma A (L^2 - c^2) = W_r L + 2W_b L - 2F_{B_b} L$$

and

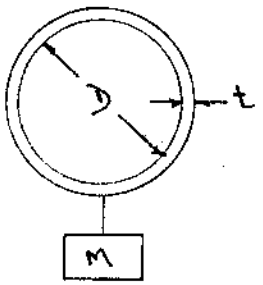
$$c = \left[L^2 - \frac{1}{\gamma A}\left(W_r L + 2W_b L - 2F_{B_b} L\right)\right]^{1/2} = \frac{a}{\sin\theta}$$

$$c = \left[(10)^2\,ft^2 - \frac{ft^3}{62.4\,lbf} \times \frac{1}{3\,in^2}\left(3\,lbf \times 10\,ft + 2 \times 67\,lbf \times 10\,ft - 2 \times 62.4\,\frac{lbf}{ft^3} \times 1\,ft^3 \times 10\,ft\right) \times 144\,\frac{in^2}{ft^2}\right]^{1/2}$$

$$c = \left[6.18\right]^{1/2} = 2.48\,ft$$

$$\sin\theta = \frac{a}{c} = \frac{1.00\,ft}{2.48\,ft} = 0.403 \quad \therefore \theta = 23.8^\circ$$

Given: A pressurized helium balloon is to be designed to lift a payload of mass, M, to an altitude of 40 km, where $P = 3.0$ mbar and $T = -25°C$.
The balloon skin has a specific gravity, S.G = 1.28 and thickness, $t = 0.015$ mm
The gage pressure of the helium is 0.45 mbar. The allowable tensile stress in the balloon skin is $\sigma = 62$ MN/m²



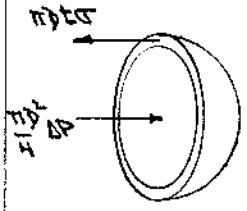
Find: (a) Maximum balloon diameter
(b) Payload, M

Solution:

Basic equation: $\Sigma \vec{F} = m\vec{a} = 0$

Assumptions: (1) static equilibrium at altitude
(2) air and helium exhibit ideal gas behavior

The balloon diameter is limited by tensile stress

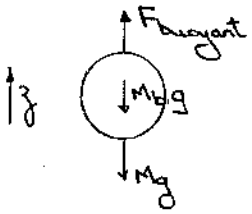


$$\Sigma F = 0 = \frac{\pi D^2}{4}\Delta P - \pi D t \sigma$$

$$D_{max} = \frac{4t\sigma}{\Delta P}$$

$$D_{max} = 4 \times 1.50\times10^{-5}\,m \times 62\times10^{6}\,\frac{N}{m^2} \times 0.45\times10^{-3}\,bar \times \frac{bar}{10^5\,\frac{N}{m^2}}$$

$$D_{max} = 82.7\ m \longleftarrow D_{max}$$



$$\Sigma F_z = 0 = F_{buoy} - M_{He}g - M_b g - Mg \qquad M_{He} = \rho_{He} \forall$$

$$F_{buoy} - M_{He}g = (\rho_{air} - \rho_{He})g\forall = (\rho_{air} - \rho_{He})g\frac{\pi D^3}{6}$$

$$M_b = \rho_s \forall_s = \rho_s A_s t_s = \rho_s \pi D^2 t$$

$$\therefore M = \frac{F_{buoy}}{g} - M_b = (\rho_{air} - \rho_{He})\frac{\pi D^3}{6} - \rho_s \pi D^2 t$$

$$M = \pi D^2 \left[(\rho_{air} - \rho_{He})\frac{D}{6} - \rho_s t\right]$$

From ideal gas law

$$\rho_{air} = \frac{P}{RT} = 3.0\times10^{-3}\,bar \times \frac{kg\cdot K}{287\,J} \times \frac{1}{248\,K} \times \frac{10^5\,Pa}{1\,bar} \times \frac{N}{Pa\cdot m^2} \times \frac{1\,J}{N\cdot m} = 4.21\times10^{-3}\ kg/m^3$$

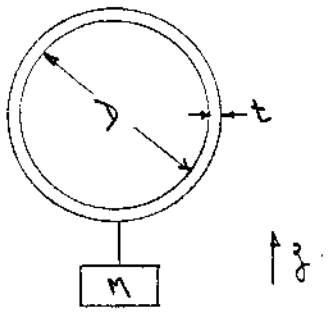
$$\rho_{He} = \frac{(P+\Delta P)}{RT} = 3.45\times10^{-3}\,bar \times \frac{kg\cdot K}{2080\,J} \times \frac{1}{248\,K} \times \frac{10^5\,Pa}{1\,bar} \times \frac{N}{Pa\cdot m^2} \times \frac{1\,J}{N\cdot m} = 6.69\times10^{-4}\ kg/m^3$$

Then,

$$M = \pi(82.7)^2\,m^2\left[(42.1-6.69)\times10^{-4}\,\frac{kg}{m^3} \times \frac{82.7\,m}{6} - 1.28\times999\,\frac{kg}{m^3} \times 1.5\times10^{-5}\,m\right]$$

$$M = 637\ kg \longleftarrow M$$

Given: Spherical balloon of diameter D, and skin thickness, $t = 0.013$ mm, filled with helium lifted a payload of mass $M = 230$ kg to an altitude of 49 km. At altitude, $P = 0.95$ mbar and $T = -20°C$. The helium temperature is $-10°C$. The specific gravity of the skin material is 1.28



Find: The diameter and mass of the balloon

Solution: Basic equation $\Sigma \vec{F} = m\vec{a} = 0$

Assumptions: (1) static equilibrium at altitude of 49 km
(2) air and helium exhibit ideal gas behavior.

$$\Sigma F_z = 0 = F_{buoy} - M_{He}g - M_s g - Mg = \rho_{air} g \forall_b - \rho_{He} g \forall_b - \rho_s \forall_s - Mg$$

$$0 = \forall_b(\rho_{air} - \rho_{He}) - \rho_s A_s t - M = \frac{\pi}{3} R^3 (\rho_{air} - \rho_{He}) - \rho_s 4\pi R^2 t - M$$

$$0 = \frac{\pi D^3}{6}(\rho_{air} - \rho_{He}) - \rho_s \pi D^2 t - M$$

This is a cubic equation which requires an iterative solution

$$\pi D^2\left[\frac{D}{6}(\rho_{air} - \rho_{He}) - \rho_s t\right] - M = 0 \qquad \text{Solving for } D.$$

$$D = \frac{6}{(\rho_{air} - \rho_{He})}\left[\frac{M}{\pi D^2} + \rho_s t\right] = 6\left[\frac{M}{\pi D^2(\rho_{air} - \rho_{He})} + \frac{\rho_s t}{(\rho_{air} - \rho_{He})}\right]$$

From the ideal gas law,

$$\rho_{air} = \frac{P}{RT} = 0.95\times10^{-3}\,\text{bar} \times \frac{\text{kg·K}}{287\,\text{J}} \times \frac{1}{253\,\text{K}} \times \frac{10^5\,\text{Pa}}{1\,\text{bar}} \times \frac{\text{N}}{\text{Pa·m}^2} \times \frac{\text{N·m}}{\text{J}} = 1.31\times10^{-3}\ \frac{\text{kg}}{\text{m}^3}$$

$$\rho_{He} = \frac{P}{RT} = 0.95\times10^{-3}\,\text{bar} \times \frac{\text{kg·K}}{2080} \times \frac{1}{263\,\text{K}} \times \frac{10^5\,\text{Pa}}{1\,\text{bar}} \times \frac{\text{N}}{\text{Pa·m}^2} \times \frac{\text{N·m}}{\text{J}} = 1.74\times10^{-4}\ \frac{\text{kg}}{\text{m}^3}$$

Substituting into the expression for D

$$D = 6\left[\frac{1}{\pi D^2} \times 230\,\text{kg} \times \frac{\text{m}^3}{11.4\times10^{-4}\,\text{kg}} + (1.28)999\,\frac{\text{kg}}{\text{m}^3} \times 1.3\times10^{-5}\,\text{m} \times \frac{\text{m}^3}{11.4\times10^{-4}\,\text{kg}}\right]$$

$$D = \left[\frac{38.5\times10^4}{D^2} + 87.5\right] \qquad \text{where } D \text{ is in meters}$$

Organizing Calculations:

Guess D (m) =	100	120	116
RHS =	126	114	116.1

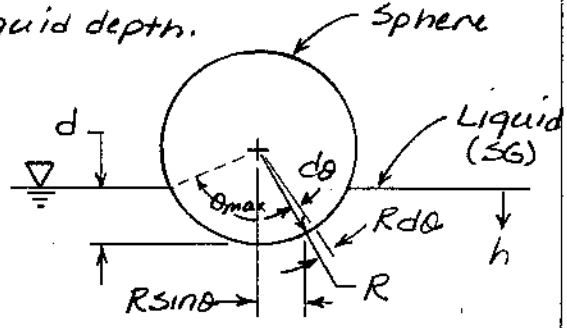
$\therefore D = 116$ m ← D

$$M_b = \rho_s \forall_s = \rho_s A_s t = \rho_s \pi D^2 t = 1.28\times999\,\frac{\text{kg}}{\text{m}^3} \times \pi(116)^2\,\text{m}^2 \times 1.3\times10^{-5}\,\text{m}$$

$$M_b = 703\ \text{kg} \leftarrow M_b$$

Given: Sphere partially immersed in liquid of specific gravity, SG.

Find: (a) Formula, algebraic expression for buoyancy force, as a function of submersion depth, d, for $0 \le d \le R$.
(b) Plot of results over range of liquid depth.



Solution: Apply fluid statics

Basic equations: $\frac{dp}{dh} = \rho g$

$dF = p\,dA$

Assumptions: (1) Static liquid
(2) Incompressible, so $p = p_0 + \rho g h$
(3) Neglect p_{atm} since it acts everywhere

Then $dF_V = \cos\theta\, p\, dA$; $p = \rho g h$; $d = h + R(1-\cos\theta)$; $h = d - R(1-\cos\theta)$

$$dA = 2\pi (R\sin\theta)\, R\, d\theta = 2\pi R^2 \sin\theta\, d\theta$$

$$dF_V = \cos\theta\, \rho g \left[d - R(1-\cos\theta)\right] 2\pi R^2 \sin\theta\, d\theta = 2\pi R^3 \left[\frac{d}{R} - (1-\cos\theta)\right] \sin\theta\cos\theta\, d\theta\, \rho g$$

Now

$$F_V = \int_A dF_V = \int_0^{\theta_{max}} 2\pi R^3 \left[\frac{d}{R} - (1-\cos\theta)\right] \sin\theta\cos\theta\, d\theta\, \rho g$$

$$F_V = 2\pi R^3 \left[(1 - d/R)\frac{\cos^2\theta}{2} - \frac{\cos^2\theta}{3}\right]_0^{\theta_{max}} \rho g \qquad ; \rho = SG\,\rho_{H_2O}$$

At θ_{max}, $\cos\theta_{max} = \frac{R-d}{R} = 1 - d/R$, so

$$F_V = 2\pi \rho g R^3 \left\{(1 - d/R)\left[(1-d/R)^2/2 - 1/2\right] - \frac{(1-d/R)^3}{3} + \frac{1}{3}\right\}$$

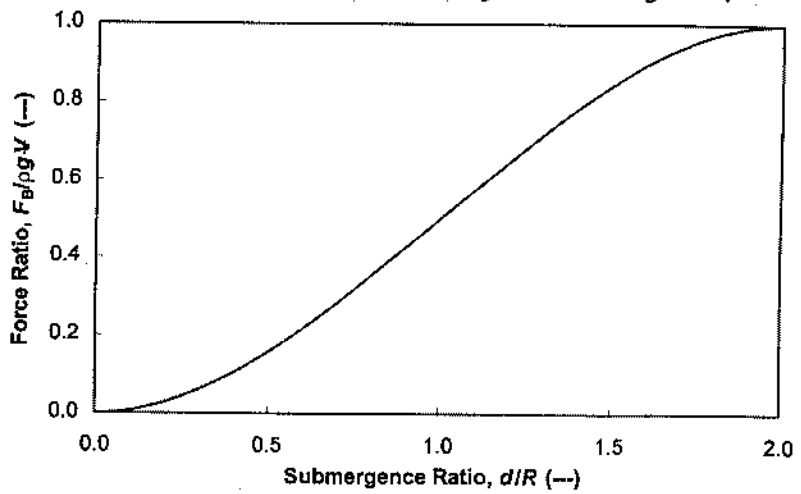
$$F_V = 2\pi \rho g R^3 \left[\frac{1}{6}(1-\frac{d}{R})^3 - \frac{1}{2}(1-\frac{d}{R}) + \frac{1}{3}\right] \qquad F_V$$

Dividing both sides by the vertical force on a fully submerged sphere,

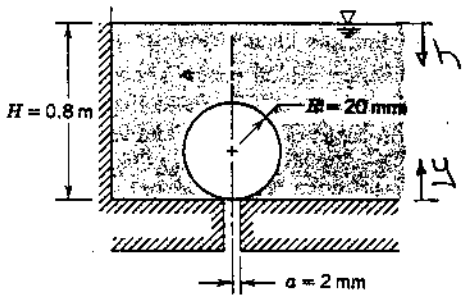
$$\frac{F_V}{\rho g \frac{4\pi R^3}{3}} = \frac{3}{2}\left[\frac{1}{6}()^3 - \frac{1}{2}() + \frac{1}{3}\right]$$

where $() = (1 - \frac{d}{R})$.

The plot is



Given: Sphere, of radius R and specific gravity SG, is submerged in a tank of water. Sphere is placed over a hole, of radius a, in the tank bottom.



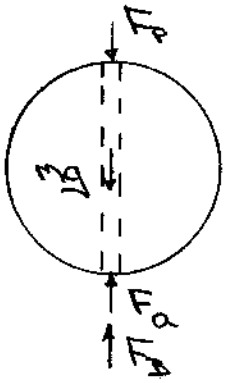
Find: (a) general expression for the range of SG for which sphere will float to the surface
(b) minimum SG required for sphere to remain in the position shown

Solution:

Basic equations: $F_{buoy} = \rho g \forall$ $\frac{dP}{dh} = \rho g$ $dF = P\,dA$

Assumptions: (1) static liquid
(2) incompressible, so $P = P_0 + \rho g h$
(3) $P = P_{atm}$ at free surface and at hole
(4) $a/R \ll 1$

Draw fbd of sphere. $\Sigma F_y = 0$



$$\Sigma F_y = 0 = F_a - F_p + F_B - mg$$

F_a = force of air on area of sphere of radius a. $F_a = P_{atm}\pi a^2$

F_p = total force on on area of sphere of radius a at depth $h = H - 2R$

$$F_p = \left[P_{atm} + \rho g(H-2R)\right]\pi a^2$$

F_B = net buoyant force on sphere excluding cylinder of radius a

$$F_B = \rho_w g \forall_{net} = \rho_w g\left[\frac{4\pi R^3}{3} - \pi a^2(2R)\right]$$

$mg = \rho_w SG\, g \frac{4\pi R^3}{3}$

Substituting,

$$0 = P_{atm}\pi a^2 - P_{atm}\pi a^2 - \rho g(H-2R)\pi a^2 + \rho_w g \frac{4\pi R^3}{3} - \rho_w g \pi a^2 2R - \rho_w SG\, g \frac{4\pi R^3}{3}$$

$$0 = -(H-2R)a^2 + \frac{4R^3}{3} - 2a^2R - SG\,\frac{4R^3}{3}$$

$$0 = -\left(\frac{H}{R} - 2\right)\left(\frac{a}{R}\right)^2 + \frac{4}{3} - 2\left(\frac{a}{R}\right)^2 - \frac{4}{3}SG$$

$$SG = 1 - \frac{3}{4}\frac{H}{R}\left(\frac{a}{R}\right)^2 \qquad \text{SG}$$

For dimensions given $\frac{a}{R} = \frac{2}{20} = 0.1$, $\frac{H}{R} = \frac{800}{20} = 40$

$$\therefore SG = 1 - \frac{3}{4} \times 40 \times (0.1)^2 = 0.70$$

For $SG \geq 0.70$ sphere will stay in position shown. SG_{min}

Open-Ended Problem Statement: A proposed ocean salvage scheme involves pumping air into "bags" placed within and around a wrecked vessel on the sea bottom. Comment on the practicality of this plan, supporting your conclusions with analyses.

Discussion: This plan has several problems that render it impractical. First, pressures at the sea bottom are very high. For example, *Titanic* was found in about 12,000 ft of seawater. The corresponding pressure is nearly 6,000 psi. Compressing air to this pressure is possible, but would require a multi-stage compressor and very high power.

Second, it would be necessary to manage the buoyancy force after the bag and object are broken loose from the sea bed and begin to rise toward the surface. Ambient pressure would decrease as the bag and artifact rise toward the surface. The air would tend to expand as the pressure decreases, thereby tending to increase the volume of the bag. The buoyancy force acting on the bag is directly proportional to the bag volume, so it would increase as the assembly rises. The bag and artifact thus would tend to accelerate as they approach the sea surface. The assembly could broach the water surface with the possibility of damaging the artifact or the assembly.

If the bag were of constant volume, the pressure inside the bag would remain essentially constant at the pressure of the sea floor, e.g., 6,000 psi for *Titanic*. As the ambient pressure decreases, the pressure differential from inside the bag to the surroundings would increase. Eventually the difference would equal sea floor pressure. This probably would cause the bag to rupture.

If the bag permitted some expansion, a control scheme would be needed to vent air from the bag during the trip to the surface to maintain a constant buoyancy force just slightly larger than the weight of the artifact in water. Then the trip to the surface could be completed at low speed without danger of broaching the surface or damaging the artifact.

13-782 500 SHEETS, FILLER, 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-398 200 SHEETS EYE-EASE® 5 SQUARE
42-387 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED White 5 SQUARE
Made in U. S. A.
National® Brand

Open-Ended Problem Statement: In the "Cartesian diver" child's toy, a miniature "diver" is immersed in a column of liquid. When a diaphragm at the top of the column is pushed down, the diver sinks to the bottom. When the diaphragm is released, the diver again rises. Explain how the toy might work.

Discussion: A possible scenario is for the toy to have a flexible bladder that contains air. Pushing down on the diaphragm at the top of the liquid column would increase the pressure at any point in the liquid. The air in the bladder would be compressed slightly as a result. The volume of the bladder, and therefore its buoyancy, would decrease, causing the diver to sink to the bottom of the liquid column.

Releasing the diaphragm would reduce the pressure in the water column. This would allow the bladder to expand again, increasing its volume and therefore the buoyancy of the diver. The increased buoyancy would permit the diver to rise to the top of the liquid column and float in a stable, partially submerged position, on the surface of the liquid.

13-782 500 SHEETS, FILLER 5 SQUARE
42-381 150 SHEETS EYE-EASE 5 SQUARE
42-382 100 SHEETS EYE-EASE 5 SQUARE
42-392 200 SHEETS EYE-EASE 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National Brand

Open-Ended Problem Statement: Consider a conical funnel held upside down and submerged slowly in a container of water. Discuss the force needed to submerge the funnel if the spout is open to the atmosphere. Compare with the force needed to submerge the funnel when the spout opening is blocked by a rubber stopper.

Discussion: Let the weight of the funnel in air be W_a. Assume the funnel is held with its spout vertical and the conical section down. Then W_a will also be vertical.

Two possible cases are with the funnel spout open to atmosphere or with the funnel spout sealed.

With the funnel spout open to atmosphere, the pressures inside and outside the funnel are equal, so no net pressure force acts on the funnel. The force needed to support the funnel will remain constant until it first contacts the water. Then a buoyancy force will act vertically upward on every element of volume located beneath the water surface.

The first contact of the funnel with the water will be at the widest part of the conical section. The buoyancy force will be caused by the volume formed by the funnel thickness and diameter as it begins to enter the water. The buoyancy force will reduce the force needed to support the funnel. The buoyancy force will increase as the depth of submergence of the funnel increases until the funnel is fully submerged. At that point the buoyancy force will be constant and equal to the weight of water displaced by the volume of the material from which the funnel is made.

If the funnel material is less dense than water, it would tend to float partially submerged in the water. The force needed to support the funnel would decrease to zero and then become negative (i.e., down) to fully submerge the funnel.

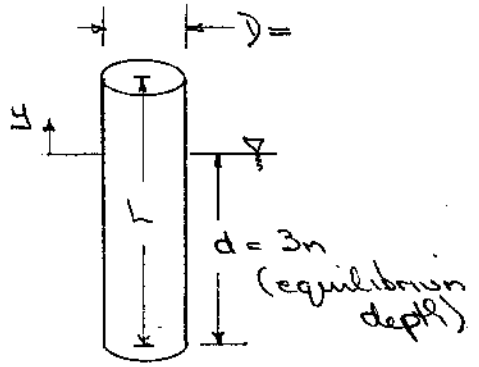
If the funnel material were more dense than water it would not tend to float even when fully submerged. The force needed to support the funnel would decrease to a minimum when the funnel became fully submerged, and then would remain constant at deeper submersion depths.

With the funnel spout sealed, air will be trapped inside the funnel. As the funnel is submerged gradually below the water surface, it will displace a volume equal to the volume of the funnel material plus the volume of trapped air. Thus its buoyancy force will be much larger than when the spout is open to atmosphere. Neglecting any change in air volume (pressures caused by submersion should be small compared to atmospheric pressure) the buoyancy force would be from the entire volume encompassed by the outside of the funnel. Finally, when fully submerged, the volume of the rubber stopper (although small) will also contribute to the total buoyancy force acting on the funnel.

500 SHEETS FILLER 5 SQUARE
50 SHEETS EYE-EASE® 5 SQUARE
100 SHEETS EYE-EASE® 5 SQUARE
200 SHEETS EYE-EASE® 5 SQUARE
100 RECYCLED WHITE 5 SQUARE
200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Cylindrical timber, $D = 0.3m$ and $L = 4m$ is weighted on lower end so it floats vertically with 3m submerged in sea water.
When displaced vertically from equilibrium position, the timber oscillates in a vertical direction upon release

Find: Estimate frequency of oscillation. (Neglect any viscous effects or water motion)



Solution:

At equilibrium

$$\Sigma F_y = 0 = F_b - mg = \rho A d - mg$$

$$\therefore m = \rho A d$$

For displacement

$$\Sigma F_y = m \frac{d^2 y}{dt^2} = m\ddot{y}$$

$$F_b - mg = m\ddot{y} \quad \text{where } F_b = \rho A (d - y)$$

$$\therefore \rho A (d - y) - mg = m\ddot{y}$$

$$\rho A d - \rho A y - \rho A d g = m\ddot{y}$$

or

$$m\ddot{y} + \rho A y = 0$$

$$\ddot{y} + \frac{\rho A}{m} y = 0 = \ddot{y} + \omega^2 y = 0$$

where $\omega^2 = \frac{\rho A g}{\rho A d} = \frac{g}{d}$

$$\omega = \left(\frac{g}{d}\right)^{1/2} = \left[9.81 \frac{m}{s^2} \times \frac{1}{3m}\right]^{1/2} = 1.81 \text{ rad/s}$$

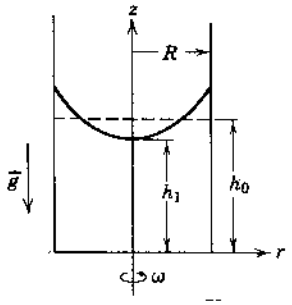
$$f = \frac{\omega}{2\pi} = 1.81 \frac{\text{rad}}{s} \times \frac{\text{cycle}}{2\pi \text{ rad}} = 0.288 \text{ cycle/s}$$

$$\tau = \frac{1}{f} = 3.47 \text{ s}$$

Given: Cylindrical container rotating as in Example Problem 3.9

$R = 0.5$ ft

$h_0 = 4$ in.



Determine: (a) value of ω such that $h_1 = 0$

(b) if solution is dependent on ρ

Solution:

In order to obtain the solution we need an expression for the shape of the free surface in terms of ω, r, and h_0

The required expression was derived in Example Problem 3.9. The equation is

$$z = h_0 - \frac{(\omega R)^2}{2g}\left[\frac{1}{2} - \left(\frac{r}{R}\right)^2\right]$$

Since $h_1 = 0$ corresponds to $z = 0$ and $r = 0$ we must determine ω such that

$$0 = h_0 - \frac{(\omega R)^2}{4g}$$

Solving for ω,

$$\omega = \frac{2}{R}\sqrt{g h_0}$$

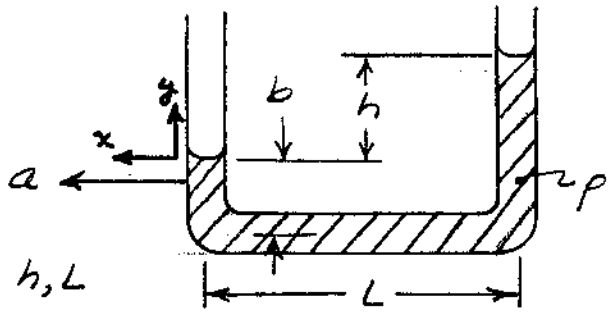
$$= \frac{2}{0.5\text{ ft}}\left(32.2\ \frac{\text{ft}}{\text{s}^2} \times 4\text{ in} \times \frac{\text{ft}}{12\text{ in}}\right)^{1/2}$$

$$= 4 \times 3.28\ \frac{1}{\text{s}}$$

$$\omega = 13.1\ \text{rad/s}$$

The solution is independent of ρ since the equation of the free surface is independent of ρ.

Given: U-tube accelerometer



Find: Acceleration in terms of h, L

Solution: Apply x, y components of hydrostatic equation.

Basic equations:

$$-\frac{\partial p}{\partial x} + \rho g_x = \rho a_x \qquad a_x = a \qquad g_x = 0$$

$$-\frac{\partial p}{\partial y} + \rho g_y = \rho a_y \qquad a_y = 0 \qquad g_y = -g$$

Assumptions: (1) Neglect sloshing
(2) Ignore corners

Then $\frac{\partial p}{\partial x} = -\rho a$, $\frac{\partial p}{\partial y} = -\rho g$. Evaluate Δp from left leg to right:

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy$$

$$\Delta p = \frac{\partial p}{\partial x} \Delta x + \frac{\partial p}{\partial y} \Delta y$$

$$= (-\rho g)(-b) + (-\rho a)(-L) + (-\rho g)(b+h)$$

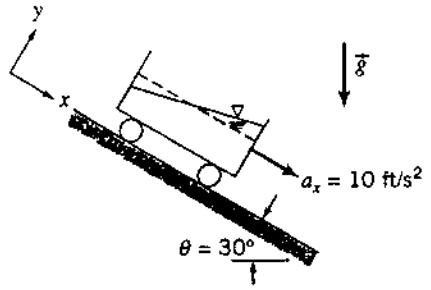
$$\Delta p = \rho a L - \rho g h = 0$$

Solving,

$$a = g\left(\frac{h}{L}\right)$$

a ←

Given: Rectangular container of water undergoing constant acceleration as shown



Determine: The slope of the free surface

Solution:

Basic equation: $-\nabla P + \rho\vec{g} = \rho\vec{a}$

Writing the component equations

$$-\frac{\partial P}{\partial x} + \rho g_x = \rho a_x$$
$$-\frac{\partial P}{\partial y} + \rho g_y = \rho a_y$$
$$-\frac{\partial P}{\partial z} + \rho g_z = \rho a_z$$

For given coordinates: $a_y = a_z = 0$, $g_y = -g\cos\theta$, $g_x = g\sin\theta$, $g_z = 0$

$$\Rightarrow \frac{\partial P}{\partial x} = \rho g\sin\theta - \rho a_x$$
$$\frac{\partial P}{\partial y} = -\rho g\cos\theta$$
$$\frac{\partial P}{\partial z} = 0$$

From the component equations we conclude that $P = P(x,y)$ then

$$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$$

Along the free surface P = constant and $dP = 0$. Hence

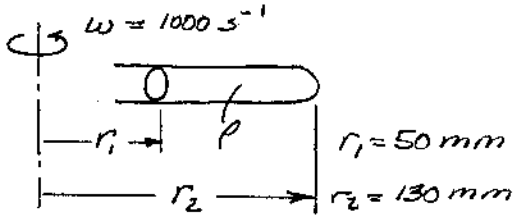
$$\left(\frac{dy}{dx}\right)_{surface} = -\frac{\partial P/\partial x}{\partial P/\partial y} = \frac{\rho g\sin\theta - \rho a_x}{\rho g\cos\theta}$$

$$= \frac{g\sin\theta - a_x}{g\cos\theta}$$

$$= \frac{32.2(0.5)\ \text{ft/s}^2 - 10\ \text{ft/s}^2}{32.2(0.866)\ \text{ft/s}^2}$$

$$\left(\frac{dy}{dx}\right)_{surface} = 0.22$$

Given: Test tube with water



Find: (a) Radial acceleration
(b) Radial pressure gradient, $\partial p/\partial r$
(c) Maximum pressure on bottom.

Solution: Apply equation for rigid-body motion

Basic equation: $-\nabla p + \rho\vec{g} = \rho\vec{a}$

(r component) $-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r$

Assumptions: (1) Rigid-body motion, so $a_r = -\frac{V^2}{r} = -\frac{(r\omega)^2}{r} = -r\omega^2$ ← a_r
(2) r horizontal, so $g_r = 0$

Then $\frac{\partial p}{\partial r} = -\rho a_r = -\rho(-r\omega^2) = \rho r\omega^2$ ← $\frac{\partial p}{\partial r}$

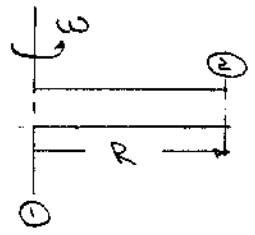
Integrating, $p_2 - p_1 = \int_1^2 \frac{\partial p}{\partial r}\,dr = \int_{r_1}^{r_2} \rho r\omega^2 = \rho\frac{r^2\omega^2}{2}\Big]_{r_1}^{r_2} = \frac{1}{2}\rho\omega^2(r_2^2 - r_1^2)$

$$p_{max} = p_2 - p_1 = \frac{1}{2} \times 999\frac{kg}{m^3} \times \frac{(1000)^2}{s^2} \times [(0.130)^2 - (0.050)^2]m^2 \times \frac{N s^2}{kg\,m} = 7.19\ MPa$$ ← p_{ma}

Given: Centrifugal micromanometer consists of pair of parallel disks that rotate to develop a radial pressure difference. There is no flow between the disks.

Find: (a) An expression for the pressure difference, ΔP, as a function of ω, R, and ρ
(b) Find ω if $\Delta P = 8\ \mu m\ H_2O$ and $R = 50\ mm$

Solution:



Basic equation: $-\nabla p + \rho \vec{g} = \rho \vec{a}$

(r component) $-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r$

Assumptions: (1) standard air between disks
(2) r horizontal, so $g_r = 0$
(3) rigid body motion, so $a_r = -\frac{V^2}{r} = -\frac{(r\omega)^2}{r} = -r\omega^2$

Then $\frac{\partial p}{\partial r} = \rho r \omega^2$ (ρ is a constant)

Separating variables and integrating, we obtain

$$\int_p^{p+\Delta p} dp = \rho\omega^2 \int_0^R r\,dr$$

$$\Delta p = \frac{\rho \omega^2 R^2}{2}$$ ΔP

Then $\omega^2 = \frac{2\Delta p}{\rho R^2}$

where $\Delta p = \rho_{H_2O} g\, \Delta h$ and $\Delta h = 8\times10^{-6}\ m$

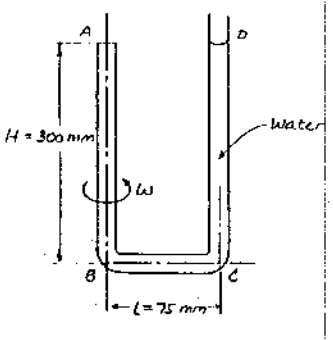
$$\omega^2 = \frac{2\rho_{H_2O} g\, \Delta h}{\rho R^2}$$

$$= 2 \times \frac{999\ kg/m^3}{1.225\ kg/m^3} \times 9.81\ \frac{m}{s^2} \times 8\times10^{-6}\ m \times \frac{1}{(0.05)^2\ m^2}$$

$$\omega^2 = 51.2\ s^{-2}$$

$$\omega = 7.16\ rad/s$$ ω

Given: U-tube, sealed at A and open to the atmosphere at D, is filled with water at T = 20 C and rotated about vertical axis AB. Dimensions are shown on the digram



Find: the maximum allowable angular speed, ω, for no cavitation.

Solution:

Basic equation: $-\nabla p + \rho\vec{g} = \rho\vec{a}$

Assumptions: (1) incompressible fluid (2) solid body rotation

Component equations $-\frac{\partial p}{\partial r} = \rho a_r = -\rho\frac{V^2}{r} = -\rho\omega^2 r$

$$\frac{\partial p}{\partial z} = -\rho g$$

Between D and C, r = constant, so $\frac{dp}{dz} = -\rho g$ and $p_D - p_C = \rho g H$ ....(1)

Between B and A, r = constant, so $\frac{dp}{dz} = -\rho g$ and $p_A - p_B = -\rho g H$ ...(2)

Between B and C, z = constant, so $\frac{dp}{dr} = \rho\omega^2 r$. Hence

$$\int_{p_B}^{p_C} dp = \int_0^L \rho\omega^2 r\, dr = \rho\omega^2 \frac{r^2}{2}\Big]_0^L = \rho\omega^2\frac{L^2}{2}$$

$$p_C - p_B = \rho\omega^2\frac{L^2}{2} \quad ....(3)$$

Since $p_D = p_{atm}$, then from Eq(1) $p_C = p_{atm} + \rho g H$

From Eq(3) $p_B = p_C - \rho\omega^2\frac{L^2}{2}$ so $p_B = p_{atm} + \rho g H - \rho\omega^2\frac{L^2}{2}$

From Eq(2) $p_A = p_B - \rho g H$ so $p_A = p_{atm} - \rho\omega^2\frac{L^2}{2}$

Thus the minimum pressure occurs at point A

At T = 20C the vapor pressure of water is $p_v = 2.34\times10^3\ N/m^2$

Solving for ω with $p_A = p_v$, we obtain

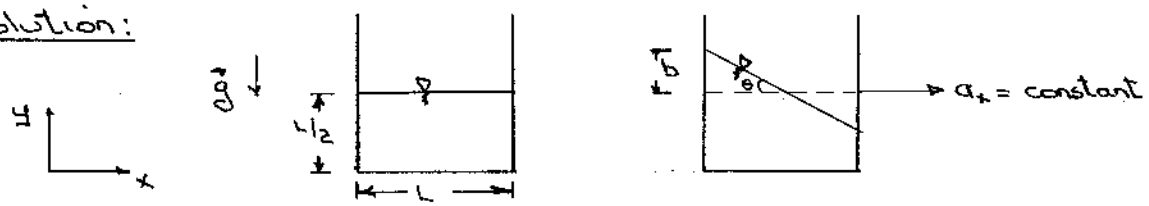
$$\omega = \left[\frac{2(p_{atm} - p_v)}{\rho L^2}\right]^{1/2} = \left[2(101.33 - 2.34)\times10^3\ \frac{N}{m^2}\times\frac{m^3}{999\ kg}\times\frac{1}{(0.075)^2\ m^2}\times\frac{kg\cdot m}{N\cdot s^2}\right]^{1/2}$$

$$\omega = 188\ rad/s$$

Given: Box, 1m × 1m × 1m, half filled with oil (SG = 0.80), subjected to a constant horizontal acceleration of 0.2g.

Determine: (a) slope of free surface
(b) pressure along bottom of box

Solution:



Basic equation: $-\nabla p + \rho\vec{g} = \rho\vec{a}$

Writing the component equations

$$-\frac{\partial p}{\partial x} + \rho g_x = \rho a_x \quad \Rightarrow \quad \frac{\partial p}{\partial x} = -\rho a_x$$

$$-\frac{\partial p}{\partial y} + \rho g_y = \rho a_y \quad \Rightarrow \quad \frac{\partial p}{\partial y} = -\rho g$$

$$-\frac{\partial p}{\partial z} + \rho g_z = \rho a_z \quad \Rightarrow \quad \frac{\partial p}{\partial z} = 0$$

From the component equations we conclude that $p = p(x,y)$

Then

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

Along the free surface p = constant and $dp = 0$. Hence

$$\left(\frac{dy}{dx}\right)_{surface} = -\frac{\partial p/\partial x}{\partial p/\partial y} = -\frac{a_x}{g} = -\frac{0.2g}{g} = -0.2$$

Since $p = p(x,y)$

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

Substituting for the partial derivatives

$$dp = -\rho a_x dx - \rho g dy$$

Integrating for ρ = constant

$$p = -\rho a_x x - \rho g y + C$$

To evaluate the constant of integration note that

$p = p_{atm}$ at $x = 0$, $y = \frac{L}{2} + b$

Hence $p_{atm} = -\rho g\left(\frac{L}{2} + b\right) + C$ and $C = p_{atm} + \rho g\left(\frac{L}{2} + b\right)$

Thus

$$p = p_{atm} - \rho a_x x + \rho g\left(\frac{L}{2} + b - y\right)$$

where $b = \frac{L}{2}\tan\theta = \frac{L}{2}\left(-\frac{dy}{dx}\right)_{surf} = \frac{L}{2}\frac{a_x}{g}$ {Note $\theta > 0$ for $\frac{dy}{dx} < 0$}

$$\therefore \quad p = p_{atm} - \rho a_x x + \rho g\left(\frac{L}{2} + \frac{L}{2}\frac{a_x}{g} - y\right)$$ {Note: This gives $p = p_{atm}$ at $x = y = \frac{L}{2}$ as it should}

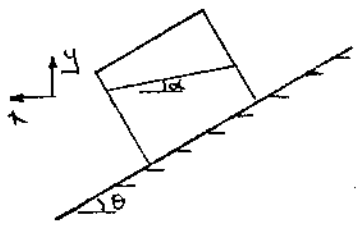
Along the bottom surface $y = 0$ and hence

$$p(x,0) = p_{atm} - \rho a_x x + \rho g\left(\frac{L}{2} + \frac{L}{2}\frac{a_x}{g}\right)$$

$$p(x,0) = 101\ kPa - (0.8)\times 999\frac{kg}{m^3}\times 0.2\times 9.81\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}\times x + \left(0.8\times 999\frac{kg}{m^3}\right)\left(9.81\frac{m}{s^2}\right)\left(\frac{1}{2} + \frac{1}{2}\times 0.2\right)m\times\frac{N\cdot s^2}{kg\cdot m}$$

$$p(x,0) = 106 - 1.57x\ kPa \quad (x \text{ in meters})$$

Given: Rectangular container of base dimensions 0.4m × 0.2m and height 0.4m is filled with water to a depth, d = 0.2m
Mass of empty container is M_c = 10 kg
Container slides down an incline, θ = 30°
Coefficient of sliding friction is 0.30



Find: The angle of the water surface relative to the horizontal

Solution:

Basic equations: $-\nabla p + \rho \vec{g} = M\vec{a}$ $\quad \Sigma \vec{F} = M\vec{a}$

Assumptions: (1) fluid moves as solid body, ie no sloshing

Writing component equations,

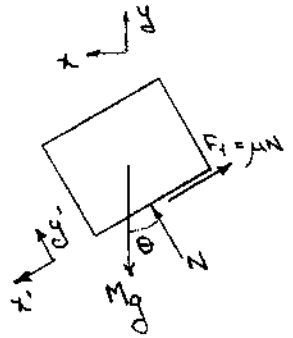
$$-\frac{\partial p}{\partial x} = \rho a_x \qquad \frac{\partial p}{\partial x} = -\rho a_x$$

$$-\frac{\partial p}{\partial y} - \rho g = \rho a_y \qquad \frac{\partial p}{\partial y} = -\rho(g + a_y)$$

$p = p(x,y)$ $\quad dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$. Along the water surface, $dp = 0$

$$\frac{dy}{dx} = -\frac{\partial p/\partial x}{\partial p/\partial y} = -\frac{a_x}{g + a_y}$$

To determine a_x and a_y consider the container and contents



$$M = M_c + M_{H_2O} = M_c + \rho \forall = 10\,kg + 999\frac{kg}{m^3} \times 0.4m \times 0.2m \times 0.2m$$

$$M = 26\,kg$$

$$\Sigma F_{y'} = 0 = N - Mg\cos\theta$$

$$N = Mg\cos\theta = 26\,kg \times 9.81\frac{m}{s^2} \times \cos 30° \times \frac{N \cdot s^2}{kg \cdot m} = 221\,N$$

$$\Sigma F_{x'} = Ma_{x'} = Mg\sin 30° - F_f = Mg\sin 30° - \mu N$$

$$a_{x'} = g\sin 30° - \mu\frac{N}{M} = 9.81\frac{m}{s^2}\sin 30° - 0.3 \times 221\,N \times \frac{1}{26\,kg} \times \frac{kg \cdot m}{N \cdot s^2}$$

$$a_{x'} = 2.36\ m/sec^2$$

Then $\quad a_x = a_{x'}\cos\theta = 2.36\frac{m}{sec^2} \times \cos 30° = 2.04\ m/s^2$

$$a_y = -a_{x'}\sin\theta = -2.36\frac{m}{sec^2} \times \sin 30° = -1.18\ m/s^2$$

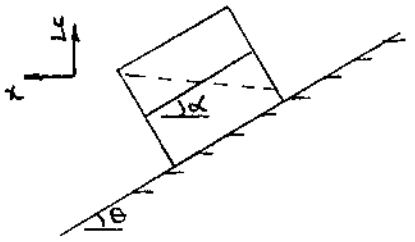
and

$$\frac{dy}{dx} = \frac{-a_x}{g + a_y} = -\frac{2.04}{9.81 - 1.18} = -0.236$$

$$\alpha = \tan^{-1} 0.236 = 13.3°$$

α

Given: Rectangular container of base dimensions 0.4m x 0.2m and height 0.4m is filled with water to a depth, d = 0.2m
Mass of empty container is M_c = 10 kg
Container slides down an incline, θ = 30°
without friction



Find: (a) The angle of the water surface relative to the horizontal.
(b) Slope of the free surface for the same acceleration up the plane

Solution:

Basic equations: $-\nabla P + \rho\vec{g} = \rho\vec{a}$ $\qquad \Sigma\vec{F} = M\vec{a}$

Assumptions: (1) fluid moves as solid body, ie. no sloshing

Writing component equations,

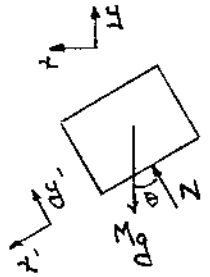
$$-\frac{\partial P}{\partial x} = \rho a_x \qquad \frac{\partial P}{\partial x} = -\rho a_x$$

$$-\frac{\partial P}{\partial y} - \rho g = \rho a_y \qquad \frac{\partial P}{\partial y} = -\rho(g + a_y)$$

$P = P(x,y)$ $\qquad dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$ $\qquad$ Along the water surface, $dP = 0$

$$\frac{dy}{dx} = \frac{-\partial P/\partial x}{\partial P/\partial y} = -\frac{a_x}{(g + a_y)}$$

For motion without friction



$$\Sigma F_{x'} = Ma_{x'} = Mg\sin\theta \qquad \therefore a_{x'} = g\sin\theta$$

$$a_x = a_{x'}\cos\theta = g\sin\theta\cos\theta$$

$$a_y = -a_{x'}\sin\theta = -g\sin^2\theta$$

$$\frac{dy}{dx} = -\frac{a_x}{(g + a_y)} = -\frac{g\sin\theta\cos\theta}{(g - g\sin^2\theta)} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

$$\frac{dy}{dx} = -\tan 30° = -0.577$$

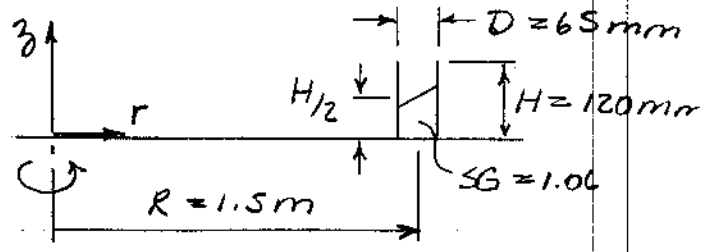
$$\alpha = \tan^{-1} 0.577 = 30° \qquad \alpha$$

For the same acceleration up the incline, $a_x = -g\sin\theta\cos\theta$ $\quad a_y = g\sin^2\theta$

$$\frac{dy}{dx} = \frac{-a_x}{(g + a_y)} = \frac{g\sin\theta\cos\theta}{(g + g\sin^2\theta)} = \frac{\sin\theta\cos\theta}{1 + \sin^2\theta} = \frac{\sin 30\cos 30}{1 + \sin^2 30}$$

$$\frac{dy}{dx} = 0.346$$

Given: Soft drink can at outer edge of merry-go-round.



Find: (a) Slope of free surface
(b) Spin rate to spill
(c) Likelihood of spilling vs. slipping

Solution: Assume rigid-body motion

Basic equation: $-\nabla p + \rho \vec{g} = \rho \vec{a}_r \qquad a_r = -\frac{V^2}{r} = -\frac{(r\omega)^2}{r} = -r\omega^2$

$$\left.\begin{aligned} -\frac{\partial p}{\partial r} + \rho g_r &= \rho a_r \\ -\frac{\partial p}{\partial z} + \rho g_z &= \rho \cancel{a_z}^{=0(3)} \end{aligned}\right\} \quad \begin{aligned} \frac{\partial p}{\partial r} &= -\rho a_r = \rho r\omega^2 \\ \frac{\partial p}{\partial z} &= +\rho g_z = -\rho g \end{aligned}$$

Assumptions: (1) Rigid-body motion, (2) $g_r = 0$, (3) $a_z = 0$, (4) $g_z = -g$

Then $p = p(r,z)$ so $dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial z} dz$

$dp = 0$ along free surface, so $\frac{dz}{dr} = -\frac{\partial p / \partial r}{\partial p / \partial z} = -\frac{\rho r\omega^2}{-\rho g} = \frac{r\omega^2}{g}$

$$\omega = 0.3 \frac{\text{rev}}{\text{sec}} \times 2\pi \frac{\text{rad}}{\text{rev}} = 1.88 \text{ rad/s}$$

$$\left.\frac{dz}{dr}\right)_{\text{surface}} = 1.5\,\text{m} \times (1.88)^2 \frac{\text{rad}^2}{\text{s}^2} \times \frac{\text{s}^2}{9.81\,\text{m}} = 0.540 \qquad \text{Slope}$$

To spill, slope must be H/D $\qquad H/D = 120/65 = 1.85$

Thus $\omega = \left[\frac{g}{r}\frac{dz}{dr}\right]^{1/4} = \left[9.81 \frac{\text{m}}{\text{s}^2} \times 1.85 \times \frac{1}{1.5\,\text{m}}\right]^{1/4} = 3.48 \text{ rad/s}$ Spill

This is nearly double the speed.

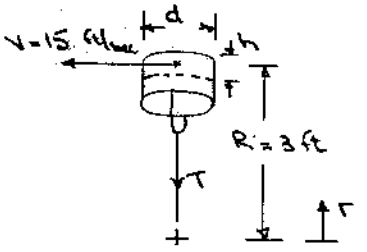
The coefficient of static friction between the can and surface is probably $\mu_s \leq 0.5$.

Thus the can would likely not spill or tip: it would slide off!

Given: Pail, 1 ft in diameter and 1 ft deep, weighs 3 lbf and contains 8 in. of water.
Pail is swung in a vertical circle of 3 ft radius and a speed of 15 ft/s.
Water moves as solid body
Point of interest is top of trajectory

Determine: (a) tension in string
(b) pressure on pail bottom from water

Solution



Assumption: center of mass of bucket and of water are located at $r = 3$ ft where $V = r\omega = 15$ ft/s

Summing forces in radial direction

$$\Sigma F_r \hat{e}_r = m_b a_{b_r} \hat{e}_r + m_w a_{w_r} \hat{e}_r$$

$$-T - (m_b + m_w) g = m_b a_{b_r} + m_w a_{w_r}$$

But $a_{b_r} = a_{w_r} = -\omega^2 r = -\frac{V^2}{r}$

$$\therefore T = \left(\frac{V^2}{r} - g\right)(m_b + m_w)$$

where

$$m_w = \rho_w \forall_w = \rho_w \frac{\pi}{4} d^2 h = 1.94 \frac{\text{slug}}{\text{ft}^3} \times \pi \times \frac{1}{4} \text{ft}^2 \times 8 \text{ in} \times \frac{\text{ft}}{12 \text{ in}} = 1.02 \text{ slug}$$

Then

$$T = \left((15)^2 \frac{\text{ft}^2}{\text{s}^2} \times \frac{1}{3 \text{ ft}} - 32.2 \frac{\text{ft}}{\text{s}^2} \right) \left(3 \text{ lbf} \cdot \frac{1}{32.2} \frac{\text{s}^2}{\text{ft}} \cdot \frac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2} + 1.02 \text{ slug} \right) \frac{\text{lbf} \cdot \text{s}^2}{\text{ft} \cdot \text{slug}}$$

$$T = 47.6 \text{ lbf} \quad \longleftarrow T$$

In the water $-\nabla p + \rho \vec{g} = \rho \vec{a}$

Writing the component in the r direction

$$-\frac{\partial p}{\partial r} - \rho g = \rho a_r = -\rho \frac{V^2}{r}$$

$$\frac{\partial p}{\partial r} = \rho \left(\frac{V^2}{r} - g \right) = 1.94 \frac{\text{slug}}{\text{ft}^3} \left((15)^2 \frac{\text{ft}^2}{\text{sec}^2} \times \frac{1}{3 \text{ ft}} - 32.2 \frac{\text{ft}}{\text{sec}^2} \right) \times \frac{\text{lbf} \cdot \text{sec}^2}{\text{ft} \cdot \text{slug}}$$

$$\frac{\partial p}{\partial r} = 83.0 \text{ lbf/ft}^3$$

Assuming that $\partial p / \partial r$ is constant throughout the water then

$$p_{\text{bottom}} \approx p_{\text{surface}} + \frac{\partial p}{\partial r} \Delta r$$

$$p_{\text{bottom}} = p_{\text{atm}} + 83.0 \frac{\text{lbf}}{\text{ft}^3} \times 8 \text{ in} \times \frac{\text{ft}}{12 \text{ in}} = p_{\text{atm}} + 55.3 \frac{\text{lbf}}{\text{ft}^2}$$

$$p_{\text{bottom}} - p_{\text{atm}} = 55.3 \text{ lbf/ft}^2 \text{ (gage)} \quad \longleftarrow$$

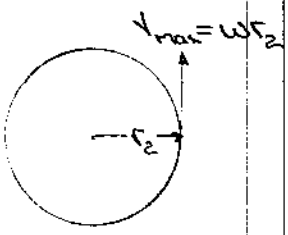
Given: Gas centrifuge, with maximum peripheral speed, $V_{max} = 300$ m/sec contains uranium hexafluoride gas ($M = 352$ kg/kgmol) at 325C.

Find: (a) Develop an expression for ratio of maximum pressure to pressure at centrifuge axis
(b) Evaluate for given conditions.

Solution:

Basic equation: $-\nabla p + \rho\vec{g} = \rho\vec{a}$ $\quad p = \rho RT$

(r component) $-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r$



Assumptions: (1) ideal gas behavior, T = constant
(2) r horizontal, so $g_r = 0$
(3) rigid body motion, so
$a_r = -\frac{V^2}{r} = -\frac{(r\omega)^2}{r} = -r\omega^2$

Then $\frac{\partial p}{\partial r} = -\rho a_r = \rho r\omega^2 = \frac{p}{RT} r\omega^2$

Separating variables and integrating, we obtain

$$\int_{p_1}^{p_2} \frac{dp}{p} = \frac{\omega^2}{RT} \int_{r_1=0}^{r_2} r\,dr = \frac{\omega^2}{RT} \frac{r_2^2}{2} \qquad V_{max} = \omega r_2$$

$$\ln \frac{p_2}{p_1} = \frac{V_{max}^2}{2RT}$$

$$\frac{p_2}{p_1} = e^{\frac{V_{max}^2}{2RT}}$$

To evaluate, $R = \frac{R_u}{M} = \frac{8314 \text{ N·m}}{\text{kgmol·K}} \times \frac{\text{kgmole}}{352 \text{ kg}} = 23.62 \frac{\text{N·m}}{\text{kg·K}}$

$$\frac{V_{max}^2}{2RT} = \frac{(300)^2}{2} \frac{\text{m}^2}{\text{s}^2} \times 23.62 \frac{\text{kg·K}}{\text{N·m}} \times \frac{1}{598 \text{ K}} \times \frac{\text{N·s}^2}{\text{kg·m}} = 3.186$$

$$\therefore \frac{p_2}{p_1} = e^{3.186} = 24.2$$

Open-Ended Problem Statement: When a water polo ball is submerged below the surface in a swimming pool and released from rest, it is observed to pop out of the water. How would you expect the height to which it rises above the water to vary with depth of submersion below the surface? Would you expect the same results for a beach ball? For a table-tennis ball?

Discussion: Separate the problem into two parts: (1) motion of the ball in water below the pool surface, and (2) motion of the ball in air above the pool surface.

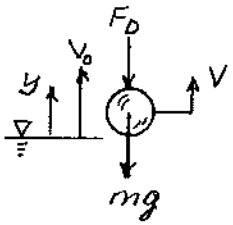
Below the pool water surface the motion of each ball is controlled by buoyancy force and inertia. For small depths of submersion ball speed upon reaching the pool surface will be small. As depth is increased, ball speed will increase until terminal speed in water is approached. For large depths, the actual depth will be irrelevant because the ball will reach terminal speed before reaching the pool water surface. All three balls are relatively light for their diameters, so terminal speed in water should be reached quickly. The depth of submersion needed to reach terminal speed should be fairly small, perhaps 1 meter or less.[1]

Buoyancy is proportional to volume and inertia is proportional to mass. The ball with the largest volume per unit mass should accelerate most quickly to terminal speed. This probably will be the beach ball, followed by the table-tennis ball and the water polo ball.

The ball with the largest diameter has the smallest frontal area per unit volume; the terminal speed should be highest for this ball. The beach ball should have the highest terminal speed, followed by the water polo ball and the table-tennis ball.

Above the pool water surface the motion of each ball is controlled by aerodynamic drag force, gravity force, and inertia (see equation below). Without aerodynamic drag, the height above the pool water surface reached by each ball would depend only its initial speed.[2] Aerodynamic drag reduces the height reached by each ball.

Aerodynamic drag force is proportional to frontal area. The heaviest ball per unit frontal area (probably the water polo ball) should reach the maximum height and the lightest ball per unit frontal area (probably the beach ball) should reach the minimum height.



$$\Sigma F_y = -F_D - mg = ma_y = m\frac{dV}{dt}$$

$$-C_D A \tfrac{1}{2}\rho V^2 - mg = m\frac{dV}{dt}, \text{ since } F_D = C_D A \tfrac{1}{2}\rho V^2$$

$$-\frac{C_D A \frac{1}{2}\rho V^2}{m} - g = \frac{dV}{dt} = V\frac{dV}{dy} \qquad (1)$$

Separating variables $\dfrac{V\,dV}{1 + \frac{C_D A \rho}{mg}\frac{V^2}{2}} = -g\,dy$

Integrating, $\dfrac{mg}{\rho C_D A} \ln\left[1 + \dfrac{\rho C_D A}{mg}\dfrac{V^2}{2}\right]_{V_0}^{0} = -\dfrac{mg}{\rho C_D A}\ln\left[1 + \dfrac{\rho C_D A}{mg}\dfrac{V_0^2}{2}\right] = -g y_{max}$

[1] The initial water depth required to reach terminal speed may be calculated using the methods of Chapter 9.

[2] The maximum height reached by a ball in air with aerodynamic drag may be calculated using the methods of Chapter 9.

Thus $y_{max} = \frac{m}{\rho C_D A} \ln\left[1 + \frac{\rho C_D A}{mg}\frac{V_0^2}{2}\right] = \frac{m}{\rho C_D A}\ln\left[1 + \frac{F_{D_0}}{mg}\right]$ (2) y_{max}

With no aerodynamic drag, Eq. 1 reduces to

$$-mg = mV\frac{dV}{dy} \quad \text{or} \quad V\,dV = -g\,dy$$

Integrating from V_0 to 0, $\left.\frac{V^2}{2}\right]_{V_0}^{0} = -g\,y_{max}$

$$y_{max} = \frac{V_0^2}{2g} \quad (3) \quad y_{max} \; (C_D = 0$$

Check the limiting value predicted by Eq. 2 as $C_D \to 0$:

$$\lim_{C_D \to 0} y_{max} = \lim_{C_D \to 0} \frac{m}{\rho C_D A}\frac{\rho C_D A}{mg}\frac{V_0^2}{2} = \frac{V_0^2}{2g} \quad \checkmark\checkmark$$

500 SHEETS FILLER 5 SQUARE
42-381 100 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-3889 200 SHEETS EYE-EASE® 5 SQUARE
42-3892 100 RECYCLED WHITE 5 SQUARE
42-3899 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

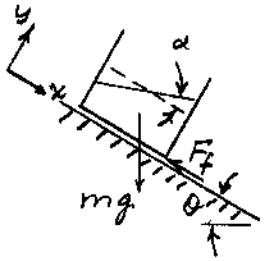
Open-Ended Problem Statement: The analysis of problem 3.98 suggests that it maybe possible to determine the coefficient of sliding friction between two surfaces by measuring the slope of the free surface in a liquid-filled container sliding down an inclined surface. Investigate the feasibility of this idea.

Discussion: A certain minimum angle of inclination would be needed to overcome static friction and start the container into motion down the incline. Once the container is in motion, the retarding force would be provided by sliding (dynamic) friction. The coefficient of dynamic friction usually is smaller than the static friction coefficient. Thus the container would continue to accelerate as it moved down the incline. This acceleration would provide a non-zero slope to the free surface of the liquid in the container.

In principle the slope could be measured and the coefficient of dynamic friction calculated. In practice several problems would arise.

To calculate dynamic friction coefficient one must assume the liquid moves as a solid body (i.e., that there is no sloshing). This condition could only be achieved if there were minimum initial disturbance and the sliding distance were long.

It would be difficult to measure the slope of the free surface of liquid in the moving container. Images made with a video camera or digital still camera might be processed to obtain the required slope information.



$\Sigma F_y = N - mg\cos\theta$; $N = mg\cos\theta$

$\Sigma F_x = mg\sin\theta - F_f = ma_x$; $F_f = \mu_k N = \mu_k mg\cos\theta$

$a_x = g\sin\theta - \mu_k g\cos\theta = g(\sin\theta - \mu_k\cos\theta)$

For static liquid $-\nabla p + \rho\vec{g} = \rho\vec{a}$

$-\frac{\partial p}{\partial x} + \rho g\sin\theta = \rho a_x = \rho g(\sin\theta - \mu_k\cos\theta)$; $\frac{\partial p}{\partial x} = \rho g\mu_k\cos\theta$

$-\frac{\partial p}{\partial y} - \rho g\cos\theta = \rho a_y^{\,0}$; $\frac{\partial p}{\partial y} = -\rho g\cos\theta$

For the free surface, $dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = 0$, so $\frac{dy}{dx} = -\frac{\partial p/\partial x}{\partial p/\partial y}$

Thus $\frac{dy}{dx} = -\frac{\rho g\mu_k\cos\theta}{-\rho g\cos\theta} = \mu_k$; $\alpha = \tan^{-1}(\mu_k)$

Since it was necessary to make the container slip on the surface,

$\theta > \tan^{-1}(\mu_s) > \tan^{-1}(\mu_k) = \alpha$

Thus $\alpha < \theta$, as shown in the sketch above.

α

Given: An automobile traveling at 90 km/hr rounds a long sweeping curve of radius $R = 250$ m. A child in the back seat of the closed car holds the string tied to a helium filled balloon. String was vertical on a straight road

Find: The magnitude and direction of the string angle as measured from the vertical.

Solution:

Basic equation: $-\nabla p + \rho \vec{g} = \rho \vec{a}$ $\qquad \Sigma \vec{F} = m\vec{a}$

Assumptions: (1) air inside car moves as solid body

Writing the component equations:

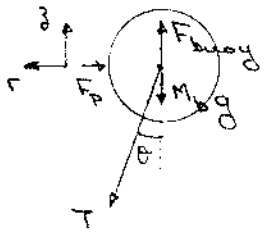
$$-\frac{\partial p}{\partial z} - \rho g = 0 \quad \text{and} \quad \frac{\partial p}{\partial z} = -\rho g$$

$$-\frac{\partial p}{\partial r} = \rho a_r = -\rho \frac{V^2}{R} \quad \text{and} \quad \frac{\partial p}{\partial r} = \rho \frac{V^2}{R}$$



The above expressions give the pressure gradient inside the car

Consider a free-body diagram of the balloon in the car.



$$\Sigma F_z = 0 = -T\cos\theta - M_b g + F_{buoy}$$

M_b = mass of balloon

$F_{buoy} = \rho_a g \forall$

$$T\cos\theta = -g(M_b - \rho_a \forall)$$

$$\Sigma F_r = M_b a_r$$

$$T\sin\theta - F_p = -M_b \frac{V^2}{R}$$

$$F_p = \int_A (P_{r+\Delta r} - P_r)\, dA = \int_A \left(P_r + \frac{\partial p}{\partial r}\Delta r - P_r\right) dA = \int \frac{\partial p}{\partial r} \Delta r\, dA = \forall \frac{\partial p}{\partial r}$$

$$T\sin\theta = F_p - M_b \frac{V^2}{R} = +\forall \frac{\partial p}{\partial r} - M_b \frac{V^2}{R} = +\rho_a \forall \frac{V^2}{R} - M_b \frac{V^2}{R} = -\frac{V^2}{R}(M_b - \rho_a \forall)$$

$$\tan\theta = \frac{T\sin\theta}{T\cos\theta} = \frac{-\frac{V^2}{R}(M_b - \rho_a \forall)}{-g(M_b - \rho_a \forall)} = \frac{V^2}{Rg} = \frac{(90)^2}{hr^2} \times \frac{1}{250\,m} \times \frac{s^2}{9.81\,m} \times \frac{10^6\,m^2}{km^2} \times \frac{hr^2}{(3600)^2\,s^2}$$

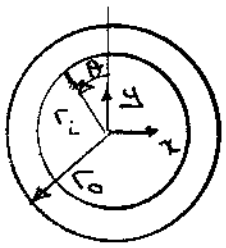
$$\tan\theta = 0.255$$

$$\theta = 14.3^\circ \text{ (toward center of curve)}$$

θ

Given: A steel liner of length $L = 2\,m$, outer radius $r_o = 0.15\,m$, and inner radius $r_i = 0.10\,m$ is to be formed in a spinning horizontal mold. To insure uniform thickness the minimum radial acceleration should be $10g$. For steel, S.G = 7.8.

Find: (a) The required angular velocity
(b) The maximum and minimum pressures on the surface of the mold.



Solution:

Basic equation: $\nabla p + \rho\vec{g} = \rho\vec{a}$

Writing component equations,

$-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r$ and $\frac{\partial p}{\partial r} = \rho g_r - \rho a_r = \rho(-g\cos\theta) - \rho(-r\omega^2) = \rho r\omega^2 - \rho g\cos\theta$

$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta = 0$ and $\frac{\partial p}{\partial \theta} = \rho g_\theta r = \rho g \sin\theta\, r$

Then, $dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial \theta}d\theta = (\rho r\omega^2 - \rho g\cos\theta)dr + \rho g r \sin\theta\, d\theta$

$\left(\frac{\partial p}{\partial r}\right)_\theta = \text{const} = \rho r\omega^2 - \rho g\cos\theta$. Since $p = p_{atm}$ at $r = r_i$, then

$p - p_{atm} = \int_{r_i}^{r} (\rho r\omega^2 - \rho g\cos\theta)\,dr + f(\theta)$ where, $f(\theta)$ is an arbitrary function

$\therefore p = p_{atm} + \rho\omega^2\left(\frac{r^2 - r_i^2}{2}\right) - \rho g\cos\theta(r - r_i) + f(\theta)$. Then,

$\frac{\partial p}{\partial \theta} = \rho g \sin\theta\,(r - r_i) + \frac{df}{d\theta} = \rho g \sin\theta\, r$

Hence, $\frac{df}{d\theta} = \rho g \sin\theta\, r_i$ and $f = -\rho g r_i \cos\theta + C$

$\therefore p = p_{atm} + \rho\omega^2\left(\frac{r^2 - r_i^2}{2}\right) - \rho g\cos\theta(r - r_i) - \rho g r_i\cos\theta + C$

At $r = r_i$, $p = p_{atm}$ for any value of θ. Hence, $C = \rho g r_i \cos\theta$ and

$$p = p_{atm} + \rho\omega^2\left(\frac{r^2 - r_i^2}{2}\right) - \rho g\cos\theta\,(r - r_i)$$

Minimum value of $a_r = 10g = r\omega^2$ occurs at r_i for given ω. Hence,

$$\omega_{min} = \left[\frac{10g}{r_i}\right]^{1/2} = \left[10 \times 9.81\,\frac{m}{s^2} \times \frac{1}{0.10\,m}\right]^{1/2} = 31.3\ rad/s \quad \omega$$

p_{max} on the surface of the mold ($r = r_o$) occurs at $\theta = \pi$

$$p_{max} - p_{atm} = \frac{\rho\omega^2}{2}(r_o^2 - r_i^2) - \rho g\cos\theta\,(r - r_i)$$

$$p_{max} - p_{atm} = \frac{1}{2} \times 7.8 \times 999\,\frac{kg}{m^3} \times \left(\frac{31.3}{s}\right)^2 \times [(0.15)^2 - (0.10)^2]\,m^2 \times \frac{N \cdot s^2}{kg \cdot m} - 7.8 \times 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2}(-1)[0.15 - 0.10]\,\frac{N \cdot s^2}{kg \cdot m}$$

$$p_{max} = 51.5\ kPa\ (gage) \quad p_{max}$$

p_{min} on the surface of the mold ($r = r_o$) occurs at $\theta = 0$

$$p_{min} - p_{atm} = \frac{\rho\omega^2}{2}(r_o^2 - r_i^2) - \rho g\cos\theta\,(r - r_i)$$

$$p_{min} - p_{atm} = \frac{1}{2} \times 7.8 \times 999\,\frac{kg}{m^3} \times \left(\frac{31.3}{s}\right)^2 [(0.15)^2 - (0.10)^2]\,m^2 \times \frac{N \cdot s^2}{kg \cdot m} - 7.8 \times 999\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2}(1)[0.15 - 0.10]\,\frac{N \cdot s^2}{kg \cdot m}$$

$$p_{min} = 43.9\ kPa\ (gage) \quad p_{min}$$

Given: Six-pack cooled from 25°C to 5°C in freezer.

Find: Change in specific entropy.

Solution: Apply the Tds equation.

Basic equation: $Tds = du + p\cancel{dv}^{\simeq 0(1)}$

Assumptions: (1) Neglect volume change
(2) Liquid properties are similar to water

Then

$$Tds = du = c_v\, dT$$

or

$$ds = c_v \frac{dT}{T}$$

Integrating,

$$s_2 - s_1 = c_v \ln\left(\frac{T_2}{T_1}\right)$$

$$= 1 \frac{kcal}{kg \cdot K} \times \ln\left(\frac{273+5}{273+25}\right) \times 4190 \frac{J}{kcal}$$

$$s_2 - s_1 = -0.291 \text{ kJ/kg.K}$$

Δs

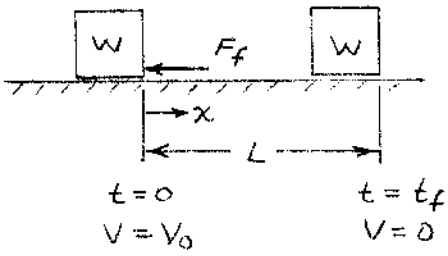
Given: Auto skids to stop in 50 meters on level road with $\mu = 0.6$.

Find: Initial speed.

Solution: Apply Newton's second law to a system (auto).

Basic equations: $\Sigma F_x = ma_x = \frac{W}{g}\frac{d^2x}{dt^2}$

Assumptions: (1) $F_f = \mu W$
(2) Neglect air resistance



Then $\Sigma F_x = -F_f = -\mu W = \frac{W}{g}\frac{d^2x}{dt^2}$

or $\frac{d^2x}{dt^2} = -\mu g$

Integrating,

$$\frac{dx}{dt} = -\mu g t + C_1 = -\mu g t + V_0 \qquad (1)$$

since $V = V_0$ at $t = 0$. Integrating again,

$$x = -\frac{1}{2}\mu g t^2 + V_0 t + C_2 = -\frac{1}{2}\mu g t^2 + V_0 t \qquad (2)$$

since $x = 0$ at $t = 0$.

Now at $x = L$, $\frac{dx}{dt} = 0$, and $t = t_f$. From Eq. 1,

$$0 = -\mu g t_f + V_0 \quad \text{or} \quad t_f = \frac{V_0}{\mu g}$$

Substituting into Eq. 2, evaluated at $t = t_f$,

$$L = -\frac{1}{2}\mu g t_f^2 + V_0 t_f = -\frac{1}{2}\mu g \frac{V_0^2}{(\mu g)^2} + V_0 \frac{V_0}{\mu g}$$

$$L = -\frac{1}{2}\frac{V_0^2}{\mu g} + \frac{V_0^2}{\mu g} = \frac{1}{2}\frac{V_0^2}{\mu g}$$

Solving, $V_0 = \sqrt{2\mu g L} = \sqrt{2(0.6)\,9.81\,\frac{m}{s^2} \times 50\,m} = 24.3\ m/s$

or

$$V_0 = 24.3\,\frac{m}{s} \times \frac{km}{1000\,m} \times 3600\,\frac{s}{hr} = 87.5\ km/hr$$

V_0

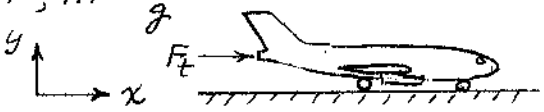
Given: Jet aircraft with $W = 715{,}000$ lbf.
Takeoff speed is $V_t = 140$ mph.
Twin engines develop 102,000 lbf thrust each.

Assume thrust is constant. Neglect resistance.

Estimate: (a) Runway length needed.
(b) Time to reach V_t.

Solution: Apply Newton's second law of motion.

$$\Sigma F_x = m\frac{dV}{dt} = mV\frac{dV}{ds} = F_t = \text{constant}\,;\; m = \frac{W}{g}$$



For distance calculation

$$mV\frac{dV}{ds} = \frac{W}{g}V\frac{dV}{ds} = F_t\,;\quad V\,dV = \frac{gF_t}{W}\,ds$$

Integrating,

$$\int V\,dV = \frac{V^2}{2} = \int \frac{gF_t}{W}\,ds = \frac{gF_t}{W}s$$

Thus

$$s = \frac{WV^2}{2gF_t}$$

$$= \frac{1}{2} \times 715{,}000\ \text{lbf}\left[140\frac{\text{mi}}{\text{hr}} \times 5280\frac{\text{ft}}{\text{mi}} \times \frac{\text{hr}}{3600\ \text{s}}\right]^2 \frac{\text{s}^2}{32.2\ \text{ft}} \times \frac{1}{2(102{,}000)\ \text{lbf}}$$

$$s = 2{,}290\ \text{ft} \qquad \leftarrow s$$

For time calculation,

$$m\frac{dV}{dt} = \frac{W}{g}\frac{dV}{dt} = F_t\,;\quad dV = \frac{gF_t}{W}\,dt$$

Integrating,

$$\int dV = V = \int \frac{gF_t}{W}\,dt = \frac{gF_t}{W}t$$

Thus

$$t = \frac{WV}{gF_t}$$

$$= 715{,}000\ \text{lbf} \times 140\frac{\text{mi}}{\text{hr}} \times \frac{\text{s}^2}{32.2\ \text{ft}} \times \frac{1}{2(102{,}000)\ \text{lbf}} \times 5280\frac{\text{ft}}{\text{mi}} \times \frac{\text{hr}}{3600\ \text{s}}$$

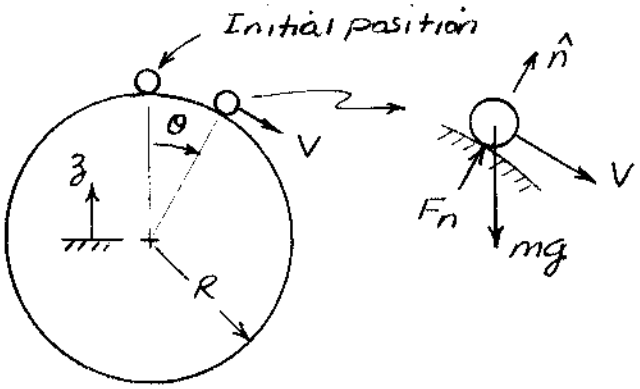
$$t = 22.4\ \text{s} \qquad \leftarrow t$$

{Aerodynamic and rolling resistance would cause these values to increase for an actual aircraft.}

Given: Small steel ball of radius, r, atop large sphere of radius, R, begins to roll. Neglect rolling and air resistance.

Find: Location where ball loses contact and becomes a projectile.

Solution: Sum forces in n direction



$$\Sigma F_n = F_n - mg\cos\theta = ma_n$$

$$a_n = -\frac{V^2}{(R+r)}$$

Contact is lost when $F_n \to 0$, or

$$-mg\cos\theta = -m\frac{V^2}{(R+r)}$$

or

$$V^2 = (R+r)g\cos\theta \qquad (1)$$

Energy must be conserved if there is no resistance. Thus

$$E = mgz + m\frac{V^2}{2} = mg(R+r)\cos\theta + m\frac{V^2}{2} = E_0 = mg(R+r)$$

Thus from energy considerations

$$V^2 = 2g(R+r)(1-\cos\theta) \qquad (2)$$

Combining Eqs. 1 and 2,

$$V^2 = 2g(R+r)(1-\cos\theta) = (R+r)g\cos\theta$$

or $\quad 2(1-\cos\theta) = 2 - 2\cos\theta = \cos\theta$

Thus $\cos\theta = \frac{2}{3}$ and $\theta = \cos^{-1}\left(\frac{2}{3}\right) = 48.2$ degrees ← θ

Given: Auditorium with volume $\forall = 1.2 \times 10^5$ ft^3 contains 6000 people. Ventilation system fails. Average heat loss per person is 300 Btu/hr.

Find: (a) increase in internal energy of air in 15 min.
(b) change in internal energy for system of people and air; account for increase in air temperature
(c) estimate rate of temperature rise.

Solution

Apply the first law of thermodynamics for a system.

Basic equation: $Q - W = \Delta E$

Assumptions: (1) no work is done, so $W = 0$
(2) stationary system, so $\Delta E = \Delta U$

(a) Consider the air in the auditorium to be the system

$$\Delta U_{air} = Q = \frac{300\ \text{Btu}}{\text{hr} \cdot \text{person}} \times 6000\ \text{persons} \times \frac{1}{4}\ \text{hr} = 4.50 \times 10^5\ \text{Btu}$$ ΔU_{air}

(b) Consider the air and people to be the system

$$\Delta U_{and} = Q_{\text{from surroundings}} = 0$$ ΔU_{air}

The increase in internal energy of the air is equal and opposite to the change in internal energy of the people

(c) To estimate the rate of temperature rise we write the first law on a rate basis

$$\dot{Q} - \dot{W} = \frac{dE}{dt}$$

Taking the air in the auditorium to be the system, then

$$\dot{Q} = \frac{dU}{dt} = M_{air} \frac{du}{dt} = M_{air} c_v \frac{dT}{dt} = \rho_{air} \forall c_v \frac{dT}{dt}$$

Assumptions: (3) air behaves as an ideal gas
(4) initial $T_{and} = 74°F$, P_{air}

$$\rho = \frac{P}{RT} = \frac{14.7\ \text{lbf/in}^2}{} \times \frac{\text{lbm}°\text{R}}{53.3\ \text{ft·lbf}} \times \frac{1}{534°\text{R}} \times \frac{144\ \text{in}^2}{\text{ft}^2} = 0.0744\ \frac{\text{lbm}}{\text{ft}^3}$$

Then

$$\frac{dT}{dt} = \frac{\dot{Q}}{\rho \forall c_v} = \frac{300\ \text{Btu}}{\text{hr·person}} \times 6000\ \text{persons} \times \frac{\text{ft}^3}{0.0744\ \text{lbm}} \times \frac{1}{1.2 \times 10^5\ \text{ft}^3} \times \frac{\text{lbm}°\text{R}}{0.171\ \text{Btu}}$$

$$\frac{dT}{dt} = 11.8\ °\text{R/hr}$$ $\frac{dT}{dt}$

Given: Air at 20°C and 1 atm compressed adiabatically, without friction, to 3 atm (abs.).

Find: Change in internal energy, in J/kg.

Solution: Apply the first law of thermodynamics. Treat the air as a system.

Basic equation: $\delta Q - \delta W = dE$ ($\delta Q = 0$ (1))

Assumptions: (1) Adiabatic process, so $\delta Q = 0$
(2) Stationary system, $dE = dU$
(3) Frictionless process, $\delta W = p\,dV = mp\,dv$
(4) Ideal gas, $pv = RT$

Then

$$\Delta U = \int dU = -\int \delta W = -\int mp\,dv$$

The problem is to relate p and v so that the integral may be evaluated. A frictionless adiabatic process is isentropic. Recall from thermodynamics that an ideal gas follows the isentropic process equation

$$pv^k = C \quad \text{where} \quad k = c_p/c_v$$

Thus $v = C^{1/k} p^{-1/k}$ and $dv = C^{1/k} \frac{1}{k} p^{-1/k-1}\,dp$. Substituting,

$$\Delta u = \frac{\Delta U}{m} = -\int p\, \frac{C^{1/k}}{k} p^{-1/k-1}\,dp = -\frac{C^{1/k}}{k}\int_{p_1}^{p_2} p^{-1/k}\,dp$$

$$= -\frac{C^{1/k}}{k}\left[\frac{1}{-\frac{1}{k}+1}\, p^{-\frac{1}{k}+1}\right]_{p_1}^{p_2} = -\frac{C^{1/k}}{k}\left[-\frac{k}{k-1}\, p^{\left(\frac{k-1}{k}\right)}\right]_{p_1}^{p_2}$$

$$\Delta u = \frac{C^{1/k}}{k-1} p_1^{\left(\frac{k-1}{k}\right)}\left[\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} - 1\right]$$

But $C^{1/k} p^{\left(\frac{k-1}{k}\right)} = C^{1/k} p^{-1/k} p = pv = RT$. Thus

$$\Delta u = \frac{RT_1}{k-1}\left[\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} - 1\right]$$

From Table A.6, $R = 287$ J/(kg·K) and $k = 1.40$ for air. Substituting,

$$\Delta u = \frac{1}{0.40} \times 287\ \frac{J}{kg\cdot K} \times (273+20)\ K\left[\left(\frac{3}{1}\right)^{\frac{1.40-1}{1.40}} - 1\right] = 77.5\ kJ/kg$$

Δu

Given: Aluminum can of soft drink cooled in refrigerator, where $T_r = 5°C$. Rate of heat transfer is modeled as

$$\dot{Q} = -k(T - T_r) \quad \text{where } k = 0.25\ W/°C$$

Mass of can and contents is equivalent to 390 g of water.

Estimate: Time to cool can from 25 to 7°C and energy removed.

Solution: Treat the can and its contents as a system. Apply the energy equation

Basic equation: $\dot{Q} - \dot{W}^{=0(1)} = \frac{dE}{dt}$; $E = U + KE + PE$

Assumptions: (1) $\dot{W} = 0$
(2) Neglect KE and PE
(3) $dU = m\,du = mC_v\,dT$

Then

$$\dot{Q} = -k(T - T_r) = \frac{dU}{dt} = mC_v\frac{dT}{dt}$$

But, since T_r = constant, $dT = d(T - T_r)$. Substituting,

$$-k(T - T_r) = mC_v\frac{d(T - T_r)}{dt}$$

Separating variables,

$$\frac{d(T - T_r)}{(T - T_r)} = -\frac{k}{mC_v}\,dt$$

Integrating,

$$\int_{T_0}^{T_f}\frac{d(T - T_r)}{(T - T_r)} = \ln(T - T_r)\Big]_{T_0}^{T_f} = \ln\left(\frac{T_f - T_r}{T_0 - T_r}\right) = -\int_0^t \frac{k}{mC_v}\,dt = -\frac{k}{mC_v}t$$

The time estimate is

$$t = -\frac{mC_v}{k}\ln\left(\frac{T_f - T_r}{T_0 - T_r}\right)$$

$$= -390\,g \times 1\,\frac{kcal}{kg\cdot K} \times \frac{C}{0.25\,W}\ln\left(\frac{7-5}{25-5}\right) \times 4190\,\frac{J}{kcal} \times \frac{W\cdot s}{J} \times \frac{kg}{1000\,g} \times \frac{hr}{3600\,s}$$

$$t = 4.18\ hr \longleftarrow t$$

The energy estimate is

$$Q = E_2 - E_1 = U_2 - U_1 = mC_v(T_2 - T_1)$$

$$Q = 390\,g \times 1\,\frac{kcal}{kg\cdot°C} \times (7-25)°C \times \frac{kg}{1000\,g} = -7.02\ kcal \longleftarrow Q$$

Given: Aluminum beverage can, $m_c = 20\,g$, $D = 65\,mm$, $H = 120\,mm$.

Maximum contents level is h_{max}, when $\forall_b = 354\,mL$ of beverage.

SG of beverage is 1.05.

Find: (a) Center of mass, y_c, vs. level, h.
(b) Level for least tendency to tip.
(c) Minimum coefficient of friction, μ_s, for full can to tip, not slide.
(d) Plot μ_s minimum for can to tip (not slide) as a function of beverage level in can.

Solution: $M_b = SG\,\rho\,\forall_b = 1.05 \times 1.0\frac{g}{cm^3} \times 354\,mL \times \frac{cm^3}{mL} = 372\,g$ (max)

$$h_{max} = \frac{\forall_b}{A} = \frac{4\forall_b}{\pi D^2} = \frac{4}{\pi} \times 354\,mL \times \frac{1}{(6.5)^2 cm^2} \times \frac{cm^3}{mL} \times \frac{10\,mm}{cm} = 107\,mm$$

At any level, $m_b = \frac{h}{h_{max}} M_b$; $m_b(g) = \frac{h(mm)}{107\,mm} \times 372\,g = 3.47\,h(mm)$

From moment considerations,

$$y_c M = \frac{h}{2} m_b + \frac{H}{2} m_c = \frac{1}{2}\left[h(3.47h) + 120(20)\right] = \frac{1}{2}(3.47h^2 + 2400)$$

$$M = m_b + m_c = 3.47h + 20$$

$$y_c = \frac{3.47h^2 + 2400}{6.94h + 40} \quad (h \text{ in mm})$$ ← y_c

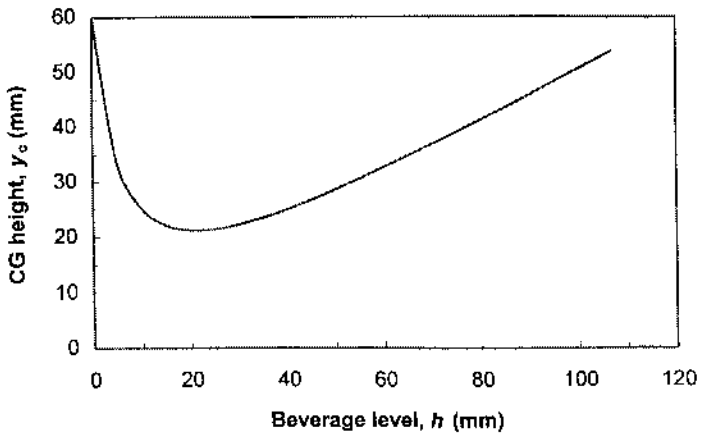
Tendency to tip will be least when y_c is a minimum. Thus

$$\frac{dy_c}{dh} = \frac{2(3.47h)}{6.94h+40} + (-1)(6.95)\frac{3.47h^2+2400}{(6.94h+40)^2} = \frac{24.1h^2 + 278h - 16{,}700}{(6.94h+40)^2} = 0$$

Using the quadratic formula,

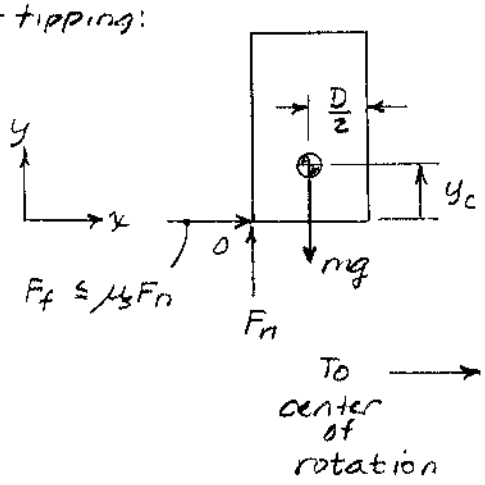
$$h\,(\text{at } y_c \text{ min}) = \frac{-278 \pm \sqrt{(278)^2 + 4(24.1)16{,}700}}{2(24.1)} = 21.2\,mm$$ ← h (y_c min)

Plotting,



Next page

Draw a free-body diagram of the can at tipping:



$$\Sigma F_x = F_f = ma_x$$

$$\Sigma F_y = F_n - mg = ma_y = 0$$

$$F_n = mg$$

Since $F_f \le \mu_s F_n$, then $\mu_s F_n \ge ma_x$

Summing moments about point O:

$$\overset{+}{\Sigma M_O} = y_c ma_x - \frac{D}{2} F_n = 0$$

or $$y_c ma_x = \frac{D}{2} F_n$$

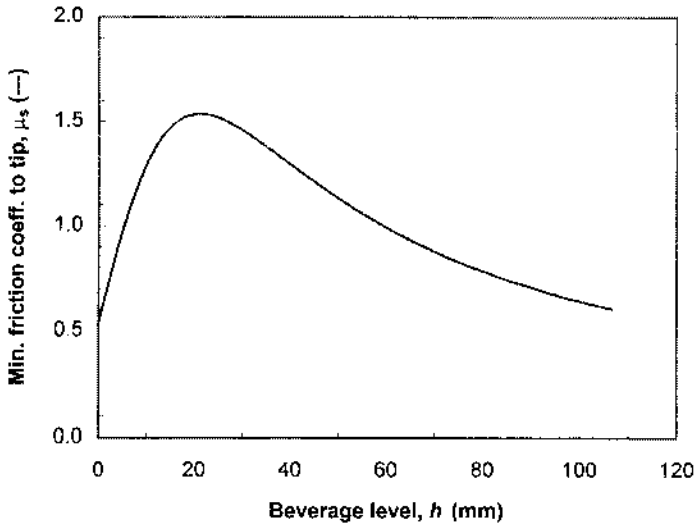
But $ma_x \le \mu_s F_n$, so

$$y_c \mu_s F_n \ge \frac{D}{2} F_n$$

Thus to tip

$$\mu_s \ge \frac{D}{2y_c}$$ μ_s

Plotting,



For the full can with $y_c = 53.8$ mm,

$$\mu_s \ge \frac{1}{2} \times 65\text{ mm} \times \frac{1}{53.8\text{ mm}} = 0.604$$ μ_s

This value is much higher than the can could develop. Therefore the can <u>will</u> <u>not</u> tip; it will slide.

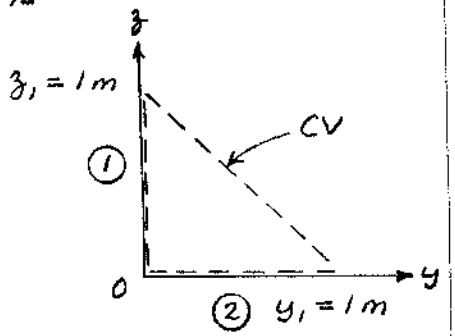
The corresponding acceleration is $a_x = \mu_s g = 0.593$ m/s^2

Given: The velocity field in the region shown is

$$\vec{V} = az\,\hat{j} + b\,\hat{k}$$

where $a = 10\,s^{-1}$ and $b = 5\,m/s$.
For depth, w, along x axis

$$d\vec{A}_1 = -w\,dz\,\hat{j} \quad \text{and} \quad d\vec{A}_2 = -w\,dy\,\hat{k}$$



Find: (a) expression for $\vec{V}\cdot d\vec{A}_1$
(b) value of $\int_{A_1} \vec{V}\cdot d\vec{A}_1$
(c) expression for $\vec{V}\cdot d\vec{A}_2$
(d) expression for $\vec{V}(\vec{V}\cdot d\vec{A}_2)$
(e) value of $\int_{A_2} \vec{V}(\vec{V}\cdot d\vec{A}_2)$

Solution: (a) $\vec{V}\cdot d\vec{A}_1 = (az\,\hat{j} + b\,\hat{k})\cdot(-w\,dz\,\hat{j}) = -awz\,dz$ ← $\vec{V}\cdot d\vec{A}_1$

(b) $\int_{A_1} \vec{V}\cdot d\vec{A}_1 = \int_0^{z_1} -awz\,dz = -aw\frac{z^2}{2}\Big]_0^{z_1} = -aw\frac{z_1^2}{2}$ ← $\int_{A_1}\vec{V}\cdot d\vec{A}_1$

(c) $\vec{V}\cdot d\vec{A}_2 = (az\,\hat{j} + b\,\hat{k})\cdot(-w\,dy\,\hat{k}) = -bw\,dy$ ← $\vec{V}\cdot d\vec{A}_2$

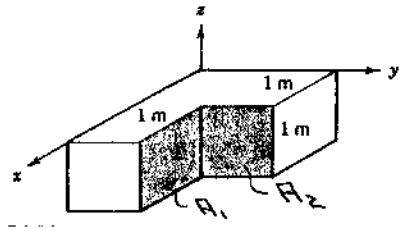
(d) $\vec{V}(\vec{V}\cdot d\vec{A}_2) = (az\,\hat{j} + b\,\hat{k})(-bw\,dy) = -abwz\,dy\,\hat{j} - b^2w\,dy\,\hat{k}$ ← $\vec{V}(\vec{V}\cdot d\vec{A}_2)$

(e) $\int_{A_2} \vec{V}(\vec{V}\cdot d\vec{A}_2) = \int_0^{y_1}(-abwz\,dy\,\hat{j} - b^2w\,dy\,\hat{k}) = -\int_0^{y_1} b^2w\,dy\,\hat{k}$,

Since $z = 0$ along A_2. Thus

$$\int_{A_2} \vec{V}(\vec{V}\cdot d\vec{A}_2) = -b^2w y_1\,\hat{k} = -(5)^2\frac{m^2}{sec^2}\times w_x\,1m\,\hat{k} = -25w\,\hat{k}\ m^3/s^2$$ (e) ←

Given: The shaded area shown, with $\vec{V} = ax\hat{i} - by\hat{j}$ where $a = b = 1\ sec^{-1}$.



Find: (a) Volume flow rate, and (b) momentum flux through the shaded area.

Solution:

For convenience, label areas A_1 and A_2. Thus

$d\vec{A}_1 = dx\,dz\,\hat{j}$ and $d\vec{A}_2 = dy\,dz\,\hat{i}$

over A_1, $y = 1$ and over A_2, $x = 1$

The volume flow rate is

$$\int \vec{V}\cdot d\vec{A} = \int_{A_1}(ax\hat{i} - by\hat{j})\cdot d\vec{A}_1 + \int_{A_2}(ax\hat{i} - by\hat{j})\cdot d\vec{A}_2$$

$$= \int_{A_1}(ax\hat{i} - by\hat{j})\cdot dx\,dz\,\hat{j} + \int_{A_2}(ax\hat{i} - by\hat{j})\cdot dy\,dz\,\hat{i}$$

$$= \int_{z=-1}^{0}\int_{x=1}^{2} -by\,dx\,dz + \int_{z=-1}^{0}\int_{y=1}^{2} ax\,dy\,dz$$

$$= \int_{z=-1}^{0}\left[-bx\right]_1^2 dz + \int_{z=-1}^{0}\left[ay\right]_1^2 dz = \int_{z=-1}^{0}\left[-b+a\right]dz$$

$$\int \vec{V}\cdot d\vec{A} = 0$$ volume flow rate

The momentum flux is

$$\int \vec{V}(\vec{V}\cdot d\vec{A}) = \int_{A_1}(ax\hat{i} - by\hat{j})\left[(ax\hat{i} - by\hat{j})\cdot d\vec{A}_1\right] + \int_{A_2}(ax\hat{i} - by\hat{j})\left[(ax\hat{i} - by\hat{j})\cdot d\vec{A}_2\right]$$

$$= \int_{A_1}(ax\hat{i} - by\hat{j})(-by\,dx\,dz) + \int_{A_2}(ax\hat{i} - by\hat{j})\,ax\,dy\,dz$$

$$= \int_{z=-1}^{0}\int_{x=1}^{2} -b(ax\hat{i} - b\hat{j})\,dx\,dz + \int_{z=-1}^{0}\int_{y=1}^{2} a(a\hat{i} - by\hat{j})\,dy\,dz$$

$$= \int_{z=-1}^{0} -b\left[a\frac{x^2}{2}\hat{i} - bx\hat{j}\right]_1^2 dz + \int_{z=-1}^{0} a\left[ay\hat{i} - b\frac{y^2}{2}\hat{j}\right]_1^2 dz$$

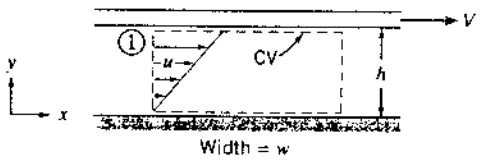
$$= \int_{z=-1}^{0} -b\left[1.5a\hat{i} - b\hat{j}\right]dz + \int_{z=-1}^{0} a\left[a\hat{i} - 1.5b\hat{j}\right]dz$$

$$= \int_{z=-1}^{0}\left\{(a^2 - 1.5ab)\hat{i} + (b^2 - 1.5ab)\hat{j}\right\}dz$$

$$= \int_{z=-1}^{0}(-0.5\hat{i} - 0.5\hat{j})\,dz = (-0.5\hat{i} - 0.5\hat{j})\,z\Big]_{-1}^{0}$$

$$\int \vec{V}(\vec{V}\cdot d\vec{A}) = -0.5\hat{i} - 0.5\hat{j}\ \ m^4/sec^2$$ momentum flux

Given: Control volume with linear velocity distribution across surface ① as shown; width = w.



Find: (a) Volume flow rate, and
(b) Momentum flux through surface ①

Solution:

The volume flow rate is $Q = \int \vec{V} \cdot d\vec{A}$

At surface ①, $\vec{V} = \frac{V}{h} y \hat{i}$ and $dA = -w\,dy\,\hat{i}$

Thus

$$Q = \int_0^h \frac{V}{h} y \hat{i} \cdot (-w\,dy\,\hat{i}) = -\frac{Vw}{h} \int_0^h y\,dy = -\frac{Vw}{h} \frac{y^2}{2} \Big|_0^h$$

$$Q = -\frac{1}{2} Vhw$$ ← Volume flow rate

The momentum flux is given by $M.F. = \int \vec{V} (\rho \vec{V} \cdot d\vec{A})$

Thus,

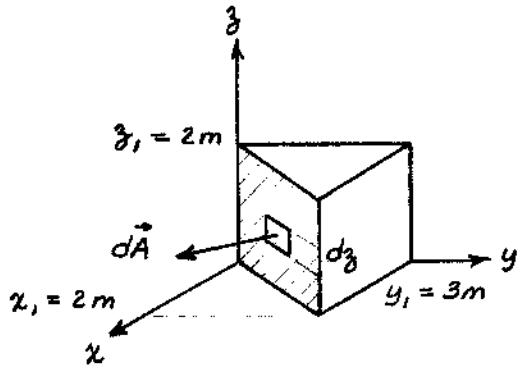
$$M.F. = \int_0^h \frac{V}{h} y \hat{i} \left(-\rho \frac{V}{h} y\,dy\right) = -\rho \frac{V^2}{h^2} \hat{i} \int_0^h y^2\,dy = -\rho \frac{V^2}{h^2} \hat{i} \frac{y^3}{3} \Big|_0^h$$

$$M.F. = -\frac{1}{3} \rho V^2 w h \hat{i}$$ ← Momentum flux

Given: The shaded area shown, with

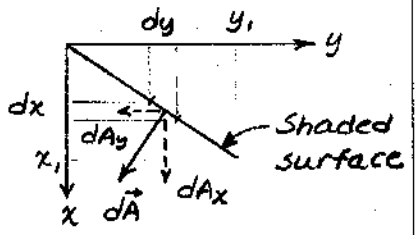
$$\vec{V} = -ax\hat{\imath} + by\hat{\jmath} + c\hat{k}$$

where $a = b = 1\,sec^{-1}$ and $c = 1\,m/s$



Find: $d\vec{A}$, $\int_A \vec{V}\cdot d\vec{A}$, $\int_A \vec{V}(\vec{V}\cdot d\vec{A})$

Solution: From sketch at right,



$$d\vec{A} = dA_x\,\hat{\imath} + dA_y(-\hat{\jmath})$$

where

$dA_x = dy\,dz$, the projection of $d\vec{A}$ on yz plane

$dA_y = dx\,dz$, the projection of $d\vec{A}$ on xz plane

Thus

$$d\vec{A} = dy\,dz\,\hat{\imath} - dx\,dz\,\hat{\jmath}$$ ← $d\vec{A}$

Along the shaded surface, $\frac{x}{y} = \frac{x_1}{y_1} = \frac{2m}{3m}$, or $y = \frac{3}{2}x$ and $x = \frac{2}{3}y$.

$$\int_A \vec{V}\cdot d\vec{A} = \int_A (-ax\hat{\imath} + by\hat{\jmath} + c\hat{k})\cdot(dy\,dz\,\hat{\imath} - dx\,dz\,\hat{\jmath}) = \int_A -ax\,dy\,dz - by\,dx\,dz$$

$$= \int_A -\frac{2}{3}ay\,dy\,dz + \int_A -\frac{3}{2}bx\,dx\,dz = -\int_{z=0}^{z_1}\int_{y=0}^{y_1}\frac{2}{3}ay\,dy\,dz - \int_{z=0}^{z_1}\int_{x=0}^{x_1}\frac{3}{2}bx\,dx\,dz$$

$$= -\int_{z=0}^{z_1}\frac{1}{3}ay_1^2\,dz - \int_{z=0}^{z_1}\frac{3}{4}bx_1^2\,dz = -\frac{1}{3}ay_1^2z_1 - \frac{3}{4}bx_1^2z_1$$

$$\int_A \vec{V}\cdot d\vec{A} = -\tfrac{1}{3}\left(\tfrac{1}{s}\right)(3)^2m^2\,2m - \tfrac{3}{4}\left(\tfrac{1}{s}\right)(2)^2m^2\,2m = -12.0\,m^3/s$$ {a scalar} ← $\int \vec{V}\cdot d$

$$\int_A \vec{V}(\vec{V}\cdot d\vec{A}) = \int_A (-ax\hat{\imath} + by\hat{\jmath} + c\hat{k})\left[(-ax\hat{\imath} + by\hat{\jmath} + c\hat{k})\cdot(dy\,dz\,\hat{\imath} - dx\,dz\,\hat{\jmath})\right]$$

$$= \int_A (-ax\hat{\imath} + by\hat{\jmath} + c\hat{k})(-ax\,dy\,dz - by\,dx\,dz)$$

Substituting $y = \frac{3}{2}x$ and $dy = \frac{3}{2}dx$, and noting that $a = b$,

$$\int_A \vec{V}(\vec{V}\cdot d\vec{A}) = \int_A (-ax\hat{\imath} + \tfrac{3}{2}ax\hat{\jmath} + c\hat{k})(-3ax\,dx\,dz)$$

$$= \int_{z=0}^{z_1}\int_{x=0}^{x_1}(3a^2x^2\hat{\imath} - \tfrac{9}{2}a^2x^2\hat{\jmath} - 3acx\,\hat{k})\,dx\,dz$$

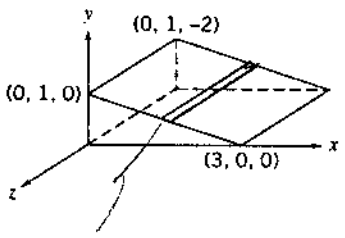
$$= \int_{z=0}^{z_1}(a^2x_1^3\hat{\imath} - \tfrac{3}{2}a^2x_1^3\hat{\jmath} - \tfrac{3}{2}acx_1^2\,\hat{k})\,dz$$

$$= a^2x_1^3z_1\,\hat{\imath} - \tfrac{3}{2}a^2x_1^3z_1\,\hat{\jmath} - \tfrac{3}{2}acx_1^2z_1\,\hat{k}$$

$$= \left(\tfrac{1}{s}\right)^2(2)^3m_x^3\,2m\,\hat{\imath} - \tfrac{3}{2}\left(\tfrac{1}{s}\right)^2(2)^3m_x^3\,2m\,\hat{\jmath} - \tfrac{3}{2}\left(\tfrac{1}{s}\right)\left(1\,\tfrac{m}{s}\right)(2)^2m_x^2\,2m\,\hat{k}$$

$$\int_A \vec{V}(\vec{V}\cdot d\vec{A}) = 16\hat{\imath} - 24\hat{\jmath} - 12\hat{k}\ \ m^4/s^2$$ {result is a vector} ← $\int \vec{V}(\vec{V}\cdot d$

Given: Velocity field $\vec{V} = ay\hat{i} + b\hat{j}$ where $a = 2\,s^{-1}$, $b = 1\,ft/s$, and dimensions are in feet



$d\vec{A} = \hat{i}\,dA_x + \hat{j}\,dA_y$

Find: (a) Volume flow rate, and
(b) momentum flux through the inclined surface

Solution:

The volume flow rate is $Q = \int \vec{V} \cdot d\vec{A}$

For the inclined surface $d\vec{A} = dA_x\hat{i} + dA_y\hat{j} = c\,dy\,\hat{i} + c\,dx\,\hat{j}$ where $c = 2\,ft$

$$Q = \int_A (ay\hat{i} + b\hat{j}) \cdot (c\,dy\,\hat{i} + c\,dx\,\hat{j}) = \int_{y=0}^{1} acy\,dy + \int_{0}^{3} bc\,dx$$

$$Q = ac\frac{y^2}{2}\Big]_0^1 + bcx\Big]_0^3 = 2\,s^{-1} \times 2\,ft \times 1\,ft^2 + 1\,\frac{ft}{s} \times 2\,ft \times 3\,ft$$

$$Q = 8\,ft^3/s$$ Volume flow rate

The momentum flux is given by $m.f. = \int \vec{V}(\rho \vec{V} \cdot d\vec{A})$

Thus

$$m.f. = \int_A (ay\hat{i} + b\hat{j})\,\rho\,(acy\,dy + bc\,dx)$$

$$m.f. = \rho\hat{i}\left[\int_0^2 acy^2\,dy + \int_0^3 abcy\,dx\right] + \rho\hat{j}\left[\int_0^3 abcy\,dy + \int_0^3 b^2c\,dx\right]$$

Along the inclined surface $y = -\frac{1}{3}x + 1$

$$m.f. = \rho\hat{i}\left\{a^2c\frac{y^3}{3}\Big]_0^1 + \int_0^3 abc\left(-\frac{x}{3}+1\right)dx\right\} + \rho\hat{j}\left\{abc\frac{y^2}{2}\Big]_0^1 + b^2cx\Big]_0^3\right\}$$

$$= \rho\hat{i}\left\{(2s^{-1})^2 \times 2ft \times \frac{ft^3}{3} + \left[abc\left(-\frac{x^2}{6}+x\right)\right]_0^3\right\} + \rho\hat{j}\left\{2s^{-1}\,\frac{ft}{s} \times 2ft \times \frac{ft^2}{2} + 1\frac{ft^2}{s^2} \times 2ft \times 3ft\right\}$$

$$= \rho\hat{i}\left\{\frac{8}{3}\frac{ft^4}{s^2} + \left[2s^{-1} \times \frac{ft}{s} \times 2ft \times 2.5\,ft^2\right]\right\} + \rho\hat{j}\left\{2\frac{ft^4}{s^2} + 6\frac{ft^4}{s^2}\right\}$$

$$m.f. = \rho\left\{7.67\hat{i} + 8\hat{j}\right\}\frac{ft^4}{s^2}$$ Momentum flux

Note: Units of ρ are $slug/ft^3$
Momentum flux has units of $slug \cdot ft/s^2$ (or lbf)

Given: Velocity distribution for laminar flow in a long circular tube

$$\vec{V} = u\hat{i} = u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i}$$

where R is the tube radius.

Evaluate: (a) the volume flow rate, and (b) the momentum flux, through a section normal to the pipe axis.

Solution: The volume flow rate is given by

$$\int_{A_{tube}} \vec{V}\cdot d\vec{A} = \int_0^R u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i}\cdot 2\pi r\,dr\,\hat{i} \qquad \{A = \pi r^2,\ dA = 2\pi r\,dr\}$$

$$= u_{max}\,2\pi\int_0^R \left[1 - \left(\frac{r}{R}\right)^2\right] r\,dr = u_{max}\,2\pi\int_0^R \left[r - \frac{r^3}{R^2}\right] dr$$

$$= u_{max}\,2\pi\left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R = u_{max}\,2\pi\left[\frac{R^2}{2} - \frac{R^2}{4}\right]$$

$$\int_{A_{tube}} \vec{V}\cdot d\vec{A} = \frac{1}{2}u_{max}\pi R^2$$ ← volume flow rate

The momentum flux is given by

$$\int_{A_{tube}} \vec{V}(\vec{V}\cdot d\vec{A}) = \int_0^R u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i}\left\{u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i}\cdot 2\pi r\,dr\,\hat{i}\right\}$$

$$= \int_0^R u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i}\left\{u_{max}\,2\pi\left[r - \frac{r^3}{R^2}\right] dr\right\}$$

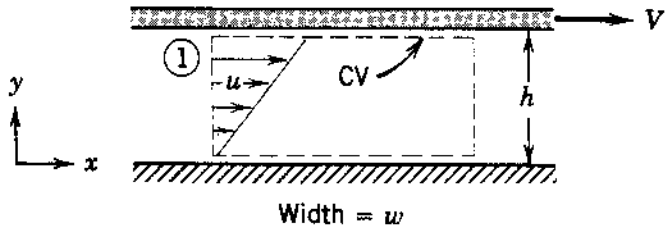
$$= u_{max}^2\,2\pi\int_0^R \left(r - \frac{2r^3}{R^2} + \frac{r^5}{R^4}\right) dr\,\hat{i}$$

$$= u_{max}^2\,2\pi\left[\frac{r^2}{2} - \frac{r^4}{2R^2} + \frac{r^6}{6R^4}\right]_0^R \hat{i}$$

$$= u_{max}^2\,2\pi R^2\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right]\hat{i}$$

$$\int_{A_{tube}} \vec{V}(\vec{V}\cdot d\vec{A}) = \frac{1}{3}u_{max}^2\pi R^2\,\hat{i}$$ ← momentum flux

Given: Flow and CV of Problem 4.11, as shown



Find: Expression for kinetic energy flux through cross-section ① of CV.

Solution: Kinetic energy flux is defined as $ke_f = \int_A \frac{V^2}{2}\rho\vec{V}\cdot d\vec{A}$

Model the velocity profile as $u = V\frac{y}{h}$. Then

$$\vec{V} = u\hat{\imath} = V\frac{y}{h}\hat{\imath}\ ;\quad V^2 = V^2\left(\frac{y}{h}\right)^2$$

Since flow is into the CV, $\vec{V}\cdot d\vec{A} = -u\,dA = -V\frac{y}{h}w\,dy$

Substituting,

$$ke_f = \int_A \frac{V^2}{2}\left(\frac{y}{h}\right)^2\left\{-\rho V\frac{y}{h}w\,dy\right\} = -\frac{\rho V^3 w}{2h^3}\int_0^h y^3\,dy$$

$$= -\frac{\rho V^3 w}{2h^3}\left[\frac{y^4}{4}\right]_0^h$$

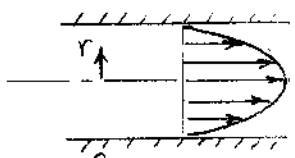
$$ke_f = -\rho V^3 \frac{wh}{8}$$ ke_f

Check dimensions:

$$[ke_f] = \frac{M}{L^3}\left(\frac{L}{t}\right)^3 LL = \frac{ML^2}{t^3}\times\frac{Ft^2}{ML} = \frac{FL}{t} = \frac{\text{Energy}}{\text{Time}}\quad \text{vv}$$

Given: Velocity profile in a circular tube,

$$\vec{V} = u\hat{i} = u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]\hat{i}$$



Find: Expression for kinetic energy flux, $ke_f = \int \frac{V^2}{2}\rho\vec{V}\cdot d\vec{A}$

Solution: $V^2 = \vec{V}\cdot\vec{V} = u_{max}^2\left[1-\left(\frac{r}{R}\right)^2\right]^2 = u_{max}^2\left[1-2\left(\frac{r}{R}\right)^2+\left(\frac{r}{R}\right)^4\right]$

$$d\vec{A} = 2\pi r\,dr\,\hat{i}$$



$$\vec{V}\cdot d\vec{A} = 2\pi r\,u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]$$

Then

$$ke_f = \int_0^R \frac{u_{max}^2}{2}\left[1-2\left(\frac{r}{R}\right)^2+\left(\frac{r}{R}\right)^4\right]\rho 2\pi r\,u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]dr$$

$$= \pi\rho u_{max}^3\int_0^R\left[1-3\left(\frac{r}{R}\right)^2+3\left(\frac{r}{R}\right)^4-\left(\frac{r}{R}\right)^6\right]r\,dr$$

$$= \pi\rho u_{max}^3 R^2\int_0^1\left[1-3\left(\frac{r}{R}\right)^2+3\left(\frac{r}{R}\right)^4-\left(\frac{r}{R}\right)^6\right]\frac{r}{R}\,d\left(\frac{r}{R}\right)$$

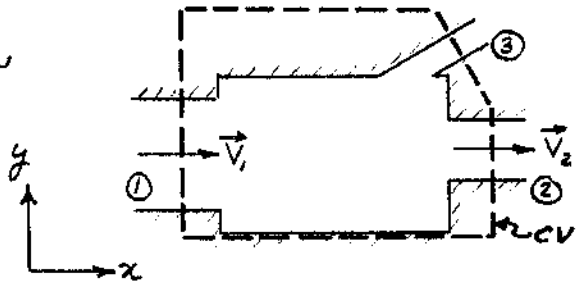
$$= \pi\rho u_{max}^3 R^2\left[\frac{1}{2}\left(\frac{r}{R}\right)^2-\frac{3}{4}\left(\frac{r}{R}\right)^4+\frac{1}{2}\left(\frac{r}{R}\right)^6-\frac{1}{8}\left(\frac{r}{R}\right)^8\right]_0^1$$

$$= \pi R^2\rho u_{max}^3\left[\frac{1}{2}-\frac{3}{4}+\frac{1}{2}-\frac{1}{8}\right]$$

$$ke_f = \frac{\pi R^2\rho u_{max}^3}{8}$$

ke_f ←

Given: Steady, incompressible flow through device shown.



$A_1 = 1\,ft^2$; $A_2 = 0.5\,ft^2$; $A_3 = 0.2\,ft^2$

$\vec{V}_1 = 10\hat{i}\ ft/s$, $\vec{V}_2 = 30\hat{i}\ ft/s$

Find: Volume flow rate through port 3.

Solution: Apply conservation of mass to CV shown

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ ($\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall = 0$ (1))

Assumptions: (1) Steady flow
(2) Uniform flow at each section
(3) Incompressible flow, ρ = constant

Then $0 = \vec{V}_1\cdot\vec{A}_1 + \vec{V}_2\cdot\vec{A}_2 + \vec{V}_3\cdot\vec{A}_3$

or $0 = -|V_1 A_1| + |V_2 A_2| + \vec{V}_3\cdot\vec{A}_3$ (flow in at ①, out at ②)

Solving,

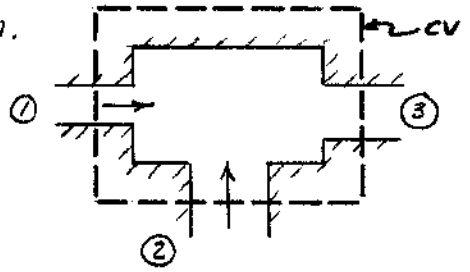
$$\vec{V}_3\cdot\vec{A}_3 = |V_1 A_1| - |V_2 A_2|$$

$$\vec{V}_3\cdot\vec{A}_3 = \left|10\tfrac{ft}{s} \times 1\,ft^2\right| - \left|30\tfrac{ft}{s} \times 0.5\,ft^2\right| = -5.00\tfrac{ft^3}{s}$$

Therefore

$Q_3 = \vec{V}_3\cdot\vec{A}_3 = -5.00\,ft^3/s$ (minus sign means <u>into</u> CV) Q_3

Given: Water flow through device shown. Flow is uniform at each port.



$A_1 = 0.1\ m^2;\ A_2 = 0.2\ m^2;\ A_3 = 0.15\ m^2$

$V_1 = 5\ \frac{m}{s}$, $V_2 = 10 + 5\cos(4\pi t)\ \frac{m}{s}$

Find: Expression for mass flow rate at section ③.

Solution: Apply continuity using CV shown.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, dV + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ (first term $= 0$ (1))

Assumptions: (1) Incompressible flow (note volume of CV is fixed)
(2) Uniform flow at each section

Then

$$0 = \rho\vec{V}_1\cdot\vec{A}_1 + \rho\vec{V}_2\cdot\vec{A}_2 + \dot{m}_3 = -|\rho V_1 A_1| - |\rho V_2 A_2| + \dot{m}_3$$

or using magnitudes,

$$\dot{m}_3 = \rho(V_1 A_1 + V_2 A_2)$$

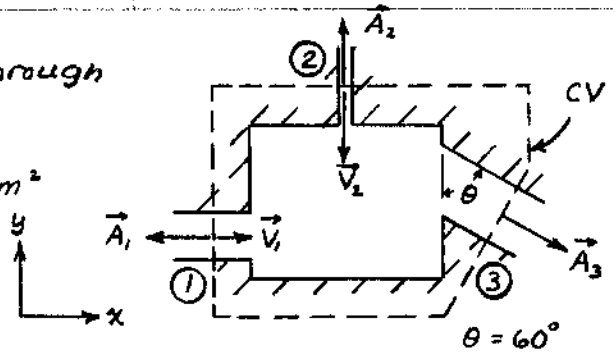
$$= 999\ \frac{kg}{m^3}\left[5\ \frac{m}{s} \times 0.1\ m^2 + (10 + 5\cos(4\pi t))\ \frac{m}{s} \times 0.2\ m^2\right]$$

$$\dot{m}_3 = 2500 + 999\cos(4\pi t)\ kg/s$$ $\dot{m}_3$

Given: Steady, incompressible flow through the device shown.

$A_1 = 0.05 m^2$, $A_2 = 0.01 m^2$, $A_3 = 0.06 m^2$

$\vec{V}_1 = 4\hat{\imath}\ m/s$, $\vec{V}_2 = -8\hat{\jmath}\ m/s$



Find: Velocity, $\vec{V}_3$

Solution: Apply conservation of mass, using CV shown.

Basic equation: $0 = \frac{d}{dt}\int_{CV} \rho\, d\forall + \int_{CS} \rho\, \vec{V}\cdot d\vec{A}$ ($\frac{d}{dt}\int_{CV} \rho\, d\forall = 0$ (1))

Assumptions: (1) Steady flow
(2) Incompressible flow, ρ = constant
(3) Uniform flow at each section

Then

$$\int_{CS} \vec{V}\cdot d\vec{A} = \vec{V}_1\cdot\vec{A}_1 + \vec{V}_2\cdot\vec{A}_2 + \vec{V}_3\cdot\vec{A}_3 = 0$$

or

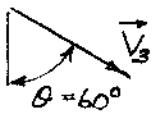
$$\vec{V}_3\cdot\vec{A}_3 = -\vec{V}_1\cdot\vec{A}_1 - \vec{V}_2\cdot\vec{A}_2 = -4\hat{\imath}\frac{m}{s}\cdot 0.05(-\hat{\imath})m^2 - (-8\hat{\jmath})\frac{m}{s}\times 0.01\hat{\jmath}\,m^2$$

$$\vec{V}_3\cdot\vec{A}_3 = 0.28\ m^3/s$$

Since $\vec{V}_3\cdot\vec{A}_3 > 0$, flow at section ③ is out of CV. Thus $\vec{V}_3\cdot\vec{A}_3 = V_3 A_3$

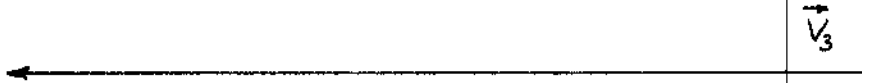
$$V_3 = \frac{1}{A_3}\times 0.28\frac{m^3}{s} = \frac{1}{0.06 m^2}\times 0.28\frac{m^3}{s} = 4.67\ m/s$$

Finally, from the geometry of the sketch,



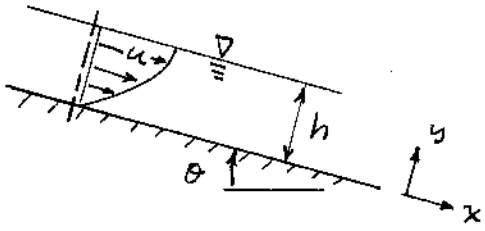
$$\vec{V}_3 = V_3\sin\theta\,\hat{\imath} - V_3\cos\theta\,\hat{\jmath} = 4.67\frac{m}{s}\times\sin 60°\,\hat{\imath} - 4.67\frac{m}{s}\times\cos 60°\,\hat{\jmath}$$

$$\vec{V}_3 = 4.04\hat{\imath} - 2.34\hat{\jmath}\ m/s$$



Given: Oil flow down inclined plane.

$$u = \frac{\rho g \sin\theta}{\mu}\left(hy - \frac{y^2}{2}\right)$$



Find: Mass flow rate per unit width.

Solution: At the dashed cross-section, $\dot{m} = \int \rho u \, dA$

$dA = w\,dy$, where w = width

$$\dot{m} = \int_0^h \rho \frac{\rho g \sin\theta}{\mu}\left(hy - \frac{y^2}{2}\right) w \, dy = \frac{\rho^2 g \sin\theta}{\mu} \int_0^h \left(hy - \frac{y^2}{2}\right) w \, dy$$

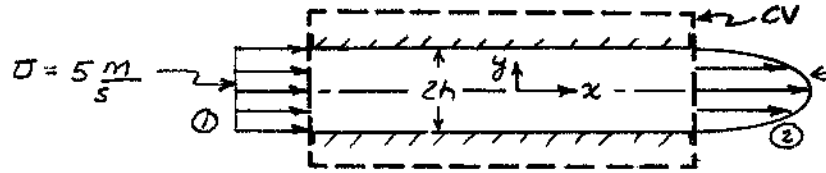
$$\dot{m} = \frac{\rho^2 g \sin\theta}{\mu} w \left[\frac{hy^2}{2} - \frac{y^3}{6}\right]_0^h = \frac{\rho^2 g \sin\theta w}{\mu} \frac{h^3}{3} = \frac{\rho^2 g \sin\theta \, w h^3}{3\mu}$$

Thus

$$\dot{m}/w = \frac{\rho^2 g \sin\theta \, h^3}{3\mu}$$

$\dot{m}/w$

Given: Water flow between parallel plates as shown.



Find: Exit centerline velocity, U_{max}.

Solution: Apply continuity using the CV shown.

Basic equation: $0 = \underbrace{\frac{\partial}{\partial t}}_{=0(1)}\int_{CV}\rho\,d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at inlet section

Then

$$0 = \vec{V}_1\cdot\vec{A}_1 + \int_{(2)}\vec{V}_2\cdot d\vec{A}_2 \;;\; \vec{V}_2 = u\hat{i},\; d\vec{A}_2 = w\,dy\,\hat{i}\;(w=\text{width})$$

$$0 = -U(2hw) + \int_{-h}^{h} u_{max}\left[1-(\frac{y}{h})^2\right]w\,dy$$

or

$$U = \frac{1}{2h}\int_{-h}^{h} u_{max}\left[1-(\frac{y}{h})^2\right]dy = \frac{u_{max}}{2}\int_{-1}^{1}\left[1-(\frac{y}{h})^2\right]d(\frac{y}{h})$$

$$U = u_{max}\int_{0}^{1}\left[1-(\frac{y}{h})^2\right]d(\frac{y}{h}) = u_{max}\left[(\frac{y}{h}) - \frac{1}{3}(\frac{y}{h})^3\right]_0^1 = \frac{2}{3}u_{max}$$

Thus

$$u_{max} = \frac{3}{2}U = \frac{3}{2}\times 5\frac{m}{s} = 7.50\ m/s$$

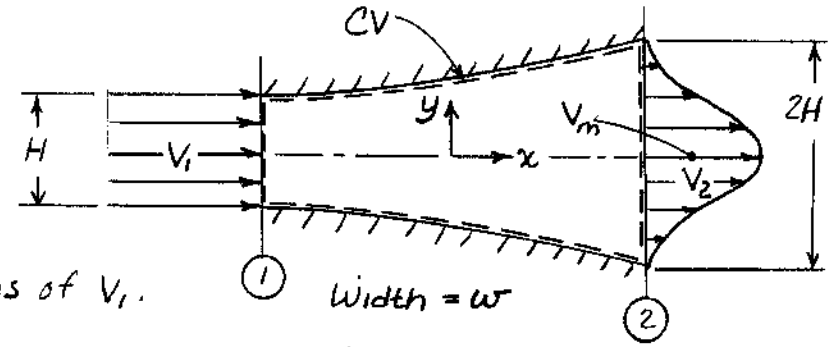
u_{ma}

{ The maximum speed at the outlet section is 3/2 that of the uniform flow speed at the inlet section. }

Given: Incompressible flow in a diverging channel, as shown.

V_1 = constant

$V_2 = V_m \cos\left(\frac{\pi y}{2H}\right)$



Find: Express V_m in terms of V_1.

Solution: Apply conservation of mass using the CV shown.

Basic equation: $0 = \underbrace{\frac{d}{dt}\int_{CV}\rho\, d\forall}_{=0\,(1)} + \int_{CS}\rho\vec{V}\cdot d\vec{A}$

Assumptions: (1) Steady flow
(2) Uniform flow at section 1
(3) Incompressible flow

Then $0 = \{-\rho V_1 A_1\} + \int_{-H}^{H}\rho V_2\, w\, dy$

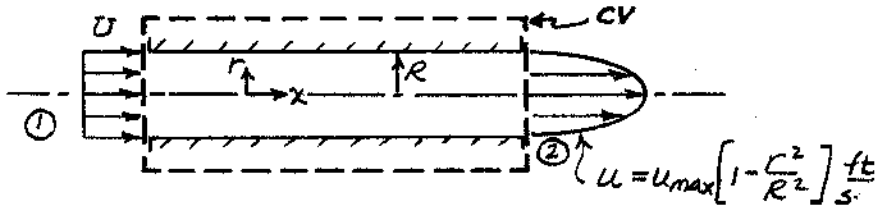
Since $A_1 = wH$, then $V_1 wH = \int_{-H}^{H} V_m \cos\left(\frac{\pi}{2}\frac{y}{H}\right) w\, dy = 2\int_0^H V_m \cos\left(\frac{\pi}{2}\frac{y}{H}\right) w\, dy$

so $V_1 H = 2V_m\left(\frac{2H}{\pi}\right)\int_0^H \cos\left(\frac{\pi}{2}\frac{y}{H}\right) d\left(\frac{\pi}{2}\frac{y}{H}\right) = \frac{4V_m H}{\pi}\left[\sin\left(\frac{\pi}{2}\frac{y}{H}\right)\right]_0^H = \frac{4V_m H}{\pi}$

Thus $V_m = \frac{\pi}{4}V_1$

V_m

Given: Water flow in a pipe as shown. $R = 3$ in. $u_{max} = 10$ ft/s



Find: Uniform inlet velocity, V.

Solution: Apply continuity using the CV shown.

Basic equation: $0 = \cancel{\frac{\partial}{\partial t}}^{=0(1)} \int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at inlet section

Then

$$0 = \vec{V}_1 \cdot \vec{A}_1 + \int_{(2)} \vec{V}_2 \cdot d\vec{A}_2 \quad ; \quad \vec{V}_2 = u\hat{i} \, , \; d\vec{A}_2 = 2\pi r \, dr \, \hat{i}$$

$$0 = -V\pi R^2 + \int_0^R u_{max}\left[1 - \frac{r^2}{R^2}\right] 2\pi r \, dr$$

or

$$V = \frac{1}{\pi R^2} \int_0^R u_{max}\left[1 - \frac{r^2}{R^2}\right] 2\pi r \, dr = 2u_{max} \int_0^1 \left[1 - \left(\frac{r}{R}\right)^2\right]\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right)$$

$$V = 20\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right]_0^1 = 5.00 \text{ ft/s}$$

V

{ The speed of the uniform inlet flow is half the maximum speed at the outlet section. }

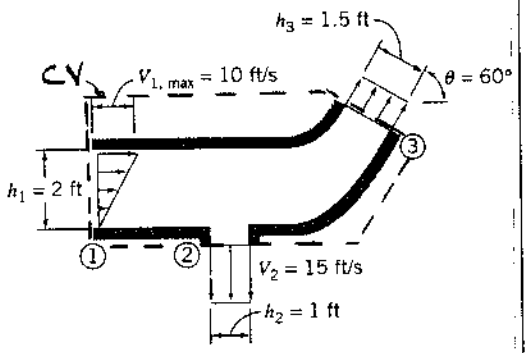
Given: Two-dimensional reducing bend as shown.

Find: Magnitude and direction of uniform velocity at section ③.

Solution: Apply conservation of mass using CV shown.

Basic equation:

$$0 = \underbrace{\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall}_{=0(1)} + \int_{CS}\rho\vec{V}\cdot d\vec{A}$$



Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at ② and ③

Then

$$0 = \int_{CS}\vec{V}\cdot d\vec{A} = \int_{A_1}\vec{V}_1\cdot d\vec{A}_1 + \vec{V}_2\cdot\vec{A}_2 + \vec{V}_3\cdot\vec{A}_3$$

or

$$\vec{V}_3\cdot\vec{A}_3 = -\int_{A_1}\vec{V}_1\cdot d\vec{A}_1 - \vec{V}_2\cdot\vec{A}_2 = +\int_0^{h_1} V_{1,max}\frac{y}{h_1}\,w\,dy - V_2 w h_2$$

$$\vec{V}_3\cdot\vec{A}_3 = V_{1,max}\,w\left[\frac{y^2}{2h_1}\right]_0^{h_1} - V_2 w h_2 = \frac{V_{1,max}\,w h_1}{2} - V_2 w h_2$$

so

$$\frac{\vec{V}_3\cdot\vec{A}_3}{w} = \frac{1}{2}\times 10\frac{ft}{s}\times 2ft - 15\frac{ft}{s}\times 1ft = -5\ ft^2/s$$

Since $\vec{V}_3\cdot\vec{A}_3 < 0$, flow at ③ is into the CV ← Direction

Thus

$$\frac{\vec{V}_3\cdot\vec{A}_3}{w} = -\frac{V_3 A_3}{w} = -\frac{V_3 w h_3}{w} = -V_3 h_3 = -5\ ft^2/s$$

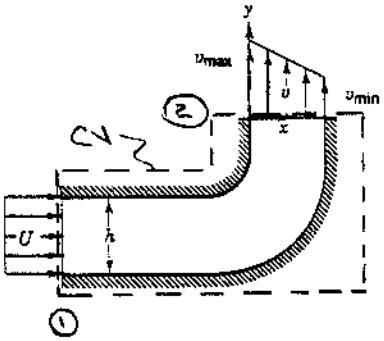
$$V_3 = \frac{1}{h_3}\times 5\frac{ft^2}{s} = \frac{1}{1.5\ ft}\times 5\frac{ft^2}{s} = 3.33\ ft/s \quad (\text{into CV})$$ ← V_3

Given: Water flow in the two-dimensional square channel shown.

$v_{max} = 2v_{min}$, $U = 7.5$ m/s , $h = 75.5$ mm

Find: v_{min}

Solution: Apply conservation of mass to the CV shown.



Basic equation:

$$0 = \cancel{\frac{\partial}{\partial t}}^{0(1)} \int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) steady flow
(2) incompressible flow
(3) uniform flow at section (1)

Then

$$0 = \vec{V}_1 \cdot \vec{A}_1 + \int \vec{V}_2 \cdot dA_2$$

$$0 = -U w h + \int_0^h v w \, dx$$

The velocity distribution across the exit at (2) is linear

$$v_2 = v_{max} - (v_{max} - v_{min}) \frac{x}{h} = 2v_{min} - v_{min} \frac{x}{h} = v_{min}\left(2 - \frac{x}{h}\right)$$

$$\therefore U w h = \int_0^h v_{min}\left(2 - \frac{x}{h}\right) w \, dx = v_{min} w \left[2x - \frac{x^2}{2h}\right]_0^h$$

$$U \cancel{w} \cancel{h} = v_{min} w \left[2h - \frac{h}{2}\right] = \frac{3}{2} v_{min} \cancel{w} \cancel{h}$$

$$\therefore v_{min} = \frac{2}{3} U = \frac{2}{3} \times 7.5 \frac{m}{s} = 5.0 \text{ m/s} \longleftarrow v_{min}$$

Given: Water flows in a porous round tube of diameter $D = 60\,mm$. At the pipe inlet the flow is uniform with $V_1 = 7.0\ m/sec$. Flow out through the porous wall is radial and axisymmetric with velocity distribution

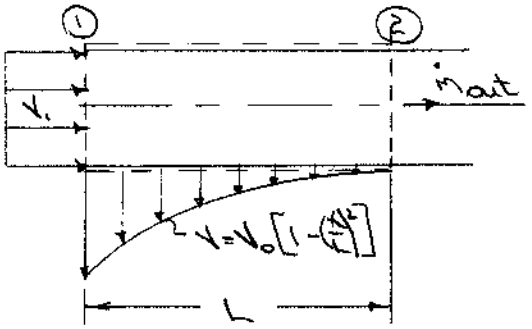
$$v = V_0\left[1 - \left(\frac{x}{L}\right)^2\right]$$

where $V_0 = 0.03\ m/s$ and $L = 0.950\,m$

Find: the mass flow rate, $\dot{m}_2$, inside the tube at $x = L$

Solution:

Basic equation: $0 = \cancel{\frac{\partial}{\partial t}\int_{CV}\rho\,dV}^{\,0(1)} + \int_{CS}\rho\vec{V}\cdot d\vec{A}$



Assumptions: (1) steady flow
(2) ρ = constant

Then

$$0 = \int_{A_1}\rho\vec{V}\cdot d\vec{A} + \int_{A_2}\rho\vec{V}\cdot d\vec{A} + \int_{A_{surf}}\rho\vec{V}\cdot d\vec{A}$$

$$= -|\rho V_1 A_1| + \dot{m}_2 + \int_0^L \rho V_0\left[1 - \left(\frac{x}{L}\right)^2\right] 2\pi R\,dx$$

$$\dot{m}_2 = \rho V_1 A_1 - 2\pi R\rho V_0\int_0^L\left[1 - \frac{x^2}{L^2}\right]dx$$

$$= \rho V_1\frac{\pi}{4}D^2 - 2\pi R\rho V_0\left[x - \frac{x^3}{3L^2}\right]_0^L$$

$$= \frac{\pi}{4}\rho V_1 D^2 - \frac{4}{3}\pi R\rho V_0 L$$

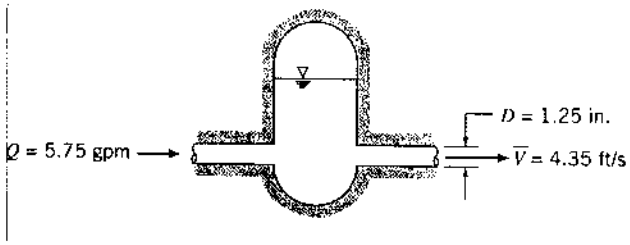
$$\dot{m}_2 = \frac{\pi}{4}\times 999\frac{kg}{m^3}\times 7.0\frac{m}{s}\times(0.06)^2 m^2 - \frac{4}{3}\pi\times 0.03m\times 999\frac{kg}{m^3}\times 0.03\frac{m}{s}\times 0.95m$$

$$\dot{m}_2 = 19.8\frac{kg}{s} - 3.6\frac{kg}{s} = 16.2\ kg/s \longleftarrow \dot{m}_{out}$$

Given: A hydraulic accumulator, designed to reduce pressure pulsations in a hydraulic system, is operating under conditions shown, at a given instant.

Find: Rate at which accumulator gains or loses hydraulic oil.

Solution:



Use the control volume shown

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{cv} \rho\, d\forall + \int_{cs} \rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) uniform flow at section ②
(2) ρ = constant

Then,

$$0 = \frac{\partial}{\partial t}(M_{cv}) + \int_{A_1}\{-1\,\rho V_1\, dA_1\} + \int_{A_2}\{1\,\rho V_2\, dA_2\}$$

But $\int_{A_1} \rho V_1\, dA_1 = \rho Q_1$ where Q = volume flowrate and $\rho = SG\,\rho_{H_2O}$

So $0 = \frac{\partial}{\partial t} M_{cv} - \rho Q_1 + \rho V_2 A_2$

or

$$\frac{\partial M_{cv}}{\partial t} = \rho(Q_1 - V_2 A_2)$$

$$= SG\,\rho_{H_2O}\left(Q_1 - V_2 \pi \left(\frac{D}{12}\right)^2\right) \quad \text{where } SG = 0.88 \text{ (Table A.2)}$$

$$= 0.88 \times 1.94 \frac{slug}{ft^3}\left[5.75 \frac{gal}{min} \times \frac{ft^3}{7.48 gal} \times \frac{min}{60 s} - 4.35 \frac{ft}{s} \times \frac{\pi}{4} \times (1.25)^2 in^2 \times \frac{ft^2}{144 in^2}\right]$$

$$\frac{\partial M_{cv}}{\partial t} = -4.14 \times 10^{-2} \frac{slug}{s} \quad \text{or} \quad -1.33 \frac{lbm}{s} \longleftarrow \frac{\partial M_{cv}}{\partial t}$$

(mass is decreasing in the CV)

Since $M_{cv} = \rho_{oil} \forall_{oil}$

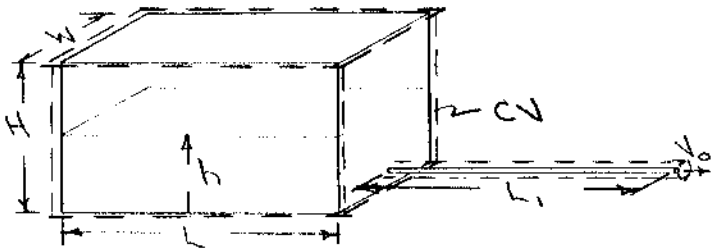
$$\frac{\partial M_{cv}}{\partial t} = \frac{\partial}{\partial t}(\rho_{oil} \forall_{oil}) = \rho_{oil} \frac{\partial \forall_{oil}}{\partial t} = SG_{oil}\,\rho_{H_2O} \frac{\partial \forall_{oil}}{\partial t}$$

$$\frac{\partial \forall_{oil}}{\partial t} = \frac{1}{SG_{oil}\,\rho_{H_2O}} \frac{\partial M_{cv}}{\partial t} = \frac{1}{0.88 \times 1.94 \frac{slugs}{ft^3}} \times (-4.14) \times 10^{-2} \frac{slug}{s}$$

$$\frac{\partial \forall_{oil}}{\partial t} = -2.43 \times 10^{-2} \frac{ft^3}{s} \quad \text{or} \quad 0.181 \frac{gal}{s} \longleftarrow \frac{\partial \forall_{oil}}{\partial t}$$

Given: Rectangular tank with dimensions $H = 230\,mm$, $W = 150\,mm$, $L = 230\,mm$, supplies water to an outlet tube of diameter, $D = 6.35\,mm$. When the tank is half full the flow in the tube is at Reynolds number $Re = 2000$. At this instant there is no water flow into the tank.

Find: the rate of change of water level in the tank at this instant.



Solution:

Apply conservation of mass to CV which includes tank and tube.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, dV + \int_{CS} \rho \vec{V}\cdot d\vec{A}$

Definition: $Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu}$

Assumptions: (1) uniform flow at exit of tube
(2) incompressible flow
(3) neglect air entering the control volume

Then, $0 = \frac{\partial}{\partial t}\left[\rho W L h + \rho \frac{\pi D^2}{4} L_1\right] + \left\{+1\left(\rho \bar{V}_o \frac{\pi D^2}{4}\right)\right\}$

$0 = WL\frac{dh}{dt} + \bar{V}_o \frac{\pi D^2}{4}$ (note L_1 = constant)

$\therefore \frac{dh}{dt} = -\frac{\bar{V}_o}{4}\frac{\pi D^2}{WL}$

To find $\bar{V}$ use the definition of Re

$\bar{V}_o = \frac{Re\,\nu}{D}$

For water at 20C $\nu = 1\times10^{-6}\ m^2/sec$ (Table A.8)

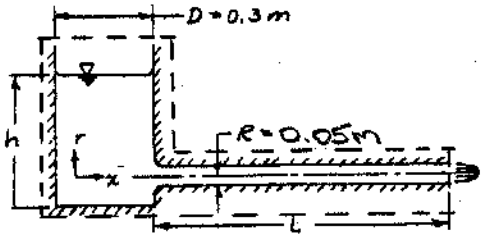
$V_o = 2000 \times 1\times10^{-6}\ \frac{m^2}{sec} \times \frac{1}{6.35\times10^{-3}\ m} = 0.315\ m/sec$

$\frac{dh}{dt} = -\frac{\bar{V}_o}{4}\frac{\pi D^2}{WL} = -\frac{0.315}{4}\frac{m}{sec} \times \frac{\pi (6.35)^2\ mm^2}{150\,mm \times 230\,mm} \times 10^3\ \frac{mm}{m}$

$\frac{dh}{dt} = -0.289\ mm/sec$ (falling) $\frac{dh}{dt}$

Given: Liquid drains from a tank through a long circular tube. Flow is laminar; velocity profile at tube discharge is given by

$$u = u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$



Find: (a) Show that $\bar{V} = 0.5\, u_{max}$ at any instant
(b) rate of change of liquid level in tank when $u_{max} = 0.155$ m/s

Solution:

(a) The average velocity $\bar{V}$ is defined as Q/A.

Since $Q = \int u\, dA$, $dA = 2\pi r\, dr$ and $A = \pi R^2$, then

$$\bar{V} = \frac{Q}{A} = \frac{1}{\pi R^2}\int_0^R u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right] 2\pi r\, dr = \frac{2u_{max}}{R^2}\int_0^R \left[1 - \left(\frac{r}{R}\right)^2\right] r\, dr$$

$$\bar{V} = \frac{2u_{max}}{R^2} R^2 \int_0^1 \left[1 - \left(\frac{r}{R}\right)^2\right]\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) = 2u_{max}\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right]_0^1$$

$$\bar{V} = \frac{1}{2} u_{max}$$ ← $\bar{V}$

(b) Apply conservation of mass to the CV shown

Basic equation: $$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, dV + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) neglect air entering the CV
(2) incompressible flow

Then

$$0 = \rho_l \frac{\partial}{\partial t} V_{CV} + \{|\rho_l \bar{V} A_e|\} = \rho_l \frac{\partial}{\partial t}\left[\frac{\pi}{4}D^2 h + L\pi R^2\right] + \rho \bar{V} \pi R^2$$

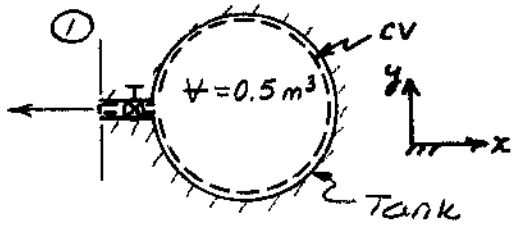
$$0 = \frac{\pi}{4}D^2 \frac{dh}{dt} + \bar{V}\pi R^2 \qquad \left(\text{note } \frac{dL}{dt} = 0\right)$$

$$\therefore \frac{dh}{dt} = -4\bar{V}\left(\frac{R}{D}\right)^2 \qquad \text{But } \bar{V} = \frac{1}{2} u_{max} \text{ and hence}$$

$$\frac{dh}{dt} = -2u_{max}\left(\frac{R}{D}\right)^2 = -2 \times 0.155 \frac{m}{s} \times \left(\frac{0.05\,m}{0.30\,m}\right)^2 \times 1000 \frac{mm}{m}$$

$$\frac{dh}{dt} = -8.61 \text{ mm/s} \quad \text{(level is falling)}$$ ← $\frac{dh}{dt}$

Given: Tank shown in sketch with air escaping.



$\vec{V}_1 = -300\,\hat{i}\ m/s$

$\vec{A}_1 = -130\,\hat{i}\ mm^2$

$T_1 = -15\,C$

$p_1 = 350\ kPa\ (abs)$

Find: Rate of change of density in tank.

Solution: Apply conservation of mass using CV shown

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$

Assume: (1) Density is uniform in tank, so $\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall = \frac{\partial}{\partial t}(\rho_t \forall)$

(2) Exit flow is uniform, so

$$\int_{CS} \rho \vec{V}\cdot d\vec{A} = \rho_1 \vec{V}_1 \cdot \vec{A}_1 = |\rho_1 V_1 A_1|$$

(3) Air is an ideal gas, $p_1 = \rho_1 R T_1$

Then $\rho_1 = \frac{p_1}{RT_1} = 3.50\times10^5 \frac{N}{m^2} \times \frac{kg\cdot K}{287\ N\cdot m} \times \frac{1}{(273-15)\ K} = 4.73\ kg/m^3$

$$\frac{\partial}{\partial t}(\rho_t \forall) = \rho_t \underbrace{\frac{\partial \forall}{\partial t}}_{0} + \forall \frac{\partial \rho_t}{\partial t} = -|\rho_1 V_1 A_1|$$

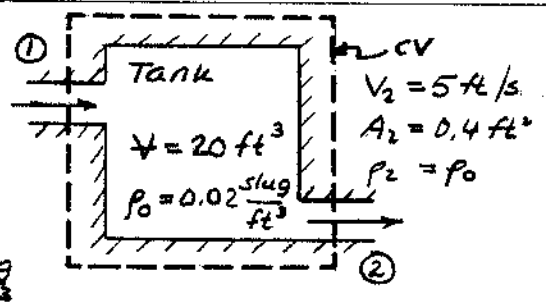
$$\frac{\partial \rho_t}{\partial t} = \frac{-|\rho_1 V_1 A_1|}{\forall} = -4.73\ \frac{kg}{m^3} \times 300\ \frac{m}{s} \times 130\ mm^2 \times \frac{1}{0.5\ m^3} \times \frac{m^2}{(1000)^2\ mm^2}$$

$$\frac{\partial \rho_t}{\partial t} = -0.369\ kg/m^3\cdot s$$

$\frac{\partial \rho_t}{\partial t}$ ←

{ Note since $\frac{\partial \rho_t}{\partial t} < 0$, mass in tank decreases as expected. }

Given: Air flow through tank with conditions shown at time, t_0.



$V_1 = 15$ ft/s
$A_1 = 0.2$ ft²
$\rho_1 = 0.03$ slug/ft³

Find: $\frac{\partial \rho}{\partial t}$ in tank at time, t_0.

Solution: Apply conservation of mass, using CV shown.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) Density is uniform in tank, so $\frac{\partial}{\partial t}\int_{CV} \rho d\forall = \frac{\partial}{\partial t}(\rho_0 \forall)$

(2) Flow is uniform at inlet and outlet sections.

Then

$$0 = \frac{\partial}{\partial t}(\rho_0 \forall) + \rho_1 \vec{V}_1 \cdot \vec{A}_1 + \rho_0 \vec{V}_2 \cdot \vec{A}_2$$

$$0 = \rho_0 \underbrace{\frac{\partial \forall}{\partial t}}_{=0} + \forall \frac{\partial \rho_0}{\partial t} - |\rho_1 V_1 A_1| + |\rho_0 V_2 A_2|$$

or

$$\frac{\partial \rho_0}{\partial t} = \frac{|\rho_1 V_1 A_1| - |\rho_0 V_2 A_2|}{\forall}$$

Substituting magnitudes

$$\frac{\partial \rho_0}{\partial t} = \frac{1}{20\ ft^3}\left[\frac{0.03\ slug}{ft^3} \times \frac{15\ ft}{s} \times 0.2\ ft^2 - \frac{0.02\ slug}{ft^3} \times \frac{5\ ft}{s} \times 0.4\ ft^2\right]$$

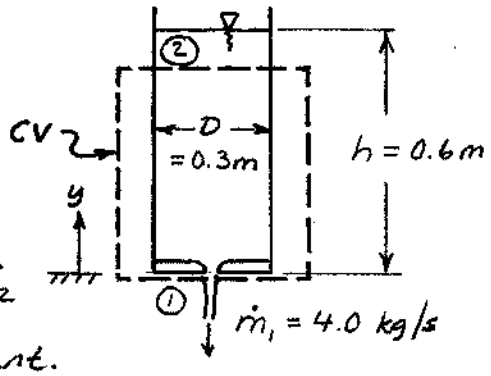
$$\frac{\partial \rho_0}{\partial t} = 2.50 \times 10^{-3}\ slug/(ft^3 \cdot s)$$

$\frac{\partial \rho_0}{\partial t}$

{Note since $\frac{\partial \rho_0}{\partial t} > 0$, mass in tank increases.}

Given: Circular tank, with D = 1 ft draining through a hole in its bottom. Fluid is water

Find: Rate of change of water level at the instant shown.



Solution: Apply conservation of mass to CV shown. Note section ② cuts below free surface, so $\vec{V}_2$ corresponds to free surface velocity; volume of CV is constant.

Basic equation: $0 = \underbrace{\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall}_{=0\,(1)} + \int_{CS} \rho \vec{V}\cdot d\vec{A}$

Assumptions: (1) Incompressible flow, so unsteady term is zero, since volume of CV is fixed
(2) Uniform flow at each section

Then

$$0 = \rho \vec{V}_1\cdot\vec{A}_1 + \rho\vec{V}_2\cdot\vec{A}_2 = \dot{m}_1 + \rho\vec{V}_2\cdot\vec{A}_2$$

and

$$\vec{V}_2\cdot\vec{A}_2 = -\frac{\dot{m}_1}{\rho} = -\frac{4.0\ kg}{s}\times\frac{m^3}{999\ kg} = -0.004\ m^3/s$$

Since $\vec{V}_2\cdot\vec{A}_2 < 0$, flow at section ② is <u>into</u> CV. Therefore

$$V_2 = \frac{|V_2 A_2|}{A_2} = \frac{0.004\ m^3}{s}\times\frac{4}{\pi}\times\frac{1}{(0.3)^2 m^2} = 0.0566\ m/s$$

The water level is <u>falling</u> at 56.6 mm/s.

$$\vec{V}_s = -V_2\downarrow = -56.6\downarrow\ mm/s$$

$\vec{V}_s$

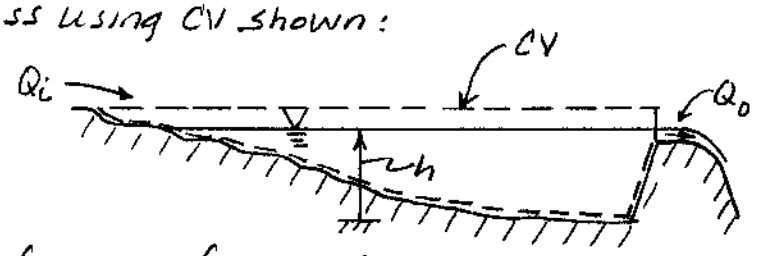
Given: Lake being drained at 2,000 cubic feet per second (cfs). Level falls at 1 ft per 8 hr. Normal flow rate is 290 cfs.

Find: (a) Actual flow rate during draining (gal/s).
(b) Estimate surface area of lake.

Solution: Convert units

$$Q = 2000 \frac{ft^3}{s} = 2000 \frac{ft^3}{s} \times 7.48 \frac{gal}{ft^3} = 1.50 \times 10^4 \, gal/s$$ Q

Apply conservation of mass using CV shown:



Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumption: (1) ρ = constant

Then $$\frac{d\forall}{dt} = A\frac{dh}{dt} = -\int_{CS} \vec{V} \cdot d\vec{A} = -Q_o + Q_i$$

$$A = -\frac{Q_o - Q_i}{dh/dt} = -\frac{\Delta Q}{dh/dt} \; ; \; \Delta Q = Q_o - Q_i$$

But $\Delta Q = 1{,}710 \, ft^3/s$ and $dh/dt = -1 \, ft/8 \, hr$, since decreasing.

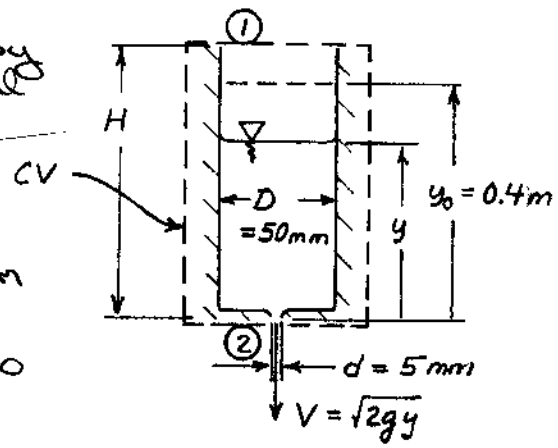
Thus $$A = -1{,}710 \frac{ft^3}{s} \times \frac{8 \, hr}{-1 \, ft} \times 3600 \frac{s}{hr} = 4.92 \times 10^7 \, ft^2$$ A

Since 1 acre = 43,600 ft^2,

$$A = 4.92 \times 10^7 \, ft^2 \times \frac{acre}{43{,}600 \, ft^2} \approx 1{,}130 \, acres$$

Since 1 square mile = 640 acres, the lake surface area is slightly less than 2 square miles!

Given: Cylindrical tank, draining by gravity as shown; initial depth is y_0



Find: Water depth at t = 12 s

Plot: (a) y/y_0 vs t for $0.1 \le y_0 \le 1$ m and $D/d = 10$

(b) y/y_0 vs t for $2 \le D/d \le 10$ and $y_0 = 0.4$ m

Solution:

Apply conservation of mass using CV shown

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) incompressible flow
(2) uniform flow at each section
(3) neglect ρ_{air} compared to ρ_{H_2O}

For the CV, $d\forall = A_t \, dy$, so

$$0 = \frac{\partial}{\partial t}\int_0^y \rho_{H_2O} A_t \, dy + \frac{\partial}{\partial t}\int_y^H \rho_{air} A_t \, dy + \{-1\rho_{air} V_1 A_1\} + \{1\rho_{H_2O} V_2 A_2\}$$

(the ρ_{air} integral ≈ 0 (3); $\rho_{air} V_1 A_1 \approx 0$ (3))

or

$$0 = \rho A_t \frac{dy}{dt} + \rho A_2 V_2 = A_t \frac{dy}{dt} + A_2\sqrt{2gy}$$

Separating variables,

$$\frac{dy}{y^{1/2}} = -\sqrt{2g}\,\frac{A_2}{A_t}\,dt$$

Integrating from y_0 at t = 0 to y at t

$$\int_{y_0}^{y} y^{-1/2}\,dy = 2\left[y^{1/2} - y_0^{1/2}\right] = -\sqrt{2g}\,\frac{A_2}{A_t}\,t$$

$$\frac{y^{1/2}}{y_0^{1/2}} = 1 - \sqrt{\frac{g}{2y_0}}\,\frac{A_2}{A_t}\,t \quad \text{or} \quad y = y_0\left[1 - \sqrt{\frac{g}{2y_0}}\left(\frac{D}{d}\right)^2 t\right]^2 \quad (1)$$

At t = 12 sec

$$y = 0.4\,m\left[1 - \left(\frac{9.81}{2}\,\frac{m}{s^2} \times \frac{1}{0.4\,m}\right)^{1/2}\left(\frac{5\,mm}{50\,mm}\right)^2 12\,s\right]^2 = 0.134\,m \quad \leftarrow y_{t=12s}$$

For D/d = 10, Eq. 1 gives

$$\frac{y}{y_0} = \left[1 - 2.215 \times 10^{-2}\; y_0^{-1/2}\; t\right]^2$$

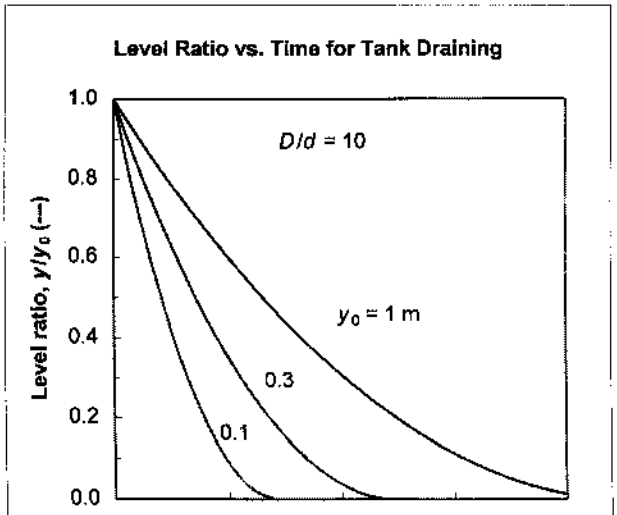
For $y_0 = 0.4$ m, Eq. 1 gives

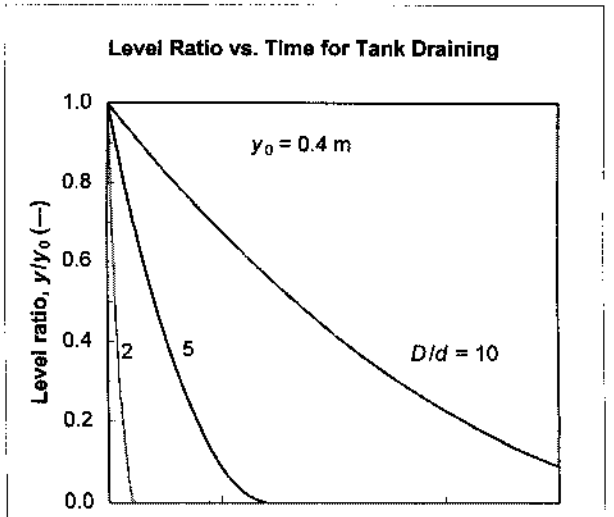
$$\frac{y}{y_0} = \left[1 - \frac{3.502}{(D/d)^2}\, t\right]^2$$

The variation of y/y_0 with t is plotted below for:

- $D/d = 10$ and $0.1 \le y_0 \le 1.0$ m
- $y_0 = 0.4$ m and $2 \le D/d \le 10$

y_0 (m) =	0.1	0.3	1	D/d (---) =	2	5	10
Time, t (s)	y/y_0 (---)	y/y_0 (---)	y/y_0 (---)	Time, t (s)	y/y_0 (---)	y/y_0 (---)	y/y_0 (---)
0	1.000	1.000	1.000	0	1.000	1.000	1.000
2	0.739	0.845	0.913	0.5	0.316	0.865	0.965
4	0.518	0.703	0.831	1	0.016	0.739	0.931
6	0.336	0.574	0.752	1.1	0.001	0.716	0.924
8	0.193	0.458	0.677	2		0.518	0.865
10	0.090	0.355	0.606	3		0.336	0.801
12	0.025	0.265	0.539	4		0.193	0.739
14	0.000	0.188	0.476	5		0.090	0.680
16		0.125	0.417	6		0.025	0.624
18		0.074	0.362	7		0.000	0.570
20		0.037	0.310	10			0.422
22		0.012	0.263	12			0.336
24		0.001	0.219	14			0.260
26			0.180	16			0.193
28			0.144	18			0.137
30			0.113	20			0.090
32			0.085	22			0.053
34			0.061	24			0.025
36			0.041	26			0.008
38			0.025	28			0.000
40			0.013				
45			0.000				

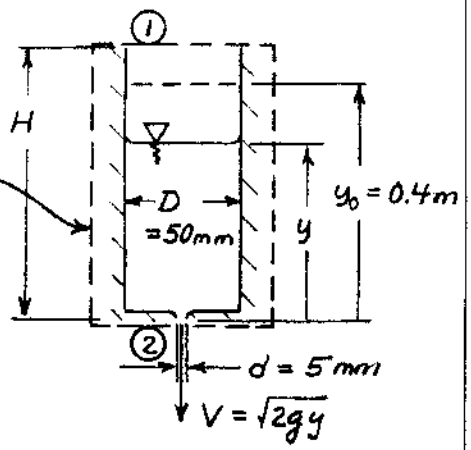




Given: Cylindrical tank, draining by gravity as shown; initial depth is y_0.

Find: Time to drain tank to depth $y = 20$ mm

Plot: Time t to drain the tank (to $y = 20$ mm) as a function of y/y_0 for $0.1 \le y_0 \le 1$ m with d/D as a parameter for $0.1 \le d/D \le 0.5$.



Solution:

Apply conservation of mass using CV shown.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) incompressible flow
(2) uniform flow at each section.
(3) neglect pair compared to ρ_{H_2O}

For the CV, $d\forall = A_t \, dy$, so

$$0 = \frac{\partial}{\partial t}\int_0^y \rho_{H_2O} A_t \, dy + \frac{\partial}{\partial t}\int_{\text{pair}}^{\approx 0 \text{ (3)}} \rho_{\text{pair}} A_t \, dy + \{-1 \, \rho_{\text{pair}} V_1 A_t\} + \{1 \, \rho_{H_2O} V_2 A_2\}$$

or

$$0 = \frac{\partial}{\partial t}\int_0^y \rho_{H_2O} A_t \, dy + \rho_{H_2O} V_2 A_2 = A_t \frac{dy}{dt} + A_2\sqrt{2gy}$$

Separating variables,

$$\frac{dy}{y^{1/2}} = -\sqrt{2g}\,\frac{A_2}{A_t}\,dt$$

Integrating from y_0 at $t = 0$ to y at t

$$\int_{y_0}^{y} \frac{dy}{y^{1/2}} = 2\left[y^{1/2} - y_0^{1/2}\right] = -\sqrt{2g}\,\frac{A_2}{A_t}\,t$$

$$-\sqrt{2g}\,\frac{A_2}{A_t}\,t = 2y_0^{1/2}\left[\left(\frac{y}{y_0}\right)^{1/2} - 1\right] \quad \text{or} \quad t = \sqrt{\frac{2y_0}{g}}\left(\frac{D}{d}\right)^2\left[1 - \left(\frac{y}{y_0}\right)^{1/2}\right] \quad (1)$$

Evaluating at $y = 20$ mm

$$t = \left[2 \times 0.4\text{m} \times \frac{s^2}{9.81\text{m}}\right]^{1/2}\left[\frac{50\text{mm}}{5\text{mm}}\right]^2\left[1 - \left(\frac{0.02\text{m}}{0.40\text{m}}\right)^{1/2}\right] = 22.2\text{s} \quad \leftarrow t_{y=20\text{mm}}$$

Time t is plotted as a function of y/y_0 ($y = 20$ mm) with d/D as a parameter.

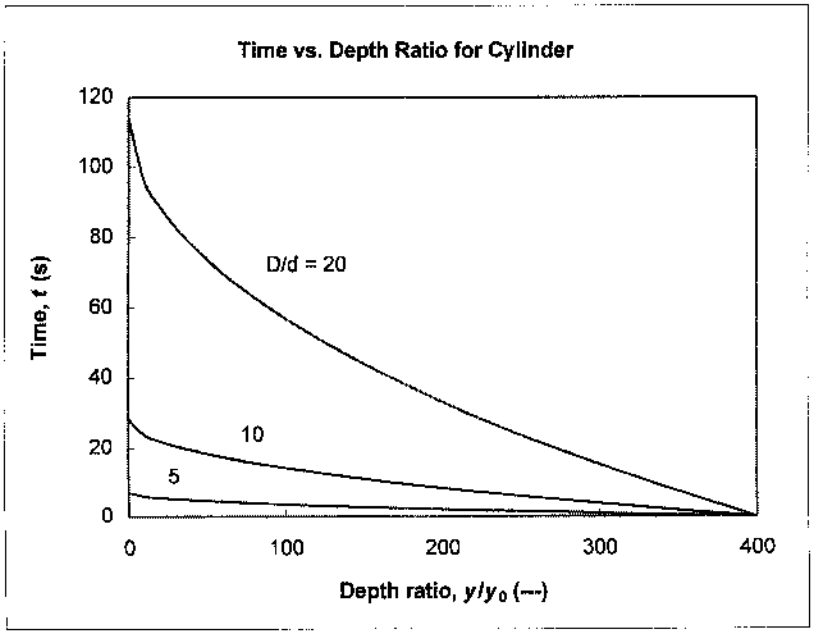
Draining of a cylindrical liquid tank:

Input Data:

Initial height:	y_0	0.4	m		
Diameter ratio:	D/d	20	10	5	---

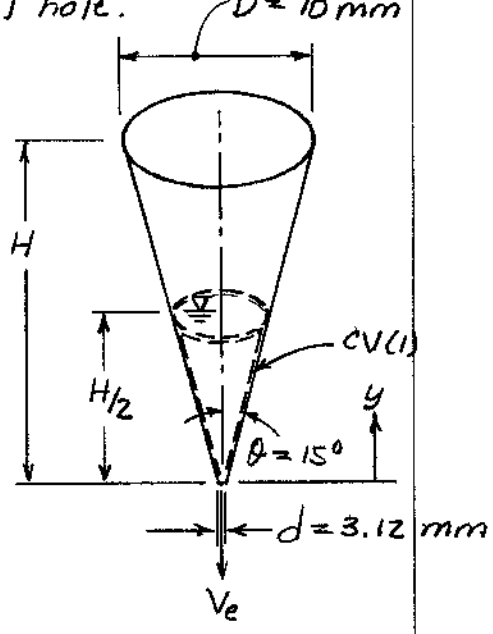
Calculated Results:

Level, y (mm)	D/d =	Time, t (s) 20	10	5
400		0	0	0
380		2.89	0.723	0.181
360		5.86	1.47	0.366
340		8.91	2.23	0.557
320		12.1	3.01	0.754
300		15.3	3.83	0.96
280		18.7	4.66	1.17
260		22.1	5.53	1.38
240		25.7	6.44	1.61
220		29.5	7.38	1.84
200		33.5	8.36	2.09
180		37.6	9.40	2.35
160		42.0	10.5	2.62
140		46.6	11.7	2.92
120		51.7	12.9	3.23
100		57.1	14.3	3.57
80		63.1	15.8	3.95
60		70.0	17.5	4.37
40		78.1	19.5	4.88
30		82.9	20.7	5.18
20		88.7	22.2	5.54
10		96.2	24.0	6.01
0		114	28.6	7.14



Given: Conical funnel draining through small hole.

$$V_e = \sqrt{2gy}$$



Find: Rate of change of surface level when $y = H/2$.

Solution: Apply conservation of mass.

(1) Choose CV with top just below surface level.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$ ($\partial/\partial t = 0$ (1))

Assumptions: (1) ρ = constant, $\forall$ = const, so $\partial/\partial t = 0$
(2) Uniform flow at each section.

For CV(1): $0 = \{-|\rho V_s A_s|\} + \{+|\rho V_e A_e|\}$ (in, out) or $V_s = V_e \frac{A_e}{A_s}$

Thus

$$V_s = V_e\left(\frac{d}{D/2}\right)^2 = \sqrt{2g\frac{H}{2}}\,4\left(\frac{d}{D}\right)^2 = 4\sqrt{gH}\left(\frac{d}{D}\right)^2 = -\frac{dy}{dt} \text{ (since } y \text{ decreases)}$$

But $\tan\theta = \frac{D/2}{H}$ so $H = \frac{D}{2\tan\theta} = \frac{0.070\ m}{2\tan 15^\circ} = 0.131\ m$

Substituting,

$$\frac{dy}{dt} = -4\sqrt{9.81\ \frac{m}{s^2}\times 0.131\ m}\left(\frac{0.00312\ m}{0.070\ m}\right)^2 1000\ \frac{mm}{m} = -9.01\ mm/s$$ ← $\frac{dy}{dt}$

Alternate solution: Choose CV(2) enclosing entire funnel.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$

Assumptions: (1) ρ = constant, but $\forall$ changes (Note: $\forall = \frac{\pi}{3}r^2h$ for a cone.)
(2) Neglect air
(3) Uniform flow at outlet section

Then $0 = \rho\frac{\partial}{\partial t}\forall_{H_2O} + \{+|\rho V_e A_e|\}$ (out) or $\frac{d\forall}{dt} = -V_e A_e$

The volume of water is $\forall = \frac{\pi}{3}r^2h = \frac{\pi}{3}(y\tan\theta)^2 y = \frac{\pi y^3\tan^2\theta}{3}$

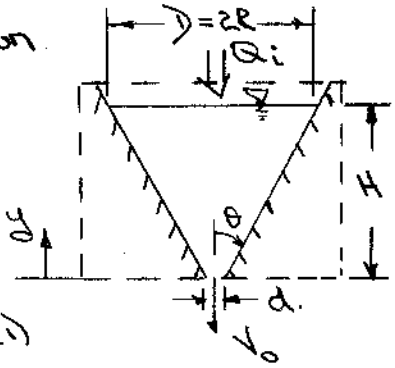
so $\frac{d\forall}{dt} = \pi y^2\tan^2\theta\frac{dy}{dt} = \pi\left(\frac{D}{4}\right)^2\frac{dy}{dt}$ and $\frac{\pi D^2}{16}\frac{dy}{dt} = -V_e A_e = -\sqrt{2gy}\,\frac{\pi d^2}{4}$

Finally, since $y = H/2$, $\frac{dy}{dt} = -4\sqrt{2gH}\left(\frac{d}{D}\right)^2$ as before.

{Note: Flow is not steady in either CV. The $\partial/\partial t$ term vanishes for CV(1) because there is no change in mass inside the CV.}

Given: Water flows into the top of a conical flask at a constant rate of $Q = 3.75 \times 10^{-7}$ m³/hr. Water drains out through the round opening of diameter $d = 7.35$ mm at the apex of the cone; the flow speed at the exit is $V = (2gy)^{1/2}$ where y is the water depth above the exit plane. At the instant of interest, the water depth $H = 36.8$ mm and the corresponding diameter $D = 29.4$ mm

Find: At the instant of interest:
(a) find the volume flow rate from the bottom of the flask
(b) evaluate the direction and rate of change of water surface level



Solution: Apply continuity to the CV shown

Basic eq.: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$

Assumptions: (1) uniform flow at each section
(2) neglect mass of air.

Then

$$0 = \rho \left(\frac{d\forall}{dt}\right)_{water} + \rho Q_{out} - \rho Q_{in} \quad \text{.....(1)}$$

$$Q_{out} = V_o A_o = (2gH)^{1/2} \frac{\pi}{4} d^2$$

$$Q_{out} = \left[2 \times 9.81 \frac{m}{s^2} \times 0.0368\, m\right]^{1/2} \frac{\pi}{4} \times (0.00735)^2\, m^2$$

$$Q_{out} = 3.61 \times 10^{-5}\ m^3/s \quad (0.130\ m^3/hr) \leftarrow Q_{out}$$

From eq. (1)

$$\left(\frac{d\forall}{dt}\right)_{water} = Q_{in} - Q_{out}$$

$$\forall = \frac{1}{3} \text{ area of base} \times \text{altitude} = \frac{1}{3}\pi R^2 y$$

Since $R = y \tan\theta$, $\forall = \frac{1}{3}\pi y^3 \tan^2\theta$

$$\frac{d\forall}{dt} = \frac{1}{3}\pi \tan^2\theta \times 3y^2 \frac{dy}{dt} = \pi y^2 \tan^2\theta \frac{dy}{dt} = \pi R^2 \frac{dy}{dt}$$

$$\therefore \frac{dy}{dt} = \frac{Q_{in} - Q_{out}}{\pi R^2} = \frac{4}{\pi D^2}(Q_{in} - Q_{out})$$

$$= \frac{4}{\pi} \times (0.0294)^2\, m^2 \left(3.75 \times 10^{-7} - 0.130\right) \frac{m^3}{hr} \times \frac{hr}{3600\, s}$$

$$\frac{dy}{dt} = -0.0532\ m/s \quad \text{(surface moves downward)} \leftarrow \frac{dy}{dt}$$

Given: Steady flow of water past a porous flat plate. Suction is constant. Velocity profile at section cd is

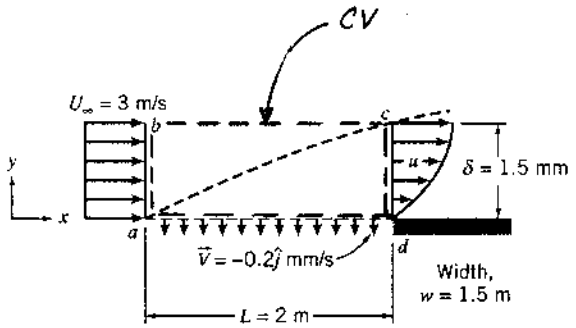
$$\frac{u}{U_\infty} = 3\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^{1.5}$$

Find: Mass flow rate across Section bc.

Solution: Apply conservation of mass using the CV shown.

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{cv} \rho\, d\forall + \int_{cs} \rho \vec{V}\cdot d\vec{A}$$



Assumptions: (1) Steady flow
(2) Incompressible flow
(3) $\vec{V} = -v_0\hat{j}$ along da

Then

$$0 = \int_{cs} \rho\vec{V}\cdot d\vec{A} = \int_{ab} \rho\vec{V}\cdot d\vec{A} + \dot{m}_{bc} + \int_{cd} \rho\vec{V}\cdot d\vec{A} + \int_{da} \rho\vec{V}\cdot d\vec{A}$$

or

$$0 = -\rho U_\infty w\delta + \dot{m}_{bc} + \int_0^\delta \rho U_\infty\left[3\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^{1.5}\right] w\, dy + \rho v_0 w L$$

Thus

$$\dot{m}_{bc} = \rho U_\infty w\delta - \rho U_\infty w\delta \int_0^1 \left[3\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^{1.5}\right] d\left(\frac{y}{\delta}\right) - \rho v_0 w L$$

$$= \rho w\left\{U_\infty\delta - U_\infty\delta\left[\frac{3}{2}\left(\frac{y}{\delta}\right)^2 - \frac{2}{2.5}\left(\frac{y}{\delta}\right)^{2.5}\right]_0^1 - v_0 L\right\}$$

$$= \rho w\left[U_\infty\delta - U_\infty\delta\left(\frac{3}{2} - \frac{2}{2.5}\right) - v_0 L\right] = \rho w(0.3 U_\infty\delta - v_0 L)$$

$$= 999\ \frac{kg}{m^3} \times 1.5\ m\left(0.3 \times 3\ \frac{m}{s} \times 0.0015\ m - 0.0002\ \frac{m}{s} \times 2\ m\right)$$

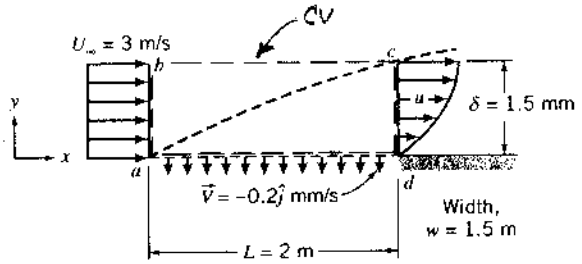
$\dot{m}_{bc} = 1.42$ kg/s ($\dot{m} > 0$, so out of CV)

$\dot{m}_{bc}$ ←

Given: Steady incompressible flow of air on porous surface shown in Fig. P4.38. Velocity profile at downstream end is parabolic. Uniform suction is applied along ad.

Find: (a) Volume flow rate across cd.
(b) Volume flow rate through porous surface (ad).
(c) Volume flow rate across bc.

Solution: Apply conservation of mass to CV shown.



Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

(the $\frac{\partial}{\partial t}$ term $= 0$ (1))

Assumptions: (1) Incompressible flow
(2) Parabolic profile at section cd: $\frac{u}{U_\infty} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$

Then $$0 = \int_{CS} \vec{V}\cdot d\vec{A} = Q_{ab} + Q_{bc} + Q_{cd} + Q_{da} \qquad (1)$$

$$Q_{cd} = \int_{cd} \vec{V}\cdot d\vec{A} = \int_0^\delta u\, w\, dy = w U_\infty \delta \int_0^1 \frac{u}{U}\, d\left(\frac{y}{\delta}\right) = w U_\infty \delta \int_0^1 \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] d\left(\frac{y}{\delta}\right)$$

$$= w U_\infty \delta \left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]_0^1 = \frac{2}{3} w \delta U_\infty$$

$$Q_{cd} = \frac{2}{3} \times 1.5\,m \times 0.0015\,m \times 3\frac{m}{s} = 4.50\times10^{-3}\,m^3/s \text{ (out of CV)} \qquad Q_{cd}$$

Flow across ad is uniform, so

$$Q_{ad} = \vec{V}\cdot\vec{A} = V\hat{j}\cdot wL(-\hat{j}) = -VwL$$

$$Q_{ad} = -0.2\frac{mm}{s} \times 1.5\,m \times 2\,m \times \frac{m}{1000\,mm} = 6.00\times10^{-4}\,m/s \text{ (out of CV)} \qquad Q_{ad}$$

Finally, from Eq. 1,

$$Q_{bc} = -Q_{ab} - Q_{cd} - Q_{da} \qquad (2)$$

But $Q_{ab} = \vec{U}_\infty \cdot \vec{A}_{ab} = U_\infty \hat{i}\cdot w\delta(-\hat{i}) = -w\delta U_\infty$

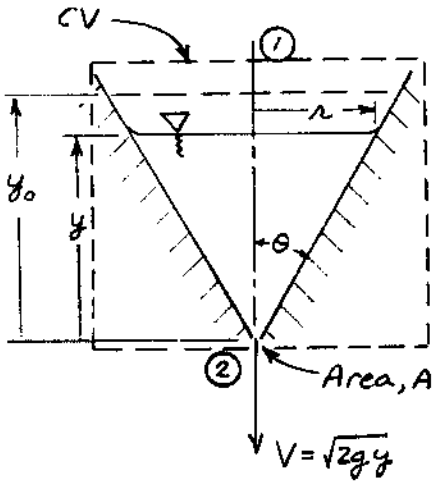
$$Q_{ab} = -1.5\,m \times 0.0015\,m \times 3\frac{m}{s} = -6.75\times10^{-3}\,m^3/s \text{ (into CV)}$$

Substituting into Eq. 2,

$$Q_{bc} = \left[-\left(-6.75\times10^{-3}\right) - 4.50\times10^{-3} - 6.00\times10^{-4}\right] m^3/s$$

$$Q_{bc} = 1.65\times10^{-3}\,m^3/s \text{ (out of CV)} \qquad Q_{bc}$$

Given: Funnel of liquid draining through a small hole of diameter $d = 5\,mm$ (area, A) as shown; y_0 is initial depth.



Find: (a) Expression for time to drain
(b) Expression for result in terms of
- initial volume $\forall_0$, and
- initial volume flow rate $Q_0 = AV_0 = A\sqrt{2gy_0}$.

Plot: t as a function of y_0 ($0.1 \le y_0 \le 1\,m$) with angle θ as a parameter for $15^\circ \le \theta \le 45^\circ$.

Solution

Apply conservation of mass using CV shown.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$

Assumptions: (1) Incompressible flow
(2) Uniform flow at each section
(3) Neglect ρ_{air} compared to ρ_{H_2O}

Then,

$$0 = \frac{\partial}{\partial t}\Big(\cancel{\int_{air}\rho_{air}\,d\forall}^{\,z=0\,(3)} + \int_{H_2O}\rho_{H_2O}\,d\forall\Big) + \cancel{\{-|\rho_{air} V_1 A_1|\}}^{\,z=0\,(3)} + \{|\rho_{H_2O} V A|\}$$

For the CV,

$$d\forall = A_s\,dy = \pi r^2\,dy = \pi (y\tan\theta)^2\,dy \quad ; \quad \forall = \pi\tan^2\theta\,\frac{y^3}{3}$$

Then

$$0 = \rho_{H_2O}\frac{\partial}{\partial t}\left(\pi\tan^2\theta\,\frac{y^3}{3}\right) + \rho_{H_2O} A\sqrt{2gy}$$

$$0 = \pi\tan^2\theta\, y^2\frac{dy}{dt} + A\sqrt{2g}\,y^{1/2}$$

Separating variables, $y^{3/2}\,dy = \dfrac{-\sqrt{2g}A}{\pi\tan^2\theta}\,dt$

Integrating from y_0 at $t=0$ to 0 at t,

$$\int_{y_0}^{0} y^{3/2}\,dy = \frac{2}{5}\left(-y_0^{5/2}\right) = -\frac{\sqrt{2g}A}{\pi\tan^2\theta}\,t$$

or

$$t = \frac{2}{5}\,\frac{\pi\tan^2\theta\, y_0^{5/2}}{\sqrt{2g}A}$$

t

But $\forall_0 = \pi \tan^2\theta \frac{y_0^3}{3}$ and $Q_0 = AV_0 = A\sqrt{2gy_0}$, so

$$t = \frac{\pi \tan^2\theta\, y_0^{5/2}}{\sqrt{2g}\, A} + \frac{y_0^{3/2}}{3\sqrt{2g}} = \frac{1}{6}\frac{\forall_0}{Q_0} \qquad t$$

Since $A = \frac{\pi}{4}d^2$, we can write

$$t = \frac{\pi \tan^2\theta\, y_0^{5/2}}{\sqrt{2g}\,\frac{\pi}{4}d^2} + \frac{y_0^{3/2}}{3\sqrt{2g}} = \frac{8}{15}\frac{\tan^2\theta\, y_0^{5/2}}{d^2\sqrt{2g}}$$

t is plotted as a function of y_0 with θ as a parameter

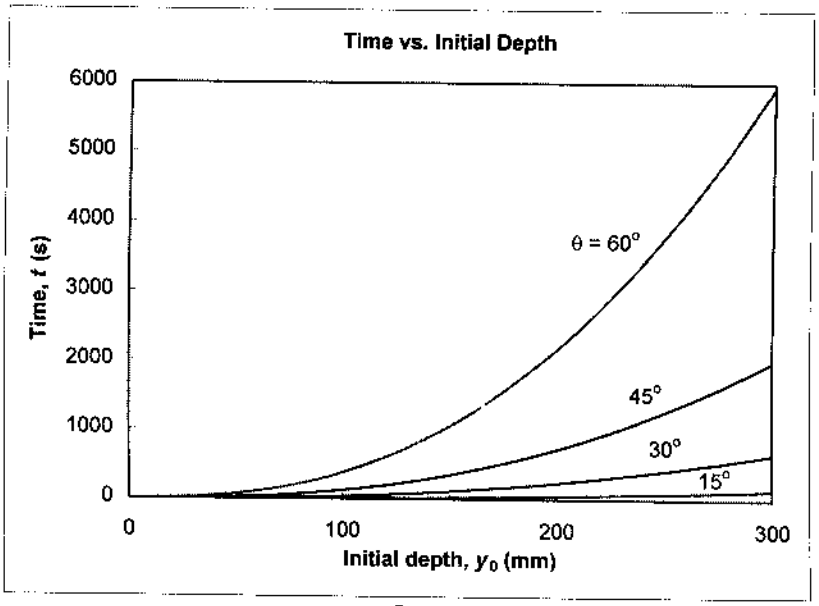
Draining of a conical liquid tank:

Input Data:

Orifice diameter: d = 3 mm

Calculated Results:

Initial Height, y_0 (mm)	Cone Half Angle, θ (deg)	Drain Time, t (s) 60	45	30	15
300		5935	1978	659	142
275		4775	1592	531	114
250		3763	1254	418	90.0
225		2891	964	321	69.2
200		2154	718	239	51.5
175		1543	514	171	36.9
150		1049	350	117	25.1
125		665	222	74	15.9
100		381	127	42	9.11
75		185	62	21	4.44
50		67	22	7	1.61
25		12	4	1	0.285
0		0	0	0	0



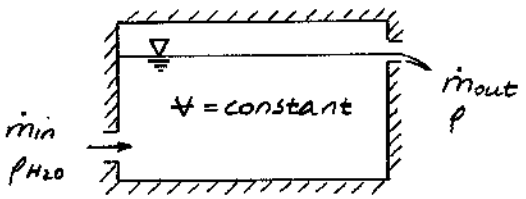
Given: Tank containing brine with steady inlet stream of water. Initial density is $\rho_i > \rho_{H_2O}$.

Find: (a) Rate of change of liquid density in tank.
(b) Time required to reach density, ρ_f, where $\rho_i > \rho_f > \rho_{H_2O}$.

Solution: Apply conservation of mass using the CV shown.

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$



Assumptions: (1) $\forall_{tank}$ = constant
(2) ρ uniform in tank
(3) Uniform flows at inlet and outlet sections

Then $V_1 A_1 = V_2 A_2$ since tank volume is constant, and

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \rho VA - \rho_{H_2O} VA = \frac{\partial}{\partial t}\rho\forall + (\rho - \rho_{H_2O})VA = \forall \frac{d\rho}{dt} + (\rho - \rho_{H_2O})VA$$

So that

$$\frac{d\rho}{dt} = -\frac{(\rho - \rho_{H_2O})\,VA}{\forall}$$ $\frac{d\rho}{dt}$

Separating variables,

$$\frac{d\rho}{\rho - \rho_{H_2O}} = -\frac{VA}{\forall}\,dt$$

Integrating from ρ_i at $t = 0$ to ρ_f at t,

$$\int_{\rho_i}^{\rho_f} \frac{d\rho}{\rho - \rho_{H_2O}} = \ln(\rho - \rho_{H_2O})\Big|_{\rho_i}^{\rho_f} = \ln\left(\frac{\rho_f - \rho_{H_2O}}{\rho_i - \rho_{H_2O}}\right) = \int_0^t -\frac{VA}{\forall}\,dt = -\frac{VA}{\forall}\,t$$

Finally,

$$t = -\frac{\forall}{VA}\ln\left(\frac{\rho_f - \rho_{H_2O}}{\rho_i - \rho_{H_2O}}\right)$$ t

{ Note that $\rho_f \to \rho_{H_2O}$ asymptotically as $t \to \infty$. }

Given: The instantaneous leakage mass flow rate $\dot{m}$ from a bicycle tire is proportional to the air density ρ in the tire and to the gage pressure p_o in the tire.
Air in the tire is nearly isothermal (because the leakage rate is slow).
The initial air pressure is $p_o = 0.60$ MPa (gage) and the initial rate of pressure loss is 1 psi/day.

Find: (a) Pressure in the tire after 30 days
(b) Accuracy of rule of thumb which says a tire loses pressure at the rate of "a pound [1 psi] a day."

Plot: the pressure as a function of time over the 30 days; show rule of thumb results for comparison.

Solution:

Apply conservation of mass to tire as the CV ($\forall$)

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ (CV boundary; $\dot{m}$ out)

Assumptions: (1) uniform properties in tire
(2) air inside CV behaves as ideal gas
(3) T = constant $\Rightarrow \forall$ = constant
(4) $\dot{m} = c(p - p_{atm})\rho$

Then we can write

$$0 = \forall \frac{\partial \rho}{\partial t} + \dot{m} = \forall \frac{\partial \rho}{\partial t} + c(p - p_{atm})\rho \qquad \text{.....(1)}$$

But $\rho = p/RT$ and $\frac{\partial \rho}{\partial t} = \frac{1}{RT}\frac{dp}{dt}$, so

$$0 = \frac{\forall}{RT}\frac{dp}{dt} + \frac{c}{RT}(p - p_{atm})$$

At $t = 0$, $p = p_o$ and $dp/dt = (dp/dt)_o$. Thus

$$0 = \forall \left(\frac{dp}{dt}\right)_o + c p_o (p_o - p_{atm}) \quad \text{and} \quad c = -\frac{\forall}{p_o(p_o - p_{atm})}\left(\frac{dp}{dt}\right)_o$$

Substituting into Eq. 1 we obtain

$$0 = \frac{dp}{dt} - \frac{p(p - p_{atm})}{p_o(p_o - p_{atm})}\left(\frac{dp}{dt}\right)_o$$

Separating variables and integrating

$$\int_{p_o}^{p} \frac{dp}{p(p - p_{atm})} = \frac{(dp/dt)_o}{p_o(p_o - p_{atm})}\int_0^t dt$$

$$\frac{1}{p_{atm}}\left[\ln \frac{p_o(p - p_{atm})}{p(p_o - p_{atm})}\right] = \frac{(dp/dt)_o}{p_o(p_o - p_{atm})}\, t$$

$$\ln\left[\frac{1 - p_{atm}/p}{1 - p_{atm}/p_o}\right] = \frac{(dp/dt)_o}{p_o(p_o/p_{atm} - 1)}\, t$$

Taking antilogs,

$$1 - \frac{p_{atm}}{p} = \left(1 - \frac{p_{atm}}{p_o}\right) e^{\left\{\frac{(dp/dt)_o}{p_o(p_o/p_{atm}-1)} t\right\}} = \left(1 - \frac{p_{atm}}{p_o}\right) e^{kt}$$

where

$$k = \frac{(dp/dt)_o}{p_o(p_o/p_{atm}-1)} = -1\,\frac{psi}{day} \times 6.895\,\frac{kPa}{psi} \times \frac{1}{701\,kPa} \frac{1}{(701/101-1)}$$

$$k = -0.00166\ day^{-1}$$

Then

$$\frac{p_{atm}}{p} = 1 - \left(\frac{p_o - p_{atm}}{p_o}\right) e^{kt}$$

and

$$p = \frac{p_{atm}}{1 - \left(\frac{p_o - p_{atm}}{p_o}\right) e^{kt}} \qquad (2)$$

Evaluating at $t = 30$ days.

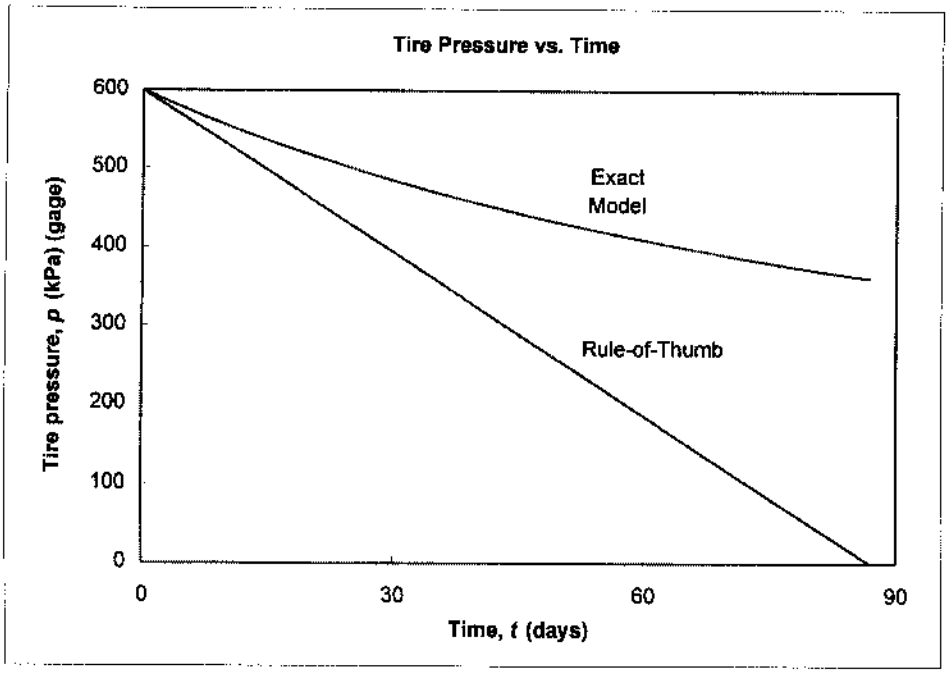
$$p = \frac{101\,kPa}{1 - \frac{600}{701} e^{-30(0.00166)}} = 544\,kPa \qquad p_{t=30\,days}$$

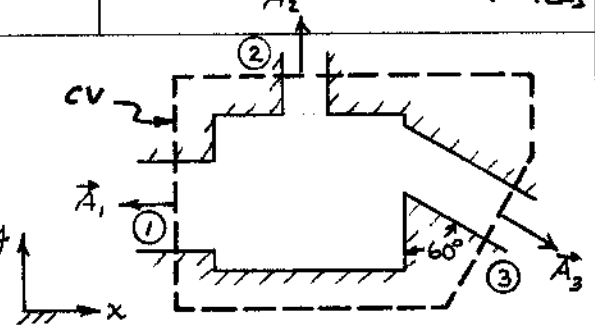
Rule of thumb gives $p = p_o - 6.895\,\frac{kPa}{day}\,t$ ..... (3)

At $t = 30$ days $p = 600\,kPa - 207\,kPa = 493\,kPa$ $p_{est.}$

The rule of thumb predicts a larger pressure loss

Results for both models are presented below





Given: Steady, incompressible flow ($\rho = 1050\ kg/m^3$) through rectangular box shown.

$A_1 = 0.05\ m^2$, $A_2 = 0.01\ m^2$, $A_3 = 0.06\ m^2$

$\vec{V}_1 = 4\,\hat{\imath}\ m/s$, $\vec{V}_2 = -8\,\hat{\jmath}\ m/s$

and, from Problem 4.19, $\vec{V}_3 = 4.04\,\hat{\imath} - 2.34\,\hat{\jmath}\ m/s$; $V_3 = 4.67\ m/s$

Find: Net rate of efflux of momentum through CV.

Solution: The net rate of momentum efflux is given by the term

$$\int_{CS} \vec{V}\,\rho\vec{V}\cdot d\vec{A}$$

Assumption: (1) Flow is uniform at each section.

Then $\int_{CS} \vec{V}\,\rho\vec{V}\cdot d\vec{A} = \vec{V}_1\,\rho\vec{V}_1\cdot\vec{A}_1 + \vec{V}_2\,\rho\vec{V}_2\cdot\vec{A}_2 + \vec{V}_3\,\rho\vec{V}_3\cdot\vec{A}_3$

or in components, since $\vec{V} = u\,\hat{\imath} + v\,\hat{\jmath}$,

$$\int_{CS} \vec{V}\,\rho\vec{V}\cdot d\vec{A} = (u_1\rho\vec{V}_1\cdot\vec{A}_1 + u_2\rho\vec{V}_2\cdot\vec{A}_2 + u_3\rho\vec{V}_3\cdot\vec{A}_3)\,\hat{\imath} + (v_1\rho\vec{V}_1\cdot\vec{A}_1 + v_2\rho\vec{V}_2\cdot\vec{A}_2 + v_3\rho\vec{V}_3\cdot\vec{A}_3)\,\hat{\jmath}$$

$$\vec{mf} = \left[u_1\{-|\rho V_1 A_1|\} + u_2\{-|\rho V_2 A_2|\} + u_3\{|\rho V_3 A_3|\}\right]\hat{\imath}$$

$u_1 = 4\ m/s$ $\quad u_2 = 0$ $\quad u_3 = 4.04\ m/s$

$$+ \left[v_1\{-|\rho V_1 A_1|\} + v_2\{-|\rho V_2 A_2|\} + v_3\{|\rho V_3 A_3|\}\right]\hat{\jmath}$$

$v_1 = 0$ $\quad v_2 = -8\ m/s$ $\quad v_3 = -2.34\ m/s$

$$= 1050\,\frac{kg}{m^3}\left[4.0\frac{m}{s}\left\{-4.0\frac{m}{s}\times 0.05\,m^2\right\} + 4.04\frac{m}{s}\left\{4.67\frac{m}{s}\times 0.06\,m^2\right\}\right]\hat{\imath}$$

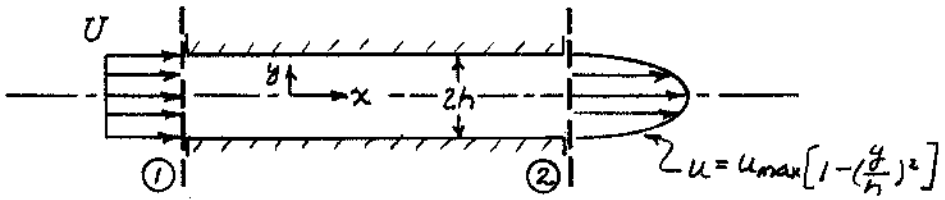
$$+ 1050\,\frac{kg}{m^3}\left[-8.0\frac{m}{s}\left\{-8.0\frac{m}{s}\times 0.01\,m^2\right\} - 2.34\frac{m}{s}\left\{4.67\frac{m}{s}\times 0.06\,m^2\right\}\right]\hat{\jmath}$$

$$= (349\,\hat{\imath} - 13.5\,\hat{\jmath})\,\frac{kg\cdot m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}$$

$$\vec{mf} = 349\,\hat{\imath} - 16.5\,\hat{\jmath}\ N$$

Ne
mf

Given: Water flow between parallel plates, as shown.



From Problem 4.21, $u_{max} = \frac{3}{2}\bar{U}$ (from continuity).

Evaluate: Ratio of x direction momentum flux at outlet to that at inlet.

Solution: The x direction momentum flux at a section is given by

$$mf_x = \int_A u\rho V dA$$

Assumptions: (1) Uniform flow at section ①
(2) Incompressible flow

Then at ① $mf_x)_1 = \int_{A_1} U\rho U dA = \rho U^2 2hw$ (w = channel depth)

At section ②, the velocity varies and we must integrate. Using $dA = w dy$,

$$mf_x)_2 = \int_{-h}^{h} u\rho u w dy = \rho w u_{max}^2 \int_{-h}^{h}\left[1-\left(\frac{y}{h}\right)^2\right]^2 dy$$

$$= \rho w h u_{max}^2 \int_{-1}^{1}\left[1-\left(\frac{y}{h}\right)^2\right]^2 d\left(\frac{y}{h}\right) = 2\rho w h u_{max}^2 \int_0^1\left[1-\left(\frac{y}{h}\right)^2\right]^2 d\left(\frac{y}{h}\right)$$

$$= 2\rho w h u_{max}^2 \int_0^1\left[1-2\left(\frac{y}{h}\right)^2+\left(\frac{y}{h}\right)^4\right] d\left(\frac{y}{h}\right)$$

$$= 2\rho w h u_{max}^2\left[\left(\frac{y}{h}\right)-\frac{2}{3}\left(\frac{y}{h}\right)^3+\frac{1}{5}\left(\frac{y}{h}\right)^5\right]_0^1$$

$$mf_x)_2 = \frac{16}{15}\rho w h u_{max}^2$$

The ratio of x direction momentum fluxes is

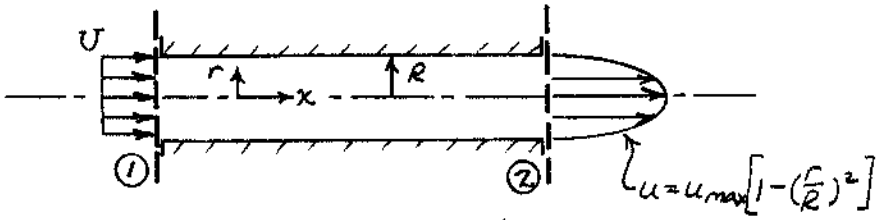
$$\frac{mf_x)_2}{mf_x)_1} = \frac{\frac{16}{15}\rho w h u_{max}^2}{2\rho w h \bar{U}^2} = \frac{8}{15}\left(\frac{u_{max}}{\bar{U}}\right)^2$$

But from Problem 4.21, $u_{max} = \frac{3}{2}\bar{U}$, so

$$\frac{mf_x)_2}{mf_x)_1} = \frac{8}{15}\left(\frac{3}{2}\right)^2 = \frac{72}{60} = \frac{6}{5} = 1.2$$

Rat

Given: Water flow through a circular pipe as shown.



From Problem 4.23, $u_{max} = 2U$ (from continuity).

Evaluate: Ratio of x direction momentum flux at outlet to that at inlet.

Solution: The x direction momentum flux at a section is given by

$$mf_x = \int_A u\rho V dA$$

Assumptions: (1) Uniform flow at section ①
(2) Incompressible flow

Then at ① $mf_x)_1 = \int_{A_1} U\rho U dA = \rho U^2 \pi R^2$

At section ② the velocity varies and we must integrate. Using $dA = 2\pi r dr$,

$$mf_x)_2 = \int_0^R u\rho u\, 2\pi r dr = 2\pi\rho u_{max}^2 \int_0^R \left[1-(\frac{r}{R})^2\right]^2 r dr$$

$$= \rho u_{max}^2 2\pi R^2 \int_0^1 \left[1-(\frac{r}{R})^2\right]^2 (\frac{r}{R})\, d(\frac{r}{R})$$

$$= \rho u_{max}^2 2\pi R^2 \int_0^1 \left[(\frac{r}{R}) - 2(\frac{r}{R})^3 + (\frac{r}{R})^5\right] d(\frac{r}{R})$$

$$= \rho u_{max}^2 2\pi R^2 \left[\frac{1}{2}(\frac{r}{R})^2 - \frac{1}{2}(\frac{r}{R})^4 + \frac{1}{6}(\frac{r}{R})^6\right]_0^1$$

or

$$mf_x)_2 = \rho u_{max}^2 \pi R^2 (\frac{1}{3})$$

The ratio of x direction momentum fluxes is

$$\frac{mf_x)_2}{mf_x)_1} = \frac{\rho u_{max}^2 \pi R^2 (\frac{1}{3})}{\rho U^2 \pi R^2} = \frac{1}{3}(\frac{u_{max}}{U})^2$$

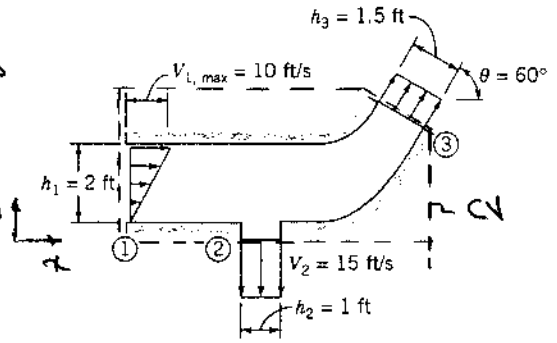
But from Problem 4.23, $u_{max} = 2U$, so

$$\frac{mf_x)_2}{mf_x)_1} = \frac{1}{3}(2)^2 = \frac{4}{3}$$

Rati

Given: Two-dimensional reducing bend shown has width $w = 3$ ft.

$V_3 = 3.33$ ft/s into CV (from Problem 4.24)



Find: Momentum flux through the bend.

Solution:

The momentum flux is defined as $m.f = \int \vec{V}(\rho \vec{V}\cdot d\vec{A})$

The net momentum flux through the CV is

$$m.f = \int_{A_1} \vec{V}(\rho \vec{V}\cdot d\vec{A}) + \int_{A_2} \vec{V}(\rho \vec{V}\cdot d\vec{A}) + \int_{A_3} \vec{V}(\rho \vec{V}\cdot d\vec{A})$$

where $\vec{V}_1 = V_{1,max}\frac{y}{h_1}\hat{i}$, $\vec{V}_2 = -V_2\hat{j}$, $\vec{V}_3 = -V_3(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

$V_{1,max} = 10$ ft/s, $V_2 = 15$ ft/s, $V_3 = 3.33$ ft/s

Assumptions: (1) incompressible flow
(2) fluid is water
(3) uniform flow at (2) and (3) (given)

$$\int_{A_1} \vec{V}(\rho \vec{V}\cdot d\vec{A}) = \int_0^{h_1} V_{1,max}\frac{y}{h_1}\hat{i}\;\rho\left\{-V_{1,max}\frac{y}{h_1}\right\}W\,dy = -\hat{i}\,\rho V_{1,max}^2\frac{W}{h_1^2}\int_0^{h_1} y^2\,dy$$

$$\int_{A_1} \vec{V}(\rho \vec{V}\cdot d\vec{A}) = -\hat{i}\,\rho V_{1,max}^2\frac{W}{h_1^2}\left.\frac{y^3}{3}\right|_0^{h_1} = -\hat{i}\,\rho V_{1,max}^2\frac{W h_1}{3} \quad \text{---- (1)}$$

$$\int_{A_2} \vec{V}(\rho \vec{V}\cdot d\vec{A}) = \vec{V}_2\,|\rho V_2 h_2 W| = -V_2\hat{j}\,|\rho V_2 h_2 W| = -\hat{j}\,\rho V_2^2 h_2 W \quad \text{---- (2)}$$

$$\int_{A_3} \vec{V}(\rho \vec{V}\cdot d\vec{A}) = \vec{V}_3(-|\rho V_3 h_3 W) = -V_3(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})(-|\rho V_3 h_3 W)$$

$$\int_{A_3} \vec{V}(\rho \vec{V}\cdot d\vec{A}) = \rho V_3^2 h_3 W(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) \quad \text{---- (3)}$$

$$m.f = \hat{i}\left[\rho V_3^2 h_3 W\cos\theta - \rho V_{1,max}^2\frac{W h_1}{3}\right] + \hat{j}\left[\rho V_3^2 h_3 W\sin\theta - \rho V_2^2 h_2 W\right]$$

$$m.f = \rho W\left\{\left[V_3^2 h_3\cos\theta - V_{1,max}^2\frac{h_1}{3}\right]\hat{i} + \left[V_3^2 h_3\sin\theta - V_2^2 h_2\right]\hat{j}\right\}$$

Evaluating

$$m.f = 1.94\frac{\text{slug}}{\text{ft}^3}\times 3\text{ ft}\times\frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}\left\{\left[(3.33)^2\frac{\text{ft}^2}{\text{s}^2}\times 1.5\text{ ft}\times\cos 60° - (10)^2\frac{\text{ft}^2}{\text{s}^2}\times\frac{2}{3}\text{ft}\right]\hat{i} + \left[(3.33)^2\frac{\text{ft}^2}{\text{s}^2}\times 1.5\text{ ft}\times\sin 60° - (15)^2\frac{\text{ft}^2}{\text{s}^2}\times 1\text{ ft}\right]\hat{j}\right\}$$

$$m.f = -340\hat{i} - 1230\hat{j}\ \text{lbf}$$

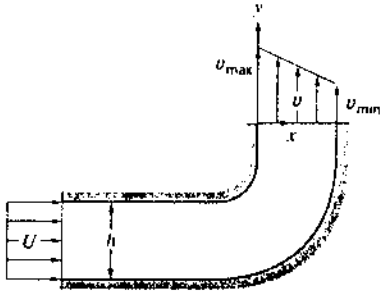
m.f.

Given: Water flow in the two-dimensional square channel shown

$U = 7.5$ m/s, $h = w = 75.5$ mm

$v_{max} = 2 v_{min}$

$v_{min} = 5.0$ m/s (from Problem 4.25)



Find: Momentum flux through the channel; comment on expected outlet pressure (relative to pressure at the inlet.

Solution:

The momentum flux is defined as $m.f. = \int \vec{V}(\rho \vec{V}\cdot d\vec{A})$

The net momentum flux through the CV is

$$m.f. = \int_{A_1} \vec{V}(\rho \vec{V}\cdot d\vec{A}) + \int_{A_2} \vec{V}(\rho \vec{V}\cdot d\vec{A})$$

where $\vec{V}_1 = U\hat{i}$, $\vec{V}_2 = \left\{v_{max} - (v_{max} - v_{min})\frac{x}{h}\right\}\hat{j}$

$$\vec{V}_2 = \left\{2v_{min} - v_{min}\frac{x}{h}\right\}\hat{j} = v_{min}\left(2 - \frac{x}{h}\right)\hat{j}$$

Assumptions: (1) incompressible flow
(2) uniform flow at ① (given).

$$\int_{A_1} \vec{V}(\rho \vec{V}\cdot d\vec{A}) = \vec{V}_1\{-|\rho V_1 A_1|\} = -\rho U^2 h^2 \hat{i} \quad \text{.....(1)}$$

$$\int_{A_2} \vec{V}(\rho \vec{V}\cdot d\vec{A}) = \int_0^h v_{min}\left(2-\frac{x}{h}\right)\hat{j}\,\rho v_{min}\left(2-\frac{x}{h}\right) h\,dx$$

$$= \hat{j}\,\rho v_{min}^2 h \int_0^h \left(4 - 4\frac{x}{h} + \frac{x^2}{h^2}\right)dx$$

$$= \hat{j}\,\rho v_{min}^2 h\left[4x - 2\frac{x^2}{h} + \frac{1}{3}\frac{x^3}{h^2}\right]_0^h = \hat{j}\,\rho v_{min}^2 h\left[4h - 2h + \frac{h}{3}\right]$$

$$= \hat{j}\,\frac{7}{3}\rho v_{min}^2 h^2$$

$$\therefore m.f. = -\rho U^2 h^2 \hat{i} + \frac{7}{3}\rho v_{min}^2 h^2 \hat{j} = \rho h^2\left[-U^2\hat{i} + \frac{7}{3}v_{min}^2\hat{j}\right]$$

Evaluating

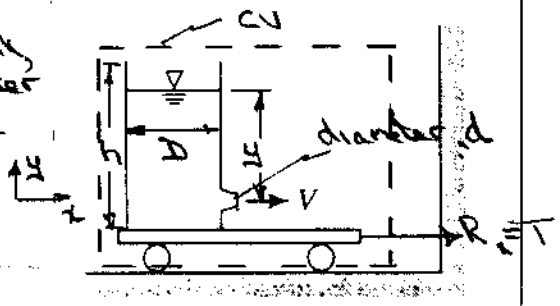
$$m.f. = 999\,\frac{kg}{m^3} \times (0.0755)^2 m^2 \left[-(7.5)^2\,\frac{m^2}{s^2}\,\hat{i} + \frac{7}{3}(5)^2\,\frac{m^2}{s^2}\,\hat{j}\right] \times \frac{N\cdot s^2}{kg\cdot m}$$

$$m.f. = -320\hat{i} + 332\hat{j}\ N \qquad \text{m.f.}$$

For viscous (real) flow friction causes a pressure drop in the direction of flow (Chapter 8)
For flow in a bend streamline curvature results in a pressure gradient normal to the flow (Chapter 6)

Given: Water discharges from tank of height $h = 1$ m and diameter $D = 0.6$ m through a nozzle of diameter $d = 10$ mm.

$V_{jet} = \sqrt{2gy}$ where y is height of free surface above the nozzle.



Find: Tension in wire holding the cart when $y = 0.8$ m.

Plot: tension in wire as a function of water depth for $0 \le y \le 0.8$ m.

Solution:

Apply the x component of the momentum equation, using the inertial CV shown.

Basic equation: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$ ($F_{Sx} = 0$ (2); $\frac{\partial}{\partial t}\int_{CV} = 0$ (3))

Assumptions: (1) There are no net pressure forces
(2) $F_{Bx} = 0$
(3) Steady flow
(4) Uniform flow across the jet

Then, $R_x = T = u\int|\rho V_j A_j| = \rho V_j^2 A = \rho\, 2gy\, \frac{\pi}{4} d^2$

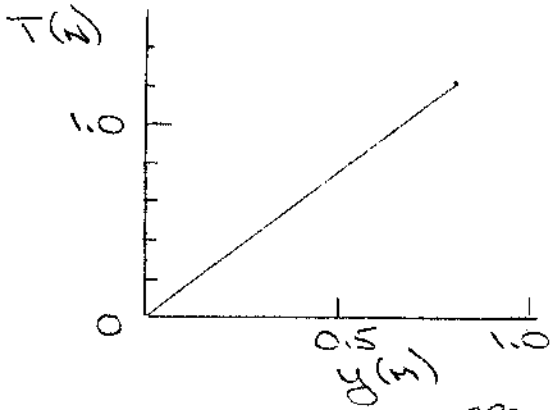
$$T = \rho g y \frac{\pi}{2} d^2 \qquad (1)$$

Evaluating for $y = 0.8$ m

$$T = 999\ \frac{kg}{m^3} \times 9.81\ \frac{m}{s^2} \times 0.8\ m \times \frac{\pi}{2} \times (0.010)^2\ m^2 \times \frac{N \cdot s^2}{kg \cdot m}$$

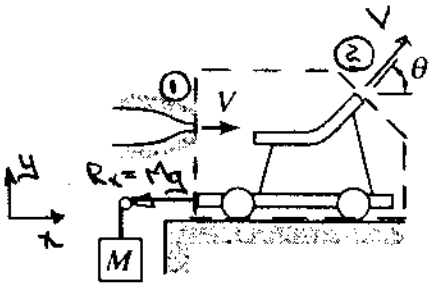
$$T = 1.23\ N$$

From Eq. 1 we see that T varies linearly with y



Given: Cart with vane, struck by water jet.

$V_j = 15$ m/s $\quad A_j = 0.05\ m^2$



Find: Mass needed to hold cart stationary for $\theta = 50°$

Plot: mass needed to hold cart stationary for $0 \le \theta \le 180$ degrees.

Solution:

Apply the x component of the momentum equation to the inertial CV shown.

Basic equation: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\,d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$ ($F_{Bx} = 0$ (2); $\frac{\partial}{\partial t} = 0$ (3))

Assumptions: (1) atmospheric pressure surrounds CV
(2) $F_{Bx} = 0$
(3) steady flow
(4) jet velocity (and area) remain constant on vane
(5) uniform flow at each section
(6) incompressible flow

Then

$$-Mg = u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho V_2 A_2|\} \quad \begin{cases} u_1 = V \ ; \ u_2 = V\cos\theta \\ V_1 = V_2 = V \ ; \ A_1 = A_2 = A \end{cases}$$

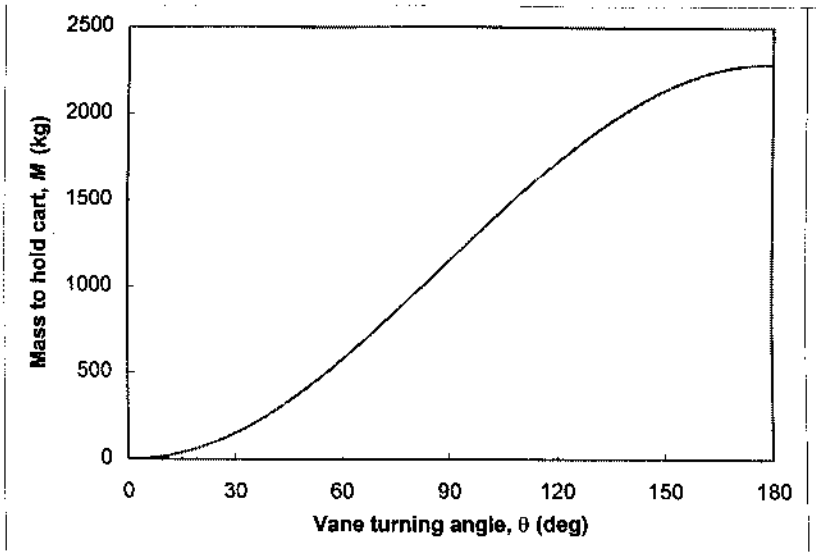
$$-Mg = V(-\rho V A) + V\cos\theta(\rho V A) = \rho V^2 A(\cos\theta - 1)$$

$$M = \frac{\rho V^2 A}{g}(1 - \cos\theta) \qquad (1)$$

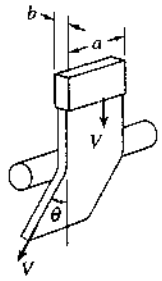
Evaluating for $\theta = 50°$

$$M = 999\frac{kg}{m^3} \times (15)^2\frac{m^2}{s^2} \times (0.05\,m^2) \times \frac{s^2}{9.81\,m}(1 - \cos 50°) = 409\ kg \leftarrow M$$

M is plotted as a function of θ



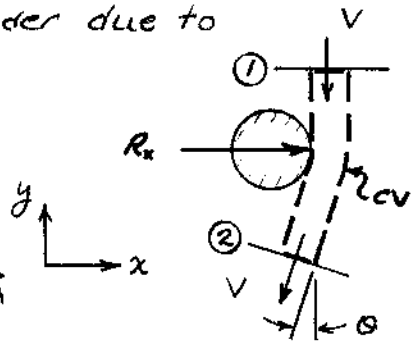
Given: Circular cylinder deflecting flat jet of water as shown.



$a = 0.5$ in., $b = 0.1$ in.

$V = 10 \frac{ft}{s}$, $\theta = 20°$

Find: Horizontal component of force on cylinder due to flowing water.



Solution: Apply x component of momentum equation to inertial CV shown.

Basic equation: $F_{Sx} + \cancel{F_{Bx}} = \cancel{\frac{\partial}{\partial t}}\int_{CV} u\rho dV + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$

($F_{Bx} = 0$ (2); $\frac{\partial}{\partial t} = 0$ (3))

Assumptions: (1) No pressure forces on jet
(2) $F_{Bx} = 0$
(3) Steady flow
(4) Incompressible flow
(5) Uniform flow at each section
(6) Jet speed and area remain constant

Then

$$R_x = u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho V_2 A_2|\}$$

$u_1 = 0$ $\qquad u_2 = -V\sin\theta$, $V_2 = V$, $A_2 = ab$

$$R_x = -\rho V^2 ab\sin\theta$$

$$= -1.94 \frac{slug}{ft^3} \times (10)^2 \frac{ft^2}{s^2} \times 0.5\,in. \times 0.1\,in. \times \frac{ft^2}{144\,in^2} \times \sin 20° \times \frac{lbf\cdot s^2}{slug\cdot ft}$$

$$R_x = -0.0230\ lbf$$

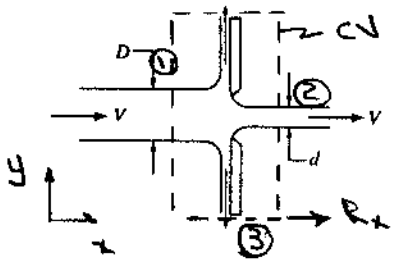
But R_x is force of cylinder on CV. Force of CV on cylinder is

$K_x = -R_x = 0.0230$ lbf (to the right) $\longleftarrow$ K_x

Given: Plate with orifice struck concentrically by water jet as shown.

Find: (a) Expression for force needed to hold the plate.
(b) Value of force for $V = 5$ m/s, $D = 100$ mm, and $d = 25$ mm.

Plot: required force as a function of diameter ratio d/D



Solution:

Apply the x component of the momentum equation to the inertial CV shown.

Basic equation: $F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\, \vec{V}\cdot d\vec{A}$ ($F_{B_x} = 0$ (2); $\frac{\partial}{\partial t} = 0$ (3))

Assumptions: (1) atmospheric pressure surrounds CV
(2) $F_{B_x} = 0$
(3) steady flow
(4) uniform flow at each section
(5) incompressible flow

Then,

$$R_x = u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho V_2 A_2|\} + u_3\{|\rho V_3 A_3|\}$$

$u_1 = V$, $A_1 = \frac{\pi}{4}D^2$ $\quad u_2 = V$, $A_2 = \frac{\pi}{4}d^2$ $\quad u_3 = 0$

and

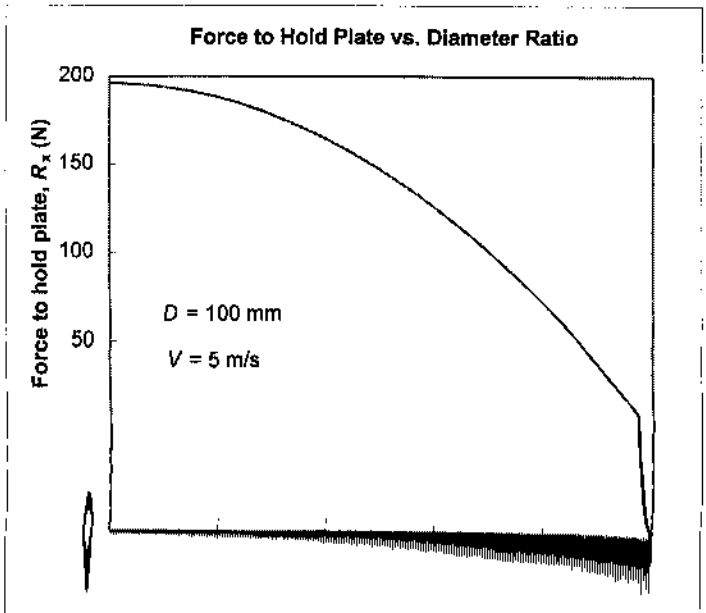
$$R_x = -\rho V^2 A_1 + \rho V^2 A_2 = \rho V^2 (A_2 - A_1) = \rho V^2 \frac{\pi}{4}(d^2 - D^2)$$

$$R_x = -\rho V^2 \frac{\pi}{4} D^2 \left[1 - \left(\frac{d}{D}\right)^2\right] \longleftarrow R_x$$

Evaluating for $d = 25$ mm

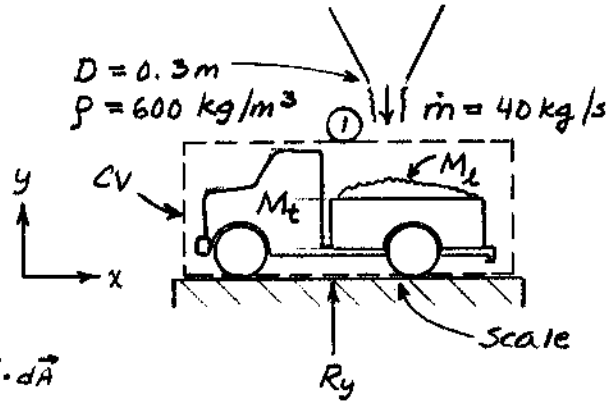
$$R_x = -\frac{\pi}{4} \times 999 \frac{kg}{m^3} (5)^2 \frac{m^2}{s^2} \times (0.10)^2 m^2 \left[1 - \left(\frac{25 mm}{100 mm}\right)^2\right] \frac{N \cdot s^2}{kg \cdot m} = -184 N \longleftarrow R_x$$

Since $R_x < 0$, it must be applied to the left. R_x is plotted as a function of d/D.



Given: Farmer purchases 675 kg of bulk grain. The grain is loaded into a pickup truck from a hopper as shown. Grain flow is terminated when the scale reading reaches the desired gross value.

Find: The true payload.



Solution: Apply the y component of momentum equation using CV shown.

Basic equation:

$$F_{sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\,d\forall + \int_{CS} v\,\rho\,\vec{V}\cdot d\vec{A} \qquad \left(\frac{\partial}{\partial t} \approx 0 \text{ (2)}\right)$$

Assumptions: (1) No net pressure force; $F_{sy} = R_y$
(2) Neglect v inside CV
Then (3) Uniform flow of grain at inlet section (1)

$$R_y - (M_t + M_\ell)g = v_1\{-|\dot{m}|\}$$

$$v_1 = -V_1 = -\frac{\dot{m}}{\rho A}$$

or

$$R_y = (M_t + M_\ell)g + \frac{\dot{m}^2}{\rho A} \quad \text{(indicated during grain flow)}$$

Loading is terminated when

$$\frac{R_y}{g} - M_t = M_\ell + \frac{\dot{m}^2}{\rho g A} = 675\ \text{kg}$$

Thus

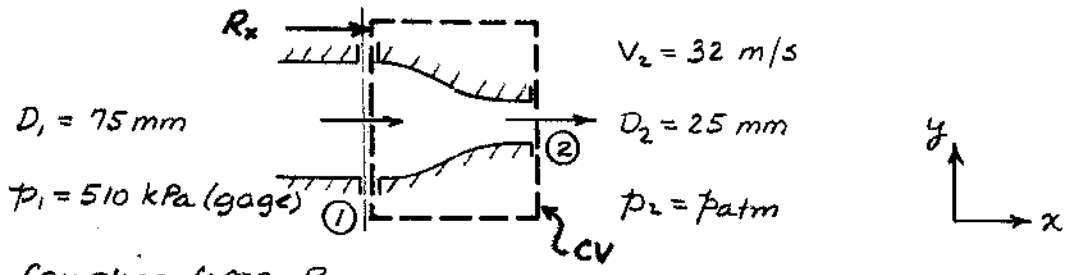
$$M_\ell = 675\ \text{kg} - \frac{\dot{m}^2}{\rho g A}$$

$$= 675\ \text{kg} - (40)^2\,\frac{\text{kg}^2}{\text{s}^2} \times \frac{\text{m}^3}{600\ \text{kg}} \times \frac{\text{s}^2}{9.81\ \text{m}} \times \frac{4}{\pi}\,\frac{1}{(0.3)^2\ \text{m}^2}$$

$$M_\ell = 671\ \text{kg}$$

M_ℓ

Given: Water flow through a fire hose and nozzle.



Find: (a) Coupling force, R_x
(b) Indicate if in tension or compression.

Solution: Apply continuity and x component of momentum equation to inertial CV shown; use gage pressures to cancel p_{atm}.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ (=0(1), =0(4), =0(1))

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Uniform flow at each section
(3) Incompressible flow
(4) $F_{Bx} = 0$

Then

$$0 = \{-|\rho V_1 A_1|\} + \{|\rho V_2 A_2|\} = -\rho V_1 A_1 + \rho V_2 A_2$$

$$V_1 = V_2 \frac{A_2}{A_1} = V_2\left(\frac{D_2}{D_1}\right)^2 = 32\frac{m}{s} \times \left(\frac{25\ mm}{75\ mm}\right)^2 = 3.56\ m/s$$

and

$$R_x + p_{1g} A_1 = u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho V_2 A_2|\}$$

$$u_1 = V_1 \qquad u_2 = V_2$$

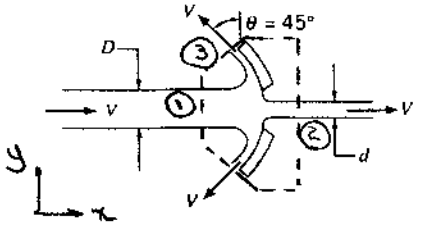
$$R_x = -p_{1g}A_1 - V_1 \rho V_1 A_1 + V_2 \rho V_2 A_2 = -p_{1g} A_1 + \rho V_2 A_2 (V_2 - V_1)$$

$$= -510\times10^3 \frac{N}{m^2} \times \frac{\pi}{4}(0.075)^2 m^2 + 999\frac{kg}{m^3} \times 32\frac{m}{s} \times \frac{\pi}{4}(0.025)^2 m^2 (32.0 - 3.56)\frac{m}{s} \times \frac{N\cdot s^2}{kg\cdot m}$$

$$R_x = -1.81\ kN$$ (i.e. force on CV is to the left) R_x

Thus the coupling must be in tension.

Given: Circular dish with central orifice struck concentrically by water jet as shown



Find: (a) Expression for force needed to hold the dish in place
(b) Value of force for $V = 5$ m/s, $D = 100$ mm, and $d = 20$ mm

Plot: required force as a function of θ $(0 \le \theta \le 90^\circ)$ with d/D as a parameter.

Solution:

Apply the x component of the momentum equation to the inertial CV shown.

Basic equation: $F_{S_x} + \cancel{F_{B_x}}^{=0(2)} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}^{=0(3)} + \int_{CS} u(\rho \vec{V}\cdot d\vec{A})$

Assumptions: (1) atmospheric pressure acts on all CV surfaces
(2) $F_{B_x} = 0$
(3) steady flow
(4) uniform flow at each section
(5) incompressible flow
(6) no change in jet speed on dish: $V_1 = V_2 = V_3 = V$

Then,

$$R_x = u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho V_2 A_2|\} + u_3\{|\rho V_3 A_3|\}$$

$u_1 = V$ $\quad A_1 = \frac{\pi}{4}D^2$ $\quad u_2 = V$ $\quad A_2 = \frac{\pi}{4}d^2$ $\quad u_3 = -V\sin\theta$ $\quad A_3 = A_1 - A_2$

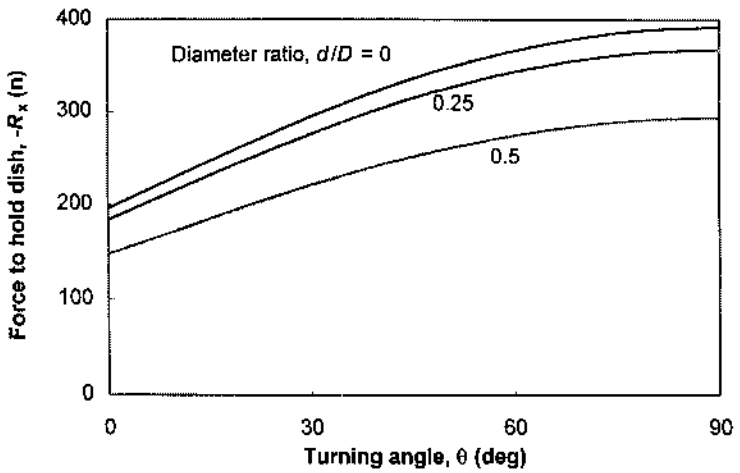
$$R_x = -\rho V^2 \frac{\pi}{4}D^2 + \rho V^2 \frac{\pi}{4}d^2 - \rho V^2 \sin\theta \frac{\pi}{4}(D^2 - d^2) = \rho V^2 \frac{\pi}{4}(1+\sin\theta)(d^2 - D^2)$$

$$R_x = -\rho V^2 \frac{\pi}{4}D^2 (1+\sin\theta)\left[1 - \left(\frac{d}{D}\right)^2\right] \longleftarrow R_x$$

Evaluating for $d = 25$ mm

$$R_x = -\frac{\pi}{4} \times 999 \frac{kg}{m^3} \times (5)^2 \frac{m^2}{s^2} \times (0.10)^2 m^2 (1+\sin 45^\circ)\left[1 - \left(\frac{25}{100}\right)^2\right] \frac{N \cdot s^2}{kg \cdot m} = -314 N \longleftarrow R_x$$

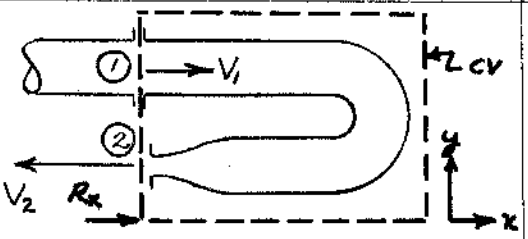
Since $R_x < 0$, it must be applied to the left. R_x is plotted as a function of θ for different values of d/D



Given: Elbow assembly shown, water flow.

$p_1 = 96$ kPa (gage), $V_1 = 3.05$ m/s

$A_1 = 2600$ mm^2, $A_2 = 650$ mm^2



Find: Horizontal force required to hold in place.

Solution: Use CV shown, apply x component of momentum eq.

Basic equation: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$ ($F_{Bx}=0$ (1), $=0$ (2))

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Incompressible flow
(4) Uniform flow at each section

Then

$$R_x + p_{1g}A_1 = u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho V_2 A_2|\}$$

$u_1 = V_1$ $\qquad$ $u_2 = -V_2$

$$R_x = -p_{1g}A_1 + \dot{m}(-V_1 - V_2)$$

From continuity, $\dot{m} = \rho V_1 A_1 = \rho V_2 A_2$; $V_2 = V_1\frac{A_1}{A_2} = 3.05\frac{m}{s}\times\frac{2600}{650} = 12.2$ m/s

$$\dot{m} = 999\frac{kg}{m^3}\times 3.05\frac{m}{s}\times 2600\,mm^2\times\frac{m^2}{10^6 mm^2} = 7.92\ kg/s$$

Thus

$$R_x = -96\times10^3\frac{N}{m^2}\times 2600\,mm^2\times\frac{m^2}{10^6 mm^2} + 7.92\frac{kg}{s}(-3.05-12.2)\frac{m}{s}\times\frac{N\cdot s^2}{kg\cdot m}$$

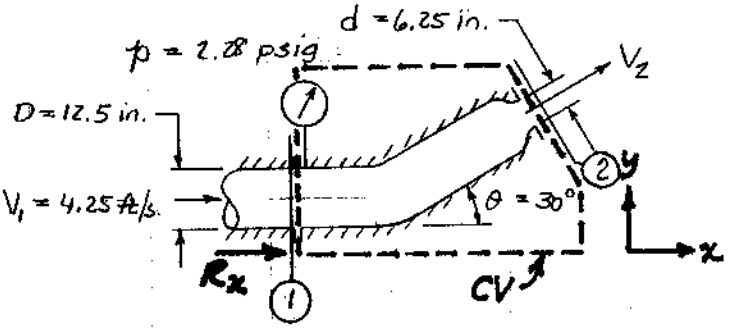
$$R_x = -370\ N \quad \text{(to the left)}$$

R_x ←

Given: Water flow through nozzle shown, discharging to p_{atm}.

Find: (a) Horizontal force component in the joint.

(b) Indicate whether joint is in tension or compression.



Solution: Apply continuity & x momentum using CV & CS shown.

Basic equation: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$ ($F_{Bx} = 0$ (1); $\frac{\partial}{\partial t} = 0$ (2))

Assumptions: (1) $F_{Bx} = 0$ (4) Uniform flow at each section
(2) Steady flow (5) Use gage pressures
(3) Incompressible

Then

$$p_{1g}A_1 + R_x = u_1\{-|\rho V_1 A_1|\} + u_2\{+|\rho V_2 A_2|\}$$

$u_1 = V_1$ $\quad$ $u_2 = V_2\cos\theta$

so $R_x = V_1(-\rho V_1 A_1) + V_2\cos\theta(+\rho V_2 A_2) - p_{1g}A_1$

From continuity, $\rho V_1 A_1 = \rho V_2 A_2$ and $V_2 = V_1 A_1/A_2 = V_1\left(\frac{D_1}{D_2}\right)^2$

$$V_2 = 4.25\frac{ft}{s}\left(\frac{12.5\ in.}{6.25\ in.}\right)^2 = 17.0\ ft/s$$

$$R_x = \rho V_1 A_1(V_2\cos\theta - V_1) - p_{1g}A_1$$

$$= 1.94\frac{slug}{ft^3}\times 4.25\frac{ft}{s}\times\frac{\pi}{4}\left(\frac{12.5}{12}\right)^2 ft^2\left[17.0\frac{ft}{s}\times\cos 30° - 4.25\frac{ft}{s}\right]\frac{lbf\cdot s^2}{slug\cdot ft}$$

$$- 2.28\frac{lbf}{in.^2}\times\frac{\pi}{4}(12.5)^2\ in.^2$$

$R_x = -206$ lbf (on CV; to left, since <0) $\quad R_x$ ←

Thus $K_x = -R_x = +206$ lbf (from CV on joint)

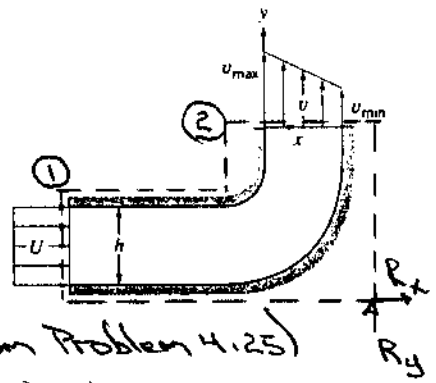
∴ joint is in tension

Given: Two-dimensional square bend shown is a segment of a larger channel, lies in horizontal plane.

$V = 7.5$ m/s, $h = w = 75.5$ mm

$p_1 = 170$ kPa(abs), $p_2 = 130$ kPa(abs)

$v_{max} = 2v_{min}$; $v_{min} = 5.0$ m/s (from Problem 4.25)



Find: Force required to hold the bend in place.

Solution:

Basic equation: $\vec{F}_s + \vec{F}_B = \cancel{\frac{\partial}{\partial t}\int_{CV} \vec{V}\,\rho\,d\forall}^{=0(1)} + \int_{CS} \vec{V}(\rho \vec{V}\cdot d\vec{A})$

Assumptions: (1) steady flow
(2) $F_{B_x} = F_{B_y} = 0$
(3) incompressible flow
(4) atmospheric pressure acts on outside surfaces.

The x-momentum equation becomes

$R_x + p_1 A_1 + \cancel{F_{B_x}}^{=0(2)} = \int_{CS} u(\rho \vec{V}\cdot d\vec{A}) = U\{-|\rho U A_1|\}$

$R_x = -p_1 A_1 - \rho U^2 A_1 = -h^2(p_1 + \rho U^2)$

$R_x = -(0.0755)^2 m^2 \left[(170-101)10^3 \frac{N}{m^2} + 999 \frac{kg}{m^3}\times(7.5)^2 \frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}\right] = -714$ N

The y-momentum equation becomes

$R_y - p_2 A_2 + \cancel{F_{B_y}}^{=0(2)} = \int_{CS} v(\rho \vec{V}\cdot d\vec{A})$

$v_2 = V_2 = v_{max} - (v_{max} - v_{min})\frac{x}{h} = 2v_{min} - v_{min}\frac{x}{h} = v_{min}\left(2 - \frac{x}{h}\right)$

$R_y - p_2 A_2 = \int_0^h v_{min}\left(2-\frac{x}{h}\right)\rho v_{min}\left(2-\frac{x}{h}\right)h\,dx$

$R_y = p_2 A_2 + \rho v_{min}^2 h \int_0^h \left(4 - 4\frac{x}{h} + \frac{x^2}{h^2}\right)dx$

$= p_2 A_2 + \rho v_{min}^2 h \left[4x - 2\frac{x^2}{h} + \frac{x^3}{3h^2}\right]_0^h$

$R_y = p_2 A_2 + \rho v_{min}^2 h\left[4h - 2h + \frac{h}{3}\right] = p_2 A_2 + \frac{7}{3}\rho v_{min}^2 h^2$

$R_y = h^2\left(p_2 + \frac{7}{3}\rho v_{min}^2\right)$

$= (0.0755)^2 m^2 \left[(130-101)10^3 \frac{N}{m^2} + \frac{7}{3}\times 999\frac{kg}{m^3}\times(5.0)^2\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}\right]$

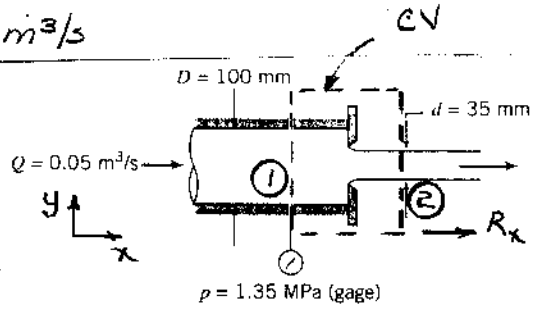
$R_y = 498$ N

$\therefore \vec{R} = -714\hat{i} + 498\hat{j}$ N ← $\vec{R}$

Given: Flat plate orifice at end of pipe, as shown.

$D = 100$ mm, $d = 35$ mm, $Q = 0.05\ m^3/s$

Neglect friction on pipe wall.



Find: Force to hold orifice plate.

Solution: Apply the x component of the momentum equation. The CV and CS are shown.

Basic equation:

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho dV + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

($F_{Bx} = 0$ (1); $\frac{\partial}{\partial t} = 0$ (2))

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Use gage pressures to cancel p_{atm}
(5) Incompressible flow

Then

$$Q = V_1 A_1 = V_2 A_2;\quad V_1 = \frac{Q}{A_1} = \frac{4Q}{\pi D_1^2} = \frac{4}{\pi} \times 0.05 \frac{m^3}{s} \times \frac{1}{(0.1)^2 m^2} = 6.37\ m/s$$

$$V_2 = V_1\left(\frac{D_1}{D_2}\right)^2 = 6.37 \frac{m}{s}\left(\frac{100}{35}\right)^2 = 52.0\ m/s$$

From momentum,

$$R_x + p_{1g} A_1 = u_1\{-\rho Q\} + u_2\{+\rho Q\} = (V_2 - V_1)\rho Q$$

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x = -p_{1g} A_1 + (V_2 - V_1)\rho Q$$

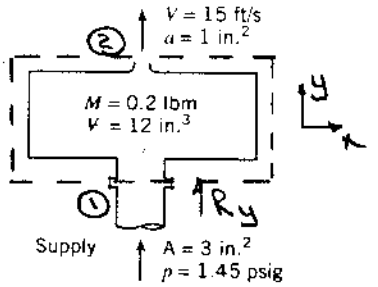
$$= -1.35 \times 10^6 \frac{N}{m^2} \times \frac{\pi}{4}(0.1)^2 m^2 + (52.0 - 6.37)\frac{m}{s} \times 999 \frac{kg}{m^3} \times 0.05 \frac{m^3}{s} \times \frac{N \cdot s^2}{kg \cdot m}$$

$R_x = -8.32$ kN (to left)

R_x

Given: Spray system, of mass $M = 0.200$ lbm and internal volume $\forall = 12$ in.^{3} operates under steady state conditions shown.

Find: the vertical force exerted on the supply pipe by the spray system.



Solution:

Apply the y component of the momentum equation to the fixed control volume shown.

Basic Equation:

$$F_{sy} + F_{By} = \cancel{\frac{\partial}{\partial t}\int_{cv} v\,\rho\,d\forall}^{a(1)} + \int_{cs} v\,\rho\,\vec{V}\cdot d\vec{A} \qquad (1)$$

Assumptions: (1) steady flow
(2) incompressible flow
(3) uniform flow at each section
(4) calculation of surface forces is simplified through use of gage pressures

From continuity, $0 = \cancel{\frac{\partial}{\partial t}\int_{cv}\rho\,d\forall}^{=0} + \int_{cs}\rho\,\vec{V}\cdot d\vec{A}$, for given conditions

$$0 = -|\rho V_1 A_1| + |\rho V_2 A_2| \quad \text{and} \quad V_1 = V_2\frac{A_2}{A_1} = V\frac{a}{A}$$

The momentum flux is

$$\int_{cs} v\,\rho\,\vec{V}\cdot d\vec{A} = v_1\{-|\rho V_1 A_1|\} + v_2\{|\rho V_2 A_2|\} = V_1(-\rho V_1 A_1) + V(\rho V a)$$

$$= V\frac{a}{A}(-\rho V a) + V(\rho V a) = \rho V^2 a\left(1 - \frac{a}{A}\right)$$

Then from eq. (1) we can write

$$R_y + p_{1g}A - \rho\forall g - Mg = \rho V^2 a\left(1 - \frac{a}{A}\right)$$

Solving for R_y,

$$R_y = -p_{1g}A + \rho\forall g + Mg + \rho V^2 a\left(1 - \frac{a}{A}\right)$$

$$= -1.45\ \frac{lbf}{in^2}\times 3\,in^2 + 1.94\ \frac{slug}{ft^3}\times 12\,in^3\times 32.2\ \frac{ft}{s^2}\times\frac{ft^3}{1728\,in^3}\times\frac{lbf\cdot s^2}{slug\cdot ft}$$

$$+ 0.2\,lbm\times 32.2\frac{ft}{s^2}\times\frac{slug}{32.2\,lbm}\times\frac{lbf\cdot s^2}{slug\cdot ft}$$

$$+ 1.94\ \frac{slug}{ft^3}\times(15)^2\frac{ft^2}{s^2}\times 1\,in^2\times\frac{ft^2}{144\,in^2}\times\frac{lbf\cdot s^2}{slug\cdot ft}\left(1 - \frac{1\,in^2}{3\,in^2}\right)$$

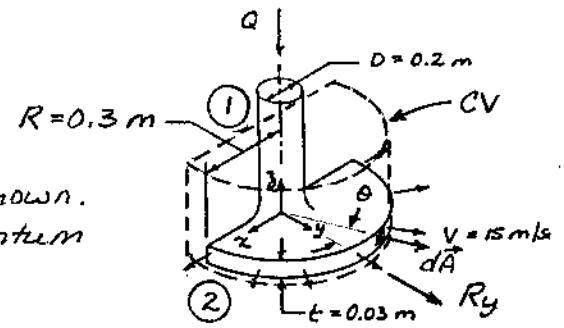
$$R_y = -1.70\ lbf$$

The force of the spray system on the supply pipe is

$$K_y = -R_y = 1.70\ lbf\ (\text{upward})$$

Given: Flow through semi-circular nozzle, as shown.

Find: (a) Volume flow rate
(b) y-component of force required to hold in place



Solution: Choose CV and coordinates shown. Apply continuity and momentum equation in y-direction.

Basic equations: $Q = \int_A \vec{V} \cdot d\vec{A}$

$$F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall + \int_{CS} v\rho \vec{V}\cdot d\vec{A}$$

($F_{By} = 0$ (2); $\frac{\partial}{\partial t} = 0$ (3))

Assumptions: (1) Flow uniform across exit section
(2) $F_{By} = 0$
(3) Steady flow

At section (2), $\vec{V}\cdot d\vec{A} = VRt\,d\theta$, since flow out of CV. Then

$$Q = \int_{-\pi/2}^{\pi/2} VRt\,d\theta = VRt\left[\theta\right]_{-\pi/2}^{\pi/2} = VRt\pi$$

$$Q = 15\frac{m}{s} \times 0.3\,m \times 0.03\,m \times \pi = 0.424\ m^3/s \qquad Q$$

From momentum

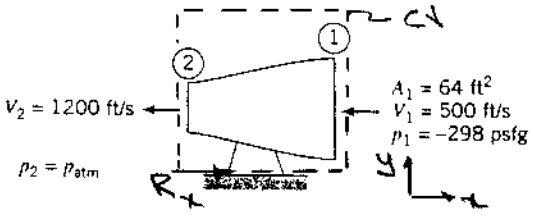
$$R_y = \int_{CS} v\rho\vec{V}\cdot d\vec{A} = \int_{A_1} v_1\left\{-\rho V_1 dA_1\right\} + \int_{A_2} v_2\left\{+\rho V_2 dA_2\right\}$$

with $v_1 = 0$ $\qquad v_2 = V\cos\theta$

$$R_y = \int_{-\pi/2}^{\pi/2} V\cos\theta\,\rho V Rt\,d\theta = \rho V^2 Rt\left[\sin\theta\right]_{-\pi/2}^{\pi/2} = 2\rho V^2 Rt$$

$$R_y = 2 \times 999\frac{kg}{m^3} \times (15)^2\frac{m^2}{s^2} \times 0.3\,m \times 0.03\,m \times \frac{N\cdot s^2}{kg\cdot m} = 4.05\ kN \qquad R_y$$

Given: Jet engine on test stand.
Fuel enters vertically
at rate $\dot{m}_{fuel} = 0.02\,\dot{m}_{air}$



Find: (a) Air flow rate
(b) Estimate of engine thrust

Solution:

Apply x-component of the momentum equation to CV shown

Basic equations: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\,d\forall + \int_{CS} u\rho\,\vec{V}\cdot d\vec{A}$

$\dot{m}_{air} = \rho_1 V_1 A_1$, $\rho = p/RT$

Assumptions: (1) $F_{Bx} = 0$
(2) steady flow
(3) uniform flow at inlet and outlet sections
(4) air behaves as ideal gas; $T_1 = 70°F$
(5) fuel enters vertically (given)

$$\rho_1 = \frac{p_1}{RT_1} = \left(14.7\frac{lbf}{in^2}\times 144\frac{in^2}{ft^2} - 298\frac{lbf}{ft^2}\right)\times\frac{lbm°R}{53.3\,ft\cdot lbf}\times\frac{1}{530°R} = 0.0644\frac{lbm}{ft^3}$$

$$\dot{m}_{air} = \rho_1 V_1 A_1 = 0.0644\frac{lbm}{ft^3}\times 500\frac{ft}{s}\times 64\,ft^2 = 2060\,lbm/s \quad \dot{m}$$

From the x-momentum equation

$$R_x - p_{1g}A_1 + p_{2g}A_2 = u_1\{-\dot{m}_1\} + u_2\{\dot{m}_2\} + u_f\{-\dot{m}_f\}$$

($p_{2g}A_2 = 0$ (2); $u_f = 0$ (5))

$u_1 = -V_1$, $u_2 = -V_2$, $\dot{m}_2 = \dot{m}_1 + \dot{m}_f$

Also thrust $T = R_x$ (force of engine on surroundings) $= -R_x$

So

$$-T - p_{1g}A_1 = \dot{m}_1 V_1 - \dot{m}_2 V_2 = \dot{m}_1 V_1 - (1.02\dot{m}_1)V_2$$

$$T = \dot{m}_1(1.02V_2 - V_1) - p_{1g}A_1$$

$$T = 2060\frac{lbm}{s}\left[1.02\times 1200\frac{ft}{s} - 500\frac{ft}{s}\right]\times\frac{slug}{32.2\,lbm}\times\frac{lbf\cdot s^2}{ft\cdot slug} - \left(-298\frac{lbf}{ft^2}\right)64\,ft^2$$

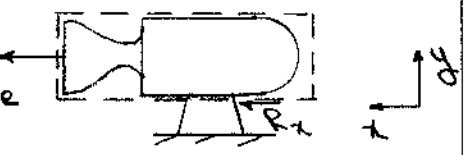
$$T = 65{,}400\,lbf \quad T$$

Given: Liquid-fueled rocket motor consumes 180 lbm/s of nitric acid as oxidizer and 70 lbm/s of aniline as fuel. Flow leaves axially at $V = 6000$ ft/s relative to nozzle and $p = 16.5$ psia. Nozzle exit diameter $D = 2$ ft. Motor run on test stand at standard sea-level.

Find: Thrust produced by the motor on test stand.

Solution: Apply x-component of momentum equation to CV shown.

Basic eq.: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$ (=0 (1); ≠0 (2))



Assumptions: (1) $F_{Bx} = 0$
(2) Neglect $\partial/\partial t$ of x momentum inside CV.
(3) uniform flow at nozzle exit

Then

$$R_x - p_{eg}A_e = V_e\dot{m}$$

where $\dot{m} = \dot{m}_{n.a} + \dot{m}_a = (180 + 70)\text{ lbm/s} = 250\text{ lbm/s}$

R_x is force from test stand on CV

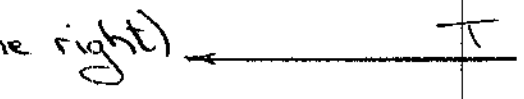
$$\therefore R_x = p_{eg}A_e + V_e\dot{m} = p_{eg}\frac{\pi}{4}D_e^2 + V_e\dot{m}$$

$$= (16.5 - 14.7)\frac{\text{lbf}}{\text{in}^2} \times \frac{\pi}{4}(2)^2\text{ft}^2 \times 144\frac{\text{in}^2}{\text{ft}^2} + 6000\frac{\text{ft}}{\text{s}} \times 250\text{ lbm} \times \frac{\text{slug}}{32.2\text{ lbm}} \times \frac{\text{lbf}\cdot\text{s}^2}{1\text{ slug}\cdot\text{ft}}$$

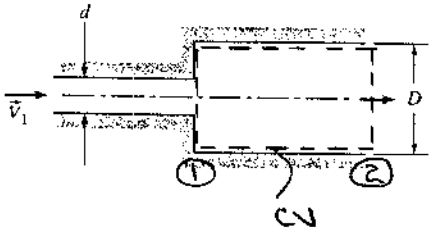
$$R_x = 814\text{ lbf} + 46{,}600\text{ lbf} = 47{,}400\text{ lbf}$$

The thrust of the motor, $T = -R_x$

$$\vec{T} = -47{,}400\,\hat{i}\text{ lbf}\quad\text{(to the right)}$$



Given: Incompressible, frictionless flow through a sudden expansion as shown



Show: Pressure rise, $\Delta p = p_2 - p_1$, is given by

$$\frac{\Delta p}{\frac{1}{2}\rho \bar{V}_1^2} = 2\left(\frac{d}{D}\right)^2\left[1-\left(\frac{d}{D}\right)^2\right]$$

Plot: the nondimensional pressure rise vs d/D to determine the optimum d/D and corresponding nondimensional pressure rise

Solution:

Apply x component of momentum equation, using fixed c.v. shown

Basic equation: $F_{Sx} + \cancel{F_{Bx}}^{=0\,(2)} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}^{=0\,(3)} + \int_{CS} u(\rho \vec{V}\cdot d\vec{A})$

Assumptions: (1) no friction, so surface force due to pressure only
(2) $F_{Bx} = 0$
(3) steady flow (4) incompressible flow (given).
(5) uniform flow at sections ① and ②
(6) uniform pressure p_1 on vertical surface of expansion

Then,

$$p_1 A_2 - p_2 A_2 = u_1\{-|\rho \bar{V}_1 A_1|\} + u_2\{|\rho \bar{V}_2 A_2|\} \qquad u_1 = \bar{V}_1,\ u_2 = \bar{V}_2$$

From continuity for uniform flow, $\dot{m} = \rho A_1 \bar{V}_1 = \rho A_2 \bar{V}_2$; $\bar{V}_2 = \bar{V}_1 \frac{A_1}{A_2}$

Thus,

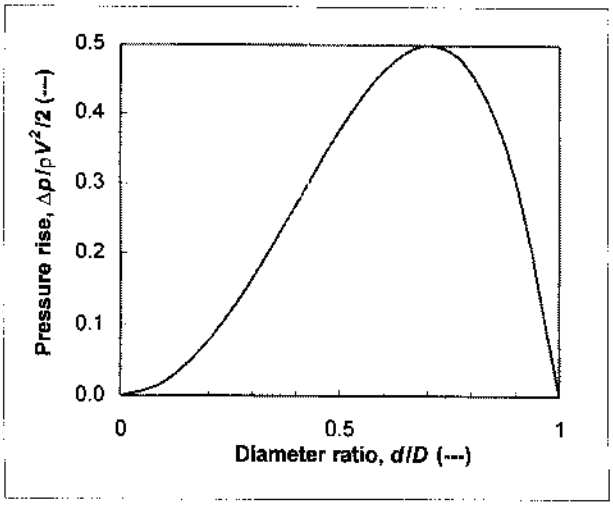
$$p_2 - p_1 = \rho \bar{V}_1 \frac{A_1}{A_2} V_1 - \rho \bar{V}_1 \frac{A_1}{A_2} V_2 = \rho \bar{V}_1 \frac{A_1}{A_2}(\bar{V}_1 - \bar{V}_2)$$

$$p_2 - p_1 = \rho \bar{V}_1^2 \frac{A_1}{A_2}\left(1 - \frac{\bar{V}_2}{\bar{V}_1}\right) = \rho \bar{V}_1^2 \frac{A_1}{A_2}\left(1 - \frac{A_1}{A_2}\right).$$

and

$$\frac{p_2 - p_1}{\frac{1}{2}\rho \bar{V}_1^2} = 2\frac{A_1}{A_2}\left(1 - \frac{A_1}{A_2}\right) = 2\left(\frac{d}{D}\right)^2\left[1-\left(\frac{d}{D}\right)^2\right] \qquad \text{Q.E.D.}$$

From the plot below we see that $\frac{\Delta p}{\frac{1}{2}\rho \bar{V}_1^2}$ has an optimum value of ≈ 0.5 at $d/D = 0.70$

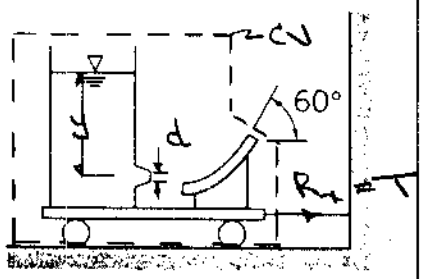


Note: As expected
- for $d = D$, $\Delta p = 0$ for straight pipe
- for $\frac{d}{D} \to 0$, $\Delta p = 0$ for free jet

Also note that the location of section ② would have to be chosen with care to make assumption (5) reasonable

Given: Water discharges from tank of height $h = 1\,m$ and diameter $D = 0.6\,m$, through a nozzle of diameter $d = 50\,mm$.

$V_{jet} = \sqrt{2gy}$ where y is height of free surface above the nozzle



Find: (a) Tension in wire holding the cart when $y = 0.8\,m$
(b) Force of the vane on the cart for $y = 0.8\,m$

Plot: Tension in the wire and force of vane on cart vs water depth ($0 \le y \le 0.8\,m$) with angle θ as a parameter (for $0 \le \theta \le 60°$).

Solution:

(a) To find the force in the wire, apply the x momentum equation to the CV which cuts the wire and encloses the cart.

Basic equation: $F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$ ($F_{B_x} = 0$ (2); $\frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV = 0$ (3))

Assumptions: (1) no net pressure force, so $F_{S_x} = R_x$
(2) $F_{B_x} = 0$
(3) steady flow
(4) uniform flow at each section of jet
(5) jet area and velocity constant across vane

Then

$$R_x = u_2\{|\rho V_2 A_2|\} = V_2\cos\theta\,\rho V_2 A_2 = \rho V^2 A\cos\theta$$

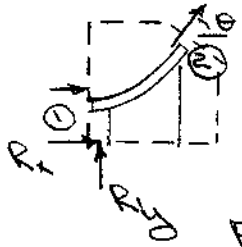
$$R_x = \rho\,2gy\,\frac{\pi}{4}d^2\cos\theta = \rho g y\cos\theta\,\frac{\pi}{2}d^2 \qquad (1)\ R_x$$

{the wire is in tension}

Evaluating for $y = 0.8\,m$

$$T = R_x = 999\,\frac{kg}{m^3}\times 9.81\,\frac{m}{s^2}\times 0.8\,m\times\cos 60°\times\frac{\pi}{2}(0.010)^2\,m^2\times\frac{N\cdot s^2}{kg\cdot m} = 0.616\,N \quad T$$

(b) To find the force of the vane on the cart we select a CV enclosing the vane.



Applying the x momentum equation

$$R_x = u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho_2 V_2 A_2|\} \qquad \begin{cases} u_1 = V \\ u_2 = V\cos\theta \end{cases}$$

$$R_x = -\rho V^2 A + \rho V^2 A\cos\theta$$

Force of vane on cart is $F_x = -R_x$

$$F_x = \rho V^2 A(1-\cos\theta) = \rho g y\,\frac{\pi}{2}d^2(1-\cos\theta) \qquad (2)$$

Applying the y momentum equation

$$R_y = v_1\{-|\rho_1 V_1 A_1|\} + v_2\{|\rho_2 V_2 A_2|\} \qquad \begin{cases} v_1 = 0 \\ v_2 = V\sin\theta \end{cases}$$

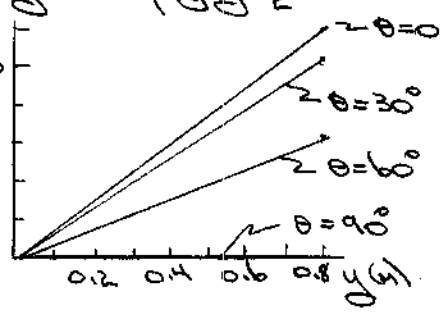
$$R_y = V\sin\theta\,\rho V A = \rho V^2 A \sin\theta = \rho^2 g y \pi \frac{d^2}{4}\sin\theta$$

$$R_y = \rho g y \pi \frac{d^2}{2}\sin\theta \qquad \text{(3)}$$

Evaluating: from Eq. 2 $R_x = 0.616$ N

from Eq. 3 $K_y = -R_y = 1.07$ N

$\therefore \vec{K} = 0.616\,\hat{i} - 1.07\,\hat{j}$ N



The tension in the wire is given by $T = \rho g y \frac{\pi d^2}{2}\cos\theta$

$\theta = 0$, $T = 1.54\,y$

$\theta = 30°$, $T = 1.33\,y$

$\theta = 60°$, $T = 0.770\,y$

$\theta = 90°$, $T = 0$

The horizontal force of the vane on the cart is given by

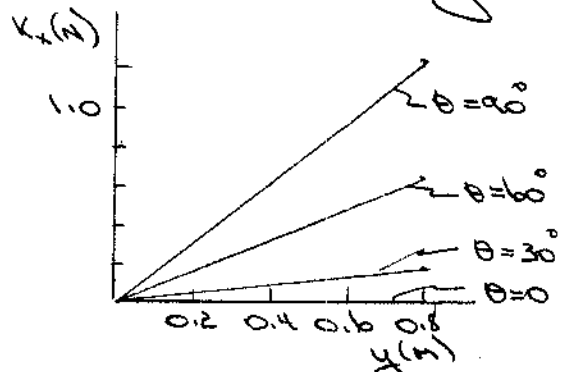
$$K_x = \rho g y \pi \frac{d^2}{2}(1-\cos\theta).$$

$\theta = 0$, $K_x = 0$

$\theta = 30°$, $K_x = 0.206\,y$

$\theta = 60°$, $K_x = 0.770\,y$

$\theta = 90°$, $K_x = 1.54\,y$



The vertical force of the vane on the cart is given by

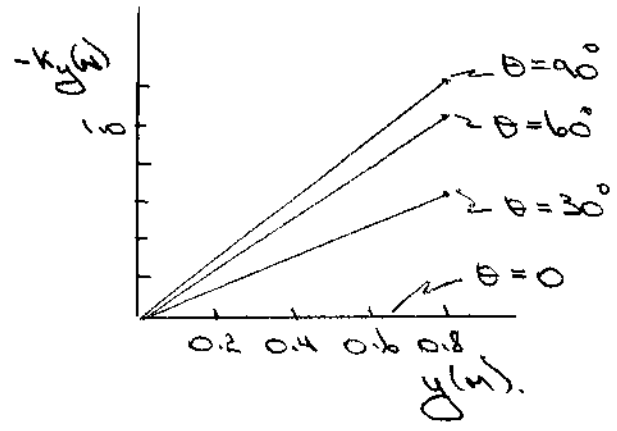
$$K_y = -\rho g y \pi \frac{d^2}{2}\sin\theta$$

$\theta = 0$, $K_y = 0$

$\theta = 30°$, $K_y = -0.770\,y$

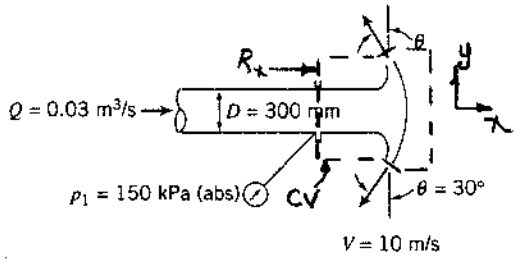
$\theta = 60°$, $K_y = -1.33\,y$

$\theta = 90°$, $K_y = -1.54\,y$



Given: Conical spray head discharging water, as shown.

Find: (a) Thickness of spray sheet at $R = 400$ mm radius.
(b) Axial force exerted on supply pipe.



Solution: Apply continuity and the x component of the momentum equation, using the CV, CS shown.

Basic equation:

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\, \vec{V}\cdot d\vec{A}$$

($F_{Bx} = 0$ (1); $\frac{\partial}{\partial t} = 0$ (2))

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow.
(3) Incompressible flow
(4) Uniform flow at each section
(5) Use gage pressure to cancel p_{atm}

From continuity,

$$V_1 = \frac{Q}{A_1} = \frac{4Q}{\pi D_1^2} = \frac{4}{\pi} \times 0.03\frac{m^3}{sec} \times \frac{1}{(0.3)^2 m^2} = 0.424 \text{ m/s}$$

Assume velocity in jet sheet is constant at $V = 10$ m/s. Then

$$Q = 2\pi R t V;\quad t = \frac{Q}{2\pi R V} = \frac{1}{2\pi} \times 0.03\frac{m^3}{s} \times \frac{1}{0.4\,m} \times \frac{s}{10\,m} \times 1000\frac{mm}{m} = 1.19 \text{ mm}$$

t

From momentum,

$$R_x + p_{1g} A_1 = u_1\{-\rho Q\} + u_2\{+\rho Q\}$$

$$u_1 = V_1 \qquad u_2 = -V\sin\theta$$

$$R_x + p_{1g} A_1 = -(V_1 + V\sin\theta)\rho Q$$

or

$$R_x = -p_{1g} A_1 - (V_1 + V\sin\theta)\rho Q$$

$$= -(150 - 101)10^3 \frac{N}{m^2} \times \frac{\pi}{4}(0.3)^2 m^2 - (0.424 + 10\sin 30°)\frac{m}{s} \times 999\frac{kg}{m^3} \times 0.03\frac{m^3}{s} \times \frac{N\cdot s^2}{kg\cdot m}$$

$$R_x = -3.63 \text{ kN}$$

But R_x is force on CV; force on supply pipe is K_x,

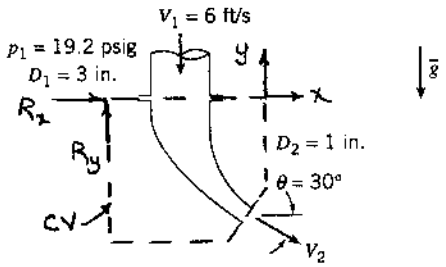
$$K_x = -R_x = 3.63 \text{ kN (to the right)}$$

K_x

Given: Curved nozzle assembly, as shown.

$W = 10$ lbf $\forall = 150$ in.^{3}

Fluid is water.



Find: Force of nozzle assembly on inlet pipe.

Solution: Apply the x and y components of the momentum equation, using the CV and coordinates shown. Use gage pressures to cancel p_{atm}.

Basic equations:

$$F_{Sx}^{=0(1)} + F_{Bx}^{=0(2)} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

$$F_{Sy} + F_{By} = \overbrace{\frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall}^{=0(3)} + \int_{CS} v\rho\vec{V}\cdot d\vec{A}$$

Assumptions: (1) $F_{Sx} = 0$
(2) $F_{Bx} = 0$
(3) Steady flow
(4) Uniform flow at each section
(5) Incompressible flow

From continuity,

$$V_1 A_1 = V_2 A_2 \;;\; V_2 = V_1\frac{A_1}{A_2} = V_1\left(\frac{D_1}{D_2}\right)^2 = 6\,\frac{ft}{s}\left(\frac{3\,in.}{1\,in.}\right)^2 = 54\ ft/s$$

From x component of momentum,

$$R_x = \overbrace{u_1\{-|\rho V_1 A_1|\}}^{=0} + u_2\{+|\rho V_2 A_2|\} = \rho V_2^2 A_2 \cos\theta$$

$$R_x = 1.94\,\frac{slug}{ft^3}\times(54)^2\frac{ft^2}{s^2}\;\frac{\pi}{4}(1)^2\,in.^2\times\cos 30°\times\frac{ft^2}{144\,in.^2}\times\frac{lbf\cdot s^2}{slug\cdot ft} = 26.7\ lbf$$

$K_x = -R_x = -26.7$ lbf (to left on pipe) K_x

From y component of momentum,

$$R_y - p_{1g}A_1 - W - \rho g\forall = v_1\{-|\rho V_1 A_1|\} + v_2\{+|\rho V_2 A_2|\} = (V_1 - V_2\sin\theta)\rho V_1 A_1$$

$$v_1 = -V_1 \qquad v_2 = -V_2\sin\theta$$

$$R_y = p_{1g}A_1 + W + \rho g\forall + (V_1 - V_2\sin\theta)\rho V_1 A_1$$

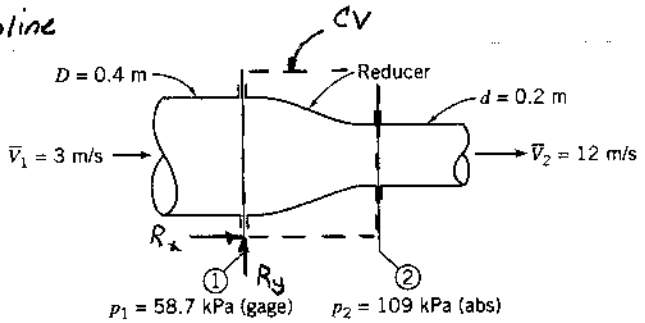
$$= 19.2\,\frac{lbf}{in.^2}\times\frac{\pi}{4}(3)^2\,in.^2 + 10\ lbf + 1.94\,\frac{slug}{ft^3}\times 32.2\,\frac{ft}{s^2}\times 150\,in.^3\times\frac{ft^3}{1728\,in.^3}\times\frac{lbf\cdot s^2}{slug\cdot ft}$$

$$+\left[6 - 54\left(\tfrac{1}{2}\right)\right]\frac{ft}{s}\times 1.94\,\frac{slug}{ft^3}\times 6\,\frac{ft}{s}\times\frac{\pi}{4}(3)^2\,in.^2\times\frac{ft^2}{144\,in.^2}\times\frac{lbf\,s^2}{slug\cdot ft}$$

$R_y = 139$ lbf ; $K_y = -R_y = -139$ lbf (down on pipe) K_y

Given: Flow through reducer in gasoline piping system, as shown.

$M = 25\ kg \quad \forall = 0.2\ m^3$



Find: Force needed to hold reducer in place.

Solution: Apply the x and y components of the momentum equation, using the CV and coordinates shown. Use gage pressures to cancel p_{atm}.

Basic equations:

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A} \qquad (=0 \text{ (1)}, =0 \text{ (2)})$$

$$F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall + \int_{CS} v\rho\vec{V}\cdot d\vec{A} \qquad (=0 \text{ (2)})$$

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Incompressible flow, SG = 0.72 {Table A.2, Appendix A}

From the x component of momentum,

$$R_x + p_{1g}A_1 - p_{2g}A_2 = u_1\{-|\rho V_1 A_1|\} + u_2\{+|\rho V_2 A_2|\} = (V_2 - V_1)\rho V_1 A_1$$

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x = p_{2g}A_2 - p_{1g}A_1 + (\bar{V}_2 - \bar{V}_1)\rho\bar{V}_1 A_1 \qquad \text{Note } \rho = SG\,\rho_{H_2O}$$

$$= (109-101)10^3\frac{N}{m^2}\times\frac{\pi}{4}(0.2)^2 m^2 - 58.7\times10^3\frac{N}{m^2}\times\frac{\pi}{4}(0.4)^2 m^2$$

$$+ (12-3)\frac{m}{s}\times(0.72)1000\frac{kg}{m^3}\times 3\frac{m}{s}\times\frac{\pi}{4}(0.4)^2 m^2\times\frac{N\cdot s^2}{kg\cdot m}$$

$R_x = -4.68\ kN$ (force must be applied to left) R_x

From the y component of momentum,

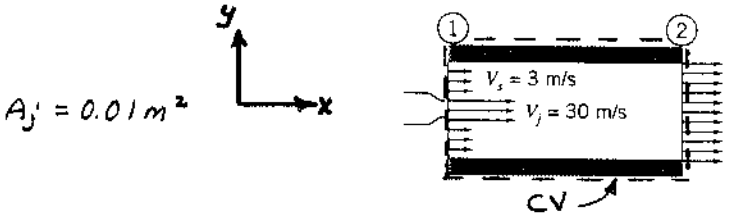
$$R_y - Mg - \rho g\forall = v_1\{-|\rho V_1 A_1|\} + v_2\{+|\rho V_2 A_2|\} \qquad (v_1 = 0,\ v_2 = 0)$$

$$R_y = Mg + \rho g\forall$$

$$= 25\ kg\times 9.81\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} + (0.72)1000\frac{kg}{m^3}\times 9.81\frac{m}{s^2}\times 0.2\ m^3\times\frac{N\cdot s^2}{kg\cdot m}$$

$R_y = 1.66\ kN$ (force must be applied up) R_y

Given: Water jet pump as shown in the sketch.



$A_j = 0.01\ m^2$ $\quad A_2 = 0.075\ m^2$

The two streams are thoroughly mixed at section (2), and the inlet pressures are the same.

Find: (a) The velocity at the pump exit
(b) The pressure rise, $p_2 - p_1$

Solution: Apply continuity and the x component of momentum to the inertial CV shown.

Basic equations:

$$0 = \underbrace{\frac{\partial}{\partial t}}_{=0(1)} \int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

$$F_{Sx} + \underbrace{F_{Bx}}_{=0(5)} = \underbrace{\frac{\partial}{\partial t}}_{=0(1)} \int_{CV} u \rho\, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) No viscous forces act on CV
(5) $F_{Bx} = 0$

Then from continuity

$$0 = \{-|\rho V_s A_s|\} + \{-|\rho V_j A_j|\} + \{|\rho V_2 A_2|\} = -\rho V_s A_s - \rho V_j A_j + \rho V_2 A_2$$

$$V_2 = \frac{1}{A_2}(V_s A_s + V_j A_j) \quad ; \quad A_s = A_2 - A_j = (0.075 - 0.01)\,m^2 = 0.065\ m^2$$

$$V_2 = \frac{1}{0.075\ m^2}\left(3\frac{m}{s} \times 0.065\ m^2 + 30\frac{m}{s} \times 0.01\ m^2\right) = 6.60\ \frac{m}{s}$$ ← V_2

and

$$p_1 A_2 - p_2 A_2 = u_s\{-|\rho V_s A_2|\} + u_j\{-|\rho V_j A_j|\} + u_2\{|\rho V_2 A_2|\}$$

$$u_s = V_s \qquad u_j = V_j \qquad u_2 = V_2$$

$$\Delta p = p_2 - p_1 = \frac{1}{A_2}\left(+\rho V_s^2 A_s + \rho V_j^2 A_j - \rho V_2^2 A_2\right) = \frac{\rho}{A_2}\left(+V_s^2 A_s + V_j^2 A_j - V_2^2 A_2\right)$$

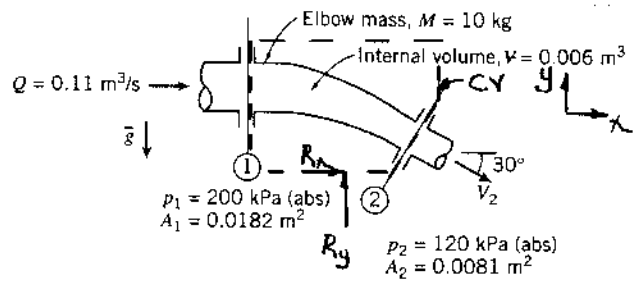
$$= 999\frac{kg}{m^3} \times \frac{1}{0.075\ m^2}\left[(3.0)^2(0.065) + (30)^2(0.01) - (6.6)^2(0.075)\right]\frac{m^4}{s^2} \times m^2 \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_2 - p_1 = 84.2\ kPa$$ ← $p_2 - p_1$

Given: Reducing elbow shown.

Fluid is water.

Find: Force components needed to keep elbow from moving.



Solution: Apply the x and y components of the momentum equation using the CS and CV shown.

Basic equations: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$ ($F_{Bx}=0$(4), $\frac{\partial}{\partial t}\int_{CV}=0$(1))

$$F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v\rho d\forall + \int_{CS} v\rho \vec{V}\cdot d\vec{A}$$ ($F_{By}=0$(1))

Assumptions: (1) Steady flow (3) Use gage pressures
(2) Uniform flow at each section (4) x horizontal

x comp: $R_x + p_1 g A_1 - p_2 g A_2 \cos\theta = u_1\{-|\rho Q|\} + u_2\{+|\rho Q|\}$

$u_1 = V_1 \qquad u_2 = V_2\cos\theta$

$R_x = (-V_1 + V_2\cos\theta)\rho Q - p_1 g A_1 + p_2 g A_2\cos\theta$

$= \left(-6.04\frac{m}{s} + 13.6\frac{m}{s}\cos 30°\right) 999\frac{kg}{m^3}$

$\times 0.11\frac{m^3}{s}\times\frac{N\cdot s^2}{kg\cdot m} - (200-101)10^3\frac{N}{m^2}\times 0.0182\,m^2 + (120-101)10^3\frac{N}{m^2}\times 0.0081\,m^2\cos 30°$

$V_1 = \frac{Q}{A_1} = 0.11\frac{m^3}{s}\times\frac{1}{0.0182\,m^2} = 6.04\frac{m}{s}$

$V_2 = \frac{Q}{A_2} = 0.11\frac{m^3}{s}\times\frac{1}{0.0081\,m^2} = 13.6\frac{m}{s}$

$R_x = +631 - 1800 + 133\ N = -1040\ N$ ← R_x

y comp: $R_y + p_2 g A_2\sin\theta - Mg - \rho\forall g = v_1\{-|\rho Q|\} + v_2\{+|\rho Q|\}$

$v_1 = 0 \qquad v_2 = -V_2\sin\theta$

$R_y = -V_2\sin\theta\,\rho Q + Mg + \rho\forall g - p_2 g A_2\sin\theta$

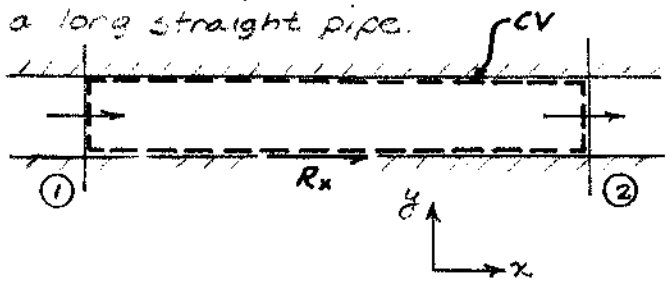
$= -13.6\frac{m}{s}\times\sin 30°\times 999\frac{kg}{m^3}\times 0.11\frac{m^3}{s}\times\frac{N\cdot s^2}{kg\cdot m} + 10\,kg\times 9.81\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}$

$+ 999\frac{kg}{m^3}\times 0.006\,m^3\times 9.81\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} - (120-101)10^3\frac{N}{m^2}\times 0.0081\,m^2\sin 30°$

$R_y = -747 + 98.1 + 58.8 - 77 = -667\ N$ ← R_y

{ R_x and R_y are the horizontal and vertical components of force that must be supplied by the adjacent pipes to keep the elbow (the control volume) from moving. }

Given: Air flow in a long straight pipe.



Find: Axial force of the air on the pipe.

Solution: Apply the x component of the momentum equation to the inertial CV shown. Also use continuity and ideal gas.

Basic equations: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$

($F_{Sx} = R_x + p_1A_1 - p_2A_2$; $u_1 = V_1$, $u_2 = V_2$)

$p = \rho RT$

$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Air behaves as an ideal gas

Then

$$R_x + p_1A_1 - p_2A_2 = u_1\{-|\rho_1V_1A_1|\} + u_2\{|\rho_2V_2A_2|\}$$

But from continuity $0 = \{-|\rho_1V_1A_1|\} + \{|\rho_2V_2A_2|\}$, so

$$R_x = p_2A_2 - p_1A_1 + u_1\{-|\rho_1V_1A_1|\} + u_2\{|\rho_1V_1A_1|\} \quad ; A_1 = A_2 = A$$

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x = (p_2 - p_1)A + (V_2 - V_1)\rho_1V_1A$$

From the ideal gas equation of state

$$\rho_1 = \frac{p_1}{RT_1} = 30\ \frac{lbf}{in^2} \times 144\ \frac{in^2}{ft^2} \times \frac{lbm\cdot °R}{53.3\ ft\cdot lbf} \times \frac{slug}{32.2\ lbm} \times \frac{1}{600°R} = 0.00420\ \frac{slug}{ft^3}$$

and

$$R_x = (11.3 - 30)\ \frac{lbf}{in^2} \times 0.5\ ft^2 \times 144\ \frac{in^2}{ft^2}$$

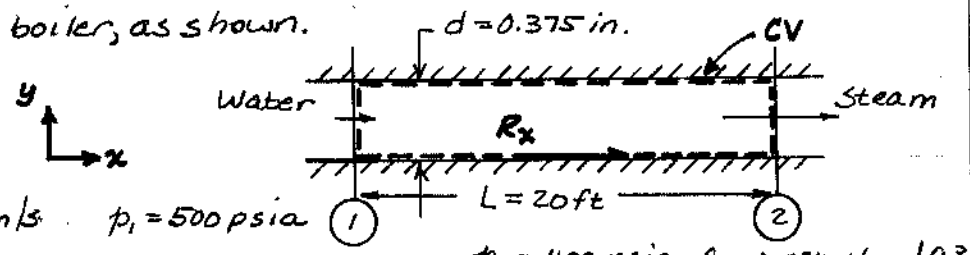
$$+ (985 - 500)\ \frac{ft}{s} \times 0.0042\ \frac{slug}{ft^3} \times 500\ \frac{ft}{s} \times 0.5\ ft^2 \times \frac{lbf\cdot s^2}{slug\cdot ft}$$

$R_x = -837\ lbf$ (this is force of the pipe wall on the CV)

The force of the gas on the pipe is then

$K_x = -R_x = 837\ lbf$ (to the right) $\leftarrow K_x$

Given: Monotube boiler, as shown.



$\dot{m} = 0.3$ lbm/s $\quad p_1 = 500$ psia $\quad p_2 = 400$ psig, $\rho_2 = 0.024$ slug/ft^3

Find: Magnitude and direction of force exerted by fluid on tube.

Solution: Apply the x component of the momentum equation, using the CV and coordinates shown.

Basic equation: $\quad F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$ $\quad (F_{Bx}=0,\ \frac{\partial}{\partial t}=0)$

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Use gage pressures to cancel patm

From continuity,

$\dot{m} = \rho_1 V_1 A_1 = \rho_2 V_2 A_2$; A = constant, so $\rho_1 V_1 = \rho_2 V_2$. Thus

$$V_1 = \frac{\dot{m}}{\rho_1 A} = 0.3\frac{\text{lbm}}{s}\times\frac{\text{ft}^3}{1.94\,\text{slug}}\times\frac{4}{\pi}\frac{1}{(0.375)^2\,\text{in.}^2}\times\frac{\text{slug}}{32.2\,\text{lbm}}\times 144\frac{\text{in.}^2}{\text{ft}^2} = 6.26\ \text{ft/s}$$

and

$$V_2 = V_1\frac{\rho_1}{\rho_2} = 6.26\frac{\text{ft}}{s}\times 1.94\frac{\text{slug}}{\text{ft}^3}\times\frac{\text{ft}^3}{0.024\,\text{slug}} = 506\ \text{ft/s}$$

From momentum,

$$R_x + p_{1g}A_1 - p_{2g}A_2 = u_1\{-\dot{m}\} + u_2\{+\dot{m}\} = (V_2 - V_1)\dot{m}$$

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x = (p_{2g} - p_{1g})A + (V_2 - V_1)\dot{m}$$

$$= [400 - (500 - 14.7)]\frac{\text{lbf}}{\text{in.}^2}\times\frac{\pi}{4}(0.375)^2\,\text{in.}^2 + (506 - 6.26)\frac{\text{ft}}{s}\times 0.3\frac{\text{lbm}}{s}\times\frac{\text{slug}}{32.2\,\text{lbm}}$$

$$\times\frac{\text{lbf}\cdot s^2}{\text{slug}\cdot\text{ft}}$$

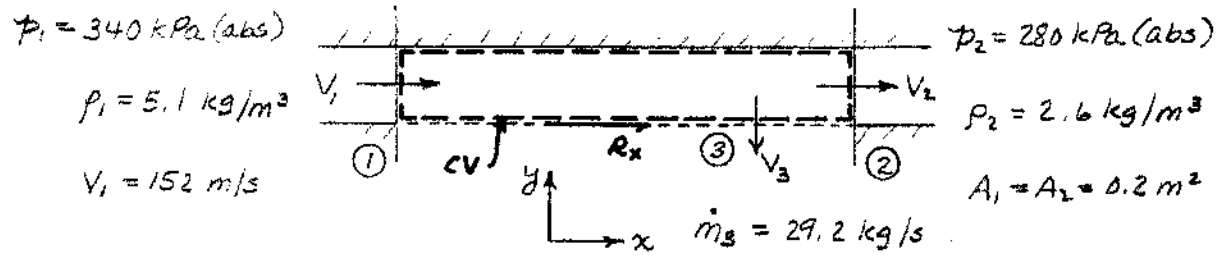
$$R_x = -4.77\ \text{lbf}$$

But R_x is force on CV; force on pipe is K_x,

$$K_x = -R_x = 4.77\ \text{lbf}\ (\text{to } \underline{\text{right}})$$

K_x

Given: Gas flow through a porous pipe of constant area.



V_3 is uniform over surface ③ and normal to pipe wall.

Find: Axial force of fluid on pipe.

Solution: Apply continuity and x component of momentum equation using inertial CV shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ (=0 (1))

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

(F_{Bx} = 0 (3); $\frac{\partial}{\partial t}$ = 0 (1))

Assumptions: (1) Steady flow
(2) Flow uniform at each section
(3) $F_{Bx} = 0$
(4) Flow at section ③ normal to wall; $u_3 = 0$

Then

$$0 = \{-|\rho_1 V_1 A|\} + \{|\rho_2 V_2 A|\} + \dot{m}_3 = -\rho_1 V_1 A + \rho_2 V_2 A + \dot{m}_3$$

$$V_2 = \frac{1}{\rho_2 A}\left[\rho_1 V_1 A - \dot{m}_3\right] = V_1 \frac{\rho_1}{\rho_2} - \frac{\dot{m}_3}{\rho_2 A}$$

$$V_2 = 152\frac{m}{s}\times 5.1\frac{kg}{m^3}\times\frac{m^3}{2.6\,kg} - 29.2\frac{kg}{s}\times\frac{m^3}{2.6\,kg}\times\frac{1}{0.2\,m^3} = 242\ m/s$$

and

$$R_x + p_1 A - p_2 A = u_1\{-|\rho_1 V_1 A|\} + u_2\{|\rho_2 V_2 A|\} + u_3\{|\dot{m}_3|\}$$ (u_3 = 0 (4))

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x = (p_2 - p_1 + \rho_2 V_2^2 - \rho_1 V_1^2)A$$

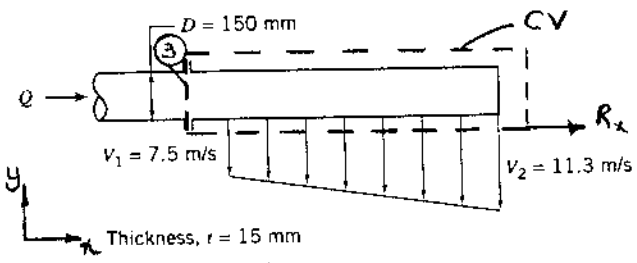
$$= \left[(280-340)10^3\frac{N}{m^2} + \left(2.6\frac{kg}{m^3}\times(242)^2\frac{m^2}{s^2} - 5.1\frac{kg}{m^3}\times(152)^2\frac{m^2}{s^2}\right)\frac{N\cdot s^2}{kg\cdot m}\right]0.2\,m^2$$

$R_x = -5.11$ kN (this is force of the duct wall on the gas)

The force of the gas on the duct wall is $K_x = -R_x = 5.11$ kN (acting to the right) ← K_x

Given: Water flow discharging nonuniformly from slot, as shown.

$p_{1g} = 30$ kPa



Find: (a) Volume flow rate.
(b) Forces to hold pipe.

Solution: Apply x, y components of momentum, using the CV, as shown.

Basic equations:

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A} \quad (F_{Bx}=0(1),\ u=0(2)) \qquad F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall + \int_{CS} v\rho\vec{V}\cdot d\vec{A} \quad (F_{By}=0(1),\ v=0(2))$$

Assumptions: (1) $F_{Bx} = F_{By} = 0$
(2) Steady flow
(3) Uniform flow at inlet section
(4) Use gage pressures to cancel p_{atm}

From continuity,

$$Q = \bar{V}A = \tfrac{1}{2}(V_1 + V_2)Lt = \frac{1}{2}(7.5+11.3)\frac{m}{s}\times 1m\times 0.015m = 0.141\ m^3/s$$ Q

$$V_3 = \frac{Q}{A_3} = 0.141\frac{m^3}{s}\times\frac{4}{\pi}\frac{1}{(0.15)^2 m^2} = 7.98\ m/s$$

From x momentum, since flow leaves slot vertically ($u=0$),

$$R_x + p_{3g}A_3 = u_3\{-\rho Q\} = -V_3\rho Q \;;\; R_x = -p_{3g}A_3 - V_3\rho Q$$

$$R_x = -30\times10^3\frac{N}{m^2}\times\frac{\pi}{4}(0.15)^2 m^2 - 7.98\frac{m}{s}\times 999\frac{kg}{m^3}\times 0.141\frac{m^3}{s}\times\frac{N\cdot s^2}{kg\cdot m}$$

$R_x = -1.65$ kN (to left) R_x

From y momentum, since $v_3 = 0$,

$$R_y = \overset{=0}{\cancel{v_3}}\{-\rho Q\} + \int_0^L v\rho V t\, dx = -\rho t\int_0^L \left(V_1 + \frac{V_2 - V_1}{L}x\right)^2 dx$$

$$= -\rho t\left[V_1^2 x + 2V_1\left(\frac{V_2-V_1}{L}\right)\frac{x^2}{2} + \left(\frac{V_2-V_1}{L}\right)^2\frac{x^3}{3}\right]_0^L$$

$$= -999\frac{kg}{m^3}\times 0.015\,m\left[(7.5)^2\frac{m^2}{s^2} + 7.5\frac{m}{s}\times(11.3-7.5)\frac{m}{s}\times\frac{1}{1m}\times(1)^2 m^2 + (11.3-7.5)^2\frac{m^2}{s^2}\times\frac{1}{(1)^2 m^2}\times\frac{(1)^3 m^3}{3}\right]$$

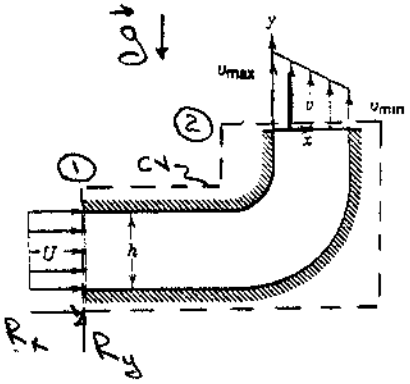
$R_y = -1.34$ kN (down) R_y

{A moment also would be required at the coupling.}

Given: Steady flow of water through square channel shown
$v_{max} = 2v_{min}$, $V = 7.5$ m/s, $p_1 = 185$ kPa (gage), $p_2 = p_{atm}$
$M_c = 2.05$ kg, $\forall_c = 0.00355$ m^3, $h = 75.5$ mm $= w$

Find: Force exerted by channel assembly on the supply duct.

Solution: Apply conservation of mass + momentum equations to the CV shown.



Basic equations:

$$0 = \cancel{\frac{\partial}{\partial t}\int_{CV}\rho\,d\forall}^{0(1)} + \int_{CS}\rho\vec{V}\cdot d\vec{A} \quad (1)$$

$$F_{S_x} + F_{B_x} = \cancel{\frac{\partial}{\partial t}\int_{CV}u\rho\,d\forall}^{0(1)} + \int_{CS}u\rho\vec{V}\cdot d\vec{A} \quad (2)$$

$$F_{S_y} + F_{B_y} = \cancel{\frac{\partial}{\partial t}\int_{CV}v\rho\,d\forall}^{0(1)} + \int_{CS}v\rho\vec{V}\cdot d\vec{A} \quad (3)$$

Assumptions:
(1) steady flow (2) incompressible flow
(3) uniform flow at inlet
(4) use gage pressures.

From continuity, $0 = \vec{V}_1\cdot\vec{A}_1 + \int\vec{V}_2\cdot d\vec{A}_2 = -VWh + \int_0^h vW\,dx$

$$\therefore Vh = \int_0^h v\,dx = \int_0^h v_{min}\left(2 - \frac{2x}{h}\right)dx = v_{min}\left[2x - \frac{2x^2}{2h}\right]_0^h = \tfrac{2}{3}v_{min}h$$

and $$v_{min} = \tfrac{3}{2}V = \tfrac{3}{2}\times 7.5 \tfrac{m}{s} = 5.0 \text{ m/s}$$

From Eq. 2,

$$R_x + p_{1g}A_1 = u_1\{-\rho VA_1\} + \cancel{\int_0^h u_2\,\rho\,v_{min}\left(2-\frac{2x}{h}\right)W\,dx}^{0} = -\rho V^2A_1$$

$$R_x = -p_{1g}A_1 - \rho V^2A_1 = -(185-101)10^3\tfrac{N}{m^2}\times(0.0755)^2m^2 - 999\tfrac{kg}{m^3}\times\left(7.5\tfrac{m}{s}\right)^2\times(0.0755)m^2$$

$$R_x = -479\text{ N} - 320\tfrac{kg\cdot m}{s^2}\cdot\tfrac{N\cdot s^2}{kg\cdot m} = -479\text{ N} - 320\text{ N} = -799\text{ N}$$

$K_x = -R_x = 799$ N (on supply duct to the right) → K_x

From Eq. 3,

$$R_y - M_cg - \rho\forall g = \cancel{v_1\{-\rho VA_1\}}^{v_1=0} + \int_0^h v_2\{\rho v_2W\,dx\}$$

$$R_y - M_cg - \rho\forall g = \int_0^h v_{min}\left(2-\frac{2x}{h}\right)\rho\,v_{min}\left(2-\frac{2x}{h}\right)W\,dx$$

$$= \rho V_{min}^2W\int_0^h\left(4 - 4\frac{2x}{h} + \frac{4x^2}{h^2}\right)dx$$

$$= \rho V_{min}^2W\left[4x - 2\frac{2x^2}{h} + \frac{4x^3}{3h^2}\right]_0^h = \rho V_{min}^2Wh\,\tfrac{8}{3}$$

$$\therefore R_y = \left[2.05\text{kg}\times 9.81\tfrac{m}{s^2} + 999\tfrac{kg}{m^3}\times(0.00355m^3)9.81\tfrac{m}{s^2} + \tfrac{8}{3}\times 999\tfrac{kg}{m^3}\times(5.0)^2\tfrac{m^2}{s^2}\times(0.0755)m^2\right]\tfrac{N\cdot s^2}{kg\cdot m}$$

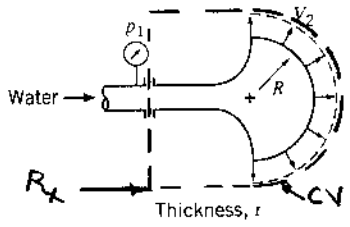
$$R_y = (20.1 + 34.8 + 332)\text{ N} = 387\text{ N (on CV)}$$

$K_y = -R_y = -387$ N (on supply duct, down) ← K_y

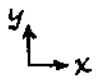
Given: Nozzle discharging flat, radial sheet of water, as shown.

Find: Axial force of nozzle on coupling.

$D_1 = 35$ mm



Solution: Apply the x component of momentum, using CV and coordinates shown.



Basic equation:

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

(F_{Sx}: $=0$ (1); $\frac{\partial}{\partial t}$: $=0$ (2))

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Use gage pressure to cancel p_{atm}

From continuity

$$Q = V_1 A_1 = V_2 A_2 = V_2 \pi R t = \pi \times 10 \frac{m}{sec} \times 0.05\, m \times 0.0015\, m = 0.00236\ m^3/s$$

$$V_1 = \frac{Q}{A_1} = \frac{4Q}{\pi D_1^2} = \frac{4}{\pi} \times 0.00236 \frac{m^3}{sec} \times \frac{1}{(0.035)^2 m^2} = 2.45\ m/s$$

From momentum

{Note $A_1 = \frac{\pi D_1^2}{4} = 0.000962\ m^2$}

$$R_x + p_{1g} A_1 = u_1\{-\rho Q\} + \int_{A_2} u_2 \rho V_2\, dA_2$$

$u_1 = V_1$ $\quad u_2 = V_2 \cos\theta$; $dA_2 = Rt\, d\theta$

$$\int_{A_2} = \int_{-\pi/2}^{\pi/2} V_2 \cos\theta\, \rho V_2 Rt\, d\theta = 2\rho V_2^2 Rt \int_0^{\pi/2} \cos\theta\, d\theta = 2\rho V_2^2 Rt$$

Thus

$$R_x = -p_{1g} A_1 - V_1 \rho Q + 2\rho V_2^2 Rt$$

$$= -(150 - 101)10^3 \frac{N}{m^2} \times 0.000962\, m^2 - 2.45 \frac{m}{sec} \times 999 \frac{kg}{m^3} \times 0.00236 \frac{m^3}{sec} \times \frac{N \cdot sec^2}{kg \cdot m}$$
$$+ 2 \times 999 \frac{kg}{m^3} \times (10)^2 \frac{m^2}{sec^2} \times 0.05\, m \times 0.0015\, m \times \frac{N \cdot sec^2}{kg \cdot m}$$

$$R_x = -37.9\ N$$

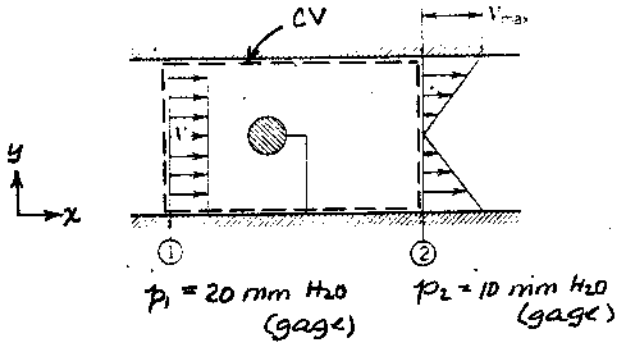
But R_x is force on CV; force on coupling is K_x,

$$K_x = -R_x = 37.9\ N \text{ (to right)}$$

K_x

Given: Small round object tested in wind tunnel. Neglect friction.

Find: (a) Mass flow rate
(b) $V_{2,max}$
(c) Drag of object



Solution:

Basic equations:

$$0 = \underbrace{\frac{\partial}{\partial t}\int_{CV}\rho\,d\forall}_{=0(1)} + \int_{CS}\rho\vec{V}\cdot d\vec{A}$$

$$F_{Sx} + \underbrace{F_{Bx}}_{=0(4)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u\rho\,d\forall}_{=0(1)} + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

$p_1 = \rho g h_1 = 999\frac{kg}{m^3}\times 9.81\frac{m}{sec^2}\times 0.02\,m = 196\,Pa\,(gage)$

$p_2 = 98.0\,Pa\,(gage)$

Assumptions: (1) Steady flow
(2) Density uniform at each section
(3) Uniform flow at section ①, so $\dot{m} = \rho V_1 A$
(4) Horizontal flow; $F_{Bx} = 0$

Then

$$\dot{m} = \rho_1 V_1 A = 1.23\frac{kg}{m^3}\times 10\frac{m}{s}\times\frac{\pi}{4}(1)^2 m^2 = 9.67\ kg/s$$ $\dot{m}$

From continuity,

$$\dot{m} = \int_{A_2}\rho_2 u_2 dA_2 = \rho_2\int_0^R V_{2,max}\frac{r}{R}2\pi r\,dr = 2\pi\rho_2 V_{2,max}R^2\int_0^1\left(\frac{r}{R}\right)^2 d\left(\frac{r}{R}\right) = \frac{2\pi}{3}\rho_2 V_{2,max}R^2$$

$$V_{2,max} = \frac{3\dot{m}}{2\pi\rho_2 R^2} = \frac{3}{2\pi}\times 9.67\frac{kg}{s}\times\frac{m^3}{1.23\,kg}\times\frac{1}{(0.5)^2 m^2} = 15.0\ m/s$$ $V_{2,max}$

From the momentum equation,

$$R_x + p_1 A - p_2 A = u_1\{-\dot{m}\} + \int_{A_2} u\rho_2 V_2\,dA_2 = -V_1\dot{m} + 2\pi\rho_2 V_{2,max}^2 R^2\int_0^1\left(\frac{r}{R}\right)^3 d\left(\frac{r}{R}\right)$$

$u_1 = V_1 \qquad u_2 = V_{2,max}\frac{r}{R}$

$$R_x = (p_2 - p_1)A - V_1\dot{m} + 2\pi\rho_2 V_{2,max}^2 R^2\left(\frac{1}{4}\right)$$

$$= (98.0 - 196)\frac{N}{m^2}\times\frac{\pi}{4}(1)^2 m^2 + \left[-10\frac{m}{s}\times 9.67\frac{kg}{s} + \frac{\pi}{2}\times 1.23\frac{kg}{m^3}\times(15)^2\frac{m^2}{s^2}\times(0.5)^2 m^2\right]\frac{N\cdot s^2}{kg\cdot m}$$

$R_x = -65.0\,N$

R_x is force to hold CV in place. CV cuts strut, so R_x is force needed to hold object. Drag of object and strut is

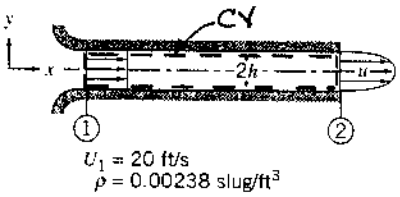
$$F_D = |R_x| = 65.0\,N$$ F_D

Given: Incompressible flow in entrance region of two-dimensional channel.

$$u_2 = u_{max}\left[1-\left(\frac{y}{h}\right)^2\right]$$

Find: (a) Maximum velocity at section ②.
(b) Pressure drop if viscous friction could be neglected.

Solution: Apply continuity and the x momentum equations. Use the CV and CS shown.



Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV}\rho\,d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} \qquad (=0(1))$$

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\,d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A} \qquad (=0(4),\ =0(1))$$

Assumptions: (1) Steady flow
(2) Uniform flow at Section ①
(3) $F_{Bx} = 0$
(4) Neglect friction at duct wall
(5) Incompressible flow

Then

$$0 = \left\{-|\rho U_1\, 2hw|\right\} + \int_{-h}^{h}\rho u_{max}\left[1-\left(\frac{y}{h}\right)^2\right]w\,dy$$

or

$$2U_1 hw = 2u_{max}wh\int_0^1\left[1-\left(\frac{y}{h}\right)^2\right]d\left(\frac{y}{h}\right) = 2u_{max}wh\left[\frac{y}{h}-\frac{1}{3}\left(\frac{y}{h}\right)^3\right]_0^1$$

Thus

$$u_{max} = \frac{3}{2}U_1 = \frac{3}{2}\times 20\,\frac{ft}{s} = 30\ ft/s$$ ← u_{max}

From the momentum equation,

$$p_1\,2hw - p_2\,2hw = u_1\left\{-\rho U_1\,2hw\right\} + \int_{-h}^{h}u_2\rho u_2\,dA_2 \rightarrow 2\int_0^h\rho u_{max}^2\left[1-\left(\frac{y}{h}\right)^2\right]^2 w\,dy$$

$$u_1 = U_1 \qquad u_2 = u_{max}\left[1-\left(\frac{y}{h}\right)^2\right]$$

or

$$p_1 - p_2 = -\rho U_1^2 + \rho u_{max}^2\int_0^1(1-\eta^2)^2\,d\eta\ ;\ \eta = \frac{y}{h}$$

But

$$\int_0^1(1-\eta^2)^2\,d\eta = \int_0^1(1-2\eta^2+\eta^4)\,d\eta = \eta-\frac{2}{3}\eta^3+\frac{1}{5}\eta^5\Big]_0^1 = \frac{15-10+3}{15} = \frac{8}{15}$$

and $u_{max}^2 = \left(\frac{3}{2}U_1\right)^2 = \frac{9}{4}U_1^2$, so

$$p_1 - p_2 = -\rho U_1^2 + \frac{9}{4}\rho U_1^2\left(\frac{8}{15}\right) = \rho U_1^2\left(\frac{6}{5}-1\right) = \frac{1}{5}\rho U_1^2$$

$$= \frac{1}{5}\times 0.00238\,\frac{slug}{ft^3}\times(20)^2\,\frac{ft^2}{s^2}\times\frac{lbf\cdot s^2}{slug\cdot ft}$$

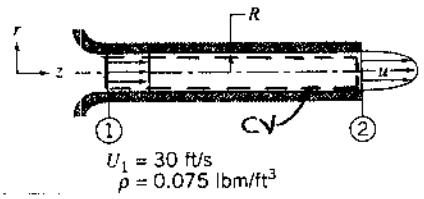
$$p_1 - p_2 = 0.190\ lbf/ft^2$$ ← $p_1 - p_2$

Given: Incompressible flow in entrance region of circular tube of radius, R.

$$u_2 = u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]$$

Find: (a) Maximum velocity at section ②.
(b) Pressure drop if viscous friction could be neglected.

Solution: Apply continuity and the x direction momentum equations. Use the CV and CS shown.



Basic equations:

$$0 = \underbrace{\frac{\partial}{\partial t}}_{=0(1)}\int_{CV}\rho\,d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$$

$$F_{Sx} + \underbrace{F_{Bx}}_{=0(3)} = \underbrace{\frac{\partial}{\partial t}}_{=0(1)}\int_{CV} u\rho\,d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Uniform flow at section ①
(3) $F_{Bx} = 0$
(4) Neglect friction at duct wall
(5) Incompressible flow

Then

$$0 = \{-|\rho U_1 \pi R^2|\} + \int_0^R \rho u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]2\pi r\,dr$$

or

$$\pi\rho U_1 R^2 = 2\pi\rho u_{max} R^2\int_0^1\left[1-\left(\frac{r}{R}\right)^2\right]\left(\frac{r}{R}\right)d\left(\frac{r}{R}\right) = 2\pi\rho u_{max} R^2\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right]_0^1$$

Thus $u_{max} = 2U_1 = 2\times 30\frac{ft}{s} = 60\ ft/s$ u_{max}

From the momentum equation,

$$p_1\pi R^2 - p_2\pi R^2 = u_1\{-\rho U_1\pi R^2\} + \int_0^R u_2\rho u_2\,dA_2 = -\rho U_1\pi R^2 + \rho u_{max}^2 2\pi R^2\int_0^1\left[1-\left(\frac{r}{R}\right)^2\right]\left(\frac{r}{R}\right)d\left(\frac{r}{R}\right)$$

$$u_1 = U_1 \qquad u_2 = u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]$$

or

$$p_1 - p_2 = -\rho U_1^2 + 2\rho u_{max}^2\int_0^1(1-\eta^2)^2\eta\,d\eta\ ;\ \eta = \frac{r}{R}$$

But $\int_0^1(1-\eta^2)\eta\,d\eta = \int_0^1(1-2\eta^2+\eta^4)\eta\,d\eta = \frac{1}{2}\eta^2 - \frac{1}{2}\eta^4 + \frac{1}{6}\eta^6\Big]_0^1 = \frac{1}{6}$

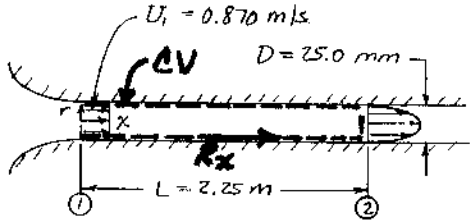
and $u_{max}^2 = (2U_1)^2 = 4U_1^2$, so

$$p_1 - p_2 = -\rho U_1^2 + \frac{8}{6}\rho U_1^2 = \rho U_1^2\left(\frac{4}{3}-1\right) = \frac{1}{3}\rho U_1^2$$

$$= \frac{1}{3}\times 0.075\ \frac{lbm}{ft^3}\times(30)^2\frac{ft^2}{s^2}\times\frac{slug}{32.2\ lbm}\times\frac{lbf\cdot s^2}{slug\cdot ft}$$

$$p_1 - p_2 = 0.699\ lbf/ft^2$$ p_1-p_2

Given: Uniform flow into, fully developed flow from duct shown.



Air

$$\frac{u(r)}{U_c} = 1 - \left(\frac{r}{R}\right)^2 \quad \text{at } ②$$

$$p_1 - p_2 = 1.92 \text{ N/m}^2$$

Find: Total force exerted by tube on the flowing air.

Solution: Apply continuity and momentum to CV, CS shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ (=0(1))

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$ (=0(4), =0(1))

Assumptions: (1) Steady flow (3) Uniform flow at inlet
(2) Incompressible flow (4) $F_{Bx} = 0$

Then $$0 = \{-|\rho U_1 A_1|\} + \int_{②} \rho u\, dA = -\rho U_1 \pi R^2 + \int_0^R \rho U_c\left[1-\left(\frac{r}{R}\right)^2\right] 2\pi r\, dr$$

$$0 = -\rho U_1 \pi R^2 + 2\rho\pi R^2 U_c \int_0^1 (1-\lambda^2)\lambda\, d\lambda \quad \text{or} \quad 0 = -U_1 + 2U_c\left[\frac{\lambda^2}{2} - \frac{\lambda^4}{4}\right]_0^1$$

Thus $0 = -U_1 + \frac{1}{2}U_c$ or $U_c = 2U_1$ $\quad (\lambda = r/R)$

From momentum $R_x + p_1 A_1 - p_2 A_2 = u_1\{-|\rho U_1 A_1|\} + \int_{②} u_2\{+\rho u_2 dA_2\}$

$u_1 = U_1 \qquad u_2 = U_c\left[1-\left(\frac{r}{R}\right)^2\right]$

so $$\int_{②} = \int_0^R U_c\left[1-\left(\frac{r}{R}\right)^2\right]\rho U_c\left[1-\left(\frac{r}{R}\right)^2\right] 2\pi r\, dr = 2\pi\rho U_c^2 R^2 \int_0^1 (1-\lambda^2)(1-\lambda^2)\lambda\, d\lambda$$

$$= 2\pi\rho U_c^2 R^2 \int_0^1 (1-2\lambda^2+\lambda^4)\lambda\, d\lambda = 2\pi\rho U_c^2 R^2\left[\frac{\lambda^2}{2} - \frac{\lambda^4}{2} + \frac{\lambda^6}{6}\right]_0^1 = \frac{1}{3}\pi\rho U_c^2 R^2$$

Substituting,

$$R_x + (p_1 - p_2)\pi R^2 = -\pi\rho U_1^2 R^2 + \frac{1}{3}\pi\rho U_c^2 R^2 = -\pi\rho U_1^2 R^2 + \frac{1}{3}\pi\rho(2U_1)^2 R^2$$

$$R_x = -(p_1 - p_2)\frac{\pi D^2}{4} + \frac{1}{3}\rho U_1^2 \frac{\pi D^2}{4}$$

$$= -1.92 \frac{N}{m^2} \times \frac{\pi}{4}(0.025)^2 m^2 + \frac{1}{3} \times 1.23 \frac{kg}{m^3} \times (0.870)^2 \frac{m^2}{s^2} \times \frac{\pi}{4}(0.025)^2 m^2 \times \frac{N\cdot s^2}{kg\cdot m}$$

$$R_x = -7.90 \times 10^{-4} \text{ N (to left on CV, since } < 0)$$

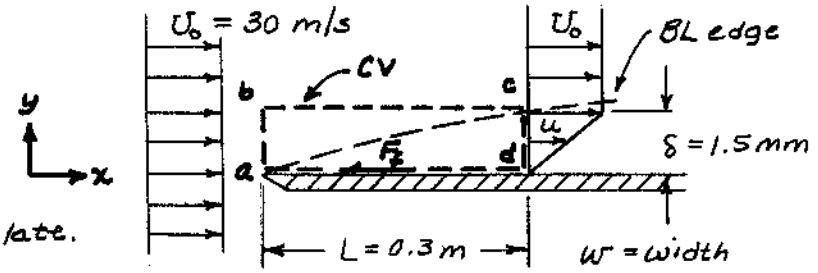
R_x ←

Given: Incompressible flow in boundary layer, as shown.

In BL: $\frac{u}{U_0} = \frac{y}{\delta}$

Standard air

Find: Horizontal force per unit width to hold plate.



Solution: Apply continuity and x component momentum. Use CV, CS shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ (=0(1))

$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$ (=0(3), =0(1))

Assumptions: (1) Steady flow
(2) No net pressure force; $F_{Sx} = -F_f$
(3) $F_{Bx} = 0$
(4) Uniform flow at section ab
(5) Incompressible flow

Then from continuity,

$$0 = \{-|\rho U_0 w\delta|\} + \dot{m}_{bc} + \left\{\left|\int_0^\delta \rho u w\, dy\right|\right\};\quad \delta = \int_0^\delta dy;\quad \dot{m}_{bc} = \int_0^\delta \rho(U_0 - u) w\, dy$$

From momentum equation,

$$-F_f = U_0\{-|\rho U_0 w\delta|\} + U_0\dot{m}_{bc} + \left\{\int_0^\delta u\rho u w\, dy\right\} = \int_0^\delta \rho\left[-U_0^2 + u^2 + U_0(U_0 - u)\right] w\, dy$$

Drag force $= F_f = \int_0^\delta \rho u(U_0 - u) w\, dy$

At cd, $\frac{u}{U_0} = \frac{y}{\delta} = \eta$; $dy = \delta\, d\left(\frac{y}{\delta}\right) = \delta\, d\eta$

$$\frac{F_f}{w} = \int_0^{y/\delta=1} \rho U_0 \frac{y}{\delta}\left(U_0 - U_0\frac{y}{\delta}\right)\delta\, d\left(\frac{y}{\delta}\right) = \rho U_0^2\delta\int_0^1 \eta(1-\eta)\, d\eta = \rho U_0^2\delta\left[\frac{\eta^2}{2} - \frac{\eta^3}{3}\right]_0^1$$

$$= \frac{\rho U_0^2\delta}{6} = \frac{1}{6}\times 1.23\frac{kg}{m^3}\times(30)^2\frac{m^2}{s^2}\times 0.0015\, m\times\frac{N\cdot s^2}{kg\cdot m}$$

$$\frac{F_f}{w} = 0.277\ N/m\ (\text{to } \underline{\text{left}})$$

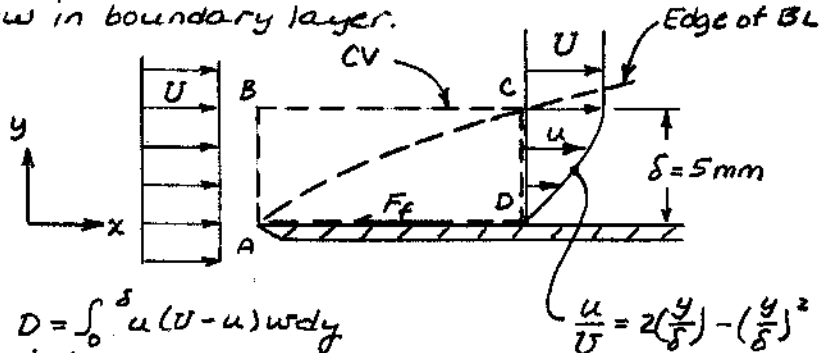
$\frac{F_f}{w}$ ←

Given: Incompressible flow in boundary layer.

$w = 0.6\ m$

$U = 30\ m/s$

$\rho = 1.24\ kg/m^3$



$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$

Find: (a) Show that drag, $D = \int_0^\delta \rho u(U-u)w\,dy$

(b) Evaluate for conditions shown.

Solution: Apply continuity and x component of momentum using CV shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV}\rho\,d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$ (=0(1))

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\,d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$ (=0(3), =0(1))

Assumptions: (1) Steady flow

(2) No net pressure force; $F_{Sx} = -F_f$

(3) $F_{Bx} = 0$

(4) Uniform flow at section (AB)

(5) Incompressible flow

Then from continuity

$$0 = \{-|\rho U w \delta|\} + \{|\int_0^\delta \rho u w\,dy|\} + \dot{m}_{BC}\ ;\ \delta = \int_0^\delta dy\ ;\ \dot{m}_{BC} = \rho\int_0^\delta (U-u)w\,dy$$

From momentum

$$-F_f = U\{-|\rho U w \delta|\} + \{\int_0^\delta \rho u^2 w\,dy|\} + U\dot{m}_{BC} = \rho\int_0^\delta [-U^2 + u^2 + U(U-u)]w\,dy$$

$$\text{Drag} = F_f = \int_0^\delta \rho u(U-u)w\,dy$$ Drag

At CD, $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 = 2\eta - \eta^2$; $dy = \delta\, d\left(\frac{y}{\delta}\right) = \delta\,d\eta$

$$\text{Drag} = \int_0^\delta \rho U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]\left(U - U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]\right)w\,dy = \rho U^2 w\delta\int_0^1 (2\eta-\eta^2)(1-2\eta+\eta^2)\,d\eta$$

$$= \rho U^2 w\delta\int_0^1 (2\eta - 5\eta^2 + 4\eta^3 - \eta^4)\,d\eta = \rho U^2 w\delta\left[\eta^2 - \frac{5}{3}\eta^3 + \eta^4 - \frac{1}{5}\eta^5\right]_0^1$$

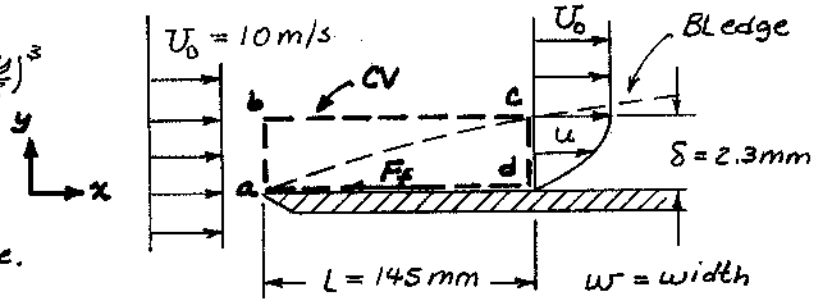
$$= \frac{2}{15}\rho U^2 w\delta$$

$$\text{Drag} = \frac{2}{15}\times 1.24\frac{kg}{m^3}\times(30)^2\frac{m^2}{s^2}\times 0.6\,m\times 0.005\,m\times\frac{N\cdot s^2}{kg\cdot m} = 0.446\,N$$ Drag

Given: Incompressible flow in boundary layer, as shown.

In BL: $\frac{u}{U_0} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$

Standard air



Find: Horizontal force per unit width to hold plate.

Solution: Apply continuity and x component momentum. Use CV, CS shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$ (=0 (1))

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$ (F_{Sx}: =0 (3); $\frac{\partial}{\partial t}$: =0 (1))

Assumptions: (1) Steady flow
(2) No net pressure force; $F_{Sx} = -F_f$
(3) $F_{Bx} = 0$
(4) Uniform flow at section ab
(5) Incompressible flow

Then from continuity,

$$0 = \left\{-|\rho U_0 w\delta|\right\} + \dot{m}_{bc} + \left\{\int_0^\delta \rho u w\, dy\right\};\quad \delta = \int_0^\delta dy;\quad \dot{m}_{bc} = \int_0^\delta \rho(U_0 - u)w\, dy$$

From momentum equation,

$$-F_f = U_0\left\{-|\rho U_0 w\delta|\right\} + U_0\dot{m}_{bc} + \left\{\int_0^\delta u\rho u w\, dy\right\} = \int_0^\delta\left[-U_0^2 + u^2 + U_0(U_0 - u)\right]w\, dy$$

$$\text{Drag force} = F_f = \int_0^\delta \rho u(U_0 - u)w\, dy = \int_0^\delta \rho U_0^2 \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right)w\, dy$$

At section cd, $\frac{u}{U_0} = \frac{3}{2}\eta - \frac{1}{2}\eta^3$; $dy = \delta\, d\eta$

$$\frac{F_f}{w} = \int_0^{\eta=1} \rho U_0^2\delta \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right)d\eta = \rho U_0^2\delta\int_0^1\left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)\left(1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3\right)d\eta$$

$$= \rho U_0^2\delta\int_0^1\left(\frac{3}{2}\eta - \frac{9}{4}\eta^2 - \frac{1}{2}\eta^3 + \frac{3}{2}\eta^4 - \frac{1}{4}\eta^6\right)d\eta$$

$$= \rho U_0^2\delta\left[\frac{3}{4}\eta^2 - \frac{3}{4}\eta^3 - \frac{1}{8}\eta^4 + \frac{3}{10}\eta^5 - \frac{1}{28}\eta^7\right]_0^1 = \rho U_0^2\delta\,(0.139)$$

$$= 0.139 \times 1.23\,\frac{kg}{m^3} \times (10)^2\,\frac{m^2}{s^2} \times 0.0023\,m \times \frac{N\cdot s^2}{kg\cdot m}$$

$$\frac{F_f}{w} = 0.0393\ N/m \quad (\text{to } \underline{\text{right}})$$

$\frac{F_f}{w}$

Problem 4.83

Open-Ended Problem Statement: Consider the water jet and vertical plate of Example Problem 4.4. An implicit assumption made in solving the problem was that the jet remains horizontal until it strikes the plate. Discuss the implications of including gravity in the solution. Specifically address the trajectory of the jet before it strikes the plate and the velocity of the liquid on the plate above and below the jet impact point. Are the dimensions of the vertical plate important? Are the properties of the liquid stream important?

Discussion: Gravity could be included in the solution relatively easily. Gravity would cause the water stream in the jet to have a downward component of velocity. The jet would have a parabolic trajectory; the jet stream would droop more as it travels farther from the nozzle.

As a first approximation flow on the vertical plate is assumed to be frictionless. Therefore there can be no friction force from the jet on the plate. Pressure forces acting on the plate have no vertical component. Consequently there can be no net vertical force on the plate.

The liquid jet will split into two streams when it strikes the plate. Each stream will have the same speed as the jet, because there is no friction force along the plate to change the liquid speed. However, the lower stream will carry more than half the flow; the upper stream will carry less than half the flow. Because no vertical force acts on the flow the flow will divide so as to give the two streams the same net vertical momentum as the jet stream striking the plate.

The vertical plate must be large enough to re-direct all the flow from the jet so that it leaves parallel to the surface of the plate. When gravity is included the plate must extend farther below the jet centerline than in the hypothetical case without gravity.

Properties of the liquid stream include its density, area, velocity, and viscosity. The horizontal force exerted on the vertical plate is proportional to density, area, and the square of the velocity of the liquid stream. The viscosity of the liquid is not important, when it is low enough so the frictionless flow approximation is a reasonable flow model.

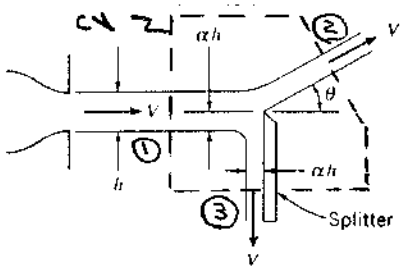
Given: Flow of flat jet over sharp-edged splitter plate, as shown. Neglect friction force between water and plate; $0 \le \alpha \le 0.5$.

Find: (a) Expression for angle θ as a function of α.
(b) Expression for force R_x needed to hold splitter plate in place.

Plot: both θ and R_x as functions of θ.

Solution

Apply the x and y components of the momentum equation to the CV shown.



Basic equations:

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u(\rho \vec{V}\cdot d\vec{A})$$

$$F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall + \int_{CS} v(\rho \vec{V}\cdot d\vec{A})$$

Assumptions: (1) no net pressure forces on CV.
(2) no friction in y direction, so $F_{Sy} = 0$
(3) neglect body forces
(4) steady flow
(5) no change in jet speed: $V_1 = V_2 = V_3 = V$
(6) uniform flow at each section

Then from the y equation

$$0 = v_1\{-|\rho V_1 A_1|\} + v_2\{|\rho V_2 A_2|\} + v_3\{|\rho V_3 A_3|\}$$

$v_1 = 0$ $\quad v_2 = V\sin\theta$ $\quad v_3 = -V$

{w is depth} $\quad A_1 = wh$ $\quad A_2 = w(1-\alpha)h$ $\quad A_3 = w\alpha h$

$$0 = 0 + \rho V^2 \sin\theta\, w(1-\alpha)h - \rho V^2 w\alpha h$$

Thus

$$\sin\theta = \frac{\rho V^2 w\alpha h}{\rho V^2 w(1-\alpha)h} = \frac{\alpha}{(1-\alpha)}\ ;\ \theta = \sin^{-1}\left(\frac{\alpha}{1-\alpha}\right) \quad \theta(\alpha)$$

From the x equation.

$$R_x = u_1\{-|\rho_1 V_1 A_1|\} + u_2\{|\rho_2 V_2 A_2|\} + u_3\{|\rho_3 V_3 A_3|\}$$

$u_1 = V$ $\quad u_2 = V\cos\theta$ $\quad u_3 = 0$

$$R_x = -\rho V^2 wh + \rho V^2 \cos\theta\, w(1-\alpha)h = \rho V^2 wh\left[\cos\theta(1-\alpha) - 1\right]$$

But $\cos\theta = (1-\sin^2\theta)^{1/2} = \left(1 - \frac{\alpha^2}{(1-\alpha)^2}\right)^{1/2} = \frac{(1-2\alpha)^{1/2}}{(1-\alpha)}$

$$\therefore R_x = -\rho V^2 wh\left[1 - (1-2\alpha)^{1/2}\right] \quad (R_x < 0;\ \text{so to left}) \quad R_x$$

{ Check: $\alpha = 0$, $R_x = 0$ ✓ ; $\alpha = \frac{1}{2}$, $R_x = -\rho V^2 wh$ ✓ }

Plots of: $\theta = \sin^{-1}\left(\frac{\alpha}{1-\alpha}\right)$ and

$$\frac{R_x}{R_{x\,\alpha=0.5}} = 1 - \sqrt{1-2\alpha}$$

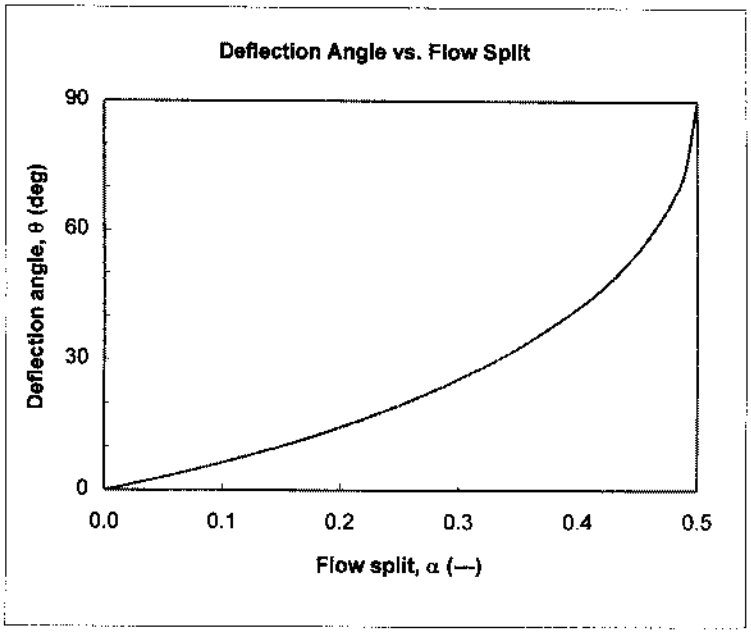
are presented below

Flow deflection by sharp-edged splitter:

α = fraction of jet intercepted by splitter

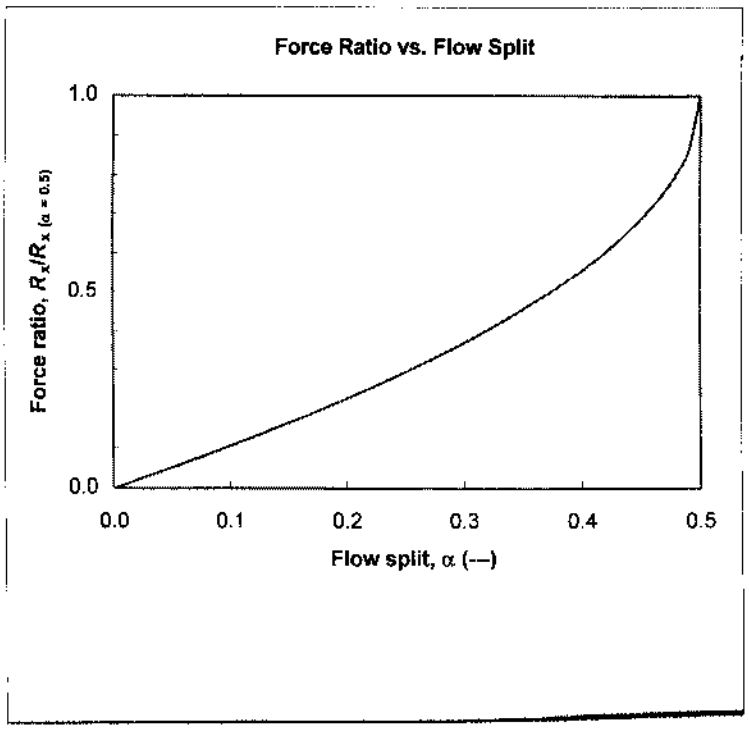
Calculated Results: Deflection angle

α (—)	θ (deg)
0	0
0.05	3.02
0.10	6.38
0.15	10.2
0.20	14.5
0.25	19.5
0.30	25.4
0.35	32.6
0.40	41.8
0.425	47.7
0.45	54.9
0.470	62.5
0.480	67.4
0.490	73.9
0.50	90.0



Calculated Results: Force over maximum force

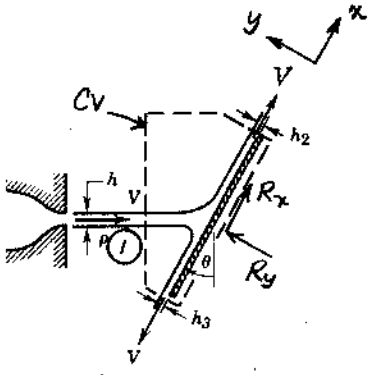
α (—)	$R_x/R_{x\,(\alpha = 0.5)}$
0	0
0.05	0.0513
0.10	0.106
0.15	0.163
0.20	0.225
0.25	0.293
0.30	0.368
0.35	0.452
0.40	0.553
0.425	0.613
0.45	0.684
0.470	0.755
0.480	0.800
0.490	0.859
0.50	1.00



Given: Plane jet striking inclined plate, as shown. No frictional force along plate surface.

Find: (a) Expression for h_2/h as a function of θ.
(b) Plot of results.
(c) Comment on limiting cases, $\theta = 0$ and $\theta = 90°$.

Solution: Apply the x component of the momentum equation using the CV and coordinates shown.



Basic equation:

$$F_{Sx}^{=0(1)} + F_{Bx}^{=0(2)} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall^{=0(3)} + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

Assumptions: (1) No surface force on CV
(2) Neglect body forces
(3) Steady flow
(4) No change in jet speed: $V_1 = V_2 = V_3 = V$
(5) Uniform flow at each section

From continuity for uniform incompressible flow $0 = -\rho V w h + \rho V w h_2 + \rho V w h_3$

or

$$h = h_2 + h_3 = h_1 \quad \text{or} \quad h_3 = h_1 - h_2$$

From momentum

$$0 = u_1\{-|\rho V w h_1|\} + u_2\{+|\rho V w h_2|\} + u_3\{+|\rho V w h_3|\}$$

$$u_1 = V\sin\theta \qquad u_2 = V \qquad u_3 = -V$$

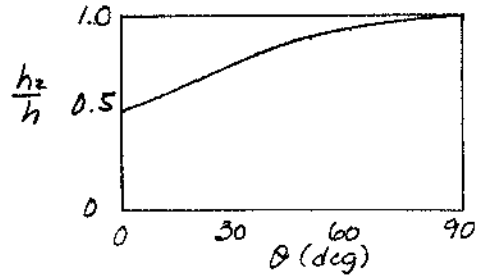
$$0 = -\rho V^2 \sin\theta\, w h_1 + \rho V^2 w h_2 - \rho V^2 w h_3$$

Substituting from continuity and simplifying

$$0 = -\sin\theta\, h_1 + h_2 - (h_1 - h_2) \quad \text{so} \quad \frac{h_2}{h} = \frac{h_2}{h_1} = \frac{1+\sin\theta}{2}$$

$\frac{h_2}{h}$

Plot:



At $\theta = 0$, $\frac{h_2}{h} = 0.5$; flow is equally split when plate is $\perp$ to jet.

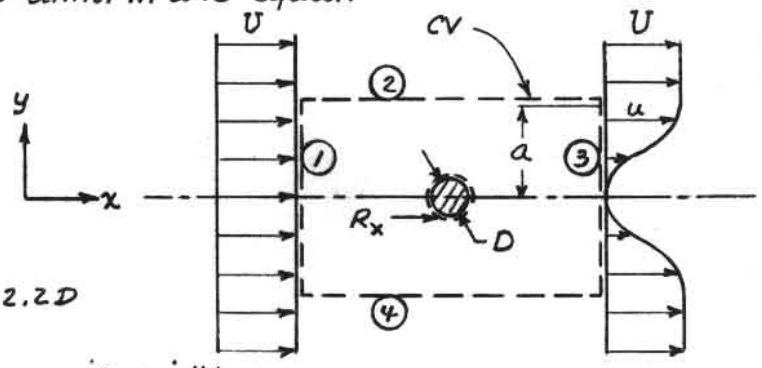
At $\theta = 90°$, $\frac{h_2}{h} = 1.0$; plate has no effect on flow.

Given: Experimentally measured velocity profile for flow over a cylinder. Pressures are uniform and equal.

$U = 50$ m/s
$\rho = 1.2$ kg/m^3
$D = 30$ mm

$u = U\sin(\frac{\pi y}{2a})$; $0 \le |y| \le a$

$u = U$ $y > a$; $a = 2.2D$



Find: Drag force on cylinder per unit width.

Solution: Apply continuity and x component of momentum using CV shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV}\rho d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$ (=0(1))

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$ (=0(2), =0(1))

Assumptions: (1) Steady flow
(2) No net pressure force; $F_{Sx} = R_x$
(3) $F_{Bx} = 0$
(4) Uniform flow at inlet section (1)
(5) Incompressible flow
(6) $u = U$ at sections (2) and (4)

From continuity

$$0 = \{-|\rho U w\, 2a|\} + \dot{m}_2 + \{|2\int_0^a \rho u w dy|\} + \dot{m}_4 \; ; \; \dot{m}_{2+4} = 2\int_0^a \rho(U-u)w dy$$

From momentum

$$R_x = U\{-|\rho U w\, 2a|\} + U\dot{m}_2 + 2\int_0^a \rho u^2 w dy + U\dot{m}_4$$

$$= U\dot{m}_{2+4} + 2\int_0^a \rho(u^2 - U^2)w dy = 2\int_0^a \rho[U(U-u) + u^2 - U^2]w dy$$

$$R_x = 2\int_0^a \rho u(u-U)w dy \quad (R_x \text{ is force of cylinder } \underline{\text{on}} \text{ CV})$$

$$F_D = -R_x = 2\int_0^a \rho u(U-u)w dy = 2\rho w U^2\int_0^a \frac{u}{U}(1-\frac{u}{U})dy \; ; \; \frac{u}{U} = \sin(\frac{\pi}{2}\frac{y}{a})$$

$$\frac{F_D}{w} = 2\rho U^2(\frac{2a}{\pi})\int_0^{\pi/2}(\sin\theta - \sin^2\theta)d\theta = \frac{4}{\pi}\rho U^2 a\left[-\cos\theta - \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \frac{4}{\pi}\rho U^2 a\left[1 - \frac{\pi}{4}\right]$$

$$\frac{F_D}{w} = \frac{4}{\pi} \times 1.20\frac{kg}{m^3} \times (50)^2\frac{m^2}{s^2} \times (2.2)0.03m\left(1-\frac{\pi}{4}\right) \times \frac{N\cdot s^2}{kg\cdot m} = 54.1\ N/m$$ ← $\frac{F_D}{w}$

{ Plus sign means force on cylinder acts to right. }

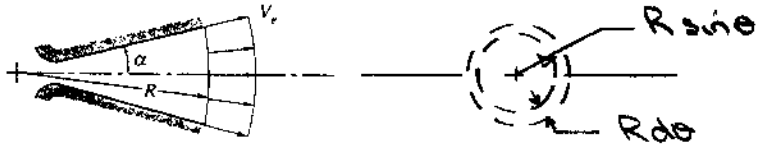
Given: Model gas flow in a propulsion nozzle as a spherical source; V_e = constant

Find: (a) Expression for axial thrust, T_a, and compare to the 1-D approximation, $T = \dot{m} V_e$
(b) Percent error for $\alpha = 15°$.

Plot: the percent error vs α for $0 \le \alpha \le 22.5°$.

Solution:

Apply definitions $\dot{m} = \int_A \rho V dA$, $T_a = \int_A u \rho V dA$. Use spherically symmetric flow.



The mass flow rate is [assuming $\rho_e \neq \rho_e(\theta)$]

$$\dot{m} = \int_A \rho V dA = \int_0^\alpha \rho_e V_e (2\pi R\sin\theta) R d\theta = 2\pi \rho_e V_e R^2 \left[-\cos\theta\right]_0^\alpha = 2\pi \rho_e V_e R^2 (1-\cos\alpha)$$

The one-dimensional approximation for thrust is then

$$T = \dot{m} V_e = 2\pi \rho_e V_e^2 R^2 (1-\cos\alpha) \quad \leftarrow T_{1\text{-}D}$$

The axial thrust is given by

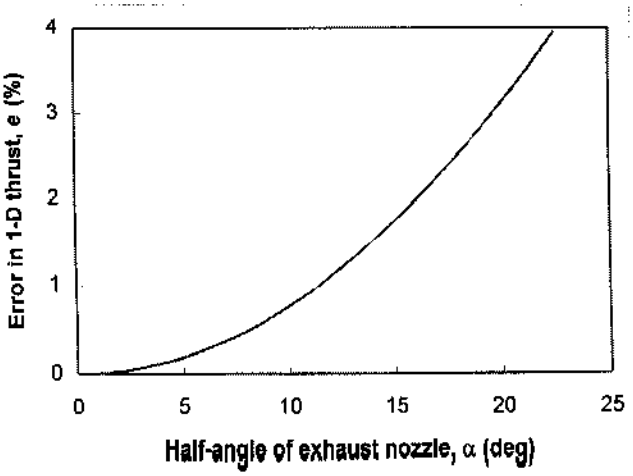
$$T_a = \int u \rho V dA = \int_0^\alpha V_e \cos\theta\, \rho_e V_e (2\pi R\sin\theta) R d\theta = 2\pi \rho_e V_e^2 R^2 \int_0^\alpha \sin\theta\cos\theta\, d\theta$$

$$T_a = 2\pi \rho_e V_e^2 R^2 \left[\frac{\sin^2\theta}{2}\right]_0^\alpha = \pi \rho_e V_e^2 R^2 \sin^2\alpha \quad \leftarrow T_a$$

The error in the one-dimensional approximation is

$$e = \frac{T_{1\text{-}D} - T_a}{T_a} = \frac{T_{1\text{-}D}}{T_a} - 1 = \frac{2\pi \rho_e V_e^2 R^2 (1-\cos\alpha)}{\pi \rho_e V_e^2 R^2 \sin^2\alpha} - 1 = \frac{2(1-\cos\alpha)}{\sin^2\alpha} - 1 \quad \text{---(1)}$$

The percent error is plotted as a function of α



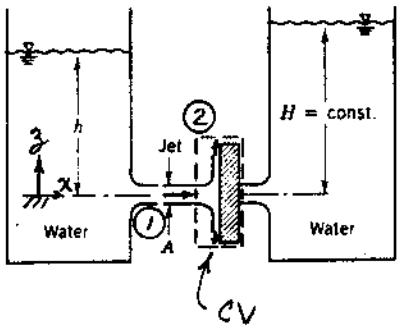
For $\alpha = 15°$

$$e_{15} = \frac{2(1-\cos 15)}{\sin^2 15°} - 1$$

$e_{15} = 0.0173$ or 1.73% $\leftarrow e_{15}$

Given: Tanks and flat plate shown.

Find: Minimum height h needed to keep plate in place.



Solution: Apply Bernoulli and momentum equations, use CV enclosing plate, as shown.

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\, \vec{V}\cdot d\vec{A}$$

($F_{Sx} = 0$ (5), $\frac{\partial}{\partial t} = 0$ (1))

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Flow along a streamline
(4) No friction
(5) $F_{Bx} = 0$

Apply Bernoulli from water surface to jet

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gh = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g(0) \quad (V_1^2 \ll V_2^2) \quad \text{so that } V_2^2 = 2gh \text{ or } V_2 = \sqrt{2gh}$$

From fluid statics, $p_{3g} = \rho g H$

From momentum

$$-p_{3g}A = -\rho g H A = u_1\{-\rho V A\} + u_2\{+\rho V A\} = -\rho V^2 A$$

$$u_1 = V \qquad u_2 = 0$$

Thus, using Bernoulli,

$$\rho g H A = \rho V^2 A = \rho(2gh)A = 2\rho g h A$$

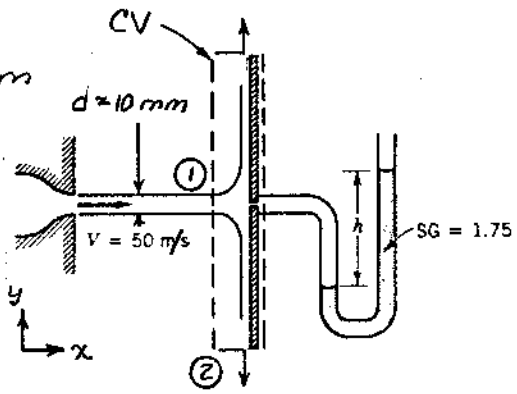
and

$$h = \frac{H}{2}$$

h

Given: Air jet striking disk of diameter, D = 200 mm, as shown.

Find: (a) Manometer deflection.
(b) Force to hold disk.



Solution: Apply Bernoulli and momentum equations. Use CV shown.

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + g\cancel{z}^{(5)} = \text{constant}$

$$F_{Sx} + \cancel{F_{Bx}}^{=0(5)} = \cancel{\frac{\partial}{\partial t}}^{=0(1)} \int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Flow along a streamline
(4) No friction
(5) $F_{Bx} = 0$; horizontal flow
(6) Uniform flow in jet

Apply Bernoulli between jet exit and stagnation point

$$\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_0}{\rho} + 0 \; ; \; p_0 - p = \frac{1}{2}\rho V^2$$

From hydrostatics, $p_0 - p = SG\, \rho_{H_2O}\, g\, \Delta h$

Thus $\Delta h = \dfrac{\frac{1}{2}\rho V^2}{SG\, \rho_{H_2O}\, g} = \dfrac{\rho V^2}{2SG\, \rho_{H_2O}\, g}$

$$\Delta h = 1.23 \frac{kg}{m^3} \times (50)^2 \frac{m^2}{s^2} \times \frac{1}{2(1.75)} \times \frac{m^3}{999\ kg} \times \frac{s^2}{9.81\ m} = 0.0896\ m \text{ or } 89.6\ mm \qquad \Delta h$$

From momentum,

$$R_x = u_1\{-\rho V A\} + u_2\{\rho V A\} = -\rho V^2 A$$

$u_1 = V \qquad u_2 = 0$

$$R_x = -1.23 \frac{kg}{m^3} \times (50)^2 \frac{m^2}{s^2} \times \frac{\pi}{4}(0.01)^2 m^2 \times \frac{N \cdot s^2}{kg \cdot m} = -0.242\ N \text{ (to left)} \qquad R_x$$

This is the force needed to hold the plate.

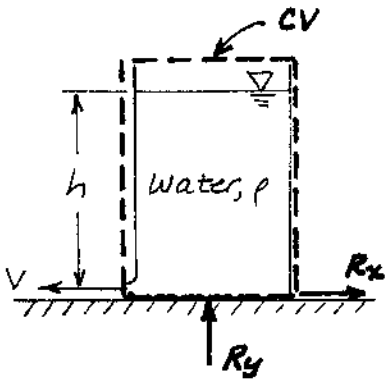
The "force" of the jet on the plate is

$K_x = -R_x = 0.242\ N$ (to right)

Given: Tank on plane surface. Jet issues from side of tank

Tank: Mass, M ; Diameter, D

Jet: Diameter, d



Find: (a) Expression for coefficient of friction that allows tank to slide.

(b) coefficient of friction for $d/D = 0.3$.

Plot: coefficient of friction as a function of D/d for $2 \le D/d \le 10$.

Solution:

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$ ($z = 0$ (2), $V = 0$ (1))

$$F_{Sx} + \cancel{F_{Bx}} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\rho d\forall} + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) flow is quasi-steady ($A_{tank} \gg A_{jet}$).

(2) incompressible flow

(3) flow along a streamline

(4) frictionless flow

(5) $F_{Bx} = 0$

(6) uniform flow at jet exit

Apply Bernoulli equation between surface and jet exit.

$$\cancel{\frac{p}{\rho}} + \frac{V_s^2}{2} + gh = \cancel{\frac{p}{\rho}} + \frac{V_j^2}{2}$$

From continuity for uniform flow $V_s A_{tank} = V_j A_j$, so

$V_s = V_j \frac{A_j}{A_t}$ and $\frac{V_j^2}{2}\left(\frac{A_j}{A_t}\right)^2 + gh = \frac{V_j^2}{2}$

Then $$V_j = \left[\frac{2gh}{1-(A_j/A_t)^2}\right]^{1/2} \quad \text{---------- (1)}$$

From the x momentum equation

$$R_x = u\{|\rho V_j A_j|\} = -V_j(\rho V_j A_j) = -\rho V_j^2 A_j \quad \text{---- (2)}$$

Combining Eqs 1 and 2

$$R_x = -\frac{2\rho g h A_j}{1-(A_j/A_t)^2}$$

The tank will slide if $|R_x| > \mu_s R_y = \mu_s (Mg + \rho g A_t h)$, or if

$$\mu_s < \frac{|R_x|}{Mg+\rho g A_t h} = \frac{2\rho g h A_j}{g(M+\rho A_t h)} \frac{1}{1-(A_j/A_t)^2}$$

If the tank mass is small compared to the mass of the water.

$$\mu_s < \frac{2A_j}{A_t}\frac{1}{1-(A_j/A_t)^2} = \frac{2(d/D)^2}{[1-(d/D)^4]} \quad \mu_s$$

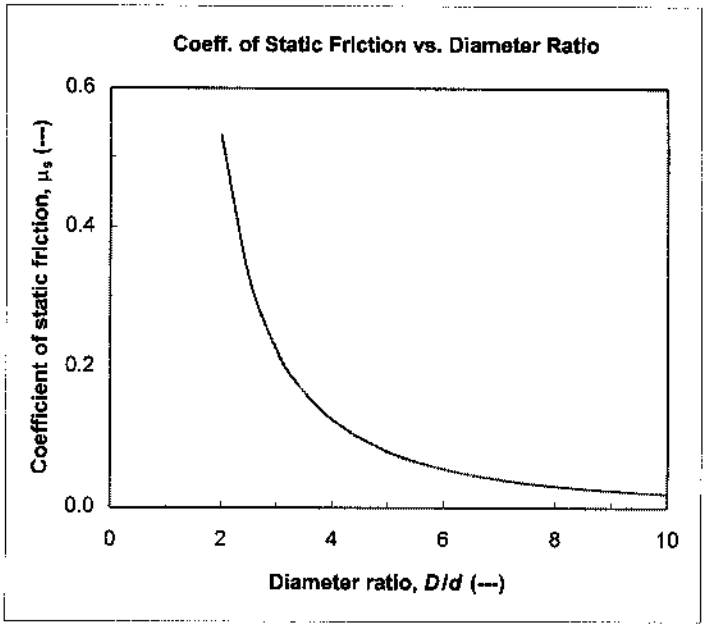
Evaluating for $d/D = 0.3$

$$\mu_s \leq \frac{2(0.3)^2}{1-(0.3)^4} \leq 0.181 \quad \mu_s$$

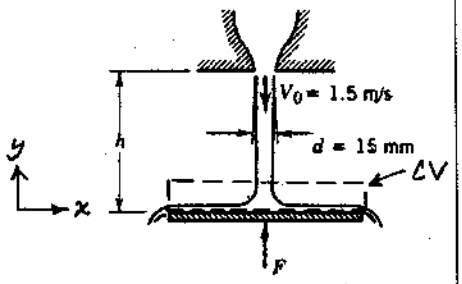
The friction coefficient is plotted as a function of D/d

Tank propelled by thrust from horizontal water jet:

Diameter Ratio, *D/d* (–)	Coefficient of Friction, μ_s (---)
2	0.533
2.5	0.328
3	0.225
3.333333	0.181
4	0.125
5	0.0801
6	0.0556
7	0.0408
8	0.0313
9	0.0247
10	0.0200



Given: Jet flowing downward, striking horizontal disk, as shown.



Find: (a) Velocity in jet at h.
(b) Expression for force to hold disk.
(c) Evaluate for h = 1.5 m.

Solution: Apply Bernoulli and momentum equations. Use CV shown.

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$ (5)

$$F_{Sz} + F_{Bz} = \frac{\partial}{\partial t}\int_{CV} w\rho\, d\forall + \int_{CS} w\rho \vec{V}\cdot d\vec{A}$$

($F_{Bz} = 0$ (6); $\frac{\partial}{\partial t} = 0$ (1))

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Flow along a streamline
(4) Frictionless flow
(5) Atmospheric pressure along jet
(6) Neglect water on plate; $F_{Bz} = 0$
(7) Uniform flow at each section

The Bernoulli equation becomes

$$\frac{V_0^2}{2} + gh = \frac{V^2}{2} + g(0) \quad \text{or} \quad V^2 = V_0^2 + 2gh\,; \quad V = \sqrt{V_0^2 + 2gh}$$ V

From the momentum equation

$$R_z = w_1\{-\rho V A\} + w_2\{+\rho V_0 A_0\} = +\rho V^2 A$$

$w_1 = -V$ $\qquad$ $w_2 = 0$

But from continuity, $\dot{m} = \rho V_0 A_0 = \rho V A$. Thus $VA = V_0 A_0$, and

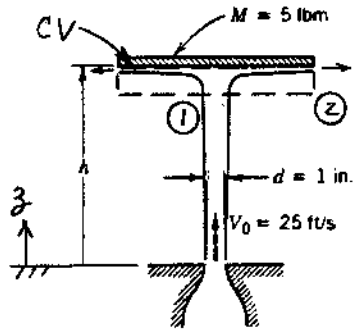
$$R_z = \rho V_0 A_0 V = \rho V_0 A_0 \sqrt{V_0^2 + 2gh}$$ R_z

At h = 1.5 m,

$$R_z = 999\,\frac{kg}{m^3} \times 1.5\,\frac{m}{s} \times \frac{\pi}{4}(0.015)^2\,m^2\left[(1.5)^2\frac{m^2}{s^2} + 2 \times 9.81\,\frac{m}{s^2} \times 1.5\,m\right]^{1/2} \frac{N \cdot s^2}{kg \cdot m}$$

$R_z = 1.49$ N (upward force) R_z

Given: Horizontal disk above jet, as shown.
No external force applied to disk.



Find: (a) Expression for jet speed, V(h).
(b) Equilibrium height for disk.

Solution: Apply Bernoulli and momentum equations. Use CS, CV shown.

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$ (5)

$$F_{fz} + F_{Bz} = \frac{\partial}{\partial t}\int_{CV} w\rho\, d\forall + \int_{CS} w\rho \vec{V}\cdot d\vec{A}$$ (1)

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Flow along a streamline
(4) Frictionless flow
(5) Atmospheric pressure along jet
(6) Neglect mass of water in CV
(7) Uniform flow at each section

The Bernoulli equation becomes

$$\frac{V_0^2}{2} + g(0) = \frac{V^2}{2} + gh \quad \text{or} \quad V^2 = V_0^2 - 2gh \;;\; V = \sqrt{V_0^2 - 2gh}$$ V

From the momentum equation,

$$-Mg = w_1\{-\rho VA\} + w_2\{+\rho VA\} = -\rho V^2 A$$

$$w_1 = V \qquad w_2 = 0$$

But from continuity, $\dot{m} = \rho V_0 A_0 = \rho VA$. Thus $VA = V_0 A_0$, and

$$Mg = \rho V_0 A_0 V = \rho V_0 A_0 \sqrt{V_0^2 - 2gh}$$

Solving for h,

$$h = \frac{1}{2g}\left[V_0^2 - \left(\frac{Mg}{\rho V_0 A_0}\right)^2\right]$$

$$= \frac{1}{2}\times\frac{s^2}{32.2\ ft}\left[(25)^2\frac{ft^2}{s^2} - \left(5\ lbm \times 32.2\frac{ft}{s^2}\times\frac{ft^3}{62.4\ lbm}\times\frac{s}{25\ ft}\times\frac{4}{\pi}\frac{1}{(1)^2 in.^2}\times\frac{144\ in.^2}{ft^2}\right)^2\right]$$

$$h = 4.15\ ft$$ h

Given: Stream of air at standard conditions strikes a curved vane. Stagnation tube with water-filled manometer in exit plane.

Find: (a) Speed of air leaving nozzle.
(b) Horizontal component of force exerted on vane by jet.
(c) Comment on each assumption used to solve this problem.

Solution: Apply the definition of stagnation pressure and the x component of the momentum equation.

By definition $p_0 = p + \frac{1}{2}\rho_{air} V^2$

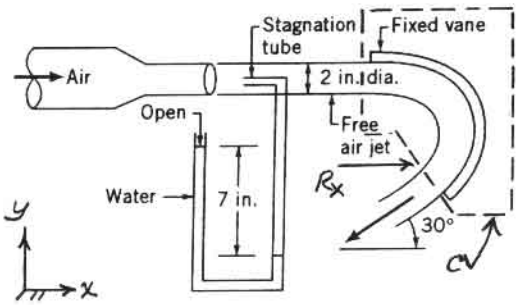
From fluid statics, $p_0 - p = \rho_{water} g \Delta h$

Combining, $\rho_{water} g \Delta h = \frac{1}{2}\rho_{air} V^2$ or $V = \sqrt{\frac{2\rho_{water} g \Delta h}{\rho_{air}}}$

$$V = \left[2 \times 1.94 \frac{slug}{ft^3} \times 32.2 \frac{ft}{s^2} \times 7\ in. \times \frac{ft^3}{0.00238\ slug} \times \frac{ft}{12\ in.}\right]^{\frac{1}{2}} = 175\ ft/s \qquad V$$

The momentum equation is

$$F_{Sx} + \overbrace{F_{Bx}}^{=0\,(2)} = \overbrace{\frac{\partial}{\partial t}}^{=0\,(3)} \int_{CV} u \rho d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$



Assumptions: (1) No net pressure force
(2) $F_{Bx} = 0$
(3) Steady flow
(4) Uniform flow
(5) Constant speed on vane

Then

$$R_x = u_1\{-\rho VA\} + u_2\{\rho VA\} = -\rho V^2 A(1 + \cos\theta)$$

$$u_1 = V \qquad u_2 = -V\cos\theta$$

$$R_x = -0.00238 \frac{slug}{ft^3} \times (175)^2 \frac{ft^2}{s^2} \times \frac{\pi}{4}\left(\frac{7}{12}\right)^2 ft^2 \times (1 + \cos 30^\circ) = -2.97\ lbf$$

Force of air on vane is $K_x = -R_x = +2.97\ lbf$ (to right) $\qquad K_x$

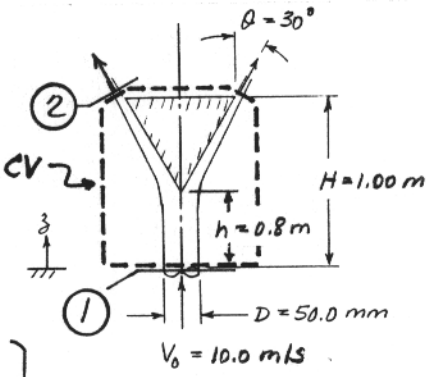
Comments on each assumption used to solve this problem:

- Frictionless flow in the nozzle is a good assumption.
- Incompressible flow is a good assumption for this low-speed flow.
- No horizontal component of body force is exact.
- No net pressure force on the control volume is exact.
- Frictionless flow along the vane is not realistic; air flow along the vane would be slowed by friction, reducing the momentum flux at the exit.

Given: Water jet supporting conical object, as shown.

Find: (a) Combined mass of cone and water, M, supported.
(b) Estimate mass of water in CV.

Solution: Apply continuity, Bernoulli, and momentum equations using CV shown.



Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV}\rho\,d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$ (=0 (1))

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$ ((6), (6))

$$F_{Sz} + F_{Bz} = \frac{\partial}{\partial t}\int_{CV} w\rho\,d\forall + \int_{CS} w\rho\vec{V}\cdot d\vec{A}$$ (F_{Sz} = 0 (6), = 0 (1))

Assumptions: (1) Steady flow
(2) No friction
(3) Flow along a streamline } required for Bernoulli
(4) Incompressible flow
(5) Uniform flow at each cross-section
(6) $F_{Sz} = 0$ since p_{atm} acts everywhere

Then

$$0 = \{-|\rho V_1 A_1|\} + \{+|\rho V_2 A_2|\} \quad \text{so} \quad V_1 A_1 = V_2 A_2$$

From Bernoulli

$$\frac{V_1^2}{2} + gz_1 = \frac{V_2^2}{2} + gz_2 = \frac{V_0^2}{2} = \frac{V_2^2}{2} + gH \;;\; V_2^2 = V_0^2 - 2gH$$

From momentum

$$F_{Bz} = \int_{CS} w\rho\vec{V}\cdot d\vec{A} = -Mg = w_1\{-|\rho V_1 A_1|\} + w_2\{+|\rho V_2 A_2|\}$$

$$w_1 = V_0 \qquad w_2 = V_2\cos\theta$$

or

$$-Mg = -V_0\rho V_1 A_1 + V_2\cos\theta\,\rho V_2 A_2 = \rho V_0 A_1(V_2\cos\theta - V_0)$$

so

$$M = \frac{(V_0 - V_2\cos\theta)\rho V_0 A_1}{g}$$

From Bernoulli

$$V_2 = (V_0^2 - 2gH)^{1/2} = \left[(10)^2\frac{m^2}{s^2} - 2\times 9.81\frac{m}{s^2}\times 1m\right]^{1/2} = 8.97\ m/s$$

Substituting

$$M = \left(10.0\frac{m}{s} - 8.97\frac{m}{s}\times\cos 30°\right)999\frac{kg}{m^3}\times 10\frac{m}{s}\times\frac{\pi}{4}(0.050)^2 m^2\times\frac{s^2}{9.81\ m}$$

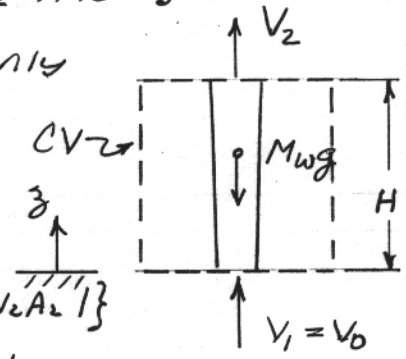
$$M = 4.46\ kg \quad \text{(total mass in CV: water + object)}$$ M

To find mass of water in CV, we have 3 options:

(1) assume area of jet is constant

$$M = \rho \forall \simeq \rho A_1 H = 999 \frac{kg}{m^3} \times \frac{\pi}{4}(0.05)^2 m_x^2 \, 1m = 1.96 \, kg$$

(2) Use a CV that encloses the free jet only



Continuity $V_1 A_1 = V_2 A_2$

Bernoulli $V_2 = (V_1^2 - 2gH)^{1/2}$

Momentum $-M_w g = w_1\{-|\rho V_1 A_1|\} + w_2\{+|\rho V_2 A_2|\}$

$w_1 = V_1 = V_0 \quad w_2 = V_2$

Substituting in momentum

$$-M_w g = V_0(-\rho V_0 A_1) + V_2(+\rho V_0 A_1) = \rho V_0 A_1 (V_2 - V_0)$$

$$M_w = \frac{\rho V_0 A_1 (V_0 - V_2)}{g}$$

$$= 999 \frac{kg}{m^3} \times 10 \frac{m}{s} \times \frac{\pi}{4}(0.05)^2 m^2 (10 - 8.97) \frac{m}{s} \times \frac{s^2}{9.81 \, m}$$

$$M_w = 2.06 \, kg \longleftarrow M_w$$

(3) Evaluate the area at each cross-section using Bernoulli and continuity, then integrate to find $\forall$.

$$VA = V_0 A_1 = (V_0^2 - 2gz)^{1/2} A = V_0 A_1 \quad \text{so} \quad A = \frac{V_0 A_1}{(V_0^2 - 2gz)^{1/2}}$$

$$\forall = \int_0^H A \, dz = \int_0^H \frac{V_0 A_1}{(V_0^2 - 2gz)^{1/2}} \, dz = A_1 \int_0^H \frac{V_0^2}{2g} \frac{1}{(1 - \frac{2gz}{V_0^2})^{1/2}} \, d\left(\frac{2gz}{V_0^2}\right)$$

This can be integrated. Let $r = 1 - 2gz/V_0^2$, so $\int = \int \frac{-dr}{r^{1/2}}$

Then $$\forall = A_1 \frac{V_0^2}{2g}\left[-2\left(1 - \frac{2gz}{V_0^2}\right)^{1/2}\right]_{z=0}^{z=h} = \frac{A_1}{g}\left[V_0^2 - V_0(V_0^2 - 2gh)^{1/2}\right]$$

and $$M_w = \rho \forall = \frac{\rho A_1 V_0 (V_0 - V_2)}{g} = 2.06 \, kg \text{ (same as (2) above)}$$

Thus the mass of the cone is $M_c = M - M_w = 2.40 \, kg$. $\longleftarrow M_c$

{Note: If V_0 were smaller or H larger, V_2 would differ more from V_0 and the jet area would increase significantly. Option (2) would still give the correct result with little effort.}

Open-Ended Problem Statement: Observe a uniform stream of water falling vertically from a water faucet onto a dinner plate held level beneath the stream. Vary the plate height. What do you observe? Vary the plate angle; comment on the changes in flow distribution and flow pattern that you observe.

Discussion: The jet falling from the faucet outlet is accelerated by gravity; the jet speed increases as it falls farther below the faucet outlet. Thus the jet is more energetic as it strikes the plate when the plate is held far below the nozzle outlet.

The flow pattern on the plate consists of a symmetric ring of a relatively thin layer of high-speed flow spreading radially outward from the jet impact point. At a certain radius the liquid layer thickens abruptly in a hydraulic jump. The continuity equation shows that flow speed must decrease when the liquid layer thickens. The thicker slow-speed layer continues radially outward and then flows off the edge of the plate.

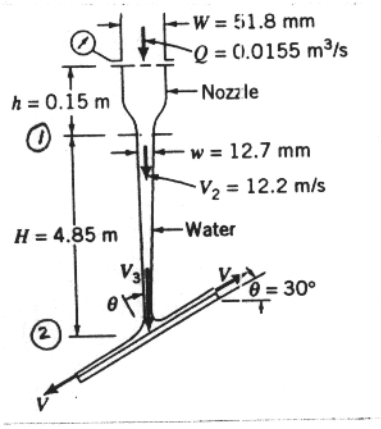
The jet striking the plate has a higher speed as the plate is moved downward in the stream. This causes the thin circular layer of high-speed flow to extend farther radially outward, moving the hydraulic jump to a larger radius.

Tipping the plate so that it is no longer horizontal destroys the symmetry of the flow pattern on the plate surface. The thin layer of high-speed flow is no longer circular. On the upper side of the tilted plate the hydraulic jump forms closer to the jet impact point. On the lower side of the tilted plate the hydraulic jump forms farther from the jet impact point. In all other respects the flow pattern is similar to that observed when the plate is held horizontal under the jet.

42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-384 200 SHEETS EYE-EASE® 5 SQUARE
42-390 RECYCLED WHITE 5 SQUARE
42-392
Made in U.S.A.
National® Brand

Given: Plane nozzle discharging water steadily, striking an inclined plate.

Neglect friction in nozzle and along plate surface.



Find: (a) Minimum gage pressure at nozzle inlet.
(b) Magnitude and direction of force exerted by water stream on plate.
(c) Sketch pressure distribution on plate. Explain why the pressure distribution is shaped as you show it.

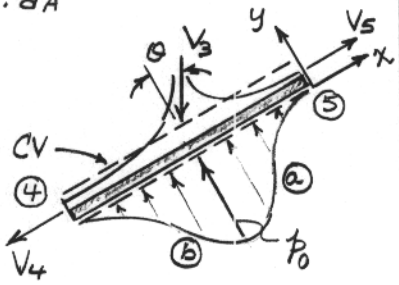
Solution: Apply continuity, Bernoulli, and momentum equations using the CV and coordinates shown.



Basic equations: $V_1 A_1 = V_2 A_2$ $\quad \frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2$

$$R_y + F_{By} = \frac{\partial}{\partial t}\int_{CV} v \rho d\forall + \int_{CS} v \rho \vec{V}\cdot d\vec{A}$$

(F_{By}: =0 (6); $\frac{\partial}{\partial t}$: =0 (3))



Assumptions: (1) Frictionless flow
(2) Incompressible flow
(3) Steady flow
(4) Flow along a streamline
(5) Uniform flow at each section

Then from continuity $V_1 = \frac{A_2}{A_1} V_2 = \frac{w}{W} V_2 = \frac{12.7\ mm}{51.8\ mm} \times 12.2 \frac{m}{s} = 2.99\ m/s$

From Bernoulli $p_{1g} = \frac{\rho}{2}(V_2^2 - V_1^2) + \rho g (z_1 - z_2) + p_{2g}^{=0}$; $z_1 - z_2 = h$

$$p_{1g} = \left[\frac{1}{2} \times 999 \frac{kg}{m^3} \times \left[(12.2)^2 - (2.99)^2\right] \frac{m^2}{s^2} - 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s} \times 0.15\,m\right] \frac{N \cdot s^2}{kg \cdot m} = 68.4\ kPa\ (gage)$$ p_{1g}

Calculate V_3 in the absense of the plate using Bernoulli ($p_2 = p_3$)

$$V_3 = \sqrt{V_2^2 + 2gH} = \left[(12.2)^2 \frac{m^2}{s^2} + 2 \times 9.81 \frac{m}{s^2} \times 4.85\,m\right]^{\frac{1}{2}} = 15.6\ m/s$$

From momentum: $R_x = 0$ since there is no friction on the plate surface.

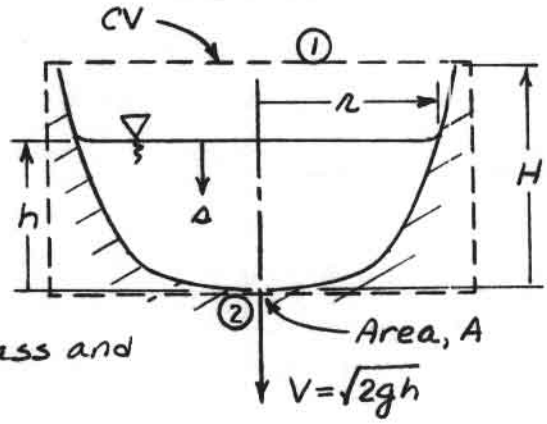
Assumptions: (6) Neglect mass of plate and of water on plate.
(7) Atmospheric pressure acts on entire CV; $F_{By} = R_y$

Then $R_y = v_3\{-\dot{m}_3\} + v_4^{0}\{+\dot{m}_4\} + v_5^{0}\{+\dot{m}_5\} = V_3 \cos\theta\, \rho Q$, since $v_3 = -V_3 \cos\theta$

$$R_y = 15.6 \frac{m}{s} \times \cos 30° \times 999 \frac{kg}{m^3} \times 0.0155 \frac{m^3}{s} \times \frac{N \cdot s^2}{kg \cdot m} = 209\ N;\ K_y = -R_y = -209\ N$$ K_y

Pressure is maximum at stagnation, minimum (p_{atm}) at (4) and (5).
Pressure at (a) is higher than at (b) because of streamline curvature.

Given: Egyptian water clock. Surface level drops at rate, Δ = constant.



Find: (a) Expression for $R(h)$.
(b) Volume needed for n hours' operation.

Solution: Apply conservation of mass and the Bernoulli equation.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Assumptions: (1) Quasi-steady flow; $\frac{\partial}{\partial t}$ small
(2) Incompressible flow
(3) Uniform flow at each cross section
(4) Flow along a streamline
(5) No friction
(6) $\rho_{air} \ll \rho_{H_2O}$

Writing Bernoulli from the liquid surface to the jet exit,

$$\frac{p_{atm}}{\rho} + \frac{\Delta^2}{2} + gh = \frac{p_{atm}}{\rho} + \frac{V^2}{2} + g(0);$$

For $\Delta \ll V$, then $V = \sqrt{2gh}$.

For the CV,

$$0 = \frac{\partial}{\partial t}\int_{\forall_{air}} \rho_{air}\, d\forall \;(\ll \rho_{H_2O}(6)) + \frac{\partial}{\partial t}\int_{\forall_{H_2O}} \rho_{H_2O}\, d\forall + \left\{-\left|\rho_{air} V_1 A_1\right|\right\} (\ll \rho_{H_2O}(6)) + \left\{\left|\rho_{H_2O} V A\right|\right\}$$

or

$$0 = \rho\frac{d\forall}{dt} + \rho V A = \rho\pi R^2\frac{dh}{dt} + \rho\sqrt{2gh}\,A = 0$$

But h decreases, so $\frac{dh}{dt} = -\Delta$. Thus

$$\pi R^2 \Delta = \sqrt{2gh}\,A \quad \text{or} \quad R = \sqrt[4]{2g}\sqrt{\frac{A}{\pi\Delta}}\,h^{1/4}$$

R

For n hours' operation, $H = n\Delta$, and

$$\forall = \int_0^H \pi R^2\, dh = \int_0^{n\Delta} \sqrt{2gh}\,\frac{A}{\Delta}\, dh = \frac{2A}{3\Delta}\sqrt{2g}\,(n\Delta)^{3/2}$$

or

$$\forall = \frac{2A\sqrt{2g}\,n^{3/2}\Delta^{1/2}}{3}$$

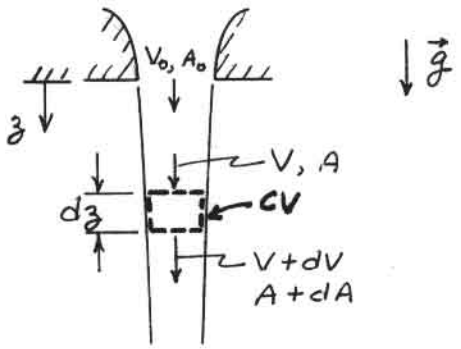
∀

Check dimensions:

$$[\forall] = L^3 = \left[A\sqrt{g}\,n^{3/2}\Delta^{1/2}\right] = L^2\,L^{1/2}\,t^{3/2}\,L^{1/2} = L^3 \;\checkmark$$

Given: Low-speed jet of incompressible liquid moving downward from nozzle.

Find: Expressions for $V(z)$, $A(z)$.
Location where $A = A_0/2$.



Solution: Apply continuity and momentum equations using CV shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{cv} \rho\, d\forall + \int_{cs} \rho \vec{V}\cdot d\vec{A} \qquad (=0(1))$$

$$F_{Sz} + F_{Bz} = \frac{\partial}{\partial t}\int_{cv} w\rho\, d\forall + \int_{cs} w\rho \vec{V}\cdot d\vec{A} \qquad (F_{Sz}=0(4,5),\ =0(1))$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) Patm acts everywhere } $F_{Sz} = 0$
(5) No friction

Then

$$0 = \int_{cs} \vec{V}\cdot d\vec{A} = \{-VA\} + \{+(V+dV)(A+dA)\}\ ;\ VA = V_0A_0 = \text{constant}$$

From momentum,

$$\rho g\left(A + \frac{dA}{2}\right)dz = V\{-\rho VA\} + (V+dV)\{-\rho(V+dV)(A+dA)\} = \rho VA\,dV$$

since $dV\,dA \ll dA$. Also, since $dA\,dz \ll dz$, the left side is $\rho g A\,dz$.
Thus

$$\rho g A\,dz = \rho V A\,dV \quad \text{or} \quad V\,dV = g\,dz$$

Integrating from V_0 at $z_0 = 0$ to V at z,

$$\int_{V_0}^{V} V\,dV = \frac{V^2}{2}\Big]_{V_0}^{V} = \frac{V^2}{2} - \frac{V_0^2}{2} = \int_{z_0}^{z} g\,dz = g(z - z_0) = gz$$

Thus

$$V^2 = V_0^2 + 2gz \quad \text{or} \quad V(z) = \sqrt{V_0^2 + 2gz} \qquad \leftarrow V(z)$$

Since $VA = V_0A_0$, $A = A_0\,\frac{V_0}{V}$

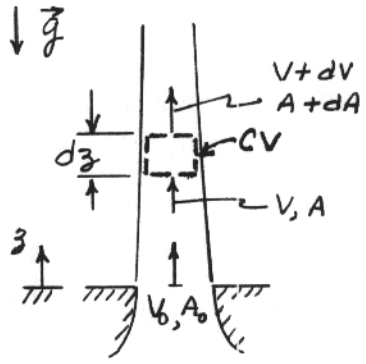
$$A(z) = A_0\frac{V_0}{\sqrt{V_0^2 + 2gz}} = \frac{A_0}{\sqrt{1 + 2gz/V_0^2}} \qquad \leftarrow A(z)$$

Solving for z,

$$z = \frac{V_0^2}{2g}\left[\left(\frac{A_0}{A}\right)^2 - 1\right]\ ;\ \text{for } \frac{A}{A_0} = \frac{1}{2},\ \frac{A_0}{A} = 2,\ \text{and}\ z_{1/2} = \frac{3V_0^2}{2g} \qquad \leftarrow z_{1/2}$$

Given: Low-speed jet of incompressible liquid moving upward from nozzle.

Find: Expressions for $V(z)$, $A(z)$.
Location where $V = 0$.



Solution: Apply continuity and momentum equation using CV shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \qquad \left(\frac{\partial}{\partial t} = 0(1)\right)$$

$$F_{Bz} + F_{Bz} = \frac{\partial}{\partial t}\int_{CV} w\rho\, d\forall + \int_{CS} w\rho \vec{V}\cdot d\vec{A} \qquad \left(F_{Bz} = 0(4,5),\ \frac{\partial}{\partial t} = 0(1)\right)$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) p_{atm} acts everywhere } $F_{Sz} = 0$
(5) No friction

Then

$$0 = \int_{CS} \vec{V}\cdot d\vec{A} = \{-VA\} + \{+(V+dV)(A+dA)\}\ ;\quad VA = V_0 A_0 = \text{constant}$$

From momentum,

$$-\rho g\left(A + \frac{dA}{2}\right)dz = V\{-\rho VA\} + (V+dV)\{+\rho(V+dV)(A+dA)\} = \rho VA\, dV$$

since $dV\,dA \ll dA$. Also, since $dA\,dz \ll dz$, the left side is $-\rho g A\, dz$. Thus

$$-\rho g A\, dz = \rho VA\, dV \quad \text{or} \quad V\,dV = -g\,dz$$

Integrating from V_0 at $z_0 = 0$ to V at z,

$$\int_{V_0}^{V} V\,dV = \frac{V^2}{2}\Big]_{V_0}^{V} = \frac{V^2}{2} - \frac{V_0^2}{2} = \int_{z_0}^{z} -g\,dz = -g(z - z_0) = -gz$$

Thus $V^2 = V_0^2 - 2gz$ or $V(z) = \sqrt{V_0^2 - 2gz}$ ← $V(z)$

Since $VA = V_0 A_0$, then $A = A_0 \frac{V_0}{V}$

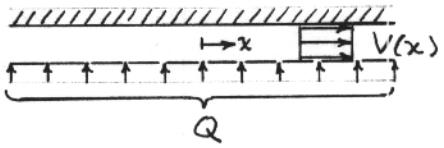
$$A(z) = A_0 \frac{V_0}{\sqrt{V_0^2 - 2gz}} = \frac{A_0}{\sqrt{1 - 2gz/V_0^2}}$$ ← $A(z)$

Solving for z at $V = 0$,

$$z = \frac{V_0^2}{2g}$$ ← z

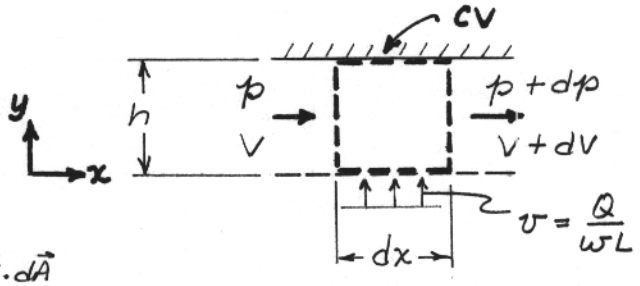
Given: Uniform flow in narrow gap between parallel plates, as shown.

Fluid in gap has only horizontal motion.



Find: Expression for $p(x)$.

Solution: Apply continuity and x component of momentum equation.



Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{cv}\rho\, d\forall + \int_{cs}\rho\vec{V}\cdot d\vec{A} \qquad \left(\frac{\partial}{\partial t}=0(1)\right)$$

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{cv} u\rho\, d\forall + \int_{cs} u\rho\vec{V}\cdot d\vec{A} \qquad \left(F_{Sx}=0(5),\ F_{Bx}=0(5),\ \frac{\partial}{\partial t}=0(1)\right)$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) Neglect friction
(5) $F_{Bx} = 0$

Then

$$0 = \int_{cs}\vec{V}\cdot d\vec{A} = \{-V w h\} + \left\{-\frac{Q}{wL} w\, dx\right\} + \{(V+dV) w h\}; \quad w h\, dV = \frac{Q}{L}dx$$

$$V = \frac{Q}{wh}\frac{x}{L} + C \ ; \ C = 0 \text{ since } V(0) = 0 \ ; \ V(x) = \frac{Q}{wh}\frac{x}{L}$$

From momentum,

$$p\, w h - (p+dp) w h = u_x\{-\rho V w h\} + u_{dx}\left\{-\rho\frac{Q}{wh} w\, dx\right\} + u_{x+dx}\{+\rho(V+dV) w h\}$$

$$u_x = V \qquad u_{dx} = 0 \qquad u_{x+dx} = V + dV$$

From continuity, $(V+dV) w h = V w h + Q\frac{dx}{L}$, so

$$-dp\, w h = -\rho V^2 w h + 0 + (V+dV)\left(V w h + Q\frac{dx}{L}\right)\rho$$

$$= -\rho V^2 w h + \rho V^2 w h + \rho V w h\, dV + V\rho Q\frac{dx}{L} + \rho Q\, dV\frac{dx}{L}$$

Neglecting products of differentials ($dV dx \ll dx$), and with $dV = \frac{Q}{wh}\frac{dx}{L}$

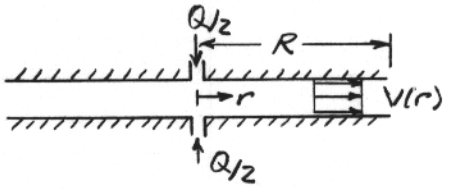
$$-dp = \rho V dV + V\frac{\rho Q}{wh}\frac{dx}{L} = \rho V\frac{Q}{wh}\frac{dx}{L} + V\frac{\rho Q}{wh}\frac{dx}{L} = 2\rho\frac{Q}{wh}\frac{x}{L}\frac{Q}{wh}\frac{dx}{L}$$

$$-dp = 2\rho\left(\frac{Q}{whL}\right)^2 x\, dx \qquad p(x) = -\rho\left(\frac{Q}{whL}\right)^2 x^2 + C$$

If $p(0) = p_0$, then $p(x) = p_0 - \rho\left(\frac{Q}{wh}\right)^2\left(\frac{x}{L}\right)^2$ $p(x)$

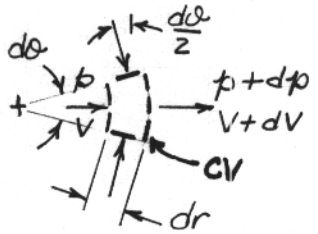
Given: Uniform flow in narrow gap between parallel disks, as shown.

Liquid in gap has only radial motion.



Find: Expression for $p(r)$.

Solution: Apply continuity and momentum equations to the differential CV shown.



Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \qquad (=0(1))$$

$$F_{sr} + F_{\beta r} = \frac{\partial}{\partial t}\int_{CV} V_r \rho\, d\forall + \int_{CS} V_r \rho \vec{V}\cdot d\vec{A} \qquad (F_{sr}=0(5),\ F_{\beta r}=0(1))$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) Neglect friction
(5) $F_{\beta r} = 0$
(6) No flow in θ direction
(7) $\sin\frac{d\theta}{2} = \frac{d\theta}{2}$

Then

$$0 = \int_{CS} \vec{V}\cdot d\vec{A} = \{-\rho V h r d\theta\} + \{\rho(V+dV)h(r+dr)d\theta\} \quad ; V_r = \text{constant}$$

For $r=R$, $Q = V_R 2\pi r h$, so $V_R = Q/2\pi R h$

From momentum,

$$p h r d\theta + 2\left(p+\frac{dp}{2}\right) h\, dr \sin\frac{d\theta}{2} - (p+dp)h(r+dr)d\theta$$

$$= V\{-\rho V h r d\theta\} + (V+dV)\{\rho(V+dV)h(r+dr)d\theta\}$$

$$p h r d\theta + p h\, dr\, d\theta + \tfrac{1}{2} dp\, h\, dr\, d\theta - (p r + p\, dr + r\, dp + dr\, dp) h\, d\theta$$

$$= dV(\rho V h r d\theta) \quad \{\text{Note terms in braces are equal.}\}$$

Assuming products of differentials are much smaller than single differentials,

$$-r\,dp\,h\,d\theta = dV(\rho V h r d\theta) \quad \text{or} \quad dp = -\rho V dV$$

Integrating, $p(r) - p(R) = -\rho\frac{V^2}{2} + \rho\frac{V_R^2}{2}$ or $p(r) - p_{atm} = \frac{\rho}{2}(V_R^2 - V^2)$

Since $V_R = \frac{Q}{2\pi R h}$, and V_r = constant, $\frac{V}{V_R} = \frac{R}{r}$, so $= \frac{\rho V_R^2}{2}\left[1 - \left(\frac{V}{V_R}\right)^2\right]$

$$p(r) - p_{atm} = \frac{\rho}{2}\left(\frac{Q}{2\pi R h}\right)^2\left[1 - \left(\frac{R}{r}\right)^2\right] \qquad p(r)$$

Note since $r < R$, that $p(r) < p_{atm}$ between the disks.

Given: Liquid falling vertically into short, horizontal, rectangular open channel. Neglect viscous effects.

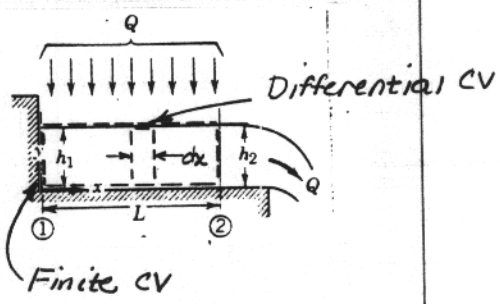
Find: (a) Expression for h_1 in terms of h_2, Q, and b.
(b) Sketch surface profile, h(x).

Solution: Apply continuity and momentum equations to (i) finite CV, and (ii) differential CV, as shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} \qquad (=0(1))$$

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A} \qquad (F_{Bx}=0(6),\ \frac{\partial}{\partial t}=0(1))$$



Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) Hydrostatic pressure distribution; $F_p(h) = \rho g b \frac{h^2}{2}$
(5) No friction on bed
(6) Horizontal bed; $F_{Bx} = 0$

Then for finite CV shown,

$$0 = \int_{CS}\vec{V}\cdot d\vec{A} = -Q + V_2 b h_2 \;;\; V_2 = \frac{Q}{b h_2}$$

From momentum

$$\rho g b\frac{h_1^2}{2} - \rho g b\frac{h_2^2}{2} = u_1\{0\} + u_2\{+\rho Q\} + u_3\{-\rho Q\}$$

$$u_2 = V_2 \qquad u_3 = 0$$

$$\frac{\rho g b}{2}(h_1^2 - h_2^2) = V_2\rho Q = \frac{Q}{b h_2}\rho Q = \frac{\rho Q^2}{b h_2} \;;\; h_1 = \sqrt{h_2^2 + \frac{2Q^2}{g b^2 h_2}}$$ ← h_1

For differential CV shown,

$$0 = \int_{CS}\vec{V}\cdot d\vec{A} = \{-Vbh\} + \{-\frac{Q}{bL}b\,dx\} + \{+(V+dV)b(h+dh)\}$$

$$0 = -\frac{Q}{L}dx + b(h\,dV + V\,dh) = -\frac{Q}{L}dx + b\,d(hV) \;;\; \frac{d(hV)}{dx} = \frac{Q}{L}$$

From momentum,

$$\rho g b\frac{h^2}{2} - \rho g b\frac{(h+dh)^2}{2} = V\{-\rho V b h\} + 0\{-\rho\frac{Q}{L}dx\} + (V+dV)\{+\rho(V+dV)b(h+dh)\}$$

Using continuity,

$$\frac{\rho g b}{2}(-2h\,dh + d\cancel{h}\,dh) = -\rho V^2 b h + (V+dV)\{\rho V b h + \rho\frac{Q}{L}dx\} \qquad (d\cancel{h}\,dh \ll dh)$$

Cont'd. →

$$-\rho g b\, h\,dh = -\rho V^2 bh + \rho V^2 bh + \rho V bh\,dV + \frac{\rho Q}{L} V\,dx + \frac{\rho Q}{L} dV\,dx$$ ($dV\,dx \ll dx$)

or

$$-ghdh = Vh\,dV + \frac{Q}{bL} V\,dx$$

From continuity, $Vh\,dV = -V^2 dh + \frac{Q}{bL} V\,dx$, so

$$-ghdh = -V^2 dh + \frac{2Q}{bL} V\,dx$$

Solving,

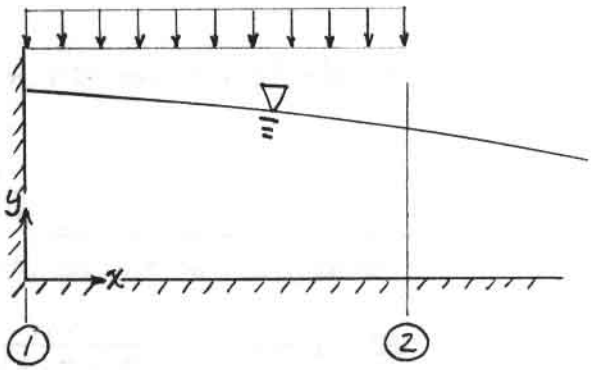
$$\frac{dh}{dx}\left(V^2 - gh\right) = \frac{2Q}{bL} V$$

$$\frac{dh}{dx} = \frac{2QV}{bL(V^2 - gh)} = \frac{2QV}{bLgh\left(V^2/gh - 1\right)}$$

From finite CV analysis, $h_1 > h_2$, so $\frac{dh}{dx} < 0$. Thus $V^2/gh < 1$. As x increases, $V \uparrow$ and $h \downarrow$. Therefore

$$\frac{V^2}{gh} \uparrow, \quad \frac{V}{h} \uparrow, \quad \text{and} \quad \left|\frac{dh}{dx}\right| \uparrow.$$

Sketch:



Given: Narrow gap between parallel disks filled with liquid.

At $t = 0^+$, upper disk begins to move downward at V_0.

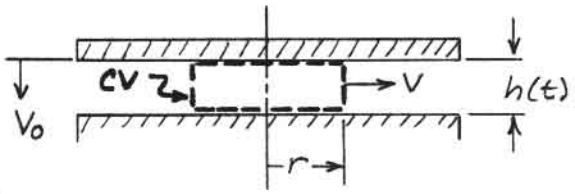
Neglect viscous effects; flow uniform in horizontal direction.

Find: Expression for velocity field, $V(r)$. Note flow is <u>not</u> steady.

<u>Solution</u>: Apply continuity, using the <u>deformable</u> CV shown.

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$



Assumptions: (1) Incompressible flow
(2) Uniform flow at each cross section

Then

$$0 = \frac{\partial}{\partial t}\int_{CV} d\forall + \int_{CS} \vec{V} \cdot d\vec{A} = \frac{\partial}{\partial t}\int_{CV} d\forall + V\,2\pi r h$$

But

$$\int_{CV} d\forall = \pi r^2 h, \quad \text{so} \quad \frac{\partial}{\partial t}\int_{CV} d\forall = \frac{\partial}{\partial t}(\pi r^2 h) = \pi r^2 \frac{dh}{dt}$$

Thus

$$0 = \pi r^2 \frac{dh}{dt} + V 2\pi r h = \pi r^2 (-V_0) + V\,2\pi r h$$

so

$$V(r) = V_0 \frac{r}{2h} \qquad V(r)$$

If V_0 is constant, so $h = h_0 - V_0 t$, and

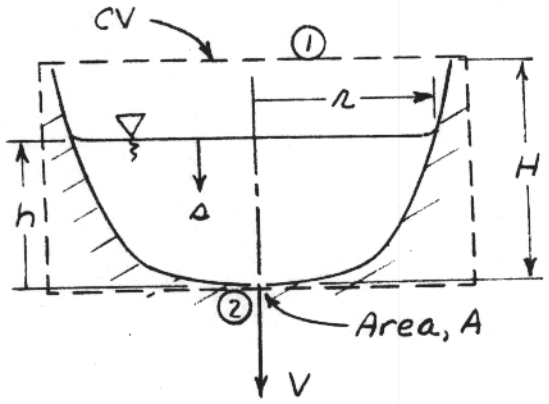
$$V(r,t) = \frac{V_0 r}{2(h_0 - V_0 t)} \quad \text{for } t < \frac{h_0}{V_0} \qquad V(r,t)$$

Open-Ended Problem Statement: Design a clepsydra (Egyptian water clock) — a vessel from which water drains by gravity through a hole in the bottom and time is indicated by the level of the remaining water. Specify the dimensions of the vessel and the size of the drain hole; indicate the amount of water needed to fill the vessel, and at what interval it must be filled. Plot the vessel shape. (This is an open-ended problem when choosing dimensions for a specific application.)

Discussion: The original Egyptian water clock was an open water-filled vessel with an orifice in the bottom. The vessel shape was designed so that the water level dropped at a constant rate during use.

Water leaves the orifice at higher speed when the water level within the vessel is high, and at lower speed when the water level within the vessel is low. The size of the orifice is constant. Thus the instantaneous volume flow rate depends on the water level in the vessel.

The rate at which the water level falls in the vessel depends on the volume flow rate and the area of the water surface. The surface area at each water level must be chosen so that the water level within the vessel decreases at a constant rate. The continuity and Bernoulli equations can be applied to determine the required vessel shape so that the water surface level drops at a constant rate.



Use the CV and notation shown (Problem 4.97):

Solution: Basic equations are

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Assumptions: (1) Quasi-steady flow
(2) Incompressible flow
(3) Uniform flow at each cross-section
(4) Flow along a streamline
(5) No friction
(6) $\rho_{air} \ll \rho_{H_2O}$

Writing Bernoulli from the liquid surface to the jet exit,

$$\frac{p_{atm}}{\rho} + \frac{a^2}{2} + gh = \frac{p_{atm}}{\rho} + \frac{V^2}{2} + g(0)$$

For $a \ll V$, then $V = \sqrt{2gh}$

For the CV,

$$0 = \frac{\partial}{\partial t}\int_{\forall_{air}} \rho_{air} \, d\forall \; (\ll \rho_{H_2O}\ (6)) + \frac{\partial}{\partial t}\int_{\forall_{H_2O}} \rho_{H_2O} \, d\forall + \left\{-|\rho_{air} V_1 A_1|\right\} \; (\ll \rho_{H_2O}\ (6)) + \left\{|\rho_{H_2O} V A|\right\}$$

42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

or $$0 = \rho \frac{d\forall}{dt} + \rho V A = \rho \pi r^2 \frac{dh}{dt} + \rho \sqrt{2gh}\, A$$

But h decreases, so $\frac{dh}{dt} = -\Delta$. Thus

$$\pi r^2 \Delta = \sqrt{2gh}\, A \quad \text{or} \quad r = \sqrt[4]{2g}\sqrt{\frac{A}{\pi \Delta}}\, h^{1/4}$$ ← r

For n hours operation, $H = n\Delta$, and

$$\forall = \int_0^H \pi r^2 dh = \int_0^{n\Delta} \sqrt{2gh}\,\frac{A}{\Delta}\, dh = \frac{2A}{3\Delta}\sqrt{2g}\,(n\Delta)^{3/2}$$

or

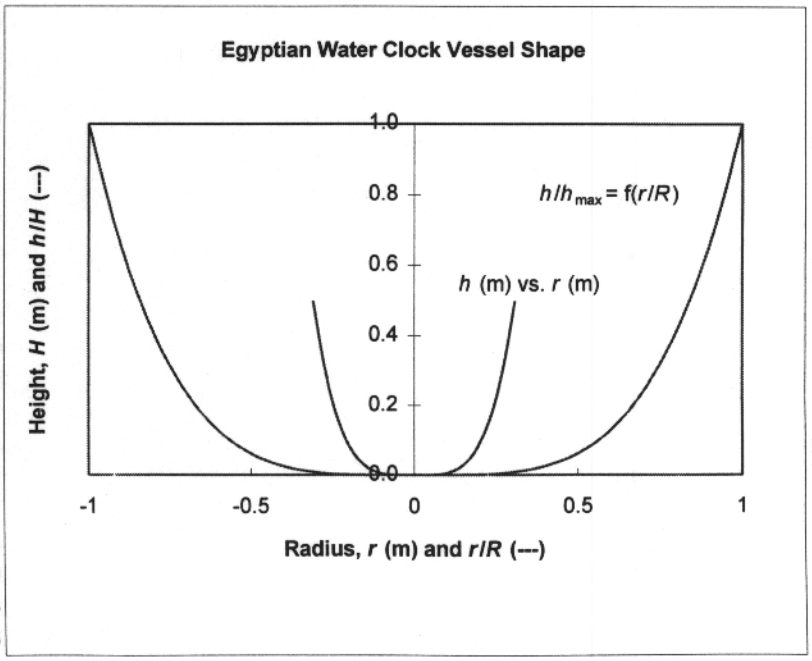
$$\forall = \frac{2A\sqrt{2g}}{3} n^{3/2} \Delta^{1/2}$$ ← ∀

Evaluating and plotting:

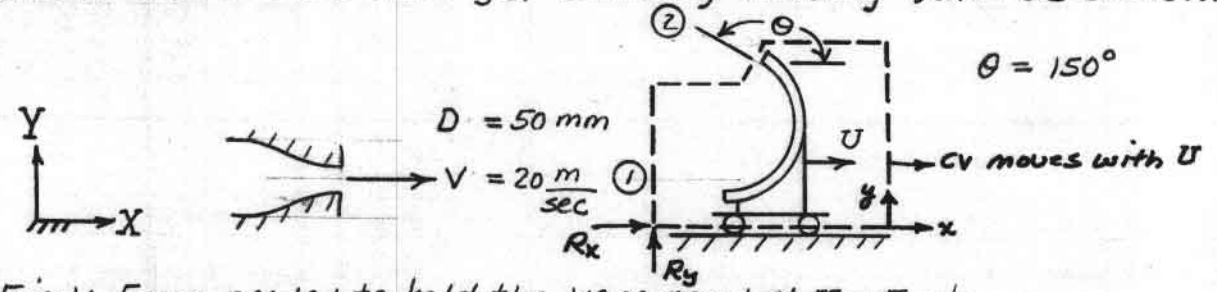
Input Parameters:

Maximum water height:	H =	0.5	m
Number of hours' duration:	n =	24	hr

Dimensionless Shape		Actual Shape	
r/R	***h/H***	***r* (m)**	***h* (m)**
-1	1.00	-0.309	0.500
-0.9	0.656	-0.278	0.328
-0.8	0.410	-0.247	0.205
-0.7	0.240	-0.216	0.120
-0.6	0.130	-0.185	0.065
-0.5	0.063	-0.155	0.031
-0.4	0.026	-0.124	0.013
-0.3	0.008	-0.093	0.004
-0.2	0.002	-0.062	0.001
-0.1	0.000	-0.031	0.000
0	0	0	0
0.1	0.000	0.031	0.000
0.2	0.002	0.062	0.001
0.3	0.008	0.093	0.004
0.4	0.026	0.124	0.013
0.5	0.063	0.155	0.031
0.6	0.130	0.185	0.065
0.7	0.240	0.216	0.120
0.8	0.410	0.247	0.205
0.9	0.656	0.278	0.328
1	1.000	0.309	0.500



Given: Water flow from jet striking moving vane as shown.



Find: Force needed to hold the vane speed at $U = 5$ m/s.

Solution: Apply momentum equation to moving CV shown.

$$\text{B.E.:}\quad F_{Sx} + \cancel{F_{Bx}}^{=0(2)} = \cancel{\frac{\partial}{\partial t}}^{=0(3)} \int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

$$F_{Sy} + \cancel{F_{By}}^{\approx 0(2)} = \cancel{\frac{\partial}{\partial t}}^{=0(3)} \int_{CV} v_{xyz}\,\rho\,d\forall + \int_{CS} v_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

Assume: (1) No pressure forces or friction, so $F_{Sx} = R_x$, $F_{Sy} = R_y$
(2) $F_{Bx} = 0$, neglect F_{By} since not given
(3) Steady flow
(4) Uniform flow at each section
(5) Relative velocity constant for jet stream crossing vane

Then

$$R_x = u_1\{-|\rho(V-U)A|\} + u_2\{|\rho(V-U)A|\} \quad ; A = \frac{\pi}{4}(0.05)^2\,m^2 = 1.96\times10^{-3}\,m^2$$

$$u_1 = V - U \qquad u_2 = (V-U)\cos\theta$$

$$R_x = \rho(V-U)^2 A(\cos\theta - 1)$$

$$R_x = 999\,\frac{kg}{m^3}(20-5)^2\frac{m^2}{s^2}\times 1.96\times10^{-3}\,m^2(\cos 150^\circ - 1)\frac{N\cdot s^2}{kg\cdot m} = -822\ N \qquad R_x \leftarrow$$

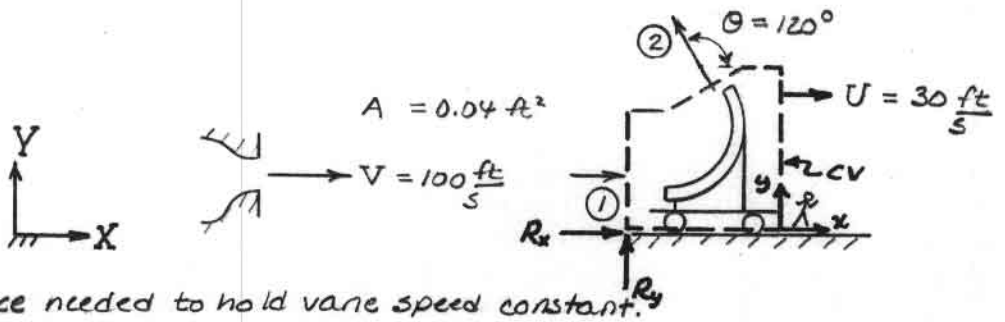
$$R_y = v_1\{-|\rho(V-U)A|\} + v_2\{|\rho(V-U)A|\}$$

$$v_1 = 0 \qquad v_2 = (V-U)\sin\theta$$

$$R_y = \rho(V-U)^2 A\sin\theta = 999\,\frac{kg}{m^3}(20-5)\frac{m^2}{s^2}\times 1.96\times10^{-3}\,m^2\times\sin 150^\circ\times\frac{N\cdot s^2}{kg\cdot m} = 220\ N \qquad R_y \leftarrow$$

{Thus a force of 822 N to the left and 220 N upward must be applied to the vane to maintain its motion at $U = 5$ m/s.}

Given: Jet of water striking a moving vane as shown.



Find: Force needed to hold vane speed constant.

Solution: Apply momentum equation using moving CV shown.

Basic equations:

$$F_{Sx} + \cancel{F_{Bx}} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

(=0 (2), =0 (3))

$$F_{Sy} + \cancel{F_{By}} = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\,d\forall + \int_{CS} v_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

(=0 (2), =0 (3))

Assumptions: (1) No pressure forces on CV; $F_{Sx} = R_x$, $F_{Sy} = R_y$
(2) $F_{Bx} = 0$; neglect F_{By}
(3) Steady flow relative to vane
(4) Flow uniform at each section
(5) Jet area and speed relative to vane are constant

The subscript xyz is a reminder that all velocities must be evaluated relative to the CV. Then

$$R_x = u_1\{-|\rho(V-U)A|\} + u_2\{|\rho(V-U)A|\}$$

$$u_1 = V - U \qquad u_2 = (V-U)\cos\theta$$

and

$$R_x = \rho(V-U)^2 A\,(\cos\theta - 1)$$

$$= 1.94\,\frac{slug}{ft^3} \times (100-30)^2\,\frac{ft^2}{s^2} \times 0.04\,ft^2 \times (\cos 120° - 1) \times \frac{lbf\cdot s^2}{slug\cdot ft}$$

$$R_x = -570\ lbf \text{ (to the left)} \qquad \leftarrow R_x$$

Also

$$R_y = v_1\{-|\rho(V-U)A|\} + v_2\{|\rho(V-U)A|\}$$

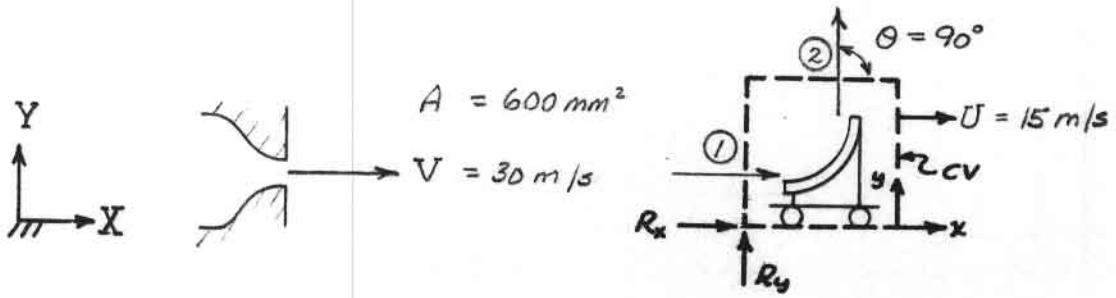
$$v_1 = 0 \qquad v_2 = (V-U)\sin\theta$$

$$R_y = \rho(V-U)^2 A\sin\theta$$

$$= 1.94\,\frac{slug}{ft^3} \times (100-30)^2\,\frac{ft^2}{s^2} \times 0.04\,ft^2 \times \sin 120° \times \frac{lbf\cdot s^2}{slug\cdot ft}$$

$$R_y = 329\ lbf \text{ (force is up)} \qquad \uparrow R_y$$

Given: Jet of water striking a moving vane as shown.



Find: Force needed to maintain the vane speed constant.

Solution: Apply momentum equation using moving CV shown.

Basic Equations: $F_{Sx} + \cancel{F_{Bx}}^{=0(2)} = \cancel{\frac{\partial}{\partial t}}^{=0(3)} \int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$

$$F_{Sy} + \cancel{F_{By}}^{=0(2)} = \cancel{\frac{\partial}{\partial t}}^{=0(3)} \int_{CV} v_{xyz}\,\rho\,d\forall + \int_{CS} v_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) No net pressure forces on CV; $F_{Sx} = R_x$, $F_{Sy} = R_y$
(2) $F_{Bx} = 0$; neglect F_{By}
(3) Steady flow relative to vane
(4) Uniform flow at each section
(5) Jet area and speed relative to vane are constant

All velocities must be relative to CV. Then

$$R_x = u_1\{-|\rho(V-U)A|\} + u_2\{|\rho(V-U)A|\}$$

$$u_1 = V - U \qquad u_2 = (V-U)\cos\theta$$

and

$$R_x = \rho(V-U)^2 A(\cos\theta - 1)$$

$$= 999\,\frac{kg}{m^3}(30-15)^2\frac{m^2}{s^2}\times 600\,mm^2\times\frac{m^2}{10^6\,mm^2}(\cos 90^\circ - 1)\frac{N\cdot s^2}{kg\cdot m}$$

$R_x = -135\,N$ (to the left) R_x

Also

$$R_y = v_1\{-|\rho(V-U)A|\} + v_2\{|\rho(V-U)A|\}$$

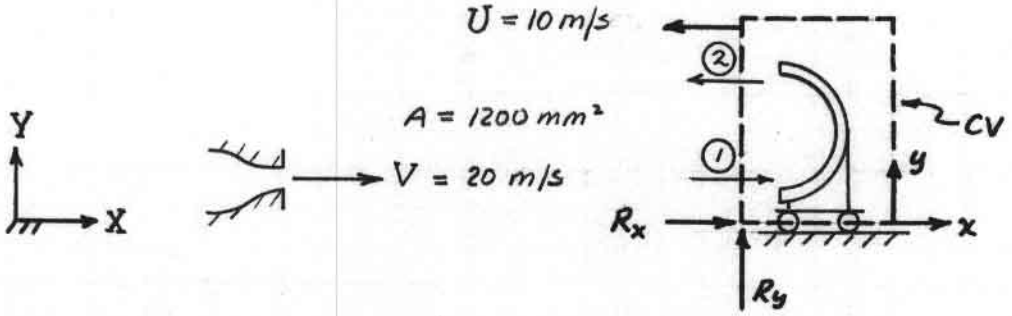
$$v_1 = 0 \qquad v_2 = (V-U)\sin\theta$$

$$R_y = \rho(V-U)^2 A\sin\theta$$

$$= 999\,\frac{kg}{m^3}(30-15)^2\frac{m^2}{s^2}\times 600\,mm^2\times\frac{m^2}{10^6\,mm^2}\times\sin 90^\circ\times\frac{N\cdot s^2}{kg\cdot m} = 135\,N$$

$R_y = 135\,N$ (force must be up) R_y

Given: Jet of oil (SG = 0.8) striking moving vane.



Find: Force needed to maintain vane speed constant.

Solution: Apply x component of momentum equation to moving CV shown.

Basic equation: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$ ($F_{Bx} = 0$; $\frac{\partial}{\partial t} = 0$)

Assumptions: (1) No net pressure force on CV; $F_{Sx} = R_x$
(2) $F_{Bx} = 0$
(3) Steady flow
(4) Flow uniform at each section
(5) Jet area and speed relative to vane are constant

The subscript xyz is a reminder that all velocities must be evaluated relative to the CV. Then

$$R_x = u_1\{-|\rho(V+U)A|\} + u_2\{|\rho(V+U)A|\}$$

$$u_1 = V + U \qquad u_2 = -(V+U)$$

and $$R_x = -\rho(V+U)^2A - \rho(V+U)^2A = -2\rho(V+U)^2A = -2SG\,\rho_{H_2O}(V+U)^2A$$

$$R_x = -2(0.8)\,999\,\frac{kg}{m^3}\,(20+10)^2\frac{m^2}{s^2} \times 1200\,mm^2 \times \frac{m^2}{10^6\,mm^2} \times \frac{N\cdot s^2}{kg\cdot m} = -1.73\,kN$$

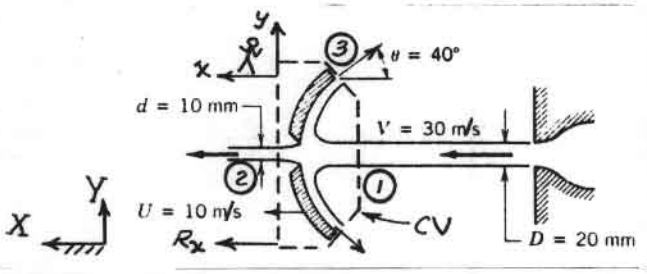
R_x ←

This force must be applied to the left on the vane.

{Note $R_y = mg$, since there are no vertical components of velocity.}

Given: Circular dish and jet moving as shown.

Find: Force required to maintain dish motion.



Solution: Apply continuity and x momentum equation to CV moving with dish as shown.

Basic equations:

$$0 = \cancel{\frac{\partial}{\partial t}}\int_{CV}\rho\,d\forall + \int_{CS}\rho\vec{V}_{xyz}\cdot d\vec{A}$$

(=0(1) =0(3) =0(1))

$$F_{Sx} + \cancel{F_{Bx}} = \cancel{\frac{\partial}{\partial t}}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow w.r.t. CV
(2) No pressure forces on CV
(3) Horizontal; $F_{Bx}=0$
(4) Uniform flow at each section
(5) No change in speed of jet relative to vane
(6) Incompressible flow

Then

$$0 = \int_{CS}\vec{V}_{xyz}\cdot d\vec{A} = (V-U)\left(-\frac{\pi D^2}{4} + \frac{\pi d^2}{4} + A_{3,4}\right)$$

$$A_{3,4} = \frac{\pi}{4}(D^2-d^2) = \frac{\pi}{4}\left[(0.020)^2-(0.010)^2\right]m^2 = 2.36\times10^{-4}\,m^2$$

From the momentum equation

$$R_x = u_1\left\{-\rho(V-U)\frac{\pi D^2}{4}\right\} + u_2\left\{+\rho(V-U)\frac{\pi d^2}{4}\right\} + u_3\left\{+\rho(V-U)A_{3,4}\right\}$$

$u_1 = V-U \qquad u_2 = V-U \qquad u_3 = -(V-U)\cos 40^\circ$

$$R_x = -\rho(V-U)^2\frac{\pi D^2}{4} + \rho(V-U)^2\frac{\pi d^2}{4} - \rho(V-U)^2\frac{\pi}{4}(D^2-d^2)\cos 40^\circ$$

$$= -\rho(V-U)^2\frac{\pi}{4}(D^2-d^2)(1+\cos 40^\circ)$$

$$= -999\frac{kg}{m^3}\times(30-10)^2\frac{m^2}{s^2}\times 2.36\times10^{-4}\,m^2(1+\cos 40^\circ)\times\frac{N\cdot s^2}{kg\cdot m}$$

$R_x = -167$ N (force must be applied to right)

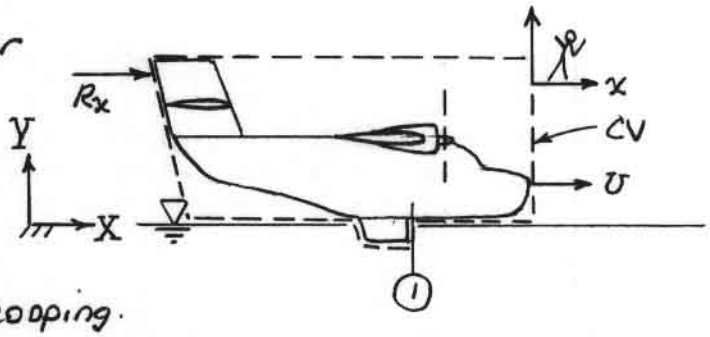
R_x

{Note: $R_y = Mg$, since there is no net momentum flux in the y-direction.}

Given: Aircraft scooping water from lake:

1620 gal in 12 sec

Find: Added thrust needed to maintain steady aircraft speed during scooping.



Solution: Use CV moving with aircraft, as shown. Apply momentum.

Basic equation: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$ ($F_{Bx}=0$ (1); ≈ 0 (2))

Assumptions: (1) Horizontal motion, so $F_{Bx} = 0$
(2) Neglect u_{xyz} within the CV
(3) Uniform flow at inlet cross-section
(4) Neglect hydrostatic pressure

Then

$$R_x = u_1\{-|\rho Q_1|\} = -U(-\rho Q) = +U\rho Q$$

$$u_1 = -U$$

From data given

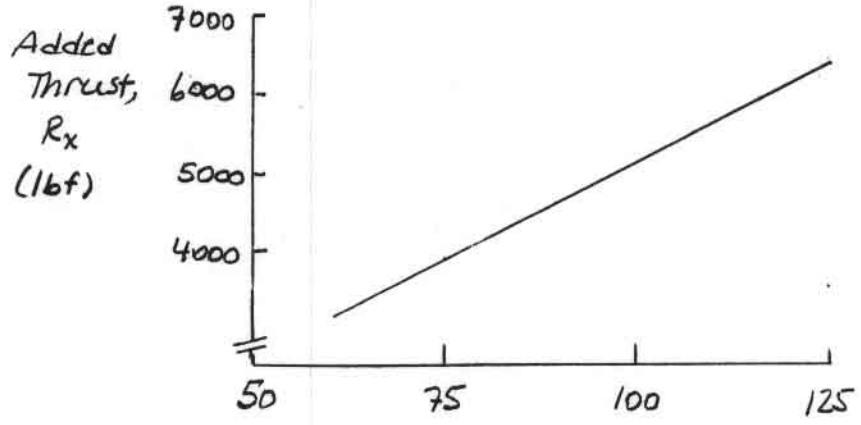
$$Q = \frac{\Delta\forall}{\Delta t} = \frac{1620\ \text{gal}}{12\ \text{sec}} \times \frac{\text{ft}^3}{7.48\ \text{gal}} = 18.0\ \text{ft}^3/\text{s}$$

For an aircraft speed of $U = 75$ mph (110 ft/s)

$$R_x = 110\frac{\text{ft}}{\text{s}} \times 1.94\frac{\text{slug}}{\text{ft}^3} \times 18.0\frac{\text{ft}^3}{\text{s}} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} = 3{,}840\ \text{lbf}$$

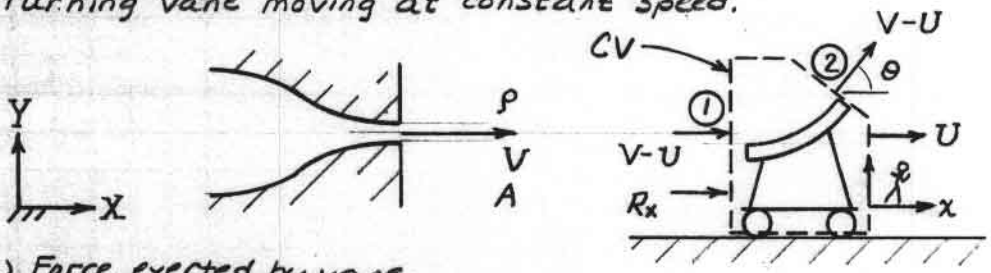
R_x ←

For a range of aircraft speeds:



{Thus at 60 mph the added thrust is 3,070 lbf, while at 125 mph the added thrust is 6,400 lbf.}

Given: Turning vane moving at constant speed.



Find: (a) Force exerted by vane.
(b) Power produced by vane.
(c) Show power maximized when $U = V/3$.

Solution: Apply x component of momentum equation to moving CV.

Basic equation: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$ ($F_{Bx}=0$ (2); $\frac{\partial}{\partial t}(\cdots)=0$ (3))

Assumptions: (1) No net pressure force; $F_{Sx} = R_x$
(2) $F_{Bx} = 0$
(3) Steady flow with respect to observer on CV
(4) Uniform flow at each section
(5) No change in jet speed (relative to vane) on vane
(6) Incompressible flow, so A = constant

Then

$$R_x = u_{xyz_1}\{-|\rho V_{xyz_1} A_1|\} + u_{xyz_2}\{|\rho V_{xyz_2} A_2|\}$$

$u_{xyz_1} = V-U$ $\qquad u_{xyz_2} = (V-U)\cos\theta$

$V_{xyz_1} = V-U$ $\qquad V_{xyz_2} = V-U$

so

$$R_x = (V-U)[-\rho(V-U)A] + (V-U)\cos\theta[\rho(V-U)A] = \rho(V-U)^2 A(\cos\theta - 1)$$

This is force exerted on vane. The force exerted by vane is

$$K_x = -R_x = \rho(V-U)^2 A(1-\cos\theta) \longleftarrow K_x$$

The power produced by the vane is

$$\dot{W}_{out} = K_x U = \rho(V-U)^2 U A(1-\cos\theta) \longleftarrow \dot{W}_{out}$$

To maximize, set $d\dot{W}_{out}/dU = 0$

$$\frac{d\dot{W}}{dU} = \rho(V-U)^2 A(1-\cos\theta) + (2)(-1)\rho(V-U)UA(1-\cos\theta) = 0$$

or

$$V - U - 2U = V - 3U = 0$$

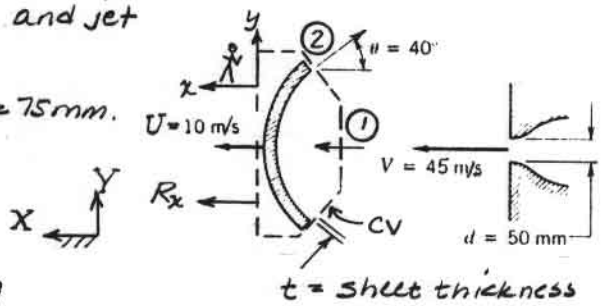
so

$$U = \frac{V}{3} \quad \text{(vane speed for maximum power output)} \longleftarrow U$$

{Note: $K_y = -R_y = -Mg - \rho(V-U)^2 A\sin\theta$, but this force does not produce power.}

Given: Circular dish with $D = 0.15$ m and jet as shown.

Find: (a) Thickness of jet sheet at $R = 75$ mm.
(b) Horizontal force required to maintain dish motion.



Solution: Apply the momentum equation to a CV moving with the dish, as shown.

Basic equation:

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

($F_{Sx} = 0$ (2); $\frac{\partial}{\partial t} = 0$ (3))

Assumptions: (1) No pressure forces
(2) Horizontal; $F_{Bx} = 0$
(3) Steady flow w.r.t. CV
(4) Uniform flow at each section
(5) Use relative velocities
(6) No change in relative velocity on the dish

Then

$$R_x = u_1\{-\rho(V-U)A\} + u_2\{+\rho(V-U)A\}$$

$$u_1 = V-U \qquad u_2 = -(V-U)\cos\theta$$

$$R_x = -\rho(V-U)^2A - \rho(V-U)^2A\cos\theta = -\rho(V-U)^2A(1+\cos\theta)$$

$$= -\frac{999\,kg}{m^3}\times(45-10)^2\frac{m^2}{s^2}\times\frac{\pi}{4}(0.050)^2m^2(1+\cos 40^\circ)\frac{N\cdot s^2}{kg\cdot m}$$

$$R_x = -4.24\ kN \text{ (force must act to right)} \qquad R_x$$

Apply conservation of mass to determine the jet sheet thickness:

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV}\rho\,d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$

Using the above assumptions, then

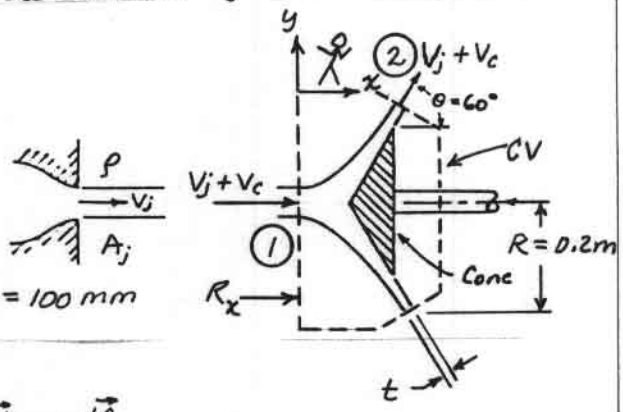
$$0 = -\rho V_1A_1 + \rho V_2A_2$$

$$V_1 = V-U\ ;\ V_2 = V-U\ ;\ A_1 = \frac{\pi d^2}{4}\ ;\ A_2 = 2\pi Rt$$

Therefore $A_1 = A_2 = \frac{\pi d^2}{4} = 2\pi Rt$, and $t = \frac{d^2}{8R}$

$$t = \frac{1}{8}\times(0.050)^2m^2\times\frac{1}{0.075\ m} = 4.17\times10^{-3}\ m \text{ or } 4.17\ mm \qquad t$$

Given: Water jet deflected by cone as shown. $V_j = 30$ m/sec, $V_c = 15$ m/sec



Find: (a) Thickness of jet sheet at $R = 200$ mm
(b) Force needed to move cone.

Solution: Apply continuity and x component of momentum.

Use moving coordinate system and control volume shown. $D_j = 100$ mm

Basic equations: $0 = \cancel{\frac{\partial}{\partial t}}^{=0(1)} \int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A}$

$$F_{Sx} + \cancel{F_{Bx}}^{=0(4)} = \cancel{\frac{\partial}{\partial t}}^{=0(1)} \int_{CV} u_{xyz} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) Steady flow (in moving reference frame)
(2) Uniform flow at each section
(3) $V_2 = V_1$ (no change in relative velocity)
(4) $F_{Bx} = 0$; Neglect F_{By}
(5) No pressure forces

Then

$$0 = \int_{A_1} \{-\rho V_1 dA_1\} + \int_{A_2} \{+\rho V_2 dA_2\} = -\rho (V_j + V_c) \frac{\pi D_j^2}{4} + \rho (V_j + V_c) 2\pi R t$$

and

$$t = \frac{D_j^2}{8R} = \frac{1}{8} \times (0.1)^2 m^2 \times \frac{1}{0.2 m} \times 1000 \frac{mm}{m} = 6.25 \text{ mm} \quad \longleftarrow t$$

Using relative velocities, momentum becomes

$$R_x = \int_{A_1} u_1 \{-\rho V_1 dA_1\} + \int_{A_2} u_2 \{+\rho V_2 dA_2\}$$

$$u_1 = V_j + V_c \qquad u_2 = (V_j + V_c)\cos\theta$$

$$V_1 = V_j + V_c \qquad V_2 = V_j + V_c$$

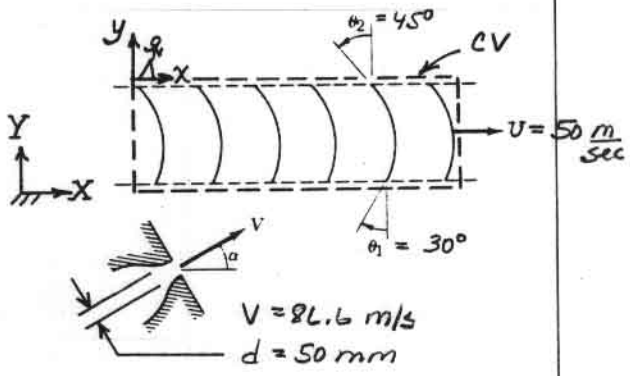
$$R_x = -\rho (V_j + V_c)^2 A_j + \rho (V_j + V_c)^2 A_j \cos\theta = (\cos\theta - 1) \rho (V_j + V_c)^2 A_j$$

$$= (\cos 60° - 1) \, 999 \frac{kg}{m^3} \times (30 + 15)^2 \frac{m^2}{s^2} \times \frac{\pi}{4} (0.1)^2 m^2 \times \frac{N \cdot s^2}{kg \cdot m}$$

$R_x = -7.94$ kN (force must be applied to left) $\longleftarrow R_x$

{Note: $R_y = Mg$ since there is no net momentum flux in the y-direction.}

Given: Series of vanes struck by continuous jet, as shown.



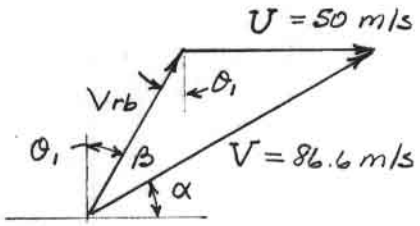
Find: (a) Nozzle angle, α.
(b) Force to hold vane speed constant.

Solution: Apply momentum equation using CV moving with vanes, as shown.

Basic equation: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$ (F_Bx = 0 (2); ∂/∂t = 0 (3))

Assumptions: (1) No pressure forces
(2) Horizontal; $F_{Bx} = 0$
(3) Steady flow w.r.t. CV
(4) Uniform flow at each section
(5) No change in relative velocity on vane
(6) Flow enters and leaves tangent to vanes

The nozzle angle may be obtained from trigonometry. The inlet velocity relationship is shown in the sketch:



From the law of sines,

$$\frac{\sin\alpha}{V_{rb}} = \frac{\sin(90+\theta_1)}{V} = \frac{\sin\beta}{U}$$

$$\beta = \sin^{-1}\left[\frac{U}{V}\sin(90+\theta_1)\right] = \sin^{-1}\left[\frac{50}{86.6}\sin(120°)\right] = 30°$$

From the sketch, $90° = \alpha + \beta + \theta_1$, so $\alpha = 90° - \beta - \theta_1 = 90° - 30° - 30° = 30°$ ← α

Also $V_{rb}\cos\theta_1 = V\sin\alpha$; $V_{rb} = V\frac{\sin\alpha}{\cos\theta_1} = 86.6\frac{m}{s}\times\frac{\sin 30°}{\cos 30°} = 50.0\ m/s$

From momentum equation (note all of $\dot{m}$ flows across vanes)

$$R_x = u_1\{-\dot{m}\} + u_2\{\dot{m}\} = V_{rb}\sin\theta(-\dot{m}) - V_{rb}\sin\theta_2(\dot{m}) = V_{rb}\dot{m}(-\sin\theta_1 - \sin\theta_2)$$

$u_1 = V_{rb}\sin\theta_1$ $u_2 = -V_{rb}\sin\theta_2$; $R_y = \dot{m}V_{rb}(-\cos\theta_1 + \cos\theta_2)$

Thus, since $\dot{m} = \rho Q$,

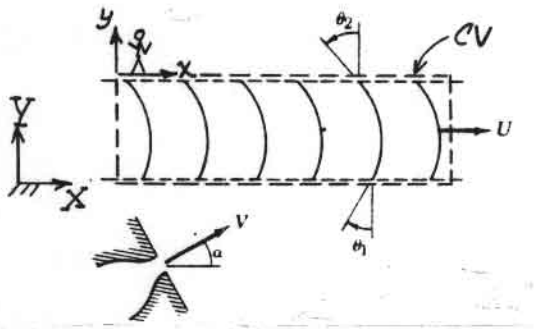
$$R_x = V_{rb}\rho Q(-\sin\theta_1 - \sin\theta_2)$$

$$= 50\frac{m}{s}\times 999\frac{kg}{m^3}\times 0.170\frac{m^3}{s}(-\sin 30° - \sin 45°)\times\frac{N\cdot s^2}{kg\cdot m}$$

$R_x = -10.3\ kN$ (to left) ← R_x

{Note: The net force on the CV in the y-direction is $R_y = -1.35\ kN$.}

Given: Series of vanes struck by continuous jet, as shown.



Find: For $\alpha \approx 0$ ($\theta_1 \approx 90°$), vane speed, U, to maximize power produced by vane.

Solution: Apply momentum equation using CV moving with vanes, as shown.

Basic equation: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$ (=0 (2), =0 (3))

Assumptions: (1) No pressure forces
(2) Horizontal; $F_{Bx} = 0$
(3) Steady flow w.r.t. CV
(4) Uniform flow at each section
(5) No change in relative velocity on vane
(6) Flow enters and leaves tangent to vanes

For $\alpha \approx 0$, $V_{rb} \approx V - U$; the momentum equation becomes

$$R_x = u_1\{-\dot{m}\} + u_2\{+\dot{m}\} = -\dot{m}(V-U) - \dot{m}(V-U)\sin\theta_2 = -\dot{m}(V-U)(1+\sin\theta_2)$$

$u_1 \approx V_{rb} \approx V - U$; $u_2 \approx -V_{rb}\sin\theta_2 \approx -(V-U)\sin\theta_2$

The vane system produces force, $K_x = -R_x$, and power $\mathcal{P} = K_x U$. Thus

$$\mathcal{P} = K_x U = -R_x U = \dot{m}(V-U)U(1+\sin\theta_2) \qquad (1)$$

To find maximum power, set $\frac{d\mathcal{P}}{dU} = 0$

$$\frac{d\mathcal{P}}{dU} = \dot{m}(-1)U(1+\sin\theta_2) + \dot{m}(V-U)(1)(1+\sin\theta_2) = \dot{m}(V-2U)(1+\sin\theta_2)$$

Thus power is maximized when $V - 2U = 0$, or $U = \frac{V}{2}$ (for $\mathcal{P}_{max}$) U

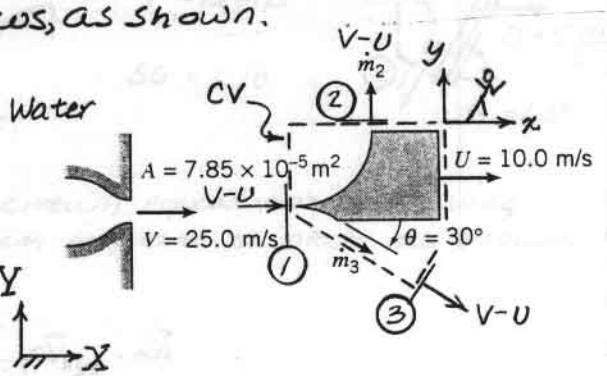
{Note from Eq. 1 that $\theta_2 \to 90°$ increases power also.}

{Note also that $K_y = -R_y = -\dot{m}V_{rb}\cos\theta_2$ but this force does not produce power.}

Given: Splitter dividing flow into two flat streams, as shown.

Find: (a) Mass flow rate ratio, $\dot{m}_2/\dot{m}_3$, so net vertical force is zero.
(b) Horizontal force need to maintain constant speed.

Solution: Apply x and y components of momentum to CV drawn with boundaries ⊥ to flows, as shown.



Basic equations:

$$F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\rho\, d\forall + \int_{CS} v_{xyz}\rho\vec{V}\cdot d\vec{A}$$

($F_{By}=0$ (1), $v_{xyz}=0$ (2), $v_{xyz}=0$ (3))

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall + \int_{CS} u_{xyz}\rho\vec{V}\cdot d\vec{A}$$

($F_{Bx}=0$, $u_{xyz}=0$ (3))

Assumptions: (1) No pressure forces
(2) Neglect mass of water on vane
(3) Steady flow w.r. to vane
(4) Uniform flow at each section
(5) No change in speed w.r. to vane

Then

$$0 = \int_{CS} v\rho\vec{V}\cdot d\vec{A} = v_1\{-\dot{m}_1\} + v_2\{+\dot{m}_2\} + v_3\{+\dot{m}_3\}$$

Measure w.r. to CV: $v_1 = 0$ $\quad v_2 = V-U$ $\quad v_3 = -(V-U)\sin\theta$

so $0 = (V-U)\dot{m}_2 - (V-U)\sin\theta\,\dot{m}_3$; $\dfrac{\dot{m}_2}{\dot{m}_3} = \sin\theta = \dfrac{1}{2}$ $\quad \dot{m}_2/\dot{m}_3$

and

$$F_{Sx} = \int_{CS} u\rho\vec{V}\cdot d\vec{A} = R_x = u_1\{-\dot{m}_1\} + u_2\{+\dot{m}_2\} + u_3\{+\dot{m}_3\}$$

Measure w.r. to CV: $u_1 = V-U$ $\quad u_2 = 0$ $\quad u_3 = (V-U)\cos\theta$

$$R_x = (V-U)(-\dot{m}_1) + (V-U)\cos\theta(\dot{m}_3) = (V-U)(\dot{m}_3\cos\theta - \dot{m}_1)$$

From continuity $0 = -\dot{m}_1 + \dot{m}_2 + \dot{m}_3 = -\dot{m}_1 + \dfrac{\dot{m}_3}{2} + \dot{m}_3$; $\dot{m}_3 = \dfrac{2}{3}\dot{m}_1$

$$R_x = (V-U)\left(\frac{2}{3}\dot{m}_1\cos\theta - \dot{m}_1\right) = (V-U)\dot{m}_1\left(\frac{2\cos\theta}{3} - 1\right)$$

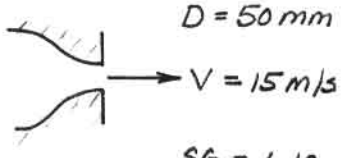
$$R_x = (25-10)\frac{m}{s} \times 999\frac{kg}{m^3} \times (25-10)\frac{m}{s} \times 7.85\times10^{-5}m^2\left(\frac{2}{3}\cos30^\circ - 1\right)\frac{N\cdot s^2}{kg\cdot m}$$

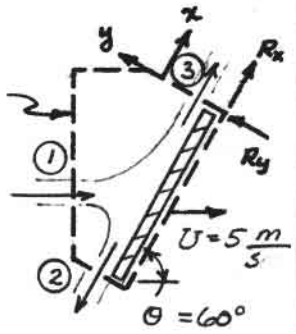
$$R_x = -7.46\ N\ \text{(to left)} \qquad R_x$$

{Force must be applied to left to maintain vane speed constant; if R_x were zero, vane would accelerate.}

Given: Jet of lye solution striking plate.

Friction force in plane of plate is negligible.





Find: Force from jet acting on plate.

Solution: Apply continuity and momentum equations to moving CV, using coordinate system in plane of plate as shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A}$ (=0(1), =0(4), =0(5), =0(1))

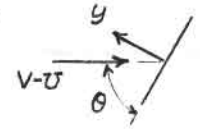
$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz} \cdot d\vec{A}$$ (=0(5), =0(1))

$$F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\, d\forall + \int_{CS} v_{xyz}\,\rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Uniform flow at each section
(3) Jet speed relative to plate remains constant
(4) $F_{Sx} = 0$
(5) Neglect body forces
(6) No net pressure forces

Subscript xyz is a reminder that all velocities must be evaluated relative to the control volume. Then $K_x = -R_x = 0$, and ← K_x

$$R_y = v_1\{-|\rho(V-U)A|\} + v_2\{|\dot{m}_2|\} + v_3\{|\dot{m}_3|\}$$



$$v_1 = -(V-U)\sin\theta \qquad v_2 = 0 \qquad v_3 = 0$$

and

$$R_y = \rho\,(V-U)^2 A \sin\theta \;;\; A = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.05)^2 m^2 = 1.96\times10^{-3} m^2$$

$$\rho = SG\,\rho_{H_2O}$$

$$R_y = (1.10)\,999\,\frac{kg}{m^3}\,(15-5)^2\frac{m^2}{s^2}\times 1.96\times10^{-3} m^2 \times \sin 60^\circ \times \frac{N\cdot s^2}{kg\cdot m} = 187\ N$$

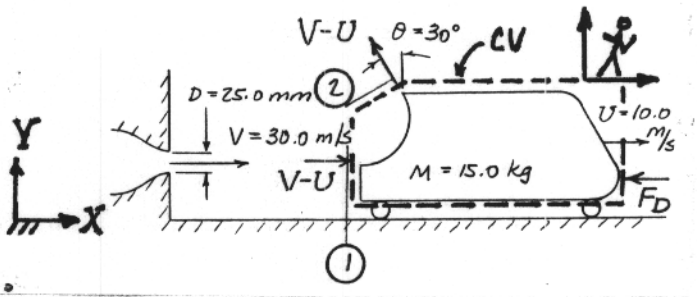
Therefore the force on the plate from all fluids is

$K_y = -187\ N$ (i.e. in the negative y direction) ← K_y

Given: Cart propelled by steady water jet, as shown.
Total resistance to motion is

$$F_D = kU^2$$

Where $k = 0.92 \frac{N \cdot s^2}{m^2}$



Find: Acceleration of cart at instant when $U = 10$ m/s.

Solution: Apply the momentum equation using CV and CS shown.

Basic equation: $F_{Sx} + F_{Bx} - \int_{CV} a_{rf_x} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho \, d\forall + \int_{CS} u_{xyz}\rho \vec{V}\cdot d\vec{A}$

($F_{Bx} = 0$ (2); $\frac{\partial}{\partial t} \approx 0$ (3))

Assumptions: (1) Only resistance is F_D; $F_{Sx} = -F_D = -kU^2$
(2) Horizontal; $F_{Bx} = 0$
(3) Neglect $\partial u/\partial t$ of mass of water in CV
(4) No change in speed w.r. to vane
(5) Uniform flow at each cross-section

Then

$$-kU^2 - a_{rf_x} M_{CV} = u_1\{-\rho(V-U)A\} + u_2\{+\rho(V-U)A\}$$

Measure u w.r. to CV: $u_1 = V-U$ $\quad u_2 = -(V-U)\sin\theta$

$$-kU^2 - a_{rf_x} M_{CV} = -\rho(V-U)^2 A - \rho(V-U)^2 A\sin\theta = -\rho(V-U)^2 A(1+\sin\theta)$$

so

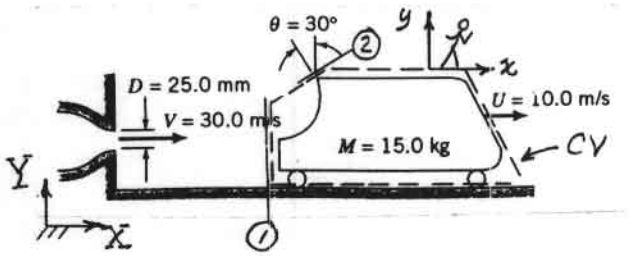
$$a_{rf_x} = \frac{1}{M}\left[\rho(V-U)^2 A(1+\sin\theta) - kU^2\right]$$

$$= \frac{1}{15\,kg}\left[999\frac{kg}{m^3}(30-10)^2\frac{m^2}{s^2}\frac{\pi}{4}(0.025)^2 m^2(1+\sin 30^\circ) - 0.92\frac{N\cdot s^2}{m^2}\times(10)^2\frac{m^2}{sec^2}\times\frac{kg\cdot m}{N\cdot s^2}\right]$$

$a_{rf_x} = 13.5$ m/s^2 (to right)

Given: Hydraulic catapult of Problem 4.118, rolling on level track with negligible resistance, speed U.

Find: Time required to accelerate from rest to $U = V/2$.



Solution: Apply x component of momentum equation to accelerating CV.

Basic equation:
$$F_{Sx}^{=0(1)} + F_{Bx}^{=0(2)} - \int_{CV} a_{rf_x}\rho\, d\forall \overset{\approx 0(3)}{=} \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall + \int_{CS} u_{xyz}\rho\, \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) $F_{Sx} = 0$, since no pressure forces, no resistance
(2) $F_{Bx} = 0$, since horizontal
(3) Neglect mass of water on vane
(4) Uniform flow in jet
(5) No change in relative velocity on vane

Then
$$-a_{rf_x} M_{CV} = u_1\{-\rho(V-U)A\} + u_2\{+\rho(V-U)A\} = -(1+\sin\theta)\rho(V-U)^2 A$$

$$u_1 = V-U \qquad u_2 = -(V-U)\sin\theta$$

so
$$\frac{dU}{dt} = \frac{\rho A(1+\sin\theta)}{M}(V-U)^2$$

To integrate, note since V = constant, $d(V-U) = -dU$, so
$$-\int_0^{V/2} \frac{d(V-U)}{(V-U)^2} = \int_0^t \frac{\rho A(1+\sin\theta)}{M}\, dt$$

or
$$\frac{1}{V-U}\Big]_{U=0}^{U=V/2} = \frac{2}{V} - \frac{1}{V} = \frac{1}{V} = \frac{\rho A(1+\sin\theta)}{M} t$$

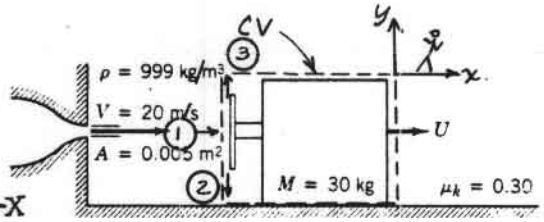
Thus
$$t = \frac{M}{\rho V A(1+\sin\theta)}$$

$$= 15.0\,kg \times \frac{m^3}{999\,kg} \times \frac{s}{30.0\,m} \times \frac{4}{\pi(0.025)^2 m^2} \times \frac{1}{(1+\sin 30°)}$$

$$t = 0.680\ s$$

t

Given: Vane/slider assembly moving under influence of jet.



Find: (a) Acceleration when $U = 10$ m/s.
(b) Terminal speed.

Solution: Apply x momentum equation to linearly accelerating CV.

Basic equation:

$$F_{Sx} + \cancel{F_{Bx}}^{=0(1)} - \int_{CV} a_{rf_x}\,\rho\,d\forall = \cancel{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall}^{\approx 0(2)} + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) Horizontal motion, so $F_{Bx} = 0$
(2) Neglect mass of liquid on vane, $u \approx 0$ on vane
(3) Uniform flow at each section
(4) Measure velocities relative to CV

Then

$$-Mg\mu_k - a_{rf_x} M = u_1\{-|\rho(V-U)A|\} + u_2\{+\dot{m}_2\} + u_3\{+\dot{m}_3\}$$

$$u_1 = V - U \qquad u_2 = 0 \qquad u_3 = 0$$

$$-Mg\mu_k - M\frac{dU}{dt} = -\rho(V-U)^2 A$$

or

$$\frac{dU}{dt} = \frac{\rho(V-U)^2 A}{M} - g\mu_k$$

$$= 999\ \frac{kg}{m^3}\,(20-10)^2\,\frac{m^2}{s^2} \times 0.005\ m^2 \times \frac{1}{30\ kg} - 9.81\,\frac{m}{s^2} \times 0.3$$

$$\frac{dU}{dt} = 13.7\ m/s^2 \quad (\text{at } U = 10\ m/s) \longleftarrow \frac{dU}{dt}$$

At terminal speed, $dU/dt = 0$ and $U = U_t$, so

$$0 = \frac{\rho(V-U_t)^2 A}{M} - g\mu_k \quad \text{or} \quad V - U_t = \sqrt{\frac{Mg\mu_k}{\rho A}}$$

and

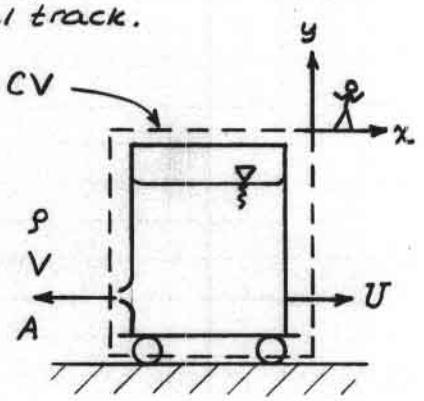
$$U_t = V - \sqrt{\frac{Mg\mu_k}{\rho A}}$$

$$= 20\,\frac{m}{s} - \left[30\ kg \times 9.81\,\frac{m}{s^2} \times 0.3 \times \frac{m^3}{999\ kg} \times \frac{1}{0.005\ m^2}\right]^{1/2}$$

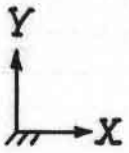
$$U_t = 15.8\ m/s \longleftarrow U_t$$

Given: Cart propelled by a horizontal liquid jet of constant speed. Neglect resistance along horizontal track.

Initial mass is M_0.



Find: A general expression for speed, U, as cart accelerates from rest.



Solution:

Apply x component of momentum equation using linearly accelerating CV shown.

Basic equation: $\overset{=0(1)}{F_{Sx}} + \overset{=0(2)}{F_{Bx}} - \int_{CV} a_{rfx}\rho d\forall = \overset{=0(3)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho d\forall} + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$

Assumptions: (1) No resistance
(2) $F_{Bx} = 0$ since track is horizontal
(3) Neglect u_{xyz} within CV
(4) Uniform flow at jet exit

Then

$$-a_{rfx} M = u\{|\rho VA|\} = -\rho V^2 A$$

$$u = -V$$

From continuity, $M = M_0 - \dot{m}t = M_0 - \rho VA\,t$. Using $a_{rfx} = \frac{dU}{dt}$,

$$\frac{dU}{dt} = \frac{\rho V^2 A}{M_0 - \rho VAt}$$

Separating variables and integrating,

$$\int_0^U dU = U = \int_0^t \frac{\rho V^2 A\, dt}{M_0 - \rho VAt} = -V\ln(M_0 - \rho VAt)\Big|_0^t = V\ln\left(\frac{M_0}{M_0 - \rho VAt}\right)$$

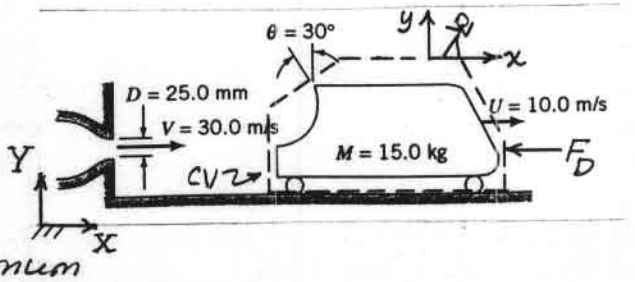
or

$$\frac{U}{V} = \ln\left(\frac{M_0}{M_0 - \rho VAt}\right)$$

$\frac{U}{V}$

Check dimensions: $[\rho VAt] = \frac{M}{L^3}\frac{L}{t}L^2 t = M$ ✓

Given: Hydraulic catapult of Problem 4.118, rolling on level track with resistance, $F_D = kU^2$, speed U, starting from rest at $t = 0$.



Find: (a) When acceleration is maximum
(b) Sketch of acceleration vs. time
(c) Value of θ to maximize acceleration, why?
(d) If U will ever reach V; explanation

Solution: Apply x component of momentum equation to accelerating CV.

Basic equation: $F_{Sx} + F_{Bx} - \int_{CV} a_{rf_x} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, d\forall + \int_{CS} u_{xyz} \rho \, \vec{V}_{xyz} \cdot d\vec{A}$

(F_{Sx}: $=0$ (1); F_{Bx}: $=0$ (2); a_{rf_x}: ≈ 0 (3))

Assumptions: (1) $F_{Sx} = -F_D = -kU^2$, where $k = 0.92\ N \cdot s^2/m^2$
(2) $F_{Bx} = 0$, since horizontal
(3) Neglect mass of water on vane
(4) Uniform flow in jet
(5) No change in relative velocity on vane

Then

$$-kU^2 - a_{rf_x} M_{CV} = u_1\{-\rho(V-U)A\} + u_2\{+\rho(V-U)A\} = -(1+\sin\theta)\rho(V-U)^2 A$$

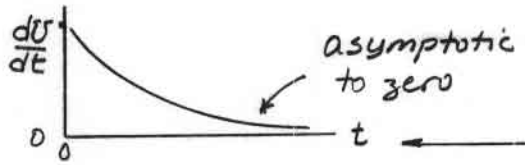
$$u_1 = V - U \qquad u_2 = -(V-U)\sin\theta$$

so

$$\frac{dU}{dt} = \frac{\rho A (1+\sin\theta)}{M}(V-U)^2 - kU^2/M \qquad (1)$$

(a) Acceleration is maximum at $t = 0$, when $U = 0$

(b) Acceleration vs. time will be



(c) From Eq. 1, dU/dt is maximum when $\theta = \pi/2$ and $\sin\theta = 1$

(d) From Eq. 1, $\frac{dU}{dt}$ will go to zero when $U < V$; this will be the terminal speed for the cart, U_t. From Eq. 1, $\frac{dU}{dt} = 0$ when

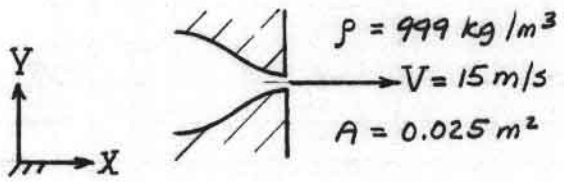
$$\rho A(1+\sin\theta)(V-U)^2 = kU^2$$

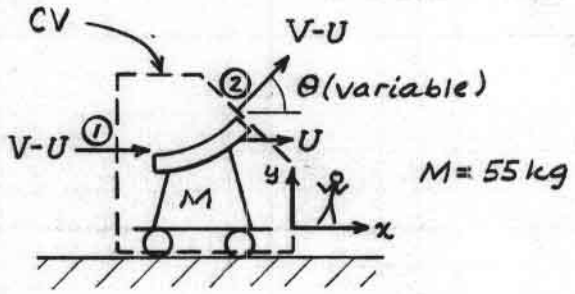
or

$$U = \frac{\left[\frac{\rho A(1+\sin\theta)}{k}\right]^{1/2}}{1+\left[\frac{\rho A(1+\sin\theta)}{k}\right]^{1/2}} V = 0.472V$$

U will be asymptotic to V.

Given: Vane/cart assembly driven by liquid jet. Motion to be controlled so that $a_{rf_x} = 1.5\ m/s^2$ by varying turning angle, θ. Neglect resistance.





Find: θ at $t = 5\ s$. Plot: $\theta(t)$ over a suitable range.

Solution: Apply x component of momentum equation, using linearly accelerating CV shown above.

Basic equation: $F_{Sx}^{=0(1)} + F_{Bx}^{=0(2)} - \int_{CV} a_{rf_x}\,\rho\,d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}} \int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$

Assumptions: (1) $F_{Sx} = 0$
(2) $F_{Bx} = 0$
(3) Neglect u and rate of change of u within CV
(4) Uniform flow at each section
(5) Jet area and speed relative to vane are constant

Then

$$-M a_{rf_x} = u_1\{-|\rho(V-U)A|\} + u_2\{|\rho(V-U)A|\}$$

$$u_1 = V-U \qquad u_2 = (V-U)\cos\theta$$

$$-M a_{rf_x} = -\rho(V-U)^2 A + \rho(V-U)^2 A\cos\theta = \rho(V-U)^2 A(\cos\theta - 1)$$

or

$$\cos\theta = 1 - \frac{M a_{rf_x}}{\rho(V-U)^2 A}$$

Since a_{rf_x} = constant, $U = a_{rf_x} t$

$$\cos\theta = 1 - \frac{M a_{rf_x}}{\rho(V - a_{rf_x} t)^2 A} \qquad (1)$$

and

$$\theta = \cos^{-1}\left[1 - \frac{M a_{rf_x}}{\rho(V - a_{rf_x} t)^2 A}\right]$$

$$= \cos^{-1}\left\{1 - 55\ kg \times 1.5\frac{m}{s} \times \frac{m^3}{999\ kg}\left[\frac{1}{15\frac{m}{s} - 1.5\frac{m}{s^2} \times 5s}\right]^2 \frac{1}{0.025\ m^2}\right\}$$

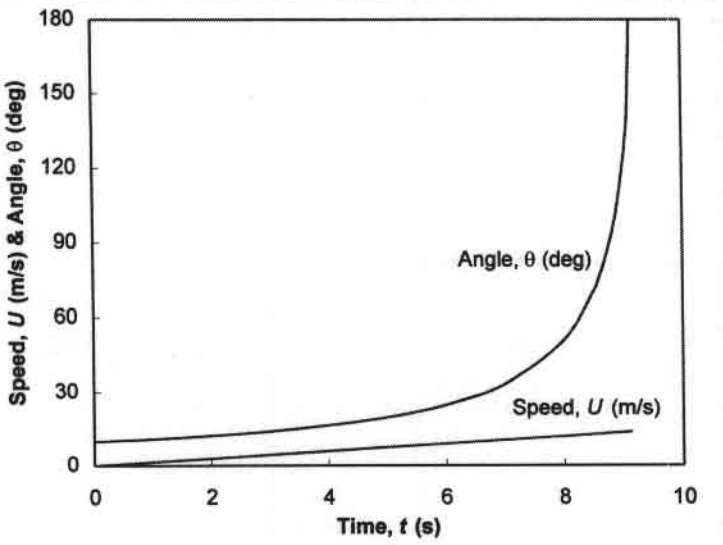
$$\theta = 19.7^\circ \quad (\text{at } t = 5s)$$

θ

Equation 1 is only valid for $\theta \le 180^\circ$. This occurs at $t \approx 9.14\ s$.

The plot is on the next page.

The plot is



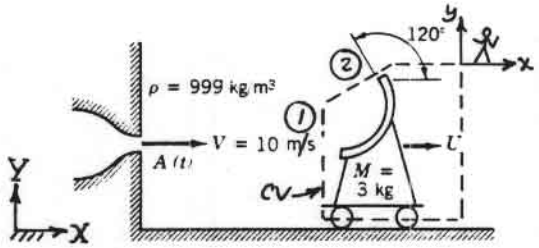
(Constant acceleration cannot be maintained after $t = 9.14$ s.)

13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Vaned cart rolling with negligible resistance.

$a_{rt_x} = 2\ m/s^2$ = constant

Jet area is A(t), programmed.



Find: (a) Expression for A(t) at cart.
(b) Sketch for $t \le 4$ s.
(c) Evaluate at $t = 2$ s.

Solution: Apply x momentum to CV with linear acceleration.

Basic equation: $F_{Bx} + F_{Bx} - \int_{CV} a_{rt_x}\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$

(F_{Bx}: =0 (1); F_{Bx}: =0 (2); $\frac{\partial}{\partial t}\int_{CV}$: ≈0 (3))

Assumptions: (1) No resistance to motion
(2) Horizontal motion, so $F_{Bx} = 0$
(3) Neglect mass of liquid in CV
(4) Uniform flow at each section
(5) All velocities measured relative to CV
(6) No change in stream area or speed on vane

Then (with $a_{rt_x} = a$)

$$-aM = u_1\{-|\rho(V-U)A|\} + u_2\{+|\rho(V-U)A|\} = -\frac{3}{2}\rho(V-U)^2 A$$

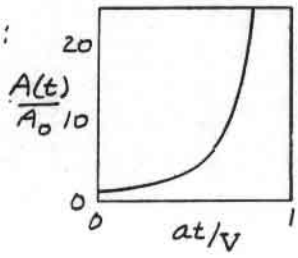
$u_1 = V - U$ $\qquad$ $u_2 = (V-U)\cos 120^\circ = -\frac{1}{2}(V-U)$

Since a = constant, $U = at$, and

$$A = A(t) = \frac{2aM}{3\rho(V-at)^2}$$ A(t)

At $t = 0$, $A(0) = A_0 = \frac{2aM}{3\rho V^2}$. Thus $\frac{A}{A_0} = \frac{1}{(1-at/V)^2}$.

Sketch:



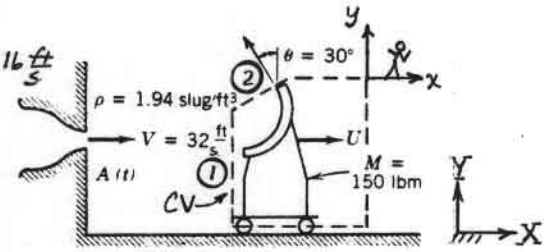
At $t = 2$ sec,

$$A = \frac{2}{3}\times\frac{2\,m}{s^2}\times 3\ kg\times\frac{m^3}{999\ kg}\frac{1}{\left[10\frac{m}{s} - \frac{2\,m}{s^2}\times 2s\right]^2}\times 10^6\frac{mm^2}{m^2} = 111\ mm^2$$ A(2)

Given: Vaned cart moving with negligible resistance.

$a_{rf_x} = 4\ ft/s^2 = $ constant; $U \le 16\ \frac{ft}{s}$

Jet area, A(t), programmed.



Find: (a) Expression for A(t) at cart.
(b) Evaluate at t = 0.
(c) Jet cutoff time, if $U_{max} = 16\ ft/s$.

Solution: Apply x momentum to CV with linear acceleration.

Basic equation: $F_{Sx} + F_{Bx} - \int_{CV} a_{rf_x}\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall + \int_{CS} u_{xyz}\rho\, \vec{V}_{xyz}\cdot d\vec{A}$

($F_{Sx} = 0$ (1), $F_{Bx} = 0$ (2), $\int_{CV} a_{rf_x}\rho\, d\forall \approx 0$ (3))

Assumptions: (1) No resistance to motion
(2) Horizontal motion, so $F_{Bx} = 0$
(3) Neglect mass of liquid in CV
(4) Uniform flow at each section
(5) All velocities measured relative to CV
(6) No change in stream area or speed on vane

Then (with $a_{rf_x} = a$),

$$-aM = u_1\{-|\rho(V-U)A\} + u_2\{+|\rho(V-U)A|\} = -\frac{3}{2}\rho(V-U)^2 A$$

$u_1 = V - U$ $\qquad u_2 = -(V-U)\sin 30° = -\frac{1}{2}(V-U)$

Since a = constant, U = at, and

$$A\,(\text{at cart}) = A(t) = \frac{2aM}{3\rho(V-at)^2} \qquad A(t)$$

At $t = 0$, $A(0) = \frac{2aM}{3\rho V^2}$

$$A(0) = \frac{2}{3}\times 4\frac{ft}{s^2}\times 150\ lbm\times\frac{ft^3}{1.94\ slug}\times\frac{s^2}{(32)^2\ ft^2}\times\frac{slug}{32.2\ lbm}\times 144\frac{in.^2}{ft^2} = 0.900\ in.^2 \qquad A(0)$$

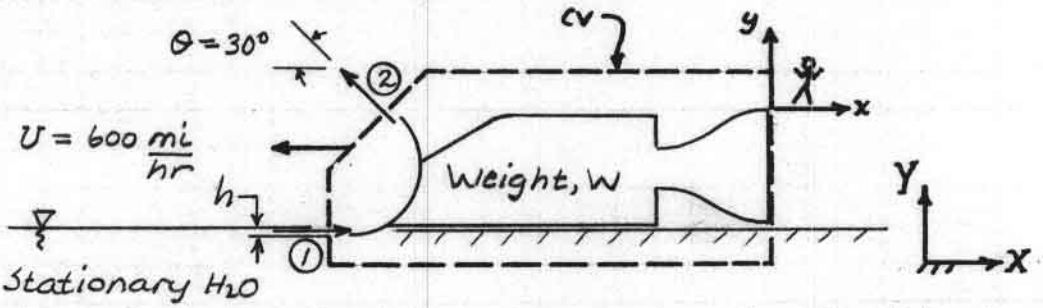
The cart reaches $U = 16\ ft/s$ at $t = U/a = 16\frac{ft}{s}\times\frac{s^2}{4ft} = 4\ s$. At that time,

$$X = U_0 t + \tfrac{1}{2}at^2 = \tfrac{1}{2}\times 4\frac{ft}{s^2}\times(4)^2 s^2 = 32\ ft$$

The jet flow must be cut off earlier, at

$$t_{co} = t - \frac{X}{V} = 4\ s - 32\ ft\times\frac{s}{32\ ft} = 3\ s \qquad t_{co}$$

Given: Rocket sled with water scoop brake. $W = 10{,}000$ lbf



Scoop immersed in trough is $w = 6$ in. wide, $h = 3$ in. deep.

Find: Time needed to decelerate to 20 mph. Plot: Speed vs. time.

Solution: Apply x component of momentum equation to linearly accelerating CV. Basic equation is

$$F_{Sx}^{=0(1)} + F_{Bx}^{=0(2)} - \int_{CV} a_{rfx}\,\rho\,d\forall = \frac{\partial}{\partial t}^{\approx 0(3)} \int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) $F_{Sx} = 0$
(2) $F_{Bx} = 0$
(3) Neglect u_{xyz} and its rate of change in CV
(4) Uniform flow at each section
(5) Speed of water relative to sled is constant

Then

$$-a_{rfx} M = u_1\{-|\rho V w h|\} + u_2\{|\rho V w h|\}\ ;\ u_1 = V,\ u_2 = -V\cos\theta$$

$$-a_{rfx}\frac{W}{g} = -\rho V^2 w h (1+\cos\theta),\ \text{or}\ a_{rfx} = \frac{\rho g V^2 w h (1+\cos\theta)}{W}$$

Now $a_{rfx} = -\dfrac{dV}{dt}$, because of coordinate choice. Thus

$$\frac{dV}{V^2} = -\frac{\gamma w h}{W}(1+\cos\theta)\,dt$$

and

$$\int_{V_i}^{V}\frac{dV}{V^2} = -\frac{1}{V} + \frac{1}{V_i} = -\frac{\gamma w h}{W}(1+\cos\theta)\,t \qquad (1)$$

Solving for t,

$$t = \left[\frac{1}{V} - \frac{1}{V_i}\right]\frac{W}{\gamma w h (1+\cos\theta)}$$

$$= \left[\frac{1}{20} - \frac{1}{600}\right]\frac{hr}{mi}\times\frac{mi}{5280\,ft}\times 3600\frac{s}{hr}\times\frac{ft^3}{62.4\,lbf}\times\frac{1}{6\,in.}\times\frac{1}{3\,in.}\times\frac{144\,in^2}{ft^2}\times\frac{10{,}000\,lbf}{1+\cos 30^\circ}$$

$$t = 22.6\ s$$

t

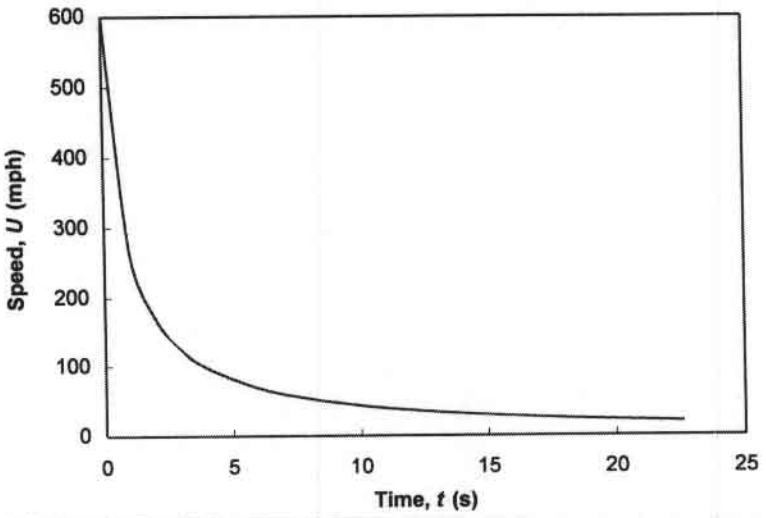
The plot is presented on the next page.

Solving Eq. 1 for U,

$$\frac{1}{U} = \frac{1}{U_i} + \frac{\gamma_w h}{W}(1+\cos\theta)t = \frac{W + \gamma_w h U_i (1+\cos\theta)t}{W U_i}$$

or $$U = \frac{W U_i}{W + \gamma_w h U_i (1+\cos\theta)t} \qquad (2)$$

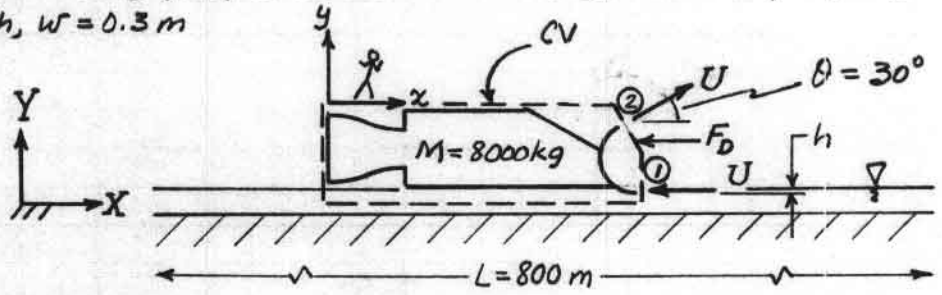
Plotting,



13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U. S. A.
National® Brand

Given: Rocket sled slowed by scoop in water trough.

Aerodynamic drag proportional to U^2. At $U_0 = 300$ m/s, $F_D = 90$ kN.
Scoop width, $w = 0.3$ m



Find: Depth of scoop immersion to slow to 100 m/s in trough length, L.

Solution: Apply x component of momentum equation using linearly accelerating CV shown.

Basic equation: $F_{Sx} + F_{Bx} - \int_{CV} a_{rfx}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$

($F_{Bx} = 0$ (1); $\frac{\partial}{\partial t}\int_{CV} \approx 0$ (2))

Assumptions: (1) $F_{Bx} = 0$
(2) Neglect rate of change of u in CV
(3) Uniform flow at each section
(4) No change in relative speed of liquid crossing scoop

Then

$$-F_D - M a_{rfx} = u_1\{-|\rho U w h|\} + u_2\{|\rho U w h|\}\ ;\ h = \text{scoop immersion}$$

$$u_1 = -U \qquad u_2 = U\cos\theta$$

But $F_D = kU^2$; $k = \frac{F_{D_0}}{U_0^2} = 90\,kN \times \frac{s^2}{(300)^2 m^2} \times \frac{10^3 N}{kN} \times \frac{kg\cdot m}{N\cdot s^2} = 1.00\,kg/m$

$$-kU^2 - M\frac{dU}{dt} = \rho U^2 w h\,(1+\cos\theta)\ ,\ \text{since } a_{rfx} = dU/dt,$$

Thus $-M\frac{dU}{dt} = [k + \rho w h\,(1+\cos\theta)]\,U^2 = -MU\frac{dU}{dX}$

or $\frac{dU}{U} = -C\,dX$, where $C = \frac{k + \rho w h\,(1+\cos\theta)}{M}$

Integrating, $\ln\frac{U}{U_0} = -CX$, so $C = -\frac{1}{X}\ln\frac{U}{U_0}$

$$C = -\frac{1}{800\,m}\ln\left(\frac{100}{300}\right) = 1.37\times10^{-3}\,m^{-1}$$

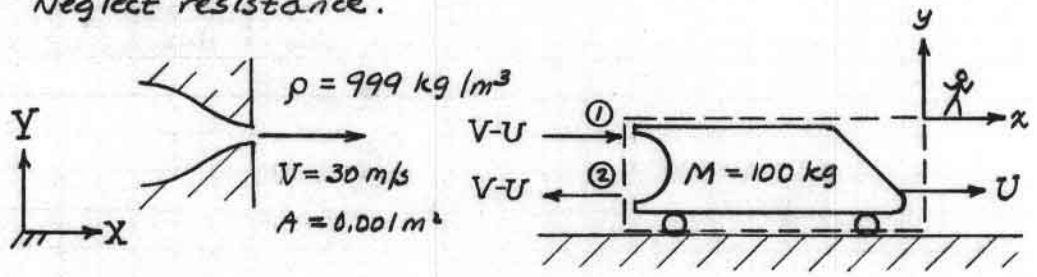
Solving for h, $h = \frac{MC - k}{\rho w(1+\cos\theta)}$

$$h = \left[8000\,kg \times \frac{1.37\times10^{-3}}{m} - 1.00\,\frac{kg}{m}\right]\frac{m^3}{999\,kg} \times \frac{1}{0.3\,m} \times \frac{1}{(1+\cos 30°)} = 0.0179\,m$$

$h = 17.9$ mm

h

Given: Vehicle accelerated from rest by a hydraulic catapult. Neglect resistance.



Find: Vehicle speed at $t = 5$ sec. Plot: Vehicle speed vs. time.

Solution: Apply x component of momentum equation using the linearly accelerating CV shown above.

Basic equation: $F_{Sx} + F_{Bx} - \int_{CV} a_{rfx}\,\rho\,d\forall = \frac{d}{dt}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$

(F_{Sx} = 0 (1), F_{Bx} = 0 (2), a_{rfx} ≈ 0 (3))

Assumptions: (1) $F_{Sx} = 0$
(2) $F_{Bx} = 0$
(3) Neglect mass of liquid and rate of change of u in CV
(4) Uniform flow at each section
(5) Jet area and speed with respect to vehicle are constant

Then

$$-Ma_{rfx} = -M\frac{dU}{dt} = u_1\{-|\rho(V-U)A|\} + u_2\{|\rho(V-U)A|\}$$

$$u_1 = V-U \qquad u_2 = -(V-U)$$

or

$$\frac{dU}{dt} = \frac{2\rho(V-U)^2A}{M}$$

Note that $dU = -d(V-U)$, and separate variables to obtain

$$-\frac{d(V-U)}{(V-U)^2} = \frac{2\rho A}{M}\,dt$$

Integrate from $U=0$ at $t=0$ to U at t,

$$\int_{V-U=V}^{V-U} -\frac{d(V-U)}{(V-U)^2} = \frac{1}{V-U}\Big]_V^{V-U} = \frac{1}{V-U} - \frac{1}{V} = \frac{V-(V-U)}{V(V-U)} = \frac{U}{V(V-U)} = \frac{2\rho A}{M}t$$

Solving,

$$U = (V-U)\frac{2\rho VA}{M}t \quad \text{or} \quad U = V\left[\frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t}\right] \qquad (1)$$

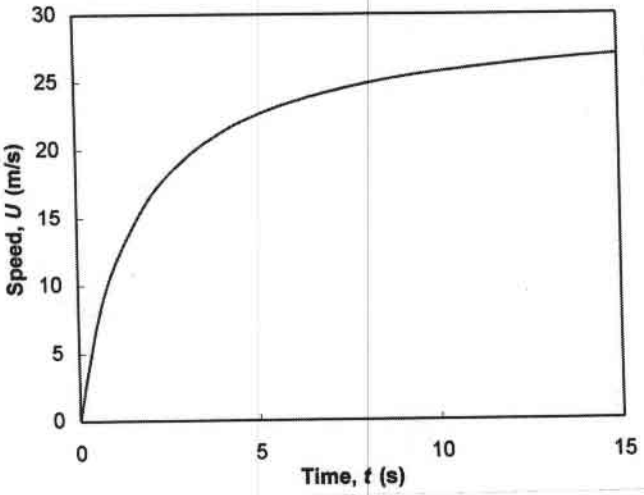
For the given conditions at $t = 5$ s,

$$\frac{2\rho VA}{M}t = 2 \times 999\,\frac{kg}{m^3} \times 30\,\frac{m}{s} \times 0.001\,m^2 \times 5s \times \frac{1}{100\,kg} = 3.00$$

$$U = 30\,\frac{m}{s}\left[\frac{3.00}{1+3.00}\right] = 22.5\ m/s \qquad U \longleftarrow$$

The plot is on the next page.

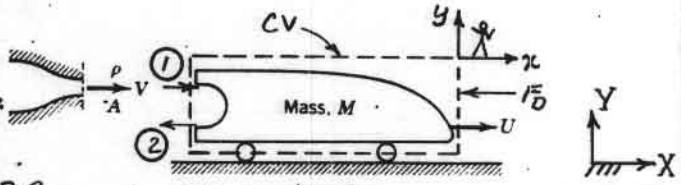
The speed vs. time plot is



13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-383 200 SHEETS EYE-EASE® 5 SQUARE
42-389 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Cart accelerated from rest by hydraulic catapult.

$F_D = kU^2$; $k = 2.0\ N\cdot s^2/m^2$



$\rho = 999\ kg/m^3$

$V = 30\ m/s$ $\quad M = 100\ kg$

$A = 0.001\ m^2$

Find: (a) Expression for acceleration in terms of speed, U.
(b) Evaluate at $U = 10\ m/s$.
(c) Fraction of U_t.

Solution: Apply x momentum for CV with linear acceleration.

Basic equation:

$$F_{Sx} + F_{Bx} - \int_{CV} a_{rf_x}\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall + \int_{CS} u_{xyz}\rho\, \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) Horizontal, $F_{Bx} = 0$
(2) Neglect mass of liquid in CV (components of u cancel)
(3) Uniform flow at each section
(4) Measure all velocities relative to the CV
(5) No change in stream area or speed on vane

Then

$$-kU^2 - a_{rf_x} M = u_1\{-|\rho(V-U)A|\} + u_2\{+|\rho(V-U)A|\} = -2\rho(V-U)^2A$$

$$u_1 = V-U \qquad u_2 = -(V-U)$$

or

$$a_{rf_x} = \frac{dU}{dt} = \frac{2\rho(V-U)^2A - kU^2}{M} \qquad a(U)$$

At $U = 10\ m/sec$

$$a_{rf_x} = \frac{2\times 999\ \frac{kg}{m^3}(30-10)^2\frac{m^2}{s^2}\times 0.001\ m^2 - 2.0\ \frac{N\cdot s^2}{m^2}\times(10)^2\frac{m^2}{s^2}\times\frac{kg\cdot m}{N\cdot s^2}}{100\ kg} = 5.99\ \frac{m}{s^2} \qquad a_{rf_x}$$

At terminal speed, $a_{rf_x} = 0$. Then $2\rho(V-U_t)^2A = kU_t^2$, or

$$V - U_t = U_t\sqrt{\frac{k}{2\rho A}}$$

Solving, $U_t = \dfrac{V}{1+\sqrt{k/2\rho A}}$

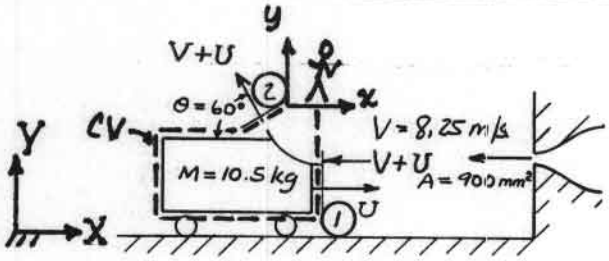
$$U_t = 30\ \frac{m}{s}\times\frac{1}{1+\left[\frac{1}{2}\times 2.0\ \frac{N\cdot s^2}{m^2}\times\frac{m^3}{999\ kg}\times\frac{1}{0.001\ m^2}\times\frac{kg\cdot m}{N\cdot s^2}\right]^{1/2}} = 15.0\ m/s$$

Finally,

$$\text{Fraction} = \frac{U}{U_t} = 10.0\ \frac{m}{s}\times\frac{s}{15.0\ m} = 0.667 \qquad \text{Fraction}$$

Given: Small vaned cart rolling on level track, struck by a water jet, as shown. At $t=0$, $U_0 = 12.5$ m/sec. Neglect air resistance and rolling resistance.

Find: (a) Time and (b) distance needed to bring cart to rest, and (c) Plot of $U(t)$, $x(t)$.



Solution: Apply x component of momentum using CS and CV shown.

Basic equation: $F_{Sx}^{=0(1)} + F_{Bx}^{=0(2)} - \int_{CV} a_{rf_x}\rho\,d\forall = \frac{\partial}{\partial t}^{\approx 0(3)} \int_{CV} u_{xyz}\rho\,d\forall + \int_{CS} u_{xyz}\rho\vec{V}\cdot d\vec{A}$

Assumptions: (1) No resistance; $F_{Sx}=0$
(2) Horizontal; $F_{Bx}=0$
(3) Neglect mass of water on vane; $\partial/\partial t \approx 0$
(4) No change in speed w.r.to vane
(5) Uniform flow at each cross-section

Then

$$-a_{rf_x} M_{CV} = u_1\{-|\rho(V+U)A|\} + u_2\{+|\rho(V+U)A|\}$$

$$a_{rf_x} = \frac{dU}{dt} \qquad u_1 = -(V+U) \qquad u_2 = -(V+U)\cos\theta \quad \text{(w.r.to CV)}$$

So $$-\frac{dU}{dt}M = \rho(V+U)^2A - \rho(V+U)^2A\cos\theta = \rho(V+U)^2A(1-\cos\theta) \qquad (1)$$

Note V = constant, so $dU = d(V+U)$. Substituting

$$-\frac{d(V+U)}{(V+U)^2} = \frac{\rho A(1-\cos\theta)}{M}dt \qquad (2)$$

Integrate from U_0 at $t=0$ to stop, when $U=0$

$$\left.\frac{1}{V+U}\right]_{U=U_0}^{U=0} = \frac{1}{V} - \frac{1}{V+U_0} = \frac{V+U_0-V}{V(V+U_0)} = \frac{U_0}{V(V+U_0)} = \frac{\rho A(1-\cos\theta)t}{M}$$

Thus $$t = \frac{U_0 M}{\rho(V+U_0)VA(1-\cos\theta)}$$

$$= 12.5\frac{m}{sec} \times 10.5\,kg \times \frac{m^3}{999\,kg} \times \frac{sec}{(12.5+8.25)m} \times \frac{sec}{8.25\,m} \times \frac{1}{900\times10^{-6}m^2} \times \frac{1}{(1-\cos 60°)}$$

$$t = 1.71 \text{ sec (to stop)} \longleftarrow \qquad t$$

To find distance note $\frac{dU}{dt} = \frac{dU}{ds}\frac{ds}{dt} = \frac{dU}{ds}U = U\frac{dU}{ds}$, so from Eq. 1

$$-U\frac{dU}{ds}M = \rho(V+U)^2A(1-\cos\theta)$$

Separating variables $$\frac{U\,dU}{(V+U)^2} = -\frac{\rho A(1-\cos\theta)}{M}ds \qquad (3)$$

Equation 3 may be integrated. Using tables, and integrating from U_0 at $t=0$ to stop (when $U=0$),

$$\int_{U_0}^{0} \frac{U\,dU}{(V+U)^2} = \left[\ln(V+U) + \frac{V}{V+U}\right]_{U_0}^{0} = \ln\left(\frac{V}{V+U_0}\right) + \frac{V}{V} - \frac{V}{V+U_0} = -\frac{\rho A(1-\cos\theta)}{M}s$$

Simplifying and solving for s,

$$s = -\frac{M}{\rho A(1-\cos\theta)}\left[\ln\left(\frac{V}{V+U_0}\right) + 1 - \frac{V}{V+U_0}\right]$$

$$= -10.5\,kg \times \frac{m^3}{999\,kg} \times \frac{1}{900\times10^{-6}\,m^2} \times \frac{1}{(1-\cos 60^\circ)}\left[\ln\left(\frac{8.25}{8.25+12.5}\right) + 1 - \frac{8.25}{8.25+12.5}\right]$$

$s = 7.47\ m$ (to stop) ← s

From Eq. 2 the general solution is

$$\int_{U_0}^{U} -\frac{d(V+U)}{(V+U)^2} = \frac{1}{V+U}\Big]_{U_0}^{U} = \frac{1}{V+U} - \frac{1}{V+U_0} = \frac{(V+U_0)-(V+U)}{(V+U)(V+U_0)} = \frac{\rho A(1-\cos\theta)t}{M} = at$$

Thus $U_0 - U = a(V+U)(V+U_0)t = aV(V+U_0)t + aU(V+U_0)t$ {Let $b = V+U_0$}

Simplifying, $U = \dfrac{U_0 - aVbt}{1+abt}$ (4) ← $U(t)$

Acceleration is found from Eq. 1

$$a_x = \frac{dU}{dt} = \frac{\rho A(1-\cos\theta)(V+U)^2}{M} = a(V+U)^2$$ ← $a_x(U)$

Integrate Eq. 4 to get $X(t)$:

$$U = \frac{dX}{dt} = \frac{U_0 - abVt}{1+abt}$$

$$dX = \frac{U_0}{1+abt}dt - \frac{abVt}{1+abt}dt$$

Integrating

$$X = \frac{U_0}{ab}\ln(1+abt)\Big]_0^t - \frac{V}{ab}\int_0^t \frac{x}{1+x}dx = \left[\frac{U_0}{ab}\ln(1+abt) - \frac{V}{ab}(1+abt-\ln(1+abt))\right]_0^t$$

$$X = \frac{U_0}{ab}\ln(1+abt) - \frac{V}{ab}\left[abt - \ln(1+abt)\right]$$ ← $X(t)$

Numerical values and plots are on the next page.

National® Brand 42-381 50 SHEETS EYE-EASE® 5 SQUARE 42-382 100 SHEETS EYE-EASE® 5 SQUARE 42-389 200 SHEETS EYE-EASE® 5 SQUARE 42-392 RECYCLED WHITE 5 SQUARE Made in U. S. A.

Acceleration, Velocity, and Position of Cart vs. Time:

Input Parameters:

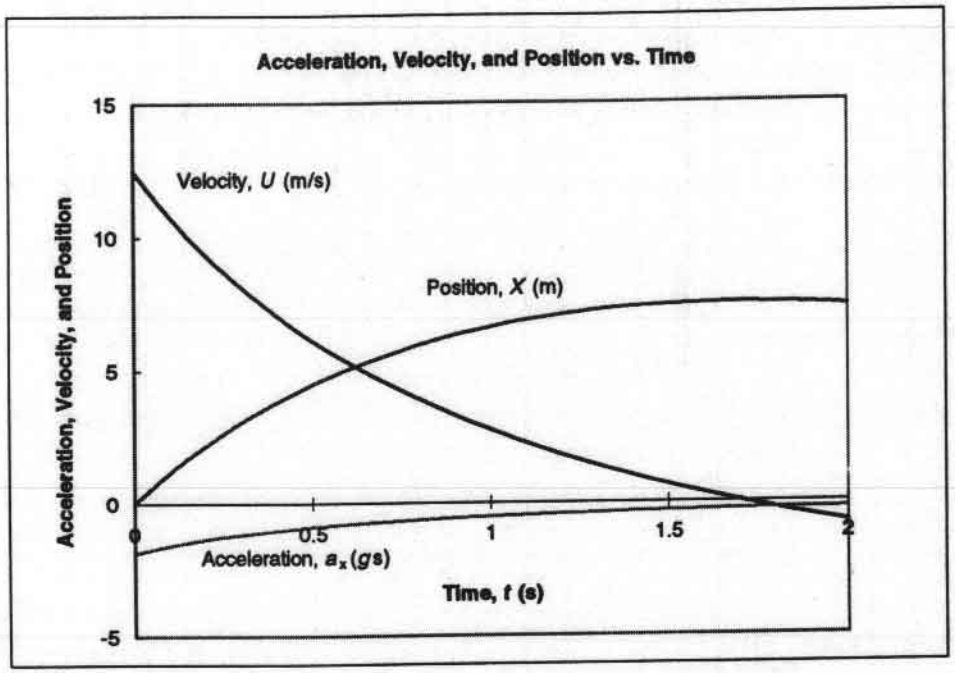
A =	900	mm^2	9.00E-04	m^2
M =	10.5	kg		
U_0 =	12.5	m/s		
V =	8.25	m/s		
θ =	60	degrees	1.047	rad
ρ =	999	kg/m^3		

Calculated Parameters:

a =	0.0428	m^{-1}
b =	20.75	m/s

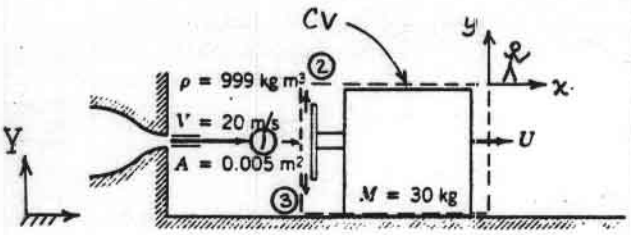
Calculated Results:

Time, t (s)	Velocity, U (m/s)	Accel., a_x (m/s)	Accel., a_x (gs)	Position, X (m)
0	12.5	-18.4	-1.88	0.00
0.1	10.8	-15.5	-1.58	1.16
0.2	9.37	-13.3	-1.35	2.17
0.3	8.13	-11.5	-1.17	3.04
0.4	7.06	-10.0	-1.02	3.80
0.5	6.12	-8.84	-0.901	4.46
0.6	5.29	-7.84	-0.800	5.03
0.7	4.54	-7.01	-0.714	5.52
0.8	3.88	-6.30	-0.642	5.94
0.9	3.28	-5.69	-0.580	6.30
1.0	2.74	-5.17	-0.527	6.60
1.1	2.24	-4.72	-0.481	6.85
1.2	1.79	-4.32	-0.440	7.05
1.3	1.38	-3.97	-0.405	7.21
1.4	0.998	-3.66	-0.373	7.33
1.5	0.646	-3.39	-0.345	7.41
1.6	0.319	-3.14	-0.320	7.46
1.7	0.0160	-2.93	-0.298	7.47
1.705	0.00000	-2.91	-0.297	7.47
1.8	-0.267	-2.73	-0.278	7.46
1.9	-0.530	-2.55	-0.260	7.42
2.0	-0.777	-2.39	-0.244	7.35



Given: Vane/slider assembly moving under influence of jet.

$F_R = kU$; $k = 7.5\ N\cdot s/m$



Find: (a) Acceleration at instant when $U = 10$ m/s.
(b) Terminal speed of slider.

Solution: Apply x momentum equation to linearly accelerating CV.

Basic equation: $F_{Sx} + F_{Bx} - \int_{CV} a_{rf_x}\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\,d\forall + \int_{CS} u_{xyz}\rho\,\vec{V}_{xyz}\cdot d\vec{A}$ ($F_{Sx}=0(1)$, $F_{Bx}\approx 0(2)$)

Assumptions: (1) Horizontal, so $F_{Bx}=0$
(2) Neglect mass of liquid on vane, $u\approx 0$ on vane
(3) Uniform flow at each section
(4) Measure velocities relative to CV

Then

$$-kU - a_{rf_x}M = u_1\{-|\rho(V-U)A|\} + u_2\{+\dot{m}_2\} + u_3\{+\dot{m}_3\}$$

$u_1 = V-U \qquad u_2 = 0 \qquad u_3 = 0$

$$-kU - M\frac{dU}{dt} = -\rho(V-U)^2A$$

or

$$\frac{dU}{dt} = \frac{\rho(V-U)^2A}{M} - \frac{kU}{M}$$

$$= \frac{999\ kg}{m^3}(20-10)^2\frac{m^2}{s^2}\times 0.005\ m^2\times\frac{1}{30\ kg} - 7.5\frac{N\cdot s}{m}\times 10\frac{m}{s}\times\frac{1}{30\ kg}\times\frac{kg\cdot m}{N\cdot s^2}$$

$$\frac{dU}{dt} = 14.2\ m/s^2 \quad (\text{at } U = 10\ m/s) \leftarrow \frac{dU}{dt}$$

At terminal speed, $U = U_t$ and $dU/dt = 0$, so

$$0 = \frac{\rho(V-U)^2A}{M} - \frac{kU}{M} \quad \text{or} \quad V^2 - 2UV + U^2 - \frac{k}{\rho A}U = 0$$

$$U^2 - \left(2V + \frac{k}{\rho A}\right)U + V^2 = 0$$

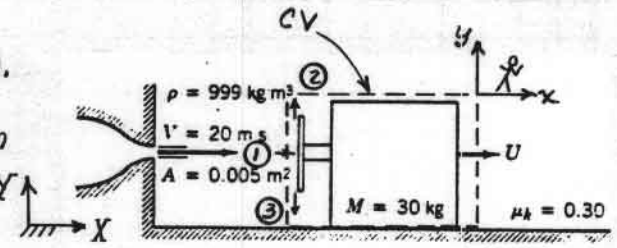
$$U = \frac{2V + k/\rho A \pm\sqrt{(2V + k/\rho A)^2 - 4V^2}}{2} = V\left\{\left(1 + \frac{k}{2\rho VA}\right) \pm\sqrt{\left(1+\frac{k}{2\rho VA}\right)^2 - 1}\right\}$$

$$1 + \frac{k}{2\rho VA} = 1 + \frac{1}{2}\times 7.5\frac{N\cdot s}{m}\times\frac{m^3}{999\ kg}\times\frac{s}{20\ m}\times\frac{1}{0.005\ m^2}\times\frac{kg\cdot m}{N\cdot s^2} = 1.0375$$

$$U = V\left\{1.0375 \pm\sqrt{(1.0375)^2 - 1}\right\} = 0.761\,V = 0.761\times 20\frac{m}{s} = 15.2\ m/s \leftarrow U_t$$

{The negative root was chosen so $U_t < V$, as required.}

Given: Vane/slider assembly moving under influence of jet (P 4.120).



Find: (a) Expressions for acceleration and speed as functions of time, (b) Terminal speed, and (c) Plots of $a_x(t)$ and $U(t)$.

Solution: Apply x momentum equation to linearly accelerating CV.

Basic equation:

$$F_{Sx} + F_{Bx} - \int_{CV} a_{rf_x} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho \, d\forall + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) Horizontal motion, so $F_{Bx} = 0$
(2) Neglect mass of liquid on vane, $u \approx 0$ on vane
(3) Uniform flow at each section
(4) Measure velocities relative to CV

Then

$$-Mg\mu_k - a_{rf_x} M = u_1\{-|\rho(V-U)A|\} + u_2\{+\dot{m}_2\} + u_3\{+\dot{m}_3\}$$

$$u_1 = V - U \qquad u_2 = 0 \qquad u_3 = 0$$

$$-Mg\mu_k - M\frac{dU}{dt} = -\rho(V-U)^2 A$$

or

$$\frac{dU}{dt} = \frac{\rho(V-U)^2 A}{M} - g\mu_k \qquad (1) \qquad a_{rf_x}(U)$$

Since V = constant, then $dU = -d(V-U)$. Letting $\beta = V - U$, separate variables to obtain

$$\frac{d\beta}{g\mu_k - \frac{\rho A}{M}\beta^2} = \frac{M}{\rho A}\,\frac{d\beta}{\frac{Mg\mu_k}{\rho A} - \beta^2} = -dt \quad \text{or} \quad \frac{d\beta}{a^2 - \beta^2} = -\frac{\rho A}{M}\,dt$$

where $a^2 = \frac{Mg\mu_k}{\rho A}$. Integrating

$$\int_{\beta=V}^{\beta=V-U} \frac{d\beta}{a^2-\beta^2} = \frac{1}{a}\tanh^{-1}\frac{\beta}{a}\Big]_V^{V-U} = \frac{1}{a}\left[\tanh^{-1}\left(\frac{V-U}{a}\right) - \tanh^{-1}\left(\frac{V}{a}\right)\right] = -\frac{\rho A}{M}t$$

Solving, $\tanh^{-1}\left(\frac{V-U}{a}\right) = \tanh^{-1}\left(\frac{V}{a}\right) - \frac{\rho A a}{M}t$; $a = \sqrt{\frac{Mg\mu_k}{\rho A}} = 4.20$ m/s

$$\frac{V-U}{a} = \tanh\left[\tanh^{-1}\left(\frac{V}{a}\right) - \rho\frac{Aa}{M}t\right]$$

or

$$U(t) = V - a\tanh\left[\tanh^{-1}\left(\frac{V}{a}\right) - \frac{\rho A a}{M}t\right] \qquad (2) \qquad U(t)$$

To find $a_{rf_x}(t)$, substitute in Eq. 1.

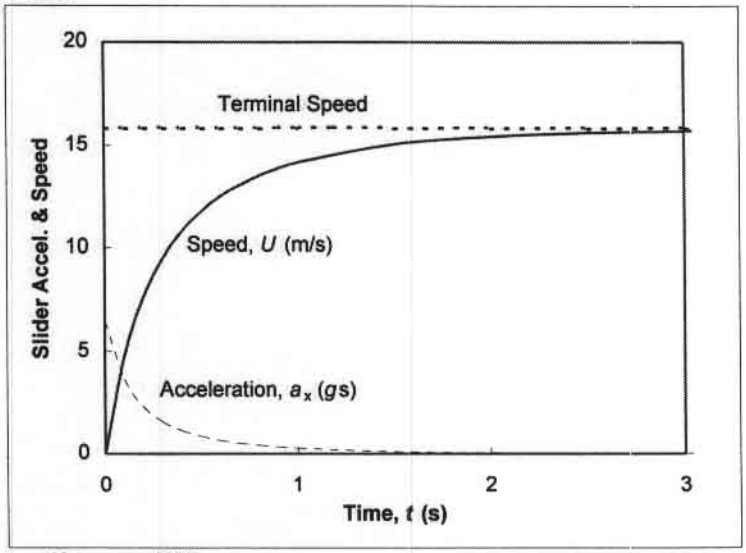
At U_t, $\frac{dU}{dt} = 0$. From Eq. 1, $U_t = V - \sqrt{\frac{Mg\mu_k}{\rho A}} = (20 - 4.20)\frac{m}{s} = 15.8$ m/s $\qquad U_t$

over

Equation 1 gives a_x as a function of U. Substitute Eq. 2 to obtain a_x as a function of time.

The results are:

Speed, Speed Ratio, and Acceleration vs. Time:



Input Parameters:

A =	0.005	m^2
g =	9.81	m/s^2
M =	30.0	kg
V =	20.0	m/s
μ_k =	0.300	---
ρ =	999	kg/m^3

Calculated Parameters:

a =	4.20	m/s
U_t =	15.8	m/s

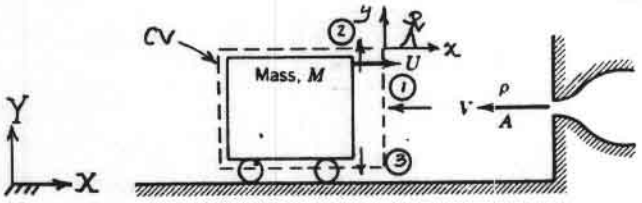
Calculated Results:

Time, t (s)	[ ]*	Speed, U (m/s)	U (m/s)	U/U_t (---)	a_x (gs)
0	0.213	0.0	15.8	0.000	6.49
0.1	0.283	4.8	15.8	0.302	3.64
0.2	0.353	7.6	15.8	0.482	2.30
0.3	0.423	9.5	15.8	0.600	1.58
0.4	0.493	10.8	15.8	0.684	1.14
0.5	0.563	11.8	15.8	0.745	0.851
0.6	0.633	12.5	15.8	0.791	0.655
0.7	0.703	13.1	15.8	0.827	0.516
0.8	0.773	13.5	15.8	0.856	0.412
0.9	0.843	13.9	15.8	0.879	0.335
1.0	0.913	14.2	15.8	0.898	0.274
1.5	1.263	15.1	15.8	0.954	0.113
2.0	1.613	15.4	15.8	0.978	0.0516
2.5	1.963	15.6	15.8	0.989	0.0246
3.0	2.313	15.7	15.8	0.995	0.0120
3.5	2.663	15.8	15.8	0.997	0.0059
4.0	3.013	15.8	15.8	0.999	0.00291
4.5	3.363	15.8	15.8	0.999	0.00144
5.0	3.713	15.8	15.8	1.000	0.000715

*[ ] = $\text{ctnh}^{-1}(V/a) - \rho A a t/M$; $a = (Mg\mu_k/\rho A)^{1/2}$

Given: Block and jet as shown.

Jet strikes block at $t > 0$.



Find: (a) Expression for acceleration.
(b) Time at which $U = 0$.

Solution: Apply x momentum equation to linearly accelerating CV.

Basic equation: $F_{Bx} + F_{Bx} - \int_{CV} a_{rf_x} \rho \, d\forall = \frac{\partial}{\partial t} \int_{CV} u_{xyz} \rho \, d\forall + \int_{CS} u_{xyz} \rho \, \vec{V}_{xyz} \cdot d\vec{A}$

($F_{Bx} = 0$ (1), $F_{Bx} = 0$ (2), ≈ 0 (3))

Assumptions: (1) No pressure or friction forces, so $F_{Bx} = 0$
(2) Horizontal, so $F_{Bx} = 0$
(3) Neglect mass of liquid in CV, $u \approx 0$ in CV
(4) Uniform flow at each section
(5) Measure velocities relative to CV

Then

$$-Ma_{rf_x} = -M\frac{dU}{dt} = u_1\{-|\rho(V+U)A|\} + u_2\{+\dot{m}_2\} + u_3\{+\dot{m}_3\}$$

$$u_1 = -(V+U) \qquad u_2 = 0 \qquad u_3 = 0$$

or

$$\frac{dU}{dt} = -\frac{\rho(V+U)^2 A}{M}$$ ← $\frac{dU}{dt}$

But, since V = constant, $dU = d(V+U)$, so

$$\frac{d(V+U)}{(V+U)^2} = -\frac{\rho A}{M} dt$$

Integrating from U_0 at $t = 0$ to $U = 0$ at t

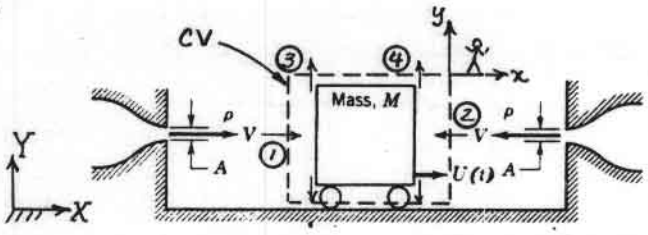
$$\int_{V+U_0}^{V} \frac{d(V+U)}{(V+U)^2} = -\frac{1}{(V+U)}\Big]_{V+U_0}^{V} = -\frac{1}{V} + \frac{1}{V+U_0} = \frac{-U_0}{V(V+U_0)} = -\frac{\rho A t}{M}$$

Solving, $t = \frac{MU_0}{\rho VA(V+U_0)} = \frac{M}{\rho VA(1+V/U_0)}$ ← t

Given: Block rolling between opposing jets, as shown.

Speed is U_0 at $t = 0$.

There is no resistance for $t > 0$.



Find: (a) Expression for acceleration, a(t).
(b) Expression for speed, U(t).

Solution: Apply x momentum to linearly accelerating CV.

Basic equation: $\overset{=0(1)\ =0(2)}{\uparrow}\quad \overset{\approx 0(3)}{\uparrow}$

$$F_{Sx} + F_{Bx} - \int_{CV} a_{rf_x}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) No pressure or friction forces, so $F_{Sx} = 0$
(2) Horizontal, so $F_{Bx} = 0$
(3) Neglect mass of liquid in CV; $u_x \approx 0$ in CV
(4) Uniform flow at each section
(5) Measure velocities relative to CV

Then

$$-a_{rf_x} M = -M\frac{dU}{dt} = u_1\{-|\rho(V-U)A|\} + u_2\{-|\rho(V+U)A|\} + u_3\{\dot{m}_3\} + u_4\{\dot{m}_4\}$$

$$u_1 = V - U \qquad u_2 = -(V+U) \qquad u_3 = 0 \qquad u_4 = 0$$

or

$$-M\frac{dU}{dt} = \rho A\left[-(V-U)^2 + (V+U)^2\right] = \rho A[4UV] = 4\rho VAU$$

Thus $$\frac{dU}{U} = -\frac{4\rho VA}{M}\,dt$$

Integrating $$\int_{U_0}^{U}\frac{dU}{U} = \ln U\Big]_{U_0}^{U} = \ln\frac{U}{U_0} = -\frac{4\rho VA}{M}t$$

or

$$U(t) = U_0\,e^{-\frac{4\rho VA}{M}t}$$ U(t)

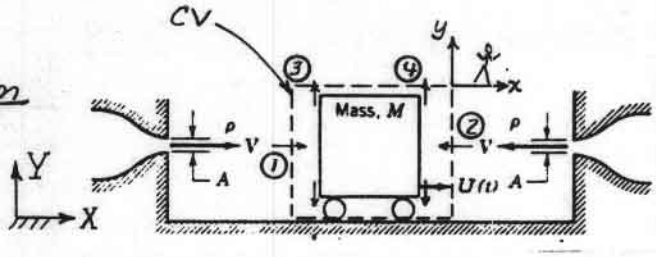
Also

$$a(t) = \frac{dU}{dt} = -\frac{4\rho VA}{M}U_0\,e^{-\frac{4\rho VA}{M}t}$$ a(t)

Given: Block rolling between opposing jets, as shown.

At $t = 0$, block moves at $U_0 = 10 \frac{m}{s}$

starting from $X = 0$.



Find: (a) Time to reduce speed to $U = 0.5$ m/s.

(b) Position at that instant.

Solution: Apply x momentum equation to linearly accelerating CV.

Basic equation: $F_{Bx} + F_{Bx} - \int_{CV} a_{rf_x} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$

($F_{Bx} = 0$ (1), $F_{Bx} = 0$ (2), $\simeq 0$ (3))

Assumptions: (1) No pressure or friction forces, so $F_{Sx} = 0$

(2) Horizontal, so $F_{Bx} = 0$

(3) Neglect mass of liquid in CV; $u \simeq 0$ in CV

(4) Uniform flow at each section

(5) Measure velocities relative to CV

Then

$$-a_{rf_x} M = -M\frac{dU}{dt} = u_1\{-|\rho(V-U)A|\} + u_2\{-|\rho(V+U)A|\} + u_3\{\dot{m}_3\} + u_4\{\dot{m}_4\}$$

$$u_1 = V - U \qquad u_2 = -(V+U) \qquad u_3 = 0 \qquad u_4 = 0$$

or

$$-M\frac{dU}{dt} = \rho A\left[-(V-U)^2 + (V+U)^2\right] = \rho A\left[4UV\right] = 4\rho V A U$$

Thus

$$\frac{dU}{U} = -\frac{4\rho V A}{M} dt$$

Integrating, $\int_{U_0}^{U} \frac{dU}{U} = \ln U\Big]_{U_0}^{U} = \ln \frac{U}{U_0} = -\frac{4\rho V A}{M} t$ (1)

Thus $t = -\frac{M}{4\rho V A} \ln \frac{U}{U_0} = -\frac{1}{4} \times \frac{M}{\rho V A} \ln \frac{0.5}{10} = 0.750 \frac{M}{\rho V A}$ t

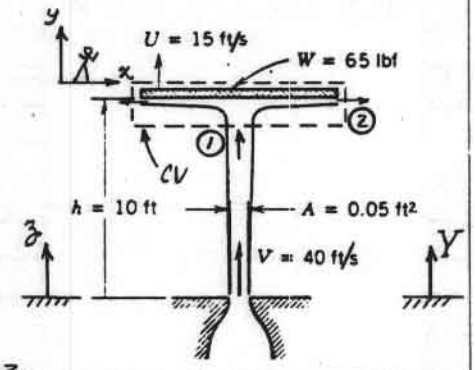
From Eq. 1, $U(t) = \frac{dX}{dt} = U_0 e^{-\frac{4\rho V A}{M} t}$

Integrating, $X = \int_0^X dX = \int_0^t U_0 e^{-\frac{4\rho V A}{M} t} dt = -\frac{M U_0}{4\rho V A} e^{-\frac{4\rho V A}{M} t}\Big]_0^t$

$$X = \frac{M U_0}{4\rho V A}\left[1 - e^{-\frac{4\rho V A}{M} t}\right] = \frac{0.95}{4} \frac{M U_0}{\rho V A} = 0.238 \frac{M U_0}{\rho V A}$$ X

Given: Vertical jet impinging on disk.

Find: Vertical acceleration of disk at the instant shown.



Solution: Apply Bernoulli equation to jet, then y momentum equation to a CV with linear acceleration.

Basic equations:

$$\frac{p_0}{\rho}^{\uparrow} + \frac{V_0^2}{2} + gz_0 = \frac{p_1}{\rho}^{\uparrow} + \frac{V_1^2}{2} + gz_1$$

(p_0/ρ: =0(6); p_1/ρ: ≈0(7))

$$F_{Sy}^{\uparrow} + F_{By} - \int_{CV} a_{rfy}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\,d\forall + \int_{CS} v_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) No friction
(4) Flow along streamline
(5) $p_0 = p_1 = p_{atm}$
} in jet

From Bernoulli,

$$V_1 = \sqrt{V_0^2 + 2g(z_0 - z_1)} = \left[40^2\,\frac{ft^2}{s^2} + 2\times 32.2\,\frac{ft}{s^2}\,(0-10)\,ft\right]^{1/2} = 30.9\ ft/s$$

(6) No pressure force on CV, $F_{Sy} = 0$
(7) Neglect mass of liquid in CV; $v \approx 0$ in CV
(8) Uniform flow at each section
(9) Measure velocities relative to CV

Then

$$-W - Ma_{rfy} = v_1\{-|\rho(V_1 - U)A_1|\} + v_2\{\dot{m}_2\} = -\rho(V_1 - U)^2 A_1$$

$$v_1 = V_1 - U \qquad v_2 = 0$$

or

$$a_{rfy} = \frac{\rho(V_1 - U)^2 A_1 - W}{M}$$

But from continuity, $V_0 A_0 = V_1 A_1$; $A_1 = A_0\,\frac{V_0}{V_1}$. Thus, since $M = W/g$,

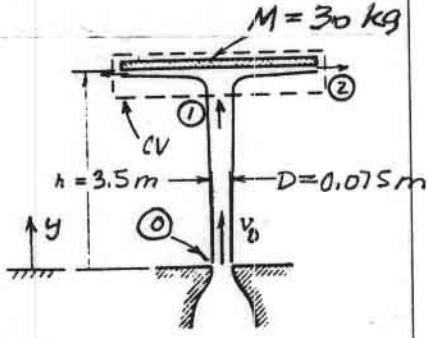
$$a_{rfy} = \frac{\rho(V_1 - U)^2 \frac{V_0}{V_1} A_0 - W}{W/g} = \left[\frac{\rho(V_1 - U)^2 V_0 A_0}{W V_1} - 1\right] g$$

$$= \left[1.94\,\frac{slug}{ft^3}\,(30.9-15)^2\,\frac{ft^2}{s^2}\times 40\,\frac{ft}{s}\times 0.05\,ft^2 \times \frac{1}{65\ lbf}\times\frac{s}{30.9\,ft}\times\frac{lbf\cdot s^2}{slug\cdot ft} - 1\right] 32.2\,\frac{ft}{s^2}$$

$$a_{rfy} = -16.5\ ft/s^2 \quad (down)$$

a_{rfy} ←

Given: Flow system of Problem 4.172, with dimensions shown at right.



Find: Plot disk mass vs. flow rate to determine the flow rate required to make $h = 3.5$ m.

Solution: Apply continuity, momentum, and Bernoulli equations using CV shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV}\rho\,d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$ (=0 (1))

$$F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v\rho\,d\forall + \int_{CS} v\rho\vec{V}\cdot d\vec{A} \quad (=0\ (1))$$

$$\frac{p_0}{\rho} + \frac{V_0^2}{2} + g\cancel{y_0}^{=0} = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gy_1$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) No friction
(4) Flow along a streamline
} in jet
(5) $p_0 = p_1 = p_{atm}$
(6) No pressure force on CV, so $F_{Sy} \simeq 0$
(7) Uniform flow at each cross-section

From momentum

$$-Mg = \int_{CS} v\rho\vec{V}\cdot d\vec{A} = v_1\{-\rho V_1 A_1\} + v_2\{+\rho V_2 A_2\} = -V_1\rho V_1 A_1$$

$v_1 = V_1 \qquad v_2 = 0$

$$M = \frac{V_1\rho V_1 A_1}{g}$$

From continuity $V_1 A_1 = V_0 A_0$

From Bernoulli $V_1 = \left[V_0^2 - 2gh\right]^{\frac{1}{2}}$

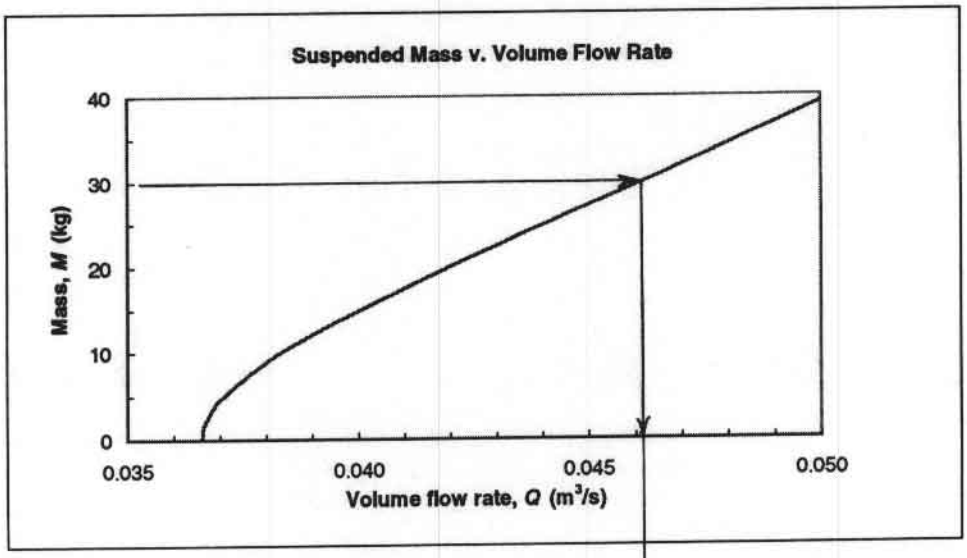
Substituting

$$M = \frac{\rho}{g}\left[V_0^2 - 2gh\right]^{\frac{1}{2}} V_0 \frac{\pi D_0^2}{4} \qquad M$$

This equation cannot be solved for V_0 directly, but it can be plotted for various values of V_0, as shown on the next page.

Disk Mass versus Volume Flow Rate (to suspend disk at h = 3.5 m):

Q (m³/s)	V_0 (m/s)	V_1 (m/s)	M (kg)
0.036609679920	8.28674	0.00	2.95E-06
0.03665	8.30	0.39	1.45
0.03667	8.30	0.48	1.78
0.03668	8.30	0.51	1.92
0.03669	8.30	0.55	2.05
0.03670	8.31	0.58	2.18
0.037	8.38	1.21	4.57
0.038	8.60	2.31	8.92
0.039	8.83	3.04	12.1
0.040	9.05	3.65	14.9
0.041	9.28	4.18	17.4
0.042	9.51	4.66	19.9
0.043	9.73	5.11	22.4
0.044	9.96	5.52	24.8
0.045	10.2	5.92	27.1
0.0462	10.5	6.38	30.0
0.047	10.6	6.67	31.9
0.048	10.9	7.03	34.3
0.049	11.1	7.37	36.8
0.050	11.3	7.71	39.2



$Q \approx 0.0462\ m^3/s$ ← Q

Given: Rocket sled on horizontal track, slowed by retro-rocket.

Initial mass $M_0 = 1620$ kg Initial speed $U_0 = 578$ m/s

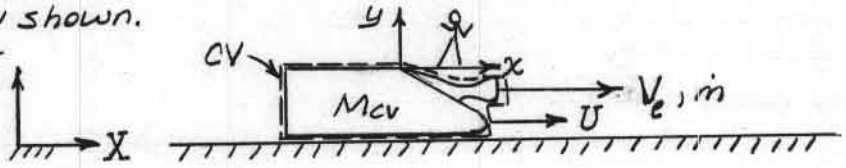
Mass flow rate $\dot{m} = 7.75$ kg/s Exhaust speed $V_e = 2540$ m/s

Firing time $t_{b0} = 23.9$ s

Neglect aerodynamic drag and rolling resistance.

Find: (a) Algebraic expression for sled speed U as a function of t.
(b) Speed at end of retro-rocket firing.

Solution: Apply x-component of momentum equation to the linearly accelerating CV shown.



From continuity,

$$M_{CV} = M_0 - \dot{m}t, \quad t < t_{b0}$$

Basic equation: $F_{Sx} + F_{Bx} - \int_{CV} a_{rfx}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$

(F_{Sx}: =0 (1); F_{Bx}: =0 (2); a_{rfx}: ≈0 (3))

Assumptions: (1) No pressure, drag, or rolling resistance, so $F_{Sx} = 0$
(2) Horizontal motion, so $F_{Bx} = 0$
(3) Neglect unsteady effects within CV
(4) Uniform flow at nozzle exit plane
(5) $p_e = p_{atm}$

Then $-a_{rfx} M_{CV} = u_e\{+\dot{m}\} = +V_e\dot{m}$ or $\frac{dU}{dt} = -\frac{V_e\dot{m}}{M_{CV}} = -\frac{V_e\dot{m}}{M_0 - \dot{m}t}$

$u_e = V_e$

Thus $dU = V_e\left(\frac{-\dot{m}\,dt}{M_0 - \dot{m}t}\right)$ and $U - U_0 = V_e \ln\left(M_0 - \dot{m}t\right)_0^t = V_e \ln\left(1 - \frac{\dot{m}t}{M_0}\right)$

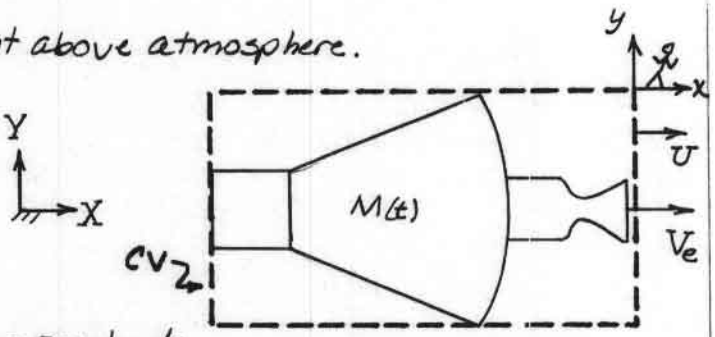
$$U(t) = U_0 + V_e \ln\left(1 - \frac{\dot{m}t}{M_0}\right) \;;\; t < t_{b0} \qquad U(t)$$

At t_{b0}, $U(t_{b0}) = 578\,\frac{m}{s} + 2540\,\frac{m}{s} \times \ln\left(1 - 7.75\,\frac{kg}{s} \times 23.9\,s \times \frac{1}{1620\,kg}\right)$

$$U(t_{b0}) = 270 \text{ m/s} \qquad U(t_{b0})$$

Given: Space capsule in level flight above atmosphere.

$U_0 = 8.05$ km/s
$M_0 = 1600$ kg
$\dot{m} = 8$ kg/s
$V_e = 2940$ m/s



Find: Time to reduce speed to $U = 5.00$ km/s.

Solution: Apply x component of momentum to CV with linear acceleration.

Basic equation: $F_{Bx} + F_{Sx} - \int_{CV} a_{rfx}\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall + \int_{CS} u_{xyz}\rho\, \vec{V}_{xyz}\cdot d\vec{A}$

($F_{Sx} = 0$ (1); $F_{Bx} = 0$ (2); $a_{rfx} = 0$ (3); $u_{xyz} \approx 0$ (4))

Assumptions: (1) No resistance; $F_{Sx} = 0$
(2) Horizontal; $F_{Bx} = 0$
(3) Use velocities measured relative to CV
(4) Neglect velocity within CV
(5) Uniform flow at exit plane with negligible p_e (given)

From continuity, $\frac{dM}{dt} = \frac{\partial}{\partial t}\int_{CV}\rho\, d\forall = -\int_{CS}\rho\vec{V}_{xyz}\cdot d\vec{A} = -\dot{m}$; $M(t) = M_0 - \dot{m}t$

From momentum, $-a_{rfx}M = -\frac{dU}{dt}(M_0 - \dot{m}t) = u_e\{+\dot{m}\} = V_e\dot{m}$

Thus $\frac{dU}{dt} = -\frac{V_e\dot{m}}{M_0 - \dot{m}t}$ $\qquad u_e = V_e$

Integrating, $U - U_0 = V_e\int_0^t \frac{-\dot{m}\,dt}{M_0 - \dot{m}t} = V_e \ln(M_0 - \dot{m}t)\Big]_0^t = V_e \ln\left(\frac{M_0 - \dot{m}t}{M_0}\right)$

Solving for t, $\frac{M_0 - \dot{m}t}{M_0} = e^{\frac{U-U_0}{V_e}}$; $M_0 - \dot{m}t = M_0 e^{U-U_0/V_e}$

$$t = \frac{M_0}{\dot{m}}\left(1 - e^{U-U_0/V_e}\right)$$

$$= 1600\text{ kg} \times \frac{s}{8\text{ kg}}\left\{1 - e^{\left[(5.00-8.05)\frac{km}{s} \times \frac{s}{2940\,m} \times 1000\frac{m}{km}\right]}\right\}$$

$t = 129$ s

t

Given: Rocket sled accelerates from rest on a level track. Initial mass, $M_0 = 600$ kg, includes fuel - $M_f = 150$ kg. The rocket motor burns fuel at rate $\dot{m} = 15$ kg/s. Exhaust gases leave nozzle uniformly and axially at atmospheric pressure with $V_e = 2900$ m/s relative to the nozzle. Neglect air and rolling resistance.

Find: (a) Maximum speed reached by the sled.
(b) Maximum acceleration of sled during the run.

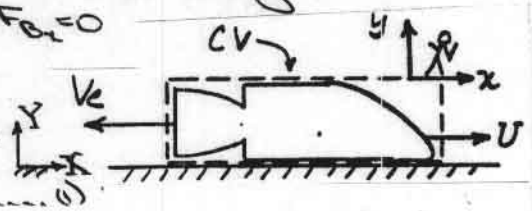
Plot: the sled speed and acceleration as functions of time.

Solution:

Apply the momentum equation to linearly accelerating CV shown.

Basic equation: $\cancel{F_{S_x}}^{=0\,(1)} + \cancel{F_{B_x}}^{=0\,(2)} - \int_{CV} a_{rx}\,\rho\,dV = \cancel{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,dV}^{=0\,(3)} + \int_{CS} u_{xyz}\,\rho\,V_{xyz}\cdot d\vec{A}$

Assumptions: (1) no net pressure forces ($P_e = P_{atm}$, given)
(2) horizontal motion, $F_{B_x} = 0$
(3) neglect $u/\partial t$ in CV
(4) uniform axial jet



From continuity, $M = M_0 - \dot{m}t$, then

$$-a_{rx}M = -\frac{dU}{dt}(M_0 - \dot{m}t) = V_e\{\dot{m}\} = -V_e\dot{m} \quad \ldots (1)$$

Separating variables,

$$dU = V_e\frac{\dot{m}\,dt}{M_0 - \dot{m}t}$$

Integrating from $U = 0$ at $t = 0$ to U at t gives

$$U = -V_e \ln(M_0 - \dot{m}t)\Big]_0^t = -V_e \ln\frac{(M_0 - \dot{m}t)}{M_0} = V_e \ln\frac{M_0}{(M_0 - \dot{m}t)} \quad \ldots (2)$$

The speed is a maximum at burnout. At burnout $M_f = 0$ and $M = M_0 - \dot{m}t = 450$ kg

At burnout, $t = \dfrac{M_f\,\text{initial}}{\dot{m}_{fuel}} = 150\text{kg} \cdot \dfrac{s}{15\text{kg}} = 10\text{ s}$

Then from Eq. 2

$$U_{max} = 2900\ \frac{m}{s} \ln\frac{600\text{ kg}}{450\text{ kg}} = 834\text{ m/s} \leftarrow U_{max}$$

From Eq. 1 the acceleration is $\dfrac{dU}{dt} = \dfrac{\dot{m}V_e}{M_0 - \dot{m}t}$

The maximum acceleration occurs at the instant prior to burnout

$$\left(\frac{dU}{dt}\right)_{max} = 15\ \frac{\text{kg}}{s} \times 2900\ \frac{m}{s} \times \frac{1}{450\text{ kg}} = 96.7\text{ m/s}^2 \leftarrow \left(\frac{dU}{dt}\right)_{max}$$

The sled speed as a function of time is

$U = V_e \ln \frac{M_0}{(M_0 - \dot{m}t)}$ for $0 \le t \le 10$ s

U = constant = 834 m/s for $t > 10$ (neglecting resistance)

The sled acceleration is given by

$\frac{dU}{dt} = \frac{\dot{m} V_e}{(M_0 - \dot{m}t)}$ for $0 \le t \le 10$ s

$\frac{dU}{dt} = 0$ for $t \ge 10$ s.

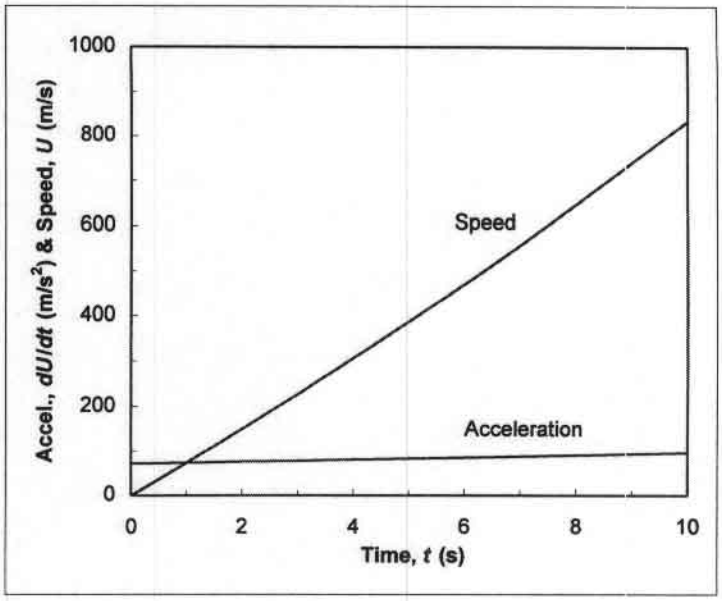
Acceleration and Velocity vs. Time for Rocket Sled:

Input Data:

M_0 =	600	kg
m(dot) =	15	kg/s
V_e =	2900	m/s

Calculated Results:

Time, t (s)	Acceleration, dU/dt (m/s^2)	Velocity, U (m/s)
0	72.5	0
1	74.4	73.4
2	76.3	149
3	78.4	226
4	80.6	306
5	82.9	387
6	85.3	471
7	87.9	558
8	90.6	647
9	93.5	739
10	96.7	834

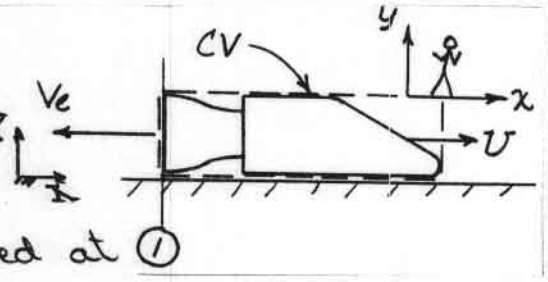


Given: Rocket sled moving on level track without resistance

Inital mass, $M_0 = 3000$ kg (includes $M_{fuel} = 1000$ kg)

$V_e = 2500$ m/s ; $p_e = p_{atm}$

Fuel consumption, $\dot{m} = 75$ kg/s



Find: Acceleration and speed of sled at ① $t = 10$ s.

Plot: sled speed and acceleration as functions of time.

Solution:

Apply x component of momentum to linearly accelerating CV; use continuity to find $M(t)$

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, dV + \int_{CS} \rho \vec{V}_{rel}\cdot d\vec{A}$

$F_{S_x} + F_{B_x} - \int_{CV} a_{rf}\,\rho\, dV = \frac{\partial}{\partial t}\int_{CV} u_{rel}\,\rho\, dV + \int_{CS} u_{rel}\,(\rho \vec{V}_{rel}\cdot d\vec{A})$

($F_{S_x} = 0$ (1), $F_{B_x} = 0$ (2), $\frac{\partial}{\partial t} = 0$ (3))

Assumptions: (1) $F_{S_x} = 0$, no resistance (given)
(2) $F_{B_x} = 0$, horizontal
(3) neglect $\partial/\partial t$ inside CV.
(4) uniform flow at nozzle ext
(5) $p_e = p_{atm}$ (given)

From continuity, $0 = \frac{\partial M}{\partial t} + \{+|\dot{m}|\} = \frac{dM}{dt} + \dot{m}$ or $dM = -\dot{m}\,dt$

Integrating, $\int_{M_0}^{M} dM = M - M_0 = \int_0^t -\dot{m}\,dt = -\dot{m}t$ or $M = M_0 - \dot{m}t$

From the momentum equation

$-a_{rf_x} M = -a_{rf_x}(M_0 - \dot{m}t) = u_1\{+|\dot{m}|\} = -V_e\dot{m}$ $\{u_1 = -V_e\}$

Thus $a_{rf_x} = \frac{dU}{dt} = \frac{V_e\dot{m}}{(M_0 - \dot{m}t)}$ ——— (1)

At $t = 10$ s

$$\frac{dU}{dt} = 2500\ \frac{m}{s} \times 75\ \frac{kg}{s^1} \times \frac{1}{3000\ kg - 75\ \frac{kg}{s^1}\times 10\ s} = 83.3\ m/s^2$$ ← a_{rf_x}

From Eq. 1, $dU = V_e \frac{\dot{m}\,dt}{M_0 - \dot{m}t}$

Integrating from $U = 0$ at $t = 0$ to U at t gives

$U = -V_e \ln(M_0 - \dot{m}t)\Big|_0^t = -V_e \ln\frac{(M_0 - \dot{m}t)}{M_0}$

$U = V_e \ln\frac{M_0}{(M_0 - \dot{m}t)}$ ——— (2)

At $t = 10$ s

$$U = 2500 \frac{m}{s} \ln \frac{3000 \text{ kg}}{3000 \text{ kg} - 75 \frac{\text{kg}}{\text{s}} \times 10 \text{ s}} = 719 \text{ m/s}$$ ← U.

Note that all fuel would be expended at $t_{b_0} = \frac{3000 \text{ kg}}{75 \text{ kg/s}} = 1000 \text{ kg} \cdot \frac{\text{s}}{75 \text{ kg}}$

i.e. at $t_{b,0} = 13.3$ s.

The sled speed as a function of time is then

$$U = V_e \ln \frac{M_0}{(M_0 - \dot{m}t)} \quad \text{for } t \le 13.3 \text{ s}$$

$$U = U_{max} = 1010 \text{ m/s} \quad \text{for } t \ge 13.3 \text{ s}$$

The sled acceleration is given by

$$\frac{dU}{dt} = \frac{\dot{m} V_e}{(M_0 - \dot{m}t)} \quad \text{for } 0 \le t \le 13.3 \text{ s}$$

$$\frac{dU}{dt} = 0 \quad \text{for } t \ge 13.3 \text{ s}$$

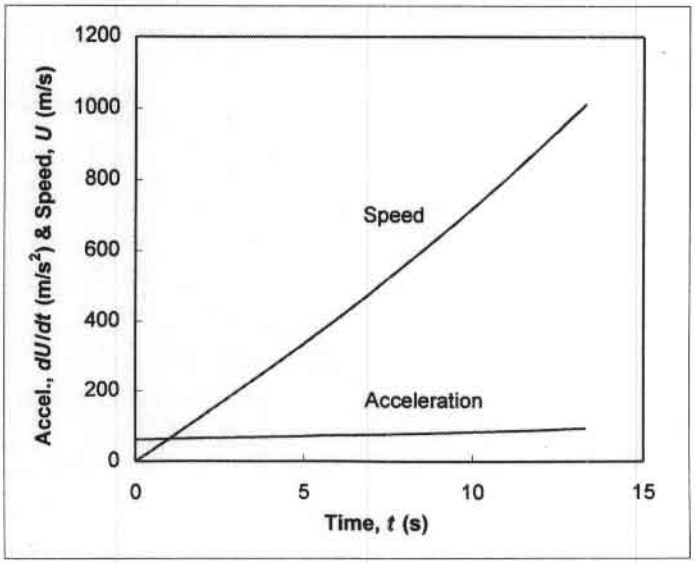
Acceleration and Speed vs. Time for Rocket Sled:

Input Data:

M_0 =	3000	kg
m(dot) =	75	kg/s
V_e =	2500	m/s

Calculated Results:

Time, t (s)	Acceleration, dU/dt (m/s^2)	Speed, U (m/s)
0	62.5	0
1	64.1	63.3
2	65.8	128
3	67.6	195
4	69.4	263
5	71.4	334
6	73.5	406
7	75.8	481
8	78.1	558
9	80.6	637
10	83.3	719
11	86.2	804
12	89.3	892
13	92.6	983
13.33	93.8	1014

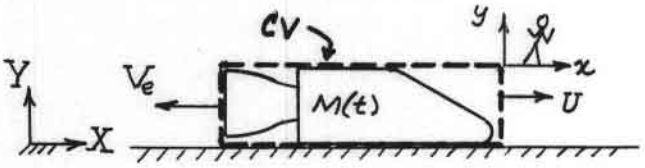


Given: Rocket sled moving on level track with no resistance.

$M_0 = 2000$ lbm

$\dot{m} = 30$ lbm/s

$V_e = 9000$ ft/s ; $p_e = p_{atm}$



Find: Minimum mass of fuel needed to accelerate sled to $U = 600$ mph.

Solution: Apply x component of momentum to CV accelerating linearly.

Basic equation: $F_{Sx} + F_{Bx} - \int_{CV} a_{rf_x}\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\,d\forall + \int_{CS} u_{xyz}\rho\,\vec{V}_{xyz}\cdot d\vec{A}$

($F_{Sx} = 0$ (1,2); $F_{Bx} = 0$ (3); ≈ 0 (5))

Assumptions: (1) No resistance } $F_{Sx} = 0$
(2) $p_e = p_{atm}$ (given)
(3) Horizontal track; $F_{Bx} = 0$
(4) Use relative velocities
(5) Neglect u and $\partial/\partial t$ within CV
(6) Uniform flow in exit plane

From continuity, $\frac{dM}{dt} = \frac{\partial}{\partial t}\int_{CV}\rho\,d\forall = -\int_{CS}\rho\,\vec{V}_{xyz}\cdot d\vec{A} = -\dot{m}$; $M = M_0 - \dot{m}t$

From momentum, $-a_{rf_x} M = -\frac{dU}{dt}(M_0 - \dot{m}t) = u_e\{+\dot{m}\} = -V_e\dot{m}$

Separating variables $u_e = -V_e$

$$dU = \frac{V_e\dot{m}\,dt}{M_0 - \dot{m}t}$$

Integrating, $U = -V_e \ln(M_0 - \dot{m}t)\Big|_0^t = V_e \ln\left(\frac{M_0}{M_0 - \dot{m}t}\right) = -V_e \ln\left(1 - \frac{\dot{m}t}{M_0}\right)$

The mass of fuel consumed is $m_f = \dot{m}t$. From the above

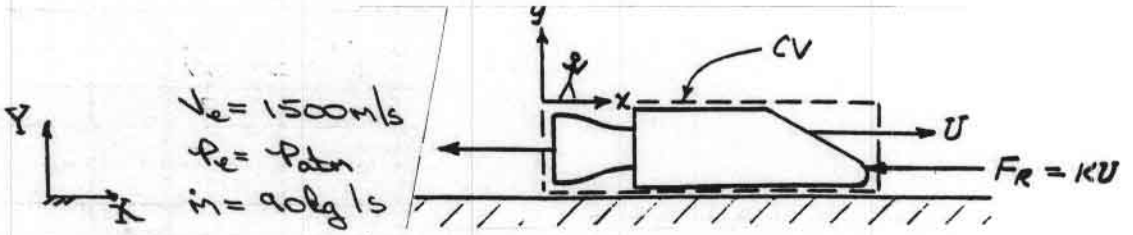
$$m_f = \dot{m}t = M_0\left(1 - e^{-U/V_e}\right)$$

$$= 2000\text{ lbm}\left[1 - e^{\left(-600\frac{mi}{hr}\times\frac{s}{9000\,ft}\times\frac{5280\,ft}{mi}\times\frac{hr}{3600\,s}\right)}\right]$$

$m_f = 186$ lbm

m_f

Given: Rocket sled with initial mass of 4 metric tons, including 1 ton of fuel. Motion resistance is given by kU where $k = 75\ N/m/s$.



Find: Sled speed 10s after starting from rest.

Plot: sled speed and acceleration as functions of time.

Solution:

Apply the x component of the momentum equation to linearly accelerating CV shown.

Basic equation: $F_{S_x} + \cancel{F_{B_x}}^{=0(2)} = \int_{CV} a_{rx}\,\rho\,dV = \cancel{\frac{\partial}{\partial t}\int_{CV} u_{xy}\rho\,dV}^{=0(3)} + \int_{CS} u_{xy}(\rho\,\vec{V}\cdot d\vec{A})$

Assumptions: (1) $p_e = p_{atm}$ (given) so $F_{S_x} = -F_R$
(2) $F_{B_x} = 0$
(3) neglect unsteady effects within CV
(4) uniform flow at exit plane.

Then

$$-F_R - a_{rx}M = u_e\{+|\dot{m}|\} = -V_e\dot{m} \qquad \{F_R = kU,\ u_e = -V_e\}$$

From continuity, $M = M_0 - \dot{m}t$. Substituting with $a_{rx} = \frac{dU}{dt}$

$$-kU - (M_0 - \dot{m}t)\frac{dU}{dt} = -V_e\dot{m}$$

$$\frac{dU}{dt} = \frac{V_e\dot{m} - kU}{M_0 - \dot{m}t} \quad \text{or} \quad \frac{dU}{V_e\dot{m} - kU} = \frac{dt}{M_0 - \dot{m}t}$$

Integrating, $\frac{1}{k}\ln(V_e\dot{m} - kU)\Big|_0^U = \frac{1}{\dot{m}}\ln(M_0 - \dot{m}t)\Big|_0^t$

and $\ln\frac{(V_e\dot{m} - kU)}{V_e\dot{m}} = \ln\left(1 - \frac{kU}{V_e\dot{m}}\right) = \frac{k}{\dot{m}}\ln\frac{(M_0 - \dot{m}t)}{M_0} = \frac{k}{\dot{m}}\ln\left(1 - \frac{\dot{m}t}{M_0}\right)$

Then $1 - \frac{kU}{V_e\dot{m}} = \left(1 - \frac{\dot{m}t}{M_0}\right)^{k/\dot{m}}$ and

$$U = \frac{V_e\dot{m}}{k}\left[1 - \left(1 - \frac{\dot{m}t}{M_0}\right)^{k/\dot{m}}\right] \qquad (1)$$

At $t = 10s$

$$U = 1500\frac{m}{s} \times 90\frac{kg}{s} \times 75\frac{N\cdot s}{m} \times \frac{kg\cdot m}{N\cdot s^2}\left[1 - \left(1 - 90\frac{kg}{s} \times 10s \times \frac{1}{4000\,kg}\right)^{\frac{75\,N\cdot s}{m}\times\frac{m}{kg}\times\frac{kg\cdot m}{N\cdot s^2}}\right]$$

$$U = 344\ m/s \qquad U$$

Note that all fuel would be expended at $t_{bo} = \frac{M_f}{\dot{m}} = \frac{1000 \text{ lbs}}{90 \text{ lbs/s}}$

i.e. at $t = 11.1$ s.

The sled speed as a function of time is then

$$v = \frac{v_e \dot{m}}{k}\left[1 - \left(1 - \frac{\dot{m}t}{M_0}\right)^{k/\dot{m}}\right] \quad \text{for } 0 \le t \le 11.1 \text{ s}$$

The speed reaches a maximum at $t = 11.1$ s and decays with time due to the motion resistance.

The sled acceleration is given by

$$\frac{dv}{dt} = \frac{v_e \dot{m} - kv}{M_0 - \dot{m}t} \quad \text{for } 0 \le t \le 11.1 \text{ s}$$

At $t \ge 11.1$ s $v_e = 0$ and

$$\frac{dv}{dt} = \frac{-kv}{M_0 - M_{fuel}}$$

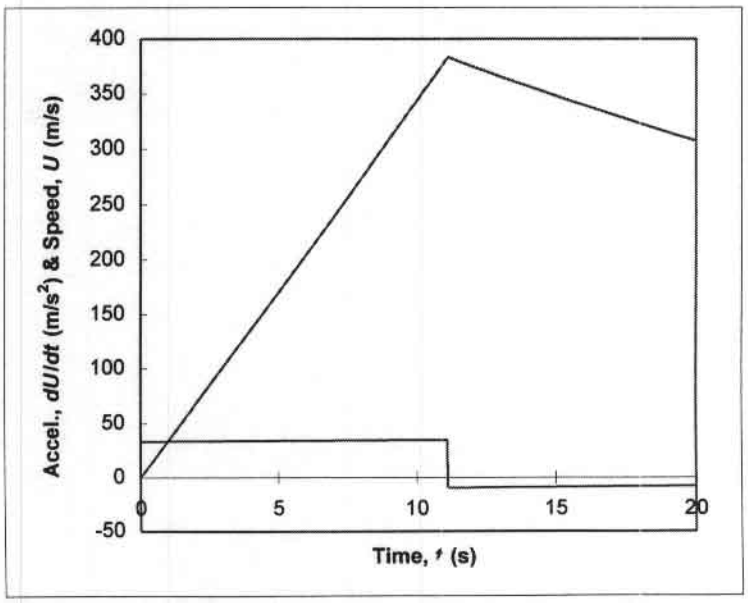
Note that for $t > t_{bo} = 11.1$ s, $\frac{dv}{dt} = -\frac{kv}{M_{bo}}$ and

$\frac{dv}{v} = -\frac{k}{M_{bo}}dt$, so $\ln\frac{v}{v_{bo}} = -\frac{k(t - t_{bo})}{M_{bo}}$

and $v = v_{bo}\, e^{-k(t - t_{bo})/M_{bo}}$

Calculated Results:

Time, t (s)	Speed, U (m/s)	Acceleration, dU/dt (m/s^2)
0	0	33.8
1	33.8	33.9
2	67.8	34.0
3	102	34.1
4	136	34.3
5	170	34.4
6	205	34.6
7	240	34.7
8	274	34.9
9	309	35.0
10	344	35.2
11	380	35.4
11.11	384	35.4
11.12	384	-9.59
12	375	-9.38
13	366	-9.15
14	357	-8.92
15	348	-8.70
16	340	-8.49
17	331	-8.28
18	323	-8.07
19	315	-7.88
20	307	-7.68



Given: Rocket launched from aircraft flying horizontally at $V_0 = 300$ m/s. Rocket accelerates to $V_f = 1.8$ km/s. Exhaust stream leaves nozzle at $V_e = 3000$ m/sec (relative to rocket) at atmospheric pressure. Neglect air resistance.

Find: (a) Algebraic expression for speed reached in horizontal flight.
(b) Minimum mass fraction needed to reach $V_f = 1.8$ km/s.

Solution: Apply x component of momentum using CV & CS shown.

Basic equation: $$F_{Sx} + F_{Bx} - \int_{CV} a_{rel_x}\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall + \int_{CS} u_{xyz}\rho\vec{V}\cdot d\vec{A}$$



Assumptions: (1) No drag; $p_e = p_{atm}$ so $F_{Sx} = 0$
(2) Horiz.; $F_{Bx} = 0$
(3) Neglect $\partial/\partial t$ in CV
(4) Constant mass flow rate; $\dot{m} = \text{const}$; $M(t) = M_0 - \dot{m}t$
(5) Uniform, axial flow at nozzle exit

Then
$$-\int a_{rel_x}\rho\, d\forall = -\frac{dU}{dt}M(t) = u_e\{+\dot{m}\} = -V_e\dot{m}$$

so
$$u_e = -V_e$$

$$\frac{dU}{dt} = \frac{V_e\dot{m}}{M(t)};\quad dU = \frac{V_e\dot{m}}{M_0 - \dot{m}t}dt = -V_e\frac{-\dot{m}\,dt}{M_0-\dot{m}t} = -V_e\frac{d(M_0-\dot{m}t)}{M_0-\dot{m}t}$$

Integrating from U_0 at $t=0$ to U at t,
$$U - U_0 = -V_e \ln(M_0-\dot{m}t)\Big]_0^t = -V_e\left[\ln(M_0-\dot{m}t) - \ln(M_0)\right] = -V_e\ln\left(\frac{M_0-\dot{m}t}{M_0}\right)$$

or
$$U = U_0 + V_e\ln\left(\frac{M_0}{M_0-\dot{m}t}\right) \qquad U(t)$$

Solving,
$$\frac{M_0-\dot{m}t}{M_0} = e^{-\frac{U-U_0}{V_e}} = 1 - \frac{\dot{m}t}{M_0};\quad \frac{\dot{m}t}{M_0} = 1 - e^{-\frac{U-U_0}{V_e}} = \text{mass fraction consumed}$$

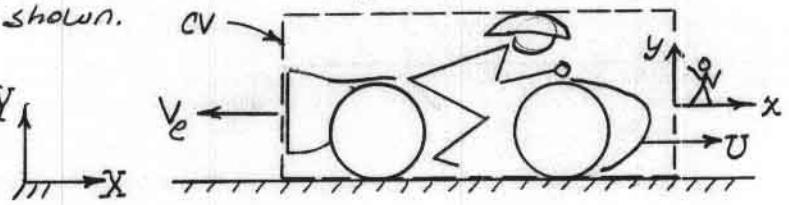
Substituting,
$$\frac{\dot{m}t}{M_0} = 1 - e^{-(1800-300)\frac{m}{s}\times\frac{s}{3000m}} = 1 - e^{-0.5} = 0.393 \qquad \text{Mass Fraction}$$

{The mass fraction calculated here is a minimum because neither air resistance nor drag due to lift were included.}

Given: Rocket-propelled motorcycle, to jump, standing start, level.
Speed needed $V_j = 875$ km/hr Rocket exhaust speed $V_e = 2510$ m/s
Total mass $M_B = 375$ kg (without fuel)

Find: Minimum fuel mass needed to reach V_j.

Solution: Apply x-component of momentum equation to linearly accelerating CV shown.



From continuity,

$M_{CV} = M_0 - \dot{m}t$

Basic equation: $F_{Bx} + F_{Bx} - \int_{CV} a_{rfx}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$

(F_{Bx} ≈ 0 (1); F_{Bx} = 0 (2); $\frac{\partial}{\partial t}\int_{CV}$ ≈ 0 (3))

Assumptions: (1) Neglect air and rolling resistance
(2) Level track, so $F_{Bx} = 0$
(3) Neglect unsteady effects within CV
(4) Uniform flow at nozzle exit plane
(5) $p_e = p_{atm}$

Then

$$-a_{rfx} M_{CV} = u_e\{+\dot{m}\} = -V_e\dot{m} \quad \text{or} \quad \frac{dU}{dt} = \frac{V_e\dot{m}}{M_{CV}} = \frac{V_e\dot{m}}{M_0 - \dot{m}t}$$

$$u_e = -V_e$$

Separating variables and integrating,

$$dU = -V_e\left(\frac{-\dot{m}\,dt}{M_0 - \dot{m}t}\right) \quad \text{or} \quad U_j = -V_e \ln(M_0 - \dot{m}t)\Big|_0^t = V_e \ln\left(\frac{M_0}{M_0 - \dot{m}t}\right)$$

But $M_0 = M_B + M_F$ and $M_F = \dot{m}t$, so

$$\frac{U_j}{V_e} = \ln\left(\frac{M_B + M_F}{M_B}\right) = \ln\left(1 + \frac{M_F}{M_B}\right); \quad 1 + \frac{M_F}{M_B} = e^{U_j/V_e}; \quad \frac{M_F}{M_B} = e^{U_j/V_e} - 1$$

Finally, $M_F = M_B\left(e^{U_j/V_e} - 1\right)$

$$M_F = 375\text{ kg} \times \exp\left[875\frac{\text{km}}{\text{hr}} \times \frac{\text{s}}{2510\text{ m}} \times 1000\frac{\text{m}}{\text{km}} \times \frac{\text{hr}}{3600\text{ s}} - 1\right]$$

$$M_F = 38.1\text{ kg}$$

M_F

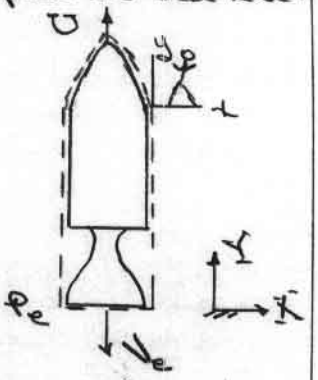
The fuel mass required is about 10 percent of the mass of the motorcycle and rider.

Given: Liquid-fueled rocket launched from pad at sea level

$M_0 = 30{,}000$ kg $\dot{m} = 2450$ kg/s

$V_e = 2270$ m/s $p_e = 66$ kPa (abs)

Exit plane diameter, $D_e = 2.6$ m



Find: acceleration at lift-off.
expression for rocket speed, $U(t)$

Solution: Apply y component of momentum equation to CV with linear acceleration

Basic equation: $F_{sy} + F_{By} - \int_{CV} a_{rf_y}\rho\,d\forall = \cancel{\frac{\partial}{\partial t}\int_{CV} v_{xyz}\rho\,d\forall}^{a(t)} + \int_{CS} v_{xyz}\rho\vec{V}\cdot d\vec{A}$

Assumptions: (1) F_{sy} due to pressure, p_{atm} assumed constant, neglect air resistance
(2) neglect rate of change of momentum inside CV
(3) uniform flow at exit

Then, $(p_e - p_{atm})A_e - Mg - a_{rf_y}M = V_e\{+\dot{m}\} = -\dot{m}V_e$

Solving for a_{rf_y},

$$a_{rf_y} = \frac{dU}{dt} = \frac{1}{M}\left[\dot{m}V_e + (p_e - p_{atm})A_e\right] - g \quad \text{...(1)}$$

$M = M(t)$. From conservation of mass $\frac{\partial}{\partial t}\int_{CV}\rho\,d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$

Then $\frac{\partial}{\partial t}\int_{CV}\rho\,d\forall = \frac{dM}{dt} = -\int_{CS}\rho\vec{V}\cdot d\vec{A} = -\dot{m}_e$ (constant)

Hence $M(t) = M_0 - \dot{m}t$, and

$$a_{rf_y} = \frac{dU}{dt} = \frac{\dot{m}V_e}{M_0 - \dot{m}t} + \frac{(p_e - p_{atm})A_e}{M_0 - \dot{m}t} - g$$

$$U = \int_0^U dU = \int_0^t \frac{\dot{m}V_e}{M_0 - \dot{m}t}\,dt + \int_0^t \frac{(p_e - p_{atm})A_e}{M_0 - \dot{m}t}\,dt - \int_0^t g\,dt$$

$$U = -V_e \ln\left[\frac{M_0 - \dot{m}t}{M_0}\right] - \frac{(p_e - p_{atm})A_e}{\dot{m}}\ln\left[\frac{M_0 - \dot{m}t}{M_0}\right] - gt$$

$$U = -\left[V_e + \frac{(p_e - p_{atm})A_e}{\dot{m}}\right]\ln\left[\frac{M_0 - \dot{m}t}{M_0}\right] - gt \longleftarrow U(t)$$

At lift-off, $t = 0$, $M = M_0$.

$$a_{rf_y} = \frac{1}{M}\left[\dot{m}V_e + (p_e - p_{atm})A_e\right] - g$$

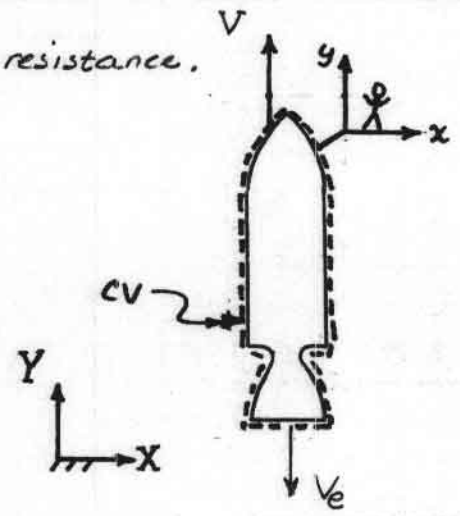
$$= \frac{1}{3\times10^4\,\text{kg}}\left[2450\frac{\text{kg}}{\text{s}}\times 2270\frac{\text{m}}{\text{s}} + (66-101)10^3\frac{\text{N}}{\text{m}^2}\times\frac{\pi}{4}(2.6)^2\text{m}^2 + \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right] - 9.81\frac{\text{m}}{\text{s}^2}$$

$a_{rf_y} = 169$ m/s² $\longleftarrow a_{rf_y}$

Given: Rocket fired vertically from rest, no resistance.

$M_0 = 200\ kg$, $\dot{m} = 10\ kg/sec$

$V_e = 2900\ m/s$, $p_e = p_{atm}$



Find: Speed at $t = 10\ s$.

Plot: Rocket speed as a function of time.

Solution: Apply y component momentum equation to accelerating CV.

Basic equation:

$$F_{By} + F_{By} - \int_{CV} a_{rfy}\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\rho\,d\forall + \int_{CS} v_{xyz}\rho\vec{V}_{xyz}\cdot d\vec{A}$$

($F_{By} = 0$ (1),(2); $\frac{\partial}{\partial t} \simeq 0$ (3))

Assumptions: (1) Neglect air resistance
(2) Exit pressure is atmospheric (given)
(3) Neglect rate of change of v_{xyz} within CV
(4) Flow is uniform at exit section
(5) All velocities are relative to CV
(6) $M = M_0 - \dot{m}t$

Then

$$F_{By} - a_{rfy}M = -Mg - a_{rfy}M = v_e\{|\dot{m}|\} = -V_e\dot{m}, \text{ since } v_e = -V_e.$$

or $$a_{rfy} = \frac{dV}{dt} = \frac{V_e\dot{m}}{M} - g \qquad (1)$$

Introducing $M = M_0 - \dot{m}t$ and separating variables

$$dV = \left(\frac{V_e\dot{m}}{M_0 - \dot{m}t} - g\right)dt$$

Integrating,

$$\int_0^V dV = V = \int_0^t \left(\frac{V_e\dot{m}}{M_0 - \dot{m}t} - g\right)dt = -V_e\ln(M_0 - \dot{m}t)\Big]_0^t - gt$$

or $$V = V_e\ln\left(\frac{M_0}{M_0 - \dot{m}t}\right) - gt \qquad (2)$$

Substituting at $t = 10\ s$,

$$V = 2900\frac{m}{s} \times \ln\left(\frac{200\ kg}{200\ kg - 10\frac{kg}{s} \times 10\ s}\right) - 9.81\frac{m}{s^2}\ 10\ s$$

$$V = 1910\ m/s$$

V

Speed as a function of time is given by Eq. 2; acceleration is given by Eq. 1.

The results are tabulated and plotted on the next page.

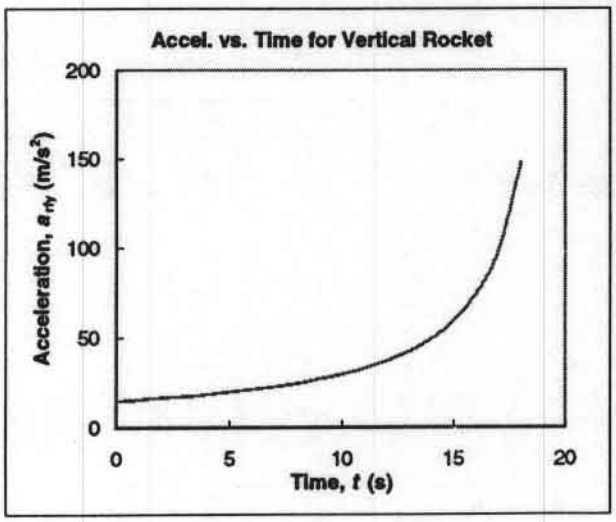
Acceleration and Speed as Functions of Time for Vertical Rocket:

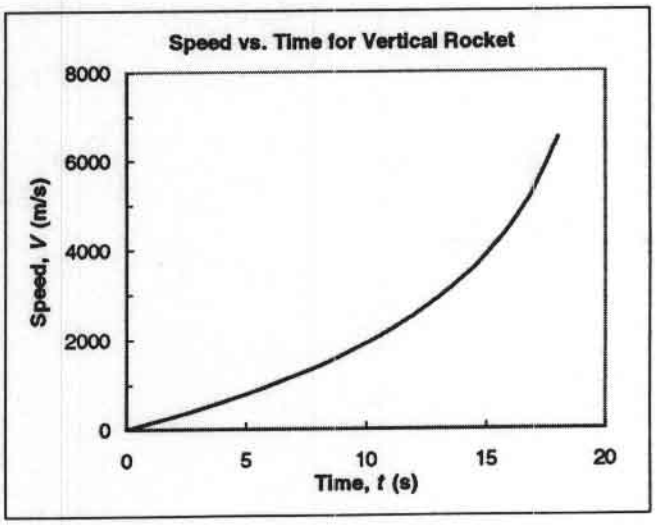
Input Data:

m(dot) =	10	kg/s	Mass flow rate
M_0 =	200	kg	Initial mass
V_e =	2900	m/s	Exhaust gas speed

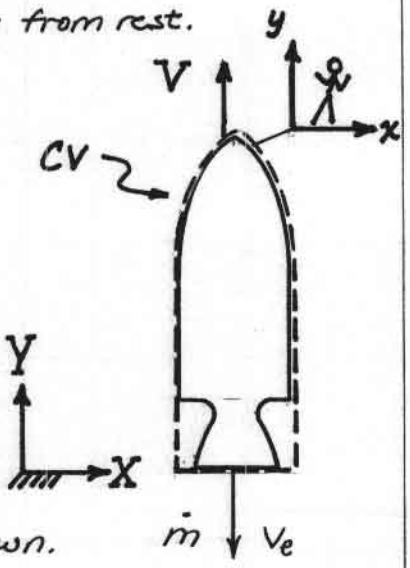
Calculated Results:

Time, t (s)	Mass, M (kg)	Mass Ratio, M/M_0 (---)	Acceleration, a_{rly} (m/s^2)	Acceleration, a_{rly} (gs)	Speed, U (m/s)
0	200	1	145	14.8	0.0
1	190	0.950	153	15.6	139
2	180	0.900	161	16.4	286
3	170	0.850	171	17.4	442
4	160	0.800	181	18.5	608
5	150	0.750	193	19.7	785
6	140	0.700	207	21.1	975
7	130	0.650	223	22.7	1181
8	120	0.600	242	24.6	1403
9	110	0.550	264	26.9	1645
10	100	0.500	290	29.6	1912
11	90	0.450	322	32.8	2208
12	80	0.400	363	37.0	2540
13	70	0.350	414	42.2	2917
14	60	0.300	483	49.3	3354
15	50	0.250	580	59.1	3873
16	40	0.200	725	73.9	4510
17	30	0.150	967	98.5	5335
18	20	0.100	1450	148	6501





Given: Home made rocket launched vertically from rest.



$M_0 = 20$ lbm, of which 15 lbm is fuel

$\dot{m} = 0.5$ lbm/s

$V_e = 6500$ ft/s (relative to rocket)

$p_e = p_{atm}$

Neglect aerodynamic drag.

Find: (a) Speed at $t = 20$ s. Plot: Speed and height as functions of time.
(b) Height at $t = 20$ s.

Solution: Apply y-component of momentum equation to accelerating CV using CS shown.

Basic equation:

$$F_{By} + F_{By} - \int_{CV} a_{rfy}\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\rho\, d\forall + \int_{CS} v_{xyz}\rho\vec{V}_{xyz}\cdot d\vec{A}$$

($F_{By} = 0$ (1); $\frac{\partial}{\partial t} \approx 0$ (2))

Assumptions: (1) Neglect air resistance; $p_e = p_{atm}$ (given)
(2) Neglect v_{xyz} and $\partial/\partial t$ within CV
(3) Uniform flow at nozzle exit section

Then

$$F_{By} - a_{rfy} M = -Mg - Ma_{rfy} = v_e\{+\dot{m}\} = -V_e\dot{m}$$

and

$$v_e = -V_e$$

$$a_{rfy} = \frac{dV}{dt} = \frac{V_e\dot{m}}{M} - g$$

Introducing $M = M_0 - \dot{m}t$ and separating variables,

$$dV = \left(\frac{V_e\dot{m}}{M_0 - \dot{m}t} - g\right)dt$$

Integrating from rest at $t = 0$

$$V = \int_0^t \left(\frac{V_e\dot{m}}{M_0 - \dot{m}t} - g\right)dt = -V_e \ln(M_0 - \dot{m}t)\Big]_0^t - gt$$

or

$$V = V_e \ln\left(\frac{M_0}{M_0 - \dot{m}t}\right) - gt \qquad (1)$$

At $t = 20$ sec,

$$V = \frac{6500\ \text{ft}}{\text{s}} \ln\left(\frac{20\ \text{lbm}}{20\ \text{lbm} - 0.5\frac{\text{lbm}}{\text{s}} \times 20\ \text{s}}\right) - \frac{32.2\ \text{ft}}{\text{s}^2} \times 20\ \text{s}$$

$$V(20\ \text{s}) = 3{,}860\ \text{ft/s} \qquad V_{20}$$

To find height, note $V = \frac{dY}{dt}$. Substitute into Eq. 1 to obtain

$$\frac{dY}{dt} = V_e \ln\left(\frac{M_0}{M_0 - \dot{m}t}\right) - gt = -V_e \ln\left(1 - \frac{\dot{m}t}{M_0}\right) - gt$$

Let $r = 1 - \frac{\dot{m}t}{M_0}$, and $dr = -\frac{\dot{m}}{M_0}\,dt$, then

$$dY = -V_e \ln r\, dt - gt\, dt = +\frac{V_e M_0}{\dot{m}} \ln r\, dr - gt\, dt$$

Integrating from $Y = 0$ at $t = 0$,

$$Y = \int_0^t \frac{V_e M_0}{\dot{m}} \ln r\, dr - \frac{1}{2}gt^2 = \frac{V_e M_0}{\dot{m}}\left[r \ln r - r\right]_0^t - \frac{1}{2}gt^2$$

$$= \frac{V_e M_0}{\dot{m}}\left\{\left(1 - \frac{\dot{m}t}{M_0}\right)\left[\ln\left(1 - \frac{\dot{m}t}{M_0}\right) - 1\right]\right\}\Bigg|_0^t - \frac{1}{2}gt^2$$

$$Y = \frac{V_e M_0}{\dot{m}}\left\{\left(1 - \frac{\dot{m}t}{M_0}\right)\left[\ln\left(1 - \frac{\dot{m}t}{M_0}\right) - 1\right] + 1\right\} - \frac{1}{2}gt^2$$

At $t = 20$ s,

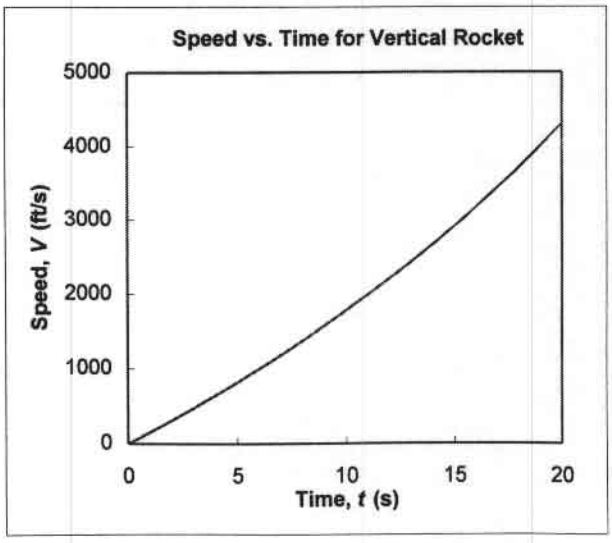
$$1 - \frac{\dot{m}t}{M_0} = 1 - 0.5\ \frac{\text{lbm}}{\text{s}} \times 20\ \text{s} \times \frac{1}{20\ \text{lbm}} = \frac{1}{2}$$

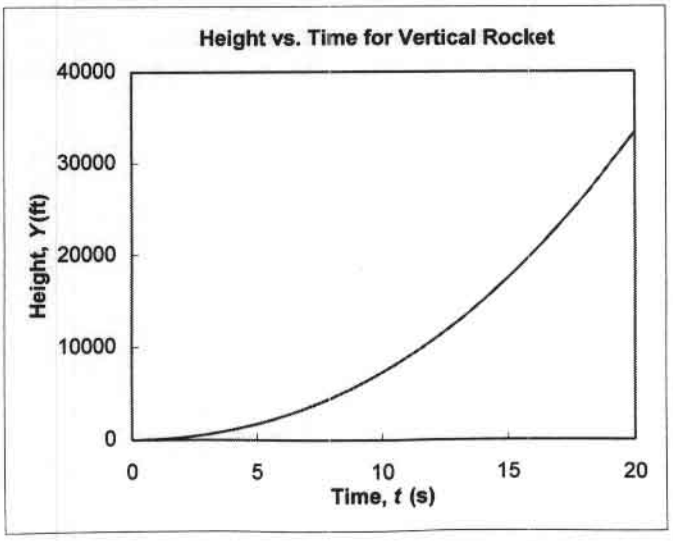
so

$$Y = \frac{6500\ \text{ft}}{\text{s}} \times 20\ \text{lbm} \times \frac{\text{s}}{0.5\ \text{lbm}}\left\{\left(\frac{1}{2}\right)\left[\ln\left(\frac{1}{2}\right) - 1\right] + 1\right\} - \frac{1}{2} \times 32.2\ \frac{\text{ft}}{\text{s}^2}\ (20)^2\ \text{s}^2$$

$$Y = 33{,}500\ \text{ft}$$

Y

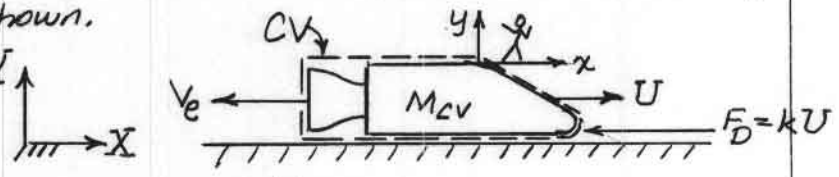




Given: Rocket sled accelerating from rest. Initial mass is 4000 kg, including 1000 kg of fuel. Motion resistance is kU, where $k = 75\ N\cdot s/m$ and U is speed of sled. Exit speed of exhaust gas is 1500 m/s (relative to the rocket); fuel burn rate is 90 kg/s.

Find: (a) Sled speed at $t = 15$ s.
(b) Maximum sled speed, and when it occurs.
(c) Initial acceleration of sled.
(d) Maximum acceleration of sled, and when it occurs.
(e) Plot sled speed and acceleration for $0 < t \le 15$ s.

Solution: Apply x-component of momentum equation to the linearly accelerating CV shown.



From continuity,

$$M_{CV} = M_0 - \dot{m}t, \quad t < t_{bo}$$

Basic equation: $F_{Sx} + F_{Bx} - \int_{CV} a_{rfx}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$

($F_{Bx} = 0$ by assumption 1; $\frac{\partial}{\partial t}\int_{CV} \approx 0$ by assumption 2)

Assumptions: (1) Horizontal, so $F_{Bx} = 0$
(2) Neglect unsteady effects within CV
(3) Uniform flow at exit plane
(4) $p_e = p_{atm}$

Then $-F_D - a_{rfx} M_{CV} = u_e\{+\dot{m}\} = -V_e\dot{m}$

$$u_e = -V_e$$

and

$$a_{rfx} = \frac{dU}{dt} = \frac{V_e\dot{m} - kU}{M_{CV}}, \quad t < t_{bo}$$

The initial acceleration is

(c)

$$\left.\frac{dU}{dt}\right)_{t=0} = \left(1500\,\frac{m}{s}\times 90\,\frac{kg}{s}\times\frac{N\cdot s^2}{kg\cdot m} - 75\,\frac{N\cdot s}{m}\times 0\,\frac{m}{s}\right)\frac{1}{4000\,kg} = 33.8\ m/s^2 \qquad \left.\frac{dU}{dt}\right)_0$$

All fuel is expended (burnout occurs) when $t_{bo} = \frac{M_{fuel}}{\dot{m}} = \frac{1000\,kg}{90\,kg/s} = 11.1$ s

Thus to determine speed at $t = 15$ s, must consider two parts:

$$0 < t < t_{bo} = 11.1\ s \qquad \frac{dU}{dt} = \frac{\dot{m}V_e - kU}{M_{CV}}$$

$$11.1\ s = t_{bo} < t \le 15\ s \qquad \frac{dU}{dt} = -\frac{kU}{M_0 - M_{fuel}}$$

To obtain speed vs. time, separate variables and integrate.

For $t < t_{bo}$ $\quad \dfrac{dU}{\dot{m}V_e - kU} = \dfrac{dt}{M_{CV}} = \dfrac{dt}{M_o - \dot{m}t}$ or $-\dfrac{1}{k}\ln(\dot{m}V_e - kU)\Big]_0^U = -\dfrac{1}{\dot{m}}\ln(M_o - \dot{m}t)\Big]_0^t$

Thus $\dfrac{1}{k}\ln\left(\dfrac{\dot{m}V_e - kU}{\dot{m}V_e}\right) = \dfrac{1}{\dot{m}}\ln\left(\dfrac{M_o - \dot{m}t}{M_o}\right)$ or $1 - \dfrac{kU}{\dot{m}V_e} = \left(1 - \dfrac{\dot{m}t}{M_o}\right)^{k/\dot{m}}$, so

$$U = \frac{\dot{m}V_e}{k}\left[1 - \left(1 - \frac{\dot{m}t}{M_o}\right)^{k/\dot{m}}\right], \quad t \le t_{bo} \qquad (1)$$

At $t = t_{bo}$, $U = \dfrac{90\,kg}{s} \times \dfrac{1500\,m}{s} \times \dfrac{m}{75\,N\cdot s} \times \dfrac{N\cdot s^2}{kg\cdot m}\left[1 - \left(1 - \dfrac{1}{4}\right)^{75\,\frac{N\cdot s}{m} \times \frac{s}{90\,kg} \times \frac{kg\cdot m}{N\cdot s^2}}\right] = 384\ m/s$ (b) U_{max}

For $t_{bo} < t < 15\,s$, $\dfrac{dU}{-kU} = \dfrac{dt}{M_{sled}}$ or $-\dfrac{1}{k}\ln U\Big]_{U_{bo}}^{U} = \dfrac{t}{M_{sled}}\Big]_{t_{bo}}^{t}$; $U = U_{bo}\,e^{-\frac{k(t - t_{bo})}{M_{sled}}}$

At $t = 15\,s$, $U = 384\,\dfrac{m}{s} \times \exp\left[-\dfrac{75\,N\cdot s}{m} \times (15 - 11.1)\,s \times \dfrac{1}{3000\,kg} \times \dfrac{kg\cdot m}{N\cdot s^2}\right] = 348\ m/s$ (a) $U_{15\,s}$

From Eq. 1, $U = \dfrac{\dot{m}V_e}{k}\left[1 - \left(1 - \dfrac{\dot{m}t}{M_o}\right)^{k/\dot{m}}\right]$, so

$$\frac{dU}{dt} = -\frac{\dot{m}V_e}{k}\left(-\frac{\dot{m}}{M_o}\right)\left(1 - \frac{\dot{m}t}{M_o}\right)^{\frac{k}{\dot{m}} - 1}\left(\frac{k}{\dot{m}}\right) = +\frac{\dot{m}V_e}{M_o}\left(1 - \frac{\dot{m}t}{M_o}\right)^{\frac{k}{\dot{m}} - 1}$$

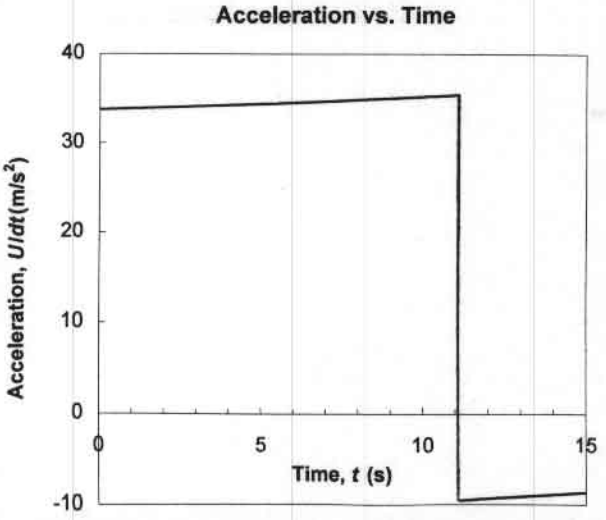
This is maximum when $t = t_{bo}$, when $\left(1 - \dfrac{\dot{m}t}{M_o}\right)^{\frac{k}{\dot{m}} - 1}$ is maximum. Thus at t_b

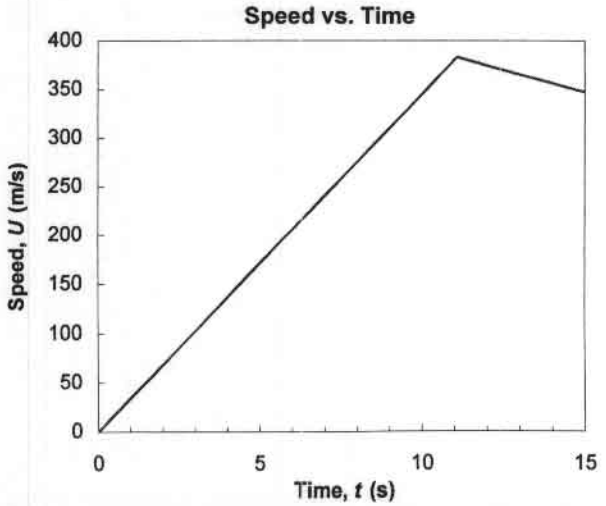
$$\frac{dU}{dt} = \frac{90\,kg}{s} \times \frac{1500\,m}{s} \times \frac{1}{4000\,kg}\left(1 - \frac{90\,kg}{s} \times 11.1\,s \times \frac{1}{4000\,kg}\right)^{75\,\frac{N\cdot s}{m} \times \frac{s}{90\,kg} \times \frac{kg\cdot m}{N\cdot s^2} - 1}$$

(d) $\left(\dfrac{dU}{dt}\right)_{max}$

$$\left(\frac{dU}{dt}\right)_{max} = 35.4\ m/s^2$$

Plotting results:





Open-Ended Problem Statement: Inflate a toy balloon with air and release it. Watch as the balloon darts about the room. Explain what causes the phenomena you see.

Discussion: Air blown into a balloon to inflate it must be compressed to overcome the skin's resistance to stretching. (Remember how hard it is to create enough pressure to "start" the inflation process!) After decreasing briefly, the required pressure seems to increase as inflation of the balloon continues.

As the balloon is inflated, the skin stretches and stores energy. When the inflated balloon is released, the stored energy in the skin forces the compressed air out the open mouth of the balloon. The expansion of the compressed air to the lower surrounding atmospheric pressure creates a high-speed jet of air, which propels the relatively light balloon initially at a high speed.

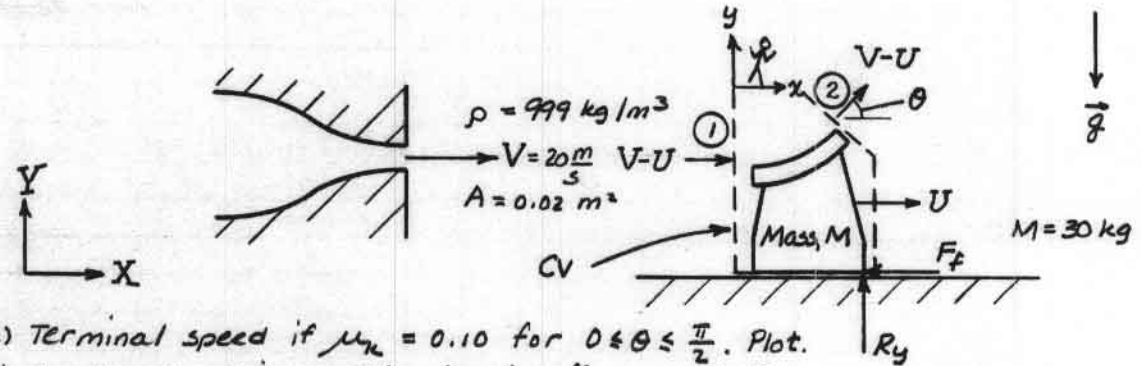
The moving balloon is unstable because it has a poor aerodynamic shape. Therefore it darts about in a random pattern. The balloon keeps moving as long as it contains pressurized air to act as a propulsion jet. However, it is not long before the energy stored in the skin is exhausted and the air in the balloon is reduced to atmospheric pressure.

When the balloon reaches atmospheric pressure it is slowed by aerodynamic drag. Finally the empty, wrinkled balloon simply falls to the floor.

Some toys that use a balloon for propulsion are available. Most have stabilizing surfaces. It is instructive to study these toys carefully to understand how each works, and why each toy is shaped the way it is.

42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 150 SHEETS EYE-EASE® 5 SQUARE
42-392 200 SHEETS RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Vane/cart assembly moving with friction under the influence of a jet, as shown.



Find: (a) Terminal speed if $\mu_k = 0.10$ for $0 \le \theta \le \frac{\pi}{2}$. Plot.
(b) Angle at which motion begins if $\mu_s = 0.15$.

Solution: From dynamics, $F_f \le \mu R_y$, so apply two components of momentum equation. Use the linearly accelerating CV shown.

Basic equations: $F_{Sx} + F_{Bx} - \int_{CV} a_{rfx}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$

($F_{Bx} = 0$ (3); $\frac{\partial}{\partial t}\int_{CV}$ ≈ 0 (2))

$F_{Sy} + F_{By} - \int_{CV} a_{rfy}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\,d\forall + \int_{CS} v_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$

($a_{rfy} = 0$ (3); $\frac{\partial}{\partial t}\int_{CV}$ ≈ 0 (2))

Assumptions: (1) No net pressure forces on assembly; $F_{Sx} = -F_f$, $F_{Sy} = R_y$
(2) Neglect mass of liquid on vane
(3) $a_{rfy} = 0$; $F_{Bx} = 0$
(4) Uniform flow at each section
(5) No change in jet area or speed relative to vane
(6) Incompressible flow

The subscript xyz reminds us to use relative velocities. Then

$$-F_f - M a_{rfx} = u_1\{-|\rho(V-U)A|\} + u_2\{|\rho(V-U)A|\}$$

$$u_1 = V-U \qquad u_2 = (V-U)\cos\theta$$

$$a_{rfx} = \frac{\rho(V-U)^2 A(1-\cos\theta) - F_f}{M} \tag{1}$$

and

$$R_y - Mg = v_1\{-|\rho(V-U)A|\} + v_2\{|\rho(V-U)A|\}$$

$$v_1 = 0 \qquad v_2 = (V-U)\sin\theta$$

$$R_y = Mg + \rho(V-U)^2 A\sin\theta \tag{2}$$

At terminal speed, $a_{rfx} = 0$ and $F_f = \mu_k R_y$. Substituting into Eq. 1,

$$0 = \frac{\rho(V-U_t)^2 A(1-\cos\theta) - \mu_k[Mg + \rho(V-U_t)^2 A\sin\theta]}{M} = \frac{\rho(V-U_t)^2 A(1-\cos\theta-\mu_k\sin\theta)}{M} - \mu_k g$$

or

$$V - U_t = \left[\frac{\mu_k Mg}{\rho A(1-\cos\theta-\mu_k\sin\theta)}\right]^{\frac{1}{2}}$$

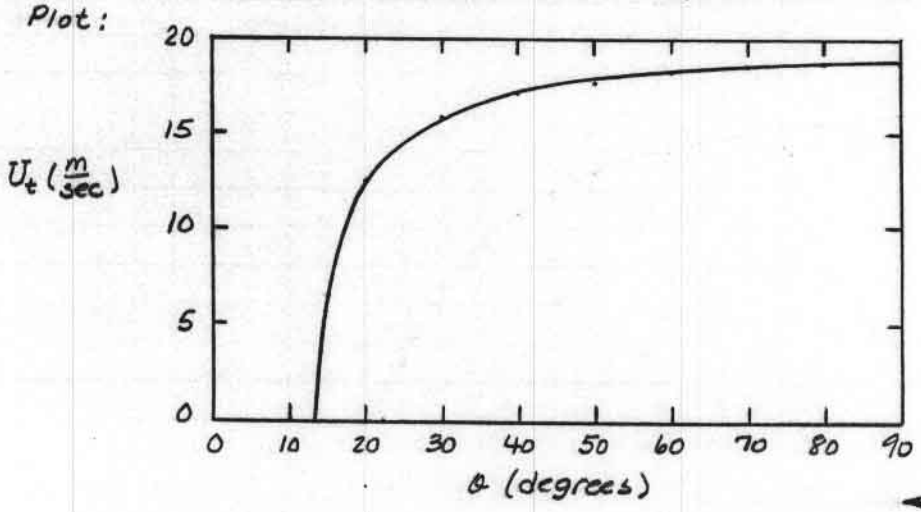
or $U_t = V - \left[\frac{\mu_k Mg}{\rho A(1-\cos\theta-\mu\sin\theta)}\right]^2$; $\frac{U_t}{V} = 1-\left[\frac{\mu_k Mg}{\rho V^2 A(1-\cos\theta-\mu\sin\theta)}\right]^{\frac{1}{2}}$ ← $\frac{U_t}{V}$

Substituting values,

$$U_t \left\{1-\left[(0.10)30\,kg \times 9.81\frac{m}{s^2}\times\frac{m^3}{999\,kg}\times\frac{s^2}{(20)^2 m^2}\times\frac{1}{0.02\,m^2}\frac{1}{(1-\cos\theta-0.1\sin\theta)}\right]^{\frac{1}{2}}\right\} 20\frac{m}{s}$$

$$U_t = 20 - 1.21\left[\frac{1}{1-\cos\theta-0.1\sin\theta}\right]^{\frac{1}{2}}\ \frac{m}{s}$$ ← U_t

Plot:



← Plot

For the static case, $F_f \le \mu_s R_y$. Substituting into Eq. 1, with $U = 0$

$$a_{rf_x} \ge \frac{\rho V^2 A(1-\cos\theta)-\mu_s[\rho V^2 A\sin\theta + Mg]}{M} = \frac{\rho V^2 A(1-\cos\theta-\mu_s\sin\theta)}{M} - g$$

When the assembly is about to move, $F_f = \mu_s R_y$ and $a_{rf_x} = 0$. Thus

$$a_{rf_x} = 0 = \frac{\rho V^2 A(1-\cos\theta-\mu_s\sin\theta)}{M} - \mu_s g$$

or

$$\cos\theta + \mu_s\sin\theta = 1 - \frac{\mu_s Mg}{\rho V^2 A}\ ;\ \cos\theta[1+\mu_s\tan\theta] = 1 - \frac{\mu_s Mg}{\rho V^2 A}$$

But $\frac{\mu_s Mg}{\rho V^2 A} = (0.15)30\,kg \times 9.81\frac{m}{s^2}\times\frac{m^3}{999\,kg}\times\frac{s^2}{(20)^2 m^2}\times\frac{1}{0.02\,m^2} = 0.00552$

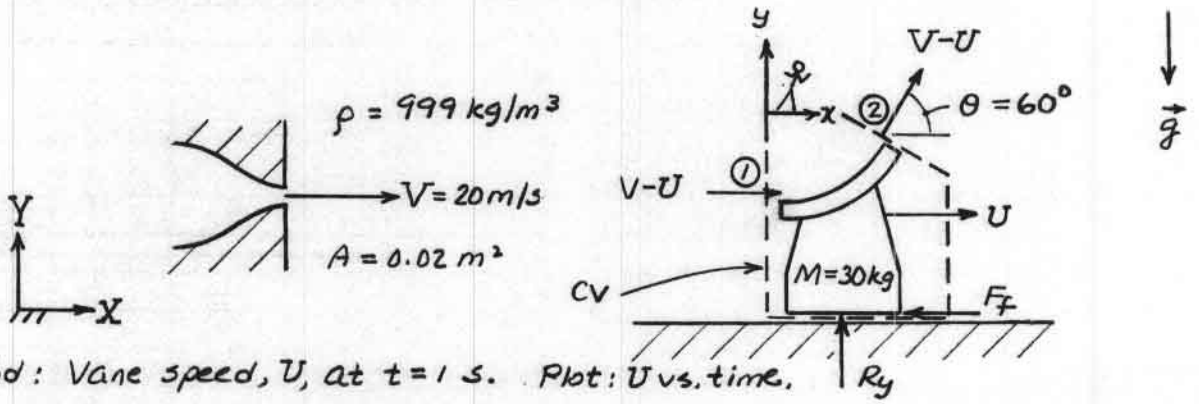
Thus

$$\cos\theta[1+\mu_s\tan\theta] = 1 - 0.00552$$

Solving by iteration, $\theta \simeq 18.9°$ (to start motion) ← θ

{ Thus the assembly would have hysteresis. If θ were varied, it would start moving at $\theta \simeq 18.9°$. Once motion began, it would continue, as θ decreased, to $\theta \simeq 13°$. The reason is because $\mu_k < \mu_s$. }

Given: Vane assembly driven by a water jet, as shown.
Coefficient of kinetic friction, $\mu_k = 0.10$.



Find: Vane speed, U, at $t = 1$ s. Plot: U vs. time.

Solution: From dynamics, $F_f = \mu_k R_y$, so apply x and y components of momentum equation. Use the linearly accelerating CV shown.

Basic equations:

$$F_{Sx} + F_{Bx} - \int_{CV} a_{rf_x}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

($F_{Bx} = 0$; ≈ 0 (3))

$$F_{Sy} + F_{By} - \int_{CV} a_{rf_y}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\,d\forall + \int_{CS} v_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

($a_{rf_y} = 0$ (4); ≈ 0 (3))

Assumptions: (1) No net pressure forces on assembly; $F_{Sx} = -F_f$, $F_{Sy} = R_y$
(2) $F_{Bx} = 0$
(3) Neglect u and v and rates of change within CV
(4) $a_{rf_y} = 0$
(5) Uniform flow at each section
(6) No change in jet speed relative to vane
(7) Incompressible flow; no change in jet area

The subscript xyz reminds us to use relative velocities. From the x equation,

$$-F_f - M a_{rf_x} = u_1\{-|\rho(V-U)A|\} + u_2\{|\rho(V-U)A|\}$$

$$u_1 = V-U \qquad u_2 = (V-U)\cos\theta$$

$$a_{rf_x} = \frac{\rho(V-U)^2 A(1-\cos\theta) - F_f}{M} \tag{1}$$

and from the y equation

$$R_y - Mg = v_1\{-|\rho(V-U)A|\} + v_2\{|\rho(V-U)A|\}$$

$$v_1 = 0 \qquad v_2 = (V-U)\sin\theta$$

$$R_y = Mg + \rho(V-U)^2 A\sin\theta$$

But $F_f = \mu_k R_y = \mu_k\left[Mg + \rho(V-U)^2 A\sin\theta\right]$. Substituting into Eq. 1,

$$a_{rf_x} = \frac{dU}{dt} = -\frac{d(V-U)}{dt} = \frac{\rho(V-U)^2 A(1-\cos\theta-\mu_k\sin\theta)}{M} - \mu_k g \tag{2}$$

Equation 2 is of the form

$$\frac{du}{a-bu^2} = dt, \quad \text{where} \quad a = \mu_k g$$

$$b = \frac{\rho A(1-\cos\theta-\mu_k\sin\theta)}{M}$$

$$u = V-U$$

The limits for integration are:

$t = 0,\ u = V-0 = V$

$t = t,\ u = V-U$

Integrating, using integral tables,

$$\int_V^{V-U} \frac{du}{a-bu^2} = \frac{1}{\sqrt{ab}}\left[\operatorname{ctnh}^{-1}\left(\sqrt{\frac{b}{a}}\,u\right)\right]_V^{V-U} = \frac{1}{\sqrt{ab}}\left\{\operatorname{ctnh}^{-1}\left[\sqrt{\frac{b}{a}}(V-U)\right]-\operatorname{ctnh}^{-1}\left[\sqrt{\frac{b}{a}}(V)\right]\right\} = t$$

Solving, $\sqrt{\frac{b}{a}}(V-U) = \operatorname{ctnh}\left\{\sqrt{ab}\,t + \operatorname{ctnh}^{-1}\left[\sqrt{\frac{b}{a}}V\right]\right\}$

or

$$U = V - \sqrt{\frac{a}{b}}\operatorname{ctnh}\left\{\sqrt{ab}\,t + \operatorname{ctnh}^{-1}\left[\sqrt{\frac{b}{a}}V\right]\right\} \qquad (3)$$

Evaluating, $a = 0.10 \times 9.81\frac{m}{s^2} = 0.981\ m/s^2$

$$b = 999\frac{kg}{m^3} \times 0.02\,m^2\,(1-\cos 60^\circ - (0.10)\sin 60^\circ)\frac{1}{30\,kg} = 0.275\ m^{-1}$$

$$\sqrt{\frac{a}{b}} = \left[0.981\frac{m}{sec^2}\times\frac{m}{0.275}\right]^{\frac{1}{2}} = 1.89\ m/s$$

$$\sqrt{ab} = \left[0.981\frac{m}{sec^2}\times\frac{0.275}{m}\right]^{\frac{1}{2}} = 0.519\ s^{-1}$$

$$\sqrt{\frac{b}{a}} = \frac{s}{1.89\,m} = 0.529\ s/m$$

$\operatorname{ctnh}^{-1}(x) = \tanh^{-1}(1/x)$

$\operatorname{ctnh}(x) = \left(1+\frac{1}{\sinh^2(x)}\right)^{1/2}$

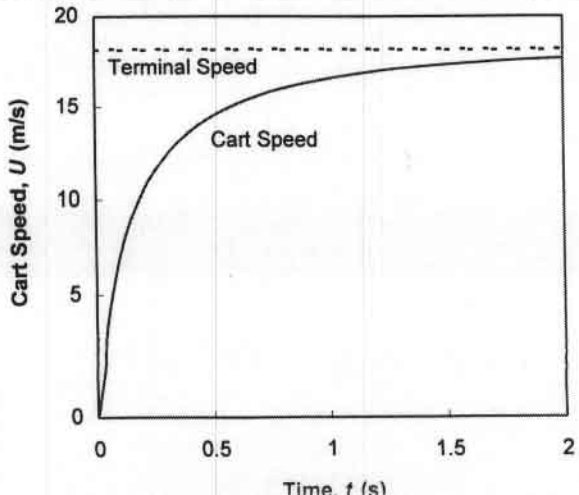
Thus

$$U = 20\frac{m}{s} - 1.89\frac{m}{s}\operatorname{ctnh}\left\{\frac{0.519}{s}\times 1\,s + \operatorname{ctnh}^{-1}\left(0.529\,s\,\frac{}{m}\times\frac{20\,m}{s}\right)\right\}$$

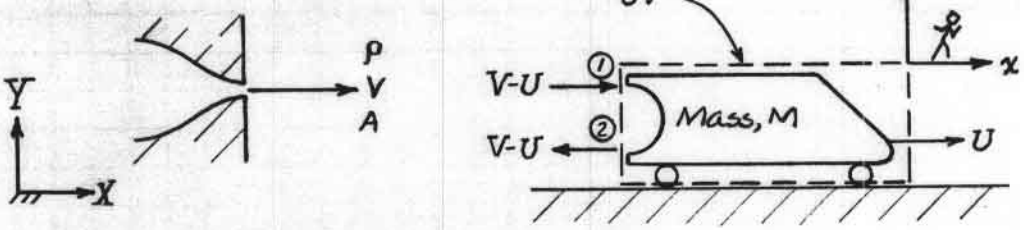
$$U = 16.5\ m/s \quad (\text{at } t = 1\ s)$$

U

The plot is:



Given: Vehicle accelerated from rest by a hydraulic catapult. Neglect resistance.



Find: (a) Expression for acceleration at any time, t.
(b) Time required to reach $U = V/2$.

Solution: Apply x component of momentum equation using linearly accelerating CV shown above.

Basic equation: $F_{Sx} + F_{Bx} - \int_{CV} a_{rf_x}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$ ($F_{Sx}=0$ (1), $F_{Bx}=0$ (2), $\simeq 0$ (3))

Assumptions: (1) $F_{Sx}=0$
(2) $F_{Bx}=0$
(3) Neglect mass of liquid and rate of change of u in CV
(4) Uniform flow at each section
(5) Jet area and speed with respect to vehicle are constant

Then

$$-Ma_{rf_x} = -M\frac{dU}{dt} = u_1\{-|\rho(V-U)A|\} + u_2\{|\rho(V-U)A|\}$$

$$u_1 = V-U \qquad u_2 = -(V-U)$$

or

$$a_{rf_x} = \frac{dU}{dt} = \frac{2\rho(V-U)^2A}{M} \quad ; \quad \frac{dU}{(V-U)^2} = \frac{2\rho A}{M}dt \quad ; -\frac{d(V-U)}{(V-U)^2} = \frac{2\rho A}{M}dt$$

To obtain $a_{rf_x}(t)$, we must first find $U(t)$. Integrating from $U=0$ at $t=0$ to U at t,

$$\int_{V-U=V}^{V-U} -\frac{d(V-U)}{(V-U)^2} = \frac{1}{V-U}\Big]_V^{V-U} = \frac{1}{V-U} - \frac{1}{V} = \frac{V-(V-U)}{V(V-U)} = \frac{2\rho A}{M}t \quad ; \quad \frac{U}{V-U} = \frac{2\rho VA}{M}t$$

Solving,

$$U = (V-U)\frac{2\rho VA}{M}t \quad , \quad U = V\frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t} \quad \text{and} \quad V-U = V\left[1-\frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t}\right]$$

Substituting,

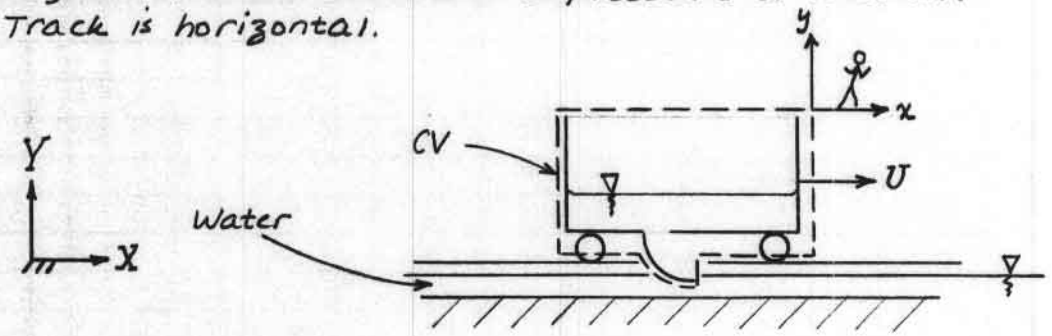
$$a_{rf_x} = \frac{2\rho V^2A}{M}\left[1-\frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t}\right]^2 = \frac{2\rho V^2A}{M}\left[\frac{1}{1+\frac{2\rho VA}{M}t}\right]^2 \qquad a_{rf_x}(t)$$

The time to reach $U = V/2$ is

$$\frac{U}{V} = \frac{1}{2} = \frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t} \quad \text{or} \quad t = \frac{M}{2\rho VA} \qquad t\left(\frac{V}{2}\right)$$

Check: $\left[\frac{M}{\rho VA}\right] = M\frac{L^3}{M}\frac{t}{L}\frac{1}{L^2} = t$ ✓ ; $\left[\frac{\rho V^2A}{M}\right] = \frac{M}{L^3}\frac{L^2}{t^2}L^2\frac{1}{M} = \frac{L}{t^2}$ ✓

Given: Moving tank slowed by lowering scoop into water trough.
Initial mass and speed are M_0 and U_0, respectively.
Neglect external forces due to pressure or friction.
Track is horizontal.



Find: (a) Apply continuity and momentum to show $U = U_0 M_0 / M$.
(b) Obtain a general expression for $U(t)$.

Solution: Apply continuity and momentum equations to linearly accelerating CV shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A}$

$$F_{Sx} + F_{Bx} - \int_{CV} a_{rfx}\, \rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\, \rho\, d\forall + \int_{CS} u_{xyz}\, \rho \vec{V}_{xyz} \cdot d\vec{A}$$

$F_{Sx} = 0$ (1); $F_{Bx} = 0$ (2); $\int_{CV} u_{xyz}\, \rho\, d\forall \simeq 0$ (3)

Assumptions: (1) $F_{Sx} = 0$
(2) $F_{Bx} = 0$
(3) Neglect u within CV
(4) Uniform flow across inlet section

From continuity

$$0 = \frac{\partial}{\partial t} M_{CV} + \{-|\rho U A|\} \quad \text{or} \quad \frac{dM}{dt} = \rho U A$$

From momentum

$$-a_{rfx} M = -\frac{dU}{dt} M = u\{-|\rho U A|\} = U \rho U A, \text{ since } u = -U$$

But from continuity, $\rho U A = \frac{dM}{dt}$, so

$$M\frac{dU}{dt} + U\frac{dM}{dt} = 0 \quad \text{or} \quad UM = \text{constant} = U_0 M_0 ; \quad U = U_0 M_0 / M$$ U

Substituting $M = M_0 U_0 / U$ into momentum, $-\frac{dU}{dt}\frac{M_0 U_0}{U} = \rho U^2 A$, or

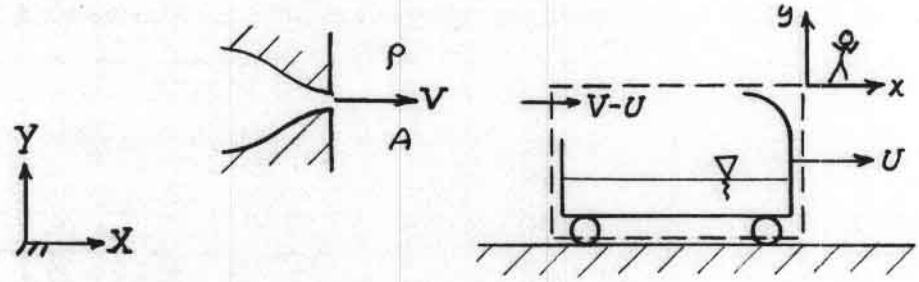
$$\frac{dU}{U^3} = -\frac{\rho A}{U_0 M_0}\, dt$$

Integrating, $\int_{U_0}^{U} \frac{dU}{U^3} = -\frac{1}{2}\frac{1}{U^2}\Big]_{U_0}^{U} = -\frac{1}{2}\left(\frac{1}{U^2} - \frac{1}{U_0^2}\right) = -\int_0^t \frac{\rho A}{U_0 M_0}\, dt = -\frac{\rho A}{U_0 M_0} t$

Solving for U,

$$U = \frac{U_0}{\left[1 + \frac{2\rho U_0 A}{M_0} t\right]^{\frac{1}{2}}}$$ U(t)

Given: Tank driven by jet along horizontal track. Neglect resistance. Acceleration is from rest. Initial mass is M_0. Track horizontal.



Find: (a) Apply continuity and momentum to show $M = M_0V/(V-U)$
(b) General expression for U/V as a function of time.

Solution: Apply continuity and x component of momentum equation to linearly accelerating CV shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho\, \vec{V}_{xyz}\cdot d\vec{A}$

$$F_{Sx} + F_{Bx} - \int_{CV} a_{rfx}\,\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

$F_{Sx} = 0$ (1), $F_{Bx} = 0$ (2), $\simeq 0$ (3)

Assumptions: (1) $F_{Sx} = 0$
(2) $F_{Bx} = 0$
(3) Neglect u within CV
(4) Uniform flow in jet

From continuity

$$0 = \frac{\partial}{\partial t} M_{CV} + \{-|\rho(V-U)A|\} \quad \text{or} \quad \frac{dM}{dt} = \rho(V-U)A$$

From momentum

$$-a_{rfx} M = -\frac{dU}{dt} M = u\{-|\rho(V-U)A|\} = (V-U)[-\rho(V-U)A]\,;\ u = V-U$$

But from continuity, $\rho(V-U)A = \frac{dM}{dt}$, and $dU = -d(V-U)$, so

$$-\frac{dU}{dt} M = \frac{d(V-U)}{dt} M = -(V-U)\frac{dM}{dt} \quad \text{or} \quad M(V-U) = \text{constant} = M_0V$$

Thus $M = M_0V/(V-U)$ ← M

Substituting into momentum, $-\frac{dU}{dt} M = \frac{d(V-U)}{dt}\frac{M_0V}{(V-U)} = -\rho(V-U)^2A$, or

$$\frac{d(V-U)}{(V-U)^3} = -\frac{\rho A}{VM_0}\,dt$$

Integrating, $\int_V^{V-U} \frac{d(V-U)}{(V-U)^3} = -\frac{1}{2}\left[\frac{1}{(V-U)^2} - \frac{1}{V^2}\right] = -\int_0^t \frac{\rho A}{VM_0}\,dt = -\frac{\rho A}{VM_0}\,t$

Solving,

$$\frac{U}{V} = \left\{1 - \frac{1}{\left[1 + \frac{2\rho VA}{M_0}t\right]^{1/2}}\right\}$$ ← U/V

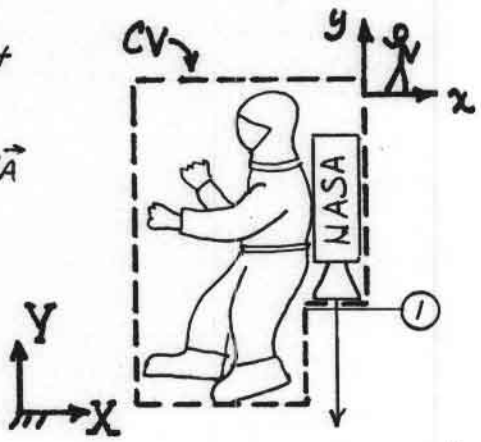
Given: Small rocket "jet pack" used to lift astronaut above Earth. Exhaust jet speed is constant but mass flow rate varies.

Find: (a) Algebraic expression for mass flow rate needed to hover.
(b) Maximum hover time.

Solution: Apply continuity and momentum using CV & CS shown.

Basic equation: $F_{By} + F_{By} - \int_{CV} a_{rf_y}\,\rho\,d\forall$ (=0(1), =0(2))

$$= \frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\,d\forall + \int_{CS} v_{xyz}\,\rho\,\vec{V}\cdot d\vec{A}$$ (≈0(3))



Assumptions: (1) Hover; $F_{sy} = 0$
(2) $a_{rf_y} = 0$
(3) Neglect $\partial/\partial t$ in CV
(4) Uniform flow exhaust

Then
$$-Mg = v_1\{+\dot{m}\}$$
$$v_1 = -V_e$$
$$-Mg = -V_e\,\dot{m}$$

$V_e = 2940$ m/s
$M_0 = 130$ kg
$M_f = 40$ kg

so
$$\dot{m} = \frac{Mg}{V_e}$$ $\dot{m}(t)$

From conservation of mass, $0 = \frac{\partial}{\partial t}\int_{CV}\rho\,d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} = \frac{dM}{dt} + \dot{m}$

so $\frac{dM}{dt} = -\dot{m} = -\frac{Mg}{V_e}$ or $\frac{dM}{M} = -\frac{g}{V_e}dt$

Integrating from M_0 at $t = 0$ to $M_0 - M_f$ at t,

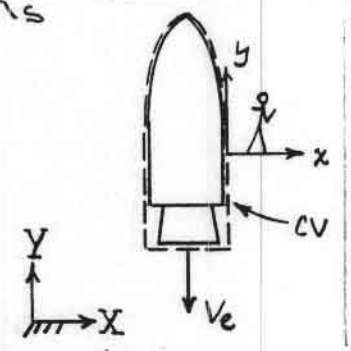
$$\int_{M_0}^{M_0-M_f}\frac{dM}{M} = \ln M\Big]_{M_0}^{M_0-M_f} = \ln\left(\frac{M_0-M_f}{M_0}\right) = \ln\left(1-\frac{M_f}{M_0}\right) = -\frac{gt}{V_e}$$

Solving for t,

$$t = -\frac{V_e}{g}\ln\left(1-\frac{M_f}{M_0}\right) = -\frac{2940\,\frac{m}{s}}{9.81\,m}\times s^2\,\ln\left(1-\frac{40\,kg}{130\,kg}\right)$$

$t = 110$ s (hover time) t

Given: Model solid propellant rocket: $M_0 = 69.6\,g$, $M_f = 12.5\,g$
Thrust, $F_t = 1.3\,lbf$; burntime, $t_b = 1.7\,s$



Find: Maximum speed and height (neglecting drag)

Plot: speed and distance traveled as functions of time

Solution

Apply the y momentum equation to analyze motion

Basic equation: $F_{sy} + F_{by} = -\int_{CV} a_{rel}\,\rho\,d\forall - \frac{\partial}{\partial t}\int_{CV} v_{rel}\,\rho\,d\forall + \int_{CS} v_{rel}(\rho\,\vec{V}\cdot d\vec{A})$ (2) (2)

Assumptions: (1) neglect pressure forces and aerodynamic drag
(2) neglect rate of change of momentum inside CV
(3) uniform flow from CV

Thrust is produced by momentum flux from CV

$R_y = -F_t = V_e \dot{m}$. Since $V_e = -v_e$, then $-F_t = -v_e \dot{m}$

$$v_e = \frac{F_t}{\dot{m}} = 1.3\,lbf \times \frac{1.7\,s}{12.5\,g} \times 4.448\,N \times \frac{kg\cdot m}{s^2} \times \frac{10^3\,g}{kg} = 786\,m/s$$

From conservation of mass, $M = M_0 - \dot{m}t$. Then from momentum

$$-Mg - M a_{rel_y} = -v_e \dot{m} \quad \text{or} \quad a_{rel_y} = \frac{dV}{dt} = \frac{v_e \dot{m}}{M_0 - \dot{m}t} - g$$

$$\frac{dV}{dt} = \frac{v_e \dot{m}}{M_0 - \dot{m}t} - g \quad \text{or} \quad dV = \left(\frac{v_e \dot{m}}{M_0 - \dot{m}t} - g\right) dt$$

At $t = t_b$, $\dot{m} = 0$. Max velocity occurs at $t = t_b = 1.7\,s$

Integrating between $V = 0$ at $t = 0$ and V at $t = t_b$

$$V = v_e \ln\left[M_0/(M_0 - \dot{m}t)\right] - gt \qquad \text{for } 0 \le t \le t_b \quad \text{---(1)}$$

Evaluating at $t = t_b = 1.7\,s$ with $\dot{m} = M_f/t_b = 7.35 \times 10^{-3}\,kg/s$

$$V_{max} = 786\,\frac{m}{s} \ln\left[\frac{69.6\,g}{(69.6 - 12.5)\,g}\right] - 9.81\,\frac{m}{s^2} \times 1.7\,s = 139\,m/s \leftarrow V_{max}$$

To obtain $Y = Y(t)$, set $V = \frac{dY}{dt}$ in Eq. 1, separate variables and integrate from $Y = 0$ at $t = 0$ to Y at $t = t_b$.

$$Y = \frac{v_e M_0}{\dot{m}}\left\{\left(1 - \frac{\dot{m}t}{M_0}\right)\left[\ln\left(1 - \frac{\dot{m}t}{M_0}\right) - 1\right] + 1\right\} - \frac{1}{2}gt^2 \qquad \text{for } 0 \le t \le t_b \quad \text{---(2)}$$

Evaluating at $t = t_b = 1.7\,s$

$$Y = 786\,\frac{m}{s} \times 69.6\,g \times \frac{s}{7.35\,g}\left\{\left(1 - \frac{12.5}{69.6}\right)\left[\ln\left(1 - \frac{12.5}{69.6}\right) - 1\right] + 1\right\} - \frac{1}{2} \times 9.81\,\frac{m}{s^2} \times (1.7)^2\,s^2$$

$$Y_{t=t_b} = 114\,m$$

After burnout the rocket will "coast" until its kinetic energy is converted to potential energy. In the absence of aerodynamic drag

$$mgy_b + \tfrac{1}{2}mV_b^2 = mgy_{max} \quad \therefore \quad y_{max} = y_b + \frac{V_{max}^2}{2g}$$

$$y_{max} = 114\,m + \frac{1}{2}\cdot\frac{(139)^2\,m^2/s^2}{9.81\,m/s^2} = 1100\,m$$



After burnout the rocket travels at $a_y = \text{const} = -g$.

$$\therefore \int_{V_{max}}^{V} dV = \int_{t_b}^{t} -g\,dt \qquad V = V_{max} - g(t - t_b)$$

$$V = 0 \text{ @ } t = \frac{V_{max}}{g} + t_b = 15.9\,s$$

Since

$$V = \frac{dy}{dt} = V_{max} - g(t - t_b), \text{ then } \int_{y_b}^{y} dy = \int_{t_b}^{t} \{V_{max} - g(t - t_b)\}\,dt$$

$$y = y_b + V_{max}(t - t_b) - \tfrac{1}{2}g(t - t_b)^2 \qquad \text{for } 1.7 \le t \le 15.9\,s.$$

Summarizing

$$V = -V_e \ln\left[\frac{m_0}{m_0 - \dot{m}t}\right] - gt \qquad \text{for } 0 \le t \le t_b = 1.7\,s$$

$$V = V_{max} - g(t - t_b) \qquad \text{for } t_b \le t \le 15.9\,s$$

$$y = \frac{V_e m_0}{\dot{m}}\left\{\left(1 - \frac{\dot{m}t}{m_0}\right)\left[\ln\left(1 - \frac{\dot{m}t}{m_0}\right) - 1\right] + 1\right\} - \tfrac{1}{2}gt^2 \qquad \text{for } 0 \le t \le t_b = 1.7\,s$$

$$y = y_b + V_{max}(t - t_b) - \tfrac{1}{2}g(t - t_b)^2 \qquad \text{for } t_b \le t \le 15.9\,s.$$

Acceleration, Speed, and Height for Model Solid-Propellant Rocket:

Input Parameters:

F_t =	1.3	lbf	Thrust
g =	9.81	m/s^2	
m_f =	12.5	g	Fuel mass
M_0 =	69.6	g	Initial mass
t_b =	1.7	s	Burn time

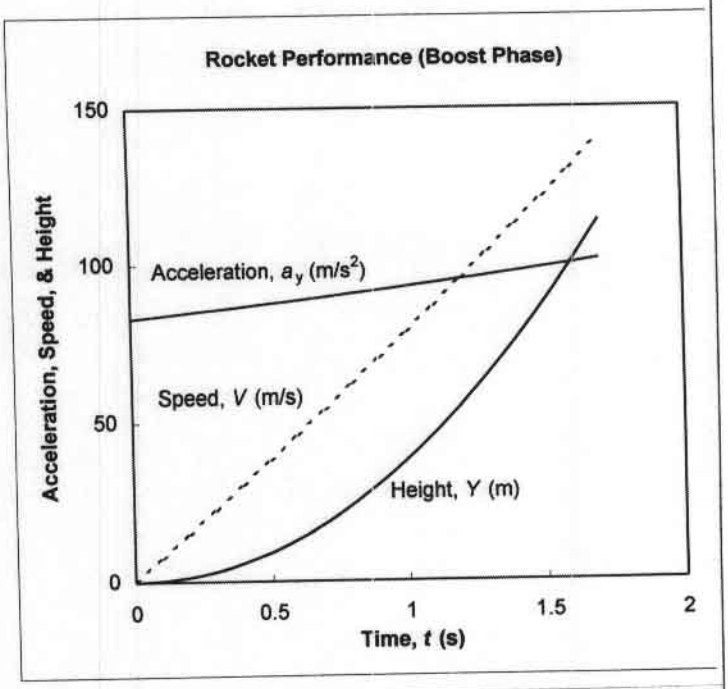
Calculated Values:

m(dot) =	0.00735	kg/s	Mass flow rate
V_e =	787	m/s	Exhaust gas velocity

13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U. S. A.
National® Brand

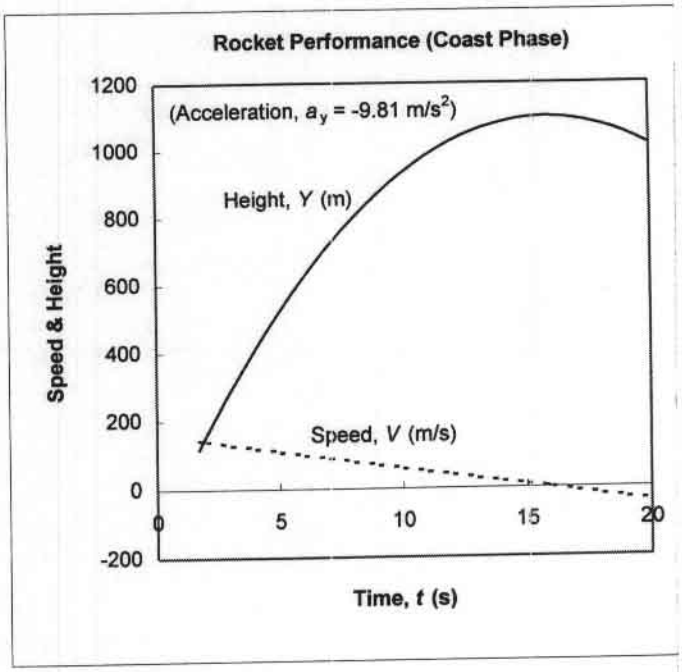
Calculated Results (Boost Phase):

Time, t (s)	[ ]	Accel, a_y (m/s^2)	Speed, V (m/s)	Height, Y (m)
0.0	1.00	83.1	0	0
0.1	0.99	84.0	7.37	0.368
0.2	0.98	84.9	14.8	1.48
0.3	0.97	85.8	22.4	3.34
0.4	0.96	86.8	30.0	5.96
0.5	0.95	87.7	37.8	9.35
0.6	0.94	88.7	45.6	13.5
0.7	0.93	89.8	53.6	18.5
0.8	0.92	90.8	61.6	24.2
0.9	0.90	91.8	69.8	30.8
1.0	0.89	92.9	78.0	38.2
1.1	0.88	94.0	86.4	46.4
1.2	0.87	95.2	94.9	55.5
1.3	0.86	96.3	103	65.4
1.4	0.85	97.5	112	76.2
1.5	0.84	98.8	121	87.8
1.6	0.83	100	130	100
1.7	0.82	101	139	114



Calculated Results (Coast Phase):

Time, t (s)	Accel, a_y (m/s^2)	Speed, V (m/s)	Height, Y (m)
1.7	-9.81	139	114
2	-9.81	136	155
3	-9.81	126	286
4	-9.81	117	408
5	-9.81	107	519
6	-9.81	97	621
7	-9.81	87	713
8	-9.81	77	795
9	-9.81	67	868
10	-9.81	58	930
11	-9.81	48	983
12	-9.81	38	1026
13	-9.81	28	1059
14	-9.81	18	1082
15	-9.81	9	1096
15.88	-9.81	0.000	1100
16	-9.81	-1	1099
17	-9.81	-11	1093
18	-9.81	-21	1077
19	-9.81	-31	1052
20	-9.81	-40	1016



Open-Ended Problem Statement: Several toy manufacturers sell water "rockets" that consist of plastic tanks to be partially filled with water and then pressurized with air. Upon release, the compressed air forces water out the nozzle rapidly, propelling the rocket. You are asked to help specify optimum conditions for this water-jet propulsion system. To simplify the analysis, consider horizontal motion only. Perform the analysis and design needed to define the acceleration performance of the compressed air/water-propelled rocket. Identify the fraction of tank volume that initially should be filled with compressed air to achieve optimum performance (i.e., maximum speed from the water charge). Describe the effect of varying the initial air pressure in the tank.

Discussion: The process may be modeled as a polytropic expansion of the trapped air which forces water out the jet nozzle, causing the "rocket" to accelerate. The polytropic exponent may be varied to model anything from an isothermal expansion process ($n = 1$) to an adiabatic expansion process ($n = k$), which is more likely to be an accurate model for the sudden expansion of the air.

Speed of the water jet leaving the "rocket" is proportional to the square root of the pressure difference between the tank and atmosphere.

Qualitatively it is apparent that the smaller the initial volume fraction of trapped air, the larger will be the expansion ratio of the air, and the more rapid will be the pressure reduction as the air expands. This will cause the water jet speed to drop rapidly. The combination of low water jet speed and relatively large mass of water will produce sluggish acceleration.

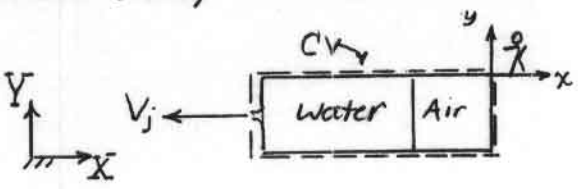
Increasing the initial volume fraction of air will reduce the expansion ratio, so higher pressure will be maintained longer in the tank and the water jet will maintain higher speed longer. This combined with the relatively small mass of water in the tank will produce rapid acceleration.

If the initial volume fraction of air is too large, all water will be expended before the air pressure is reduced significantly. In this situation, some of the stored energy of the air will be dissipated in a relatively ineffective air jet. Consequently, for any initial pressure in the tank, there is an optimum initial air fraction.

This problem cannot be solved in closed form because of the varying air pressure, mass flow rate, and mass of water in the tank;. it can only be solved numerically. One possible integration scheme is to increment time and solve for all properties of the system at each instant. The drawback to this scheme is that the water is unlikely to be exhausted at an even increment of time. A second scheme is to increment the volume of water remaining and solve for properties using the average flow rate during the interval. This scheme is outlined below.

Model the air/water jet-propelled "rocket" using the CV and coordinates shown.

First choose dimensions and mass of "rocket" to be simulated:



Input Data:

Jet diameter:	D_j =	0.003	m
Tank diameter:	D_t =	0.035	m
Tank length:	L =	0.1	m
Tank mass:	M_t =	0.01	kg
Polytropic exponent:	n =	1.4	---

Next choose initial conditions for the simulation (see sample calculations below):

Initial Conditions:

Air fraction in tank:	$\alpha =$	0.5	---
Tank pressure:	$p_0 =$	200	kPa (gage)
Volume increment:	$\Delta\alpha =$	0.02	---

Compute reference parameters:

Calculated Parameters:

Jet area:	$A_j =$	7.07E-06	m^2
Tank volume:	$\forall_t =$	9.62E-05	m^3
Initial air volume:	$\forall_0 =$	4.81E-05	m^3
Initial water mass:	$M_0 =$	0.0481	kg

(These are used in the spreadsheet below.)

Then decrease the water fraction in the tank by $\Delta\alpha$:

Calculated Results:

Water Fraction, $\forall_w/\forall_t$ (---)	Gage Pressure, p (kPa)	Water Mass, M_w (kg)	Jet Speed, V_j (m/s)	Flow Rate, dm/dt (kg/s)	Time Interval, Δt (s)	Current Time, t (s)	"Rocket" Accel., a (m/s^2)	"Rocket" Speed, U (m/s)
0.50	200	0.0481	20.0	0.141	0	0	48.7	0
0.48	184	0.0461	19.2	0.135	0.0139	0.0139	47.5	0.668

The computation is made as follows:

(1) Decrease α by $\Delta\alpha$

(2) Compute p from $p = p_0\left(\frac{\forall_0}{\forall}\right)^n$

$$p = (200+101.325)\ kPa\left(\frac{0.50}{0.52}\right)^{1.4} - 101.325 = 183.9\ kPa\ (gage)$$

(3) Use Bernoulli to calculate jet speed

$$V_j = \sqrt{\frac{2\Delta p}{\rho}} = \left[2 \times 183.9\times10^3 \frac{N}{m^2} \times \frac{m^3}{999\ kg} \times \frac{kg\cdot m}{N\cdot s^2}\right]^{\frac{1}{2}} = 19.10\ m/s^*$$

(4) Calculate water mass using α.

(5) Use conservation of mass to compute mass flow rate

$$\dot m = \rho V_j A_j = 999\frac{kg}{m^3} \times 19.10\frac{m}{s} \times 7.07\times10^{-6} m^2 = 0.1349\ kg/s$$

(6) Use the average mass flow rate during the interval to approximate Δt:

$$\Delta t = \frac{\Delta m}{dm/dt} \approx \frac{\Delta m}{\bar{\dot m}} = (0.0481-0.0461)kg \times \frac{s}{0.138\ kg} = 0.01449\ s^*$$

(7) Use momentum to compute acceleration (note $M = M_w + M_t$):

$$a_{rf_x} = \frac{\dot m V_j}{M} = 0.135\frac{kg}{s} \times 19.2\ \frac{m}{s} \times \frac{1}{0.0461+0.0100\ kg} = 46.2\ m/s^{2*}$$

(8) Finally: use average acceleration to get speed

$$U = U_0 + \bar a \Delta t = 0 + 48.1\frac{m}{s^2} \times 0.0139\ s = 0.669\ m/s^*$$

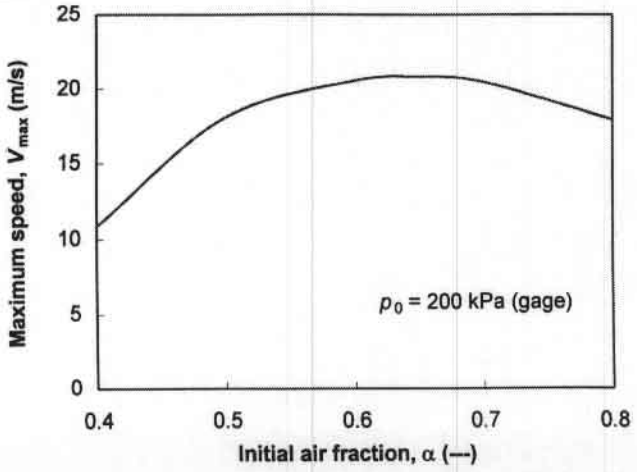
*Note effect of roundoff error.

Repeat these calculations until water is depleted or air pressure falls to zero, as shown below:

Water Fraction, V_w/V_t (---)	Gage Pressure, p (kPa)	Water Mass, M_w (kg)	Jet Speed, V_j (m/s)	Flow Rate, dm/dt (kg/s)	Time Interval, Δt (s)	Current Time, t (s)	"Rocket" Accel., a (m/s^2)	"Rocket" Speed, U (m/s)
0.50	200	0.0481	20.0	0.141	0	0	48.7	0
0.48	184	0.0461	19.2	0.135	0.0139	0.0139	47.5	0.668
0.46	169	0.0442	18.4	0.130	0.0145	0.0284	45.2	1.34
0.44	156	0.0423	17.7	0.125	0.0151	0.0435	43.1	2.01
0.42	143	0.0404	16.9	0.120	0.0157	0.0592	41.2	2.67
0.40	132	0.0384	16.3	0.115	0.0164	0.0756	39.4	3.33
0.38	122	0.0365	15.6	0.110	0.0171	0.0927	37.8	3.99
0.36	112	0.0346	15.0	0.106	0.0178	0.110	36.2	4.65
0.34	103	0.0327	14.4	0.101	0.0186	0.129	34.8	5.31
0.32	94.6	0.0308	13.8	0.0972	0.0194	0.148	33.5	5.97
0.30	86.8	0.0288	13.2	0.0931	0.0202	0.169	32.2	6.63
0.28	79.5	0.0269	12.6	0.0891	0.0211	0.190	31.0	7.30
0.26	72.7	0.0250	12.1	0.0852	0.0221	0.212	29.9	7.97
0.24	66.3	0.0231	11.5	0.0814	0.0231	0.235	28.9	8.65
0.22	60.4	0.0211	11.0	0.0776	0.0242	0.259	27.9	9.34
0.20	54.7	0.0192	10.5	0.0739	0.0254	0.284	26.9	10.0
0.18	49.4	0.0173	9.95	0.0702	0.0267	0.311	26.0	10.7
0.16	44.4	0.0154	9.43	0.0666	0.0281	0.339	25.2	11.5
0.14	39.7	0.0135	8.92	0.0630	0.0297	0.369	24.3	12.2
0.12	35.2	0.0115	8.40	0.0593	0.0314	0.400	23.5	12.9
0.10	31.0	0.00961	7.88	0.0556	0.0334	0.434	22.7	13.7
0.08	27.0	0.00769	7.35	0.0519	0.0357	0.469	22.0	14.5
0.06	23.2	0.00577	6.81	0.0481	0.0384	0.508	21.2	15.3
0.04	19.6	0.00384	6.26	0.0442	0.0416	0.550	20.4	16.2
0.02	16.1	0.00192	5.68	0.0401	0.0456	0.595	19.5	17.1
0.00	12.9	0.0000	5.07	0.0358	0.0506	0.646	18.6	18.1

In this simulation, the water is depleted when $t \approx 0.65$ s; $V_{max} = 18.1$ m/s.

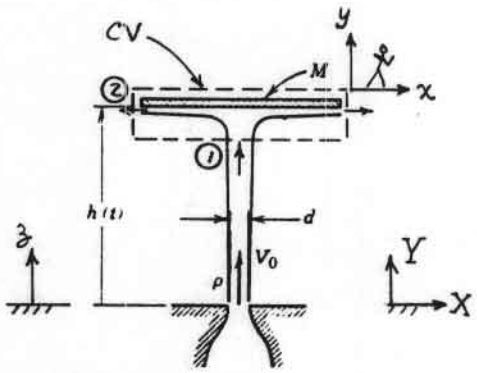
Varying the initial air fraction produces the following:



For this combination of parameters, a peak speed of about 20.8 m/s is attained with an initial air fraction of about 0.66.

Given: Vertical jet impinging on disk.

Disk is unconstrained vertically.



Find: (a) Differential equation for h(t), if disk released from $H > h_0$, where h_0 is equilibrium height.
(b) Sketch h(t) and explain.

Solution: Apply Bernoulli equation to jet, then y momentum equation to CV with linear acceleration.

Basic equations:

$$\frac{p_0}{\rho} + \frac{V_0^2}{2} + g z_0 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 \qquad (p_0 = p_1 = 0,\ z_0 = 0)$$

$$F_{sy} + F_{By} - \int_{CV} a_{rfy}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\,d\forall + \int_{CS} v_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A} \qquad (F_{sy} = 0,\ \approx 0)$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) No friction
(4) Flow along a streamline
} in jet

(5) $p_1 = p_0 = p_{atm}$

(6) No pressure force on CV, so $F_{sy} = 0$
(7) Neglect mass of liquid in CV and $v \approx 0$ in CV
(8) Uniform flow at each section
(9) Measure velocities relative to CV

From momentum

$$-(M + m_w)g - a_{rfy}(M + m_w) = v_1\{-|\rho(V_1 - U)A_1|\} + v_2\{\dot{m}_2\} \qquad (m_w \approx 0,\ a_{rfy} \approx 0)$$

$$v_1 = V_1 - U \qquad v_2 \approx 0$$

With $a_{rfy} = \frac{d^2h}{dt^2}$, $U = \frac{dh}{dt}$, then

$$-Mg - M\frac{d^2h}{dt^2} = -\rho\left(V_1 - \frac{dh}{dt}\right)^2 A_1$$

But from Bernoulli, $\frac{V_1^2}{2} = \frac{V_0^2}{2} - g z_1$, so $V_1 = \sqrt{V_0^2 - 2gh}$, since $z_1 = h(t)$

Also from continuity, $V_1 A_1 = V_0 A_0$, so $A_1 = A_0 V_0 / V_1$. Substituting

$$\frac{d^2h}{dt^2} = \rho\left(\sqrt{V_0^2 - 2gh} - \frac{dh}{dt}\right)^2 \frac{A_0 V_0}{M\sqrt{V_0^2 - 2gh}} - g \qquad \ddot{h}(t)$$

At equilibrium height, $h = h_0$, $\frac{dh}{dt} = 0$, and $\frac{d^2h}{dt^2} = 0$. Then

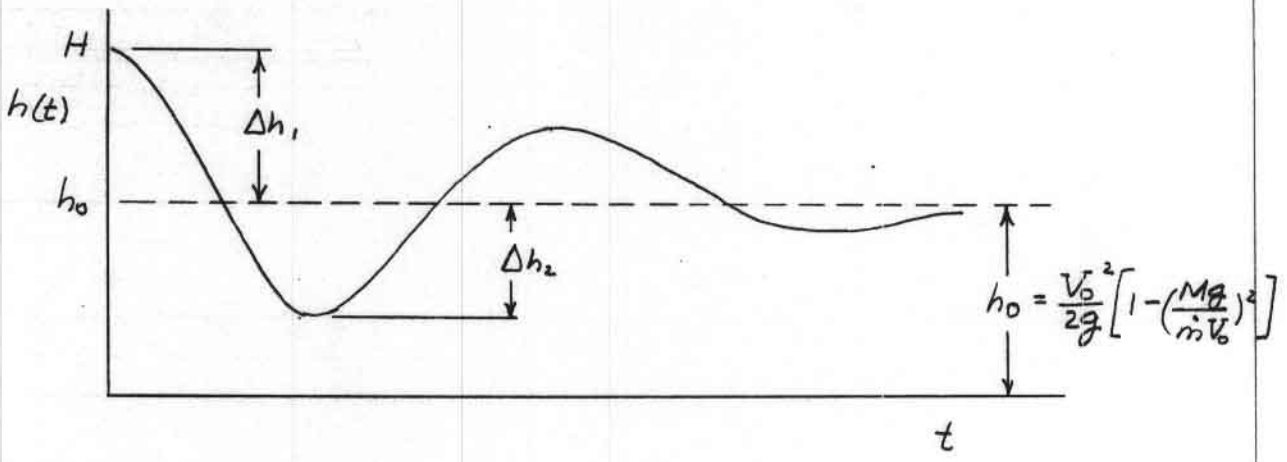
$$\rho\sqrt{V_0^2 - 2gh_0}\,A_0 V_0 - Mg = 0$$

Thus $V_0^2 - 2gh_0 = \left(\frac{Mg}{\rho V_0 A_0}\right)^2$

This may be solved to obtain

$$h_0 = \frac{V_0^2}{2g}\left[1-\left(\frac{Mg}{\rho V_0^2 A_0}\right)^2\right] = \frac{V_0^2}{2g}\left[1-\left(\frac{Mg}{\dot{m}V_0}\right)^2\right]$$

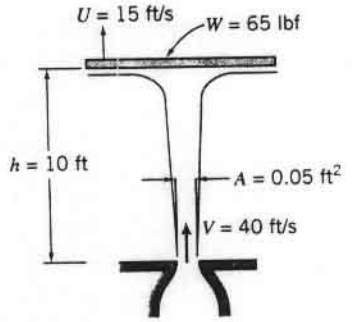
When released, $H > h_0$, and $dh/dt = 0$. Because the equation for d^2h/dt^2 is nonlinear, oscillations will occur. The expected behavior is sketched below:



Notes: (1) Expect oscillations
(2) $\Delta h_3 < \Delta h_2 < \Delta h_1$ due to nonlinear equation

Given: Configuration of vertical jet striking horizontal disc shown in P*4.136.

Assume disk released from rest at $h = 10$ ft above nozzle exit plane.



Find: (a) Solve for subsequent motion of disc.
(b) Steady-state height.

Solution: As in Problem 4.159, apply Bernoulli to jet, then y-momentum to CV with linear acceleration.

Basic equations: $\frac{p_0}{\rho} + \frac{V_0^2}{2} + g\cancel{z_0}^{=0} = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1$,

$$\cancel{F_{sy}}^{=0\,(6)} + F_{By} - \int_{CV} a_{rfy}\rho\, d\forall = \cancel{\frac{\partial}{\partial t}}^{\approx 0\,(7)} \int_{CV} v_{xyz}\rho\, d\forall + \int_{CS} v_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) Steady
(2) Incompressible flow
(3) No friction
(4) Flow along a streamline } in jet
(5) $p_1 = p_0 = p_{atm}$
(6) No pressure force on CV, so $F_{By} = 0$
(7) Neglect mass of water and $v \approx 0$ in CV
(8) Uniform flow at each cross-section
(9) Measure velocities relative to CV

From momentum,

$$-(M + \cancel{m_{(w)}}^{\approx 0(7)})g - a_{rfy}(M + \cancel{m_{(w)}}^{\approx 0(7)}) = v_1\{-\rho(V_1 - U)A_1\} + \cancel{v_2}^{=0}\{\dot{m}_2\}$$

With $a_{rfy} = \frac{d^2h}{dt^2}$ and $v = \frac{dh}{dt}$, then $v_1 = V_1 - U$ $\quad v_2 = 0$

$$-Mg - M\frac{d^2h}{dt^2} = -\rho(V_1 - \frac{dh}{dt})^2 A_1$$

But from Bernoulli, $\frac{V_1^2}{2} = \frac{V_0^2}{2} - gz_1$, so $V_1 = \sqrt{V_0^2 - 2gh}$, since $z_1 = h(t)$

Also from continuity $V_1A_1 = V_0A_0$ so $A_1 = A_0\frac{V_0}{V_1}$. Substituting

$$\frac{d^2h}{dt^2} = \rho\left(\sqrt{V_0^2 - 2gh} - \frac{dh}{dt}\right)^2 \frac{A_0V_0}{M\sqrt{V_0^2 - 2gh}} - g \qquad (1) \quad \ddot{h}(t)$$

This may be solved numerically (next page).

At equilibrium height, $h = h_0(ss)$, $\dot{h}(t) = 0$, and $\ddot{h}(t) = 0$. Then

$$\rho\sqrt{V_0^2 - 2gh_0}\,A_0V_0 - Mg = 0 \quad \text{or} \quad V_0^2 - 2gh_0 = \left(\frac{Mg}{\rho V_0A_0}\right)^2$$

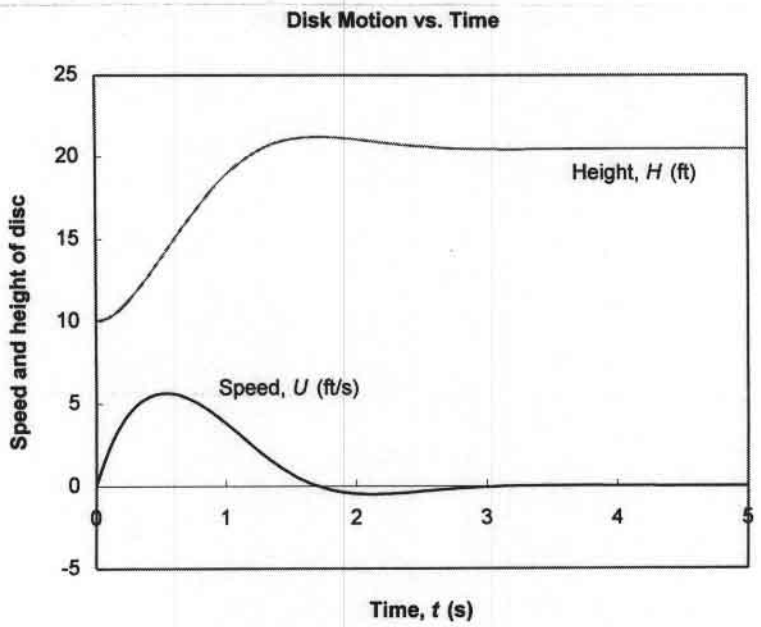
so $h_0 = \frac{V_0^2}{2g}\left[1 - \left(\frac{Mg}{\rho V_0A_0}\right)^2\right]$ $\quad h_0(ss)$

From the given data

$$h_o(ss) = \frac{1}{2} \times (40)^2 \frac{ft^2}{s^2} \times \frac{s^2}{32.2\,ft}\left[1 - \left(65\,lbf \times \frac{ft^3}{1.94\,slug} \times \frac{s^2}{(40)^2 ft^2} \times \frac{1}{0.05\,ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right)^2\right] = 20.49\,ft \quad h_o(ss)$$

Several methods may be used to integrate Eq. 1 numerically. The 4th-order Runge-Kutta method works well with a spreadsheet (next page).

The results are plotted in the figure below:



The analysis shows the motion overshoots, then the oscillation damps out and the disc settles at its steady-state height of 20.49 ft.

Problem *4.160 (cont'd.)

Analysis of disc motion (using 4th-order Runge-Kutta numerical integration):

Input Data:

A_0 =	0.050	ft^2	Nozzle area
g =	32.2	ft/s^2	Acceleration of gravity
V_0 =	40.0	ft/s	Jet speed at nozzle
W =	65.0	lbf	Weight of disc
Δt =	0.10	s	Time step size
ρ =	1.94	$slug/ft^3$	Density of water in jet

Initial Conditions:

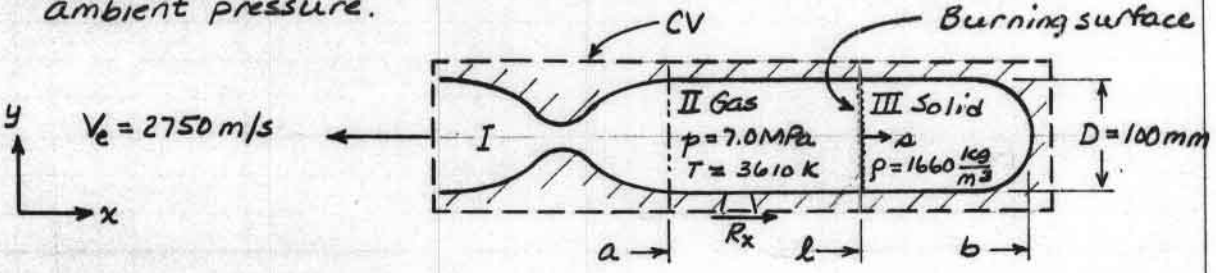
H_0 =	10.0	ft	Initial height of disc
U =	0.0	ft/s	Initial speed of disc

Calculated Values:

$H_{0\,(ss)}$ =	20.49	ft	Steady-state height
V_1 =	30.9	ft/s	Initial speed of jet reaching disc
$V_{1\,(ss)}$ =	16.75	ft/s	Steady-state jet speed
$z_{1\,(max)}$ =	24.8	ft	Maximum height reached by jet

Time Step, n	Time, t	Initial Speed, U	Initial Height, H	Jet Speed, V_1	Accel, a_y	ΔU_1	ΔU_2	ΔU_3	ΔU_4	Final Speed, U	ΔH	Final Height, H
(---)	(s)	(ft/s)	(ft)	(ft/s)	(ft/s^2)	(ft/s)	(ft/s)	(ft/s)	(ft/s)	(ft/s)	(ft)	(ft)
Initial Conditions:										0.0		10.0
0	0.0	0.00	10.00	30.92	27.23	2.72	2.21	2.31	1.87	2.27	0.23	10.23
1	0.1	2.27	10.23	30.68	18.37	1.84	1.52	1.57	1.29	3.82	0.61	10.84
2	0.2	3.82	10.84	30.04	11.78	1.18	0.98	1.01	0.84	4.82	0.86	11.70
3	0.3	4.82	11.70	29.09	6.72	0.67	0.56	0.58	0.49	5.40	1.02	12.72
4	0.4	5.40	12.72	27.94	2.75	0.28	0.23	0.24	0.20	5.64	1.10	13.83
5	0.5	5.64	13.83	26.64	-0.37	-0.04	-0.03	-0.03	-0.03	5.60	1.12	14.95
6	0.6	5.60	14.95	25.24	-2.83	-0.28	-0.24	-0.25	-0.21	5.36	1.10	16.05
7	0.7	5.36	16.05	23.80	-4.73	-0.47	-0.40	-0.41	-0.35	4.95	1.03	17.08
8	0.8	4.95	17.08	22.37	-6.14	-0.61	-0.52	-0.54	-0.45	4.42	0.94	18.01
9	0.9	4.42	18.01	20.97	-7.09	-0.71	-0.60	-0.62	-0.52	3.81	0.82	18.84
10	1.0	3.81	18.84	19.67	-7.62	-0.76	-0.64	-0.66	-0.55	3.16	0.70	19.53
11	1.1	3.16	19.53	18.49	-7.75	-0.78	-0.65	-0.67	-0.56	2.49	0.57	20.10
12	1.2	2.49	20.10	17.48	-7.50	-0.75	-0.63	-0.65	-0.53	1.86	0.44	20.53
13	1.3	1.86	20.53	16.66	-6.92	-0.69	-0.57	-0.59	-0.48	1.27	0.31	20.85
14	1.4	1.27	20.85	16.04	-6.06	-0.61	-0.50	-0.52	-0.42	0.76	0.20	21.05
15	1.5	0.76	21.05	15.63	-5.02	-0.50	-0.41	-0.43	-0.34	0.34	0.11	21.16
16	1.6	0.34	21.16	15.40	-3.90	-0.39	-0.32	-0.33	-0.26	0.02	0.04	21.20
17	1.7	0.02	21.20	15.33	-2.82	-0.28	-0.23	-0.24	-0.19	-0.21	-0.02	21.18
18	1.8	-0.21	21.18	15.37	-1.83	-0.18	-0.15	-0.15	-0.12	-0.37	-0.06	21.12
19	1.9	-0.37	21.12	15.49	-1.01	-0.10	-0.08	-0.08	-0.07	-0.45	-0.08	21.04
20	2.0	-0.45	21.04	15.66	-0.35	-0.04	-0.03	-0.03	-0.02	-0.48	-0.09	20.95
21	2.1	-0.48	20.95	15.85	0.13	0.01	0.01	0.01	0.01	-0.47	-0.09	20.85
22	2.2	-0.47	20.85	16.04	0.45	0.05	0.04	0.04	0.03	-0.43	-0.09	20.76
23	2.3	-0.43	20.76	16.22	0.65	0.06	0.05	0.05	0.04	-0.38	-0.08	20.68
24	2.4	-0.38	20.68	16.38	0.74	0.07	0.06	0.06	0.05	-0.32	-0.07	20.61
25	2.5	-0.32	20.61	16.51	0.76	0.08	0.06	0.06	0.05	-0.25	-0.06	20.55
26	2.6	-0.25	20.55	16.62	0.73	0.07	0.06	0.06	0.05	-0.19	-0.04	20.51
27	2.7	-0.19	20.51	16.71	0.66	0.07	0.05	0.06	0.04	-0.14	-0.03	20.48
28	2.8	-0.14	20.48	16.77	0.57	0.06	0.05	0.05	0.04	-0.09	-0.02	20.45
29	2.9	-0.09	20.45	16.82	0.47	0.05	0.04	0.04	0.03	-0.05	-0.01	20.44
30	3.0	-0.05	20.44	16.84	0.37	0.04	0.03	0.03	0.03	-0.02	-0.01	20.43
31	3.1	-0.02	20.43	16.86	0.28	0.03	0.02	0.02	0.02	0.00	0.00	20.43
32	3.2	0.00	20.43	16.86	0.20	0.02	0.02	0.02	0.01	0.02	0.00	20.43
33	3.3	0.02	20.43	16.86	0.13	0.01	0.01	0.01	0.01	0.03	0.00	20.44
34	3.4	0.03	20.44	16.85	0.07	0.01	0.01	0.01	0.00	0.04	0.01	20.44
35	3.5	0.04	20.44	16.83	0.02	0.00	0.00	0.00	0.00	0.04	0.01	20.45
36	3.6	0.04	20.45	16.82	-0.01	0.00	0.00	0.00	0.00	0.04	0.01	20.46
37	3.7	0.04	20.46	16.81	-0.03	0.00	0.00	0.00	0.00	0.03	0.01	20.47
38	3.8	0.03	20.47	16.79	-0.05	0.00	0.00	0.00	0.00	0.03	0.01	20.47
39	3.9	0.03	20.47	16.78	-0.06	-0.01	0.00	0.00	0.00	0.02	0.01	20.48
40	4.0	0.02	20.48	16.77	-0.06	-0.01	0.00	0.00	0.00	0.02	0.00	20.48
41	4.1	0.02	20.48	16.76	-0.06	-0.01	0.00	0.00	0.00	0.01	0.00	20.48
42	4.2	0.01	20.48	16.76	-0.05	-0.01	0.00	0.00	0.00	0.01	0.00	20.49
43	4.3	0.01	20.49	16.75	-0.04	0.00	0.00	0.00	0.00	0.01	0.00	20.49
44	4.4	0.01	20.49	16.75	-0.04	0.00	0.00	0.00	0.00	0.00	0.00	20.49
45	4.5	0.00	20.49	16.75	-0.03	0.00	0.00	0.00	0.00	0.00	0.00	20.49

Given: Small solid fuel rocket motor on test stand. The fuel burns uniformly at $s = 12.7$ mm/s. Exhaust gases leave at ambient pressure.



Treat combustion products as ideal gas with molecular mass, $M_m = 25.8$.

Find: (a) Evaluate rate of change of mass and of linear momentum within rocket motor.
(b) Express rate of change of momentum as a percentage of thrust.

Solution: Apply continuity and x component of momentum equations using fixed CV shown.

Basic equations: $$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

($F_{Bx} = 0$ (2))

Assumptions: (1) No net pressure force; $F_{Sx} = R_x$
(2) $F_{Bx} = 0$
(3) All properties constant at each point, except at surface where combustion takes place
(4) Uniform flow at exit section

The continuity equation becomes

$$0 = \frac{\partial}{\partial t}\int_I \rho \, d\forall + \frac{\partial}{\partial t}\int_a^{\ell} \rho_g A \, dx + \frac{\partial}{\partial t}\int_{\ell}^{b} \rho_f A \, dx + \{|\rho_e V_e A_e|\}$$

($\int_I \rho \, d\forall = 0$ (3))

$$0 = \frac{\partial}{\partial t}[\rho_g A(\ell - a)] + \frac{\partial}{\partial t}[\rho_f A(b - \ell)] + \dot{m}_e = (\rho_g - \rho_f) A \frac{d\ell}{dt} + \dot{m}_e$$

or

$$\dot{m}_e = (\rho_f - \rho_g) A \frac{d\ell}{dt} = (\rho_f - \rho_g) A s$$

For an ideal gas,

$$\rho_g = \frac{p_g}{R T_g} = \frac{p_g M_m}{R_u T_g} = 7.0 \times 10^6 \frac{N}{m^2} \times 25.8 \frac{kg}{mol} \times \frac{mol \cdot K}{8314 \, N \cdot m} \times \frac{1}{3610 \, K} = 6.02 \, kg/m^3$$

so

$$\dot{m}_e = (1660 - 6) \frac{kg}{m^3} \times \frac{\pi}{4}(0.1)^2 m^2 \times 0.0127 \frac{m}{s} = 0.165 \, kg/s$$

Mass flow is out, so $\frac{\partial M_{CV}}{\partial t} = -0.165$ kg/s

$\frac{\partial M_{CV}}{\partial t}$

From the momentum equation,

$$R_x = \frac{\partial}{\partial t}\int_I u\rho\, d\forall + \frac{\partial}{\partial t}\int_a^\ell u_g \rho_g A\, dx + \frac{\partial}{\partial t}\int_\ell^b \cancel{u_f \rho_f A\, dx}^{=0} + u_e\{|\rho_e V_e A_e|\}$$

($\frac{\partial}{\partial t}\int_I u\rho\, d\forall = 0$ (3).)

$$= \frac{\partial}{\partial t}\left[u_g \rho_g A(\ell - a)\right] + u_e \dot{m}_e \quad ; \quad u_g = -V_g \text{ and } u_e = -V_e$$

$$R_x = -\rho_g V_g A \frac{d\ell}{dt} - V_e \dot{m}_e = -\rho_g V_g A\, a - V_e \dot{m}_e$$

But from continuity, $\rho_g V_g A = \dot{m}_e$, since no mass accumulates in region I of the CV. Thus

$$R_x = -\dot{m}_e (V_e + a)$$

R_x is the force on the CV. The thrust is

$$K_x = \text{Thrust} = -R_x = \dot{m}_e (V_e + a)$$

$$K_x = 0.165 \frac{kg}{s}\,(2750 + 0.0127)\frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m} = 454\ N$$

The rate of change of linear momentum within the CV is

$$\frac{\partial P_{x\,cv}}{\partial t} = -\dot{m}_e a = -0.165 \frac{kg}{s} \times 0.0127 \frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m} = -2.10\ mN \qquad \frac{\partial P_{x\,cv}}{\partial t}$$

The ratio of rate of change of linear momentum to thrust is

$$\frac{\frac{\partial P_{x\,cv}}{\partial t}}{K_x} = \frac{-\dot{m}_e a}{\dot{m}_e (V_e + a)} = -\frac{a}{(V_e + a)} = -\frac{0.0127 \frac{m}{s}}{(2750 + 0.0127)\frac{m}{s}} = -4.62 \times 10^{-6}$$

or

$$\frac{\frac{\partial P_{x\,cv}}{\partial t}}{K_x} = -4.62 \times 10^{-4}\ \text{percent} \qquad \text{Ratio}$$

{ Neglecting the unsteady momentum term in the analysis of this rocket motor would cause an error of approximately 1 part in 217,000. The assumption that $\partial P_{x\,cv}/\partial t \simeq 0$ is certainly justified for engineering work. }

Problem *4.162

Open-Ended Problem Statement: Analyze the design and optimize the performance of a cart propelled along a horizontal track by a water jet that issues under gravity from an open cylindrical tank carried on board the cart. (A water-jet-propelled cart is shown in the diagram for Problem 4.121.) Neglect any change in slope of the liquid free surface in the tank during acceleration. Analyze the motion of the cart along a horizontal track, assuming it starts from rest and begins to accelerate when water starts to flow from the jet. Derive algebraic equations or solve numerically for the acceleration and speed of the cart as functions of time. Present results as plots of acceleration and speed versus time, neglecting the mass of the tank. Determine the dimensions of a tank of minimum mass required to accelerate the cart from rest along a horizontal track to a specified speed in a specified time interval.

Discussion: This problem solution consists of two parts. The first is to analyze the acceleration and velocity of a cart propelled by a gravity-driven water jet. The second is to optimize the dimensions of the cart and jet to accelerate to a specified speed in a specified time interval.

To analyze the problem, apply conservation of mass and the Bernoulli equation to the draining of the tank, then apply the x component of the momentum equation for a control volume to analyze the resulting linear acceleration. A representative plot of the results is presented below.

To optimize the performance of the water-jet-propelled cart, manipulate the solution dimensions until the best performance is attained.

Input Data:

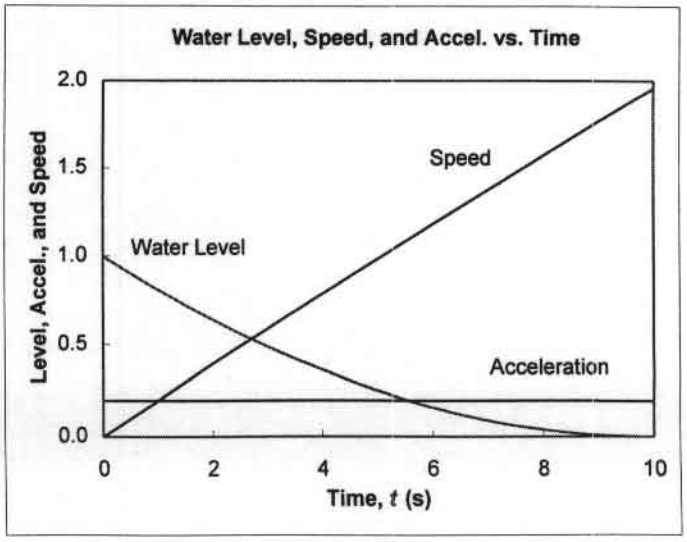
d =	10	mm	Diameter of water jet
D =	100	mm	Diameter of tank
g =	9.81	ft/s^2	Acceleration of gravity
H =	150	mm	Height of tank
M_t =	0.001	kg	Mass of tank
ρ =	999	kg/m^3	Density of water

Calculated Parameters:

a =	0.029	(---)	(a^2 =) Ratio of mass of tank to initial mass of water
b =	0.0572	s^{-1}	Geometric parameter of solution
M_0 =	1.18	kg	Initial mass of water in tank
β =	0.1	(---)	Ratio of jet diameter to tank diameter

Calculated Results:

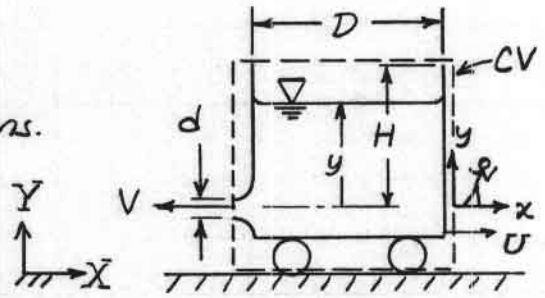
Time, t (s)	Level Ratio, y/H (---)	Accel., a_x (m/s^2)	Velocity, U (m/s)
0	1	0.196	0
1	0.810	0.196	0.196
2	0.640	0.196	0.392
3	0.490	0.196	0.588
4	0.360	0.196	0.784
5	0.250	0.196	0.980
6	0.160	0.196	1.176
7	0.0900	0.196	1.37
8	0.0400	0.196	1.57
9	0.0100	0.196	1.76
10	0	0.195	1.96



Given: Cart, propelled by water jet, accelerates along horizontal track.

Find: (a) Analyze motion, derive algebraic equations for acceleration and speed of cart as functions of time
(b) Plot acceleration and speed vs. time.

Solution: Apply conservation of mass, Bernoulli, and momentum equations.



Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$\frac{p_j}{\rho} + \frac{V_j^2}{2} + g y_j = \frac{p}{\rho} + \frac{V^2}{2} + gy \qquad (p_j = 0(8),\ y_j = 0(9))$$

M_t = mass of tank, cart

$\beta = \frac{d}{D}$

$$F_{Bx} + F_{Sx} - \int_{CV} a_{rf_x}\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A} \qquad (F_{Bx} = 0(10),\ F_{Sx} = 0(11))$$

Assumptions: (1) Uniform flow from exit jet (2) Neglect air in CV

$$0 = \frac{\partial}{\partial t}(\rho A_t y) + \{+|\rho V_j A_j|\} = \rho A_t \frac{dy}{dt} + \rho V_j A_j = -\rho A_t V + \rho V_j A_j \qquad (1)$$

Thus $V = V_j \frac{A_j}{A_t} = V_j\left(\frac{d}{D}\right)^2 = \beta^2 V_j$ (2)

(3) No slope to free surface (given)
(4) Quasi-steady flow
(5) Frictionless flow
(6) Incompressible flow
(7) Flow along a streamline
(8) $p = p_j = p_{atm}$
(9) $y_j = 0$

From Bernoulli, $\frac{V_j^2}{2} = \frac{V^2}{2} + gy$ or $V_j^2 - V^2 = 2gy$

Substituting from (2), $V_j^2 - \beta^4 V_j^2 = V_j^2(1-\beta^4) = 2gy$; $V_j^2 = \frac{2gy}{(1-\beta^4)}$ (3)

Substituting into (1), $\frac{dy}{dt} = -\beta^2 V_j = -\beta^2\frac{\sqrt{2gy}}{(1-\beta^4)}$ or $\frac{dy}{y^{1/2}} = -\frac{\beta^2\sqrt{2g}}{1-\beta^4}\,dt$

Integrating, $2y^{1/2}\Big]_{y_0}^{y} = -\frac{\beta^2\sqrt{2g}}{(1-\beta^4)}t$ or $y^{1/2} - y_0^{1/2} = -\frac{\beta^2\sqrt{2g}}{2(1-\beta^4)}t$

Thus $\left(\frac{y}{y_0}\right)^{1/2} = 1 - \left[\frac{g\beta^4}{2y_0(1-\beta^4)}\right]^{1/2} t = 1 - bt$; $b = \left[\frac{g\beta^4}{2y_0(1-\beta^4)}\right]^{1/2}$ (4)

From momentum (10) $F_{Sx} = 0$; no resistance
(11) $F_{Bx} = 0$; horizontal motion
(12) $u \approx 0$ in CV, so $\partial/\partial t \approx 0$

Then

$$-a_{rf_x} M(t) = u_j \{+|\rho V_j A_j|\} = -\rho V_j^2 A_j \qquad (5)$$

$$a_{rf_x} = \frac{dU}{dt} \qquad u_j = -V_j$$

But from (4), $M(t) = M_t + \rho A_t y = M_t + \rho A_t y_0 (1-bt)^2$

From (3), $V_j^2 = \frac{2gy}{1-\beta^4} = \frac{2g}{1-\beta^4} y_0 (1-bt)^2$

Substituting into (5)

$$\frac{dU}{dt}\left[M_t + \rho A_t y_0 (1-bt)^2\right] = \rho A_j \frac{2g}{1-\beta^4} y_0 (1-bt)^2 = \rho A_t y_0 \frac{2g\beta^2}{1-\beta^4}(1-bt)^2$$

Define M_0 = initial mass of water $= \rho A_t y_0$. Then

$$\frac{dU}{dt}\left[M_t + M_0(1-bt)^2\right] = M_0 \frac{2g\beta^2}{1-\beta^4}(1-bt)^2$$

or

$$\frac{dU}{dt} = \frac{2g\beta^2}{1-\beta^4} \frac{M_0(1-bt)^2}{M_t + M_0(1-bt)^2} \qquad (6) \quad \frac{dU}{dt}(t)$$

To integrate, let $r = 1-bt$, $dr = -b\,dt$, and $a^2 = M_t/M_0$. Then

$$U = \int_0^U dU = \frac{2g\beta^2}{1-\beta^4}\left(-\frac{1}{b}\right)\int_0^t \frac{r^2}{a^2+r^2}\,dr = -\frac{2g\beta^2}{1-\beta^4}\frac{1}{b}\left[r - a\tan^{-1}\left(\frac{r}{a}\right)\right]_0^t$$

$$= -\frac{2g\beta^2}{1-\beta^4}\frac{1}{b}\left[(1-bt) - a\tan^{-1}\left(\frac{1-bt}{a}\right)\right]_0^t$$

$$U = -\frac{2g\beta^2}{1-\beta^4}\frac{1}{b}\left[(1-bt) - a\tan^{-1}\left(\frac{1-bt}{a}\right) - 1 + a\tan^{-1}\left(\frac{1}{a}\right)\right]$$

Simplifying, then

$$U = \frac{2g\beta^2}{1-\beta^4}\left\{t + \frac{a}{b}\left[\tan^{-1}\left(\frac{1-bt}{a}\right) - \tan^{-1}\left(\frac{1}{a}\right)\right]\right\} \qquad (7) \quad U(t)$$

$$a^2 = \frac{M_t}{M_0}\,; \quad b = \left[\frac{g\beta^4}{2y_0(1-\beta^4)}\right]^{1/2}$$

Given: Cart, propelled by water jet, accelerating on horizontal track.

$$\frac{dU}{dt} = \frac{2g\beta^2}{1-\beta^4}\,\frac{(1-bt)^2}{a^2+(1-bt)^2} \qquad (1)$$

$$U(t) = \frac{2g\beta^2}{1-\beta^4}\left\{t + \frac{a}{b}\left[\tan^{-1}\left(\frac{1-bt}{a}\right) - \tan^{-1}\left(\frac{1}{a}\right)\right]\right\} \qquad (2)$$

$$\beta = \frac{d}{D}\,,\quad a^2 = \frac{Mt}{M_0}\,,\quad b = \left[\frac{g\beta^4}{2y_0(1-\beta^4)}\right]^{1/2}$$

Find: (a) Shape for tank of minimum mass for given volume.
(b) Minimum water volume to reach $U = 2.5$ m/sec in $t = 25$ sec.

Solution: Mass of tank is $M = \rho_t A_S t$, where t = thickness of wall

$$A_S = A_{bottom} + A_{cylinder} = \pi\frac{D^2}{4} + \pi DH$$

Since volume is $\forall = \frac{\pi D^2}{4}H$, then $H = \frac{4\forall}{\pi D^2}$, and

$$A_S = \frac{\pi D^2}{4} + \pi D\left(\frac{4\forall}{\pi D^2}\right) = \frac{\pi D^2}{4} + \frac{4\forall}{D}$$

To minimize, set $dA_S/dD = 0$

$$\frac{dA_S}{dD} = \frac{\pi D}{2} + (-1)\frac{4\forall}{D^2} = 0 \quad \text{so} \quad D^3 = \frac{8\forall}{\pi} \quad \text{or} \quad D = \left(\frac{8\forall}{\pi}\right)^{1/3} \qquad (3)\ D_{opt}$$

Then $\forall = \frac{\pi D^2 H}{4} = \frac{\pi D^3}{8}$ so $\frac{H}{D} = \frac{1}{2}$ (4) $\left(\frac{H}{D}\right)_{opt}$

The tank mass per volume for optimum H/D is

$$m = \frac{M}{\forall} = \frac{\rho_t\left(\frac{\pi D^2}{4} + \pi DH\right)t}{\frac{\pi D^2}{4}H} = \rho_t\left(\frac{t}{H} + \frac{4t}{D}\right) = \rho_t\,\frac{t}{H}\left(1 + 4\,\frac{H}{D}\right) = 3\rho_t\,\frac{t}{H}$$

Therefore mass depends on $\rho_t t$ for a given volume. The minimum mass is achieved for the smallest combination of ρ_t and t.

$$a^2 = \frac{Mt}{M_0} = \frac{Mt}{\rho\forall} = \frac{3\rho_t}{\rho}\,\frac{t}{H} = 3SG\left(\frac{t}{H}\right) \qquad (5)$$

which still depends on volume, since it contains H.

The best solution strategy seems to be: pick $\forall$, calculate H, D, β, a, and b, then plot $U(t)$.

Open-Ended Problem Statement: A classroom demonstration of linear momentum is planned, using a water-jet propulsion system for a cart traveling on a horizontal linear air track. The track is 5 m long, and the cart mass is 155 g. The objective of the design is to obtain the best performance for the cart, using 1 L of water contained in an open cylindrical tank made from plastic sheet with density of 0.0819 g/cm^2. For stability, the maximum height of the water tank cannot exceed 0.5 m. The diameter of the smoothly rounded water jet may not exceed 10 percent of the tank diameter. Determine the best dimensions for the tank and the water jet by modeling the system performance. Plot acceleration, velocity, and distance as functions of time. Find the optimum dimensions of the water tank and jet opening from the tank. Discuss the limitations on your analysis. Discuss how the assumptions affect the predicted performance of the cart. Would the actual performance of the cart be better or worse than predicted? Why? What factors account for the difference(s)?

Discussion: This solution is an extension of Problem *4.162. The analyses for tank level, acceleration, and velocity are identical; please refer to the solution for Problem *4.162 for equations describing each of these variables as functions of time.

One new feature of this problem is computation of distance traveled. Equation 7 of Problem *4.162 could be integrated in closed form to provide an equation for distance traveled as a function of time. However, the integral would be messy, and it would provide little insight into the dependence on key parameters. Consequently, a numerical analysis has been chosen in this problem. The results are presented in the plots and spreadsheet on the next page.

We have chosen to define velocity as the output to be maximized.

A second new feature of this problem is the geometric constraints: the maximum track length is 5 m. Intuitively jet diameter should be chosen as the largest possible fraction of tank diameter for optimum performance. Using the spreadsheet to vary $\beta = d/D$ verifies that this is the case. Therefore we have used the maximum allowable ratio, $\beta = 0.1$, for all computations.

Tank height should be a factor in performance. Intuition suggests that increasing tank height should improve performance. Using the spreadsheet shows a very weak dependence on tank height. Performance is best at smaller tank heights, corresponding to the minimum tank mass.

As tank height is decreased, diameter increases because tank volume is held constant. Since diameter ratio is constant, then jet diameter increases with decreasing tank height. This effect almost overshadows the effect of tank height.

The principal limitations on the analysis are the assumptions of negligible motion resistance and no slope to the free surface of water in the tank. Actual performance of the cart would likely be less than predicted because of motion resistance.

Distance is modeled as

$$x_{i+1} = x_i + v_i \Delta t + \frac{1}{2} a_{x,i} \Delta t^2$$

The accuracy of this model for position is consistent with the accuracy of modeling the water-jet propulsion system.

Analysis of Cart Propelled by Gravity-Driven Water Jet:

Input Data:

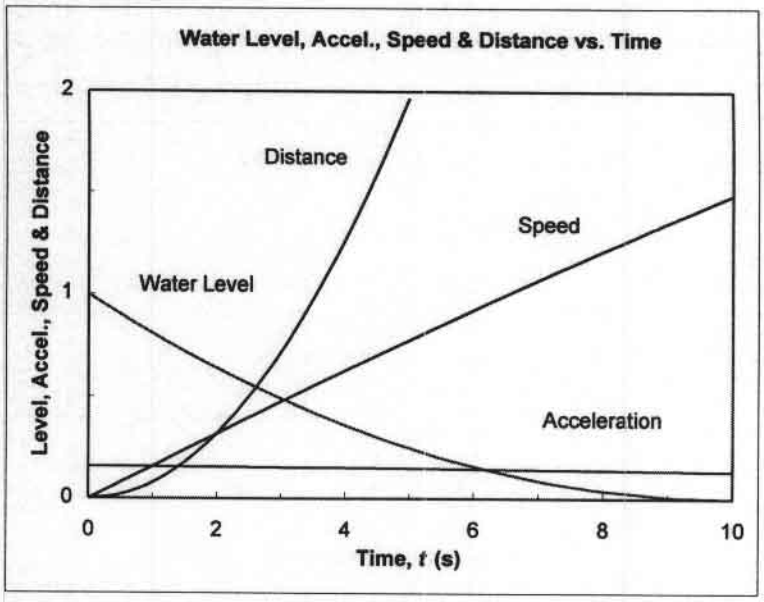
g =	9.81	m/s^2	Acceleration of gravity
H =	500	mm	Height of tank
M_c =	0.155	kg	Mass of cart
$\forall$ =	1.00	L	Tank volume
β =	0.100	(---)	Ratio of jet diameter to tank diameter
ρ =	999	kg/m^3	Density of water
ρ'' =	0.819	kg/m^2	(Area) density of tank material

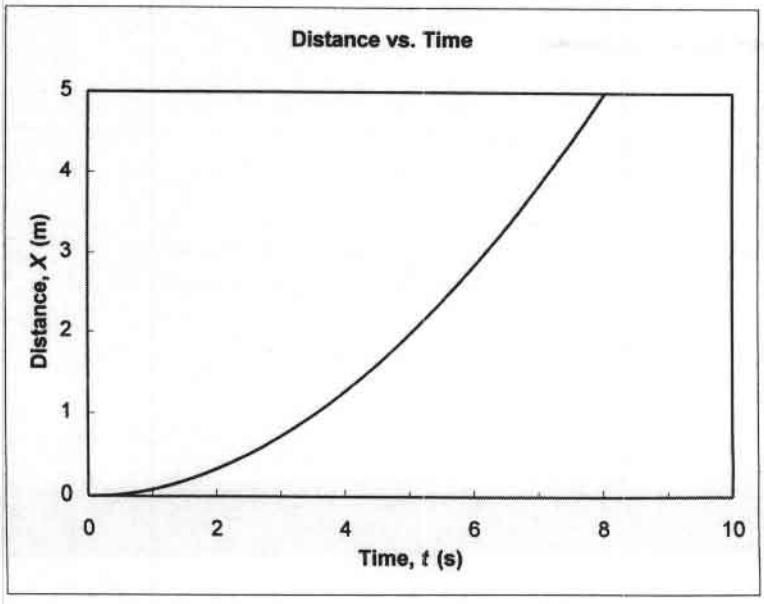
Calculated Parameters:

a =	0.471	(---)	(a^2 =) Ratio of mass of tank to initial mass of water
b =	0.0313	s^{-1}	Geometric parameter of solution
d =	5.05	mm	Diameter of water jet
D =	50.5	mm	Diameter of tank
M_0 =	1.00	kg	Initial mass of water in tank
M_p =	0.0666	kg	Mass of plastic in tank
M_t =	0.222	kg	Mass of plastic tank plus cart

Calculated Results:

Time, t	Level, y/H	Accel., a_x	Velocity, U	Position, X
(s)	(---)	(m/s^2)	(m/s)	(m)
0	1	0.161	0	0
0.5	0.903	0.160	0.080	0.0201
1.0	0.810	0.159	0.160	0.080
1.5	0.723	0.158	0.239	0.180
2.0	0.640	0.157	0.317	0.319
2.5	0.563	0.156	0.395	0.497
3.0	0.490	0.154	0.473	0.714
3.5	0.423	0.153	0.550	0.97
4.0	0.360	0.152	0.626	1.26
4.5	0.303	0.151	0.702	1.60
5.0	0.250	0.150	0.777	1.97
5.5	0.203	0.148	0.852	2.37
6.0	0.160	0.147	0.925	2.82
6.5	0.123	0.145	1.00	3.30
7.0	0.0900	0.144	1.07	3.82
7.5	0.0625	0.142	1.14	4.37
8.0	0.0400	0.141	1.21	4.96
8.03	0.0388	0.141	1.22	5.00
9.0	0.0100	0.137	1.35	
9.5	0.0025	0.135	1.42	
10.0	0.0000	0.133	1.49	





Problem *4.164

Open-Ended Problem Statement: The capability of the Aircraft Landing Loads and Traction Facility at NASA's Langley Research Center is to be upgraded. The facility consists of a rail-mounted carriage propelled by a water jet issuing from a pressurized tank. (The setup is identical in concept to the hydraulic catapult of Problem 4.118.) The 49,000 kg carriage must accelerate to 220 knots in 122 m. (The vane turning angle is 170°.) Identify a range of water jet sizes and speeds needed to accomplish this performance. Specify the recommended operating pressure for the water jet system and determine the shape and estimated size of tankage to contain the pressurized water.

Discussion: The analysis of Example Problem 4.11 forms the basis for the solution outlined below. Use a control volume attached to and moving with the carriage to analyze the motion. Neglect aerodynamic and rolling resistance to obtain a best-case solution. Solve the resulting differential equation of motion for carriage speed and position as functions of time, and for speed as a function of position along the rails.

Computing equations are summarized and results tabulated below. As shown in Example Problem 4.11, analysis of the carriage motion results in the differential equation

$$\frac{dU}{dt} = \frac{\rho (V_j - U)^2 (1-\cos\theta)}{M} \tag{1}$$

Integrating with respect to time gives carriage speed versus time

$$U = V_j \frac{bt}{1+bt} \tag{2}$$

where parameter b is

$$b = \frac{\rho V_j A_j (1-\cos\theta)}{M} \tag{3}$$

Equation 2 is integrated to obtain carriage position versus time

$$x = V_j \left[t - \frac{\ln(1+bt)}{b} \right] \tag{4}$$

Substitute $dU/dt = UdU/dx$ and integrate Eq. 1 for distance traveled versus carriage speed

$$x = \frac{V_j}{b} \left[\ln(1 - U/V_j) + \frac{1}{1 - U/V_j} - 1 \right] \tag{5}$$

Relate jet speed to water tank pressure using the Bernoulli equation

$$V_j = \sqrt{2\Delta p/\rho} \tag{6}$$

The required volume of water is computed as follows:

1. Assume a range of tank pressures.
2. Compute the jet speed corresponding to each tank pressure from Eq. 6.
3. Solve for parameter b from Eq. 5 using the known maximum speed and specified distance.
4. Obtain jet area from Eq. 3.
5. Compute the time required to accelerate the carriage from Eq. 2.
6. Calculate jet diameter from jet area.
7. Compute the required volume of water from the product of mass flow rate and acceleration time.

13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U. S. A.
National® Brand

The optimum operating pressure requires the least costly tankage. (Assume the most efficient spherical shape for pressurized tankage and constant tank pressure during acceleration.) Tankage calculations are organized as follows:

1. Obtain tank diameter from tank volume.
2. Calculate wall thickness from a force balance on the thin wall of the tank.
3. Calculate steel volume from tank surface area and wall thickness.
4. Assume steel cost is proportional to steel volume.

Sample Calculation: assume $p = 6000$ psig

$$V_j = \left[2 \times 6000 \frac{lbf}{in.^2} \times \frac{ft^3}{1.94\ slug} \times 144 \frac{in.^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right]^{\frac{1}{2}} = 944\ ft/s \; ; \; \frac{U}{V_j} = \frac{371}{944} = 0.393$$

$$b = 944 \frac{ft}{s} \times \frac{1}{400\ ft}\left[\ln(1-0.393) + \frac{1}{1-0.393} - 1\right] = 0.350\ s^{-1}$$

$$A_j = \frac{bM}{\rho V_j (1-\cos\theta)} = \frac{0.350}{s} \times 3350\ slug \times \frac{ft^3}{1.94\ slug} \times \frac{s}{944\ ft} \times \frac{1}{(1-\cos 170^\circ)} = 0.323\ ft^2$$

$$D = \sqrt{\frac{4A}{\pi}} = \left[\frac{4}{\pi} \times 0.323\ ft^2 \times 144 \frac{in.^2}{ft^2}\right]^{\frac{1}{2}} = 7.69\ in.$$

$$t = \frac{1}{b}\left(\frac{U/V_j}{1-U/V_j}\right) = \frac{s}{0.350} \times \frac{0.393}{1-0.393} = 1.85\ s$$

$$Q = V_j A = 944 \frac{ft}{s} \times 0.323\ ft^2 \times 7.48 \frac{gal}{ft^3} = 2280\ gal/s$$

$$\forall = Qt = 2280 \frac{gal}{s} \times 1.85\ s = 4220\ gal$$

$$D = (6\forall/\pi)^{1/3} = \left(\frac{6}{\pi} \times 4220\ gal \times \frac{ft^3}{7.48\ gal}\right)^{1/3} = 10.3\ ft$$

$$\Delta p \frac{\pi D^2}{4} = \pi D t \; ; \; t = \frac{pD}{4S} = \frac{1}{4} \times 6000 \frac{lbf}{in.^2} \times 10.3\ ft \times \frac{in.^2}{40{,}000\ lbf} \times 12 \frac{in.}{ft} = 4.64\ in.$$

$$\forall_{steel} = \pi D^2 t = \pi \times (10.3)^2 ft^2 \times 4.64\ in. \times \frac{ft}{12\ in.} = 129\ ft^3$$

Discussion: The results show the steel volume plummets as tank pressure is raised, with a broad minimum between 3,000 and 4,000 psig.

Problem *4.164 (cont'd.)

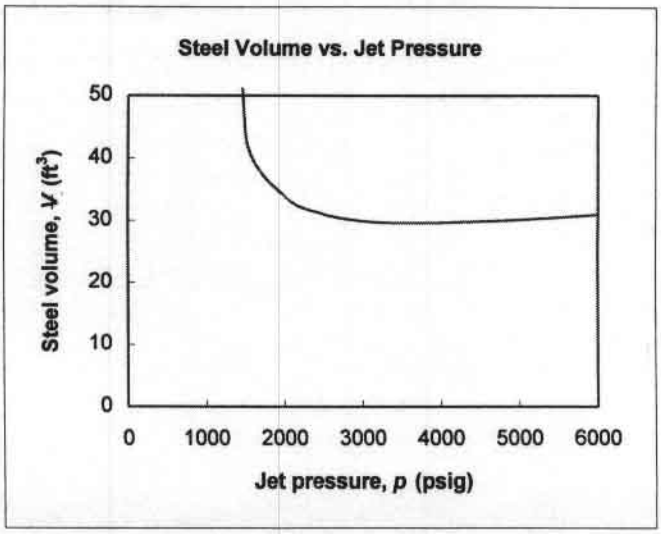
Input Data:

M =	49000	kg	3355	slug
U =	220	kt	371.3	ft/s
X =	122	m	400.3	ft
θ =	170	degrees		

Calculated Results:

Jet Pressure (psig)	Jet Speed (ft/s)	Parameter b (s^{-1})	Jet Area (ft^2)	Jet Diameter (in.)	Flow Rate (gal/s)	Flow Time (s)	Water Volume (gal)
6000	944	0.351	0.324	7.70	2285	1.85	4227
5500	904	0.380	0.367	8.20	2477	1.84	4546
5000	862	0.417	0.421	8.79	2715	1.82	4936
4500	817	0.463	0.494	9.51	3019	1.80	5426
4000	771	0.525	0.593	10.4	3419	1.77	6061
3500	721	0.610	0.737	11.6	3973	1.74	6924
3000	667	0.736	0.961	13.3	4797	1.70	8174
2500	609	0.944	1.35	15.7	6155	1.65	10172
2000	545	1.35	2.17	19.9	8830	1.58	13942
1500	472	2.53	4.67	29.3	16490	1.46	24061
1000	385	22.4	50.6	96.3	145835	1.19	173113

Jet Pressure (psig)	Water Volume (gal)	Tank Diameter (ft)	Wall Thickness (in.)	Steel Volume (ft^3)	Steel Mass (ton)
6000	4227	10.3	4.6	127.2	30.9
5500	4546	10.5	4.3	125.4	30.5
5000	4936	10.8	4.1	123.7	30.1
4500	5426	11.1	3.8	122.4	29.8
4000	6061	11.6	3.5	121.5	29.6
3500	6924	12.1	3.2	121.5	29.6
3000	8174	12.8	2.9	122.9	29.9
2500	10172	13.7	2.6	127.5	31.0
2000	13942	15.3	2.3	139.8	34.0
1500	24061	18.3	2.1	180.9	44.0
1000	173113	35.4	2.7	867.9	211.2



13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Problem *4.165

Open-Ended Problem Statement: A software package developed to accompany this text may be downloaded from the Wiley website (see the Preface). One module simulates performance of a vertical rocket. Five parameters may be varied:

M_f = mass of fuel, kg
m_e = mass flow rate of fuel, kg/s
V_e = speed of rocket exhaust, m/s
C_D = aerodynamic drag coefficient, dimensionless
ρ = air density, kg/m^3 (constant or variable with altitude)

In the simulation, fuel is half the total mass of the rocket at launch. The diameter is scaled to rocket mass and a length-to-diameter ratio of 5. The simulation plots acceleration, speed, and altitude as functions of time. The results may be explored in detail by moving a cursor along the time axis. Summary results of the five most recent simulations are retained for comparison. Verify the predictions of the simulation for the idealized case of zero aerodynamic drag. Use the simulation to explore the effect of aerodynamic drag on rocket performance, in the constant-density atmosphere. Extend the idealized analysis of Example Problem 4.12 to calculate the rocket altitude as a function of time for zero aerodynamic drag. Analyze the *coast* phase of rocket trajectory — following burnout — for the case of zero aerodynamic drag. Verify the simulation predictions for these cases. Plot rocket acceleration, speed, and altitude versus time, in a constant-density atmosphere, with $C_D = 0.375$. Discuss key features of the performance plot; explain the curve shapes. Would the actual performance of the rocket be better or worse than predicted? Why? What factors account for the difference(s)?

Discussion: Simulation and analytical results are compared in the following paragraphs and spreadsheet. (It is impossible to compare the simulation and analysis directly. The simulation is based on the initial mass being half fuel. The mass of fuel is not specified in Example Problem 4.12; it was assumed to be 50 kg. An analytical solution is not possible when aerodynamic drag is included. Therefore qualitative discussion is given below, for the case of zero drag, considering acceleration, speed, and altitude variations with time.)

Acceleration increases with time as fuel is consumed and vehicle mass decreases. The acceleration curve is discontinuous in both slope and value at burnout, when all fuel has been consumed. Acceleration is constant at negative one "g" after burnout.

Speed also increases with time as fuel is consumed. The speed versus time curve is discontinuous in slope at burnout, as required by the discontinuity in acceleration. Maximum speed is reached at burnout; the vehicle slows under the influence of gravity following burnout.

Altitude also increases with time. Altitude continues to increase after burnout, but at a slower rate. For the conditions of this simulation, the maximum height of about 8700 m is reached between 45 and 50 seconds of flight, more than 35 seconds after burnout. The time at which maximum altitude is reached corresponds to the time when velocity changes from positive to negative.

Performance of an actual rocket propelled vehicle would be different from that predicted by the simulation for several reasons, including (1) inaccuracy in choosing the vehicle drag coefficient, (2) unknown effects of Mach number on drag coefficient, (3) errors in modeling atmospheric density, and (4) significant variation in gravity with altitude at the heights reached by some rocket propelled vehicles.

13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-387 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

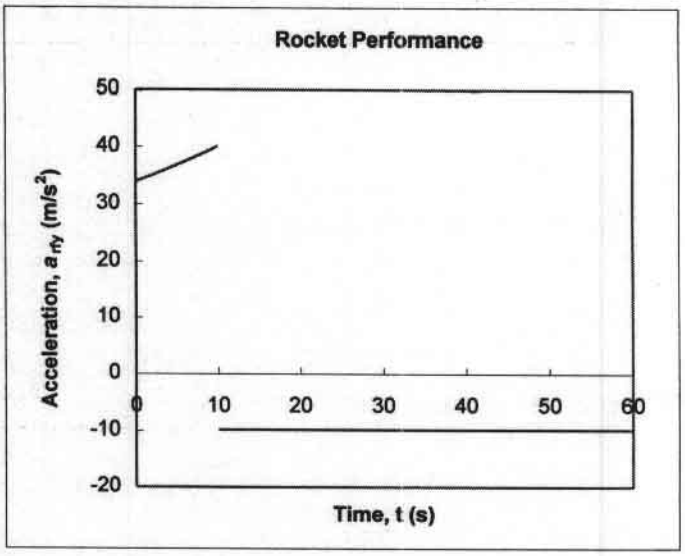
Performance Analysis of a Vertically Directed Rocket (Example Problem 4.12 extended):

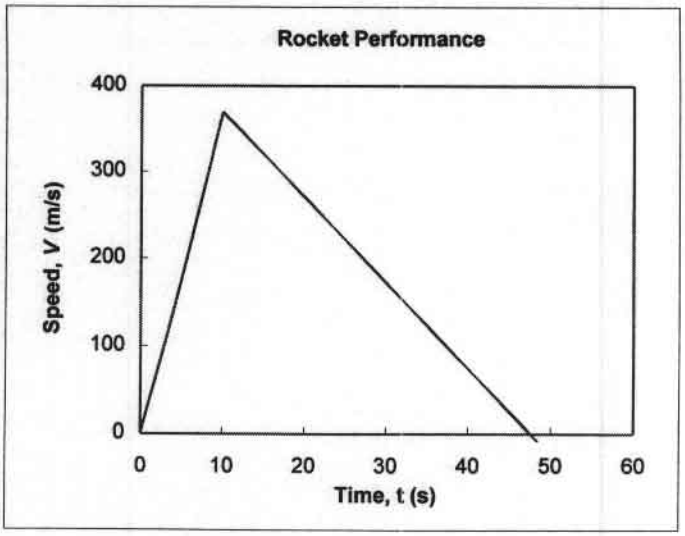
Input Data:

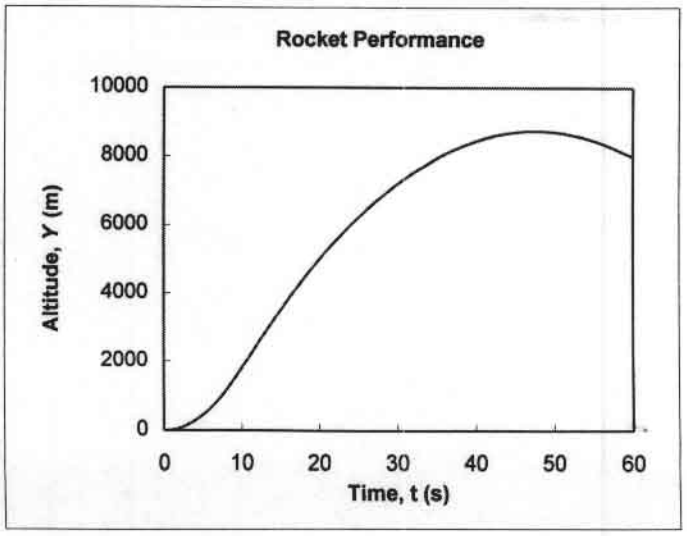
g =	9.81	m/s^2
M_f =	50	kg
M_0 =	400	kg
m_e =	5	kg/s
t_{bo} =	10.0	s
V_e =	3500	m/s

Calculated Results:

Time, t (s)	Accel., a_{rfy} (m/s^2)	Speed, V (m/s)	Height, Y (m)
0	33.9	0	0
2	35.1	69.0	68.6
4	36.2	140	278
6	37.5	214	631
8	38.8	290	1135
9.999	40.2	369	1794
10			
10	-9.81	369	1795
15	-9.81	320	3518
20	-9.81	271	4996
25	-9.81	222	6229
30	-9.81	173	7217
35	-9.81	124	7959
40	-9.81	75	8456
45	-9.81	26	8708
50	-9.81	-23	8715
55	-9.81	-72	8476
60	-9.81	-121	7992

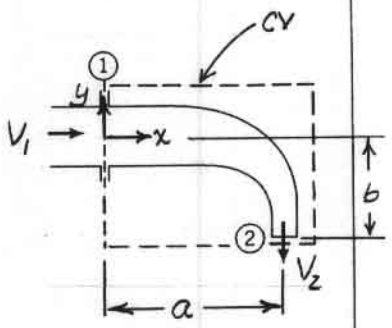






Given: The 90° reducing elbow of Example Problem 4.7 discharges to atmosphere. Section ② is located 0.3 m to the right of section ①.

Find: Estimate the moment exerted by the flange on the elbow.



Solution: Apply moment of momentum, using the CV and CS shown.

From Example Problem 4.7, $\vec{V}_2 = -16\hat{j}$ m/s, $A_1 = 0.01$ m²

Steady flow, $A_2 = 0.0025$ m²

Basic equation (fixed CV):

$$\vec{r}\times\vec{F}_S + \int_{CV}\vec{r}\times\vec{g}\rho\,d\forall + \vec{T}_{shaft} = \frac{d}{dt}\int_{CV}\vec{r}\times\vec{V}\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}\rho\vec{V}\cdot d\vec{A}$$

($\vec{r}\times\vec{g}$: ≈0 (1); $\vec{T}_{shaft}$: =0 (2); $\frac{d}{dt}$: =0 (3))

Assumptions: (1) Neglect body forces
(2) No shafts, so $\vec{T}_{shaft} = 0$
(3) Steady flow (given)
(4) Uniform flow at each cross section
(5) Incompressible flow

Then

$$\vec{M}_{flange} = \vec{r}\times\vec{F}_S\big)_{flange} = \vec{r}_1\times\vec{V}_1\{-\rho V_1 A_1\} + \vec{r}_2\times\vec{V}_2\{+\rho V_2 A_2\} \qquad (1)$$

$\vec{r}_1 = 0$

$\left.\begin{aligned}\vec{r}_2 &= a\hat{i} - b\hat{j}\\ \vec{V}_2 &= -V_2\hat{j}\end{aligned}\right\}\ \vec{r}_2\times\vec{V}_2 = -aV_2\hat{k} + 0$

Substituting into Eq. 1,

$$\vec{M}_{flange} = -aV_2\hat{k}\{+\rho V_2 A_2\} = -a\rho V_2^2 A_2\hat{k}$$

$$= 0.3\,m \times 999\,\frac{kg}{m^3} \times (16)^2\,\frac{m^2}{s^2} \times 0.0025\,m^2 \times \frac{N\cdot s^2}{kg\cdot m}\,(-\hat{k})$$

$$\vec{M}_{flange} = -192\,\hat{k}\ N\cdot m$$

$\vec{M}_{flange}$

This is the torque that must be exerted on the CV by the flange.

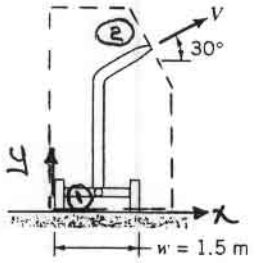
{ Since $\vec{M}_{flange}$ is in the $-\hat{k}$ direction, it must act cw in the xy-plane. }

Given: Irrigation sprinkler mounted on cart

$V = 40\ m/s$ $\theta = 30°$

$D = 50mm$ Flow is water

$h = 3\ m$ $M = 350\ kg$



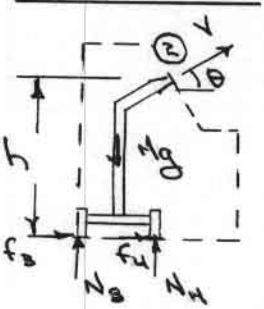
Find: (a) Magnitude of moment which tends to overturn the cart

(b) Value of V to cause impending motion; nature of impending motion.

(c) Effect on jet inclination of results

Plot: Jet velocity as a function of θ for the case of impending motion

Solution:



Apply moment of momentum equation, using fixed CV shown at left. Origin of coordinates is on ground at left wheel of cart. With this coordinate system counterclockwise moments are positive (about the z axis).

Basic equation:

$$\vec{r} \times \vec{F}_s + \int_{CV} \vec{r} \times \vec{g}\rho dV + \vec{T}_s = \frac{\partial}{\partial t}\int_{CV} \vec{r} \times \vec{V}\rho dV + \int_{CS} \vec{r} \times \vec{V}(\rho \vec{V} \cdot d\vec{A})$$

($\vec{T}_s = 0$ (1); $\frac{\partial}{\partial t} = 0$ (2))

Assumptions:

(1) $T_s = 0$

(2) steady flow

(3) uniform flow at nozzle outlet

(4) neglect $\vec{r} \times \vec{V}$ of inlet flow.

(5) center of mass located at $x = w/2$

(6) nozzle length is short; coordinates of nozzle exit are $(x_2, y_2) = (w/2, h)$

Then

$$\vec{r}_1 \times \vec{F}_s + \vec{r}_1 \times M\vec{g} = \vec{r}_1 \times \vec{V}_1\{-|\rho_1 A_1|\} + \vec{r}_2 \times \vec{V}_2\{|\rho_2 V_2 A_2|\} \quad (v_1 = 0)$$

$$\vec{r}_2 = \frac{w}{2}\hat{i} + h\hat{j} \qquad \vec{V}_2 = V(\cos\theta \hat{i} + \sin\theta \hat{j}).$$

and

$$W N_A \hat{k} - \frac{w}{2} Mg\hat{k} = \frac{w}{2} V \sin\theta\, \dot{m}_2 \hat{k} - h V \cos\theta\, \dot{m}\hat{k}$$

$$W N_A - \frac{w}{2} Mg = \dot{m}_2 V\left[\frac{w}{2}\sin\theta - h\cos\theta\right] \quad \text{------(1)}$$

Rewriting Eq. 1 in the form $\Sigma M_z = 0$ {for static equilibrium}

$$W N_A - \frac{w}{2} Mg + \dot{m}_2 V\left[h\cos\theta - \frac{w}{2}\sin\theta\right] = 0 \quad \text{---(2)}$$

The last term in Eq 2 is the moment (due to the jet) which tends to overturn the cart.

Evaluating, $\dot{m}_2 = \rho A_2 V_2 = \rho \frac{\pi}{4} D_2^2 V_2$

$\dot{m}_2 = 999 \frac{kg}{m^3} \times \frac{\pi}{4} (0.05)^2 m^2 \times 40 \frac{m}{s} = 78.5 \frac{kg}{s}$

Then with $V_2 = 40 m/s$

Moment from jet $= 78.5 \frac{kg}{s} \times 40 \frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m} \left[3m \cos 30° - \frac{1.5}{2} m \sin 30°\right]$

Moment$_{jet}$ = 6.98 kN·m ← Moment$_{jet}$

For the case of impending tipping (about point 3)

$N_4 \to 0$ and from Eq. 1

$$-\frac{W}{2} Mg + \dot{m}_2 V \left[h \cos\theta - \frac{W}{2} \sin\theta\right] = 0$$

To solve for V_2, write $\dot{m} = \rho A_2 V_2$

$$V_2^2 = \frac{WMg}{2\rho A_2 \left[h \cos\theta - \frac{W}{2} \sin\theta\right]} \quad (3)$$

$$V_2^2 = 1.5 m \times 350 kg \times 9.81 \frac{m}{s^2} \times 999 \frac{kg}{m^3} \times \frac{1}{1.96 \times 10^{-3} m^2} \times \frac{1}{(3\cos 30° - 0.75 \sin 30°) m}$$

$V_2^2 = 592 \, m^2/s^2$ $\therefore V_2 = 24.3 \, m/s$ ← V_2

Thus, the maximum speed allowable without tipping is less than the value suggested.

The impending motion will be tipping since $f_3 < \mu N_3$

From the x momentum equation $f_3 = \dot{m} V_2 \cos\theta$

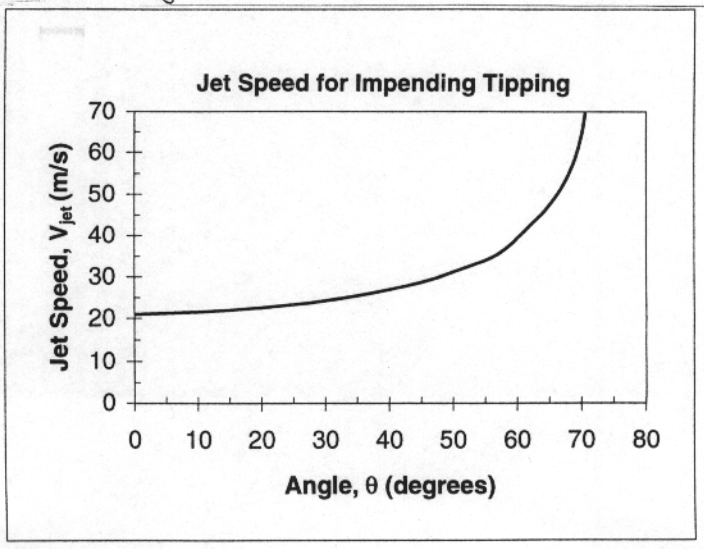
From the y momentum equation $N_3 = Mg + \dot{m} V_2 \sin\theta$

For tipping $\mu > 0.377$

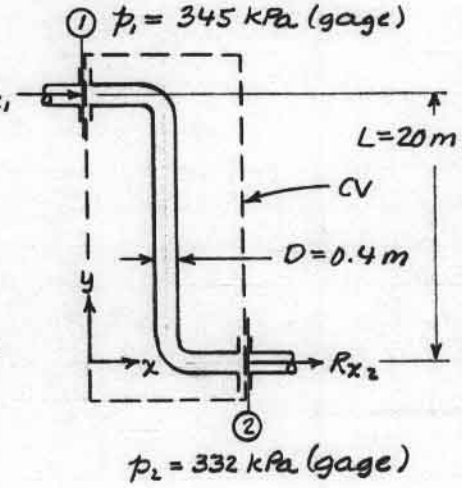
From Eq. 2 we see that as θ increases the tendency to tip decreases

For impending motion from Eq. 3.

$$V = \left\{\frac{WMg}{2\rho A_2 \left[h \cos\theta - \frac{W}{2} \sin\theta\right]}\right\}^{1/2}$$



Given: Crude oil (SG = 0.95) flow through a pipe assembly in the horizontal configuration shown.



Find: Force and torque exerted by assembly on its supports.

Solution: No momentum components exist in the y direction. Apply x component of linear momentum and the moment of momentum equations using the CV shown. Location of coordinates is arbitrary; for simplicity, choose as shown.

Basic equations: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$ (F_{Bx}: =0(1); $\frac{\partial}{\partial t}$: =0(2))

$$\vec{r}\times\vec{F}_S + \int_{CV}\vec{r}\times\vec{g}\,\rho d\forall + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}\rho d\forall + \int_{CS}\vec{r}\times\vec{V}\,\rho\vec{V}\cdot d\vec{A}$$

($\int_{CV}\vec{r}\times\vec{g}\,\rho d\forall$: =0(4); $\vec{T}_{shaft}$: =0(5); $\frac{\partial}{\partial t}$: =0(2))

Assumptions: (1) $F_{Bx} = 0$; $\vec{g}$ acts in z direction
(2) Steady flow
(3) Uniform flow at each section
(4) No z component of $\vec{r}\times\vec{g}$
(5) $\vec{T}_{shaft} = 0$

$$A = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.4)^2 m^2 = 0.126\ m^2$$

From momentum equation,

$$R_{x_1} + R_{x_2} + p_1 A - p_2 A = u_1\{-\dot{m}\} + u_2\{\dot{m}\} = 0\ ;\quad R_{x_1} + R_{x_2} = (p_2 - p_1)A$$

From moment of momentum,

$$\vec{r}_1\times(R_{x_1} + p_1 A)\hat{\imath} + \vec{r}_2\times(R_{x_2} - p_2 A)\hat{\imath} = \vec{r}_1\times V_1\hat{\imath}\{-\dot{m}\} + \vec{r}_2\times V_2\hat{\imath}\{\dot{m}\}\ ;\ \vec{r}_1 = L\hat{\jmath},\ \vec{r}_1\times\hat{\imath} = -L\hat{k}$$

($\vec{r}_2\times(R_{x_2} - p_2 A)\hat{\imath}$: =0; $\vec{r}_2\times V_2\hat{\imath}\{\dot{m}\}$: =0)

$$-L(R_{x_1} + p_1 A)\hat{k} = -LV_1(-\dot{m})\hat{k} = LV_1\dot{m}\,\hat{k} = L\frac{Q}{A}(\rho Q)\hat{k} = L\rho\frac{Q^2}{A}\hat{k}$$

$$R_{x_1} = -\rho\frac{Q^2}{A} - p_1 A = -0.95\times 999\frac{kg}{m^3}\times(0.58)^2\frac{m^6}{s^2}\times\frac{1}{0.126\,m^2}\times\frac{N\cdot s^2}{kg\cdot m} - 3.45\times10^5\frac{N}{m^2}\times 0.126\,m^2 = -46.0\ kN$$

$$R_{x_2} = (p_2 - p_1)A - R_{x_1} = p_2 A - p_1 A + \rho\frac{Q^2}{A} + p_1 A = p_2 A + \rho\frac{Q^2}{A}$$

$$= 3.32\times10^5\frac{N}{m^2}\times 0.126\,m^2 + 0.95\times 999\frac{kg}{m^3}\times(0.58)^2\frac{m^6}{s^2}\times\frac{1}{0.126\,m^2}\times\frac{N\cdot s^2}{kg\cdot m} = 44.4\ kN$$

$$\vec{r}\times\vec{F}_S = \vec{r}_1\times R_{x_1}\hat{\imath} = L\hat{\jmath}\times R_{x_1}\hat{\imath} = -LR_{x_1}\hat{k} = -20\,m\times(-46.0)kN\,\hat{k} = 920\,\hat{k}\ kN\cdot m$$

These are forces and torque on CV. The corresponding reactions are:

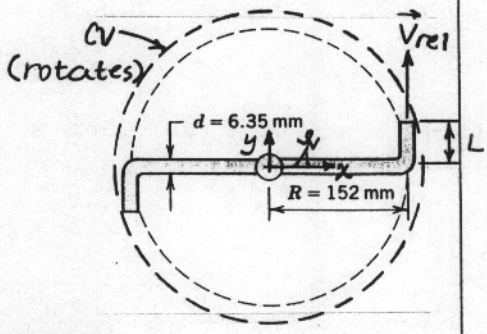
$$K_{x_1} = -R_{x_1} = 46.0\ kN,\quad K_{x_2} = -R_{x_2} = -44.4\ kN$$ ← Force

$$\vec{M} = -\vec{r}\times\vec{F}_S = -920\,\hat{k}\ kN\cdot m$$ (i.e. clockwise) ← Torque

Given: Simplified lawn sprinkler rotating in horizontal plane, $Q = 4.5$ gal/min.

Water discharges horizontally from jets.

Neglect pivot friction, inertia of sprinkler.



Find: (a) Torque needed to hold at $\omega = 0$.
(b) Angular acceleration when torque is removed.

Solution: Choose rotating CV. Apply angular momentum principle, Eq. 4.53.

Basic equation: $\vec{r} \times \vec{F}_s^{\,=0(1)} + \int_{CV} \vec{r} \times \vec{g}^{\,=0(2)} \rho \, d\forall + \vec{T}_{shaft}$

$$- \int_{CV} \vec{r} \times [2\vec{\omega}^{\,=0(3)} \times \vec{V}_{xyz} + \vec{\omega}^{\,=0(3)} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}] \rho \, d\forall$$

$$= \frac{\partial}{\partial t}^{=0(4)} \int_{CV} \vec{r} \times \vec{V}_{xyz} \rho \, d\forall + \int_{CS} \vec{r} \times \vec{V}_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) No surface forces
(2) Body torques cancel
(3) Sprinkler stationary, $\vec{\omega} = 0$
(4) Steady flow
(5) Uniform flow at each section
(6) $L \ll R$

Analyze right arm of sprinkler. From geometry $\vec{r} = r\hat{i}$ in CV, $\vec{r} = R\hat{i}$ at jet. Then

$$T\hat{k} - \int_{CV} \vec{r} \times (\vec{\omega} \times \vec{r}) \rho \, d\forall = R\hat{i} \times V\hat{j} \, \rho \frac{Q}{2} = \rho \frac{QRV}{2} \hat{k} = \frac{\dot{m} RV}{2} \hat{k}$$

$$r\hat{i} \times (\omega \hat{k} \times r\hat{i}) = r\hat{i} \times \omega r \hat{j} = \omega r^2 \hat{k} \; ; \; \int_{CV} = \omega \frac{R^3}{3} \rho A \, \hat{k}$$

Dropping $\hat{k}$, $T - \frac{\omega \rho A R^3}{3} = \frac{\dot{m} RV}{2}$. When arm is stationary, $\omega = 0$, and

$$T = \frac{\dot{m} RV}{2} \qquad \dot{m} = \rho Q = 999 \frac{kg}{m^3} \times 4.5 \frac{gal}{min} \times 231 \frac{in.^3}{gal} \times \frac{(0.0254)^3 m^3}{in.^3} \times \frac{min}{60 s} = 0.284 \frac{kg}{s}$$

$$V = \frac{Q}{2A} = \frac{2Q}{\pi d^2} = \frac{2}{\pi} \times 2.84 \times 10^{-4} \frac{m^3}{s} \times \frac{1}{(0.00635)^2 m^2} = 4.48 \; m/s$$

$$T = \frac{1}{2} \times 0.284 \frac{kg}{sec} \times 0.152 \, m \times 4.48 \frac{m}{sec} = 0.0967 \; N \cdot m \; (\text{per arm})$$

For two arms, $T_2 = 2T = 2 \times 0.0967 \; N \cdot m = 0.193 \; N \cdot m$ $\quad T_2$

When torque is removed, angular acceleration would be the same for each arm. Thus

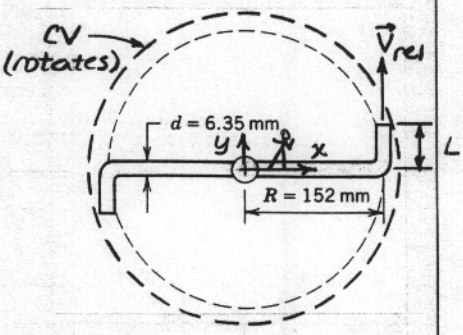
$$\dot{\omega} = \frac{\dot{m} RV}{2} \times \frac{3}{\rho A R^3} = \frac{3 \dot{m} V}{2 \rho A R^2}$$

$$\dot{\omega} = \frac{3}{2} \times 0.284 \frac{kg}{s} \times 4.48 \frac{m}{s} \times \frac{m^3}{999 \, kg} \times \frac{4}{\pi (0.00635)^2 m^2} \times \frac{1}{(0.152)^2 m^2} = 2610 \; rad/s^2 \qquad \dot{\omega}$$

Given: Simplified lawn sprinkler rotating in horizontal plane, $Q = 4.5$ gal/min.

Water discharges horizontally from jets.

Neglect pivot friction, inertia of sprinkler.



Find: (a) Derive a differential equation for angular speed as a function of time.
(b) Evaluate steady-state angular speed.

Solution: Choose rotating CV. Apply angular momentum principle, Eq. 4.53.

Basic equation: $\vec{r} \times \vec{F}_S^{=0(1)} + \int_{CV} \vec{r} \times \vec{g}^{=0(2)} \rho \, d\forall + \vec{T}_{shaft}^{=0(3)}$

$$- \int_{CV} \vec{r} \times \left[2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}^{=0(4)}) + \dot{\vec{\omega}} \times \vec{r} \right] \rho \, d\forall$$

$$= \frac{\partial}{\partial t}^{=0(5)} \int_{CV} \vec{r} \times \vec{V}_{xyz} \rho \, d\forall + \int_{CS} \vec{r} \times \vec{V}_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) $\vec{F}_S = 0$, (2) Body torques cancel, (3) $\vec{T}_{shaft} = 0$, (4) No $\hat{k}$ component of centripetal acceleration, (5) Steady flow, (6) $L \ll R$

Analyze right arm of sprinkler. From geometry, $\vec{r} = r\hat{\imath}$ in CV, $\vec{r} = R\hat{\imath}$ at jet. Then

$$- \int_{CV} r\hat{\imath} \times \left[2\omega\hat{k} \times V\hat{\imath} + \dot{\omega}\hat{k} \times r\hat{\imath} \right] \rho A \, dr = R\hat{\imath} \times V\hat{\jmath} \, \rho \frac{Q}{2} = \rho \frac{QRV}{2} \hat{k}$$

$$r\hat{\imath} \times \left[2\omega V(+\hat{\jmath}) + \dot{\omega} r(+\hat{\jmath}) \right] = (2\omega r V + \dot{\omega} r^2)(+\hat{k}); \quad - \int_{CV} = -(\omega R^2 V + \dot{\omega} \frac{R^3}{3}) \rho A$$

Dropping $\hat{k}$,

$$-\omega \rho V A R^2 - \dot{\omega} \frac{\rho A R^3}{3} = \rho \frac{QRV}{2} \quad \text{or} \quad \dot{\omega} = \frac{3}{\rho A R^3} \left[-\omega \rho V A R^2 - \rho \frac{QRV}{2} \right] \qquad \text{O.D.E.}$$

Thus $\frac{d\omega}{dt} = -a - b\omega$, where $a = \frac{3}{\rho A R^3} \rho \frac{QRV}{2} = \frac{3}{2} \frac{QV}{AR^2} = \frac{3V^2}{R^2}$, $b = \frac{3 \rho V A R^2}{\rho A R^3} = \frac{3V}{R}$

$\frac{d\omega}{dt} = 0$ when $-a - b\omega_{max} = 0$, i.e., when $\omega_{max} = -a/b$. ↑ Note $V = \frac{Q}{2A}$ (one arm)

$$Q = 4.5 \frac{gal}{min} \times 231 \frac{in.^3}{gal} \times (0.0254)^3 \frac{m^3}{in.^3} \times \frac{min}{60\ s} = 2.84 \times 10^{-4}\ m^3/s$$

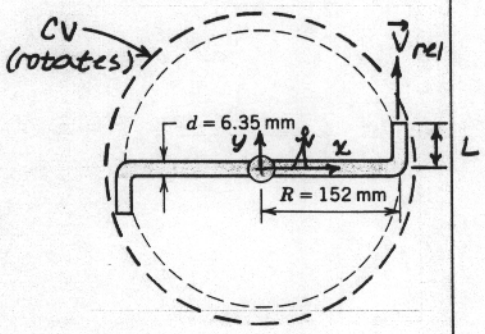
$$\omega_{max} = -\frac{a}{b} = -\frac{3V^2}{R^2} \times \frac{R}{3V} = -\frac{V}{R} = -\frac{4.48\ m/s}{} \times \frac{1}{0.152\ m} = -29.5\ rad/s \quad (-281\ rpm) \qquad \omega_{max}$$

{ Note it is not necessary to solve the differential equation to find ω_{max}. }

Given: Simplified lawn sprinkler rotating in horizontal plane, $Q = 4.5$ gal/min.

Water discharges horizontally from jets.

Neglect inertia of sprinkler; $T_f = 0.045$ ft·lbf



Find: (a) Derive a differential equation for angular speed as a function of time.
(b) Evaluate steady-state angular speed.

Solution: Choose rotating CV. Apply angular momentum principle, Eq. 4.53.

Basic equation: $\vec{r}\times\vec{F}_S + \int_{CV}\vec{r}\times\vec{g}\,\rho\,d\forall + \vec{T}_{shaft}$

$$-\int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \dot{\vec{\omega}}\times\vec{r}\right]\rho\,d\forall$$

$$= \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) $\vec{F}_S = 0$, (2) Body torques cancel, (3) $\vec{T}_{shaft} = 0.045$ ft·lbf, (4) No $\hat{k}$ component of centripetal acceleration, (5) steady flow, (6) $L \ll R$.

Analyze right arm of sprinkler. From geometry, $\vec{r} = r\hat{\imath}$ in CV, $\vec{r} = R\hat{\imath}$ at jet. Then

$$-\int_{CV} r\hat{\imath}\times\left[2\omega\hat{k}\times V\hat{\imath} + \dot{\omega}\hat{k}\times r\hat{\imath}\right]\rho A\,dr = R\hat{\imath}\times V\hat{\jmath}\,\rho\frac{Q}{2} = \rho\frac{QRV}{2}\hat{k}$$

$$r\hat{\imath}\times\left[2\omega V\hat{\jmath} + \dot{\omega}r\hat{\jmath}\right] = (2\omega Vr + \dot{\omega}r^2)\hat{k}\,;\quad -\int_{CV} = -\left(\omega VR^2 + \dot{\omega}\frac{R^3}{3}\right)\rho A\,\hat{k}$$

For both arms, dropping $\hat{k}$, $\{T = 0.045\ \text{ft·lbf} = 0.0610\ \text{N·m}\}$

$$T - 2\omega\rho VAR^2 - 2\dot{\omega}\frac{\rho AR^3}{3} = \rho QRV \quad\text{or}\quad \dot{\omega} = \frac{3}{2\rho AR^3}\left[T - \rho QRV - 2\omega\rho VAR^2\right]$$ D.D.E.

Thus $\frac{d\omega}{dt} = a - b\omega$, where $a = \frac{3}{2\rho AR^3}(T - \rho QRV)$, $b = \frac{3}{2\rho AR^3}\times 2\rho VAR^2 = 3\frac{V}{R}$

The steady-state speed occurs when $\frac{d\omega}{dt} = 0$, i.e. when $\omega_{max} = \frac{a}{b}$

$$Q = 4.5\frac{\text{gal}}{\text{min}}\times 231\frac{\text{in}^3}{\text{gal}}\times\frac{(0.0254)^3\,\text{m}^3}{\text{in}^3}\times\frac{\text{min}}{60\,\text{s}} = 2.84\times10^{-4}\ \text{m}^3/\text{s}\,;\quad A = \frac{\pi d^2}{4} = 3.17\times10^{-5}\ \text{m}^2$$

From the D.D.E., $\omega_{max} = \dfrac{T - \rho QRV}{2\rho VAR^2}$

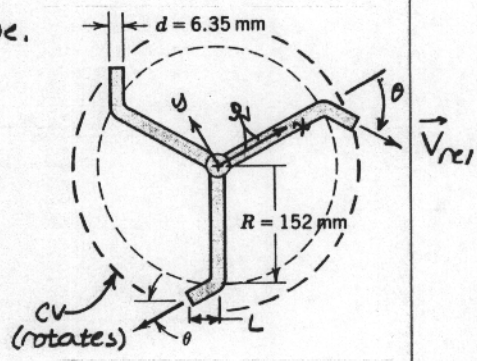
$$\omega_{max} = \frac{1}{2}\left[0.0610\ \text{N·m}\times\frac{\text{kg·m}}{\text{N·s}^2} - 999\frac{\text{kg}}{\text{m}^3}\times 2.84\times10^{-4}\frac{\text{m}^3}{\text{s}}\times 0.152\ \text{m}\times 4.48\frac{\text{m}}{\text{s}}\right]\frac{\text{m}^3}{999\,\text{kg}}\times\frac{\text{s}}{4.48\,\text{m}}$$

$$\times\frac{1}{3.17\times10^{-5}\,\text{m}^2}\times\frac{1}{(0.152)^2\,\text{m}^2}$$

$\omega_{max} = -20.2$ rad/s $(-193$ rpm) ω_{max}

Given: Lawn sprinkler rotating in horizontal plane.

Neglect friction. $Q = 68$ L/min



Find: Steady-state angular speed for $\theta = 30°$.

Plot: Steady-state angular speed for $0 \le \theta \le 90°$.

Solution: Choose rotating CV. Apply angular momentum principle, Eq. 4.53.

Basic equation: $\vec{r} \times \vec{F}_s^{\,=0(1)} + \int_{CV} \vec{r} \times \vec{g}^{\,=0(2)} \rho d\forall + \vec{T}_{shaft}^{\,=0(3)}$

$$- \int_{CV} \vec{r} \times \left[2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r})^{=0(5)} + \dot{\vec{\omega}}^{=0(6)} \times \vec{r} \right] \rho d\forall$$

$$= \frac{\partial}{\partial t}^{=0(6)} \int_{CV} \vec{r} \times \vec{V}_{xyz} \rho d\forall + \int_{CS} \vec{r} \times \vec{V}_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) $\vec{F}_s = 0$, (2) Body torques cancel, (3) $\vec{T}_{shaft} = 0$, (4) Neglect aerodynamic drag, (5) No $\hat{k}$ component of centripetal acceleration, (6) Steady flow, (7) $L \ll R$

Analyze one arm of sprinkler. From geometry, $\vec{r} = r\hat{\imath}$ in CV, $\vec{r} = R\hat{\imath}$ at jet. Then

$$- \int_{CV} \vec{r} \times \left[2\vec{\omega} \times \vec{V}_{xyz} \right] \rho d\forall = R\hat{\imath} \times (-V \sin\theta \hat{\jmath}) \rho \frac{Q}{3} = -\rho \frac{QRV}{3} \sin\theta \, \hat{k}$$

$$r\hat{\imath} \times (2\omega\hat{k} \times V\hat{\imath}) = 2\omega V r \hat{k} \; ; \; - \int_{CV} = -\omega V R^2 \rho A \hat{k}$$

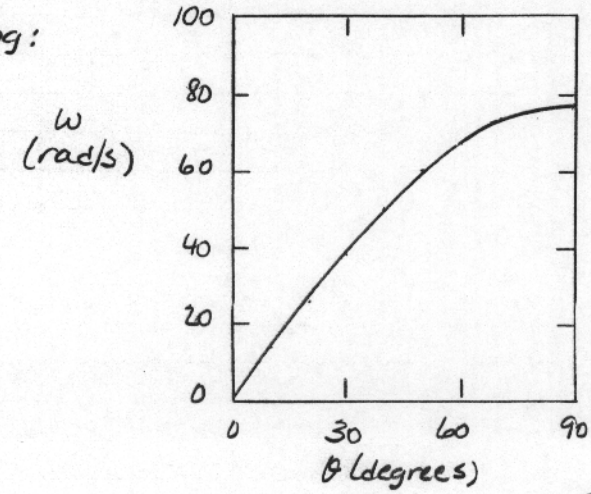
Dropping $\hat{k}$, $-\omega V R^2 \rho A = -\rho \frac{QRV}{3} \sin\theta$, so with $VA = Q/3$,

$$\omega = \frac{V}{R} \sin\theta \; ; \; V = \frac{Q}{3A} = \frac{4Q}{3\pi d^2} = \frac{4}{3\pi} \times 68 \times 10^{-3} \frac{m^3}{min} \times \frac{1}{(0.00635)^2 m^2} \times \frac{min}{60 s} = 11.9 \text{ m/s}$$

$$\omega = 11.9 \frac{m}{s} \times \frac{1}{0.152 m} \times \sin\theta = 78.3 \sin\theta \text{ rad/s}$$

ω

Plotting:



For $\theta = 30°$,

$\omega = 78.3 \sin 30°$

$\omega = 39.1$ rad/s

ω (θ = 30°)

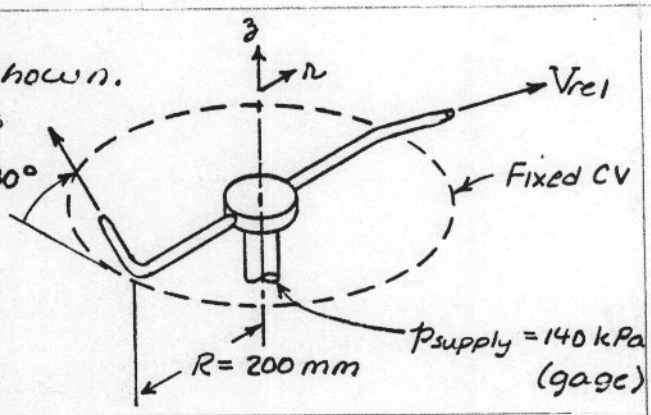
Plot

Given: Small lawn sprinkler as shown.

$V_{rel} = 17$ m/s

Friction torque at pivot is $T_f = 0.18$ N·m.

Flowrate is $Q = 4.0$ liter/min.



Find: Torque to hold stationary.

Solution: Apply moment of momentum using fixed CV enclosing sprinkler arms.

Basic equation:

$$\vec{r} \times \vec{F}_s + \int_{CV} \vec{r} \times \vec{g}\,\rho\,d\forall + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV} \vec{r} \times \vec{V}\,\rho\,d\forall + \int_{cs} \vec{r} \times \vec{V}\,\rho\,\vec{V}\cdot d\vec{A}$$

($\vec{r} \times \vec{F}_s = 0$ (1); $\vec{r} \times \vec{g} = 0$ (2); $\frac{\partial}{\partial t}\int_{CV} = 0$ (3))

Assumptions: (1) Neglect torque due to surface forces
(2) Torques due to body forces cancel by symmetry
(3) Steady flow
(4) Uniform flow leaving each jet

Then

$$-T_f \hat{k} = (\vec{r} \times \vec{V})_{in}\{-\rho Q\} + 2(\vec{r} \times \vec{V})_{jet}\left\{\tfrac{1}{2}\rho Q\right\}$$

$(\vec{r} \times \vec{V})_{in} \simeq 0$

$\vec{r} = R\hat{i}_r$

$\vec{V} = (R\omega - V_{rel}\cos\alpha)\hat{i}_\theta + V_{rel}\sin\alpha\,\hat{i}_z$

The absolute velocity of the jet leaving sprinkler is $\vec{V} = V_{rel}[\cos\alpha(-\hat{i}_\theta) + \sin\alpha(\hat{i}_z)]$

Then $(\vec{r} \times \vec{V})_z = \left\{R\hat{i}_r \times V_{rel}[\cos\alpha(-\hat{i}_\theta) + \sin\alpha(\hat{i}_z)]\right\}_z = \left\{RV_{rel}\cos\alpha(-\hat{i}_z) + RV_{rel}\sin\alpha(-\hat{i}_\theta)\right\}_z$

$(\vec{r} \times \vec{V})_z = -RV_{rel}\cos\alpha$

Substituting, $T_{shaft} = T_{ext} - T_f = 2(-RV_{rel}\cos\alpha)\left(\tfrac{1}{2}\rho Q\right)$

Thus $T_{ext} = T_f - \rho Q R V_{rel}\cos\alpha$

$$= 0.18\ \text{N·m} - \frac{999\ \text{kg}}{\text{m}^3} \times \frac{4\ \text{L}}{\text{min}} \times 0.2\ \text{m} \times \frac{17\ \text{m}}{\text{s}} \times 0.866 \times \frac{\text{m}^3}{1000\ \text{L}} \times \frac{\text{min}}{60\ \text{s}} \times \frac{\text{N·s}^2}{\text{kg·m}}$$

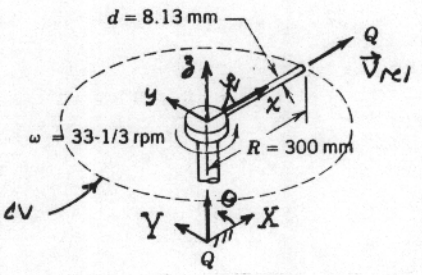
$T_{ext} = -0.0161$ N·m (to hold sprinkler stationary)

T_{ext}

{ Since $T_{ext} < 0$, it must be applied in the minus z direction to oppose motion. }

Given: Single rotating tube with water.

$Q = 13.8$ L/min



Find: Torque that must be applied to maintain steady rotation using:
(a) Rotating control volume.
(b) Fixed control volume.

Solution: Apply angular momentum principle, $\left\{\omega = 33\frac{1}{3}\frac{rev}{min} = 3.49\ rad/s\right\}$

(a) Rotating CV: use relative velocities, Eq. 4.53:

Basic equation: $\vec{r}\times\vec{F}_S^{\,=0(1)} + \int_{CV}\vec{r}\times\vec{g}^{\,=0(2)}\rho d\forall + \vec{T}_{shaft}$

$$-\int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz}^{\,=0(3)} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \dot{\vec{\omega}}^{\,=0(4)}\times\vec{r}\right]\rho d\forall$$

$$= \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}^{\,=0(5)}\rho d\forall + \int_{CS}\vec{r}\times\vec{V}_{xyz}^{\,=0(6)}\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) $\vec{F}_S = 0$, (2) Body torques cancel, (3) No $\hat{k}$ in centripetal accel,
(4) $\dot{\vec{\omega}} = 0$, (5) steady flow, (6) $\vec{r}\times\vec{V} = 0$

Then

$$T_{shaft}\,\hat{k} = \int_{CV}\vec{r}\times(2\vec{\omega}\times\vec{V})\rho d\forall = \int_0^R r\hat{i}\times(2\omega\hat{k}\times V\hat{i})\rho A\,dr = \omega\rho V A R^2\hat{k} = \omega\rho Q R^2\hat{k}$$

$$T_{shaft} = 3.49\frac{rad}{s}\times 999\frac{kg}{m^3}\times 13.8\times10^{-3}\frac{m^3}{min}\times(0.3)^2 m^2\times\frac{min}{60s}\times\frac{N\cdot s^2}{kg\cdot m} = 0.0722\ N\cdot m \quad T$$

(b) Fixed control volume: use absolute velocities, Eq. 4.47:

Basic equation: $\vec{r}\times\vec{F}_S^{\,=0(1)} + \int_{CV}\vec{r}\times\vec{g}^{\,=0(2)}\rho d\forall + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}\rho d\forall + \int_{CS}\vec{r}\times\vec{V}\rho\vec{V}_{xyz}\cdot d\vec{A}$

Relative to fixed coordinates XY, $\vec{r} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

$$\vec{V} = V(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) + r\omega(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

$$\vec{r}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r\cos\theta & r\sin\theta & 0 \\ V\cos\theta - r\omega\sin\theta & V\sin\theta + r\omega\cos\theta & 0 \end{vmatrix} = \hat{k}\,(rV\sin\theta\cos\theta + \omega r^2\cos^2\theta - rV\sin\theta\cos\theta + \omega^2 r^2\sin^2\theta) = \omega r^2\hat{k}$$

Thus $\partial/\partial t = 0$ and $\int_{CS}\vec{r}\times\vec{V}\rho\vec{V}_{xyz}\cdot d\vec{A} = \omega R^2\hat{k}\{+\rho Q\} = \omega\rho Q R^2\hat{k}$ and

$T_{shaft}\,\hat{k} = \omega\rho Q R^2\hat{k}$ (as before); $T = 0.0722\ N\cdot m$ T

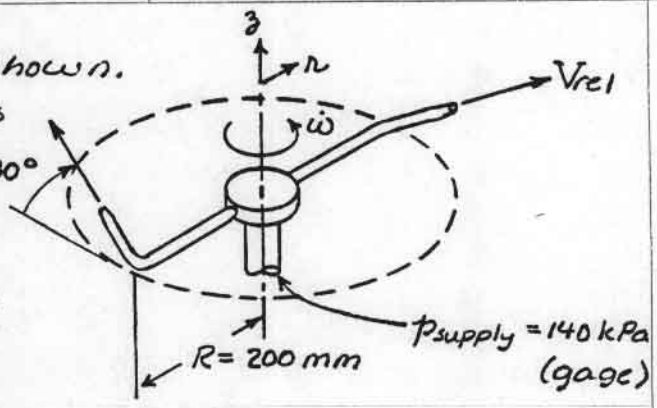
{Note that when applied correctly, either choice of CV produces the same result.}

Given: Small lawn sprinkler as shown.

$V_{rel} = 17$ m/s

Friction torque at pivot is zero. $I = 0.1$ kg·m²

Flowrate is $Q = 4.0$ liter/min.



Find: Initial angular acceleration from rest.

Solution: Apply moment of momentum using fixed CV enclosing sprinkler arms.

Basic equation:

$$\vec{r} \times \vec{F}_S + \int_{CV} \vec{r} \times \vec{g}\,\rho\,d\forall + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV} \vec{r} \times \vec{V}\,\rho\,d\forall + \int_{CS} \vec{r} \times \vec{V}\,\rho\,\vec{V}\cdot d\vec{A}$$

($\vec{r} \times \vec{F}_S \approx 0$ (1); $\int_{CV} \vec{r} \times \vec{g}\,\rho\,d\forall = 0$ (2); $\frac{\partial}{\partial t}\int_{CV} \vec{r} \times \vec{V}\,\rho\,d\forall = 0$ (3))

Assumptions: (1) Neglect torque due to surface forces
(2) Torques due to body forces cancel by symmetry
(3) Steady flow
(4) Uniform flow leaving each jet

Then

$$-T_f\,\hat{k} = (\vec{r} \times \vec{V})_{in}\{-\rho Q\} + 2(\vec{r} \times \vec{V})_{jet}\left\{\tfrac{1}{2}\rho Q\right\}$$

$(\vec{r} \times \vec{V})_{in} \approx 0$

$\vec{r} = R\,\hat{e}_r$

$\vec{V} = (R\omega - V_{rel}\cos\alpha)\hat{e}_\theta + V_{rel}\sin\alpha\,\hat{e}_z$

The jet leaves the sprinkler at $\vec{V}(abs) = V_{rel}[\cos\alpha(-\hat{e}_\theta) + \sin\alpha(\hat{e}_z)]$

Then $\vec{r} \times \vec{V} = R\hat{e}_r \times V_{rel}[\cos\alpha(-\hat{e}_\theta) + \sin\alpha(\hat{e}_z)] = RV_{rel}[\cos\alpha(-\hat{e}_z) + \sin\alpha(-\hat{e}_\theta)]$

Summing moments on the rotor, $\Sigma \vec{M} = I\dot{\vec{\omega}}$. Thus

$$\dot{\omega} = \frac{\Sigma T}{I} = \frac{\rho Q R V_{rel}\cos\alpha - T_f}{I}$$

$$= \left[\frac{999\,kg}{m^3}\times\frac{4\,L}{min}\times 0.2\,m\times\frac{17\,m}{s}\times 0.866\times\frac{m^3}{1000\,L}\times\frac{min}{60\,s} - 0.18\,N\cdot m\times\frac{kg\cdot m}{N\cdot s^2}\right]\frac{1}{0.1\,kg\cdot m^2}$$

$\dot{\omega} = 0.161$ rad/s² $\dot{\omega}$

{It is not necessary to use a rotating CV, because at the instant considered, $\vec{\omega} = 0$ and I is known.}

Open-Ended Problem Statement: When a garden hose is used to fill a bucket, water in the bucket may develop a swirling motion. Why does this happen? How could the amount of swirl be calculated approximately?

Discussion: Frequently when filling a bucket the hose is held so that the water stream entering the bucket is not vertical. If, in addition, the water stream is off-center in the bucket, then flow entering the bucket has a tangential component of velocity, a swirl component.

The tangential component of the water velocity entering the bucket has a moment-of-momentum (swirl) with respect to a control volume drawn around the stationary bucket. This entering swirl can only be reduced by a torque acting to oppose it. Viscous forces among fluid layers will tend to transfer swirl to other layers so that eventually all of the water in the bucket has a swirling motion.

Swirl in the bucket may be influenced by viscosity. The swirl may tend to nearly a rigid-body motion to minimize viscous forces between annular layers of water in the bucket. The rigid-body motion assumption may be a reasonable model to calculate the total angular momentum (moment-of-momentum) of the water in the bucket.

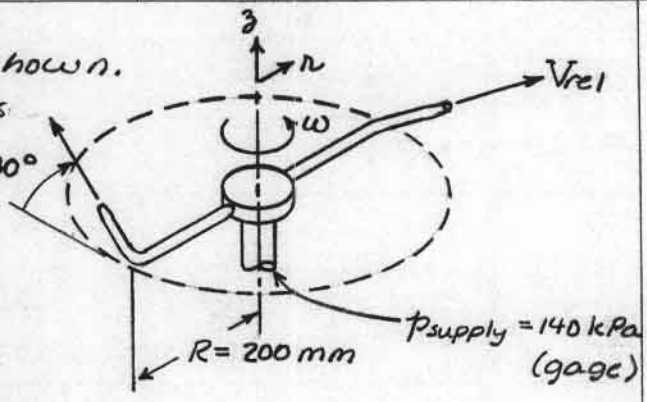
50 SHEETS EYE EASE® 5 SQUARE
100 SHEETS EYE EASE® 5 SQUARE
200 SHEETS EYE EASE® 5 SQUARE
RECYCLED WHITE 5 SQUARE
National® Brand

Given: Small lawn sprinkler as shown.

$V_{rel} = 17 \text{ m/s}$

Friction torque at pivot is $T_f = 0.18 \text{ N·m}$.

Flowrate is $Q = 4.0 \text{ liter/min}$.



Find: (a) Steady speed of rotation.
(b) Area covered by spray.

Solution: Apply moment of momentum using fixed CV enclosing sprinkler arms.

Basic equation:

$$\vec{r} \times \vec{F}_S + \int_{CV} \vec{r} \times \vec{g}\,\rho\,d\forall + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV} \vec{r} \times \vec{V}\,\rho\,d\forall + \int_{CS} \vec{r} \times \vec{V}\,\rho\,\vec{V}\cdot d\vec{A}$$

($\vec{r} \times \vec{F}_S \approx 0$ (1); $\int_{CV} \vec{r} \times \vec{g}\,\rho\,d\forall = 0$ (2); $\frac{\partial}{\partial t}\int_{CV} \vec{r} \times \vec{V}\,\rho\,d\forall = 0$ (3))

Assumptions: (1) Neglect torque due to surface forces
(2) Torques due to body forces cancel by symmetry
(3) Steady flow
(4) Uniform flow leaving each jet

Then

$$-T_f \hat{k} = (\vec{r} \times \vec{V})_{in}\{-\rho Q\} + 2(\vec{r} \times \vec{V})_{jet}\left\{\tfrac{1}{2}\rho Q\right\}$$

$(\vec{r} \times \vec{V})_{in} \approx 0$

$\vec{r} = R\,\hat{i}_r$

$\vec{V} = (R\omega - V_{rel}\cos\alpha)\hat{i}_\theta + V_{rel}\sin\alpha\,\hat{i}_z$

$(\vec{r} \times \vec{V})_z = R(R\omega - V_{rel}\cos\alpha)$

or

$$-T_f = R(R\omega - V_{rel}\cos\alpha)\rho Q$$

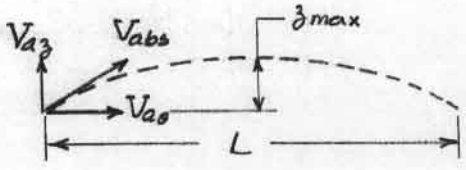
Thus

$$\omega = \frac{V_{rel}\cos\alpha}{R} - \frac{T_f}{\rho Q R^2}$$

$$= 17\frac{m}{s} \times \frac{\cos 30°}{0.2 m} - 0.18 N·m \times \frac{m^3}{999 kg} \times \frac{min}{4.0 \ell} \times \frac{1}{(0.2)^2 m^2} \times \frac{60 s}{min} \times \frac{10^3 \ell}{m^3} \times \frac{kg·m}{N·s^2}$$

$$\omega = 6.04 \frac{rad}{s} \quad \text{or} \quad 57.7 \text{ rpm}$$ ω

Treat the spray outside each nozzle as moving without air resistance:



For each particle, $\frac{dV_{az}}{dt} = -g$, so $V_z = V_{z0} - gt$

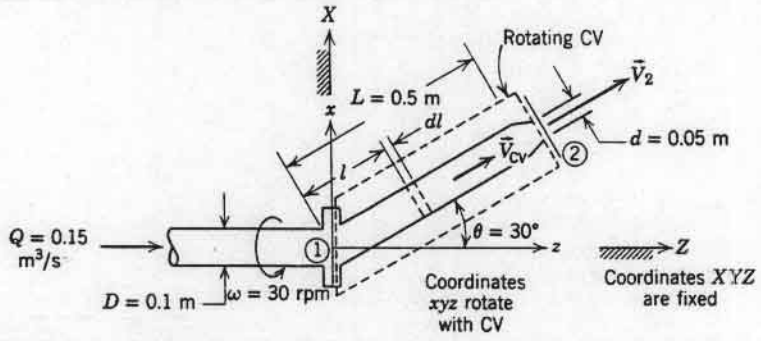
At z_{max}, $V_z = 0$, so $t = \frac{V_{z0}}{g}$; flight time is $2t$.

$$L = 2t\,V_{ax} = \frac{2V_{az}}{g}V_{ax} = \frac{2V_{rel}\sin\alpha(V_{rel}\cos\alpha - R\omega)}{g}$$

$$L = 2 \times 17\frac{m}{s} \times \sin 30° \left(17\frac{m}{s} \times \cos 30° - 0.2 m \times 6.04\frac{rad}{s}\right)\frac{s^2}{9.81 m} = 23.4 m$$

$R_{spray} = \sqrt{R^2 + L^2} = 23.4 m$; $A_{spray} = \pi R_{spray}^2 = \pi \times (23.4)^2 m^2 = 1720 m^2$ Aspra

Given: Nozzle assembly rotating steadily, as shown in the sketch.



Find: (a) Torque required to drive the nozzle assembly
(b) Reaction torques at the flange.

Solution: Apply the moment of momentum equation to the rotating CV shown.

Basic equation: $\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\,d\forall \;(\approx 0(2)) + \vec{T}_{shaft}$

$$-\int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \dot{\vec{\omega}}\times\vec{r}\right]\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

($\dot{\vec{\omega}}\times\vec{r}$: =0(3); $\frac{\partial}{\partial t}$: =0(5); $\vec{r}\times\vec{V}_{xyz}$: =0(7))

Assumptions: (1) Let $\vec{T}_{cv}$ represent all torques acting on the CV
(2) Neglect torque due to body force
(3) Constant angular speed
(4) Neglect mass of arm compared to water inside
(5) Steady flow in CV
(6) Neglect nozzle length compared to L
(7) $\vec{r}$ colinear with $\vec{V}$, so $\vec{r}\times\vec{V}_{xyz}=0$

Then
$$\vec{T}_{cv} = \int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r})\right]\rho\,d\forall$$

Since $\vec{\omega} = \omega\hat{k}$ and $\vec{r} = l(\sin\theta\hat{\imath} + \cos\theta\hat{k})$, then

$$\vec{\omega}\times\vec{r} = \omega l\sin\theta\,\hat{\jmath}$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \omega\hat{k}\times\omega l\sin\theta\,\hat{\jmath} = \omega^2 l\sin\theta\,(-\hat{\imath})$$

and $\vec{r}\times[\vec{\omega}\times(\vec{\omega}\times\vec{r})] = l(\sin\theta\hat{\imath}+\cos\theta\hat{k})\times\omega^2 l\sin\theta\,(-\hat{\imath}) = \omega^2 l^2\sin\theta\cos\theta\,(-\hat{\jmath})$

Since $\vec{V}_{xyz} = V_{cv}(\sin\theta\,\hat{\imath} + \cos\theta\,\hat{k})$, then

$$2\vec{\omega}\times\vec{V}_{xyz} = 2\omega\hat{k}\times V_{cv}(\sin\theta\hat{\imath}+\cos\theta\hat{k}) = 2\omega V_{cv}\sin\theta\,\hat{\jmath}$$

and $\vec{r}\times[2\vec{\omega}\times\vec{V}_{xyz}] = l(\sin\theta\hat{\imath}+\cos\theta\hat{k})\times 2\omega V_{cv}\sin\theta\hat{\jmath} = 2\omega l V_{cv}\sin^2\theta\,\hat{k}$

$$+ 2\omega l V_{cv}\sin\theta\cos\theta\,(-\hat{\imath})$$

Substituting and introducing $d\forall = A\,dl$,

$$\vec{F}_{cv} = \int_0^L (-2\omega l V_{cv}\sin\theta\cos\theta\,\hat{i} - \omega^2 l^2 \sin\theta\cos\theta\,\hat{j} + 2\omega l V_{cv}\sin^2\theta\,\hat{k})\rho A\,dl$$

$$\vec{F}_{cv} = \left[-\omega L^2 V_{cv}\sin\theta\cos\theta\,\hat{i} - \frac{\omega^2 L^3}{3}\sin\theta\cos\theta\,\hat{j} + \omega L^2 V_{cv}\sin^2\theta\,\hat{k}\right]\rho A$$

The shaft torque needed to maintain steady rotation of the assembly is

$$T_{shaft} = T_{cv_z} = \omega L^2 V_{cv}\sin^2\theta\,\rho A = \omega L^2 \frac{Q}{A}\sin^2\theta\,\rho A = \rho Q \omega L^2 \sin^2\theta$$

$$= 999\frac{kg}{m^3}\times 0.15\frac{m^3}{s}\times 30\frac{rev}{min}\times(0.5)^2 m^2\times(0.5)^2\times 2\pi\frac{rad}{rev}\times\frac{min}{60\,s}\times\frac{N\cdot s^2}{kg\cdot m}$$

$$T_{shaft} = 29.4\ N\cdot m \longleftarrow T_{shaft}$$

The reaction moments acting on the flange are

$$M_x = -T_{cv_x} = \omega L^2 V_{cv}\sin\theta\cos\theta\,\rho A = \rho Q \omega L^2 \sin\theta\cos\theta$$

$$= 999\frac{kg}{m^3}\times 0.15\frac{m^3}{s}\times 30\frac{rev}{min}\times(0.5)^2 m^2\times(0.5)(0.866)\,2\pi\frac{rad}{rev}\times\frac{min}{60\,s}\times\frac{N\cdot s^2}{kg\cdot m}$$

$$M_x = 51.0\ N\cdot m \text{ (applied to flange by CV)} \longleftarrow M_x$$

$$M_y = -T_{cv_y} = \frac{1}{3}\rho\omega^2 L^3 A\sin\theta\cos\theta$$

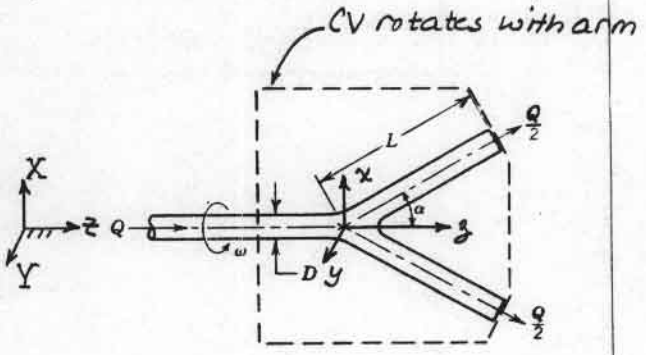
$$= \frac{1}{3}\times 999\frac{kg}{m^3}\left[30\frac{rev}{min}\times 2\pi\frac{rad}{rev}\times\frac{min}{60\,s}\right]^2(0.5)^3 m^3\times\frac{\pi}{4}(0.1)^2 m^2\times(0.5)(0.866)\times\frac{N\cdot s^2}{kg\cdot m}$$

$$M_y = 1.40\ N\cdot m \text{ (applied to flange by CV)} \longleftarrow M_y$$

{ Torques due to the masses of water, tube, and nozzle must be considered in the overall design. }

Given: Branched pipe with symmetrical legs as shown.

Angular momentum zero at inlet, relative to nonrotating frame.



Find: (a) External torque expression
(b) Additional torque to produce angular acceleration of $\dot{\omega}$.

Solution: Apply moment of momentum equation using rotating CV.

Basic equation: $\vec{r}\times\vec{F}_S + \int_{CV}\vec{r}\times\vec{g}\,\rho\,d\forall + \vec{T}_{shaft}$ ($\vec{r}\times\vec{F}_S$: =0(1); $\vec{r}\times\vec{g}$: =0(2))

$$-\int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz}+\vec{\omega}\times(\vec{\omega}\times\vec{r})+\dot{\vec{\omega}}\times\vec{r}\right]\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

($\frac{\partial}{\partial t}$: =0(3); $\vec{r}\times\vec{V}_{xyz}\,\rho\,\vec{V}_{xyz}$: =0(4))

Assumptions: (1) No surface forces
(2) Body-forces produce no torque about axis (symmetry)
(3) Flow steady in rotating frame
(4) $\vec{r}$ and $\vec{V}_{xyz}$ are colinear: $\vec{r}\times\vec{V}_{xyz}=0$

Then
$$\vec{T}_{shaft} = \int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz}+\vec{\omega}\times(\vec{\omega}\times\vec{r})+\dot{\vec{\omega}}\times\vec{r}\right]\rho\,d\forall$$

Using the coordinates above, $\vec{\omega}=\omega\hat{k}$ $\qquad \dot{\vec{\omega}}=\dot{\omega}\hat{k}$

$$\vec{r} = r(\cos\alpha\,\hat{k}+\sin\alpha\,\hat{\imath}) \quad \text{(upper tube)}$$

$$\vec{V}_{xyz} = \frac{Q}{2A}(\cos\alpha\,\hat{k}+\sin\alpha\,\hat{\imath}) \quad \text{(upper tube)};\ A=\frac{\pi D^2}{4}$$

and
$$\dot{\vec{\omega}}\times\vec{r} = \dot{\omega}r\sin\alpha\,\hat{\jmath}$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \omega\hat{k}\times\omega r\sin\alpha\,\hat{\jmath} = -\omega^2 r\sin\alpha\,\hat{\imath}$$

$$2\vec{\omega}\times\vec{V}_{xyz} = 2\omega\frac{Q}{2A}\sin\alpha\,\hat{\jmath} = \frac{\omega Q}{A}\sin\alpha\,\hat{\jmath}$$

Thus for the upper tube,

$$\vec{T}_{shaft} = \int_0^L\left\{r(\cos\alpha\,\hat{k}+\sin\alpha\,\hat{\imath})\times\left[\left(\frac{\omega Q}{A}+\dot{\omega}r\right)\sin\alpha\,\hat{\jmath}-\omega^2 r\sin\alpha\,\hat{\imath}\right]\right\}\rho A\,dr$$

$$= \int_0^L\left[\left(\frac{r\omega Q}{A}+\dot{\omega}r^2\right)(\sin\alpha\cos\alpha)\,\hat{\imath}+\left(\frac{r\omega Q}{A}+\dot{\omega}r^2\right)\sin^2\alpha\,\hat{k}+\omega^2 r^2\sin\alpha\cos\alpha(-\hat{\jmath})\right]\rho A\,dr$$

$$\vec{T}_{shaft\,(upper)} = \left(\frac{L^2\omega Q}{2A}+\frac{\dot{\omega}L^3}{3}\right)\sin\alpha\cos\alpha\,\hat{\imath}+\left(\frac{L^2\omega Q}{2A}+\frac{\dot{\omega}L^3}{2}\right)\sin^2\alpha\,\hat{k}+\frac{\omega^2 L^3}{3}\sin\alpha\cos\alpha(-\hat{\jmath})\Big]\rho A$$

For the lower tube, $\vec{\omega} = \omega\hat{k}$ $\qquad \dot{\vec{\omega}} = \dot{\omega}\hat{k}$

$$\vec{r} = r(\cos\alpha\,\hat{k} - \sin\alpha\,\hat{\imath}) \quad \text{(lower tube)}$$

$$\vec{V}_{xyz} = \frac{Q}{2A}(\cos\alpha\,\hat{k} - \sin\alpha\,\hat{\jmath}) \quad \text{(lower tube)}$$

and

$$\dot{\vec{\omega}} \times \vec{r} = -r\dot{\omega}\sin\alpha\,\hat{\jmath}$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \omega\hat{k} \times (-r\omega\sin\alpha\,\hat{\jmath}) = r\omega^2\sin\alpha\,\hat{\imath}$$

$$2\vec{\omega} \times \vec{V}_{xyz} = 2\omega\frac{Q}{2A}(-\sin\alpha)(\hat{\jmath}) = -\frac{\omega Q}{A}\sin\alpha\,\hat{\jmath}$$

Thus for the lower tube,

$$\vec{T}_{shaft} = \int_0^L \left\{ r(\cos\alpha\,\hat{k} - \sin\alpha\,\hat{\imath}) \times \left[\left(\frac{r\omega Q}{A} + r\dot{\omega} \right)\sin\alpha(-\hat{\jmath}) + r\omega^2\sin\alpha \right] \right\} \rho A\,dr$$

$$= \int_0^L \left[\left(\frac{r\omega Q}{A} + r^2\dot{\omega} \right)\sin\alpha\cos\alpha(-\hat{\imath}) + \left(\frac{r\omega Q}{A} + r^2\dot{\omega} \right)\sin^2\alpha\,\hat{k} + r^2\omega^2\sin\alpha\cos\alpha\,\hat{\jmath} \right] \rho A\,dr$$

$$\vec{T}_{shaft\,(lower)} = \left[\left(\frac{L^2\omega Q}{2A} + \frac{L^3\dot{\omega}}{3} \right)\sin\alpha\cos\alpha\,\hat{\imath} + \left(\frac{L^2\omega Q}{2A} + \frac{L^3\dot{\omega}}{3} \right)\sin^2\alpha\,\hat{k} + \frac{L^3\omega^2}{3}\sin\alpha\cos\alpha\,\hat{\jmath} \right] \rho A$$

Summing these expressions gives

$$\vec{T}_{shaft\,(total)} = \left(\frac{L^2\omega Q}{A} + \frac{2L^3\dot{\omega}}{3} \right)\sin^2\alpha\,\rho A\,\hat{k}$$

Thus the steady-state portion of the torque is

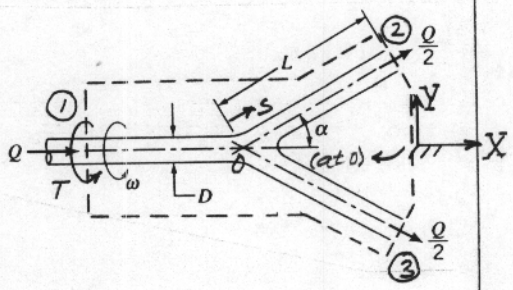
$$\vec{T}_{shaft\,(steady\ state)} = \left(\frac{L^2\omega Q}{A} \right)\sin^2\alpha\,\rho A\hat{k} = L^2\rho\omega Q\sin^2\alpha\,\hat{k}$$ Steady

The additional torque needed to provide angular acceleration, $\dot{\omega}$, is

$$\vec{T}_{shaft\,(acceleration)} = \frac{2L^3\rho\dot{\omega}A}{3}\sin^2\alpha\,\hat{k}$$ Accel

{ Torques of individual tubes about the x and y axes are reacted internally; they must be considered in design of the tube. }

(b) Using fixed CV:



Basic equation: $\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\,d\forall + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}\,\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}\,\rho\,\vec{V}\cdot d\vec{A}$

($\vec{r}\times\vec{F}_s = 0$ (1); $\vec{r}\times\vec{g}$ term $= 0$ (2); $\frac{\partial}{\partial t}$ term $= 0$ (3))

Assumptions: (1) No surface forces
(2) Body forces symmetric (no moment about X axis)
(3) No change in angular momentum within CV w.r.to time
(4) Symmetry in two branches
(5) Uniform flow at each cross-section

Then $\vec{T}_s = T\hat{I} = \vec{r}_1\times\vec{V}_1\{-\rho Q\} + \vec{r}_2\times\vec{V}_2\{+\rho\frac{Q}{2}\} + \vec{r}_3\times\vec{V}_3\{+\rho\frac{Q}{2}\} = 2\vec{r}_2\times\vec{V}_1\{\rho\frac{Q}{2}\}$

$\vec{r}_1 = 0 \qquad \vec{r}_2 = L\sin\alpha\,\hat{J}\;;\; \vec{V}_2 = wr_2\,\hat{K}\;;\; \vec{r}_2\times\vec{V}_2 = wL^2\sin^2\alpha\,\hat{I}$

or

$$T_{ss} = \rho wQL^2\sin^2\alpha \quad \text{(steady-state torque)} \qquad T_{ss}$$

The torque required for acceleration is $T_{acc} = I\dot{w}$, where $I = \int r^2\,dm$

For one leg of the branch, $I = \int r^2\,dm = \int_0^L (s\sin\alpha)^2\rho A\,ds = \frac{\rho AL^3}{3}\sin^2\alpha$

(6) Neglect mass of pipe

For both sides, $I = \frac{2\rho AL^3}{3}\sin^2\alpha$.

Thus

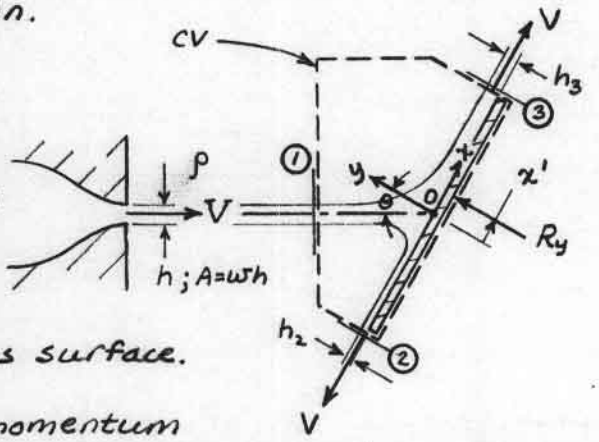
$$T_{acc} = \frac{2\rho\dot{w}AL^3}{3}\sin^2\alpha \quad \text{(torque required for angular acceleration)} \qquad T_{acc}$$

The total torque that must be applied is

$$T = T_{ss} + T_{acc} = \rho wQL^2\sin^2\alpha + \frac{2\rho\dot{w}AL^3}{3}\sin^2\alpha \qquad T_{total}$$

Given: Thin sheet of liquid, of width, w, and thickness, h, striking inclined flat plate, as shown.

Neglect any viscous effects.



Find: (a) Magnitude and line of action of resultant force as functions of θ.

(b) Equilibrium angle of plate if force is applied at point O, where jet centerline intersects surface.

Solution: Apply continuity, linear momentum and moment of momentum using CV and coordinates shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ (=0(1))

$F_{Sx} + F_{\not{B}x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$ (=0(4), =0(5), =0(1))

$F_{Sy} + F_{\not{B}y} = \frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall + \int_{CS} v\rho \vec{V}\cdot d\vec{A}$ (=0(5), =0(1))

$\vec{r}\times\vec{F}_S + \int_{CV} \vec{r}\times\vec{g}\,\rho\, d\forall + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV} \vec{r}\times\vec{V}\rho\, d\forall + \int_{CS} \vec{r}\times\vec{V}\rho\vec{V}\cdot d\vec{A}$ (=0(5), =0(6), =0(1))

Assumptions: (1) Steady flow
(2) Uniform flow at each section
(3) No net pressure forces; $F_{Sx} = R_x$, $F_{Sy} = R_y$
(4) No viscous effects; $R_x = 0$ and $V_1 = V_2 = V_3 = V$
(5) Neglect body forces and torques
(6) $\vec{T}_{shaft} = 0$
(7) Incompressible flow, ρ = constant

Then from continuity,

$$0 = \{-|\rho V w h_1|\} + \{|\rho V w h_2|\} + \{|\rho V w h_3|\} \quad \text{or} \quad h_1 = h_2 + h_3 = h \qquad (1)$$

From x momentum

$$0 = u_1\{-|\rho V w h_1|\} + u_2\{|\rho V w h_2|\} + u_3\{|\rho V w h_3|\}$$

$$u_1 = V\sin\theta \qquad u_2 = -V \qquad u_3 = V$$

$$0 = \rho V^2 w(-h_1\sin\theta - h_2 + h_3) \quad \text{or} \quad h_3 - h_2 = h_1\sin\theta = h\sin\theta \qquad (2)$$

Combining Eqs. 1 and 2, $h_2 = h\left(\frac{1-\sin\theta}{2}\right)$ (3)

$$h_3 = h\left(\frac{1+\sin\theta}{2}\right) \qquad (4)$$

From y momentum, $R_y = v_1\{-|\rho V w h_1|\} + v_2\{|\rho V w h_2|\} + v_3\{|\rho V w h_3|\}$

$$v_1 = -V\cos\theta \qquad v_2 = 0 \qquad v_3 = 0$$

$$R_y = \rho V^2 w h \cos\theta \tag{5}$$ R_y

From moment of momentum,

$$\vec{r}' \times \vec{F}_s = \vec{r}_1 \times \vec{V}_1\{-|\rho V w h_1|\} + \vec{r}_2 \times \vec{V}_2\{|\rho V w h_2|\} + \vec{r}_3 \times \vec{V}_3\{|\rho V w h_3|\}$$

$\vec{r}' = x'\hat{\imath}$ $\quad \vec{r}_1 \times \vec{V}_1 = 0$ $\quad \vec{r}_2 = \frac{h_2}{2}\hat{\jmath}$ $\quad \vec{r}_3 = \frac{h_3}{2}\hat{\jmath}$

$\vec{F}_s = R_y\hat{\jmath}$ $\quad \vec{V}_2 = -V\hat{\imath}$ $\quad \vec{V}_3 = V\hat{\imath}$

$\vec{r}' \times \vec{F}_s = x' R_y \hat{k}$ $\quad \vec{r}_2 \times \vec{V}_2 = \frac{h_2 V}{2}\hat{k}$ $\quad \vec{r}_3 \times \vec{V}_3 = -\frac{h_3 V}{2}\hat{k}$

Combining and dropping $\hat{k}$,

$$x' R_y = \tfrac{1}{2}\rho V^2 w h_2^2 - \tfrac{1}{2}\rho V^2 w h_3^2 = \tfrac{1}{2}\rho V^2 w (h_2^2 - h_3^2)$$

or

$$x' = \frac{\rho V^2 w (h_2^2 - h_3^2)}{2R_y} = \frac{\rho V^2 w (h_2 + h_3)(h_2 - h_3)}{2R_y}$$

Substituting from Eqs. 3, 4 and 5,

$$x' = \frac{\rho V^2 w h^2 \left(\frac{1-\sin\theta}{2} + \frac{1+\sin\theta}{2}\right)\left(\frac{1-\sin\theta}{2} - \frac{1+\sin\theta}{2}\right)}{2\rho V^2 w h \cos\theta} = \frac{h(-\sin\theta)}{2\cos\theta}$$

or

$$x' = -\frac{h}{2}\tan\theta \tag{6}$$ x'

Note that $x' < 0$. This means that R_y must be applied <u>below</u> point 0.

If R_y is applied <u>at</u> point 0, then $x' = 0$. For equilibrium, from Eq. 6, $\theta = 0$. Thus if force is applied at point 0, plate will be in equilibrium when <u>perpendicular</u> to jet.

Given: The rotating lawn sprinkler of Example Problem 4.14.

Find: (a) Jet angle α for maximum speed of rotation.
(b) What jet angle will provide the maximum area of coverage by the spray?
(c) Draw a velocity diagram to show the absolute velocity of the water jet leaving the nozzle.
(d) What governs the steady rotational speed of the sprinkler?
(e) Does the rotational speed of the sprinkler affect the area covered by the spray?
(f) How would you estimate the area of coverage?
(g) For fixed α, what might be done to increase or reduce the area covered by the spray?

Solution: The results of Example Problem 4.14 were computed assuming steady flow of water and constant frictional retarding torque at the sprinkler pivot.

$$T_f = R(V_{rel}\cos\alpha - \omega R)\rho Q$$

From these results,

$$\omega = \frac{V_{rel}\cos\alpha}{R} - \frac{T_f}{\rho Q R^2}$$

Thus rotational speed of the sprinkler increases as cos α increases, i.e., as α decreases. The maximum rotational speed occurs when $\alpha = 0$. Then $\cos\alpha = 1$ and the rotational speed is

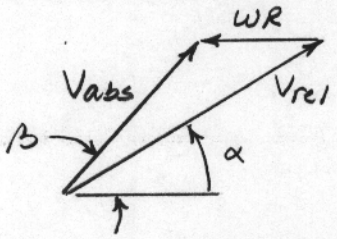
$$\omega = \frac{V_{rel}}{R} - \frac{T_f}{\rho Q R^2}$$

For the conditions of Example Problem 4.14 the maximum rotational speed is

$$\omega = 4.97\frac{m}{s}\times\frac{1}{0.150\,m} - 0.0718\,N\cdot m\times\frac{m^3}{994\,kg}\times\frac{min}{7.5\,L}\times\frac{1}{(0.150)^2 m^2}\times\frac{1000\,L}{m^3}\times\frac{60\,s}{min} = 7.58\ rad/s \quad \omega_{max}$$

The steady rotation speed ω of the sprinkler is governed by torque T_f and angle α.

Maximum coverage by the spray occurs when the "carry" of each jet stream is the longest. When aerodynamic drag on the stream is neglected, maximum carry occurs when the absolute velocity of the stream leaves the sprinkler at β = 45°, as shown in the velocity diagram below.



Note $\vec{V}_{abs} = \vec{V}_{rel} - \omega R\,\hat{t}_\theta$

Both the magnitude and direction of $\vec{V}_{abs}$ vary with ω!

For ω = 0, the relative velocity angle α and absolute velocity angle β are equal. Therefore maximum carry occurs when α =45° (see graph on next page).

Any rotation rate ω reduces the magnitude V_{abs} and increases the angle β of the absolute velocity leaving the sprinkler jet. When ω > 0, then β > α, so for maximum carry α must be less than 45°. Consequently rotation reduces the carry of the stream and the area of coverage; at specified α the area of coverage decreases with increasing ω.

For the conditions of Example Problem 4.14 (ω = 30 rpm), optimum carry occurs at α ≈ 42°, and the coverage area is reduced from approximately 20 m^2 with a fixed sprinkler to 15 m^2 with 30 rpm rotation. If the rotation speed is increased (by decreasing pivot friction or decreasing nozzle angle α), coverage area may be reduced still further, to 9 m^2 or less.

$A \approx \pi (x_{max})^2$

Analysis of Ground Area Covered by Rotating Lawn Sprinkler:

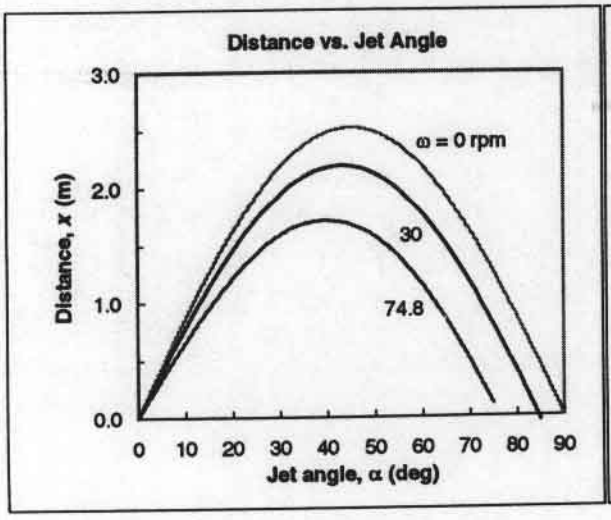
Variables:
A = ground area covered by spray stream
x = ground distance reached by spray stream
α = angle of jet above ground plane
β = angle of absolute velocity above ground plane

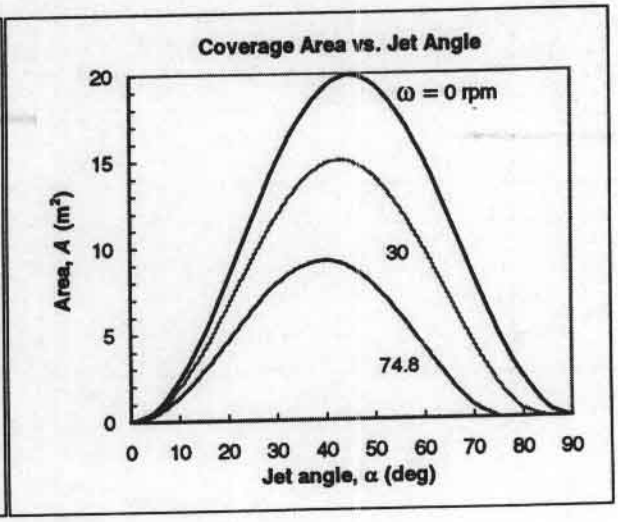
Input Data:
R = 0.150 m
V_{rel} = 4.97 m/s (Q = 7.5 L/min)

Results:

ω (rpm) =	0	30	74.8
ωR (m/s) =	0	0.471	1.17

α (deg)	x_{max} (m)	A (m^2)	x_{max} (m)	A (m^2)	x_{max} (m)	A (m^2)
0	0.00	0.00	0.00	0.00	0.00	0.00
5	0.437	0.601	0.396	0.492	0.333	0.349
10	0.861	2.33	0.778	1.90	0.654	1.35
15	1.26	4.98	1.14	4.05	0.951	2.84
20	1.62	8.23	1.46	6.65	1.21	4.61
25	1.93	11.7	1.73	9.37	1.43	6.39
30	2.18	14.9	1.94	11.8	1.59	7.90
35	2.37	17.6	2.09	13.8	1.68	8.90
40	2.48	19.3	2.17	14.8	1.71	9.23
45	2.52	19.9	2.18	14.9	1.68	8.83
50	2.48	19.3	2.11	14.0	1.57	7.72
55	2.37	17.6	1.97	12.3	1.39	6.08
60	2.18	14.9	1.77	9.81	1.15	4.15
65	1.93	11.7	1.50	7.03	0.850	2.269
70	1.62	8.23	1.17	4.30	0.500	0.785
75	1.26	4.98	0.798	2.00	0.109	0.037
78	1.02	3.30	0.557	0.975		
80	0.861	2.33	0.391	0.480		
85	0.437	0.601	-0.04	0.00		
90	0.00	0.00				



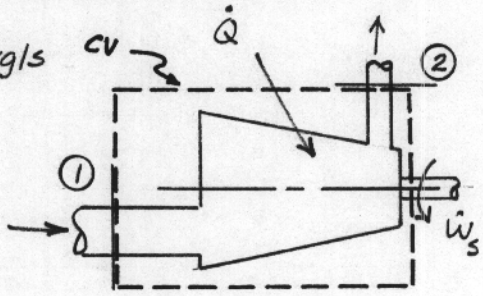


Given: Compressor, $\dot{m} = 1.0$ kg/s

$p_1 = 101$ kPa (abs)

$T_1 = 288$ K

$V_1 = 75$ m/s



$p_2 = 200$ kPa (abs)

$T_2 = 345$ K

$V_2 = 125$ m/s

$\frac{dQ}{dm} = -18$ kJ/kg

Find: Power required.

Solution: Apply first law of thermodynamics, using CV shown.

B.E. $\dot{Q} - \dot{W}_s - \dot{W}_{shear} = \frac{\partial}{\partial t}\int_{CV} e\rho d\forall + \int_{CS}(e + pv)\rho\vec{V}\cdot d\vec{A}$

($\dot{W}_{shear} = 0$ (1); $\frac{\partial}{\partial t} = 0$ (2))

Assume: (1) $\dot{W}_{shear} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Neglect Δz
(5) Ideal gas, $p = \rho RT$, $\Delta h = c_p \Delta T$; $c_p = 1.00$ kJ/kg·K
(6) From continuity, $\dot{m}_1 = \dot{m}_2 = \dot{m}$

Then

$$\dot{Q} - \dot{W}_s = \left(u_2 + \frac{V_2^2}{2} + g\underbrace{z_2}_{\approx 0(4)} + p_2 v_2\right)\{|\dot{m}|\} + \left(u_1 + \frac{V_1^2}{2} + g\underbrace{z_1}_{\approx 0(4)} + p_1 v_1\right)\{-|\dot{m}|\}$$

Note that $h = u + pv$, and $\dot{Q} = \dot{m}\frac{dQ}{dm}$, so

$$\dot{W}_{in} = -\dot{W}_s = \dot{m}\left(\frac{V_2^2 - V_1^2}{2} + h_2 - h_1 - \frac{dQ}{dm}\right) = \dot{m}\left[\frac{V_2^2 - V_1^2}{2} + c_p(T_2 - T_1) - \frac{dQ}{dm}\right]$$

or

$$\dot{W}_{in} = 1.0\frac{kg}{s}\left\{\frac{1}{2}\left[(125)^2 - (75)^2\right]\frac{m^2}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \times \frac{kJ}{1000 N\cdot m}\right.$$

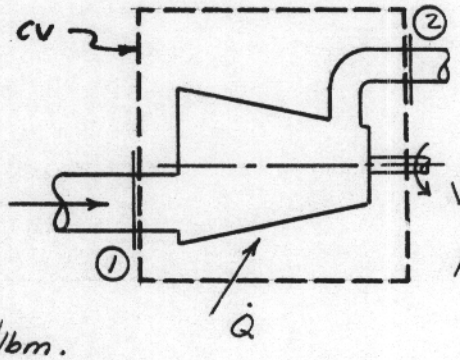
$$\left. + 1.00\frac{kJ}{kg\cdot K}(345 - 288)K - \left(-18\frac{kJ}{kg}\right)\right\}\frac{kW\cdot s}{kJ}$$

$$\dot{W}_{in} = 80.0 \text{ kW}$$

$\dot{W}_{in}$

Given: Compressor operating at conditions shown. $p_2 = 70$ psia

$\dot{m} = 20 \frac{lbm}{s}$, $p_1 = 14$ psia, $T_1 = 80°F$, $V_1 \simeq 0$



$T_2 = 500°F$, $V_2 = 500 \frac{ft}{s}$, $\dot{W}_{in} = 3200$ hp, Fluid is air.

Find: Heat transfer, in Btu/lbm.

Solution: Apply energy equation to CV shown.

Basic equations: $p = \rho RT$, $\Delta h = c_p \Delta T$

$$\dot{Q} - \dot{W}_s - \cancel{\dot{W}_{shear}}^{=0(2)} - \cancel{\dot{W}_{other}}^{=0(2)} = \cancel{\frac{\partial}{\partial t}}^{=0(3)} \int_{CV} e\rho dV + \int_{CS} \left(u + pv + \frac{V^2}{2} + \cancel{gz}^{=0(5)}\right) \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Ideal gas, constant specific heat
(2) $\dot{W}_{shear} = 0$ by choice of CV; $\dot{W}_{other} = 0$
(3) Steady flow
(4) Uniform flow at each section
(5) Neglect Δz
(6) $V_1 \simeq 0$

By definition $h \equiv u + pv$, so

$$\dot{Q} - \dot{W}_s = \left(h_1 + \cancel{\frac{V_1^2}{2}}^{\simeq 0(6)}\right)\{-|\dot{m}|\} + \left(h_2 + \frac{V_2^2}{2}\right)\{|\dot{m}|\} = \dot{m}\left[\frac{V_2^2}{2} + c_p(T_2 - T_1)\right]$$

or

$$\frac{\delta Q}{dm} = \frac{\dot{Q}}{\dot{m}} = \frac{\dot{W}_s}{\dot{m}} + \frac{V_2^2}{2} + c_p(T_2 - T_1)$$

Noting $\dot{W}_s = -3200$ hp, so

$$\frac{\delta Q}{dm} = -3200\ hp \times \frac{2545\ Btu}{hp \cdot hr} \times \frac{s}{20\ lbm} \times \frac{hr}{3600\ s} + 0.240 \frac{Btu}{lbm \cdot °F} \times (500 - 80)°F$$

$$+ \frac{(500)^2\ ft^2}{2\ s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \frac{slug}{32.2\ lbm} \times \frac{Btu}{778\ ft \cdot lbf}$$

$$\frac{\delta Q}{dm} = -7.32\ Btu/lbm \quad \leftarrow \frac{\delta Q}{dm}$$

Therefore heat transfer is <u>out of</u> CV, since $\delta Q/dm < 0$. The rate of heat transfer is

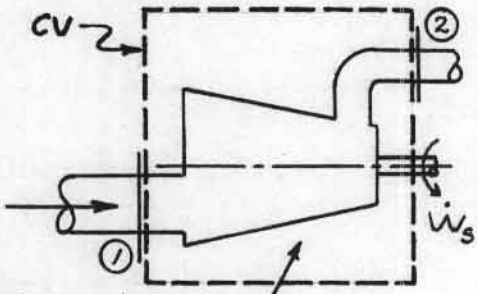
$$\dot{Q} = -7.32 \frac{Btu}{lbm} \times 20 \frac{lbm}{s} = -146\ Btu/s \quad \leftarrow \dot{Q}$$

Given: Flow through turbomachine shown. Fluid is air.

$\dot{m} = 0.8$ kg/s

$T_1 = 288$ K

$p_1 = 101$ kPa (abs)



$T_2 = 130°C$

$p_2 = 500$ kPa (gage)

$V_2 = 100$ m/s

$V_1 \simeq 0$ (from atmosphere)

$\dot{Q} = 0$

Find: Shaft work interaction with surroundings.

Solution: Apply energy equation, using CV shown.

Basic equations: $p = \rho RT$, $\Delta h = c_p \Delta T$

$$\overset{=0(1)}{\dot{Q}} - \dot{W}_s - \overset{=0(2)}{\dot{W}_{shear}} - \overset{=0(2)}{\dot{W}_{other}} = \overset{=0(3)}{\frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall} + \int_{CS}\left(u + pv + \frac{V^2}{2} + \overset{\simeq 0(5)}{gz}\right)\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Ideal gas, constant specific heat
(2) $\dot{W}_{shear} = 0$ by choice of CV; $\dot{W}_{other} = 0$
(3) Steady flow
(4) Uniform flow at each section
(5) Neglect Δz
(6) $V_1 \simeq 0$
(7) $\dot{Q} = 0$

By definition, $h \equiv u + pv$, so

$$-\dot{W}_s = \left(h_1 + \overset{\simeq 0(1)}{\frac{V_1^2}{2}}\right)\{-|\dot{m}|\} + \left(h_2 + \frac{V_2^2}{2}\right)\{|\dot{m}|\} = \dot{m}\left(h_2 - h_1 + \frac{V_2^2}{2}\right)$$

or

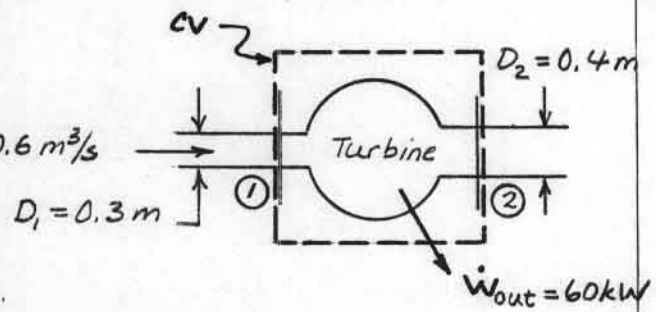
$$-\dot{W}_s = \dot{m}\left(h_2 - h_1 + \frac{V_2^2}{2}\right) = \dot{m}\left[c_p(T_2 - T_1) + \frac{V_2^2}{2}\right]$$

$$= 0.8\ \frac{kg}{s}\left[1.00\ \frac{kJ}{kg\cdot K}(403 - 288)\,K + (100)^2\ \frac{m^2}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \times \frac{kJ}{10^3\ N\cdot m}\right]\frac{kW\cdot s}{kJ}$$

$-\dot{W}_s = 96.0$ kW or $\dot{W}_s = -96.0$ kW $\dot{W}_s$ ←

{Power is into CV because $\dot{W}_s < 0$.}

Given: Turbine operating on water.



Find: Pressure drop across turbine.

Solution: Apply continuity, energy equations, using CV shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ ($=0(1)$)

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall + \int_{CS} \left(u + \frac{V^2}{2} + gz + pv\right)\rho \vec{V}\cdot d\vec{A}$$

($\dot{Q}$: $=0(4)$; $\dot{W}_s$: $=0(5)$; $\dot{W}_{shear}$: $=0(5)$; $\dot{W}_{other}$: $=0(1)$; u: $\approx 0(6)$; gz: $\approx 0(7)$)

Assumptions: (1) Steady flow
(2) Uniform flow at each section
(3) Incompressible flow
(4) $\dot{Q} = 0$
(5) $\dot{W}_{shear} = 0$ by choice of CV; $\dot{W}_{other} = 0$
(6) Neglect Δu
(7) Neglect Δz

Then

$$0 = \{-|\rho V_1 A_1|\} + \{|\rho V_2 A_2|\} \quad \text{or} \quad V_2 = V_1 \frac{A_1}{A_2} = V_1\left(\frac{D_1}{D_2}\right)^2$$

and

$$-\dot{W}_s = \left(\frac{V_1^2}{2} + p_1 v\right)\{-|\rho V_1 A_1|\} + \left(\frac{V_2^2}{2} + p_2 v\right)\{|\rho V_2 A_2|\}$$

$$-\dot{W}_s = -\left[\frac{V_1^2 - V_2^2}{2} + (p_1 - p_2)v\right]\rho Q = -\left\{\frac{V_1^2}{2}\left[1 - \left(\frac{D_1}{D_2}\right)^4\right] + (p_1 - p_2)v\right\}\rho Q$$

or

$$p_1 - p_2 = \frac{1}{v}\left\{\frac{\dot{W}_s}{\rho Q} - \frac{V_1^2}{2}\left[1 - \left(\frac{D_1}{D_2}\right)^4\right]\right\} = \frac{\dot{W}_s}{Q} - \frac{\rho V_1^2}{2}\left[1 - \left(\frac{D_1}{D_2}\right)^4\right]$$

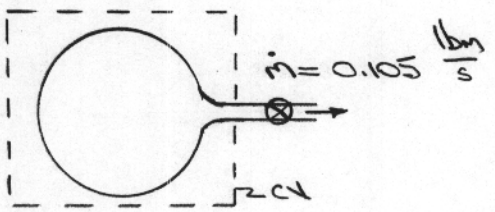
But $V_1 = \frac{Q}{A_1} = \frac{0.6\ m^3}{s} \times \frac{4}{\pi (0.3)^2 m^2} = 8.49\ m/s$, and $\dot{W}_s = \dot{W}_{out} = 60\ kW$, so

$$p_1 - p_2 = (60\ kW)\frac{10^3\ N\cdot m}{kW\cdot s} \times \frac{s}{0.6\ m^3} - \frac{1}{2} \times 999\ \frac{kg}{m^3} \times (8.49)^2 \frac{m^2}{s^2}\left[1 - \left(\frac{0.3}{0.4}\right)^4\right]\frac{N\cdot s^2}{kg\cdot m}$$

$$p_1 - p_2 = 75.4\ kPa$$

$p_1 - p_2$

Given: Pressure bottle, $\forall = 10\ ft^3$
contains compressed air at
$p = 3000$ psia, $T = 140°F$
At $t=0$, $\dot{m} = 0.105\ lbm/s$



Find: $\partial T/\partial t$ at $t=0$

Solution: Use CV shown

Basic equations:

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{oth} = \frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho\vec{V}\cdot d\vec{A}$$

$$e = u + \frac{V^2}{2} + gz$$

Assumptions: (1) $\dot{Q} = 0$ (insulated)
(2) $\dot{W}_s = 0$
(3) $\dot{W}_{shear} = \dot{W}_{oth} = 0$
(4) neglect V^2
(5) neglect z
(6) perfect gas, $u = c_v T$
(7) properties uniform in bottle and at exit

From continuity, $0 = \frac{\partial M_{CV}}{\partial t} + \dot{m}$ $\quad\therefore\quad \frac{\partial M_{CV}}{\partial t} = -\dot{m}$

From the first law,

$$0 = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \left(u + \frac{p}{\rho}\right)\dot{m}$$

$$= u\frac{\partial M}{\partial t} + M\frac{\partial u}{\partial t} + \left(u + \frac{p}{\rho}\right)\dot{m}$$

$$0 = u(-\dot{m}) + M c_v \frac{\partial T}{\partial t} + \left(u + \frac{p}{\rho}\right)\dot{m}$$

Thus,

$$\frac{\partial T}{\partial t} = -\frac{\dot{m}\,p/\rho}{M c_v} = \frac{-\dot{m}\,p}{\rho\forall c_v p} = -\frac{\dot{m}\,p}{\forall c_v \rho^2}$$

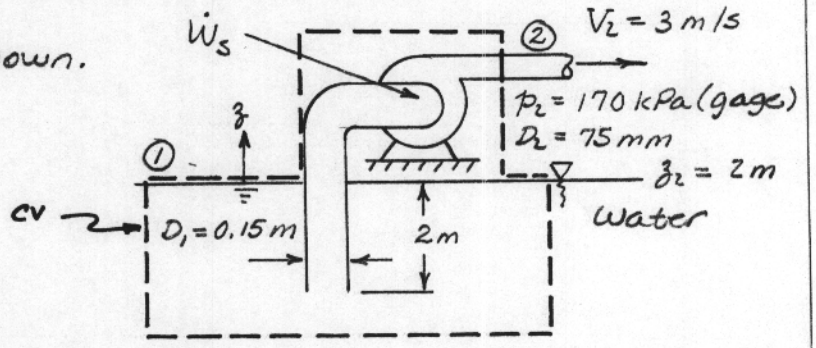
where $\rho = \frac{p}{RT} = 3000\frac{lbf}{in^2} \times \frac{144\ in^2}{ft^2} \times \frac{lbm\cdot R}{53.3\ ft\cdot lbf} \times \frac{1}{600R} = 13.5\ \frac{lbm}{ft^3}$

$$\therefore \frac{\partial T}{\partial t} = -0.1\frac{lbm}{s} \times 3000\frac{lbf}{in^2} \times 144\frac{in^2}{ft^2} \times \frac{1}{10\ ft^3} \times 0.171\frac{Btu}{lbm\cdot R} \times \frac{778\ ft\cdot lbf}{Btu} \times (13.5)^2\frac{ft^6}{lbm^2}$$

$$\frac{\partial T}{\partial t} = -0.178\ °R/s$$

Given: Pump system as shown.

$\eta_{pump} = 0.75$



Find: Power required.

Solution: Apply first law to CV shown, noting that flow enters with negligible velocity at section ①.

Basic equation:

$$\dot{Q} - \dot{W}_{shaft} - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall + \int_{CS}\left(e + \frac{p}{\rho}\right)\rho\vec{V}\cdot d\vec{A}$$

($\dot{W}_{shear}$: $=0$ (1); $\dot{W}_{other}$: $=0$ (1); $\frac{\partial}{\partial t}\int_{CV}$: $=0$ (2))

$$e = u + \frac{V^2}{2} + gz$$

Assumptions: (1) $\dot{W}_{shear} = \dot{W}_{other} = 0$
(2) Steady flow
(3) $V_1 \simeq 0$
(4) $z_1 = 0$
(5) $p_1 = 0$ (gage)
(6) Uniform flow at each section
(7) Incompressible flow; $V_1 A_1 = V_2 A_2$

Then ($\simeq 0$ (3); $=0$ (4); $=0$ (5))

$$\dot{Q} - \dot{W}_s = \left(u_1 + \frac{V_1^2}{2} + gz_1 + \frac{p_1}{\rho}\right)\{-\dot{m}\} + \left(u_2 + \frac{V_2^2}{2} + gz_2 + \frac{p_2}{\rho}\right)\{\dot{m}\}$$

or

$$-\dot{W}_s = \dot{m}\left[\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \left(u_2 - u_1 - \frac{\delta Q}{dm}\right)\right]$$

Obtain the ideal or minimum power input by neglecting thermal effects.

Thus

$$-\dot{W}_{s,ideal} = \dot{m}\left[\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2\right]$$

For the system,

$$\dot{m} = \rho V_2 A_2 = 999\,\frac{kg}{m^3} \times 3\,\frac{m}{s} \times \frac{\pi}{4}(0.075)^2 m^2 = 13.2\ kg/s$$

and

$$-\dot{W}_{s,ideal} = 13.2\,\frac{kg}{s}\left[1.70\times10^5\,\frac{N}{m^2}\times\frac{m^3}{999\,kg} + \frac{1}{2}(3)^2\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} + 9.81\,\frac{m}{s^2}\times 2m\times\frac{N\cdot s^2}{kg\cdot m}\right]$$

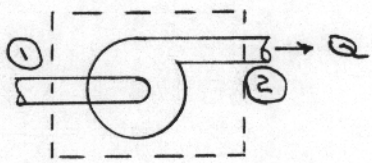
$$\dot{W}_{s,ideal} = -2560\,\frac{N\cdot m}{s}\times\frac{kW\cdot s}{10^3\,N\cdot m} = -2.56\ kW$$

Finally

$$\dot{W}_{s,actual} = \frac{\dot{W}_{s,ideal}}{\eta} = \frac{-2.56\ kW}{0.75} = -3.41\ kW$$

$\dot{W}_{s,actua}$ ←

Given: Centrifugal water pump operating under conditions as follows:



$D_1 = D_2 = 4$ in. $Q = 300$ gpm

$p_1 = 8$ in Hg (vacuum), $p_2 = 35$ psig $z_1 = z_2$

$\mathcal{P}_{input} = 9.1$ hp

Find: pump efficiency.

Solution: Apply the energy equation to the CV shown. Neglect all losses to find the energy added to the fluid.

Basic equations: $\eta = \dfrac{\dot{W}_s}{\mathcal{P}_{in}}$ where $\dot{W}_s$ = power into fluid

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho\, \vec{V}\cdot d\vec{A}$$

(1) $\dot{Q} = 0$; (2) $\dot{W}_{shear} = 0$ (by choice of CV); $\dot{W}_{other} = 0$; (3) steady flow; (4) neglect Δu; (5) $\Delta z = 0$; (6) incompressible flow; (7) uniform flow at inlet and outlet

Then $$-\dot{W}_s = \left(p_1 v_1 + \frac{V_1^2}{2}\right)\{-\dot{m}\} + \left(p_2 v_2 + \frac{V_2^2}{2}\right)\{\dot{m}\}$$

Since $\dot{m} = \rho Q$ and $V_1 = V_2$ (from continuity)

$$-\dot{W}_s = \rho Q(p_2 v_2 - p_1 v_1) = Q(p_2 - p_1)$$

$p_1 = \rho g h = SG\,\rho_{H_2O}\, g h$

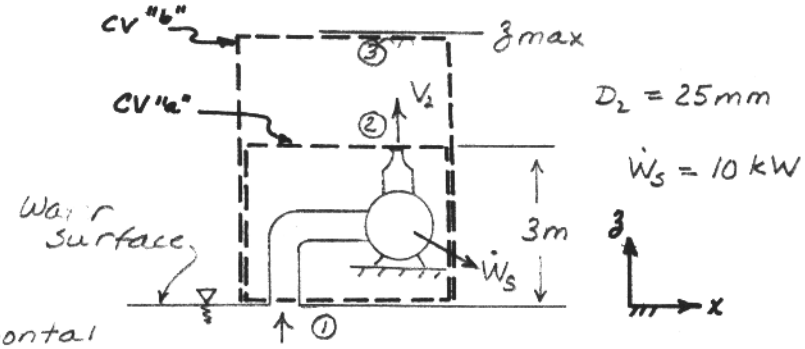
$$p_1 = 13.6 \times 1.94 \frac{slug}{ft^3} \times 32.2 \frac{ft}{s^2} \times (-8\,in) \times \frac{ft}{12\,in} \times \frac{lbf \cdot s^2}{ft \cdot slug} \times \frac{ft^2}{144\,in^2} = -3.93 \text{ psig}$$

$$\therefore -\dot{W}_s = 300 \frac{gal}{min} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} \times [35 - (-3.93)]\frac{lbf}{in^2} \times \frac{144\,in^2}{ft^2} \times \frac{hp \cdot s}{550\,ft \cdot lbf}$$

$\dot{W}_s = -6.81$ hp (negative sign indicates energy added).

Then $$\eta = \frac{\dot{W}_s}{\mathcal{P}_{in}} = \frac{6.81}{9.1} = 0.748 \text{ or } 74.8 \text{ percent}$$

Given: Fire boat



Find: (a) Q_2
(b) z_{max}
(c) Force if horizontal

Solution: Apply first law to CV "a" shown above

Basic equation: $\dot{Q}^{(1)} - \dot{W}_s - \dot{W}^{=0(2)}_{shear} - \dot{W}^{=0(2)}_{other} = \frac{\partial}{\partial t}\int_{CV} e\rho d\forall + \int_{CS}(e+pv)\rho\vec{V}\cdot d\vec{A}^{=0(3)}$

Assume: (1) Neglect losses, i.e. $u_2 - u_1 - \frac{dQ}{dm} \approx 0$ $\qquad e = u^{(1)} + \frac{V^2}{2} + gz$
(2) $\dot{W}_{shear} = \dot{W}_{other} = 0$
(3) Steady flow
(4) Uniform flow at each section
(5) Neglect V_1
(6) $z_1 = 0$
(7) Incompressible flow, $v_2 = v_1 = v$, $\dot{m}_1 = \dot{m}_2 = \dot{m}_3$
(8) $p_2 = p_1 = p_{atm} = 0$ gage

Then

$$\dot{Q} - \dot{W}_s = \left(u_1 + \frac{V_1^2}{2}^{\approx 0(5)} + gz_1^{=0(6)} + p_1v_1^{=0(8)}\right)(-\dot{m}) + \left(u_2 + \frac{V_2^2}{2} + gz_2 + p_2v_2^{=0(8)}\right)(\dot{m})$$

or

$$-\dot{W}_s = \left(\frac{V_2^2}{2} + gz_2\right)\dot{m} \quad ; \quad \dot{m} = \rho V_2 A_2$$

Note that this equation contains V_2 to the third power, so that it cannot be solved directly. As a first approximation, neglect z_2:

$$-\dot{W}_{in} \simeq \left(\frac{V_2^2}{2}\right)\rho V_2 A_2 = \frac{1}{2}\rho V_2^3 A_2 \; ; \quad A_2 = \frac{\pi}{4}D_2^2 = \frac{\pi}{4}(0.025)^2 m^2 = 4.91\times10^{-4} m^2$$

$$V_2 \simeq \left[\frac{-2\dot{W}_s}{\rho A_2}\right]^{\frac{1}{3}} = \left[2 \times 10\,kW \times \frac{m^3}{999\,kg} \times \frac{1}{4.91\times10^{-4} m^2} \times 10^3 \frac{N\cdot m}{kW\cdot s} \times \frac{kg\cdot m}{N\cdot s^2}\right]^{\frac{1}{3}} = 34.4\ m/s$$

Comparing terms,

$$\frac{V_2^2}{2} \simeq \frac{(34.4)^2}{2} = 592\ m^2/s^2 \; ; \quad gz_2 = 9.81\frac{m}{s^2} \times 3m = 29.4\ m^2/s^2, \text{ about 5 percent}$$

Therefore this value of V_2 is about $\frac{5}{3}$ percent too large. Assume

$V_2 \simeq 33.8$ m/sec, and $Q = V_2 A_2 = 33.8\frac{m}{s} \times 4.91\times10^{-4} m^2 = 0.0166\ m^3/s$ Q

To compute z_{max}, apply first law to CV "b" using above assumptions, plus

(9) $V_3 = 0$

Then $-\dot{W}_s = \left(\frac{V_2^2}{2}^{=0(9)} + g z_{max}\right)\dot{m}$; $\dot{m} = \rho Q = 999\frac{kg}{m^3} \times 0.0166\frac{m^3}{s} = 16.6\ kg/s$

or

$$z_{max} = \frac{-\dot{W}_s}{g\dot{m}} = 10\,kW \times \frac{s^2}{9.81\,m} \times \frac{s}{16.6\,kg} \times \frac{10^3\,N\cdot m}{kW\cdot s} \times \frac{kg\cdot m}{N\cdot s^2} = 61.4\ m \qquad z_{max}$$

To find horizontal force, apply x component of momentum equation using CV "a", with flow at ② leaving horizontally.

Basic equation:

$$F_{Sx} + F_{Bx}^{=0(11)} = \frac{\partial}{\partial t}^{=0(3)}\int_{CV} u\rho\,d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

Assumptions: (10) No net pressure force on CV; $F_{Sx} = R_x$

(11) $F_{Bx} = 0$

Then

$$R_x = u_1\{-\dot{m}_1\} + u_2\{\dot{m}\} = \dot{m}V_2, \text{ since } u_1 = 0$$

or

$$K_x = -\dot{m}V_2 = -16.6\frac{kg}{s} \times 33.8\frac{m}{s} \times \frac{N\cdot s^2}{kg\cdot m} = -561\ N \qquad K_x$$

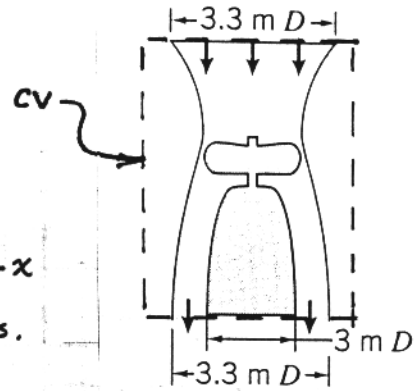
{Minus sign indicates reaction on boat is opposite from stream direction.}

Given: Helicopter-type craft hovering.

Mass, $M = 1500\ kg$

Assume atmospheric pressure at outlet, and treat as steady, uniform, incompressible flow.

Assume air is at standard conditions.



Find: (a) Speed of air leaving craft.
(b) Minimum power required.

Solution: Use inertial CV and coordinates shown. Apply continuity and momentum to determine V_2, then apply energy to find power.

Basic equations: $p = \rho RT$; $\Delta h = c_p \Delta T$; $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$

$$0 = \underbrace{\frac{\partial}{\partial t}}_{=0(2)} \int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

$$F_{Sz} + F_{Bz} = \underbrace{\frac{\partial}{\partial t}}_{=0(2)} \int_{CV} w \rho \, d\forall + \int_{CS} w \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Air is an ideal gas, c_p = constant
(2) Steady flow
(3) Incompressible flow
(4) Uniform flow at each section
(5) Uniform pressure at inlet; $F_{Sz} = (p_{atm} - p_1) A_1 = -p_{1g} A_1$

Then

$$\rho = \frac{p}{RT} = 1.01 \times 10^5 \frac{N}{m^2} \times \frac{kg \cdot K}{287\ N \cdot m} \times \frac{1}{288\ K} = 1.22\ kg/m^3$$

and from continuity

$$0 = \{-|\rho V_1 A_1|\} + \{|\rho V_2 A_2|\} = \rho (V_2 A_2 - V_1 A_1) \quad \text{or} \quad V_1 = V_2 \left(\frac{A_2}{A_1}\right)$$

Now $A_1 = \frac{\pi}{4} D_0^2 = \frac{\pi}{4} (3.3)^2 m^2 = 8.55\ m^2$

$$A_2 = \frac{\pi}{4} (D_0^2 - D_i^2) = \frac{\pi}{4} \left[(3.3)^2 - (3.0)^2\right] m^2 = 1.48\ m^2$$

From momentum

$$-p_{1g} A_1 - Mg = w_1 \{-|\rho V_1 A_1|\} + w_2 \{|\rho V_2 A_2|\}$$

$$w_1 = -V_1 \qquad w_2 = -V_2 \qquad \text{and} \quad \rho V_1 A_1 = \rho V_2 A_2$$

$$-p_{1g} A_1 - Mg = V_1 \rho V_2 A_2 - V_2 \rho V_2 A_2 = -\rho V_2 A_2 (V_2 - V_1)$$

For steady, incompressible flow without friction, along a streamline from atmosphere to ①, Bernoulli gives, neglecting Δz,

$$p_{atm} + \tfrac{1}{2}\rho V_0^2 + g\rho z_0 = p_1 + \tfrac{1}{2}\rho V_1^2 + g\rho z_1 \quad \text{so} \quad p_{1g} = -\tfrac{1}{2}\rho V_1^2$$

Using continuity, $p_{1g}A_1 = -\tfrac{1}{2}\rho V_1^2 A_1 = -\tfrac{1}{2}\rho V_2 A_2 V_1 = -\tfrac{1}{2}\rho V_2^2 A_2 \frac{A_2}{A_1}$

Substituting into the momentum equation and using continuity,

$$\tfrac{1}{2}\rho V_2^2 A_2 \frac{A_2}{A_1} - Mg = -\rho V_2^2 A_2\left(1 - \frac{V_1}{V_2}\right) = -\rho V_2^2 A_2\left(1 - \frac{A_2}{A_1}\right) \quad \text{or} \quad Mg = \rho V_2^2 A_2\left(1 - \tfrac{1}{2}\frac{A_2}{A_1}\right)$$

Thus

$$V_2 = \sqrt{\frac{Mg}{\rho A_2\left(1 - \frac{1}{2}\frac{A_2}{A_1}\right)}} = \left[1500\,\text{kg} \times 9.81\,\frac{\text{m}}{\text{s}^2} \times \frac{\text{m}^3}{1.22\,\text{kg}} \times \frac{1}{1.48\,\text{m}^2} \frac{1}{\left(1 - \frac{1}{2}\frac{1.48}{8.55}\right)}\right]^{\frac{1}{2}} = 94.5\ \text{m/s}$$

V_2

Basic equation:

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall + \int_{CS}\left(u + p\upsilon + \frac{V^2}{2} + gz\right)\rho\vec{V}\cdot d\vec{A}$$

($\dot{W}_{shear} = 0$ (6), $\dot{W}_{other} = 0$ (6), steady (2), $p\upsilon$ = constant (7), neglect Δz (8))

Additional assumptions: (6) $\dot{W}_{shear} = \dot{W}_{other} = 0$
(7) $p\upsilon$ = constant
(8) Neglect Δz

Then

$$-\dot{W}_s = \left(u_1 + \frac{V_1^2}{2}\right)\{-|\dot{m}|\} + \left(u_2 + \frac{V_2^2}{2}\right)\{|\dot{m}|\} - \dot{Q}$$

$$-\dot{W}_s = \dot{m}\left(\frac{V_2^2 - V_1^2}{2}\right) + \dot{m}\left(u_2 - u_1 - \frac{dQ}{dm}\right)$$

The term $\left(u_2 - u_1 - \frac{dQ}{dm}\right)$ represents nonmechanical energy. The minimum possible work would be attained when the nonmechanical energy is zero. Thus

$$-\dot{W}_s)_{min} = \dot{m}\left(\frac{V_2^2 - V_1^2}{2}\right) = \dot{m}\frac{V_2^2}{2}\left[1 - \left(\frac{V_1}{V_2}\right)^2\right] = \frac{\rho A_2 V_2^3}{2}\left[1 - \left(\frac{A_2}{A_1}\right)^2\right]$$

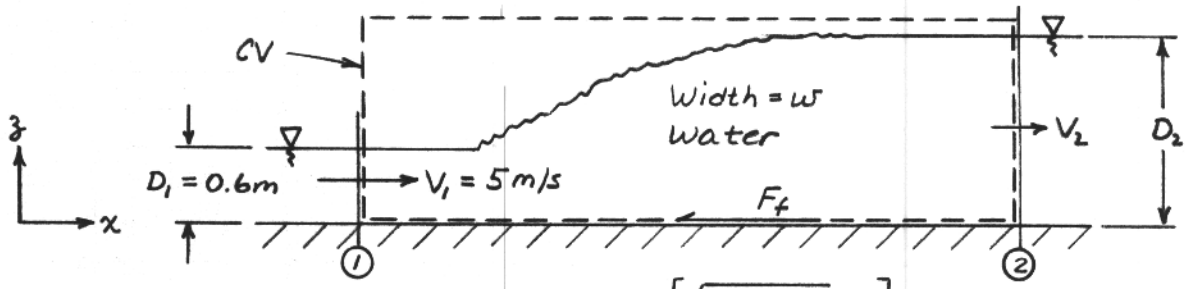
$$-\dot{W}_s = \frac{1}{2} \times 1.22\,\frac{\text{kg}}{\text{m}^3} \times 1.48\,\text{m}^2 \times (94.5)^3\,\frac{\text{m}^3}{\text{s}^3}\left[1 - \left(\frac{1.48}{8.55}\right)^2\right]\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} \times \frac{\text{kW}\cdot\text{s}}{10^3\,\text{N}\cdot\text{m}}$$

$$\dot{W}_s)_{min} = -739\ \text{kW (input)}$$

$\dot{W}_s$

{ The power required for hovering in a real craft would be greater due to flow losses, nonuniformities, etc. }

Given: Liquid flow in a wide, horizontal open channel, as shown.



Find: (a) Show that in general, $D_2 = \frac{D_1}{2}\left[\sqrt{1+\frac{8V_1^2}{gD_1}} - 1\right]$

(b) Change in mechanical energy across hydraulic jump.
(c) Temperature rise if no heat transfer.

Solution: Apply continuity, x component of momentum, and energy equations using CV shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV}\rho\,d\forall + \int_{CS}\rho\,\vec{V}\cdot d\vec{A}$ (=0(1))

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} V_x\rho\,d\forall + \int_{CS} V_x\rho\,\vec{V}\cdot d\vec{A} \quad (F_{Bx}=0(8),\ \frac{\partial}{\partial t}=0(1))$$

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\rho\,d\forall + \int_{CS}(e+pv)\rho\,\vec{V}\cdot d\vec{A}\ ; \quad (\dot{Q}=0(6),\ \dot{W}_s=0(7),\ \dot{W}_{shear}=0(7),\ \dot{W}_{other}=0(7),\ \frac{\partial}{\partial t}=0(1))$$

$$e = u + \frac{V^2}{2} + gz$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) Hydrostatic pressure distribution at sections ①, ②, so $p = \rho g(D-z)$
(5) Neglect friction force, F_f, on CV
(6) $\dot{Q} = 0$
(7) $\dot{W}_s = \dot{W}_{shear} = \dot{W}_{other} = 0$
(8) $F_{Bx} = 0$, since channel is horizontal

From continuity,

$$0 = \{-|\rho V_1 A_1|\} + \{|\rho V_2 A_2|\} = -\rho V_1 w D_1 + \rho V_2 w D_2\ ;\ V_1 D_1 = V_2 D_2$$

From momentum,

$$F_{Sx} = \underbrace{\rho g\frac{D_1}{2} w D_1 - \rho g\frac{D_2}{2} w D_2}_{\text{hydrostatic forces}} = V_{x_1}\{-|\rho V_1 w D_1|\} + V_{x_2}\{|\rho V_2 w D_2|\} \qquad V_{x_1} = V_1 \qquad V_{x_2} = V_2$$

or

$$\frac{\rho g}{2}(D_1^2 - D_2^2) = V_1 D_1 (V_2 - V_1) = V_1^2 D_1\left(\frac{V_2}{V_1} - 1\right) = V_1^2 D_1\left(\frac{D_1}{D_2} - 1\right)$$

or

$$\frac{\rho g}{2}(D_1 + D_2)\cancel{(D_1 - D_2)} = V_1^2\frac{D_1}{D_2}\cancel{(D_1 - D_2)}$$

Thus $\frac{gD_1}{2}\left(1+\frac{D_2}{D_1}\right) = V_1^2\frac{D_1}{D_2}$ or $\frac{D_2}{D_1}\left(1+\frac{D_2}{D_1}\right) = \frac{2V_1^2}{gD_1}$ or $\left(\frac{D_2}{D_1}\right)^2+\frac{D_2}{D_1}-\frac{2V_1^2}{gD_1}=0$

Using the quadratic equation,

$$\frac{D_2}{D_1} = \frac{1}{2}\left[-1 \pm \sqrt{1+\frac{8V_1^2}{gD_1}}\right] \quad \text{or} \quad D_2 = \frac{D_1}{2}\left[\sqrt{1+\frac{8V_1^2}{gD_1}}-1\right]$$ D_2

Solving for D_2

$$D_2 = \frac{1}{2}\times 0.6m\left[\sqrt{1+8\times\frac{(5)^2m^2}{s^2}\times\frac{s^2}{9.81m}\times\frac{1}{0.6m}}-1\right] = 1.47\ m$$

$$V_2 = \frac{D_1}{D_2}V_1 = \frac{0.6}{1.47}\times 5\frac{m}{s} = 2.04\ m/s$$

From the energy equation, with $e_{mech} = \frac{V^2}{2}+gz+\frac{p}{\rho}$, and $dA = w\,dz$, the mechanical energy fluxes are

$$mef_1 = \int_0^{D_1}\left[\frac{V_1^2}{2}+gz+\frac{1}{\rho}\rho g(D-z)\right]\rho V_1 w\,dz = \left(\frac{V_1^2}{2}+gD_1\right)\rho V_1 w D_1$$

$$mef_2 = \int_0^{D_2}\left[\frac{V_2^2}{2}+gz+\frac{1}{\rho}\rho g(D-z)\right]\rho V_2 w\,dz = \left(\frac{V_2^2}{2}+gD_2\right)\rho V_2 w D_2$$

and

$$\Delta mef = mef_2 - mef_1 = \left[\frac{V_2^2-V_1^2}{2}+g(D_2-D_1)\right]\rho V_1 w D_1, \text{ since } V_1D_1 = V_2D_2$$

Thus $\frac{\Delta mef}{\dot m} = \frac{1}{2}\left[V_2^2-V_1^2+2g(D_2-D_1)\right]$

$$\frac{\Delta mef}{\dot m} = \frac{1}{2}\left[(2.04)^2\frac{m^2}{s^2}-(5)^2\frac{m^2}{s^2}+2\times 9.81\frac{m}{s^2}(1.47-0.6)m\right]\frac{N\cdot s^2}{kg\cdot m} = -1.88\ N\cdot m/kg$$ $\frac{\Delta mef}{\dot m}$

From the energy equation,

$$0 = \left[u_1+\frac{V_1^2}{2}+gz+\frac{1}{\rho}\rho g(D-z)\right]\{-1\rho V_1 w D_1\}$$
$$+\left[u_2+\frac{V_2^2}{2}+gz+\frac{1}{\rho}\rho g(D-z)\right]\{1\rho V_2 w D_2\}$$

or

$$0 = (u_2-u_1)\dot m + \Delta mef$$

Thus

$$u_2-u_1 = c_v(T_2-T_1) = -\frac{\Delta mef}{\dot m}$$

$$\Delta T = T_2-T_1 = -\frac{\Delta mef}{\dot m c_v} = -\left(-1.88\frac{N\cdot m}{kg}\right)\frac{kg\cdot K}{1\,kcal}\times\frac{kcal}{4187\ J} = 4.49\times10^{-4}K$$ ΔT

{This small temperature change would be almost impossible to measure.}

Given: Velocity fields listed below.

Find: Which are possible two-dimensional, incompressible flow cases?

Solution: Apply the continuity equation in differential form.

Basic equation: $\frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \cancel{\frac{\partial}{\partial z}\rho w}^{=0(1)} + \cancel{\frac{\partial \rho}{\partial t}}^{=0(2)} = 0$

Assumptions: (1) Two-dimensional flow, $\vec{V} = \vec{V}(x,y)$, so $\frac{\partial}{\partial z} = 0$
(2) Incompressible flow

ρ = constant, so $\frac{\partial \rho}{\partial t} = 0$, $\frac{\partial \rho}{\partial (\text{distance})} = 0$

Then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{is criterion.}$$

(a) $u = x+y, v = x-y$; $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 - 1 = 0$, so <u>possible</u>

(b) $u = x + 2y, v = x^2 - y^2$; $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 - 2y \neq 0$, so $\rho \neq$ constant

(c) $u = 4x + y, v = x - y^2$; $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 4 - 2y \neq 0$, so $\rho \neq$ constant

(d) $u = xt + 2y, v = x^2 - yt^2$; $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = t - t^2 \neq 0$, so $\rho \neq$ constant

(e) $u = xt^2, v = xyt + y^2$; $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = t^2 + xt + 2y$, so $\rho \neq$ constant

Given: Velocity fields listed below.

Find: Which are possible two-dimensional, incompressible flow cases?

Solution: Apply the continuity equation in differential form.

Basic equation: $\frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w + \frac{\partial \rho}{\partial t} = 0$ (terms: $=0(1)$, $=0(2)$)

Assumptions: (1) Two-dimensional flow, $\vec{V} = \vec{V}(x,y)$, so $\frac{\partial}{\partial z} = 0$

(2) Incompressible flow,

$$\rho = \text{constant, so } \frac{\partial \rho}{\partial t} = 0, \quad \frac{\partial \rho}{\partial (\text{distance})} = 0$$

Then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{is criterion.}$$

Velocity Field	$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$	Possible?
(a) $u = 2x^2 + y^2$, $v = x^3 - x(y^2 - 2y)$	$4x - x(2y-2) \neq 0$	No
(b) $u = 2xy - x^2 + y$, $v = 2xy - y^2 + x^2$	$2y - 2x + 2x - 2y = 0$	Yes
(c) $u = xt + 2y$, $v = xt^2 - yt$	$t - t = 0$	Yes
(d) $u = (x+2y)xt$, $v = (2x-y)yt$	$2xt + 2yt + 2xt - 2yt$ $\neq 0$	No

Given: Velocity profiles listed below.

Find: Which are possible three-dimensional, incompressible cases?

Solution: Apply the continuity equation in differential form.

Basic equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

Assumption: Incompressible flow

Field	Terms	Sum	Possible
(a) $u = x + y + z^2$	$\frac{\partial u}{\partial x} = 1$		
$v = x - y + z$	$\frac{\partial v}{\partial y} = -1$	0	Yes
$w = 2xy + y^2 + 4$	$\frac{\partial w}{\partial z} = 0$		
(b) $u = xyzt$	$\frac{\partial u}{\partial x} = yzt$		
$v = -xyzt^2$	$\frac{\partial v}{\partial y} = -xzt^2$	0	Yes
$w = \frac{z^2}{2}(xt^2 - yt)$	$\frac{\partial w}{\partial z} = xzt^2 - yzt$		
(c) $u = y^2 + 2xz$	$\frac{\partial u}{\partial x} = 2z$		
$v = -2yz + x^2yz$	$\frac{\partial v}{\partial y} = -2z + x^2z$	$\neq 0$	No
$w = \frac{x^2z^2}{2} + x^3y^4$	$\frac{\partial w}{\partial z} = x^2z$		

Given: Velocity field $u = Ax + By + Cz$

$v = Dx + Ey + Fz$

$w = Gx + Hy + Jz$

Find: The relationship among coefficients A thru J for this to be an incompressible flow field.

Solution: Flow must satisfy differential form of continuity.

Basic equation: $\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \cancel{\frac{\partial \rho}{\partial t}}^{=0} = 0$

Assumption: Incompressible flow, so $\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial x} = 0$

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

For the given flow field, $\frac{\partial u}{\partial x} = A$, $\frac{\partial v}{\partial y} = E$, $\frac{\partial w}{\partial z} = J$. Thus

$A + E + J = 0$, and

B, C, D, F, G, H are arbitrary

Given: Flow in xy plane, $v = y^2 - 2x + 2y$, steady.

Find: (a) Possible x component for ρ = constant.
(b) Is it also valid for unsteady flow with $\rho = c$?
(c) Number of possible x components.

Solution:

Basic equation: $\nabla \cdot \rho\vec{V} + \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w + \frac{\partial \rho}{\partial t} = 0$

($\frac{\partial}{\partial z}\rho w = 0$ (1); $\frac{\partial \rho}{\partial t} = 0$ (2))

Assume: (1) Flow in xy plane, $\frac{\partial}{\partial z} = 0$
(2) ρ = constant

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ or $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$

$$-\frac{\partial v}{\partial y} = -\frac{\partial}{\partial y}(y^2 - 2x + 2y) = -(2y + 2) = -2y - 2$$

Integrating,

$$u = \int \frac{\partial u}{\partial x}\,dx = \int -\frac{\partial v}{\partial y}\,dx = \int (-2y - 2)\,dx = -2yx - 2x + f(y)$$ u

The basic equation reduces to the same form for unsteady flow with ρ = constant. Therefore it is also valid for unsteady flow.

There are an infinite number of possible x components, since $f(y)$ is arbitrary. The simplest would be to choose $f(y) = 0$.

Given: Flow in xy plane, $u = Ax(y-B)$, where $A = 2\,m^{-1}\cdot s^{-1}$, $B = 1\,m$, and coordinates are measured in meters.

Find: (a) Possible y component for steady, incompressible flow.
(b) If result is valid for unsteady, incompressible flow.
(c) Number of possible y components

Solution:

Basic equation: $\nabla \cdot \rho\vec{V} + \frac{\partial \rho}{\partial t} = 0 = \frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w + \frac{\partial \rho}{\partial t}$ (=0 (1), =0 (2))

Assumptions: (1) flow in xy plane (given), $\frac{\partial}{\partial z} = 0$
(2) ρ = constant (given)

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ or $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$

and $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} Ax(y-B) = -A(y-B)$

Integrating

$$v = \int \frac{\partial v}{\partial y}\,dy = -A\int (y-B)\,dy = -A\left(\frac{y^2}{2} - By\right) + f(x) \quad \longleftarrow v$$

The basic equation reduces to the same form for unsteady flow (as with steady flow). Hence the result is also valid for unsteady flow. (b)

There are an infinite number of possible y components, since $f(x)$ is arbitrary. The simplest is obtained with $f(x) = 0$. (c)

13-782 500 SHEETS FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-388 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Steady, incompressible flow field in the xy plane has an x component of velocity given by

$u = \frac{A}{x}$, where $A = 2\ m^2/s$ and x is in meters.

Find: the simplest y component of velocity for this flow field.

Solution:

Apply the continuity equation for the conditions given

Basic equation: $\nabla \cdot \rho \vec{V} + \frac{\partial \rho}{\partial t} = 0$

For steady flow $\frac{\partial \rho}{\partial t} = 0$ and for two-dimensional flow in the xy plane, $\frac{\partial w}{\partial z}() = 0$. Thus the basic equation reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Then

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{A}{x}\right) = \frac{A}{x^2}$$

and

$$v = \int \frac{\partial v}{\partial y}\, dy + f(x) = \int \frac{A}{x^2}\, dy + f(x) = \frac{Ay}{x^2} + f(x)$$

The simplest y component of velocity is obtained with $f(x) = 0$

$$\therefore v = \frac{Ay}{x^2}$$

Given: The y component of velocity for a steady, incompressible flow in the xy plane is

$v = Ay/x^2$, where $A = 2\ m^2/s$, xy in m.

Find: simplest y component.

Solution: Apply differential form of conservation of mass

For incompressible flow, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. Thus $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = -\frac{A}{x^2}$

Integrating, $u = \frac{A}{x} + f(y)$. The simplest form is for $f(y) = 0$.

Thus

$$u = \frac{A}{x}$$

u

and

$$\vec{V} = \frac{A}{x}\hat{\imath} + \frac{Ay}{x^2}\hat{\jmath}$$

$\vec{V}$

13-782 500 SHEETS FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Approximate profile for laminar boundary layer

$$u = C\,U\,\frac{y}{x^{1/2}}$$

Find: (a) Show simplest v is $v = \frac{U}{4}\frac{y}{x}$

(b) Evaluate maximum value of v/U where $\delta = 5\,mm$, $x = 0.5\,m$.

Solution: Apply continuity for incompressible flow

Basic equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}} = 0$ (2-D flow)

Thus

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\left(-\tfrac{1}{2}\right) C U \frac{y}{x^{3/2}}$$

$$v = \int \frac{\partial v}{\partial y}\,dy + f(x) = \int \tfrac{1}{2} C U \frac{y}{x^{3/2}}\,dy + f(x) = \tfrac{1}{4} C U \frac{y^2}{x^{3/2}} + f(x)$$

or

$$v = \frac{U}{4}\frac{y}{x} \qquad [f(x) = 0 \text{ since } v = 0 \text{ along } y = 0]$$

From

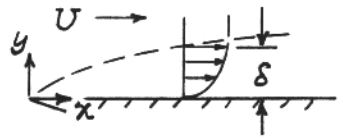
$$\frac{v}{U} = \frac{1}{4}\frac{y}{x}$$

maximum value occurs at $y = \delta$. At the location given,

$$\left.\frac{v}{U}\right)_{max} = \frac{1}{4}\frac{\delta}{x} = \frac{1}{4}\frac{0.005\,m}{0.5\,m} = 0.0025$$

Given: Laminar boundary layer, parabolic approximate profile.

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \qquad \delta = cx^{1/2}$$



Find: Show $\frac{v}{U} = \frac{\delta}{x}\left[\frac{1}{2}\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]$ for incompressible flow.

Plot: $\frac{v}{U}$ vs. $\frac{y}{\delta}$, evaluate max. at $x = 0.5$ m, if $\delta = 5$ mm.

Solution: Apply conservation of mass for incompressible flow.

Basic equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ ($\partial w/\partial z = 0$ (2))

Assumptions: (1) Incompressible flow (ρ = const)
(2) $w = 0$

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$; $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$; $v = \int_0^y -\frac{\partial u}{\partial x}\,dy + f(x)$ ($f(x) = 0$ simplest)

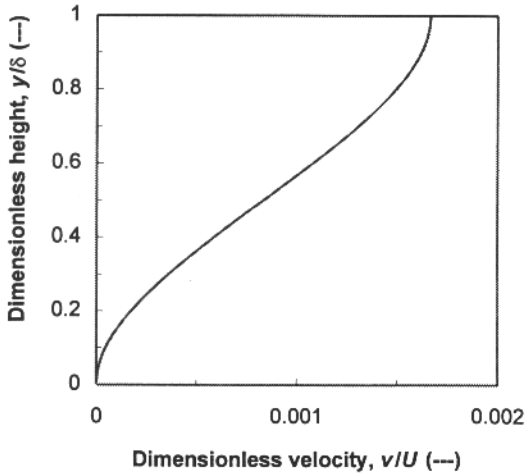
From the given profile

$$\frac{\partial u}{\partial x} = 2Uy(-1)\frac{1}{\delta^2}\frac{d\delta}{dx} - Uy^2(-2)\frac{1}{\delta^3}\frac{d\delta}{dx} = 2U\frac{d\delta}{dx}\left(\frac{y^2}{\delta^3} - \frac{y}{\delta^2}\right)$$

Since $\delta = cx^{1/2}$, $\frac{d\delta}{dx} = \frac{1}{2}cx^{-1/2} = \frac{cx^{1/2}}{2x} = \frac{\delta}{2x}$, so $\frac{\partial u}{\partial x} = U\frac{\delta}{x}\left(\frac{y^2}{\delta^3} - \frac{y}{\delta^2}\right)$

Integrating, $\frac{v}{U} = \frac{\delta}{x}\int_0^y \left(\frac{y}{\delta^2} - \frac{y^2}{\delta^3}\right)dy = \frac{\delta}{x}\left[\frac{1}{2}\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]$ ← $\frac{v}{U}$

Plotting shows:



Maximum occurs at $\left(\frac{y}{\delta}\right) = 1$

$$\left(\frac{v}{U}\right)_{max} = \left(\frac{v}{U}\right)_{\frac{y}{\delta}=1} = \frac{\delta}{x}\left[\frac{1}{2}(1)^2 - \frac{1}{3}(1)^2\right] = \frac{\delta}{6x}$$ ← $\left(\frac{v}{U}\right)_{ma}$

Evaluating, $\left(\frac{v}{U}\right)_{max} = \frac{1}{6} \times 0.005\,m \times \frac{1}{0.5\,m} = 0.00167$ or 0.167 percent ← $\left(\frac{v}{U}\right)_{ma}$

Given: Approximation for x component of velocity in laminar boundary layer

$$u = U\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) \qquad \text{where } \delta = cx^{1/2}$$

Show: $\frac{v}{U} = \frac{\delta}{\pi x}\left[\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) + \frac{\pi}{2}\frac{y}{\delta}\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - 1\right]$ for incompressible flow.

Plot: v/U vs. y/δ to locate maximum value of v/U; evaluate at location where $x = 0.5$ m and $\delta = 5$ mm.

Solution: Apply differential continuity for incompressible flow.

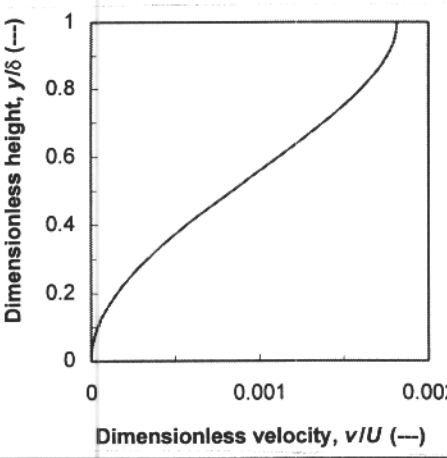
Basic equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ ($\partial w/\partial z = 0$ (2-D flow))

Thus $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial \delta}\frac{d\delta}{dx} = -\left(\frac{\pi y}{2}\right)\left(-\frac{1}{\delta^2}\right)\cos\left(\frac{\pi y}{2\delta}\right)\frac{U}{2}cx^{-1/2} = \frac{U}{2x}\left(\frac{\pi y}{2\delta}\right)\cos\left(\frac{\pi y}{2\delta}\right)$

Integrating, $v = \int_0^y \frac{\partial v}{\partial y}\,dy + f(x) = \int_0^y \frac{U}{2x}\left(\frac{\pi}{2}\frac{y}{\delta}\right)\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)dy + f(x)$

$$v = \frac{2\delta}{\pi}\frac{U}{2x}\int_0^{\frac{\pi}{2}\frac{y}{\delta}} r\cos r\,dr + f(x) = \frac{\delta}{\pi}\frac{U}{x}\left[\cos r + r\sin r\right]_0^{\frac{\pi}{2}\frac{y}{\delta}} + f(x)^{0}$$

$$\frac{v}{U} = \frac{1}{\pi}\frac{\delta}{x}\left[\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) + \left(\frac{\pi}{2}\frac{y}{\delta}\right)\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - 1\right]$$



This expression is a maximum at $y = \delta$; where

$$\frac{v}{U} = \frac{1}{\pi}\frac{\delta}{x}\left[\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) - 1\right] = \frac{\delta}{\pi x}\left(\frac{\pi}{2} - 1\right)$$

and $$\left.\frac{v}{U}\right)_{max} = 0.182\,\frac{\delta}{x}$$

At the location given

$$\left.\frac{v}{U}\right)_{max} = 0.182 \times 0.005\ \text{m} \times \frac{1}{0.5\ \text{m}} = 0.00182 \quad \text{or} \quad 0.182 \text{ percent}$$

Given: Flow in xy plane, $v = -Bxy^3$ where $B = 0.2\ m^3 \cdot s^{-1}$ and coordinates are measured in meters; steady, $\rho = c$.

Find: (a) Simplest x component of velocity.
(b) Equation of streamlines.

Plot: streamlines through points (1,4) and (2,4).

Solution:

Basic equation: $\nabla \cdot \rho \vec{V} + \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w + \frac{\partial \rho}{\partial t}$ (=0 (1), =0 (2))

Assumptions: (1) flow is the xy plane (given), $\frac{\partial}{\partial z} = 0$
(2) ρ = constant (given).

Then, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ or $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$

and $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}(-Bxy^3) = 3Bxy^2$

Integrating,

$$u = \int \frac{\partial u}{\partial x}\,dx = \int 3Bxy^2 = \frac{3}{2}Bx^2y^2 + f(y).$$

The simplest expression is obtained with $f(y) = 0$

$$\therefore u = \frac{3}{2}Bx^2y^2 \quad \leftarrow u$$

The equation of the streamlines is

$$\left(\frac{dy}{dx}\right)_{\text{sl}} = \frac{v}{u} = \frac{-Bxy^3}{\frac{3}{2}Bx^2y^2} = \frac{-2y}{3x}$$

Separating variables & integrating

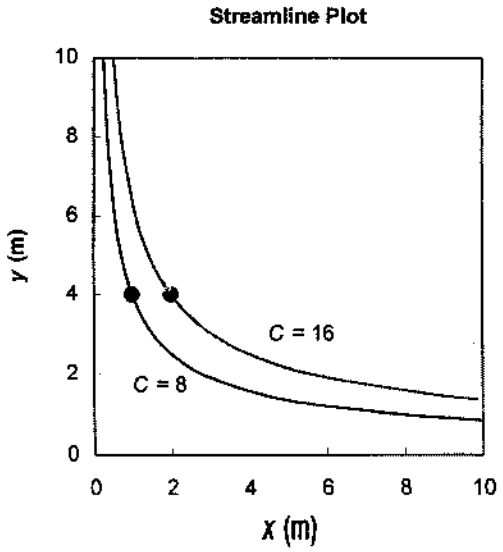
$$\frac{dy}{y^{3/2}} + \frac{dx}{x} = 0$$

$$-2y^{-1/2} + \ln x = \ln c$$

$$xy^{3/2} = C \quad \text{Streamline}$$

pt (1,4) $xy^{3/2} = 8$

pt (2,4) $xy^{3/2} = 16$



Given: Flow in xy plane, $u = Ax^2y^2$ where $A = 0.3\ m^{-3}\cdot s^{-1}$, and coordinates are measured in meters

Find: (a) Possible y component for steady, incompressible flow
(b) If result is valid for unsteady, incompressible flow
(c) Number of possible y components
(d) Equation of streamlines for simplest value of v.

Plot: streamlines through points (1,4) and (2,4)

Solution:

Basic equation: $\nabla \cdot \rho\vec{V} + \frac{\partial \rho}{\partial t} = 0 = \frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w + \frac{\partial \rho}{\partial t}$ (=0(1), =0(2), =0)

Assumptions: (1) flow is xy plane (given), $w = 0$
(2) ρ = constant (given)

Then, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ or $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial}{\partial x}(Ax^2y^2) = -2Axy^2$

Integrating $v = \int \frac{\partial v}{\partial y}\,dy = -\int 2Axy^2 = -\frac{2}{3}Axy^3 + f(x)$ ← v

The basic equation reduces to the same form for unsteady flow. Hence the result is also valid for unsteady flow. (b)

There are an infinite number of possible y components, since f(x) is arbitrary. The simplest is obtained with $f(x) = 0$. (c)

The equation of the streamline is

$$\left.\frac{dy}{dx}\right|_{s.l.} = \frac{v}{u} = -\frac{\frac{2}{3}Axy^3}{Ax^2y^2} = -\frac{2y}{3x}$$

Separating variables & integrating

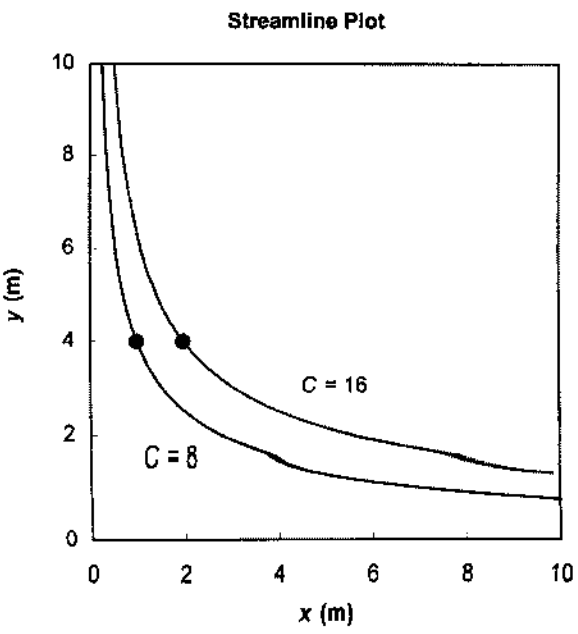
$$\frac{3}{2}\frac{dy}{y} + \frac{dx}{x} = 0$$

$$\ln y^{3/2} + \ln x = \ln C$$

$$xy^{3/2} = C$$ ← Streamline

Pt (1,4) $xy^{3/2} = 8$

(2,4) $xy^{3/2} = 16$



Problem 5.14

Given: Conservation of mass.

Find: Identical result to Eq. 5.1a by expanding products of density and velocity in Taylor series.

Solution: Use diagram of Fig. 5.1:

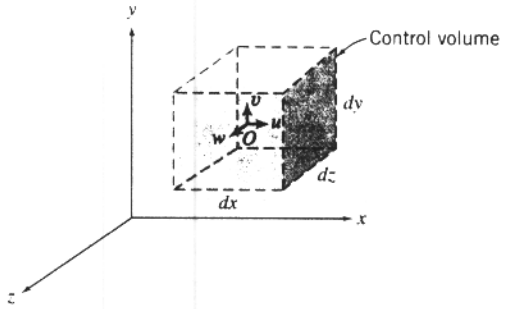


Fig. 5.1 Differential control volume in rectangular coordinates

Apply conservation of mass, using a Taylor series expansion of products. Evaluate derivatives at 0.

For the x direction the mass flux is

$$\dot{m}_x = \rho u \, dA = \rho u \, dx \, dy$$

At the right face

$$\dot{m}_{x+dx/2} = \rho u \, dy \, dz + \frac{\partial}{\partial x} \rho u \frac{dx}{2} dy dz \quad \text{(out of CV)}$$

At the left face

$$\dot{m}_{x-dx/2} = \rho u \, dy dz + \frac{\partial}{\partial x} \rho u \left(-\frac{dx}{2}\right) dy dz \quad \text{(into CV)}$$

The net mass flux is "out" minus "in," so

$$\dot{m}_x(\text{net}) = \dot{m}_{x+dx/2} - \dot{m}_{x-dx/2} = \frac{\partial}{\partial x} \rho u \, dx dy dz$$

Summing terms for x, y, and z, and including $\frac{\partial \rho}{\partial t} dx dy dz$, we get

$$0 = \frac{\partial}{\partial x} \rho u + \frac{\partial}{\partial y} \rho v + \frac{\partial}{\partial z} \rho w + \frac{\partial \rho}{\partial t}$$

Problem 5.15

Open-Ended Problem Statement: Consider a water stream from a jet of an oscillating lawn sprinkler. Describe the corresponding pathline and streakline.

Discussion: Because the sprinkler jets oscillate, this is an unsteady flow. Therefore pathlines and streaklines need not coincide.

A *pathline* is a line tracing the path of an individual fluid particle. The path of each fluid particle is determined by the jet angle and the speed at which the particle leaves the jet.

Once a particle leaves the jet it is subject to gravity and drag forces. If aerodynamic drag were negligible, the path of each water particle would be parabolic. The horizontal speed of the particle would remain constant throughout its trajectory. The vertical speed would be slowed by gravity until reaching peak height, then it would become increasingly negative until the particle strikes the ground. The effect of aerodynamic drag is to reduce the particle speed. With drag the particle will not rise as far vertically nor travel as far horizontally. At each instant the particle trajectory will be lower and closer to the jet compared to the no-friction case. The trajectory after the particle reaches its peak height will be steeper than in the no-friction case.

A *streamline* is a line drawn in the flow that is tangent everywhere to the velocity vectors of the fluid motion. It is difficult to visualize the streamlines for an unsteady flow field because they may move laterally. However, the streamline pattern may be drawn at any instant.

A *streakline* is the locus of the present locations of fluid particles that passed a reference point at previous times. As an example, choose the exit of a jet as the reference point. Imagine marking particles that pass the jet exit at a given instant and at uniform time intervals later. The first particle will travel farthest from the jet exit and on the lowest trajectory; the last particle will be located right at the jet exit. The curve joining the present positions of the particles will resemble a spiral whose radius increases with distance from the jet opening.

13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-390 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Velocity fields listed below.

Find: Which are possible incompressible flow cases?

Solution: Apply the continuity equation in differential form.

Basic equation: $\frac{1}{r}\frac{\partial r\rho V_r}{\partial r} + \frac{1}{r}\frac{\partial \rho V_\theta}{\partial \theta} + \frac{\partial \rho V_z}{\partial z} + \frac{\partial \rho}{\partial t} = 0$ ($\frac{\partial \rho V_z}{\partial z} = 0$ (1), $\frac{\partial \rho}{\partial t} = 0$ (2))

Assumptions: (1) Two-dimensional flow, so $\frac{\partial}{\partial z} = 0$
(2) Incompressible flow

ρ = constant, so $\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial (\text{distance})} = 0$

Then
$$\frac{1}{r}\frac{\partial r V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} = 0$$

or
$$\frac{\partial r V_r}{\partial r} + \frac{\partial V_\theta}{\partial \theta} = 0 \quad \text{is the criterion.}$$

Field	V_r	V_θ	$\frac{\partial r V_r}{\partial r}$	$\frac{\partial V_\theta}{\partial \theta}$	$\frac{\partial r V_r}{\partial r} + \frac{\partial V_\theta}{\partial \theta}$	Possible?
(a)	$U\cos\theta$	$-U\sin\theta$	$U\cos\theta$	$-U\cos\theta$	0	Yes
(b)	$-\frac{q}{2\pi r}$	$\frac{K}{2\pi r}$	0	0	0	Yes
(c)	$U\cos\theta\left[1-(\frac{a}{r})^2\right]$*	$-U\sin\theta\left[1+(\frac{a}{r})^2\right]$	$U\cos\theta\left[1+(\frac{a}{r})^2\right]$	$-U\cos\theta\left[1+(\frac{a}{r})^2\right]$	0	Yes

* Note if $V_r = U\cos\theta\left[1-(\frac{a}{r})^2\right]$, then $rV_r = U\cos\theta\left[r - \frac{a^2}{r}\right]$

and $\frac{\partial r V_r}{\partial r} = U\cos\theta\left[1 + \frac{a^2}{r^2}\right] = U\cos\theta\left[1+(\frac{a}{r})^2\right]$

Given: Incompressible flow in $r\theta$ plane with $V_r = -\frac{\Lambda \cos\theta}{r^2}$

Find: (a) A possible component, V_θ
(b) How many possible θ components are there?

Solution: Velocity field must satisfy the differential continuity equation.

Basic equation: $\frac{1}{r}\frac{\partial r\rho V_r}{\partial r} + \frac{1}{r}\frac{\partial \rho V_\theta}{\partial \theta} + \frac{\partial \rho V_z}{\partial z} + \frac{\partial \rho}{\partial t} = 0$ ($\frac{\partial \rho V_z}{\partial z} = 0$ (1), $\frac{\partial \rho}{\partial t} = 0$ (2))

Assumptions: (1) Flow in the $r\theta$ plane, so $\frac{\partial}{\partial z} = 0$
(2) Incompressible flow

ρ = constant, so $\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial \ell} = 0$

Then $\frac{1}{r}\frac{\partial rV_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} = 0$ or $\frac{\partial V_\theta}{\partial \theta} = -\frac{\partial rV_r}{\partial r}$

Solving for V_θ,

$$V_\theta = \int -\frac{\partial rV_r}{\partial r}\, d\theta + f(r)$$

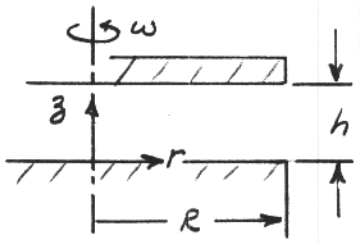
Since $V_r = -\frac{\Lambda \cos\theta}{r^2}$, $rV_r = -\frac{\Lambda \cos\theta}{r}$ and $\frac{\partial rV_r}{\partial r} = \frac{\Lambda \cos\theta}{r^2}$

Thus

$$V_\theta = \int -\frac{\Lambda \cos\theta}{r^2}\, d\theta + f(r) = -\frac{\Lambda \sin\theta}{r^2} + f(r)$$ ← V_θ

There are an infinite number of solutions for V_θ, one for each choice of $f(r)$.

Given: Flow between parallel disks as shown.



Velocity is purely tangential.
No-slip condition is satisfied, so velocity varies linearly with z.

Find: Expression for velocity field.

Solution: A general velocity field would be

$$\vec{V} = V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{k}$$

but velocity is purely tangential, so $V_r = V_z = 0$. Then we seek

$$V_\theta = V_\theta(r, \theta, z)$$

By symmetry, $\frac{\partial V_\theta}{\partial \theta} = 0$, so

$$V_\theta = V_\theta(r, z)$$

Since the variation with z is linear, $V_\theta = z f(r) + C$ at most, that is

$$\frac{\partial V_\theta}{\partial z} = f(r)$$

at most.

Along the surface $z = 0$, $V_\theta = 0$, so $C = 0$.

Along the surface $z = h$, $V_\theta = \omega r$, so

$$V_\theta(z = h) = \omega r = h f(r)$$

or

$$f(r) = \frac{\omega r}{h}$$

and

$$V_\theta = \omega r \frac{z}{h}$$

Thus

$$\vec{V} = \omega r \frac{z}{h} \hat{e}_\theta$$

$\vec{V}$

Given: Velocity field $\vec{V} = \frac{A}{r}\hat{e}_r + \frac{A}{r}\hat{e}_\theta$

where $A = 0.25\ m^2/s$.

Find: (a) If a possible incompressible flow?
(b) Streamline through $(r,\theta) = 1m, 0$
(c) Compare with pathline through same point.

Solution: Incompressible flow must satisfy differential continuity.

Basic equation: $\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(V_\theta) = 0$

For the given flow,

$\frac{\partial}{\partial r}(rV_r) = \frac{\partial}{\partial r}(A) = 0$ and $\frac{\partial}{\partial \theta}(V_\theta) = \frac{\partial}{\partial \theta}\left(\frac{A}{r}\right) = 0$ ∴ Incompressible ←

A streamline must satisfy $\frac{dr}{V_r} = \frac{r\,d\theta}{V_\theta}$. Thus $\frac{dr}{r} = \frac{V_r}{V_\theta}d\theta = d\theta$.

Integrating $\int \frac{dr}{r} = \ln r = \int d\theta = \theta + \ln C$ or $r = Ce^\theta$.

For $r = 1m$ at $\theta = 0$, $C = 1$. The streamline is $r = e^\theta$ Streamlin ←

Flow is steady, so pathline should be the same. For a pathline

$\frac{dr_p}{dt} = V_r$ or $\frac{dr}{dt} = \frac{A}{r}$ or $r\,dr = A\,dt$, so $\frac{r^2}{2} = At + C_1$

But $r = 1$ at $t = 0$, so $\frac{1}{2} = 0 + C_1$ and $C_1 = 1/2$, so $\frac{r^2}{2} = At + \frac{1}{2}$

and $r^2 = 2At + 1$

$\frac{r\,d\theta_p}{dt} = V_\theta$ or $\frac{r\,d\theta}{dt} = \frac{A}{r}$ or $d\theta = \frac{A}{r^2}dt = \frac{A}{2At+1}dt$, so $\theta = \frac{1}{2}\ln(2At+1) + C$

But $\theta = 0$ at $t = 0$, so $0 = \frac{1}{2}\ln(1) + C_2$ and $C_2 = 0$. Thus

$\theta = \frac{1}{2}\ln(2At+1) = \frac{1}{2}\ln r^2 = \ln r$ or $r = e^\theta$ Pathline ←

Given: Definition of ∇ in cylindrical coordinates.

Obtain: $\nabla \cdot \rho\vec{V}$ in cylindrical coordinates (use hint on page 202).

Show result is identical to Eq. 5.2.

Solution: The definition of ∇ in cylindrical coordinates is

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z} \qquad (3.21)$$

Note $\rho\vec{V} = \rho(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{k} V_z)$

Hint: $\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$, and $\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$ (p. 202)

Substituting $\nabla \cdot \rho\vec{V} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \rho(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{k} V_z)$

$$\nabla \cdot \rho\vec{V} = \hat{e}_r \cdot \frac{\partial}{\partial r}\rho(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{k} V_z)$$
$$+ \hat{e}_\theta \cdot \frac{\partial}{\partial \theta}\rho(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{k} V_z)$$
$$+ \hat{k} \cdot \frac{\partial}{\partial z}\rho(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{k} V_z)$$

$$= \hat{e}_r \cdot \hat{e}_r \frac{\partial}{\partial r}\rho V_r + \hat{e}_\theta \cdot \frac{1}{r}\frac{\partial \hat{e}_r}{\partial \theta}\rho V_r + \hat{e}_\theta \cdot \hat{e}_r \frac{\partial}{\partial \theta}\rho V_r$$
$$+ \hat{e}_\theta \cdot \frac{\partial \hat{e}_\theta}{\partial \theta}\rho V_\theta + \hat{e}_\theta \cdot \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\rho V_\theta + \hat{k} \cdot \hat{k}\frac{\partial}{\partial z}\rho V_z$$

$$\nabla \cdot \rho\vec{V} = \frac{\partial}{\partial r}\rho V_r + \rho\frac{V_r}{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\rho V_\theta + \frac{\partial}{\partial z}\rho V_z$$

Combining the first two terms, $\frac{\partial}{\partial r}\rho V_r + \rho\frac{V_r}{r} = \frac{1}{r}\frac{\partial}{\partial r} r\rho V_r$, as may be verified by differentiation. Substituting

$$\nabla \cdot \rho\vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z)$$

This result is identical to the corresponding terms in Eq. 5.2.

Open-Ended Problem Statement: Consider a water stream from a nozzle attached to a rotating lawn sprinkler. Describe the corresponding pathline, streamline, and streakline.

Discussion: The rotating motion of the sprinkler jets makes this an unsteady flow. Therefore pathlines, streamlines, and streaklines need not coincide.

A *pathline* is a line tracing the path of an individual fluid particle. The trajectory of each particle depends on the absolute velocity with which it leaves the jet. Thus the path of each fluid particle is determined by the jet angle, the speed at which the particle leaves the jet, and the speed with which the sprinkler is rotating.

Once a particle leaves the jet it is subject to gravity and drag forces. The path of each water particle would be parabolic if aerodynamic drag were negligible. The absolute horizontal speed of the particle would remain constant throughout its trajectory. The particle would be slowed by gravity until reaching peak height, then its vertical speed would become increasingly negative until the particle strikes the ground. Aerodynamic drag reduces the particle speed. With drag the particle will not rise as far vertically nor travel as far horizontally. At each instant the particle trajectory will be lower and closer to the jet and the trajectory after the particle reaches its peak height will be steeper compared to the no-friction case.

A *streamline* is a line drawn in the flow that is tangent everywhere to the velocity vectors of the fluid motion. When unsteady effects are negligible, the streamline on which a given fluid particle lies is coincident with the pathline for the same particle. Flow unsteadiness creates different pathlines for particles that leave the sprinkler nozzle at different instants. It is difficult to visualize streamlines for an unsteady flow field because they may move laterally. The term "streamline" has little meaning for a rotating sprinkler with discrete jets.

A *streakline* is the locus of the present positions of fluid particles that passed a reference point at previous times. As an example, choose the exit of a jet as the reference point. Imagine marking particles that pass the jet exit at a given instant and at uniform time intervals later. The first particle will travel farthest from the jet exit where it was emitted; the last particle will be located right at the exit. In plan view the curve joining the positions of several particles will resemble a spiral with tighter radius close to the present position of the jet.

13-782 500 SHEETS FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 150 SHEETS EYE-EASE® 5 SQUARE
42-383 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Velocity field for viscometric flow of Example Problem 5.7

$$\vec{V} = U\frac{y}{h}\hat{\imath}$$

Find: (a) Stream function
(b) Locate streamline that divides flow rate equally.

Solution: Flow is incompressible, so stream function can be derived.

$$\frac{\partial \psi}{\partial y} = u = U\frac{y}{h}, \text{ so } \psi = \int \frac{\partial \psi}{\partial y}dy + f(x) = \int \frac{Uy}{h}dy + f(x) = \frac{Uy^2}{2h} + f(x)$$

Let $\psi = 0$ at $y = 0$, so $f(x) = 0$

$$\psi = \frac{Uy^2}{2h}$$ ← ψ

Stream function is maximum at $y = h$.

$$\psi_{max} = \frac{Uh^2}{2h} = \frac{Uh}{2}; \quad Q/w = \psi_{max} - \psi_{min} = \frac{Uh}{2} - 0 = \frac{Uh}{2}$$

$$\psi_{Q/2} = \frac{1}{2}\psi_{max} = \frac{Uh}{4} = \frac{Uy^2}{2h}$$

Thus $$y^2 = \frac{2h}{U}\frac{Uh}{4} = \frac{h^2}{2} \quad \text{so} \quad y = \frac{h}{\sqrt{2}}$$ ← $\psi_{Q/2}$

Given: Velocity field $\vec{V} = (x^2 - y^2)\hat{\imath} - 2xy\hat{\jmath}$

Find: Corresponding family of stream functions.

Solution: Ψ may be defined only if flow is incompressible.

Basic equations: $\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$ ($\partial\rho w/\partial z = 0(1)$, $\partial\rho/\partial t = 0(2)$)

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}$$

Assumptions: (1) $\vec{V} = \vec{V}(x, y)$, so $\frac{\partial}{\partial z} = 0$

(2) ρ = constant, so $\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial x} = 0$

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2x - 2x = 0$, so flow is incompressible.

Thus

$$u = x^2 - y^2 = \frac{\partial \Psi}{\partial y}; \quad \Psi = \int u\,dy + f(x) = x^2 y - \tfrac{1}{3}y^3 + f(x)$$

$$v = -2xy = -\frac{\partial \Psi}{\partial x}; \quad \Psi = \int -v\,dx + g(y) = x^2 y + g(y)$$

Comparing these two expressions for Ψ, we see that

$$f(x) = 0 \text{ and } g(y) = -\tfrac{1}{3}y^3$$

so $\Psi = x^2 y - \frac{1}{3}y^3$ Ψ

Given: Velocity field in cylindrical coordinates

$$\vec{V} = \frac{A}{r}\hat{e}_r + \frac{A}{r}\hat{e}_\theta$$

with $A = 0.25\ m^2/s$.

Find: (a) Is this a possible incompressible flow?
(b) If so, evaluate stream function.
If not, find rate of change of density.

Solution: If incompressible, flow must satisfy differential continuity.

Basic equation: $\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(V_\theta) = 0$

Thus $\frac{1}{r}\frac{\partial}{\partial r}(r\frac{A}{r}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\frac{A}{r}) = 0 + 0 = 0$ ∴ Flow is incompressible

From the definition of ψ, $V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$ and $V_\theta = -\frac{\partial \psi}{\partial r}$

$$\frac{\partial \psi}{\partial \theta} = rV_r = A \; ; \; \psi = A\theta + f(r)$$

$$\frac{\partial \psi}{\partial r} = -V_\theta = -\frac{A}{r} \; ; \; \psi = -A\ln r + g(\theta)$$

Comparing,

$$\psi = A\theta - A\ln r$$

ψ

Given: Stream function for an incompressible flow field,

$$\psi = -Ur\sin\theta + \frac{q}{2\pi}\theta$$

Find: (a) An expression for the velocity field.
(b) Points where $|\vec{V}| = 0$.
(c) Show $\psi = 0$ where $|\vec{V}| = 0$.

Solution: The velocity components are given by

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = -U\cos\theta + \frac{q}{2\pi r}$$

$$V_\theta = -\frac{\partial\psi}{\partial r} = U\sin\theta$$

so $\vec{V} = V_r\hat{\imath}_r + V_\theta\hat{\imath}_\theta = \left(-U\cos\theta + \frac{q}{2\pi r}\right)\hat{\imath}_r + U\sin\theta\,\hat{\imath}_\theta$ ← $\vec{V}$

Now $|\vec{V}| = (V_r^2 + V_\theta^2)^{1/2} = 0$ only when both V_r and V_θ are zero.

From the component equations, $V_\theta = 0$ for $\theta = 0, \pi$. When $V_r = 0$,

$$r = \frac{q}{2\pi U\cos\theta}$$

For $r > 0$, then $V_r = 0$ for $\theta = 0$, and $r = \frac{q}{2\pi U}$.

Stagnation point ($|\vec{V}| = 0$) occurs at $(r,\theta) = \left(\frac{q}{2\pi U}, 0\right)$ ← $|\vec{V}| =$

Substituting, $\psi_{stagnation} = -Ur\sin\theta + \frac{q}{2\pi}\theta\Big]_{r=\frac{q}{2\pi U},\,\theta=0}$

or $\psi_{stagnation} = 0$ ← ψ_{st}

Given: Flow with velocity components

$$u = 0, \quad v = -y^3 - 4z, \quad w = 3y^2 z$$

Find: (a) Is this one-, two- or three-dimensional?
(b) Incompressible?
(c) Stream function, if possible

Solution: $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k} = \vec{V}(y,z)$

Velocity field is a function of two space coordinates. Therefore flow is two-dimensional. ← 2-D

If incompressible, it must satisfy differential continuity equation.

Basic equation: $\frac{\partial \rho u}{\partial x}^{=0(1)} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t}^{=0(2)} = 0$

Assumptions: (1) Two-dimensional flow, so $\frac{\partial}{\partial x} = 0$
(2) Incompressible flow

ρ = constant, so $\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial z} = 0$

Then

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -3y^2 + 3y^2 = 0 \quad \therefore \text{ Flow is incompressible}$$ ← $\rho = c$

For incompressible flow in yz plane, $\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ will be satisfied identically if

$$v = \frac{\partial \psi}{\partial z} \quad \text{and} \quad w = -\frac{\partial \psi}{\partial y}$$

(Then continuity becomes $\frac{\partial^2 \psi}{\partial z \partial y} - \frac{\partial^2 \psi}{\partial y \partial z} = 0$.)

Thus $\psi = \int v\, dz + f(y) = -y^3 z - 2z^2 + f(y)$

and $\psi = \int -w\, dy + g(z) = -y^3 z + g(z)$

Comparing these two expressions, we see $f(y) = 0$ and $g(z) = -2z^2$:

$$\psi = -y^3 z - 2z^2$$ ← ψ

Given: An incompressible, frictionless flow specified by

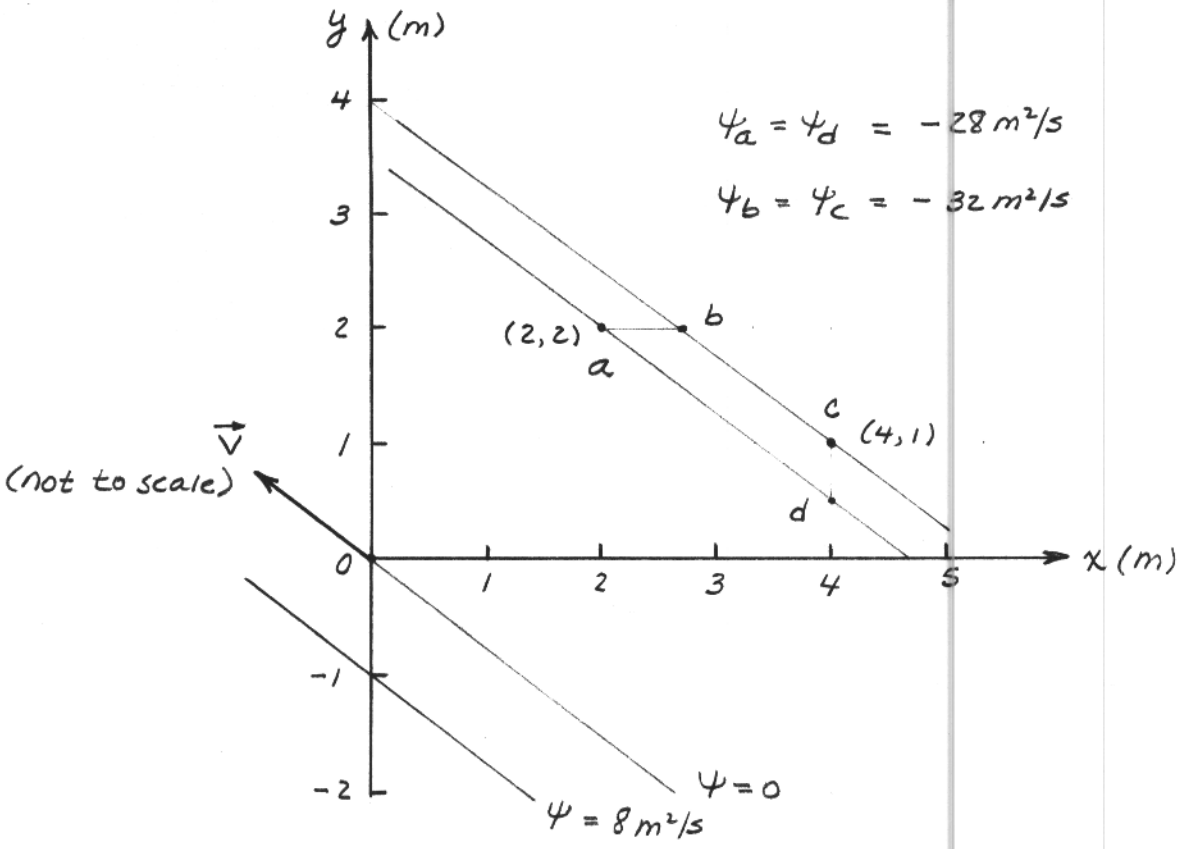
$$\psi = -6Ax - 8Ay \quad ; \quad x, y \text{ in meters}, \; A = 1 \text{ m/s}$$

Find: (a) Sketch streamlines $\psi = 0$ and $\psi = 8 \text{ m}^2/\text{s}$.
(b) Velocity vector at (0,0).
(c) Flowrate between streamlines passing through points (2,2) and (4,1).

Solution: Streamlines are lines ψ = constant.

For $\psi = 0$, $0 = -6Ax - 8Ay$ or $y = -\frac{3}{4}x$

For $\psi = 8$, $8 = -6Ax - 8Ay$ or $y = -\frac{3}{4}x - \frac{1}{8} \times \frac{8 \text{ m}^2}{s} \times \frac{s}{1 \text{ m}} = -\frac{3}{4}x - 1 \text{ m}$



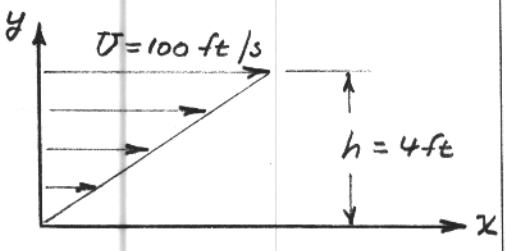
$$u = \frac{\partial \psi}{\partial y} = -8A; \quad v = -\frac{\partial \psi}{\partial x} = 6A, \text{ so } \vec{V} = -8A\hat{i} + 6A\hat{j} \quad \leftarrow \vec{V}$$

$$Q = \int_{x=a}^{x=b} v\,dx = \int_{x=a}^{x=b} -\frac{\partial \psi}{\partial x}\,dx = \int_{\psi_a}^{\psi_b} -d\psi = \psi_a - \psi_b = 4 \text{ m}^2/\text{s}, \text{ i.e.} \uparrow$$

$$Q = \int_{y=d}^{y=c} u\,dy = \int_{y=d}^{y=c} \frac{\partial \psi}{\partial y}\,dy = \int_{\psi_d}^{\psi_c} d\psi = \psi_c - \psi_d = -4 \text{ m}^2/\text{s}, \text{ i.e.} \leftarrow$$

Thus $Q = 4 \text{ m}^3/\text{s}$ per meter of depth $\leftarrow Q$

Given: Parallel one-dimensional flow in x direction with linear variation in velocity.



Find: (a) An expression for ψ.
(b) y coordinate below which half of flow passes.

Solution: Represent the velocity profile by $u = U(\frac{y}{h})$,

where $U = 100 \frac{ft}{s}$, $h = 4 ft$.

Note that $u = \frac{\partial \psi}{\partial y}$, so

$$\psi = \int u\,dy + f(x) = \frac{Uy^2}{2h} + f(x)$$

Also $v = -\frac{\partial \psi}{\partial x}$, but $v = 0$, so

$$\psi = \int -v\,dx + g(y) = g(y)$$

Comparing these expressions, we find $f(x) = 0$ and $g(y) = \frac{Uy^2}{2h}$, so

$$\psi = \frac{Uy^2}{2h}$$ ψ

For the whole profile, $0 < y < h$, the flowrate is

$$Q = \int_0^h u\,dy = h\int_0^1 u\,d(\frac{y}{h}) = hU\int_0^1 (\frac{y}{h})\,d(\frac{y}{h}) = \frac{hU}{2}$$

For half the flowrate, up to y^*

$$\frac{Q}{2} = \int_0^{y^*} u\,dy = \frac{U}{h}\int_0^{y^*} y\,dy = \frac{Uy^{*2}}{2h} = \frac{hU}{4} \quad \text{or} \quad y^{*2} = \frac{h^2}{2}$$

so

$$y^* = \frac{h}{\sqrt{2}} = 2.83 ft$$ y*

Given: Linear approximation to boundary layer velocity profile

$$u = U\frac{y}{\delta}$$

Find: (a) stream function for the flow field
(b) location of streamlines at one-quarter and one-half the total flow rate in the boundary layer.

Solution: For 2-D incompressible flow, ψ satisfies

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u = \frac{\partial \psi}{\partial y} = U\frac{y}{\delta} \qquad \therefore \psi = \int \frac{\partial \psi}{\partial y}\,dy + f(x) = \int U\frac{y}{\delta}\,dy + f(x)$$

Thus $\psi = \frac{U}{2\delta}y^2 + f(x)$

Let $\psi = 0$ along $y = 0$, so $f(x) = 0$ and $\psi = \frac{U}{2\delta}y^2$ ← ψ

The total flow rate within the boundary layer is

$$Q_{1/2} = \psi(\delta) - \psi(0) = \frac{1}{2}U\delta$$

At $\frac{1}{4}$ of total, $\psi - \psi_0 = \frac{U}{2\delta}y^2 = \frac{1}{4}\left(\frac{1}{2}U\delta\right)$

$$\therefore \left(\frac{y}{\delta}\right)^2 = \frac{1}{4} \qquad \text{and} \quad \frac{y}{\delta} = \frac{1}{2}$$ ← $\frac{1}{4}$ @ $\frac{1}{2}$

At $\frac{1}{2}$ of total, $\psi - \psi_0 = \frac{U}{2\delta}y^2 = \frac{1}{2}\left(\frac{1}{2}U\delta\right)$

$$\therefore \left(\frac{y}{\delta}\right)^2 = \frac{1}{2} \qquad \text{and} \quad \frac{y}{\delta} = \sqrt{\frac{1}{2}} = 0.707$$ ← $\frac{1}{2}$ @ 0.707

Given: Parabolic approximation to boundary layer velocity profile

$$u = U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]$$

Find: (a) stream function for the flow field
(b) location of streamlines at one-quarter and one-half the total flow rate in the boundary layer.

Solution: For 2-D incompressible flow, ψ satisfies

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u = \frac{\partial \psi}{\partial y} = U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]$$

$$\therefore \psi = \int \frac{\partial \psi}{\partial y}\,dy + f(x) = U\int\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]dy + f(x)$$

$$\psi = U\left[\frac{y^2}{\delta} - \frac{y^3}{3\delta^2}\right] + f(x)$$

Let $\psi = 0$ along $y = 0$, so $f(x) = 0$ and $\psi = U\delta\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]$

The total flow rate within the boundary layer is

$$Q = \psi(\delta) - \psi(0) = U\delta\left[1 - \frac{1}{3}\right] = \frac{2}{3}U\delta$$

At $\frac{1}{4}$ of total, $\psi - \psi_0 = U\delta\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right] = \frac{1}{4}\left(\frac{2}{3}U\delta\right)$

$$\therefore \left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3 = \frac{1}{6} = 0.167$$

Trial and error solution gives $\frac{y}{\delta} = 0.442$ @ $\frac{1}{4}$

At $\frac{1}{2}$ of total, $\psi - \psi_0 = U\delta\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right] = \frac{1}{2}\left(\frac{2}{3}U\delta\right)$

$$\therefore \left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3 = \frac{1}{3} = 0.333$$

Trial and error solution gives $\frac{y}{\delta} = 0.652$ @ $\frac{1}{2}$

Given: Sinusoidal approximation to boundary layer velocity profile

$$u = U\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

Find: Locate streamlines at quarter and half total flow rate.

Solution: Flow is incompressible so ψ may be derived.

$$u = \frac{\partial\psi}{\partial y} = U\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right);\quad \psi = \int\frac{\partial\psi}{\partial y}dy + f(x) = \int U\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)dy + f(x)$$

Thus $\psi = -\frac{2\delta U}{\pi}\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) + f(x)$

Let $\psi = 0$ along $y = 0$, so $f(x) = 0$ $\quad \psi = -\frac{2\delta U}{\pi}\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)$ ← ψ

The total flow rate is $\frac{Q}{w} = \psi(\delta) - \psi(0) = -\frac{2\delta U}{\pi}\cos\left(\frac{\pi}{2}\right) + \frac{2\delta U}{\pi}\cos(0) = \frac{2\delta U}{\pi}$

At 1/4 of total, $\psi - \psi_0 = \frac{2\delta U}{\pi}\left[1 - \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right] = \frac{1}{4}\frac{2\delta U}{\pi} = \frac{\delta U}{2\pi}$

$1 - \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) = \frac{\pi}{2\delta U}\frac{\delta U}{2\pi} = \frac{1}{4}$; $\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) = \frac{3}{4}$; $\frac{y}{\delta} = 0.460$ ← $\frac{1}{4}\frac{Q}{w}$

At 1/2 of total, $\psi - \psi_0 = \frac{2\delta U}{\pi}\left[1 - \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right] = \frac{1}{2}\frac{2\delta U}{\pi} = \frac{\delta U}{\pi}$

$1 - \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) = \frac{\pi}{2\delta U}\frac{\delta U}{\pi} = \frac{1}{2}$; $\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) = \frac{1}{2}$; $\frac{y}{\delta} = 0.667$ ← $\frac{1}{2}\frac{Q}{w}$

42 381 50 SHEETS 5 SQUARE
42 382 100 SHEETS 5 SQUARE
42 389 200 SHEETS 5 SQUARE
NATIONAL

Given: Rigid-body motion in Example Problem 5.6

$$\vec{V} = r\omega\,\hat{e}_\theta \qquad \omega = 0.5 \text{ rad/s}$$

Find: (a) Obtain the stream function for this flow.
(b) Evaluate the volume flow rate per unit depth between $r_1 = 0.05$ m and $r_2 = 0.07$ m.
(c) Sketch the velocity profile along a line of constant θ.
(d) Check the volume flow rate calculated from the stream function by integrating the velocity profile along this line.

Solution: From the definition of ψ, $\frac{\partial \psi}{\partial r} = -V_\theta = -r\omega$

Thus $\psi = \int \frac{\partial \psi}{\partial r}\,dr + f(\theta) = \int -r\omega\,dr + f(\theta) = -\frac{1}{2}r^2\omega + f(\theta)$

But $V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r}f'(\theta) = 0 \quad \therefore f(\theta) = C$

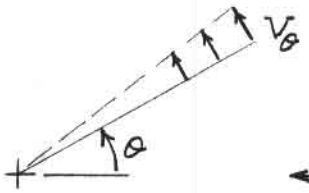
and $\psi = -\frac{1}{2}r^2\omega + C$ ← ψ

The volume flow rate per unit depth is

$$\frac{Q}{b} = \psi(r_2) - \psi(r_1) = -\frac{1}{2}r_2^2\omega + C - \left[-\frac{1}{2}r_1^2\omega + C\right] = \frac{\omega}{2}(r_1^2 - r_2^2)$$

$$\frac{Q}{b} = \frac{1}{2} \times 0.5 \frac{\text{rad}}{\text{s}}\left[(0.05)^2 - (0.07)^2\right] \text{m}^2 = -0.000600 \text{ m}^3/\text{s} / \text{m}$$ ← Q/b

Because $Q/b < 0$, flow is in the direction of $\hat{e}_\theta$.



Along θ = constant, V_θ varies linearly: ← Plot

From the linear velocity variation, $V_\theta = \omega r$

Thus $\frac{Q}{b} = \int_{r_1}^{r_2} V_\theta\,dr = \int_{r_1}^{r_2} r\omega\,dr = \frac{1}{2}r^2\omega\Big]_{r_1}^{r_2} = \frac{\omega}{2}(r_2^2 - r_1^2)$ ← Q/b

From the sketch, this flow is in the direction of $\hat{e}_\theta$.

Comparing the expressions for Q/b shows they are the same except for sign.

Given: Velocity field for a free vortex from Example Problem 5.6:

$$\vec{V} = \frac{C}{r}\hat{e}_\theta \qquad C = 0.5\ m^2/sec$$

Find: (a) Obtain the stream function for this flow.
(b) Evaluate the volume flow rate per unit depth between $r_1 = 0.05\ m$ and $r_2 = 0.7\ m$.
(c) Sketch the velocity profile along a line of constant θ.
(d) Check the volume flow rate calculated from the stream function by integrating the velocity profile along this line.

Solution: From the definition of ψ, $\frac{\partial \psi}{\partial r} = -V_\theta = -\frac{C}{r}$

Thus $\psi = \int \frac{\partial \psi}{\partial r} dr + f(\theta) = \int -\frac{C}{r} dr + f(\theta) = -C \ln r + f(\theta)$

But $V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r} f'(\theta) = 0$. Therefore $f(\theta) = \text{constant} = C_1$, and

$$\psi = -C \ln r + C_1 \qquad \psi$$

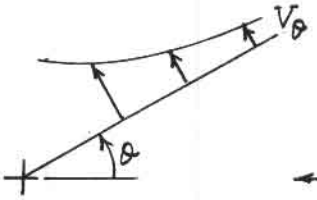
The volume flow rate per unit depth is

$$\frac{Q}{b} = \psi(r_2) - \psi(r_1) = -C \ln r_2 + C_1 - \left[-C \ln r_1 + C_1\right] = C(\ln r_1 - \ln r_2) = C \ln\left(\frac{r_1}{r_2}\right)$$

$$\frac{Q}{b} = 0.5 \frac{m^2}{s} \times \ln\left(\frac{0.05\ m}{0.07\ m}\right) = -0.168\ m^3/s\ /\ m \qquad Q/b$$

Because $Q/b < 0$, flow is in the direction of $\hat{e}_\theta$.

Along θ = constant, V_θ varies inversely with r:



Plot

From the expression for $\vec{V}$, $V_\theta = \frac{C}{r}$. Thus

$$\frac{Q}{b} = \int_{r_1}^{r_2} V_\theta\, dr = \int_{r_1}^{r_2} \frac{C}{r} dr = C \ln\left(\frac{r_2}{r_1}\right) \qquad Q/b$$

From the sketch, this flow is in the direction of $\hat{e}_\theta$.

Comparing shows that the expressions for Q/b are the same except for sign.

Given: Velocity field in the xy plane

$$\vec{V} = A x y \hat{\imath} - \frac{A y^2}{2}\hat{\jmath}, \text{ where } A = 0.25\ m^{-1}s^{-1} \text{ and } x, y \text{ in } m.$$

Find: (a) Is this a possible incompressible flow?
(b) Acceleration of a fluid particle at $(x, y) = (2, 1)$.

Solution:

Basic equations: $0 = \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t}$ ($=0(2)$, $=0(1)$)

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$ ($=0(2)$, $=0(1)$)

Assumptions: (1) Steady flow
(2) Incompressible flow (ρ = constant)
(3) Flow in xy plane: $w = 0$

Continuity becomes

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(Axy) + \frac{\partial}{\partial y}\left(-\frac{Ay^2}{2}\right) = Ay - Ay = 0 \quad \checkmark$$

Flow is incompressible! $\rho = c$

Compute acceleration components:

$$x: \quad a_{px} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = (Axy)(Ay) + \left(-\frac{Ay^2}{2}\right)(Ax) = \frac{A^2xy^2}{2}$$

$$y: \quad a_{py} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = (Axy)(0) + \left(-\frac{Ay^2}{2}\right)(-Ay) = \frac{A^2y^3}{2}$$

Thus $\vec{a}_p = \frac{A^2xy^2}{2}\hat{\imath} + \frac{A^2y^3}{2}\hat{\jmath}$

At $(x, y) = (2, 1)$

$$\vec{a}_p(2,1) = \frac{1}{2} \times \frac{(0.25)^2}{m^2 \cdot s^2} \times 2\ m \times (1)^2 m^2 \hat{\imath} + \frac{1}{2} \times \frac{(0.25)^2}{m^2 \cdot s^2} \times (1)^3 m^3 \hat{\jmath}$$

$$\vec{a}_p(2,1) = 0.0625\,\hat{\imath} + 0.0313\,\hat{\jmath}\ m/s^2$$ $\vec{a}_p(2,1)$

Given: Flow field $\vec{V} = xy^2\hat{\imath} - \frac{1}{3}y^3\hat{\jmath} + xy\hat{k}$

Find: (a) Dimensions.
(b) If possible incompressible flow.
(c) Acceleration of particle at point $(x,y,z) = (1,2,3)$.

Solution: Apply continuity, use substantial derivative.

Basic equations: $\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$ (terms: $=0(1)$, $=0(2)$)

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$ ($=0(1)$, $=0(2)$)

Assumptions: (1) Two-dimensional flow, $\vec{V} = \vec{V}(x,y)$, so $\partial/\partial z = 0$
(2) Incompressible flow
(3) Steady flow, $\vec{V} \neq \vec{V}(t)$

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = y^2 - y^2 = 0$ Flow is a possible incompressible case. $\rho =$

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} \quad ; \quad \frac{\partial \vec{V}}{\partial x} = y^2\hat{\imath} + y\hat{k} \; ; \quad \frac{\partial \vec{V}}{\partial y} = 2xy\hat{\imath} - y^2\hat{\jmath} + x\hat{k}$$

$$= (xy^2)(y^2\hat{\imath} + y\hat{k}) + (-\tfrac{1}{3}y^3)(2xy\hat{\imath} - y^2\hat{\jmath} + x\hat{k})$$

$$= \hat{\imath}\left(xy^4 - \tfrac{2}{3}xy^4\right) + \hat{\jmath}\left(\tfrac{1}{3}y^5\right) + \hat{k}\left(xy^3 - \tfrac{1}{3}xy^3\right)$$

$$\vec{a}_p = \hat{\imath}\left(\tfrac{1}{3}xy^4\right) + \hat{\jmath}\left(\tfrac{1}{3}y^5\right) + \hat{k}\left(\tfrac{2}{3}xy^3\right)$$

At $(x,y,z) = (1,2,3)$

$$\vec{a}_p = \hat{\imath}\left[\tfrac{1}{3}(1)(16)\right] + \hat{\jmath}\left[\tfrac{1}{3}(32)\right] + \hat{k}\left[\tfrac{2}{3}(1)(8)\right] = \frac{16}{3}\hat{\imath} + \frac{32}{3}\hat{\jmath} + \frac{16}{3}\hat{k}$$ $\vec{a}_p$

($\vec{a}_p$ will be in m/s^2)

Given: Flow field $\vec{V} = ax^2y\hat{i} - by\hat{j} + cz^2\hat{k}$; $a = 1/m^2 \cdot s$, $b = 3/s$, $c = 2/m \cdot s$

Find: (a) Dimensions of flow field.
(b) If possible incompressible flow.
(c) Acceleration of a particle at $(x, y, z) = (3, 1, 2)$.

Solution: Apply continuity, use substantial derivative.

Basic equations: $\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$ ($\frac{\partial \rho}{\partial t} = 0$)

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \quad (\frac{\partial \vec{V}}{\partial t} = 0)$$

Assumption: Incompressible flow, ρ = constant

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ is criterion.

Note $\vec{V} = \vec{V}(x, y, z)$, so flow is three-dimensional, and ← 3-D

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 2xy - 3 + 4z \neq 0$$

Flow cannot be incompressible. ← $\rho \neq$

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}; \quad \frac{\partial \vec{V}}{\partial x} = 2axy\hat{i}, \quad \frac{\partial \vec{V}}{\partial y} = ax^2\hat{i} - b\hat{j}, \quad \frac{\partial \vec{V}}{\partial z} = 2cz\hat{k}$$

$$= (ax^2y)(2axy\hat{i}) + (-by)(ax^2\hat{i} - b\hat{j}) + (cz^2)(2cz\hat{k})$$

$$\vec{a}_p = \hat{i}(2a^2x^3y^2 - abx^2y) + \hat{j}(b^2y) + \hat{k}(2c^2z^3)$$

At $(x, y, z) = (3, 1, 2)$,

$$\vec{a}_p = \hat{i}\left[2 \times \frac{(1)^2}{m^4 \cdot s^2} \times (3)^3 m^3 \times (1)^2 m^2 - \frac{1}{m^2 \cdot s} \times \frac{3}{s} \times (3)^2 m^2 \times 1m\right] + \hat{j}\left[\frac{(3)^2}{s^2} \times 1m\right] + \hat{k}\left[2 \times \frac{(2)^2}{m^2 \cdot s^2} \times (2)^3 m^3\right]$$

$$\vec{a}_p = 27\hat{i} + 9\hat{j} + 64\hat{k}\ \frac{m}{s^2}$$ ← $\vec{a}_p$

Given: Velocity field (within a laminar boundary layer) is given by $\vec{V} = A\sqrt{\frac{U}{x}}\left(\hat{i} + \frac{y}{4x}\hat{j}\right)$

where $A = 141\ m^{-1/2}$

$U = 0.240\ m/s$

Find: (a) Show that this velocity field represents a possible incompressible flow

(b) Calculate $\vec{a}$ of particle at $(x,y) = (0.5m, 5mm)$

(c) Slope of streamline through point $(0.5m, .5mm)$

Solution:

From given velocity field $\vec{V} = \vec{V}(x,y)$, $w=0$, flow is steady

(a) Check conservation of mass for ρ = constant

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}}^{\,0} = 0$$

$u = A\sqrt{\frac{U}{x}}$, $\frac{\partial u}{\partial x} = -\frac{1}{2}AU\,x^{-3/2}$

$v = AU\frac{y}{x^{3/2}}$, $\frac{\partial v}{\partial y} = \frac{1}{x^{1/2}}AU\,x^{-1/2}$

$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ ∴ incompressible ← Q.E.D.

(b) $\vec{a} = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + \cancel{w\frac{\partial \vec{V}}{\partial z}}^{\,0} + \cancel{\frac{\partial \vec{V}}{\partial t}}^{\,0}$

$a_{p_x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$; $\frac{\partial u}{\partial y} = AU\frac{1}{x^{1/2}}$

$a_{p_x} = A\sqrt{\frac{U}{x}}\left(-\frac{1}{2}AU\frac{y}{x^{3/2}}\right) + AU\frac{y}{x^{3/2}}\left(AU\frac{1}{x^{1/2}}\right)$

$a_{p_x} = -\frac{1}{2}A^2U^2\frac{y}{x^2} + A^2U^2\frac{y}{x^2} = -\frac{1}{4}\left(\frac{AUy}{x}\right)^2$

$a_{p_x} = -\frac{1}{4}\left[\frac{141}{m^{1/2}} \times 0.240\frac{m}{s} \times \frac{0.005m}{0.5m}\right]^2 = -0.0286\ m/s^2$ ← —

$a_{p_y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$; $\frac{\partial v}{\partial x} = -\frac{1}{8}AU\frac{y}{x^{3/2}}$

$= AU\frac{y}{x^{1/2}}\left(-\frac{1}{8}AU\frac{y}{x^{3/2}}\right) + AU\frac{y}{x^{3/2}}\left(\frac{1}{2}AU\frac{y}{x^{3/2}}\right)$

$= -\frac{1}{8}A^2U^2\frac{y^3}{x^3} + \frac{1}{8}A^2U^2\frac{y^3}{x^3} = -\frac{1}{4}A^2U^2\frac{y^3}{x^3}$

$a_{p_y} = -\frac{1}{4}\left(\frac{141}{m^{1/2}} \times 0.240\frac{m}{s}\right)^2\left(\frac{0.005m}{0.5m}\right)^3 = -2.86\times10^{-4}\ m/s^2$ ← —

$\therefore \vec{a}_p = -2.86\left(10^{-2}\hat{i} + 10^{-4}\hat{j}\right)\ m/s^2$ ← $\vec{a}_p$

The slope of the streamline is given by

$\left(\frac{dy}{dx}\right)_s = \frac{v}{u} = \frac{y}{4x} = \frac{5\times10^{-3}m}{4\times0.5m} = 0.0025$ ← $\left(\frac{dy}{dx}\right)_s$

Given: Incompressible, two-dimensional flow field with $w=0$, has a y component of velocity given by

$$v = -Axy$$

where units of v are m/s; x and y are in meters and A is a dimensional constant.

Find: (a) the dimensions of the constant A
(b) the simplest x component of velocity for this flow field.
(c) the acceleration of a fluid particle at the point $(x,y)=(1,2)$

Solution:

(a) Since $v = -Axy$, then the dimensions of A, $[A]$, are given by

$$[A] = \left[\frac{v}{xy}\right] = \frac{L/t}{L \cdot L} = \frac{1}{Lt}$$ [A]

(b) Apply the continuity equation for the conditions given

Basic equation: $\nabla \cdot \rho\vec{V} + \frac{\partial \rho}{\partial t} = 0$

For incompressible flow, $\frac{\partial \rho}{\partial t} = 0$. Thus with $w=0$, the basic equation reduces to $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Then, $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = -\frac{\partial}{\partial y}(-Axy) = Ax$

and

$$u = \int \frac{\partial u}{\partial x}\,dx + f(y) = \int Ax\,dx + f(y) = \frac{1}{2}Ax^2 + f(y)$$

The simplest x component of velocity is obtained with $f(y)=0$

$$\therefore \quad u = \frac{1}{2}Ax^2$$ u

(c) The acceleration of a fluid particle is given by

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\cancel{\frac{\partial \vec{V}}{\partial z}}^{\,0} + \cancel{\frac{\partial \vec{V}}{\partial t}}^{\,0}$$

$$\vec{a}_p = \frac{1}{2}Ax^2\frac{\partial}{\partial x}\left[\frac{1}{2}Ax^2\hat{i} - Axy\hat{j}\right] - Axy\frac{\partial}{\partial y}\left[\frac{1}{2}Ax^2\hat{i} - Axy\hat{j}\right]$$

$$\vec{a}_p = \frac{1}{2}Ax^2\left[Ax\hat{i} - Ay\hat{j}\right] - Axy\left[-Ax\hat{j}\right] = \frac{1}{2}A^2x^3\hat{i} + \frac{1}{2}A^2x^2y\hat{j}$$

At the point $(x,y) = (1,2)$

$$\vec{a}_p = \frac{1}{2}A^2(1)^3\hat{i} + \frac{1}{2}A^2(1)^2(2)\hat{j} = A^2\left[\frac{1}{2}\hat{i} + \hat{j}\right]$$ $\vec{a}_p$

Given: The x component of velocity in a steady, incompressible flow in the xy plane is

$$u = \frac{A}{x} \quad \text{where } A = 2\ m^2/s, \text{ and } x \text{ is in } m$$

Find: (a) Show the simplest y component is $v = Ay/x^2$
(b) Evaluate acceleration of a fluid particle at $(x,y) = (1,3)$.

Solution: Apply conservation of mass for incompressible flow.

Basic equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ ($\partial w/\partial z = 0$ (2))

Assumptions: (1) Incompressible flow
(2) Flow in xy plane ($w = 0$ and $\partial/\partial z = 0$)

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$; $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$; $v = \int \frac{\partial v}{\partial y}\,dy + f(x)$ ($f(x) = 0$ (simplest))

$$v = \int -\frac{\partial u}{\partial x}\,dy = \int -\left(-\frac{A}{x^2}\right)dy = \frac{Ay}{x^2}$$ v

The acceleration components for steady flow are

$$a_{px} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \left(\frac{A}{x}\right)\left(-\frac{A}{x^2}\right) + \left(\frac{Ay}{x^2}\right)(0) = -\frac{A^2}{x^3}$$

$$a_{py} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \left(\frac{A}{x}\right)\left(-2\frac{Ay}{x^3}\right) + \left(\frac{Ay}{x^2}\right)\left(\frac{A}{x^2}\right) = -\frac{2A^2y}{x^4} + \frac{A^2y}{x^4} = -\frac{A^2y}{x^4}$$

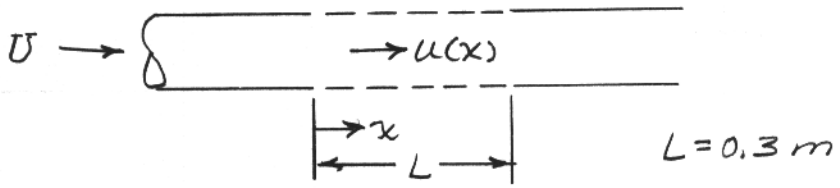
At $(x,y) = (1,3)$:

$$a_{px} = -\frac{(2)^2 m^4}{s^2} \times \frac{1}{(1)^3 m^3} = -4\ m/s^2$$

$$a_{py} = -\frac{(2)^2 m^4}{s^2} \times 3\,m \times \frac{1}{(1)^4 m^4} = -12\ m/s^2$$

Thus $\vec{a}_p(1,3) = -4\hat{i} - 12\hat{j}\ m/s^2$ $\vec{a}_p$ (1,3)

Given: Duct flow with inviscid liquid, ρ = constant.



$$u(x) = U(1 - x/2L) \qquad U = 5\ m/s$$

Find: Expression for acceleration along ℓ.

Solution: Computing equation

$$a_{px} = u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial u}{\partial y}}^{=0(1)} + \cancel{w\frac{\partial u}{\partial z}}^{=0(1)} + \cancel{\frac{\partial u}{\partial t}}^{=0(2)}$$

Assumptions: (1) Along ℓ $v = w = 0$

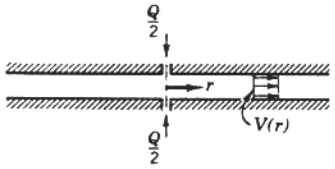
(2) Steady flow

Then

$$a_{px} = u\frac{\partial u}{\partial x} = U\left(1 - \frac{x}{2L}\right)U\left(-\frac{1}{2L}\right) = -\frac{U^2}{2L}\left(1 - \frac{x}{2L}\right)$$

a_{px} ←

Given: Incompressible flow between parallel plates as shown.



Find: (a) Show $V_r = \frac{Q}{2\pi r h}$

(b) Acceleration in gap.

Solution: Apply conservation of mass

Basic equation: $\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(V_\theta) + \frac{\partial}{\partial z}V_z = 0$ (V_θ: $=0(1)$, V_z: $=0(2)$)

Assumptions: (1) $V_\theta = 0$

(2) $V_z = 0$

Then

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) = 0 \quad \text{or} \quad rV_r = C \quad \text{or} \quad V_r = \frac{C}{r} \text{ is form of solution.}$$

The volume flow rate is $Q = 2\pi r h V_r$, so $V_r = \frac{Q}{2\pi r h}$ ← V_r

Because $V_\theta = 0$, $a_\theta = 0$. The radial acceleration is

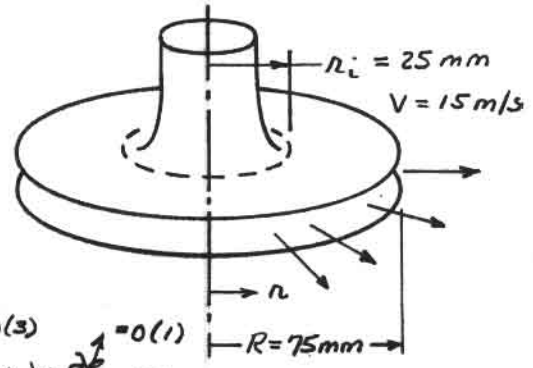
$$a_r = V_r\frac{\partial V_r}{\partial r} = \frac{Q}{2\pi r h}\left[(-1)\frac{Q}{2\pi r^2 h}\right] = -\left(\frac{Q}{2\pi h}\right)^2\frac{1}{r^3}$$

Thus

$$\vec{a}_p = -\left(\frac{Q}{2\pi h}\right)^2\frac{1}{r^3}\hat{e}_r$$ ← $\vec{a}_p$

The above expressions are valid only for $r > 0$.

Given: Incompressible, inviscid flow of air between parallel disks.



Find: (a) Simplify continuity.
(b) Show $\vec{V} = V(R/r)\hat{e}_r$, $r_i < r < R$
(c) Calculate acceleration of a particle at $r = r_i, R$.

Solution: Apply continuity equation and substantial derivative

Basic equations: $\frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) + \frac{\partial \rho}{\partial t} = 0$ ($=0(2)$, $=0(3)$, $=0(1)$)

$$a_r = V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + \frac{\partial V_r}{\partial z} - \frac{V_\theta}{r} + \frac{\partial V_r}{\partial t}$$ ($=0(2)$, $=0(3)$, $=0(2)$, $=0(4)$)

Assumptions: (1) Incompressible flow, ρ = constant
(2) Radial flow, $V_\theta = 0$
(3) Uniform flow at each radial location, $\partial/\partial z = 0$
(4) Steady flow

Then

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) = 0 \quad \text{or} \quad rV_r = \text{constant} = RV \; ; \; V_r = V\frac{R}{r}$$

so that $\vec{V} = V\frac{R}{r}\hat{e}_r$ ← $\vec{V}$

The radial acceleration of a fluid particle is

$$a_r = V_r\frac{\partial V_r}{\partial r} = V\frac{R}{r}(VR)\left(-\frac{1}{r^2}\right) = -\frac{V^2R^2}{r^3} = -\frac{V^2}{R}\left(\frac{R}{r}\right)^3$$

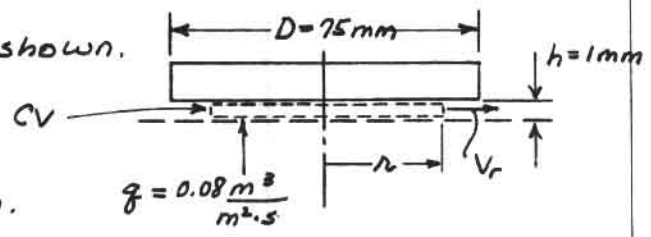
At $r = r_i = 25\,mm$,

$$a_r = -\frac{(15)^2 m^2}{s^2} \times \frac{1}{0.075\,m}\left(\frac{75}{25}\right)^3 = -81.0\,\frac{km}{s^2}$$ ← $a_r(r_i)$

At $r = R = 75\,mm$

$$a_r = -\frac{(15)^2 m^2}{s^2} \times \frac{1}{0.075\,m}\left(\frac{75}{75}\right)^3 = -3.00\,\frac{km}{s^2}$$ ← $a_r(R)$

Given: Air hockey puck, modeled as shown.



Find: (a) An expression for $V_r(r)$.
(b) Magnitude and location of maximum radial acceleration.

Solution: Apply the continuity equation using a control volume of radius, r, as shown.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ (=0(1))

Assumptions: (1) Steady flow
(2) Uniform radial flow at each radial section
(3) Incompressible flow
(4) Uniformly distributed flow over table surface

Then

$$0 = \{-|\rho q \pi r^2|\} + \{|\rho V_r 2\pi r h|\}$$

or

$$V_r = q\frac{r}{2h}$$ V_r

The radial acceleration is

$$a_r = V_r\frac{\partial V_r}{\partial r} + \cancel{\frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta}}^{=0(5)} + V_z\cancel{\frac{\partial V_r}{\partial z}}^{=0(2)} - \cancel{\frac{V_\theta^2}{r}}^{=0(5)} + \cancel{\frac{\partial V_r}{\partial t}}^{=0(1)}$$

(5) By symmetry, $V_\theta = 0$

Thus

$$a_r = V_r\frac{\partial V_r}{\partial r} = \left(q\frac{r}{2h}\right)\left(q\frac{1}{2h}\right) = \frac{q^2 r}{4h^2}$$

The maximum radial acceleration occurs at $r = D/2$, where

$$a_r = \frac{q^2 D}{8h^2} = \frac{1}{8}\times(0.08)^2 m^6 \frac{}{m^4 \cdot s^2} \times 0.075\, m \times \frac{1}{(0.001)^2 m^2} = 60.0\ m/s^2$$ $a_{r,max}$

Given: Temperature variation, $T = T_0 - \alpha e^{-x/L} \sin\left(\frac{2\pi t}{\tau}\right)$

Particle moves at speed U = constant

Find: Rate of change of T experienced by particle.

Solution: $T = T(x,t)$

$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial t} dt$$

$$\left.\frac{dT}{dt}\right)_{particle} = \frac{\partial T}{\partial x} \left.\frac{dx}{dt}\right)_{particle} + \frac{\partial T}{\partial t}$$

or

$$\frac{DT}{Dt} = \left.\frac{dT}{dt}\right)_{particle} = u\frac{\partial T}{\partial x} + \frac{\partial T}{\partial t}$$

For the given data, $u = U$ = constant

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left[T_0 - \alpha e^{-x/L} \sin\left(\frac{2\pi t}{\tau}\right)\right] = \frac{\alpha}{L} e^{-x/L} \sin\left(\frac{2\pi t}{\tau}\right)$$

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}\left[T_0 - \alpha e^{-x/L} \sin\left(\frac{2\pi t}{\tau}\right)\right] = -\frac{2\pi}{\tau} \alpha e^{-x/L} \cos\left(\frac{2\pi t}{\tau}\right)$$

Substituting,

$$\frac{DT}{Dt} = \left[\frac{U}{L} \sin\left(\frac{2\pi t}{\tau}\right) - \frac{2\pi}{\tau} \cos\left(\frac{2\pi t}{\tau}\right)\right] \alpha e^{-x/L} \quad \text{deg/s}$$

$\frac{DT}{Dt}$ ←

Given: Aircraft flying north with velocity component $u = 300$ mph is climbing at rate, $v = 3000$ ft/min. The rate of temperature change with vertical distance y is $\partial T/\partial y = -3°F/1000$ ft. The variation of temperature with position x is $\partial T/\partial x = -1°F/\text{mile}$

Find: the rate of temperature change shown by a recorder on board the aircraft

Solution: Apply the substantial derivative concept

Basic equation: $\frac{DT}{Dt} = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \cancel{\frac{\partial T}{\partial t}}^{\,0}$

Substituting numerical values,

$$\frac{DT}{Dt} = 300\frac{\text{mile}}{\text{hr}} \times -\frac{1°F}{\text{mile}} \times \frac{\text{hr}}{60\,\text{min}} + 3000\frac{\text{ft}}{\text{min}} \times -\frac{3°F}{1000\,\text{ft}}$$

$$\frac{DT}{Dt} = (-5-9)°F/\text{min} = -14°F/\text{min} \longleftarrow \frac{DT}{Dt}$$

Given: Instruments on board an aircraft flying through a cold front give the following information:
- rate of change of temperature is -0.5F/min
- air speed = 300 knots
- rate of climb = 3500 ft/min

Front is stationary and vertically uniform

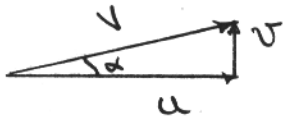
Find: rate of change of temperature with respect to horizontal distance through the cold front

Solution: Apply the substantial derivative concept

Basic equation: $\frac{DT}{Dt} = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \frac{\partial T}{\partial t}$ ($v\frac{\partial T}{\partial y} = 0$: vertically uniform; $\frac{\partial T}{\partial t} = 0$: stationary front)

$\frac{DT}{Dt} = -0.5\,F/min$. Need to find $\frac{\partial T}{\partial x}$

Velocity picture



$$V = 300\frac{nm}{hr}\times\frac{6080\,ft}{nm}\times\frac{hr}{3600\,s} = 507\frac{ft}{s}$$

$$v = 3500\frac{ft}{min}\times\frac{min}{60\,s} = 58.3\,ft/s$$

Then $\alpha = \sin^{-1}\frac{v}{V} = \sin^{-1}\frac{58.3}{507} = 6.60°$

and $u = V\cos\alpha = 507\frac{ft}{sec}\cos 6.60° = 504\,ft/s$

$$\therefore \frac{\partial T}{\partial x} = \frac{1}{u}\frac{DT}{Dt} = \frac{s}{504\,ft}\times\frac{-0.5°F}{min}\times\frac{min}{60\,s}\times\frac{5280\,ft}{mi}$$

$$\frac{\partial T}{\partial x} = -0.0873°F/mile$$

Given: Sediment concentration rates in a river after a rainfall are:

$$\frac{\partial c}{\partial t} = 100 \frac{ppm}{hr}, \quad \frac{\partial c}{\partial x} = 50 \frac{ppm}{mi} \text{ (downstream)}$$

Stream speed is $u_s = 0.5$ mph, where a boat is used to survey concentration.

Boat speed is $V_b = 2.5$ mph.

Find: (a) Calculate rates of change of sediment concentration observed when boat travels upstream, drifts with the current, or travels downstream.

(b) Explain physically why the observed rates differ.

Solution: Apply substantial derivative concept

Basic equation: $\frac{Dc}{Dt} = u\frac{\partial c}{\partial x} + \frac{\partial c}{\partial t}$

To obtain rate of change seen from boat, set $u = u_B$.

(i) For travel upstream, $u_B = u_s - V_b = 0.5 - 2.5 = -2.0$ mph

$$\frac{Dc}{Dt}(up) = -2.0 \frac{mi}{hr} \times 50 \frac{ppm}{mi} + 100 \frac{ppm}{hr} = 0.00 \text{ ppm/hr}$$ up

(ii) For drifting, $u_B = u_s + 0 = 0.5$ mph

$$\frac{Dc}{Dt}(drift) = 0.5 \frac{mi}{hr} \times 50 \frac{ppm}{mi} + 100 \frac{ppm}{hr} = 125 \text{ ppm/hr}$$ drift

(iii) For travel downstream, $u_B = u_s + V_b = 0.5 + 2.5 = 3.0$ mph

$$\frac{Dc}{Dt}(down) = 3.0 \frac{mi}{hr} \times 50 \frac{ppm}{mi} + 100 \frac{ppm}{hr} = 250 \text{ ppm/hr}$$ down

Physically the observed rates of change differ because the observer is convected through the flow. The convective change may add to or subtract from the local rate of change.

Expand $(\vec{V}\cdot\nabla)\vec{V}$ in rectangular coordinates to obtain the convective acceleration of a fluid particle. Verify the results given in Eqs 5.11

Solution:

In rectangular coordinates $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$, $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$

$$(\vec{V}\cdot\nabla)\vec{V} = \left[(u\hat{i}+v\hat{j}+w\hat{k})\cdot\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\right](u\hat{i}+v\hat{j}+w\hat{k})$$

$$= \left[u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right](u\hat{i}+v\hat{j}+w\hat{k})$$

$$(\vec{V}\cdot\nabla)\vec{V} = \underbrace{\left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right\}}_{①}\hat{i} + \underbrace{\left\{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right\}}_{②}\hat{j} + \underbrace{\left\{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right\}}_{③}\hat{k}$$

Term ① is x component of convective acceleration

Eq 5.11a $$a_{xp} = \left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right\} + \frac{\partial u}{\partial t}$$

Term ② is the y component of convective acceleration

Eq 5.11b $$a_{yp} = \left\{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right\} + \frac{\partial v}{\partial t}$$

Term ③ is the z component of convective acceleration

Eq 5.11c $$a_{zp} = \left\{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right\} + \frac{\partial w}{\partial t}$$

500 SHEETS, FILLER 5 SQUARE
50 SHEETS EYE-EASE® 5 SQUARE
100 SHEETS EYE-EASE® 5 SQUARE
100 SHEETS JUMBO 5 SQUARE
100 RECYCLED WHITE 5 SQUARE
200 RECYCLED WHITE 5 SQUARE
200 RECYCLED WHITE 5 SQUARE
13-782 42-381 42-382 42-383 42-392 42-393
Made in U.S.A.
National® Brand

Given: Steady, two-dimensional velocity field, $\vec{V} = Ax\hat{i} - Ay\hat{j}$; $A = 1\ s^{-1}$, coordinates measured in meters

Show: that streamlines are hyperbolas, $xy = C$

Find: (a) Expression for acceleration.
(b) Particle acceleration at $(x,y) = (\frac{1}{2},2)$, $(1,1)$ and $(2,\frac{1}{2})$

Plot: Streamlines corresponding to $C = 0, 1$, and $2\ m^2$; show acceleration vectors on the plot.

Solution:

Along a streamline, $\frac{dy}{dx} = \frac{v}{u} = \frac{-y}{x}$ or $\frac{dy}{y} + \frac{dx}{x} = 0$

Integrating we obtain $\ln y + \ln x = \ln C$ and $xy = C$ — Streamline

The acceleration of a particle is

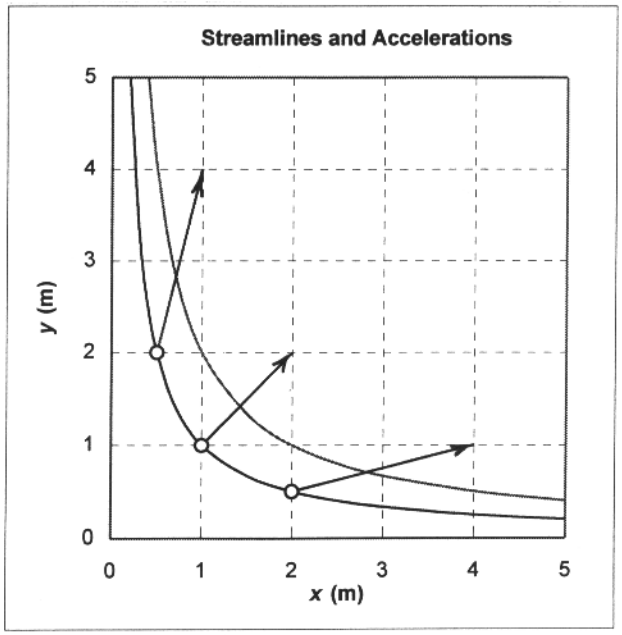
$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \quad \{w=0 \text{ and steady flow}\}$$

$$\vec{a}_p = Ax(A\hat{i}) - (Ay)(-A\hat{j}) = A^2(x\hat{i} + y\hat{j}) \quad \vec{a}_p$$

$$\vec{a}_p)_{\frac{1}{2},2} = \frac{1}{2}\hat{i} + 2\hat{j}\ m/s^2 \qquad \vec{a}_p)_{1,1} = \hat{i} + \hat{j}\ m/s^2$$

$$\vec{a}_p)_{2,\frac{1}{2}} = 2\hat{i} + \frac{1}{2}\hat{j}\ m/s^2 \quad \Big\} \vec{a}_p$$

Plot:



Given: Velocity field $\vec{V} = (Ax+B)\hat{i} - Ay\hat{j}$ where $A = 1\,s^{-1}$, $B = 3\,ft\,s^{-1}$ and coordinates are measured in ft.

Find: (a) If velocity field represents an incompressible flow.
(b) Acceleration of fluid particle at $(x,y) = 3,2$.

Plot: a few streamlines in the xy plane; show acceleration vector on the plot.

Solution:

For incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ $(w = 0)$

$\frac{\partial u}{\partial x} = A \qquad \frac{\partial v}{\partial y} = -A \qquad \therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = A - A = 0$ ← possible $\rho = c$ flow

$\vec{a}_p = \frac{d\vec{V}}{dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$

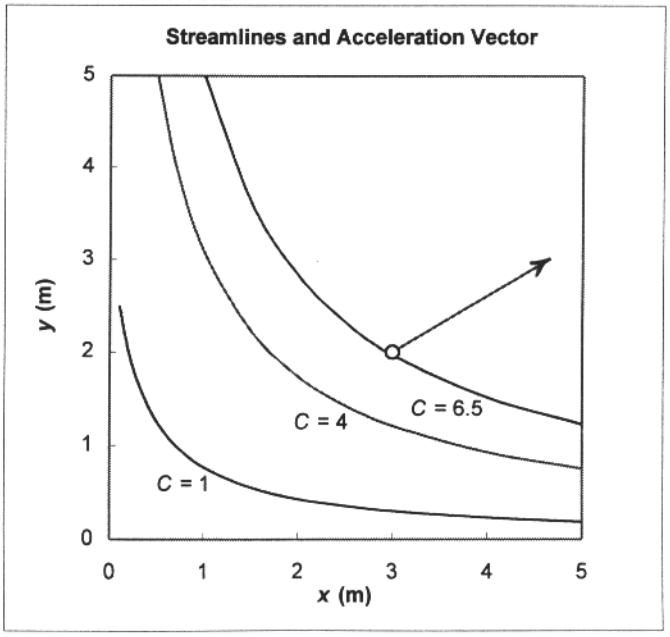
$\vec{a}_p = (Ax+B)A\hat{i} - (Ay)(-A\hat{j}) = A(Ax+B)\hat{i} + A^2 y\hat{j}$ At point (3,2)

$\vec{a}_p = \frac{1}{s}\left[\frac{1}{s} \times 3 + 3\,\frac{ft}{s}\right]\hat{i} + \left(\frac{1}{s}\right)^2 \times 2\,ft\,\hat{j} = 330\hat{i} + 200\hat{j}\ ft/s^2$ ← $\vec{a}_p$

The slope of a streamline is $\frac{dy}{dx} = \frac{v}{u} = \frac{-Ay}{Ax+B}$

Thus $\frac{dy}{dx} = -\frac{y}{x+B/A}$ or $\frac{dy}{y} + \frac{dx}{x+B/A} = 0$

and $\ln y + \ln(x + B/A) = \ln C$ or $(x + B/A)y = C$



Given: Velocity field represented by

$$\vec{V} = (Ax - B)\hat{\imath} + Cy\hat{\jmath} + Dt\,\hat{k} \qquad (x, y \text{ in } m)$$

where $A = 2\ s^{-1}$, $B = 4\ m/s$, and $D = 5\ m/s^2$

Find: (a) Proper value of C for incompressible flow.
(b) Acceleration of particle at $(x,y) = (3,2)$.
(c) Sketch streamlines in xy plane.

Solution: For incompressible flow, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$. Since $w = Dt$, $\partial w/\partial z = 0$, and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial y} = C = -\frac{\partial u}{\partial x} = -A = -2s^{-1}$$ ← C

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$

$$\vec{a}_p = (Ax - B)(A\hat{\imath}) + (Cy)(C\hat{\jmath}) + (Dt)(0) + D\hat{k}$$

$$\vec{a}_p(3,2) = \left(\frac{2}{s} \times 3m - \frac{4m}{s}\right)\left(\frac{2}{s}\right)\hat{\imath} + \left(\frac{-2}{s} \times 2m\right)\left(-\frac{2}{s}\right)\hat{\jmath} + \frac{5m}{s^2}\,\hat{k}$$

$$\vec{a}_p(3,2) = 4\hat{\imath} + 8\hat{\jmath} + 5\hat{k} \quad m/s^2$$ ← $\vec{a}_p(3,2)$

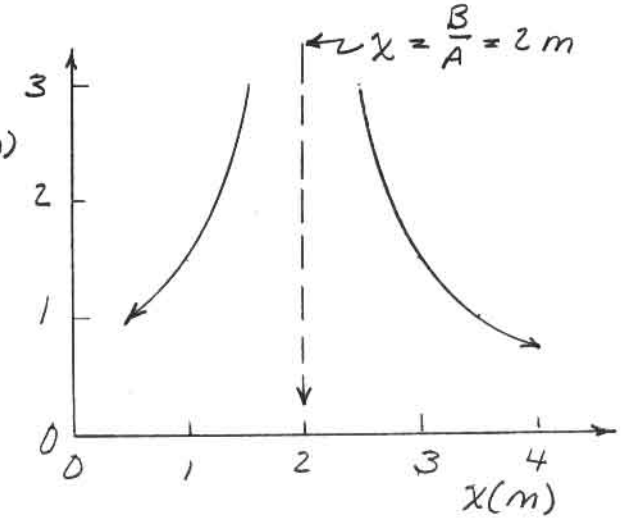
In the xy plane, streamlines are $\frac{dy}{dx} = \frac{v}{u} = \frac{Cy}{Ax - B}$. Thus

$$\frac{dx}{Ax - B} = \frac{dy}{Cy} \quad \text{or} \quad \frac{dx}{Ax - B} = -\frac{dy}{Ay} \quad \text{or} \quad \frac{dx}{x - B/A} + \frac{dy}{y} = 0$$

Integrating

$$\ln(x - B/A) + \ln y = \ln C_0$$

$$(x - \tfrac{B}{A})\,y = \text{const}$$ ←



Given: Velocity field $\vec{V} = (Ax - B)\hat{i} - Ay\hat{j}$; $A = 0.2\ s^{-1}$, $B = 0.6\ s^{-1}$, x in m.

Find: (a) General expression for acceleration of a fluid particle.
(b) Acceleration at $(x,y) = (0, 4/3), (1, 2)$, and $(2, 4)$.
(c) Plot of streamlines.
(d) Acceleration vectors on plot.

Solution: Note $w = 0$ and flow is steady. Then

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = (Ax - B)A\hat{i} + (-Ay)(-A)\hat{j} = (A^2x - AB)\hat{i} + A^2y\hat{j} \qquad \vec{a}_p$$

At $(x, y) = (0, 4/3)$, $\vec{a}_p = -0.12\hat{i} + 0.0533\hat{j}$ m/s²

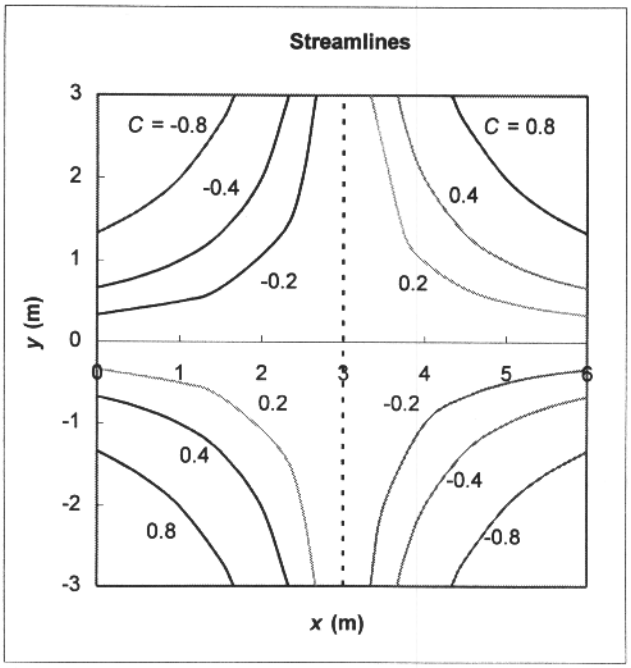
$(1, 2)$, $\vec{a}_p = -0.08\hat{i} + 0.0800\hat{j}$ m/s²

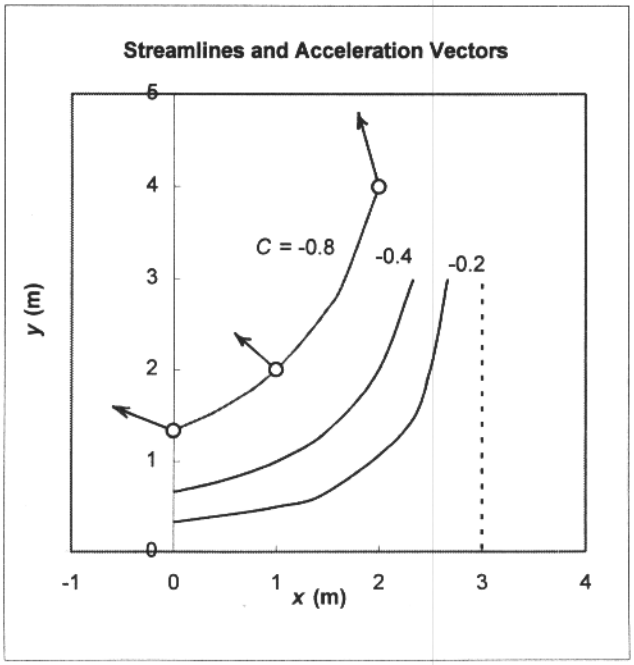
$(2, 4)$, $\vec{a}_p = -0.04\hat{i} + 0.160\hat{j}$ m/s² $\qquad \vec{a}_p$

streamlines are $\frac{dx}{u} = \frac{dy}{y} = \frac{dx}{Ax - B} = \frac{dy}{-Ay}$. Integrating,

$$\frac{1}{A}\ln(Ax - B) + \frac{1}{A}\ln y = \frac{1}{A}\ln C \quad \text{or} \quad (Ax - B)y = C$$

The plots are:





Given: Air flowing downward toward infinite horizontal flat plate.
Velocity field is

$$\vec{V} = (ax\hat{i} - ay\hat{j})(2+\cos\omega t)\ ;\ a = 3\ s^{-1},\ \omega = \pi\ s^{-1}$$

Find: (a) Expression for streamline at $t = 1.5$ s.
(b) Plot of streamline through $(x,y) = (2,4)$ at this instant.
(c) Velocity vector
(d) Vectors representing local, convective, and total acceleration.

Solution: Streamline is $\frac{dx}{u} = \frac{dy}{v}$, or $\frac{dx}{x} + \frac{dy}{y} = 0$ or $xy = C$

At point $(x,y) = (2,4)$, $C = 2m \times 4m = 8m^2$; $xy = 8m^2$ ← Streamline

The plot is shown below. Note $u = ax\hat{i}[2+\cos\omega t]$, $v = -ay\hat{j}[2+\cos\omega t]$

At $(x,y,t) = (2m, 4m, 1.5s)$, $\vec{V} = (6\hat{i} - 12\hat{j})(2+0) = 12\hat{i} - 24\hat{j}$ ← $\vec{V}$

The local acceleration components at $(x,y,t) = (2m, 4m, 1.5s)$ are

$$a_{x,local} = \frac{\partial u}{\partial t} = ax\hat{i}(-\omega\sin\omega t) = \frac{3}{s} \times 2m \times \left(\frac{-\pi}{s}\right) \times \sin\left(\frac{3\pi}{2}\right) = 6\pi\hat{i}\ m/s^2$$

$$a_{y,local} = \frac{\partial v}{\partial t} = -ay\hat{j}(-\omega\sin\omega t) = \frac{3}{s} \times 4m \times \left(\frac{-\pi}{s}\right) \times \sin\left(\frac{3\pi}{2}\right) = -12\pi\hat{j}\ m/s^2$$ ← Local

The convective acceleration components at $(x,y,t) = (2m, 4m, 1.5s)$ are

$$a_{x,conv} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = ax(a\hat{i})[2+\cos\tfrac{3\pi}{2}]^2 = (3)(2)(3)[2]^2\hat{i} = 72\hat{i}$$

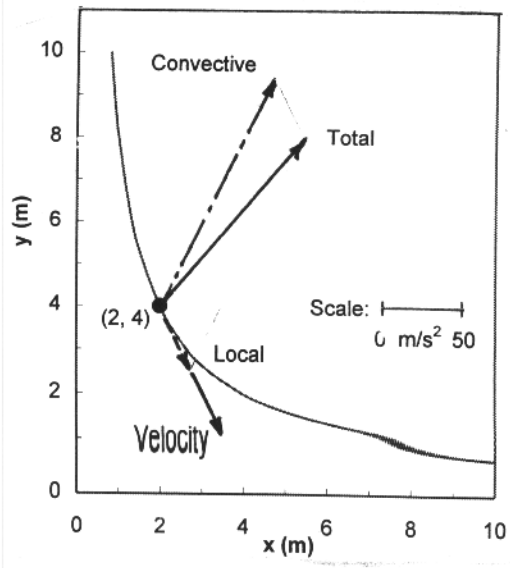
$$a_{y,conv} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = (-ay)(-a\hat{j})[2+\cos\tfrac{3\pi}{2}]^2 = 4a^2y\hat{j} = 4(3)^2 4\hat{j} = 144\hat{j}$$ ← Convective

The total acceleration is the sum of the convective and local values:

$$a_{x,total} = a_{x,conv} + a_{x,local} = (72 + 6\pi)\hat{i} = 90.8\hat{i}\ m/s^2$$

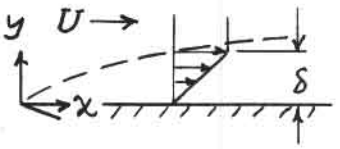
$$a_{y,total} = a_{y,conv} + a_{y,local} = (144 - 12\pi)\hat{j} = 106\ \hat{j}\ m/s^2$$ ← Total

The plot is



Given: Laminar boundary layer, linear approximate profile.

$$\frac{u}{U} = \frac{y}{\delta} \qquad \delta = Cx^{1/2}$$



From Problem 5.7, $v = \frac{uy}{4x} = U\frac{y^2}{4x\delta}$

Find: (a) x and y components of acceleration of a fluid particle.
(b) Locate maximum values.
(c) Ratio, $a_{x,max}/a_{y,max}$.

Solution: Basic equations: $a_{px} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \cancel{w\frac{\partial u}{\partial z}}^{=0(1)} + \cancel{\frac{\partial u}{\partial t}}^{=0(2)}$

$$a_{py} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \cancel{w\frac{\partial v}{\partial z}}^{=0(1)} + \cancel{\frac{\partial v}{\partial t}}^{=0(2)}$$

Assumptions: (1) w and $\partial/\partial z$ zero, (2) steady flow. $\frac{d\delta}{dx} = \frac{1}{2}Cx^{-1/2} = \frac{\delta}{2x}$

$$u = U\frac{y}{\delta}\ ;\ \frac{\partial u}{\partial x} = Uy\left(-\frac{1}{\delta^2}\right)\frac{d\delta}{dx} = -Uy\frac{1}{\delta^2}\frac{\delta}{2x} = -\frac{Uy}{2x\delta}\ ;\ \frac{\partial u}{\partial y} = \frac{U}{\delta}$$

$$v = U\frac{y^2}{4x\delta}\ ;\ \frac{\partial v}{\partial x} = \frac{Uy^2}{4}\left(-\frac{1}{x^2\delta} - \frac{1}{x\delta^2}\frac{d\delta}{dx}\right) = -\frac{3Uy^2}{8x^2\delta}\ ;\ \frac{\partial v}{\partial y} = \frac{Uy}{2x\delta}$$

Thus

$$a_{px} = \left(U\frac{y}{\delta}\right)\left(-\frac{Uy}{2x\delta}\right) + \left(U\frac{y^2}{4x\delta}\right)\left(\frac{U}{\delta}\right) = -\frac{U^2}{2x}\left(\frac{y}{\delta}\right)^2 + \frac{U^2}{4x}\left(\frac{y}{\delta}\right)^2 = -\frac{U^2}{4x}\left(\frac{y}{\delta}\right)^2 \qquad a_{px}$$

$$a_{py} = \left(U\frac{y}{\delta}\right)\left(-\frac{3Uy^2}{8x^2\delta}\right) + \left(U\frac{y^2}{4x\delta}\right)\left(U\frac{y}{2x\delta}\right) = -\frac{3U^2}{8x}\left(\frac{y}{x}\right)\left(\frac{y}{\delta}\right)^2 + \frac{U^2}{8x}\left(\frac{y}{x}\right)\left(\frac{y}{\delta}\right)^2$$

$$a_{py} = -\frac{U^2}{4x}\left(\frac{y}{x}\right)\left(\frac{y}{\delta}\right)^2 \qquad a_{py}$$

Maximum values are at $y = \delta$

$$a_{px,max} = -\frac{U^2}{4x} \qquad (max)\ a_{px}$$

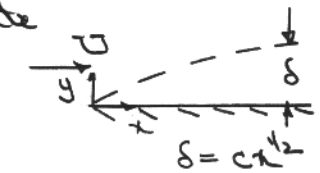
$$a_{py,max} = -\frac{U^2}{4x}\left(\frac{\delta}{x}\right) \qquad (max)\ a_{py}$$

Thus $\frac{a_{px,max}}{a_{py,max}} = \frac{x}{\delta}$

At $x = 0.5\,m$, $\delta = 5\,mm$, $\frac{a_{px,max}}{a_{py,max}} = \frac{0.5\,m}{0.005\,m} = 100$ Ratio

Given: Laminar boundary layer on a flat plate

$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$, $\frac{v}{U} = \frac{1}{4}\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]$



Show: that $a_x = -\frac{U^2}{x}\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3 + \frac{1}{3}\left(\frac{y}{\delta}\right)^4\right]$

Find: Value of y/δ for which a_x is a maximum at given x; evaluate $a_{x\,max}$ for $U = 8.7$ m/s and $x = 0.5$ m.

Plot: nondimensional value of a_x versus y/δ

Solution:

$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$ $\quad u = U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] = U\left[2\eta - \eta^2\right]$ where $\eta = y/\delta$

$\frac{\partial u}{\partial x} = \frac{du}{d\eta}\frac{d\eta}{dx} = U\left[2 - 2\eta\right]\left(-\frac{y}{\delta^2}\right)\frac{d\delta}{dx}$ $\quad \frac{d\delta}{dx} = \frac{1}{2}c\,x^{-1/2}$

$\frac{\partial u}{\partial x} = U\left[2 - 2\eta\right]\left(-\frac{\eta}{\delta}\right)\frac{1}{2}c\,x^{-1/2} = U\left[2\eta - \eta^2\right]\left(-\frac{1}{2}\right)\frac{1}{x}$

$\frac{\partial u}{\partial x} = -U\left[2 - 2\eta\right]\frac{\eta}{2x} = -\frac{U(\eta - \eta^2)}{x}$ ← – –

$\frac{\partial u}{\partial y} = U\left[\frac{2}{\delta} - \frac{2y}{\delta^2}\right] = \left[\frac{2}{\delta} - \frac{2\eta}{\delta}\right]U = \frac{2U}{\delta}(1 - \eta)$ ← – –

Substituting into the expression for $a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$

$a_x = U\left[2\eta - \eta^2\right]U\left(-\frac{\eta}{x}\right) + U\left[\frac{1}{3}\eta^2 - \frac{1}{3}\eta^3\right]\frac{2U}{\delta}(1 - \eta)$

$= \frac{U^2}{x}\left(-2\eta^2 + 3\eta^3 - \eta^4\right) + \frac{U^2}{x}\left(\eta^3 - \frac{1}{3}\eta^4 + \frac{1}{3}\eta^5\right)$

$= \frac{U^2}{x}\left(-2\eta^2 + 3\eta^3 - \eta^4\right) + \frac{U^2}{x}\left(\eta^2 - \frac{5}{3}\eta^3 + \frac{1}{3}\eta^4\right)$

$a_x = \frac{U^2}{x}\left(-\eta^2 + \frac{4}{3}\eta^3 - \frac{1}{3}\eta^4\right) = -\frac{U^2}{x}\left[\left(\frac{y}{\delta}\right)^2 - \frac{4}{3}\left(\frac{y}{\delta}\right)^3 + \frac{1}{3}\left(\frac{y}{\delta}\right)^4\right]$ ← a_x

To find value of $\eta(= y/\delta)$ for which a_x is a maximum, set

$\frac{da_x}{d\eta} = 0 = \frac{U^2}{x}\left(-2\eta + 4\eta^2 - \frac{4}{3}\eta^3\right) = \frac{U^2}{x}\eta\left(-2 + 4\eta - \frac{4}{3}\eta^2\right)$

At $\eta = 0$, $y/\delta = 0$ and $a_x = 0$

For $\left(-2 + 4\eta - \frac{4}{3}\eta^2\right) = 0$ or $\eta^2 - 3\eta + \frac{3}{2} = 0$

$\eta = \frac{+3 \pm \sqrt{(3)^2 - 4(1)\frac{3}{2}}}{2} = \frac{3 \pm \sqrt{3}}{2}$

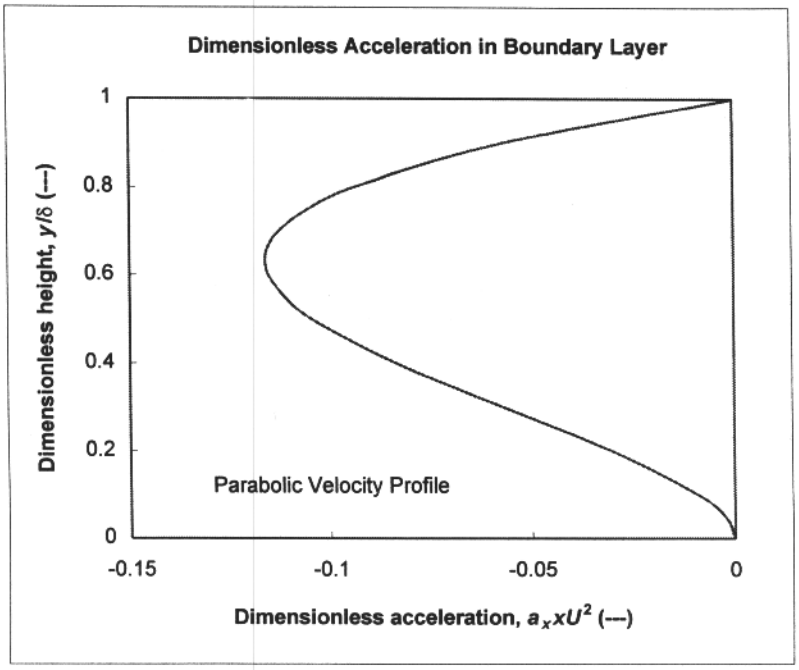
Choose $0 < \eta < 1$ (within $0 \le y \le \delta$) $\therefore \eta = 0.634$ ← y/δ

At $\eta = 0.634$

$a_x = (8.7)^2\frac{m^2}{s^2} \times \frac{1}{0.5\,m}\left[-(0.634)^2 + \frac{4}{3}(0.634)^3 - \frac{1}{3}(0.634)^4\right] = -17.6\,m^2/s^2$ ← a_x

The nondimensional value of a_x is given by

$$\frac{a_x}{U^2/L} = -\eta^2 + \frac{4}{3}\eta^3 - \frac{1}{3}\eta^4 \qquad \text{where } \eta = y/\delta.$$



13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U. S. A.
National® Brand

Given: Laminar boundary layer on a flat plate. (Problem 5.11)

$\frac{u}{U} = \sin\frac{\pi y}{2\delta}$, $\delta = c x^{1/2}$

$\frac{v}{U} = \frac{\pi}{4}\frac{\delta}{x}\left[\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) + \left(\frac{\pi}{2}\frac{y}{\delta}\right)\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - 1\right]$



Find: Expression for a_{xp} and a_{yp}

Plot: a_x and a_y as functions of y/δ for $U = 8.7$ m/s and $x = 0.5$ m; determine maximum values at locations at which maxima occur.

Solution:

Basic equations: $a_{px} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$ ----(1)

$a_{py} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$ ----(2)

Assumptions: (1) steady flow
(2) w and $\partial/\partial z = 0$

Let $\eta = \frac{y}{\delta}$; $\eta = \eta(x, y)$ $\qquad \frac{y}{\delta} = \frac{y}{cx^{1/2}}$; $\delta = cx^{1/2}$, $\frac{d\delta}{dx} = \frac{1}{2}cx^{-1/2} = \frac{\delta}{2x}$

$\frac{\partial\eta}{\partial x} = \frac{\partial\eta}{\partial\delta}\frac{d\delta}{dx} = \frac{\pi}{2}\left(-\frac{1}{\delta^2}\right)\frac{\delta}{2x}y = -\frac{\pi}{4}\left(\frac{y}{\delta}\right)$

$u = U\sin\eta$

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial x} = U\cos\eta\left(-\frac{\pi}{2x}\frac{y}{\delta}\right) = -\frac{U}{2x}\left(\frac{\pi}{2}\frac{y}{\delta}\right)\cos\eta = -\frac{U}{2x}\eta\cos\eta$ ----(3)

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial y} = U\cos\eta\frac{\pi}{2\delta} = \frac{U\pi}{2\delta}\cos\eta$ ----(4)

$v = U\frac{\pi}{4}\frac{\delta}{x}(\cos\eta + \eta\sin\eta - 1)$. Differentiating using product rule

$\frac{\partial v}{\partial x} = \frac{\pi U}{4}\left(-\frac{1}{x^2}\frac{\delta}{\pi/2} - \frac{1}{x}\frac{\delta}{x^2}\right)(\cos\eta + \eta\sin\eta - 1) + \frac{\pi U}{4}\frac{\delta}{x}\left(-\sin\eta + \eta\cos\eta + \sin\eta\right)\frac{\partial\eta}{\partial x}$

$= \frac{\pi U}{4}\left(-\frac{1}{x}\frac{\delta}{x} - \frac{\delta}{x^2}\right)(\cos\eta + \eta\sin\eta - 1) + \frac{\pi U}{4}\frac{\delta}{x}\eta\cos\eta\left(-\frac{\pi}{4}\right)\frac{\delta}{x}$

$\frac{\partial v}{\partial x} = -\frac{\pi U}{4}\frac{\delta}{x^2}(\cos\eta + \eta\sin\eta - 1) - \frac{U\delta}{4x^2}\left(\frac{\pi}{2}\right)\eta\cos\eta$ ----(5)

$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial\eta}\frac{\partial\eta}{\partial y} = \frac{\pi U}{4}\frac{\delta}{x}\left(-\sin\eta + \eta\cos\eta + \sin\eta\right)\frac{\pi}{2\delta} = \frac{\pi U}{8x}\eta\cos\eta$ ----(6)

Substituting into Eq. 1,

$a_x = U\sin\eta\left(-\frac{U}{2x}\eta\cos\eta\right) + \frac{\pi U}{4}\frac{\delta}{x}(\cos\eta - \eta\sin\eta - 1)\frac{U\pi}{2\delta}\cos\eta$

$a_x = -\frac{U^2}{2x}\eta\sin\eta\cos\eta + \frac{U^2}{2x}(\cos\eta - \eta\sin\eta - 1)\cos\eta$

$a_x = \frac{U^2}{2x}\cos\eta\left[\cos\eta - \eta\sin\eta - 1 - \eta\sin\eta\right]$

$$a_x = \frac{U^2}{2x}\cos\eta(\cos\eta - 1) = -\frac{U^2}{2x}\cos\eta(1-\cos\eta) \qquad a_x$$

Substituting into Eq. 2.

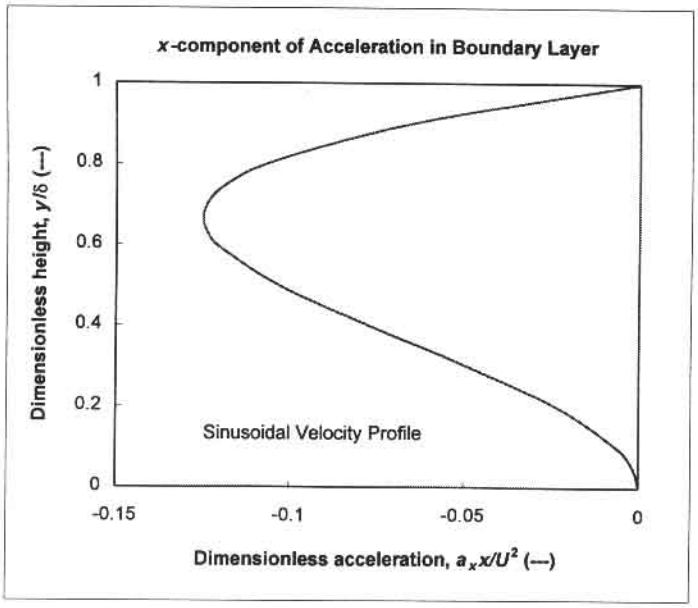
$$a_y = U\sin\eta\left[-\frac{\pi U}{2x^2}\frac{\delta}{2}(\cos\eta + \eta\sin\eta - 1) - \frac{U\delta}{4x^2}\left(\frac{\pi}{\delta}y\right)\eta\cos\eta\right] + \frac{\pi U}{2x}\frac{\delta}{2}(\cos\eta + \eta\sin\eta - 1)\frac{U}{2x}\eta\cos\eta$$

$$a_y = \frac{U^2\delta}{\pi/2\,x^2}\left\{\left[-\sin\eta(\cos\eta + \eta\sin\eta - 1) - \frac{\pi}{2}\left(\frac{y}{\delta}\right)\eta\cos\eta\sin\eta\right] + \eta\cos\eta(\cos\eta + \eta\sin\eta - 1)\right\}$$

$$a_y = \frac{U^2\delta}{\pi/2\,x^2}\left\{-\sin\eta(\cos\eta + \eta\sin\eta - 1) - \frac{\pi}{2}\left(\frac{y}{\delta}\right)\eta\cos\eta\sin\eta + \eta\cos\eta(\cos\eta + \eta\sin\eta - 1)\right\} \qquad a_y$$

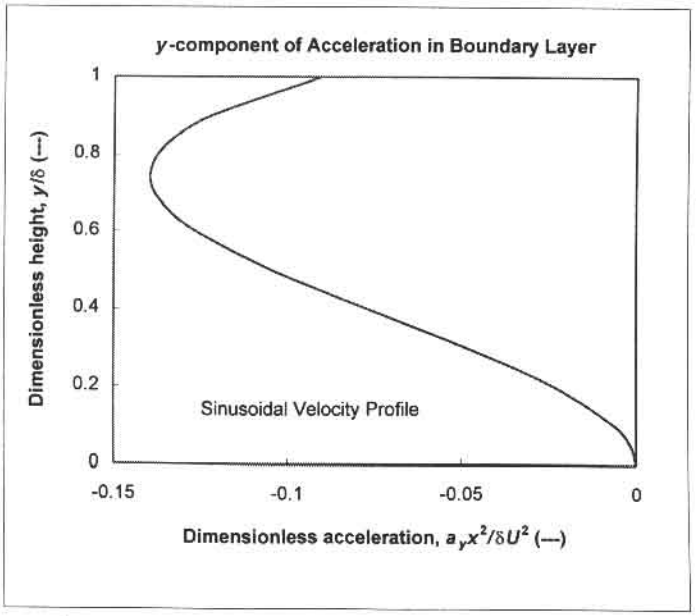
x-component:

η (—)	$a_x x/U^2$ (---)	y/δ (—)
0	0.000	0
0.0785	-0.00154	0.05
0.157	-0.00608	0.1
0.314	-0.0233	0.2
0.471	-0.0486	0.3
0.628	-0.0773	0.4
0.785	-0.104	0.5
0.942	-0.121	0.6
0.974	-0.123	0.62
1.02	-0.125	0.65
1.07	-0.125	0.68
1.10	-0.124	0.7
1.13	-0.122	0.72
1.18	-0.118	0.75
1.26	-0.107	0.8
1.41	-0.0660	0.9
1.57	0.000	1



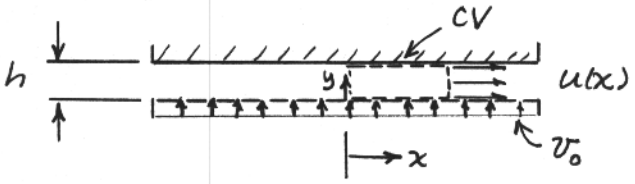
y-component:

η (—)	$a_y x^2/\delta U^2$ (---)	y/δ (—)
0	0.000	0
0.0785	-0.00154	0.05
0.157	-0.00607	0.1
0.314	-0.0232	0.2
0.471	-0.0482	0.3
0.628	-0.0770	0.4
0.785	-0.105	0.5
0.942	-0.126	0.6
1.02	-0.134	0.65
1.10	-0.138	0.7
1.13	-0.139	0.72
1.18	-0.140	0.75
1.26	-0.137	0.8
1.34	-0.132	0.85
1.41	-0.122	0.9
1.57	-0.0908	1



Note: a_y is normalized with x^2/δ and a_x is normalized with x. Thus $a_y = 0\left(\frac{\delta}{x}a_x\right) \approx 0.001\,a_x$

Given: Air flow through porous surface into narrow gap.



Find: (a) Show $u(x) = v_0 x/h$
(b) Component, v
(c) Acceleration of a fluid particle in the gap.

Solution: Apply conservation of mass to CV shown.

Basic equation: $0 = \cancel{\frac{\partial}{\partial t}}^{=0(1)} \int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section

Then

$$0 = \{ -xwv_0 \} + \{ hw\,u(x) \} \quad \text{or} \quad u(x) = v_0 \frac{x}{h}$$ $u(x)$

Apply differential form to find v:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \frac{\partial u}{\partial x} = \frac{v_0}{h}$$

$$v - v_0 = \int_0^y \frac{\partial v}{\partial y} dy + f(x) = \int_0^y -\frac{v_0}{h} dy + f(x) = -\frac{v_0 y}{h} + f(x)$$

or

$$v = v_0 \left(1 - \frac{y}{h}\right) \quad [f(x) = 0 \text{ since } v = v_0 = \text{const along } y = 0]$$ $v(y)$

$$a_{px} = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\cancel{\frac{\partial u}{\partial y}}^{=0(3),\,=0} + \cancel{w\frac{\partial u}{\partial z}}^{=0} + \cancel{\frac{\partial u}{\partial t}}^{=0(1)}$$

$$a_{px} = \left(v_0 \frac{x}{h}\right)\left(\frac{v_0}{h}\right) = \frac{v_0^2 x}{h^2}$$ a_{px}

$$a_{py} = u\cancel{\frac{\partial v}{\partial x}} + v\frac{\partial v}{\partial y} + \cancel{w\frac{\partial v}{\partial z}} + \cancel{\frac{\partial v}{\partial t}}$$

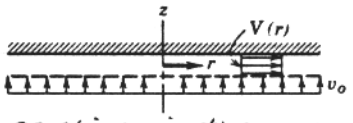
$$a_{py} = v_0\left(1 - \frac{y}{h}\right)\left(-\frac{v_0}{h}\right) = \frac{v_0^2}{h}\left(\frac{y}{h} - 1\right)$$ a_{py}

Thus

$$\vec{a}_p = a_{px}\hat{i} + a_{py}\hat{j} = \frac{v_0^2 x}{h^2}\hat{i} + \frac{v_0^2}{h}\left(\frac{y}{h} - 1\right)\hat{j}$$ $\vec{a}_p$

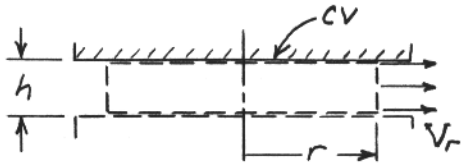
42:381 50 SHEETS 5 SQUARE
42:382 100 SHEETS 5 SQUARE
42:389 200 SHEETS 5 SQUARE
Made in U.S.A.
NATIONAL

Given: Flow between parallel disks through porous surface.



Find: (a) Show $V_r = v_0 r/2h$
(b) V_z, if $v_0 \ll V_r$
(c) Components of acceleration for a fluid particle in the gap.

Solution: Apply CV form of continuity to finite CV shown.



Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$ ($\frac{\partial}{\partial t} = 0(1)$)

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section

Then

$$0 = \{-|\rho v_0 \pi r^2|\} + \{+|\rho V_r 2\pi r h|\} \quad \text{or} \quad V_r = \frac{v_0 r}{2h}$$

V_r

Apply differential form of conservation of mass for incompressible flow.

Basic equation: $\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(V_\theta) + \frac{\partial}{\partial z}V_z = 0$ ($V_\theta = 0(1)$)

Assumptions: (4) $V_\theta = 0$ by symmetry
(5) $V_r = v_0 r/2h$ from above

Then

$$\frac{\partial V_z}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(rV_r) = -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{v_0 r^2}{2h}\right) = -\frac{1}{r}\left(\frac{v_0 r}{h}\right) = -\frac{v_0}{h}$$

Integrating,

$$V_z = -\frac{v_0 z}{h} + f(r)$$

Boundary conditions are $V_z = v_0$ at $z = 0$, $V_z = 0$ at $z = h$

Thus from first BC, $f(r) = v_0 = \text{constant}$, so

$$V_z = v_0\left(1 - \frac{z}{h}\right)$$

V_z

The r component of acceleration is

$$a_r = V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial\theta} + V_z\frac{\partial V_r}{\partial z} + \frac{\partial V_r}{\partial t} = \left(\frac{v_0 r}{2h}\right)\left(\frac{v_0}{2h}\right) = \left(\frac{v_0}{2h}\right)^2 r$$

($\frac{\partial V_r}{\partial\theta} = 0$, $\frac{\partial V_r}{\partial t} = 0$)

a_r

The z component is

$$a_z = V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial\theta} + V_z\frac{\partial V_z}{\partial z} + \frac{\partial V_z}{\partial t} = v_0\left(1 - \frac{z}{h}\right)\left(-\frac{v_0}{h}\right) = \frac{v_0^2}{h}\left(\frac{z}{h} - 1\right)$$

($\frac{\partial V_z}{\partial r} = 0$, $\frac{\partial V_z}{\partial\theta} = 0$, $\frac{\partial V_z}{\partial t} = 0$)

a_z

Given: Steady, inviscid flow over a circular cylinder of radius R.

$$\vec{V} = U\cos\theta\left[1-\left(\frac{R}{r}\right)^2\right]\hat{e}_r - U\sin\theta\left[1+\left(\frac{R}{r}\right)^2\right]\hat{e}_\theta$$

Find: (a) Expression for acceleration of particle moving along $\theta=\pi$
(b) Expression for acceleration of particle moving along $r=R$
(c) Locations at which accelerations a_r and a_θ reach maximum and minimum values.

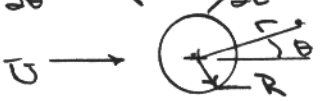
Plot: a_r as a function of R/r for $\theta=\pi$ and as a function of θ for $r=R$; plot a_θ as a function of θ for $r=R$

Solution:

Basic equations: $a_r = V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + \frac{\partial V_r}{\partial t}$ ($=0$ (1))

$a_\theta = V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + \frac{\partial V_\theta}{\partial t}$ ($=0$ (1))

Assumptions: (1) steady flow



Along $\theta=\pi$

- $\cos\theta = -1$, $\sin\theta = 0$, so $V_\theta = 0$ and $V_r = -U\left[1-\left(\frac{R}{r}\right)^2\right]$

Then

$$a_r = V_r\frac{\partial V_r}{\partial r} = -U\left[1-\left(\frac{R}{r}\right)^2\right](-U)(-2)\left(-\frac{R^2}{r^3}\right) = \frac{2U^2}{R}\left[1-\left(\frac{R}{r}\right)^2\right]\left(\frac{R}{r}\right)^3$$

a_r, a_θ } $\theta=\pi$

$a_\theta = 0$

To determine location of maximum a_r, let $\frac{R}{r}=\eta$ and evaluate $\frac{da_r}{d\eta}$

$$a_r = \frac{2U^2}{R}\left[1-\eta^2\right]\eta^3 = \frac{2U^2}{R}\left[\eta^3-\eta^5\right]$$

$$\frac{da_r}{d\eta} = \frac{2U^2}{R}\left[3\eta^2-5\eta^4\right]$$. Thus $\frac{da_r}{d\eta}=0$ at $\eta^2 = \frac{3}{5}$ or $\eta = 0.775$

Thus, $a_{r\,max}$ occurs at $r = R/0.775 = 1.29R$ ← r_{amax}

$$a_{r\,max} = \frac{2U^2}{R}(0.775)^3\left[1-(0.775)^2\right] = 0.372\,\frac{U^2}{R}\ @\ r=1.29R$$

Since $a_\theta = 0$, $\vec{a}_{max} = a_{r\,max}\hat{e}_r = 0.372\frac{U^2}{R}\hat{e}_r\ @\ r=1.29R$

Along $r=R$

- $r=R$, $V_r=0$ and $V_\theta = -2U\sin\theta$

$$a_r = -\frac{V_\theta^2}{r} = -\frac{(-2U\sin\theta)^2}{R} = -\frac{4U^2}{R}\sin^2\theta$$

$$a_\theta = \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial\theta} = \left(-\frac{2U\sin\theta}{R}\right)\left(-\frac{2U\cos\theta}{R}\right) = \frac{4U^2}{R^2}\sin\theta\cos\theta$$

a_r, a_θ } $r=R$

a_r has maximum negative value at $\theta = \pm\pi/2$
has minimum value (of zero) at $\theta = 0, \pi$

a_θ has maximum values at $\theta = \pm\pi/4,\ 3\pi/4$
has minimum values at $\theta = 0, \pm\pi/2, \pi$

The acceleration magnitude is

$$|\vec{a}| = \left[a_r^2 + a_\theta^2\right]^{1/2} = \left[\left(\frac{-4U^2}{R}\right)^2 \sin^4\theta + \left(\frac{4U^2}{R}\right)^2 \sin^2\theta\cos^2\theta\right]^{1/2} = \frac{4U^2}{R}\sin\theta$$

• This is a maximum at $\theta = \pm\pi/2$.

Thus

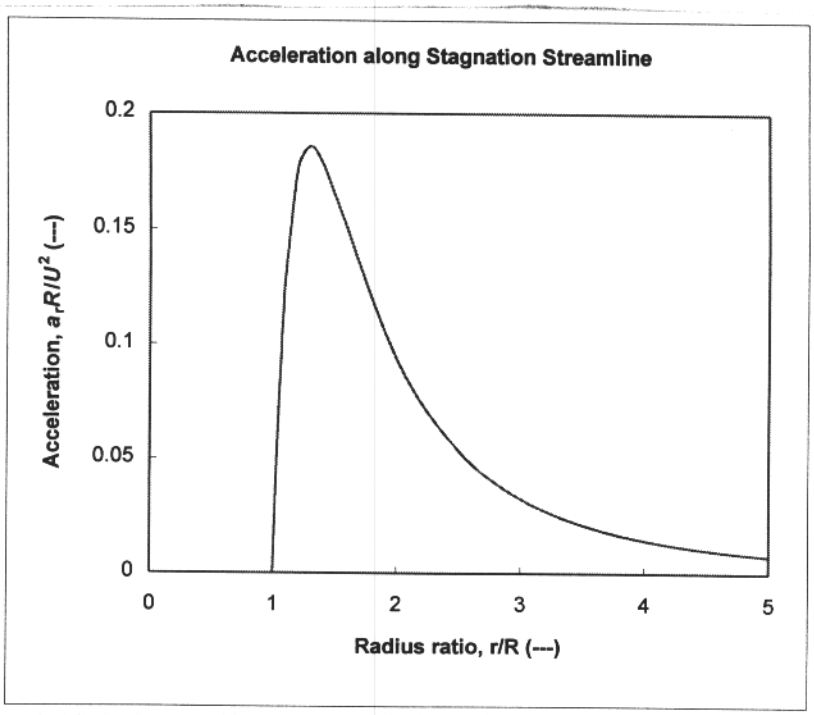
$$\vec{a}_{max} = \pm 4\frac{U^2}{R} \quad \text{at } \theta = \pm\pi/2.$$

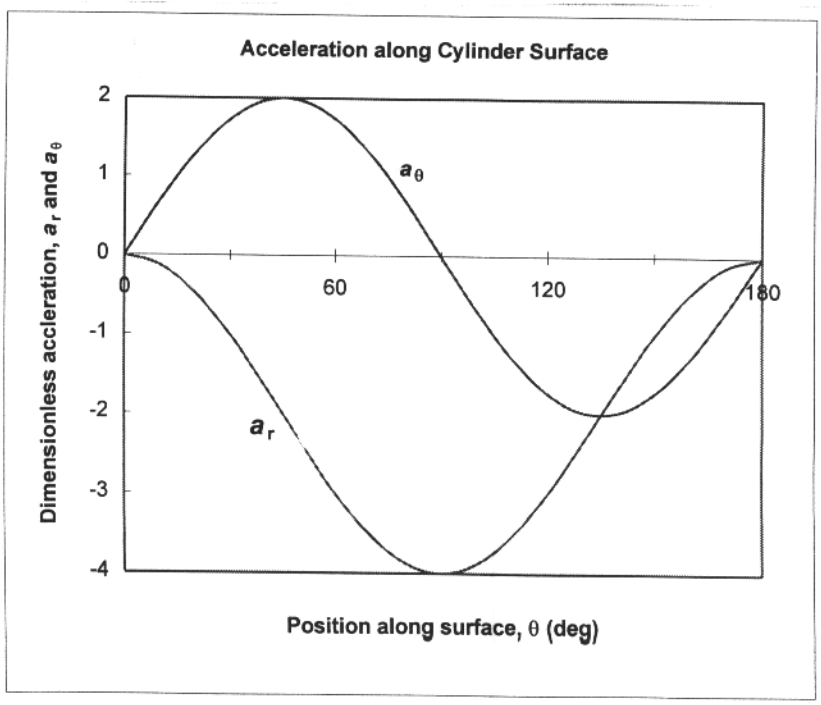
Plots:

(1) $\theta = \pi$ $\quad a_r = \frac{2U^2}{R}\left(\frac{R}{r}\right)^3\left[1-\left(\frac{R}{r}\right)^2\right]$; $\quad \frac{a_r}{U^2/R} = \left(\frac{R}{r}\right)^3\left[1-\left(\frac{R}{r}\right)^2\right]$

(2) $r = R$. $\quad a_r = -4\frac{U^2}{R}\sin^2\theta$; $\quad \frac{a_r}{U^2/R} = -4\sin^2\theta$

$a_\theta = 4\frac{U^2}{R}\sin\theta\cos\theta$; $\quad \frac{a_\theta}{U^2/R} = 4\sin\theta\cos\theta$



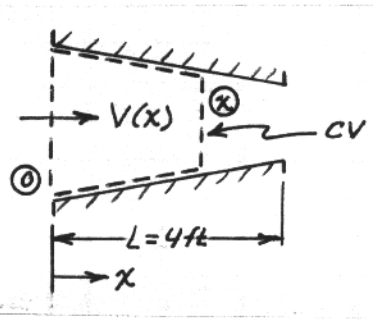


Given: Incompressible flow through nozzle

$A = A_0(1-bx)$; $A_0 = 1\ ft^2$, $b = 0.1\ ft^{-1}$

$U = c(1+at)$; $c = 6\ ft/s$, $a = 2\ s^{-1}$

$L = 4\ ft$.



Find: Acceleration of fluid particle on centerline at $x = L/2$ for $t = 0, 0.5\ s$.

Plot: a_x on the centerline as a function of x/L for $t = 0, 0.5\ s$.

Solution:

Use CV shown with conservation to obtain an expression for the variation of the centerline velocity with distance x along the nozzle

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ (the first term $=0$ (1), the second $=0$ (3))

$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$ ($v = 0$ (2), $w = 0$ (3))

Assumptions: (1) ρ = constant (given)

(2) $v = 0$ along centerline

(3) $\partial/\partial z = 0$

Then from continuity, $0 = \{-1\rho V_0 A_0\} + \{1\rho V A\} = -\rho V_0 A_0 + \rho V A$

and $V = V_0 \frac{A_0}{A} = \frac{c(1+at)}{1-bx} = u$. Along centerline $\vec{V} = u\hat{i}$

$$\frac{\partial \vec{V}}{\partial x} = \frac{\partial}{\partial x} u\hat{i} = -\frac{c(1+at)}{(1-bx)^2}(-b)\hat{i} = \frac{cb(1+at)}{(1-bx)^2}\hat{i}$$

$$\frac{\partial \vec{V}}{\partial t} = \frac{\partial}{\partial t} u\hat{i} = \frac{ac}{(1-bx)}\hat{i}$$

. Substituting we obtain

$$\vec{a}_p = \frac{c(1+at)}{(1-bx)}\frac{cb(1+at)}{(1-bx)^2}\hat{i} + \frac{ac}{(1-bx)}\hat{i} = \left[\frac{bc^2(1+at)^2}{(1-bx)^3} + \frac{ac}{(1-bx)}\right]\hat{i} \leftarrow \vec{a}_p$$

At $x = L/2 = 2\ ft$ and $t = 0$

$$\vec{a}_p = \left[0.1\ ft \times (10)^2 \frac{ft^2}{s^2}\frac{1}{[1-0.1(2)]^3} + \frac{2}{s}\times\frac{6\ ft}{s}\times\frac{1}{[1-0.1(2)]}\right]\hat{i}$$

$$\vec{a}_p = 44.5\,\hat{i}\ ft/s^2 \quad \leftarrow \vec{a}_p\left(x=\tfrac{L}{2}, t=0\right)$$

At $x = L/2 = 2\ ft$ and $t = 0.5\ s$

$$\vec{a}_p = \left[0.1\ ft \times (10)^2 \frac{ft^2}{s^2}\frac{[1+2\,s^{-1}(0.5\,s)]^2}{[1-0.1(2)]^3} + \frac{2}{s}\times\frac{6\ ft}{s}\times\frac{1}{[1-0.1(2)]}\right]\hat{i}$$

$$\vec{a}_p = 103\,\hat{i}\ ft/s^2 \quad \leftarrow \vec{a}_p\left(x=\tfrac{L}{2}, t=0.5\,s\right)$$

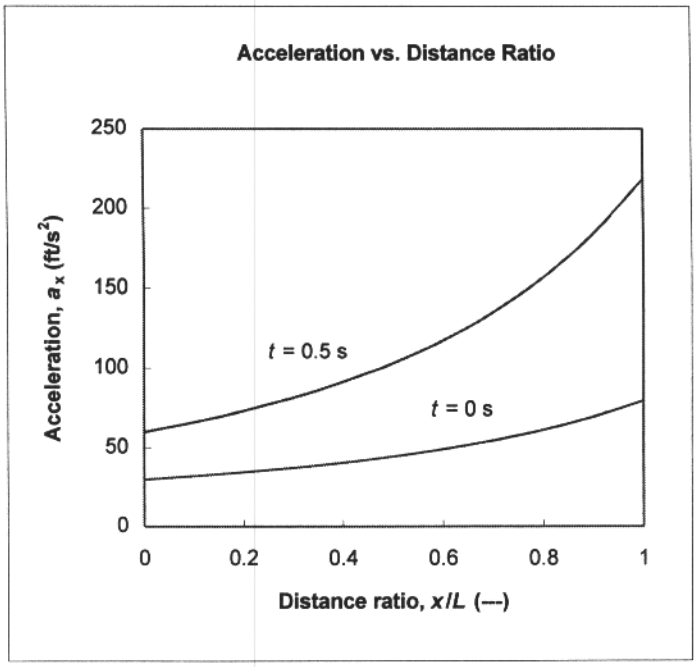
$$a_x = \frac{bc^2(1+at)^2}{(1-bx)^3} + \frac{ac}{(1-bx)} = \frac{bc^2(1+at)^2}{\left[1-bL\left(\frac{x}{L}\right)\right]^3} + \frac{ac}{\left[1-bL\left(\frac{x}{L}\right)\right]}$$

At $t=0$

$$a_x = \frac{10}{\left[1-0.4\left(\frac{x}{L}\right)\right]^3} + \frac{20}{\left[1-0.4\left(\frac{x}{L}\right)\right]} \quad \text{for } 0 \le x/L \le 1$$

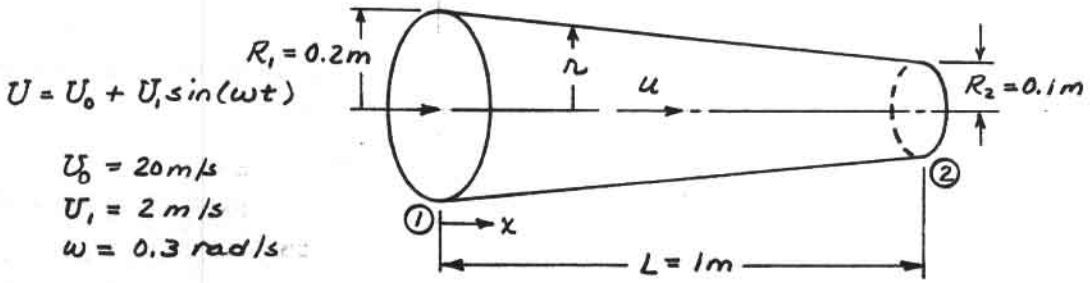
At $t=0.5s$

$$a_x = \frac{40}{\left[1-0.4\left(\frac{x}{L}\right)\right]^3} + \frac{20}{\left[1-0.4\left(\frac{x}{L}\right)\right]} \quad \text{for } 0 \le x/L \le 1$$



13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: One-dimensional, incompressible flow through circular channel.



Find: (a) The acceleration of a particle at the channel exit.
(b) Plot as a function of time for a complete cycle.

Solution: The acceleration of a particle in one-dimensional flow is

$$a_x = u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}$$

From continuity, $u = U\frac{A_1}{A} = U\frac{R_1^2}{r^2}$

From geometry, $r = R_1 - (R_1 - R_2)\frac{x}{L} = R_1 - \Delta R\frac{x}{L}$, so

$$u = U\frac{R_1^2}{(R_1 - \Delta R\frac{x}{L})^2} = \left[U_0 + U_1\sin(\omega t)\right]\frac{1}{\left[1 - \frac{\Delta R}{R_1}(\frac{x}{L})\right]^2}$$

Thus

$$a_x = \left[U_0 + U_1\sin(\omega t)\right]\frac{1}{\left[1 - \frac{\Delta R}{R_1}(\frac{x}{L})\right]^2}\left[U_0 + U_1\sin(\omega t)\right](-2)(-\frac{\Delta R}{R_1 L})\frac{1}{\left[1 - \frac{\Delta R}{R_1}(\frac{x}{L})\right]^3}$$

$$+ \frac{\omega U_1\cos(\omega t)}{\left[1 - \frac{\Delta R}{R_1}(\frac{x}{L})\right]^2}$$

$$a_x = \frac{2\Delta R}{R_1 L}\frac{\left[U_0 + U_1\sin(\omega t)\right]^2}{\left[1 - \frac{\Delta R}{R_1}(\frac{x}{L})\right]^5} + \frac{\omega U_1\cos(\omega t)}{\left[1 - \frac{\Delta R}{R_1}(\frac{x}{L})\right]^2}$$

At $x/L = 1$, $\left[1 - \frac{\Delta R}{R_1}(\frac{x}{L})\right] = 1 - \frac{0.1m}{0.2m} = 0.5$, so

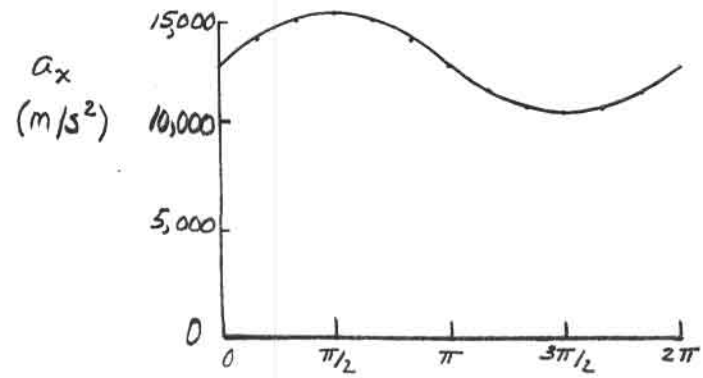
$$a_x = 2 \times 0.1m \times \frac{1}{0.2m} \times \frac{1}{1m}\left[20 + 2\sin(\omega t)\right]^2 \frac{m^2}{s^2} \times \frac{1}{(0.5)^5} + 0.3\frac{rad}{s} \times \frac{2m}{s} \times \cos(\omega t) \times \frac{1}{(0.5)^2}$$

or

$$a_x\,(m/sec^2) = 32\left[20 + 2\sin(\omega t)\right]^2 + 2.4\cos(\omega t) \quad (\text{at } x = L)$$

a_x

Plot:



Plot

Given: Steady, two-dimensional velocity field of Problem 5.47,

$$\vec{V} = Ax\,\hat{\imath} - Ay\,\hat{\jmath} \;;\; A = 1\,s^{-1}$$

Find: (a) Expressions for particle coordinates, $x_p = f_1(t)$ and $y_p = f_2(t)$.
(b) Time required for particle to travel from $(x_0, y_0) = (\frac{1}{2}, 2)$ to $(x, y) = (1, 1)$ and $(2, \frac{1}{2})$.
(c) Compare acceleration determined from $f_1(t)$ and $f_2(t)$ with those found in Problem 5.49.

Solution: For the given flow, $u = Ax$ and $v = -Ay$. Thus

$$u_p = \frac{df_1}{dt} = Ax_p = Af_1 \text{, or } \frac{df_1}{f_1} = A\,dt$$

Integrating from x_0 to f_1,

$$\int_{x_0}^{f_1} \frac{df_1}{f_1} = \ln f_1 \Big]_{x_0}^{f_1} = \ln\left(\frac{f_1}{x_0}\right) = At \text{, or } f_1 = x_0 e^{At}$$ $f_1(t)$

Likewise $v_p = \frac{df_2}{dt} = -Ay_p = -Af_2$, or $\frac{df_2}{f_2} = -A\,dt$

Integrating from y_0 to f_2,

$$\int_{y_0}^{f_2} \frac{df_2}{f_2} = \ln f_2 \Big]_{y_0}^{f_2} = \ln\left(\frac{f_2}{y_0}\right) = -At \text{ or } f_2 = y_0 e^{-At}$$ $f_2(t)$

For a particle initially at $(\frac{1}{2}, 2)$, $x_0 = \frac{1}{2}$ and $y_0 = 2$

To reach the point $(x, y) = (1, 1)$, $e^{At} = \frac{x}{x_0} = 2$, so $t = \frac{\ln 2}{A} = 0.693$ sec

$$e^{-At} = \frac{y}{y_0} = \frac{1}{2}, \text{ so } t = \frac{-\ln\frac{1}{2}}{A} = 0.693 \text{ sec}$$ $t(1,1)$

To reach the point $(x, y) = (2, \frac{1}{2})$, $e^{At} = \frac{x}{x_0} = 4$, so $t = \frac{\ln 4}{A} = 1.39$ sec

$$e^{-At} = \frac{y}{y_0} = \frac{1}{4}, \text{ so } t = \frac{-\ln\frac{1}{4}}{A} = 1.39 \text{ sec}$$ $t(2,\frac{1}{2})$

The acceleration components are

$$a_{px} = \frac{d^2f_1}{dt^2} = x_0 A^2 e^{At} = x_0 A^2 \frac{f_1}{x_0} = A^2 f_1$$

$$a_{py} = \frac{d^2f_2}{dt^2} = y_0 A^2 e^{-At} = y_0 A^2 \frac{f_2}{y_0} = A^2 f_2$$

At $(x, y) = (1, 1)$

$$\vec{a}_p = a_{px}\hat{\imath} + a_{py}\hat{\jmath} = \frac{(1)^2}{s^2} \times 1\,m\,\hat{\imath} + \frac{(1)^2}{s^2} \times 1\,m\,\hat{\jmath} = (\hat{\imath} + \hat{\jmath})\frac{m}{s^2}$$ $\vec{a}_p(1,1)$

At $(x, y) = (2, \frac{1}{2})$

$$\vec{a}_p = \frac{(1)^2}{s^2} \times 2m\,\hat{\imath} + \frac{(1)^2}{s^2} \times \frac{1}{2}m\,\hat{\jmath} = (2\hat{\imath} + \frac{1}{2}\hat{\jmath})\frac{m}{s^2}$$ $\vec{a}_p(2,\frac{1}{2})$

These are identical to the accelerations found in Problem 5.49.

Given: Velocity field $\vec{V} = (Ax - B)\hat{i} - Ay\hat{j}$, where $A = 0.2\ s^{-1}$, $B = 0.6\ m \cdot s^{-1}$ and coordinates are in meters

Find: (a) General expression for acceleration of fluid particle
(b) Normal and tangential components of acceleration at $(x,y) = (0, 4/3)$, $(1,2)$, and $(2,4)$

Plot: a few streamlines and show acceleration vectors on plot

Solution:

Note that $w = 0$ and flow is steady. Then

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = (Ax - B)A\hat{i} + (-Ay)(-A\hat{j}) = (A^2x - AB)\hat{i} + A^2y\hat{j}$$

The tangential component of acceleration, a_t, is given by

$$a_t = \vec{a}\cdot\hat{e}_t \quad \text{where} \quad \hat{e}_t = \frac{\vec{V}}{|\vec{V}|} = \frac{(Ax - B)\hat{i} - Ay\hat{j}}{[(Ax - B)^2 + (-Ay)^2]^{1/2}}$$

The normal component of acceleration, a_n, is then

$a_n = \vec{a}\cdot\hat{e}_n$ where $\hat{e}_n$ is perpendicular to $\hat{e}_t$.

The computing equations (for each point) are

$$\vec{a} = (A^2x - AB)\hat{i} + A^2y\hat{j} = (0.04x - 0.12)\hat{i} + 0.04y\hat{j} \quad \text{---(1)}$$

$$\vec{V} = (Ax - B)\hat{i} - Ay\hat{j} = (0.2x - 0.6)\hat{i} - 0.2y\hat{j} \quad \text{---(2)}$$

Point $(0, 4/3)$

$\vec{V} = -0.6\hat{i} - 0.267\hat{j}$, $\hat{e}_t = -0.914\hat{i} - 0.407\hat{j}$, $\hat{e}_n = -0.407\hat{i} + 0.914\hat{j}$

$\vec{a} = -0.12\hat{i} + 0.0533\hat{j}\ m/s^2$

$a_t = \vec{a}\cdot\hat{e}_t = (-0.12\hat{i} + 0.0533\hat{j})\cdot(-0.914\hat{i} - 0.407\hat{j}) = 0.0880\ m/s^2$ } a_t

$a_n = \vec{a}\cdot\hat{e}_n = (-0.12\hat{i} + 0.0533\hat{j})\cdot(-0.407\hat{i} + 0.914\hat{j}) = 0.0976\ m/s^2$ } a_n $(0, 4/3)$

Point $(1,2)$

$\vec{V} = +0.4\hat{i} - 0.4\hat{j}$, $\hat{e}_t = (-0.707\hat{i} - 0.707\hat{j})$, $\hat{e}_n = -0.707\hat{i} + 0.707\hat{j}$

$\vec{a} = -0.08\hat{i} + 0.08\hat{j}\ m/s^2$

$a_t = \vec{a}\cdot\hat{e}_t = (-0.08\hat{i} + 0.08\hat{j})\cdot(-0.707\hat{i} - 0.707\hat{j}) = 0\ m/s^2$ } a_t

$a_n = \vec{a}\cdot\hat{e}_t = (-0.08\hat{i} + 0.08\hat{j})\cdot(-0.707\hat{i} + 0.707\hat{j}) = 0.113\ m/s^2$ } a_n $(1,2)$

Point $(2,4)$

$\vec{V} = -0.2\hat{i} - 0.80\hat{j}$, $\hat{e}_t = -0.243\hat{i} - 0.970\hat{j}$, $\hat{e}_n = -0.970\hat{i} + 0.243\hat{j}$

$\vec{a} = -0.04\hat{i} + 0.16\hat{j}\ m/s^2$

$a_t = \vec{a}\cdot\hat{e}_t = (-0.04\hat{i} + 0.16\hat{j})\cdot(-0.243\hat{i} - 0.970\hat{j}) = -0.145\ m/s^2$ } a_t

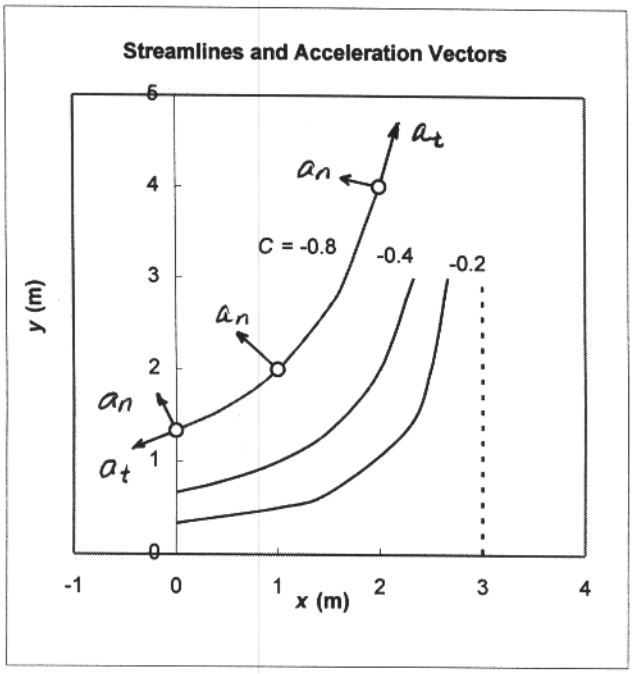
$a_n = \vec{a}\cdot\hat{e}_n = (-0.04\hat{i} + 0.16\hat{j})\cdot(-0.970\hat{i} + 0.243\hat{j}) = 0.0777\ m/s^2$ } a_n $(2,4)$

The slope of a streamline is given by $\left(\frac{dy}{dx}\right)_s = \frac{v}{u} = \frac{-Ay}{(Ax-B)}$

or $\frac{dx}{(Ax-B)} + \frac{dy}{Ay} = 0$

Integrating $\frac{1}{A}\ln(Ax-B) + \frac{1}{A}\ln y = \text{constant} = \frac{1}{A}\ln c$.

$\therefore (Ax-B)y = c$ or $y = c/(Ax-B)$.



500 SHEETS, FILLER 5 SQUARE
50 SHEETS EYE-EASE® 5 SQUARE
100 SHEETS EYE-EASE® 5 SQUARE
200 SHEETS EYE-EASE® 5 SQUARE
100 RECYCLED WHITE 5 SQUARE
200 RECYCLED WHITE 5 SQUARE
13-782
42-381
42-382
42-389
42-392
42-399
Made in U.S.A.
National® Brand

Expand $(\vec{V}\cdot\nabla)\vec{V}$ in cylindrical coordinates to obtain the convective acceleration of a fluid particle. Verify the results given in Eqs. 5.12.

Recall $\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$ and $\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$

Solution:

In cylindrical coordinates $\nabla = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}$

$\vec{V} = V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}$

$$(\vec{V}\cdot\nabla)\vec{V} = \left[V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right]\cdot\left[\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}\right]\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right)$$

$$= \left[V_r\frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial \theta} + V_z\frac{\partial}{\partial z}\right]\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right)$$

$$= V_r\frac{\partial}{\partial r}V_r\hat{e}_r + \frac{V_\theta}{r}\frac{\partial}{\partial \theta}V_r\hat{e}_r + V_z\frac{\partial}{\partial z}(V_r\hat{e}_r)$$
$$+ V_r\frac{\partial}{\partial r}V_\theta\hat{e}_\theta + \frac{V_\theta}{r}\frac{\partial}{\partial \theta}V_\theta\hat{e}_\theta + V_z\frac{\partial}{\partial z}V_\theta\hat{e}_\theta$$
$$+ V_r\frac{\partial}{\partial r}V_z\hat{k} + \frac{V_\theta}{r}\frac{\partial}{\partial \theta}V_z\hat{k} + V_z\frac{\partial}{\partial z}V_z\hat{k}$$

$$= \hat{e}_r\left\{V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z}\right\}_{①} + V_r\frac{\partial}{\partial r}\left(\frac{\partial \hat{e}_r}{\partial \theta}\right)V_\theta + \frac{V_\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial \hat{e}_r}{\partial \theta}\right)V_\theta + V_z\frac{\partial}{\partial z}\left(\frac{\partial \hat{e}_r}{\partial \theta}\right)V_\theta$$

$$+ \hat{e}_\theta\left\{V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z}\right\}_{②} + \hat{k}\left\{V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}\right\}_{③}$$

$$(\vec{V}\cdot\nabla)\vec{V} = \hat{e}_r\left\{V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z}\right\}$$
$$+ \hat{e}_\theta\left\{V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + V_z\frac{\partial V_\theta}{\partial z}\right\}$$
$$+ \hat{k}\left\{V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}\right\}.$$

Term ① is the r component of convective acceleration

Eq. 5.12a $$a_{rp} = \left\{V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z}\right\} + \frac{\partial V_r}{\partial t}$$

Term ② is the θ component of convective acceleration

Eq. 5.12b $$a_{\theta p} = \left\{V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + V_z\frac{\partial V_\theta}{\partial z}\right\} + \frac{\partial V_\theta}{\partial t}$$

Term ③ is the z component of convective acceleration

Eq. 5.12c $$a_{zp} = \left\{V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}\right\} + \frac{\partial V_z}{\partial t}$$

500 SHEETS; FILLER 5 SQUARE
50 SHEETS EYE-EASE® 5 SQUARE
100 SHEETS EYE-EASE® 5 SQUARE
200 SHEETS EYE-EASE® 5 SQUARE
100 RECYCLED WHITE 5 SQUARE
200 RECYCLED WHITE 5 SQUARE
13-782 42-381 42-382 42-389 42-392 42-399
Made in U.S.A.
National® Brand

Given: Velocity field $\vec{V} = 10x\hat{\imath} - 10y\hat{\jmath} + 30\hat{k}$

Determine if the field is: (a) Incompressible.
(b) Irrotational.

Solution: Apply continuity and irrotationality condition.

Basic equations: $\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \cancel{\frac{\partial \rho w}{\partial z}}^{=0(1)} + \cancel{\frac{\partial \rho}{\partial t}}^{=0(2)} = 0$

$\nabla \times \vec{V} = 0$ (if irrotational)

Assumptions: (1) $\vec{V} = \vec{V}(x,y)$, so $\frac{\partial}{\partial z} = 0$

(2) Incompressible flow, so ρ = constant

Then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 10 - 10 = 0$$

Flow is a possible incompressible flow. ←

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix} = \hat{\imath}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\nabla \times \vec{V} = \hat{\imath}(0-0) + \hat{\jmath}(0-0) + \hat{k}(0-0) = 0$$

Flow is irrotational. ←

Given: Velocity field $\vec{V} = (4x^2 + 3y)\hat{\imath} + (3x - 2y)\hat{\jmath}$

Determine if the field is: (a) A possible incompressible flow.
(b) Irrotational.

Solution: Apply continuity and irrotationality condition.

Basic equations: $\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$

$\nabla \times \vec{V} = 0$ (if irrotational)

Assumptions: (1) $\vec{V} = \vec{V}(x, y)$, so $\frac{\partial}{\partial z} = 0$

(2) Incompressible flow, so ρ = constant

Then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 8x - 2 \neq 0$$

Flow cannot be incompressible. ← $\rho \neq c$

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix} = \hat{\imath}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\nabla \times \vec{V} = \hat{\imath}(0 - 0) + \hat{\jmath}(0 - 0) + \hat{k}(3 - 3) = \hat{k}(0) = 0$$

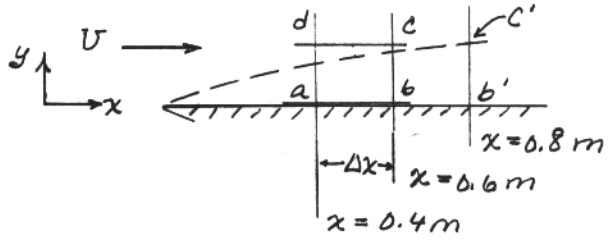
Flow is irrotational. ←

Given: Sinusoidal approximation to boundary-layer velocity profile,

$$u = U\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) \quad \text{where } \delta = 5\,mm \text{ at } x = 0.5\,m \text{ (Problem 5.11)}$$

Neglect vertical component of velocity. $U = 0.5\,m/s$.

Find: (a) Circulation about contour bounded by $x = 0.4\,m$, $x = 0.6\,m$, $y = 0$, and $y = 8\,mm$.
(b) Result if evaluated $\Delta x = 0.2\,m$ further downstream?



Solution: Evaluate circulation

Defining equation:

$$\Gamma = \oint \vec{V}\cdot d\vec{s}$$

From the definition

$$\Gamma = \int_{ab} \vec{V}\cdot d\vec{s}^{\,(u=0)} + \int_{bc} \vec{V}\cdot d\vec{s}^{\,(u\perp dy)} + \int_{cd} \vec{V}\cdot d\vec{s} + \int_{da} \vec{V}\cdot d\vec{s}^{\,(u\perp dy)} = \int_0^{\Delta x} U\hat{i}\cdot dx(-\hat{i})$$

$$\Gamma = -U\Delta x = -5\frac{m}{sec}\times 0.2\,m = -0.100\,m^2/sec$$ ⟵ Γ

At the downstream location, since $\delta = cx^{1/2}$

$$\delta' = \delta\left(\frac{x}{x'}\right)^{1/2} = 5\,mm\left(\frac{0.8}{0.5}\right)^{1/2} = 6.32\,mm$$

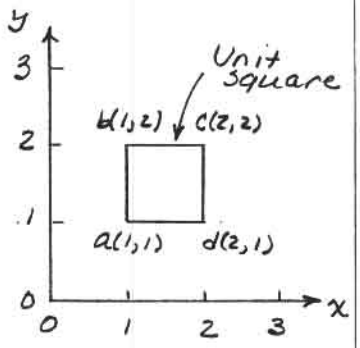
Point c' is also outside the boundary layer. Consequently the integral along c'c will be the same as along cd. Thus

$$\Gamma_{bb'c'c} = \Gamma_{abcd}$$ ⟵ Γ

Given: Velocity field for flow in a rectangular "corner,"

$$\vec{V} = Ax\hat{i} - Ay\hat{j} \quad \text{with } A = 0.35\ s^{-1}$$

as in Example Problem 5.8.



Find: Circulation about unit square shown.

Solution: Evaluate circulation

Defining equation:

$$\Gamma = \oint \vec{V}\cdot d\vec{s}$$

The dot product is $\vec{V}\cdot d\vec{s} = (Ax\hat{i} - Ay\hat{j})\cdot(dx\hat{i} + dy\hat{j}) = Ax\,dx - Ay\,dy$.

For the contour shown, $dy = 0$ along ad and cb, and $dx = 0$ along ba and dc. Thus

$$\Gamma = \int_a^d Ax\,dx + \int_d^c -Ay\,dy + \int_c^b Ax\,dx + \int_b^a -Ay\,dy$$

$$= \frac{Ax^2}{2}\Big]_{x_a}^{x_d} - \frac{Ay^2}{2}\Big]_{y_d}^{y_c} + \frac{Ax^2}{2}\Big]_{x_c}^{x_b} - \frac{Ay^2}{2}\Big]_{y_b}^{y_a}$$

$$= \frac{A}{2}(x_d^2 - x_a^2 + x_b^2 - x_c^2) - \frac{A}{2}(y_c^2 - y_d^2 + y_a^2 - y_b^2)$$

$\Gamma = 0$ (since $x_a = x_b$ and $x_c = x_d$

$y_a = y_d$ and $y_b = y_c$) ← Γ

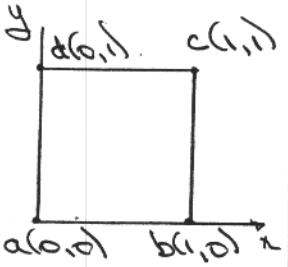
{ This result is to be expected, since flow is irrotational ($\nabla\times\vec{V} = 0$). From Stokes' Theorem (Eq. 5.18),

$$\Gamma = \int_A (\nabla\times\vec{V})_z\,dA = 0$$ }

Given: Two dimensional flow field $\vec{V} = Axy\,\hat{i} + By^2\,\hat{j}$, where $A = 1\ m^{-1}\cdot s^{-1}$, $B = -\frac{1}{2}\ m^{-1}\cdot s^{-1}$ and coordinates are measured in meters

Show: velocity field represents a possible incompressible flow

Find: (a) Rotation at point $(x,y) = (1,1)$
(b) Circulation about curve shown



Solution:

For incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

For given flow field.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(Axy) + \frac{\partial}{\partial y}(By^2) = Ay + 2By = (1)y + 2\left(-\tfrac{1}{2}\right)y = 0$$ ✓

The fluid rotation is defined as $\vec{\omega} = \frac{1}{2}\nabla\times\vec{V}$

$$\vec{\omega} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Axy & By^2 & 0 \end{vmatrix} = -\frac{1}{2}Ax\,\hat{k}$$

$$\vec{\omega}_{1,1} = -\frac{1}{2}\times 1\ m^{-1}\cdot s^{-1}\times 1\ m\,\hat{k} = -0.5\,\hat{k}\ \text{rad/s}$$ $\vec{\omega}_{1,1}$

The circulation is defined as $\Gamma = \oint \vec{V}\cdot d\vec{s}$

For the contour shown with $\vec{V} = Axy\,\hat{i} + By^2\,\hat{j}$

$$\Gamma = \int_a^b u\,dx + \int_b^c v\,dy + \int_c^d u(-dx) + \int_d^a v(-dy)$$

$u = 0$ along ab.

$$\Gamma = \int_0^1 By^2\,dy + \int_1^0 Axy\,dx + \int_1^0 By^2\,dy \qquad \{y = 1 \text{ along cd}\}$$

$$\Gamma = \tfrac{1}{3}By^3\Big]_0^1 + \tfrac{1}{2}Ax^2\Big]_1^0 + \tfrac{1}{3}By^3\Big]_1^0$$

$$\Gamma = \tfrac{1}{3}B - \tfrac{1}{2}A - \tfrac{1}{3}B = -\tfrac{1}{2}A = -\tfrac{1}{2}\ m^2/s$$ Γ

Given: Flow field represented by $\psi = 10xy + 17$

Determine if the flow is: (a) A possible incompressible flow.
(b) Irrotational

Solution: Apply continuity and the irrotationality condition.

Basic Equations: $\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$ ($\partial\rho w/\partial z = 0$, $\partial\rho/\partial t = 0$)

$\nabla \times \vec{V} = 0$ (if irrotational)

Assumptions: (1) $\psi = \psi(x,y)$ so $\partial/\partial z = 0$, $w = 0$

(2) Incompressible flow, so ρ = constant

Then
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

But
$$u = \frac{\partial \psi}{\partial y} = 10x \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} = -10y$$

So
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 10 - 10 = 0$$

Flow could be incompressible. ←

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & 0 \\ u & v & 0 \end{vmatrix} = \hat{i}(0) + \hat{j}(0) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\nabla \times \vec{V} = \hat{k}(0-0) = 0$$

Flow is irrotational. ←

Given: Velocity field for motion in x direction with constant shear.
The shear rate is

$$\frac{\partial u}{\partial y} = A \quad \text{where } A = 0.1\ s^{-1}$$

Find: (a) Expression for $\vec{V}$
(b) Rate of rotation
(c) Stream function.

Solution: The velocity field is

$$\vec{V} = u\hat{\imath} = \left[\int \frac{\partial u}{\partial y}\,dy + f(x)\right]\hat{\imath} = [Ay + f(x)]\,\hat{\imath}$$ $\vec{V}$

Fluid rotation is given by

$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = \frac{1}{2}\left(\cancel{\frac{\partial v}{\partial x}}^{\,0} - \frac{\partial u}{\partial y}\right)\hat{k} = -\frac{1}{2}\frac{\partial u}{\partial y}\hat{k} = -\frac{A}{2}\hat{k} = -0.05\ s^{-1}\ \hat{k}$$ $\vec{\omega}$

From the definition of the stream function,

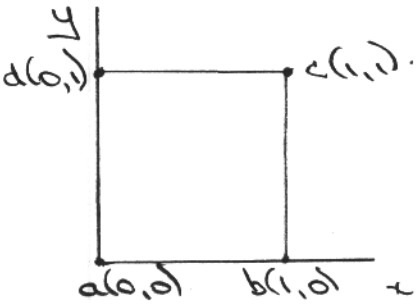
$$u = \frac{\partial \psi}{\partial y} \quad \text{so} \quad \frac{\partial \psi}{\partial y} = Ay + f(x) \quad \text{and} \quad \psi = \frac{1}{2}Ay^2 + f(x)y + g(x)$$

$$v = -\frac{\partial \psi}{\partial x} = f'(x)\,y + g'(x) = 0$$

Thus $f'(x) = 0$ and $g'(x) = 0$, and

$$\psi = \frac{1}{2}Ay^2 + C$$ C

Given: Velocity field $\vec{V} = Axy\,\hat{i} + By^2\,\hat{j}$, where $A = 4\ m^{-1}\cdot s^{-1}$, $B = -2\ m^{-1}\cdot s^{-1}$ and coordinates are in meters.



Find: (a) Fluid rotation
(b) Circulation about "curve" shown
(c) Stream function

Plot: several streamlines in first quadrant.

Solution:

(a) The fluid rotation is given by

$$\omega = \frac{1}{2}\nabla\times\vec{V} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ Axy & By^2 & 0 \end{vmatrix} = \hat{k}\,\frac{1}{2}(-Ax) = -\frac{1}{2}\times 4\times 1\ \hat{k} = -2\hat{k}\ (s^{-1})$$

13

(b) The circulation is defined as $\Gamma = \oint \vec{V}\cdot d\vec{s}$

For the contour shown with $\vec{V} = Axy\,\hat{i} + By^2\,\hat{j}$

$$\Gamma = \int_a^b \cancel{Axy}^{\,y=0}\,dx + \int_b^c By^2\,dy + \int_c^d Axy\,dx + \int_d^a By^2\,dy$$

$$\Gamma = \int_0^1 By^2\,dy + \int_1^0 Ax\,dx + \int_1^0 By^2\,dy = B\frac{y^3}{3}\Big]_0^1 + A\frac{x^2}{2}\Big]_1^0 + B\frac{y^3}{3}\Big]_1^0$$

$$\Gamma = \frac{1}{3}B - \frac{1}{2}A - \frac{1}{3}B = -\frac{1}{2}A = -2\ m^2/s$$

Γ

(c) For incompressible flow $u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial x}$.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = Ay + 2By = Ay + 2(-2)y = 0 \quad \therefore \text{ incompressible}$$

Thus $u = Axy = \frac{\partial\psi}{\partial y}$ and

$$\psi = \int Axy\,dy + f(x)$$

$$\psi = \frac{1}{2}Ax y^2 + f(x)$$

Also,

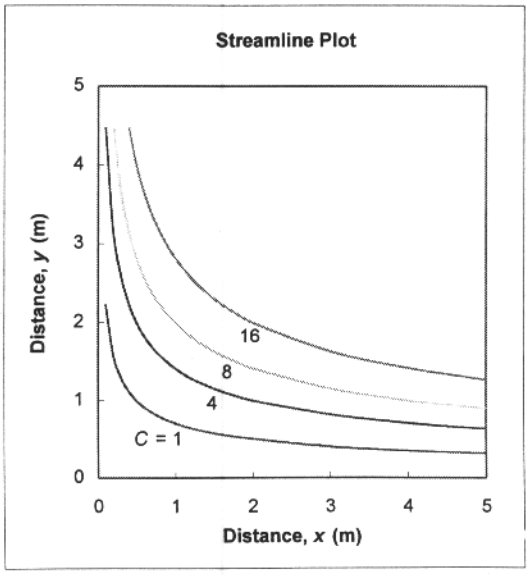
$$v = -\frac{\partial\psi}{\partial x} = -\frac{1}{2}Ay^2 - \frac{df}{dx} = By^2$$

$$\therefore \frac{df}{dx} = -\frac{1}{2}Ay^2 - By^2 = -2y^2 + 2y^2 = 0$$

Hence f = constant.

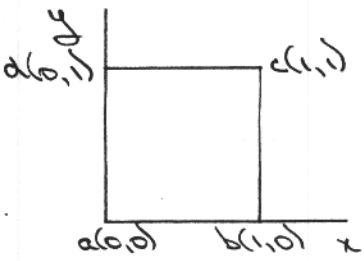
Taking $f = 0$ gives

$$\psi = \frac{1}{2}Axy^2 = 2xy^2$$



ψ

Given: Velocity field, $\vec{V} = (Ay+B)\hat{i} + A\hat{j}$, where $A = 6\ s^{-1}$, $B = 3\ m\cdot s^{-1}$ and coordinates are in meters.



Find: (a) An expression for the stream function.
(b) Circulation about "curve" shown.

Plot: several streamlines (including stagnation streamline) in the first quadrant.

Solution

For incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(Ay+B) + \frac{\partial}{\partial y}(A) = 0+0 = 0$ ∴ incompressible.

For $u = Ay + B = \frac{\partial \psi}{\partial y}$ and $\psi = \int (Ay+B)\,dy + f(x) = \frac{1}{2}Ay^2 + By + f(x)$

and $v = -\frac{\partial \psi}{\partial x} = -\frac{df}{dx} = Ax$ and $f(x) = -\frac{1}{2}Ax^2 + \text{constant}$ (set = 0)

∴ $\psi = \frac{1}{2}A(y^2 - x^2) + By$ ← ψ

Several streamlines are plotted below. The stagnation point (where $\vec{V} = 0$) is at $x = 0$, $y = -B/A = -0.5\ m$.

The circulation is defined as $\Gamma = \oint \vec{V} \cdot d\vec{s}$

For the contour shown with $\vec{V} = (Ay+B)\hat{i} + Ax\hat{j}$

$\Gamma = \int_a^b u\,dx + \int_b^c v\,dy + \int_c^d u\,dx + \int_d^a v\,dy$ (last term = 0)

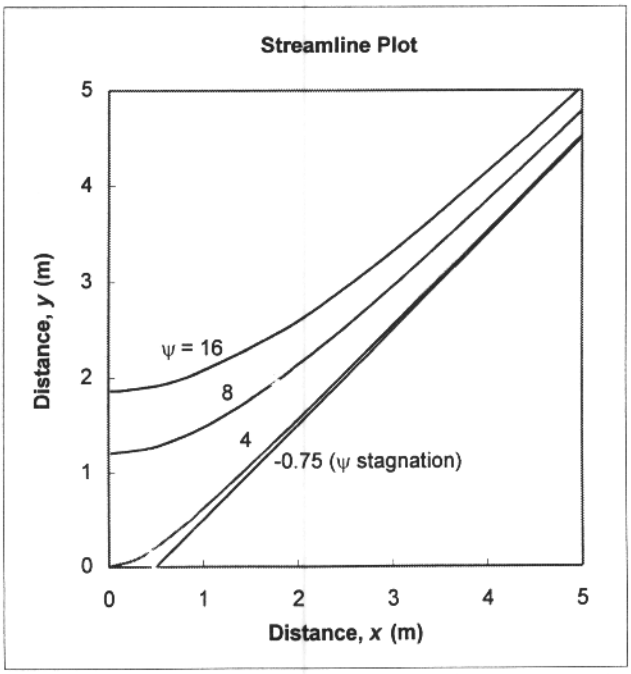
$\Gamma = \int_0^1 B\,dx + \int_0^1 A\,dy + \int_1^0 (A+B)\,dx$

{ x = 1 from b to c
y = 1 " c to d }

$\Gamma = Bx\Big|_0^1 + Ay\Big|_0^1 + (A+B)x\Big|_1^0$

$\Gamma = B + A - (A+B)$

$\Gamma = 0$ ← Γ



Note: The flow is irrotational, i.e. $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = 0$ and hence we would expect $\Gamma = 0$

At stagnation, $\psi(x,y) = \psi(0,-0.5)$

$\psi(x,y) = 3[(-0.5)^2 - 0] + 3(-0.5) = -3/4$

Given: Flow field represented by $\psi = Axy + Ay^2$; $A = 1\ s^{-1}$

Find: (a) Show that this represents a possible incompressible flow field.
(b) Evaluate the rotation of the flow.
(c) Plot a few streamlines in the upper half plane.

Solution: For incompressible flow, $\nabla \cdot \vec{V} = 0$

The velocity field is determined from the stream function

$$u = \partial\psi/\partial y = Ax + 2Ay$$
$$v = -\partial\psi/\partial x = -Ay$$
$$\vec{V} = A\{(x+2y)\hat{i} - y\hat{j}\}$$

Then

$$\nabla \cdot \vec{V} = \frac{\partial}{\partial x}A(x+2y) - \frac{\partial}{\partial y}(Ay) = A - A = 0$$ ◀ Q.E.D.

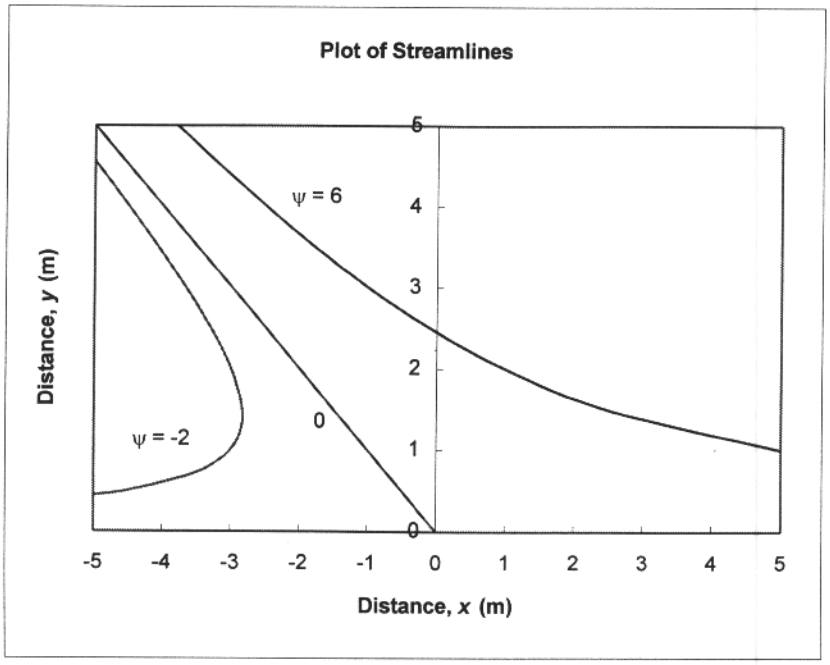
The rotation is given by $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$

$$\vec{\omega} = \frac{1}{2}\left[\frac{\partial}{\partial x}(-Ay) - \frac{\partial}{\partial y}A(x+2y)\right]\hat{k} = \frac{1}{2}[0 - 2A]\hat{k} = -A\hat{k}$$

$$\vec{\omega} = -\hat{k}\ \text{rad/s}$$ ◀ $\vec{\omega}$

To plot a few streamlines, $\psi = Axy + Ay^2$, note that for a given streamline

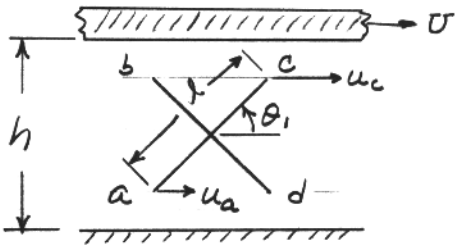
$$x = \frac{\psi}{Ay} - y$$



Given: Viscometric flow of Example Problem 5.7, $\vec{V} = U(y/h)\hat{i}$, where $U = 4$ mm/s and $h = 4$ mm.

Find: (a) Average rate of rotation of two line segments at ± 45°
(b) Show that this is the same as in the Example.

Solution: Consider lines shown:

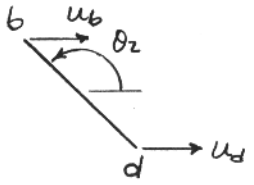


$$u_c = u_a + \frac{\partial u}{\partial y}(\ell \sin\theta_1)$$

$$-\omega_{ac} = \frac{(u_c - u_a)\sin\theta_1}{\ell} \quad \left\{\begin{array}{l}\text{Component } \perp \\ \text{to } \ell \text{ is } u\sin\theta_1.\end{array}\right\}$$

$$-\omega_{ac} = \frac{\frac{\partial u}{\partial y}(\ell\sin\theta_1)\sin\theta_1}{\ell} = \frac{\partial u}{\partial y}\sin^2\theta_1 = \frac{U}{h}\sin^2\theta_1$$

Sketch showing θ_2:



$$u_b = u_d + \frac{\partial u}{\partial y}(\ell\sin\theta_2)$$

$$-\omega_{bd} = \frac{(u_b - u_d)\sin\theta_2}{\ell} \quad \left\{\begin{array}{l}\text{Component } \perp \\ \text{to } \ell \text{ is } u\sin\theta_2.\end{array}\right\}$$

$$-\omega_{bd} = \frac{\frac{\partial u}{\partial y}(\ell\sin\theta_2)\sin\theta_2}{\ell} = \frac{\partial u}{\partial y}\sin^2\theta_2 = \frac{U}{h}\sin^2\theta_2$$

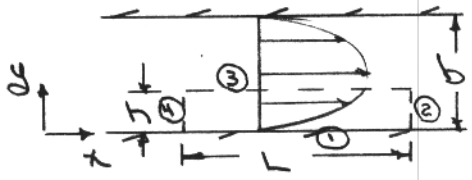
$$\omega\,(+\circlearrowleft) = \frac{1}{2}(\omega_{ac} + \omega_{bd}) = -\frac{1}{2}\frac{U}{h}(\sin^2\theta_1 + \sin^2\theta_2) = -\frac{1}{2}\frac{U}{h}(\sin^2 45° + \sin^2 135°)$$

$$= -\frac{1}{2}\frac{U}{h}\left[\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2\right] = -\frac{1}{2}\frac{U}{h} \qquad \omega$$

$$\omega = -\frac{1}{2} \times \frac{4\text{ mm}}{\text{sec}} \times \frac{1}{4\text{ mm}} = -0.5\ \text{s}^{-1} \qquad \omega$$

Given: Flow between parallel plates. Velocity field given by

$$u = U\left(\frac{y}{b}\right)\left[1 - \frac{y}{b}\right]$$



Find: (a) expression for circulation about a closed contour of height h and length L.
(b) evaluate for $h = b/2$ and $h = b$.
(c) show that same result is obtained from area integral of Stokes Theorem (Eq. 5.8).

Solution:

Basic equations: $\Gamma = \oint \vec{V}\cdot d\vec{s} = \int_A (\nabla \times \vec{V})_3\, dA$

Then, $\Gamma = \int_1^2 \vec{V}\cdot d\vec{s} + \int_2^3 \vec{V}\cdot d\vec{s} + \int_3^4 \vec{V}\cdot d\vec{s} + \int_4^1 \vec{V}\cdot d\vec{s}$

$$= \int_0^L U\frac{y}{b}\left(1 - \frac{y}{b}\right) dx$$

$$\Gamma = -UL\frac{h}{b}\left(1 - \frac{h}{b}\right)$$

For $h = y = b/2$, $\Gamma = -\frac{UL}{4}$

$h = y = b$, $\Gamma = 0$

From Stokes Theorem,

$$\Gamma = \int_A (\nabla \times \vec{V})_3\, dA = \int_A \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA = \int_A -U\left(\frac{1}{b} - \frac{2y}{b^2}\right) dA$$

$$\Gamma = -U\left(\left(\frac{1}{b} - \frac{2y}{b^2}\right) L\, dy = -UL\left[\frac{y}{b} - \frac{y^2}{b^2}\right]_0^h\right.$$

$$\Gamma = -UL\left[\frac{h}{b} - \frac{h^2}{b^2}\right] = -UL\frac{h}{b}\left(1 - \frac{h}{b}\right)$$

Given: Flow field represented by $\psi = x^2 - y^2$

Find: corresponding velocity field

Show: that flowfield is irrotational

Plot: several streamlines and illustrate the velocity field

Solution:

Apply definition of ψ and irrotationality condition:

Computing equations: $u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$

$\vec{\omega} = \nabla \times \vec{V} = 0$

From the given $\psi = x^2 - y^2$

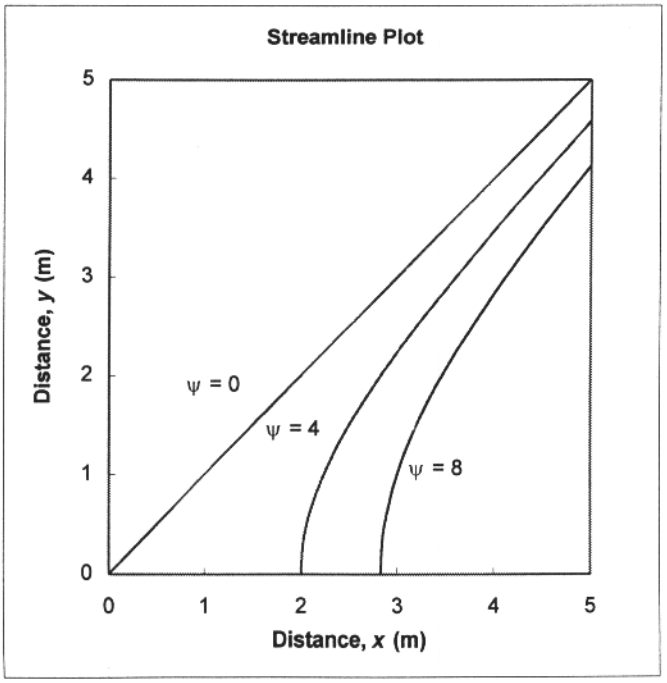
$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(x^2 - y^2) = -2y$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(x^2 - y^2) = -2x$$

$$\vec{V} = u\hat{i} + v\hat{j} = -2y\hat{i} - 2x\hat{j}$$ ← V

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2y & -2x & 0 \end{vmatrix} = \hat{k}(-2 - (-2)) = 0$$

Since $\vec{\omega} = \nabla \times \vec{V} = 0$ flow is irrotational ← $\vec{\omega} = 0$



Given: Velocity field $\vec{V} = -\frac{q}{2\pi r}\hat{e}_r + \frac{K}{2\pi r}\hat{e}_\theta$ approximates a tornado.

Is it irrotational? Obtain the stream function.

Solution: Apply irrotationality condition.

Basic equation: $\nabla \times \vec{V} = 0$ (if irrotational)

It makes sense to work in cylindrical coordinates, where

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}$$

But flow is in the $r\theta$ plane, so $\frac{\partial}{\partial z} = 0$. Then

$$\nabla \times \vec{V} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right) \times \left(V_r \hat{e}_r + V_\theta \hat{e}_\theta\right)$$

$$= \hat{e}_r \times \left(\frac{\partial V_r}{\partial r}\hat{e}_r + \frac{\partial V_\theta}{\partial r}\hat{e}_\theta\right)$$

$$+ \hat{e}_\theta \frac{1}{r} \times \left(\frac{\partial V_r}{\partial \theta}\hat{e}_r + V_r \frac{\partial \hat{e}_r}{\partial \theta} + \frac{\partial V_\theta}{\partial \theta}\hat{e}_\theta + V_\theta \frac{\partial \hat{e}_\theta}{\partial \theta}\right)$$

($\hat{e}_\theta$ and $-\hat{e}_r$)

$$\nabla \times \vec{V} = \hat{k}\left(\frac{\partial V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{V_\theta}{r}\right) = \hat{k}\frac{1}{r}\left(\frac{\partial r V_\theta}{\partial r} - \frac{\partial V_r}{\partial \theta}\right)$$

For the given flow field, $\vec{V} = \vec{V}(r)$, so

$$\nabla \times \vec{V} = \hat{k}\frac{1}{r}\frac{\partial r V_\theta}{\partial r} = \hat{k}\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{k}{2\pi}\right) \equiv 0$$

Flow is irrotational. ←

$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$; $\frac{1}{r}\frac{\partial \psi}{\partial \theta} = -\frac{q}{2\pi r}$; $\frac{\partial \psi}{\partial \theta} = \frac{-q}{2\pi}$; $\psi = \frac{-q}{2\pi}\theta + f(r)$

$V_\theta = -\frac{\partial \psi}{\partial r}$; $\frac{\partial \psi}{\partial r} = -\frac{K}{2\pi r}$; $\psi = -\frac{K}{2\pi}\ln r + g(\theta)$

Comparing, $\psi = \frac{-q}{2\pi}\theta - \frac{K}{2\pi}\ln r$ ← ψ

Given: Velocity profile for fully developed flow in a circular tube is

$$v_z = v_{max}\left[1-(r/R)^2\right]$$

Find: (a) rates of linear and angular deformation for this flow.
(b) expression for the vorticit vector, $\vec{\zeta}$

Solution:

Computing equations: B.1 and B.2 of Appendix B

Volume dilation rate $= \nabla\cdot\vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial}{\partial z}v_z = 0$

Rates of linear deformation in each of the three coordinate directions r, θ, z are zero. ← linear def

Angular deformation in the:

$r\theta$ plane is $r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta} = 0$

θz plane is $\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial\theta} = 0$

zr plane is $\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} = -v_{max}\frac{2r}{R^2}$ ← Angular def.

The vorticity vector is given by $\vec{\zeta} = \nabla\times\vec{V}$

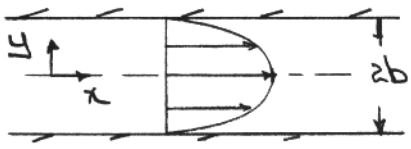
In cylindrical coordinates.

$$\nabla\times\vec{V} = \hat{e}_r\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta} - \frac{\partial v_\theta}{\partial z}\right) + \hat{e}_\theta\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right) + \hat{k}\left(\frac{r}{r}\frac{\partial(rv_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right) \quad (5.16)$$

$$\vec{\zeta} = \nabla\times\vec{V} = \hat{e}_\theta\, v_{max}\frac{2r}{R^2}$$ ← $\vec{\zeta}$

Given: Flow between parallel plates. Velocity field given by

$$u = u_{max}\left[1 - \left(\frac{y}{b}\right)^2\right]$$



Find: (a) rates of linear and angular deformation
(b) expression for the vorticity vector, $\vec{\zeta}$
(c) location of maximum vorticity

Solution:

The rate of linear deformation is zero since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = 0$

The rate of angular deformation in the xy plane is

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -\frac{2y\,u_{max}}{b^2}$$

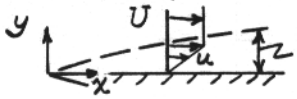
The vorticity vector is given by $\vec{\zeta} = \nabla \times \vec{V}$

$$\vec{\zeta} = \hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\vec{\zeta} = -\frac{\partial u}{\partial y}\hat{k} = \frac{2y\,u_{max}}{b^2}\hat{k}$$

The vorticity is a maximum at $y = \pm b$

Given: Linear approximate velocity profile in boundary layer.



$\delta = cx^{1/2}$; $u = U\frac{y}{\delta} = \frac{Uy}{cx^{1/2}}$; $v = \frac{uy}{4x} = \frac{Uy^2}{4cx^{3/2}}$

Find: (a) Express rotation, find maximum.
(b) Express angular deformation, locate maximum.
(c) Express linear deformation, locate maximum.
(d) Express shear force per unit volume, locate maximum.

Solution: Work in xy plane.

Computing equations: $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ $\quad -\frac{d\gamma}{dt} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$

Linear def: $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$

Evaluating partial derivatives,

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\frac{Uy}{cx^{3/2}} \qquad \frac{\partial u}{\partial y} = \frac{U}{cx^{1/2}} \qquad \frac{\partial v}{\partial x} = -\frac{3}{8}\frac{Uy^2}{cx^{5/2}} \qquad \frac{\partial v}{\partial y} = \frac{1}{2}\frac{Uy}{cx^{3/2}}$$

Then

$$\omega_z = \frac{1}{2}\left[-\frac{3}{8}\frac{Uy^2}{cx^{5/2}} - \frac{U}{cx^{1/2}}\right] = -\frac{U}{2cx^{1/2}}\left[1 + \frac{3}{8}\left(\frac{y}{x}\right)^2\right] \quad (\max \text{ at } y=\delta) \longleftarrow \omega_z$$

$$-\frac{d\gamma}{dt} = -\frac{3}{8}\frac{Uy^2}{cx^{5/2}} + \frac{U}{cx^{1/2}} = \frac{U}{cx^{1/2}}\left[1 - \frac{3}{8}\left(\frac{y}{x}\right)^2\right] \quad (\max \text{ at } y=0) \longleftarrow -\frac{d\gamma}{dt}$$

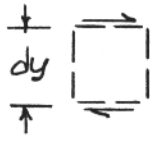
$$\frac{\partial u}{\partial x} = -\frac{1}{2}\frac{Uy}{cx^{3/2}} = -\frac{U}{2cx^{1/2}}\left(\frac{y}{x}\right) \quad (\max \text{ at } y=\delta)$$

$$\frac{\partial v}{\partial y} = +\frac{1}{2}\frac{Uy}{cx^{3/2}} = +\frac{U}{2cx^{1/2}}\left(\frac{y}{x}\right) \quad (\max \text{ at } y=\delta) \longleftarrow \text{Lin Def}$$

} sum = 0

Shear stress is $\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \mu\left(\frac{U}{cx^{1/2}} - \frac{3}{8}\frac{Uy^2}{cx^{5/2}}\right) = \frac{\mu U}{cx^{1/2}}\left[1 - \frac{3}{8}\left(\frac{y}{x}\right)^2\right]$

Net shear force on a fluid element is $d\tau\, dx\, dz$



$$d\tau = \frac{\partial \tau}{\partial y}dy = \frac{\mu U}{cx^{1/2}}\left(-\frac{3}{8}\frac{2y}{x^2}\right)dy = -\frac{3\mu Uy}{4cx^{5/2}}dy$$

Shear stress per volume is $\frac{dF}{d\forall} = -\frac{3\mu U}{4cx^{3/2}}\left(\frac{y}{x}\right)$ $(\max \text{ at } y=\delta)$ $\longleftarrow \frac{dF}{d\forall}$

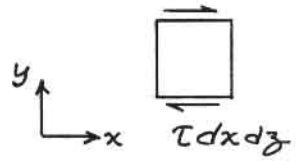
Given: x component of velocity in laminar boundary layer in water

$$u = U\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) \qquad U = 3\ m/s,\quad \delta = 2\ mm$$

y component is much smaller than u.

Find: (a) Expression for net shear force per unit volume in x direction.
(b) Maximum value for this flow

Solution: Consider a small element of fluid



Then

$$dF_{shear,x} = (\tau + d\tau)dxdz - \tau dxdz$$

$$= d\tau\, dxdz = \frac{d\tau}{dy}dxdydz$$

and

$$\frac{dF_{s,x}}{d\forall} = \frac{d\tau}{dy} = \frac{d}{dy}\left(\mu\frac{du}{dy}\right) = \mu\frac{d^2u}{dy^2}$$

From the given profile,

$$\frac{du}{dy} = \frac{\pi U}{2\delta}\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

and

$$\frac{d^2u}{dy^2} = U\left(\frac{\pi}{2\delta}\right)^2\left(-\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right)$$

The maximum value occurs when $y = \delta$, when

$$\frac{dF_{sx,max}}{d\forall} = -\mu U\left(\frac{\pi}{2\delta}\right)^2 \longleftarrow \frac{dF_{3x}}{d\forall}$$

$$= -1\times10^{-3}\frac{N\cdot sec}{m^2}\times 3\frac{m}{sec}\left(\frac{\pi}{2}\frac{1}{0.002\ m}\right)^2 = -1.85\times10^{3}\ N/m^3$$

$$\frac{dF_{sx,max}}{d\forall} = -1.85\ kN/m^3 \longleftarrow \frac{dF_{sx}}{d\forall}$$

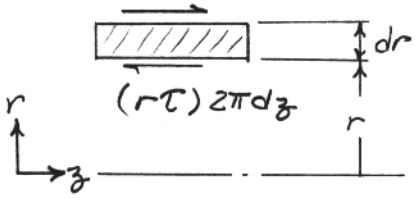
Given: Velocity profile for fully developed laminar flow in a tube

$$\frac{u}{u_{max}} = 1-\left(\frac{r}{R}\right)^2$$

where $u_{max} = 10$ ft/s, $R = 3$ in., fluid is water.

Find: (a) Expression for shear force per unit volume in z direction.
(b) Maximum value for these conditions.

Solution: Consider a differential element: $\left[r\tau + \frac{d}{dr}(r\tau)dr\right]2\pi dz$



Then

$$dF_{shear,z} = \left[r\tau + \frac{d}{dr}(r\tau)dr\right]2\pi dz - r\tau\, 2\pi dz$$

$$= \frac{d}{dr}(r\tau)\,2\pi dr dz$$

Since $d\forall = 2\pi r dr dz$, then

$$\frac{dF_{sz}}{d\forall} = \frac{1}{2\pi r dr dz}\frac{d}{dr}(r\tau)\,2\pi dr dz = \frac{1}{r}\frac{d}{dr}(r\tau)$$

In cylindrical coordinates, $\tau_{rz} = \mu\frac{du}{dr}$. For the given profile

$$\tau = \tau_{rz} = \mu\frac{du}{dr} = -\mu u_{max}\frac{2r}{R^2}$$

Substituting

$$\frac{dF_{sz}}{d\forall} = \frac{1}{r}\frac{d}{dr}\left[r\left(-\frac{2\mu u_{max} r}{R^2}\right)\right] = \frac{1}{r}\frac{d}{dr}\left[-\frac{2\mu u_{max} r^2}{R^2}\right] = \frac{1}{r}\left[\frac{-4\mu u_{max} r}{R^2}\right]$$

$$\frac{dF_{sz}}{d\forall} = -\frac{4\mu u_{max}}{R^2} = \text{constant} \longleftarrow \frac{dF_{sz}}{d\forall}$$

Evaluating,

$$\frac{dF_{sz}}{d\forall} = -4\times10^{-3}\frac{N\cdot s}{m^2}\times 10\frac{ft}{s}\times\frac{1}{(0.25)^2 ft^2}\times(0.3048)^2\frac{m^2}{ft^2}\times\frac{1\,lbf}{4.448\,N}$$

$$\frac{dF_{sz}}{d\forall} = -0.0134\ lbf/ft^3 \longleftarrow$$

Given: Incompressible flow field, $\vec{V} = (Ax + By)\hat{i} - Ay\hat{j}$

where: $A = 1\ s^{-1}$

$B = 2\ s^{-1}$

coordinates x, y, are in meters

Find: (a) Magnitude of $\vec{a}_p$ at location (1,2)

(b) Direction of $\vec{a}_p$ at location (1,2)

(c) Pressure gradient at (1,2) if $g = -g\hat{j}$

Solution:

Basic equation: $\frac{D\vec{V}}{Dt} = \vec{a}_p = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$

Substituting the given velocity field into the equation for $\vec{a}_p$

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = (Ax+By)\frac{\partial}{\partial x}\left[(Ax+By)\hat{i} - Ay\hat{j}\right] - Ay\frac{\partial}{\partial y}\left[(Ax+By)\hat{i} - Ay\hat{j}\right]$$

$$= (Ax+By)\,A\hat{i} - Ay\left[B\hat{i} - A\hat{j}\right]$$

$$\vec{a}_p = A^2x\hat{i} + A^2y\hat{j} = A^2\left[x\hat{i} + y\hat{j}\right]$$

At location (1,2)

$$\vec{a}_p = \frac{1}{s^2}\left[\hat{i} + 2\hat{j}\right] m = \hat{i} + 2\hat{j}\ \ m/s^2$$

$$|\vec{a}_p| = \sqrt{a_x^2 + a_y^2} = \sqrt{(1)^2 + (2)^2}\ \ m/s^2 = 2.24\ m/s^2 \quad \longleftarrow |\vec{a}_p|_{(1,2)}$$

$\vec{a}$, a_y, $\Delta\theta$, a_x

$$\theta = \tan^{-1}\frac{a_y}{a_x} = \tan^{-1} 2 = 63.4° \quad \longleftarrow \theta_{(1,2)}$$

Assume frictionless flow

$$\nabla p = -\rho g\hat{j} - \rho\vec{a}_p = -\rho(g\hat{j} + \vec{a}_p)$$

$$= -999\ \frac{kg}{m^3} \times (9.81\hat{j} + \hat{i} + 2\hat{j})\ \frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m}$$

$$\nabla p = -1.0\hat{i} - 11.8\hat{j}\ \ kN/m^2/m \quad \longleftarrow \nabla p_{(1,2)}$$

Note: $\nabla\cdot\vec{V} = 0$ as required for incompressible flow

Given: Flow field $\vec{V} = Axy\hat{i} - By^2\hat{j}$, where $A = 2\ ft^{-1}\cdot s^{-1}$, $B = 1\ ft^{-1}\cdot s^{-1}$ and coordinates are measured in ft; $\rho = 2\ slug/ft^3$, gravity acts in negative y direction.

Find: (a) Acceleration of fluid particle at $(x,y) = 1,1$.
(b) Pressure gradient at (1,1)

Solution.

Basic equations: $\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + \cancel{w\frac{\partial \vec{V}}{\partial z}}^{=0} + \cancel{\frac{\partial \vec{V}}{\partial t}}^{0}$

$$\rho\vec{g} - \nabla p = \rho\frac{\partial \vec{V}}{\partial t}$$

Assumptions: (1) frictionless flow.

$$\vec{a}_p = \frac{d\vec{V}}{dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = Axy\frac{\partial}{\partial x}(Axy\hat{i} - By^2\hat{j}) - By^2\frac{\partial}{\partial y}(Axy\hat{i} - By^2\hat{j})$$

$$\vec{a}_p = Axy(Ay\hat{i}) - By^2(Ax\hat{i} - 2By\hat{j})$$

$$\vec{a}_p = \hat{i}(A^2xy^2 - ABxy^2) + \hat{j}2By^3 = Axy^2(A-B)\hat{i} + 2By^3\hat{j}$$

At location (1,1)

$$\vec{a}_p = \frac{2}{ft\cdot s}\times 1ft\times 1ft^2\left(\frac{2-1}{ft\cdot s}\right)\hat{i} + 2\times\frac{1}{ft^2\cdot s^2}\times 1ft^3\hat{j} = 2\hat{i} + 2\hat{j}\ ft/s^2$$

$\vec{a}_p$

$$\nabla p = \rho\vec{g} - \rho\vec{a}_p = -\rho g\hat{j} - \rho\vec{a}_p = -\rho(g\hat{j} + \vec{a}_p)$$

$$= -2\frac{slug}{ft^3}\left(32.2\hat{j} + 2\hat{i} + 2\hat{j}\right)\frac{ft}{s^2}\times\frac{lbf\cdot s^2}{slug\cdot ft}$$

$$\nabla p = -4\hat{i} - 68.4\hat{j}\ lbf/ft^2/ft$$

∇p

Given: Flow of water described by the velocity field,

$$\vec{V} = (Ax + Bt)\hat{i} + (-Ay + Bt)\hat{j}$$

where: $A = 10\ s^{-1}$

$B = 5\ ft \cdot s^{-2}$

coordinates x, y are in ft

Find: (a) Acceleration of particle at location (1,5) and $t = 10$ s.

(b) $\frac{\partial p}{\partial x}$ at same conditions if $\vec{g} = -g\hat{j}$

Solution:

Basic equations: $\rho\vec{g} - \nabla p = \rho\frac{d\vec{V}}{dt}$

$$\frac{d\vec{V}}{dt} = \vec{a}_p = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

Assumptions: (1) frictionless flow

(2) ρ = constant = 1.94 slug/ft^3

Substituting the velocity field into the equation for $\vec{a}_p$,

$$\vec{a}_p = \frac{\partial}{\partial t}\left[(Ax+Bt)\hat{i} + (-Ay+Bt)\hat{j}\right] + (Ax+Bt)\frac{\partial}{\partial x}\left[(Ax+Bt)\hat{i} + (-Ay+Bt)\hat{j}\right]$$
$$+ (-Ay+Bt)\frac{\partial}{\partial y}\left[(Ax+Bt)\hat{i} + (-Ay+Bt)\hat{j}\right]$$

$$\vec{a}_p = \left[B\hat{i} + B\hat{j}\right] + (Ax+Bt)\left[A\hat{i}\right] + (-Ay+Bt)\left[-A\hat{j}\right]$$

$$\vec{a}_p = \hat{i}\left[B + A(Ax+Bt)\right] + \hat{j}\left[B - A(-Ay+Bt)\right]$$

At location (1,5) for $t = 10$ s

$$\vec{a}_p = \hat{i}\left[5\frac{ft}{s^2} + \frac{10}{s}\left(\frac{10}{s}\times 1ft + 5\frac{ft}{s^2}\times 10s\right)\right] + \hat{j}\left[5\frac{ft}{s^2} - \frac{10}{s}\left(-\frac{10}{s}\times 5ft + 5\frac{ft}{s^2}\times 10sec\right)\right]$$

$$\vec{a}_p = 605\hat{i} + 5\hat{j}\ \ ft/s^2 \qquad \vec{a}_p$$

Writing the x component of Euler's equation

$$\cancel{\rho g_x}^{\,0} - \frac{\partial p}{\partial x} = \rho a_{p_x}$$

At location (1,5) for $t = 10$ sec

$$\frac{\partial p}{\partial x} = -\rho a_x = -1.94\frac{slug}{ft^3}\times 605\frac{ft}{s^2}\times\frac{lbf\cdot s^2}{slug\ ft} = -1170\frac{lbf/ft^2}{ft} \qquad \partial p/\partial x$$

Note: $\nabla\cdot\vec{V} = 0$ as required for incompressible flow

Given: Velocity field, $\vec{V} = (Ax - By)t\,\hat{i} - (Ay + Bx)t\,\hat{j}$

where $A = 1\ s^{-2}$

$B = 2\ s^{-2}$

coordinates x, y are in meters

Fluid density is $\rho = 1500\ kg/m^3$. Body forces are negligible

Find: ∇P at location (1,2) at $t = 1$ s.

Solution:

Basic equations: $\cancel{\rho \vec{g}}^{\,0} - \nabla P = \rho \frac{D\vec{V}}{Dt}$

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

Assumptions: (1) frictionless flow

Substituting for the velocity field in the equation for $\frac{D\vec{V}}{Dt}$,

$$\frac{D\vec{V}}{Dt} = \frac{\partial}{\partial t}\left[(Ax - By)t\,\hat{i} - (Ay + Bx)t\,\hat{j}\right] + (Ax - By)t\frac{\partial}{\partial x}\left[(Ax - By)t\,\hat{i} - (Ay + Bx)t\,\hat{j}\right]$$
$$-(Ay + Bx)t\frac{\partial}{\partial y}\left[(Ax - By)t\,\hat{i} - (Ay + Bx)t\,\hat{j}\right]$$

$$= \left[(Ax - By)\hat{i} - (Ay + Bx)\hat{j}\right] + (Ax - By)t\left[At\,\hat{i} - Bt\,\hat{j}\right] - (Ay + Bx)t\left[-Bt\,\hat{i} - At\,\hat{j}\right]$$

$$= \hat{i}\left\{Ax - By + A^2xt^2 - \cancel{AByt^2} + \cancel{AByt^2} + B^2xt^2\right\} + \hat{j}\left\{-Ay - Bx - \cancel{ABxt^2} + B^2yt^2 + A^2yt^2 + \cancel{ABxt^2}\right\}$$

$$\frac{D\vec{V}}{Dt} = \hat{i}\left\{Ax - By + xt^2(A^2 + B^2)\right\} + \hat{j}\left\{-Ay - Bx + yt^2(A^2 + B^2)\right\}$$

Then,

$$\nabla P = -\rho\frac{D\vec{V}}{Dt} = -\rho\left[\hat{i}\left\{Ax - By + xt^2(A^2 + B^2)\right\} + \hat{j}\left\{-Ay - Bx + yt^2(A^2 + B^2)\right\}\right]$$

At location (1,2) at $t = 1$ s

$$\nabla P = -1500\frac{kg}{m^3}\left[\hat{i}\left\{\frac{1}{s^2} \times 1m - \frac{2}{s^2} \times 2 + 1m \times 1s^2\left(\frac{(1)^2 + (2)^2}{s^4}\right)\right\}\right.$$
$$\left. + \hat{j}\left\{-\frac{1}{s^2} \times 2m - \frac{2}{s^2} \times 1m + 2m \times 1s^2 \times \left(\frac{(1)^2 + (2)^2}{s^4}\right)\right\}\right]\ \frac{N \cdot s^2}{kg \cdot m}$$

$$\nabla P = -\left(3.0\,\hat{i} + 9.0\,\hat{j}\right)\frac{kN/m^2}{m}$$ ∇P

Note: $\nabla \cdot \vec{V} = 0$ as required for incompressible flow

Given: Velocity field, $\vec{V} = (Axy - Bx^2)\hat{i} + (Axy - By^2)\hat{j}$

where: $A = 2\ ft^{-1}\cdot s^{-1}$

$B = 1\ ft^{-1}\cdot s^{-1}$

coordinates x, y are in ft

Fluid density, $\rho = 2\ slug/ft^3$. Body force $\vec{g} = -g\hat{j}$

Find: (a) Acceleration of fluid particle at (1,1)

(b) Pressure gradient at (1,1)

Solution:

Basic equations: $\vec{B} - \nabla p = \rho \frac{D\vec{V}}{Dt}$

$$\frac{D\vec{V}}{Dt} = \vec{a}_p = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

Assumptions: (1) frictionless flow

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = (Axy - Bx^2)\frac{\partial}{\partial x}\left[(Axy - Bx^2)\hat{i} + (Axy - By^2)\hat{j}\right] + (Axy - By^2)\frac{\partial}{\partial y}[\quad]$$

$$= (Axy - Bx^2)\left[(Ay - 2Bx)\hat{i} + Ay\hat{j}\right] + (Axy - By^2)\left[Ax\hat{i} + (Ax - 2By)\hat{j}\right]$$

$$\vec{a}_p = \hat{i}\left[(Axy - Bx^2)(Ay - 2Bx) + Ax(Axy - By^2)\right] + \hat{j}\left[(Axy - Bx^2)Ay + (Axy - By^2)(Ax - 2By)\right]$$

At location (1,1)

$$\vec{a}_p = \hat{i}\left[(2-1)\frac{ft}{s^2}\times(2-2)\frac{1}{s} + \frac{2}{s}(2-1)\frac{ft}{s}\right] + \hat{j}\left[(2-1)\frac{ft}{s}\times\frac{2}{s} + (2-1)\frac{ft}{s}(2-2)\frac{1}{s}\right]$$

$$\vec{a}_p = 2\hat{i} + 2\hat{j}\ \ ft/s^2 \qquad \vec{a}_p(1,1)$$

$$\rho\vec{B} - \nabla p = \rho\frac{D\vec{V}}{Dt} = \rho\vec{a}_p$$

$$\nabla p = \rho\vec{B} - \rho\vec{a}_p = \rho(\vec{B} - \vec{a}_p) = \rho(-g\hat{j} - \vec{a}_p) = -\rho(g\hat{j} + \vec{a}_p)$$

At location (1,1)

$$\nabla p = -2\ \frac{slug}{ft^3}\left[32.2\hat{j} + 2\hat{i} + 2\hat{j}\right]\frac{ft}{s^2}\times\frac{lbf\cdot s^2}{ft\cdot slug}\Big] = -\left[4\hat{i} + 68.4\hat{j}\right]\frac{lbf/ft^2}{ft} \qquad \nabla p(1,1)$$

Note: For incompressible flow, $\nabla\cdot\vec{V} = 0$

$$\nabla\cdot\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = Ay - 2Bx + Ax - 2By$$

$$\nabla\cdot\vec{V} = (A - 2B)(x + y) = 0$$

Hence given velocity field represents a possible incompressible flow

Given: x component of velocity for incompressible flow in the xy plane is

$$u = Ax \quad \text{where } A = 2\ s^{-1} \text{ and } x \text{ in } m$$

Pressure is $p_0 = 190$ kPa(gage) at $(x,y) = (0,0)$.
Density is $\rho = 1.50\ kg/m^3$; z is vertical; neglect viscosity.

Find: (a) Simplest y component of velocity.
(b) Acceleration at point $(x,y) = (2,1)$.
(c) Pressure gradient at same point.
(d) Pressure distribution along x axis.

Solution: For 2-D incompressible flow, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, so $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$

$$v = \int \frac{\partial v}{\partial y}\,dy + f(x) = \int -A\,dy + f(x) = -Ay + f(x)$$

For simplest case, $f(x) = 0$, and $v = -Ay$ ← v

Acceleration is $\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y}$; $\vec{V} = Ax\hat{\imath} - Ay\hat{\jmath}$

$$\vec{a}_p = (Ax)\,A\hat{\imath} + (-Ay)(-A)\hat{\jmath} = A^2x\hat{\imath} + A^2y\hat{\jmath}$$

At (1,2), $\vec{a}_p(1,2) = \frac{2^2}{s^2} \times 2m\,\hat{\imath} + \frac{2^2}{s^2} \times 1m\,\hat{\jmath} = 8\hat{\imath} + 4\hat{\jmath}\ m/s^2$ ← $\vec{a}_p(2,1)$

To find pressure gradient, apply Euler's equation ($\mu = 0$):

BE $\quad -\nabla p + \rho\vec{g} = \rho\vec{a}_p$

$$\nabla p = \rho\vec{g} - \rho\vec{a}_p = \rho(-g\hat{k}) - \rho(8\hat{\imath} + 4\hat{\jmath}) = -\rho(8\hat{\imath} + 4\hat{\jmath} + g\hat{k})$$

$$\nabla p = -1.50\frac{kg}{m^3}\left(8\hat{\imath} + 4\hat{\jmath} + 9.81\hat{k}\right)\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m}$$

$$\nabla p = -12\hat{\imath} - 6\hat{\jmath} - 14.7\hat{k}\ N/m^3$$ ← ∇p

Along the x axis, $y = 0$, and $\vec{a}_p = A^2x\hat{\imath}$. Thus

$$\nabla p = \rho\vec{g} - \rho\vec{a}_p = \rho(-g\hat{k}) - \rho A^2x\hat{\imath} \quad \text{so} \quad \frac{\partial p}{\partial x} = -\rho A^2x$$

Thus along the x axis $dp = \frac{\partial p}{\partial x}\,dx$. Integrating,

$$p(x) - p_0 = \int_0^x dp = \int_0^x -\rho A^2x\,dx = -\rho A^2\frac{x^2}{2}\Big]_0^x = -\rho\frac{A^2x^2}{2}$$

Finally

$$p(x) = p_0 - \rho\frac{A^2x^2}{2} = 190\frac{N}{m^2} - \frac{1}{2} \times 1.50\frac{kg}{m^3} \times \frac{(2)^2}{s^2} \times (x)^2m^2 \times \frac{N\cdot s^2}{kg\cdot m}$$

$$p(x) = 190 - 3x^2 \text{ Pa(gage)} \quad (x \text{ in } m)$$ ← p(x)

Given: Frictionless, incompressible flow field with

$$\vec{V} = Ax\,\hat{i} - Ay\,\hat{j}$$

$$\vec{g} = -g\,\hat{k}$$

At $(0,0,0)$ $p = p_0$

Find: Expression for the pressure field $p(x,y,z)$

Solution:

Basic equations: $\rho\vec{g} - \nabla p = \rho\frac{d\vec{V}}{dt}$

$$\frac{d\vec{V}}{dt} = \frac{\partial\vec{V}}{\partial t} + u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}$$

$$\nabla p = \rho\left(\vec{g} - \frac{d\vec{V}}{dt}\right) = \rho\left(-g\hat{k} - u\frac{\partial\vec{V}}{\partial x} - v\frac{\partial\vec{V}}{\partial y}\right)$$

$$= -\rho\left[g\hat{k} + Ax(A\hat{i}) - Ay(-A\hat{j})\right]$$

$$\nabla p = -\rho\left[A^2x\,\hat{i} + A^2y\,\hat{j} + g\hat{k}\right]$$

$$\hat{i}\frac{\partial p}{\partial x} + \hat{j}\frac{\partial p}{\partial y} + \hat{k}\frac{\partial p}{\partial z} = -\rho\left[A^2x\,\hat{i} + A^2y\,\hat{j} + g\hat{k}\right]$$

$$\frac{\partial p}{\partial x} = -\rho A^2x \qquad \frac{\partial p}{\partial y} = -\rho A^2y \qquad \frac{\partial p}{\partial z} = -\rho g$$

$$p = p(x,y,z)$$

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz = -\rho A^2x\,dx - \rho A^2y\,dy - \rho g\,dz$$

* $$p - p_0 = \int_{p_0}^{p} dp = -\int_0^x \rho A^2x\,dx - \int_0^y \rho A^2y\,dy - \int_0^z \rho g\,dz$$

$$p - p_0 = -\rho\left[\frac{A^2x^2}{2} + \frac{A^2y^2}{2} + gz\right]$$

$$p = p_0 - \rho\left[\frac{A^2x^2}{2} + \frac{A^2y^2}{2} + gz\right]$$ ← p

* We can integrate to obtain Δp between any two points in the flow field if, and only if, the integral of the right hand side is independent of the path of integration. This is true for the present case.

Given: The velocity distribution in a steady, 2-D flow field in the xy plane is given by $\vec{V} = (A+Bx)\hat{i} + (C-Dy)\hat{j}$, where $A = 2\ s^{-1}$, $B = -3\ s^{-1}$, $C = 3\ m \cdot s^{-1}$, and the body force distribution is $\vec{g} = -g\hat{k}$.

Find: (a) Does the velocity field represent the flow of an incompressible fluid?
(b) Find the stagnation point of the flow field.
(c) Obtain an expression for the pressure gradient.
(d) Evaluate Δp between origin and point (1,3) if $\rho = 1.2\ kg/m^3$

Solution:

(a) Apply the continuity equation, $\frac{\partial \rho}{\partial t} + \nabla \cdot \rho\vec{V} = 0$, for the given conditions. If ρ = constant, then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 = \frac{\partial}{\partial x}(2x-5) + \frac{\partial}{\partial y}(3-2y) = 2-2 = 0 \ \checkmark$$

∴ velocity field represents an incompressible flow

(b) At the stagnation point, $\vec{V} = 0$. For $\vec{V} = 0$, then

$u = 2x - 5 = 0$ and $v = (3-2y) = 0$

Thus stagnation point is at $(x,y) = \left(\frac{5}{2}, \frac{3}{2}\right)$

(c) Euler's equation, $\rho\vec{g} - \nabla p = \rho\frac{D\vec{V}}{Dt}$, can be used to obtain an expression for the pressure gradient

$$\nabla p = \rho\vec{g} - \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \rho\left[\frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}\right]$$

(steady; $w = 0$)

$$\nabla p = \rho\left[\vec{g} - u\frac{\partial \vec{V}}{\partial x} - v\frac{\partial \vec{V}}{\partial y}\right] = \rho\left[-g\hat{k} - (2x-5)2\hat{i} - (3-2y)(-2\hat{j})\right]$$

$$\nabla p = -\rho\left[(4x-10)\hat{i} + (4y-6)\hat{j} + g\hat{k}\right]$$ ∇p

(d) Since $p = p(x,y,z)$ we can write

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz = -\rho(4x-10)dx - \rho(4y-6)dy - \rho g\,dz$$

We can integrate to obtain Δp between any two points in the field if, and only if, the integral of the right hand side is independent of the path of integration. This is true for the present case.

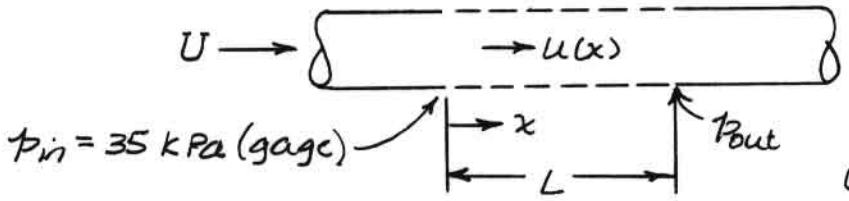
$$\therefore p_{1,3} - p_{0,0} = -\rho\left\{\int_0^1 (4x-10)dx + \int_0^3 (4y-6)dy\right\} = -\rho\left\{\left[2x^2-10x\right]_0^1 + \left[2y^2-6y\right]_0^3\right\}$$

$$= -\rho\{-8-0\} = 8\rho$$

$$p_{1,3} - p_{0,0} = 8\ \frac{m^2}{s^2} \cdot 1.2\ \frac{kg}{m^3} \cdot \frac{N \cdot s^2}{kg \cdot m} = 9.6\ N/m^2$$ Δp

Problem 6.9

Given: Porous pipe with liquid ($\mu = 0$, $\rho = 900\ kg/m^3$)



$V = 5\ m/s$

$L = 0.3\ m$

$u(x) = V(1 - x/2L)$

Find: (a) Expression for acceleration along ℓ.
(b) Expression for pressure gradient along ℓ.
(c) Evaluate p_{out}

Solution: Computing equations (acceleration and Euler in x-direction)

$$a_x = u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial u}{\partial y}}^{=0(1)} + \cancel{w\frac{\partial u}{\partial z}}^{=0(1)} + \cancel{\frac{\partial u}{\partial t}}^{=0(2)}; \quad \cancel{\rho g_x}^{=0(3)} - \frac{\partial p}{\partial x} = \rho a_x$$

Assumptions: (1) $v = w = 0$ along ℓ
(2) steady flow
(3) $g_x = 0$

Then

$$a_x = u\frac{\partial u}{\partial x} = V\left(1 - \frac{x}{2L}\right)V\left(-\frac{1}{2L}\right) = -\frac{V^2}{2L}\left(1 - \frac{x}{2L}\right) \qquad a_x$$

From Euler

$$\frac{\partial p}{\partial x} = \frac{dp}{dx} = -\rho a_x = \rho\frac{V^2}{2L}\left(1 - \frac{x}{2L}\right) \qquad \frac{dp}{dx}$$

Integrating,

$$p_{out} - p_{in} = \int_0^L \frac{dp}{dx}\,dx = \rho\frac{V^2}{2L}\int_0^L \left(1 - \frac{x}{2L}\right)dx = \rho\frac{V^2}{2L}\left(x - \frac{x^2}{4L}\right)\Big]_0^L$$

or

$$p_{out} = p_{in} + \rho\frac{V^2}{2L}\left(\frac{3}{4}L\right) = p_{in} + \frac{3}{8}\rho V^2$$

$$= 35\ kPa + \frac{3}{8} \times 900\ \frac{kg}{m^3} \times (5)^2\ \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_{out} = 43.4\ kPa\ (gage) \qquad p_{out}$$

Given: Liquid, ρ = constant and negligible viscosity, is pumped at total volume flow rate, Q, through two small holes into the narrow gap between closely space parallel plates. The liquid flowing away from the holes has only radial motion. Flow may be assumed uniform at any section.

(a) Show that $v_r = Q/2\pi rh$, where h is the spacing between the plates.

(b) Obtain an expression for a_r and $\partial p/\partial r$

Solution:



Apply the conservation of mass to a CV with outer edge at r.

Basic equation: $0 = \cancel{\frac{\partial}{\partial t}\int_{CV}\rho\,d\forall}^{0(1)} + \int_{CS}\rho\vec{V}\cdot d\vec{A}$

Assumptions: (1) steady flow
(2) incompressible flow
(3) uniform flow at each section

Then

$$0 = \int_{CS}\vec{V}\cdot d\vec{A} = -2\times\frac{Q}{2} + v_r\,2\pi rh$$

and $$v_r = \frac{Q}{2\pi rh}$$ ← v_r

From Eq. 6.4a

$$g_r - \frac{1}{\rho}\frac{\partial p}{\partial r} = a_r = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}$$

Since $v_r = v_r(r)$ and $v_\theta = 0$, then

$$a_r = v_r\frac{\partial v_r}{\partial r} = \frac{Q}{2\pi rh}\left[\frac{Q}{2\pi h}\left(-\frac{1}{r^2}\right)\right] = -\left(\frac{Q}{2\pi rh}\right)^2\frac{1}{r}$$

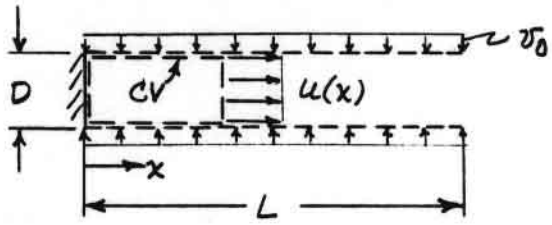
$$a_r = -\frac{v_r^2}{r}$$ ← a_r

Since $g_r = 0$, then

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = a_r$$

$$\frac{\partial p}{\partial r} = -\rho a_r = \rho\frac{v_r^2}{r}$$ ← $\frac{\partial p}{\partial r}$

Given: Circular tube with porous wall; incompressible flow, uniform in x direction.



Find: (a) Algebraic expression for a_{px} at x.
(b) Pressure gradient at x.
(c) Integrate to obtain p at x = 0.

Solution: Apply conservation of mass using the CV shown.

Basic equations: $0 = \underbrace{\frac{\partial}{\partial t}}_{=0(1)} \int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

$$a_{px} = u\frac{\partial u}{\partial x} + \underbrace{v}_{\approx 0(5)}\frac{\partial u}{\partial y} + \underbrace{w}_{\approx 0(5)}\frac{\partial u}{\partial z} + \underbrace{\frac{\partial u}{\partial t}}_{=0(1)} \; ; \; -\frac{\partial p}{\partial x} + \rho g_x = \rho a_{px}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each cross-section
(4) Horizontal; $g_x = 0$
(5) $v \approx 0$ in channel ($w \approx 0$ too)
(6) Inviscid flow

Then

$$\int \vec{V} \cdot d\vec{A} = \left\{-|v_0 \pi D x|\right\} + \left\{+|u \frac{\pi D^2}{4}|\right\} = 0 \quad \text{or} \quad u(x) = 4v_0 \frac{x}{D}$$

and

$$a_{px} = 4v_0 \frac{x}{D}\left(4v_0 \frac{1}{D}\right) = 16 v_0^2 \frac{x}{D^2}$$ a_{px}

From the Euler equation,

$$-\frac{\partial p}{\partial x} = \rho a_{px} \quad \text{so} \quad \frac{\partial p}{\partial x} = -\rho a_{px} = -16\rho v_0^2 \frac{x}{D^2}$$ $\frac{\partial p}{\partial x}$

Since $v \approx w \approx 0$, then $p(x)$ and $dp = \frac{\partial p}{\partial x} dx$. Integrating

$$\int_0^L dp = p_L - p(0) = \int_0^L -16\rho v_0^2 \frac{x}{D^2} dx = -\frac{16\rho v_0^2}{D^2}\frac{x^2}{2}\Big]_0^L = -\frac{8\rho v_0^2 L^2}{D^2}$$

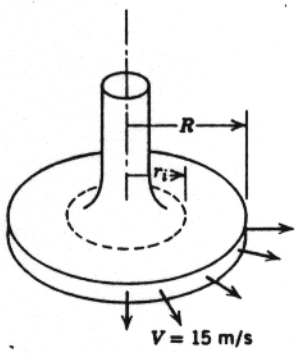
Thus, since $p_L = p_{atm}$, the gage pressure at x = 0 is

$$p(0) = 8\rho v_0^2 \left(\frac{L}{D}\right)^2$$ $p(0)$

Given: Steady, incompressible flow of air between parallel discs as shown

$\vec{V} = V\frac{R}{r}\hat{e}_r$ for $r_i \le r \le R$

where $V = 15$ m/s, $R = 75$ mm, $r_i = R/2$



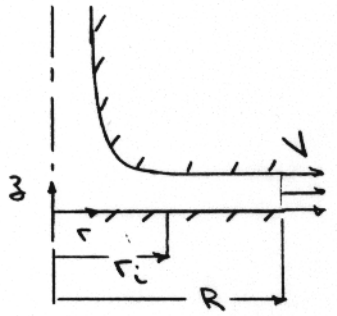
Find: magnitude and direction of the net pressure force that acts on the upper plate between r_i and R.

Solution:

Basic equations: $\rho\vec{g} - \nabla p = \rho\frac{d\vec{V}}{dt}$ $\qquad \vec{F} = -\int p\, d\vec{A}$

Assumptions: (1) incompressible flow
(2) steady flow
(3) frictionless flow
(4) uniform flow at each section.

To determine the pressure distribution $p(r)$, apply Eulers equation in the r direction



$$-\frac{\partial p}{\partial r} + \cancel{\rho g_r} = \rho a_r = \rho V_r\frac{\partial V_r}{\partial r}$$

$$\frac{\partial p}{\partial r} = -\rho V_r\frac{\partial V_r}{\partial r} = -\rho V\frac{R}{r}\frac{\partial}{\partial r}\left(V\frac{R}{r}\right) = \rho V^2\frac{R}{r}\frac{R}{r^2}$$

$$\frac{dp}{dr} = \rho V^2 R^2 r^{-3}$$

$$dp = \rho V^2 R^2 r^{-3}\, dr$$

Integrating we obtain

$$p - p_{atm} = \int_{p_{atm}}^{p} dp = \rho V^2 R^2\int_R^r r^{-3}\, dr = \rho V^2 R^2\left[-\frac{1}{2r^2}\right]_R^r = \frac{1}{2}\rho V^2 R^2\left[\frac{1}{R^2} - \frac{1}{r^2}\right]$$

Then

$$F_z = \int (p - p_{atm})\, dA = \int_{R/2}^{R}\frac{1}{2}\rho V^2 R^2\left[\frac{1}{R^2} - \frac{1}{r^2}\right]2\pi r\, dr = \rho V^2 R^2\pi\left[\frac{r^2}{2R^2} - \ln r\right]_{R/2}^{R}$$

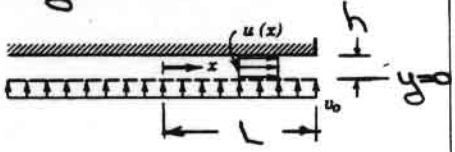
$$= \rho V^2 R^2\pi\left[\frac{1}{2R^2}\left(R^2 - \frac{R^2}{4}\right) - \ln\frac{R}{R/2}\right] = \rho V^2 R^2\pi\left[0.375 - \ln 2\right] = -0.318\pi\rho V^2 R^2$$

$$= -0.318\pi \times 1.23\ \frac{kg}{m^3} \times (15)^2\ \frac{m^2}{s^2} \times (0.075)^2\ m^2 \times \frac{N\cdot s^2}{kg\cdot m}$$

$F_z = -1.56$ N ($F_z < 0$, so force acts down) $\qquad F_z$

Given: Air flows into the narrow gap between closely spaced parallel plates through a porous surface as shown. The uniform velocity in the x direction is $u = v_0 x/h$. Assume the flow is incompressible with $\rho = 1.23\ kg/m^3$ and that friction is negligible.

$v_0 = 15\ mm/s$, $L = 22\ mm$, $h = 1.2\ mm$



Find: (a) the pressure gradient at the point (L, h)
(b) an equation for the flow streamlines in the cavity

Solution:

Euler's equation, $\rho \vec{g} - \nabla p = \rho \frac{D\vec{V}}{Dt}$, can be used to determine the pressure gradient for incompressible frictionless flow.

We need first to determine the velocity field. With $u = v_0 x/h$, for 2-D, incompressible flow we can use the continuity equation to determine v.

Since $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, then $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{v_0 x}{h}\right) = -\frac{v_0}{h}$

Then $v = \int \frac{\partial v}{\partial y}\, dy + f(x) = -\frac{v_0}{h} y + f(x)$

But $v = v_0$ at $y = 0$ and hence $f(x) = v_0$ and $v = v_0\left(1 - \frac{y}{h}\right)$

Then $\nabla p = \rho \vec{g} - \rho \frac{D\vec{V}}{Dt} = \rho\left[\vec{g} - u\frac{\partial \vec{V}}{\partial x} - v\frac{\partial \vec{V}}{\partial y}\right] = \rho\left[-g\vec{j} - \frac{v_0 x}{h}\left(\frac{v_0}{h}\vec{i}\right) - v_0\left(1 - \frac{y}{h}\right)\left(-\frac{v_0}{h}\vec{j}\right)\right]$

$\nabla p = \rho\left[-g\vec{j} - \frac{v_0^2 x}{h^2}\vec{i} - \frac{v_0^2}{h}\left(1 - \frac{y}{h}\right)\vec{j}\right]$

At the point $(x, y) = (L, h)$

$\nabla p = \rho\left[-\frac{v_0^2 L}{h^2}\vec{i} - g\vec{j}\right]$

$= 1.23\ \frac{kg}{m^3}\left[-\frac{(15)^2}{s^2} \times 0.022\ m \times \frac{1}{(1.2)^2\ mm^2}\vec{i} - 9.81\ \frac{m}{s^2}\vec{j}\right] \cdot \frac{N \cdot s^2}{kg \cdot m}$

$\nabla p\big]_{L,h} = -4.23\vec{i} - 12.1\vec{j}\ N/m^3$ ← ∇p

(b) The slope of the streamlines is given by $\frac{dy}{dx} = \frac{v}{u}$

$\therefore \frac{dy}{dx} = \frac{v_0\left(1 - \frac{y}{h}\right)}{\frac{v_0 x}{h}}$ and separating variables, we can write

$\frac{d\left(\frac{y}{h}\right)}{\left(1 - \frac{y}{h}\right)} = \frac{d\left(\frac{x}{h}\right)}{\frac{x}{h}}$ then integrating we obtain

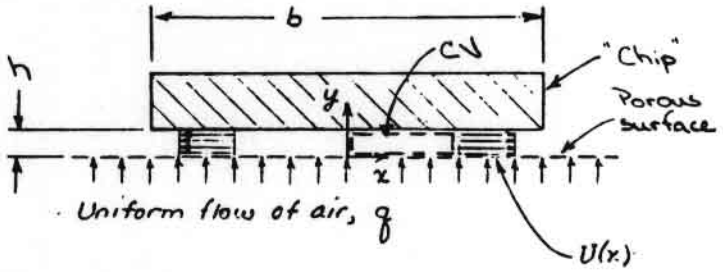
$-\ln\left(1 - \frac{y}{h}\right) = \ln\frac{x}{h} - \ln C$

or $\frac{x}{h}\left(1 - \frac{y}{h}\right) = \text{constant}$ ←

Given: Rectangular "chip" floats on thin layer of air of thickness, $h = 0.5$ mm above a porous surface as shown. Chip width $b = 20$ mm; length L (perpendicular to diagram) $>> b$; no flow in z direction. Flow in x direction under chip may be assumed uniform; ρ = constant; neglect frictional effects

Find: (a) Use a suitably chosen CV to show $U(x) = qx/h$ in the gap
(b) Find an expression for $\vec{a}_p$ in the gap
(c) Estimate the maximum value of a_p
(d) Obtain an expression for $\partial p/\partial x$
(e) Sketch the pressure distribution under the chip
(f) Is the net pressure force on the chip directed up or down?
(g) Estimate the mass per unit length of the chip if $q = 0.06$ m³/sec/m²

Solution:



Assumptions:
(1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) uniform flow at porous surface and in the gap at x.

(a) Apply continuity equation to CV, $0 = \cancel{\frac{\partial}{\partial t}\int_{CV}\rho\,d\forall}^{0(1)} + \int_{CS}\rho\vec{V}\cdot d\vec{A}$

$$0 = \{-1\,\rho q x L\} + \{+1\,\rho U h L} \quad \text{or} \quad U = q\frac{x}{h} \longleftarrow U(x)$$

(b) Apply the substantial derivative definition

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + \cancel{w\frac{\partial \vec{V}}{\partial z}}^{0(1)} + \cancel{\frac{\partial \vec{V}}{\partial t}}^{0(1)}$$

Obtain v from differential continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$\therefore \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{q}{h} \quad \text{and} \quad v - v_0 = \int_0^y -\frac{q}{h}\,dy + f(x) = -\frac{q}{h}y + f(x)$$

or $v = q\left(1 - \frac{y}{h}\right)$ [$f(x) = 0$ since $v = v_0 = q$ = const. along $y = 0$]

$$a_{p_x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{q}{h}x\left(\frac{q}{h}\right) + q\left(1 - \frac{y}{h}\right)\left(-\frac{q}{h}\right) = \frac{q^2}{h^2}y$$

$$a_{p_y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{q}{h}x\left(-\frac{q}{h}\right) + q\left(1 - \frac{y}{h}\right)\left(-\frac{q}{h}\right) = \frac{q^2}{h^2}\left(-1 + \frac{y}{h}\right)$$

$$\vec{a}_p = \frac{q^2}{h^2}x\,\hat{i} + \frac{q^2}{h^2}\left(-1 + \frac{y}{h}\right)\hat{j}$$

(c) The magnitude of $|\vec{a}_p| = \frac{q^2}{h^2}\left[\left(\frac{x}{h}\right)^2 + \left(-1 + \frac{y}{h}\right)^2\right]^{1/2}$ is a maximum at $x = L$, $y = 0$

$$|\vec{a}_p|_{max} = \frac{q^2}{h^2}\left[\left(\frac{L}{h}\right)^2 + 1\right]^{1/2} = 144\ \text{m/s}^2 \longleftarrow |\vec{a}_p|_{max}$$

(d) To obtain $\partial p/\partial x$ write the x component of the Euler equation

$$-\frac{\partial p}{\partial x} + \rho g_x = \rho a_x \quad \therefore \quad \frac{\partial p}{\partial x} = -\rho a_x = -\frac{\rho q^2 x}{h^2}$$

$\partial p/\partial x$

(e) To obtain an expression for the pressure distribution, $p(x)$ we need to separate variables and integrate noting that $p = p_{atm}$ at $x = b/2$. Thus

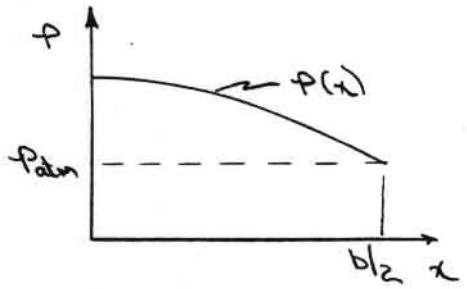
$$p - p_{atm} = \int_{b/2}^{x} \frac{\partial p}{\partial x}\, dx = -\int_{b/2}^{x} \frac{\rho q^2}{h^2} x = -\left.\frac{\rho q^2 x^2}{2 h^2}\right]_{b/2}^{x}$$

$$p - p_{atm} = \frac{\rho q^2}{2h^2}\left[\left(\frac{b}{2}\right)^2 - x^2\right] = \frac{\rho q^2 b^2}{8h^2}\left[1 - \left(\frac{2x}{b}\right)^2\right]$$

$$p = p_{atm} + \frac{\rho q^2 b^2}{8h^2}\left[1 - \left(\frac{2x}{b}\right)^2\right]$$

$p(x)$

(f) The net pressure force on the chip is up. Note that the pressure on the chip is greater than p_{atm} over the entire chip surface



(g) To estimate the mass per unit weight of the chip we must determine the net pressure force on the chip

$$F_{net} = \int_A (p - p_{atm})\, dA = 2\int_0^{b/2} \frac{\rho q^2 b^2}{8h^2}\left[1 - \left(\frac{2x}{b}\right)^2\right] L\, dx$$

$$= \frac{\rho q^2 b^2 L}{4h^2}\left[x - \frac{4x^3}{3b^2}\right]_0^{b/2} = \frac{\rho q^2 b^2 L}{4h^2}\left[\frac{b}{2} - \frac{4}{3}\frac{b}{2}\right]$$

$$F_{net} = \frac{\rho q^2 b^3 L}{12 h^2}$$

The weight of the chip, $W = Mg$, must be balanced by the net pressure force. Hence

$$Mg = F_{net} = \frac{\rho q^2 b^3 L}{12 h^2}$$

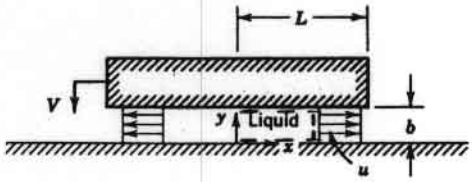
$$\frac{M}{L} = \frac{\rho q^2 b^3}{12 h^2 g}$$

$$= \frac{1}{12} \times 1.23 \frac{kg}{m^3} \times (0.06)^2 \frac{m^6}{s^2 \cdot m^4} \times (0.02)^3 m^3 \times \frac{1}{(0.0005)^2 m^2 \cdot 9.81 \frac{m}{s^2}}$$

$$\frac{M}{L} = 1.20 \times 10^{-3} \text{ kg/m}$$

mass/length

Given: Upper plane surface moving downward at constant speed V causes incompressible liquid layer to be squeezed between surfaces as shown. Depth w is z direction and $w \gg L$.



Find: (a) Show that $u = Vx/b$ within the gap ($b = b_0 - Vt$)
(b) expression for a_x
(c) $\partial p/\partial x$
(d) $p(x)$
(e) net pressure force on upper surface

Solution:

Basic equations: $0 = \frac{\partial}{\partial t}\int_{cv} \rho\, d\forall + \int_{cs} \rho \vec{V}\cdot d\vec{A}$

$-\nabla p + \rho\vec{g} = \rho \frac{D\vec{V}}{Dt}$ $\qquad \vec{F} = -\int p\, d\vec{A}$

(a) For the deformable CV shown

$$0 = \frac{\partial}{\partial t}\int_0^y \rho w x\, dy + \rho u w y = \rho w x \frac{dy}{dt} + \rho u w y$$

But $dy/dt = -V$ and hence $u = \frac{V}{y}\frac{dy}{dt}x$

If $y = b_0$ at $t = 0$, then $y = b = b_0 - Vt$ at any time t

$$\therefore u = \frac{Vx}{b} \qquad u(t)$$

(b) $a_x = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\cancel{\frac{\partial u}{\partial y}}^{0} + w\cancel{\frac{\partial u}{\partial z}}^{0} + \frac{\partial u}{\partial t}$

Assumptions: (1) $u \neq u(y)$, $w = 0$

$$a_x = \frac{Vx}{b}\left(\frac{V}{b}\right) + \frac{\partial}{\partial b}\frac{\partial b}{\partial t} = \frac{V^2x}{b^2} + \left(-\frac{Vx}{b^2}\right)(-V) = \frac{2V^2x}{b^2} \qquad a_x$$

(c) From Euler's equation in the x direction with $g_x = 0$

$$\frac{\partial p}{\partial x} = -\rho a_x = -\frac{2\rho V^2 x}{b^2} \qquad \frac{\partial p}{\partial x}$$

(d) $p - p_{atm} = \int_L^x \frac{\partial p}{\partial x}\, dx = \int_L^x -\frac{2\rho V^2}{b^2}x\, dx = -\frac{\rho V^2 x^2}{b^2}\Big]_L^x = \frac{\rho V^2 L^2}{b^2}\left[1 - \left(\frac{x}{L}\right)^2\right]$ $\qquad p(x)$

(e) $F_y = \int_A (p - p_{atm})\, dA = 2\int_0^L \frac{\rho V^2 L^2}{b^2}\left[1 - \left(\frac{x}{L}\right)^2\right] w\, dx$

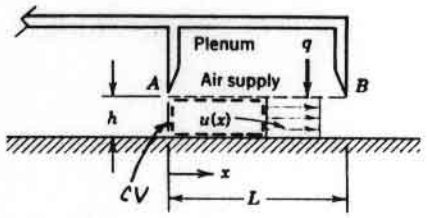
$$= 2\int_0^1 \frac{\rho V^2 L^3}{b^2}\left[1 - \left(\frac{x}{L}\right)^2\right] w\, d\left(\frac{x}{L}\right) = \frac{2\rho V^2 L^3 w}{b^2}\left[\left(\frac{x}{L}\right) - \frac{1}{3}\left(\frac{x}{L}\right)^3\right]_0^1$$

$$F_y = \frac{4\rho V^2 L^3 w}{3b^2} \quad \text{(upward, since } F_y > 0\text{)} \qquad F_y$$

Given: Load pallet supported by air:

Flow is incompressible, uniform, and frictionless; $h \ll L$.

No flow across plane at $x = 0$.



Find: (a) Use a suitable CV to show $u(x) = qx/h$ in the gap.
(b) Calculate the acceleration of a fluid particle in the gap.
(c) Evaluate the pressure gradient, $\partial p/\partial x$.
(d) Sketch the pressure distribution; indicate pressure at $x = L$.

Solution: Choose a CV in the gap, from 0 to x, as shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ (=0 (1))

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \qquad -\frac{\partial p}{\partial x} + \rho g_x = \rho a_x$$

(=0 (3)) (=0 (4)) (=0 (1)) (=0 (5))

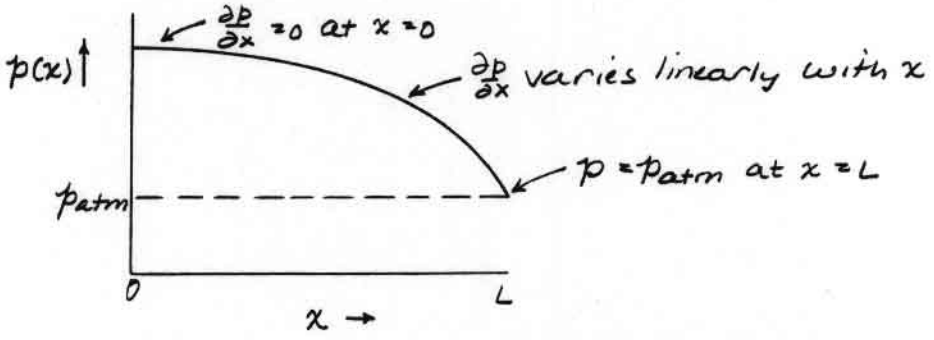
Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) No variation with z
(5) Horizontal, so $g_x = 0$

From continuity, $0 = \{-|\rho q w x|\} + \{+|\rho u(x)\, w h|\}$ so $u(x) = q\frac{x}{h}$ ← $u(x)$

The acceleration is $a_{p_x} = (q\frac{x}{h})(q\frac{1}{h}) = \frac{q^2 x}{h^2}$ ← a_{px}

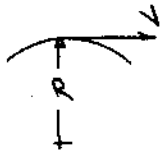
The pressure gradient is $\frac{\partial p}{\partial x} = -\rho a_{p_x} = -\frac{\rho q^2 x}{h^2}$ ← $\frac{\partial p}{\partial x}$

Sketching:



← Sketc

Given: Air at 20 psia, 100°F flows around a smooth corner
Velocity = 150 ft/s
Radius of curvature of streamline is 3 in.



Find: (a) magnitude of centripetal acceleration in G's
(b) pressure gradient, $\frac{\partial p}{\partial r}$

Solution:

Basic equations: $\rho \vec{g} - \nabla p = \rho \frac{d\vec{V}}{dt}$ ..... (1)

$\frac{d\vec{V}}{dt} = \vec{a}_p$ ..... (2) $\quad p = \rho RT$ ..... (3)

Assumptions: (1) ρ = constant
(2) frictionless flow
(3) $\vec{g} = -g\hat{k}$

Writing the r component of equation (1)

$$g_r^{\,0} - \frac{1}{\rho}\frac{\partial p}{\partial r} = a_r = \frac{\partial V_r}{\partial t}^{\,0} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta}^{\,0} + V_z\frac{\partial V_r}{\partial z}^{\,0} - \frac{V_\theta^2}{r}$$

$$a_r = -\frac{V_\theta^2}{r} \qquad a_r\big|_{@} = -\frac{V_\theta^2}{r}\Big|_{@} = -\frac{(150)^2\ ft^2}{s^2} \times \frac{1}{3\ in} \times \frac{12\ in}{ft} \times \frac{s^2}{32.2\ ft}$$

$$a_r\big|_{@} = -2800\ G's$$

Also

$$\frac{\partial p}{\partial r} = \rho \frac{V_\theta^2}{r}$$

where $\rho = \frac{p}{RT} = \frac{20\ lbf}{in^2} \times \frac{lbm\ °R}{53.3\ ft\text{-}lbf} \times \frac{1}{560°R} \times \frac{144\ in^2}{ft^2} \times \frac{slug}{32.2\ lbm}$

$$\rho = 0.003\ slug/ft^3$$

$$\frac{\partial p}{\partial r} = \rho \frac{V_\theta^2}{r} = 0.003\ \frac{slug}{ft^3} \times \frac{(150)^2\ ft^2}{s^2} \times \frac{1}{3\ in} \times \frac{12\ in}{ft} \times \frac{lbf\text{-}s^2}{ft\text{-}slug}$$

$$\frac{\partial p}{\partial r} = 270\ \frac{lbf/ft^2}{ft}$$

Given: The velocity field for steady, frictionless, incompressible flow (from right to left) over a stationary circular cylinder of radius, a, is given by

$$\vec{V} = U\left[\left(\frac{a}{r}\right)^2 - 1\right]\cos\theta\,\hat{e}_r + U\left[\left(\frac{a}{r}\right)^2 + 1\right]\sin\theta\,\hat{e}_\theta$$

Consider flow along the streamline forming the cylinder surface, i.e. $r = a$

Find: The pressure gradient along cylinder surface
Plot $V(r)$ along $\theta = \pi/2$ for $r > a$.

Solution:

Basic equation: $\rho\vec{g} - \nabla p = \rho\frac{d\vec{V}}{dt}$

Assumptions: (1) neglect body force

Along the surface, $r = a$, $\vec{V} = 2U\sin\theta\,\hat{e}_\theta$

Computing equations:

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = \cancel{\frac{\partial V_r}{\partial t}}^{\,0} + V_r\cancel{\frac{\partial V_r}{\partial r}}^{\,0} + \frac{V_\theta}{r}\cancel{\frac{\partial V_r}{\partial \theta}}^{\,0} + V_z\cancel{\frac{\partial V_r}{\partial z}}^{\,0} - \frac{V_\theta^2}{r}$$

$$-\frac{1}{\rho r}\frac{\partial p}{\partial \theta} = \cancel{\frac{\partial V_\theta}{\partial t}}^{\,0} + V_r\cancel{\frac{\partial V_\theta}{\partial r}}^{\,0} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\cancel{\frac{\partial V_\theta}{\partial z}}^{\,0} + \frac{V_r V_\theta}{r}^{\,0}$$

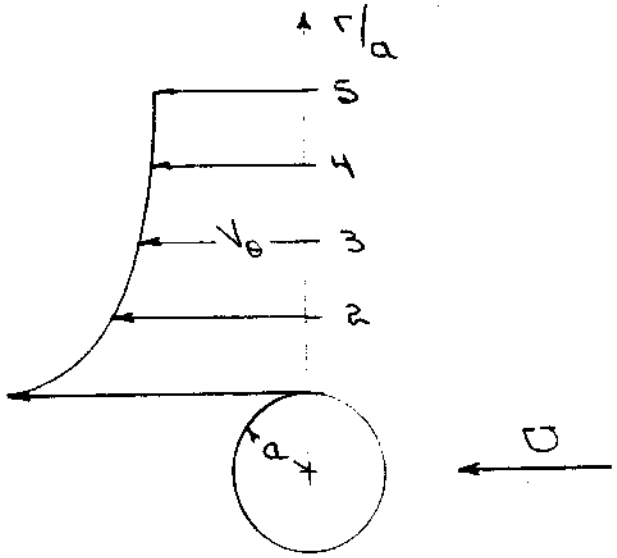
$$\frac{\partial p}{\partial r} = \rho\frac{V_\theta^2}{r} = \rho\frac{[2U\sin\theta]^2}{a} = \frac{4U^2\rho}{a}\sin^2\theta$$

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = -\rho\frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} = -\rho\frac{[2U\sin\theta]}{a}(2U\cos\theta) = -\frac{4U^2\rho}{a}\sin\theta\cos\theta$$

$$\nabla p = \hat{e}_r\frac{\partial p}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial p}{\partial \theta} = \frac{4\rho U^2}{a}\sin\theta\,(\hat{e}_r\sin\theta - \hat{e}_\theta\cos\theta)$$ ← ∇p

Along $\theta = \frac{\pi}{2}$, $\vec{V} = U\left[\left(\frac{a}{r}\right)^2 + 1\right]\hat{e}_\theta$

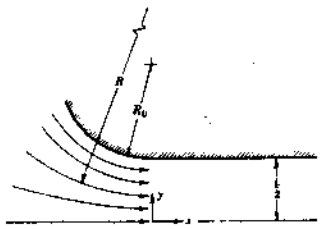
$\frac{r}{a}$	V_θ
1	$2U$
2	$1.25U$
3	$1.111U$
4	$1.063U$
5	$1.04U$



Given: Radius of curvature of streamlines at wind tunnel inlet is modeled as

$$R = \frac{L/2}{\pi} R_0$$

Speed along each streamline assumed constant at $V = 20\ m/s$; $L = 0.15\ m$, $R_0 = 0.6\ m$



Find: Δp between $y=0$ and tunnel wall ($y = L/2$)

Solution:

Basic equation: $\frac{\partial p}{\partial n} = \rho \frac{V^2}{R}$

Assumptions: (1) steady flow (2) frictionless flow
(3) neglect body forces
(4) constant speed along each streamline

At the inlet section, $p = p(y)$

$$\therefore \frac{dp}{dn} = -\frac{dp}{dy} = \rho \frac{V^2}{R} = \rho \frac{V^2}{\frac{L/2}{\pi} R_0}$$

$$\therefore dp = -\frac{\rho V^2}{R_0 L} 2\pi y\, dy$$

$$p_{L/2} - p_0 = \int_0^{L/2} dp = -\frac{2\rho V^2}{R_0 L} \int_0^{L/2} y\, dy = -\frac{2\rho V^2}{R_0 L} \left[\frac{y^2}{2}\right]_0^{L/2}$$

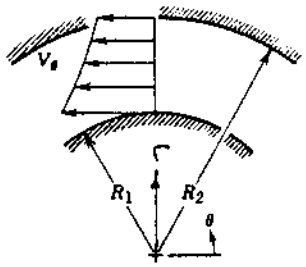
$$p_{L/2} - p_0 = -\frac{\rho V^2}{R_0 L} \frac{L^2}{4} = -\frac{\rho V L}{4 R_0}$$

$$p_{L/2} - p_0 = -1.225 \frac{kg}{m^3} \times \left(20 \frac{m}{s}\right)^2 \times 0.15\,m \times \frac{1}{4} \times \frac{1}{0.6\,m} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_{L/2} - p_0 = -30.6\ N/m^2$$

$p_{L/2} - p_0$

Given: Velocity variation at midsection of 180° bend is given by $rV_\theta = \text{constant}$. Cross section of the bend is square.



Find: Derive an equation for the pressure difference, $p_2 - p_1$. (Express the answer in terms of $\dot{m}$, ρ, R_1, R_2, and the depth of the bend, h.)

Solution:

Assumptions: (1) Frictionless flow (Euler's equations apply)
(2) ρ = constant
(3) $V_\theta = V_\theta(r)$ only
(4) streamlines are circular in the bend.

Apply Euler's "n" equation, $\frac{1}{\rho}\frac{\partial p}{\partial n} = \frac{V^2}{R}$

Then we can write

$$\frac{dp}{dr} = \rho\frac{V^2}{R} = \rho\frac{V_\theta^2}{r} \qquad \text{where } V_\theta = \frac{C}{r}$$

Separating variables, $dp = \rho\frac{V_\theta^2}{r}dr = \rho\frac{C^2}{r^3}dr$

$$p_2 - p_1 = \int_{p_1}^{p_2} dp = \rho C^2\int_{R_1}^{R_2}\frac{dr}{r^3} = \rho C^2\left(-\frac{1}{2}\right)\left[r^{-2}\right]_{R_1}^{R_2}$$

$$p_2 - p_1 = -\frac{1}{2}\rho C^2\left[\frac{1}{R_2^2} - \frac{1}{R_1^2}\right] = -\frac{1}{2}\rho C^2\left[\frac{R_1^2 - R_2^2}{R_1^2 R_2^2}\right]$$

$$p_2 - p_1 = \frac{1}{2}\rho C^2\frac{(R_2^2 - R_1^2)}{R_1^2 R_2^2}$$

The constant, C, can be written in terms of the mass flow rate, $\dot{m}$.

$$\dot{m} = \int\rho\vec{V}\cdot d\vec{A} = \int\rho V_\theta h\,dr = \rho h\int_{R_1}^{R_2}\frac{C}{r}dr = \rho h C\left[\ln r\right]_{R_1}^{R_2} = \rho h C \ln\frac{R_2}{R_1}$$

Solving for C, $C = \dfrac{\dot{m}}{\rho h \ln\frac{R_2}{R_1}}$

Substituting into the expression for $p_2 - p_1$,

$$p_2 - p_1 = \frac{1}{2}\rho\frac{\dot{m}^2}{\rho^2 h^2}\frac{1}{\left(\ln\frac{R_2}{R_1}\right)^2}\frac{[R_2^2 - R_1^2]}{R_1^2 R_2^2}$$

$$p_2 - p_1 = \frac{\dot{m}^2}{2\rho h^2}\frac{1}{\left(\ln\frac{R_2}{R_1}\right)^2}\frac{[R_2^2 - R_1^2]}{R_1^2 R_2^2}$$

$p_2 - p_1$

Given: Velocity field $\vec{V} = (Ax+B)\hat{i} - Ay\hat{j}$ where $A = 1\ s^{-1}$, $B = 2\ m/s$ and coordinates are measured in meters.

Show: that streamlines are given by $(x + B/A)y = \text{constant}$.
Plot: streamlines through points (x,y) = (1,1), (1,2), (2,2).

Find: (a) velocity vector + acceleration vector at (1,2); show these on streamline plot
(b) component of $\vec{a}_p$ along the streamline at (1,2); express as a vector.
(c) pressure gradient along streamline at (1,2) for air
(d) relative value of pressure at points (1,1) + (2,2).

Solution:

The slope of a streamline is $\left(\frac{dy}{dx}\right)_{s.l.} = \frac{v}{u} = \frac{-Ay}{Ax+B} = \frac{-y}{x+B/A}$

Then $\frac{dy}{y} + \frac{dx}{x+B/A} = 0$ and $\ln y + \ln(x + B/A) = \ln c$

and $(x + B/A)y = \text{constant}$ ← Streamlines

For (1,1) $(x+2)y = 3$
(1,2) $(x+2)y = 6$
(2,2) $(x+2)y = 8$
} these streamlines are shown in the plot at the end of the problem solution

The particle acceleration $\vec{a}_p = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y}$ (1) (2)

Assumptions: (1) steady flow (given)
(2) 2-D flow (given) $\vec{V} \neq \vec{V}(z)$.

$\vec{a}_p = (Ax+B)\frac{\partial}{\partial x}\left[(Ax+B)\hat{i} - Ay\hat{j}\right] - Ay\frac{\partial}{\partial y}\left[(Ax+B)\hat{i} - Ay\hat{j}\right]$

$\vec{a}_p = (Ax+B)A\hat{i} - Ay(-A\hat{j}) = A(Ax+B)\hat{i} + A^2y\hat{j}$

At point (1,2).

$\vec{a}_p = \frac{1}{s}\left(\frac{1}{s}\cdot 1m + 2\frac{m}{s}\right)\hat{i} + \frac{1}{s^2}\cdot 2m\,\hat{j} = 3\hat{i} + 2\hat{j}\ m/s^2$ ← $\vec{a}_{(1,2)}$

$\vec{V} = \left(\frac{1}{s}\cdot 1m + 2\frac{m}{s}\right)\hat{i} - \frac{1}{s}2m\,\hat{j} = 3\hat{i} - 2\hat{j}\ m/s$ ← $\vec{V}_{(1,2)}$

$\vec{V}$ and $\vec{a}$ are shown on the streamline plot

(b) The component of $\vec{a}_p$ along (tangent to) the streamline is given by $a_t = \vec{a}_p \cdot \hat{e}_t$ where $\hat{e}_t = \frac{\vec{V}}{|V|}$

Thus $\hat{e}_t = \frac{3\hat{i} - 2\hat{j}}{[3^2 + (-2)^2]^{1/2}} = 0.832\hat{i} - 0.555\hat{j}$

and

$a_t = \vec{a}_p \cdot \hat{e}_t = (3\hat{i}+2\hat{j})\ m/s^2 \cdot (0.832\hat{i} - 0.555\hat{j}) = 1.39\ m/s^2$

$\vec{a}_t = 1.39\,\hat{e}_t = 1.16\hat{i} - 0.771\hat{j}\ m/s^2$ $\vec{a}_{t(1,2)}$

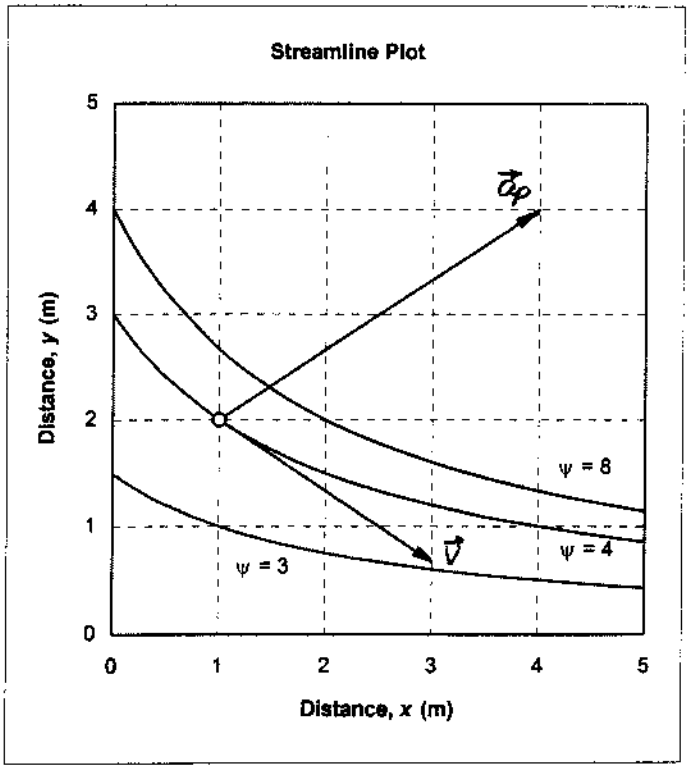
For frictionless flow, Euler's equation along a streamline (neglecting gravity, i.e. assuming flow is horizontal plane is

$$\frac{\partial p}{\partial s} = -\rho \frac{\partial V}{\partial s} = -\rho a_t = -1.23\ \frac{kg}{m^3} \times 1.39\ \frac{m}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$\frac{\partial p}{\partial s} = -1.71\ N/m^2/m$$ $\left(\frac{\partial p}{\partial s}\right)_{(1,2)}$

Looking at the streamline we would expect $p(2,2)$ to be less than $p(1,1)$ due to streamline curvature; Euler's equation normal to a streamline says

$\partial p/\partial n = \rho V^2/R$



Given: Velocity field $\vec{V} = (A/x)\hat{i} + (Ay/x^2)\hat{j}$, where $A = 7.25\ m^2/s$ and coordinates are measured in meters.

Show: that this field represents a possible incompressible flow.

Find: (a) equation of streamline through $(x,y) = (2,1)$
(b) expression for particle acceleration
(c) component of acceleration along the streamline at $(2,1)$

Solution:

For incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{A}{x}\right) + \frac{\partial}{\partial y}\left(\frac{Ay}{x^2}\right) = -\frac{A}{x^2} + \frac{A}{x^2} = 0$ ✓ $\therefore \rho = \text{constant}$

The slope of a streamline is given by $\left(\frac{dy}{dx}\right)_{s.l.} = \frac{v}{u} = \frac{y}{x}$

So $\frac{dy}{y} - \frac{dx}{x} = 0$ and $\ln y - \ln x = \ln c$

Thus $y/x = c$. Through $(2,1)$ $c = \frac{1}{2}$

$\therefore y = \frac{1}{2}x$ ← Streamline

The particle acceleration is given by

$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$

$\vec{a}_p = \frac{A}{x}\left[-\frac{A}{x^2}\hat{i} - \frac{A}{x^2}\hat{j}\right] + \frac{Ay}{x^2}\left[\frac{A}{x^2}\hat{j}\right] = -\frac{A^2}{x^3}\hat{i} - \frac{A^2 y}{x^4}\hat{j}$

$\vec{a}_p = -\frac{A^2}{x^3}\left[\hat{i} + \frac{y}{x}\hat{j}\right]$ ← $\vec{a}_p$

The component of $\vec{a}_p$ along a streamline is given by

$a_t = \vec{a} \cdot \hat{e}_t$ where $\hat{e}_t = \vec{V}/|\vec{V}|$

At point $(2,1)$

$\vec{V} = A\left[\frac{1}{x}\hat{i} + \frac{y}{x^2}\hat{j}\right] = 7.25\frac{m^2}{s}\left[\frac{1}{2}\frac{1}{m}\hat{i} + \frac{1}{(2)^2}\frac{1}{m^2}\hat{j}\right] = 3.625\hat{i} + 1.813\hat{j}\ m/s$

$\hat{e}_t = 0.894\hat{i} + 0.447\hat{j}$

$\vec{a}_p = -(7.25)^2\frac{1}{s^2}\frac{1}{m}\times\frac{1}{2^3}\frac{1}{m^3}\left[\hat{i} + \frac{1}{2}\hat{j}\right] = -6.57\hat{i} - 3.285\hat{j}$

$\vec{a}_t = (-6.57\hat{i} - 3.285\hat{j}) \cdot (0.894\hat{i} + 0.447\hat{j})$

$\vec{a}_t = -7.34\ m/s^2$ ← a_t

Note: Since the streamlines are straight there is no acceleration normal to the streamlines; $a_t = |\vec{a}_p|$

Problem 6.23

Given: Velocity field $\vec{V} = Axy\hat{\imath} + By^2\hat{\jmath}$; $A = 0.2\ m^{-1}\cdot s^{-1}$

B = constant

Find: (a) Value and units for B for incompressible flow.
(b) Acceleration of a fluid particle at point $(x, y) = (2, 1)$.
(c) Component of particle acceleration normal to velocity vector at this point.

Solution: Apply conservation of mass. For ρ = constant, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

But $w = 0$, so $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. For this field, $\frac{\partial u}{\partial x} = Ay$ and $\frac{\partial v}{\partial y} = 2By$

Thus $Ay + 2By = 0$, or $B = -\frac{A}{2} = -0.1\ m^{-1}\cdot s^{-1}$ ← B

Acceleration of a particle is given by (since $\vec{V} = \vec{V}(x, y)$ only),

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = (Axy)\,Ay\hat{\imath} + (By^2)(Ax\hat{\imath} + 2By\hat{\jmath})$$

$$\vec{a}_p = A^2xy^2\hat{\imath} - \frac{A^2}{2}xy^2\hat{\imath} + \frac{A^2}{2}y^3\hat{\jmath} = \frac{A^2}{2}\left(xy^2\hat{\imath} + y^3\hat{\jmath}\right)$$

At point $(x, y) = (2, 1)$ the acceleration is

$$\vec{a}_p = \frac{1}{2}\times(0.2)^2\frac{1}{m^2\cdot s^2}\left[2\,m\times(1)^2m^2\hat{\imath} + (1)^3m^3\hat{\jmath}\right] = 0.04\hat{\imath} + 0.02\hat{\jmath}\ \ m/s^2$$ ← $\vec{a}_p$ (1, 2)

The velocity at point $(x, y) = (2, 1)$ is

$$\vec{V} = \frac{0.2}{m\cdot s}\times 2\,m\times 1\,m\hat{\imath} - \frac{0.1}{m\cdot s}\times(1)^2m^2\hat{\jmath} = 0.40\hat{\imath} - 0.10\hat{\jmath}\ \ m/s$$

The unit vectors tangent and normal to the velocity vector are

$$\hat{e}_t = \frac{\vec{V}}{|\vec{V}|} = \frac{0.40\hat{\imath} - 0.10\hat{\jmath}}{[(0.40)^2 + (0.10)^2]^{1/2}} = \frac{0.40\hat{\imath} - 0.10\hat{\jmath}}{0.412} = 0.971\hat{\imath} - 0.243\hat{\jmath}$$

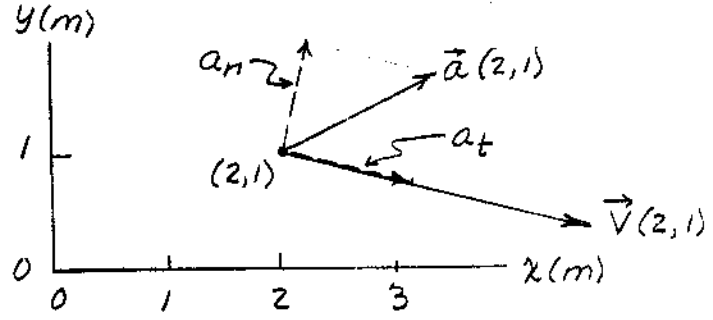
and

$$\hat{e}_n = 0.243\hat{\imath} + 0.971\hat{\jmath}$$

Thus $a_n = \vec{a}\cdot\vec{e}_n = (0.04\hat{\imath} + 0.02\hat{\jmath})\frac{m}{s^2}\cdot(0.243\hat{\imath} + 0.971\hat{\jmath})$

$$a_n = [0.04(0.243) + 0.02(0.971)]\frac{m}{s^2} = 0.0291\ m/s^2$$ ← a_n

Plotting:



Given: The y component of velocity in a 2-D, incompressible flow field is

$v = -Axy$ where $A = 1\ m^{-1}s^{-1}$ and coordinates are in meters; $w = 0$ and $\partial/\partial z = 0$.

Find: (a) acceleration of fluid particle at $(x,y) = (1,2)$
(b) radius of curvature of streamline at $(1,2)$.

Plot: streamline through $(1,2)$; show velocity and acceleration vectors on the plot.

Solution:

For 2-D incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, so $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$

$$u = \int \frac{\partial u}{\partial x}\,dx + f(y) = \int -\frac{\partial v}{\partial y}\,dx + f(y) = -\int (-Ax)\,dx + f(y) = \frac{Ax^2}{2} + f(y)$$

Choose the simplest solution, $f(y) = 0$, so $u = \frac{Ax^2}{2}$. Hence

$$\vec{V} = \frac{Ax^2}{2}\hat{i} - Axy\hat{j} = A\left(\frac{x^2}{2}\hat{i} - xy\hat{j}\right)$$

The acceleration of a fluid particle is

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = \frac{Ax^2}{2}(Ax\hat{i} - Ay\hat{j}) - Axy(-Ax\hat{j})$$

$$\vec{a}_p = \frac{A^2}{2}x^3\hat{i} + A^2x^2y\hat{j} = \frac{A^2}{2}(x^3\hat{i} + 2x^2y\hat{j})$$

At the point $(1,2)$

$$\vec{a}_p = \frac{1}{2}\times(1)\ \frac{1}{m^2s^2}\left[(1)^3 m^3\hat{i} + (1)^2(2)\,m^3\hat{j}\right] = 0.5\hat{i} + 2\hat{j}\ m/s^2 \quad \leftarrow \vec{a}(1,2)$$

$$\vec{V} = \frac{1}{m\cdot s}\left[\frac{1}{2}(1)^2 m^2\hat{i} - (1)(2)\,m^2\hat{j}\right] = 0.5\hat{i} - 2\hat{j}\ m/s$$

The unit vector tangent to the streamline is

$$\hat{e}_t = \frac{\vec{V}}{|\vec{V}|} = \frac{0.5\hat{i} - 2\hat{j}}{[(0.5)^2 + (-2)^2]^{1/2}} = 0.243\hat{i} - 0.970\hat{j}$$

The unit vector normal to the streamline is

$$\hat{e}_n = \hat{e}_t \times \hat{k} = (0.243\hat{i} - 0.970\hat{j}) \times \hat{k} = -0.970\hat{i} - 0.243\hat{j}$$

The normal component of acceleration is

$$a_n = -\frac{V^2}{R} = \vec{a}\cdot\hat{e}_n = (0.5\hat{i} + 2\hat{j})\cdot(-0.970\hat{i} - 0.243\hat{j})$$

$$-\frac{V^2}{R} = -0.728\ m/s^2$$

$$R = \frac{V^2}{0.728} = \frac{4.25}{0.728}\ \frac{m^2/s^2}{m/s^2} = 5.84\ m \quad \leftarrow R(1,2)$$

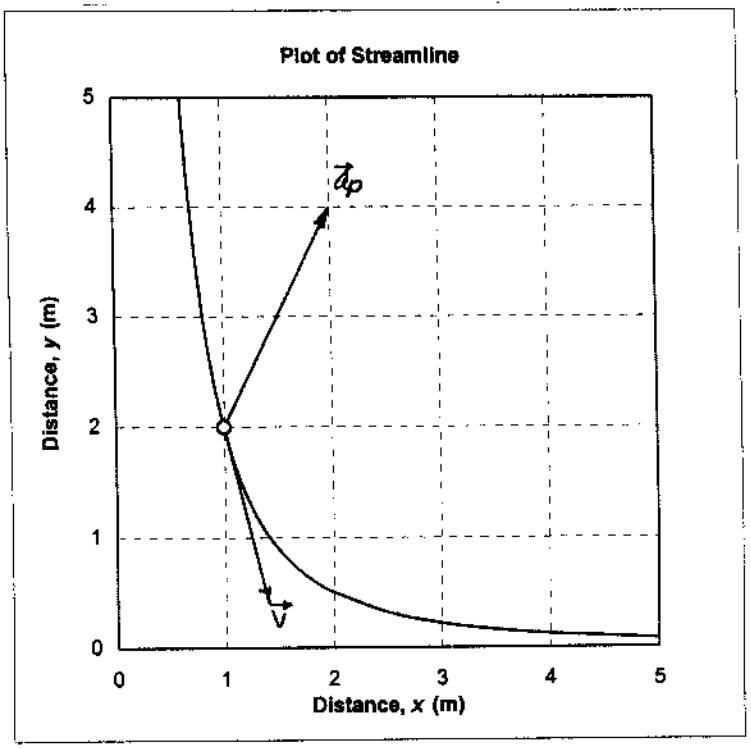
The slope of the streamlines is given by

$$\left(\frac{dy}{dx}\right)_{s\ell} = \frac{v}{u} = \frac{-Axy}{Ax^2/2} = -\frac{2y}{x}$$

Thus $\frac{dy}{y} + 2\frac{dx}{x} = 0$ and $\ln y + \ln x^2 = \ln c$ or

$$x^2 y = c$$

The equation of the streamline through (1,2) is $x^2 y = 2$



13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-388 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: The x component of velocity in a 2-D, incompressible flow field is
$u = Ax^2$ where $A = 1\ ft^{-1}\cdot s^{-1}$ and coordinates are in ft; $w = 0$ and $\partial/\partial z = 0$.

Find: (a) acceleration of fluid particle at $(x,y) = (1,2)$
(b) radius of curvature of streamline at (1,2)

Plot: streamline through (1,2); show velocity and acceleration vectors on the plot.

Solution:

For 2-D incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, so $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$

$$v = \int \frac{\partial v}{\partial y}\,dy + f(x) = \int -\frac{\partial u}{\partial x}\,dy + f(x) = -\int 2Ax\,dy + f(x) = -2Axy + f(x)$$

Choose the simplest solution, $f(x) = 0$, so $v = -2Axy$. Hence

$$\vec{V} = Ax^2\hat{i} - 2Axy\hat{j} = A\left[x^2\hat{i} - 2xy\hat{j}\right]$$

The acceleration of a fluid particle is

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = Ax^2\left[A\left(2x\hat{i} - 2y\hat{j}\right)\right] - 2Axy\left[-2Ax\hat{j}\right]$$

$$\vec{a}_p = 2A^2x^3\hat{i} + 2A^2x^2y\hat{j} = 2A^2x^2\left[x\hat{i} + y\hat{j}\right]$$

At the point (1,2)

$$\vec{a}_p = 2\times\frac{(1)^2}{ft^2 s^2}\times(1)^2\left[1\,ft\,\hat{i} + 2\,ft\,\hat{j}\right] = 2\hat{i} + 4\hat{j}\ ft/s^2 \quad \longleftarrow \vec{a}_{(1,2)}$$

$$\vec{V} = \frac{1}{ft/s}\left[(1)^2(1\,ft)^2\hat{i} - 2(1\,ft)(2\,ft)\hat{j}\right] = \hat{i} - 4\hat{j}\ ft/s$$

The unit vector tangent to the streamline is

$$\hat{e}_t = \frac{\vec{V}}{|\vec{V}|} = \frac{\hat{i} - 4\hat{j}}{[(1)^2 + (-4)^2]^{1/2}} = 0.243\hat{i} - 0.970\hat{j}$$

The unit vector normal to the streamline is

$$\hat{e}_n = \hat{e}_t \times \hat{k} = (0.243\hat{i} - 0.970\hat{j}) \times \hat{k} = -0.970\hat{i} - 0.243\hat{j}$$

The normal component of acceleration is

$$a_n = -\frac{V^2}{R} = \vec{a}\cdot\hat{e}_n = (2\hat{i} + 4\hat{j})\cdot(-0.970\hat{i} - 0.243\hat{j})$$

$$-\frac{V^2}{R} = -2.91\ ft/s^2$$

$$R = \frac{V^2}{2.91} = \frac{17\ ft^2/s^2}{2.91\ ft/s^2} = 5.84\ ft \quad \longleftarrow R$$

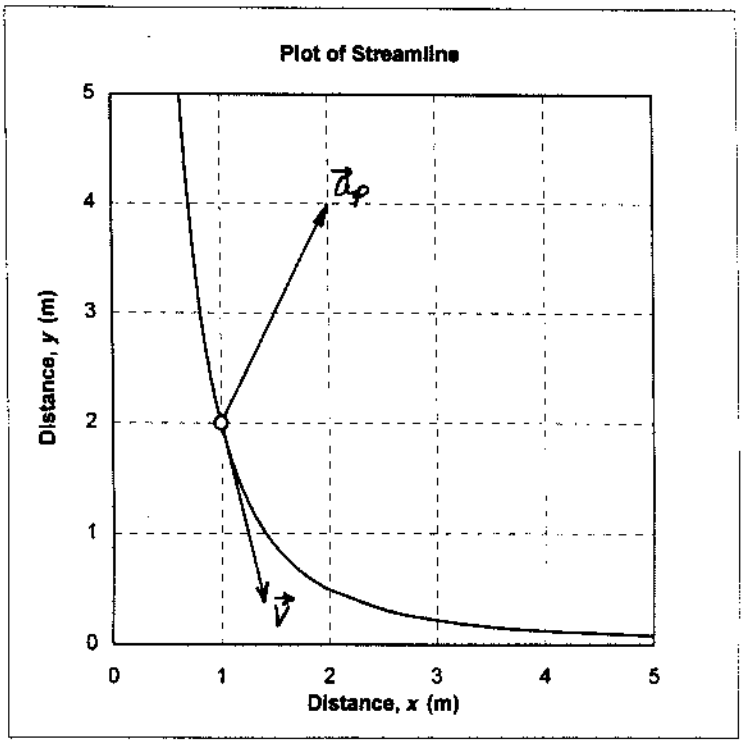
The slope of the streamline is given by

$$\left.\frac{dy}{dx}\right|_{s.l.} = \frac{v}{u} = \frac{-2Axy}{Ax^2} = \frac{-2y}{x}$$

Thus $\frac{dy}{y} + \frac{2\,dt}{t} = 0$ and $\ln y + \ln t^2 = \ln c$ or

$t^2 y = c$

The equation of the streamline through (1, 2) is $t^2 y = 2$.



13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-992 100 RECYCLED WHITE 5 SQUARE
42-999 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Problem 6.26

Given: Incompressible, 2-D flow with $u = Axy$, $w = 0$; $A = 2\ ft^{-1}s^{-1}$

Find: (a) Acceleration of particle at $(x,y) = (2,1)$.
(b) Radius of curvature of streamline at that point.
(c) Plot streamline; show velocity vector and acceleration vector.

Solution: For two-d. incompressible flow, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, so

$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -Ay$; Integrating, $v = -\frac{1}{2}Ay^2$; $\vec{V} = Axy\hat{\imath} - \frac{1}{2}Ay^2\hat{\jmath}$.

The acceleration is

$$a_{p_x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = (Axy)(Ay) + (-\tfrac{1}{2}Ay^2)(Ax) = \tfrac{1}{2}A^2xy^2$$

$$a_{p_y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = (Axy)(0) + (-\tfrac{1}{2}Ay^2)(-Ay) = \tfrac{1}{2}A^2y^3$$

$$\vec{a}_p = \tfrac{1}{2}A^2xy^2\hat{\imath} + \tfrac{1}{2}A^2y^3\hat{\jmath}\ ;\ \text{at } (2,1)\ \ \vec{a}_p = 4\hat{\imath} + 2\hat{\jmath}\ (ft/s^2)$$ $\vec{a}_p$

Note $a_n = \frac{V^2}{R}$, so $R = \frac{V^2}{a_n}$, where a_n is acceleration normal to $\vec{V}$

At $(2,1)$, $\vec{V} = 4\hat{\imath} - 1\hat{\jmath}$ ft/s, so $V^2 = (4)^2 + (1)^2 = 17\ ft^2/s^2$

To find a_n, dot $\vec{a}_p$ with $\hat{e}_n$, the unit normal vector. To find $\hat{e}_n$, set

$$\hat{e}_n = -\frac{v}{V}\hat{\imath} + \frac{u}{V}\hat{\jmath} = \frac{1}{\sqrt{17}}\hat{\imath} + \frac{4}{\sqrt{17}}\hat{\jmath}$$

$$a_n = \hat{e}_n \cdot \hat{a}_p = \frac{4}{\sqrt{17}} + \frac{8}{\sqrt{17}} = \frac{12}{\sqrt{17}} = 2.91\ ft/s^2$$

Substituting

$$R = \frac{V^2}{a_n} = 17\ \frac{ft^2}{s^2} \times \frac{s^2}{2.91\ ft} = 5.84\ ft$$ R

The streamline is $\frac{dx}{u} = \frac{dy}{v} = \frac{dx}{Axy} = \frac{dy}{-\frac{1}{2}Ay^2}$ or $\frac{dx}{x} + 2\frac{dy}{y} = 0$

Integrating, $\ln x + 2\ln y = \ln C$ or $xy^2 = C$

For $(x,y) = (2,1)$, then $C = 2\ ft^3$

The plot and streamlines are on the following page.

Components of Velocity and Acceleration:

Input Parameters:

A = 2 $ft^{-1}s^{-1}$

Calculated Values:

C = 2 ft^3

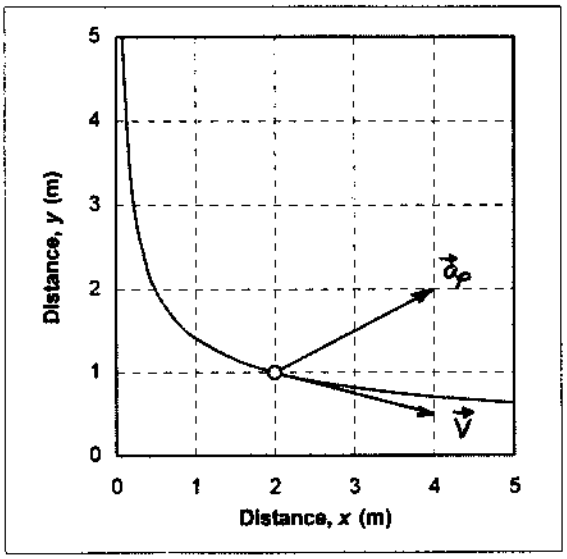
Coord. x	Coord. y	Velocity, V_x	Velocity, V_y	Velocity, V	Accel., a_x	Accel., a_y	Accel., a	Normal Accel., a_n
0.08	5.00							
0.2	3.16							
0.4	2.24							
0.5	2.00	2.00	-4.00	4.47	2.00	16.0	16.1	8.94
0.6	1.83							
0.8	1.58							
1.0	1.41	2.83	-2.00	3.46	2.83	5.66	6.32	6.25
1.5	1.15	3.46	-1.33	3.71	3.46	3.08	4.63	4.12
2.0	1.00	4.00	-1.00	4.12	4.00	2.00	4.47	2.91
2.5	0.89	4.47	-0.80	4.54	4.47	1.43	4.70	2.20
3.0	0.82	4.90	-0.67	4.94	4.90	1.09	5.02	1.74
3.5	0.76	5.29	-0.57	5.32	5.29	0.86	5.36	1.43
4.0	0.71	5.66	-0.50	5.68	5.66	0.71	5.70	1.20
4.5	0.67	6.00	-0.44	6.02	6.00	0.59	6.03	1.03
5.0	0.63	6.32	-0.40	6.34	6.32	0.51	6.34	0.90

Acceleration:

2	1
4	2

Velocity:

2	1
4	0.5



Given: The y component of velocity in a 2-D, incompressible flowfield is

$v = Ay^2$ where $A = 1\ m^{-1}\cdot s^{-1}$ and coordinates are in meters; $w = 0$ and $\partial/\partial z = 0$

Find: (a) acceleration of fluid particle at $(x,y) = (2,1)$
(b) radius of curvature of streamline at $(2,1)$

Plot: streamline through $(2,1)$; show velocity and acceleration vectors on the plot.

Solution:

For 2-D incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, so $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$

$$u = \int \frac{\partial u}{\partial x}\,dx + f(y) = \int -\frac{\partial v}{\partial y}\,dx + f(y) = \int -2Ay\,dx + f(y) = -2Axy + f(y)$$

Choose the simplest solution, $f(y) = 0$, so $u = -2Axy$. Hence

$$\vec{V} = -2Axy\,\hat{i} + Ay^2\,\hat{j} = A\left[-2xy\,\hat{i} + y^2\,\hat{j}\right]$$

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = -2Axy\left[-2Ay\,\hat{i}\right] + Ay^2\left[-2Ax\,\hat{i} + 2Ay\,\hat{j}\right]$$

$$\vec{a}_p = 2A^2xy^2\,\hat{i} + 2A^2y^3\,\hat{j} = 2Ay^2\left[x\,\hat{i} + y\,\hat{j}\right]$$

At the point $(2,1)$

$$\vec{a}_p = 2\times\frac{1}{m\cdot s}\times(1)^2\,m^2\left[2m\,\hat{i} + 1m\,\hat{j}\right] = 4\,\hat{i} + 2\,\hat{j}\ m/s^2 \qquad \vec{a}(2,1)$$

$$\vec{V} = \frac{1}{m\cdot s}\left[-2(2m)(1m)\,\hat{i} + (1)^2\,m^2\,\hat{j}\right] = -4\,\hat{i} + \hat{j}\ m/s$$

The unit vector tangent to the streamline is

$$\hat{e}_t = \frac{\vec{V}}{|\vec{V}|} = \frac{-4\,\hat{i} + \hat{j}}{\left[(-4)^2 + (1)^2\right]^{1/2}} = -0.970\,\hat{i} + 0.243\,\hat{j}$$

The unit vector normal to the streamline is

$$\hat{e}_n = \hat{k}\times\hat{e}_t = \hat{k}\times(-0.970\,\hat{i} + 0.243\,\hat{j}) = -0.243\,\hat{i} - 0.970\,\hat{j}$$

The normal component of acceleration is

$$a_n = -\frac{V^2}{R} = \vec{a}\cdot\hat{e}_n = (4\,\hat{i} + 2\,\hat{j})\cdot(-0.243\,\hat{i} - 0.970\,\hat{j})$$

$$-\frac{V^2}{R} = -2.91\ m/s^2$$

$$R = \frac{V^2}{2.91} = \frac{17\ m^2/s^2}{2.91\ m/s^2} = 5.84\ m \qquad R(1,2)$$

The slope of the streamline is given by

$$\left(\frac{dy}{dx}\right)_{s.l.} = \frac{v}{u} = \frac{Ay^2}{-2Axy} = -\frac{y}{2x}$$

Thus, $2\frac{dy}{y} + \frac{dx}{x} = 0$ and $\ln y^2 + \ln x = \ln C$ or

$xy^2 = C$

The equation of the streamline through (2,1) is $xy^2 = 2$

Components of Velocity and Acceleration:

Input Parameters:

$A =$ 2 $ft^{-1}s^{-1}$

Calculated Values:

$C =$ 2 ft^3

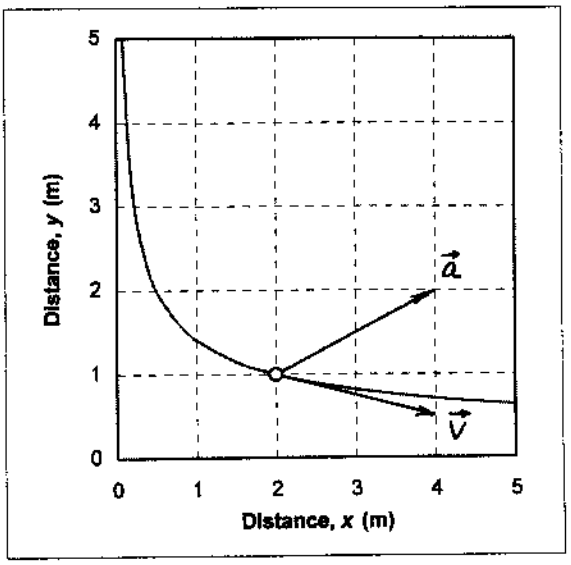
Coord. x	Coord. y	Velocity, V_x	Velocity, V_y	Velocity, V	Accel., a_x	Accel., a_y	Accel., a	Normal Accel., a_n
0.08	5.00							
0.2	3.16							
0.4	2.24							
0.5	2.00	2.00	-4.00	4.47	2.00	16.0	16.1	8.94
0.6	1.83							
0.8	1.58							
1.0	1.41	2.83	-2.00	3.46	2.83	5.66	6.32	6.25
1.5	1.15	3.46	-1.33	3.71	3.46	3.08	4.63	4.12
2.0	1.00	4.00	-1.00	4.12	4.00	2.00	4.47	2.91
2.5	0.89	4.47	-0.80	4.54	4.47	1.43	4.70	2.20
3.0	0.82	4.90	-0.67	4.94	4.90	1.09	5.02	1.74
3.5	0.76	5.29	-0.57	5.32	5.29	0.86	5.36	1.43
4.0	0.71	5.66	-0.50	5.68	5.66	0.71	5.70	1.20
4.5	0.67	6.00	-0.44	6.02	6.00	0.59	6.03	1.03
5.0	0.63	6.32	-0.40	6.34	6.32	0.51	6.34	0.90

Acceleration:

2	1
4	2

Velocity:

2	1
4	0.5



Given: Velocity field $\vec{V} = Ax\hat{i} - By\hat{j}$, where $A = 2\,m^{-1}\cdot s^{-1}$, $B = 4\,m^{-1}\cdot s^{-1}$ and coordinates are measured in meters

Show: that this is a possible incompressible flow

Find: (a) equation of streamline through point $(x,y) = (1,2)$
(b) expression for the acceleration of a fluid particle.
(c) radius of curvature of streamline at (1,2)

Solution:

For 2-D incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

For this flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2Ax - By = 2(2)x - 4x = 0 \quad \therefore \rho = \text{const.}$

The slope of the streamline is given by

$$\left(\frac{dy}{dx}\right)_{s.l.} = \frac{v}{u} = -\frac{By}{Ax^2} = -\frac{By}{Ax} = -\frac{4y}{2x} = -\frac{2y}{x}$$

Thus $\frac{dy}{y} + 2\frac{dx}{x} = 0$ and $\ln y + \ln x^2 = \ln c$ or $x^2 y = c$

The streamline through point (1,2) is $x^2 y = 2$ ← streamline

The acceleration of a fluid particle is

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = Ax\left[2Ax\hat{i} - By\hat{j}\right] - By\left[-Bx\hat{j}\right]$$

$$\vec{a}_p = 2A^2x^3\hat{i} + \left[B^2x^2y - ABx^2y\right]\hat{j} = 2A^2x^3\hat{i} + Bx^2y(B-A)\hat{j}$$ ← $\vec{a}_p$

At the point (1,2)

$$\vec{a}_p = 2\times\frac{(2)^2}{m^2/s^2}\times(1)^3m^3\hat{i} + \frac{4}{m/s}\times(1)^2m^2\times 2m\left[4-2\right]\frac{1}{m\cdot s}\hat{j} = 8\hat{i} + 16\hat{j}\ m/s^2$$

$$\vec{V} = \frac{2}{m/s}\times(1)m^2\hat{i} - \frac{4}{m/s}\times(1m)\times(2m)\hat{j} = 2\hat{i} - 8\hat{j}\ m/s$$

The unit vector tangent to the streamline is

$$\hat{e}_t = \frac{\vec{V}}{|\vec{V}|} = \frac{2\hat{i} - 8\hat{j}}{[(2)^2 + (-8)^2]^{1/2}} = 0.243\hat{i} - 0.970\hat{j}$$

The unit vector normal to the streamline is

$$\hat{e}_n = \hat{e}_t \times \hat{k} = (0.243\hat{i} - 0.970\hat{j}) \times \hat{k} = -0.970\hat{i} - 0.243\hat{j}$$

The normal component of acceleration is

$$a_n = \vec{a}\cdot\hat{e}_n = (8\hat{i} + 16\hat{j})\cdot(-0.970\hat{i} - 0.243\hat{j}) = -11.6\ m/s^2$$

$$a_n = -\frac{V^2}{R} = -11.6 \quad \therefore \quad R = \frac{V^2}{11.6} = \frac{68\ m^2/s^2}{11.6\ m/s^2}$$

$$R = 5.86\ m$$ ← R(1,2)

Given: Flow of water with speed $V = 3\ m/s$

Find: Dynamic pressure, expressed in mm of mercury.

Solution:

Dynamic pressure is $p_d = \frac{1}{2}\rho V^2$

From hydrostatics, $p_d = \rho_{Hg} g \Delta h$

$$\therefore \Delta h = \frac{\rho V^2}{2\rho_{Hg} g} = \frac{V^2}{2 SG_{Hg} g}$$

$$= \frac{1}{2} \times \frac{(3)^2\ m^2/s^2}{} \times \frac{1}{13.6} \times \frac{9.81\ s^2/m}{} \times \frac{1000\ mm}{m}$$

$$\Delta h = 33.7\ mm\ Hg$$

Δh

Given: Standard air

Find: Dynamic pressure that corresponds to V = 100 km/hr

Solution: Dynamic pressure is $p_{dyn} = \frac{1}{2}\rho V^2$

For standard air, $\rho = 1.23\ kg/m^3$

Then $p_{dyn} = \frac{1}{2} \times 1.23 \frac{kg}{m^3} \times (100)^2 \frac{(km)^2}{(hr)^2} \times (1000)^2 \frac{m^2}{(km)^2} \times \frac{(hr)^2}{(3600)^2 s^2} \times \frac{N \cdot s^2}{kg \cdot m}$

$p_{dyn} = 475\ N/m^2$ ← p_{dyn}

This may be expressed conveniently as a water column height.

$p_{dyn} = \rho_{water} g h_{dyn}$

$h_{dyn} = \frac{p_{dyn}}{\rho_w g} = 475 \frac{N}{m^2} \times \frac{m^3}{999\ kg} \times \frac{s^2}{9.81\ m} \times \frac{kg \cdot m}{N \cdot s^2}$

$h_{dyn} = 0.0484\ m$ or $48.4\ mm$ ← h_{dyn}

50 SHEETS EYE-EASE® 5 SQUARE
100 SHEETS EYE-EASE® 5 SQUARE
200 SHEETS EYE-EASE® 5 SQUARE
RECYCLED WHITE 5 SQUARE
42-381
42-382
42-389
42-392
Made in U.S.A.
National® Brand

Given: Air at $p = 99.1$ kPa, $T = 23°C$, is to be accelerated to create an air speed of $V = 25$ m/s.

Find: minimum reduction in static pressure (in mm water) required

Solution:

The decrease in pressure $p_0 - p = \frac{1}{2}\rho_{air} V^2$

From hydrostatics $\Delta p = \rho_{H_2O} g \Delta h$

$$\therefore \Delta h = \frac{1}{2}\frac{\rho_{air} V^2}{\rho_{H_2O} g}$$

$$\rho_{air} = \frac{p}{RT} = 99.1 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287 N \cdot m} \times \frac{1}{296 K} = 1.17 \ kg/m^3$$

$$\Delta h = \frac{1}{2} \times \frac{1.17 \ kg/m^3}{999 \ kg/m^3} \times (25)^2 \frac{m^2}{s^2} \times 9.81 \frac{s^2}{m} \times 10^3 \frac{mm}{m}$$

$$\Delta h = 37.3 \ mm \ H_2O$$

Δh

Given: Pitot static probe is used to measure speed in standard air.

$V = 100$ m/s

Find: Manometer deflection in mm H_2O, corresponding to given conditions.

Solution:

Manometer reads $P_0 - P$ in mm of H_2O.

Basic equations: $\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant}$ for flow

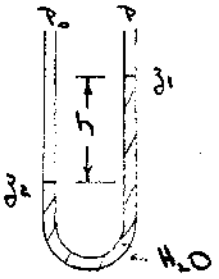
$\frac{dP}{dz} = -\rho g$ for manometer

Assumptions: (1) steady flow
(2) incompressible flow
(3) flow along a streamline
(4) frictionless deceleration to P_0
(5) ρ = constant for manometer

From the Bernoulli equation

$$\frac{P_0}{\rho} = \frac{P}{\rho} + \frac{V^2}{2}$$

$$P_0 - P = \frac{\rho V^2}{2}$$



For the manometer, $dP = -\rho g\, dz$

$$P_0 - P = \int_P^{P_0} dP = -\rho g (z_2 - z_1) = \rho g h$$

Then,

$$\rho_{H_2O} g h = \rho_{air} \frac{V^2}{2}$$

and

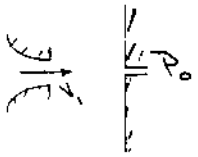
$$h = \frac{\rho_{air}}{\rho_{H_2O}} \frac{V^2}{2g} = \frac{1.23}{999} \times (100)^2 \frac{m^2}{s^2} \times \frac{1}{2} \times 9.81 \frac{s^2}{m} \times \frac{10^3\ mm}{m} = 628\ mm$$

h

Given: Air discharging from a nozzle impinges on a wall as shown

$P_1 = 14.7$ psia $T_1 = 40°F$

$P_0 = 0.14$ in Hg gage



Find: the speed, V_1

Solution:

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$ for flow

$\frac{dP}{dh} = \gamma$ for manometer reading P_0

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) γ = constant for manometer
(6) air behaves as an ideal gas

From the Bernoulli equation

$$\frac{P_0}{\rho} = \frac{P_1}{\rho} + \frac{V_1^2}{2}$$

$$P_0 - P_1 = \frac{1}{2}\rho V_1^2$$

For the manometer $dP = \gamma\, dh$

$\therefore P_0 - P_{atm} = \gamma \Delta h$ where $\Delta h = 0.14$ in Hg

Since $P_1 = P_{atm}$

$$P_0 - P_1 = \gamma \Delta h$$

$$\therefore \gamma \Delta h = \frac{1}{2}\rho V_1^2 \quad \text{and} \quad V_1 = \sqrt{\frac{2\gamma \Delta h}{\rho}}$$

where

$$\rho = \frac{P}{RT} = 14.7 \frac{\text{lbf}}{\text{in}^2} \times \frac{144 \text{ in}^2}{\text{ft}^2} \times \frac{\text{lbm}\cdot°R}{53.3 \text{ ft}\cdot\text{lbf}} \times \frac{1}{500°R} \times \frac{\text{slug}}{32.2 \text{ lbm}} = 0.00247 \frac{\text{slug}}{\text{ft}^3}$$

$$V_1 = \sqrt{\frac{2\gamma \Delta h}{\rho}}$$

$$= \left[2 \times 13.6 \times 62.4 \frac{\text{lbf}}{\text{ft}^3} \times 0.14 \text{ in} \times \frac{\text{ft}}{12 \text{ in}} \times \frac{\text{ft}^3}{0.00247 \text{ slug}} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}\right]^{1/2}$$

$$V_1 = 89.5 \text{ ft/s}$$

Given: High-pressure hydraulic system subject to small leak.

Plot: Jet speed of a leak vs system pressure for system pressures up to 40 MPa gage; explain how a high-speed jet of hydraulic fluid can cause injury

Solution:

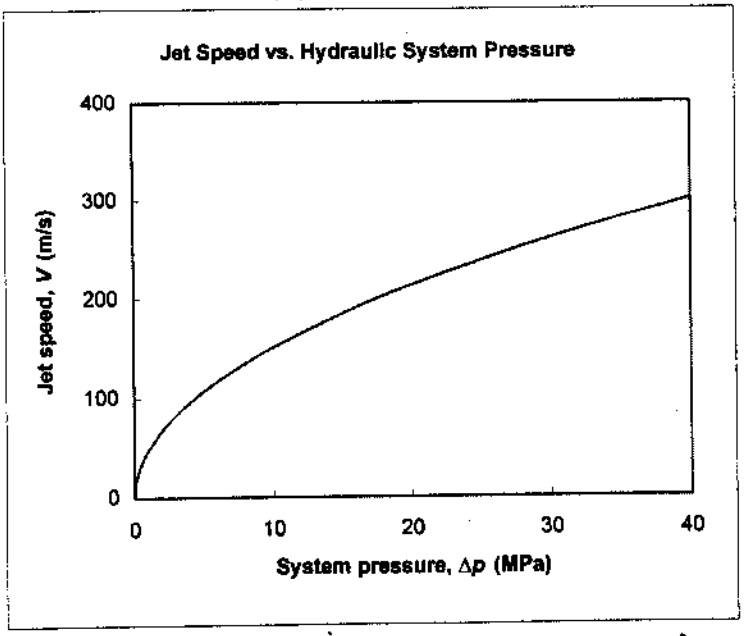
Basic equation: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline

The Bernoulli equation gives

$$V = \left[\frac{2(p_0 - p_{atm})}{\rho} \right]^{1/2}$$

From Table A.2 (Appendix A) for lubricating oil SG = 0.88



The high stagnation pressure ruptures the skin causing the jet to penetrate the tissue

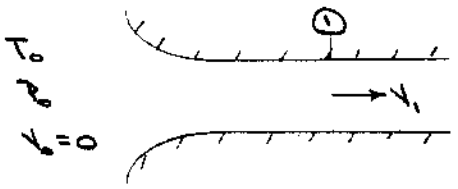
Given: Air flow in open circuit wind tunnel as shown.

$p_{atm} - p_1 = 45\ mm\ H_2O$

$T_o = 25C$

$p_o = p_{atm}$

Consider air to be incompressible.



Find: Air speed in tunnel at section (1)

Solution:

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) air behaves as an ideal gas
(6) stagnation pressure = p_{atm}

From the Bernoulli equation, $\frac{p_o}{\rho} = \frac{p_1}{\rho} + \frac{V_1^2}{2}$

$$p_o - p_1 = p_{atm} - p_1 = \frac{1}{2}\rho V_1^2$$

$$V_1 = \left[\frac{2(p_{atm} - p_1)}{\rho}\right]^{1/2}$$

From the manometer reading $p_{atm} - p_1 = \rho_{H_2O} g h$ then

$$V_1 = \left[\frac{2\rho_{H_2O} g h}{\rho}\right]^{1/2}$$

From the ideal gas equation of state

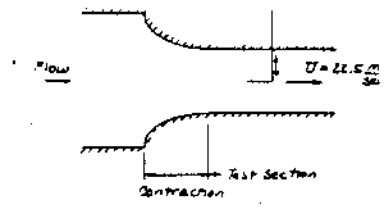
$$\rho = \frac{p}{RT} = 100 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287\ N \cdot m} \times \frac{1}{298\ K} = 1.17\ kg/m^3$$

$$V_1 = \left[\frac{2\rho_{H_2O}}{\rho} g h\right]^{1/2} = \left[2 \times \frac{999}{1.17} \times 9.81\ \frac{m}{s^2} \times 0.045\ m\right]^{1/2} = 27.5\ m/s \longleftarrow V_1$$

Given: Wind tunnel with inlet and test section as shown.

$U = 22.5$ m/s , $p_{o_b} = -6.0$ mm H_2O gage

$p_a = 99.1$ kPa(abs) , $T_a = 25C$



Find: (a) $p_{dynamic}$ on tunnel centerline
(b) p_{static} " " "
(c) compare p_{static} at tunnel wall with that measured at centerline

Solution:

(a) By definition $p_{dyn} = \frac{1}{2}\rho U^2$

Assume: (1) air behaves as an ideal gas, and (2) incompressible flow

Then $\rho = \frac{p}{RT} = 99.1\times10^3 \frac{N}{m^2} \times \frac{kg\,K}{287\,N\cdot m} \times \frac{1}{(273+25)K} = 1.17\ kg/m^3$

and $p_{dyn} = \frac{1}{2}\rho U^2 = \frac{1}{2} \times 1.17 \frac{kg}{m^3} \times (22.5)^2 \frac{m^2}{s^2} \cdot \frac{N\cdot s^2}{kg\cdot m} = 296\ N/m^2$ ← p_{dyn}

(b) By definition $p_o = p_s + p_{dyn}$

$\therefore p_s = p_o - p_{dyn}$ where $p_o = -6$ mm H_2O gage

Then $p_o - p_a = \rho g \Delta h = 999 \frac{kg}{m^3} \cdot 9.81 \frac{m}{s^2} \cdot -6\times10^{-3} m \cdot \frac{N\cdot s^2}{kg\cdot m}$

$p_{o_{gage}} = -58.8\ N/m^2$

$\therefore p_s = p_o - p_{dyn} = -58.8 - 296 = -355\ N/m^2$ gage ← p_{static}

{ or $p_s = -36.2$ mm H_2O (gage) }

(c) Streamlines in the test section should be straight. Then in the test section the variation of static pressure is given by $\frac{\partial p}{\partial n} = 0$ and $p_{wall} = p_{centerline}$ ←

In the contraction section the streamlines are curved. The variation of static pressure normal to the streamlines is given by $\frac{\partial p}{\partial n} = \rho \frac{V^2}{R}$

and consequently the static pressure increases toward the centerline, i.e. $p_{wall} < p_{centerline}$

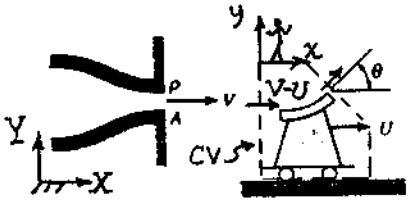
Given: Wheeled cart of Problem 4.106:

$V = 40$ m/s

$A = 25$ mm^2

Water, no friction on vane, $\theta = 120°$

Vane accelerates to the right



Find: At instant when $U = 15$ m/s,

(a) Stagnation pressure leaving nozzle, relative to fixed observer.

(b) Stagnation pressure leaving nozzle, relative to observer on vane.

(c) Absolute velocity of jet leaving vane.

(d) Stagnation pressure of jet leaving vane, relative to fixed observer.

(e) How would viscous forces increase, decrease, or leave unchanged the stagnation pressure in (d). How can you justify this?

Solution: Stagnation pressure is $p_0 = p + \frac{1}{2}\rho V^2$ or $p_0 - p = \frac{1}{2}\rho V^2$

At jet, $p_{0j} = \frac{1}{2}\rho V^2 = \frac{1}{2} \times 999 \frac{kg}{m^3} \times (40)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 799$ kPa (gage) $\quad p_{0j}$

At cart, $p_{0,rel} = \frac{1}{2}\rho (V-U)^2 = \frac{1}{2} \times 999 \frac{kg}{m^3} \times (40-15)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 312$ kPa (gage) $\quad p_{0,rel}$

Leaving vane, $\vec{V}_{abs} = U\hat{\imath} + (V-U)(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$

$$\vec{V}_{abs} = [U + (V-U)\cos\theta]\hat{\imath} + (V-U)\sin\theta\,\hat{\jmath}$$

$$= \left[15 \frac{m}{s} + (40-15)\frac{m}{s} \times \left(-\frac{1}{2}\right)\right]\hat{\imath} + (40-15)\frac{m}{s} \times 0.866\,\hat{\jmath}$$

$$\vec{V}_{abs} = 2.5\hat{\imath} + 21.7\hat{\jmath} \text{ m/s} \quad \vec{V}_{abs}$$

The magnitude $|\vec{V}_{abs}| = \left[(2.5)^2 + (21.7)^2\right]^{1/2}$ m/s $= 21.8$ m/s

Leaving vane, $p_0 = \frac{1}{2}\rho |\vec{V}_{abs}|^2$, relative to a fixed observer. Thus

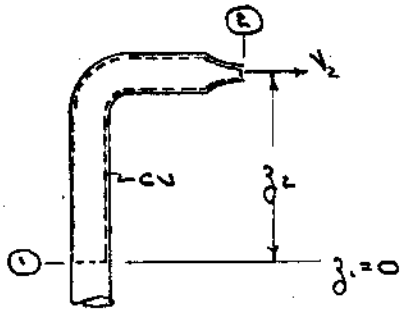
$$p_{0,fixed} = \frac{1}{2} \times 999 \frac{kg}{m^3} \times (21.8)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 237 \text{ kPa (gage)} \quad p_{0,fixed}$$

{ The corresponding absolute pressures are 900, 413, and 338 kPa (abs). }

Discussion: Viscous forces would slow the jet speed relative to the vane. The jet would enter the vane with relative speed $(V - U)$; it would leave the vane with speed $\alpha\,(V - U)$, where $\alpha < 1$.

Friction would reduce both components of relative velocity leaving the vane. The absolute velocity of the jet leaving the vane, as seen by a fixed observer, would decrease. Thus the stagnation pressure of the flow leaving the vane, relative to a fixed observer, would decrease.

Given: Steady flow of water through elbow and nozzle as shown



$D_1 = 0.1\,m$ $D_2 = 0.05\,m$

$p_2 = p_{atm}$ $V_2 = 20\,m/s$

$z_1 = 0$ $z_2 = 4\,m$

Find: Gage pressure, p_1

Solution: Apply continuity to cv. shown to determine V_1; the Bernoulli equation is then applied along a streamline from ① to ② to determine p_1.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{cv} \rho\, d\forall + \int \rho \vec{V}\cdot d\vec{A}$ ($\frac{\partial}{\partial t} = 0$)

$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2$ ($z_1 = 0$)

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) $p_2\,\text{gage} = 0$
(6) $z_1 = 0$

From the continuity equation, $0 = -|\rho V_1 A_1| + |\rho V_2 A_2|$

then, $V_1^2 = \left(\frac{A_2}{A_1}\right)^2 V_2^2 = \left(\frac{D_2}{D_1}\right)^4 V_2^2$

From the Bernoulli equation

$$p_1 = \rho\left[\frac{V_2^2}{2} - \frac{V_1^2}{2} + g z_2\right] = \rho\left[\frac{V_2^2}{2}\left(1 - \frac{V_1^2}{V_2^2}\right) + g z_2\right] = \rho\left[\frac{V_2^2}{2}\left(1 - \left(\frac{D_2}{D_1}\right)^4\right) + g z_2\right]$$

$$p_1 = 999\,\frac{kg}{m^3}\left[\frac{1}{2}\times(20)^2\,\frac{m^2}{s^2}\times\left(1 - \left(\frac{1}{2}\right)^4\right) + 9.81\,\frac{m}{s^2}\times 4\,m\right]\times\frac{N\cdot s^2}{kg\cdot m}$$

$p_1 = 227\,kN/m^2 = 227\,kPa$ (gage) p_1

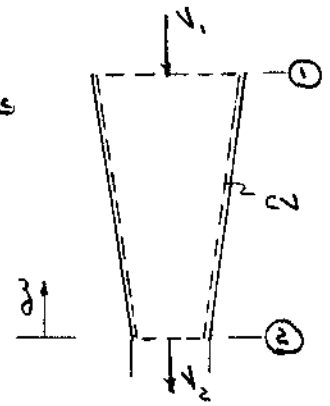
Given: Water flow in a circular duct

$D_1 = 0.3\,m$ $P_1 = 260\,kPa(gage)$ $\vec{V}_1 = -3\hat{k}\ m/s$

$z_1 = 10\,m$

$z_2 = 0$ $D_2 = 0.15\,m$

Frictional effects may be neglected.



Find: Pressure, P_2

Solution: Apply continuity to cv shown to determine V_2; the Bernoulli equation is then applied along a streamline from ① to ② to determine P_2

Basic equations: $0 = \frac{\partial}{\partial t}\int_{cv}\rho\,d\forall + \int_{cs}\rho\vec{V}\cdot d\vec{A}$ (a(1))

$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) uniform flow at sections ① and ②

From the continuity equation

$$0 = -|\rho V_1 A_1| + |\rho V_2 A_2|$$

Then, $V_2 = \frac{A_1}{A_2}V_1 = \left(\frac{D_1}{D_2}\right)^2 V_1 = \left(\frac{0.3}{0.15}\right)^2 \times 3\frac{m}{s} = 12\ m/s$

From the Bernoulli equation,

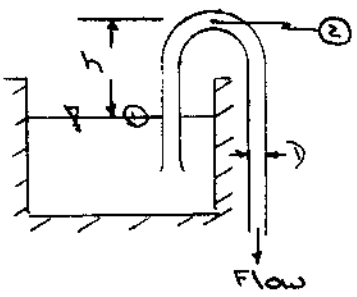
$$P_2 = P_1 + \frac{\rho}{2}(V_1^2 - V_2^2) + \rho g(z_1 - z_2)$$

$$= 260\frac{kN}{m^2} + \frac{1}{2}\times 999\frac{kg}{m^3}\times\left[(3)^2 - (12)^2\right]\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} + 999\frac{kg}{m^3}\times 9.81\frac{m}{s^2}\times 10\,m\times\frac{N\cdot s^2}{kg\cdot m}$$

$P_2 = 291\ kN/m^2 = 291\ kPa\ (gage)$ P_2

Given: Water flow through siphon as shown

$Q = 0.03\ m^3/sec$, $T = 20°C$, $D = 75\ mm$



Find: Maximum allowable height, h, such that P_2 is above the vapor pressure of the water

Solution: Apply the Bernoulli equation along the streamline between locations ① and ② to determine h after employing the definition of volume flowrate to determine the flow speed in the tube

Basic equations: $Q = \int \vec{V} \cdot d\vec{A}$ (Q is volume flow rate)

$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + g z_2$$

(assumption 6: $V_1 \approx 0$; assumption 5: $z_1 = 0$)

Assumptions:
(1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) $z_1 = 0$
(6) $V_1 \approx 0$
(7) uniform flow in the tube

From the definition of Q and assumption 7, $Q = V_2 A_2$, and

$$V_2 = \frac{Q}{A_2} = \frac{4Q}{\pi D_2^2} = \frac{4}{\pi} \times 0.03 \frac{m^3}{s} \times \left(75 \times 10^{-3}\right)^{-2} m^2 = 6.79\ m/s$$

From the Bernoulli equation,

$$h = z_2 = \frac{1}{g}\left[\frac{P_1 - P_2}{\rho} - \frac{V_2^2}{2}\right]$$

For water at 20°C, $P_{vapor} = P_2 = 2.33\ kPa$ Then

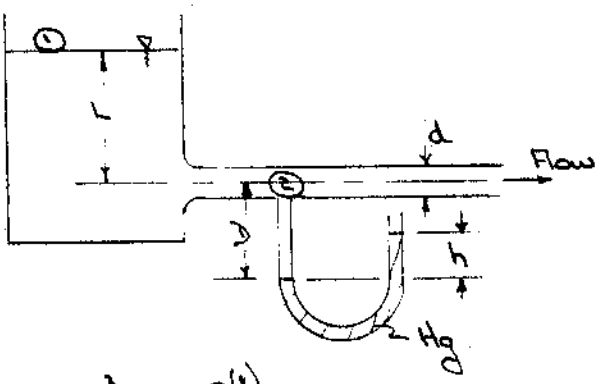
$$h = \frac{1}{g}\left[\frac{P_1 - P_2}{\rho} - \frac{V_2^2}{2}\right] = \frac{s^2}{9.81\ m}\left[(101 - 2.33) \times 10^3 \frac{N}{m^2} \times \frac{m^3}{999\ kg} \times \frac{kg \cdot m}{N \cdot s^2} - \frac{1}{2} \times (6.79)^2 \frac{m^2}{s^2}\right]$$

$$h = 7.72\ m$$

Given: Water flow from a large tank as shown

$L = 12$ ft $D = 2$ ft $d = 2$ in

$h = 6$"



Find: (a) Velocity in discharge pipe
(b) Rate of discharge

Solution:

Basic equations: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$ (5) (4)

$$Q = \int u\,dA$$

Assumptions:
(1) steady flow
(2) incompressible flow
(3) no friction
(4) flow along a streamline
(5) $V_1 \approx 0$, for large tank
(6) $p_1 = p_{atm}$
(7) uniform flow at section ②
(8) $z_2 = 0$

From the Bernoulli equation, $V_2 = \left[2\left(\frac{p_1 - p_2}{\rho}\right) + 2gL\right]^{1/2} = \left[2\left(\frac{p_{atm} - p_2}{\rho}\right) + 2gL\right]^{1/2}$

From the conditions of the manometer,

$$p_{atm} + \gamma_{Hg} h - \gamma_{H_2O} D = p_2 \quad \text{and} \quad p_{atm} - p_2 = \gamma_{H_2O} D - \gamma_{Hg} h$$

Substituting into the expression for V_2,

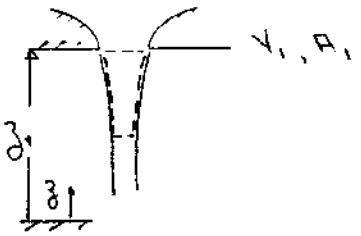
$$V_2 = \left[\frac{2}{\rho}\left(\gamma_{H_2O} D - \gamma_{Hg} h\right) + 2gL\right]^{1/2} = \left[\frac{2}{\rho}\gamma_{H_2O}\left(D - SG_{Hg} h\right) + 2gL\right]^{1/2} = \left[2g\left(D - SG_{Hg} h + L\right)\right]^{1/2}$$

$$V_2 = \left[2 \times 32.2 \frac{ft}{s^2} \times \left(2 ft - 13.6 \times \frac{1}{2} ft + 12 ft\right)\right]^{1/2} = 21.5 \text{ ft/s} \qquad \leftarrow V_2$$

$$Q = \int u\,dA = V_2 A_2 \quad \text{(for uniform flow at ②)}$$

$$Q = V_2 \frac{\pi d^2}{4} = 21.5 \frac{ft}{s} \times \frac{\pi}{4} \times \left(\frac{2}{12}\right)^2 ft^2 = 0.469 \text{ ft}^3/\text{s} \qquad \leftarrow Q$$

Given: Liquid stream leaving a nozzle pointing downward as shown
Assume uniform flow
Neglect friction



Find: Variation in jet area for $z < z_0$

Solution:

Basic equations: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p}{\rho} + \frac{V^2}{2} + gz$

$$0 = \frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) $p = p_1 = p_{atm}$
(6) uniform flow at a section

From the Bernoulli equation

$$V^2 = V_1^2 + 2g(z_1 - z)$$

From the continuity equation

$$0 = \int_{cs} \rho \vec{V} \cdot d\vec{A} = -\{1\rho V_1 A_1 1\} + \{1\rho V A 1\}$$

and

$$V_1 A_1 = VA \quad \text{or} \quad V = V_1 \frac{A_1}{A}$$

Thus

$$V_1^2 \left(\frac{A_1}{A}\right)^2 = V_1^2 + 2g(z_1 - z)$$

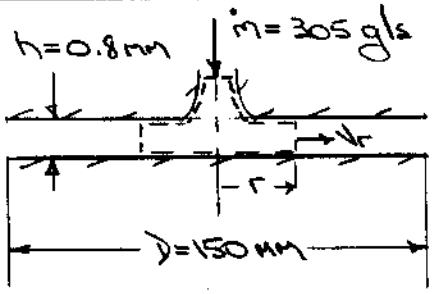
Solving for A,

$$A = A_1 \sqrt{\frac{1}{1 + \frac{2g(z_1 - z)}{V_1^2}}}$$

$A(z)$

{ Note: jet area decreases as z decreases, owing to the higher velocity }

Given: Water flow between parallel disks discharging to atmosphere as shown.



Find: (a) theoretical static pressure between the disks at $r = 50$ mm.
(b) in actual laboratory situation, would the pressure be above or below the theoretical value?

Solution:

Basic equations: $0 = \frac{\partial}{\partial t}\int_{cv} \rho\, dV + \int_{cs} \rho \vec{V}\cdot d\vec{A}$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Assumptions: (1) steady flow
(2) incompressible flow
(3) flow along a streamline
(4) neglect friction
(5) uniform flow at each section

Apply continuity to the CV shown

$$0 = \{-\dot{m}\} + \{\rho V_r\, 2\pi r h\} \quad \text{so} \quad V = \frac{\dot{m}}{2\pi\rho r h}$$

$$V_1 = V_{r=50mm} = \frac{1}{2\pi} \times 0.305\,\frac{kg}{s} \times \frac{m^3}{999\,kg} \times \frac{1}{0.050\,m} \times \frac{1}{8\times10^{-3}\,m} = 1.21\ m/s$$

$$V_2 = V_{r=e} = \frac{1}{2\pi} \times 0.305\,\frac{kg}{s} \times \frac{m^3}{999\,kg} \times \frac{1}{0.075\,m} \times \frac{1}{8\times10^{-4}\,m} = 0.810\ m/s$$

From the Bernoulli equation

$$p_1 - p_2 = p_{r=50m} - p_{atm} = \frac{1}{2}\rho V_2^2 - \frac{1}{2}\rho V_1^2 = \frac{\rho}{2}\left(V_2^2 - V_1^2\right)$$

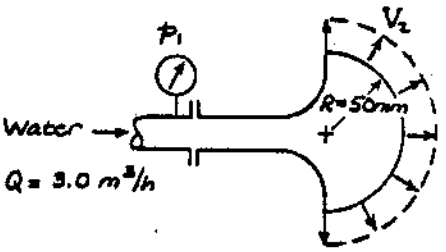
$$p_{r=50mm} = \frac{1}{2} \times 999\,\frac{kg}{m^3}\left[(0.810)^2 - (1.21)^2\right]\frac{m^2}{s^2} \times \frac{N\cdot s^2}{kg\cdot m}$$

$$p_{r=50mm} = -404\ N/m^2\ (gage)$$ $p_{r=50mm}$

Friction would cause a pressure drop in the flow direction. Since the discharge pressure is fixed at p_{atm}, the measured pressure would be greater than the theoretical value.

Given: Spray nozzle designed to produce a flat radial sheet of water of thickness, $t = 0.5$ mm. The inlet pipe has diameter, $D_1 = 20$ mm.

Find: Evaluate the minimum pressure, p_1, to produce volume flow rate, $Q = 3.0\ m^3/hr$ through the nozzle.



Solution:

The minimum pressure occurs when friction is neglected, and so we apply the Bernoulli equation

Basic equation: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + g\cancel{z_1} = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g\cancel{z_2}$

Assumptions: (1) steady flow
(2) incompressible flow
(3) no friction
(4) flow along a streamline
(5) $z_1 = z_2$
(6) $p_2 = p_{atm}$
(7) uniform flow at ① & ②

Then $p_{1g} = p_1 - p_{atm} = \frac{\rho}{2}(V_2^2 - V_1^2)$

From continuity,

$$V_1 = \frac{Q}{A_1} = \frac{Q}{\pi D_1^2/4} = \frac{4}{\pi} \times 3.0 \frac{m^3}{hr} \times \frac{1}{(0.020)^2 m^2} \times \frac{hr}{3600 s} = 2.65 \frac{m}{s}$$

$$V_2 = \frac{Q}{A_2} = \frac{Q}{\pi D t} = \frac{1}{\pi} \times 3.0 \frac{m^3}{hr} \times \frac{1}{0.05 m} \times 0.0005 m \times \frac{hr}{3600 s} = 10.6 \frac{m}{s}$$

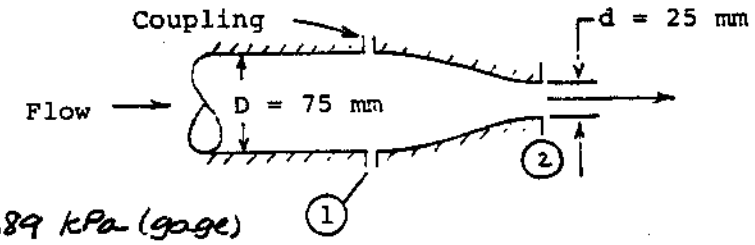
and

$$p_{1g} = \frac{1}{2}\rho(V_2^2 - V_1^2) = \frac{1}{2} \times 999 \frac{kg}{m^3}\left[(10.6)^2 - (2.65)^2\right]\frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 52.6\ kPa\ (gage)$$

p_{1g}

The actual pressure would be slightly higher.

Given: Fire hose nozzle shown.



$p_1 = 689$ kPa (gage)

Find: Maximum flow rate that could be delivered.

Solution: Apply Bernoulli equation. Assumptions needed are:

(1) Steady
(2) Incompressible
(3) Frictionless
(4) Flow along streamline
(5) Neglect Δz
(6) Uniform at ① and ②

Then $\frac{p_1}{\rho} + \frac{V_1^2}{2} + g\cancel{z_1} = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g\cancel{z_2}$

But $V_1 A_1 = V_2 A_2$, so $V_1 = V_2 \frac{A_2}{A_1} = V_2 \left(\frac{d}{D}\right)^2$; $V_1^2 = V_2^2 \left(\frac{d}{D}\right)^4$

$$V_2 = \left[\frac{2(p_1 - p_2)}{\rho\left[1 - (d/D)^4\right]}\right]^{1/2}$$

$$= \left[2 \times 689 \times 10^3 \frac{N}{m^2} \times \frac{m^3}{999\,kg} \times \frac{1}{1-(1/3)^4} \times \frac{kg \cdot m}{N \cdot s^2}\right]^{\frac{1}{2}}$$

$V_2 = 37.4$ m/s

$$Q = V_2 A_2 = 37.4 \frac{m}{s} \times \frac{\pi}{4}(0.025)^2 m^2 \times 3600 \frac{s^2}{hr^2} = 66.1\ m^3/hr$$

Q

Given: Steady, frictionless, incompressible air flow over a wing as shown

$P_1 = 10$ psia
$T_1 = 40°F$
$V_1 = 200$ ft/s
$P_2 = -0.40$ psig



Find: V_2

Solution: Apply the Bernoulli equation along the streamline from the upstream conditions through point (2)

Basic equations: $\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$

$P = \rho RT$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) ideal gas
(6) neglect Δz

Then from the Bernoulli equation

$$V_2^2 = V_1^2 + \frac{2}{\rho}(P_1 - P_2)$$

where $\rho = \frac{P}{RT} = \frac{10\ lbf/in^2}{53.3\ ft\text{-}lbf/lbm\text{-}°R} \times \frac{1}{500°R} \times \frac{slug}{32.2\ lbm} \times \frac{144\ in^2/ft^2}{} = 1.68 \times 10^{-3}\ slug/ft^3$

$$V_2^2 = (200)^2\ ft^2/s^2 + 2 \times \frac{ft^3}{1.68 \times 10^{-3}\ slug} \times 0.40\ lbf/in^2 \times \frac{144\ in^2/ft^2}{} \times \frac{slug\text{-}ft}{lbf\text{-}s^2}$$

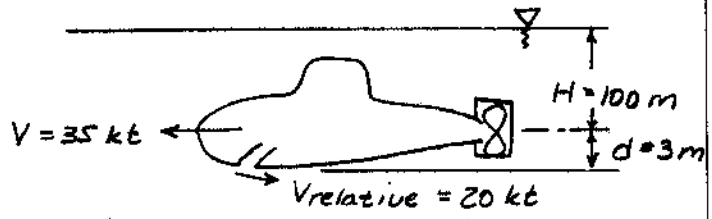
$$V_2^2 = 109{,}000\ ft^2/s^2$$

$$V_2 = 330\ ft/s$$

V_2

{ Note: this is about the upper limit on velocity for the assumption of incompressible flow to be valid }

Given: Submarine as shown.
At cooling inlet,
$V_{relative} = 20$ kt.



Find: Static pressure at inlet.

Solution: To apply Bernoulli equation, we must have a steady flow.
Flow appears steady to an observer on the boat.

Basic equation:

$$\frac{p_\infty}{\rho} + \frac{V_\infty^2}{2} + g z_\infty^{(5)} = \frac{p_i}{\rho} + \frac{V_i^2}{2} + g z_i^{(5)}$$



Assumptions: (1) Steady flow (seen by observer)
(2) Incompressible flow of sea water, SG = 1.025
(3) No friction
(4) Flow along a streamline
(5) Neglect Δz

Solving,

$$p_i - p_\infty = \frac{1}{2}\rho V_\infty^2 - \frac{1}{2}\rho V_i^2 = \frac{1}{2}\rho V_\infty^2\left[1 - \left(\frac{V_i}{V_\infty}\right)^2\right]$$

$$V_\infty = 35\ \text{kt} \times \frac{\text{nm}}{\text{kt}\cdot\text{hr}} \times \frac{6076\ \text{ft}}{\text{nm}} \times 0.305\ \frac{\text{m}}{\text{ft}} \times \frac{\text{hr}}{3600\ \text{s}} = 18.0\ \text{m/s}$$

$$\rho = SG\,\rho_{H_2O} = (1.025)\,1000\ \frac{\text{kg}}{\text{m}^3} = 1025\ \text{kg/m}^3$$

$$p_i - p_\infty = \frac{1}{2} \times 1025\ \frac{\text{kg}}{\text{m}^3} \times (18.0)^2\ \frac{\text{m}^2}{\text{s}^2}\left[1 - \left(\frac{20}{35}\right)^2\right] \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$p_i - p_\infty = 112\ \text{kPa}$$

$$p_\infty = \rho g(H+d) = 1025\ \frac{\text{kg}}{\text{m}^3} \times 9.81\ \frac{\text{m}}{\text{s}^2} \times 103\ \text{m} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} = 1.04 \times 10^6\ \text{Pa or } 1040\ \text{kPa}$$

$$p_i = p_i - p_\infty + p_\infty = 112\ \text{kPa} + 1040\ \text{kPa} = 1150\ \text{kPa (gage)} \leftarrow p_i$$

Given: Mercury barometer carried in car on windless day.

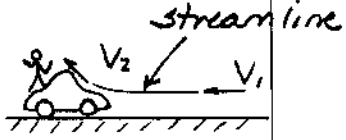
Outside: $T = 20°C$, $h_{bar} = 761$ mm Hg (corrected)

Inside: $V = 105$ km/hr, window open, $h_{bar} = 756$ mm Hg

Find: (a) Explain what is happening.
(b) Local speed of air flow past window, relative to car.

Solution: (a) Air speed relative to car is higher than in the freestream, thus lowering the pressure at window

(b) Apply the Bernoulli equation in frame seen by an observer on the car:



Basic equation: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$

Assumptions: (1) Steady flow (seen by observer on car)
(2) Incompressible flow
(3) Neglect friction
(4) Flow along a streamline
(5) Neglect Δz

Then $V_2^2 = \left[V_1^2 + 2\left(\frac{p_1 - p_2}{\rho}\right)\right]$ or $V_2 = \left[V_1^2 + 2\frac{(p_1 - p_2)}{\rho}\right]^{1/2}$ (1)

From fluid statics

$$p_1 - p_2 = \rho g (h_1 - h_2) = SG_{H_2O}\, g\, \Delta h$$

$$= 13.6 \times 1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.005\ m \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_1 - p_2 = 667\ N/m^2$$

and from ideal gas

$$\rho = \frac{p}{RT} = 13.6 \times 1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.761 m \times \frac{kg \cdot K}{287\ N \cdot m} \times \frac{1}{(273+20)K} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$\rho = 1.21\ kg/m^3$$

Substituting into Eq. 1

$$V_2 = \left[\left(105 \frac{km}{hr} \times 1000 \frac{m}{km} \times \frac{hr}{3600\ s}\right)^2 + 2 \times 667 \frac{N}{m^2} \times \frac{m^2}{1.21\ kg} \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2}$$

$V_2 = 44.2$ m/s (159 km/hr) relative to car ← V_2

Given: Pitot static tube, connected to differential manometer in a car, is held out the window of the car moving at 60 mi/hr. Tube faces upstream.

Find: (a) manometer reading if there is no wind
(b) manometer reading if car is traveling into a head wind of 20 mph

Solution: To the person in the car, the flow appears steady

Basic equation: $\frac{p}{\rho} + \frac{V^2}{2} + g\cancel{z} = \frac{p_0}{\rho} + \cancel{\frac{V^2}{2}}^{=0} + g\cancel{z_0}$

Assumptions: (1) steady flow
(2) incompressible flow
(3) flow along a streamline
(4) frictionless flow
(5) $\Delta z = 0$

Then, $\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_0}{\rho}$ or $p_0 - p = \frac{1}{2}\rho V^2$

For the manometer $\Delta p = \rho_m g \Delta h$

Thus $\rho_m g \Delta h = \frac{1}{2}\rho V^2$ and $\Delta h = \frac{\rho V^2}{2\rho_m g}$

Assuming the manometer fluid is water, then

(a) For no wind, $V = 60 \text{ mph} = 88 \text{ ft/s}$

$$\Delta h = \frac{1}{2} \times \frac{0.00238 \text{ slug/ft}^3}{1.94 \text{ slug/ft}^3} \times \frac{(88)^2 \text{ ft}^2}{\text{s}^2} \times \frac{\text{s}^2}{32.2 \text{ ft}} \times \frac{12 \text{ in}}{\text{ft}}$$

$$\Delta h = 1.77 \text{ in. } H_2O$$ Ans

(b) With a 20 mph headwind, $V = 80 \text{ mph} = 117 \text{ ft/s}$

$$\Delta h = 1.77 \text{ in } H_2O \times \left(\frac{80}{60}\right)^2 = 3.15 \text{ in. } H_2O$$ Ans

Given: Indianapolis race car, $V_\infty = 98.3$ m/s, on a straightaway.

Air inlet at location where $V = 25.5$ m/s along body surface.

Find: (a) Static pressure at inlet location.
(b) Express pressure rise as a fraction of the dynamic pressure.

Solution: Apply the Bernoulli equation, relative to the auto.

Basic equation: $\frac{p_\infty}{\rho} + \frac{V_\infty^2}{2} + g\cancel{z_\infty} = \frac{p}{\rho} + \frac{V^2}{2} + g\cancel{z}$

Assumptions: (1) Steady flow (as seen by observer on auto)
(2) Incompressible flow ($V_\infty < 100$ m/sec)
(3) No friction
(4) Flow along a streamline
(5) Neglect changes in z
(6) Standard air: $\rho = 1.23$ kg/m^3

Then

$$p - p_\infty = \frac{1}{2}\rho V_\infty^2 - \frac{1}{2}\rho V^2 = \frac{1}{2}\rho V_\infty^2\left[1 - \left(\frac{V}{V_\infty}\right)^2\right] = q\left[1 - \left(\frac{V}{V_\infty}\right)^2\right]$$

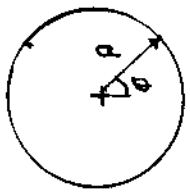
$$q = \frac{1}{2}\rho V_\infty^2 = \frac{1}{2} \times 1.23\,\frac{kg}{m^3} \times (98.3)^2\,\frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 5.94 \text{ kPa}$$

$$\frac{\Delta p}{q} = 1 - \left(\frac{V}{V_\infty}\right)^2 = 1 - \left(\frac{25.5}{98.3}\right)^2 = 0.933 \quad \Delta p/q$$

and $\Delta p = 0.933\,q = 0.933 \times 5.94 \text{ kPa} = 5.54 \text{ kPa}$ $p - p$

Given: Steady, frictionless, incompressible flow over a stationary cylinder of radius, a.

$$\vec{V} = -U\left[1-\left(\frac{a}{r}\right)^2\right]\cos\theta\,\hat{e}_r - U\left[1+\left(\frac{a}{r}\right)^2\right]\sin\theta\,\hat{e}_\theta$$



Find: a) expression for pressure distribution along streamline forming cylinder, $r=a$.
b) locations on cylinder where $p = p_\infty$.

Solution:

Basic equation: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$

Assumptions: (1) steady flow (given).
(2) incompressible flow (given)
(3) frictionless flow (given)
(4) flow along a streamline.

Along the cylinder surface $r=a$ and $\vec{V} = -2U\sin\theta\,\hat{e}_\theta$

Applying the Bernoulli equation along the streamline $r=a$,

$$\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_\infty}{\rho} + \frac{U^2}{2}$$

$$p = p_\infty + \tfrac{1}{2}\rho(U^2 - V^2) = p_\infty + \tfrac{1}{2}\rho(U^2 - 4U^2\sin^2\theta)$$

$$p = p_\infty + \tfrac{1}{2}\rho U^2(1 - 4\sin^2\theta) \qquad p$$

For $p = p_\infty$, $1 - 4\sin^2\theta = 0$ and $\sin\theta = \pm 0.5$

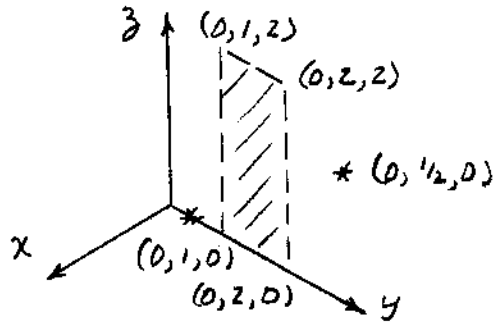
$\therefore \theta = 30^\circ, 150^\circ, 210^\circ, 330^\circ$ $\qquad \theta$

Given: Flow field, $\vec{V} = Ax\hat{i} - Ay\hat{j}$, where $A = 3\,s^{-1}$, xy in feet.

$$\vec{g} = -g\hat{k}$$

$$p^* = 250\ lbf/ft^2$$

Water, $\rho = 1.94\ slug/ft^3$



Find: Force on shaded surface.

Solution: Surface is in yz plane, along streamlines. Apply Bernoulli equation for steady, incompressible, frictionless flow.

Basic equation: $p + \frac{1}{2}\rho V^2 + \rho g z = C = \text{constant}$

From flow field, $V^2 = u^2 + v^2 = (Ax)^2 + (-Ay)^2 = A^2(x^2 + y^2)$

Evaluate constant: $C = p^* + \frac{1}{2}\rho A^2(\cancel{x^{*2}}^{0} + y^{*2}) + \rho g \cancel{z^*}^{0} = p^* + \frac{1}{2}\rho A^2 (y^*)^2$

$$C = 250\ \frac{lbf}{ft^2} + \frac{1}{2} \times 1.94\ \frac{slug}{ft^3} \times \frac{(3)^2}{s^2} \times \left(\frac{1}{2}\right)^2 ft^2 \times \frac{lbf \cdot s^2}{slug \cdot ft} = 252\ lbf/ft^2$$ C

The force is $F = \int p\,dA = \int p\,dy\,dz$. At any location $p = C - \frac{1}{2}\rho V^2 - \rho g z$

$$F = \int C\,dy\,dz - \int \tfrac{1}{2}\rho A^2 (\cancel{x^2}^{0} + y^2)\,dy\,dz - \int \rho g z\,dy\,dz$$

$$= CA - \frac{1}{2}\rho A^2 (z_2 - z_1) \int_{y_1}^{y_2} y^2\,dy - \rho g (y_2 - y_1) \int_{z_1}^{z_2} z\,dz$$

$$= CA - \frac{1}{2}\rho A^2 (z_2 - z_1)\left[\frac{y_2^3 - y_1^3}{3}\right] - \rho g (y_2 - y_1)\left[\frac{z_2^2 - z_1^2}{2}\right]$$

$$= 252\ \frac{lbf}{ft^2} \times 2\ ft^2 - \frac{1}{2} \times 1.94\ \frac{slug}{ft^3} \times \frac{(3)^2}{s^2} (2-0)\,ft \times \frac{(2)^3 - (1)^3}{3}\,ft^3 \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$- 1.94\ \frac{slug}{ft^3} \times 32.2\ \frac{ft}{s^2} (2-1)\,ft \times \frac{(2)^2 - (0)^2}{2}\,ft^2 \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

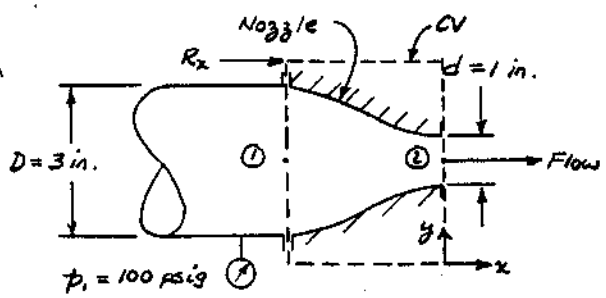
$$F = 504 - 40.7 - 125\ lbf = 338\ lbf$$ F

Given: A fire nozzle is attached to a hose of inside diameter, $D = 3$ in. The smoothly contoured nozzle is designed to operate at an inlet water pressure of $p_1 = 100$ psig. The outlet diameter is $d = 1$ in.

Find: (a) The design flow rate of the nozzle, in gpm
(b) The force required to hold the nozzle in place

Solution:

(a) To determine the design flow rate we apply the continuity equation and the Bernoulli equation.



Assume: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) neglect Δy
(6) uniform flow at each section

From the continuity equation $A_1 V_1 = A_2 V_2$ $\therefore V_1 = V_2 \frac{A_2}{A_1} = V_2 \left(\frac{d}{D}\right)^2$

Bernoulli equation $\frac{p_1}{\rho} + \frac{V_1^2}{2} + g\cancel{y_1} = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g\cancel{y_2}$

Then substituting for V_1 with $p_2 = p_{atm} = 0$ (gage)

$$\frac{p_{1g}}{\rho} + \frac{V_2^2}{2}\left(\frac{d}{D}\right)^4 = \frac{V_2^2}{2} \quad \text{and} \quad V_2 = \left\{\frac{2\,p_{1g}}{\rho\left[1 - \left(\frac{d}{D}\right)^4\right]}\right\}^{1/2}$$

Substituting numerical values

$$V_2 = \left\{2 \times 100 \frac{lbf}{in^2} \times \frac{ft^3}{1.94\,slug} \times \frac{1}{\left[1 - \left(\frac{1}{3}\right)^4\right]} \times 144 \frac{in^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right\}^{1/2} = 123 \text{ ft/s}$$

and

$$Q = A_2 V_2 = \frac{\pi d_2^2}{4} V_2 = \frac{\pi}{4}\left(\frac{1}{12}\right)^2 ft^2 \times 123 \frac{ft}{s} \times 7.48 \frac{gal}{ft^3} \times 60 \frac{s}{min} = 301 \text{ gpm} \quad \textcircled{Q}$$

(b) Apply the x component of the momentum equation to the CV shown

$$F_{Sx} + F_{Bx} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV}^{\,0} + \int u\,\rho\,\vec{V}\cdot d\vec{A}$$

$$R_x + p_{1g}A_1 - \cancel{p_{2g}A_2}^{\,0} = u_1\{-(\rho V_1 A_1)\} + u_2\{+(\rho V_2 A_2)\}$$

$u_1 = V_1 \qquad u_2 = V_2$

$$R_x = -p_{1g}A_1 + \rho V_2 A_2 V_2 - \rho V_1 A_1 V_1 = -p_{1g}A_1 + \rho Q(V_2 - V_1)$$

$$R_x = -p_{1g}A_1 + \rho Q V_2\left(1 - \frac{V_1}{V_2}\right) = -p_{1g}A_1 + \rho A_2 V_2^2\left[1 - \left(\frac{d}{D}\right)^2\right]$$

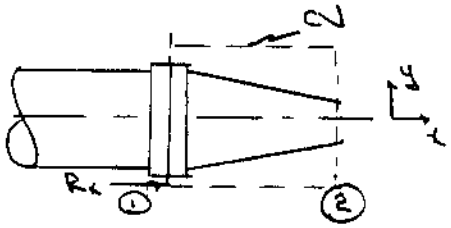
$$\therefore R_x = -100 \frac{lbf}{in^2} \times \frac{\pi}{4}(3)^2 in^2 + 1.94 \frac{slug}{ft^3} \times 301 \frac{gal}{min} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} \times 123 \frac{ft}{s} \left[1 - \left(\frac{1}{3}\right)^2\right] \times \frac{lbf \cdot s^2}{ft \cdot slug}$$

$$R_x = -707 \text{ lbf} + 142 \text{ lbf} = -565 \text{ lbf} \quad \textcircled{R_x}$$

The coupling is in tension.

Given: Water flows steadily through a pipe with diameter $D = 3.25$ in. and discharges through a nozzle ($d = 1.25$ in.) to atmosphere. The flow rate is $Q = 24.5$ gal/min.

Find: (a) the minimum static pressure required in the pipe to produce this flowrate
(b) the horizontal force of the nozzle assembly on the pipe flange.



Solution:

Apply the Bernoulli equation along the central streamline between sections (1) and (2)

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g\cancel{z_1} = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g\cancel{z_2}$$

Assumptions: (1) steady flow (2) incompressible flow
(3) friction less flow (4) flow along a streamline
(5) $z_1 = z_2$ (6) uniform flow at each section

Then $p_1 = p_2 + \frac{\rho}{2}(V_2^2 - V_1^2) = p_2 + \frac{\rho V_2^2}{2}\left[1 - \left(\frac{V_1}{V_2}\right)^2\right]$

$p_2 = p_{atm}$ and from continuity, $A_2 V_2 = A_1 V_1$.

$$\therefore p_{1g} = \frac{\rho}{2} V_2^2 \left[1 - \left(\frac{A_2}{A_1}\right)^2\right] = \frac{\rho V_2^2}{2}\left[1 - \left(\frac{D_2}{D_1}\right)^4\right]$$

$$V_2 = \frac{Q}{A} = \frac{4Q}{\pi d^2} = \frac{4}{\pi} \times 24.5 \frac{\text{gal}}{\text{min}} \times 7.48 \frac{\text{ft}^3}{\text{gal}} \times \frac{1}{60} \frac{\text{min}}{\text{s}} \times (1.25)^2 \text{in}^2 \times \frac{1}{144} \frac{\text{ft}^2}{\text{in}^2}$$

$V_2 = 6.41$ ft/s and

$$p_{1g} = \frac{1}{2} \times 1.94 \frac{\text{slug}}{\text{ft}^3} \times (6.41)^2 \frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}} \left[1 - \left(\frac{1.25}{3.25}\right)^4\right] = 39.0 \text{ psfgage}$$ p_1

(b) Apply the x momentum equation to the CV

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, dV + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

$$R_x + p_{1g}A_1 = u_1\{-\dot{m}\} + u_2\{\dot{m}\} = -V_1\dot{m} + V_2\dot{m}$$

$$R_x = -p_{1g}A_1 + \dot{m}(V_2 - V_1) = -p_{1g}A_1 + \rho Q V_2\left(1 - \frac{V_1}{V_2}\right)$$

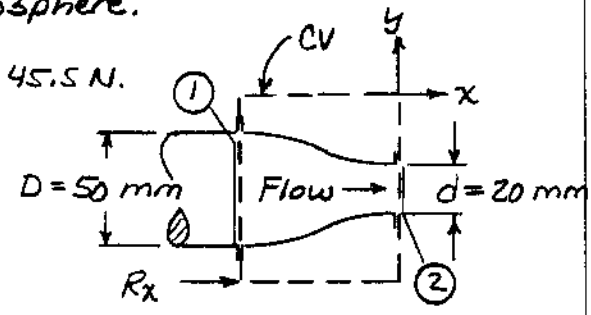
$$= -39 \frac{\text{lbf}}{\text{ft}^2} \times \frac{\pi}{4}\left(\frac{3.25}{12}\right)^2 \text{ft}^2 + 1.94 \frac{\text{slug}}{\text{ft}^3} \times 24.5 \frac{\text{gal}}{\text{min}} \times \frac{\text{ft}^3}{7.48\text{gal}} \times \frac{\text{min}}{60\text{s}} \times 6.41 \frac{\text{ft}}{\text{s}}\left[1 - \left(\frac{1.25}{3.25}\right)^2\right] \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}$$

$R_x = -2.25 + 0.58 = -1.67$ lbf

Force of nozzle on flange $F_x = -R_x = 1.67$ lbf F_x

Given: Nozzle coupled to straight pipe by flanges, bolts. Water flow discharges to atmosphere.

For steady, inviscid flow, $R_x = -45.5$ N.



Find: Volume flow rate.

Solution: Apply continuity, x momentum, and Bernoulli.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$ (=0(1))

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 \quad (p_2 = 0(7),\ z_1 = z_2 = 0(6))$$

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A} \quad (F_{Bx} = 0(6),\ \frac{\partial}{\partial t} = 0(1))$$

Assumptions: (1) Steady flow
(2) Uniform flow at each section
(3) Flow along a streamline
(4) Incompressible flow
(5) No friction
(6) Horizontal, $F_{Bx} = 0$, $z_1 = z_2$
(7) Use gage pressures

Then

$$0 = \{-V_1 A_1\} + \{+V_2 A_2\}\ ;\ V_2 = V_1\frac{A_1}{A_2} = V_1\left(\frac{D}{d}\right)^2\ ;\ Q = V_1 A_1 = V_2 A_2$$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{V_2^2}{2}\ ;\ p_1 = \rho\left(\frac{V_2^2}{2} - \frac{V_1^2}{2}\right) = \rho\frac{V_1^2}{2}\left[\left(\frac{V_2}{V_1}\right)^2 - 1\right] = \rho\frac{V_1^2}{2}\left[\left(\frac{D}{d}\right)^4 - 1\right]$$

$$R_x + p_1 A_1 - p_2 A_2 = u_1\{-|\rho V_1 A_1|\} + u_2\{+|\rho V_2 A_2|\} = \rho V_1 A_1 (V_2 - V_1) \quad (p_2 = 0(7))$$

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x + A_1\rho\frac{V_1^2}{2}\left[\left(\frac{D}{d}\right)^4 - 1\right] = \rho V_1^2 A_1\left(\frac{V_2}{V_1} - 1\right) = \rho V_1^2 A\left[\left(\frac{D}{d}\right)^2 - 1\right]$$

Thus

$$V_1^2 = \frac{-2R_x}{\rho A_1}\frac{1}{\left(\frac{D}{d}\right)^4 - 2\left(\frac{D}{d}\right)^2 + 1} \quad \text{so} \quad V_1 = \sqrt{\frac{-2R_x}{\rho A_1}}\frac{1}{\left(\frac{D}{d}\right)^2 - 1}$$

$$V_1 = \left[-2_x\text{-}45.5\,N \times \frac{m^3}{999\,kg} \times \frac{4}{\pi(0.050)^2 m^2} \times \frac{kg\cdot m}{N\cdot s^2}\right]^{\frac{1}{2}} \frac{1}{\left(\frac{50}{20}\right)^2 - 1} = 1.30\ m/s$$

Finally,

$$Q = V_1 A_1 = 1.30\frac{m}{s} \times \frac{\pi}{4}(0.050)^2 m^2 = 2.55 \times 10^{-3}\ m^3/s \qquad Q$$

{Note: It is necessary to recognize that $R_x < 0$ for a nozzle, see Example Problem 4.7.}

Given: Steady flow of water through elbow in horizontal plane.

Find: (a) Gage pressure at ①.
(b) x component of force exerted by elbow on supply pipe.

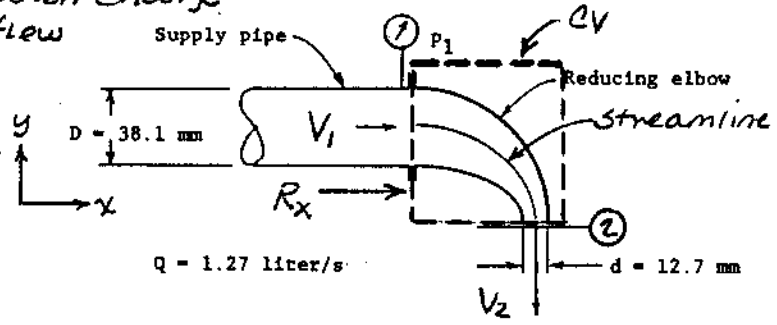
Solution: Apply Bernoulli and momentum equations using streamline and CV shown.

Basic equation: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + g\cancel{z_1} = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g\cancel{z_2}$

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

($F_{Bx} = 0$ (6); $\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall = 0$ (1))

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Neglect friction
(4) Flow along a streamline
(5) Neglect elevation change
(6) Horizontal flow
(7) $p_2 = p_{atm}$
(8) Uniform flow at each section



Then $p_{1\,gage} = \frac{\rho}{2}(V_2^2 - V_1^2)$ ...

From continuity,

$$V_1 = \frac{Q}{A_1} = \frac{4Q}{\pi D_1^2}$$

$$V_1 = \frac{4}{\pi} \times 1.27 \frac{L}{s} \times \frac{1}{(0.0381)^2 m^2} \times \frac{m^3}{1000 L} = 1.11 \text{ m/s}$$

and $V_1 A_1 = V_2 A_2$

$$V_2 = V_1 \frac{A_1}{A_2} = V_1 \left(\frac{D_1}{D_2}\right)^2 = 1.11 \frac{m}{s}\left(\frac{38.1}{12.7}\right)^2 = 9.99 \text{ m/s}$$

Thus

$$p_{1g} = \frac{1}{2} \times 999 \frac{kg}{m^3}\left[(9.99)^2 - (1.11)^2\right]\frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 49.2 \text{ kPa (gage)} \qquad p_{1g}$$

From momentum

$$R_x + p_{1g} A_1 = u_1\{-\dot{m}\} + u_2\{+\dot{m}\} = -\dot{m} V_1 = -\rho Q V_1$$

$$u_1 = V_1 \qquad u_2 = 0$$

$$R_x = -p_{1g} A_1 - \dot{m} V_1 = -49.2 \times 10^3 \frac{N}{m^2} \times \frac{\pi}{4}(0.0381)^2 m^2 - 999 \frac{kg}{m^3} \times 0.00127 \frac{m^3}{s} \times 1.11 \frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m}$$

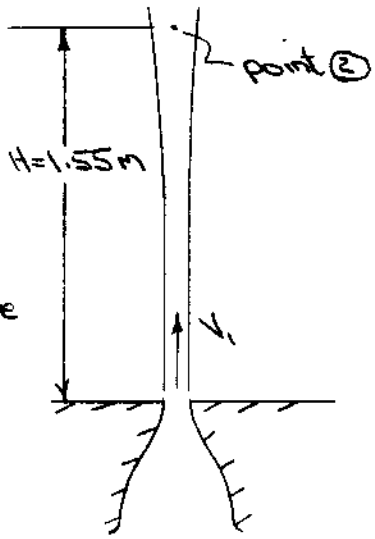
$$R_x = -57.5 \text{ N (force on CV)}$$

The force on the supply pipe is

$$K_x = -R_x = 57.5 \text{ N (on pipe to right)} \qquad K_x$$

Given: A water jet is directed upward from a well-designed nozzle of area $A_1 = 600\ mm^2$; $V_1 = 6.3\ m/s$. The flow is steady and liquid stream does not break up. Point ② is $H = 1.55\ m$ above nozzle exit

Find: (a) V_2 (b) p_{o_2}
(c) force on flat plate placed normal to the flow at ②
(d) Sketch pressure distribution on the plate



Solution: Apply Bernoulli and then y-momentum equation

Basic eq.: $\cancel{\frac{p_1}{\rho}} + \cancel{\frac{V_1^2}{2}}^{\,0} + gz_1 = \cancel{\frac{p_2}{\rho}}^{\,0} + \frac{V_2^2}{2} + gz_2$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) $p_1 = p_2 = p_{atm}$

Then

$$V_2 = \left[V_1^2 + 2g(z_1 - z_2)\right]^{1/2}$$

$$V_2 = \left[(6.3)^2 \frac{m^2}{s^2} + 2 \times 9.81 \frac{m}{s^2}(-1.55\ m)\right]^{1/2}$$

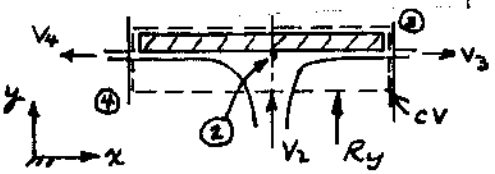
$$V_2 = 3.05\ m/s \quad \longleftarrow V_2$$

By definition, $p_{o_2} = p_2 + \frac{1}{2}\rho V_2^2 = p_{atm} + \frac{1}{2}\rho V_2^2$, so

$$p_{o_2 gage} = \frac{1}{2} \times 999 \frac{kg}{m^3} \times (3.05)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 4.65\ kPa(g) \quad \longleftarrow p_{o_2}$$

Apply y-momentum equation to CV surrounding plate

Basic eq.: $F_{sy} + \cancel{F_{By}}^{\,0} = \cancel{\frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall}^{\,0} + \int_{CS} v\rho\, \vec{V}\cdot d\vec{A}$



Assumptions: (6) neglect mass in CV
(7) V_2 enters CV uniformly
(8) $v_3 = v_4 = 0$

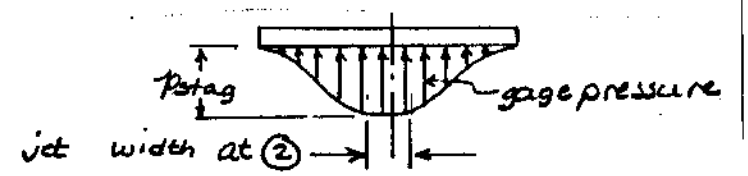
Then

$$R_y = v_2\{-\rho V_1 A\} + \cancel{v_3}^{\,0=(8)}\{\dot{m}_3\} + \cancel{v_4}^{\,0=(8)}\{\dot{m}_4\} = -\rho V_1 A_1 V_2 \quad \text{and}$$

$$K_y = -R_y = \rho V_1 A_1 V_2 = 999 \frac{kg}{m^3} \times 6.3 \frac{m}{s} \times 600\ mm^2 \times 3.05 \frac{m}{s} \times \left(\frac{1\ m}{1000\ mm}\right)^2 \times \frac{N}{kg \cdot m/s^2}$$

$$K_y = 11.5\ N \ (\text{force up}) \quad \longleftarrow K_y$$

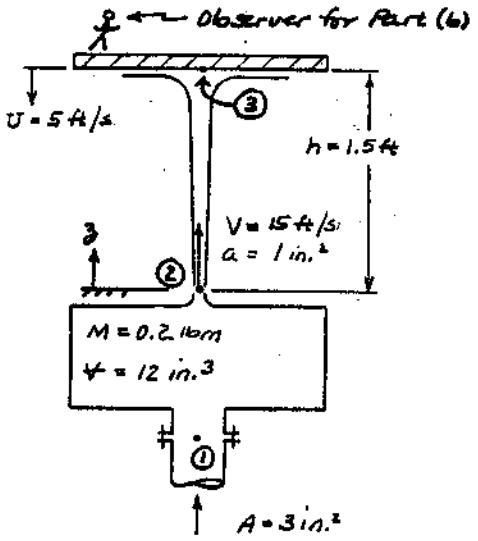
The pressure distribution on the plate is as shown.



Given: A flat object moves downward, at speed $U = 5$ ft/sec, into the water jet of the spray system shown. The spray system, of mass $M = 0.200$ lbm and internal volume $\forall = 12$ in^3, operates under steady conditions

Find: (a) the minimum supply pressure required to produce the jet of the spray system.
(b) the maximum pressure exerted by the jet on the object when the object is at $z = 1.5$ ft.

Solution:



(a) The minimum pressure occurs when friction is neglected, and so we apply the Bernoulli equation

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g\cancel{z_1}^{(5)} = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g\cancel{z_2}^{(5)}$$

Assume: (1) steady flow
(2) incompressible flow
(3) no friction
(4) flow along a streamline
(5) neglect z_2, z_1
(6) $p_2 = p_{atm}$
(7) uniform flow at ① & ②

Then

$$p_{1g} = p_1 - p_{atm} = \frac{\rho}{2}(V_2^2 - V_1^2) = \frac{\rho V_2^2}{2}\left[1 - \left(\frac{V_1}{V_2}\right)^2\right]$$

From continuity, $A_1 V_1 = A_2 V_2$, and $\frac{V_1}{V_2} = \frac{A_2}{A_1} = \frac{a}{A}$. Then,

$$p_{1g} = \frac{\rho V_2^2}{2}\left[1 - \left(\frac{a}{A}\right)^2\right] = \frac{1}{2}(1.94)\frac{\text{slug}}{\text{ft}^3}(15)^2\frac{\text{ft}^2}{\text{s}^2}\left[1 - \left(\frac{1}{3}\right)^2\right]\frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}\cdot\frac{\text{ft}^2}{144\,\text{in}^2} = 1.35\ \text{psig}$$ p_{1g}

Frictional effects would cause this value to be higher

(b) The maximum pressure of the jet on the object is the stagnation pressure

$$p_0 = p + \frac{1}{2}\rho V_r^2$$

where V is the velocity of the impinging jet relative to the object

At $z = 1.5$ ft, the jet velocity, V_4, in the absence of the object can be calculated from $\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 = \frac{p_4}{\rho} + \frac{V_4^2}{2} + gz_4$

$$V_4 = \left[V_2^2 - 2g(z_4 - z_2)\right]^{1/2} = \left[(15)^2\frac{\text{ft}^2}{\text{s}^2} - 2\cdot 32.2\frac{\text{ft}}{\text{s}^2}(1.5)\text{ft}\right]^{1/2} = 11.3\ \text{ft/s}$$

Then

$$V_{rel} = V_4 - (-U) = (11.3 + 5)\ \text{ft/s} = 16.3\ \text{ft/s}$$

and

$$p_0 - p_{atm} = p_{0g} = \frac{1}{2}\rho V_r^2 = \frac{1}{2}(1.94)\frac{\text{slug}}{\text{ft}^3}(16.3)^2\frac{\text{ft}^2}{\text{s}^2}\cdot\frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}\cdot\frac{\text{ft}^2}{144\,\text{in}^2} = 1.79\ \text{psig}$$ p_{0g}

(c) To determine the force of the water on the object we apply the $\hat{j}$ component of the momentum equation to the CV shown



$$F_{S_y} + \cancel{F_{B_y}}^{=0\,(a)} = \frac{\partial}{\partial t}\int_{CV} w_{rel_{xyz}}\,\rho\,d\forall + \int_{CS} w_{rel_{xyz}}\,(\rho \vec{V}_{rel_{xyz}}\cdot d\vec{A})$$

Assumptions: (8) neglect $\frac{\partial}{\partial t}\int_{CV}$

(9) neglect body forces

(10) uniform radial flow at ⑤

(11) uniform vertical flow at ④ with $z_4 = 1.5$ ft

Then $-F_1 = -w_{4_{xyz}} \left| \rho V_{4_{xyz}} A_4 \right|$

where F_1 is applied force necessary to maintain motion of plate at constant speed U

$$V_{4_{xyz}} = V_4 - (-U) = V_4 + U$$

$$w_{4_{xyz}} = V_{4_{xyz}} = V_4 + U$$

$$\therefore F_1 = \rho (V_4 + U)^2 A_4$$

From continuity $A_2 V_2 = A_4 V_4$

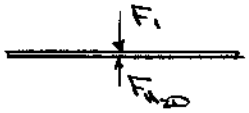
and $A_4 = \frac{V_2}{V_4} A_2 = \frac{15}{11.3} \cdot 1\,in^2 = 1.33\,in^2$

Then

$$F_1 = \rho (V_4 + U)^2 A_4 = 1.94 \frac{slug}{ft^3} (11.3 + 5)^2 \frac{ft^2}{s^2} \cdot 1.33\,in^2 \cdot \frac{ft^2}{144\,in^2} \cdot \frac{lbf \cdot s^2}{ft \cdot slug}$$

$F_1 = 4.76$ lbf (in the direction shown)

Since the plate is moving at constant speed, then



$$\Sigma \vec{F}_{plate} = m\vec{a} = 0 \quad \text{and}$$

neglecting the weight of the plate then

$$F_{H_2O} = F_1 = 4.76 \text{ lbf}$$

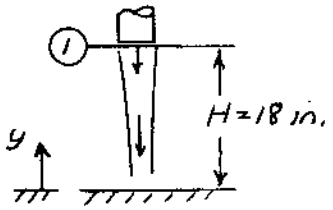
$$\vec{F}_{H_2O} = 4.76\,\hat{j} \text{ lbf}$$

Given: Water flow from a kitchen faucet of 0.5 in. diameter at 2 gpm. Bottom of sink is 18 in. below faucet outlet.

Find: (a) If area of stream increases, decreases, or remains constant, and why.
(b) Expression for cross-section vs. y, measured above bottom.
(c) Force on plate held horizontal; variation with height, and why?

Solution: The water stream is accelerated by gravity. The area of the stream will decrease toward the sink bottom, because less area is needed to carry the same flow rate.

Apply Bernoulli to steady, incompressible, frictionless flow along a streamline:



Basic equation: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p}{\rho} + \frac{V^2}{2} + gz$

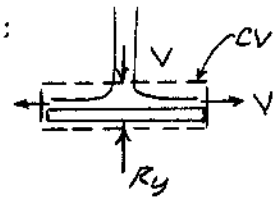
But $p_1 = p = p_{atm}$, so

$$\frac{V_1^2}{2} + gH = \frac{V^2}{2} + gy \quad ; \quad V = \left[V_1^2 + 2g(H-y)\right]^{1/2}$$

For uniform flow, continuity reduces to $V_1 A_1 = VA$

$$A = A_1 \frac{V_1}{V} = A_1 \frac{V_1}{\left[V_1^2 + 2g(H-y)\right]^{1/2}} = \frac{A_1}{\left[1 + \frac{2g}{V_1^2}(H-y)\right]^{1/2}} \qquad A(y)$$

Predict force on plate from y component of momentum:



Basic equation: $F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall + \int_{CS} v\rho\, \vec{V}\cdot d\vec{A}$ ($F_{By} = 0$ (SP))

Since uniform, $R_y - W = v\{-\rho V_1 A_1\} = -V\{-\rho Q\} = +V\rho Q$

$$v = -V$$

Thus $R_y = W + V\rho Q$ $\qquad R_y$

Since V increases as y decreases, R_y varies in the same manner.

Open-Ended Problem Statement: An old "parlor trick" uses an empty thread spool and a playing card. The playing card is placed against the bottom of the spool. Contrary to intuition, when one blows downward, through the central hole in the spool, the card is not blown away. Instead it is "sucked" up against the spool. Explain.

Discussion: The secret to this "parlor trick" lies in the velocity distribution, and hence the pressure distribution, that exists between the spool and the playing card.

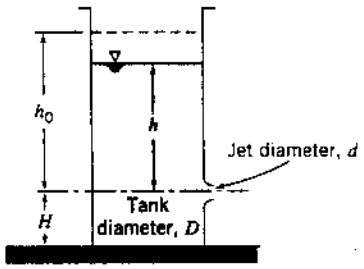
Neglect viscous effects for the purpose of initial discussion. Consider the space between the end of the spool and the playing card as a pair of parallel disks. Air from the hole in the spool enters the annular space surrounding the hole, then flows radially outward between the parallel disks. For a given flow rate of air the edge of the hole is the cross-section of minimum flow area and therefore the location of maximum air speed.

After entering the space between the parallel disks, air flows radially outward. The flow area becomes larger as the radius increases. Thus the air slows and its pressure increases. The largest flow area, slowest air speed, and highest pressure between the disks occur at the outer periphery of the spool where the air is discharged from an annular area.

The air leaving the annular space between the disk and card must be at atmospheric pressure. This is the location of the highest pressure in the space between the parallel disks. Therefore pressure at smaller radii between the disks must be lower, and hence the pressure between the disks is sub-atmospheric. Pressure above the card is less than atmospheric pressure; pressure beneath the card is atmospheric. Each portion of the card experiences a pressure difference acting upward. This causes a net pressure force to act upward on the whole card. The upward pressure force acting on the card tends to keep it from blowing off the spool when air is introduced through the central hole in the spool.

Viscous effects are present in the narrow space between the disk and card. However, they only reduce the pressure rise as the air flows outward, they do not dominate the flow behavior.

Given: Tank shown has well-rounded nozzle. At time t=0, water level is h_0



Find: expression for h/h_0 as a function of time.

Plot: (a) h/h_0 vs t for $D/d = 10$, with h_0 as a parameter for $0.1 \le h/h_0 \le 1$

(b) h/h_0 vs t for $h_0 = 1$ m, with D/d as a parameter for $2 \le D/d \le 10$

Solution:

Apply the Bernoulli equation along a streamline between the surface and the jet

Basic equation: $\frac{p_s}{\rho} + \frac{V_s^2}{2} + gz_s = \frac{p_j}{\rho} + \frac{V_j^2}{2} + gz_j$ (s) (j)

Assumptions: (1) quasi-steady flow, ie neglect acceleration in tank.
(2) incompressible flow
(3) neglect frictional effects
(4) flow along a streamline
(5) $p_s = p_j = p_{atm}$

From continuity, $V_s A_s = V_j A_j$ or $V_j = V_s \frac{A_s}{A_j} = V_s \left(\frac{D}{d}\right)^2$

Solving,

$$\frac{V_s^2}{2} - \frac{V_j^2}{2} = \frac{V_s^2}{2}\left[1 - \left(\frac{V_j}{V_s}\right)^2\right] = g(z_j - z_s) = g[H - (H+h)] = -gh$$

Then

$$V_s = \left[\frac{2gh}{(V_j/V_s)^2 - 1}\right]^{1/2} = \left[\frac{2gh}{(A_s/A_j)^2 - 1}\right]^{1/2} = \left[\frac{2gh}{(D/d)^4 - 1}\right]^{1/2} = -\frac{dh}{dt}$$

Separating variables,

$$\frac{dh}{h^{1/2}} = -\left[\frac{2g}{(D/d)^4 - 1}\right]^{1/2} dt$$

Integrating,

$$2h^{1/2} = -\left[\frac{2g}{(D/d)^4 - 1}\right]^{1/2} t + c$$

At $t = 0$, $h = h_0$, so $c = 2h_0^{1/2}$ and

$$h = \left\{h_0^{1/2} - \frac{1}{2}\left[\frac{2g}{(D/d)^4 - 1}\right]^{1/2} t\right\}^2$$

Nondimensionalize (divide by h_0) to obtain

$$\frac{h}{h_0} = \left\{1 - \sqrt{\frac{g}{2h_0\{(D/d)^4 - 1\}}}\, t\right\}^2 \qquad h/h_0$$

Draining of a cylindrical liquid tank:

Plot of h/h_0 vs. t for $0.1 < h_0 < 1$ m

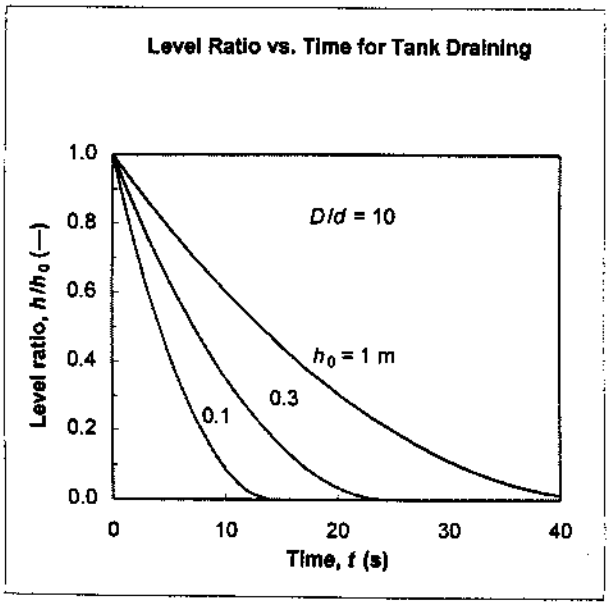
Input Data: D = 50 mm; d = 5 mm

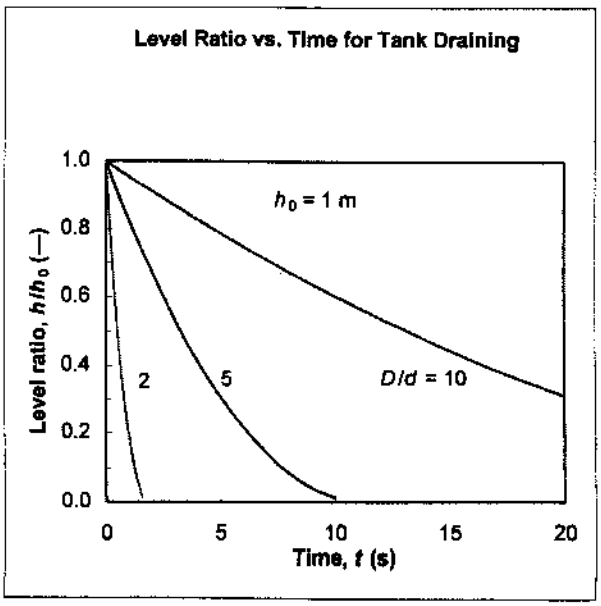
h_0 (m) =	0.1	0.3	1
Time, t (s)	**h/h_0 (—)**	**h/h_0 (—)**	**h/h_0 (—)**
0	1.00	1.00	1.00
2	0.739	0.845	0.913
4	0.518	0.703	0.831
6	0.336	0.574	0.752
8	0.193	0.458	0.677
10	0.090	0.355	0.606
12	0.025	0.265	0.539
14	0.000	0.188	0.476
16		0.125	0.417
18		0.074	0.362
20		0.037	0.310
22		0.012	0.263
24		0.001	0.219
26			0.180
28			0.144
30			0.113
32			0.085
34			0.061
36			0.041
38			0.025
40			0.013
45			0.000

Plot of h/h_0 vs. t for $10 < D/d < 2$

h_0 = 1 m

D/d (—) =	2	5	10
Time, t (s)	**h/h_0 (—)**	**h/h_0 (—)**	**h/h_0 (—)**
0	1.00	1.00	1.00
0.5	0.523	0.913	0.978
1	0.199	0.831	0.956
1.5	0.029	0.752	0.935
1.6	0.013	0.737	0.930
3		0.539	0.872
4		0.417	0.831
5		0.310	0.791
6		0.219	0.752
7		0.144	0.714
8		0.085	0.677
9		0.041	0.641
10		0.013	0.606
12			0.539
14			0.476
16			0.417
18			0.362
20			0.310

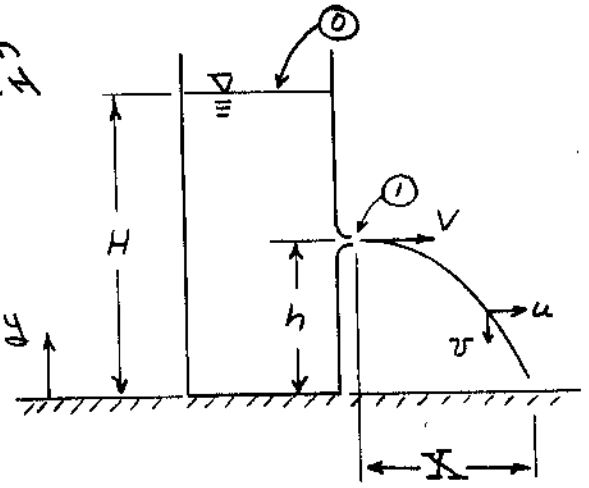




Given: Water level in tank shown is maintained at height H

Find: Elevation h to maximize range, R, of jet

Plot: R/H as a function of h/H



Solution:

Apply Bernoulli equation between tank surface and jet

Basic equation: $\cancel{\frac{p_0}{\rho}} + \cancel{\frac{V_0^2}{2}}^{0} + gy_0 = \cancel{\frac{p_1}{\rho}} + \frac{V_1^2}{2} + gy_1$

Assumptions: (1) steady flow (2) incompressible flow (3) flow along streamline (4) no friction

Then $gH = \frac{V_1^2}{2} + gh$ or $V = \sqrt{2g(H-h)}$

Assume no air resistance in the stream. Then u = constant, and $R = ut = \sqrt{2g(H-h)}\,t$ _ _ _ _ _ _ _ (1)

The only force acting on the stream is gravity

$\sum F_y = -mg = ma_y = m\frac{dv}{dt}$; thus $\frac{dv}{dt} = -g$

Integrating we obtain $v = \cancel{v_0}^{0} - gt$ and

$y = y_0 + \cancel{v_0}^{0} t - \frac{1}{2}gt^2$

Solving for t, $t = \left[\frac{2(y_0 - y)}{g}\right]^{1/2}$

The time of flight is then $t = \sqrt{\frac{2y_0}{g}} = \sqrt{\frac{2H}{g}}$

Substituting into Eq. 1:

$R = \sqrt{2g(H-h)}\sqrt{\frac{2H}{g}} = 2\sqrt{h(H-h)}$ _ _ _ _ _ _ (2)

R will be maximized when $h(H-h)$ is maximized, or when

$\frac{d}{dh}[h(H-h)] = 0 = (H-h) + h(-1) = H - 2h$ or $h = H/2$ ← h

The corresponding range is

$R = 2\sqrt{\frac{H}{2}\cdot\frac{H}{2}} = H$

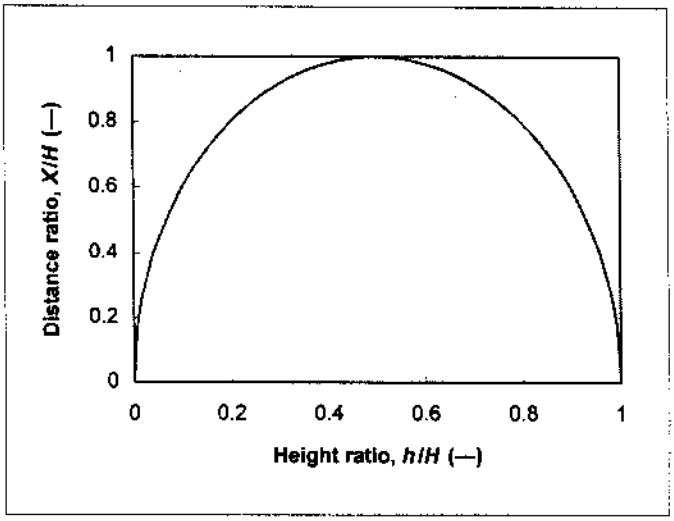
From Eq. 2

$$\frac{x}{H} = 2\sqrt{\frac{h}{H}\left(1 - \frac{h}{H}\right)}$$

x/H is plotted as a function of h/H.

Throw of a Jet from Orifice in Side of a Tank versus Height:

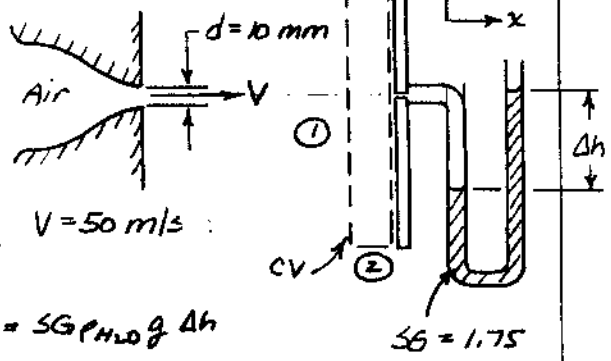
h/H (—)	*X/H* (—)
0.00	0.000
0.01	0.199
0.03	0.341
0.05	0.436
0.10	0.600
0.15	0.714
0.20	0.800
0.25	0.866
0.30	0.917
0.35	0.954
0.40	0.980
0.45	0.995
0.50	1.000
0.55	0.995
0.60	0.980
0.65	0.954
0.70	0.917
0.75	0.866
0.80	0.800
0.85	0.714
0.90	0.600
0.95	0.436
0.97	0.341
0.99	0.199
1.00	0.000



13 78/ 500 SHEETS, FILLER, 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-398 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Air jet, disk of diameter D = 200 mm, and manometer.

Find: (a) Δh
(b) Force on disk
(c) Sketch pressure distribution.



Solution: Apply Bernoulli, hydrostatic, and x component of momentum.

Basic equations: $\frac{p_0}{\rho} = \frac{p}{\rho} + \frac{V^2}{2}$ $\qquad \Delta p = SG\,\rho_{H_2O}\,g\,\Delta h$

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\,d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Flow along a streamline
(4) No friction
(5) Static liquid in manometer
(6) $F_{Bx} = 0$
(7) Uniform flow at each section

Then

$$\Delta p = p_0 - p = \frac{1}{2}\rho V^2 = SG\,\rho_{H_2O}\,g\,\Delta h$$

$$\Delta h = \frac{\rho V^2}{2SG\rho_{H_2O}g} = \frac{1}{2}\times 1.23\frac{kg}{m^3}\times(50)^2\frac{m^2}{s^2}\times\frac{m^3}{(1.75)999\,kg}\times\frac{s^2}{9.81\,m} = 0.0896\ m\ (89.6\ mm)$$ Δh

and

$$R_x = u_1\{-\rho VA\} + u_2\{\rho VA\} = -\rho V^2 A$$

$u_1 = V \qquad u_2 = 0$

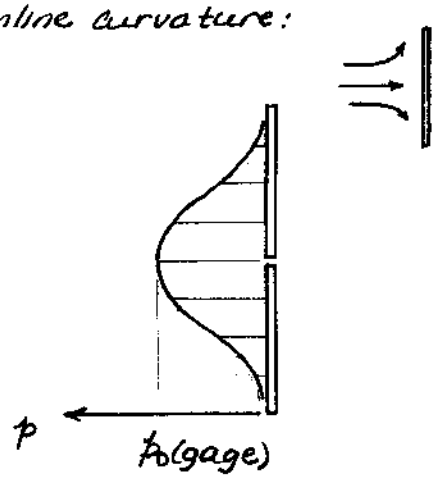
or

$$K_x = -R_x = \rho V^2 A = 1.23\frac{kg}{m^3}\times(50)^2\frac{m^2}{s^2}\times\frac{\pi}{4}(0.010)^2 m^2\times\frac{N\cdot s^2}{kg\cdot m} = 0.242\ N$$ K_x

The pressure distribution is caused by streamline curvature:

Pressure is p_0 (gage) at center.

Pressure is zero (gage) at edges.



Given: An "air hockey" puck may be modeled as a circular disc supported by a layer of air that issues from multiple tiny holes in the game table. Assume puck floats at a distance, h = 1 mm above the table from which air flows at a volumetric flowrate per unit area of table, $q = 0.08\ m^3/m^2 \cdot s$. Puck diameter D = 75 mm.



Find: (a) Expression for the pressure distribution p(r)
(b) Estimate the mass of the puck.

Solution

Basic equations: $0 = \int_{cv} \frac{\partial \rho}{\partial t}\, d\forall + \int_{cs} \rho \vec{V} \cdot d\vec{A}$

$-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r$

$a_r = \frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z \frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}$

$F_{pr} = \int_A (p - p_{atm})\, dA$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) at any r, flow is radial and uniform
(5) $g_r = 0$

Applying continuity to the control volume shown

$0 = |\rho V_r 2\pi r h| - |q \pi r^2|$ and hence $V_r = q\frac{r}{2h}$

From Euler's equation

$\frac{\partial p}{\partial r} = \frac{dp}{dr} = -\rho a_r = -\rho V_r \frac{\partial V_r}{\partial r} = -\rho\, q\frac{r}{2h}\,\frac{q}{2h} = -\rho \frac{q^2 r}{4h^2}$

$p - p_{atm} = \int_{p_{atm}}^{p} dp = -\int_R^r \frac{\rho q^2 r}{4h^2} = -\frac{\rho q^2 r^2}{8h^2}\Big]_R^r = \frac{\rho q^2}{8h^2}(R^2 - r^2)$ ← (p − p_atm)

The net pressure force on the puck is

$F_p = \int_A (p - p_{atm})\, dA = \frac{\rho q^2}{8h^2}\int_0^R (R^2 - r^2)\, 2\pi r\, dr = \frac{\rho q^2 \pi}{4h^2}\left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_0^R$

$F_p = \frac{1}{16}\pi \frac{\rho q^2 R^4}{h^2} = \frac{1}{16}\pi \times 1.23\frac{kg}{m^3} \times (0.08)^2 \frac{m^4}{s^2} \times \frac{(0.0375)^4}{(0.001)^2}\frac{m^2}{m^2} \times \frac{N \cdot s^2}{kg \cdot m} = 3.06\ mN$

Then, since $\sum F_z = 0 = Mg - F_p$

$M = \frac{F_p}{g} = 3.06\times10^{-3} N \times \frac{s^2}{9.81\ m} \times \frac{kg \cdot m}{N \cdot s^2} \times \frac{10^3\ g}{kg} = 0.312\ grams$ ← M

Given: Inflatable "bubble" structure modeled as circular semi-cylinder

diameter $D = 30\ m$

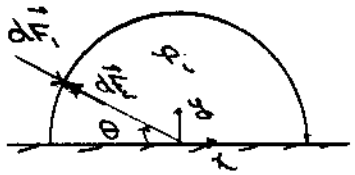
length $L = 60\ m$

Pressure inside is $p_i = p_\infty + \Delta p$

where $\Delta p = \rho_{H_2O} g\, \Delta h$ and $\Delta h = 10\ mm$

Pressure distribution over outer surface is given by

$$\frac{p - p_\infty}{\frac{1}{2}\rho V_\omega^2} = 1 - 4\sin^2\theta \qquad V_\omega = 50\ km/hr$$



Find: net vertical force exerted on the structure

Solution:

The force due to pressure is $F = \int p\, dA$.

The vertical component of dF_1 is $dF_{1_V} = -p\, dA \sin\theta = -p\, R L\, d\theta \sin\theta$

The vertical component of dF_2 is $dF_{2_V} = p_i\, dA \sin\theta = p_i R L\, d\theta \sin\theta$

Then, neglecting end effects

$$dF_{V_{net}} = (p_i - p) R L \sin\theta\, d\theta = (p_\infty + \Delta p - p) R L \sin\theta\, d\theta$$

$$F_V = \int dF_V = \int_0^\pi \left[\Delta p - (p - p_\infty)\right] R L \sin\theta\, d\theta$$

$$= \int_0^\pi \left[\Delta p - \tfrac{1}{2}\rho V_\omega^2 (1 - 4\sin^2\theta)\right] R L \sin\theta\, d\theta$$

$$= R L \left\{ \Delta p \left[-\cos\theta\right]_0^\pi - \tfrac{1}{2}\rho V_\omega^2 \left[-\cos\theta + 4\left(\cos\theta - \tfrac{\cos^3\theta}{3}\right)\right]_0^\pi \right\}$$

$$= R L \left\{ 2\Delta p - \tfrac{1}{2}\rho V_\omega^2 \left[2 + 4\left(-2 + \tfrac{2}{3}\right)\right] \right\}$$

$$F_V = R L \left\{ 2\Delta p + \tfrac{8}{3}\rho V_\omega^2 \right\} = R L \left\{ 2\rho_{H_2O} g\, \Delta h + \tfrac{8}{3}\rho V_\omega^2 \right\}$$

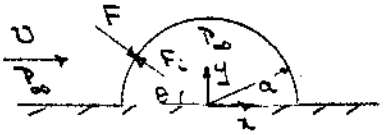
$$F_V = 15\,m \times 60\,m \left\{ 2 \times 999\,\tfrac{kg}{m^3} \times 9.81\,\tfrac{m}{s^2} \times 0.01\,m + \tfrac{8}{3} \times 1.23\,\tfrac{kg}{m^3} \times (50)^2\,\tfrac{km^2}{hr^2} \times 10^6\,\tfrac{hr^2}{km^2} \times \left(\tfrac{1}{3600}\right)^2 \tfrac{hr^2}{s^2} \right\} \times \tfrac{N \cdot s^2}{kg \cdot m}$$

$$F_{V_{net}} = 532\ kN \qquad \leftarrow F_{V_{net}}$$

Given: Flow over a Quonset hut may be approximated by the velocity field

$$\vec{V} = U\left[1-\left(\frac{a}{r}\right)^2\right]\cos\theta\,\hat{e}_r - U\left[1+\left(\frac{a}{r}\right)^2\right]\sin\theta\,\hat{e}_\theta$$

with $0 \le \theta \le 2\pi$



The hut has a diameter, $D = 6m$, and a length, $L = 18m$

During a storm, $U = 100$ km/hr, $p_\infty = 720$ mm Hg, $T_\infty = 5°C$

Find: The net force tending to lift the hut off its foundation.

Solution:

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{const}$ $\qquad F = \int p\,dA$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline

Along the top half of the cylinder, $r = a$ and $\vec{V} = -2U\sin\theta\,\hat{i}_\theta$, $0 \le \theta \le \pi$

Applying the Bernoulli equation along the streamline ($r = a$)

$$\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_0}{\rho} + \frac{V_0^2}{2}$$

$$p - p_0 = \frac{\rho}{2}(V_0^2 - V^2) = \frac{\rho}{2}(U^2 - 4U^2\sin^2\theta) = \frac{\rho U^2}{2}(1 - 4\sin^2\theta)$$

$$F_{R_x} = \int_A (p_0 - p)\,dA\sin\theta = \int_0^\pi (p_0 - p)\sin\theta\, L\,a\,d\theta$$

$$= \int_0^\pi \frac{\rho U^2}{2}(4\sin^2\theta - 1)\sin\theta\, L\,a\,d\theta = \frac{\rho U^2}{2} aL\left\{4\left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^\pi + \cos\theta\Big]_0^\pi\right\}$$

$$= \frac{\rho U^2}{2} aL\left\{4\left[\left(-\frac{1}{3}+1\right) - \left(\frac{1}{3}-1\right)\right] + (-1-1)\right\}$$

$$F_{R_x} = \frac{\rho U^2}{2} aL\left(\frac{10}{3}\right) = \frac{5}{3}\rho U^2 aL$$

From the ideal gas equation of state

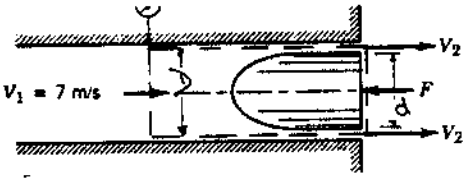
$$\rho = \frac{p}{RT} = 720\text{ mm Hg} \times \frac{\text{atm}}{760\text{ mm}} \times 1.01\times10^5\,\frac{N}{m^2\cdot\text{atm}} \times 287\,\frac{\text{kg}\cdot K}{N\cdot m} \times \frac{1}{278K} = 1.20\ \text{kg/m}^3$$

$$F_{R_x} = \frac{5}{3}\rho U^2 aL = \frac{5}{3} \times 1.20\,\frac{\text{kg}}{m^3} \times (10^5)^2\,\frac{m^2}{hr^2} \times \frac{hr^2}{(3600)^2 s^2} \times 3m \times 18m \times \frac{N\cdot s^2}{\text{kg}\cdot m}$$

$$F_{R_x} = 83.3\text{ kN} \longleftarrow F_R$$

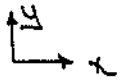
Comment: The actual pressure distribution over the rear portion of the hut is not modelled well by ideal flow. The force calculated here is lower than the actual force.

Given: Low speed water flow through a circular tube of diameter, $D = 50mm$. Smoothly contoured plug of diameter, $d = 40mm$, is held in the end of the tube where the water discharges to the atmosphere.
Frictional effects are to be neglected. Velocity profiles may be assumed uniform at each section.



Find: (a) pressure measured by the gage shown.
(b) force required to hold plug

Solution:



Basic equation: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + g\cancel{z_1} = \cancel{\frac{p_2}{\rho}}^{0\,(gage)} + \frac{V_2^2}{2} + g\cancel{z_2}$

Assumptions: (1) steady flow
(2) incompressible flow
(3) no friction
(4) flow along a streamline
(5) $\Delta z = 0$

From the Bernoulli equation $p_1 = \frac{1}{2}\rho(V_2^2 - V_1^2)$

From continuity for uniform flow, $V_1 A_1 = V_2 A_2$

$$\therefore V_2 = V_1 \frac{A_1}{A_2} = V_1 \frac{D^2}{D^2 - d^2} = V_1 \frac{1}{1-(d/D)^2} = 7\,\frac{m}{s} \times \frac{1}{1-(0.8)^2} = 19.4\ m/s$$

and

$$p_1 = \frac{1}{2}\rho(V_2^2 - V_1^2) = \frac{1}{2} \times 999\,\frac{kg}{m^3}\left[(19.4)^2 - (7)^2\right]\frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 164\ kPa\ (gage) \leftarrow p_1$$

To determine the force required to hold the plug, apply the x-component of the momentum equation to the CV shown.

$$F_{S_x} + \cancel{F_{B_x}}^{=0} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}^{=0\ a(1)} + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

$$p_{1_g}A_1 - F = u_1\{-\dot{m}\} + u_2\{\dot{m}\} = \dot{m}(u_2 - u_1) = \rho V_1 A_1 (V_2 - V_1)$$

$$F = p_{1_g}A_1 - \rho V_1 A_1 (V_2 - V_1)$$

$$= 164 \times 10^3\,\frac{N}{m^2} \times \frac{\pi}{4}(0.05)^2 m^2 - 999\,\frac{kg}{m^3} \times 7\,\frac{m}{s} \times \frac{\pi}{4}(0.05)^2 m^2 (19.4 - 7)\,\frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F = 322N - 170N = 152\ N \text{ (in direction shown)} \leftarrow F$$

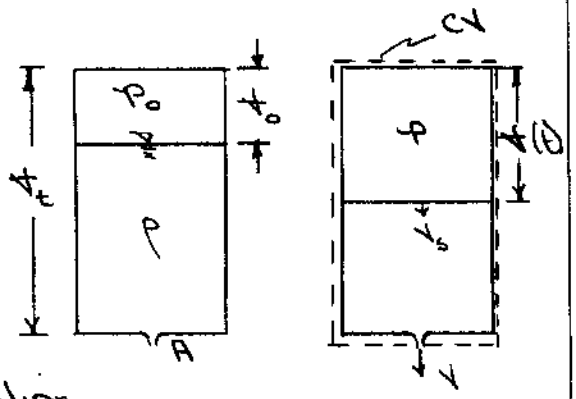
Given: High-pressure air forces a stream of water from a tiny, rounded orifice, of area A, in a tank. The air expands slowly so the expansion may be considered isothermal.

Find: (a) algebraic expression for $\dot{m}$ leaving the tank
(b) " " " dM/dt in tank
(c) expression for $M_w(t)$
(d) sketch $M_w(t)$

Solution:

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{const}$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$



Assumptions: (1) quasi steady flow $V_s \ll V$
(2) frictionless
(3) incompressible
(4) flow along a streamline
(5) uniform flow at outlet
(6) neglect gravity
(7) p_0 is p_{atm} $\therefore p_{abs} = p_{gage}$

Apply Bernoulli equation between liquid surface and orifice

$$V_j = \left[\frac{2(p - p_{atm})}{\rho}\right]^{1/2} = \sqrt{\frac{2p}{\rho}}$$

$$\dot{m} = \rho A V_j = \rho A \sqrt{\frac{2p}{\rho}} = \sqrt{2\rho p}\, A \longleftarrow \dot{m}$$

Rate of change of mass in tank is $\frac{dM}{dt} = \frac{\partial}{\partial t}\int \rho \, d\forall$

$$\frac{dM}{dt} = \rho_w \frac{d\forall_w}{dt} = -\rho_w \frac{d\forall_{air}}{dt} \quad (\forall_t = \forall_{air} + \forall_w) \longleftarrow \frac{dM}{dt}$$

For isothermal flow, $\frac{p}{\rho} = RT = \text{constant} = \frac{p_0}{\rho_0}$

where ρ is the air density and $\rho = m_{air}/\forall_{air}$

thus $p\forall = p_0 \forall_0$ or $p = p_0 \frac{\forall_0}{\forall}$

From continuity

$$0 = \rho_w \frac{d\forall_w}{dt} + \dot{m}$$

and

$$0 = -\rho_w \frac{d\forall_{air}}{dt} + \sqrt{2p\rho_w}\, A$$

$$\frac{d\forall}{dt} = \sqrt{\frac{2p}{\rho_w}} = \sqrt{\frac{2p_0 \forall_0}{\rho_w \forall}}$$

Separating variables, $h^{1/2}\,dh = \sqrt{\frac{2p_0}{\rho}}\,h_0\,A\,dt$

Integrating $\frac{2}{3}h^{3/2}\Big]_{h_0}^{h} = \sqrt{\frac{2p_0}{\rho_w}}\,h_0\,At$

$$\frac{2}{3}\left(h^{3/2} - h_0^{3/2}\right) = \frac{2}{3}h_0^{3/2}\left[\left(\frac{h}{h_0}\right)^{3/2} - 1\right] = \sqrt{\frac{2p_0}{\rho_w}}\,h_0\,At$$

Then $$\left(\frac{h}{h_0}\right)^{3/2} = \left[1 + \frac{2}{3}\frac{1}{h_0^{3/2}}\sqrt{\frac{2p_0}{\rho_w}}\,h_0\,At\right]$$

$$\frac{h}{h_0} = \left[1 + 1.11\sqrt{\frac{2p_0}{\rho_w}}\,\frac{At}{h_0}\right]^{2/3}$$

But $m_w = \rho_w(V_t - V) = \rho h_0\left\{\frac{V_t}{Ah_0} - \frac{Ah}{Ah_0}\right\}$

$$\therefore m_w = \rho_w h_0\left\{\frac{V_t}{Ah_0} - \left[1 + 1.11\sqrt{\frac{2p_0}{\rho_w}}\,\frac{At}{h_0}\right]^{2/3}\right\}$$

m_w

Clearly the mass of water in the tank decreases with time. The rate of decrease is equal to $\dot{m}_{out}$. Since $\dot{m}$ is a function of p, $\dot{m}$ decreases with time and the slope of $m_w(t)$ vs. time decreases with increasing time



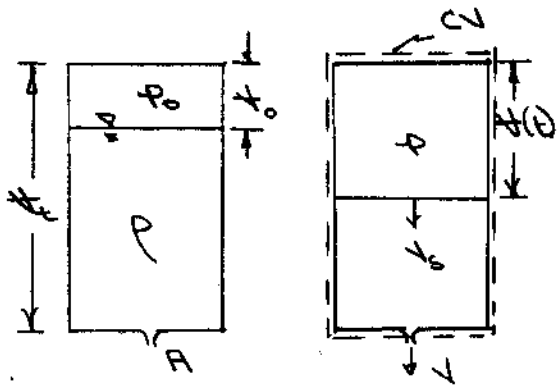
Given: High-pressure air forces a stream of water from a tiny rounded orifice, of area A, in a tank. The air expands rapidly so the expansion may be treated as adiabatic.

Find: (a) algebraic expression for $\dot{m}$ leaving the tank
(b) " " dm/dt in the tank
(c) expression for $M_w(t)$; sketch M_w vs t

Solution:

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{const}$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$



Assumptions: (1) quasi steady flow $V_s \ll V$
(2) frictionless
(3) incompressible
(4) flow along a streamline
(5) uniform flow at outlet
(6) neglect gravity
(7) p & p_{atm} $\therefore p_{abs} = p_{gage}$

Apply Bernoulli equation between liquid surface and orifice

$$V_j = \left[\frac{2(p - p_{atm})}{\rho}\right]^{1/2} = \sqrt{\frac{2p}{\rho}}$$

$$\dot{m} = \rho A V_j = \rho A \sqrt{\frac{2p}{\rho}} = \sqrt{2\rho p}\, A \quad \leftarrow \dot{m}$$

Rate of change of mass in tank is $\frac{dM}{dt} = \frac{\partial}{\partial t}\int \rho\, d\forall$

$$\frac{dM}{dt} = \rho_w \frac{d\forall_w}{dt} = -\rho_w \frac{d\forall_{air}}{dt} \quad (\forall_t = \forall_{air} + \forall_w) \quad \leftarrow \frac{dM}{dt}$$

For adiabatic expansion of air $p/\rho^k = \text{constant}$

Since mass of air is constant, $p_0 \forall_0^k = p \forall^k$.

From continuity, $-\rho_w \frac{d\forall_{air}}{dt} + \sqrt{2\rho\rho_w}\, A = 0$

$$\frac{d\forall_{air}}{dt} = \frac{A\sqrt{2}}{\rho_w^{1/2}} p^{1/2} = A\sqrt{\frac{2}{\rho_w}}\left[\frac{p_0 \forall_0^k}{\forall^k}\right]^{1/2} = A\sqrt{\frac{2 p_0 \forall_0^k}{\rho_w}}\, \forall^{-k/2}$$

$$\forall^{k/2}\, d\forall = A\sqrt{\frac{2 p_0 \forall_0^k}{\rho_w}}\, dt = c\, dt \quad \text{where } c = A\sqrt{\frac{2 p_0 \forall_0^k}{\rho_w}}$$

Integrating

$$\left(\frac{2}{k+2}\right) \forall^{\frac{k}{2}+1} \Big]_{\forall_0}^{\forall} = ct$$

$$A_t^{\left(\frac{k+2}{2}\right)} - A_0^{\left(\frac{k+2}{2}\right)} = \frac{k+2}{2} C t$$

$$\left(\frac{A_t}{A_0}\right)^{\left(\frac{k+2}{2}\right)} = 1 + \frac{k+2}{2} C t + \frac{1}{A_0^{\left(\frac{k+2}{2}\right)}}$$

$$= 1 + \frac{(k+2)}{2} A \left[\frac{2\rho_0}{\rho_\omega} \frac{1}{A_0^{k+2}}\right]^{1/2} + \left[\frac{1}{A_0^{\left(\frac{k+2}{2}\right)}}\right]^{1/2} t$$

$$\left(\frac{A_t}{A_0}\right)^{\frac{k+2}{2}} = 1 + \frac{(k+2)}{2} A \left[\frac{2\rho_0}{\rho_\omega} \frac{1}{A_0^{k+2}}\right]^{1/2} t = 1 + \frac{(k+2)}{2} \frac{A}{A_0} \left[\frac{2\rho_0}{\rho_\omega}\right]^{1/2} t$$

$$\frac{A_t}{A_0} = \left[1 + \frac{A}{A_0} \sqrt{\frac{2\rho_0}{\rho_\omega}} \left(\frac{k+2}{2}\right) t\right]^{2/k+2}$$

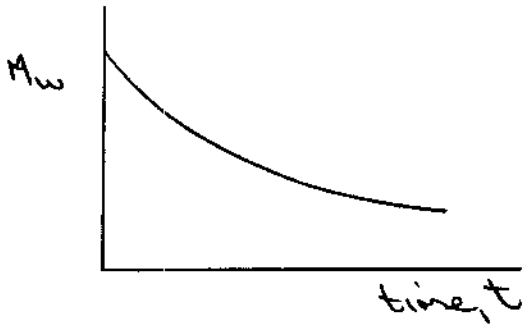
$$M_\omega = \rho_\omega (A_t - V_b) = \rho A_0 \left\{\frac{A_t}{A_0} - \frac{V_b}{A_0}\right\}.$$

$$M_\omega = \rho_\omega A_0 \left\{\frac{A_t}{A_0} - \left[1 + \frac{A}{A_0} \sqrt{\frac{2\rho_0}{\rho_\omega}} \left(\frac{k+2}{2}\right) t\right]^{2/k+2}\right\}$$

$$M_\omega = \rho_\omega A_0 \left\{\frac{A_t}{A_0} - \left[1 + 1.20 \sqrt{\frac{2\rho_0}{\rho_\omega}} \frac{A}{A_0} t\right]^{0.588}\right\}$$

M_ω

Clearly the mass of water in the tank decreases with time. The rate of decrease is equal to $\dot{m}_{out}$. Since $\dot{m}$ is a function of ρ, $\dot{m}$ decreases with time and the slope of $M_\omega(t)$ vs time decreases with increasing time.



6-3.4 [5]

Open-Ended Problem Statement: Describe the pressure distribution on the exterior of a multi-story building in a steady wind. Identify the locations of the maximum and minimum pressures on the outside of the building. Discuss the effect of these pressures on infiltration of outside air into the building.

Discussion: A multi-story building acts as a bluff-body obstruction in a thick atmospheric boundary layer. The boundary-layer velocity profile causes the air speed near the top of the building to be highest and that toward the ground to be lower.

Obstruction of air flow by the building causes regions of stagnation pressure on upwind surfaces. The stagnation pressure is highest where the air speed is highest. Therefore the maximum surface pressure occurs near the roof on the upwind side of the building. Minimum pressure on the upwind surface of the building occurs near the ground where the air speed is lowest.

The minimum pressure on the entire building will likely be in the low-speed, low-pressure wake region on the downwind side of the building.

Static pressure inside the building will tend to be an average of all the surface pressures that act on the outside of the building. It is never possible to seal all openings completely. Therefore air will tend to infiltrate into the building in regions where the outside surface pressure is above the interior pressure, and will tend to pass out of the building in regions where the outside surface pressure is below the interior pressure. Thus generally air will tend to move through the building from the upper floors toward the lower floors, and from the upwind side to the downwind side.

12-782 500 SHEETS FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Problem 6.71

Open-Ended Problem Statement: Imagine a garden hose with a stream of water flowing out through a nozzle. Explain why the end of the hose may be unstable when held a half meter or so from the nozzle end.

Discussion: Water flowing out of the nozzle tends to exert a thrust force on the end of the hose. The thrust force is aligned with the flow from the nozzle and is directed toward the hose.

Any misalignment of the hose will lead to a tendency for the thrust force to bend the hose further. This will quickly become unstable, with the result that the free end of the hose will "flail" about, spraying water from the nozzle in all directions.

This instability phenomenon can be demonstrated easily in the backyard. However, it will tend to do least damage when the person demonstrating it is wearing a bathing suit!

13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-390 100 SHEETS WIDE RULE
42-399 200 RECYCLED WHITE 5 SQUARE
National® Brand

6-3.4[5]

Open-Ended Problem Statement: An aspirator provides suction by using a stream of water flowing through a venturi. Analyze the shape and dimensions of such a device. Comment on any limitations on its use.

Discussion: The basic shape of the aspirator channel should be a converging nozzle section to reduce pressure followed by a diverging diffuser section to promote pressure recovery. The basic shape is that of a venturi flow meter.

If the diffuser exhausts to atmosphere, the exit pressure will be atmospheric. The pressure rise in the diffuser will cause the pressure at the diffuser inlet (venturi throat) to be below atmospheric.

A small tube can be brought in from the side of the throat to aspirate another liquid or gas into the throat as a result of the reduced pressure there.

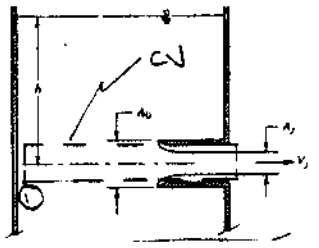
The following comments can be made about limitations on the aspirator:

1. It is desirable to minimize the area of the aspirator tube compared to the flow area of the venturi throat. This minimizes the disturbance of the main flow through the venturi and promotes the best possible pressure recovery in the diffuser.
2. It is desirable to avoid cavitation in the throat of the venturi. Cavitation alters the effective shape of the flow channel and destroys the pressure recovery in the diffuser. To avoid cavitation, the reduced pressure must always be above the vapor pressure of the driver liquid.
3. It is desirable to limit the flow rate of gas into the venturi throat. A large amount of gas can alter the flow pattern and adversely affect pressure recovery in the diffuser.

The best combination of specific dimensions could be determined experimentally by a systematic study of aspirator performance. A good starting point probably would be to use dimensions similar to those of a commercially available venturi flow meter.

13-782 500 SHEETS FILLER 5 SQUARE
42-391 50 SHEETS EYE EASE® 5 SQUARE
42-392 100 SHEETS EYE EASE® 5 SQUARE
42-393 200 SHEETS EYE EASE® 5 SQUARE
42-398 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National Brand

Given: Reentrant orifice in the side of a large tank. Pressure along the tank walls is essentially hydrostatic.



Find: The contraction coefficient, $C_c = A_j / A_o$

Solution:

Apply the x-component of the momentum equation to the CV shown

$$F_{Sx} + \cancel{F_{Bx}} = \cancel{\frac{\partial}{\partial t}}^{0(1)} \int_{CV} u\,\rho\,d\forall + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: (1) steady flow
(2) uniform flow at jet exit.
(3) hydrostatic pressure variation across CS ①, $V_1 \approx 0$
(4) momentum flux across horizontal portion of CS is negligible.
(5) ρ = constant

Then

$$\int_{A_o} p\,dA_1 = \dot{m} V_j = \rho V_j A_j V_j = \rho A_j V_j^2$$

$$p_1 A_o = \rho g h A_o = \rho A_j V_j^2$$

$$\therefore \frac{A_o}{A_j} = \frac{V_j^2}{gh}$$

Apply the Bernoulli equation along the central streamline from ① to the jet exit.

$$\frac{p_1}{\rho} + \cancel{\frac{V_1^2}{2}}^{0(3)} + \cancel{g z_1}^{0(gage)} = \cancel{\frac{p_j}{\rho}} + \frac{V_j^2}{2} + g z_j$$

Assumptions: (6) frictionless flow

$$p_1 = \rho g h = \rho \frac{V_j^2}{2}$$

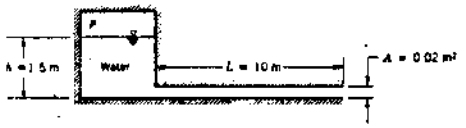
$$\therefore \frac{V_j^2}{gh} = 2$$

and

$$\frac{A_o}{A_j} = \frac{V_j^2}{gh} = 2$$

$$\therefore C_c = \frac{A_j}{A_o} = \frac{1}{2}$$ ← C_c

Given: Compressed air is used to accelerate water in tube. Velocity in tube is uniform at any section.
$V = 2 ft/s$ $dV/dt = 2.5 m/s^2$



Find: Pressure in tank for given conditions

Solution:

Basic equation: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V_s}{\partial t} ds$

Assumptions: (1) frictionless flow (2) incompressible flow
(3) flow along a streamline.
(4) $p_2 = p_{atm}$
(5) $V_1 \approx 0$

$p_{1g} = p_1 - p_{atm} = \rho\left[\frac{V_2^2}{2} - g(z_1 - z_2)\right] + \rho\int_1^2 \frac{\partial V_s}{\partial t} ds$

From continuity, for incompressible flow in a constant area tube, $V_2 = V_s = V$.

$\therefore p_{1g} = \rho\left[\frac{V_2^2}{2} - g(z_1 - z_2) + \left(\frac{dV}{dt}\right)\int_0^L ds\right]$

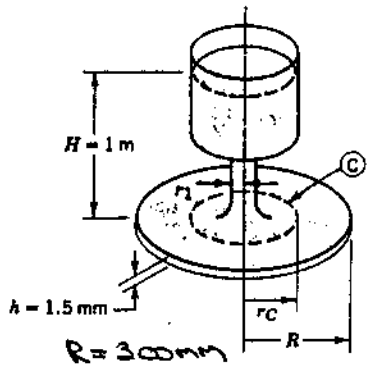
$= \rho\left[\frac{V_2^2}{2} - g(z_1 - z_2) + \left(\frac{dV}{dt}\right)L\right]$

$= 999 \frac{kg}{m^3}\left[\frac{1}{2}\left(\frac{2 ft}{s}\right)^2 - 9.81 \frac{m}{s^2} \times 1.5 m + 2.5 \frac{m}{s^2} \times 10 m\right] \times \frac{N \cdot s^2}{kg \cdot m}$

$p_{1g} = 12.3 kN/m^2$

Given: Flow between parallel disks shown is started from rest at t=0. The reservoir level is maintained constant; $r_1 = 50\,mm$.

Find: Rate of change of volume flow, dQ/dt, at $t=0$



Solution:

Apply the unsteady Bernoulli equation from the surface to the exit.

$$\frac{p_s}{\rho} + \frac{V_s^2}{2} + gz_s = \frac{p_e}{\rho} + \frac{V_e^2}{2} + gz_e + \int_1^2 \frac{\partial V_s}{\partial t}\,ds$$

$$gH = \frac{V_e^2}{2} + \int_1^2 \frac{\partial V_s}{\partial t}\,ds$$

Assumptions: (1) frictionless flow
(2) incompressible flow
(3) flow along a streamline.

For uniform flow at any section between the plates, for $r \geq r_1$, the volume flow rate is given by

$$Q = \int \vec{V}\cdot d\vec{A} = V_r\, 2\pi r h \quad \text{and} \quad V_r = \frac{Q}{2\pi r h}$$

At the exit $V_e = Q/2\pi r_1 h$

Assume that the rate of change of fluid velocity in the reservoir (out to $r = r_1$) is negligible. Then

$$\int_1^2 \frac{\partial V_s}{\partial t}\,ds = \frac{\partial}{\partial t}\int_1^2 V_r\,dr = \frac{\partial}{\partial t}\int_1^2 \frac{Q}{2\pi h}\frac{dr}{r} = \frac{\ln R/r_1}{2\pi h}\frac{dQ}{dt}$$

Then substituting into the unsteady Bernoulli equation, we obtain

$$gH = \frac{Q^2}{8\pi^2 R^2 h^2} + \frac{\ln R/r_1}{2\pi h}\frac{dQ}{dt}$$

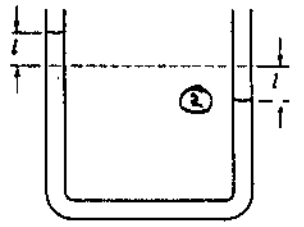
At $t=0$, $Q=0$ and

$$\left.\frac{dQ}{dt}\right|_{t=0} = \frac{2\pi h g H}{\ln R/r_1}$$

$$= 2\pi \times 0.0015\,m \times 9.81\,\frac{m}{s^2} \times 1\,m \times \frac{1}{\ln \frac{300}{50}}$$

$$\left.\frac{dQ}{dt}\right|_{t=0} = 0.0516\ m^3/s/s$$

Given: U-tube manometer of constant area as shown
Manometer fluid is initially deflected and then released



Find: a differential equation for l as a function of time

Solution

Basic equation: $\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V_s}{\partial t}\, ds$

Assumptions: (1) incompressible flow
(2) frictionless flow
(3) flow along a streamline

Since $P_1 = P_2 = P_{atm}$ and $V_1^2 = V_2^2$, then

$$g(z_1 - z_2) = \int_1^2 \frac{\partial V_s}{\partial t}\, ds$$

Let L = total length of column
l = deflection

Then $ds = dL$
$V_s = V = \frac{dl}{dt}$

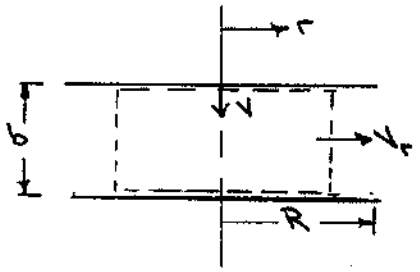
$$\therefore 2gl = \int_1^2 \frac{\partial V}{\partial t}\, dL = \frac{\partial V}{\partial t} \int_1^2 dL = L \frac{\partial V}{\partial t}$$

Since $V = -\frac{dl}{dt}$

$$2gl = L \frac{\partial V}{\partial t} = -L \frac{d^2 l}{dt^2}$$

Finally $\frac{d^2 l}{dt^2} + \frac{2g}{L} l = 0$

Given: Two circular discs of radius, R, are separated by a distance, b.
Upper disc moves toward the lower one at speed, V
Fluid between discs is incompressible and is squeezed out radially
Assume frictionless flow and uniform radial flow and any radial section
Pressure surrounding disc is at Patm



Find: gage pressure at r=0

Solution:

Basic equation: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V_s}{\partial t}\, ds$

$$0 = \frac{\partial}{\partial t}\int_{cv} \rho\, d\forall + \int_{cs} \rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) incompressible flow
(2) frictionless flow
(3) flow along a streamline
(4) uniform radial flow at any r
(5) neglect elevation changes.

Then,

$$0 = \frac{\partial}{\partial t}\int_{cv} \rho\, d\forall + \int_{cs} \rho \vec{V}\cdot d\vec{A} = \frac{\partial}{\partial t}(\rho \pi r^2 b) + \rho V_r 2\pi r b.$$

$$= \rho\pi r^2 \frac{\partial b}{\partial t} + \rho V_r 2\pi r b. \qquad \text{But } \frac{\partial b}{\partial t} = -V$$

$$\therefore 0 = -\rho\pi r^2 V + \rho V_r 2\pi r b \qquad \text{and } V_r = V\frac{r}{2b}$$

Applying the Bernoulli equation between point ① (r=r) and point ② (r=R)

$$p_1 - p_2 = \frac{\rho}{2}\left[V_2^2 - V_1^2\right] + \int_r^R \rho \frac{\partial V_r}{\partial t}\, dr \qquad \text{Now, } \frac{\partial V_r}{\partial t} = \frac{\partial}{\partial t}\left(V\frac{r}{2b}\right) = \frac{rV}{2}\left(-\frac{1}{b^2}\frac{db}{dt}\right) = \frac{V^2 r}{2b^2}$$

$$= \frac{\rho}{2}\left[\left(\frac{VR}{2b}\right)^2 - \left(\frac{Vr}{2b}\right)^2\right] + \int_r^R \rho \frac{V^2 r}{2b^2}\, dr$$

$$= \frac{\rho V^2}{8b^2}\left[R^2 - r^2\right] + \frac{\rho V^2}{4b^2} r^2\Big]_r^R = \frac{\rho V^2}{8b^2}\left[R^2 - r^2\right] + \frac{\rho V^2}{4b^2}\left[R^2 - r^2\right]$$

$$p_1 - p_{atm} = \frac{3}{8}\frac{\rho V^2}{b^2}\left[R^2 - r^2\right] = \frac{3}{8}\frac{\rho V^2 R^2}{b^2}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

When r = 0 $\quad p_1 = p_c$

$$\therefore p_c - p_{atm} = \frac{3}{8}\frac{\rho V^2 R^2}{b^2}$$

Given: A cylindrical tank of diameter, $D = 50mm$, drains through an opening, $d = 5mm$, in the body of the tank. If the flow is assumed to be quasi-steady, the speed of the liquid leaving the tank may be approximated by $V = \sqrt{2gy}$, where y is the height from the tank bottom to the free surface.

Find: Using the Bernoulli equation for unsteady flow along a streamline, evaluate the minimum diameter ratio, D/d, required to justify the assumption that flow from the tank is quasi-steady.

Solution:

For incompressible, frictionless flow along a streamline, the unsteady Bernoulli equation is

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gy_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gy_2 + \int_1^2 \frac{\partial V}{\partial t}\, ds$$

$p_1 = p_2 = p_{atm}$, $y_2 = 0$

From continuity $V_1 A_1 = V_2 A_2 = V_j A_j$

$$\therefore \frac{1}{2} V_j^2 \left(\frac{A_j}{A_1}\right)^2 + gy_1 = \frac{1}{2} V_j^2 + \int_1^2 \frac{\partial V}{\partial t}\, ds$$

or,

$$gy_1 = \frac{1}{2} V_j^2 \left[1 - \left(\frac{A_j}{A_1}\right)^2\right] + \int_1^2 \frac{\partial V}{\partial t}\, ds$$

If we assume quasi-steady flow, we say that $\int_1^2 \frac{\partial V}{\partial t}\, ds$ is negligible and hence $\frac{2gy}{V_j^2\left[1 - AR^2\right]} = 1$ where $AR = \frac{A_j}{A_1}$

Now, $\int_1^2 \frac{\partial V}{\partial t} \approx L \frac{dV}{dt} = L \frac{dV}{dH} = L \frac{d}{dH}\left(V_j \frac{A_j}{A_1}\right) = L \frac{A_j}{A_1} \frac{dV_j}{dH}$

Thus for the assumption to be reasonable we must have

$$\left| L \frac{A_j}{A_1} \frac{dV_j}{dH} \right| \ll gy \quad \text{or} \quad \left| \frac{A_j}{A_1} \frac{dV_j}{dH} \right| \ll \frac{g}{L}$$

Under the assumption of quasi-steady flow

$$V_j = \left[2gy \frac{1}{(1 - AR^2)}\right]^{1/2} \quad \text{where } AR = A_j/A_1$$

then,

$$\frac{dV_j}{dH} = \sqrt{\frac{2g}{(1-AR^2)}}\, \frac{-1}{2} y^{-1/2} \frac{dy}{dH} = \frac{dy}{dH} \sqrt{\frac{g}{2y(1-AR^2)}}$$

Since $\frac{dy}{dH} = -V_j = -V_j \frac{A_j}{A_1}$, then

$$\frac{dV_j}{dH} = -V_j \frac{A_j}{A_1} \sqrt{\frac{g}{2y(1-AR^2)}} = -\frac{A_j}{A_1} \sqrt{\frac{V_j^2 (1-AR^2)}{2gy}}\, \frac{g}{(1-AR^2)}$$

and

$$\frac{dV_j}{dH} = -\frac{A_j}{A_1} \frac{g}{(1-AR^2)}$$

For $\left| \frac{A_i}{A_1} \frac{dV_i}{dt} \right| \ll g$, then $\left(\frac{A_i}{A_1}\right)^2 \frac{1}{(1-AR^2)} \ll 1$

If we take

$$\left(\frac{A_i}{A_1}\right)^2 \frac{1}{(1-AR^2)} \approx 0.01$$

then,

$$\left(\frac{A_i}{A_1}\right)^2 = 0.01(1-AR^2) = 0.01\left[1-\left(\frac{A_i}{A_1}\right)^2\right]$$

and

$$1.01\left(\frac{A_i}{A_1}\right)^2 = 0.01$$

$$\frac{A_i}{A_1} = 0.0995$$

or

$$\frac{D_i}{D_1} = \left(\frac{A_i}{A_1}\right)^{1/2} = 0.32. \longleftarrow$$

In problem 4.34, $D_i/D_1 = d/D = 0.1$ and hence the assumption of quasi-steady flow is valid.

42-381 50 SHEETS 5 SQUARE
42-382 100 SHEETS 5 SQUARE
42-389 200 SHEETS 5 SQUARE
NATIONAL

Given: Two vortex flows with velocity fields

$$\vec{V}_1 = \omega r \hat{e}_\theta \qquad \vec{V}_2 = \frac{K}{2\pi r}\hat{e}_\theta$$

Determine: if the Bernoulli equation can be applied between different radii for each flow.

Solution: Since $V_r = 0$, the streamlines are concentric circles. In order for it to be possible to apply the Bernoulli equation between different radii, it is necessary that the flow be irrotational.

Basic equation: $\vec{\omega} = \frac{1}{2}\nabla\times\vec{V}$

Flow (1)

$$\nabla\times\vec{V}_1 = \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{k}\frac{\partial}{\partial z}\right)\times \omega r\hat{e}_\theta$$

$$= \hat{e}_r\times\hat{e}_\theta\frac{\partial}{\partial r}(\omega r) + \hat{e}_r\times\omega r\frac{\partial\hat{e}_\theta}{\partial r}^{0} + \hat{e}_\theta\times\hat{e}_\theta\frac{1}{r}\frac{\partial(\omega r)}{\partial\theta}^{0} + \hat{e}_\theta\times\frac{\omega r}{r}\frac{\partial\hat{e}_\theta}{\partial\theta}$$

$$= \hat{k}\,\omega + \hat{e}_\theta\times\omega(-\hat{e}_r)$$

$$\nabla\times\vec{V}_1 = 2\omega\hat{k}$$

∴ Flow (1) is rotational and Bernoulli equation cannot be applied between different radii.

Flow (2)

$$\nabla\times\vec{V}_2 = \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{k}\frac{\partial}{\partial z}\right)\times\frac{K}{2\pi r}\hat{e}_\theta$$

$$= \hat{e}_r\times\hat{e}_\theta\frac{\partial}{\partial r}\left(\frac{K}{2\pi r}\right) + \hat{e}_r\times\left(\frac{K}{2\pi r}\right)\frac{\partial\hat{e}_\theta}{\partial r}^{0} + \hat{e}_\theta\times\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{K}{2\pi r}\right) + \hat{e}_\theta\times\frac{1}{r}\left(\frac{K}{2\pi r}\right)\frac{\partial\hat{e}_\theta}{\partial\theta}$$

$$= -\hat{k}\,\frac{K}{2\pi r^2} + \hat{e}_\theta\,\frac{K}{2\pi r^2}\times(-\hat{e}_r)$$

$$= -\hat{k}\,\frac{K}{2\pi r^2} + \hat{k}\,\frac{K}{2\pi r^2}$$

$$\nabla\times\vec{V}_2 = 0$$

Since the flow field is irrotational, Bernoulli equation can be applied between different radii if the flow is also incompressible and frictionless.

Given: Two vortex flows with velocity fields

$$\vec{V}_1 = \omega r \hat{e}_\theta \qquad \vec{V}_2 = \frac{K}{2\pi r}\hat{e}_\theta$$

Determine: if the Bernoulli equation can be applied between different radii for each flow.

Solution: Since $V_r = 0$, the streamlines are concentric circles. In order for it to be possible to apply the Bernoulli equation between different radii, it is necessary that the flow be irrotational.

Basic equation: $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V}$

Flow (1)

$$\nabla \times \vec{V}_1 = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}\right) \times \omega r \hat{e}_\theta$$

$$= \hat{e}_r \times \hat{e}_\theta \frac{\partial}{\partial r}(\omega r) + \hat{e}_r \times \omega r \frac{\partial \hat{e}_\theta}{\partial r}^{0} + \hat{e}_\theta \times \hat{e}_\theta \frac{1}{r}\frac{\partial(\omega r)}{\partial \theta}^{0} + \hat{e}_\theta \times \frac{\omega r}{r}\frac{\partial \hat{e}_\theta}{\partial \theta}$$

$$= \hat{k}\,\omega + \hat{e}_\theta \times \omega(-\hat{e}_r)$$

$$\nabla \times \vec{V}_1 = 2\omega \hat{k}$$

∴ Flow (1) is rotational and Bernoulli equation cannot be applied between different radii.

Flow (2)

$$\nabla \times \vec{V}_2 = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}\right) \times \frac{K}{2\pi r}\hat{e}_\theta$$

$$= \hat{e}_r \times \hat{e}_\theta \frac{\partial}{\partial r}\left(\frac{K}{2\pi r}\right) + \hat{e}_r \times \left(\frac{K}{2\pi r}\right)\frac{\partial \hat{e}_\theta}{\partial r}^{0} + \hat{e}_\theta \times \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{K}{2\pi r}\right)^{0} + \hat{e}_\theta \times \frac{1}{r}\left(\frac{K}{2\pi r}\right)\frac{\partial \hat{e}_\theta}{\partial \theta}$$

$$= -\hat{k}\,\frac{K}{2\pi r^2} + \hat{e}_\theta \frac{K}{2\pi r} \times (-\hat{e}_r)$$

$$= -\hat{k}\,\frac{K}{2\pi r^2} + \hat{k}\,\frac{K}{2\pi r^2}$$

$$\nabla \times \vec{V}_2 = 0$$

Since the flow field is irrotational, Bernoulli equation can be applied between different radii if the flow is also incompressible and frictionless

Given: Flow field represented by $\psi = Ax^2y$; $A = 2.5\ ft^{-1}\cdot s^{-1}$, $\rho = 2.45\ slug/ft^3$.

Find: (a) Is the flow irrotational?
(b) If possible, determine $p_1 - p_2$ if $(x_1, y_1) = (1, 4)$ and $(x_2, y_2) = (2, 1)$

Solution:

The velocity field is determined from the stream function

$$u = \frac{\partial \psi}{\partial y} = Ax^2$$
$$v = -\frac{\partial \psi}{\partial x} = -2Axy$$
$$\therefore \vec{V} = Ax^2\hat{i} - 2Axy\hat{j}$$

Since $w = 0$ and $\frac{\partial w}{\partial z} = 0$, then

$$\nabla \times \vec{V} = \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \hat{k}(-2Ay) \neq 0$$

$\therefore$ flow is not irrotational ←

Note: For irrotational flow, $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$

For $\psi = Ax^2y$, $\nabla^2\psi = 2Ay \neq 0$ $\therefore$ flow is rotational

Since the flow is rotational, points ① and ② must be on the same streamline to apply the Bernoulli equation between the two points.

$$\psi_{x_1 y_1} = A(1)^2(4) = 4A \quad , \quad \psi_{x_2 y_2} = A(2)^2(1) = 4A$$

Hence,

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Assume flow in horizontal plane, i.e. $z_1 = z_2$

$$\vec{V}_1 = Ax_1^2\hat{i} - 2Ax_1y_1\hat{j} = 2.5\ \frac{1}{m\cdot s}\left[1^2\ m^2\hat{i} - 2\times 1\ m\times 4\ m\hat{j}\right] = 2.5\hat{i} - 20\hat{j}\ \frac{m}{s}$$

$$\vec{V}_2 = Ax_2^2\hat{i} - 2Ax_2y_2\hat{j} = \frac{2.5}{m\cdot s}\left[(2)^2\ m^2\hat{i} - 2\times 2\ m\times 1\ m\hat{j}\right] = 10\hat{i} - 10\hat{j}\ \frac{m}{s}$$

Thus $V_1^2 = 406\ m^2/s^2$ $\quad V_2^2 = 200\ m^2/s^2$

and

$$p_1 - p_2 = \rho\left[\frac{V_2^2}{2} - \frac{V_1^2}{2}\right] = \frac{\rho}{2}(V_2^2 - V_1^2)$$

$$= \frac{1}{2}\times 2.45\ \frac{slug}{ft^3}(200 - 406)\ \frac{ft^2}{s^2} \times \frac{lbf\cdot s^2}{ft\cdot slug}$$

$$p_1 - p_2 = -252\ lbf/ft^2$$

Given: Two-dimensional flow represented by the velocity field $\vec{V} = (Ax - By)t\,\hat{i} - (Bx + Ay)t\,\hat{j}$, where $A = 1 s^{-2}$, $B = 2 s^{-2}$, t is in s, and coordinates are in meters.

Find: (a) Is this a possible incompressible flow?
(b) Is the flow steady or unsteady?
(c) Show that the flow is irrotational
(d) Derive an expression for the velocity potential

Solution: For incompressible flow, $\nabla \cdot \vec{V} = 0$

For given flow $\nabla \cdot \vec{V} = \frac{\partial u}{\partial x}(Ax - By)t - \frac{\partial v}{\partial y}(Bx + Ay)t = At - At = 0$

$\therefore$ velocity field represents a possible incompressible flow

The flow is unsteady since $\vec{V} = \vec{V}(x, y, t)$

The rotation is given by $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$

$\vec{\omega} = \frac{1}{2}\left[\frac{\partial v}{\partial x} - (Bx + Ay)t - \frac{\partial u}{\partial y}(Ax - By)t\right] = -Bt + Bt = 0$

$\vec{\omega} = 0$, so flow is irrotational

From the definition of the velocity potential, $\vec{V} = -\nabla\phi$

$u = -\frac{\partial \phi}{\partial x}$ and $\phi = \int -u\,dx + f(y,t) = \int -(Ax - By)t\,dx + f(y,t)$

$$\phi = \left(-A\frac{x^2}{2} + Bxy\right)t + f(y,t)$$

$v = -\frac{\partial \phi}{\partial y}$ and $\phi = \int -v\,dy + g(x,t) = \int (Bx + Ay)t\,dy + g(x,t)$

$$\phi = \left(Bxy + A\frac{y^2}{2}\right)t + g(x,t)$$

Comparing the two expressions for ϕ we conclude

$$f(y,t) = \frac{A}{2}y^2 t \quad \text{and} \quad g(x,t) = -\frac{A}{2}x^2 t$$

Hence,

$$\phi = \left\{\frac{A}{2}(y^2 - x^2) + Bxy\right\}t$$

ϕ

Given: Flow field represented by $\psi = Ax^2y - By^3$, where $A = 1\ m^{-1}\cdot s^{-1}$, $B = \frac{1}{3}\ m^{-1}\cdot s^{-1}$, and coordinates are in meters.

Find: an expression for the velocity potential, ϕ

Solution:

The velocity field is determined from the stream function

$$u = \partial\psi/\partial y = Ax^2 - 3By^2$$
$$v = -\partial\psi/\partial x = -2Axy$$
$$\therefore \vec{V} = (Ax^2 - 3By^2)\hat{i} - 2Axy\hat{j}$$

The rotation is given by $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

$$\omega_z = \frac{1}{2}(-2Ay + 6By) = \frac{1}{2}\left(-2\times 1\, y + 6\times\frac{1}{3}y\right) = 0$$

Since $\omega_z = 0$, the flow is irrotational and $\vec{V} = -\nabla\phi$

Then

$u = -\frac{\partial\phi}{\partial x}$ and $\phi = \int -u\,dx + f(y) = \int (-Ax^2 + 3By^2)\,dx + f(y)$

$$\phi = -\frac{A}{3}x^3 + 3Bxy^2 + f(y)$$

$v = -\frac{\partial\phi}{\partial y}$ and $\phi = \int -v\,dy + g(x) = \int 2Axy\,dy + g(x)$

$$\phi = Axy^2 + g(x)$$

Comparing the two expressions for ϕ we

- note that $Axy^2 = 3Bxy^2$ $\quad (A=1,\ B=\frac{1}{3})$
- conclude that $g(x) = -\frac{A}{3}x^3$, $f(y) = 0$

Hence $\phi = Axy^2 - \frac{A}{3}x^3$ or $\phi = 3Bxy^2 - \frac{A}{3}x^3$ $\qquad \phi$

Given: Flow field represented by $\psi = x^2 - y^2$

Find: (a) the velocity field
(b) show that the flow field is irrotational
(c) the potential function, ϕ

Solution:

The velocity field is determined from the stream function.

$$u = \frac{\partial \psi}{\partial y} = -2y$$
$$v = -\frac{\partial \psi}{\partial x} = -2x$$
$$\therefore \vec{V} = -2y\hat{i} - 2x\hat{j} \quad \text{V}$$

If the flow is irrotational, then $\nabla \times \vec{V} = 0$

Since $w = 0$ and $\frac{\partial w}{\partial z} = 0$,

$$\nabla \times \vec{V} = \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \hat{k}\left[-2 - (-2)\right] = 0$$ Flow is irrotational

From $\vec{V} = -\nabla\phi$

$$u = -\frac{\partial \phi}{\partial x} \quad \text{and} \quad \phi = \int -u\,dx + f(y) = 2xy + f(y)$$
$$v = -\frac{\partial \phi}{\partial y} \quad \text{and} \quad \phi = \int -v\,dy + g(x) = 2xy + g(x)$$

Comparing these expressions, we see that neither contains a function of x only or a function of y only.

Thus $f(y) = g(x) = 0$ and

$$\phi = 2xy$$ ϕ

Given: Flow field represented by the potential function,

$$\phi = x^2 - y^2$$

Find: (a) Verify that this is an incompressible flow.
(b) Corresponding stream function

Solution:

The velocity field is given by $\vec{V} = -\nabla\phi$

$$\vec{V} = -\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 - y^2) = -2x\hat{i} + 2y\hat{j}$$

If the flow is incompressible, then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(-2x) + \frac{\partial}{\partial y}(2y) = -2 + 2 = 0$$

$\therefore$ flow is incompressible

From the definition of ψ, $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$

Thus,

$$u = \frac{\partial \psi}{\partial y} = -2x \qquad \psi = \int -2x\,dy + f(x) = -2xy + f(x)$$

Then

$$v = 2y = -\frac{\partial \psi}{\partial x} = 2y + \frac{df}{dx}$$

and $\frac{df}{dx} = 0$ or $f =$ constant

$$\therefore \psi = -2xy + C$$

Taking $C = 0$, then $\psi = -2xy$ ψ

Given: Flow field represented by the potential function,

$$\Phi = Ax^2 + Bxy - Ay^2$$

Find: a) Verify that the flow is incompressible
b) Determine the corresponding stream function, ψ

Solution:

The velocity field is given by $\vec{V} = -\nabla\Phi$

$$\vec{V} = -\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(Ax^2 + Bxy - Ay^2) = -\vec{i}(2Ax + By) - \vec{j}(Bx - 2Ay)$$

If the flow is incompressible, then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(-)(2Ax + By) + \frac{\partial}{\partial y}(-)(Bx - 2Ay) = -2A + 2A = 0$$

$\therefore$ flow is incompressible

From the definition of ψ, $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$

Thus,

$$u = -2Ax - By = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \psi = -\int (2Ax + By)\,dy + f(x)$$

$$\psi = -2Axy - B\frac{y^2}{2} + f(x)$$

Then,

$$v = -Bx + 2Ay = -\frac{\partial \psi}{\partial x} = 2Ay - \frac{df}{dx}$$

and $-\frac{df}{dx} = -Bx$ or $f = \frac{1}{2}Bx^2 + \text{constant}$

$$\therefore \psi = -2Axy - B\frac{y^2}{2} + B\frac{x^2}{2} + \text{constant}$$

Setting the constant equal to zero, we obtain

$$\psi = \frac{B}{2}(x^2 - y^2) - 2Axy \qquad \psi$$

Given: Flow field represented by the velocity potential $\phi = Ax + Bx^2 - By^2$, where $A = 1\ m \cdot s^{-1}$, $B = 1\ s^{-1}$, and coordinates are measured in meters.

Find: (a) expression for the velocity field
(b) stream function
(c) pressure difference between points $(x_1, y_1) = (0,0)$ and $(x_2, y_2) = (1,2)$

Solution

The velocity field is determined from the velocity potential

$$u = -\partial\phi/\partial x = -A - 2Bx$$
$$v = -\partial\phi/\partial y = 2By$$
$$\Rightarrow \vec{V} = -(A + 2Bx)\hat{i} + 2By\hat{j} \quad \text{V}$$

From the definition of the stream function, $u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial x}$

Then

$$\psi = \int u\,dy + f(x) = \int -(A + 2Bx)\,dy + f(x)$$
$$\psi = -Ay - 2Bxy + f(x)$$

Also,

$$\psi = \int -v\,dx + g(y) = \int -2By\,dx + g(y)$$
$$\psi = -2Bxy + g(y)$$

Comparing the two expressions for ψ we conclude

$$f(x) = 0 \quad , \quad g(y) = -Ay$$
$$\therefore \psi = -(Ay + 2Bxy) \quad \text{ψ}$$

Since $\nabla^2\phi = 2B - 2B = 0$, the flow is irrotational and the Bernoulli equation can be applied between any two points in the flow field

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 \qquad \{\text{Assume } \rho = \text{constant},\ z_1 = z_2\}$$

$$\vec{V}(0,0) = -A\hat{i} = -\hat{i}\ m/s \qquad V_{0,0} = 1\ m/s$$
$$\vec{V}(1,2) = -(A + 2B)\hat{i} + 4B\hat{j} = -3\hat{i} + 4\hat{j}\ m/s \qquad \therefore V_{1,2} = 5\ m/s$$

$$\therefore p_1 - p_2 = \rho\left(\frac{V_2^2}{2} - \frac{V_1^2}{2}\right) = \frac{\rho}{2}(V_2^2 - V_1^2)$$

Assume fluid is water

$$p_1 - p_2 = \frac{1}{2} \times 999\ \frac{kg}{m^3}\ (25 - 1)\ m^2/s^2 \times \frac{N \cdot s^2}{kg \cdot m} = 12\ kN/m^2$$

Given: Flow field represented by the velocity potential $\phi = Ax^2 - By^2$, where $A = \frac{1}{3}\ m^{-1}\cdot s^{-1}$, $B = 1\ m^{-1}\cdot s^{-1}$, and the coordinates are measured in meters.

Find: (a) expression for the magnitude of the velocity vector
(b) the stream function.

Plot: streamlines $\psi = -4, 0, 4\ m^2/s$ for $t \geq 0$; plot sufficient ϕ lines to demonstrate orthogonality of ϕ and ψ lines.

Solution:

The velocity field is determined from the velocity potential

$$u = -\partial\phi/\partial x = -2Ax = -2x \qquad v = -\partial\phi/\partial y = -(-2By) = 2y$$

$$\vec{V} = -2x\,\vec{i} + 2y\,\vec{j}$$

$$V = \left[u^2 + v^2\right]^{1/2} = \left[4x^2 + 4y^2\right]^{1/2} = \left[4(x^2 + y^2)\right]^{1/2}$$

$$V = 2\left(x^2 + y^2\right)^{1/2} \qquad \leftarrow V$$

The stream function is defined such that $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$

Then, $\psi = \int u\,dy + f(x) = \int -2x\,dy + f(x) = -2xy + f(x)$ ---(1)

Also, $\psi = \int -v\,dx + g(y) = \int -2y\,dx + g(y) = -2xy + g(y)$ ---(2)

Comparing the two expressions for ψ, we

- note that $-2xy = -2Axy$ $(B = 1,\ A = 1/3)$, and
- conclude that $f(x) = 0$ and $g(y) = 0$

$\therefore \psi = -2xy$ or $\psi = -2Axy = -\frac{2}{3}xy$ $\leftarrow \psi$

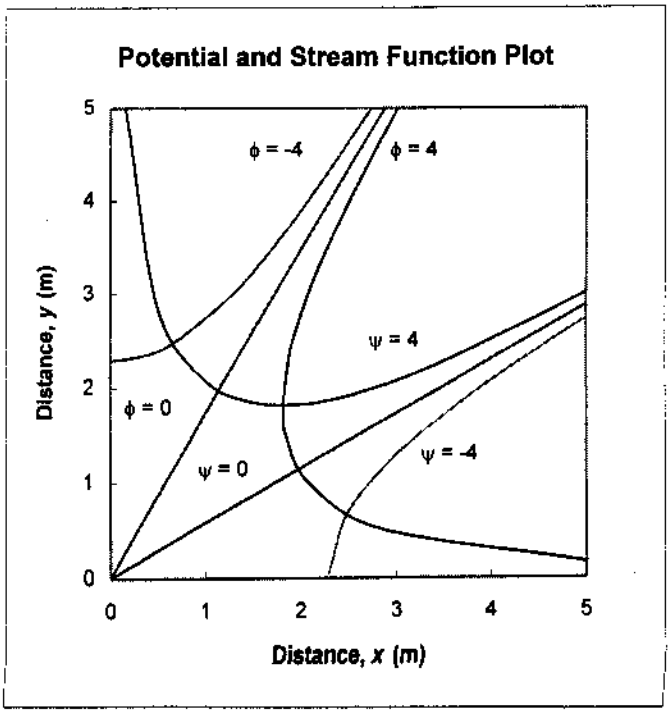
With $A = 1/3$, $B = 1$

$$\phi = Ax^2 - By^2 = \frac{1}{3}x^2 - y^2 = x\left(\frac{x}{3} - \frac{y^2}{x}\right)$$

For $\psi = 0$, $x = 0$ or $y = 0$ is $\pm x$

For $\psi = -4$, $y^2 = \frac{3}{2}x^2 + 4\frac{x}{2}$

For $\psi = 4$, $y^2 = \frac{3}{2}x^2 - 4\frac{x}{2}$



Given: Incompressible flow field represented by $\psi = 3Ax^2y - Ay^3$ where $A = 1\ m^{-1}\cdot s^{-1}$

Show: that this flow field is irrotational

Find: the velocity potential ϕ

Plot: a few streamlines in the first quadrant; plot sufficient ϕ lines to demonstrate orthogonality of ϕ and ψ lines.

Solution:

For a 2-D incompressible, irrotational flow $\nabla^2\psi = 0$ (6.30)

For this flow field:

$$\nabla^2\psi = \frac{\partial^2}{\partial x^2}(3Ax^2y - Ay^3) + \frac{\partial^2}{\partial y^2}(3Ax^2y - Ay^3) = 6Ay - 6Ay = 0$$ ← irrotational

The velocity field is given by $\vec{V} = u\hat{i} + v\hat{j}$

$$u = \frac{\partial\psi}{\partial y} = 3Ax^2 - 3Ay^2 = 3A(x^2 - y^2)$$
$$v = -\frac{\partial\psi}{\partial x} = -6Axy$$
$$\vec{V} = 3A(x^2 - y^2)\hat{i} - 6Axy\hat{j}$$

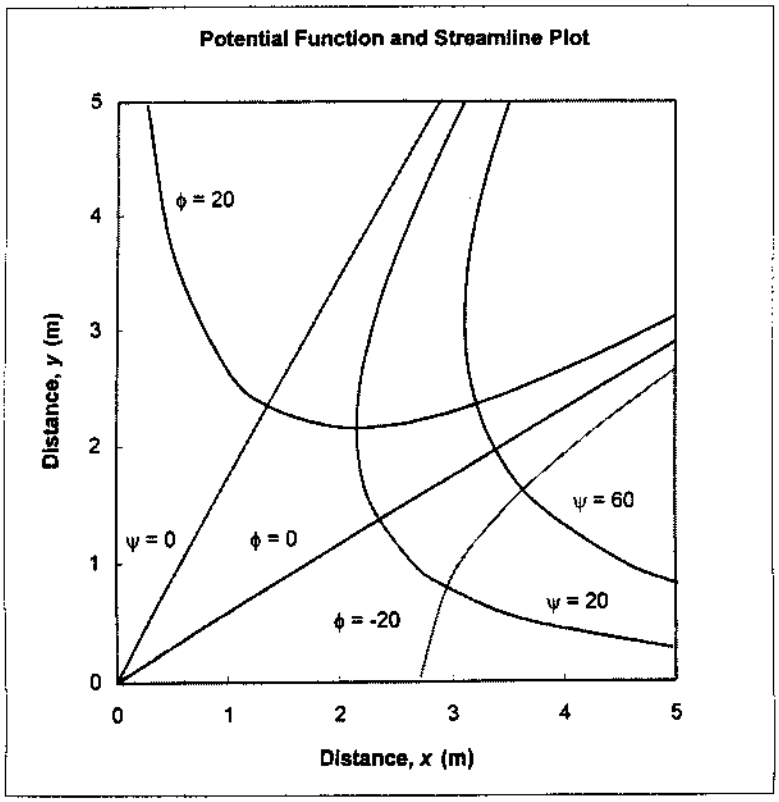
The velocity potential is defined such that $u = -\frac{\partial\phi}{\partial x}$, $v = -\frac{\partial\phi}{\partial y}$

Then, $\phi = -\int u\,dx + f(y) = -\int 3A(x^2 - y^2)\,dx + f(y) = -Ax^3 + 3Axy^2 + f(y)$ ..... (1)

Also, $\phi = -\int v\,dy + g(x) = \int 6Axy\,dy + g(x) = 3Axy^2 + g(x)$ ..... (2)

Equating expressions for ϕ (Eqs 1 and 2) we see that

$g(x) = -Ax^3$ and $f(y) = 0$ $\therefore \phi = 3Axy^2 - Ax^3$ ← ϕ



Given: Two-dimensional, inviscid flow with velocity field $\vec{V} = (A+1B)\hat{i} + (C-Ay)\hat{j}$, where $A = 3s^{-1}$, $B = 6 m/s$, $C = 4 m/s$ and the coordinates are measured in meters. The body force distribution is $\vec{B} = -g\hat{k}$; $\rho = 825 kg/m^3$.

Find: (a) if this is a possible incompressible flow
(b) stagnation point(s) of the flow field
(c) if the flow is irrotational
(d) the velocity potential (if one exists)
(e) pressure difference between origin and point $(x,y,z) = (2,2,2)$

Plot: a few streamlines in the upper half plane

Solution:

For incompressible flow $\nabla \cdot \vec{V} = 0$. For this flow

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x}(A+1B) + \frac{\partial v}{\partial y}(C-Ay) = A - A = 0$$

$\therefore$ velocity field represents possible incompressible flow.

At the stagnation point $u = v = 0$. $(\vec{V} = 0)$

$$u = 0 = (A+1B) \quad \therefore \quad x = -B/A = \frac{-6 m \cdot s^{-1}}{3 s^{-1}} = -2 m$$

$$v = 0 = (C-Ay) \quad \therefore \quad y = C/A = \frac{4 m \cdot s^{-1}}{3 s^{-1}} = 4/3 m$$

Stagnation point is at $(x,y) = (-2, 4/3) m$.

The fluid rotation (for a 2-D flow) is given by $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

For this flow $\omega_z = \frac{1}{2}\left[\frac{\partial (C-Ay)}{\partial x} - \frac{\partial (A+1B)}{\partial y}\right] = 0$

$\therefore$ flow is irrotational

Ref. $\vec{V} = -\nabla\phi$ and $u = -\partial\phi/\partial x$ and $v = -\partial\phi/\partial y$.

and

$$\phi = \int -u\,dx + f(y) = -\int (A+1B)\,dx + f(y) = -\frac{Ax^2}{2} - Bx + f(y) \quad \text{--- (1)}$$

Also

$$\phi = -\int v\,dy + g(x) = -\int (C - Ay)\,dy + g(x) = \frac{Ay^2}{2} - Cy + g(x) \quad \text{--- (2)}$$

Equating the two expressions for ϕ (Eqs 1 and 2) we note that

$$g(x) = -\left(\frac{Ax^2}{2} + Bx\right) \quad \text{and} \quad f(y) = \frac{Ay^2}{2} - Cy.$$

$$\therefore \quad \phi = \frac{A}{2}(y^2 - x^2) - Bx - Cy$$

Since the flow is irrotational we can apply the Bernoulli equation between any two points in the flow field

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

At point 1, $(0,0,0)$, $\vec{V} = B\hat{i} + C\hat{j} = 6\hat{i} + 4\hat{j}$ m/s, $V_1^2 = 52 m^2/s^2$

At point 2 (2,2,2) $\vec{V}_2 = [35^{-1} \times 2m + 6m/s]\hat{i} + [4m/s - 35^{-1} \times 2m]\hat{j}$

$\vec{V}_2 = 12\hat{i} - 2\hat{j}$ m/s , $V_2^2 = 148$ m^2/s^2

$p_1 - p_2 = \rho\left(\frac{V_2^2 - V_1^2}{2}\right) + \rho g(z_2 - z_1) = \rho\left[\frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1)\right]$

$= 825 \frac{kg}{m^3} \times \left[\frac{1}{2}(148 - 52)\frac{m^2}{s^2} + 9.81\frac{m}{s^2} \times (2m)\right] \times \frac{N}{kg \cdot m/s^2}$

$p_1 - p_2 = 55.8$ kPa ← Δp

The stream function is defined such that $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

For, $\psi = \int u\,dy + f(x) = \int (A x + B)\,dy + f(x) = Axy + By + f(x)$ ----(1)

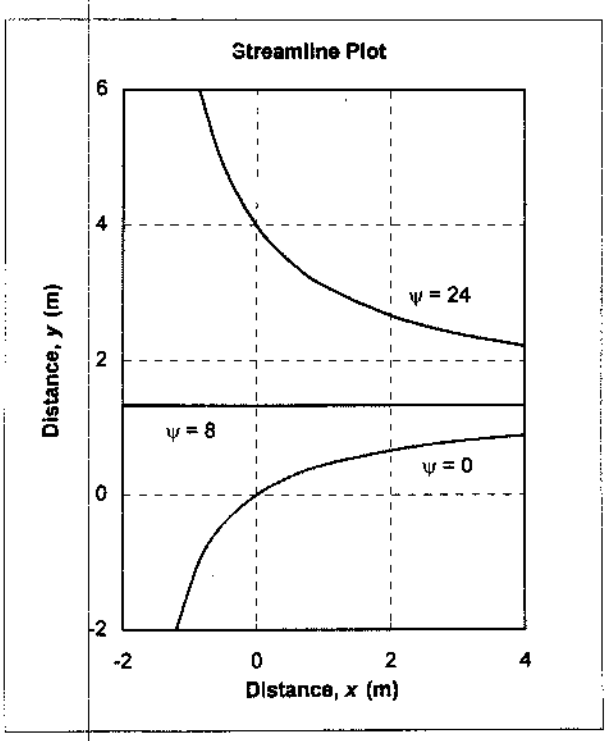
Also, $\psi = -\int v\,dx + g(y) = \int (-C + Ay)\,dx + g(y) = -Cx + Axy + g(y)$ ----(2)

Equating the two expressions for ψ (Eqs 1 and 2) we note that

$f(x) = -Cx$, $g(y) = By$ and $\therefore \psi = Axy + By - Cx$ ← ψ

The stagnation streamline goes through the stagnation point (-2, 4) m

$\psi_{stag} = 35^{-1} \times (-2m) \times 4m + 6m \cdot s^{-1} \times 4m - 4m \cdot s^{-1} \times (-2m) = 8$ m^2/s ← ψ_{stag}



Given: Irrotational flow represented by $\psi = Bxy$, where $B = 0.25\ s^{-1}$ and the coordinates are measured in meters

Find: (a) the rate of flow between points $(x_1, y_1) = (2,2)$ and $(x_2, y_2) = (3,3)$
(b) the velocity potential for this flow

Plot: (a) several streamlines in the first quadrant; indicate direction of flow
(b) some lines of constant ϕ

Solution:

The volume flow rate (per unit depth) between points ① and ② is given by

$$q_{12} = \psi_2 - \psi_1 = B[x_2 y_2 - x_1 y_1] = 0.25\ s^{-1}[3m \times 3m - 2m \times 2m]$$

$$q_{12} = 1.25\ m^3/s/m \qquad \underline{\qquad q_{12} \qquad}$$

The velocity field is determined from the stream function

$$u = \partial\psi/\partial y = Bx \qquad v = -\partial\psi/\partial x = -By \qquad \therefore \vec{V} = Bx\hat{i} - By\hat{j}$$

For irrotational flow $\vec{V} = -\nabla\phi$ and $u = -\partial\phi/\partial x$, $v = -\partial\phi/\partial y$

and

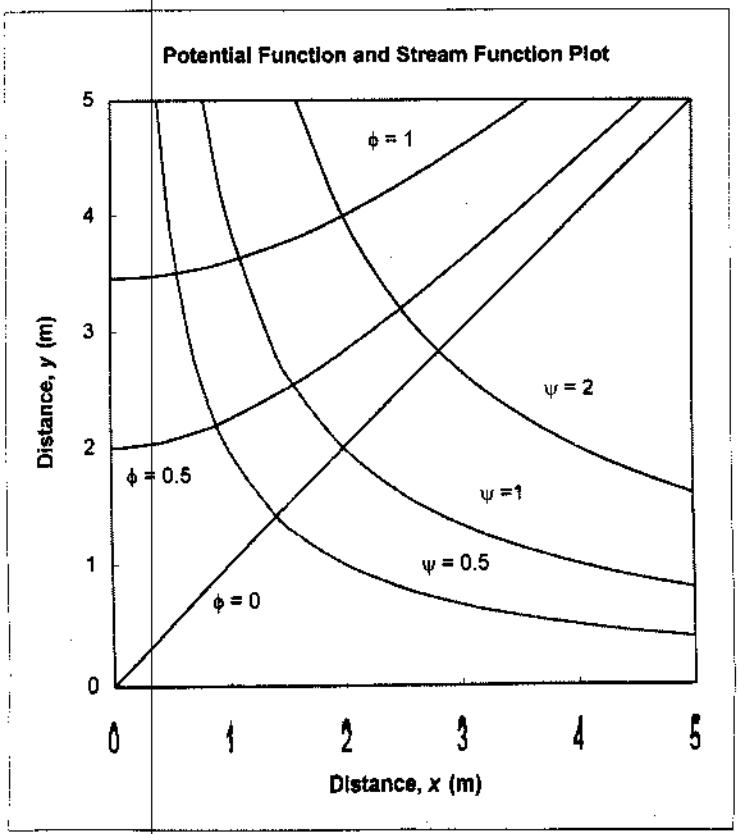
$$\phi = -\int u\,dx + f(y) = -\int Bx\,dx + f(y) = -\tfrac{1}{2}Bx^2 + f(y) \quad \text{---(1)}$$

Also

$$\phi = -\int v\,dy + g(x) = \int By\,dy + g(x) = \tfrac{1}{2}By^2 + g(x) \quad \text{---(2)}$$

Equating expressions for ϕ (Eqs 1 and 2) we conclude that

$$f(y) = \tfrac{1}{2}By^2, \quad g(x) = -\tfrac{1}{2}Bx^2 \quad \text{and} \quad \phi = \tfrac{1}{2}B(y^2 - x^2) \qquad \underline{\qquad \phi \qquad}$$



Given: Flow past a circular cylinder of Example Problem 6.11.

Find: (a) Show that $v_r = 0$ along the lines $(r, \theta) = (r, \pm\pi/2)$
(b) Plot v_θ/U versus r for $r \geq a$ along line $(r, \pi/2)$
(c) Find distance beyond which the influence of the cylinder on the velocity is less that 1% of U

Solution

From Example Problem 6.11

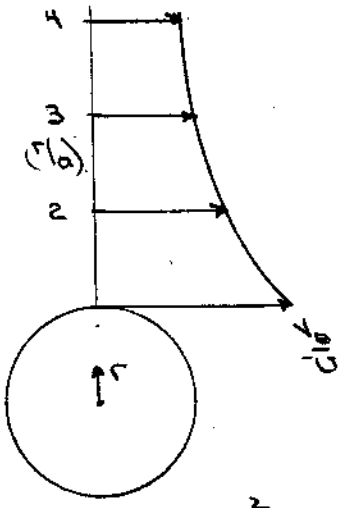
$$\vec{V} = \left(-\frac{\Lambda\cos\theta}{r^2} + U\cos\theta\right)\hat{e}_r + \left(-\frac{\Lambda\sin\theta}{r^2} - U\sin\theta\right)\hat{e}_\theta \quad \text{.... (1)}$$

Then $v_r = \left(-\frac{\Lambda}{r^2} + U\right)\cos\theta$ For $\theta = \pm\frac{\pi}{2}$, $\cos\theta = 0$ and $v_r = 0$ ←

$v_\theta = -\left(\frac{\Lambda}{r^2} + U\right)\sin\theta$, but $\frac{\Lambda}{U} = a^2$

$\therefore v_\theta = -\left(\frac{a^2}{r^2} + 1\right)U\sin\theta$ For $\theta = \pi/2$.

$$\frac{v_\theta}{U} = -\left(1 + \frac{a^2}{r^2}\right)$$



$$\vec{V} = U\cos\theta\left(1 - \frac{a^2}{r^2}\right)\hat{e}_r - U\sin\theta\left(1 + \frac{a^2}{r^2}\right)\hat{e}_\theta$$

For $\theta = \pi/2$

$$\frac{v_\theta}{U} = 1 + \frac{a^2}{r^2}$$

If $\frac{v_\theta}{U} = 1.01$ then $\frac{a^2}{r^2} = 0.01$ or $\frac{a}{r} = 0.1$

$\therefore \frac{v_\theta}{U} < 1\%$ for $r > 10\,a$ ←

Problem 6.92

Given: Consider flow around a circular cylinder with freestream velocity from right to left and a counterclockwise free vortex.

Find: Show that the lift force on the cylinder (per unit width) can be expressed as $F_L = -\rho U\Gamma$, as illustrated in Example Problem 6.12.

Discussion: The only change in this flow from the flow of Example Problem 6.12 is that the directions of the freestream velocity and the vortex are changed. This changes the sign of the freestream velocity from U to $-U$ and the sign of the vortex strength from K to $-K$. Consequently the signs of both terms in the equation for lift are changed. Therefore the direction of the lift force remains unchanged.

The analysis of Example Problem 6.12 (see page 282) shows that only the term involving the vortex strength contributes to the lift force. Therefore the expression for lift obtained with the changed freestream velocity and vortex strength is identical to that derived in Example Problem 6.12. Thus the general solution of Example Problem 6.12 holds for any orientation of the freestream and vortex velocities. For the present case, $F_L = -\rho U\Gamma$, as shown for the general case in Example Problem 6.12.

Given: A tornado is modelled by the superposition of a sink (strength, $q = 2800\ m^2/sec$) and a free vortex (strength, $K = 5600\ m^2/sec$)

Find: (a) Expressions for ψ and ϕ
(b) Estimate the radius beyond which the flow may be treated as incompressible.
(c) Find the gage pressure at that radius.

Solution:

$$\psi = \psi_{si} + \psi_{vo} = -\frac{q}{2\pi}\theta - \frac{K}{2\pi}\ln r \qquad \psi$$

$$\phi = \phi_{si} + \phi_{vo} = \frac{q}{2\pi}\ln r - \frac{K}{2\pi}\theta \qquad \phi$$

$\vec{V} = V_r \hat{i}_r + V_\theta \hat{i}_\theta$ $\qquad V_{r_{si}} = -\frac{q}{2\pi r}$, $V_{r_{vo}} = 0$; $V_{\theta_{si}} = 0$, $V_{\theta_{vo}} = \frac{K}{2\pi r}$

$\therefore \vec{V} = -\frac{q}{2\pi r}\hat{i}_r + \frac{K}{2\pi r}\hat{i}_\theta$

Then

$$V = (V_r^2 + V_\theta^2)^{1/2} = \left[\left(-\frac{q}{2\pi}\right)^2 + \left(\frac{K}{2\pi}\right)^2\right]^{1/2}\frac{1}{r}$$

For incompressible flow $M \le 0.3$. For standard air this corresponds to $V < 102\ m/sec$

Then, for incompressible flow

$$V = 102\ m/sec < \left[q^2 + K^2\right]^{1/2}\frac{1}{2\pi r}$$

or

$$r > \left[q^2 + K^2\right]^{1/2}\frac{1}{2\pi}\cdot\frac{s}{102\ m} = \left[(2800)^2 + (5600)^2\right]^{1/2}\frac{m^2}{s^2}\times\frac{1}{2\pi}\times\frac{s}{102\ m}$$

$$r > 9.77\ m \qquad r$$

To determine the gage pressure at this radius, apply the Bernoulli equation for irrotational flow

$$\frac{p_\infty}{\rho} + \frac{V_\infty^2}{2} + g z_\infty = \frac{p}{\rho} + \frac{V^2}{2} + g z \qquad \text{assume } \Delta z = 0$$

Then

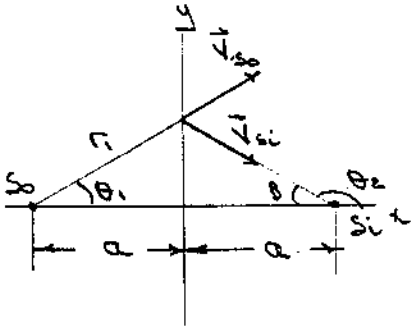
$$p_{gage} = p - p_\infty = -\frac{1}{2}\rho V^2 = -\frac{1}{2}\times 1.225\frac{kg}{m^3}\times(102)^2\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}$$

$$p_{gage} = -6.37\ kPa \quad \text{(for standard air)} \qquad p_{gage}$$

Given: A source and a sink of equal strength (q_0) are located on the x axis at $x=-a$ and $x=a$, respectively.

Find: (a) expressions for ψ and ϕ for the flow field
(b) points on the y axis between which half the total volume flow rate passes.

Solution:



$$\psi = \psi_{so} + \psi_{si} = \frac{q_0}{2\pi}\theta_1 - \frac{q_0}{2\pi}\theta_2 = \frac{q_0}{2\pi}(\theta_1 - \theta_2) \quad \psi$$

$$\phi = \phi_{so} + \phi_{si} = -\frac{q_0}{2\pi}\ln r_1 + \frac{q_0}{2\pi}\ln r_2 = \frac{q_0}{2\pi}\ln\frac{r_2}{r_1} \quad \phi$$

$$u = u_{so} + u_{si}$$

$$u_{so} = v_{r_1}\cos\theta_1 = \frac{q_0}{2\pi r_1}\cos\theta_1$$

$$u_{si} = |v_{r_2}|\cos\beta = \frac{q_0}{2\pi r_2}\cos\beta$$

Along the line $x=0$, $\beta = \theta_1$ and hence $u = 2u_{so} = \frac{q_0}{\pi r_1}\cos\theta_1$

Also $\cos\theta_1 = \frac{a}{r_1}$ and $r_1^2 = a^2 + y^2$

$$\therefore u = \frac{q_0}{\pi}\frac{a}{(a^2+y^2)}$$

The volume flow across the y axis is then (for a unit depth)

$$Q = \int u\,dy = \int_{-\infty}^{\infty} \frac{q_0}{\pi}\frac{a}{(a^2+y^2)}\,dy = \frac{q_0}{\pi}\,a\,\frac{1}{a}\arctan\left(\frac{y}{a}\right)\Big]_{-\infty}^{\infty}$$

$$Q = \frac{q_0}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = q_0$$

Then one half the flowrate, $q_0/2$, must passed between $y = \pm b$

$$\frac{q_0}{2} = \int_{-b}^{b} u\,dy = \int_{-b}^{b} \frac{q_0}{\pi}\frac{a}{(a^2+y^2)}\,dy = \frac{q_0}{\pi}\,a\,\frac{1}{a}\arctan\left(\frac{y}{a}\right)\Big]_{-b}^{b}$$

$$= \frac{q_0}{\pi}\left[\arctan\left(\frac{b}{a}\right) - \arctan\left(\frac{-b}{a}\right)\right] = \frac{q_0}{\pi}\,2\arctan\left(\frac{b}{a}\right)$$

$$\frac{q_0}{2} = \frac{q_0}{\pi}\,2\arctan\left(\frac{b}{a}\right)$$

$$\therefore \arctan\left(\frac{b}{a}\right) = \frac{\pi}{4}$$

or $\frac{b}{a} = \tan\frac{\pi}{4} = 1.0$ and $b = a$

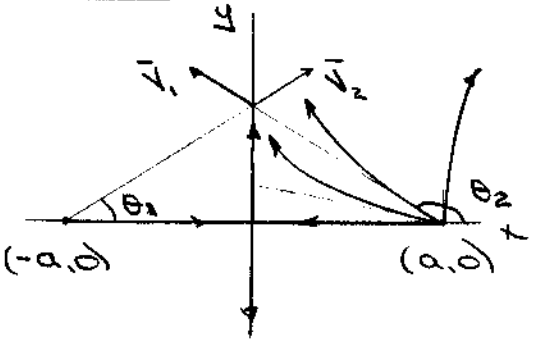
Thus, one half the total volume flow passes between the points $y = -a$ and $y = a$ on the y axis.

Given: Incompressible, inviscid flow formed by superposition of two sources of equal strength q located along the x axis at $x = \pm a$

Find: (a) Sketch the streamlines
(b) Indicate location of stagnation point(s)
(c) Evaluate ψ at the point $(x,y) = (a,a)$
(d) the volume flowrate between the y axis and the point $(x,y) = (a,a)$
(e) If pressure at stagnation point is p_0, obtain expressions for static and dynamic pressures at $(x,y) = (a,a)$

Solution The stream function for the flow is given by

$$\psi = \psi_1 + \psi_2 = \frac{q}{2\pi}\theta_1 + \frac{q}{2\pi}\theta_2$$



The velocity from a source is $V_r = \frac{q}{2\pi r}$

At any point the velocity is the vector sum of the components from each source

Stagnation occurs at the origin

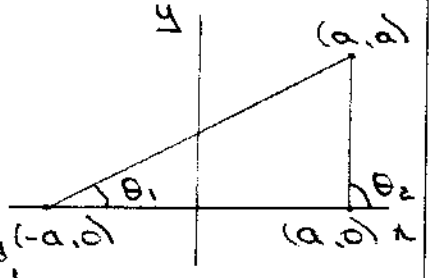
- x axis is a streamline as is y axis
- radial streamlines of a single source are bent as a result of second source
- streamline pattern is symmetric with respect to both x axis and y axis

(c) $\psi(a,a) = \psi_1(a,a) + \psi_2(a,a)$

$$\psi(a,a) = \frac{q}{2\pi}\theta_1 + \frac{q}{2\pi}\theta_2$$

$$\theta_1 = \tan^{-1}\frac{a}{2a} = 26.6°$$

$$\theta_2 = 90$$



$$\psi(a,a) = \frac{q}{2\pi}(\theta_1 + \theta_2) = \frac{q}{2\pi}(26.6 + 90)\,\text{deg} \times \frac{\pi\,\text{rad}}{180\,\text{deg}}$$

$$\psi(a,a) = 0.324\,q$$ ← $\psi(a,a)$

(d) On the y axis, $\theta_1 + \theta_2 = \pi$

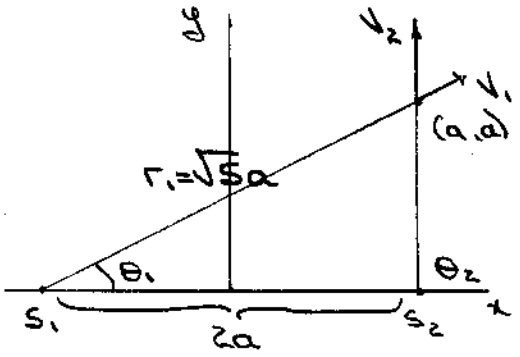
$$\therefore \psi = \frac{q}{2\pi}(\theta_1 + \theta_2) = \frac{q}{2\pi} \times \pi = \frac{q}{2}$$

Thus

$$\frac{\Psi}{b} = \Delta\psi = \psi(0,0) - \psi(a,a) = \frac{q}{2} - 0.324\,q = 0.176\,q$$ ← Ψ/b

Flow per unit depth, b, is upward since $\psi(0,0) > \psi(a,a)$

The dynamic pressure is $p_{dyn} = \frac{1}{2}\rho V^2$. Thus we need to find V^2 at $(x,y) = a,a$



At (a,a)

$$V_1 = \frac{q}{2\pi r_1} = \frac{q}{2\pi\sqrt{5}a}$$

$$V_2 = \frac{q}{2\pi r_2} = \frac{q}{4\pi a}$$

$$\sin\theta_1 = \frac{1}{\sqrt{5}} \quad \cos\theta_1 = \frac{2}{\sqrt{5}}$$

$$\sin\theta_2 = 1 \quad \cos\theta_2 = 0$$

Then

$$u_{a,a} = V_1\cos\theta_1 = \frac{q}{2\pi\sqrt{5}a}\cdot\frac{2}{\sqrt{5}} = 0.200\,\frac{q}{\pi a}$$

$$v_{a,a} = V_1\sin\theta + V_2 = \frac{q}{2\pi\sqrt{5}a}\cdot\frac{1}{\sqrt{5}} + \frac{q}{2\pi a} = 0.600\,\frac{q}{\pi a}$$

$$\therefore V_{a,a}^2 = u^2 + v^2 = \left[(0.200)^2 + (0.600)^2\right]\left(\frac{q}{\pi a}\right)^2 = 0.400\,\frac{q^2}{\pi^2 a^2}$$

$$p_{dyn} = \frac{1}{2}\rho V^2 = \frac{1}{2}\rho\left(0.400\,\frac{q^2}{\pi^2 a^2}\right) = \frac{0.20\rho q^2}{\pi^2 a^2}$$ p_{dyn}

The flow is incompressible, inviscid, and irrotational, so the Bernoulli equation can be applied between any two points in the flow field

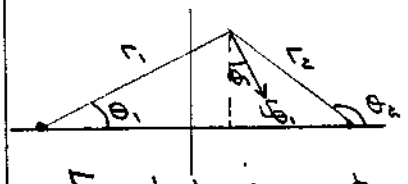
$$p_0 = p + \frac{1}{2}\rho V^2 \quad \text{so} \quad p = p_0 - \frac{1}{2}\rho V^2$$

$$p = p_0 - \frac{0.10\rho q^2}{\pi^2 a^2}$$ p

Given: Vortex pair formed by:
(i) clockwise vortex (strength, K) located at (-a, 0)
(ii) counterclockwise vortex (strength, K) located at (a, 0)

Find: (a) expressions for ψ, ϕ, $\vec{V}$
(b) pressure distribution along the y axis

Solution:



$$\psi = \psi_{cw} + \psi_{ccw} = \frac{K}{2\pi}\ln r_1 - \frac{K}{2\pi}\ln r_2 = \frac{K}{2\pi}\ln\frac{r_1}{r_2} \quad \psi$$

$$\phi = \phi_{cw} + \phi_{ccw} = -\frac{K}{2\pi}\theta_1 - \frac{K}{2\pi}\theta_2 = \frac{K}{2\pi}(\theta_1 - \theta_2) \quad \phi$$

For clockwise vortex, $u = v_{\theta_1}\sin\theta_1 = \frac{K}{2\pi r_1}\sin\theta_1$

$v = -v_{\theta_1}\cos\theta_1 = -\frac{K}{2\pi r_1}\cos\theta_1$

For counterclockwise vortex, $u = -\frac{K}{2\pi r_2}\sin\theta_2$

$v = \frac{K}{2\pi r_2}\cos\theta_2$

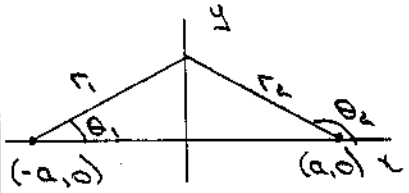
$$\therefore \vec{V} = \frac{K}{2\pi}\left(\frac{\sin\theta_1}{r_1} - \frac{\sin\theta_2}{r_2}\right)\hat{\imath} + \frac{K}{2\pi}\left(\frac{\cos\theta_2}{r_2} - \frac{\cos\theta_1}{r_1}\right)\hat{\jmath} \quad \vec{V}$$

Along the y axis,

$x = 0$, $r_1 = r_2$, $\theta_2 = \pi - \theta_1$

$\sin\theta_2 = \sin\theta_1$

$\cos\theta_2 = -\cos\theta_1$



$$\therefore \vec{V} = \frac{K}{2\pi}\left(-\frac{\cos\theta_1}{r_1} - \frac{\cos\theta_1}{r_1}\right)\hat{\jmath} = -\frac{K}{\pi}\frac{\cos\theta_1}{r_1}\hat{\jmath}$$

Since $\cos\theta_1 = \frac{a}{r_1}$ and $r_1^2 = a^2 + y^2$, then

$$\vec{V} = -\frac{K}{\pi}\frac{a}{r_1^2}\hat{\jmath} = -\frac{K}{\pi}\frac{a}{(a^2+y^2)}\hat{\jmath}$$

To determine the pressure distribution along the y axis, apply the Bernoulli equation

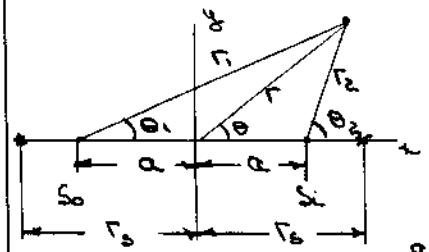
$$\frac{p_\infty}{\rho} + \frac{V_\infty^2}{2}^{\,0} = \frac{p}{\rho} + \frac{V^2}{2}$$

$$\therefore p - p_\infty = -\frac{1}{2}\rho V^2 = -\frac{1}{2}\rho\left[\frac{K}{\pi}\frac{a}{(a^2+y^2)}\right]^2$$

Given: Flow past a Rankine body is formed from the superposition of a uniform flow ($U = 20 m/s$) in the $+x$ direction and a source and a sink of equal strengths ($q = 3\pi\ m^2/s$) located on the x axis at $x = -a$ and $x = a$, respectively.

Find: (a) expressions for ψ, ϕ, and $\vec{V}$
(b) the value of ψ = constant on the stagnation streamline.
(c) the stagnation points if $a = 0.3 m$.

Solution:



$$\psi = \psi_{So} + \psi_{Si} + \psi_{uf} = \frac{q}{2\pi}\theta_1 - \frac{q}{2\pi}\theta_2 + Uy$$

$$\psi = \frac{q}{2\pi}(\theta_1 - \theta_2) + Ur\sin\theta \quad \leftarrow \psi$$

$$\phi = \phi_{So} + \phi_{Si} + \phi_{uf} = -\frac{q}{2\pi}\ln r_1 + \frac{q}{2\pi}\ln r_2 - Ux$$

$$\phi = \frac{q}{2\pi}\ln\frac{r_2}{r_1} - Ur\cos\theta \quad \leftarrow \phi$$

$$u = u_{So} + u_{Si} + u_{uf} = \frac{q}{2\pi r_1}\cos\theta_1 - \frac{q}{2\pi r_2}\cos\theta_2 + U$$

$$v = v_{So} + v_{Si} + v_{uf} = \frac{q}{2\pi r_1}\sin\theta_1 - \frac{q}{2\pi r_2}\sin\theta_2$$

$$\vec{V} = u\hat{i} + v\hat{j} = \left\{\frac{q}{2\pi}\left(\frac{\cos\theta_1}{r_1} - \frac{\cos\theta_2}{r_2}\right) + U\right\}\hat{i} + \frac{q}{2\pi}\left(\frac{\sin\theta_1}{r_1} - \frac{\sin\theta_2}{r_2}\right)\hat{j} \quad \leftarrow \vec{V}$$

At stagnation point $\vec{V} = 0$ $\quad y = 0 \quad \theta_1 = \theta_2 = 0$

$r_2 = r_s - a$, $r_1 = r_s + a$

$$\therefore u = 0 = \frac{q}{2\pi}\left(\frac{1}{r_s + a} - \frac{1}{r_s - a}\right) + U = \frac{q}{2\pi}\left[\frac{(r_s - a) - (r_s + a)}{(r_s^2 - a^2)}\right] + U$$

$$0 = -\frac{q}{\pi}\frac{a}{(r_s^2 - a^2)} + U \quad \text{or} \quad (r_s^2 - a^2) = \frac{q}{\pi U}a$$

$$r_s = \left(a^2 + \frac{q}{\pi U}a\right)^{1/2} = a\left(1 + \frac{q}{\pi U a}\right)^{1/2}$$

For $a = 0.3 m$

$$r = 0.3 m\left[1 + \frac{3\pi}{\pi}\frac{m^2}{s}\frac{s}{20 m}\frac{1}{0.3 m}\right]^{1/2} = 0.367 m$$

Stagnation points located at $\theta = 0, \pi$ $\quad r = 0.367 m$

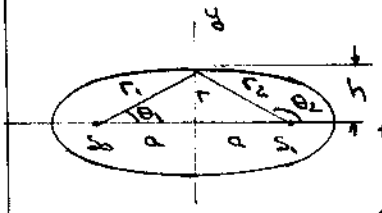
Since $\psi = \frac{q}{2\pi}(\theta_1 - \theta_2) + Uy$ and $\theta_1 = \theta_2$ & $y = 0$ at stagnation

$$\psi_{stag} = 0$$

Given: Flow past a Rankine body is formed from the superposition of a uniform flow ($U = 20$ m/s) in the $+x$ direction, and a source and a sink of equal strengths ($q = 3\pi$ m^2/s) located on the x axis at $x = -a$ and $x = a$, respectively

Find: (a) the half width of the body
(b) $\vec{V}$ and p at the points $(0, \pm h)$

Solution:



$$\psi = \psi_{sink} + \psi_{source} + \psi_{uniform} = \frac{q}{2\pi}(\theta_1 - \theta_2) + U r \sin\theta$$

At stagnation point $\theta_1 = \theta_2$ and $\theta = 0, \pi$.

$\therefore \psi_{stag} = 0$ and equation of stag. streamline is

$$0 = \frac{q}{2\pi}(\theta_1 - \theta_2) + U r \sin\theta$$

or $$r = \frac{\frac{q}{2\pi}(\theta_2 - \theta_1)}{U \sin\theta}$$

At half width, $\theta = \frac{\pi}{2}$, $\theta_2 = \pi - \theta_1$, and $r = h = \frac{\frac{q}{2\pi}\left[(\pi - \theta_1) - \theta_1\right]}{U}$

$$\therefore hU = \frac{q}{2\pi}\left[\pi - 2\theta_1\right] = \frac{q}{2}\cdot\frac{1}{2} - \frac{q}{\pi}\theta_1 \quad \text{or} \quad \theta_1 = \frac{\pi}{2} - \frac{Uh}{q}\pi$$

Since $h = a\tan\theta_1$,

$$\frac{h}{a} = \tan\left(\frac{\pi}{2} - \frac{Uh}{q}\pi\right) = \cot\left(\frac{Uh}{q}\pi\right)$$

Substituting values, $\frac{h}{0.3} = \cot\left(\frac{20}{3}\pi h\right)$. Trial and error solution gives

$h = 0.1615$ m ← h

The velocity field is given by $\vec{V} = \hat{i}u + \hat{j}v$

$$\vec{V} = \left\{\frac{q}{2\pi}\left(\frac{\cos\theta_1}{r_1} - \frac{\cos\theta_2}{r_2}\right) + U\right\}\hat{i} + \frac{q}{2\pi}\left(\frac{\sin\theta_1}{r_1} - \frac{\sin\theta_2}{r_2}\right)\hat{j}$$

At $(0, h)$, $r_1 = r_2$, $\theta_2 = \pi - \theta_1$ $\therefore \sin\theta_2 = \sin\theta_1$, $\cos\theta_2 = -\cos\theta_1$

and $$\vec{V} = \left(\frac{q}{\pi}\frac{\cos\theta_1}{r_1} + U\right)\hat{i}$$

$$\theta_1 = \tan^{-1}\frac{h}{a} = \tan^{-1}\frac{0.1615}{0.3} = 28.3^\circ \qquad r_1 = \left[a^2 + h^2\right]^{1/2} = \left[0.3^2 + 0.1615^2\right]^{1/2} = 0.341\text{ m}$$

$$\vec{V} = \left(\frac{q}{\pi}\frac{\cos\theta_1}{r_1} + U\right)\hat{i} = \left(3\pi\ \frac{\text{m}^2}{\text{s}}\times\frac{\cos 28.3^\circ}{0.341\text{ m}} + 20\ \frac{\text{m}}{\text{s}}\right)\hat{i} = 44.3\,\hat{i}\ \text{m/s}$$ ← V

To find the gage pressure apply the Bernoulli equation between the point at conditions at ∞

$$\frac{p_\infty}{\rho} + \frac{U^2}{2} = \frac{p}{\rho} + \frac{V^2}{2}$$

$$p_{gage} = p - p_\infty = \frac{1}{2}\rho\left(U^2 - V^2\right) = \frac{1}{2}\times 1.225\ \frac{\text{kg}}{\text{m}^3}\left[(20)^2 - (44.3)^2\right]\frac{\text{m}^2}{\text{s}^2}\times\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$p_{gage} = -957\ \text{N/m}^2$$ ← p_{gage}

Given: Flow field formed by superposition of a uniform flow in the + x direction ($U = 10$ m/s) and a counterclockwise vortex, with strength $K = 4\pi$ m²/s, located at the origin

Find: (a) ψ, ϕ, and $\vec{V}$ for the flow field
(b) stagnation point(s)

Plot: streamlines that pass through: (i) $\theta = \pi/2$ at $r = 1$ and 0.5, (ii) stagnation pt. and (iii) $\theta = -\pi/2$ at $r = 0.5$ and 1.

Solution:



$$\psi = \psi_{u,r} + \psi_v = Uy - \frac{K}{2\pi}\ln r = Ur\sin\theta - \frac{K}{2\pi}\ln r \qquad \psi$$

$$\phi = \phi_{u,r} + \phi_v = -Ux - \frac{K}{2\pi}\theta = -Ur\cos\theta - \frac{K}{2\pi}\theta \qquad \phi$$

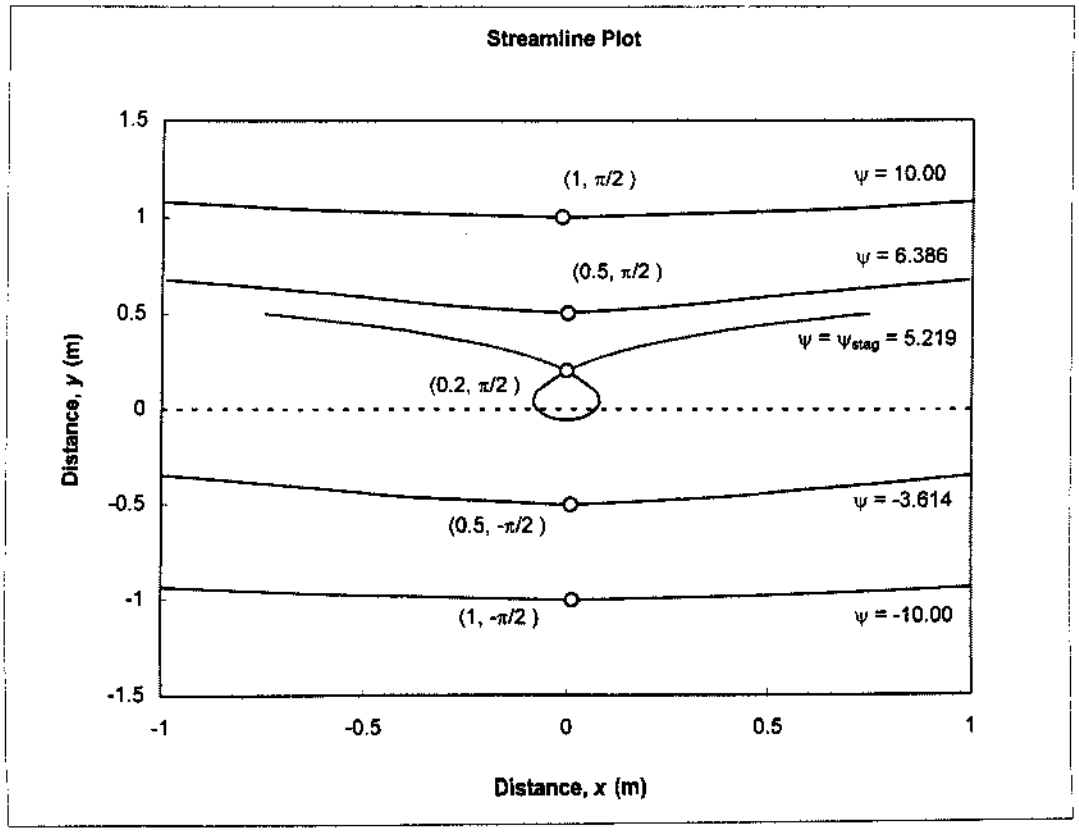
$$V_r = -\frac{\partial\phi}{\partial r} = U\cos\theta \quad , \quad V_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta + \frac{K}{2\pi r}$$

$$\vec{V} = U\cos\theta\,\hat{e}_r + \left(\frac{K}{2\pi r} - U\sin\theta\right)\hat{e}_\theta \qquad \vec{V}$$

At stagnation point, $\vec{V} = 0$

$V_r = 0$ at $\theta = \pm\frac{\pi}{2}$; $V_\theta = 0$ at $r = \frac{K}{2\pi U\sin\theta}$

$\therefore \vec{V} = 0$ at $(r, \theta) = \frac{K}{2\pi U}, \pi/2$ ← Stagnation

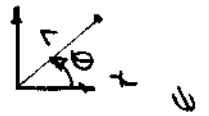


Given: Flow field obtained by superposing a uniform flow in the +x direction ($U = 25$ m/s) and a source (of strength q) at the origin. Stagnation point is at $x = -0.5$ m.

Find: (a) expressions for u, v, V
(b) source strength, q.

Plot: the stagnation streamline; locate branches of stagnation streamline far downstream.

Solution:



$$u = u_{u.f} + u_{so} = U_y + \frac{q}{2\pi}\theta = U r\sin\theta + \frac{q}{2\pi}\theta$$

$$\phi = \phi_{u.f} + \phi_{so} = -Ux - \frac{q}{2\pi}\ln r = -Ur\cos\theta - \frac{q}{2\pi}\ln r$$

$$u = u_{u.f} + u_{so};\ u_{u.f} = U;\ u_{so} = v_r\cos\theta = \frac{q}{2\pi r}\cos\theta \quad \therefore u = U + \frac{q}{2\pi}\frac{x}{x^2+y^2}$$

$$v = v_{u.f} + v_{so};\ v_{u.f} = 0;\ v_{so} = v_r\sin\theta = \frac{q}{2\pi r}\sin\theta \quad \therefore v = \frac{q}{2\pi}\frac{y}{x^2+y^2}$$

$$\vec{V} = u\hat{i} + v\hat{j} = \left\{U + \frac{q}{2\pi}\left(\frac{x}{x^2+y^2}\right)\right\}\hat{i} + \frac{q}{2\pi}\left(\frac{y}{x^2+y^2}\right)\hat{j}$$

At the stagnation point $\vec{V} = 0$ $x = -0.5$ m $y = 0$ ($v = 0$).

For $u = 0 = U + \frac{q}{2\pi}\left(\frac{x}{x^2+y^2}\right)$ $\therefore q = -2\pi U x_{stag}$

$$q = -2\pi \times 25\ \frac{m}{s} \times (-0.5\ m) = 25\pi\ m^2/s$$

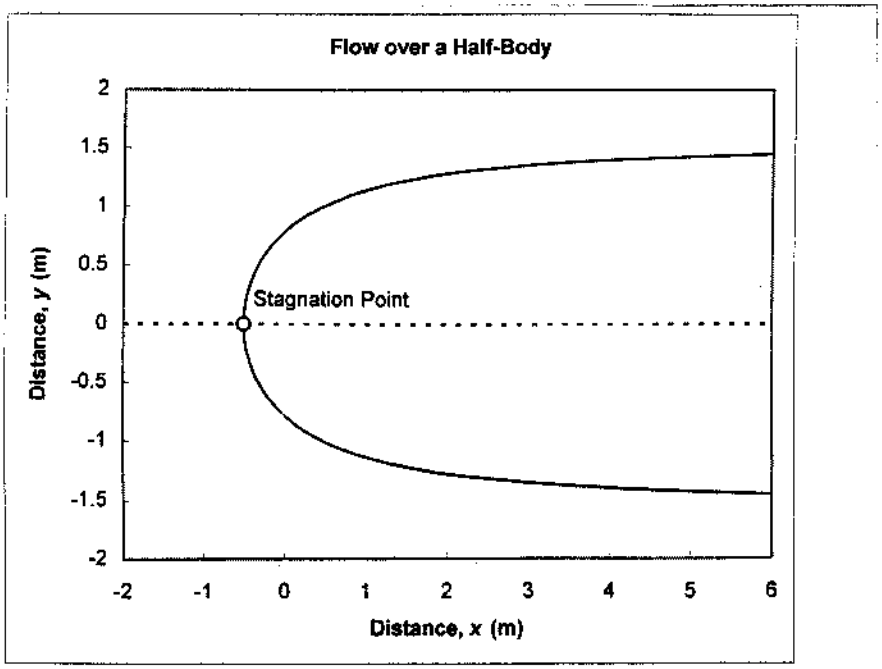
At the stagnation point, $\theta = \pi$ $\therefore U_{stag} = \frac{q}{2\pi}\theta = \frac{q}{2}$

The equation of the stagnation streamline is then

$$q/2 = Ur\sin\theta + \frac{q}{2\pi}\theta \quad \text{and} \quad r = \frac{q(\pi-\theta)}{2\pi U\sin\theta}$$

At $\theta = \pi/2$, $r = \frac{q}{4U} = 25\pi\ \frac{m^2}{s} \times \frac{s}{2\cdot 25\ m} = \pi/4$

For downstream $\theta \to 0$ and the y coordinate of the body $y = r\sin\theta = \frac{q(\pi-\theta)}{2\pi U}\sin\theta$ approaches $\frac{q}{2U} = \frac{25\pi}{2\cdot 25} = \frac{\pi}{2}$ m $\quad y_{\theta\to 0}$



Given: Flow field obtained by combining a uniform flow in the +x direction ($U = 30$ m/s) and a source (of strength $q = 150$ m²/s) located at the origin.

Plot: the ratio of the local velocity V to the free stream velocity U as a function of θ along the stagnation streamline.

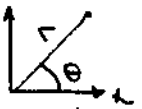
Find: (a) points on the stagnation streamline where the velocity reaches its maximum value

(b) gage pressure at this location if $\rho = 1.2$ kg/m³

Solution:

Superposition of a uniform flow and source gives flow around a half body.

$$\psi = \psi_{uni} + \psi_{so} = Uy + \frac{q}{2\pi}\theta = Ur\sin\theta + \frac{q}{2\pi}\theta \quad \text{---(1)}$$



$$u = u_{uni} + u_{so};\ u_{uni} = U;\ u_{so} = v_r\cos\theta = \frac{q}{2\pi r}\frac{x}{r} \quad \therefore u = U + \frac{q}{2\pi}\frac{x}{x^2+y^2}$$

$$v = v_{uni} + v_{so};\ v_{uni} = 0;\ v_{so} = v_r\sin\theta = \frac{q}{2\pi r}\frac{y}{r} \quad \therefore v = \frac{q}{2\pi}\frac{y}{x^2+y^2}$$

$$\therefore V = \sqrt{u^2+v^2} = \left(U + \frac{q}{2\pi}\frac{x}{x^2+y^2}\right)^2 + \left(\frac{q}{2\pi}\frac{y}{x^2+y^2}\right)^2 \quad \text{---(2)}$$

Now,

$$V^2 = u^2+v^2 = \left(U + \frac{q}{2\pi r}\cos\theta\right)^2 + \left(\frac{q}{2\pi r}\sin\theta\right)^2$$

$$= U^2 + \left(\frac{q}{2\pi r}\right)^2\cos^2\theta + \frac{Uq}{\pi r}\cos\theta + \left(\frac{q}{2\pi r}\right)^2\sin^2\theta$$

$$V^2 = U^2 + \left(\frac{q}{2\pi r}\right)^2 + \frac{Uq}{\pi r}\cos\theta \quad \text{---(3)}$$

To determine the equation of the stagnation streamline, we locate the stagnation point ($V = 0$). From Eq. 2 $y = 0$ and

$$U + \frac{q}{2\pi}\frac{x}{x^2} = 0 = U + \frac{q}{2\pi(x_{stag})} \quad \text{and} \quad x_{stag} = -\frac{q}{2\pi U}$$

$$x_{stag} = -\frac{q}{2\pi U} = -\frac{150\ \text{m}^2/\text{s}}{2\pi \times 30\ \text{m/s}} = -0.796\ \text{m}$$

At the stagnation point $y = 0$ and $\theta = \pi$. From Eq. 1 $\psi_{stag} = \frac{q}{2}$

The equation of the stagnation streamline is then

$$\psi_{stag} = \frac{q}{2} = Ur\sin\theta + \frac{q}{2\pi}\theta \quad \text{. Solving for } r, \text{ we obtain}$$

$$r = \frac{1}{U\sin\theta}\left(\frac{q}{2} - \frac{q}{2\pi}\theta\right) = \frac{q(\pi-\theta)}{2\pi U\sin\theta} \quad \text{---(4)}$$

Substituting this value of r into the expression for V^2 [Eq. 3] we obtain

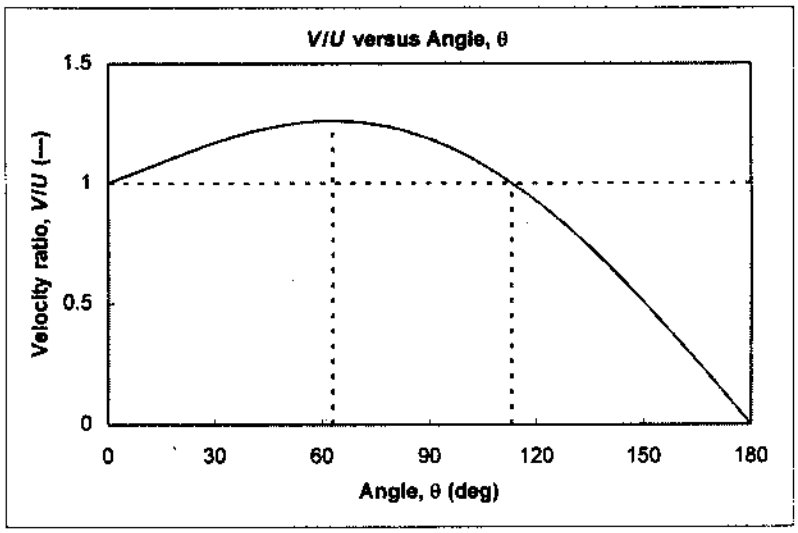
$$V^2 = U^2 + \left[\frac{q}{\pi}\times\frac{2\pi U\sin\theta}{q(\pi-\theta)}\right]^2 + \frac{Uq\cos\theta}{\pi}\times\frac{2\pi U\sin\theta}{q(\pi-\theta)}$$

$$V^2 = U^2 + \frac{U^2\sin^2\theta}{(\pi-\theta)^2} + \frac{2U^2\sin\theta\cos\theta}{(\pi-\theta)} = U^2\left[1 + \frac{\sin^2\theta}{(\pi-\theta)^2} + \frac{2\sin\theta\cos\theta}{(\pi-\theta)}\right]$$

Along the stagnation streamline

$$\frac{V}{U} = \left[1 + \frac{\sin^2\theta}{(\pi-\theta)^2} + \frac{2\sin\theta\cos\theta}{(\pi-\theta)}\right]^{1/2} \quad \text{---(5)}$$

V/U is plotted as a function of θ



From the plot we see that V/U is a maximum at $\theta = 63°$ (also at $\theta = 297°$ from symmetry with respect to the x axis)

At $\theta = 63°$, Eq 5 gives $\left(\frac{V}{U}\right)_{max} = 1.26$

, Eq 4 gives $r = \frac{(30\ m^2/s)}{2\pi\ m/s} \times \frac{(\pi - 0.35\pi)}{\sin 63°} \times \frac{s}{30\ m} = 1.82\ m$

Thus $V = V_{max}$ at $r = 1.82\ m$ and $\theta = 63°, 297°$ $\quad (r,\theta)_{Vmax}$

To determine the gage pressure at this point, write the Bernoulli equation between a point upstream and the point of maximum velocity.

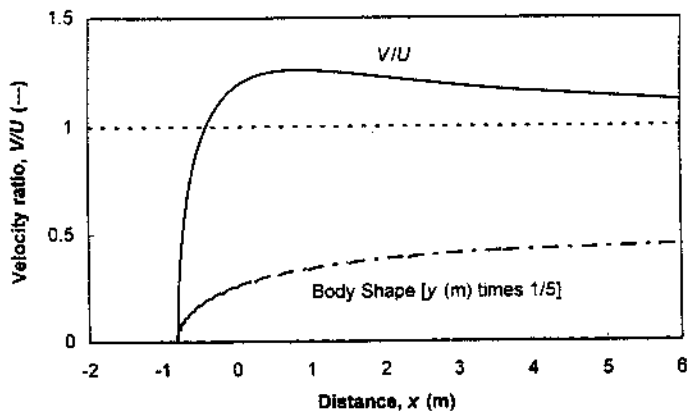
$$\frac{p_\infty}{\rho} + \frac{U^2}{2} = \frac{p}{\rho} + \frac{V_{max}^2}{2}$$

$$\therefore p - p_\infty = \frac{\rho}{2}\left[U^2 - V_{max}^2\right] = \frac{1}{2}\rho U^2\left[1 - \left(\frac{V_{max}}{U}\right)^2\right]$$

$$= \frac{1}{2} \times 1.2\ \frac{kg}{m^3} \times (30)^2\ \frac{m^2}{s^2}\left[1 - (1.26)^2\right] \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p - p_\infty = 317\ N/m^2 \quad \underline{\qquad\qquad} \quad p_{gage}$$

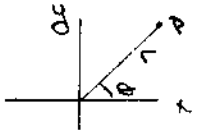
Note: From the plot we see that $V/U = 1.0$, and hence $p = p_\infty$, at $\theta = 113°$. The corresponding r is $1.01\ m$.



Given: Flow field formed by combining a uniform flow in the +x direction ($U = 50 m/s$) and a sink (of strength, $q = 90 m^2/s$) at the origin.

Find: The net force per unit depth needed to hold in place (in standard air) the surface shape formed by the stagnation streamline

Solution:



$$\psi = \psi_U + \psi_{si} = U y - \frac{q}{2\pi}\theta = U r \sin\theta - \frac{q}{2\pi}\theta \quad ----(1)$$

$$u = u_U + u_{si} \; ; \; u_U = U, \; u_{si} = -v_r \cos\theta = -\frac{q}{2\pi r}\frac{x}{r} \quad \therefore u = U - \frac{q}{2\pi}\frac{x}{r^2}$$

$$v = v_U + v_{si} \; ; \; v_U = 0, \; v_{si} = -v_r \sin\theta = -\frac{q}{2\pi r}\frac{y}{r} \quad \therefore v = -\frac{q}{2\pi}\frac{y}{r^2}$$

$$\therefore \vec{V} = u\hat{i} + v\hat{j} = \left(U - \frac{q}{2\pi}\frac{x}{r^2}\right)\hat{i} - \frac{q}{2\pi}\frac{y}{r^2}\hat{j}$$

At the stagnation point, $\vec{V} = 0$

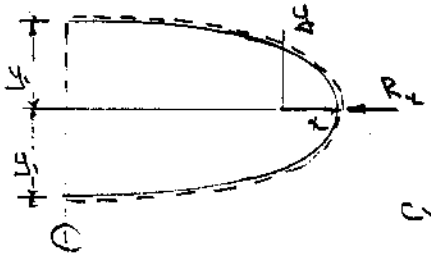
$\therefore -\frac{q}{2\pi}\frac{y}{r^2} = 0$, i.e. $y = 0$. Also $U - \frac{q}{2\pi}\frac{x}{r^2} = 0 \quad \therefore U = \frac{q}{2\pi}\left(\frac{1}{x}\right)$

and $x_{stag} = \frac{q}{2\pi U} = \frac{90 m^2/s}{2\pi} \times \frac{s}{50 m} = 0.286 m$

At stagnation point, $y = 0$ and $\theta = 0$. From eq (1) then $\psi_{stag} = 0$
The equation of the stagnation streamline is then,

$$\psi = 0 = U r \sin\theta - \frac{q}{2\pi}\theta \quad \text{or} \quad r_{stag} = \frac{q\theta}{2\pi U \sin\theta}$$

Since $y = r\sin\theta$, then along the stagnation streamline $y = \frac{q\theta}{2\pi U}$.
For upstream, $\theta \to \pi$ and $y = y_1 \to \frac{q}{2U}$.
The surface shape formed by the stagnation streamline is then as follows:



There is no flow across this streamline. The flow in through the left face must be equal to the flow (q) which leaves through the sink at the origin.

Applying the x momentum equation to the CV shown. R_x is force required to hold shape in place

$$-R_x = \int u \rho \vec{V} \cdot d\vec{A} = -U\dot{m}_1 = -U\rho q b$$

$$\therefore \frac{R_x}{b} = \rho q U$$

For standard air $\rho = 1.225 kg/m^3$ and

$$\frac{R_x}{b} = 1.225 \frac{kg}{m^3} \times 90 \frac{m^2}{s} \times 50 \frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m} = 5.51 kN/m$$

$$\therefore R_x/b = -5.51 \hat{i} \; kN/m \qquad R_x/b$$

Given: At low speeds, drag is independent of fluid density.

$$F = F(\mu, V, D)$$

Find: Appropriate dimensionless parameters.

Solution: Apply Buckingham Π procedure.

① $F \quad \mu \quad V \quad D$ $\qquad n = 4$ parameters

② Select primary dimensions M, L, t.

③ $F \quad \mu \quad V \quad D$

$\frac{ML}{t^2} \quad \frac{M}{Lt} \quad \frac{L}{t} \quad L$ $\qquad r = 3$ primary dimensions

④ $\mu, V, D \quad m = r = 3$ repeating parameters

⑤ Then $n - m = 1$ dimensionless group will result. Setting up a dimensional equation,

$$\Pi_1 = \mu^a V^b D^c F$$

$$= \left(\frac{M}{Lt}\right)^a \left(\frac{L}{t}\right)^b (L)^c \frac{ML}{t^2} = M^0 L^0 t^0$$

Summing exponents,

$M: \quad a + 1 = 0$	$a = -1$	
$L: \quad -a + b + c + 1 = 0$	$c = -1$	$\therefore \Pi_1 = \frac{F}{\mu V D}$
$t: \quad -a - b - 2 = 0$	$b = -1$	

Π_1

⑥ Check, using F, L, t primary dimensions.

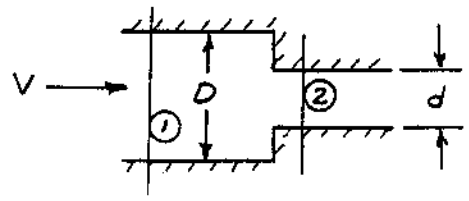
$$\Pi_1 = F \frac{L^2}{Ft} \frac{t}{L} \frac{1}{L} = [1] \quad \checkmark$$

Since the procedure produces only one dimensionless group, it must be a constant. Thus

$$\Pi_1 = \frac{F}{\mu V D} \quad \text{or} \quad F \propto \mu V D$$

Given: Flow through a sudden contraction.

$$\Delta p = p_1 - p_2 = f(\rho, \mu, V, D, d)$$



Find: Dimensionless parameters.

Solution: Choose ρ, V, and D as repeating variables.

① $\Delta p \quad \rho \quad \mu \quad V \quad D \quad d$ $\quad n = 6$ parameters

② Select primary dimensions M, L, t

③ $\Delta p \quad \rho \quad \mu \quad V \quad D \quad d$

$\frac{M}{Lt^2} \quad \frac{M}{L^3} \quad \frac{M}{Lt} \quad \frac{L}{t} \quad L \quad L$ $\quad r = 3$ primary dimensions

④ ρ, V, D $\quad m = r = 3$ repeating parameters

⑤ Then $n - m = 3$ dimensionless groups will result. Setting up dimensional equations,

$$\Pi_1 = \rho^a V^b D^c \Delta p = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \left(\frac{M}{Lt^2}\right) = M^0 L^0 t^0$$

Summing exponents,

M: $a + 1 = 0 \quad a = -1$

L: $-3a + b + c - 1 = 0$

t: $-b - 2 = 0 \quad b = -2$

$c = 1 - b + 3a = 0$

$\therefore \Pi_1 = \frac{\Delta p}{\rho V^2}$

$$\Pi_2 = \rho^a V^b D^c \mu = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \left(\frac{M}{Lt}\right) = M^0 L^0 t^0$$

Summing exponents,

M: $a + 1 = 0 \quad a = -1$

L: $-3a + b + c - 1 = 0$

t: $-b - 1 = 0 \quad b = -1$

$c = 1 - b + 3a = -1$

$\therefore \Pi_2 = \frac{\mu}{\rho V D}$

$$\Pi_3 = \rho^a V^b D^c d = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c L = M^0 L^0 t^0$$

M: $a + 0 = 0 \quad a = 0$

L: $-3a + b + c + 1 = 0$

t: $-b + 0 = 0 \quad b = 0$

$\Big\}\ c = -1 \ ; \ \Pi_3 = \frac{d}{D}$

Thus $\Pi_1 = f(\Pi_2, \Pi_3)$ or $\frac{\Delta p}{\rho V^2} = f\left(\frac{\mu}{\rho V D}, \frac{d}{D}\right)$ $\quad \frac{\Delta p}{\rho V^2}$

⑥ Check, using F, L, t $\quad \Pi_1 = \frac{F}{L^2} \frac{L^4}{Ft^2} \frac{t^2}{L^2} = [1]$ ✓, $\Pi_2 = Re = [1]$ ✓, $\Pi_3 = \frac{L}{L} = [1]$ ✓

Given: The boundary layer thickness, δ, on a smooth flat plate in incompressible flow without pressure gradient is a function of U (free stream velocity), ρ, μ, and x (distance)

Find: suitable dimensionless parameters

Solution: Apply Buckingham π-theorem

① δ U ρ μ x $n = 5$ parameters

② Select M, L, t as primary dimensions

③

δ	U	ρ	μ	x
L	$\frac{L}{t}$	$\frac{M}{L^3}$	$\frac{M}{Lt}$	L

$r = 3$ primary dimensions

④ ρ, U, x $n = r = 3$ repeating parameters

⑤ Then $n - r = 2$ dimensionless groups will result.

Setting up dimensional equations.

$\pi_1 = \rho^a U^b x^c \delta$

$M^0 L^0 t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c L$

Equating exponents,

M: $0 = a$ $\therefore a = 0$

L: $0 = -3a + b + c + 1$ $c = -1$

t: $0 = -b$ $\therefore b = 0$

$\therefore \pi_1 = \frac{\delta}{x}$

$\pi_2 = \rho^a U^b x^c \mu$

$M^0 L^0 t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \frac{M}{Lt}$

Equating exponents,

M: $0 = a + 1$ $\therefore a = -1$

L: $0 = -3a + b + c - 1$ $c = -1$

t: $0 = -b - 1$ $\therefore b = -1$

$\pi_2 = \frac{\mu}{\rho U x}$

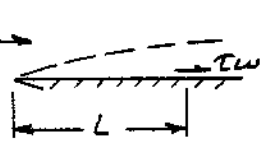
and $\frac{\delta}{x} = f\left(\frac{\rho U x}{\mu}\right)$ $\frac{\delta}{x}$

⑥ Check using F, L, t dimensions

$\pi_1 = \frac{L}{L} = [1]$ ✓

$\pi_2 = \frac{Ft}{L^2} \cdot \frac{L^4}{Ft^2} \frac{t}{L} \frac{1}{L} = [1]$ ✓

Given: Wall shear stress, τ_w, in a boundary layer, depends on ρ, μ, L, and U.



Find: (a) Dimensionless groups.
(b) Express the functional relationship.

Solution: Step ① $\tau_w \quad \rho \quad \mu \quad L \quad U \qquad n = 5$

Step ② Choose M, L, t. $\tau_w = \frac{F}{L^2} \times \frac{ML}{Ft^2} = \frac{M}{Lt^2}$

Step ③ $\frac{M}{Lt^2} \quad \frac{M}{L^3} \quad \frac{M}{Lt} \quad L \quad \frac{L}{t} \qquad r = 3$

Step ④ Select ρ, L, U

Step ⑤ $\pi_1 = \tau_w \rho^a L^b U^c = \frac{M}{Lt^2}\left(\frac{M}{L^3}\right)^a (L)^b \left(\frac{L}{t}\right)^c = M^0 L^0 t^0$

$M: 0 = 1 + a \quad a = -1$
$L: 0 = -1 - 3a + b + c \quad b = 3a - c + 1 = 0$
$t: 0 = -2 - c \quad c = -2$

$\Big\} \pi_1 = \frac{\tau_w}{\rho U^2}$ ← π_1

$\pi_2 = \mu \rho^a L^b U^c = \frac{M}{Lt}\left(\frac{M}{L^3}\right)^a (L)^b \left(\frac{L}{t}\right)^c = M^0 L^0 t^0$

$M: 0 = 1 + a \quad a = -1$
$L: 0 = -1 - 3a + b + c \quad b = 3a - c + 1 = -1$
$t: 0 = -1 - c \quad c = -1$

$\Big\} \pi_2 = \frac{\mu}{\rho U L}$ ← π_2

Step ⑥: Check using F, L, t: $\rho = \frac{M}{L^3} \times \frac{Ft^2}{ML} = \frac{Ft^2}{L^4}$

$$\pi_1 = \frac{\tau_w}{\rho U^2} = \frac{F}{L^2} \frac{L^4}{Ft^2} \frac{t^2}{L^2} = \frac{FL^4t^2}{FL^4t^2} = 1 \quad \checkmark\checkmark$$

$$\pi_2 = \frac{\mu}{\rho U L} = \frac{Ft}{L^2} \frac{L^4}{Ft^2} \frac{t}{L} \frac{1}{L} = \frac{FL^4t^2}{FL^4t^2} = 1 \quad \checkmark\checkmark$$

The functional relationship is

$\pi_1 = f(\pi_2)$ ← f

Given: The mean velocity, $\bar{u}$, for turbulent pipe or boundary layer flow, may be correlated in terms of the wall shear stress, τ_w, distance from the wall, y, and fluid properties, ρ and μ.

Find: (a) dimensionless parameter containing $\bar{u}$ and one containing y that are suitable for organizing experimental data
(b) show that the result may be written as

$$\frac{\bar{u}}{u_*} = f\left(\frac{yu_*}{\nu}\right) \quad \text{where } u_* = (\tau_w/\rho)^{1/2}$$

Solution: Apply the Buckingham π-Theorem

① $\bar{u}$ τ_w y ρ μ $\quad n = 5$ parameters

② Select M, L, t as primary dimensions

③

$\bar{u}$	τ_w	y	ρ	μ
L/t	M/Lt^2	L	M/L^3	M/Lt

④ τ_w, y, ρ $\quad r = 3$ repeating parameters

⑤ Then $n - r = 2$ dimensionless groups will result

Setting up dimensional equations

$\pi_1 = \tau_w^a y^b \rho^c \bar{u}$

$M^0L^0t^0 = \left(\frac{M}{Lt^2}\right)^a L^b \left(\frac{M}{L^3}\right)^c \frac{L}{t}$

Summing exponents

M: $a + c = 0 \quad \therefore a = -c$

L: $-a + b - 3c + 1 = 0$

t: $-2a - 1 = 0 \quad \therefore a = -1/2$

$a = -1/2, \; c = 1/2, \; b = 0$

$$\pi_1 = \frac{\bar{u}\,\rho^{1/2}}{\tau_w^{1/2}} = \frac{\bar{u}}{\sqrt{\tau_w/\rho}}$$

$\pi_2 = \tau_w^a y^b \rho^c \mu$

$M^0L^0t^0 = \left(\frac{M}{Lt^2}\right)^a L^b \left(\frac{M}{L^3}\right)^c \frac{M}{Lt}$

Summing exponents

M: $a + c + 1 = 0 \quad \therefore c = -a - 1$

L: $-a + b - 3c - 1 = 0$

t: $-2a - 1 = 0 \quad \therefore a = -1/2$

$a = -1/2, \; c = -1/2, \; b = -1$

$$\pi_2 = \frac{\mu}{\tau_w^{1/2}\rho^{1/2}y} = \frac{\mu}{\rho y\sqrt{\tau_w/\rho}}$$

$$\pi_1 = f(\pi_2) \quad \text{or} \quad \frac{\bar{u}}{\sqrt{\tau_w/\rho}} = f\left(\frac{\mu}{\rho y\sqrt{\tau_w/\rho}}\right)$$

Since $\sqrt{\tau_w/\rho} = u_*$, then

$$\frac{\bar{u}}{u_*} = f\left(\frac{\mu}{\rho y u_*}\right) = f\left(\frac{\nu}{y u_*}\right) = f\left(\frac{yu_*}{\nu}\right) \quad \leftarrow \frac{\bar{u}}{u_*}$$

Given: Volume flow rate, Q, over a weir is a function of: upstream height, h, gravity, g, and channel width, b.

Find: Expression for Q (using dimensional analysis)

Solution: Apply Buckingham Π-theorem

① List Q h g b $n = 4$ parameters

② Choose F, L, t as primary dimensions

③ Dimensions $\frac{L^3}{t}$ L $\frac{L}{t^2}$ L

④ Repeating variables g, h $m = r = 2$

⑤ Then $n - m = 2$ dimensionless groups will result

Setting up dimensional equations

$\Pi_1 = g^a h^b Q$

$L^0 t^0 = \left(\frac{L}{t^2}\right)^a L^b \left(\frac{L^3}{t}\right)$

Equating exponents

L: $0 = a + b + 3$

t: $0 = -2a - 1$

$\therefore a = -\frac{1}{2}$

$b = -2\frac{1}{2}$

$\therefore \Pi_1 = \frac{Q}{g^{\frac{1}{2}} h^{2.5}}$

$\Pi_1 = \frac{Q}{h^2 \sqrt{gh}}$

$\Pi_2 = g^a h^c b$

$L^0 t^0 = \left(\frac{L}{t^2}\right)^a L^c L$

Equating exponents

L: $0 = a + c + 1$

t: $0 = -2a$

$\therefore a = 0$

$c = -1$

$\Pi_2 = \frac{b}{h}$

(This is obvious by inspection)

Then $\frac{Q}{h^2\sqrt{gh}} = f\left(\frac{b}{h}\right)$

$Q = h^2 \sqrt{gh}\, f\left(\frac{b}{h}\right)$ ← Q

Given: Velocity, V, of a free surface gravity wave in deep water is a function of λ (wavelength), γ, ρ, and g

Find: Dependence of V on other variables.

Solution: Apply Buckingham Π-Theorem

① $V \quad \lambda \quad \gamma \quad \rho \quad g$ $\quad n = 5$ parameters

② Select M, L, t as primary dimensions

③

V	λ	γ	ρ	g
t^{-1}	L	L	$M L^{-3}$	$t^{-2} L$

$r = 3$ primary dimensions

④ ρ, γ, g $\quad m = r = 3$ repeating parameters

⑤ Then $n - m = 2$ dimensionless groups will result

Setting up dimensional equations

$\Pi_1 = \rho^a \gamma^b g^c V$

$M^0 L^0 t^0 = \left(\frac{M}{L^3}\right)^a L^b \left(\frac{L}{t^2}\right)^c \frac{L}{t}$

Summing exponents,

M: $a = 0$

L: $-3a + b + c + 1 = 0$

t: $-2c - 1 = 0$

i.e $a = 0$

$c = -\frac{1}{2}$

$b = 3a - c - 1 = -\frac{1}{2}$

$\therefore \Pi_1 = \frac{V}{\sqrt{g\gamma}}$

$\Pi_2 = \rho^a \gamma^b g^c \lambda$

$M^0 L^0 t^0 = \left(\frac{M}{L^3}\right)^a L^b \left(\frac{L}{t^2}\right)^c L$

Summing exponents,

M: $a = 0$

L: $-3a + b + c + 1 = 0$

t: $-2c = 0$

i.e $a = 0$

$c = 0$

$b = 3a - c - 1 = -1$

$\therefore \Pi_2 = \frac{\lambda}{\gamma}$

Thus $\frac{V}{\sqrt{g\gamma}} = f\left(\frac{\lambda}{\gamma}\right)$ or $V = \sqrt{g\gamma}\, f\left(\frac{\lambda}{\gamma}\right)$ ← V

⑥ Check using F, L, t

$\Pi_1 = \frac{L}{t} \cdot \left(\frac{t^2}{L}\right)^{1/2} = [1]^V$

$\Pi_2 = \frac{L}{L} = [1]$

Given: Capillary waves form on a liquid free surface. The speed of the wave is a function of σ (surface tension), λ (the wave length) and ρ

Find: The wave speed as a function of the variables

Solution: Apply Buckingham Π - theorem

① $V \quad \sigma \quad \lambda \quad \rho$ $\quad n = 4$ parameters

② Select M, L, t as primary dimensions

③

V	σ	λ	ρ
$L t^{-1}$	$\frac{M}{t^2}$	L	$\frac{M}{L^3}$

$r = 3$ primary dimensions

④ σ, λ, ρ $\quad n = r = 3$ repeating parameters

⑤ Then $n - m = 1$ dimensionless group will result

Setting up dimensional equation

$$\Pi_1 = \sigma^a \lambda^b \rho^c V$$

$$M^0 L^0 t^0 = \left(\frac{M}{t^2}\right)^a L^b \left(\frac{M}{L^3}\right)^c L t^{-1}$$

Summing exponents

M: $a + c = 0 \qquad c = -a = \frac{1}{2}$

L: $b - 3c + 1 = 0 \qquad b = 3c - 1 = \frac{1}{2}$

t: $-2a - 1 = 0 \qquad \therefore \quad a = -\frac{1}{2}$

$$\therefore \Pi_1 = \left(\frac{\rho \lambda^2}{\sigma t^2}\right)^{\frac{1}{2}} V = \text{constant} \qquad \therefore V \propto \sqrt{\frac{\sigma}{\rho \lambda}}$$

V

⑥ Check using F, L, t

$$\Pi_1 = \left(\frac{F t^2}{L^4} \cdot L \cdot \frac{L^3}{F t^2}\right)^{1/2} \frac{L}{t^{-1}} = [1] \checkmark$$

Given: Load-carrying capacity, W, (of a journal bearing) depends on: diameter, D; length, l; clearance, c; angular speed, ω; lubricant viscosity, μ

Find: Dimensionless parameters that characterize the problem.

Solution: Apply Buckingham π - theorem

① List W D l c ω μ $n=6$ parameters

② Choose F, L, t as primary dimensions

③ Dimensions F L L L $\frac{1}{t}$ $\frac{Ft}{L^2}$

④ Repeating variables D, ω, μ $m = r = 3$

⑤ Then $n-m = 3$ dimensionless groups will result

By inspection, $\pi_1 = \frac{l}{D}$ $\pi_2 = \frac{c}{D}$

Set up dimensional equation to determine π_3

$$\pi_3 = D^a \omega^b \mu^e W$$

$$F^0 L^0 t^0 = L^a \left(\frac{1}{t}\right)^b \left(\frac{Ft}{L^2}\right)^e F$$

Equating exponents:

F $\quad 0 = e + 1 \quad \therefore e = -1$

L $\quad 0 = a - 2e \quad \therefore a = -2$

t $\quad 0 = -b + e \quad \therefore b = -1$

and

$$\pi_3 = \frac{W}{D^2 \omega \mu}$$

⑥ Check using M, L, t dimensions

$$\pi_3 = \frac{ML}{t^2} \cdot \frac{1}{L^2} \cdot t \cdot \frac{Lt}{M} = [1]^0$$

$$\therefore \frac{W}{D^2 \omega \mu} = f\left(\frac{l}{D}, \frac{c}{D}\right)$$

W

Given: The torque, T, on a disc rotating near a fixed surface depends on R (disc radius), μ, h (spacing from surface) and ω.

Find: Functional relationship for T.

Solution: Apply Buckingham-Π theorem

① T R μ h ω $n=5$ parameters

② Select F, L, t as primary dimensions

③

T	R	μ	h	ω
FL	L	$\frac{Ft}{L^2}$	L	$\frac{1}{t}$

④ R, μ, ω $m=r=3$ repeating parameters

⑤ Then $n-m=2$ dimensionless groups will result.

Setting up dimensional equations

$\Pi_1 = R^a \mu^b \omega^c T$

$F^0 L^0 t^0 = L^a \left(\frac{Ft}{L^2}\right)^b \left(\frac{1}{t}\right)^c FL$

Equating exponents,

F: $0 = b+1 \quad \therefore b=-1$

L: $0 = a-2b+1 \quad \therefore a=-3$

t: $0 = b-c \quad \therefore c=-1$

$\therefore \Pi_1 = \frac{T}{R^3 \mu \omega}$

$\Pi_2 = R^a \mu^b \omega^c h$

$F^0 L^0 t^0 = L^a \left(\frac{Ft}{L^2}\right)^b \left(\frac{1}{t}\right)^c L$

Equating exponents,

F: $0 = b$

L: $0 = a-2b+1 \quad \therefore a=-1$

t: $0 = -c$

$\therefore \Pi_2 = \frac{h}{R}$

$\therefore \frac{T}{R^3 \mu \omega} = f\left(\frac{h}{R}\right)$ T

⑥ Check using M, L, t dimensions

$\Pi_1 = \frac{M/L^2}{t^2} \cdot \frac{1}{L^3} \cdot \frac{Lt}{M} \cdot t = [1]$ ✓

$\Pi_2 = \frac{L}{L} = [1]$ ✓

Given: Power per unit cross-sectional area, E, transmitted by a sound wave, depends on wave speed, V, amplitude, r, frequency, n, and medium density, ρ.

Find: General form of dependence of E on the other variables.

Solution: Step ① E V r n ρ $n = 5$

Step ② Choose M, L, t. $E = \frac{P}{L^2} = \frac{FL}{t} \times \frac{1}{L^2} = \frac{F}{Lt} \times \frac{ML}{Ft^2} = \frac{M}{t^3}$

Step ③ $\frac{M}{t^3}$ $\frac{L}{t}$ L $\frac{1}{t}$ $\frac{M}{L^3}$ $r = 3$

Step ④ Choose ρ, V, r

Step ⑤ $\Pi_1 = \rho^a V^b r^c E = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \frac{M}{t^3} = M^0 L^0 t^0$

$M: a + 1 = 0 \quad a = -1$
$L: -3a + b + c = 0 \quad c = 3a - b = 3(-1) - (-3) = 0$
$t: -b - 3 = 0 \quad b = -3$

$\Pi_1 = \frac{E}{\rho V^3}$ Π_1 ←

$\Pi_2 = \rho^a V^b r^c n = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \frac{1}{t} = M^0 L^0 t^0$

$M: a + 0 = 0 \quad a = 0$
$L: -3a + b + c = 0 \quad c = 3a - b = 3(0) - (-1) = 1$
$t: -b - 1 = 0 \quad b = -1$

$\Pi_2 = \frac{nr}{V}$ Π_2 ←

Step ⑥ Check using FLt: $\rho = \frac{M}{L^3} \times \frac{Ft^2}{ML} = \frac{Ft^2}{L^4}$

$\Pi_1 = \frac{E}{\rho V^3} = \frac{FL}{tL^2} \frac{L^4}{Ft^2} \frac{t^3}{L^3} = \frac{FL^5 t^3}{FL^5 t^3} = 1$ ✓✓

$\Pi_2 = \frac{nr}{V} = \frac{1}{t} L \times \frac{t}{L} = \frac{Lt}{Lt} = 1$ ✓✓

Given: Power, $\mathcal{P}$, required to drive a fan depends on ρ, Q, D and ω.

Find: Dependence of $\mathcal{P}$ on other parameters.

Solution: Apply Buckingham Π procedure.

① $\mathcal{P}$ ρ Q D ω $\quad n=5$ parameters

② Choose primary dimensions M, L, t

③

$\mathcal{P}$	ρ	Q	D	ω	
$\frac{ML^2}{t^3}$	$\frac{M}{L^3}$	$\frac{L^3}{t}$	L	$\frac{1}{t}$	$r=3$ primary dimensions

④ ρ, D, ω $\quad m=r=3$ repeating parameters

⑤ Then $n-m=2$ dimensionless groups will result. Setting up dimensional equations,

$\Pi_1 = \rho^a D^b \omega^c \mathcal{P}$

$= \left(\frac{M}{L^3}\right)^a (L)^b \left(\frac{1}{t}\right)^c \left(\frac{ML^2}{t^3}\right) = M^0 L^0 t^0$

Summing exponents,

$M: a+1=0 \quad a=-1$

$L: -3a+b+2=0 \quad b=-5$

$t: -c-3=0 \quad c=-3$

$\therefore \Pi_1 = \dfrac{\mathcal{P}}{\rho D^5 \omega^3}$

$\Pi_2 = \rho^d D^e \omega^f Q$

$= \left(\frac{M}{L^3}\right)^d (L)^e \left(\frac{1}{t}\right)^f \left(\frac{L^3}{t}\right) = M^0 L^0 t^0$

Summing exponents,

$M: d+0=0 \quad d=0$

$L: -3d+e+3=0 \quad e=-3$

$t: -f-1=0 \quad f=-1$

$\therefore \Pi_2 = \dfrac{Q}{D^3 \omega}$

⑥ Check using primary dimensions F, L, t

$\Pi_1 = \frac{FL}{t} \frac{L^4}{Ft^2} \frac{1}{L^5} t^3 = [1]$ ✓ $\qquad \Pi_2 = \frac{L^3}{t} \frac{1}{L^3} t = [1]$ ✓

Thus $\Pi_1 = f(\Pi_2)$, or $\dfrac{\mathcal{P}}{\rho D^5 \omega^3} = f\left(\dfrac{Q}{D^3 \omega}\right)$ ← $\mathcal{P}$

Given: Draining of a tank from initial level, h_0.

Time, t, depends on tank diameter, D, orifice diameter, d, acceleration of gravity, g, density, ρ, and viscosity, μ.

Find: (a) Number of dimensionless parameters
(b) Number of repeating variables.
(c) Π-parameter containing viscosity.

Solution: Step ① t h_0 D d g ρ μ

Step ② Choose MLt system $(n = 7)$

Step ③ t L L L $\frac{L}{t^2}$ $\frac{M}{L^3}$ $\frac{M}{Lt}$

$(r = 3)$

Then $n - r = 7 - 3 = 4$ parameters will result. Π's

Step ④ $r = 3$, so choose 3 variables: ρ, d, g

Step ⑤ $\Pi_1 = \rho^a d^b g^c \mu = \left(\frac{M}{L^3}\right)^a L^b \left(\frac{L}{t^2}\right)^c \frac{M}{Lt} = M^0 L^0 t^0$

$M: a + 1 = 0 \quad a = -1$
$L: -3a + b + c - 1 = 0 \quad b = 3a - c + 1 = 3(-1) - \left(-\frac{1}{2}\right) + 1$
$t: -2c - 1 = 0 \quad c = -\frac{1}{2} \quad b = -\frac{3}{2}$

$$\Pi_1 = \frac{\mu}{\rho d^{3/2} g^{1/2}} \qquad \Pi_1$$

Step ⑥ Check, using FLt system.

$$\mu = \frac{Ft}{L^2}; \quad \rho = \frac{M}{L^3} \times \frac{Ft^2}{ML} = \frac{Ft^2}{L^4}$$

$$\Pi_1 = \frac{Ft}{L^2} \frac{L^4}{Ft^2} \frac{1}{L^{3/2}} \frac{t}{L^{1/2}} = \frac{FL^4 t^2}{FL^4 t^2} = 1 \quad \checkmark\checkmark$$

Given: Water is drained from a tank of diameter D, through a smoothly rounded drain hole of diameter d. The initial mass flow rate, $\dot{m}$, from the tank is written in functional form as

$$\dot{m} = \dot{m}(h_0, D, d, g, \rho, \mu)$$

where h_0 is the initial water depth in the tank
g is the acceleration of gravity
and μ are fluid properties.

Find: (a) the number of dimensionless groups required to correlate the data
(b) the number of repeating variables that must be selected to determine the dimensionless parameters.
(c) the π parameter that contains the fluid viscosity, μ.

Solution: Apply the Buckingham π - theorem

① List: $\dot{m}$ h_0 D d g ρ μ $n=7$ parameters

② Select M, L, t as primary dimensions

③ Dimensions $\dot{m}: M t^{-1}$ $h_0: L$ $D: L$ $d: L$ $g: L t^{-2}$ $\rho: M L^{-3}$ $\mu: M L^{-1} t^{-1}$ $r=3$ prim dim

④ Choose repeating variables ρ, d, g $m=3$ repeating parameters ← m

$\therefore$ expect $n-m = 7-3 = 4$ dimensionless parameters ← n

⑤ $\pi_1 = \rho^a d^b g^c \mu$

$M^0 L^0 t^0 = \left(\frac{M}{L^3}\right)^a L^b \left(\frac{L}{t^2}\right)^c \frac{M}{Lt}$

t: $0 = -2c - 1 \quad \therefore c = -\frac{1}{2}$

M: $0 = a + 1 \quad \therefore a = -1$

L: $0 = -3a + b + c - 1 \quad \therefore b = 3a - c + 1 = -\frac{3}{2}$

$$\pi_1 = \frac{\mu}{\rho d^{3/2} g^{1/2}}$$ ← π

⑥ Check $\pi_1 = \frac{Ft}{L^2} \cdot \frac{L^4}{Ft^2} \cdot \frac{1}{L^{3/2}} \cdot \frac{t^{1/2}}{L^{1/2}} = [1]$ ✓

Given: Continuous belt moving vertically through a viscous liquid bath

The volume rate of liquid loss, Q, is a function of μ, ρ, g, h (thickness of liquid layer), and V

Find: form of dependence of Q on other variables.

Solution: Apply Buckingham π-theorem.

① $Q \quad \mu \quad \rho \quad g \quad h \quad V$ $\quad n = 6$ parameters

② Select M, L, t as primary dimensions

③ $Q: t^{-1}L^3 \quad \mu: \frac{M}{Lt} \quad \rho: \frac{M}{L^3} \quad g: \frac{L}{t^2} \quad h: L \quad V: \frac{L}{t}$ $\quad r = 3$ primary dimensions

④ $\rho, V, h \quad n = r = 3$ repeating parameters

⑤ Then $n - m = 3$ dimensionless groups will result.

Setting up dimensional equations.

$\Pi_1 = \rho^a V^b h^c Q$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c t^{-1}L^3$

Equating exponents,

M: $0 = a$

L: $0 = -3a + b + c + 3$

t: $0 = -b - 1$

i.e. $a = 0$, $b = -1$, $c = -2$

$\therefore \Pi_1 = \frac{Q}{Vh^2}$

$\Pi_2 = \rho^a V^b h^c \mu$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \frac{M}{Lt}$

Equating exponents,

M: $0 = a + 1$

L: $0 = -3a + b + c - 1$

t: $0 = -b - 1$

i.e. $a = -1$, $b = -1$, $c = -1$

$\therefore \Pi_2 = \frac{\mu}{\rho V h}$

$\Pi_3 = \rho^a V^b h^c g$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \frac{L}{t^2}$

Equating exponents,

M: $0 = a$

L: $0 = -3a + b + c + 1$

t: $0 = -b - 2$

i.e. $a = 0$, $b = -2$, $c = 1$

$\therefore \Pi_3 = \frac{gh}{V^2}$

Then $\frac{Q}{Vh^2} = f\left(\frac{\rho V h}{\mu}, \frac{gh}{V^2}\right)$ Q

⑥ Check using F, L, t dimensions

$\Pi_1 = \frac{L^3}{t} \cdot \frac{t}{L} \cdot \frac{1}{L^2} = [1]^V$ $\quad \Pi_2 = \frac{Ft}{L^2} \cdot \frac{L^4}{Ft^2} \cdot \frac{t}{L} \cdot \frac{1}{L} = [1]^V$ $\quad \Pi_3 = \frac{L}{t^2} \cdot L \cdot \frac{t^2}{L^2} = [1]^V$

Given: Resistance, R, of a flat plat immersed in a fluid depends on ρ, μ, V, b (width), and h (height)

Find: a convenient set of coordinates for organizing data

Solution: Apply Buckingham π - Theorem

① $R \quad \rho \quad \mu \quad V \quad b \quad h$ $\quad n = 6$ parameters

② Select M, L, t as primary dimensions

③

R	ρ	μ	V	b	h	
$\frac{ML}{t^2}$	$\frac{M}{L^3}$	$\frac{M}{Lt}$	$\frac{L}{t}$	L	L	$r = 3$ primary dimensions

④ ρ, V, b $\quad n = r = 3$ repeating parameters

⑤ Then $n - m = 3$ dimensionless groups will result

Setting up dimensional equations

$\Pi_1 = \rho^a V^c b^d R$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^c L^d \frac{ML}{t^2}$

Summing exponents,

M: $a + 1 = 0$

L: $-3a + c + d + 1 = 0$

t: $-c - 2 = 0$

i.e. $a = -1$

$c = -2$

$d = 3a - c - 1 = -2$

$\therefore \Pi_1 = \frac{R}{\rho V^2 b^2}$

$\Pi_2 = \rho^a V^c b^d \mu$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^c L^d \frac{M}{Lt}$

Summing exponents,

M: $a + 1 = 0$

L: $-3a + c + d - 1 = 0$

t: $-c - 1 = 0$

i.e. $a = -1$

$c = -1$

$d = 3a - c + 1 = -1$

$\therefore \Pi_2 = \frac{\mu}{\rho V b}$

$\Pi_3 = \rho^a V^c b^d h$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^c L^d L$

Summing exponents,

M: $a = 0$

L: $-3a + c + d + 1 = 0$

t: $-c = 0$

i.e $a = 0$

$c = 0$

$d = 3a - c - 1 = -1$

$\therefore \Pi_3 = \frac{h}{b}$

Thus convenient coordinates are $\frac{R}{\rho V^2 b^2}, \frac{\mu}{\rho V b}, \frac{h}{b}$

⑥ Check using F, L, t dimensions

$\Pi_1 = F^0 \cdot \frac{L^4}{Ft^2} \frac{t^2}{L^2} \frac{1}{L^2} = [1]$ ✓

$\Pi_2 = \frac{1}{Re} = [1]$ ✓

$\Pi_3 = \frac{L}{L} = [1]$ ✓

Given: Diameter, d, of liquid droplets formed in fuel injection process is a function of ρ, μ, σ (surface tension), $V, \mathcal{D}$.

Find: (a) number of dimensionless ratios required to characterize the process
(b) the dimensionless ratios.

Solution: Apply Buckingham π-Theorem

① $d \quad \rho \quad \mu \quad \sigma \quad V \quad \mathcal{D}$ $\quad n = 6$ parameters

② Select M, L, t as primary dimensions

③

d	ρ	μ	σ	V	$\mathcal{D}$
L	$\frac{M}{L^3}$	$\frac{M}{Lt}$	$\frac{M}{t^2}$	$\frac{L}{t}$	L

$r = 3$ primary dimensions

④ $\rho, \mathcal{D}, V$ $\quad n = r = 3$ repeating parameters

⑤ Then $n - m = 3$ dimensionless groups will result.

Setting up dimensional equations

$\pi_1 = \rho^a \mathcal{D}^b V^c d$

$M^0 L^0 t^0 = \left(\frac{M}{L^3}\right)^a L^b \left(\frac{L}{t}\right)^c L$

Summing exponents,

M: $a = 0$
L: $-3a + b + c + 1 = 0$
t: $-c = 0$

i.e. $a = 0$, $c = 0$, $b = -1$

$\therefore \pi_1 = \frac{d}{\mathcal{D}}$

$\pi_2 = \rho^a \mathcal{D}^b V^c \mu$

$M^0 L^0 t^0 = \left(\frac{M}{L^3}\right)^a L^b \left(\frac{L}{t}\right)^c \frac{M}{Lt}$

Summing exponents

M: $a + 1 = 0$
L: $-3a + b + c - 1 = 0$
t: $-c - 1 = 0$

i.e. $a = -1$, $c = -1$, $b = 3a - c + 1 = -1$

$\therefore \pi_2 = \frac{\mu}{\rho V \mathcal{D}}$

$\pi_3 = \rho^a \mathcal{D}^b V^c \sigma$

$M^0 L^0 t^0 = \left(\frac{M}{L^3}\right)^a L^b \left(\frac{L}{t}\right)^c \frac{M}{t^2}$

Summing exponents

M: $a + 1 = 0$
L: $-3a + b + c = 0$
t: $-c - 2 = 0$

i.e. $a = -1$, $c = -2$, $b = 3a - c = -1$

$\therefore \pi_3 = \frac{\sigma}{\rho \mathcal{D} V^2}$

⑥ Check using F, L, t dimensions

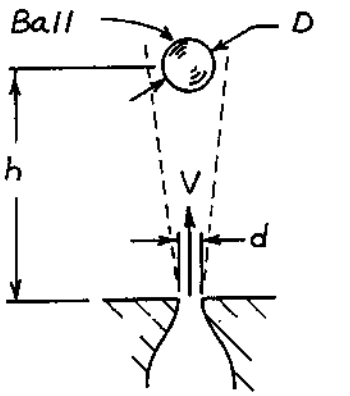
$\pi_1 = \frac{L}{L} = [1]$ ✓

$\pi_2 = \frac{Ft}{L^2} \cdot \frac{L^4}{Ft^2} \cdot \frac{t}{L} \cdot \frac{1}{L} = [1]$ ✓

$\pi_3 = \frac{F}{L^2} \cdot \frac{L^4}{Ft^2} \cdot \frac{1}{L} \cdot \frac{t^2}{L^2} = [1]$ ✓

Given: Ball in jet

$$h = h(d, D, \rho, V, \mu, W)$$



Find: Pi parameters

Solution: Apply Buckingham procedure

① $h \quad d \quad D \quad \rho \quad V \quad \mu \quad W \quad n = 7$

② M, L, t

③ $L \quad L \quad L \quad \frac{M}{L^3} \quad \frac{L}{t} \quad \frac{M}{Lt} \quad \frac{ML}{t^2} \quad m = 3$ $\quad n - m = 7 - 3 = 4$ parameters

④ Choose ρ, V, d as repeating parameters.

⑤ $\rho^a V^b d^c W = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \frac{ML}{t^2} = M^0 L^0 t^0$

$M: \; a + 1 = 0 \qquad a = -1$

$L: \; -3a + b + c + 1 = 0 \qquad c = -2$

$t: \; -b - 2 = 0 \qquad b = -2$

$\Pi_1 = \frac{W}{\rho V^2 d^2}$ Π_1

⑥ Check: $F \times \frac{L^4}{Ft^2} \frac{t^2}{L^2} \frac{1}{L^2} = 1$ ✓✓

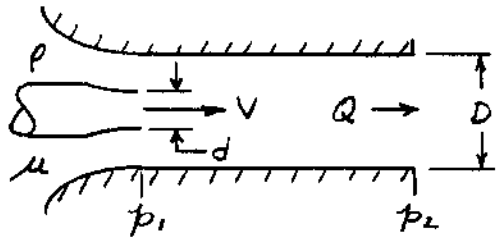
$$\Pi_2 = \rho^a V^b d^0 \mu = \frac{\mu}{\rho V d}$$

$$\Pi_3 = \rho^a V^b d^c h = \frac{h}{d}$$

$$\Pi_4 = \rho^a V^b d^c D = \frac{D}{d}$$

Given: Jet pump shown.

$$\Delta p = f(\rho, V, d, D, \mu, Q)$$



Find: (a) Number of independent dimensionless parameters.
(b) Obtain parameters containing (1) Q and (2) μ.

Solution: Apply the Buckingham π-theorem.

① Δp ρ V d D μ Q $n = 7$ parameters

② Select M, L, t as primary dimensions.

③ $\frac{M}{Lt^2}$ $\frac{M}{L^3}$ $\frac{L}{t}$ L L $\frac{M}{Lt}$ $\frac{L^3}{t}$ $r = 3$ primary dimensions

$m = r = 3$

④ Choose ρ, V, d as repeating variables.

⑤ Then $n - m = 7 - 3 = 4$ dimensionless parameters result. ← $n-m$

Set up dimensional equations:

$\Pi_1 = \rho^a V^b d^c Q$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \frac{L^3}{t}$

Equating exponents:

M: $a = 0$ $a = 0$
L: $-3a + b + c + 3 = 0$
t: $-b - 1 = 0$ $b = -1$

$\therefore c = 3a - b - 3 = -2$

$\Pi_1 = \frac{Q}{Vd^2}$

$\Pi_2 = \rho^a V^b d^c \mu$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \frac{M}{Lt}$

Equating exponents:

M: $a + 1 = 0$ $a = -1$
L: $-3a + b + c - 1 = 0$
t: $-b - 1 = 0$ $b = -1$

$\therefore c = 1 + 3a - b = -1$

$\Pi_2 = \frac{\mu}{\rho V d}$

For completeness,

$\Pi_3 = \rho^a V^b d^c \Delta p = \frac{\Delta p}{\rho V^2}$

$\Pi_4 = \rho^a V^b d^c D = \frac{D}{d}$

← Π's

⑥ Check using FLt dimensions: $\mu = Ft/L^2$, $\rho = \frac{M}{L^3}\frac{Ft^2}{ML} = Ft^2/L^4$

$\Pi_1 = \frac{L^3}{t}\frac{t}{L}\frac{1}{L^3} = [1]$ vv

$\Pi_2 = \frac{Ft}{L^2}\frac{L^4}{Ft^2}\frac{t}{L}\frac{1}{L} = [1]$ vv

The functional relationship would be

$$\Pi_3 = f(\Pi_1, \Pi_2, \Pi_4)$$

$$\frac{\Delta p}{\rho V^2} = f\left(\frac{Q}{Vd^2}, \frac{\mu}{\rho V d}, \frac{D}{d}\right)$$

Given: Pressurized tank drained through a smooth nozzle, area A.

$$\dot{m} = \dot{m}(\Delta p, h, \rho, A, g)$$

Find: (a) Number of independent dimensionless parameters.
(b) Obtain the parameters.
(c) State the functional relationship for $\dot{m}$.

Solution: Apply the Buckingham Π-theorem.

(1) $\dot{m}$ Δp h ρ A g $n=6$ parameters

(2) Select M, L, t as primary dimensions

(3) $\frac{M}{t}$ $\frac{M}{Lt^2}$ L $\frac{M}{L^3}$ L^2 $\frac{L}{t^2}$ $r=3$ primary dimensions

$m = r = 3$

(4) Choose ρ, A, g as repeating parameters.

(5) Then $n-m = 6-3 = 3$ dimensionless parameters result. ← n-m

Set up dimensional equations:

$\Pi_1 = \rho^a A^b g^c \dot{m}$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a (L^2)^b \left(\frac{L}{t^2}\right)^c \frac{M}{t}$

Equating exponents:

M: $a + 1 = 0$ $\quad a = -1$

L: $-3a + 2b + c = 0$

t: $-2c - 1 = 0$ $\quad c = -\frac{1}{2}$

$\therefore b = \frac{1}{2}(3a - c) = -\frac{5}{4}$

$\Pi_1 = \dfrac{\dot{m}}{\rho A^{5/4} g^{1/2}}$

$\Pi_2 = \rho^a A^b g^c \Delta p$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a (L^2)^b \left(\frac{L}{t^2}\right)^c \frac{M}{Lt^2}$

Equating exponents:

M: $a + 1 = 0$ $\quad a = -1$

L: $-3a + 2b + c - 1 = 0$

t: $-2c - 2 = 0$ $\quad c = -1$

$\therefore b = \frac{1}{2}(1 + 3a - c) = -\frac{1}{2}$

$\Pi_2 = \dfrac{\Delta p}{\rho A^{1/2} g}$

$\Pi_3 = \rho^a A^b g^c h$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a (L^2)^b \left(\frac{L}{t^2}\right)^c L$

Equating exponents:

M: $a = 0$ $\quad a = 0$

L: $-3a + 2b + c + 1 = 0$

t: $-2c + 0 = 0$ $\quad c = 0$

$\therefore b = \frac{1}{2}(-1 + 3a - c) = -\frac{1}{2}$

$\Pi_3 = \dfrac{h}{A^{1/2}}$ ← Π_3

(6) Check using FLt dimensions: $\dot{m} = \frac{M}{t}\frac{Ft^2}{ML} = \frac{Ft}{L}$; $\rho = \frac{M}{L^3}\frac{Ft^2}{ML} = \frac{Ft^2}{L^4}$

$\Pi_1 = \frac{Ft}{L}\frac{L^4}{Ft^2}\frac{1}{L^{5/2}}\frac{t}{L^{1/2}} = [1]$ vv

$\Pi_2 = \frac{F}{L^2}\frac{L^4}{Ft^2}\frac{1}{L}\frac{t^2}{L} = [1]$ vv

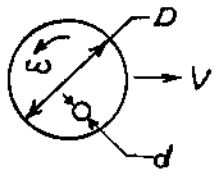
$\Pi_3 = \frac{L}{L} = [1]$ vv

Thus $\Pi_1 = f(\Pi_2, \Pi_3)$ $\quad \dfrac{\dot{m}}{\rho A^{5/4} g^{1/2}} = f\left(\dfrac{\Delta p}{\rho A^{1/2} g}, \dfrac{h}{A^{1/2}}\right)$

or $\dot{m} = \rho A^{5/4} g^{1/2} f\left(\dfrac{\Delta p}{\rho A^{1/2} g}, \dfrac{h}{A^{1/2}}\right)$ ← $\dot{m}$

Given: Aerodynamic torque on spinning ball,

$$T = f(V, \rho, \mu, D, \omega, d)$$



Find: Dimensionless parameters

Solution: Apply Buckingham procedure.

① List: $T \quad V \quad \rho \quad \mu \quad D \quad \omega \quad d \quad n = 7$

② Choose M, L, t

③ $\frac{ML^2}{t^2} \quad \frac{L}{t} \quad \frac{M}{L^3} \quad \frac{M}{Lt} \quad L \quad \frac{1}{t} \quad L \quad m = 3$

④ Choose ρ, V, D $\qquad n - m = 4$ parameters

⑤ $\pi_1 = \rho^a V^b D^c T = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \frac{ML^2}{t^2} = M^0 L^0 t^0$

$M: \; a + 1 = 0 \qquad a = -1$

$L: \; -3a + b + c + 2 = 0 \qquad c = -3 \qquad \pi_1 = \frac{T}{\rho V^2 D^3}$ ← π_1

$t: \; -b - 2 = 0 \qquad b = -2$

⑥ Check: $\pi_1 = FL \cdot \frac{L^4}{Ft^2} \cdot \frac{t^2}{L^2} \cdot \frac{1}{L^3} = 1$ ✓✓

$$\pi_2 = \frac{\mu}{\rho V D}$$

$$\pi_3 = \frac{\omega D}{V}$$

$$\pi_4 = \frac{d}{D}$$

$$\pi_1 = f(\pi_2, \pi_3, \pi_4)$$

$$\frac{T}{\rho V^2 D^3} = f\left(\frac{\mu}{\rho V D}, \frac{\omega D}{V}, \frac{d}{D}\right)$$ ← T

Given: Thrust, F_t, of a marine propeller is thought to depend on: ρ (water density), $\mathcal{D}$ (diameter), V (speed of advance), g (acceleration of gravity), ω (angular speed of propeller), p (pressure in the liquid), and μ (liquid viscosity)

Find: Dimensionless parameters that characterize propeller performance.

Solution: Apply Buckingham Π-Theorem

① List: F_t ρ $\mathcal{D}$ V g ω p μ $(n=8)$

② Choose M, L, t as primary dimensions

③ Dimensions: $\frac{ML}{t^2}$ $\frac{M}{L^3}$ L $\frac{L}{t}$ $\frac{L}{t^2}$ $\frac{1}{t}$ $\frac{M}{Lt^2}$ $\frac{M}{Lt}$

④ Repeating variables $\rho, V, \mathcal{D}$ $m=r=3$

⑤ Then $n-m=5$ dimensionless groups will result

Setting up dimensional equations

$\Pi_1 = \rho^a V^b \mathcal{D}^c F_t$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \frac{ML}{t^2}$

M: $0 = a+1$ → $a=-1$
t: $0 = -b-2$ → $b=-2$
L: $0 = -3a+b+c+1$ → $c=-2$

$\therefore \Pi_1 = \frac{F_t}{\rho V^2 \mathcal{D}^2}$

$\Pi_2 = \rho^a V^b \mathcal{D}^c g$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \left(\frac{L}{t^2}\right)$

M: $0 = a$ → $a=0$
t: $0 = -b-2$ → $b=-2$
L: $0 = -3a+b+c+1$ → $c=1$

$\therefore \Pi_2 = \frac{g\mathcal{D}}{V^2}$

$\Pi_3 = \rho^a V^b \mathcal{D}^c \omega$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \left(\frac{1}{t}\right)$

M: $0 = a$ → $a=0$
t: $0 = -b-1$ → $b=-1$
L: $0 = -3a+b+c$ → $c=1$

$\therefore \Pi_3 = \frac{\omega\mathcal{D}}{V}$

$\Pi_4 = \rho^a V^b \mathcal{D}^c p$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \frac{M}{Lt^2}$

M: $0 = a+1$ → $a=-1$
t: $0 = -b-2$ → $b=-2$
L: $0 = -3a+b+c-1$ → $c=0$

$\therefore \Pi_4 = \frac{p}{\rho V^2}$

$\Pi_5 = \rho^a V^b \mathcal{D}^c \mu$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \frac{M}{Lt}$

M: $0 = a+1$ → $a=-1$
t: $0 = -b-1$ → $b=-1$
L: $0 = -3a+b+c-1$ → $c=-1$

$\therefore \Pi_5 = \frac{\mu}{\rho V \mathcal{D}}$

Dimensionless parameters are $\frac{F_t}{\rho V^2 \mathcal{D}^2}$, $\frac{g\mathcal{D}}{V^2}$, $\frac{\omega\mathcal{D}}{V}$, $\frac{p}{\rho V^2}$, $\frac{\mu}{\rho V \mathcal{D}}$

⑥ Check using F, L, t

$\Pi_1 = F \times \frac{L^4}{Ft^2} \times \frac{t^2}{L^2} \times \frac{1}{L^2} = [\,]$ ✓ $\quad \Pi_2 = \frac{L}{t^2} \times L \times \frac{t^2}{L^2} = [\,]$ ✓

$\Pi_3 = \frac{1}{t} \times L \times \frac{t}{L} = [\,]$, $\quad \Pi_4 = \frac{F}{L^2} \times \frac{L^4}{Ft^2} \times \frac{t^2}{L^2} = [\,]$

$\Pi_5 = \frac{1}{Re} = [\,]$ ✓

Given: Power loss, $\mathcal{P}$, depends on: length, ℓ, diameter, D; clearance, c; angular speed, ω; viscosity, μ; mean pressure, p.

Find: (a) Dimensionless parameters that characterize the problem
(b) Functional form of dependence of $\mathcal{P}$ on these parameters

Solution: Apply Buckingham π - theorem

① $\mathcal{P}$ ℓ D c ω μ p $n=7$ parameters

② Select F, L, t as primary dimensions

③

$\mathcal{P}$	ℓ	D	c	ω	μ	p
$\frac{FL}{t}$	L	L	L	$\frac{1}{t}$	$\frac{Ft}{L^2}$	$\frac{F}{L^2}$

④ D, ω, p $m=r=3$ repeating parameters

⑤ Then $n-m=4$ dimensionless groups will result.

Setting up dimensional equations

$\pi_1 = D^a \omega^b p^e \mathcal{P}$

$F^0L^0t^0 = L^a\left(\frac{1}{t}\right)^b\left(\frac{F}{L^2}\right)^e \frac{FL}{t}$

Equating exponents,

F: $0 = e+1$
L: $0 = a-2e+1$
t: $0 = -b-1$

$\therefore e=-1$, $a=-3$, $b=-1$

$\pi_1 = \frac{\mathcal{P}}{p\omega D^3}$

$\pi_2 = D^a \omega^b p^e \ell$

$F^0L^0t^0 = L^a\left(\frac{1}{t}\right)^b\left(\frac{F}{L^2}\right)^e L$

Equating exponents,

F: $0 = e$
L: $0 = a-2e+1$
t: $0 = -b$

$\therefore e=0$, $a=-1$, $b=0$

$\pi_2 = \frac{\ell}{D}$

$\pi_3 = D^a \omega^b p^e c$

$F^0L^0t^0 = L^a\left(\frac{1}{t}\right)^b\left(\frac{F}{L^2}\right)^e L$

Equating exponents,

F: $0 = e$
L: $0 = a-2e+1$
t: $0 = -b$

$\therefore e=0$, $a=-1$, $b=0$

$\pi_3 = \frac{c}{D}$

$\pi_4 = D^a \omega^b p^e \mu$

$F^0L^0t^0 = L^a\left(\frac{1}{t}\right)^b\left(\frac{F}{L^2}\right)^e \frac{Ft}{L^2}$

Equating exponents,

F: $0 = e+1$
L: $0 = a-2e-2$
t: $0 = -b+1$

$\therefore e=-1$, $a=0$, $b=1$

$\pi_4 = \frac{\mu\omega}{p}$

Then, $\frac{\mathcal{P}}{p\omega D^3} = f\left(\frac{\mu\omega}{p}, \frac{\ell}{D}, \frac{c}{D}\right)$ ← $\mathcal{P}$

⑥ Check using M, L, t dimensions

$\pi_1 = \frac{ML^2}{t^3} \times \frac{L^2t^2}{M} \times t \times \frac{1}{L^3} = [1]$

$\pi_2 = \frac{L}{L} = [1]$ $\quad \pi_3 = \frac{L}{L} = [1]$

$\pi_4 = \frac{M}{Lt} \times \frac{1}{t} \times \frac{Lt^2}{M} = [1]$

Given: Power, $\mathcal{P}$, required to drive a propeller is a function of V, D, ω (angular velocity), μ, ρ, and c (speed of sound)

Find: (a) number of dimensionless groups required to characterize situation
(b) the dimensionless groups

Solution: Apply Buckingham Π-theorem

① $\mathcal{P}$ V D ω μ ρ c $\quad n=7$ parameters

② Select M, L, t as primary dimensions

③

$\mathcal{P}$	V	D	ω	μ	ρ	c
$\frac{ML^2}{t^3}$	$\frac{L}{t}$	L	$\frac{1}{t}$	$\frac{M}{Lt}$	$\frac{M}{L^3}$	$\frac{L}{t}$

$r=3$ primary dimensions

④ V, D, ρ $\quad m=r=3$ repeating parameters

⑤ Then $n-m = 4$ dimensionless groups will result.

Setting up dimensional equations:

$\Pi_1 = V^a D^b \rho^c \mathcal{P}$

$M^0L^0t^0 = \left(\frac{L}{t}\right)^a L^b \left(\frac{M}{L^3}\right)^c \frac{ML^2}{t^3}$

Summing exponents,

M: $c+1=0 \quad \therefore c=-1$

L: $a+b-3c+2=0$

t: $-a-3=0 \quad \therefore a=-3$

$b = 3c-2-a = -2$

$\therefore \Pi_1 = \frac{\mathcal{P}}{\rho D^2 V^3}$

$\Pi_2 = V^a D^b \rho^c \omega$

$M^0L^0t^0 = \left(\frac{L}{t}\right)^a L^b \left(\frac{M}{L^3}\right)^c \frac{1}{t}$

Summing exponents,

M: $c=0$

L: $a+b-3c=0$

t: $-a-1=0 \quad \therefore a=-1$

$b = 3c-a = 1$

$\therefore \Pi_2 = \frac{\omega D}{V}$

$\Pi_3 = V^a D^b \rho^c \mu$

$M^0L^0t^0 = \left(\frac{L}{t}\right)^a L^b \left(\frac{M}{L^3}\right)^c \frac{M}{Lt}$

Summing exponents,

M: $c+1=0 \quad \therefore c=-1$

L: $a+b-3c-1=0$

t: $-a-1=0 \quad \therefore a=-1$

$b = 3c+1-a = -1$

$\therefore \Pi_3 = \frac{\mu}{\rho D V}$

$\Pi_4 = V^a D^b \rho^c c$

$M^0L^0t^0 = \left(\frac{L}{t}\right)^a L^b \left(\frac{M}{L^3}\right)^c \frac{L}{t}$

Summing exponents,

M: $c=0$

L: $a+b-3c+1=0$

t: $-a-1=0 \quad \therefore a=-1$

$b = 3c-a-1 = 0$

$\therefore \Pi_4 = \frac{c}{V}$

Dimensionless groups are: $\frac{\mathcal{P}}{\rho D^2 V^3}, \frac{\omega D}{V}, \frac{\mu}{\rho D V}, \frac{c}{V}$

⑥ Check using F, L, t

$\Pi_1 = \frac{FL}{t} \cdot \frac{t^4}{FL^2} \frac{1}{L^2} \frac{t^3}{L^3} = [1]^V$

$\Pi_2 = \frac{1}{t} L \frac{t}{L} = [1]^V$

$\Pi_3 = \frac{1}{Re} = [1]^V$

$\Pi_4 = \frac{L}{t} \cdot \frac{t}{L} = [1]^V$

Given: Breakup of liquid jet into droplets, as it enters a surrounding gas.

Find: (a) Properties expected to control
(b) Dimensionless parameters that result

Solution: Strength of jet will depend on ρ_L, V, and D, may be μ.

Droplet diameter d will also depend on surface tension σ, together with ρ_G.

Thus expect $d = f(\rho_L, V, D, \sigma, \mu, \rho_G)$ ← $n = 7$

① d ρ_L V D σ μ ρ_G $n = 7$

② Choose M, L, t as primary dimensions ($r = 3$ primary dimensions)

③ L $\frac{M}{L^3}$ $\frac{L}{t}$ L $\frac{M}{t^2}$ $\frac{M}{Lt}$ $\frac{M}{L^3}$

④ ρ_L, V, D $m = r = 3$ repeating parameters

⑤ Then expect $n - m = 4$ dimensionless groups.

$\Pi_1 = \rho_L^a V^b D^c \mu = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \frac{M}{Lt}$

$M: a + 1 = 0 \quad a = -1$
$L: -3a + b + c - 1 = 0$
$t: -b - 1 = 0 \quad b = -1$
$c = 3a - b + 1 = -1$

$\Pi_1 = \frac{\mu}{\rho_L V D}$

$\Pi_2 = \rho_L^a V^b D^c \sigma = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \frac{M}{t^2}$

$M: a + 1 = 0 \quad a = -1$
$L: -3a + b + c = 0$
$t: -b - 2 = 0 \quad b = -2$
$c = 3a - b = -3 + 2 = -1$

$\Pi_2 = \frac{\sigma}{\rho_L V^2 D}$

$\Pi_3 = \rho_L^a V^b D^c d$

By inspection, $\Pi_3 = \frac{d}{D}$

$\Pi_4 = \rho_L^a V^b D^c \rho_G$

By inspection, $\Pi_4 = \frac{\rho_G}{\rho_L}$

Thus $\Pi_3 = f(\Pi_1, \Pi_2, \Pi_4)$ or $\frac{d}{D} = f\left(\frac{\mu}{\rho_L V D}, \frac{\sigma}{\rho_L V^2 D}, \frac{\rho_G}{\rho_L}\right)$ ← $\frac{d}{D}$

⑥ Check, using F, L, t $\Pi_1 = \frac{Ft}{L^2} \frac{L^4}{Ft^2} \frac{t}{L} \frac{1}{L} = [1]$ ✓ ; $\Pi_2 = \frac{1}{Re_L} = [1]$ ✓

$\Pi_3 = \frac{L}{L} = [1]$ ✓ $\quad \Pi_4 = \frac{Ft^2}{L^4} \frac{L^4}{Ft^2} = [1]$ ✓

Given: Fan-assisted convection oven; $\dot{Q}$ = heat transfer rate (energy/time).

$$\dot{Q} = f(c_p, \Theta, L, \rho, \mu, V)$$

Find: (a) Number of basic dimensions included in these variables.
(b) Number of π-parameters.
(c) Obtain the parameters.

Solution: Apply the Buckingham π-theorem.

① $\dot{Q}$ c_p Θ L ρ μ V $n = 7$ parameters

② Select F, L, t, T (temperature) as primary dimensions.

③ $\frac{FL}{t}$ $\frac{L^2}{t^2T}$ T L $\frac{Ft^2}{L^4}$ $\frac{Ft}{L^2}$ $\frac{L}{t}$ $r = 4$ primary dimensions

④ Choose ρ, V, L, Θ as repeating parameters. $m = r = 4$

⑤ Then $n - m = 7 - 4 = 3$ dimensionless parameters result. $n-m$

Set up dimensional equations:

$\pi_1 = \rho^a V^b L^c \Theta^d \dot{Q}$

$F^0L^0t^0T^0 = \left(\frac{Ft^2}{L^4}\right)^a \left(\frac{L}{t}\right)^b (L)^c (T)^d \frac{FL}{t}$

Equating exponents:

$F: a + 1 = 0 \quad a = -1$
$L: -4a + b + c + 1 = 0$
$t: 2a - b - 1 = 0 \quad b = -3$
$T: d = 0$

$\therefore c = -1 + 4a - b = -2$

$\pi_1 = \frac{\dot{Q}}{\rho V^3 L^2}$

$\pi_2 = \rho^a V^b L^c \Theta^d c_p$

$F^0L^0t^0T^0 = \left(\frac{Ft^2}{L^4}\right)^a \left(\frac{L}{t}\right)^b (L)^c (T)^d \frac{L^2}{t^2T}$

Equating exponents:

$F: a = 0 \quad a = 0$
$L: -4a + b + c + 2 = 0$
$t: 2a - b - 2 = 0 \quad b = -2$
$T: d - 1 = 0 \quad d = 1$

$\therefore c = -2 + 4a - b = 0$

$\pi_2 = \frac{c_p \Theta}{V^2}$

$\pi_3 = \rho^a V^b L^c \Theta^d \mu$

$F^0L^0t^0T^0 = \left(\frac{Ft^2}{L^4}\right)^a \left(\frac{L}{t}\right)^b (L)^c (T)^d \frac{Ft}{L^2}$

Equating exponents:

$F: a + 1 = 0 \quad a = -1$
$L: -4a + b + c - 2 = 0$
$t: 2a - b + 1 = 0 \quad b = -1$
$T: d = 0 \quad d = 0$

$\therefore c = 2 + 4a - b = -1$

$\pi_3 = \frac{\mu}{\rho V L}$

⑥ Check, using MLtT dimensions: $\dot{Q} = ML^2/t^3$; $\mu = M/Lt$

$\pi_1 = \frac{ML^2L^3}{t^3M}\frac{t^3}{L^3}\frac{1}{L^2} = [1]$ vv

$\pi_2 = \frac{L^2}{t^2T} T \frac{t^2}{L^2} = [1]$ vv

$\pi_3 = \frac{M}{Lt}\frac{L^3}{M}\frac{t}{L}\frac{1}{L} = [1]$ vv

Thus $\pi_1 = f(\pi_2, \pi_3)$ $\quad \frac{\dot{Q}}{\rho V^3 L^2} = f\left(\frac{c_p\Theta}{V^2}, \frac{\mu}{\rho V L}\right)$

or $\dot{Q} = \rho V^3 L^2 f\left(\frac{c_p\Theta}{V^2}, \frac{\mu}{\rho V L}\right)$ $\dot{Q}$

Given: Water hammer caused by sudden closure of valve in pipeline.

$$p_{max} = f(\rho, V_0, E_v)$$

Find: (a) How many dimensionless groups needed to characterize?
(b) Functional relationship in terms of Π groups.

Solution: Step ①: List p_{max} ρ V_0 E_v $n = 4$

Step ②: Choose M, L, t

Step ③: $\frac{M}{Lt^2}$ $\frac{M}{L^3}$ $\frac{L}{t}$ $\frac{M}{Lt^2}$

Check dimensional matrix:

	p_{max}	ρ	V_0	E_v
M	1	1	0	1
L	-1	-3	1	-1
t	-2	0	-1	-2

For this matrix, $r = 2$

Step ④: Choose ρ, V_0

Step ⑤: $\Pi_1 = \rho^a V_0^b p_{max} = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b \frac{M}{Lt^2} = M^0 L^0 t^0$

$M: a + 1 = 0 \quad a = -1$
$L: -3a + b - 1 = 0$
$t: -b - 2 = 0 \quad b = -2$ ✓

$$\Pi_1 = \frac{p_{max}}{\rho V_0^2} \quad \Pi_1$$

$$\Pi_2 = \rho^a V_0^b E_v = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b \frac{M}{Lt^2} = M^0 L^0 t^0$$

By inspection

$$\Pi_2 = \frac{E_v}{\rho V_0^2} \quad \Pi_2$$

Step ⑥: Check using FLt: $\rho = \frac{M}{L^3} \times \frac{Ft^2}{ML} = \frac{Ft^2}{L^4}$

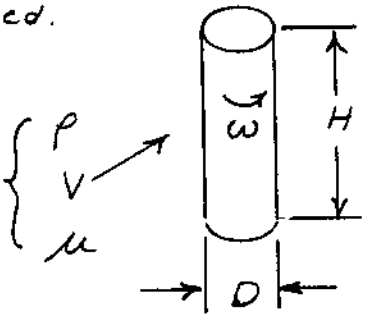
$$\Pi_1 = \frac{F}{L^2} \frac{L^4}{Ft^2} \frac{t^2}{L^2} = \frac{FL^4t^2}{FL^4t^2} = 1 \quad \checkmark\checkmark$$

The functional relationship is $\Pi_1 = f(\Pi_2)$. Thus

$$\frac{p_{max}}{\rho V_0^2} = f\left(\frac{E_v}{\rho V_0^2}\right) \quad f$$

Given: Vessel to be powered by rotating cylinder. Model to be tested to estimate power needed to rotate cylinder.

Find: (a) Parameters that should be included.
(b) Important dimensionless groups



Solution: $\mathcal{P} = f(\rho, \omega, D, \mu, H, V)$

① ρ ω D μ H V $\mathcal{P}$ $n = 7$

② Choose M, L, t as primary dimensions

③ $\frac{M}{L^3}$ $\frac{1}{t}$ L $\frac{M}{Lt}$ L $\frac{L}{t}$ $\frac{ML^2}{t^3}$ $r = 3$ primary dimensions

④ ρ, ω, D $m = 3$ $m = r = 3$ repeating parameters

⑤ Then expect $n - m = 4$ dimensionless groups

$\Pi_1 = \rho^a \omega^b D^c \mathcal{P} = \left(\frac{M}{L^3}\right)^a \left(\frac{1}{t}\right)^b (L)^c \frac{ML^2}{t^3}$

$M: a + 1 = 0 \quad a = -1$
$L: -3a + c + 2 = 0 \quad c = -5$
$t: -b - 3 = 0 \quad b = -3$

$\Pi_1 = \frac{\mathcal{P}}{\rho \omega^3 D^5}$ Π_1

$\Pi_2 = \rho^a \omega^b D^c V = \left(\frac{M}{L^3}\right)^a \left(\frac{1}{t}\right)^b (L)^c \frac{L}{t}$

$M: a + 0 = 0 \quad a = 0$
$L: -3a + c + 1 = 0 \quad c = -1$
$t: -b - 1 = 0 \quad b = -1$

$\Pi_2 = \frac{V}{\omega D}$ Π_2

$\Pi_3 = \rho^a \omega^b D^c H$

By inspection $\Pi_3 = \frac{H}{D}$ Π_3

$\Pi_4 = \rho^a \omega^b D^c \mu = \left(\frac{M}{L^3}\right)^a \left(\frac{1}{t}\right)^b (L)^c \frac{M}{Lt}$

$M: a + 1 = 0 \quad a = -1$
$L: -3a + c - 1 = 0 \quad c = -2$
$t: -b - 1 = 0 \quad b = -1$

$\Pi_4 = \frac{\mu}{\rho \omega D^2}$

Thus $\Pi_1 = f(\Pi_2, \Pi_3, \Pi_4)$ or $\frac{\mathcal{P}}{\rho \omega^3 D^5} = f\left(\frac{V}{\omega D}, \frac{H}{D}, \frac{\mu}{\rho \omega D^2}\right)$ $\frac{\mathcal{P}}{\rho \omega^3 D^5}$

⑥ Check, using F, L, t

$\Pi_1 = \frac{FL}{t} \frac{L^4}{Ft^2} \frac{t^3}{1} \frac{L^5}{1} = [1]$ ✓

$\Pi_2 = \frac{L}{t} \frac{t}{1} \frac{1}{L} = [1]$ ✓

$\Pi_3 = \frac{L}{L} = [1]$ ✓

$\Pi_4 = \frac{Ft}{L^2} \frac{L^4}{Ft^2} \frac{t}{1} \frac{1}{L^2} = [1]$ ✓

Given: Desire to match Reynolds number in two flows: one of air and one of water, using the same size model.

Find: Which flow must have the higher speed, and by how much.

Solution: Set $Re_w = \frac{\rho_w V_w L_w}{\mu_w} = Re_a = \frac{\rho_a V_a L_a}{\mu_a}$

Since $L_w = L_a$, then $\frac{V_a}{V_w} = \frac{\rho_w}{\rho_a}\frac{\mu_a}{\mu_w} = \frac{\nu_a}{\nu_w}$

From Tables A.8 and A.10, at 20°C, $\nu_w = 1.00 \times 10^{-6}\ m^2/s$ and $\nu_a = 1.51 \times 10^{-5}\ m^2/s$.

Thus $\frac{V_a}{V_w} = 1.51 \times 10^{-5} \frac{m^2}{s} \times \frac{s}{1.00 \times 10^{-6} m^2} = 15.1$

Therefore V_a must be larger than V_w. ← V_a

In fact, to match Re,

$V_a = 15.1\ V_w$ ← V_a

Given: Airship to operate at 20 m/sec in standard air.
Model built to 1/20 scale tested at same air temperature.
Model is tested at 75 m/sec

Find: (a) Criterion for dynamic similarity.
(b) Wind tunnel pressure.
(c) Prototype drag if drag force on model is 250 N.

Solution

Dimensional analysis predicts $\frac{F}{\rho V^2 L^2} = f\left(\frac{\rho V L}{\mu}\right)$

Consequently for similarity, $\left(\frac{\rho V L}{\mu}\right)_m = \left(\frac{\rho V L}{\mu}\right)_p$

Since L is fixed, and $\mu_p = \mu_m$ (because T is the same)

$$\rho_m = \rho_p \frac{V_p}{V_m}\frac{L_p}{L_m}\frac{\mu_m}{\mu_p} = \rho_p \frac{20}{75}(20)(1) = 5.33\,\rho_p$$

From ideal gas law, $P = \rho RT$

$$\therefore \frac{P_m}{P_p} = \frac{\rho_m}{\rho_p} = 5.33 \quad \text{and} \quad P_m = 5.33 P_p = 5.33 \times 101\ kPa = 5.39 \times 10^5\ Pa \qquad P_m$$

From the force ratios,

$$F_p = F_m \frac{\rho_p}{\rho_m}\frac{V_p^2}{V_m^2}\frac{L_p^2}{L_m^2} = F_m \frac{1}{5.33}\left(\frac{20}{75}\right)^2(20)^2 = 5.34\,F_m$$

Thus

$$F_p = 5.34 F_m = 5.34 \times 250\,N = 1.34\,kN \qquad F_p$$

Given: Measurements of drag force are made on a model car in a towing tank filled with freshwater; $L_m/L_p = 1/5$. The dimensionless force ratio becomes constant at model test speeds above $V_m = 4$ m/s. At this speed the drag force on the model is $F_{D_m} = 182$ N

Find: (a) State conditions required to assure dynamic similarity between model and prototype
(b) Determine required speed ratio V_m/V_p to assure dynamically similar conditions
(c) Calculate expected prototype drag when operating in air at speed, $V_p = 90$ km/hr

Solution:

(a) The flows must be geometrically and kinematically similar, and have equal Reynolds numbers to be dynamically similar
- geometric similarity requires true model in all respects
- kinematic similarity requires same flow pattern, i.e. no free-surface effects or cavitation.
- The problem may be stated as $F_D = f(\rho, V, L, \mu)$. Dimensional analysis gives

$$\frac{F_D}{\rho V^2 L^2} = f\left(\frac{\mu}{\rho V L}\right) = g(Re)$$

(b) Matching Reynolds numbers between model & prototype flows gives $\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p}$ Assume $T = 20^\circ C$

$$\frac{V_m}{V_p} = \frac{\nu_m}{\nu_p} \times \frac{L_p}{L_m} = 1 \times 10^{-6} \frac{s}{m^2} \times 1.51 \times 10^{-5} \frac{s}{m^2} \times 5 = 0.331 \qquad \frac{V_m}{V_p}$$

(c) For dynamically similar conditions, $\left(\frac{F_D}{\rho V^2 L^2}\right)_m = \left(\frac{F_D}{\rho V^2 L^2}\right)_p$

$$\therefore F_{D_p} = F_{D_m} \frac{\rho_p}{\rho_m} \times \left(\frac{V_p}{V_m}\right)^2 \times \left(\frac{L_m}{L_p}\right)^2$$

$$= 182\,N \times \frac{1.20}{999} \times \left(\frac{90\,km}{hr} \times \frac{1000\,m}{km} \times \frac{hr}{3600\,s} \times \frac{s}{4\,m}\right)^2 (5)^2$$

$$F_{D_p} = 214\,N \qquad F_{D_p}$$

Given: Prototype torpedo, $D = 533$ mm, $l = 6.7$ m operates in water at a speed of 28 m/s. Model (1/5 scale) is to be tested in a wind tunnel. Maximum wind tunnel speed is 110 m/sec; $T = 20°C$; pressure is variable.
At dynamically similar test conditions, $F_{D\,model} = 618$ N

Find: (a) required wind tunnel pressure for dynamically similar test
(b) expected drag force on prototype

Solution:

Assume $F_D = F_D(V, D, \rho, \mu)$. From the Buckingham π-theorem, for $n = 5$, with $m = r = 3$, we would expect two dimensionless groups.

$$\frac{F}{\rho V^2 D^2} = f\left(\frac{\rho V D}{\mu}\right)$$

To obtain dynamically similar model test, $\left(\frac{\rho V D}{\mu}\right)_m = \left(\frac{\rho V D}{\mu}\right)_p$

$$\therefore \rho_m = \rho_p \frac{V_p}{V_m} \frac{D_p}{D_m} \frac{\mu_m}{\mu_p}$$

For air at 20°C $\mu_m = 1.81 \times 10^{-5}$ N·s/m²
water at 20°C $\mu_p = 1 \times 10^{-3}$ N·s/m²

$$\rho_m = 998 \frac{kg}{m^3} \times \frac{28}{110} \times 5 \times \frac{1.81 \times 10^{-5}}{1 \times 10^{-3}} = 23.0 \frac{kg}{m^3}$$

From the ideal gas equation of state,

$$p = \rho_m R T_m = 23.0 \frac{kg}{m^3} \times 287 \frac{N \cdot m}{kg \cdot K} \times 293 K = 1.93 \text{ MPa (abs)} \quad \leftarrow p$$

For dynamically similar flows,

$$\left(\frac{F_D}{\rho V^2 D^2}\right)_m = \left(\frac{F_D}{\rho V^2 D^2}\right)_p$$

$$\therefore F_{D_p} = F_{D_m} \frac{\rho_p}{\rho_m} \left(\frac{V_p}{V_m}\right)^2 \left(\frac{D_p}{D_m}\right)^2$$

$$= 618 \text{ N} \times \frac{998}{23.0} \left(\frac{28}{110}\right)^2 (5)^2$$

$$F_{D_p} = 43.4 \text{ kN} \quad \leftarrow F_{D_p}$$

Given: Drag force, F, of an airfoil at zero angle of attack is a function of ρ, μ, V, and L.

Model test conditions:

$\frac{L_m}{L_p} = \frac{1}{10}$ $\quad Re_m = 5.5 \times 10^6$ based on chord length

$T = 15^\circ C$, $P = 10$ atmospheres

Prototype data: chord length, $L = 2\,m$

$T = 15^\circ C$ $\quad P = 101\ kPa$

Find: (a) velocity, V_m, of model test

(b) corresponding prototype velocity

Solution

Dimensional analysis predicts $\frac{F}{\rho V^2 L^2} = f\left(\frac{\rho V L}{\mu}\right)$

$Re_m = \left(\frac{\rho V L}{\mu}\right)_m$ and hence $V_m = \frac{Re_m \mu_m}{\rho_m L_m}$

To determine ρ_m assume air behaves as an ideal gas.

$\rho_m = \frac{P_m}{RT_m} = 10 \times 101 \times 10^3 \frac{N}{m^2} \times \frac{kg\ K}{287\ N \cdot m} \times \frac{1}{288\ K} = 12.2\ kg/m^3$

From Table A.10, Appendix A, $\mu_m = 1.79 \times 10^{-5}\ N \cdot s/m^2$

$V_m = \frac{Re_m \mu_m}{\rho_m L_m} = 5.5 \times 10^6 \times 1.79 \times 10^{-5} \frac{N \cdot sec}{m^2} \times \frac{m^3}{12.2\ kg} \times \frac{1}{0.2\,m} \times \frac{kg \cdot m}{N \cdot sec^2}$

$V_m = 40.3\ m/s$ ← V_m

For dynamic similarity $\left(\frac{\rho V L}{\mu}\right)_m = \left(\frac{\rho V L}{\mu}\right)_p$

$V_p = V_m \frac{\mu_p}{\mu_m} \frac{\rho_m}{\rho_p} \frac{L_m}{L_p} = V_m \frac{\mu_p}{\mu_m} \frac{P_m}{P_p} \frac{T_p}{T_m} \frac{L_m}{L_p}$

$V_p = 40.3 \frac{m}{s} \times (1) \times (10) \times (1) \times \left(\frac{1}{10}\right) = 40.3\ m/s$ ← V_p

Given: Airplane wing with chord length, $l = 5$ ft and span, $s = 30$ ft, is designed to move through standard air at speed, $V = 230$ ft/s. A model (1/10 scale) is to be tested in a water tunnel.

Find: (a) speed necessary in water tunnel to achieve dynamic similarity.
(b) ratio of forces measured in the model flow to those on the prototype airfoil.

Solution:

For an airfoil at a given angle of attack, we would expect the forces (e.g. drag) to be depended on l, s, V, ρ, and μ.

$F = F(l, s, V, \rho, \mu)$ From the Buckingham π theorem, with $n=6$, and $m = r = 3$, we would expect three dimensionless parameters.

$$\frac{F}{\rho V^2 l s} = f\left(\frac{\rho V l}{\mu}, \frac{s}{l}\right)$$

Thus for dynamically similar flows over geometrically similar airfoils (at the same angle of attack), then

$$\left(\frac{\rho V l}{\mu}\right)_m = \left(\frac{\rho V l}{\mu}\right)_p \qquad \text{(Assume } T = 59^\circ \text{F)}.$$

$$V_m = V_p \frac{\rho_p}{\rho_m} \frac{l_p}{l_m} \frac{\mu_m}{\mu_p} = 230 \frac{\text{ft}}{\text{sec}} \times \frac{0.00238}{1.94} \times 10 \times \frac{2.37\times10^{-5}}{3.74\times10^{-7}}$$

$$V_m = 179 \text{ ft/s}$$

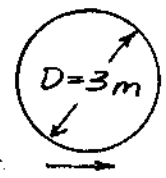
For dynamically similar flows,

$$\left(\frac{F}{\rho V^2 l s}\right)_m = \left(\frac{F}{\rho V^2 l s}\right)_p$$

$$\therefore \frac{F_m}{F_p} = \frac{\rho_m}{\rho_p}\left(\frac{V_m}{V_p}\right)^2 \frac{l_m s_m}{l_p s_p} = \frac{1.94}{0.00238} \times \left(\frac{179}{230}\right)^2 \frac{1}{10} \times \frac{1}{10} = 4.94$$

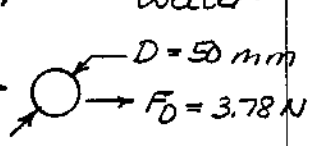
This speed is high. The tunnel would have to be pressurized to minimize the chances of cavitation

Given: Model test of weather balloon. Full-scale:



$$F_D = f(\rho, V, D, \mu, c)$$

Find: (a) Model test speed.
(b) Drag force on full-scale balloon.



Solution: Apply Buckingham procedure to obtain

$$\frac{F_D}{\rho V^2 D^2} = f\left(\frac{\mu}{\rho V D}, \frac{V}{c}\right) = f(Re, M)$$

For similarity $Re_p = Re_m$ and $M_p = M_m$. (Mach number criterion satisfied automatically because $M \approx 0$.) Assume $T = 20°C$.

$$Re_p = \frac{V_p D_p}{\nu_p} = Re_m = \frac{V_m D_m}{\nu_m}$$

{ Water (Table A.8)
{ Air (Table A.10)

$$V_m = V_p \frac{\nu_m}{\nu_p} \frac{D_p}{D_m} = 1.5\frac{m}{s} \times 1\times10^{-6}\frac{m^2}{s} \times \frac{s}{1.51\times10^{-5} m^2} \times \frac{3\ m}{0.05\ m}$$

$$V_m = 5.96\ m/s \longleftarrow V_m$$

Then $\left(\frac{F_D}{\rho V^2 D^2}\right)_m = \left(\frac{F_D}{\rho V^2 D^2}\right)_p$

$$F_{Dp} = F_{Dm} \frac{\rho_p}{\rho_m} \frac{V_p^2}{V_m^2} \frac{D_p^2}{D_m^2}$$

$$= 3.78\ N \times 1.23\frac{kg}{m^3} \times \frac{m^3}{999\ kg} \times \left(\frac{1.5\ m/s}{5.96\ m/s}\right)^2 \times \left(\frac{3.0\ m}{0.05\ m}\right)^2$$

$$F_{Dp} = 1.06\ N \longleftarrow F_{Dp}$$

Given: Fluid dynamic characteristics of a golf ball are to be tested using a model in a wind tunnel.
dependent variables: F_D, F_L
independent variables should include $w \cdot d$ (dimple depth)
Golf pro can hit prototype ($D = 1.68$ in) at $V = 240$ ft/s and $\omega = 9000$ rpm. Prototype is to be modeled in wind tunnel with $V = 80$ ft/s.

Find: (a) suitable dimensionless parameters
(b) required diameter of model
(c) required rotational speed of model

Solution: Assume the functional dependence to be given by

$$F_D = F_D(D, V, \omega, d, \rho, \mu) \quad \text{and} \quad F_L = F_L(D, V, \omega, d, \rho, \mu)$$

From the Buckingham π-theorem, for $n = 7$ and $m = r = 3$, we would expect four dimensionless groups

$$\frac{F_D}{\rho V^2 D^2} = f\left(\frac{\rho V D}{\mu}, \frac{\omega D}{V}, \frac{d}{D}\right) \text{ and } \frac{F_L}{\rho V^2 D^2} = g\left(\frac{\rho V D}{\mu}, \frac{\omega D}{V}, \frac{d}{D}\right) \quad \pi\text{'s}$$

To determine the required diameter of the model,

$$\left(\frac{\rho V D}{\mu}\right)_m = \left(\frac{\rho V D}{\mu}\right)_p \quad \therefore D_m = \frac{\rho_p}{\rho_m} \frac{V_p}{V_m} \frac{\mu_m}{\mu_p} D_p = 1 \times \frac{240}{80} \times 1 \times D_p$$

$$D_m = 3 D_p = 3 \times 1.68 \text{ in} = 5.04 \text{ in.} \quad D_m$$

To determine the required rotational speed of the model,

$$\left(\frac{\omega D}{V}\right)_m = \left(\frac{\omega D}{V}\right)_p \quad \therefore \omega_m = \omega_p \frac{D_p}{D_m} \frac{V_m}{V_p} = \omega_p \frac{1}{3} \times \frac{80}{240} = \frac{1}{9} \omega_p$$

$$\omega_m = \frac{1}{9} \omega_p = \frac{1}{9} \times 9000 \text{ rpm} = 1000 \text{ rpm} \quad \omega_m$$

Given: Flight characteristics of a Frisbee are to be determined via a model test.

dependent parameters: F_D, F_L

independent parameters should include $w \cdot h$ (roughness height).

Test is to be performed (using air) on a model (¼ scale), which is to be geometrically, kinematically, and dynamically similar to the prototype. For prototype, $V_p = 20$ ft/s, $\omega_p = 100$ rpm

Find: (a) suitable dimensionless parameters.
(b) values of V_m and ω_m.

Solution: Assume the functional dependence is given by

$F_D = F_D(\ell, V, \omega, h, \rho, \mu)$ and $F_L = F_L(\ell, V, \omega, h, \rho, \mu)$

From the Buckingham π-theorem, for $n = 7$ and $m = r = 3$, we would expect four dimensionless groups

$$\frac{F_D}{\rho V^2 \ell^2} = f\left(\frac{\rho V \ell}{\mu}, \frac{\omega \ell}{V}, \frac{h}{\ell}\right) \quad \text{and} \quad \frac{F_L}{\rho V^2 \ell^2} = g\left(\frac{\rho V \ell}{\mu}, \frac{\omega \ell}{V}, \frac{h}{\ell}\right)$$

To determine the required air speed, V_m,

$$\left(\frac{\rho V \ell}{\mu}\right)_m = \left(\frac{\rho V \ell}{\mu}\right)_p \qquad \therefore V_m = V_p \frac{\rho_p}{\rho_m} \frac{\ell_p}{\ell_m} \frac{\mu_m}{\mu_p} = V_p (1) \times 4 \times 1 = 4 V_p$$

$$V_m = 4 \times 20 \frac{\text{ft}}{\text{s}} = 80 \text{ ft/s}$$

To determine the required rotational speed, ω_m,

$$\left(\frac{\omega \ell}{V}\right)_m = \left(\frac{\omega \ell}{V}\right)_p \qquad \therefore \omega_m = \omega_p \frac{\ell_p}{\ell_m} \frac{V_m}{V_p} = \omega_p \times 4 \times 4 = 16 \omega_p$$

$$\omega_m = 16 \times 100 \text{ rpm} = 1600 \text{ rpm}$$

Given: Model of hydrofoil boat (1:20 scale) is to be tested in water at 30°F. Prototype operates at speed of 60 knots in water at 45°F.
To model cavitation correctly, cavitation number must be duplicated.

Find: ambient pressure at which model test must be run.

Solution:

To duplicate the Froude number between model and prototype requires

$$\frac{V_m}{\sqrt{gL_m}} = \frac{V_p}{\sqrt{gL_p}} \quad \text{or} \quad \frac{V_m}{V_p} = \left(\frac{L_m}{L_p}\right)^{1/2} = \frac{1}{\sqrt{20}}$$

and $V_m = \frac{1}{\sqrt{20}} V_p = \frac{1}{\sqrt{20}} 60 \text{ knot} = 13.4 \text{ knot}$

For $Ca_m = Ca_p$, then

$$\left(\frac{p - p_v}{\frac{1}{2}\rho V^2}\right)_m = \left(\frac{p - p_v}{\frac{1}{2}\rho V^2}\right)_p$$

or

$$p_m = p_{v_m} + (p - p_v)_p \frac{V_m^2}{V_p^2} \qquad \text{(assuming } \rho_m \approx \rho_p\text{)}$$

and

$$p_m = p_{v_m} + (p - p_v)_p \cdot \frac{1}{20}$$

From the Table 9.1, at $T = 130°F$ $p_{v_m} = 2.23$ psia
$T = 45°F$ $p_{v_p} = 0.15$ psia

$$\therefore p_m = 2.23 \text{ psia} + (14.7 - 0.15) \text{ psia} \cdot \frac{1}{20}$$

$$p_m = 2.96 \text{ psia}$$ p_m

Given: SAE 10W oil at 80F flows in a horizontal pipe of diameter, $D = 1$ in. at an average speed $\bar{V} = 3$ ft/sec. The pressure drop, Δp, is 65.3 psig over a length of 500 ft.
Water at 60°F flows through the same pipe under dynamically similar conditions.

Find: (a) the average speed of the water.
(b) the corresponding pressure drop.

Solution:

From Example Problem 7.2, we learn that pressure drop data for flow in a pipe are correlated by the functional relationship

$$\frac{\Delta p}{\rho \bar{V}^2} = f\left(\frac{\mu}{\rho \bar{V} D}, \frac{\ell}{D}, \frac{\varepsilon}{D}\right)$$

For water flow and oil flow in the same pipe to be dynamically similar requires that

$$\left(\frac{\mu}{\rho \bar{V} D}\right)_{H_2O} = \left(\frac{\mu}{\rho \bar{V} D}\right)_{oil}$$

or $$\bar{V}_{H_2O} = \left(\frac{\mu}{\rho}\right)_{H_2O} \left(\frac{\rho}{\mu}\right)_{oil} \bar{V}_{oil} = \frac{\nu_{H_2O}}{\nu_{oil}} \times \bar{V}_{oil}$$

From Fig A.3 ν_{oil} at 80°F (26.7°C) $= 7 \times 10^{-5}$ m²/s $= 7.53 \times 10^{-4}$ ft²/s

From Table A.7 ν_{H_2O} at 60°F $= 1.21 \times 10^{-5}$ ft²/s

$$\therefore \bar{V}_{H_2O} = 1.21 \times 10^{-5} \text{ ft}^2/\text{s} \times \frac{1}{7.53 \times 10^{-4} \text{ ft}^2/\text{s}} \times 3 \frac{\text{ft}}{\text{sec}} = 0.0482 \text{ ft/sec} \quad \bar{V}_{H_2O}$$

Then

$$\left(\frac{\Delta p}{\rho \bar{V}^2}\right)_{oil} = \left(\frac{\Delta p}{\rho \bar{V}^2}\right)_{H_2O}$$

$$\Delta p_{H_2O} = \frac{\rho_{H_2O}}{\rho_{oil}} \times \frac{\bar{V}_{H_2O}^2}{\bar{V}_{oil}^2} \times \Delta p_{oil}$$

From Table A.2 Appendix A, S.G lubricating oil = 0.88

$$\therefore \Delta p_{H_2O} = \frac{1}{0.88} \times \frac{(0.0482)^2}{3^2} \times 65.3 \text{ psig} = 0.019 \text{ psig} \quad \Delta p_{H_2O}$$

Given: The drag force on a circular cylinder immersed in a water flow can be expressed as

$$F_D = f(D, \ell, V, \rho, \mu)$$

The static pressure distribution on a circular cylinder can be expressed in terms of the dimensionless pressure coefficient

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho V^2}$$

At the location of minimum static pressure on the cylinder surface, $C_p = -2.4$. The onset of cavitation occurs at $Ca = 0.5$

Find: (a) expression for dimensionless drag force
(b) an estimate of maximum speed V at which cylinder could be towed in water (at p_{atm}) without causing cavitation

Solution:

$F_D = f(D, \ell, V, \rho, \mu)$. From the Buckingham π-theorem, for $n = 6$, with $m = r = 3$, we would expect three dimensionless groups.

$$\frac{F}{\rho V^2 D^2} = f_1\left(\frac{\ell}{D}, \frac{\rho V D}{\mu}\right)$$

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho V^2} \qquad Ca = \frac{p - p_v}{\frac{1}{2}\rho V^2}$$

For $C_{p_{min}} = -2.4$, $p_{min} - p_\infty = \frac{1}{2}\rho V_{max}^2 (C_{p_{min}})$ $\therefore p_{min} = p_\infty + \frac{1}{2}\rho V_{max}^2 C_{p_{min}}$

For $Ca = \frac{1}{2}$, $p_{min} - p_v = \frac{1}{2}\rho V_{max}^2 Ca$ $\therefore p_{min} = p_v + \frac{1}{2}\rho V_{max}^2 Ca$

Equating expressions for p_{min},

$$p_\infty + \tfrac{1}{2}\rho V_{max}^2 C_{p_{min}} = p_v + \tfrac{1}{2}\rho V_{max}^2 Ca$$

$$\tfrac{1}{2}\rho V_{max}^2 \left[Ca - C_{p_{min}}\right] = p_\infty - p_v$$

$$V_{max} = \left\{\frac{2(p_\infty - p_v)}{\rho\left[Ca - C_{p_{min}}\right]}\right\}^{1/2}$$

For water at 68°F (Table A.7), $p_v = 0.339$ psia

$$\therefore V_{max} = \left\{2 \times (14.7 - 0.339)\frac{lbf}{in^2} \times \frac{ft^3}{1.94\,slug} \times \frac{1}{[0.5 - (-2.4)]} \times \frac{144\,in^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right\}^{1/2}$$

$$V_{max} = 27.1\ ft/s \quad (8.26\ m/s) \longleftarrow$$

Given: $\frac{1}{8}$-scale model of tractor-trailer rig tested in pressurized wind tunnel.

$W = 0.305\ m$ $\quad V = 75.0\ m/s$
$H = 0.476\ m$ $\quad F_D = 128\ N$
$L = 2.48\ m$ $\quad \rho = 3.23\ kg/m^3$

Find: (a) Aerodynamic drag coefficient of model.
(b) Compare Reynolds number for model with prototype at $V = 55$ mph.
(c) Aerodynamic drag on prototype at $V = 55$ mph, with headwind, $V_w = 10$ mph.

Solution: Defining equations: $F_D = C_D A \frac{1}{2}\rho V^2$; $Re = \frac{\rho V L}{\mu}$

Then $C_{Dm} = \dfrac{F_{Dm}}{\frac{1}{2}\rho_m V_m^2 A_m}$

Assume $A_m = W_m H_m = 0.305\ m \times 0.476\ m = 0.145\ m^2$

$$C_{Dm} = 2 \times 128\ N \times \frac{m^3}{3.23\ kg} \times \frac{s^2}{(25)^2 m^2} \times \frac{1}{0.145\ m^2} \times \frac{kg \cdot m}{N \cdot s^2} = 0.0972$$ C_{Dm}

$$\frac{Re_m}{Re_p} = \frac{\rho_m V_m L_m}{\mu_m} \times \frac{\mu_p}{\rho_p V_p L_p} = \frac{\rho_m}{\rho_p}\frac{V_m}{V_p}\frac{L_m}{L_p} \quad (\text{assume air: } \mu_m = \mu_p)$$

For the prototype, $V_p = 55\frac{mi}{hr} \times 5280\frac{ft}{mi} \times \frac{hr}{3600\ s} \times 0.305\frac{m}{ft} = 24.6\ m/s$

$$\frac{Re_m}{Re_p} = \left(\frac{3.23}{1.23}\right)\left(\frac{250}{24.6}\right)\left(\frac{1}{8}\right) = 1.00 \quad \therefore Re_m = Re_p$$ Re

Since $Re_m = Re_p$, then $C_{Dp} = C_{Dm}$, assuming geometric and kinematic similarity, so

$$F_{Dp} = C_{Dp} A_p \tfrac{1}{2}\rho_p (V_p + V_w)^2$$

With $V_w = 10$ mph, $V_p + V_w = \frac{60}{55} \times 24.6\ m/s = 29.1\ m/s$

Thus

$$F_{Dp} = 0.0972 \times (8)^2 0.145\ m^2 \times \frac{1}{2} \times 1.23\frac{kg}{m^3}(29.1)^2\frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$F_{Dp} = 470\ N$ F_{Dp}

Given: Automobile (prototype) to travel at 100 km/hr through standard air. Model, $L_m/L_p = \frac{1}{5}$, to be tested in water. The lowest pressure coefficient is $C_p = -1.4$ at the location of minimum static pressure on the surface. Onset of cavitation occurs at $C_a = 0.5$

Find: (a) factors necessary to ensure kinematic similarity in tests.
(b) water speed to be used.
(c) corresponding ratio of drag forces
(d) minimum tunnel pressure to avoid cavitation.

Solution:

To assure kinematic similarity:
(1) model and prototype must be geometrically similar
(2) model must be submerged in flow to avoid surface effects.
(3) cavitation effects must be absent in model test

To determine model test speed, note that flows will be dynamically similar if

$$Re_m = Re_p, \text{ i.e. } \left(\frac{\rho V L}{\mu}\right)_m = \left(\frac{\rho V L}{\mu}\right)_p \quad \text{or} \quad \left(\frac{V L}{\nu}\right)_m = \left(\frac{V L}{\nu}\right)_p$$

Hence, $V_m = V_p \frac{\nu_m}{\nu_p} \frac{L_p}{L_m}$. For standard air, $\nu_p = 1.46 \times 10^{-5}\ m^2/s$

Assume water at 20°C. Table 9.8, Appendix H gives $\nu_m = 1.0 \times 10^{-6}\ m^2/s$

$$V_m = 100 \frac{km}{hr} \times \frac{hr}{3600 s} \times \frac{1.0 \times 10^{-6}}{1.46 \times 10^{-5}} \times 5 \times 10^{3}\ \frac{m}{km} = 9.51\ m/s \qquad V_m$$

Then $\left(\frac{F_D}{\rho V^2 L^2}\right)_m = \left(\frac{F_D}{\rho V^2 L^2}\right)_p$ and $\frac{F_{Dp}}{F_{Dm}} = \frac{(\rho V^2 L^2)_p}{(\rho V^2 L^2)_m}$

$$\frac{F_{Dp}}{F_{Dm}} = \frac{1.23}{999} \times \left(\frac{27.8}{9.51}\right)^2 \times \left(\frac{5}{1}\right)^2 = 0.262 \qquad F_{Dp}/F_{Dm}$$

For $C_a = 0.5$, then $\frac{p - p_v}{\frac{1}{2}\rho V^2} = 0.5$ and local pressure $p = p_v + \frac{1}{2}\rho V^2$

For water at 20°C, $p_v = 2.34\ kPa$ and

$$p_m = 2.34\ kPa + \frac{1}{2} \times 999 \frac{kg}{m^3} \times (9.51)^2 \frac{m^2}{s^2} \cdot \frac{N \cdot s^2}{kg \cdot m} = 24.9\ kPa$$

For $C_{p_{min}} = -1.4 = \frac{p_{min} - p_\infty}{\frac{1}{2}\rho V^2}$, then

$$p_\infty = p_{min} + 0.7 \rho V^2$$

$$p_\infty = 24.9\ kPa + 0.7 \times 999 \frac{kg}{m^3} \times (9.51)^2 \frac{m^2}{s^2} = 88.1\ kPa \qquad p_\infty$$

Given: The frequency, f, of vortex shedding from the rear of a bluff cylinder is a function of ρ, V, d, μ

Two cylinders in standard air, $\frac{d_1}{d_2} = 2$

Find: (a) functional relationship for f, using dimensional analysis
(b) V_1/V_2 for dynamic similarity
(c) f_1/f_2

Solution: Apply Buckingham Π theorem.

① $f \quad \rho \quad V \quad d \quad \mu$ $\quad n = 5$ parameters

② Select M, L, t as primary dimensions

③

f	ρ	V	d	μ
t^{-1}	$FL^{-4}t^2$	Lt^{-1}	L	$FL^{-2}t$

$r = 3$ primary dimensions

④ ρ, V, d $\quad m = r = 3$ repeating parameters

⑤ Then $n - m = 2$ dimensionless groups will result

Setting up dimensional equations

$\Pi_1 = \rho^a V^b d^c f$

$M^0L^0t^0 = \left(\frac{FL^{-4}t^2}{}\right)^a \left(\frac{Lt^{-1}}{}\right)^b L^c \, t^{-1}$

Equating exponents,

M: $0 = a$

L: $0 = -3a + b + c \quad c = 1$

t: $0 = -b - 1 \quad \therefore b = -1$

$\therefore \Pi_1 = \frac{fd}{V}$

$\Pi_2 = \rho^a V^b d^c \mu$

$M^0L^0t^0 = \left(\frac{FL^{-4}t^2}{}\right)^a \left(\frac{Lt^{-1}}{}\right)^b L^c \, FL^{-2}t$

Equating exponents,

M: $0 = a + 1 \quad \therefore a = -1$

L: $0 = -3a + b + c - 1 \quad c = -1$

t: $0 = -b - 1 \quad \therefore b = -1$

$\therefore \Pi_2 = \frac{\mu}{\rho V d}$

⑥ Check using F, L, t dimensions

$\Pi_1 = \frac{1}{t} \cdot L \cdot \frac{t}{L} = [1]$ ✓

$\Pi_2 = \frac{Ft}{L^2} \cdot \frac{L^4}{Ft^2} \cdot \frac{t}{L} \cdot \frac{1}{L} = [1]$ ✓

$\therefore \frac{fd}{V} = g\left(\frac{\rho V d}{\mu}\right)$ $\qquad \frac{fd}{V}$

To achieve dynamic similarity between geometrically similar flows, we must duplicate all but one of the dimensionless groups

$\left(\frac{\rho V d}{\mu}\right)_1 = \left(\frac{\rho V d}{\mu}\right)_2 \Rightarrow \frac{V_1}{V_2} = \frac{\rho_2}{\rho_1} \frac{\mu_1}{\mu_2} \frac{d_2}{d_1} = 1 \times 1 \times \frac{1}{2} = \frac{1}{2}$ $\qquad V_1/V_2$

If $\left(\frac{\rho V d}{\mu}\right)_1 = \left(\frac{\rho V d}{\mu}\right)_2$, then $\left(\frac{fd}{V}\right)_1 = \left(\frac{fd}{V}\right)_2$

and $\frac{f_1}{f_2} = \frac{V_1}{V_2} \frac{d_2}{d_1} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ $\qquad f_1/f_2$

Given: Model test of tractor-trailer rig in standard air.

$F_D = f(A, V, \rho, \mu)$; scale is 1:4 ; $A_m = 0.625\ m^2$

At $V_m = 89.6$ m/s, $F_D = 2.46$ kN

Find: (a) Dimensionless parameters.
(b) Conditions for dynamic similarity.
(c) Drag force on prototype at $V_p = 22.4$ m/s (no wind).
(d) Power to overcome aero drag.

Solution: ① F_D A V ρ μ | ② $M\ L\ t$

③ $\frac{ML}{t^2}$ L^2 $\frac{L}{t}$ $\frac{M}{L^3}$ $\frac{M}{Lt}$ | ④ $\rho V A$

⑤ $\Pi_1 = \rho^a V^b A^c F_D = M^0 L^0 t^0$ $\qquad \Pi_2 = \rho^a V^b A^c \mu$

$M: a + 1 = 0$	$a = -1$	$M: a + 1 = 0$	$a = -1$
$L: -3a + b + 2c + 1 = 0$	$c = -1$	$L: -3a + b + 2c - 1 = 0$	$c = -\frac{1}{2}$
$t: -b - 2 = 0$	$b = -2$	$t: -b - 1 = 0$	$b = -1$

$\Pi_1 = \frac{F_D}{\rho V^2 A}$ $\qquad \Pi_2 = \frac{\mu}{\rho V A^{1/2}}$ $\qquad \Pi_1, \Pi_2$ ←

⑥ $\Pi_1 = F \times \frac{L^4}{Ft^2} \times \frac{t^2}{L^2} \times \frac{1}{L^2} = 1$ ✓✓ $\qquad \Pi_2 = \frac{Ft}{L^2} \times \frac{L^4}{Ft^2} \times \frac{t}{L} \times \frac{1}{L} = 1$ ✓✓

For dynamic similarity, must have geometric and kinematic similarity and $Re_m = Re_p$. Then $\left(\frac{F_D}{\rho V^2 A}\right)_m = \left(\frac{F_D}{\rho V^2 A}\right)_p$

For the prototype,

$$F_{Dp} = F_{Dm} \frac{\rho_p}{\rho_m} \left(\frac{V_p}{V_m}\right)^2 \frac{A_p}{A_m} = F_{Dm} \left(\frac{1.23}{1.23}\right)\left(\frac{22.4}{89.6}\right)^2 (4)^2 = F_{Dm} = 2.46\ \text{kN} \qquad F_{Dp} \leftarrow$$

The power requirement is

$$P = F_{Dp} V_p = 2.46\ \text{kN} \times 22.4\ \frac{\text{m}}{\text{s}} \times \frac{\text{W·s}}{\text{N·m}} = 55.1\ \text{kW}\ (73.9\ \text{hp}) \qquad P \leftarrow$$

Given: A 1:50 scale model of a submarine is to be tested in a water tunnel. The drag force is given by

$$F_D = f(V, \rho, \mu, \forall)$$

where $\forall$ is the volume of the model

Find: (a) a set of dimensionless parameters suitable for organizing data.

(b) the drag of prototype at $V_p = 27$ knots if $F_m = 13$ N at $V_m = 10$ knots.

Solution: Apply Buckingham Π-Theorem

① $F_D \quad V \quad \rho \quad \mu \quad \forall$ $\qquad n = 5$ parameters.

② Choose M, L, t

③ $\frac{ML}{t^2} \quad \frac{L}{t} \quad \frac{M}{L^3} \quad \frac{L}{Lt} \quad L^0$ $\qquad r = 3$ primary dimensions

④ Choose $\rho, V, \forall$ $\qquad m = 3$ repeating parameters

$\therefore$ expect $n - m = 2$ dimensionless groups

⑤ $\Pi_1 = \rho^a V^b \forall^c \mu$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L^3)^c \frac{M}{Lt}$

M: $0 = a + 1 \quad \therefore a = -1$

t: $0 = -b - 1 \quad \therefore b = -1$

L: $0 = -3a + b + 3c - 1 \quad \therefore c = -\frac{1}{3}$

$$\Pi_1 = \frac{\mu}{\rho V \forall^{1/3}}$$

$\Pi_2 = \rho^a V^b \forall^c F_D$

$M^0L^0t^0 = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L^3)^c \frac{ML}{t^2}$

M: $0 = a + 1 \quad \therefore a = -1$

t: $0 = -b - 2 \quad \therefore b = -2$

L: $0 = -3a + b + 3c + 1 \quad \therefore c = -\frac{2}{3}$

$$\Pi_2 = \frac{F_D}{\rho V^2 \forall^{2/3}}$$

⑥ Check

$\Pi_1 = \frac{Ft}{L^2} \cdot \frac{L^4}{Ft^2} \cdot \frac{t}{L} \cdot \frac{1}{L} = [1]$ ✓

$\Pi_2 = F \cdot \frac{L^4}{Ft^2} \cdot \frac{1}{L^2 t^2} \cdot \frac{1}{L^2} = [1]$ ✓

$$\therefore \frac{F_D}{\rho V^2 \forall^{2/3}} = f\left(\frac{\rho V \forall^{1/3}}{\mu}\right)$$ (a)

(b) For dynamic similarity, we must match Reynolds numbers. Since both flows use the same fluid, then

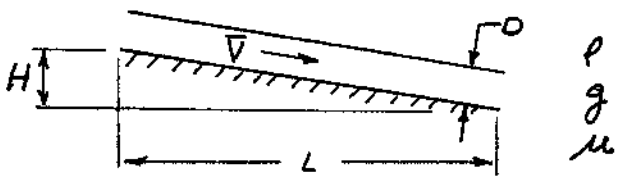
$$V_m \forall_m^{1/3} = V_p \forall_p^{1/3} \quad \text{or} \quad V_m = V_p \left(\frac{\forall_p}{\forall_m}\right)^{1/3} = 27 \text{ knots } (50) = 1350 \text{ kt}$$

This speed cannot be achieved. We would test at the highest speed attainable and assume that $C_D = F_D / \rho V^2 \forall^{2/3}$ = constant. With this assumption

$$F_{Dp} = F_{Dm} \left(\frac{V_p}{V_m}\right)^2 \left(\frac{\forall_p}{\forall_m}\right)^{2/3} = 13 \text{ N} \times \left(\frac{27}{10}\right)^2 \times \left(50\right)^2 = 237 \text{ kN}$$ F_{Dp}

Given: Model glacier using glycerine. Assume ice is Newtonian and 10^6 x as viscous.

$D = 15\ m$
$H = 1.5\ m$
$L = 1850\ m$ } model



In lab test, model instructor reappears in $\tau = 9.6\ hr$.

Find: (a) Develop suitable dimensionless parameters.
(b) Estimate time when instructor will reappear.

Solution: ① $\bar{V}$ ρ g μ D H L $n=7$

② MLt $\frac{L}{t}$ $\frac{M}{L^3}$ $\frac{L}{t^2}$ $\frac{M}{Lt}$ L L L $m=r=3$

④ Choose ρ, g, D as repeating variables: $n-m = 7-3 = 4$ parameters

⑤ $\Pi_1 = \rho^a g^b D^c \bar{V} = M^0 L^0 t^0$

M: $a+0=0$ | $a=0$
L: $-3a+b+c+1=0$ | $c=-b-1=-\frac{1}{2}$
t: $-2b-1=0$ | $b=-\frac{1}{2}$

$\Pi_1 = \dfrac{\bar{V}}{\sqrt{gD}}$ (Froude no.)

$\Pi_2 = \rho^a g^b D^c \mu = M^0 L^0 t^0$

M: $a+1=0$ $a=-1$
L: $-3a+b+c-1=0$ $c=3a-b+1=-\frac{3}{2}$
t: $-2b-1=0$ $b=-\frac{1}{2}$

$\Pi_2 = \dfrac{\mu}{\rho g^{1/2} D^{3/2}} \sim \dfrac{\mu}{\rho\sqrt{gD}\,D}$ (Reynolds no)

$\Pi_3 = \frac{H}{D}$, $\Pi_4 = \frac{L}{D}$ (by inspection)

⑥ Check: obvious from forms above. $\Pi_1 = f(\Pi_2, \Pi_3, \Pi_4)$ ← geometric similarity

For dynamic similarity, $\Pi_{2m} = \Pi_{2p} = \dfrac{\mu_m}{\rho_m g_m^{1/2} D_m^{3/2}} = \dfrac{\mu_p}{\rho_p g_p^{1/2} D_p^{3/2}}$, so

$$\frac{D_m}{D_p} = \left(\frac{\mu_m}{\mu_p}\frac{\rho_p}{\rho_m}\right)^{2/3} = \left(\frac{\mu_m}{\mu_p}\frac{SG_p}{SG_m}\right)^{2/3} = \left(\frac{1}{10^6}\times\frac{0.92}{1.26}\right)^{2/3} = 8.11\times10^{-5}$$

{ $SG_{ice} = 0.92$ (Table A.1)
$SG_{glycerin} = 1.26$ (A.2)

So $\frac{L_m}{L_p} = 8.11\times10^{-5}$; $L_m = 8.11\times10^{-5} L_p = 8.11\times10^{-5} \times 1850\ m = 0.150\ m$

From Π_1, $\dfrac{\bar{V}_m}{\bar{V}_p} = \sqrt{\dfrac{D_m}{D_p}} = 9.00\times10^{-3}$

The time to reappear is $\tau = L/\bar{V}$, so $\tau_p = L_p/\bar{V}_p$, $\tau_m = L_m/\bar{V}_m$

$$\frac{\tau_p}{\tau_m} = \frac{L_p}{L_m}\frac{\bar{V}_m}{\bar{V}_p} = \frac{D_p}{D_m}\sqrt{\frac{D_m}{D_p}} = \sqrt{\frac{D_p}{D_m}} = \frac{1}{9.00\times10^{-3}} = 111$$

Thus $\tau_p = 111\,\tau_m = 111 \times 9.6\ hr = 1070\ hr$ (~45 days) ← τ_p

{ The instructor will reappear before the semester ends! }

Given: Submarine model (1:30 scale) to be tested in fresh water under two conditions:
(1) on the surface at 20 kt (prototype)
(2) far below the surface at 0.5 kt (prototype)

Find: (a) Speed for model test on surface
(b) Speed for model test submerged
(c) Ratio of full-scale to model drag force.

Solution: On the surface, match the Froude number, $Fr = \frac{V}{\sqrt{gL}}$

Thus $Fr_m = \frac{V_m}{\sqrt{g L_m}} = Fr_p = \frac{V_p}{\sqrt{g L_p}}$ or $V_m = V_p \sqrt{\frac{L_m}{L_p}}$

For 1:30 scale,

$$V_m = 20\,kt \sqrt{\frac{1}{30}} = 3.65\,kt$$

$$V_m = 3.65 \frac{nm}{hr} \times 1852 \frac{m}{nm} \times \frac{hr}{3600\,s} = 1.88\,m/s \quad \leftarrow V_m$$

Submerged, match the Reynolds number, $Re = \frac{\rho V L}{\mu} = \frac{VL}{\nu}$

Thus $Re_m = \frac{V_m L_m}{\nu_m} = Re_p = \frac{V_p L_p}{\nu_p}$ or $V_m = V_p \frac{L_p}{L_m} \frac{\nu_m}{\nu_p}$

From Table A.2, for seawater, SG = 1.025 and $\mu = 1.08 \times 10^{-3}\,N \cdot s/m^2$ at 20°C, Thus

$$\nu_p = \frac{\mu_p}{\rho_p} = \frac{\mu_p}{SG\,\rho_{H_2O}} = 1.08 \times 10^{-3} \frac{N \cdot s}{m^2} \times \frac{m^3}{(1.025)\,1000\,kg} \times \frac{kg \cdot m}{N \cdot s^2} = 1.05 \times 10^{-6} \frac{m^2}{s}$$

From Table A.8, fresh water at 20°C has $\nu = 1.00 \times 10^{-6}\,m^2/s$.

For 1:30 scale

$$V_m = 0.5\,kt \times \frac{30}{1} \times 1.00 \times 10^{-6} \frac{m^2}{s} \times \frac{s}{1.05 \times 10^{-6}\,m^2} = 14.3\,kt$$

$$V_m = 14.3 \frac{nm}{hr} \times 1852 \frac{m}{nm} \times \frac{hr}{3600\,s} = 7.36\,m/s \quad \leftarrow V_m$$

Under dynamically similar conditions the drag coefficients, $C_D = \frac{F_D}{\rho V^2 A}$, will be identical. Thus

$$\frac{F_p}{\rho_p V_p^2 L_p^2} = \frac{F_m}{\rho_m V_m^2 L_m^2} \quad \text{or} \quad F_p = F_m \frac{\rho_p}{\rho_m} \frac{V_p^2}{V_m^2} \frac{L_p^2}{L_m^2}$$

$$F_p = F_m \frac{1.025}{0.999} \times \left(\frac{0.5}{14.3}\right)^2 \left(\frac{30}{1}\right)^2 = 1.13 \text{ (submerged)}, 2.71 \times 10^4 \text{ (surface)} \quad \leftarrow F_p/F_m$$

Given: A model ($\frac{1}{10}$ scale) of a tractor-trailer rig is tested in a wind tunnel; $A_m = 1.08\ ft^2$. For $V_m = 250\ ft/s$, $F_{D_m} = 76.3\ lbf$

Find: (a) drag coefficient for the model.
(b) F_{D_p} at $V_p = 55\ mi/hr$ if $C_{D_p} = C_{D_m}$
(c) V_m if $V_p = 55\ mi/hr$.
(d) Is answer to part (c) reasonable.

Solution:

$C_D = \dfrac{F_D}{\frac{1}{2}\rho V^2 A}$ For the model assuming air at STP,

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} = \frac{76.3\ lbf}{0.5 \times 0.00237 \frac{slug}{ft^3} \times (250)^2 ft^2/s^2 \times 1.08 ft^2 \times \frac{1}{1} \frac{slug \cdot ft}{lbf \cdot s^2}}$$

$$C_{D_m} = 0.951 \quad \longleftarrow C_{D_m}$$

$F_{D_p} = \frac{1}{2}\rho V_p^2 A_p C_{D_p}$ $\quad C_{D_p} = C_{D_m} = 0.951 \quad A_p = \left(\frac{L_p}{L_m}\right)^2 A_m = 100 A_m$

$$F_{D_p} = \frac{1}{2} \times 0.00237 \frac{slug}{ft^3} \times \left(55 \frac{mi}{hr} \times 5280 \frac{ft}{mi} \times \frac{hr}{3600 s}\right)^2 \times 100 \times 1.08 ft^2 \times 0.951 \times \frac{lbf \cdot s^2}{ft \cdot slug}$$

$$F_{D_p} = 794\ lbf \quad \longleftarrow F_{D_p}$$

For dynamic similarity between model and prototype

$$\left(\frac{\rho V L}{\mu}\right)_m = \left(\frac{\rho V L}{\mu}\right)_p \quad \text{or} \quad V_m = V_p \frac{\rho_p}{\rho_m} \frac{L_p}{L_m} \frac{\mu_m}{\mu_p} = V_p \times 1 \times 10 \times 1$$

$$V_m = 10 V_p = 550\ mi/hr \quad \longleftarrow V_m$$

$$V_m = 550 \frac{mi}{hr} \times 5280 \frac{ft}{mi} \times \frac{hr}{3600 s} = 807\ ft/s$$

For air at standard conditions, the speed of sound, $c = \sqrt{kRT}$

$$c = \left(1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot R} \times 519 R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right)^{1/2} = 1117\ ft/s$$

$$M = \frac{V}{c} = \frac{807}{1117} = 0.72$$

At this value of M, compressibility would be important in the model test. Thus, the speed is not practical. $\longleftarrow$

Given: Recommended procedures for wind tunnel tests of trucks + buses suggest:

$A_{model} / A_{test\ section} < 0.05$

$h_{model} / h_{test\ section} < 0.30$ (h = height)

$w_{model\ at\ max\ yaw\ (20°)} / w_{test\ sect} < 0.30$ (w = projected width)

$V_{max} < 300$ ft/s

Wind tunnel test section is $h = 1.5$ ft, $w = 2$ ft.

Prototype has: $h = 13.5$ ft, $w = 8$ ft, Length = 65 ft.

Find: (a) scale ratio of largest model that meets the recommended criteria.

(b) Use results of Ex. Prob 7.5 to assess whether an adequate value of Re can be achieved in the test facility.

Solution:

Let s = scale ratio. Then $h_m = s h_p$, $w_m = s w_p$, $l_m = s l_p$

(1) Height criteria

$h_m = 0.30 h_{test\ section} = 0.3(1.5\text{ ft}) = 0.45$ ft

$$s = \frac{h_m}{h_p} = \frac{0.45\text{ ft}}{13.5\text{ ft}} = 0.0333 \quad \left\{ \frac{1}{s} = 30 \right\}$$

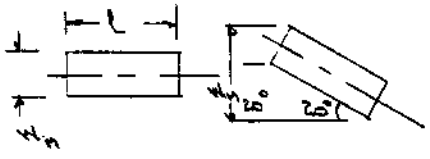
(2) Frontal area criteria

$$A_{model} = 0.05 A_{test\ sect} = 0.05 \times 1.5\text{ ft} \times 2\text{ ft} = 0.15\text{ ft}^2$$

$$A_{model} = s^2 A_p = s^2[13.5\text{ ft} \times 8\text{ ft}] = s^2(108)\text{ ft}^2 = 0.15$$

$$\therefore s = \left(\frac{0.15}{108}\right)^{1/2} = 0.0373 \quad \left\{ \frac{1}{s} = 26.8 \right\}$$

(3) width criteria



$$w_{m_{20°}} = l_m \sin 20° + w_m \cos 20°$$

$$= s(l_p \sin 20° + w_p \cos 20°)$$

$$w_{m_{20°}} = s[65 \sin 20° + 8 \cos 20°]\text{ ft} = 29.7\,s\text{ ft}$$

From constraint, $w_{m_{20°}} = 0.30 w_{test\ sect} = 0.30(2\text{ ft}) = 0.6$ ft

$\therefore 0.6\text{ ft} = 29.7\,s$ ft and $s = 0.0202 \quad \left\{ \frac{1}{s} = 49.5 \right\}$

The width criteria is the most stringent $\therefore s = \frac{1}{50}$

Model = $\frac{1}{50}$ Prototype

From Ex. Prob 7.5, C_D = const for $Re > 4 \times 10^5$

with $Re = \frac{\rho V w}{\mu} = \frac{V w}{\nu}$ Standard air $\nu = 1.57 \times 10^{-4}$ ft/s

For current model test, $Re = 300\frac{\text{ft}}{\text{sec}} \times \left(\frac{1}{50} \times 8\text{ ft}\right) \times \frac{s}{1.57 \times 10^{-4}\text{ ft}} = 3.06 \times 10^5$

$\therefore$ Adequate Re cannot be achieved

Given: Circular container partially filled with water is rotated about its axis at constant angular velocity, ω.
The velocity v_θ is a function of: location, r, time from start, t, angular velocity, ω, density, ρ, and viscosity, μ.
Water is replaced with honey and cylinder is rotated at the same value of ω.

Find: (a) dimensionless parameters that characterize the problem.
(b) Determine whether honey will attain steady state motion as quickly as water.
(c) Explain why Re would not be an important parameter in scaling the steady state motion of the liquid.

Solution:

$$v_\theta = v_\theta(\omega, r, t, \rho, \mu)$$

From the Buckingham π-theorem, for $n=6$ and $m=r=3$, we would expect three dimensionless groups.

$$\frac{v_\theta}{\omega r} = f\left(\frac{\mu}{\rho \omega r^2}, \omega t\right) \qquad \frac{v_\theta}{\omega r}$$

From the above results $\pi_2 = \frac{\mu}{\rho \omega r^2}$ contains the fluid properties ρ, μ.

$\pi_3 = \omega t$ contains the time t

$$\pi_2 \pi_3 = \frac{\mu}{\rho \omega r^2}\omega t = \frac{\mu t}{\rho r^2} = \frac{\nu t}{r^2} \qquad \text{where } \nu = \mu/\rho$$

For steady flow at the same radius

$$\left(\frac{\nu t}{r^2}\right)_{\text{Honey}} = \left(\frac{\nu t}{r^2}\right)_{\text{water}}$$

$$\therefore t_H = \frac{\nu_{\text{water}}}{\nu_{\text{Honey}}} t_{\text{water}}$$

Since $\nu_{\text{Honey}} > \nu_{\text{water}}$ ($\mu_{\text{Honey}} > \mu_{\text{water}}$ and $\rho_H \approx \rho_w$)

$$t_H < t_{\text{water}} \qquad t_H$$

At steady state conditions, we have solid body rotation there are no viscous forces. Hence Re is not important.

Given: Power, P, to drive a fan depends on ρ, Q, D, and ω.

Condition	D(in.)	Q (cfm)	ω(rpm)
1	8	800	2400
2	16	?	1850

Find: Volume flow rate at condition 2, for dynamic similarity.

Solution: step ① P ρ Q D ω

step ② MLt ③: $\frac{ML^2}{t^3}$ $\frac{M}{L^3}$ $\frac{L^3}{t}$ L $\frac{1}{t}$ ④ ρ, ω, D

⑤ $\Pi_1 = \rho^a \omega^b D^c P = M^0 L^0 t^0$

M: $a + 1 = 0$ | $a = -1$

L: $-3a + c + 2 = 0$ | $c = 3a - 2 = -5$

t: $-b - 3 = 0$ | $b = -3$

$\Pi_1 = \frac{P}{\rho \omega^3 D^5}$

$\Pi_2 = \rho^a \omega^b D^c Q = M^0 L^0 t^0$

M: $a + 0 = 0$ | $a = 0$

L: $-3a + c + 3 = 0$ | $c = -3$

t: $-b - 1 = 0$ | $b = -1$

$\Pi_2 = \frac{Q}{\omega D^3}$

⑥ $\Pi_1 = \frac{FL}{t} \times \frac{L^4}{Ft^2} \times t^3 \times \frac{1}{L^5} = \frac{FL^5t^3}{FL^5t^2} = 1$ ✓✓

$\Pi_2 = \frac{L^3}{t} \times t \times \frac{1}{L^3} = \frac{L^3 t}{L^3 t} = 1$ ✓✓

Thus $\Pi_1 = f(\Pi_2)$ or $\frac{P}{\rho \omega^3 D^5} = f\left(\frac{Q}{\omega D^3}\right)$

For dynamic similarity, need geometric and kinematic similarity and

$$\frac{Q_1}{\omega_1 D_1^3} = \frac{Q_2}{\omega_2 D_2^3}$$

Thus

$$Q_2 = Q_1 \frac{\omega_2}{\omega_1}\left(\frac{D_2}{D_1}\right)^3 = 800 \text{ cfm} \frac{1850 \text{ rpm}}{2400 \text{ rpm}} \left(\frac{16 \text{ in.}}{8 \text{ in.}}\right)^3 = 4930 \text{ cfm}$$ Q_2 ←

Given: Information relating to geometrically similar model test of centrifugal pump:

	Variable	Prototype	Model
Pressure Rise	Δp		29.3 kPa
Volume Flow Rate	Q	1.25 m³/min	
Density	ρ	800 kg/m³	999 kg/m³
Angular Speed	ω	183 rad/s	367 rad/s
Diameter	D	150 mm	50 mm

Find: Missing values for dynamically similar conditions.

Solution: Apply Buckingham Π-theorem. Assume $\Delta p = f(Q, \rho, \omega, D)$

① Δp Q ρ ω D $n = 5$ parameters

② Choose M, L, t as fundamental dimensions.

③ $\frac{M}{Lt^2}$ $\frac{L^3}{t}$ $\frac{M}{L^3}$ $\frac{L}{t}$ L $r = 3$ primary dimensions

④ Let ρ, ω, and D be repeating variables. $m = r = 3$

⑤ Then $n - m = 5 - 3 = 2$ dimensionless parameters result. ⑥ Check:

$$\Pi_1 = \rho^a \omega^b D^c \Delta p = \left(\frac{M}{L^3}\right)^a \left(\frac{1}{t}\right)^b (L)^c \frac{M}{Lt^2} = M^0 L^0 t^0$$

$$\left.\begin{array}{ll} M: & a + 1 = 0 \quad a = -1 \\ L: & -3a + c - 1 = 0 \quad c = -2 \\ t: & -b - 2 = 0 \quad b = -2 \end{array}\right\} \Pi_1 = \frac{\Delta p}{\rho \omega^2 D^2}$$

$$\Pi_1 = \frac{F}{L^2 F t^2} \frac{L^4}{1} \frac{t^2}{1} \frac{1}{L^2} = [1] \text{ vv}$$

$$\Pi_2 = \rho^a \omega^b D^c Q = \left(\frac{M}{L^3}\right)^a \left(\frac{1}{t}\right)^b (L)^c \frac{L^3}{t} = M^0 L^0 t^0$$

$$\left.\begin{array}{ll} M: & a = 0 \quad a = 0 \\ L: & -3a + c + 3 = 0 \quad c = -3 \\ t: & -b - 1 = 0 \quad b = -1 \end{array}\right\} \Pi_2 = \frac{Q}{\omega D^3}$$

$$\Pi_2 = \frac{L^3}{t} \frac{t}{1} \frac{1}{L^3} = [1] \text{ vv}$$

Thus $\Pi_1 = f(\Pi_2)$ for this situation. Flows are geometrically similar. Assume kinematic similarity. Then for dynamic similarity, if $\Pi_{2m} = \Pi_{2p}$ then $\Pi_{1m} = \Pi_{1p}$.

$$\Pi_{2m} = \frac{Q_m}{\omega_m D_m^3} = \Pi_{2p} = \frac{Q_p}{\omega_p D_p^3}; \quad Q_m = Q_p \left(\frac{\omega_m}{\omega_p}\right)\left(\frac{D_m}{D_p}\right)^3 = Q_p \left(\frac{367}{183}\right)\left(\frac{50}{150}\right)^3 = 0.0743\, Q_p$$

$$Q_m = 0.0743 \times 1.25 \frac{m^3}{min} = 0.0928 \text{ m}^3/\text{min} \qquad Q_m$$

$$\Pi_{1m} = \frac{\Delta p_m}{\rho_m \omega_m^2 D_m^2} = \Pi_{1p} = \frac{\Delta p_p}{\rho_p \omega_p^2 D_p^2}; \quad \Delta p_p = \Delta p_m \frac{\rho_p}{\rho_m} \left(\frac{\omega_p}{\omega_m}\right)^2 \left(\frac{D_p}{D_m}\right)^2$$

$$\Delta p_p = \Delta p_m \left(\frac{800}{999}\right)\left(\frac{183}{367}\right)^2 \left(\frac{150}{50}\right)^2 = 1.79 \times 29.3 \text{ kPa} = 52.5 \text{ kPa} \qquad \Delta p_p$$

{This result neglects any effect of viscosity.}

Given: Axial-flow pump:
$Q = 25$ ft³/s (water) $h = 150$ ft·lbf/slug
$D = 1$ ft $\omega = 500$ rpm
Model:
$\mathcal{P} = 3$ hp, $\omega = 1000$ rpm

Find: For similar performance between prototype and model, calculate the head, volume flow rate, and the diameter of the model

Solution:

$$\frac{h}{\omega^2 D^2} = f_1\left(\frac{Q}{\omega D^3}, \frac{\rho \omega D^2}{\mu}\right) \quad \text{and} \quad \frac{\mathcal{P}}{\rho \omega^3 D^5} = f_2\left(\frac{Q}{\omega D^3}, \frac{\rho \omega D^2}{\mu}\right)$$

Neglecting viscous effects,

if $\left(\frac{Q}{\omega D^3}\right)_m = \left(\frac{Q}{\omega D^3}\right)_p$ then $\left(\frac{h}{\omega^2 D^2}\right)_m = \left(\frac{h}{\omega^2 D^2}\right)_p$

and $\left(\frac{\mathcal{P}}{\rho \omega^3 D^5}\right)_m = \left(\frac{\mathcal{P}}{\rho \omega^3 D^5}\right)_p$

Q

$$\text{if} \quad \frac{Q_m}{\omega_m D_m^3} = \frac{Q_p}{\omega_p D_p^3} \quad \text{or} \quad \frac{Q_m}{Q_p} = \frac{\omega_m}{\omega_p}\frac{D_m^3}{D_p^3} = \frac{1000}{500}\left(\frac{D_m}{D_p}\right)^3 = 2\left(\frac{D_m}{D_p}\right)^3 \qquad \text{----- (1)}$$

$$\text{then} \quad \frac{h_m}{\omega_m^2 D_m^2} = \frac{h_p}{\omega_p^2 D_p^2} \quad \text{or} \quad \frac{h_m}{h_p} = \frac{\omega_m^2}{\omega_p^2}\frac{D_m^2}{D_p^2} = \frac{(1000)^2}{(500)^2}\left(\frac{D_m}{D_p}\right)^2 = 4\left(\frac{D_m}{D_p}\right)^2 \qquad \text{----- (2)}$$

$$\text{and} \quad \frac{\mathcal{P}_m}{\mathcal{P}_p} = \frac{\rho_m}{\rho_p}\frac{\omega_m^3}{\omega_p^3}\frac{D_m^5}{D_p^5} = \left(\frac{1000}{500}\right)^3\left(\frac{D_m}{D_p}\right)^5 = 8\left(\frac{D_m}{D_p}\right)^5 \qquad \text{----- (3)}$$

We can determine $\mathcal{P}_p$ from the energy equation applied to the prototype. From footnote, page 316

$$\mathcal{P}_p = \rho Q h = 1.94 \frac{\text{slug}}{\text{ft}^3} \times 25 \frac{\text{ft}^3}{\text{s}} \times 150 \frac{\text{ft·lbf}}{\text{slug}} \times \frac{1}{550} \frac{\text{hp}}{\text{ft·lbf}}$$

$$\mathcal{P}_p = 13.2 \text{ hp}$$

From Eq. 3 $\quad \frac{\mathcal{P}_m}{\mathcal{P}_p} = 8\left(\frac{D_m}{D_p}\right)^5 \quad \therefore \quad \frac{D_m}{D_p} = \left[\frac{1}{8}\frac{\mathcal{P}_m}{\mathcal{P}_p}\right]^{1/5} = \left[\frac{1}{8} \times \frac{3}{13.2}\right]^{1/5} = 0.491$

$$\therefore D_m = 0.491 D_p = 0.491 \text{ ft} \longleftarrow D_m$$

From Eq. 1 $\quad Q_m = 2\left(\frac{D_m}{D_p}\right)^3 Q_p = 2(0.491)^3 \times 25 \frac{\text{ft}^3}{\text{s}} = 5.92 \text{ ft}^3/\text{s} \longleftarrow Q_m$

From Eq. 2 $\quad h_m = 4\left(\frac{D_m}{D_p}\right)^2 h_p = 4(0.491)^2 \times 150 \frac{\text{ft·lbf}}{\text{slug}} = 145 \frac{\text{ft·lbf}}{\text{slug}} \longleftarrow h_m$

Given: For a marine propeller (from prob 7.22) the thrust force, F_t, is
$F_t = F_t(\rho, D, V, g, \omega, \Phi, \mu)$
Neglecting viscous effects, and pressure, then
$F_t = F_t(\rho, D, V, g, \omega)$
Assume that torque, T, and power, $\mathcal{P}$, depend on same parameters
$T = T(\rho, D, V, g, \omega)$
$\mathcal{P} = \mathcal{P}(\rho, D, V, g, \omega)$

Find: Derive scaling "laws" for propellers that relate F_t, T, and $\mathcal{P}$ to other variables.

Solution: Apply Buckingham π-theorem

① ρ D V g ω F_t T $\mathcal{P}$

② Choose F, L, t as primary dimensions

③ $\frac{Ft^2}{L}$ L $\frac{L}{t}$ $\frac{L}{t^2}$ $\frac{1}{t}$ F FL $\frac{FL}{t}$

④ Repeating variables ρ, ω, D

⑤ Then $j = n - m = 5$ dimensionless groups (2 independent, 3 dependent)

Setting up dimensional equations

$\Pi_1 = \rho^a \omega^b D^c V$

$F^0 L^0 t^0 = \left(\frac{Ft^2}{L}\right)^a \left(\frac{1}{t}\right)^b L^c \frac{L}{t}$

F: $0 = a$ → $a = 0$
t: $0 = 2a - b - 1$ → $b = -1$
L: $0 = -4a + c + 1$ → $c = -1$

$\therefore \Pi_1 = \frac{V}{\omega D}$

$\Pi_2 = \rho^a \omega^b D^c g$

$F^0 L^0 t^0 = \left(\frac{Ft^2}{L}\right)^a \left(\frac{1}{t}\right)^b L^c \frac{L}{t^2}$

F: $0 = a$ → $a = 0$
t: $0 = 2a - b - 2$ → $b = -2$
L: $0 = -4a + c + 1$ → $c = -1$

$\therefore \Pi_2 = \frac{g}{\omega^2 D}$

$\Pi_3 = \rho^a \omega^b D^c F_t$

$F^0 L^0 t^0 = \left(\frac{Ft^2}{L}\right)^a \left(\frac{1}{t}\right)^b L^c F$

F: $0 = a + 1$ → $a = -1$
t: $0 = 2a - b$ → $b = -2$
L: $0 = -4a + c$ → $c = -4$

$\therefore \Pi_3 = \frac{F_t}{\rho \omega^2 D^4}$

$\Pi_4 = \rho^a \omega^b D^c T$

$F^0 L^0 t^0 = \left(\frac{Ft^2}{L}\right)^a \left(\frac{1}{t}\right)^b L^c FL$

F: $0 = a + 1$ → $a = -1$
t: $0 = 2a - b$ → $b = -2$
L: $0 = -4a + c + 1$ → $c = -5$

$\therefore \Pi_4 = \frac{T}{\rho \omega^2 D^5}$

$\Pi_5 = \rho^a \omega^b D^c \mathcal{P}$

$F^0 L^0 t^0 = \left(\frac{Ft^2}{L}\right)^a \left(\frac{1}{t}\right)^b L^c \frac{FL}{t}$

F: $0 = a + 1$ → $a = -1$
t: $0 = 2a - b - 1$ → $b = -3$
L: $0 = -4a + c + 1$ → $c = -5$

$\therefore \Pi_5 = \frac{\mathcal{P}}{\rho \omega^3 D^5}$

Then scaling "laws" are

$$\frac{F_t}{\rho \omega^2 D^4} = f_1\left(\frac{V}{\omega D}, \frac{g}{\omega^2 D}\right)$$

$$\frac{T}{\rho \omega^2 D^5} = f_2\left(\frac{V}{\omega D}, \frac{g}{\omega^2 D}\right)$$

$$\frac{\mathcal{P}}{\rho \omega^3 D^5} = f_3\left(\frac{V}{\omega D}, \frac{g}{\omega^2 D}\right)$$

Given: Thrust and torque of propeller depend on D, ω, V, μ, ρ

Model:		Prototype:	
D	= 2 ft	D	= 20 ft
ω	= 2000 rpm	ω	= ?
V	= 150 ft/s	V	= 400 ft/s
F_t	= 25 lbf (thrust)	F_t	= ?
T	= 7.5 lbf-ft	T	= ?

Find: (a) ω, (b) F_t and (c) T for prototype, neglecting effects of viscosity, under dynamically similar conditions.

Solution: There are two problems here. (1) Determine $F_t = f_1(D, \omega, V, \mu, \rho)$ and (2) $T = f_2(D, \omega, V, \mu, \rho)$. Since μ is to be ignored, do not select it as a repeating parameter. Instead, select D, ω, ρ as repeating variables.

(1) $F_t = f_1(D, \omega, V, \mu)$

① $F_t \quad D \quad \omega \quad V \quad \mu \quad \rho \qquad n = 6$ parameters

② Select F, L, t as primary dimensions.

③

F_t	D	ω	V	μ	ρ	
F	L	$\frac{1}{t}$	$\frac{L}{t}$	$\frac{Ft}{L^2}$	$\frac{Ft^2}{L^4}$	$r = 3$ primary dimensions

④ Choose D, ω, V $\qquad m = r = 3$ repeating parameters

⑤ Then $n - m = 3$ dimensionless groups will result. Setting up dimensional equations,

$\Pi_1 = D^a \omega^b \rho^c F_t$

$= (L)^a \left(\frac{1}{t}\right)^b \left(\frac{Ft^2}{L^4}\right)^c F = F^0 L^0 t^0$

$F: \; c + 1 = 0 \qquad c = -1$
$L: \; a - 4c = 0 \qquad a = -4$
$t: \; -b + 2c = 0 \qquad b = -2$

$\Pi_1 = \frac{F_t}{\rho \omega^2 D^4}$

$\Pi_2 = D^a \omega^b \rho^c V$

$= (L)^a \left(\frac{1}{t}\right)^b \left(\frac{Ft^2}{L^4}\right)^c \frac{L}{t} = F^0 L^0 t^0$

$F: \; c = 0 \qquad c = 0$
$L: \; a - 4c + 1 = 0 \qquad a = -1$
$t: \; -b + c - 1 = 0 \qquad b = -1$

$\Pi_2 = \frac{V}{\omega D}$

$\Pi_3 = D^a \omega^b \rho^c \mu = (L)^a \left(\frac{1}{t}\right)^b \left(\frac{Ft^2}{L^4}\right)^c \frac{Ft}{L^2} = F^0 L^0 t^0$

$F: \; c + 1 = 0 \qquad c = -1$
$L: \; a - 4c - 2 = 0 \qquad a = 4c + 2 = -2$
$t: \; -b + 2c + 1 = 0 \qquad b = 2c + 1 = -1$

$\Pi_3 = \frac{\mu}{\rho \omega D^2}$

Then $\Pi_1 = f_1(\Pi_2, \Pi_3)$ or $\frac{F_t}{\rho \omega^2 D^4} = f_1\left(\frac{V}{\omega D}, \frac{\mu}{\rho \omega D^2}\right)$

If viscous effects are neglected, then $\Pi_1 = g_1(\Pi_2)$ or $\frac{F_t}{\rho\omega^2 D^4} = g_1\left(\frac{V}{\omega D}\right)$

For dynamic similarity, $\Pi_2)_{model} = \Pi_2)_{prototype}$, or

$$\frac{V_m}{\omega_m D_m} = \frac{V_p}{\omega_p D_p}$$

Thus $\omega_p = \omega_m \frac{V_p}{V_m}\frac{D_m}{D_p} = (2000\,rpm)\left(\frac{400}{450}\right)\left(\frac{1}{10}\right) = 533\,rpm$ ω_p ←

When $\Pi_2)_{model} = \Pi_2)_{prototype}$, then neglecting μ, $\Pi_1)_{model} = \Pi_1)_{prototype}$, or

$$\frac{F_{tm}}{\rho_m \omega_m^2 D_m^4} = \frac{F_{tp}}{\rho_p \omega_p^2 D_p^4} \quad ; \text{ assume } \rho_m = \rho_p$$

Then $F_{tp} = F_{tm}\left(\frac{\omega_p}{\omega_m}\right)^2\left(\frac{D_p}{D_m}\right)^4 = 25\,lbf\left(\frac{533}{2000}\right)^2(10)^4 = 17{,}800\,lbf$ F_{tp} ←

(2) The analysis of Π_2 and Π_3 for the second problem is identical to that for problem (1). Combining T with D, ω and ρ gives

$$\Pi_4 = D^a \omega^b \rho^c T = (L)^a\left(\frac{1}{t}\right)^b\left(\frac{Ft^2}{L^4}\right)^c(FL) = M^0L^0t^0$$

$$\left.\begin{array}{lll} F: & c+1=0 & c=-1 \\ L: & a-4c+1=0 & a=4c-1=-5 \\ t: & -b+2c=0 & b=2c=-2 \end{array}\right\} \quad \Pi_4 = \frac{T}{\rho\omega^2 D^5}$$

Thus $\Pi_4 = f_2(\Pi_2, \Pi_3)$ or neglecting μ, $\Pi_4 = g_2(\Pi_2)$. For dynamic similarity, $\Pi_4)_{model} = \Pi_4)_{prototype}$, or

$$\frac{T_m}{\rho_m \omega_m^2 D_m^5} = \frac{T_p}{\rho_p \omega_p^2 D_p^5} \quad ; \text{ assume } \rho_m = \rho_p$$

Then $T_p = T_m\left(\frac{\omega_p}{\omega_m}\right)^2\left(\frac{D_p}{D_m}\right)^5 = 7.5\,ft\text{-}lbf\left(\frac{533}{2000}\right)^2(10)^5 = 53{,}300\,ft\text{-}lbf$ T_p ←

⑥ Check, using M, L, t:

$$\Pi_1 = \frac{ML}{t^2}\frac{L^3}{M}\frac{t^2}{1}\frac{1}{L^4} = [1] \checkmark$$

$$\Pi_2 = \frac{L}{t}\frac{t}{1}\frac{1}{L} = [1] \checkmark$$

$$\Pi_3 = \frac{M}{Lt}\frac{L^3}{M}\frac{t}{1}\frac{1}{L^2} = [1] \checkmark$$

$$\Pi_4 = \frac{ML^2}{t^2}\frac{L^3}{M}\frac{t^2}{1}\frac{1}{L^5} = [1] \checkmark$$

Given: Data for a model test of a marine propeller

	Model	Prototype
Diameter	16 in	24 ft
Angular speed	960 rpm	240 rpm
Speed of advance	20 ft/s	
Thrust	35.1 lbf	
Torque	120 in·lbf	
Fluid	fresh water	sea water
Temp	65°F	59°F

Find: (a) for dynamically similar tests, calculate V, F_t, T of prototype (use scaling laws of prob. 7.54)
(b) thrust power produced, power input, and efficiency of prototype propeller

Solution:

From problem 7.54, the scaling laws for a propeller are

$$\frac{F_t}{\rho\omega^2 D^4} = f_1\left(\frac{V}{\omega D}, \frac{\omega D}{\nu}\right); \quad \frac{T}{\rho\omega^2 D^5} = f_2\left(\frac{V}{\omega D}, \frac{\omega D}{\nu}\right); \quad \frac{\mathcal{P}}{\rho\omega^3 D^5} = f_3\left(\frac{V}{\omega D}, \frac{\omega D}{\nu}\right)$$

$$\left(\frac{V}{\omega D}\right)_p = \left(\frac{V}{\omega D}\right)_m \quad \therefore \quad V_p = \frac{\omega_p}{\omega_m}\frac{D_p}{D_m}V_m = \frac{240}{960}\times\frac{24}{1.5}\times 20\ \frac{ft}{s} = 80\ ft/s \qquad V_p$$

$$\left(\frac{F_t}{\rho\omega^2 D^4}\right)_p = \left(\frac{F_t}{\rho\omega^2 D^4}\right)_m \quad \therefore \quad F_{t_p} = \frac{\rho_p}{\rho_m}\left(\frac{\omega_p}{\omega_m}\right)^2\left(\frac{D_p}{D_m}\right)^4 F_{t_m} = 1.025\times\left(\frac{1}{4}\right)^2\left(16\right)^4\times 35.1\,lbf = 1.47\times10^5\ lbf \qquad F_{t_p}$$

$$\left(\frac{T}{\rho\omega^2 D^5}\right)_p = \left(\frac{T}{\rho\omega^2 D^5}\right)_m \quad \therefore T_p = \frac{\rho_p}{\rho_m}\left(\frac{\omega_p}{\omega_m}\right)^2\left(\frac{D_p}{D_m}\right)^5 T_m = 1.025\left(\frac{1}{4}\right)^2\left(16\right)^5 10\,ft\cdot lbf = 6.72\times10^5\ ft\cdot lbf \qquad T_p$$

The thrust power for the prototype is given by

$$\text{Thrust power} = F_t V = 1.47\times10^5\,lbf\times 80\frac{ft}{s}\times\frac{s\cdot hp}{550\,ft\cdot lbf} = 2.14\times10^4\ hp \qquad \mathcal{P}_t$$

The power input for the prototype is given by

$$\text{Power input} = T\omega = 6.72\times10^5\,ft\cdot lbf\times 240\frac{rev}{min}\times\frac{2\pi\,rad}{1\,rev}\times\frac{min}{60\,s}\times\frac{s\cdot hp}{550\,ft\cdot lbf}$$

$$\text{Power input} = 3.07\times10^4\ hp \qquad \mathcal{P}_{in}$$

The efficiency of the prototype is given by

$$\text{Eff} = \frac{\text{Thrust power}}{\text{Power input}} = \frac{2.14\times10^4}{3.07\times10^4} = 0.697 \qquad \eta$$

Given: The kinetic energy ratio is a figure of merit defined as the ratio of kinetic energy flux in a wind tunnel test section to the drive power.

Find: an estimate of the kinetic energy ratio for the 40×80 wind tunnel at NASA-Ames.

Solution:

From text (p. 319). for NASA-Ames tunnel:

$$A = 40\text{ft} \times 80\text{ft} = 3200\text{ft}^2 \quad , \quad P = 125{,}000\text{ hp}$$

$$V_{max} = 300\ \frac{\text{km}}{\text{hr}} \times \frac{0.6080\text{ ft}}{1\text{ km}} \times \frac{\text{hr}}{3600\text{ s}} = 507\ \text{ft/s}$$

$$\text{K.E ratio} = \frac{\text{K.E flux}}{\text{Power in}} = \frac{\frac{1}{2}\rho V^3}{P} = \frac{\rho V A V^2}{2P} = \frac{\rho V^3 A}{2P}$$

Assuming standard air,

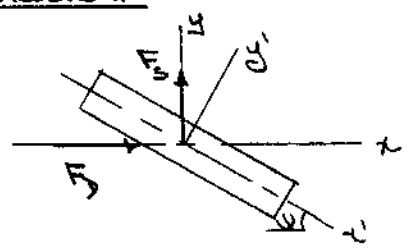
$$\text{K.E. ratio} = \frac{1}{2} \times 0.00238\ \frac{\text{slug}}{\text{ft}^3} \times (507)^3\ \frac{\text{ft}^3}{\text{s}^3} \times 3200\text{ft}^2 \times \frac{1}{125{,}000\text{hp}} \times \frac{\text{hp}\cdot\text{s}}{550\text{ft}\cdot\text{lbf}} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

K.E ratio = 7.22

Given: In tests of models in wind tunnels, forces and moments are measured relative to an axis system aligned with tunnel centerline. Forces on highway vehicle are expressed most usefully in a coordinate system reference to the body axis. Assume that in model test yaw angle is ψ.

Find: Develop equations to calculate body-axis drag and side force from data measured in wind tunnel coordinates

Solution: $\vec{F}$ has two components, $\vec{F} = F_D \hat{i} + F_S \hat{j}$



Plan view of tunnel
- xy is tunnel coordinate system
- x'y' is body-axis coordinate system
- ψ is yaw angle

To obtain body-axis drag, $F_{x'}$, and side force, $F_{y'}$, resolve $\vec{F}$ into components parallel and perpendicular to body-axis, x'

$F_{x'}, F_{y'}$

$$F_{x'} = F_D \cos\psi - F_S \sin\psi \qquad F_{y'} = F_D \sin\psi + F_S \cos\psi$$

Check: $|\vec{F}| = [F_D^2 + F_S^2]^{1/2} = [F_{x'}^2 + F_{y'}^2]^{1/2}$

$$= [F_D^2 \cos^2\psi - 2F_D F_S \cos\psi \sin\psi + F_S^2 \sin^2\psi + F_D^2 \sin^2\psi + 2F_D F_S \cos\psi \sin\psi + F_S^2 \cos^2\psi]^{1/2}$$

$$|\vec{F}| = [F_D^2 + F_S^2]^{1/2} \quad \checkmark$$

Given: Wind tunnel test of 1:16 model bus in standard air.

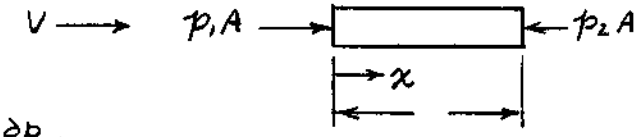
$W = 152$ mm	$V = 26.5$ m/s	Pressure gradient:
$H = 200$ mm	$F_D = 6.09$ N (measured)	$\frac{dp}{dx} = -11.8$ N/m²/m
$L = 762$ mm		

Find: (a) Estimate the horizontal buoyancy correction.
(b) Calculate the corrected model drag coefficient.
(c) Evaluate the drag force on the prototype at 100 km/hr on a calm day.

Solution: Apply definitions

Computing equations: $C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$ Assume $A = WH$

The buoyancy force will be



$$F_B = p_1 A - p_2 A = (p_1 - p_2)A$$

But $p_2 = p_1 + \frac{\partial p}{\partial x}\Delta x + \cdots \approx p_1 + \frac{\partial p}{\partial x} L$

Therefore $p_1 - p_2 = -\frac{\partial p}{\partial x} L$, and $F_B \approx -\frac{\partial p}{\partial x} LA = -\frac{\partial p}{\partial x} LWH$

$$F_B \approx -(-11.8)\frac{N}{m^3} \times 0.762\ m \times 0.152\ m \times 0.200\ m = 0.273\ N \text{ (to right)}$$ F_B

The corrected drag force is

$$F_{Dc} = F_{Dm} - F_B = (6.09 - 0.273)\ N = 5.82\ N$$

The corrected model drag coefficient is

$$C_{Dm} = \frac{F_{Dc}}{\frac{1}{2}\rho V^2 A} = 2 \times 5.82\ N \times \frac{m^3}{1.23\ kg} \times \frac{s^2}{(26.5)^2 m^2} \times \frac{1}{(0.200)(0.152)\ m^2} \times \frac{kg \cdot m}{N \cdot s^2} = 0.443$$ C_D

Assume the test was conducted at high enough Reynolds number so $C_{Dp} = C_{Dm}$. Then

$$F_{Dp} = C_{Dp} A_p \tfrac{1}{2}\rho V_p^2$$

$$= \frac{1}{2} \times 0.443 \times 0.200(16)\ m \times 0.152(16)\ m \times 1.23\ \frac{kg}{m^3} \times \left[100\ \frac{km}{hr} \times 1000\ \frac{m}{km} \times \frac{hr}{3600\ s}\right]^2 \frac{N \cdot s^2}{kg \cdot m}$$

$$F_{Dp} = 1.64\ kN \text{ (prototype at 100 km/hr)}$$ F_{Dp}

{Rolling resistance must be included to obtain the total tractive effort needed to propel the full-scale vehicle.}

Given: A 1:16 scale model of a 20 m long truck is tested in a wind tunnel at speed $V_m = 80$ m/s. The axial pressure gradient at this speed is $dh/dx = -1.2$ mm H_2O/m. The frontal area of the prototype is $A_p = 10\ m^2$. $C_D = 0.85$

Find: (a) Estimate the horizontal buoyancy correction
(b) Express the correction as a fraction of the measured C_D.

Solution:

The horizontal buoyancy force, F_B, is the difference in the pressure force between the front and back of the model due to the pressure gradient in the tunnel

$$F_B = (p_2 - p_1)A = \rho_0 g \frac{dh}{dx} L_m A_m \qquad (\Delta p = \rho w g \Delta h)$$

$$L_m = \frac{L_p}{16} \qquad A_m = \frac{A_p}{(16)^2}$$

$$\therefore F_B = 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times (-1.2) \times 10^{-3} \frac{m}{m} \times \frac{20\ m}{16} \times \frac{10\ m^2}{(16)^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_B = -0.574\ N$$

The horizontal buoyancy correction should be added to the measured drag force on the model

The measured drag force on the model is given by

$$F_{D_m} = \frac{1}{2} \rho V^2 A_m C_D = \frac{1}{2} \rho V^2 \frac{A_p}{(16)^2} C_D$$

Assume air at standard conditions, $\rho = 1.23\ kg/m^3$

$$F_{D_m} = \frac{1}{2} \times 1.23 \frac{kg}{m^3} \times (80)^2 \frac{m^2}{s^2} \times \frac{10\ m^2}{(16)^2} \times 0.85 \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_{D_m} = 131\ N$$

$$\frac{F_B}{F_{D_m}} = \frac{-0.574}{131} = -4.38 \times 10^{-3} = -0.44\%$$

Given: The propagation speed of small amplitude waves in a region of uniform depth is given by

$$c^2 = \left(\frac{\sigma}{\rho}\frac{2\pi}{\lambda} + \frac{g\lambda}{2\pi}\right)\tanh\frac{2\pi h}{\lambda}$$

where h is the depth of the undisturbed liquid
λ is the wavelength.

Find: Obtain the dimensionless groups and determine the conditions for similarity. (Use L as a characteristic length and V_0 as a characteristic velocity)

Solution:

$$c^2 = \left(\frac{\sigma}{\rho}\frac{2\pi}{\lambda} + \frac{g\lambda}{2\pi}\right)\tanh\frac{2\pi h}{\lambda}$$

To nondimensionalize the equation, all lengths are divided by L and all velocities are divided by V_0.

Denoting nondimensional quantities by an asterisk, then

$$\lambda^* = \frac{\lambda}{L} \qquad h^* = \frac{h}{L} \qquad c^* = \frac{c}{V_0}$$

Then

$$c^{*2}V_0^2 = \left(\frac{\sigma}{\rho L}\frac{2\pi}{\lambda^*} + \frac{gL\lambda^*}{2\pi}\right)\tanh\frac{2\pi h^* L}{\lambda^* L}$$

$$c^{*2} = \left(\frac{\sigma}{\rho L V_0^2}\frac{2\pi}{\lambda^*} + \frac{gL}{V_0^2}\frac{\lambda^*}{2\pi}\right)\tanh\frac{2\pi h^*}{\lambda^*}$$

$\therefore$ Dimensionless groups are $\frac{\sigma}{\rho L V_0^2}$, $\frac{gL}{V_0^2}$

For similarity between flows, these two parameters must be equal for the two flows.

Given: The slope of the free surface of a steady wave in one-dimensional flow in a shallow liquid layer is described by the equation

$$\frac{\partial h}{\partial x} = -\frac{u}{g}\frac{\partial u}{\partial x}$$

Find: Nondimensionalize the equation (using length scale, L, and velocity scale, V_0)
Obtain the dimensionless groups that characterize this flow and determine the conditions for similarity.

Solution:

To nondimensionalize the equation, all lengths are divided by the reference length, L, and all velocities are divided by the reference velocity, V_0.

Denoting the nondimensional quantities by an asterisk,

$$h^* = \frac{h}{L}, \quad x^* = \frac{x}{L}, \quad u^* = \frac{u}{V_0}$$

Substituting into the governing equation

$$\frac{\partial(h^* L)}{\partial(x^* L)} = -\frac{V_0 u^*}{g}\frac{\partial(V_0 u^*)}{\partial(L x^*)}$$

$$\frac{\partial h^*}{\partial x^*} = -\frac{V_0^2}{gL}\frac{\partial u^*}{\partial x^*}$$

The dimensionless group is $\frac{V_0^2}{gL}$. This is the square of the Froude number. Hence for two flows to be similar, the Froude number must be the same in both flows.

Given: One-dimensional, unsteady flow in a thin liquid layer is described by the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x}$$

Find: Nondimensionalize the equation (using length scale, L, and velocity scale, V_0)
Obtain the dimensionless groups that characterize this flow and determine the conditions for similarity.

Solution:

To nondimensionalize the equation, all lengths are divided by the reference length, L, velocity is divided by the reference velocity, V_0, and time is divided by the ratio, L/V_0.

Denoting the nondimensional quantities by an asterisk,

$$x^* = \frac{x}{L}, \quad h^* = \frac{h}{L}, \quad u^* = \frac{u}{V_0}, \quad t^* = \frac{t}{L/V_0}$$

Substituting into the governing equation

$$\frac{\partial(V_0 u^*)}{\partial\left(L t^*/V_0\right)} + u^* V_0 \frac{\partial(V_0 u^*)}{\partial(x^* L)} = -g\frac{\partial(h^* L)}{\partial(x^* L)}$$

$$\frac{V_0^2}{L}\frac{\partial u^*}{\partial t^*} + \frac{V_0^2}{L} u^*\frac{\partial u^*}{\partial x^*} = -g\frac{\partial h^*}{\partial x^*}$$

Multiplying through by L/V_0^2,

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} = -\frac{gL}{V_0^2}\frac{\partial h^*}{\partial x^*}$$

The dimensionless group is $\frac{gL}{V_0^2}$. This is one over the square of the Froude number. Hence for two flows to be similar, the Froude number must be the same in both flows.

Given: For steady, incompressible, two-dimensional flow, the Prandtl boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{(1)}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \qquad \text{(2)}$$

Find: Nondimensionalize these equations (using L and V_0 as characteristic length and velocity, respectively) and identify the resulting similarity parameters.

Solution:

Denoting nondimensional quantities by an asterisk.

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad u^* = \frac{u}{V_0}, \quad v^* = \frac{v}{V_0}$$

Substituting into Eq. 1, we obtain

$$\frac{\partial(u^*V_0)}{\partial(x^*L)} + \frac{\partial(v^*V_0)}{\partial(y^*L)} = 0 = \frac{V_0}{L}\frac{\partial u^*}{\partial x^*} + \frac{V_0}{L}\frac{\partial v^*}{\partial y^*}$$

or

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

Consider each term in Eq. 2

$$u\frac{\partial u}{\partial x} = u^*V_0\frac{\partial(u^*V_0)}{\partial(x^*L)} = \frac{V_0^2}{L}u^*\frac{\partial u^*}{\partial x^*}$$

$$v\frac{\partial u}{\partial y} = v^*V_0\frac{\partial(u^*V_0)}{\partial(y^*L)} = \frac{V_0^2}{L}v^*\frac{\partial u^*}{\partial y^*}$$

Leave 2nd term as is for the moment

$$\nu\frac{\partial^2 u}{\partial y^2} = \nu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \nu\frac{\partial}{\partial y}\left(\frac{\partial u^*V_0}{\partial y^*L}\right) = \nu\frac{V_0}{L}\frac{\partial}{\partial y}\left(\frac{\partial u^*}{\partial y^*}\right)\frac{V_0}{L} = \nu\frac{V_0}{L^2}\frac{\partial^2 u^*}{\partial y^{*2}}$$

Substituting into Eq. 2

$$\frac{V_0^2}{L}u^*\frac{\partial u^*}{\partial x^*} + \frac{V_0^2}{L}v^*\frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{V_0}{L^2}\frac{\partial^2 u^*}{\partial y^{*2}}$$

Multiplying through by $\frac{L}{V_0^2}$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{L}{\rho V_0^2}\frac{\partial p}{\partial x} + \frac{\nu}{V_0 L}\frac{\partial^2 u^*}{\partial y^{*2}} = -\frac{L}{\rho V_0^2}\frac{\partial(p^*\rho V_0^2)}{\partial(x^*L)} + \frac{\nu}{V_0 L}\frac{\partial^2 u^*}{\partial y^{*2}}$$

Define the non-dimensional pressure $p^* = \frac{p}{\rho V_0^2}$, then

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\nu}{V_0 L}\frac{\partial^2 u^*}{\partial y^{*2}}$$

The similarity parameter is $\frac{\nu}{V_0 L} = \frac{1}{Re}$

7-6 [4]

Open-Ended Problem Statement: During a recent stay at a motel, a hanging lamp was observed to oscillate in the air stream from the air conditioning unit. Explain why this might occur.

Discussion: Minor fluctuations occur in the speed and direction of the air blowing from the air conditioning unit. These tend to move the hanging lamp from the vertical, steady-state position.

If the fluctuations in air flow speed and direction are large enough, they can cause significant random motions of the hanging lamp.

If the fluctuations in air flow speed and direction contain a periodic frequency content that is close to the natural frequency of the lamp's motion, they can excite the resonant frequency, leading to quite large oscillations in the lamp motion. These periodic motions may occur in combination with the smaller, random motions.

500 SHEETS, FILLER 5 SQUARE
50 SHEETS EYE-EASE® 5 SQUARE
100 SHEETS EYE-EASE® 5 SQUARE
200 SHEETS EYE-EASE® 5 SQUARE
100 RECYCLED WHITE 5 SQUARE
200 RECYCLED WHITE 5 SQUARE
National® Brand

Open-Ended Problem Statement: Frequently one observes a flag on a pole "flapping" in the wind. Explain why this occurs. What dimensionless parameters might characterize the phenomenon? Why?

Discussion: The natural wind contains significant fluctuations in air speed and direction. These fluctuations tend to disturb the flag from an initially plane position.

When the flag is bent or curved from the plane position, the flow nearby must follow its contour. Flow over a convex curved surface tends to be faster, and have lower pressure, than flow over a concave curved surface. The resulting pressure forces tend to exaggerate the curvature of the flag. The result is a seemingly random, "flapping" motion of the flag.

The rope or chain used to raise the flag may also flap in the wind. It is much more likely to exhibit a periodic motion than the flag itself. The rope is quite close to the flag pole, where it is influenced by any vortices shed from the pole. If the Reynolds number is such that periodic vortices are shed from the pole, they will tend to make the rope move with the same frequency. This accounts for the periodic thump of a rope or clank of a chain against the pole.

The vortex shedding phenomenon is characterized by the Strouhal number, $St = fD/V_\infty$, where f is the vortex shedding frequency, D the pole diameter, and V_∞ the wind speed. The Strouhal number is constant at approximately 0.2 over a broad range of Reynolds number.

Open-Ended Problem Statement: Explore the variation in wave propagation speed given by the equation of Problem 7.61 for a free-surface flow of water. Find the operating depth to minimize the speed of capillary waves (waves with small wavelength, also called *ripples*). First assume wavelength is much smaller than water depth. Then explore the effect of depth. What depth do you recommend for a water table used to visualize compressible-flow wave phenomena? What is the effect of reducing surface tension by adding a surfactant?

Discussion: The equation given in Problem 7.61 contains three terms. The first term contains surface tension and gives a speed inversely proportional to wavelength. This term will be important when small wavelengths are considered.

The second term contains gravity and gives a speed proportional to wavelength. This term will be important when long wavelengths are considered.

The argument of the hyperbolic tangent is proportional to water depth and inversely proportional to wavelength. For small wavelengths, this term should approach unity since the hyperbolic tangent of a large number approaches one.

See the spreadsheet for numerical values and a plot.

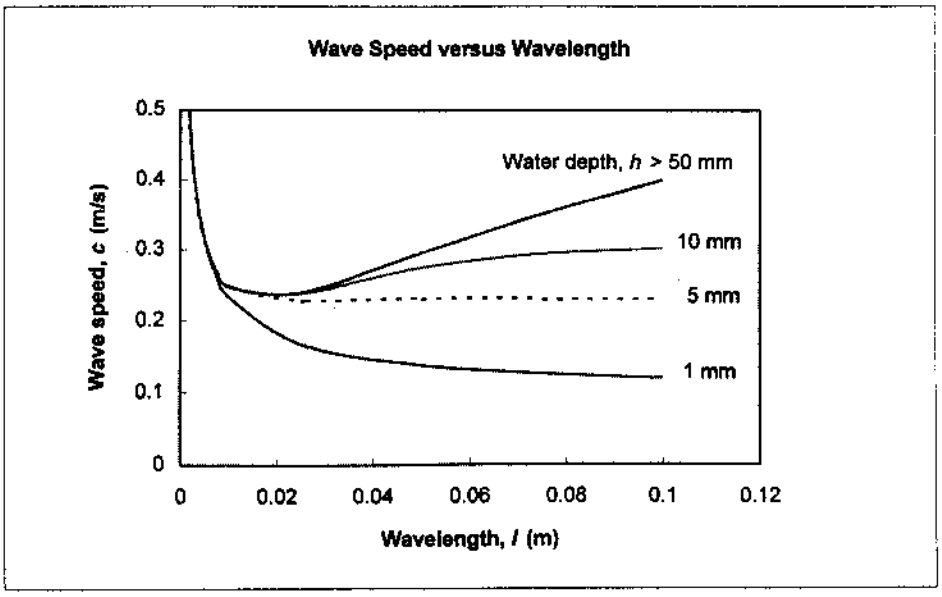
13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE 5 SQUARE
42-382 150 SHEETS EYE-EASE 5 SQUARE
42-386 200 SHEETS EYE-EASE 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National Brand

Input Parameters:

g =	9.81	m/s^2	Acceleration of gravity
h =	0.01	m	Liquid depth (for hyperbolic tangent calculation)
ρ =	999	kg/m^3	Liquid density
σ =	0.0728	N/m	Surface tension

Calculated Values:

	h (m) =	0.001	0.005	0.01	0.05	0.1	0.5
Wavelength, λ (m)	**tanh (---) (h = 10 mm)**	**Wave Speed, c (m/s)**					
0.00185	1.00	0.500	0.500	0.500	0.500	0.500	0.500
0.003	1.00	0.396	0.397	0.397	0.397	0.397	0.397
0.005	1.00	0.313	0.315	0.315	0.315	0.315	0.315
0.0075	1.00	0.263	0.270	0.270	0.270	0.270	0.270
0.01	1.00	0.233	0.248	0.248	0.248	0.248	0.248
0.025	0.987	0.167	0.227	0.238	0.239	0.239	0.239
0.05	0.850	0.138	0.229	0.275	0.295	0.295	0.295
0.075	0.685	0.126	0.229	0.294	0.351	0.351	0.351
0.1	0.557	0.120	0.228	0.303	0.400	0.401	0.401
0.2	0.304	0.110	0.226	0.312	0.537	0.560	0.561
0.5	0.125	0.104	0.223	0.314	0.660	0.815	0.884
0.75	0.0836	0.102	0.223	0.314	0.681	0.896	1.08
1	0.0627	0.101	0.222	0.314	0.690	0.933	1.25
2	0.0314	0.100	0.222	0.314	0.698	0.975	1.69
5	0.0126	0.100	0.222	0.313	0.700	0.988	2.09
7.5	0.00838	0.0994	0.222	0.313	0.700	0.989	2.15
10	0.00628	0.0993	0.222	0.313	0.700	0.990	2.18
Froude Speed, $(gh)^{1/2}$ (m/s)		0.0990	0.221	0.313	0.700	0.990	2.21



Given: Standard air entering a 0.3 m diameter duct.

$$Q = 2.0 \frac{m^3}{min}$$

Find: (a) Whether flow is laminar or turbulent.
(b) Approximate entrance length for fully developed flow.

Solution: The Reynolds number may be written in terms of the volume flowrate. Thus since $\bar{V} = Q/A = 4Q/\pi D^2$, then

$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu} = \frac{Q D}{A \nu} = \frac{4}{\pi D^2} \frac{Q D}{\nu} = \frac{4Q}{\pi \nu D}$$

At 15 C, $\nu = 1.46 \times 10^{-5} \frac{m^2}{s}$ for air, from Table A.10. Thus

$$Re = \frac{4}{\pi} \times 2.0 \frac{m^3}{min} \times \frac{s}{1.46 \times 10^{-5} m^2} \times \frac{1}{0.3 m} \times \frac{min}{60 s} = 9690$$

Since $Re > 2300$, flow is turbulent.

According to Sec. 8-1, the entrance length may be between 25 and 40 diameters. Thus, since $D = 0.3 m$,

$$7.5 m < L < 12.0 m$$

Given: Flow in a circular tube.

Transition to turbulence occurs at $Re \approx 2300$.

Find: Prepare a log-log plot of mean velocity versus tube diameter. Plot lines corresponding to $Re = 2300$ for (a) standard air, and (b) water at 15C.

Solution: $Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu}$

For standard air (Table A.10), $\nu = 1.46 \, m^2/s$

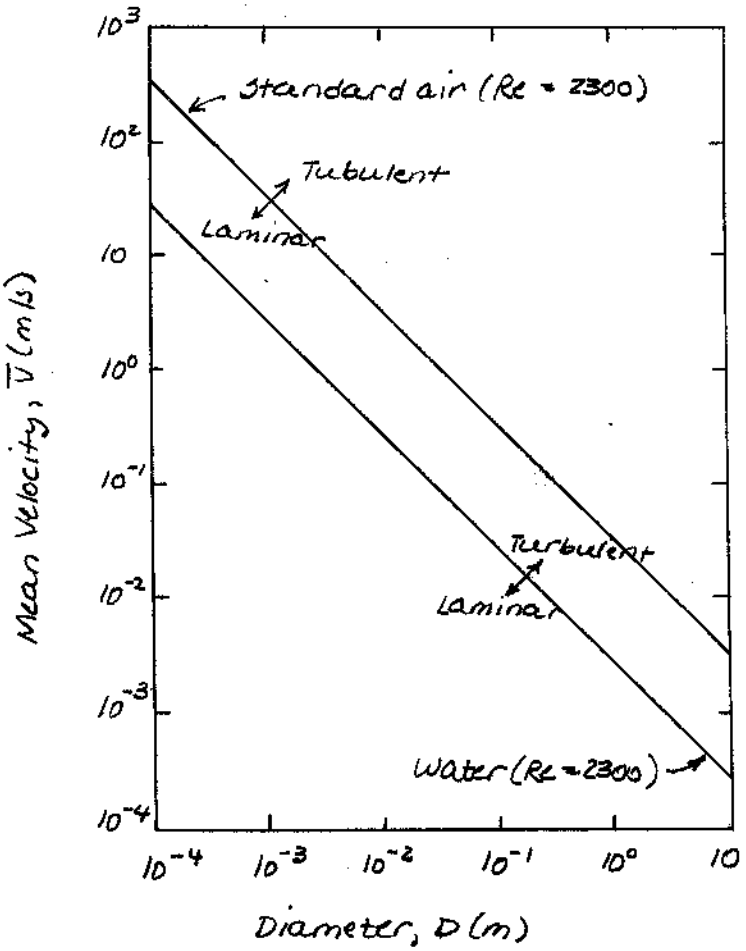
$$\bar{V} = \frac{Re\,\nu}{D} = 2300 \times 1.46 \times 10^{-5} \frac{m^2}{s} \times \frac{1}{10^{-4} m} = 336 \, m/s \quad \text{at } D = 10^{-4} m$$

$$\bar{V} = \frac{Re\,\nu}{D} = 2300 \times 1.46 \times 10^{-5} \frac{m^2}{s} \times \frac{1}{1 m} = 0.0336 \, m/s \quad \text{at } D = 1 m$$

For water at 15°C (Table A.8), $\nu = 1.14 \times 10^{-6} \, m^2/s$

$$\bar{V} = \frac{Re\,\nu}{D} = 2300 \times 1.14 \times 10^{-6} \frac{m^2}{s} \times \frac{1}{10^{-4} m} = 26.2 \, m/s \quad \text{at } D = 10^{-4} m$$

$$\bar{V} = \frac{Re\,\nu}{D} = 2300 \times 1.14 \times 10^{-6} \frac{m^2}{s} \times \frac{1}{1 m} = 0.00262 \, m/s \quad \text{at } D = 1 m$$



Given: For flow in a circular tube, transition from laminar to turbulent flow occurs at $Re \approx 2300$

Find: Prepare a log-log plot of volume flow rate, Q, versus tube diameter, D. Plot lines corresponding to $Re = 2300$ for (a) standard air, and (b) water at 15C

Solution:

Definition $Re = \frac{\rho \bar{V} D}{\mu}$. Also $\bar{V} = \frac{Q}{A}$, $A = \frac{\pi D^2}{4}$, $\nu = \frac{\mu}{\rho}$

$$\therefore Re = \frac{\rho}{\mu} \frac{Q}{A} D = \frac{1}{\nu} \frac{Q 4}{\pi D^2} D = \frac{4Q}{\pi \nu D} \quad \text{and} \quad Q = \frac{\pi \nu D Re}{4}$$

For standard air (Table A.6), $\nu = 1.46 \times 10^{-5}\ m^2/s$

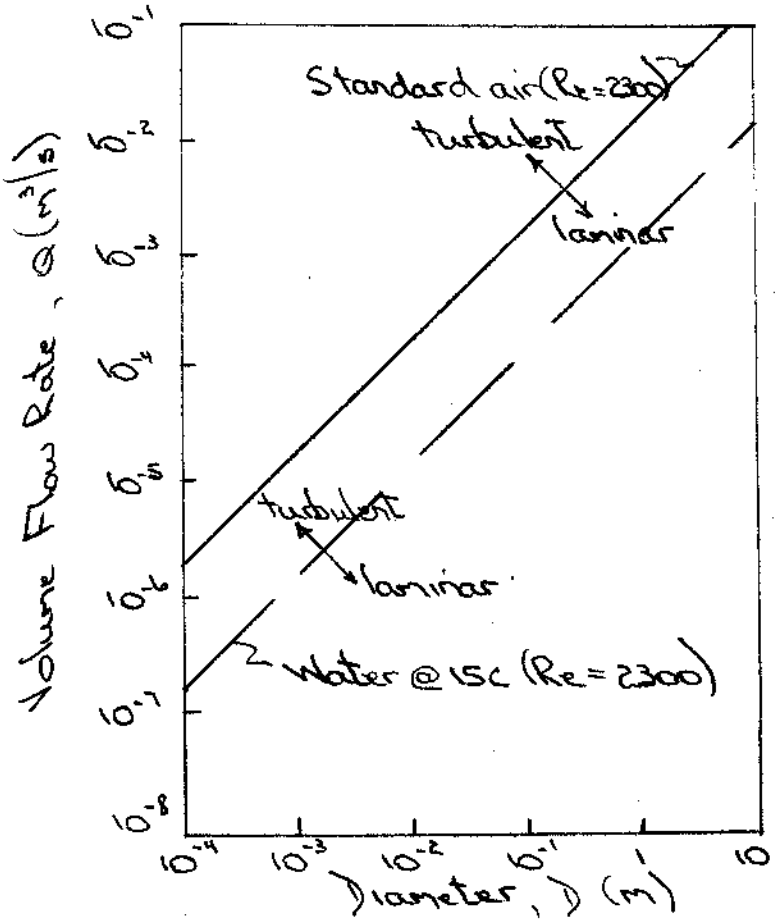
At $D = 1\ m$, $Q = \frac{\pi \nu D Re}{4} = \frac{\pi}{4} \times 1.46 \times 10^{-5} \frac{m^2}{s} \times 1\ m \times 2300 = 2.62 \times 10^{-2}\ m^3/s$

At $D = 10^{-4}\ m$, $Q = 2.62 \times 10^{-6}\ m^3/s$

For water at 15°C (Table A.8), $\nu = 1.14 \times 10^{-6}\ m^2/s$

At $D = 1\ m$, $Q = \frac{\pi \nu D Re}{4} = \frac{\pi}{4} \times 1.14 \times 10^{-6} \frac{m^2}{s} \times 1\ m \times 2300 = 2.06 \times 10^{-3}\ m^3/s$

At $D = 10^{-4}\ m$, $Q = 2.06 \times 10^{-7}\ m^3/s$



Given: For flow in a circular tube, transition from laminar to turbulent flow occurs at $Re \approx 2300$.

Find: Prepare a log-log plot of mass flow rate, $\dot{m}$, versus tube diameter, D. Plot lines corresponding to $Re = 2300$ for (a) standard air, and (b) water at 15C.

Solution:

Definition $Re = \frac{\rho \bar{V} D}{\mu}$ Also $\bar{V} = \frac{\dot{m}}{\rho A}$, $A = \frac{\pi}{4} D^2$

$\therefore Re = \frac{\rho}{\mu} \times \frac{\dot{m}}{\rho \frac{\pi}{4} D^2} \times D = \frac{4\dot{m}}{\pi \mu D}$ and $\dot{m} = \frac{\pi \mu D}{4} Re$

For standard air (Table A.10), $\mu = 1.79 \times 10^{-5}$ kg/m·s

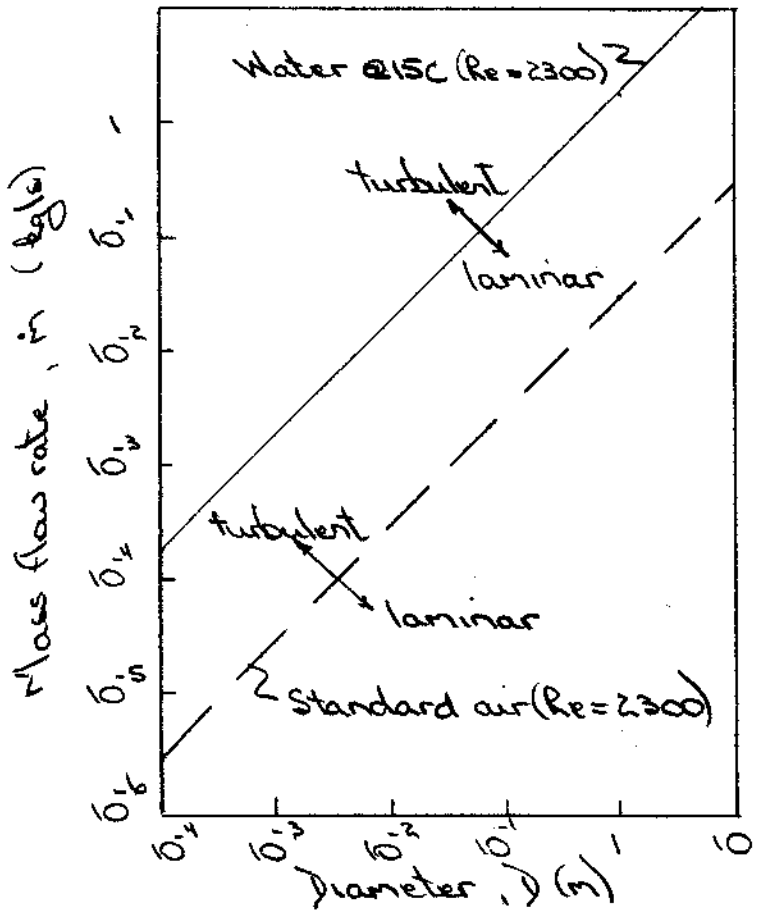
At $D = 10^{-4}$ m, $\dot{m} = \frac{\pi \mu D}{4} Re = \frac{\pi}{4} \times 1.79 \times 10^{-5} \frac{kg}{m \cdot s} \times 10^{-4} m \times 2300 = 3.23 \times 10^{-6}$ kg/s

At $D = 1$ m, $\dot{m} = 3.23 \times 10^{-2}$ kg/s

For water at 15C (Table 9.8), $\mu = 1.14 \times 10^{-3}$ kg/m·s

At $D = 10^{-4}$ m, $\dot{m} = \frac{\pi \mu D}{4} Re = \frac{\pi}{4} \times 1.14 \times 10^{-3} \frac{kg}{m \cdot s} \times 10^{-4} m \times 2300 = 2.06 \times 10^{-4}$ kg/s

At $D = 1$ m, $\dot{m} = 2.06$ kg/s



Given: Smooth tube with $D = 12.7$ mm.

Find: (a) Maximum volume flow rate for laminar flow of water.
(b) Maximum average speed for laminar flow of air.
(c) Entrance lengths.

Solution: Computing equations: $Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu}$; $\frac{L_e}{D} = 0.06\, Re$

Data (Table A.10): Air $\nu = 1.46 \times 10^{-5}\ m^2/s$ (STP)

(Table A.8): Water $\nu = 1.00 \times 10^{-6}\ m^2/s$ (20C)

For laminar flow, $Re \leq 2300$, so maximum is 2300.

$$Q = \bar{V} A \qquad \bar{V} = \frac{\nu Re}{D} \qquad A = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.0127)^2 m^2 = 1.27 \times 10^{-4}\ m^2$$

For water

$$\bar{V} = 1 \times 10^{-6} \frac{m^2}{s} \times 2300 \times \frac{1}{0.0127\ m} = 0.181\ m/s$$

$$Q = 0.181 \frac{m}{s} \times 1.27 \times 10^{-4}\ m^2 = 2.29 \times 10^{-5}\ m^3/s$$ ← Water Q

$$Q = 2.29 \times 10^{-5} \frac{m^3}{s} \times 1000 \frac{L}{m^3} \times 60 \frac{s}{min} = 1.38\ L/min$$

For air

$$\bar{V} = 1.46 \times 10^{-5} \frac{m^2}{s} \times 2300 \times \frac{1}{0.0127\ m} = 2.64\ m/s$$ ← Air $\bar{V}$

Entrance length depends only on Re, so it's same for both:

$$L_e = 0.06\, Re\, D = 0.06 \times 2300 \times 0.0127\ m = 1.75\ m$$ ← L_e

Given: Incompressible flow in a circular channel.
$Re = 1800$ in a section where the channel diameter is $D = 10\ mm$

Find: (i) general expression for Re in terms of
(a) volume flow rate, Q, and channel diameter, D
(b) mass flow rate, $\dot{m}$, and channel diameter, D.
(ii) Re for same flow rate and $D = 6\ mm$

Solution:

Assume steady, incompressible flow

Definitions: $Re = \frac{\rho D \bar{V}}{\mu}$, $Q = A\bar{V}$, $\dot{m} = \rho A \bar{V}$ and $A = \frac{\pi D^2}{4}$

Then,

$$Re = \frac{\rho D \bar{V}}{\mu} = \frac{\rho D}{\mu}\frac{Q}{A} = \frac{\rho D}{\mu}\frac{Q}{\frac{\pi D^2}{4}} = \frac{4Q}{\pi D}\frac{\rho}{\mu} = \frac{4Q}{\pi D \nu}$$ ← Re

Also

$$Re = \frac{\rho D \bar{V}}{\mu} = \frac{D}{\mu}\rho\frac{\bar{V}A}{A} = \frac{D}{\mu}\frac{\dot{m}}{\frac{\pi D^2}{4}} = \frac{4\dot{m}}{\pi D \mu}$$ ← Re

From Eq (i) a

$$Q = \frac{\pi D \nu Re}{4}$$

Then for same flow rate in sections with different channel diameter,

$$D_1 Re_1 = D_2 Re_2$$

$$Re_2 = \frac{D_1}{D_2} Re_1 = \frac{10\ mm}{6\ mm} \times 1800 = 3000$$ ← Re_2

Problem 8.7

Given: Laminar flow in the entrance section of a pipe shown schematically in Fig. 8.1.

Find: Sketch centerline velocity, static pressure, and wall shear stress as functions of distance along the pipe. Explain significant features of the plots, comparing them with fully developed flow. Can the Bernoulli equation be applied anywhere in the flow field? If so, where? Explain briefly.

Discussion: The centerline velocity, static pressure, and wall shear stress variations are sketched on the next page. Each variation sketch is aligned vertically with the corresponding sections of the developing pipe flow in Fig. 8.1.

Boundary layers grow on the tube wall, reducing the velocity near the wall. The velocity reduction becomes more pronounced farther downstream. Consequently the centerline velocity must increase in the streamwise direction to carry the same mass flow rate across each section of the tube. (When laminar flow becomes fully developed, the centerline velocity becomes twice the average velocity at any cross-section.)

Frictional effects are concentrated within the boundary layers. The boundary layers do not join at the tube centerline for some distance along the tube. Therefore in the center region outside the boundary layers flow may still be considered to behave as though it were inviscid.

Flow outside the boundary layers is steady, frictionless, incompressible, and along a streamline. These are the restrictions required to apply the Bernoulli equation. Therefore the Bernoulli equation may be applied as a reasonable model for the actual flow outside the boundary layers. The Bernoulli equation predicts that pressure decreases as flow speed increases.

After the boundary layers merge at the centerline of the channel the entire flow is affected by friction. Therefore it is no longer possible to apply the Bernoulli equation.

When flow becomes fully developed the rate of change of pressure with distance becomes constant. In the entrance region the pressure falls more rapidly; the increased pressure gradient is caused by increased shear stress at the wall (larger than for fully developed flow) and by the developing velocity profile, which causes momentum flux to increase.

In fully developed flow the pressure curve becomes linear; the pressure drops the same amount for each length along the tube. The pressure distribution curve at the end of the entrance length becomes asymptotic to the linear variation for fully developed flow.

The wall shear stress initially is large, because the boundary layers are thin. The shear stress decreases as the boundary layers become thicker. At the end of the entrance length the shear stress asymptotically approaches the constant value for fully developed flow.

13-732 500 SHEETS FILLER 5 SQUARE
42-381 500 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-388 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
12-389 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

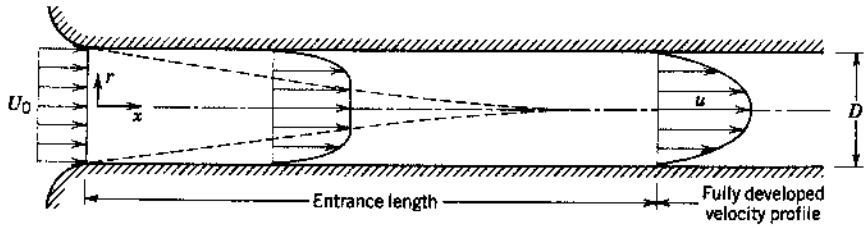
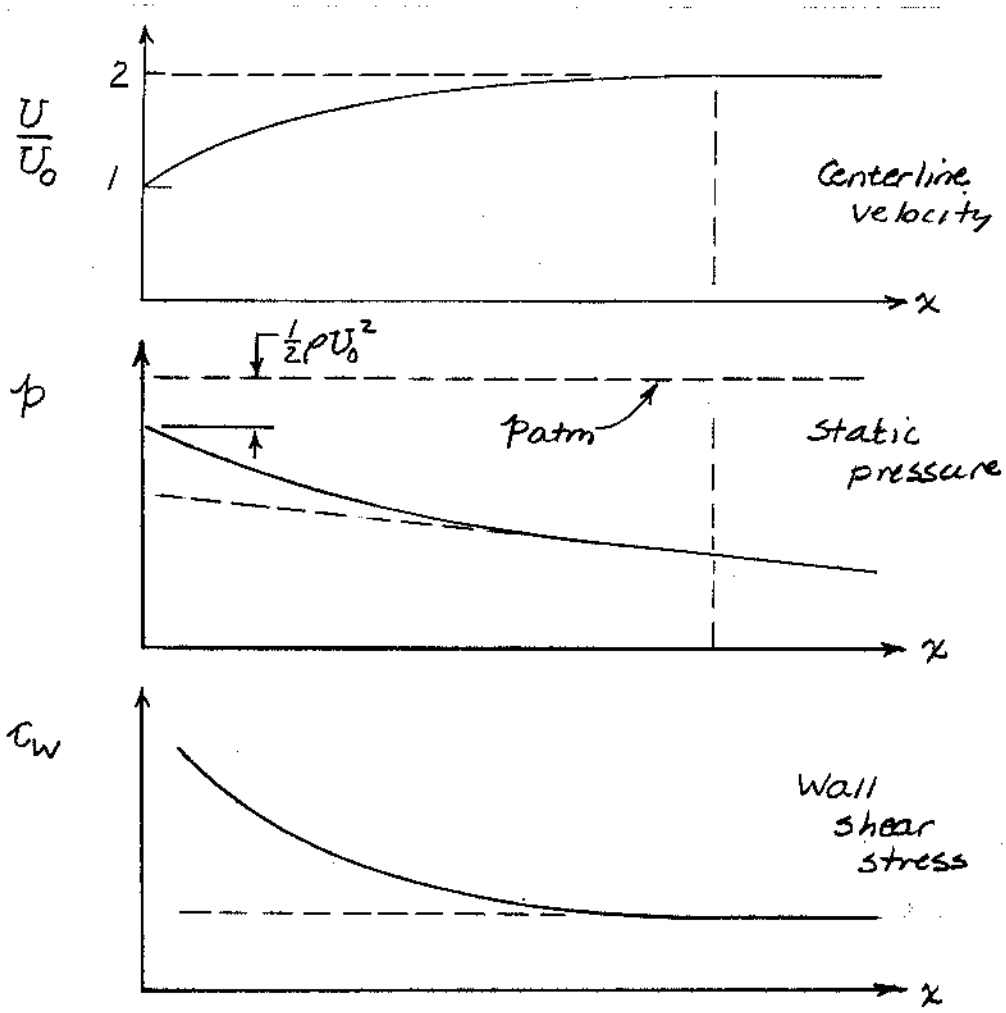
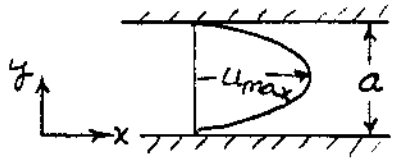


Fig. 8.1 Flow in the entrance region of a pipe.



Given: Velocity profile for flow between stationary parallel plates,

$$u = ay(h-y)$$



Find: Ratio $\bar{V}/u_{max}$.

Solution: First find u_{max}, by setting $\frac{du}{dy} = 0$.

$$\frac{du}{dy} = a(h-y) + ay(-1) = a(h-2y) \; ; \; \frac{du}{dy} = 0 \text{ at } y = \frac{h}{2}$$

$$u_{max} = u\left(\frac{h}{2}\right) = a\frac{h}{2}\left(h - \frac{h}{2}\right) = a\frac{h^2}{4}$$

From the definition of $\bar{V}$,

$$\bar{V} = \frac{Q}{A} = \frac{1}{A}\int_A u\,dA = \frac{1}{h}\int_0^h u\,dy = \frac{a}{h}\int_0^h y(h-y)\,dy$$

$$\bar{V} = \frac{a}{h}\left[\frac{1}{2}hy^2 - \frac{1}{3}y^3\right]_0^h = a\frac{h^2}{6}$$

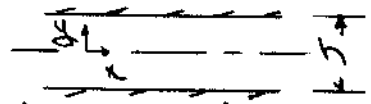
and

$$\frac{\bar{V}}{u_{max}} = \frac{ah^2}{6}\,\frac{4}{ah^2} = \frac{2}{3}$$

$\frac{\bar{V}}{u_{ma}}$

Given: Incompressible flow between parallel plates with

$$u = u_{max}(Ay^2 + By + C)$$



Find: (a) A, B, C using appropriate boundary conditions
(b) Q per unit depth
(c) $\bar{V}/u_{max}$

Solution:

(a) Available boundary conditions: (1) $y = h/2$, $u = 0$
(2) $y = -h/2$, $u = 0$
(3) $\frac{\partial u}{\partial y} = 0$, $u = u_{max}$

From B.C (3), $u_{max} = u_{max}(0 + 0 + C)$ $\therefore C = 1$

From B.C (1), $0 = u_{max}\left(A\frac{h^2}{4} + B\frac{h}{2} + 1\right)$ .....(i)

From B.C (2), $0 = u_{max}\left(A\frac{h^2}{4} - B\frac{h}{2} + 1\right)$ .....(ii)

Subtracting eq.(ii) from eq.(i) gives $0 = Bh$ $\therefore B = 0$

Adding eqs. (i) and (ii) gives $0 = u_{max}\left(A\frac{h^2}{2} + 2\right)$

then $A = -\frac{4}{h^2} = -\left(\frac{2}{h/2}\right)^2$ {units m^{-2}}

Thus, $u = u_{max}\left[-\left(\frac{2}{h/2}\right)y^2 + 1\right] = u_{max}\left[1 - \left(\frac{y}{h/2}\right)^2\right]$ u

(b) $Q = \int u\, dA$

For uniform depth b, $dA = b\,dy$

$$Q = \int_{-h/2}^{h/2} u b\, dy = 2\int_0^{h/2} u b\, dy = 2b\int_0^{h/2} u_{max}\left[1 - \left(\frac{y}{h/2}\right)^2\right] dy$$

$$Q = 2u_{max}\left[y - \frac{1}{3}\frac{y^3}{(h/2)^2}\right]_0^{h/2} = 2u_{max}\left[\frac{h}{2} - \frac{1}{3}\left(\frac{h}{2}\right)\right]$$

$$\frac{Q}{b} = 2u_{max}\frac{2}{3}\left(\frac{h}{2}\right) = \frac{2}{3}u_{max}h$$ Q/b

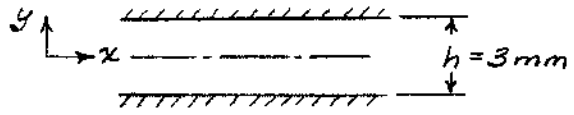
(c) Since $Q = \bar{V}A = \bar{V}bh$

$$\frac{Q}{b} = \bar{V}h = \frac{2}{3}u_{max}h$$

and $$\frac{\bar{V}}{u_{max}} = \frac{2}{3}$$ $\bar{V}/u_{max}$

Given: Laminar, fully developed flow between parallel plates

$\mu = 0.5 \frac{N \cdot s}{m^2}$; $\frac{\partial p}{\partial x} = -1200 \frac{N}{m^3}$



Find: (a) Shear stress on upper plate.
(b) Volume flow rate per unit width.

Solution: From Eq. 8.7 with $a = h$,

$$u = -\frac{h^2}{8\mu}\frac{\partial p}{\partial x}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

Then

$$\tau_{yx} = \mu\frac{du}{dy} = -\frac{h^2}{8}\frac{\partial p}{\partial x}\left(-\frac{8y}{h^2}\right) = y\frac{\partial p}{\partial x}$$

At upper surface, $y = h/2$, and

$$\tau_{yx} = \frac{0.003\ m}{2} \times -1200 \frac{N}{m^3} = -1.80\ N/m^2$$

The upper plate is a negative y surface. Thus since $\tau_{yx} < 0$, stress acts to right, in +x direction. τ_{yx} ←

The volume flow rate is

$$Q = \int_A u\,dA = \int_{-h/2}^{h/2} u\,b\,dy = 2\int_0^{h/2} u\,b\,dy = 2\left(\frac{h}{2}\right)b\int_0^1 u\,d\left(\frac{2y}{h}\right)$$

or

$$\frac{Q}{b} = h\int_0^1 u\,d\eta \quad \text{where } \eta = \frac{2y}{h} \text{ and } u = -\frac{h^2}{8\mu}\frac{\partial p}{\partial x}(1-\eta^2)$$

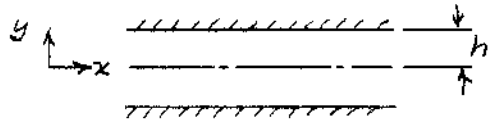
Thus $$\frac{Q}{b} = h\int_0^1 -\frac{h^2}{8\mu}\frac{\partial p}{\partial x}(1-\eta^2)\,d\eta = -\frac{h^3}{8\mu}\frac{\partial p}{\partial x}\left(\eta - \frac{1}{3}\eta^3\right)\Big|_0^1 = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x}$$

$$\frac{Q}{b} = -\frac{1}{12} \times (0.003)^3 m^3 \times \frac{m^2}{0.5\ N \cdot s} \times -1200 \frac{N}{m^3} = 5.40 \times 10^{-6}\ m^2/s$$ Q/b ←

Note $u > 0$, so flow is from left to right.

Given: Fully developed laminar flow between parallel plates.

$\mu = 0.01 \frac{lbf \cdot s}{ft^2}$; $\frac{\partial p}{\partial x} = -8 \frac{lbf}{ft^3}$



Find: (a) Shear stress on upper plate.
(b) Volume flow rate per unit width.

Solution: From Eq. 8.17 with $a = 2h$, $u = -\frac{h^2}{2\mu}\frac{\partial p}{\partial x}\left[1-\left(\frac{y}{h}\right)^2\right]$

Then $\tau_{yx} = \mu \frac{du}{dy} = -\frac{h^2}{2}\frac{\partial p}{\partial x}\left(-\frac{2y}{h^2}\right) = y\frac{\partial p}{\partial x}$

At upper surface, $y = h$, and

$$\tau_{yx} = 0.06\,in. \times -8 \frac{lbf}{ft^3} \times \frac{ft}{12\,in.} = -0.0400\ lbf/ft^2$$

The upper plate is a negative y surface. Thus since $\tau_{yx} < 0$, stress acts to right, in +x direction. ← τ_{yx}

The volume flow rate is

$$Q = \int_A u\,dA = \int_{-h}^{h} u\,b\,dy = 2\int_0^h u b\,dy = 2hb\int_0^1 u\,d(y/h)$$

or

$$\frac{Q}{b} = 2h\int_0^1 u\,d\eta \quad \text{where } \eta = y/h \text{ and } u = -\frac{h^2}{2\mu}\frac{\partial p}{\partial x}(1-\eta^2)$$

Thus

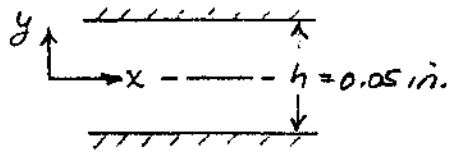
$$\frac{Q}{b} = 2h\int_0^1 -\frac{h^2}{2\mu}\frac{\partial p}{\partial x}(1-\eta^2)\,d\eta = -\frac{h^3}{\mu}\frac{\partial p}{\partial x}\left(\eta - \tfrac{1}{3}\eta^3\right)\Big|_0^1 = -\frac{2h^3}{3\mu}\frac{\partial p}{\partial x}$$

$$\frac{Q}{b} = -\frac{2}{3} \times \left(\frac{0.06}{12}\right)^3 ft^3 \times \frac{ft^2}{0.01\ lbf \cdot s} \times -8 \frac{lbf}{ft^3} = 6.67 \times 10^{-5} ft^2/s$$ ← Q/b

Note $u > 0$, so flow is from left to right.

Given: Fully developed laminar flow between parallel plates.

$\mu = 2.40 \times 10^{-5} \frac{lbf \cdot s}{ft^2}$; $\frac{\partial p}{\partial x} = -4 \frac{lbf}{ft^3}$



Find: (a) Derive and plot equation for shear stress versus y.
(b) Maximum shear stress.

Solution: From Eq. 8.7, with $a = h$, $u = -\frac{h^2}{8\mu} \frac{\partial p}{\partial x} \left[1 - \left(\frac{2y}{h}\right)^2\right]$

By symmetry, the origin for y must be located at the channel centerline. Apply Newton's law of viscosity.

$$\tau_{yx} = \mu \frac{du}{dy}$$

Assumption: Newtonian fluid

Then $$\tau_{yx} = \mu \frac{d}{dy}\left\{-\frac{h^2}{8\mu} \frac{\partial p}{\partial x}\left[1 - \left(\frac{2y}{h}\right)^2\right]\right\} = y \frac{\partial p}{\partial x}$$ τ_{yx}

For $u > 0$, $\partial p/\partial x < 0$. Thus $\tau_{yx} < 0$ for $y > 0$ and $\tau_{yx} > 0$ for $y < 0$.

On the upper plate (a minus y surface), $\tau_{yx} < 0$, so shear stress acts to the right.

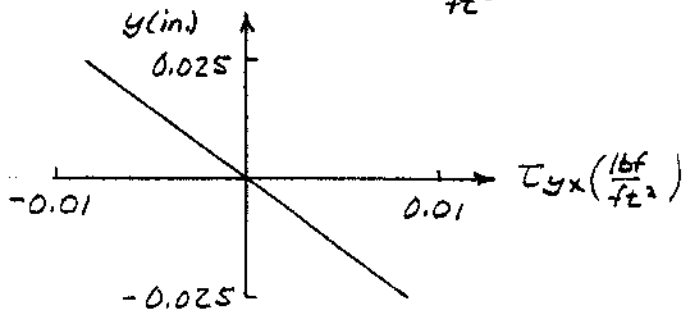
On the lower plate (a plus y surface), $\tau_{yx} > 0$, so shear stress acts to the right.

The maximum stress occurs when $y = \pm h/2$. Thus

$$\tau_{max} = \tau_{yx}\left(\frac{h}{2}\right) = \frac{h}{2} \frac{\partial p}{\partial x} = \frac{1}{2} \times 0.05 in. \times \frac{ft.}{12 in.} \times \left(-4.0 \frac{lbf}{ft^3}\right) = -0.00835 \frac{lbf}{ft^2}$$ τ_{max}

or $$\tau_{max} = \tau_{yx}\left(-\frac{h}{2}\right) = 0.00835 \frac{lbf}{ft^2}$$

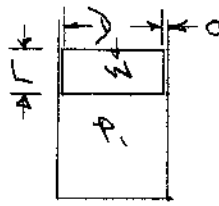
Plot:



Given: Hydraulic jack supports a load of 20,000 lbf.
piston diameter $D = 4.0$ in
piston length $L = 4.8$ in
radial clearance $a = 2 \times 10^{-3}$ in
Fluid has viscosity of SAE 30 oil at 80°F

Find: Leakage rate of fluid past the piston

Solution:



Model the flow as steady, fully-developed laminar flow between stationary parallel plates, i.e. neglect motion of the piston.

Then the leakage flow rate can be evaluated from Eq 8.6c

$$\frac{Q}{L} = \frac{a^3 \Delta p}{12 \mu L}$$

Where $L = \pi D$

$\mu = 3 \times 10^{-1} \times 2.089 \times 10^{-2} = 6.27 \times 10^{-3}$ lbf·s/ft² (at T = 26.7°C, Fig. A.2)

$\Delta p = p_1 - p_{atm}$ and $p_1 = \frac{W}{A_1} = \frac{4W}{\pi D^2}$

$$p_1 = \frac{4}{\pi} \times \frac{2 \times 10^4 \text{ lb}}{} \times \left(\frac{1}{4 \text{ in}}\right)^2 = 1592 \text{ lbf/in}^2$$

$$\therefore Q = \frac{\pi D a^3 \Delta p}{12 \mu L} = \frac{\pi}{12} \times 4 \text{ in} \times (2 \times 10^{-3} \text{ in})^3 (1592 - 14.7) \frac{\text{lb}}{\text{in}^2} \times 6.27 \times 10^{-3} \frac{\text{ft}^2}{\text{lbf}\cdot\text{s}} \times \frac{1}{4.8 \text{ in}} \times \frac{\text{ft}}{12 \text{ in}}$$

$$Q = 3.66 \times 10^{-5} \frac{\text{ft}^3}{\text{s}} \times \left(\frac{12 \text{ in}}{\text{ft}}\right)^3 = 6.32 \times 10^{-2} \text{ in}^3/\text{s}$$

Check $Re = \frac{\rho V l}{\mu} = \frac{V l}{\nu}$ $\quad \nu = 3.0 \times 10^{-3}$ ft²/s (Fig. A.3).

$$V = \frac{Q}{A} = \frac{Q}{\pi D a} = \frac{6.32 \times 10^{-2}}{\pi} \frac{\text{in}^3}{\text{s}} \times \frac{1}{4 \text{ in}} \times \frac{1}{2 \text{ in} \times 10^{-3}} = 2.52 \text{ in/s}$$

$$Re = \frac{V l}{\nu} = 2 \times 10^{-3} \text{ in} \times 2.52 \frac{\text{in}}{\text{s}} \times \frac{1}{3 \times 10^{-3}} \frac{\text{s}}{\text{ft}^2} \times \frac{\text{ft}^2}{144 \text{ in}^2}$$

$Re = 0.012$ ∴ flow definitely laminar. ✓

Piston moving down at speed U displaces liquid at rate Q, where

$$Q = \frac{\pi D^2}{4} U$$

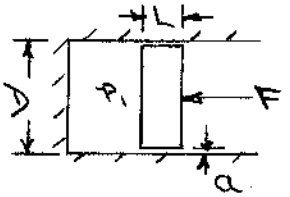
Then $U = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 6.32 \times 10^{-2} \frac{\text{in}^3}{\text{s}} \times \left(\frac{1}{4 \text{ in}}\right)^2 = 5.0 \times 10^{-3}$ in/s

Since $\frac{U}{V} = \frac{5 \times 10^{-3}}{2.52} = 2 \times 10^{-3}$, motion of piston can be neglected. ✓

Given: Oil is confined in a cylinder of diameter, $D = 4$ in, by a piston with a radial clearance, $a = 1.0 \times 10^{-3}$ in, and a length, $L = 2$ in. A steady force, $F = 5000$ lbf is applied to the piston. The oil has properties of SAE 30 oil at 120°F.

Find: Leakage rate of oil past the piston

Solution:



Model the flow as steady, fully-developed laminar flow between stationary parallel plates, i.e. neglect motion of the piston.

Then, the leakage flow rate can be evaluated from Eq. 8.6c

$$\frac{Q}{\ell} = \frac{a^3 \Delta p}{12 \mu L}$$

where $\ell = \pi D$

μ (from Fig A.2) @ $T = 120°F$ (49°C) is

$$\mu = 5.9 \times 10^{-2} \times 2.089 \times 10^{-2} = 1.23 \times 10^{-3} \ \text{lbf·s/ft}^2$$

$\Delta p = p_1 - p_{atm}$ and $\Delta p = \frac{F}{A} = \frac{4F}{\pi D^2} = \frac{4}{\pi} \times 5000 \text{lbf} \times \frac{1}{(4 \text{in})^2} = 398 \ \text{psi}$

Then

$$Q = \frac{\pi D a^3 \Delta p}{12 \mu L} = \frac{\pi}{12} \times 4 \text{in} \times (1 \times 10^{-3} \text{in})^3 \times 398 \frac{\text{lbf}}{\text{in}^2} \times 1.23 \times 10^{-3} \frac{\text{ft}^2}{\text{lbf·s}} \times \frac{1}{2 \text{in}} \times \frac{144 \text{in}^2}{\text{ft}^2}$$

$$Q = 2.44 \times 10^{-2} \ \text{in}^3/\text{s}$$ Q

Check $Re = \frac{\rho a \bar{V}}{\mu} = \frac{a \bar{V}}{\nu}$

$$\bar{V} = \frac{Q}{A} = \frac{Q}{a \ell} = \frac{Q}{a \pi D} = \frac{1}{\pi} \times 2.44 \times 10^{-2} \frac{\text{in}^3}{\text{s}} \times \frac{1}{1 \times 10^{-3} \text{in}} \times \frac{1}{4 \text{in}} = 1.94 \ \text{in/s}$$

From Fig. A.3 at $T = 49°C$, $\nu = 6 \times 10^{-5} \times 10.76 \ \text{ft}^2/\text{s} = 6.46 \times 10^{-4} \ \text{ft}^2/\text{s}$

$$\therefore Re = \frac{a \bar{V}}{\nu} = 10^{-3} \text{in} \times 1.94 \frac{\text{in}}{\text{s}} \times \frac{1}{6.46 \times 10^{-4}} \frac{\text{s}}{\text{ft}^2} \times \frac{\text{ft}^2}{144 \text{in}^2} = 0.0209$$

and flow is definitely laminar

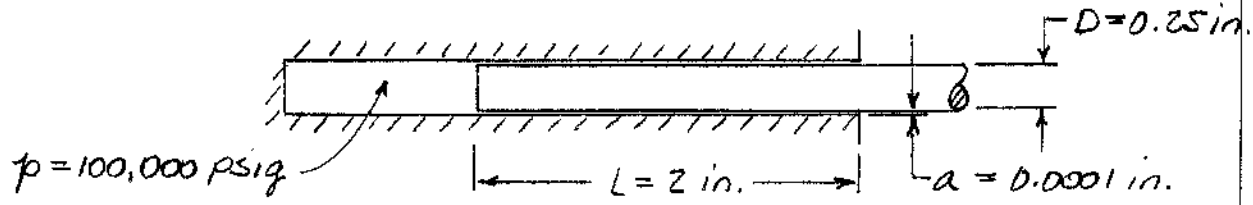
Piston moving down at speed V displaces liquid at rate, Q, where

$$Q = \frac{\pi D^2}{4} V$$

Then, $V = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 2.44 \times 10^{-2} \frac{\text{in}^3}{\text{s}} \times \frac{1}{(4 \text{in})^2} = 1.94 \times 10^{-3} \ \text{in/s}$

Since $\frac{V}{\bar{V}} = \frac{1.94 \times 10^{-3}}{1.94} = 10^{-3}$, motion of piston can be neglected.

Given: Piston-cylinder device with SAE 10W oil at 100°F.



Find: Leakage flow rate.

Solution: Computing equation: $\frac{Q}{\ell} = \frac{a^3 \Delta p}{12 \mu L}$ (8.6c)

Assumptions: (1) Laminar flow
(2) Fully developed flow ($L \gg a$)

For SAE 10W oil at 100°F (37.8°C), $\mu = 3.8 \times 10^{-2} \frac{N \cdot s}{m^2}$ (Fig. A.2)

$$\mu = 0.038 \frac{N \cdot sec}{m^2} \times \frac{1 lbf}{4.448 N} \times (0.305)^2 \frac{m^2}{ft^2} = 7.95 \times 10^{-4} \frac{lbf \cdot s}{ft^2}$$

For this configuration, $\ell = \pi D$, since $a \ll D$. Then

$$Q = \frac{a^3 \Delta p \ell}{12 \mu L} = \frac{\pi a^3 \Delta p D}{12 \mu L}$$

$$= \frac{\pi}{12} \times (10^{-4})^3 in.^3 \times 100{,}000 \frac{lbf}{in.^2} \times 0.25 in. \times \frac{ft^2}{7.95 \times 10^{-4} lbf \cdot s} \times \frac{1}{2 in.} \times 144 \frac{in.^2}{ft^2}$$

$$Q = 5.93 \times 10^{-4} in.^3/s$$ Q

Check Re to assure laminar flow

$$\bar{V} = \frac{Q}{A} = \frac{Q}{\pi D a} = \frac{1}{\pi} \times 5.93 \times 10^{-4} \frac{in.^3}{s} \times \frac{1}{0.25 in.} \times \frac{1}{0.0001 in.} = 7.55 in./s$$

SG = 0.88 (Table A.2); $\rho = SG \rho_{H_2O}$

$$Re = \frac{\rho \bar{V} a}{\mu} = \frac{SG \rho_{H_2O} \bar{V} a}{\mu}$$

$$= 0.88 (1.94 \frac{slug}{ft^3}) 7.55 \frac{in.}{s} \times 10^{-4} in. \times \frac{ft^2}{7.95 \times 10^{-4} lbf \cdot s} \times \frac{16f \cdot s^2}{slug \cdot ft} \times \frac{ft^2}{144 in.^2}$$

$Re = 0.0113 \ll 2300$ so flow is definitely laminar!

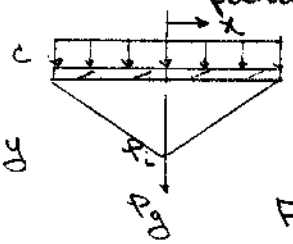


Given: Hydrostatic bearing is to support a load of $c = 3600$ lbf per ft. of length perpendicular to the diagram. Bearing is supplied with SAE 30 oil at 100°F and 100 psig.

Find: (a) Required width of the bearing pad
(b) Resulting pressure gradient
(c) Gap height, h, for $\frac{Q}{L} = 6.0 \times 10^{-4}$ ft³/min/ft

Solution:

Assume steady, fully developed, laminar flow between infinite, parallel plates.



Then the pressure over the bearing is linear, varying from p_i at $x=0$ to 0 gage at $x = W/2$

Let b = length perpendicular to diagram.

From the freebody diagram of the pad, $\Sigma F_y = 0$

$$\therefore cb = 2\int p_g\,dA = 2\int_0^{W/2} p_g\,b\,dx = 2\int_0^{W/2} p_i\left(1 - \frac{x}{W/2}\right)b\,dx$$

$$c = 2\int_0^{W/2} p_i\left(1 - \frac{2x}{W}\right)dx = 2\left[p_i\left(x - \frac{1}{W}x^2\right)\right]_0^{W/2} = 2p_i\left(\frac{W}{2} - \frac{W}{4}\right) = p_i\frac{W}{2}$$

$$\therefore W = \frac{2c}{p_i} = 2 \times \frac{3600\ \text{lb}}{\text{ft}} \times \frac{1}{100\ \text{lb/in}^2} \times \frac{\text{ft}^2}{144\ \text{in}^2} = 0.50\ \text{ft} \quad \leftarrow W$$

The corresponding pressure gradient, $\frac{dp}{dx}$, is given by

$$\frac{dp}{dx} = -\frac{\Delta p}{\Delta x} = -\frac{2\Delta p}{W} = -2 \times \frac{100\ \text{lb/in}^2}{1} \times \frac{1}{0.50\ \text{ft}} = -400\ \text{psi/ft} \quad \leftarrow \frac{dp}{dx}$$

The flow rate is given by Eq 8.6b $\quad \frac{Q}{L} = -\frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)h^3$

Then $$h = \left[-\frac{12\mu\left(Q/L\right)}{\partial p/\partial x}\right]^{1/3}$$

From Fig. A.2, $\mu = 2.30 \times 10^{-3}$ lbf-s/ft²

$$h = \left[12 \times 2.30 \times 10^{-3}\ \frac{\text{lbf-s}}{\text{ft}^2} \times 6.0 \times 10^{-4}\ \frac{\text{ft}^2}{\text{min}} \times \frac{1}{-400\ \text{lbf/in}^2} \times \frac{\text{min}}{60\ \text{s}} \times \frac{12\ \text{in}}{\text{ft}}\right]^{1/3} = 2.02 \times 10^{-3}\ \text{in} \quad \leftarrow h$$

Check Re

$$\text{Re} = \frac{\rho V h}{\mu} = \frac{2}{\nu}\frac{Q}{A} = \frac{h}{\nu}\frac{Q}{bh} = \frac{1}{\nu}\left(\frac{Q}{L}\right)$$

From Fig. A.3 $\nu = 1.29 \times 10^{-3}$ ft²/s

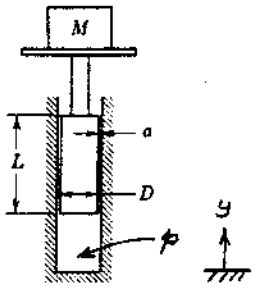
$$\therefore \text{Re} = \frac{1}{\nu}\left(\frac{Q}{L}\right) = \frac{1}{1.29 \times 10^{-3}}\ \frac{\text{s}}{\text{ft}^2} \times 6.0 \times 10^{-4}\ \frac{\text{ft}^2}{\text{min}} \times \frac{\text{min}}{60\ \text{s}} = 7.75 \times 10^{-3}$$

$\therefore$ Flow is definitely laminar.

Given: Piston-cylinder device, as shown.

$D = 6$ mm $L = 25$ mm

Liquid is SAE-30 oil at 20°C.



Find: (a) M to develop $p = 1.5$ MPa (gage)
(b) Leakage flow rate in terms of a
(c) Maximum a to provide < 1 mm/min movement.

Solution: The mass may be found from a force balance on the piston.

$$\Sigma F_y = \frac{\pi D^2}{4}(p - p_{atm}) - Mg = 0 \quad \text{so} \quad M = \frac{\pi D^2}{4g} p_{gage}$$

$$M = \frac{\pi}{4} \times (0.006)^2 m^2 \times 1.5 \times 10^6 \frac{N}{m^2} \times \frac{s^2}{9.81 m} \times \frac{kg \cdot m}{N \cdot s^2} = 4.32 \text{ kg} \quad \text{M}$$

The leakage flow rate may be evaluated for flow between flat plates. From Eq. 8.6c, neglecting motion of the piston,

$$\frac{Q}{l} = \frac{a^3 \Delta p}{12 \mu L} \quad \text{or, since } l = \pi D, \quad Q = \frac{\pi}{12} \frac{a^3 \Delta p D}{\mu L} \sim a^3 \quad \text{Q}$$

The piston, moving downward at speed, v, displaces liquid at rate

$$Q = \frac{\pi D^2}{4} v = \frac{\pi}{4} (0.006)^2 m^2 \times 0.001 \frac{m}{min} \times \frac{min}{60 s} = 4.71 \times 10^{-10} m^3/s$$

Then, with $\mu = 0.42$ N·sec/m² (at 20°C, Fig. A.2),

$$a = \left[\frac{12 \mu Q L}{\pi D \Delta p} \right]^{1/3} = \left[\frac{12}{\pi} \times 0.42 \frac{N \cdot s}{m^2} \times 4.71 \times 10^{-10} \frac{m^3}{s} \times 0.025 m \times \frac{1}{0.006 m} \times \frac{m^2}{1.5 \times 10^6 N} \right]^{1/3}$$

$$a = 1.28 \times 10^{-5} \text{ m} \quad (12.8 \mu m) \quad \text{a}$$

Check assumptions: $\bar{V} = \frac{Q}{A} = \frac{Q}{\pi D a} = \frac{1}{\pi} \times 4.71 \times 10^{-10} \frac{m^3}{s} \times \frac{1}{0.006 m} \times \frac{1}{1.28 \times 10^{-5} m} = 1.95 \frac{mm}{s}$

Thus $\frac{v}{\bar{V}} = 1 \frac{mm}{min} \times \frac{sec}{1.95 mm} \times \frac{min}{60 s} = 0.00855 < 0.01$

Therefore piston motion is negligible.

Also $Re = \frac{\bar{V} a}{\nu}$; $\nu = \frac{\mu}{\rho} = \frac{\mu}{SG \rho_{H_2O}}$. From Table A.2 (Appendix A), SG = 0.92

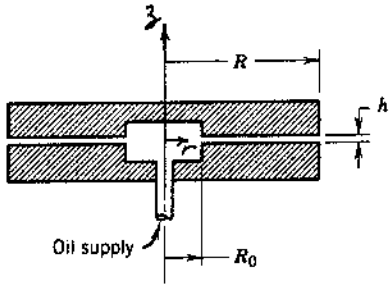
$$\nu = 0.42 \frac{N \cdot s}{m^2} \times \frac{m^3}{(0.92) 1000 kg} \times \frac{kg \cdot m}{N \cdot s^2} = 4.57 \times 10^{-4} m^2/s .$$

$$Re = 1.95 \times 10^{-3} \frac{m}{s} \times 1.28 \times 10^{-5} m \times \frac{s}{4.57 \times 10^{-4} m^2} = 5.46 \times 10^{-5} \ll 1$$

Therefore flow is surely laminar!

Given: Viscous flow in narrow gap between parallel disks, as shown.

Flow rate is Q, accelerations are small.
Velocity profile same as fully developed.



Find: (a) Expression for $\bar{V}(r)$, (b) dp/dr in gap
(c) Expression for $p(r)$.
(d) Show net force to hold upper plate is

$$F = \frac{3\mu Q R^2}{h^3}\left[1-\left(\frac{R_0}{R}\right)^2\right]$$

Solution: From the definition of mean velocity, $Q = \bar{V} 2\pi r h$ so $\bar{V} = \frac{Q}{2\pi r h}$ $\bar{V}(r)$

The pressure change with radius can be evaluated by analogy to Eq. 8.66

$$\frac{Q}{\ell} = -\frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)h^3 \quad \text{with } \ell = 2\pi r \quad \text{so} \quad \frac{Q}{2\pi r} = -\frac{1}{12\mu}\left(\frac{\partial p}{\partial r}\right)h^3$$

Thus

$$\frac{dp}{dr} = -\frac{6\mu Q}{\pi h^3 r}$$ $\frac{dp}{dr}$

Integrating to find $p(r)$,

$$\int_p^{p_{atm}} dp = p_{atm} - p = \int_r^R -\frac{6\mu Q}{\pi h^3 r}\,dr = -\frac{6\mu Q}{\pi h^3}\ln r\Big]_r^R = \frac{6\mu Q}{\pi h^3}\ln(r/R)$$

Thus $p(r) = p_{atm} - \frac{6\mu Q}{\pi h^3}\ln(r/R)$ $(R_0 < r < R)$; $p = p_0$ $r < R_0$ $p(r)$

The force on the upper plate is $dF_z = (p(r) - p_{atm})2\pi r\,dr$

Integrating and using gage pressures (note $p_{0g} = -\frac{6\mu Q}{\pi h^3}\ln\left(\frac{R_0}{R}\right)$)

$$F_z = p_0\pi R_0^2 + \int_{R_0}^R p(r)\,2\pi r\,dr = p_0\pi R_0^2 + 2\pi R^2\int_{R_0/R}^1 p(r)\left(\frac{r}{R}\right)d\left(\frac{r}{R}\right)$$

$$= p_0\pi R_0^2 + 2\pi R^2\int_{R_0/R}^1 -\frac{6\mu Q}{\pi h^3}\ln\left(\frac{r}{R}\right)\left(\frac{r}{R}\right)d\left(\frac{r}{R}\right) = p_0\pi R_0^2 - \frac{12\mu Q R^2}{h^3}\left(\frac{r}{R}\right)^2\left[\frac{1}{2}\ln\left(\frac{r}{R}\right) - \frac{1}{4}\right]_{R_0/R}^1$$

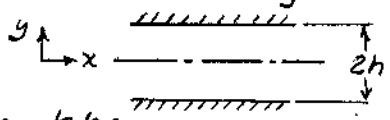
$$= p_0\pi R_0^2 - \frac{12\mu Q R^2}{h^3}\left\{(1)\left[\frac{1}{2}(0) - \frac{1}{4}\right] - \left(\frac{R_0}{R}\right)^2\left[\frac{1}{2}\ln\left(\frac{R_0}{R}\right) - \frac{1}{4}\right]\right\}$$

$$= -\frac{6\mu Q R^2}{h^3}\left(\frac{R_0}{R}\right)^2\ln\left(\frac{R_0}{R}\right) - \frac{6\mu Q R^2}{h^3}\left[-\frac{1}{2} - \left(\frac{R_0}{R}\right)^2\ln\left(\frac{R_0}{R}\right) + \frac{1}{2}\left(\frac{R_0}{R}\right)^2\right]$$

$$F_z = \frac{3\mu Q R^2}{h^3}\left[1-\left(\frac{R_0}{R}\right)^2\right]$$ F_z

Given: Power-law model for non-Newtonian liquid, $\tau_{yx} = k\left(\frac{du}{dy}\right)^n$

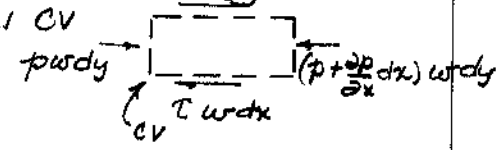
Find: Show $u = \left(\frac{h}{k}\frac{\Delta p}{L}\right)^{1/n}\frac{nh}{n+1}\left[1-\left(\frac{y}{h}\right)^{\frac{n+1}{n}}\right]$



for fully developed laminar flow between plates.

Plot: Profiles u/U vs. y/h for $n = 0.7$, 1.0, and 1.3 ($U = u_{max}$).

Solution: Apply momentum equation to differential CV



Basic equation: $\sum F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\,d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$ (=0(1), =0(2), =0(3))

Assumptions: (1) Horizontal flow
(2) Steady flow
(3) Fully developed flow

Then

$$p\,w\,dy + \left(\tau + \frac{\partial\tau}{\partial y}dy\right)w\,dx - \left(p+\frac{\partial p}{\partial x}dx\right)w\,dy - \tau\,w\,dx = 0 \quad \text{or} \quad \frac{\partial\tau}{\partial y} = \frac{\partial p}{\partial x}$$

Since $\tau = \tau(y)$ and $p = p(x)$, then $\frac{d\tau}{dy} = \frac{\partial p}{\partial x} = \text{constant}$ and $\tau = y\frac{\partial p}{\partial x}$ or

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^n = y\frac{\partial p}{\partial x} = -y\frac{\Delta p}{L}$$

Thus $$\frac{du}{dy} = -\left(\frac{1}{k}\frac{\Delta p}{L}\right)^{1/n} y^{1/n}$$

Integrating

$$u = -\left(\frac{1}{k}\frac{\Delta p}{L}\right)^{1/n}\frac{1}{1/n+1}\,y^{1/n+1} + C = -\left(\frac{1}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}}\frac{n}{n+1}\,y^{\frac{n+1}{n}} + C$$

But $u = 0$ at $y = h$, so

$$C = \left(\frac{1}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}}\frac{n}{n+1}\,h^{\frac{n+1}{n}}$$

and

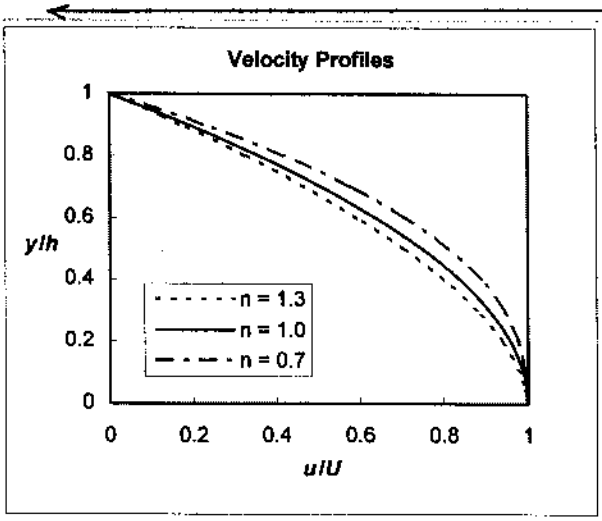
$$u = \left(\frac{1}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}}\frac{n}{n+1}\,h^{\frac{n+1}{n}}\left[1-\left(\frac{y}{h}\right)^{\frac{n+1}{n}}\right]$$

or

$$u = \left(\frac{h}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}}\frac{nh}{n+1}\left[1-\left(\frac{y}{h}\right)^{\frac{n+1}{n}}\right]$$

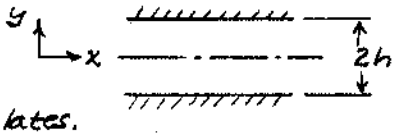
u

	n = 0.7	n = 1.0	n = 1.3
y/h	u/U	u/U	u/U
0	1	1	1
0.03	1.000	0.999	0.998
0.06	0.999	0.996	0.993
0.1	0.996	0.990	0.983
0.2	0.980	0.960	0.942
0.3	0.946	0.910	0.881
0.4	0.892	0.840	0.802
0.5	0.814	0.750	0.707
0.6	0.711	0.640	0.595
0.7	0.580	0.510	0.468
0.8	0.418	0.360	0.326
0.9	0.226	0.190	0.170
1	0	0	0



Given: Power-law model for non-Newtonian liquid, $\tau_{yx} = k\left(\frac{du}{dy}\right)^n$

Find: Show $\frac{u}{\bar{V}} = \frac{2n+1}{n+1}\left[1-\left(\frac{y}{h}\right)^{\frac{n+1}{n}}\right]$

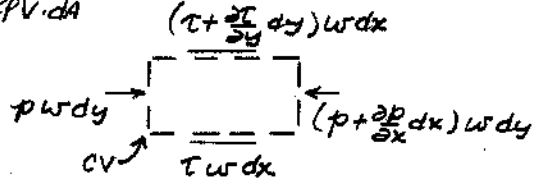


for fully developed laminar flow between plates.

Plot: Profiles $u/\bar{V}$ vs. y/h for $n = 0.7$, 1.0, and 1.3 ($\bar{V} = u_{avg}$).

Solution: Apply momentum equation to differential CV

Basic equation: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\,d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$ (=0(1), =0(2), =0(3))



Assumptions: (1) Horizontal flow
(2) Steady flow
(3) Fully developed flow

Then $p w dx + (\tau + \frac{\partial \tau}{\partial y}dy)w dx - (p+\frac{\partial p}{\partial x}dx)w dy - \tau w dx = 0$ or $\frac{\partial \tau}{\partial y} = \frac{\partial p}{\partial x}$

Since $\tau = \tau(y)$ and $p = p(x)$, then $\frac{d\tau}{dy} = \frac{\partial p}{\partial x} = \text{constant}$ and $\tau = y\frac{\partial p}{\partial x}$ or

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^n = y\frac{\partial p}{\partial x} = -y\frac{\Delta p}{L}$$

Thus $\frac{du}{dy} = -\left(\frac{1}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}} y^{\frac{1}{n}}$

Integrating

$$u = -\left(\frac{1}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}} \frac{1}{\frac{1}{n}+1} y^{\frac{1}{n}+1} + c = -\left(\frac{1}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}} \frac{n}{n+1} y^{\frac{n+1}{n}} + c$$

But $u = 0$ at $y = h$, so $c = \left(\frac{1}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}} \frac{n}{n+1} h^{\frac{n+1}{n}}$

and

$$u = \left(\frac{1}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}} \frac{n}{n+1} h^{\frac{n+1}{n}}\left[1-\left(\frac{y}{h}\right)^{\frac{n+1}{n}}\right] = \left(\frac{h}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}} \frac{nh}{n+1}\left[1-\left(\frac{y}{h}\right)^{\frac{n+1}{n}}\right]$$

The volume flow rate is

$$Q = \bar{V}A = \bar{V}(w\,2h) = \int_A u\,dA = \int_{-h}^{h} u\,w\,dy = 2wh\int_0^1 u\,d\left(\frac{y}{h}\right)$$

or

$$\bar{V} = \int_0^1 u\,d\left(\frac{y}{h}\right) = \left(\frac{h}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}} \frac{nh}{n+1}\int_0^1\left[1-\left(\frac{y}{h}\right)^{\frac{n+1}{n}}\right] d\left(\frac{y}{h}\right) = \left(\frac{h}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}} \frac{nh}{n+1}\left[\left(\frac{y}{h}\right) - \frac{1}{\frac{n+1}{n}+1}\left(\frac{y}{h}\right)^{\frac{n+1}{n}+1}\right]_0^1$$

$$\bar{V} = \left(\frac{h}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}} \frac{nh}{n+1}\left[1-\frac{1}{\frac{2n+1}{n}}\right] = \left(\frac{h}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}} \frac{nh}{n+1}\frac{n+1}{2n+1} = \left(\frac{h}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}} \frac{nh}{2n+1}$$

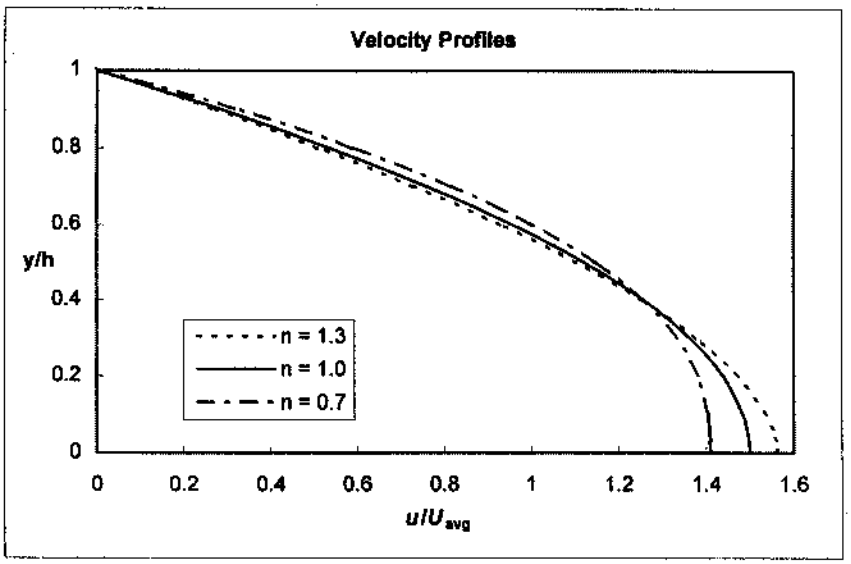
Thus

$$\frac{u}{\bar{V}} = \frac{2n+1}{n+1}\left[1-\left(\frac{y}{h}\right)^{\frac{n+1}{n}}\right] \quad \leftarrow u/\bar{V}$$

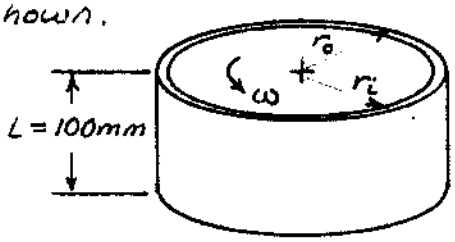
and

$$Q = \bar{V}\,2wh = \left(\frac{h}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}} \frac{2nwh^2}{2n+1} \quad \leftarrow Q$$

	n = 0.7	n = 1.0	n = 1.3
y/h	***u/U***	***u/U***	***u/U***
0	1.41	1.50	1.57
0.03	1.41	1.50	1.56
0.06	1.41	1.49	1.55
0.1	1.41	1.49	1.54
0.2	1.38	1.44	1.47
0.3	1.34	1.37	1.38
0.4	1.26	1.26	1.26
0.5	1.15	1.13	1.11
0.6	1.00	0.960	0.931
0.7	0.818	0.765	0.732
0.8	0.591	0.540	0.510
0.9	0.319	0.285	0.266
1	0	0	0



Given: Sealed journal bearing rotating as shown.



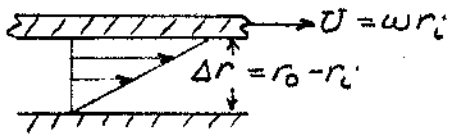
$r_o = 26\,mm$, $r_i = 25\,mm$

Gap contains oil in laminar motion with linear velocity profile.

$\omega = 2800\,rpm$ and Torque, $T = 0.2\,N\cdot m$

Find: (a) Viscosity of oil
(b) Will torque increase or decrease with time? Why?

Solution: "Unfold" bearing since gap is small, and consider as flow between parallel plates. Apply Newton's law of viscosity.



Basic equation: $\tau_{yx} = \mu \frac{du}{dy}$

Assumption: Linear velocity profile

Then $\tau_{yx} = \mu \frac{U}{\Delta r} = \frac{\mu \omega r_i}{\Delta r}$

and

$$T = r_i \left(2\pi r_i L \tau_{yx}\right) = 2\pi r_i^2 L \tau_{yx} = \frac{2\pi \mu \omega r_i^3 L}{\Delta r}$$

Solving, $\mu = \frac{\Delta r\, T}{2\pi \omega r_i^3 L}$

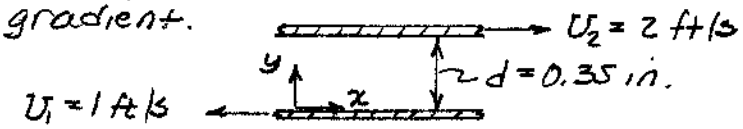
$$\mu = \frac{1}{2\pi} \times 0.001\,m \times 0.2\,N\cdot m \times \frac{min}{2800\,rev} \times \frac{1}{(0.025)^3 m^3} \times \frac{1}{0.1\,m} \times \frac{rev}{2\pi\,rad} \times 60\,\frac{s}{min}$$

$$\mu = 0.0695\ N\cdot s / m^2$$

μ

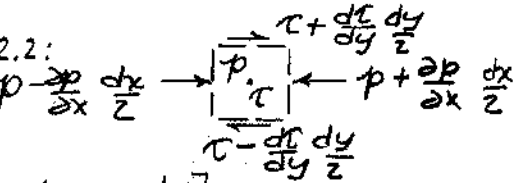
Bearing is sealed, so oil temperature will increase as energy is dissipated by friction. For liquids, μ decreases as T increases. Thus torque will <u>decrease</u>, since it is proportional to μ.

Given: Fully developed laminar flow between parallel plates with no pressure gradient.



Find: (a) Expression for velocity profile in gap.
(b) Volume flow rate per unit depth passing cross-section.

Solution: Use analysis of Section 8-2.2:



Sum forces in x direction:

$$\left[\tau + \frac{d\tau}{dy}\frac{dy}{2} - \left(\tau - \frac{d\tau}{dy}\frac{dy}{2}\right)\right] b\,dx + \left[p - \frac{\partial p}{\partial x}\frac{dx}{2} - \left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)\right] b\,dy = 0$$

Simplifying $\frac{d\tau}{dy} - \frac{dp}{dx} = 0$ so $\mu \frac{d^2u}{dy^2} = 0$

Integrating twice $u = C_1 y + C_2$

Boundary conditions: $y = 0$, $u = -U_1$; $C_2 = -U_1$

$y = d$, $u = U_2$; $U_2 = C_1 d - U_1$, so $C_1 = \frac{U_1 + U_2}{d}$

Thus $u = (U_1 + U_2)\frac{y}{d} - U_1$

$u(\text{m/sec}) = 3\frac{y}{d} - 1$ ← Profile $u(y)$

The volume flow rate is

$$Q = \int_A u\,dA = \int_0^d u\,b\,dy = \int_0^d \left[(U_1 + U_2)\frac{y}{d} - U_1\right] b\,dy = b\left[(U_1 + U_2)\frac{y^2}{2d} - U_1 y\right]_0^d$$

$$Q = b\left[(U_1 + U_2)\frac{d}{2} - U_1 d\right] = b(U_2 - U_1)\frac{d}{2} = bd\left(\frac{U_2 - U_1}{2}\right)$$

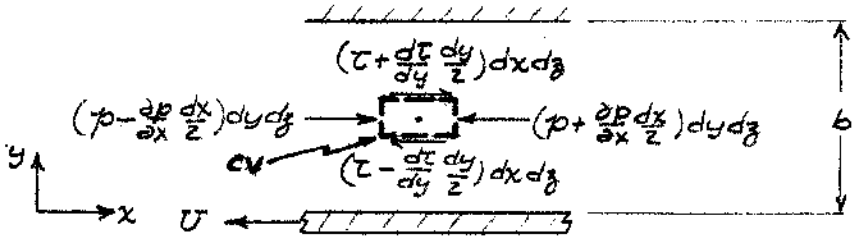
So

$$\frac{Q}{b} = \frac{1}{2} \times 0.35\ \text{in.} \times (2-1)\frac{\text{ft}}{\text{s}} \times \frac{\text{ft}}{12\ \text{in.}} = 0.0146\ \text{ft}^3/\text{s}\ /\ \text{ft}$$ ← $\frac{Q}{b}$

Given: Water at 60°C flows between large flat plates.

$U = 0.3$ m/s

$b = 3$ mm



Find: Pressure gradient required for zero net flow at a section.

Solution: Apply momentum equation using CV and coordinates shown.

Basic equations: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho dV + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$, $\tau = \tau_{yx} = \mu\frac{du}{dy}$

($F_{Sx} = 0$ (1), $F_{Bx} = 0$ (2), $\frac{\partial}{\partial t}\int_{CV} u\rho dV = 0$ (3))

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Fully-developed flow
(4) Newtonian fluid

Then $F_{Sx} = 0$. Substituting the force terms (see page 315 for details) gives

$$\frac{\partial p}{\partial x} = \frac{d\tau_{yx}}{dy} = \frac{d}{dy}\left(\mu\frac{du}{dy}\right) = \mu\frac{d^2u}{dy^2} \quad \text{or} \quad \frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

Integrating twice,

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + C_1 y + C_2$$

To evaluate the constants C_1 and C_2, we must use the boundary conditions. At $y = 0$, $u = -U$, so $C_2 = -U$. At $y = b$, $u = 0$, so

$$0 = \frac{1}{2\mu}\frac{\partial p}{\partial x}b^2 + C_1 b - U \quad \text{or} \quad C_1 = \frac{U}{b} - \frac{1}{2\mu}\frac{\partial p}{\partial x}b$$

Thus

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - by) + U\left(\frac{y}{b} - 1\right)$$

To find the flowrate, we integrate

$$\frac{Q}{w} = \int_0^b u\,dy = \int_0^b \left[\frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - by) + U\left(\frac{y}{b} - 1\right)\right]dy = -\frac{1}{12\mu}\frac{\partial p}{\partial x}b^3 - \frac{Ub}{2}$$

For $Q = 0$, with $\mu = 4.63\times10^{-4}\ \frac{N\cdot s}{m^2}$ from Table A.8,

$$\frac{\partial p}{\partial x} = -\frac{6U\mu}{b^2} = -6\times\frac{0.3\ m}{s}\times 4.63\times10^{-4}\frac{N\cdot s}{m^2}\times\frac{1}{(0.003)^2 m^2} = -92.6\ N/m^2\cdot m$$

$\frac{\partial p}{\partial x}$

Thus pressure must <u>decrease</u> in x direction for zero net flowrate. ←

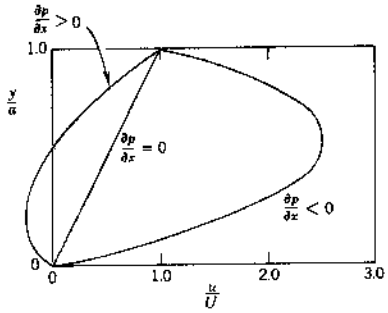
Given: Dimensionless velocity profile for fully developed laminar flow between infinite parallel plates with upper plate moving at constant speed U (Fig. 8.5).

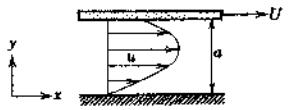
Find: Plot the shear stress as a function of nondimensional distance y/a for each of the three cases shown in the figure.

Solution: The velocity profile (Eq. 8.8) may be nondimensionalized as

$$u = \frac{Uy}{a} + \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right] \quad \text{or} \quad \frac{u}{U} = \frac{y}{a} + \frac{a^2}{2\mu U}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right]$$

The parameter $C = \frac{a^2}{2\mu U}\left(\frac{\partial p}{\partial x}\right)$ controls the profile shape. For zero pressure gradient the profile is linear. The favorable pressure gradient has $C \approx -8$ and the adverse pressure gradient has $C \approx +2$, as shown below.





From Eq. 8.9a,

$$\tau_{yx} = \mu\frac{U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{y}{a} - \frac{1}{2}\right] \quad \text{or} \quad \frac{\tau_{yx}}{\mu U/a} = 1 + \frac{a^2}{\mu U}\left(\frac{\partial p}{\partial x}\right)\left[\frac{y}{a} - \frac{1}{2}\right]$$

Introducing C,

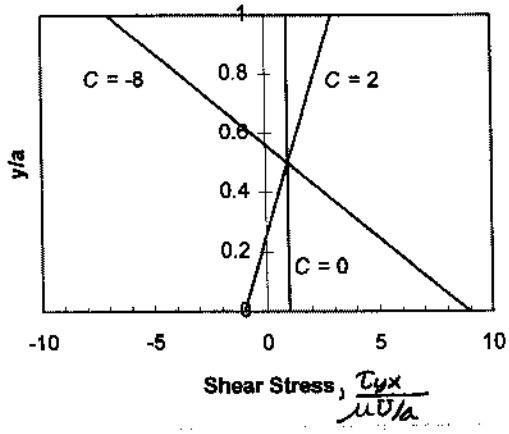
$$\frac{\tau_{yx}}{\mu U/a} = 1 + C\left[2\left(\frac{y}{a}\right) - 1\right]$$

$\frac{\tau_{yx}}{\mu U/a}$

This expression is linear; it may be plotted as shown below:

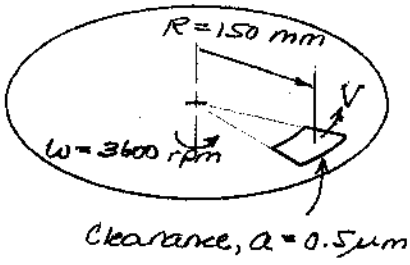
Tabulated values are:

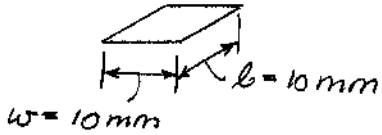
C	y/a	$\frac{a\tau_{yx}}{\mu U}$
0	0	1
	1	1
+2	0	−1
	1	3
−8	0	9
	1	−7



Plot

Given: Record-write head for computer disk-storage system.





Find: (a) Reynolds number in gap
(b) Viscous shear stress
(c) Power to overcome viscous shear.

Solution: $V = R\omega = 0.15\ m \times 3600\ \frac{rev}{min} \times 2\pi\ \frac{rad}{rev} \times \frac{min}{60\ sec} = 56.5\ m/s$

$Re = \frac{\rho V a}{\mu} = \frac{Va}{\nu} = 56.5\ \frac{m}{s} \times 0.5 \times 10^{-6}\ m \times \frac{s}{1.46 \times 10^{-5}\ m^2} = 1.94$ Re

(Table A.10 at T = 15°C)

$\tau = \mu \frac{du}{dy} = \mu \frac{V}{a}$ for small gap

Assuming standard conditions, $\mu = 1.79 \times 10^{-5}\ kg/m \cdot s$

$\tau = 1.79 \times 10^{-5}\ \frac{kg}{m \cdot s} \times 56.5\ \frac{m}{s} \times \frac{1}{0.5 \times 10^{-6}\ m} = 2.02\ kN/m^2$ τ

The force is $F = \tau A = \tau w l$, and the torque is $T = FR = \tau w l R$.

The power dissipation rate is

$P = T\omega = \tau l w R \omega$

$= 2.02 \times 10^3\ \frac{N}{m^2} \times 0.01\ m \times 0.01\ m \times 0.150\ m \times 3600\ \frac{rev}{min} \times 2\pi\ \frac{rad}{rev} \times \frac{min}{60\ s} \times \frac{W \cdot s}{N \cdot m}$

$P = 11.4\ W$ P

Given: Steady, incompressible, fully developed laminar flow down an incline (at angle θ).
Velocity profile (Example Problem 5.9) is

$$u = \frac{\rho g \sin\theta}{\mu}\left(hy - y^2/2\right)$$

Find: Kinematic viscosity ν of liquid for $h = 0.8$ mm, $\theta = 30°$ and $u_{max} = 15.7$ mm/s

Plot: the velocity profile

Solution:

$$u = \frac{\rho g \sin\theta}{\mu}\left(hy - y^2/2\right) = \frac{g \sin\theta}{\nu}\left(hy - y^2/2\right)$$

$u = u_{max}$ at $y = h$

$$\therefore u_{max} = \frac{g \sin\theta}{\nu}\left(h^2 - h^2/2\right) = \frac{g \sin\theta h^2}{2\nu}$$

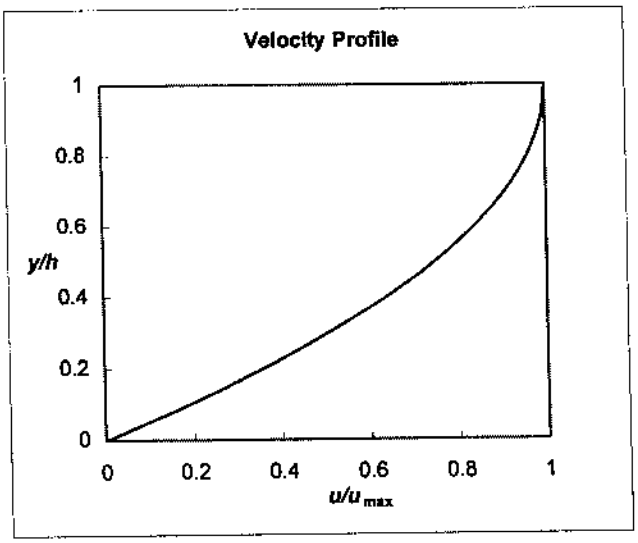
and

$$\nu = \frac{g \sin\theta h^2}{2 u_{max}} = \frac{\sin 30°}{2} \times 9.81 \frac{m}{s^2} \times (0.8 \times 10^{-3} m)^2 \times \frac{s}{15.7 \times 10^{-3} m}$$

$$\nu = 1.00 \times 10^{-4} \ m^2/s \quad \leftarrow \nu$$

Plot $\frac{u}{u_{max}} = \frac{g \sin\theta}{\nu}\left(hy - y^2/2\right) \times \frac{2\nu}{g \sin\theta h^2} = 2\frac{y}{h} - \left(\frac{y}{h}\right)^2$

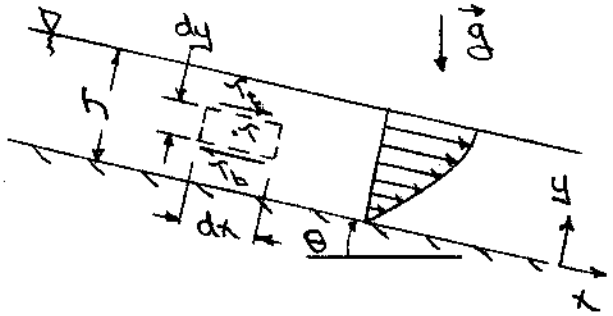
u/u_{max}	y/h
0	0
0.0396	0.02
0.098	0.05
0.190	0.1
0.360	0.2
0.510	0.3
0.640	0.4
0.750	0.5
0.840	0.6
0.910	0.7
0.960	0.8
0.990	0.9
1.00	1.0



Given: Fully developed, laminar flow of an incompressible liquid down an inclined surface. The thickness, h, of the liquid layer is constant.

Find: (a) the velocity profile by use of a suitably chosen differential control volume. (b) volume flow rate, Q/w

Solution: Flow is fully developed, so $u = u(y)$ and $\tau = \tau(y)$.



Expand τ in a Taylor series about the center of the differential CV

$$\tau_t = \tau + \frac{d\tau}{dy}\frac{dy}{2}$$

$$\tau_b = \tau + \frac{d\tau}{dy}\left(-\frac{dy}{2}\right)$$

The boundary conditions on the velocity profile are:

@ $y = 0$, $u = 0$ (no slip).

@ $y = h$, $\frac{du}{dy} = 0$ (no shear stress)

Apply the x component of the momentum equation to the differential CV shown

$$F_{Sx} + F_{Bx} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\,\rho\,d\forall}^{0(1)} + \cancel{\int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}}^{0(2)}$$

Assumptions: (1) steady flow
(2) fully developed flow, so u and τ are functions of y only
(3) no variation of pressure in the x direction

Then

$$F_{Sx} + F_{Bx} = 0 = \left(\tau + \frac{d\tau}{dy}\frac{dy}{2}\right)dx\,dz - \left(\tau - \frac{d\tau}{dy}\frac{dy}{2}\right)dx\,dz + \rho g \sin\theta\, dx\,dy\,dz$$

or

$$\frac{d\tau}{dy} = -\rho g \sin\theta$$

Integrating, $\tau = -\rho g \sin\theta\, y + C_1$

But $\tau = 0$ @ $y = h$, $\therefore C_1 = \rho g \sin\theta\, h$, and

$$\frac{du}{dy} = \frac{\rho g \sin\theta}{\mu}(h - y)$$

Integrating again, $u = \frac{\rho g \sin\theta}{\mu}\left(hy - \frac{y^2}{2}\right) + C_2$

At $y = 0$, $u = 0$, so $C_2 = 0$ and hence

$$u = \frac{\rho g \sin\theta}{\mu}\left(hy - \frac{y^2}{2}\right) \longleftarrow u$$

$$Q/w = \int_0^h u\,dy = \frac{\rho g \sin\theta}{\mu}\int_0^h \left(hy - \frac{y^2}{2}\right)dy = \frac{\rho g \sin\theta}{\mu}\left[\frac{hy^2}{2} - \frac{y^3}{6}\right]_0^h$$

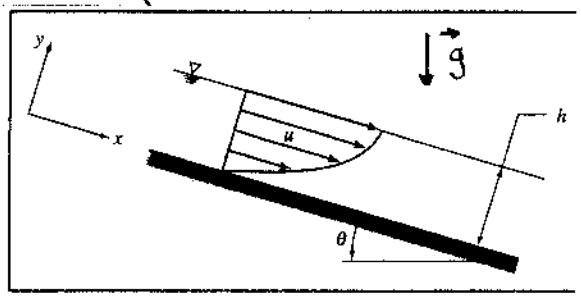
$$Q/w = \rho g \sin\theta\, h^3/3\mu \longleftarrow Q/w$$

Given: Velocity distribution for flow of a thin viscous film down a plane surface, inclined at angle θ is given (from Example 5.9) as

$$u = \frac{\rho g \sin\theta}{\mu}\left(hy - y^2/2\right)$$

$h = 5.63\ mm$, SG liquid $= 1.26$, $\mu = 1.40\ N\cdot s/m^2$

Find: (a) expression for shear stress distribution in film
(b) maximum shear stress in film; indicate direction
(c) volume flow rate (mm^3/s) per mm of width
(d) film Re based on $\bar{V}$



Solution:

(a) $\tau_{yx} = \mu \frac{du}{dy} = \rho g \sin\theta\,(h - y)$

(b) τ_{yx} is a maximum at $y = 0$

$$\tau_{yx_{max}} = \rho g \sin\theta\, h = SG\,\rho_{H_2O}\, g \sin\theta\, h$$

$$\tau_{yx_{max}} = 1.26\times10^3\ \frac{kg}{m^3}\times 9.81\ \frac{m}{s^2}\times \sin 30^\circ \times 5.63\times10^{-3}\ m = 34.8\ N/m^2$$ $\tau_{yx_{max}}$

- stress on wall (+y surface) is in + x direction
- " " film (-y surface) is in - x direction.

(c) $$Q = \int \vec{u}\cdot d\vec{A} = \int_0^h \frac{\rho g \sin\theta}{\mu}\left(hy - y^2/2\right) w\, dy$$

$$= \frac{\rho g \sin\theta}{\mu} w\left[\frac{hy^2}{2} - \frac{y^3}{6}\right]_0^h = \frac{\rho g \sin\theta\, w h^3}{3\mu}$$

$$\frac{Q}{w} = \frac{1}{3}\times 1.26\times10^3\ \frac{kg}{m^3}\times 9.81\ \frac{m}{s^2}\times \sin 30^\circ \times (5.63\ mm)^3 \times \frac{1}{1.40\ N\cdot s/m^2}\times\frac{N\cdot s^2}{kg\cdot m}\times 10^3\ \frac{mm}{m}$$

$$\frac{Q}{w} = 263\ mm^3/s/mm \qquad Q/w$$

(d) $$\bar{V} = \frac{Q}{A} = \frac{Q}{wh} = 263\ \frac{mm^2}{s}\times\frac{1}{5.63\ mm} = 46.7\ mm/s$$

$$Re = \frac{\rho \bar{V} h}{\mu} = 1.26\times10^3\ \frac{kg}{m^3}\times 46.7\times10^{-3}\ \frac{m}{s}\times 5.63\times10^{-3}\ m\times\frac{1}{1.40\ N\cdot s/m^2}\times\frac{N\cdot s^2}{kg\cdot m}$$

$$Re = 0.236 \qquad Re$$

Given: Fully developed flow between parallel plates with the upper plate moving (Fig. 8.5). $U = 2$ m/s; $a = 2.5$ mm.

Find: (a) Q/ℓ for $\partial p/\partial x = 0$
(b) τ_{yx} at $y = 0$ for $\partial p/\partial x = 0$
(c) Plot τ_{yx} vs. y for $\partial p/\partial x = 0$
(d) Will Q increase or decrease if $\partial p/\partial x > 0$?
(e) $\partial p/\partial x$ for $\tau_{yx} = 0$ at $y = 0.25\,a$, if fluid is air
(f) Plot τ_{yx} vs. y for this case.

Solution: The velocity profile is given by Eq. 8.8: $u = \frac{Uy}{a} + \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right]$

(a) For $\frac{\partial p}{\partial x} = 0$, $u = \frac{Uy}{a}$; $\frac{Q}{\ell} = \int_0^a u\,dy = \int_0^a \frac{Uy}{a}\,dy = \frac{U}{a}\frac{y^2}{2}\Big]_0^a = \frac{Ua}{2}$

$$\frac{Q}{\ell} = \frac{1}{2} \times 2\,\frac{m}{s} \times 0.0025\,m = 0.00250\ m^3/s/m$$ ← Q/ℓ

(b) $\tau_{yx} = \mu\frac{du}{dy}$; for $\frac{\partial p}{\partial x} = 0$, $\tau = \frac{\mu U}{a}$. For air at STP, $\mu = 1.79 \times 10^{-5}\ N\cdot s/m^2$,

$$\tau_{yx} = 1.79\times10^{-5}\,\frac{N\cdot s}{m^2} \times 2\,\frac{m}{s} \times \frac{1}{0.0025\,m} = 0.0143\ N/m^2$$ ← τ_{yx}

(c) Shear stress is constant for $\frac{\partial p}{\partial x} = 0$; see plot below.

(d) Q will decrease if $\frac{\partial p}{\partial x} > 0$; Q will increase if $\frac{\partial p}{\partial x} < 0$. ← Q

The shear stress is given by Eq. 8.9a: $\tau_{yx} = \frac{\mu U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{y}{a} - \frac{1}{2}\right]$

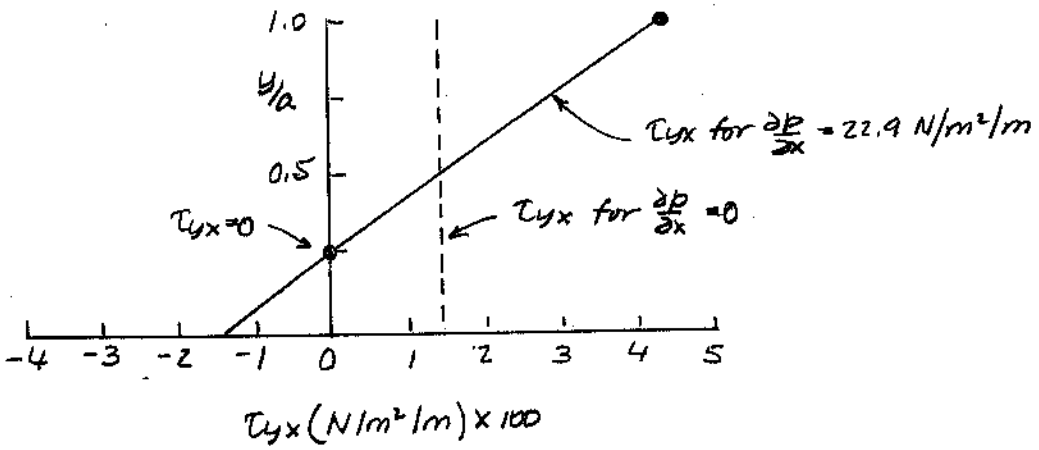
(e) For $\tau = 0$ at $y = 0.25\,a$, $0 = \frac{\mu U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{1}{4} - \frac{1}{2}\right]$ or $\frac{\partial p}{\partial x} = \frac{4\mu U}{a^2}$

$$\frac{\partial p}{\partial x} = 4 \times 1.79\times10^{-5}\,\frac{N\cdot s}{m^2} \times 2\,\frac{m}{s} \times \frac{1}{(0.0025)^2\,m^2} = 22.9\ N/m^2/m$$ ← $\frac{\partial p}{\partial x}$

(f) To plot, calculate τ_{yx} at $y = a$:

$$\tau_{yx} = \frac{\mu U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[1 - \frac{1}{2}\right] = \frac{\mu U}{a} + \frac{a}{2}\left(\frac{\partial p}{\partial x}\right) = 0.0143\,\frac{N}{m^2} + \frac{1}{2} \times 0.0025\,m \times 22.9\,\frac{N}{m^3} = 0.0429\,\frac{N}{m^2}$$

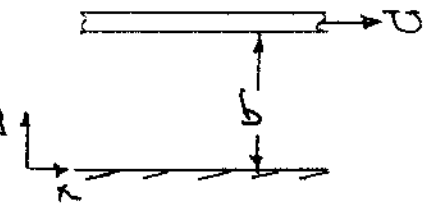
Plotting:



Given: Water at 60°F flows between parallel plates as shown.

$U = 1$ ft/s $b = 0.01$ ft

$\partial p/\partial x = -1.20$ lbf/ft³/ft



Find: (a) location of point of maximum velocity
(b) value of u_{max}
(c) volume of flow passing a section in 1 s.

Plot: the velocity and shear stress distributions.

Solution:

Computing equation: $u = \frac{U}{b}y + \frac{b^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{b}\right)^2 - \frac{y}{b}\right]$ (8.8)

To locate u_{max}, set $du/dy = 0$

$$\frac{du}{dy} = 0 = \frac{U}{b} + \frac{b^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\frac{2y}{b^2} - \frac{1}{b}\right] = \frac{U}{b} + \frac{1}{2\mu}\frac{\partial p}{\partial x}(2y - b)$$

when $y = \frac{b}{2} - \frac{\mu U}{b(\partial p/\partial x)}$ From Table A.7, $\mu = 2.34\times10^{-5}$ lb·s/ft²

$$y = \frac{0.01\text{ ft}}{2} - \frac{2.34\times10^{-5}\ \frac{\text{lbf·s}}{\text{ft}^2}\times 1\ \frac{\text{ft}}{\text{s}}}{0.01\text{ ft}\times(-1.20)\ \frac{\text{lbf}}{\text{ft}^3}} =$$

$u = u_{max}$ at $y = 0.00695$ ft ← y for u_{max}

So

$$u_{max} = \frac{U}{b}y + \frac{b^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{b}\right)^2 - \frac{y}{b}\right] \quad \text{where } y = 0.695\times10^{-2}\text{ ft}$$

$$= \frac{1\text{ ft}}{\text{s}}\times\frac{0.695}{1.0} + \frac{(0.01\text{ ft})^2}{2}\times 2.34\times10^{-5}\ \frac{\text{lbf·s}}{\text{ft}^2}\times\left(-1.20\ \frac{\text{lbf}}{\text{ft}^3}\right)\left[(0.695)^2 - (0.695)\right]$$

$u_{max} = 1.24$ ft/s ← u_{max}

To find the volume of flow, evaluate $\forall = Q\Delta t$

$$Q = \int_A u\,dA = \int_0^b u\,w\,dy = w\int_0^b\left[\frac{U}{b}y + \frac{b^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left\{\left(\frac{y}{b}\right)^2 - \frac{y}{b}\right\}\right]dy$$

$$Q/w = \left[\frac{U}{2b}y^2 + \frac{b^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left\{\frac{y^3}{3b^2} - \frac{y^2}{2b}\right\}\right]_0^b = \frac{Ub}{2} + \frac{b^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left\{\frac{b}{3} - \frac{b}{2}\right\}$$

$$Q/w = \frac{Ub}{2} - \frac{b^3}{12\mu}\left(\frac{\partial p}{\partial x}\right) = \frac{1}{2}\times\frac{1\text{ ft}}{\text{s}}\times 0.01\text{ ft} - \frac{(0.01\text{ ft})^3}{12}\times 2.34\times10^{-5}\ \frac{\text{ft}^2}{\text{lbf·s}}\left(-1.20\ \frac{\text{lbf}}{\text{ft}^3}\right)$$

$$Q/w = 9.27\times10^{-3}\ \text{ft}^2/\text{s}$$

$$\forall/w = \frac{\forall}{w}\Delta t = 9.27\times10^{-3}\ \frac{\text{ft}^2}{\text{s}}\times 10\text{ s} = 0.0927\text{ ft}^3/\text{ft}$$ ← $\forall/w$

The velocity distribution is given by

$$\frac{u}{U} = \frac{y}{b} + \frac{1}{2\mu}\left(\frac{dp}{dx}\right)\left[\left(\frac{y}{b}\right)^2 - \frac{y}{b}\right]$$

$$\frac{u}{U} = \frac{y}{b} - 2.56\left[\left(\frac{y}{b}\right)^2 - \frac{y}{b}\right] \leftarrow \text{---}$$

The shear stress is $\tau_{yx} = \mu \frac{du}{dy} = \frac{\mu}{b}\frac{du}{d(y/b)}$

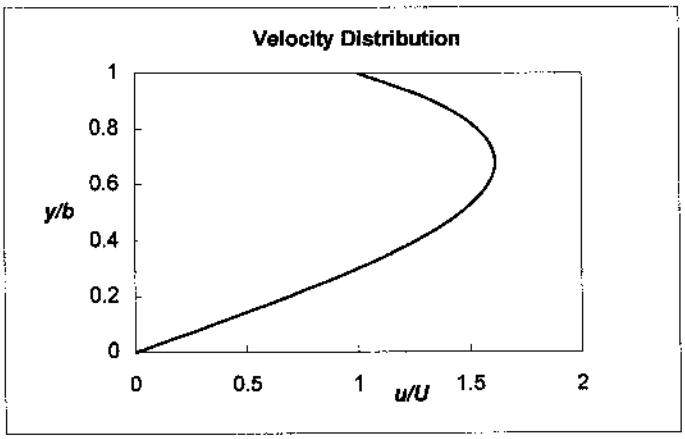
$$\tau_{yx} = \frac{\mu U}{b} + \frac{1}{2b}\left(\frac{dp}{dx}\right)\left[2\left(\frac{y}{b}\right) - 1\right]$$

$$= 2.34\times10^{-5}\ \frac{lbf\cdot s}{ft^2}\times\frac{1\ ft}{s}\times\frac{1}{0.014} + \frac{0.01}{2}\left(-1.20\ \frac{lbf}{ft^3}\right)\left[2\left(\frac{y}{b}\right) - 1\right]$$

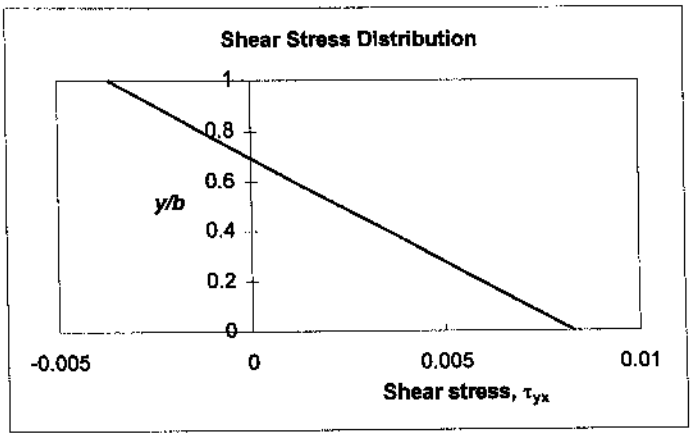
$$\tau_{yx} = 2.34\times10^{-3}\ \frac{lbf}{ft^2} - 6.00\times10^{-3}\ \frac{lbf}{ft^2}\left[2\left(\frac{y}{b}\right) - 1\right] \leftarrow \text{---}$$

The velocity and shear stress distributions are plotted below.

u/U	y/b
0	0
0.353	0.1
0.692	0.2
0.999	0.3
1.26	0.4
1.46	0.5
1.58	0.6
1.61	0.7
1.54	0.8
1.34	0.9
1.00	1

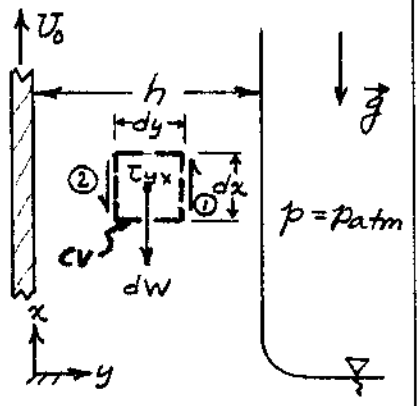


τ_{yx}	y/b
0.00834	0
0.00714	0.1
0.00594	0.2
0.00474	0.3
0.00354	0.4
0.00234	0.5
0.00114	0.6
-0.00006	0.7
-0.00126	0.8
-0.00246	0.9
-0.00366	1.0



Given: Belt moving steadily through bath as shown.

Assume zero shear at film/air surface, and no pressure forces.



Find: (a) Boundary conditions for velocity at $y=0$, $y=h$.
(b) Velocity profile.

Solution: Choose CV dxdydz as shown.
Apply x component of momentum equation.

Basic equations:

$$F_{Sx} + F_{Bx} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\,d\forall}^{=0(2)} + \cancel{\int_{CS} u\rho\vec{V}\cdot d\vec{A}}^{=0(3)} \quad ; \quad \tau_{yx} = \mu\frac{du}{dy} = \tau$$

Assumptions: (1) F_{Sx} due to shear forces only
(2) Steady flow
(3) Fully-developed flow

Then

$$F_{Sx} + F_{Bx} = F_① - F_② + F_{Bx} = \left(\tau + \frac{d\tau}{dy}\frac{dy}{2}\right)dxdz - \left(\tau - \frac{d\tau}{dy}\frac{dy}{2}\right)dxdz - \rho g\,dxdydz = 0$$

or $\frac{d\tau}{dy} = \rho g$. Integrating

$$\tau = \rho g y + c_1 = \mu\frac{du}{dy} \quad \text{or} \quad \frac{du}{dy} = \frac{\rho g y}{\mu} + \frac{c_1}{\mu}$$. Integrating again,

$$u = \frac{\rho g y^2}{2\mu} + \frac{c_1}{\mu}y + c_2$$

To evaluate the constants c_1 and c_2, apply the boundary conditions:

At $y=0$, $u = U_0$, so $c_2 = U_0$

At $y=h$, $\tau = 0$, so $\frac{du}{dy} = 0$, and $c_1 = -\rho g h$ ← BC

Substituting,

$$u = \frac{\rho g y^2}{2\mu} - \frac{\rho g h y}{\mu} + U_0 = \frac{\rho g}{\mu}\left(\frac{y^2}{2} - hy\right) + U_0$$ ← u

{Note that at $y=h$,

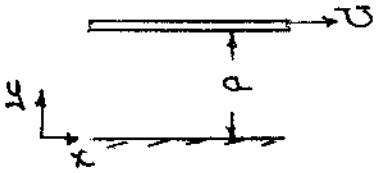
$$u = \frac{\rho g}{\mu}\left(-\frac{h^2}{2}\right) + U_0 \neq 0$$

Thus the solution is determined only when U_0 and h are known.}

Given: Velocity profile for fully developed laminar flow of air between parallel plates

$$u = U\frac{y}{a} + \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right]$$

$U = 3\ m/s \qquad a = 2.5\ mm$



Find: (a) pressure gradient for which net flow is zero; sketch expected $u(y)$ and $\tau_{yx}(y)$.

(b) expected $u(y)$ and $\tau_{yx}(y)$ for case where $u = 2U$ at $y/a = 0.5$

Solution:

Computing equations:

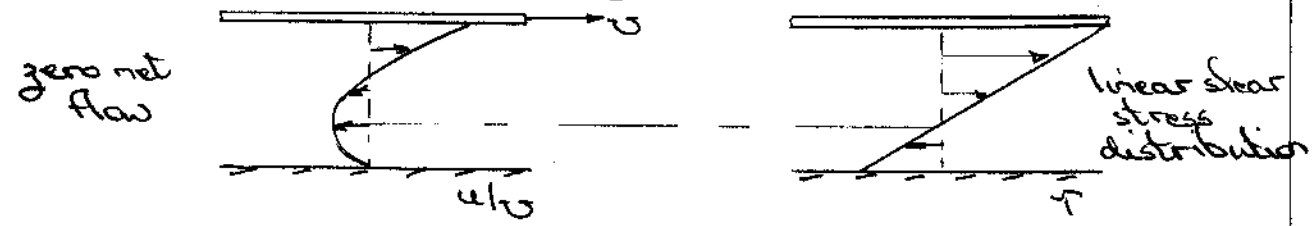
$$Q/\ell = \frac{Ua}{2} - \frac{a^3}{12\mu}\left(\frac{\partial p}{\partial x}\right) \qquad (8.9b)$$

$$\tau_{yx} = \mu\frac{U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{y}{a} - \frac{1}{2}\right] \qquad (8.9a)$$

For $Q = 0$, from Eq. 8.9b (assuming $T = 15°C$)

$$\frac{\partial p}{\partial x} = \frac{6\mu U}{a^2} = 6 \times 1.79\times10^{-5}\ \frac{N\cdot s}{m^2} \times 3\ \frac{m}{s} \times \frac{1}{(2.5\times10^{-3}\ m)^2} = 51.6\ N/m^2/m \qquad \left(\frac{\partial p}{\partial x}\right)$$

For this adverse pressure gradient



For $u = 2U$ at $y/a = 0.5$

$$2U = 0.5U + \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\frac{1}{2} - \frac{1}{2}\right] \quad \text{and} \quad \frac{3}{2}U = -\frac{a^2}{8\mu}\left(\frac{\partial p}{\partial x}\right)$$

$$\frac{\partial p}{\partial x} = -\frac{12U\mu}{a^2} = -12 \times \frac{3}{5}\ \frac{m}{s} \times 1.79\times10^{-5}\ \frac{N\cdot s}{m^2} \times \frac{1}{(2.5\times10^{-3}\ m)^2} = -103\ N/m^2/m$$

$$\tau = \mu\frac{U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{y}{a} - \frac{1}{2}\right] \qquad \{\text{shear stress is linear}\}$$

$y = 0 \quad \tau = \mu\frac{U}{a} - \frac{a}{2}\left(\frac{\partial p}{\partial x}\right) = 1.79\times10^{-5}\ \frac{N\cdot s}{m^2} \times \frac{3}{5}\ \frac{m}{s} \times \frac{1}{2.5\times10^{-3}\ m} - \frac{2.5\times10^{-3}\ m}{2}\left(-103\ \frac{N}{m^2}\right) = 15\times10^{-2}\ N/m^2$

$y = a \quad \tau = \mu\frac{U}{a} + \frac{a}{2}\left(\frac{\partial p}{\partial x}\right) = -10.8\times10^{-2}\ N/m^2$

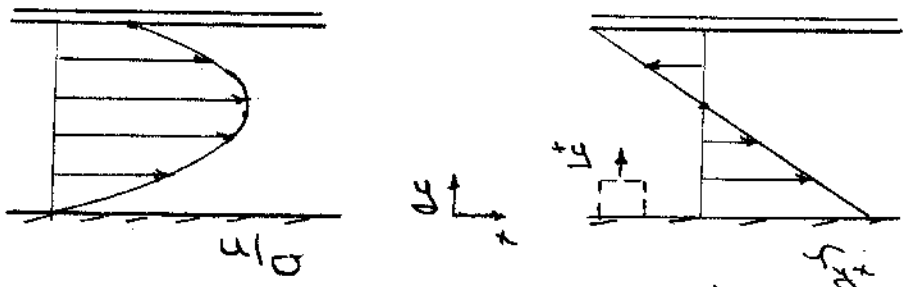
Note that the point of zero shear stress is not at $y/a = 0.5$ and hence $y/a = 0.5$ is not the location of maximum velocity. Maximum velocity occurs at $y/a > 0.5$.

To find the location of zero shear set $\tau_{yx}=0$. Then

$$0 = \frac{\mu U}{a} + a\left(\frac{2p}{\partial x}\right)\left(\frac{y}{a}-\frac{1}{2}\right) \quad \text{and} \quad \frac{y}{a} = \frac{1}{2} - \frac{\mu U}{a^2\left(\frac{\partial p}{\partial x}\right)}.$$

$$\frac{y}{a} = 0.5 - 1.79\times10^{-5}\,\frac{N\cdot s}{m^2}\times\frac{3m}{s}\times\frac{1}{(2.5\times10^{-3}\,m)^2}\times\frac{1}{(-69)\frac{N}{m^3}} = 0.583$$

For this case ($u=2U$ at $y/a=0.5$) the velocity and shear stress distributions would be as follows.



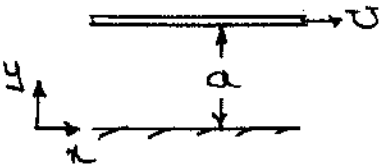
The shear stress is positive ($du/dy>0$) below $y/a = 0.583$; positive stress acts in positive x direction on a positive y surface.

The shear stress is negative ($du/dy<0$) above $y/a = 0.583$; negative stress acts in the negative x direction on a positive y surface.

Given: Velocity profile for fully developed laminar flow of water between parallel plates

$$u = \frac{U}{a}y + \frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right)^2 - \frac{y}{a}\right]$$

$U = 3\ m/s \qquad a = 2.5\ mm$



Find: (a) Volume flow rate for zero pressure gradient.
(b) Shear stress on lower plate; sketch $\tau(y)$
(c) effect of mild adverse pressure gradient on Q
(d) pressure gradient for zero shear at $y/a = 0.25$; sketch $\tau(y)$.

Solution:

Computing equations: $\tau_{yx} = \mu\frac{U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{1}{a}\frac{y}{a} - \frac{1}{2}\right]$ (8.9a)

$$Q/\ell = \frac{Ua}{2} - \frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^3 \quad (8.9b)$$

For $\partial p/\partial x = 0$, $Q/\ell = \frac{Ua}{2} = \frac{1}{2}\times 3\frac{m}{s}\times 2.5\times10^{-3}\ m = 3.75\times10^{-3}\ m^3/s/m$ Q

The shear stress is $\tau_{yx} = \mu\frac{U}{a}$ {At 15C, $\mu = 1.14\times10^{-3}\ N\cdot s/m^2$ (Table A.8}

$$\tau_{yx} = 1.14\times10^{-3}\ \frac{N\cdot s}{m^2}\times 3\frac{m}{s}\times\frac{1}{2.5\times10^{-3}\ m} = 1.37\ N/m^2$$ τ_{yx}

The shear stress is constant across the channel (curve 1 below)

For $\partial p/\partial x > 0$, Eq. 8.9b indicates that Q will decrease

For $\tau = 0$ at $y/a = 0.25$

$$\tau_{yx} = 0 = \mu\frac{U}{a} + a\left(\frac{\partial p}{\partial x}\right)\left[\frac{1}{4} - \frac{1}{2}\right] = \mu\frac{U}{a} - \frac{a}{4}\left(\frac{\partial p}{\partial x}\right)$$

$$\frac{\partial p}{\partial x} = \frac{4\mu U}{a^2} = 4\times1.14\times10^{-3}\ \frac{N\cdot s}{m^2}\times 3\frac{m}{s}\times\frac{1}{(2.5\times10^{-3}\ m)^2} = 2.19\ kN/m^2/m$$ $\frac{\partial p}{\partial x}$

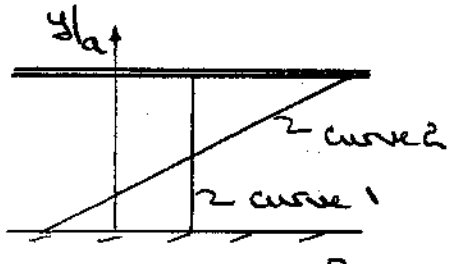
For this pressure gradient

$$\tau_{yx} = 1.14\times10^{-3}\ \frac{N\cdot s}{m^2}\times 3\frac{m}{s}\times\frac{1}{2.5\times10^{-3}\ m} + 2.5\times10^{-3}\ m\times 2.19\times10^{3}\ \frac{N}{m^3}\left[\frac{y}{a} - 0.5\right]$$

$$\tau_{yx} = 1.37\ N/m^2 + 5.48\ \frac{N}{m^2}\left(\frac{y}{a} - 0.5\right)$$

$\tau_{yx}|_{y=0} = -1.37\ N/m^2$

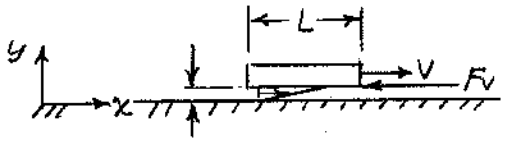
$\tau_{yx}|_{y=a} = 4.11\ N/m^2$ } curve 2



Given: Microchip supported on air film, on a horizontal surface. Chips are $L = 11.7$ mm long, $w = 9.35$ mm wide, and have mass $m = 0.325$ g. The air film is $h = 0.125$ mm thick. The initial speed of the chips is $V_0 = 1.75$ mm/s; they slow from viscous shear.

Find: (a) Differential equation for chip motion during deceleration.
(b) Time required for chip to lose 5 percent of V_0.
(c) Sketch of chip speed vs. time, with labels and comments.

Solution: Apply Newton's law of viscosity



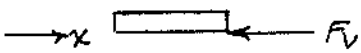
Basic equations: $\tau_{yx} = \mu \dfrac{du}{dy}$

$$F_V = \tau A \qquad \Sigma F = ma_x$$

Assume: (1) Newtonian fluid (3) Air at STP
(2) Linear velocity profile in narrow gap

Then $$\tau_{yx} = \mu \frac{du}{dy} = \mu \frac{V}{h}; \quad F_V = \tau A = \mu \frac{V}{h} wL = \frac{\mu V w L}{h}$$

The free-body diagram for the chip is



$$\Sigma F_x = -F_V = -\frac{\mu V w L}{h} = m\frac{dV}{dt}; \quad \frac{dV}{V} = -\frac{\mu w L}{mh}\,dt$$ ← D.E.

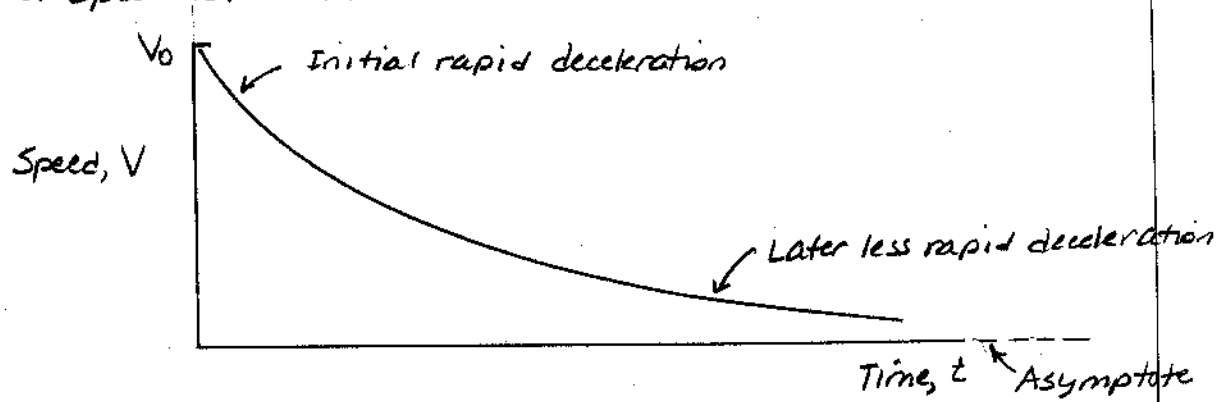
Integrating, $$\int_{V_0}^{V} \frac{dV}{V} = \ln \frac{V}{V_0} = -\frac{\mu w L}{mh} t$$

Thus $$t = -\frac{mh}{\mu w L} \ln \frac{V}{V_0}$$

$$t = -0.325\,g \times 0.125\,mm \times \frac{m \cdot s}{1.79 \times 10^{-5}\,kg} \times \frac{1}{9.35\,mm} \times \frac{1}{11.7\,mm} \times \ln 0.95 \times \frac{kg}{1000\,g} \times \frac{1000\,mm}{m}$$

$$t = 1.06\,s$$ ← t

The plot of speed vs. time is:



Problem 8.35

8-2.2 [4]

Given: Free-surface waves begin to form on a laminar liquid film flowing down an inclined surface whenever the Reynolds number, based on mass flow per unit width of film, is larger than about 33.

Find: Estimate of the maximum thickness of a laminar film of water that remains free from waves while flowing down a vertical surface.

Solution: The mass flow rate is $\dot{m} = \rho \bar{V} A = \rho \bar{V} w \delta$, so $\dot{m}/w = \rho \bar{V} \delta$.

Thus
$$Re = \frac{\rho \bar{V} \delta}{\mu} = \frac{\bar{V}\delta}{\nu} = 33 \text{ (maximum)}$$

From Example Problem 8.3 (pp. 343-345),

$$\bar{V} = \frac{\rho g \delta^2}{3\mu}$$

Thus
$$\frac{\rho \bar{V} \delta}{\mu} = \frac{\rho^2 g \delta^3}{3\mu^2} = 33$$

Solving for δ,

$$\delta = \left[\frac{99\mu^2}{\rho^2 g}\right]^{1/3}$$

At $T = 20°C$, $\mu = 1.00 \times 10^{-3}$ kg/m·s and $\rho = 998$ kg/m³ (Table A.8). Substituting,

$$\delta = \left[99 \times (1.00\times10^{-3})^2 \frac{kg^2}{m^2 \cdot s^2} \times \frac{m^6}{(998)^2 kg^2} \times \frac{s^2}{9.81 m}\right]^{1/3}$$

$$\delta = 2.16 \times 10^{-4} \text{ m} \quad \text{or} \quad 0.216 \text{ mm}$$

δ_{max}

Problem 8.36

Open-Ended Problem Statement: Hold a flat sheet of paper 50 to 75 mm above a smooth desktop. Propel the sheet smoothly parallel to the desk surface as you release it. Comment on the motion you observe. Explain the fluid dynamic phenomena involved in the motion.

Discussion: After some practice, one can release the sheet so that it continues to move parallel to the desktop for a considerable distance before finally slowing and stopping. The slowing of the paper sheet is so gradual that the motion appears to be almost frictionless.

The thin layer of air trapped under the paper sheet acts to "lubricate" the motion as the sheet moves parallel to the tabletop. Kinetic sliding friction between the sheet and the desktop is prevented by the fluid layer. Instead the motion is resisted by the much smaller viscous shear stress caused by the motion of the sheet (see Section 8-2.2). Thus the sheet appears to move across the desktop almost without friction.

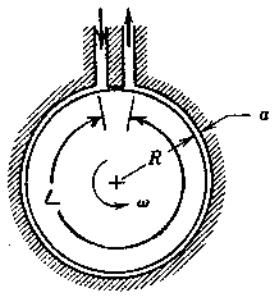
The same phenomena are involved in hydrodynamic lubrication. Detailed analysis of lubrication is beyond the scope of this text, but contact between two solid surfaces can be prevented, even with large normal loads, by properly shaping the clearance space between the two surfaces. To analyze the phenomenon, the Navier-Stokes equations for incompressible flow (Equations 5.27) are simplified further to a "thin layer" form. These equations are used to predict the load carrying capacity of a lubricated bearing.

The NCFMF video *Low-Reynolds-Number Flows* shows further examples of flows in which viscous effects are dominant.

13-/42 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-382 100 RECYCLED WHITE 5 SQUARE
42-389 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National Brand

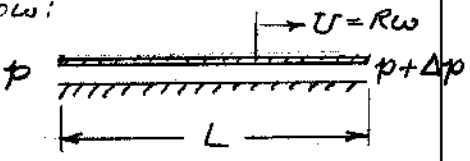
Given: Viscous-shear pump, as shown.

b = width normal to diagram; $a \ll R$



Find: Performance characteristics
(a) Pressure differential
(b) Input power
(c) Efficiency
as functions of volume flow rate.

Solution: Since $a \ll R$, unwrap to form flow between parallel plates.
Apply Eqs. 8.9 to fully developed flow:



Volume flow rate is $\frac{Q}{b} = \frac{Ua}{2} - \frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^3$

Substituting $U = R\omega$ and $\frac{\partial p}{\partial x} = \frac{\Delta p}{L}$, then

$$\Delta p = \frac{12\mu L}{a^3}\left(\frac{\omega R a}{2} - \frac{Q}{b}\right) = \frac{6\mu L R\omega}{a^2}\left(1 - \frac{2Q}{abR\omega}\right) \qquad \Delta p$$

Torque is $T = \tau R(bL) = RLb\tau$. Power is $P = T\omega$. From Eq. 8.9a, at $y = a$,

$$P = RLb\omega\left[\frac{\mu R\omega}{a} + \frac{\Delta p}{L}\frac{a}{2}\right] = RLb\omega\left[\frac{\mu R\omega}{a} + \frac{6\mu L R\omega}{a^2}\left(1 - \frac{2Q}{abR\omega}\right)\frac{a}{2L}\right]$$

$$P = RLb\omega\left[\frac{\mu R\omega}{a}\left(4 - \frac{6Q}{abR\omega}\right)\right] = \frac{\mu L b (R\omega)^2}{a}\left(4 - \frac{6Q}{abR\omega}\right) \qquad P$$

Output power is $Q\Delta p$, so efficiency is

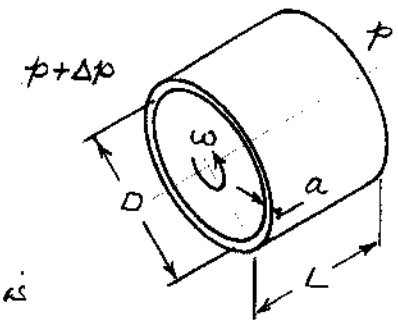
$$\eta = \frac{Q\Delta p}{P} = \frac{6\mu Q L R\omega}{a^2}\left(1 - \frac{2Q}{abR\omega}\right)\frac{a}{\mu L b (R\omega)^2}\frac{1}{\left(4 - \frac{6Q}{abR\omega}\right)}$$

$$\eta = \frac{6Q}{abR\omega}\frac{\left(1 - \frac{2Q}{abR\omega}\right)}{\left(4 - \frac{6Q}{abR\omega}\right)} \qquad \eta$$

Given: Annular gap seal as shown.

Power required to pump oil, P_p.

Power to overcome viscous dissipation, P_V.



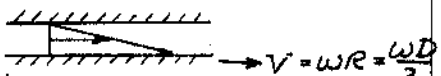
Find: (a) Expressions for P_p, P_V

(b) Show total power minimized when a is chosen so that $P_V = 3P_p$.

Solution: Apply Eqs. 8.6 and 8.9 for flow between parallel plates.

Assumptions: (1) $a \ll D$, so unfold to flat plates

(2) No pressure gradient circumferentially



The viscous power is the product of viscous torque times ω:

$$P_V = T\omega = \tau(2\pi RL)R\omega = \mu\frac{V}{a}(2\pi\frac{D}{2}L)\frac{D}{2}\omega = \mu\frac{\omega D}{2a}\pi DL\frac{D}{2}\omega = \frac{\pi\mu\omega^2D^3L}{4a} \qquad P_V$$

The pump power is the product of flow rate times pressure drop.

$$P_p = Q\Delta p$$

From Eq. 8.6c, $Q = \frac{la^3\Delta p}{12\mu L} = \frac{\pi Da^3\Delta p}{12\mu L}$, so $P_p = \frac{\pi Da^3\Delta p^2}{12\mu L}$ $\qquad P_p$

The total power required is $P_T = P_V + P_p = \frac{\pi\mu\omega^2D^3L}{4a} + \frac{\pi Da^3\Delta p^2}{12\mu L}$

It may be minimized by setting $\frac{dP_T}{da} = 0$. Thus

$$\frac{dP_T}{da} = -\frac{\pi\mu\omega^2D^3L}{4a^2} + \frac{\pi Da^2\Delta p^2}{4\mu L} = 0 \qquad (1)$$

This can be written

$$\frac{dP_T}{da} = -\frac{1}{a}P_V + \frac{3}{a}P_p = 0$$

which is satisfied when $3P_p - P_V = 0$ or $P_V = 3P_p$ Optimu

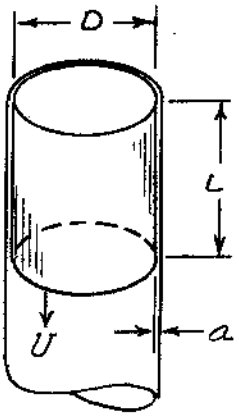
Equation 1 also can be solved for a at optimum conditions:

$$a^4 = \frac{\mu^2\omega^2D^2L^2}{\Delta p^2} \quad \text{or} \quad a^2 = \frac{\mu\omega DL}{\Delta p} \quad \text{or} \quad \frac{a}{D} = \sqrt{\frac{\mu\omega L}{D\Delta p}} \text{ (optimum)}$$

Given: "Viscous timer," consisting of a cylindrical mass inside a circular tube filled with viscous liquid, creating a narrow annular gap.

Find: (a) The flow field created when the mass falls under gravity.
(b) Whether this would make a satisfactory timer, and if so, for what range of time intervals.
(c) Effect of temperature change on measured time interval.

Solution: Apply conservation of mass to a CV enclosing the cylinder and the moving mass:



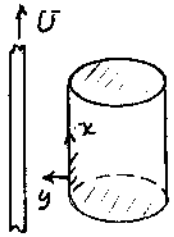
Then: $Q = U\frac{\pi D^2}{4} = \bar{V}\,\pi D a = \bar{V} l a$ (1)

Assume: (1) Gap is narrow, $a \ll D$
(2) Unroll gap so flat, $l = \pi D$
(3) Steady flow
(4) Fully developed laminar flow

Under these assumptions, the flow field in the gap is that for flow between parallel plates with one plate moving.

Flow Field

Place coordinates on the moving mass:

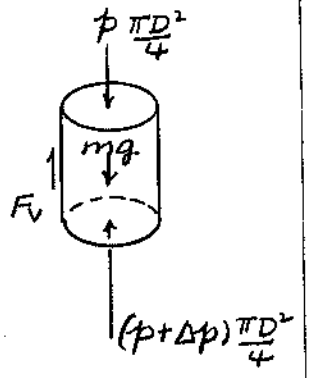


Then the volume flow rate (Eq. 8.9b) is

$$\frac{Q}{l} = \frac{Q}{\pi D} = \frac{Ua}{2} - \frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^3$$

But $\frac{\partial p}{\partial x} = -\frac{\Delta p_v}{L}$, where Δp_v is the pressure drop driving viscous flow, so

$$\frac{Q}{l} = \frac{Ua}{2} - \frac{1}{12\mu}\left(-\frac{\Delta p_v}{L}\right)a^3 = \frac{Ua}{2} + \frac{\Delta p_v a^3}{12\mu L} \quad (2)$$



The pressure change across the moving mass is

$$\Delta p = \rho_l g L + \Delta p_v \quad (3)$$

Summing forces on the moving mass gives

$$\Sigma F_x = \Delta p\,\frac{\pi D^2}{4} - mg + F_v = m\frac{dU}{dt}^{\,=0\,(3)}$$

But $mg = \rho_m \frac{\pi D^2}{4} L$ and $F_v = \tau_s \pi D L$

From Eq. 8.9a, $\tau_s = \mu\frac{U}{a} - \frac{a}{2}\left(\frac{\partial p}{\partial x}\right) = \mu\frac{U}{a} + \frac{a}{2}\frac{\Delta p_v}{L}$

Substituting, $\Delta p\,\frac{\pi D^2}{4} - \rho_m \frac{\pi D^2}{4} L g + \left[\mu\frac{U}{a} + \frac{a}{2}\frac{\Delta p_v}{L}\right]\pi D L = 0$

or $\Delta p = \rho_m g L - \left[\mu\frac{U}{a} + \frac{a}{2}\frac{\Delta p_v}{L}\right]\frac{4L}{D}$ (4)

Combining Eqs. 1 and 2 gives $\frac{UD}{4} = \frac{Ua}{2} + \frac{\Delta p_v a^3}{12\mu L}$

Thus $\Delta p_v = \frac{12\mu L}{a^3}\left[\frac{UD}{4} - \frac{Ua}{2}\right] = \frac{3\mu U L D}{a^3}$ (5) Δp_v

($a \ll D$)

Combining Eqs. 3 and 4 gives $\Delta p = \rho_\ell g L + \Delta p_v = \rho_m g L - \left[\mu\frac{U}{a} + \frac{a}{2}\frac{\Delta p_v}{L}\right]\frac{4L}{D}$

Using Eq. 5,

$$\rho_\ell g L + \frac{3\mu U L D}{a^3} = \rho_m g L - \mu\frac{U}{a}\frac{4L}{D} - \frac{a}{2}\frac{3\mu U L D}{L a^3}\frac{4L}{D}$$

Simplifying and re-arranging,

$$(\rho_m - \rho_\ell) g L = \frac{3\mu U L D}{a^3} + \frac{4\mu U L}{aD} + \frac{6\mu U L}{a^2} \approx \frac{3\mu U L D}{a^3}$$

Finally, using $\rho = SG\,\rho_{H_2O}$,

$$U = \frac{(SG_m - SG_\ell)\rho_{H_2O} g a^3}{3\mu D}$$ U

The time interval for the mass to move distance H is

$$\Delta t = \frac{H}{U} = \frac{3\mu D}{(SG_m - SG_\ell)\rho_{H_2O} g a^3}$$ (6) Δt

Equation 6 shows that the time interval for the mass to fall any distance H is proportional to liquid viscosity μ and inversely proportional to gap width a cubed. A temperature change would affect the diameter of the measuring tube and the diameter of the falling mass. A temperature change also would affect the viscosity of the liquid in the tube.

Speed of the falling mass is proportional to the cube of gap width. If the coefficient of thermal expansion of the falling mass were greater than that of the glass measuring tube (which seems likely), then the width of the annular gap would decrease with increasing temperature. This would tend to slow the falling mass. The total amount of thermal expansion would depend on the diameter of the mass and tube. The effect on gap width would be greater, the larger the tube diameter compared to the initial gap width.

It might be possible to "tailor" the thermal expansion coefficient of the cylinder, by using a suitable material, to closely match that of the falling mass. Then there would be no differential thermal expansion between the mass and tube, and changes in temperature would not affect the gap width.

Speed of the falling mass is inversely proportional to liquid viscosity. Liquid viscosity decreases sharply as temperature increases (the viscosity of SAE 30 oil drops more than 10 percent as its temperature increases from 20°C to 25°C, see Fig. A.2). This would tend to increase the speed of the falling mass.

The entire device could be maintained at constant temperature.

Problem 8.40

8-2.2 [5]

Open-Ended Design Problem: Automotive design is tending toward all-wheel drive to improve vehicle performance and safety when traction is poor. An all-wheel drive vehicle must have an interaxle differential to allow operation on dry roads. Numerous vehicles are being built using multiplate viscous drives for interaxle differentials. Perform the analysis and design needed to define the torque transmitted by the differential for a given speed difference, in terms of the design parameters. Identify suitable dimensions for a viscous differential to transmit a torque of 150 N · m at a speed loss of 125 rpm, using lubricant with the properties of SAE 30 oil. Discuss how to find the minimum material cost for the differential, if the plate cost per square meter is constant.

Solution: From Problem 2.45, $dT = r\,dF = r\tau\,dA$

But $\tau = \mu\frac{du}{dy} = \mu\frac{u}{h} = \mu\frac{r\Delta\omega}{h}$; $dA = 2\pi r\,dr$

Thus $dT = r\mu\frac{r\Delta\omega}{h}2\pi r\,dr = \frac{2\pi\mu\Delta\omega}{h}r^3\,dr$; $T = \frac{\pi\mu\Delta\omega}{2h}\left[R_o^4 - R_i^4\right]$

or $T = \frac{\pi\mu\Delta\omega}{2h}R^4(1-\alpha^4)$ where $\alpha = R_i/R$ ← T

This value is per gap. Each rotor has 2 gaps to a housing. For n gaps

$$T_n = \frac{n\pi\mu\Delta\omega}{2h}R^4(1-\alpha^4) \qquad (1)$$ ← T_n

From Eq. 1, assuming $\mu = 0.18$ kg/m·s (Fig. A.2) and $\alpha = \frac{1}{2}$, so $1-\alpha^4 = 1-\frac{1}{16} \approx 1$, then

$$\frac{nR^4}{h} = \frac{2T_n}{\pi\mu\Delta\omega} = \frac{2}{\pi}\times 150\ \text{N·m}\times\frac{\text{m}^2}{0.18\ \text{N·s}}\times\frac{\text{min}}{125\ \text{rev}}\times\frac{\text{rev}}{2\pi\ \text{rad}}\times 60\frac{\text{s}}{\text{min}} = 40.5\ \text{m}^3 = C$$

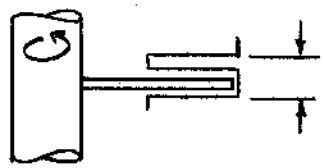
or

$$R^4 = C\frac{h}{n}$$

For $n = 100$ and $h = 0.2$ mm, $R^4 = 40.5\ \text{m}^3 \times 0.0002\ \text{m} \times \frac{1}{100} = 8.11\times10^{-5}\ \text{m}^4$

$R = \left[8.11\times10^{-5}\right]^{1/4}\text{m} = 0.0949\ \text{m}$ (or $D = 190$ mm) ← R

The stack length might be



≈ 2.5 mm for $n = 2$, or 125 mm for $n = 100$

Given: Fully developed laminar flow in a pipe, with

$$u = -\frac{R^2}{4\mu}\frac{\partial p}{\partial x}\left[1-\left(\frac{r}{R}\right)^2\right]$$

Find: Radius from pipe axis at which u equals the average velocity, $\bar{V}$.

Solution: First determine $\bar{V}$.

$$\bar{V} = \frac{Q}{A} = \frac{1}{\pi R^2}\int_A u\,dA = \frac{1}{\pi R^2}\int_0^R \left\{-\frac{R^2}{4\mu}\frac{\partial p}{\partial x}\left[1-\left(\frac{r}{R}\right)^2\right]\right\} 2\pi r\,dr$$

$$= -\frac{R^2}{2\mu}\frac{\partial p}{\partial x}\int_0^1 \left[1-\left(\frac{r}{R}\right)^2\right]\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) = -\frac{R^2}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right]_0^1$$

$$\bar{V} = -\frac{R^2}{8\mu}\frac{\partial p}{\partial x}$$

Then $u = \bar{V}$ when

$$u = -\frac{R^2}{4\mu}\frac{\partial p}{\partial x}\left[1-\left(\frac{r}{R}\right)^2\right] = \bar{V} = -\frac{R^2}{8\mu}\frac{\partial p}{\partial x}$$

or $$1-\left(\frac{r}{R}\right)^2 = \frac{1}{2}$$

or $$\left(\frac{r}{R}\right)^2 = \frac{1}{2}$$

$$r = \frac{R}{\sqrt{2}} = 0.707R$$ ← r

Given: Water and SAE 10W oil flowing at 40°C through a 6 mm tube.

Find, for each fluid:

(a) The maximum flowrate for laminar flow.
(b) The corresponding pressure gradient.

Solution: Laminar flow is expected for $Re \le 2300$. Expressing this in terms of flowrate,

$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu} = \frac{Q D}{A \nu} = \frac{4}{\pi D^2} \frac{Q D}{\nu} = \frac{4Q}{\pi \nu D} \quad \text{or} \quad Q = \frac{\pi \nu D Re}{4}$$

Thus

$$Q_{max} = \frac{\pi \nu D Re_{max}}{4} = \frac{\pi}{4} \times 2300 \times 0.006\, m \times \nu \frac{m^2}{s} = 10.8\, \nu \left(\frac{m^3}{s}\right)$$

Also, $Q = -\frac{\pi R^4}{8\mu} \frac{\partial p}{\partial x}$ for laminar flow, according to Eq. 8.136. Then

$$\frac{\partial p}{\partial x} = -\frac{8 \mu Q}{\pi R^4} = -\frac{128\, \mu Q}{\pi D^4}$$

so

$$\frac{\partial p}{\partial x} = -\frac{128}{\pi} \times \mu \frac{N \cdot s}{m^2} \times Q \frac{m^3}{s} \times \frac{1}{(0.006)^4 m^4} = -3.14 \times 10^{10} \mu Q \left(\frac{N}{m^3}\right)$$

Using data from Appendix A, at 40°C,

Fluid	$\nu \left(\frac{m^2}{s}\right)$	$Q \left(\frac{m^3}{s}\right)$	$\mu \left(\frac{N \cdot s}{m^2}\right)$	$\mu Q (N \cdot m)$	$\frac{\partial p}{\partial x} \left(\frac{N}{m^3}\right)$
Water	6.57×10^{-7}	7.10×10^{-6}	6.51×10^{-4}	4.62×10^{-9}	-145
SAE 10W oil	3.8×10^{-5}	4.10×10^{-4}	3.4×10^{-2}	1.39×10^{-5}	-4.36×10^{5}

$\frac{\partial p}{\partial x}$

Q_{ma}

{Note $Q \sim \nu = \frac{\mu}{\rho}$ and $\frac{\partial p}{\partial x} \sim \mu Q \sim \frac{\mu^2}{\rho}$.}

Given: Viscosity of water is to be determined by measuring pressure drop and flow rate through tygon tubing of length, $L = 50$ ft and diameter, $D = 0.125 \pm 0.010$ in.

Find: (a) Maximum volume flow rate at which flow would be laminar
(b) Pressure drop corresponding to this Q
(c) Estimate of experimental uncertainty in "measured" viscosity
(d) Way in which set up might be improved.

Solution:

Assume: steady, fully developed laminar flow in the tube

Flow is expected to remain laminar up to $Re = 2300$.

$$Re = \frac{\rho D \bar{V}}{\mu} = \frac{D}{\nu}\frac{Q}{A} = \frac{D}{\nu}\frac{Q}{\pi D^2/4} = \frac{4Q}{\pi D \nu}$$

To determine Q, we need to know ν. Assume $T = 70°F$. Then

$\nu = 1.05 \times 10^{-5}$ ft²/s (Table A.7)

$\mu = 2.04 \times 10^{-5}$ lbf·s/ft² (Table A.7)

$$\therefore Q_{max} = \frac{\pi D \nu Re}{4} = \frac{\pi}{4} \times 0.125\,\text{in} \times 1.08\times10^{-5}\,\frac{\text{ft}^2}{\text{sec}} \times 2300 \times \frac{\text{ft}}{12\,\text{in}} = 2.03\times10^{-4}\,\frac{\text{ft}^3}{\text{sec}}$$ ← Q_{max}

The corresponding pressure drop, $\Delta p = p_1 - p_2$, can be determined from Eq. 8.13c

$$Q = \frac{\pi}{128}\frac{\Delta p D^4}{\mu L} \quad \text{.....(8.13c)}$$

$$\therefore \Delta p = \frac{128 \mu L Q}{\pi D^4} = \frac{128}{\pi} \times 2.04\times10^{-5}\,\frac{\text{lbf·s}}{\text{ft}^2} \times 50\,\text{ft} \times 2.03\times10^{-4}\,\frac{\text{ft}^3}{\text{s}} \times \frac{1}{(0.125\,\text{in})^4}\left(\frac{12\,\text{in}}{\text{ft}}\right)^4$$

$$\Delta p = 716\ \text{lbf/ft}^2 = 4.97\ \text{lbf/in}^2$$ ← Δp

Equation 8.13c can be used to determine μ from measurements of Δp and Q. Thus

$$\mu = \frac{\pi}{128}\frac{\Delta p D^4}{L Q} \quad \text{or} \quad \mu = \mu(\Delta p, D, L, Q)$$

From uncertainty analysis

$$u_\mu = \pm\left[\left(\frac{\Delta p}{\mu}\frac{\partial \mu}{\partial \Delta p}u_{\Delta p}\right)^2 + \left(\frac{D}{\mu}\frac{\partial \mu}{\partial D}u_D\right)^2 + \left(\frac{L}{\mu}\frac{\partial \mu}{\partial L}u_L\right)^2 + \left(\frac{Q}{\mu}\frac{\partial \mu}{\partial Q}u_Q\right)^2\right]^{1/2}$$

Evaluating, $\frac{\Delta p}{\mu}\frac{\partial \mu}{\partial \Delta p} = \frac{\Delta p}{\mu}\frac{\pi}{128}\frac{D^4}{LQ} = 1$, $\frac{D}{\mu}\frac{\partial \mu}{\partial D} = \frac{D}{\mu}\frac{4D^3}{\pi}\frac{\pi}{128}\frac{\Delta p}{LQ} = 4$

$\frac{L}{\mu}\frac{\partial \mu}{\partial L} = \frac{L}{\mu}(-1)\frac{\pi}{128}\frac{\Delta p D^4}{LQ^2} = -1$, $\frac{Q}{\mu}\frac{\partial \mu}{\partial Q} = \frac{Q}{\mu}(-1)\frac{\pi}{128}\frac{\Delta p D^4}{LQ^2} = -1$

Thus $u_\mu = \left[(u_{\Delta p})^2 + (4u_D)^2 + (-u_L)^2 + (-u_Q)^2\right]^{1/2}$

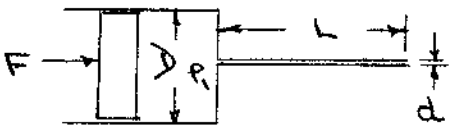
Since $u_D = \frac{\delta D}{D} = \pm\frac{0.01}{0.125} = \pm 8\%$, $u_\mu \approx 4u_D = 32\%$ ← u_μ

The set up could be improved by reducing u_D. Use somewhat larger diameter tube and/or more uniform diameter tube.

Given: Hypodermic needle of diameter, $d = 0.100$ mm and length $L = 25.0$ mm is attached to a syringe of diameter, $D = 10.0$ mm. The syringe is filled with saline solution of viscosity, $\mu = 5\mu_{H_2O}$. The maximum force on the plunger is $F = 45.0$ N.

Find: the maximum flow rate at which saline can be delivered.

Solution:



Model the flow as steady, fully developed laminar flow in a circular tube.

Assume: (1) discharge is to p_{atm}
(2) fluid is at $T = 20$C.

Then, the volume flow rate Q can be evaluated from Eq. 8.13c

$$Q = \frac{\pi \Delta p D^4}{128 \mu L}$$

where $\Delta p = p_1 - p_{atm}$ and $\Delta p = \frac{F}{A} = \frac{4F}{\pi D^2} = \frac{4}{\pi} \times 45\,N \times \frac{1}{(0.01\,m)^2} = 573\,kPa$

$\mu = 5\mu_{H_2O}$ and from Table A.8, $\mu_{H_2O} = 1 \times 10^{-3}$ kg/m·s

Then $Q = \frac{\pi \Delta p D^4}{128 \mu L} = \frac{\pi}{128} \times 573 \times 10^3 \frac{N}{m^2} \times (10^{-2}\,m)^4 \times \frac{1}{5 \times 10^{-3}} \frac{m \cdot s}{kg} \times \frac{1}{0.025\,m} \times \frac{kg \cdot m}{N \cdot s^2}$

$Q = 11.3\ m^3/s$ ◀ Q

Check Re

$$Re = \frac{\rho D \bar{V}}{\mu} = \frac{\rho D}{\mu} \frac{Q}{\pi/4 D^2} = \frac{\rho D}{\mu} \frac{4Q}{\pi D^2} = \frac{4 \rho Q}{\pi \mu D}$$

Assume $\rho_{saline} = \rho_{H_2O}$, then

$$Re = \frac{4}{\pi} \frac{\rho Q}{\mu D} = \frac{4}{\pi} \times 999 \frac{kg}{m^3} \times 9.27 \times 10^{-9} \frac{m^3}{s} \times \frac{1}{5 \times 10^{-3}} \frac{m \cdot s}{kg} \times \frac{1}{0.01\,m}$$

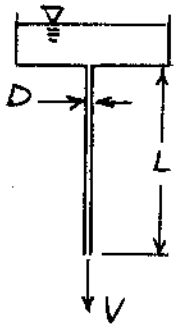
$Re = 23.8$ (flow is definitely laminar)

Given: Commercial viscometer, $D = 0.31$ mm and $L = 73$ mm.

Size chosen so $\Delta t \simeq 200$ s, and $u_{\Delta \forall}$ is negligible.

Find: Estimate the maximum uncertainty in D to allow measurement of μ within ± 1 percent.

Solution: Computing equation: $Q = \dfrac{\pi \Delta p D^4}{128 \mu L}$ (8.130)



Assumptions: (1) Laminar flow
(2) Fully developed flow

Solving for viscosity, $\mu = \dfrac{\pi \Delta p D^4}{128 Q L}$

But $\Delta p \sim \rho g L$ and $Q \sim \Delta \forall / \Delta t$. Thus

$$\mu \sim \rho g L \frac{\Delta t}{\Delta \forall} \frac{D^4}{L} = \frac{\rho g \Delta t D^4}{\Delta \forall}$$

From Appendix E,

$$u_\mu = \pm \left[(u_{\Delta t})^2 + (4 u_D)^2 + (u_{\Delta \forall})^2 \right]^{1/2}$$

Neglecting $u_{\Delta t}$ and $u_{\Delta \forall}$ compared to u_D,

$$u_\mu \simeq \pm \left[(4 u_D)^2 \right]^{1/2} = \pm 4 u_D = \pm 4 \frac{\delta D}{D}$$

Thus

$$\delta D = \pm \frac{u_\mu}{4} D = \pm \frac{0.01}{4} \times 0.31 \text{ mm} \times \frac{\text{m}}{1000 \text{ mm}}$$

$$\delta D = \pm 0.775 \, \mu\text{m} \qquad \delta D$$

{Such a small tolerance would be impossible to hold in any manufacturing operation. Therefore capillary viscometers are calibrated using a liquid of known viscosity in the range of interest.}

Given: Resistance to fluid flow, $R \equiv \frac{\Delta p}{Q}$

Tube 50 mm long and 0.25 mm inside diameter.

Find: (a) Show that for laminar flow, $R = \frac{128 \mu L}{\pi D^4}$

(b) Maximum Δp for laminar flow in given tube at 40°C for (1) kerosine and (2) castor oil.

Solution: Apply results developed in section 8-4 for laminar flow

Computing equation: $\Delta p = \frac{128 \mu Q L}{\pi D^4}$

Assumptions: (1) Laminar flow, $Re \leq 2300$
(2) Fully developed flow

Then $R = \frac{\Delta p}{Q} = \frac{128 \mu L}{\pi D^4}$ ← R

To calculate Δp, first write Q in terms of Re

$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{\rho D}{\mu} \frac{Q}{A} = \frac{4 \rho Q}{\pi \mu D} \quad \text{or} \quad Q = \frac{\pi \mu D Re}{4 \rho}$$

and

$$\Delta p = \frac{128 \mu Q L}{\pi D^4} = \frac{128 \mu L}{\pi D^4} \frac{\pi \mu D Re}{4 \rho} = 32 Re \frac{L}{D^3} \frac{\mu^2}{\rho}$$

Flow may be laminar up to $Re = 2300$, so

$$\Delta p_{max} = 32 (2300)\, 0.050\, m \times \frac{1}{(0.00025)^3 m^3} \times \mu^2 \frac{N^2 s^2}{m^2} \times \frac{m^3}{\rho\, kg} \times \frac{kg \cdot m}{N \cdot s^2}$$

$$\Delta p_{max}\,(Pa) = 2.36 \times 10^{14} \frac{[\mu\,(N \cdot s/m^2)]^2}{\rho\,(kg/m^3)}$$

Using data from Table A.2 (assume T = 40°C) and Fig. A.2,

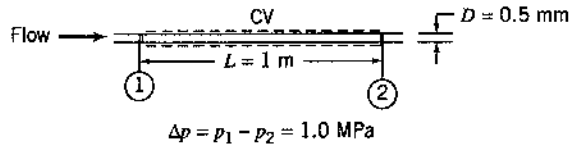
Fluid	$\mu (N \cdot s/m^2)$	$\rho (kg/m^3)$	Δp_{max}
Kerosine	1.1×10^{-3}	819	349 kPa
Castor oil	0.24	968	14.0 GPa

← Δp_{max}

{ For castor oil, Δp_{max} is about 140,000 atmospheres. The chance for turbulent flow is slim, indeed! }

Given: Capillary viscometer of Example Problem 8.4. $Q = 880\ mm^3/s$.

Least counts are: ± 0.01 MPa for Δp, ± 0.01 mm for D, $\pm 5\ mm^3/s$ for Q, and ± 1.00 mm for L. SG of test liquid is 0.82.



Find: (a) Effect of tube diameter on experimental uncertainty.
(b) If proper choice of diameter can minimize uncertainty.

Solution: Viscosity is given by $\mu = \frac{\pi \Delta p D^4}{128 L Q}$, so the uncertainty is

$$u_\mu = \pm \left[u_{\Delta p}^2 + (4u_D)^2 + u_L^2 + u_Q^2 \right]^{\frac{1}{2}} \qquad (1)$$

$$= \pm \left[\left(\pm \frac{0.01}{1.00}\right)^2 + \left(\pm 4\ \frac{0.01}{0.50}\right)^2 + \left(\pm \frac{0.001}{1.00}\right)^2 + \left(\pm \frac{5}{880}\right)^2 \right]^{\frac{1}{2}}$$

$$= \pm \left[(\pm 0.01)^2 + (\pm 0.08)^2 + (\pm 0.001)^2 + (\pm 0.00568)^2 \right]^{\frac{1}{2}}$$

$4u_D$ is the largest influence on u_μ ← u_D

$u_{\Delta p}$ is the second largest influence on u_μ

$$u_\mu = \pm \left[0.00653 \right]^{\frac{1}{2}} = \pm 0.0808 \text{ or } \pm 8.08 \text{ percent}$$ ← u_μ

To reduce u_D, increase diameter. Then reduce Q to maintain Re = constant

$Re = \frac{\rho \bar{V} D}{\mu}$, so $\bar{V}D$ = constant. But $\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2}$, so $\bar{V}D \sim \frac{Q}{D}$ = constant

But Δp will be affected. $\Delta p = \frac{128 \mu L Q}{\pi D^4}$, so $\Delta p \sim \frac{Q}{D^4} \sim \frac{1}{D^3}$ (since $\frac{Q}{D}$ = const.)

Representative values may be computed as follows:

D (mm)	Q (mm^3/s)	Δp (MPa)	$u_{\Delta p}$ (—)	$4u_D$ (—)	u_L (—)	u_Q (—)	u_μ (—)
0.50	880	1.00	0.0100	0.0800	0.001	0.00568	0.0808
0.55	968	0.751	0.0133	0.0727	0.001	0.00517	0.0741
0.60	1056	0.579	0.0173	0.0667	0.001	0.00473	0.0690
0.65	1144	0.455	0.0220	0.0615	0.001	0.00437	0.0655
0.70	1232	0.364	0.0274	0.0571	0.001	0.00406	0.0635
0.75	1320	0.296	0.0338	0.0533	0.001	0.00379	0.0632
0.80	1408	0.244	0.0410	0.0500	0.001	0.00355	0.0647
0.85	1496	0.204	0.0491	0.0471	0.001	0.00334	0.0681
0.90	1584	0.171	0.0583	0.0444	0.001	0.00316	0.0734
0.95	1672	0.146	0.0686	0.0421	0.001	0.00299	0.0805
1.00	1760	0.125	0.0800	0.0400	0.001	0.00284	0.0895

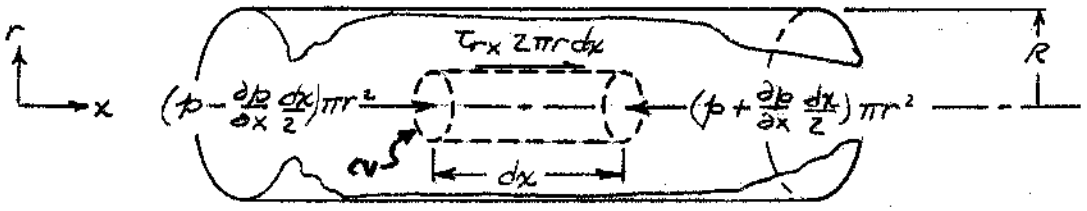
← u_μ (min.)

The uncertainty in D drops quickly as D increases. Although $u_{\Delta p}$ increases, there is a diameter that minimizes u_μ.

The optimum diameter is $D \approx 0.75$ mm. ← D_{opt}

(Note that the entrance length would increase, since $L_e/D = 0.06\ Re$.)

Given: Fully-developed laminar flow in a circular pipe, with cylindrical control volume as shown.



Find: (a) Forces acting on CV.
(b) Expression for velocity distribution.

Solution: The forces on a CV of radius r are shown above.

Apply the x component of momentum to CV shown.

Basic equations: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$, $\tau_{rx} = \mu\frac{du}{dr}$

(F_{Bx} = 0 (1); $\frac{\partial}{\partial t}$ = 0 (2); $\int_{CS}$ = 0 (3))

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Fully-developed flow

Then

$$F_{Sx} = (p-\frac{\partial p}{\partial x}\frac{dx}{2})\pi r^2 + \tau_{rx}\, 2\pi r\, dx - (p+\frac{\partial p}{\partial x}\frac{dx}{2})\pi r^2 = 0$$

Cancelling and combining terms,

$$-r\frac{\partial p}{\partial x} + 2\tau_{rx} = 0 \quad \text{or} \quad \tau_{rx} = \mu\frac{du}{dr} = \frac{r}{2}\frac{\partial p}{\partial x}$$

Thus $$\frac{du}{dr} = \frac{r}{2\mu}\frac{\partial p}{\partial x}$$

and

$$u = \frac{r^2}{4\mu}\frac{\partial p}{\partial x} + C_1$$

To evaluate C_1, apply the boundary condition $u = 0$ at $r = R$. Thus

$$C_1 = -\frac{R^2}{4\mu}\frac{\partial p}{\partial x}$$

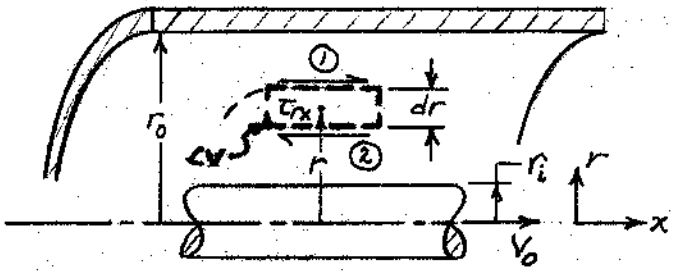
and

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x}(r^2 - R^2) = -\frac{R^2}{4\mu}\frac{\partial p}{\partial x}\left[1-\left(\frac{r}{R}\right)^2\right]$$ ← u

which is identical to Eq. 8.12.

Given: Fully-developed laminar flow in an annulus as shown. The inner section is stationary; the outer moves at V_0.

Assume $\frac{\partial p}{\partial x} = 0$.



Find: (a) $\tau(r)$ in terms of C_1.
(b) $V(r)$ in terms of C_1, C_2.
(c) Evaluate C_1, C_2.

Solution: Apply x component of momentum equation, using annular CV shown.

Basic Equations: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$; $\tau_{rx} = \mu \frac{du}{dr} = \tau$

($F_{Sx} + F_{Bx}$: F_{Bx} = 0(1); $\frac{\partial}{\partial t}\int_{CV} u\rho d\forall$ = 0(2); $\int_{CS} u\rho \vec{V}\cdot d\vec{A}$ = 0(3))

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Fully-developed flow

Then
$$F_{Sx} = F_① - F_② = \left(\tau + \frac{d\tau}{dr}\frac{dr}{2}\right)2\pi\left(r+\frac{dr}{2}\right)dx - \left(\tau - \frac{d\tau}{dr}\frac{dr}{2}\right)2\pi\left(r-\frac{dr}{2}\right)dx = 0$$

Neglecting products of differentials, this reduces to

$$\tau + r\frac{d\tau}{dr} = 0 \quad \text{or} \quad \frac{d}{dr}(r\tau) = 0$$

Thus $r\tau = C_1$ or $\tau = \frac{C_1}{r}$ ← $\tau(r)$

But $\tau = \mu\frac{du}{dr}$, so $\frac{du}{dr} = \frac{C_1}{\mu r}$

and $u = \frac{C_1}{\mu}\ln r + C_2$ ← $u(r)$

To evaluate constants C_1 and C_2, use boundary conditions.

At $r = r_i$, $u = V_0$, so $V_0 = \frac{C_1}{\mu}\ln r_i + C_2$

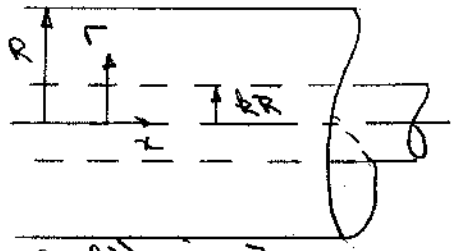
At $r = r_0$, $u = 0$, so $0 = \frac{C_1}{\mu}\ln r_0 + C_2$ and $C_2 = -\frac{C_1}{\mu}\ln r_0$

Thus, subtracting, $V_0 = \frac{C_1}{\mu}\ln\left(\frac{r_i}{r_0}\right)$ or $C_1 = \frac{\mu V_0}{\ln(r_i/r_0)}$ so $C_2 = -\frac{V_0 \ln r_0}{\ln(r_i/r_0)}$

Finally

$$u = \frac{V_0}{\ln(r_i/r_0)}(\ln r - \ln r_i) = V_0\frac{\ln(r/r_0)}{\ln(r_i/r_0)}$$ ← $u(r)$

Given: Fully developed laminar flow with pressure gradient, $\frac{dp}{dx}$, in the annulus shown



(a) Show that the velocity profile is given by $u = -\frac{R^2}{4\mu}\left(\frac{dp}{dx}\right)\left[1-\left(\frac{r}{R}\right)^2 + \frac{(1-k^2)}{\ln(1/k)}\ln\frac{r}{R}\right]$

(b) Obtain an expression for the location ($\alpha = r/R$) of maximum u as a function of k.

(c) Plot α vs k.

(d) Compare limiting case, $k \to 0$, with flow in circular pipe.

Solution: We may use the results of the differential control volume analysis of Section 8-3 to write

$$u = \frac{r^2}{4\mu}\frac{dp}{dx} + \frac{C_1}{\mu}\ln r + C_2 \quad \text{----(1)}$$

The boundary conditions are $u=0$ at $r=R$

$u=0$ at $r=kR$

Substituting the boundary condition

$$0 = \frac{R^2}{4\mu}\frac{dp}{dx} + \frac{C_1}{\mu}\ln R + C_2 \quad \text{----(2)}$$

$$0 = \frac{k^2R^2}{4\mu}\frac{dp}{dx} + \frac{C_1}{\mu}\ln kR + C_2 \quad \text{----(3)}$$

Subtracting, $0 = \frac{R^2}{4\mu}\frac{dp}{dx}(1-k^2) + \frac{C_1}{\mu}(\ln R - \ln kR)$

$$\therefore C_1 = -\frac{R^2}{4}\frac{dp}{dx}\frac{(1-k^2)}{\ln(1/k)}$$

From Eq. 2

$$C_2 = -\frac{R^2}{4\mu}\frac{dp}{dx} + \frac{R^2}{4\mu}\frac{dp}{dx}\frac{(1-k^2)}{\ln(1/k)}\ln R$$

Substituting for C_1 and C_2 into Eq. 1 gives

$$u = \frac{r^2}{4\mu}\frac{dp}{dx} - \frac{R^2}{4\mu}\frac{dp}{dx}\frac{(1-k^2)}{\ln(1/k)}\ln r - \frac{R^2}{4\mu}\frac{dp}{dx} + \frac{R^2}{4\mu}\frac{dp}{dx}\frac{(1-k^2)}{\ln(1/k)}\ln R$$

$$u = \frac{1}{4\mu}\frac{dp}{dx}\left[r^2 - R^2 - \frac{R^2(1-k^2)}{\ln(1/k)}(\ln r - \ln R)\right]$$

$$u = -\frac{R^2}{4\mu}\frac{dp}{dx}\left[1-\left(\frac{r}{R}\right)^2 + \frac{(1-k^2)}{\ln(1/k)}\ln\frac{r}{R}\right] \quad \leftarrow u$$

To locate max u, set $\tau_{rx} = \mu\frac{du}{dr} = 0$

$$\tau_{rx} = \mu\frac{du}{dr} = -\frac{R^2}{4}\frac{dp}{dx}\left[-\frac{2r}{R^2} + \frac{(1-k^2)}{\ln(1/k)}\frac{1}{r}\right]$$

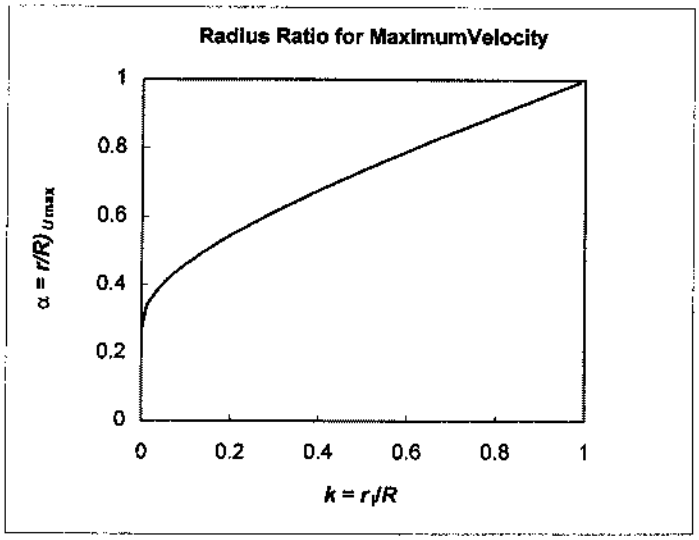
Set $\tau_{r\theta} = 0$ at $r = \alpha R$. Then

$$0 = -\frac{2\alpha R}{R^2} + \frac{(1-k^2)}{\ln(1/k)}\frac{1}{\alpha R}$$

and

$$\alpha = \left[\frac{1}{2}\frac{(1-k^2)}{\ln(1/k)}\right]^{1/2} \quad \alpha = \alpha(k)$$

$k = r_i/R$	$\alpha = r/R)_{U\max}$
0.001	0.269
0.01	0.329
0.02	0.357
0.05	0.408
0.08	0.444
0.1	0.464
0.2	0.546
0.4	0.677
0.6	0.791
0.8	0.898
0.95	0.975
0.99	0.995

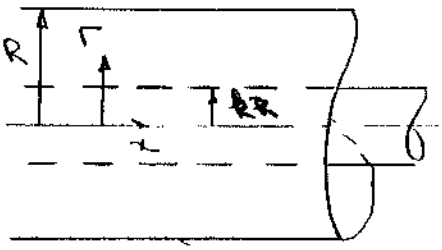


For $k \to 0$, $\alpha \to 0$ and $u = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)\left[1-\left(\frac{r}{R}\right)^2\right]$

This agrees with the results for flow in a circular pipe

As $k \to 1.0$, $\alpha \to 1.0$ and the flow behaves like flow between stationary infinite parallel plates.

Given: Fully developed laminar flow in the annulus shown, with pressure gradient $\partial p/\partial x$. The velocity profile is given by



$$u = -\frac{R^2}{4\mu}\frac{\partial p}{\partial x}\left[1-\left(\frac{r}{R}\right)^2 + \frac{(1-k^2)}{\ln(1/k)}\ln\frac{r}{R}\right]$$

(a) Show that the volume flow rate is given by

$$Q = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x}\left[(1-k^4) - \frac{(1-k^2)^2}{\ln(1/k)}\right]$$

(b) Obtain an expression for the average velocity

(c) Compare limiting case, $k \to 0$, with flow in a circular pipe.

Solution: The volume flow rate is given by

$$Q = \int u\,dA = \int_{kR}^{R} u\,2\pi r\,dr = 2\pi\int_{kR}^{R} u\,r\,dr$$

$$= 2\pi\left(-\frac{R^2}{4\mu}\frac{\partial p}{\partial x}\right)\int_{kR}^{R}\left[r - \frac{r^3}{R^2} + \frac{(1-k^2)}{\ln(1/k)}\,r\ln\frac{r}{R}\right]dr$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\int_{k}^{1}\left[\frac{r}{R} - \left(\frac{r}{R}\right)^3 + \frac{(1-k^2)}{\ln(1/k)}\,\frac{r}{R}\ln\frac{r}{R}\right]d\left(\frac{r}{R}\right)$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4 + \frac{(1-k^2)}{\ln(1/k)}\left\{\left(\frac{r}{R}\right)^2\left[\frac{1}{2}\ln\left(\frac{r}{R}\right) - \frac{1}{4}\right]\right\}\right]_k^1$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1}{2} - \frac{k^2}{2} - \frac{1}{4} + \frac{k^4}{4} + \frac{(1-k^2)}{\ln(1/k)}\left\{-\frac{1}{4} - k^2\left[\frac{1}{2}\ln k - \frac{1}{4}\right]\right\}\right]$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1}{4} - \frac{k^2}{2} + \frac{k^4}{4} + \frac{(1-k^2)}{\ln(1/k)}\left\{-\frac{1}{4} + \frac{k^2}{4} - k^2\frac{1}{2}\ln k\right\}\right]$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1-2k^2+k^4}{4} + \frac{(1-k^2)}{\ln(1/k)}\,\frac{(k^2-1)}{4} - \frac{(1-k^2)}{\ln(1/k)}\,k^2\,\frac{1}{2}\ln k\right]$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1-2k^2+k^4}{4} - \frac{(1-k^2)^2}{4\ln(1/k)} + \frac{k^4-k^4}{2}\right]$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1-2k^2+k^4+2k^2-2k^4}{4} - \frac{(1-k^2)^2}{4\ln(1/k)}\right]$$

$$Q = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x}\left[(1-k^4) - \frac{(1-k^2)^2}{\ln(1/k)}\right] \quad \blacksquare\ Q$$

The average velocity, $V = \dfrac{Q}{\pi R^2}$

The area is given by

$$A = \int dA = \int 2\pi r\, dr = 2\pi R^2 \int_k^1 \frac{r}{R}\, d\left(\frac{r}{R}\right)$$

$$A = 2\pi R^2 \left[\frac{1}{2}\left(\frac{r}{R}\right)^2\right]_k^1 = 2\pi R^2 \cdot \frac{1}{2}(1-k^2) = \pi R^2(1-k^2)$$

Thus

$$V' = \frac{Q}{A} = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x} \cdot \frac{1}{\pi R^2}\left[\frac{(1-k^4)}{(1-k^2)} - \frac{(1-k^2)}{\ln(1/k)}\right]$$

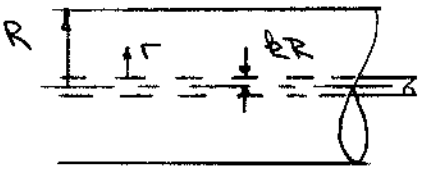
$$V' = -\frac{R^2}{8\mu}\frac{\partial p}{\partial x}\left[\frac{(1-k^4)}{(1-k^2)} - \frac{(1-k^2)}{\ln(1/k)}\right] \longleftarrow \bar{V}$$

For $k \to 0$

$$Q = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x} \quad \text{and} \quad \bar{V} = -\frac{R^2}{8\mu}\frac{\partial p}{\partial x}$$

These agree with the results for flow in a circular pipe.

Given: Fully developed laminar flow in a circular pipe is converted to flow in an annulus by insertion of a thin wire along the centerline



(a) Use results of Problem 8.51 to obtain an expression for the percent change in pressure drop as a function of radius ratio k

(b) Plot percent change in ΔP vs k for $0.001 \le k \le 0.10$

Solution: The results of problem 8.48 give

$$Q = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x}\left[(1-k^4) - \frac{(1-k^2)^2}{\ln(1/k)}\right]$$

Thus

$$\frac{\Delta P}{L} = -\frac{\partial p}{\partial x} = \frac{8\mu Q}{\pi R^4} \times \frac{1}{\left[(1-k^4) - \frac{(1-k^2)^2}{\ln(1/k)}\right]}$$

For $k=0$, $\quad \frac{\Delta P}{L} = \frac{8\mu Q}{\pi R^4}$

$$\text{Percent change} = \frac{\Delta P/L - \Delta P/L|_{k=0}}{\Delta P/L|_{k=0}} = \frac{1}{\left[(1-k^4) - \frac{(1-k^2)^2}{\ln(1/k)}\right]} - 1$$

$$\% \text{ change} = \frac{1 - \left[(1-k^4) - \frac{(1-k^2)}{\ln(1/k)}\right]}{\left[(1-k^4) - \frac{(1-k^2)^2}{\ln(1/k)}\right]}$$

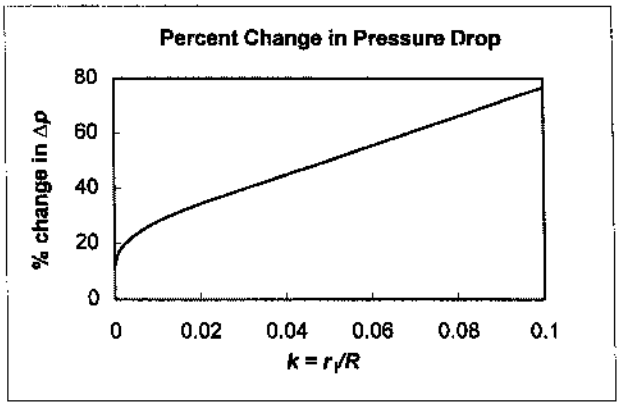
For small k,

$$\% \text{ change} = \frac{1 - \left[1 - \frac{1}{\ln(1/k)}\right]}{\left[1 - \frac{1}{\ln(1/k)}\right]} = \frac{1 - \left[1 + \frac{1}{\ln k}\right]}{\left[1 + \frac{1}{\ln k}\right]} = \frac{-\frac{1}{\ln k}}{\left[1 + \frac{1}{\ln k}\right]}$$

$$\% \text{ change} = -\frac{1}{\ln k\left(1 + \frac{1}{\ln k}\right)} \times 100$$

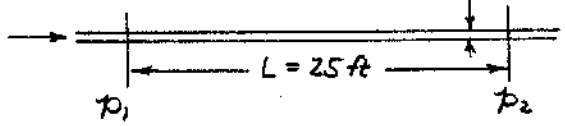
% change

$k = r_i/R$	% change in Δp
0.0001	12.2
0.0002	13.3
0.0005	15.1
0.001	16.9
0.002	19.2
0.005	23.3
0.01	27.7
0.02	34.3
0.05	50.1
0.1	76.8



The plot shows that even the smallest of wires causes a significant increase in pressure drop for a given flow rate.

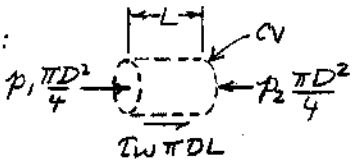
Given: Horizontal pipe with fully developed turbulent flow.



$\Delta p = p_1 - p_2 = 3$ psi

Find: Wall shear stress, τ_w.

Solution: Apply momentum equation to cylindrical CV:



Basic equation: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$ (=0(1), =0(2), =0(3))

Assumptions: (1) Horizontal pipe
(2) Steady flow
(3) Fully developed flow

Then

$$F_{Sx} = p_1 \frac{\pi D^2}{4} + \tau_w \pi D L - p_2 \frac{\pi D^2}{4} = 0 \quad \text{or} \quad \tau_w = \frac{p_2 - p_1}{4}\frac{D}{L} = -\frac{\Delta p D}{4L}$$

or

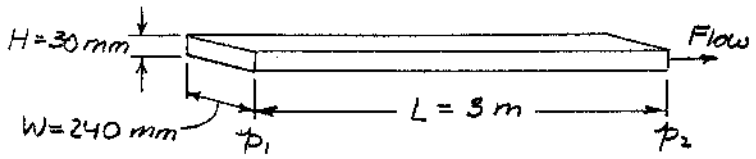
$$\tau_w = -\frac{1}{4} \times 3 \frac{\text{lbf}}{\text{in.}^2} \times 6 \text{ in.} \times \frac{1}{25 \text{ ft}} \times 12 \frac{\text{in.}}{\text{ft}} = -2.16 \text{ lbf/ft}^2$$

τ_w

Since $\tau_w < 0$, it acts to left on fluid, to right on pipe wall.

Given: Horizontal rectangular channel with fully developed flow of water:

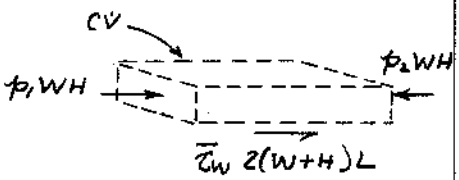
$\Delta p = p_1 - p_2 = 1.78$ kPa



Find: Average wall shear stress, $\bar{\tau}_w$.

Solution: Apply momentum equation to CV inside duct surface $\longmapsto x$

Basic equation: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$ (=0(1), =0(2), =0(3))



Assumptions: (1) Horizontal channel
(2) Steady flow
(3) Fully developed flow

Then

$$F_{Sx} = p_1 WH + \bar{\tau}_w\, 2(W+H)L - p_2 WH = 0 \quad \text{or} \quad \bar{\tau}_w = (p_2 - p_1)\frac{WH}{2(W+H)L} = -\Delta p \frac{H}{2\left(1+\frac{H}{W}\right)L}$$

or

$$\bar{\tau}_w = -\frac{1.78\ \text{kPa}}{2} \times 0.03\ \text{m} \times \frac{1}{(1+30/240)} \times \frac{1}{3\ \text{m}} = -7.91\ \text{Pa}\ (-7.91\ \text{N/m}^2)$$

$\bar{\tau}_w$ ←

Since $\tau_w < 0$, it acts to left on fluid, to right on channel wall.

Given: Flow of kerosene at 70°F in a smooth tube with $D = 1$ in. The Reynolds number is 4000, so flow can be either laminar or turbulent. The pressure gradient in these two cases is

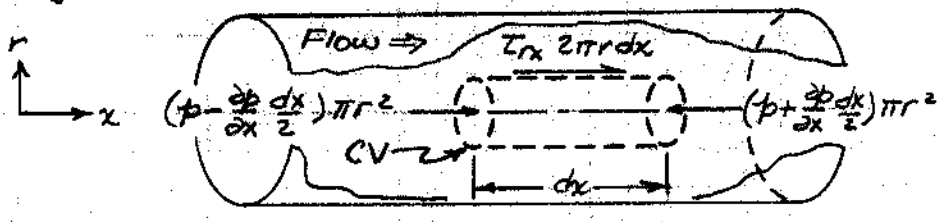
$$\text{Laminar } \frac{\partial p}{\partial x} = -0.20 \frac{\text{lbf}}{\text{ft}^3}, \quad \text{Turbulent } \frac{\partial p}{\partial x} = -0.50 \frac{\text{lbf}}{\text{ft}^3}$$

Plot: Variation of shear stress as a function of radius for both flow conditions.

Solution: Apply the momentum equation to a cylindrical control volume to relate shear stress to $\partial p/\partial x$.

Basic equation: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$ ($F_{Sx} = \mathcal{O}(1)$, $F_{Bx} = \mathcal{O}(2)$, $\int_{CS} = \mathcal{O}(3)$)

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Fully-developed flow



Then $F_{Sx} = 0$

Substituting,

$$F_{Sx} = \left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right)\pi r^2 + \tau_{rx}\, 2\pi r dx - \left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)\pi r^2 = 0$$

Cancelling and combining terms,

$$-r\frac{\partial p}{\partial x} + 2\tau_{rx} = 0 \quad \text{or} \quad \tau_{rx} = \frac{r}{2}\frac{\partial p}{\partial x}$$ ← τ_{rx}

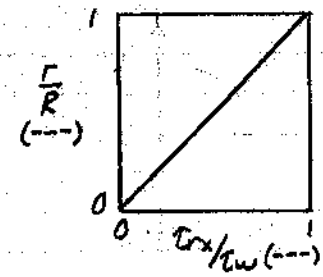
Thus the shear stress is <u>proportional to radius</u> for <u>both</u> laminar and turbulent flow. The maximum magnitude is the value at the tube wall, where $r = R = D/2$. Thus

$$\tau_{rx})_{max} = \frac{D}{4}\frac{\partial p}{\partial x} = \frac{1\text{ in.}}{4} \times \frac{\text{ft}}{12\text{ in.}} \times (-0.20)\frac{\text{lbf}}{\text{ft}^3} = -4.17 \times 10^{-3} \frac{\text{lbf}}{\text{ft}^2} \text{ (laminar)}$$

and

$$\tau_{rx})_{max} = \frac{1\text{ in.}}{4} \times \frac{\text{ft}}{12\text{ in.}} \times (-0.50)\frac{\text{lbf}}{\text{ft}^3} = -1.04 \times 10^{-2} \frac{\text{lbf}}{\text{ft}^2} \text{ (turbulent)}$$

Plotting,



Plot ←

Given: Liquid with viscosity and density of water in laminar flow in a smooth capillary tube. $D = 0.25$ mm, $L = 50$ mm.

Find: (a) Maximum volume flow rate.
(b) Pressure drop to produce this flow rate.
(c) Corresponding wall shear stress.

Solution: Flow will be laminar for $Re < 2300$.

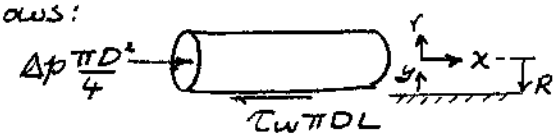
$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu} = \frac{Q}{A}\frac{D}{\nu} = \frac{4Q}{\pi D^2}\frac{D}{\nu} = \frac{4Q}{\pi \nu D} < 2300$$

Thus (at $T = 20°C$)

$$Q < \frac{2300 \pi \nu D}{4} = \frac{2300\pi}{4} \times 1.0 \times 10^{-6} \frac{m^2}{s} \times 0.00025\, m = 4.52 \times 10^{-7} m^3/s \quad Q$$

(This flow rate corresponds to 27.1 mL/min.)

A force balance on a fluid element shows:



$$\Sigma F_x = \Delta p \frac{\pi D^2}{4} - \tau_w \pi D L = 0$$

or

$$\Delta p = \tau_w \frac{4L}{D}$$

For laminar pipe flow, $u = u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$, from Eq. 8.14. Thus

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right)_{y=0} = -\mu \left.\frac{\partial u}{\partial r}\right)_{r=R} = -\mu u_{max}\left(-\frac{2r}{R^2}\right)_{r=R} = \frac{2\mu u_{max}}{R}$$

But $u_{max} = 2\bar{V}$, so $\tau_w = \frac{2\mu 2\bar{V}}{D/2} = \frac{8\mu\bar{V}}{D} = 8\rho\nu\frac{\bar{V}}{D}$

Also

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 4.52 \times 10^{-7} \frac{m^3}{s} \times \frac{1}{(0.00025)^2 m^2} = 9.21\ m/s$$

Thus

$$\tau_w = 8 \times 999 \frac{kg}{m^3} \times 1.0 \times 10^{-6} \frac{m^2}{s} \times 9.21 \frac{m}{s} \times \frac{1}{0.00025\, m} \times \frac{N \cdot s^2}{kg \cdot m} = 294\ N/m^2\ (294\ Pa) \quad \tau_w$$

and

$$\Delta p = 4 \times 0.05\, m \times \frac{1}{0.00025 m} \times 294 \frac{N}{m^2} = 235\ kPa \quad \Delta p$$

Problem 8.57

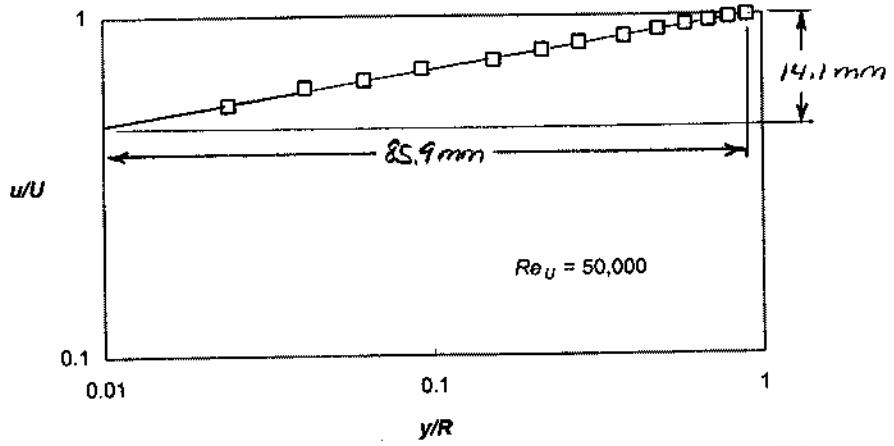
Given: Data measured by Laufer in fully developed turbulent pipe flow at $Re_U = 50{,}000$ are shown below.

Find: (a) Power-law exponent obtained graphically from the data plotted on log-log coordinates.

(b) Comparison with power-law exponent obtained from least-squares fit to the data using the results of Problem 8.63.

Solution: The data plot is shown below:

y/R	u/U
0.898	0.996
0.794	0.981
0.691	0.963
0.588	0.937
0.486	0.907
0.383	0.866
0.280	0.831
0.216	0.792
0.154	0.742
0.093	0.700
0.062	0.650
0.041	0.619
0.024	0.551



To obtain the power-law exponent from the plot, fair a straight line through the data by eye. Measure the horizontal and vertical changes to define the slope of the line. Then scale the results so the horizontal and vertical scales of each decade change on the plot are the same. (For this plot, the horizontal scale is 44.0 mm per decade; the vertical scale is 45.2 mm per decade.)

$$\frac{1}{n} = \frac{14.1\ mm}{45.2\ mm} \times \frac{44.0\ mm}{85.9\ mm} = 0.160\ ;\ n = 6.26$$ (From graph) n

Calculate the power-law exponent using the method of least squares (results of Problem 8.63):

ln y/R	ln u/U	ln (y/R) x ln (u/U)	$(\ln y/R)^2$
-0.108	-0.00401	0.000431	0.0116
-0.231	-0.01918	0.00442	0.0532
-0.370	-0.0377	0.0139	0.137
-0.531	-0.0651	0.0346	0.282
-0.722	-0.0976	0.0704	0.521
-0.960	-0.144	0.138	0.921
-1.27	-0.185	0.236	1.62
-1.53	-0.233	0.357	2.35
-1.87	-0.298	0.558	3.50
-2.38	-0.357	0.847	5.64
-2.78	-0.431	1.20	7.73
-3.19	-0.480	1.53	10.2
-3.73	-0.596	2.22	13.9
	Sums:	7.21	46.9

$$\frac{1}{n} = \frac{\Sigma[\ln(u/U)][\ln(y/R)]}{\Sigma[\ln(y/R)^2]} = \frac{7.21}{46.9}$$

$$\frac{1}{n} = 0.154$$

$$n = \frac{1}{0.154} = 6.49$$ (From values) n

Comparison shows good agreement between the measured and calculated power-law exponents. (The measured result is very sensitive to minor changes in the line faired through the data.)

Problem 8.58

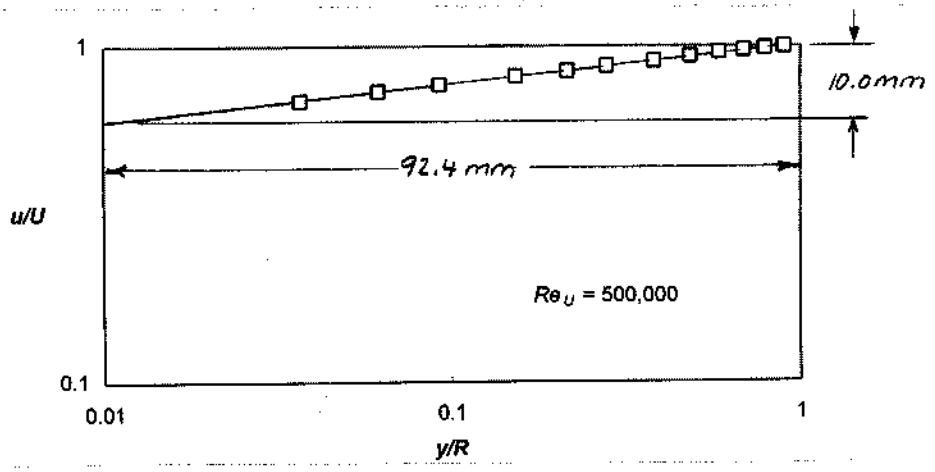
Given: Data measured by Laufer in fully developed turbulent pipe flow at $Re_U = 500{,}000$ are shown below.

Find: (a) Power-law exponent obtained graphically from the data plotted on log-log coordinates.

(b) Comparison with power-law exponent obtained from least-squares fit to the data using the results of Problem 8.63.

Solution: The data plot is shown below:

y/R	u/U
0.898	0.997
0.794	0.988
0.691	0.975
0.588	0.959
0.486	0.934
0.383	0.908
0.280	0.874
0.216	0.847
0.154	0.818
0.093	0.771
0.062	0.736
0.037	0.690



To obtain the power-law exponent from the plot, fair a straight line through the data by eye. Measure the horizontal and vertical changes to define the slope of the line. Then scale the results so the horizontal and vertical scales of each decade change on the plot are the same. (For this plot, the horizontal scale is 46.2 mm per decade; the vertical scale is 44.8 mm per decade.)

$$\frac{1}{n} = \frac{10.0\ mm}{44.8\ mm} \times \frac{46.2\ mm}{92.4\ mm} = 0.112 \; ; \; n = 8.93 \qquad \text{(From graph)} \quad n$$

Calculate the power-law exponent using the method of least squares (results of Problem 8.63):

ln y/R	ln u/U	ln (y/R) x ln (u/U)	(ln y/R)²
-0.108	-0.00300	0.000323	0.0116
-0.231	-0.0121	0.00278	0.0532
-0.370	-0.0253	0.00936	0.137
-0.531	-0.0419	0.0222	0.282
-0.722	-0.0683	0.0493	0.521
-0.960	-0.0965	0.0926	0.921
-1.27	-0.135	0.171	1.62
-1.53	-0.166	0.254	2.35
-1.87	-0.201	0.376	3.50
-2.38	-0.260	0.618	5.64
-2.78	-0.307	0.852	7.73
-3.30	-0.371	1.22	10.9
	Sums:	3.67	33.6

$$\frac{1}{n} = \frac{\Sigma[\ln(u/U)][\ln(y/R)]}{\Sigma[\ln(y/R)]^2} = \frac{3.67}{33.6}$$

$$\frac{1}{n} = 0.109$$

$$n = \frac{1}{0.109} = 9.17 \qquad \text{(From values)} \quad n$$

Comparison shows good agreement between the measured and calculated power-law exponents. (The measured result is very sensitive to minor changes in the line faired through the data.)

Given: Velocity profiles for pipe flow

$\frac{u}{U} = \left(1 - \frac{r}{R}\right)^{\frac{1}{n}}$ (turbulent); $\frac{u}{U} = 1 - \left(\frac{r}{R}\right)^2$ (laminar)

Find: (a) value of r/R at which $u = \bar{V}$ for each profile.

Plot: r/R vs n for $6 \le n \le 10$.

Solution:

Definition: $\bar{V} = \frac{Q}{A} = \frac{1}{A}\int u\,dA$

For laminar flow, $\bar{V} = \frac{1}{\pi R^2}\int_0^R U\left[1 - \left(\frac{r}{R}\right)^2\right] 2\pi r\,dr = 2U\int_0^1 \left[1 - \left(\frac{r}{R}\right)^2\right]\frac{r}{R}\,d\left(\frac{r}{R}\right)$

$\bar{V} = 2U\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right]_0^1 = \frac{U}{2}$

Thus $u = \bar{V}$ when $1 - \left(\frac{r}{R}\right)^2 = \frac{U}{2} = \frac{1}{2}$ or $\frac{r}{R} = 0.707$ laminar

For turbulent flow, $\bar{V} = \frac{1}{\pi R^2}\int_0^R U\left(1 - \frac{r}{R}\right)^{\frac{1}{n}} 2\pi r\,dr$

$\bar{V} = 2U\int_0^1 \left(1 - \frac{r}{R}\right)^{\frac{1}{n}} \frac{r}{R}\,d\left(\frac{r}{R}\right)$

To integrate let $m = 1 - \frac{r}{R}$. Then $\frac{r}{R} = 1 - m$, $d\left(\frac{r}{R}\right) = -dm$

and $\bar{V} = 2U\int_0^1 m^{\frac{1}{n}}(1-m)(-dm) = 2U\int_0^1 \left(m^{\frac{1}{n}} - m^{\frac{1}{n}+1}\right)dm$

$= 2U\left[\frac{n}{n+1}m^{\frac{1}{n}+1} - \frac{n}{2n+1}m^{\frac{1}{n}+\frac{n}{n}}\right]_0^1 = 2U\left[\frac{n}{n+1} - \frac{n}{2n+1}\right]$

$\bar{V} = 2U\left[\frac{n(2n+1) - n(n+1)}{(n+1)(2n+1)}\right] = U\frac{2n^2}{(n+1)(2n+1)}$ ..... (8.24)

For $n = 7$, $\bar{V} = U\frac{2(7)^2}{8 \times 15} = 0.817U$

Thus $u = \bar{V}$ when $\left(1 - \frac{r}{R}\right)^{1/7} = 0.817$ or $\frac{r}{R} = 1 - (0.817)^7 = 0.758$ turb

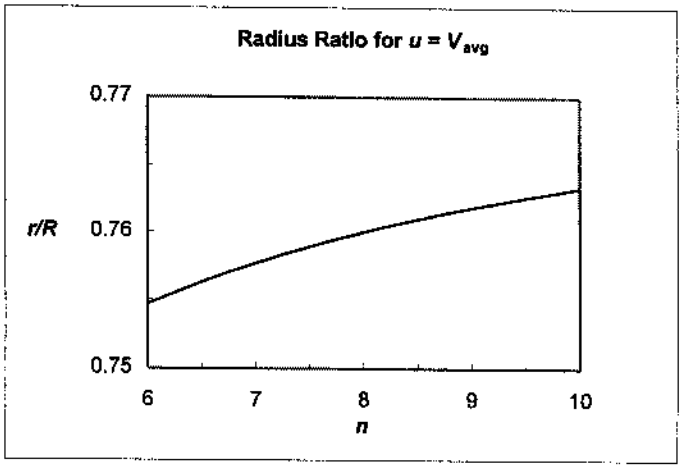
From Eq 8.24, $u = \bar{V}$ when

$\left(1 - \frac{r}{R}\right)^{\frac{1}{n}} = \frac{2n^2}{(n+1)(2n+1)}$

or

$\frac{r}{R} = 1 - \left[\frac{2n^2}{(n+1)(2n+1)}\right]^n$

r/R is plotted vs n.



Given: Empirical fit for velocity profile in turbulent flow through smooth pipes

$$\frac{u}{U} = \left(1 - \frac{r}{R}\right)^{1/n} \quad \text{where } n = -1.7 + 1.8 \log Re_U$$

Show: that $\frac{\bar{V}}{U} = \frac{2n^2}{(n+1)(2n+1)}$

Plot: $\frac{\bar{V}}{U}$ vs Re_U

Solution:

Definition: $\bar{V} = \frac{Q}{A} = \frac{1}{A}\int u\,dA = \frac{1}{\pi R^2}\int_0^R U\left(1-\frac{r}{R}\right)^{1/n} 2\pi r\,dr$

$$\bar{V} = 2U\int_0^1 \left(1-\frac{r}{R}\right)^{1/n} \frac{r}{R}\,d\left(\frac{r}{R}\right)$$

To integrate, let $\eta = 1 - \frac{r}{R}$, so $\frac{r}{R} = 1-\eta$ and $d\left(\frac{r}{R}\right) = -d\eta$

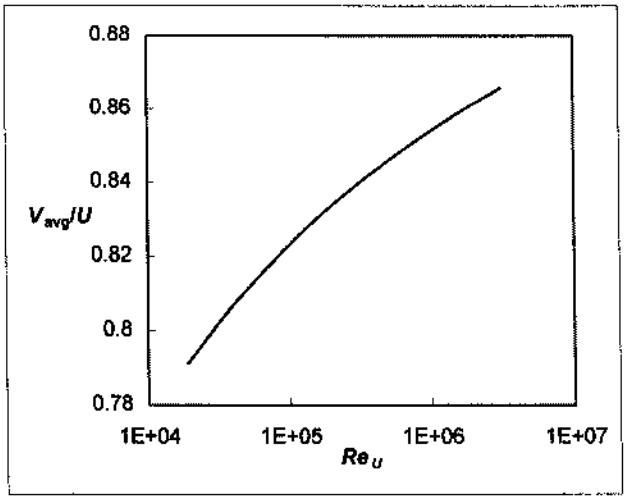
$$\bar{V} = 2U\int_1^0 \eta^{\frac{1}{n}}(1-\eta)(-d\eta) = 2U\int_0^1 \eta^{\frac{1}{n}}(1-\eta)\,d\eta$$

$$\frac{\bar{V}}{U} = 2\left[\frac{n}{(n+1)}\eta^{\frac{n+1}{n}} - \frac{n}{(2n+1)}\eta^{\frac{2n+1}{n}}\right]_0^1 = 2\left[\frac{n}{(n+1)} - \frac{n}{(2n+1)}\right]$$

$$\frac{\bar{V}}{U} = 2\left[\frac{(2n+1)n - (n+1)n}{(n+1)(2n+1)}\right] = \frac{2n^2}{(n+1)(2n+1)} \qquad (8.24)$$

To plot $\frac{\bar{V}}{U}$ vs Re_U, $n = -1.7 + 1.8\log Re_U$ (8.23)

n	V_{avg}/U	Re_U
6.0	0.791	1.90E+04
6.5	0.805	3.60E+04
7.0	0.817	6.85E+04
7.5	0.827	1.29E+05
8.0	0.837	2.45E+05
8.5	0.845	4.65E+05
9.0	0.853	8.80E+05
9.5	0.860	1.67E+06
10.0	0.866	3.16E+06



Given: Power-law exponent n as a function of Re_U and ratio $\bar{V}/U$ as a function of n.

$$n = -1.7 + 1.8 \log Re_U \qquad (8.23)$$

$$\frac{\bar{V}}{U} = \frac{2n^2}{(n+1)(2n+1)} \qquad (8.24)$$

Plot: $\bar{V}/U$ vs $Re_{\bar{V}}$

Solution:

Prepare a Table of values

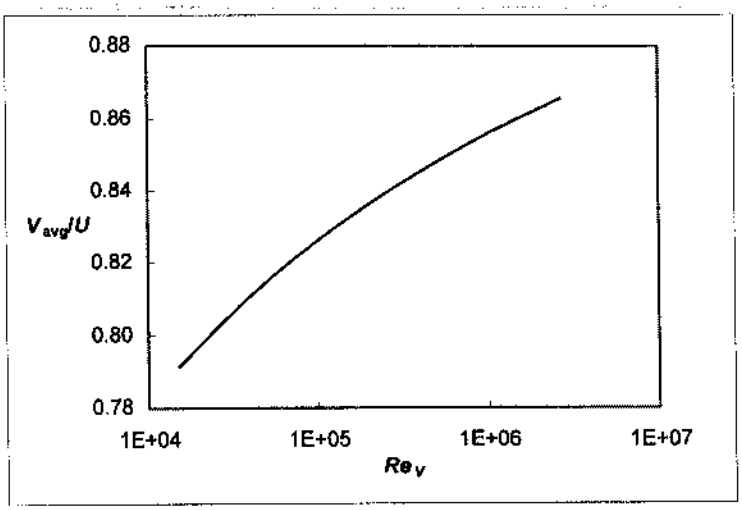
Re_U

n from Eq 8.23

$\bar{V}/U$ from Eq. 8.24

$Re_{\bar{V}} = \frac{\bar{V}}{U} \times Re_U$

Re_U	n	$Re_{\bar{V}}$	V_{avg}/U
1.90E+04	6.00	1.50E+04	0.791
3.60E+04	6.50	2.90E+04	0.805
6.85E+04	7.00	5.59E+04	0.817
1.29E+05	7.50	1.07E+05	0.827
2.45E+05	8.00	2.05E+05	0.837
4.65E+05	8.50	3.93E+05	0.845
8.80E+05	9.00	7.50E+05	0.853
1.67E+06	9.50	1.44E+06	0.860
3.16E+06	10.0	2.74E+06	0.866



Given: Velocity profiles for pipe flow:
$\frac{u}{U} = 1 - \left(\frac{r}{R}\right)^2$ (laminar); $\frac{u}{U} = \left(1 - \frac{r}{R}\right)^{1/n}$ (turbulent)
Momentum coefficient, β, where $\beta \dot{m} V = \int_A u\,\rho u\,dA$

Find: (a) β for laminar profile.
(b) β for turbulent profile with $n = 7$

Plot: β vs n for turbulent profile over range $6 \le n \le 10$, and compare with laminar profile

Solution:

$$\beta = \frac{\dot{m}V}{\rho V^2 A} \int_A u\,\rho u\,dA = \frac{1}{\rho V^2 \pi R^2} \int_0^R u\,\rho u\, 2\pi r\,dr$$

Noting that $\frac{u}{U} = f\left(\frac{r}{R}\right)$

$$\beta = \frac{1}{\pi R^2}\left[\frac{U}{V}\right]^2 \int_0^R \left(\frac{u}{U}\right)^2 2\pi r\,dr = 2\left[\frac{U}{V}\right]^2 \int_0^1 \left(\frac{u}{U}\right)^2 \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) \quad \text{--- (1)}$$

For laminar flow, $\frac{u}{U} = 1 - \left(\frac{r}{R}\right)^2$, so $\left(\frac{u}{U}\right)^2 = 1 - 2\left(\frac{r}{R}\right)^2 + \left(\frac{r}{R}\right)^4$, and

$$\beta = 2\left[\frac{U}{V}\right]^2 \int_0^1 \left[\left(\frac{r}{R}\right) - 2\left(\frac{r}{R}\right)^3 + \left(\frac{r}{R}\right)^5\right] d\left(\frac{r}{R}\right) = 2\left[\frac{U}{V}\right]^2 \left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right]$$

$$\beta = \frac{1}{3}\left[\frac{U}{V}\right]^2. \quad \text{For this case } U = 2V \text{ so}$$

$$\beta = \frac{1}{3}[2]^2 = \frac{4}{3} \qquad \beta_{laminar}$$

For turbulent flow, $\frac{u}{U} = \left(1 - \frac{r}{R}\right)^{1/n}$, so $\left(\frac{u}{U}\right)^2 = \left(1 - \frac{r}{R}\right)^{2/n}$, and

$$\beta = 2\left[\frac{U}{V}\right]^2 \int_0^1 \left(1 - \frac{r}{R}\right)^{2/n} \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right)$$

To integrate, let $m = 1 - \frac{r}{R}$. Then $\frac{r}{R} = 1 - m$, $d\left(\frac{r}{R}\right) = -dm$, so

$$\beta = 2\left[\frac{U}{V}\right]^2 \int_1^0 m^{2/n}(1-m)(-dm) = 2\left[\frac{U}{V}\right]^2 \int_0^1 \left(m^{2/n} - m^{\frac{2}{n}+1}\right) dm$$

$$\beta = 2\left[\frac{U}{V}\right]^2 \left[\frac{n}{n+2} m^{\frac{2}{n}+1} - \frac{n}{2n+2} m^{\frac{2}{n}+2}\right]_0^1 = 2\left[\frac{U}{V}\right]^2 \left[\frac{n}{n+2} - \frac{n}{2n+2}\right]$$

$$\beta = 2\left[\frac{U}{V}\right]^2 \left[\frac{(2n+2)n - (n+2)n}{(n+2)(2n+2)}\right] = 2\left[\frac{U}{V}\right]^2 \left[\frac{n^2}{(n+2)(2n+2)}\right] \quad \text{--- (2)}$$

From Eq. 8.24, $\frac{U}{V} = \frac{2n^2}{(n+1)(2n+1)}$

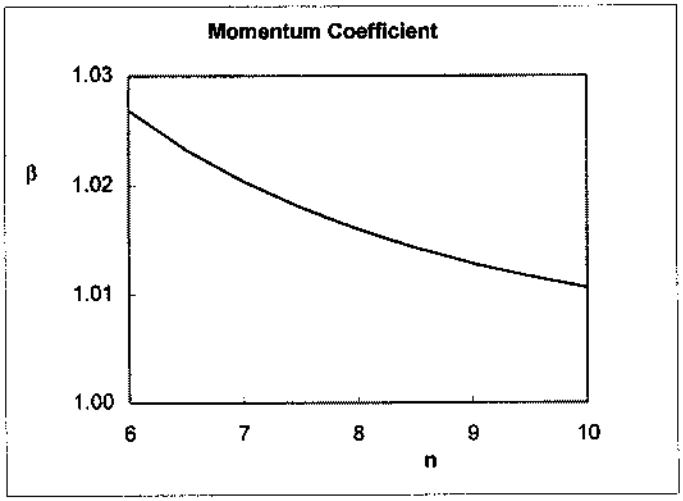
For $n = 7$, $\frac{U}{V} = 0.817$, so

$$\beta = \left[\frac{1}{0.817}\right]^2 \frac{2(7)^2}{(9)(15)} = 1.02 \qquad \beta_{turb}$$

To plot β vs n

- $C_T = \dfrac{2n^2}{(n+1)(2n+1)}$ ..... (8.24)
- $\beta = \left[\dfrac{1}{\Delta U_0}\right]^2 \dfrac{n^2}{(n+2)(n+1)}$
- $\beta = \dfrac{(n+1)(2n+1)^2}{4n^2(n+2)}$

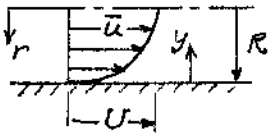
n	β
6.0	1.027
6.5	1.023
7.0	1.020
7.5	1.018
8.0	1.016
8.5	1.014
9.0	1.013
9.5	1.012
10.0	1.011



13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Empirical power-law velocity profile for turbulent pipe flow,

$$\frac{\bar{u}}{U} = \left(1 - \frac{r}{R}\right)^{1/n} = \left(\frac{y}{R}\right)^{1/n}$$

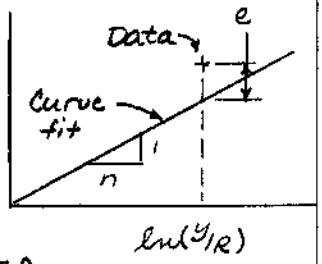


Find: Using method of least squares, show that a best-fit value of n may be calculated from measured data as

$$\frac{1}{n} = \frac{\Sigma \ln(\bar{u}/U)\ln(y/R)}{\Sigma[\ln(y/R)]^2}$$

Solution: Express the profile as $\ln(\bar{u}/U) = \frac{1}{n}\ln(y/R)$. Express the error between the curve fit and a measured point as e, where

$$e = \ln(\bar{u}/U)_m - \ln(\bar{u}/U) = \ln(\bar{u}/U)_m - \frac{1}{n}\ln(y/R)_m$$



Then

$$e^2 = \left[\ln(\bar{u}/U)_m\right]^2 - \frac{2}{n}\ln(\bar{u}/U)_m \ln(y/R)_m + \frac{1}{n^2}\left[\ln(y/R)_m\right]^2$$

The mean-square error is minimized by setting $\partial e^2/\partial n = 0$.

Thus

$$\frac{\partial e^2}{\partial n} = +\frac{2}{n^2}\ln(\bar{u}/U)_m \ln(y/R)_m - \frac{2}{n^3}\left[\ln(y/R)_m\right]^2 = 0$$

or

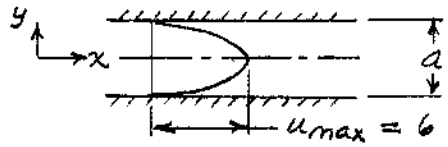
$$n = \frac{\Sigma\left[\ln(y/R)_m\right]^2}{\Sigma \ln(\bar{u}/U)_m \ln(y/R)_m}$$

or

$$\frac{1}{n} = \frac{\Sigma \ln(\bar{u}/U)_m \ln(y/R)_m}{\Sigma\left[\ln(y/R)_m\right]^2}$$

$\frac{1}{n}$

Given: Fully developed laminar flow of water between parallel plates.



Find: Kinetic energy flux at a cross-section.

Solution: From the analysis of Section 8-2, for laminar flow between plates,

$$\frac{u}{u_{max}} = 1-\left(\frac{y}{a/2}\right)^2$$

Then $$kef = \int_A \frac{V^2}{2}\rho V dA = \int_A \frac{\rho}{2}u^3 dA = \int_{-a/2}^{a/2}\frac{\rho}{2}u^3 w\,dy = \int_0^{a/2}\rho u^3 w\,dy$$

$$kef = \rho u_{max}^3 w\int_0^{a/2}\left[1-\left(\frac{y}{a/2}\right)^2\right]^3 dy = \frac{\rho u_{max}^3 w a}{2}\int_0^1 (1-\eta^2)^3 d\eta \quad \text{where } \eta = \frac{y}{a/2}$$

Evaluating,

$$(1-\eta^2)^3 = 1-3\eta^2+3\eta^4-\eta^6$$

The integral is

$$\int_0^1 (1-\eta^2)^3 d\eta = \left[\eta-\frac{3}{3}\eta^3+\frac{3}{5}\eta^5-\frac{1}{7}\eta^7\right]_0^1 = \frac{3}{5}-\frac{1}{7} = \frac{21-5}{35} = \frac{16}{35}$$

Finally,

$$kef = \frac{\rho u_{max}^3 w a}{2}\frac{16}{35} = \frac{8\rho u_{max}^3 w a}{35}$$

$$= \frac{8}{35}\times 999\frac{kg}{m^3}\times(6)^3\frac{m^3}{s^3}\times 0.03\,m\times 0.0002\,m\times\frac{N\cdot s^2}{kg\cdot m}$$

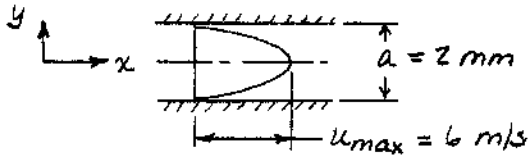
$$kef = 0.296\ N\cdot m/s \quad (0.296\ W)$$ ← kef

Check Reynolds number:

$$Re = \frac{\bar{V}a}{\nu} = \frac{2}{3}\frac{u_{max}a}{\nu} = \frac{2}{3}\times 6\frac{m}{s}\times 0.0002\,m\times\frac{s}{1.00\times10^{-6}\,m^2} = 800$$

$Re = 800 \ll 1500$, so flow is laminar.

Given: Flow of Problem 8.63: laminar flow of water between parallel plates.



Find: Kinetic energy coefficient, α

Solution: Apply definition of kinetic energy coefficient,

$$\alpha = \frac{\int_A \rho V^3 dA}{\dot{m}\bar{V}^2} \qquad \dot{m} = \rho \bar{V} A \tag{8.25b}$$

From the analysis of Section 8-2, for flow between parallel plates,

$$u = u_{max}\left[1-\left(\frac{y}{a/2}\right)^2\right] = \frac{3}{2}\bar{V}\left[1-\left(\frac{y}{a/2}\right)^2\right] \quad \text{since } u_{max} = \frac{3}{2}\bar{V} \tag{8.6c}$$

Substituting into Eq. 8.26b,

$$\alpha = \frac{\int_A \rho V^3 dA}{\dot{m}\bar{V}^2} = \frac{\int_A \rho u^3 dA}{\rho \bar{V} A \bar{V}^2} = \frac{1}{A}\int_A \left(\frac{u}{\bar{V}}\right)^3 dA = \frac{1}{wa}\int_{-a/2}^{a/2}\left(\frac{u}{\bar{V}}\right)^3 w\,dy = \frac{2}{a}\int_0^{a/2}\left(\frac{u}{\bar{V}}\right)^3 dy$$

Then

$$\alpha = \frac{2}{a}\frac{a}{2}\int_0^1 \left(\frac{u}{u_{max}}\right)^3\left(\frac{u_{max}}{\bar{V}}\right)^3 d\left(\frac{y}{a/2}\right) = \left(\frac{3}{2}\right)^3\int_0^1 (1-\eta^2)^3 d\eta \quad \text{where } \eta = \frac{y}{a/2}.$$

Evaluating,

$$(1-\eta^2)^3 = 1 - 3\eta^2 + 3\eta^4 - \eta^6$$

The integral is

$$\int_0^1 (1-\eta^2)^3 d\eta = \left[\eta - \frac{3}{3}\eta^3 + \frac{3}{5}\eta^5 - \frac{1}{7}\eta^7\right]_0^1 = \frac{3}{5} - \frac{1}{7} = \frac{21-5}{35} = \frac{16}{35}$$

Substituting,

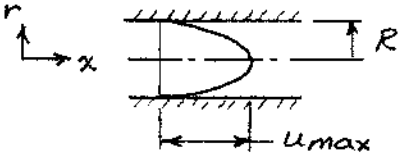
$$\alpha = \left(\frac{3}{2}\right)^3\int_0^1 (1-\eta^2)^3 d\eta = \frac{27}{8}\frac{16}{35} = \frac{54}{35} = 1.54$$

α

Check: From Eq. 8.6e, $\bar{V} = 4$ m/sec

$$\alpha = \frac{kef}{\dot{m}\frac{\bar{V}^2}{2}} = \frac{2\,kef}{\rho \bar{V}^3 A} = 2 \times 0.296 \frac{N\cdot m}{s} \times \frac{m^3}{999\,kg} \times \frac{s^3}{(4)^3 m^3} \times \frac{1}{(0.03)(0.0002)m^2} \times \frac{kg\cdot m}{N\cdot s^2} = 1.54 \checkmark\checkmark$$

Given: Fully developed laminar flow in a circular tube.



Find: Kinetic energy coefficient, α

Solution: Apply definition of kinetic energy coefficient,

$$\alpha = \frac{\int_A \rho V^3 dA}{\dot{m}\bar{V}^2}, \quad \dot{m} = \rho \bar{V} A \qquad (8.26b)$$

From the analysis of Section 8-3, for flow in a circular tube,

$$u = u_{max}\left[1-\left(\frac{r}{R}\right)^2\right] = 2\bar{V}\left[1-\left(\frac{r}{R}\right)^2\right] \quad \text{since} \quad u_{max} = 2\bar{V}$$

Substituting into Eq. 8.25b,

$$\alpha = \frac{\int_A \rho V^3 dA}{\dot{m}\bar{V}^2} = \frac{\int_A \rho u^3 dA}{\rho \bar{V} A \bar{V}^2} = \frac{1}{A}\int_A \left(\frac{u}{\bar{V}}\right)^3 dA = \frac{1}{\pi R^2}\int_0^R \left(\frac{u}{\bar{V}}\right)^3 2\pi r dr = 2\int_0^1 \left(\frac{u}{\bar{V}}\right)^3 \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right)$$

Then

$$\alpha = 2\int_0^1 \left(\frac{u}{u_{max}}\right)^3 \left(\frac{u_{max}}{\bar{V}}\right)^3 \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) = 2(2)^3 \int_0^1 (1-\eta^2)^3 \eta \, d\eta \quad \text{where } \eta = \frac{r}{R}$$

Evaluating,

$$(1-\eta^2)^3 \eta = \eta - 3\eta^3 + 3\eta^5 - \eta^7$$

The integral is

$$\int_0^1 (1-\eta^2)^3 \eta \, d\eta = \left[\frac{\eta^2}{2} - \frac{3}{4}\eta^4 + \frac{3}{6}\eta^6 - \frac{1}{8}\eta^8\right]_0^1 = \frac{1}{2} - \frac{3}{4} + \frac{1}{2} - \frac{1}{8} = \frac{1}{8}$$

Substituting,

$$\alpha = 16\int_0^1 (1-\eta^2)^3 \eta \, d\eta = 16 \times \frac{1}{8} = 2$$

α

Given: "Power law" velocity profile for turbulent flow, $\frac{u}{U} = \left(1-\frac{r}{R}\right)^{1/n}$

Kinetic energy coefficient, α, where $\alpha = \frac{1}{\frac{1}{2}\rho V^2} \int \frac{1}{2}\rho u^2 \, dA$.

Show: that $\alpha = \left[\frac{1+n}{2+n}\right]^3 \frac{2n^2}{(3+n)(3+2n)}$

Evaluate α for $n=7$

Plot: (a) α vs n, and (b) α vs Re_V, for turbulent power law profiles over the range $6 \le n \le 10$.

Solution:

From the definition of α,

$$\alpha = \frac{1}{\frac{1}{2}V^2} \int \frac{1}{2} u^2 \, dA = \frac{1}{\rho \pi R^2 \frac{1}{2} V^2} \int_0^R U^2 \left(1-\frac{r}{R}\right)^{2/n} \rho U \left(1-\frac{r}{R}\right)^{1/n} 2\pi r \, dr$$

$$= \frac{2\rho\pi R^2 U^3}{\rho \pi R^2 V^3} \int_0^1 \left(1-\frac{r}{R}\right)^{3/n} \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) = 2\left[\frac{U}{V}\right]^3 \int_0^1 \left(1-\frac{r}{R}\right)^{3/n} \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right)$$

To integrate, let $m = 1-\frac{r}{R}$. Then $\frac{r}{R} = 1-m$, and $d\left(\frac{r}{R}\right) = -dm$, so

$$\alpha = 2\left[\frac{U}{V}\right]^3 \int_1^0 m^{3/n} (1-m)(-dm) = 2\left[\frac{U}{V}\right]^3 \int_0^1 m^{3/n} (1-m)\, dm$$

$$\alpha = 2\left[\frac{U}{V}\right]^3 \left[\frac{n}{(n+3)} m^{\frac{n+3}{n}} - \frac{n}{(2n+3)} m^{\frac{2n+3}{n}}\right]_0^1$$

$$= 2\left[\frac{U}{V}\right]^3 \left[\frac{n}{(n+3)} - \frac{n}{(2n+3)}\right] = 2\left[\frac{U}{V}\right]^3 \left[\frac{(2n+3)n - n(n+3)}{(n+3)(2n+3)}\right]$$

$$\alpha = \left[\frac{U}{V}\right]^3 \frac{2n^2}{(n+3)(2n+3)} \quad \alpha$$

From Eq 8.24, $\frac{U}{V} = \frac{2n^2}{(n+1)(2n+1)}$

Then for $n=7$ $\frac{U}{V} = \frac{2(7)^2}{(8)(15)} = 0.817$ and

$$\alpha = \left[\frac{1}{0.817}\right]^3 \frac{2(7)^2}{(10)(17)} = 1.058 \quad \alpha_{n=7}$$

To plot α vs n

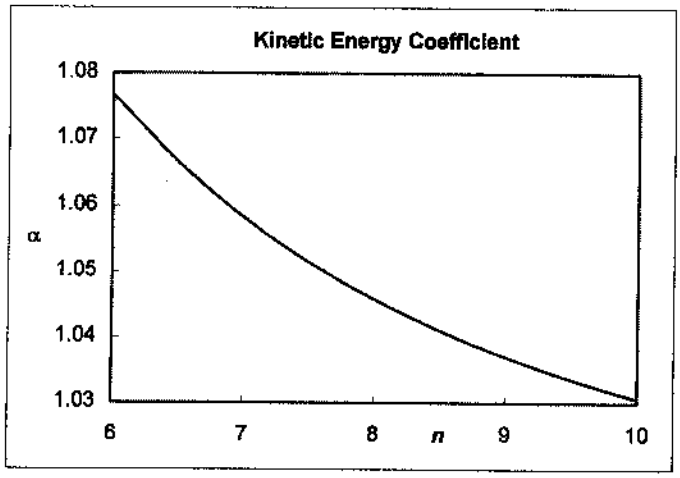
- $\frac{U}{V} = \frac{2n^2}{(n+1)(2n+1)}$ and $\alpha = \left[\frac{U}{V}\right]^3 \frac{2n^2}{(n+3)(2n+3)}$

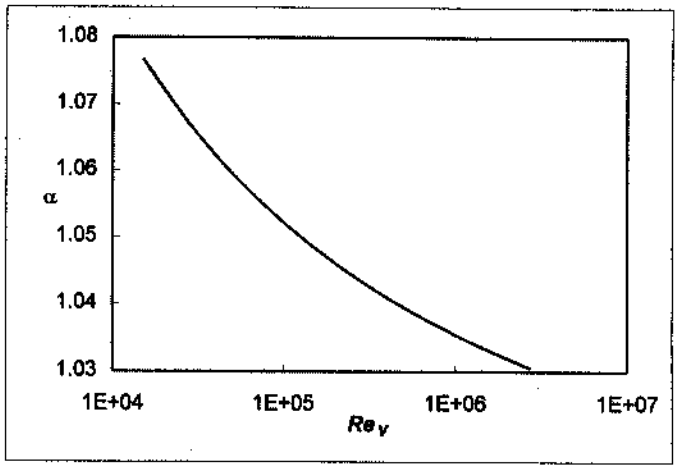
To plot α vs Re_V

- $n = -1.7 + 1.8 \log Re_U$ and $Re_V = \frac{U}{V} Re_U$

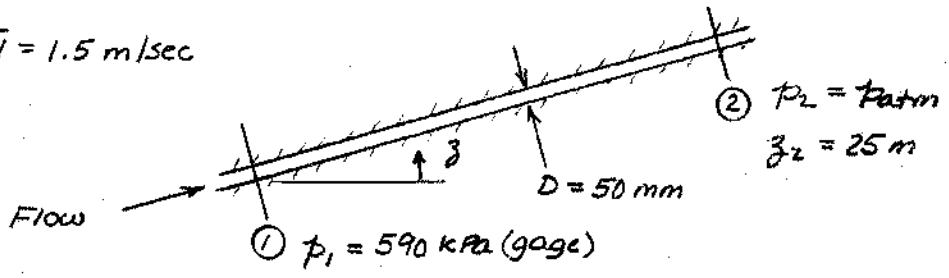
The plots are shown below.

Re_U	n	V_{avg}/U	α	Re_V
1.90E+4	6.0	0.791	1.077	1.50E+4
3.60E+4	6.5	0.805	1.067	2.90E+4
6.85E+4	7.0	0.817	1.058	5.59E+4
1.29E+5	7.5	0.827	1.052	1.07E+5
2.45E+5	8.0	0.837	1.046	2.05E+5
4.65E+5	8.5	0.845	1.041	3.93E+5
8.80E+5	9.0	0.853	1.037	7.50E+5
1.67E+6	9.5	0.860	1.034	1.44E+6
3.16E+6	10.0	0.866	1.031	2.74E+6





Given: Flow of water through a pipe with the conditions shown.



Find: Head loss between inlet and outlet of pipe.

Solution: Apply the energy equation for steady, incompressible flow.

Basic equation: $\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2 + h_{LT}$

($\alpha_1 = 1$ (1), $\bar{V}_1 = 0$, $z_1 = 0$, $p_2 = 0$ (2), $\alpha_2 = 1$ (1))

Assumptions: (1) Fully developed flow, so $\alpha_1 = \alpha_2$
(2) $p_2 = 0$ psig

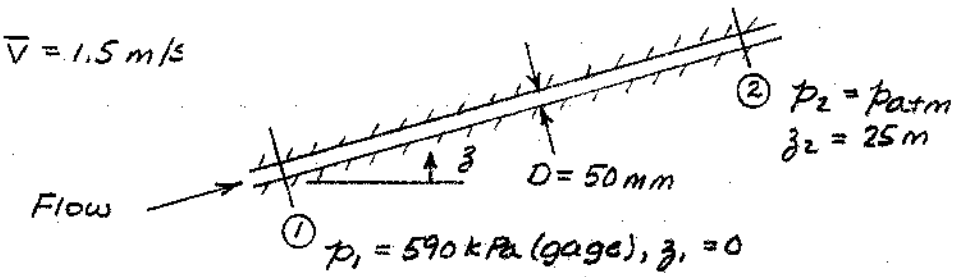
Then

$$h_{LT} = \frac{p_1}{\rho} - gz_2 = 5.90\times10^5\frac{N}{m^2}\times\frac{m^3}{999\,kg} - 9.81\frac{m}{s^2}\times 25\,m\times\frac{N\cdot s^2}{kg\cdot m}$$

$$h_{LT} = 345\ J/kg$$

h_{LT}

Given: Flow of water through a pipe at the conditions shown.



Flow conditions are held constant, but pipe is placed on a horizontal surface.

Find: Inlet pressure for new conditions.

Solution: Apply the energy equation for steady, incompressible pipe

Basic equation: $\frac{p_1}{\rho} + \alpha_1 \frac{\overline{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \alpha_2 \frac{\overline{V}_2^2}{2} + g z_2 + h_{LT}$ (1) $z_1 = 0$; (2) $p_2 = 0$; (1) $\overline{V}_1 = 0$

Assumptions: (1) Fully developed flow without area change, $\alpha_1 \frac{\overline{V}_1^2}{2} = \alpha_2 \frac{\overline{V}_2^2}{2}$

(2) $p_2 = 0$ psig

Then

$$h_{LT} = \frac{p_1}{\rho} - g z_2 = 5.90 \times 10^5 \frac{N}{m^2} \times \frac{m^3}{999\ kg} - 9.81 \frac{m}{s^2} \times 25\ m \times \frac{N \cdot s^2}{kg \cdot m}$$

$$h_{LT} = 345\ N \cdot m/kg$$

Changing the orientation of the pipe will not affect h_{LT}. To find the new p_1, apply the energy equation again, using previous assumptions.

Basic equation: $\frac{p_1}{\rho} + \alpha_1 \frac{\overline{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \alpha_2 \frac{\overline{V}_2^2}{2} + g z_2 + h_{LT}$ (1) $\overline{V}_1 = 0$; (2) $p_2 = 0$; (1) $z_1 = 0$

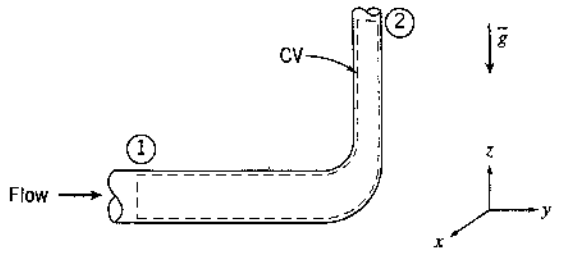
Then

$$p_1 = \rho h_{LT} = 999 \frac{kg}{m^3} \times 345 \frac{N \cdot m}{kg} = 3.45 \times 10^5 \frac{N}{m^2} \text{ or } 345\ kPa\ (gage)$$

New p_1

Given: Flow in configuration shown:

Section	p (psig)	$\bar{V}$ (ft/s)	z (ft)
1	10.2	5.5	7.5
2	6.5	11.2	10.5



Find: (a) Head loss (in ft)
(b) Head loss in energy per unit mass.

Solution: Apply the energy equation for pipe flow, Eq. 8.30:

Computing Equation: $\left(\frac{p_1}{\rho g} + \alpha_1 \frac{\bar{V}_1^2}{2g} + z_1\right) - \left(\frac{p_2}{\rho g} + \alpha_2 \frac{\bar{V}_2^2}{2g} + z_2\right) = \frac{h_{eT}}{g} = H_{eT}$

Assumptions: (1) Steady flow
(2) Incompressible flow (water)
(3) α_1 and α_2 approximately 1

Then

$$H_{eT} = \frac{p_1 - p_2}{\rho g} + \frac{\bar{V}_1^2 - \bar{V}_2^2}{2g} + z_1 - z_2$$

$$= (10.2 - 6.5)\frac{lbf}{in.^2} \times \frac{ft^3}{1.94\,slug} \times \frac{s^2}{32.2\,ft} \times 144\frac{in.^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2}$$

$$+ \frac{(5.5)^2 - (11.2)^2}{2}\frac{ft^2}{s^2} \times \frac{s^2}{32.2\,ft} + (7.5 - 10.5)\,ft$$

$$H_{eT} = 4.05\ ft \longleftarrow H_{eT}$$

and

$$h_{eT} = g\,H_{eT} = 32.2\frac{ft}{s^2} \times 4.05\ ft = 130\ ft^2/s^2$$

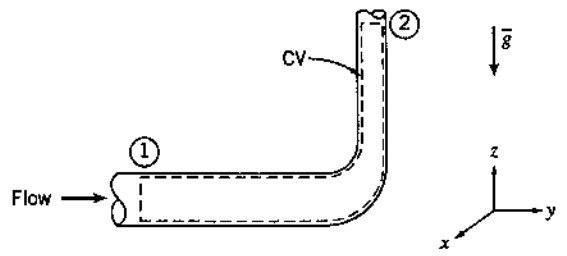
or

$$h_{eT} = 130\frac{ft^2}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} = 130\frac{lbf \cdot ft}{slug} \longleftarrow h_{eT}$$

Given: Flow through 90° reducing elbow

$H_{e_T} = 1.7$ ft, $p_1 - p_2 = 3.7$ psi

$\bar{V}_2 = 1.75\bar{V}_1$, $z_2 - z_1 = 5.5$ ft



Find: inlet velocity, $\bar{V}_1$

Solution:

Computing equation: $\left(\frac{p_1}{\rho g} + \alpha_1 \frac{\bar{V}_1^2}{2g} + z_1\right) - \left(\frac{p_2}{\rho g} + \alpha_2 \frac{\bar{V}_2^2}{2g} + z_2\right) = H_{e_T}$ (8.36)

Assumption: (1) $\alpha_1 = \alpha_2 = 1.0$, (2) fluid is water, $\rho = 1.94$ slug/ft^3

$$\bar{V}_2^2 - \bar{V}_1^2 = 2\frac{(p_1 - p_2)}{\rho} + 2g(z_1 - z_2) - g H_{e_T}$$

$$(1.75\bar{V}_1)^2 - \bar{V}_1^2 = 2 \times 3.7 \frac{\text{lbf}}{\text{in}^2} \times 144 \frac{\text{in}^2}{\text{ft}^2} \times \frac{\text{ft}^3}{1.94 \text{ slug}} \times \frac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2} + 2 \times 32.2 \frac{\text{ft}}{\text{s}^2} \times (-5.5 \text{ ft})$$

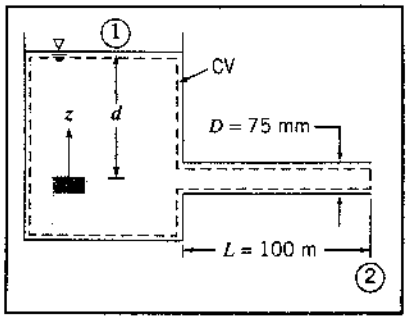
$$- 32.2 \frac{\text{ft}}{\text{s}^2} \times 1.7 \text{ ft}$$

$$2.063\bar{V}_1^2 = 140 \text{ ft}^2/\text{s}^2$$

$$\bar{V}_1 = 8.25 \text{ ft/s} \quad \longleftarrow \bar{V}_1$$

Given: Water flow from a reservoir through system shown

When $Q = 0.0067\ m^3/s$, $h_{e_T} = 2.85\ m$



Find: reservoir depth, d, to maintain this flow rate

Solution:

Computing equation: $\left(\frac{p_1}{\rho g} + \alpha_1 \frac{V_1^2}{2g} + z_1\right) - \left(\frac{p_2}{\rho g} + \alpha_2 \frac{V_2^2}{2g} + z_2\right) = h_{e_T}$ (1.30)

($\alpha_1 = \alpha(2)$)

Assumptions: (1) steady, incompressible flow

(2) $V_1 \approx 0$, $\alpha_2 \approx 1.0$

(3) $p_1 = p_2 = p_{atm}$

Then,

$$z_1 - z_2 = d = h_{e_T} + \frac{V_2^2}{2g}$$

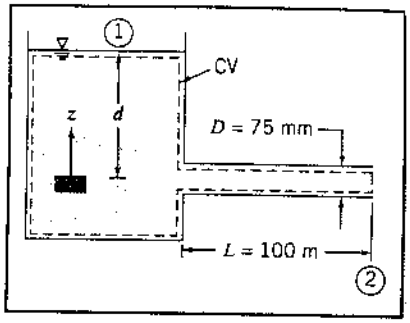
$$V_2 = \frac{Q}{A_2} = \frac{Q}{\pi D^2/4} = \frac{4}{\pi} \times 0.0067\ \frac{m^3}{s} \times \frac{1}{(0.075\ m)^2} = 1.52\ m/s$$

and

$$d = 2.85\ m + \frac{(1.52)^2}{2} \frac{m^2}{s^2} \times 9.81 \frac{s^2}{m} = 2.97\ m \quad \underline{\qquad d \qquad}$$

Given: Water flow from a reservoir through system shown.

When $d = 3.60\ m$, $H_{LT} = 1.75\ m$



Find: Volume flow rate, Q

Solution:

Computing equation: $\left(\frac{p_1}{\rho g} + \alpha_1 \frac{\bar{V}_1^2}{2g} + z_1\right) - \left(\frac{p_2}{\rho g} + \alpha_2 \frac{\bar{V}_2^2}{2g} + z_2\right) = H_{LT}$ (8.30)

($\bar{V}_1 = 0$, so $\alpha_1 \frac{\bar{V}_1^2}{2g} = 0$ (2))

Assumptions: (1) Steady, incompressible flow
(2) $\bar{V}_1 = 0$, $\alpha_2 = 1.0$
(3) $p_1 = p_2 = p_{atm}$

Then,

$$\bar{V}_2^2 = 2g\left[(z_1 - z_2) - H_{LT}\right]$$

$$\bar{V}_2 = 2 \times 9.81 \frac{m}{s^2}\left[3.60 - 1.75\right] m$$

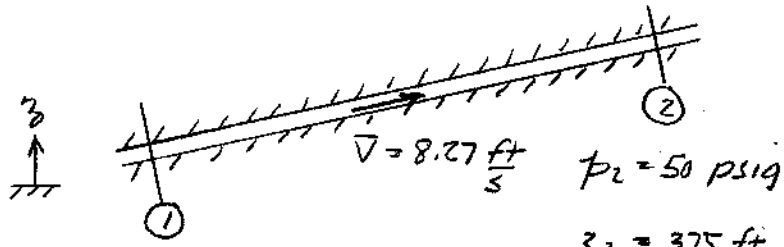
$$\bar{V}_2 = 6.03\ m/s$$

$$Q = A_2 \bar{V}_2 = \frac{\pi}{4} D^2 \bar{V} = \frac{\pi}{4} \times (0.075 m)^2 \times 6.03 \frac{m}{s} = 2.66 \times 10^{-2}\ m^3/s \quad \text{Q}$$

Given: Section of Alaskan pipeline with conditions shown.

Find: Head loss.

Solution:



Computing equation:

$$h_{lT} = \left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2}^{(1)} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2}^{(1)} + g z_2\right)$$

Assumptions: (1) Incompressible flow, so $\bar{V}_1 = \bar{V}_2$ (2) Fully developed so $\alpha_1 = \alpha_2$

(3) SG = 0.9 (Table A.2)

Then $$h_{lT} = \frac{p_1 - p_2}{SG\rho_{H_2O}} + g(z_1 - z_2)$$

$$h_{lT} = (1200 - 50)\frac{lbf}{in^2} \times 144\frac{in^2}{ft^2} \times \frac{ft^3}{(0.9)1.94\ slug} \times \frac{slug \cdot ft}{lbf \cdot s^2} + 32.2\frac{ft}{s^2}(150 - 375)ft$$

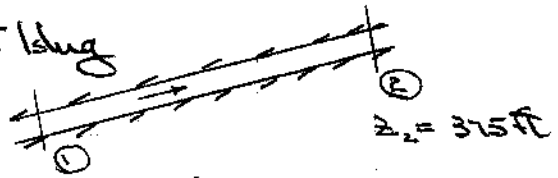
$$h_{lT} = 8.76 \times 10^4\ ft^2/s^2$$ h_{lT}

Also

$$H_{lT} = \frac{h_{lT}}{g} = 8.76 \times 10^4 \frac{ft^2}{s^2} \times \frac{s^2}{32.2\ ft} = 2{,}720\ ft$$ H_{lT}

Given: Section of Alaskan pipeline with conditions shown

$h_{e_{1\to 2}} = 74{,}400$ ft·lbf/slug



Find: outlet pressure, p_2

Solution:

Governing equation: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + gz_2\right) = h_{e_T}$ (8.29)

Assumptions: (1) incompressible flow, so $\bar{V}_1 = \bar{V}_2$
(2) fully developed so $\alpha_1 = \alpha_2$
(3) SG crude oil = 0.90 (Table A.2)

Then

$$p_2 = p_1 + \rho g(z_1 - z_2) - \rho h_{e_T}$$

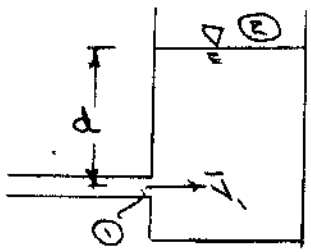
$$= 1200 \frac{\text{lbf}}{\text{ft}^2} + 0.90 \times 1.94 \frac{\text{slug}}{\text{ft}^3} \times 32.2 \frac{\text{ft}}{\text{s}^2} \times (-225\,\text{ft}) \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}} \times \frac{\text{ft}^2}{144\,\text{in}^2}$$

$$- 0.90 \times 1.94 \frac{\text{slug}}{\text{ft}^3} \times 74{,}400 \frac{\text{ft}\cdot\text{lbf}}{\text{slug}} \times \frac{\text{ft}^2}{144\,\text{in}^2}$$

$p_2 = 210$ psig ← p_2

Given: Water flows from a horizontal tube into a very large tank as shown.

$d = 8$ ft, $h_e = 21.1$ ft·lbf/slug



Find: Average flow speed in tube

Solution:

Apply definition of head loss, Eq 8.29,

$$\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + gz_2\right) = h_{eT}$$

At free surface, $\bar{V}_2 = 0$, $p_2 = p_{atm}$

At tube discharge $p_1 = \rho g d$, $z_1 = 0$. Assume $\alpha_1 = 1$

Then

$$gd + \frac{\bar{V}_1^2}{2} - gd = h_{eT}$$

$$\bar{V}_1^2 = 2h_{eT} = 2 \times 21.1 \frac{\text{ft·lbf}}{\text{slug}} \times \frac{\text{slug·ft}}{\text{lbf·s}^2} = 42.2 \text{ ft}^2/\text{s}^2$$

$$\bar{V}_1 = 6.50 \text{ ft/s} \quad \bar{V}_1$$

500 SHEETS, PILLAR 5 SQUARE
50 SHEETS EYE-EASE® 5 SQUARE
100 SHEETS EYE-EASE® 5 SQUARE
200 SHEETS EYE-EASE® 5 SQUARE
100 RECYCLED WHITE 5 SQUARE
200 RECYCLED WHITE 5 SQUARE
13-782
42-381
42-382
42-389
42-392
42-398
National® Brand
Made in U.S.A.

Given: Water flow at $Q = 3$ gpm through a horizontal 5/8 in. diameter garden hose. Pressure drop in $L = 50$ ft is 12.3 psi.

Find: Head loss

Solution: Computing equation is

$$h_{eT} = \left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right)^{(1)\,(3)} - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right)^{(1)\,(3)}$$

Assumptions: (1) Incompressible flow, so $\bar{V}_1 = \bar{V}_2$

(2) Fully developed so $\alpha_1 = \alpha_2$

(3) Horizontal, so $z_1 = z_2$

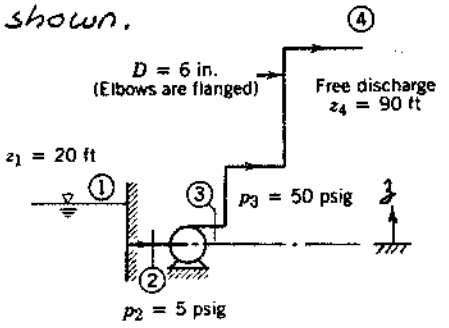
Then $$h_{eT} = \frac{p_1 - p_2}{\rho} = 12.3 \frac{lbf}{in.^2} \times \frac{ft^3}{1.94\ slug} \times 144 \frac{in.^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2}$$

$$h_{eT} = 913\ ft^2/s^2 \qquad h_{eT}$$

Also $$H_{eT} = \frac{h_{eT}}{g} = 913 \frac{ft^2}{s^2} \times \frac{s^2}{32.2\ ft} = 28.4\ ft \qquad H_{eT}$$

Given: Water pumped through flow system shown.

$Q = 2\ ft^3/s$



Find: (a) Head supplied by pump.
(b) Head loss between pump outlet and free discharge.

Solution: Apply energy equation to CV around pump for steady flow:

Computing equation: $$\dot{W}_{in} = \dot{m}\left[\left(\frac{p_3}{\rho} + \alpha_3\frac{\bar{V}_3^2}{2} + gz_3\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2\right)\right]$$

Assumptions: (1) Incompressible flow
(2) $\alpha_2\bar{V}_2^2 = \alpha_3\bar{V}_3^2 = \alpha_4\bar{V}_4^2$
(3) $z_2 = z_3$

Head is energy per unit mass (or per unit weight). On a unit mass basis,

$$\Delta h_{pump} = \frac{\dot{W}_{in}}{\dot{m}} = \frac{1}{\rho}(p_3 - p_2) = (50-5)\frac{lbf}{in.^2}\times\frac{ft^3}{1.94\,slug}\times\frac{144\,in.^2}{ft^2} = 3{,}340\ ft\cdot lbf/slug$$ Δh_{pump}

Apply energy equation for steady, incompressible pipe flow between ③, ④:

Computing equation: $$\left(\frac{p_3}{\rho} + \alpha_3\frac{\bar{V}_3^2}{2} + gz_3\right) - \left(\frac{p_4}{\rho} + \alpha_4\frac{\bar{V}_4^2}{2} + gz_4\right) = h_{LT} \qquad (8.29)$$

Assumptions: (4) $p_4 = p_{atm}$
(5) $\alpha_3\bar{V}_3^2 = \alpha_4\bar{V}_4^2$

Then
$$h_{LT} = \frac{p_3}{\rho} - gz_4 = 50\frac{lbf}{in.^2}\times\frac{ft^3}{1.94\,slug}\times 144\frac{in.^2}{ft^2} - 32.2\frac{ft}{s^2}\times 90\,ft\times\frac{lbf\cdot s^2}{slug\cdot ft}$$

$$h_{LT} = 813\ ft\cdot lbf/slug$$ h_{LT}

On a per unit weight basis,

$$\Delta H = \frac{\dot{W}_{in}}{\dot{m}g} = \frac{p_3 - p_2}{\rho g} = (50-5)\frac{lbf}{in.^2}\times\frac{ft^3}{62.4\,lbf}\times\frac{144\,in.^2}{ft^2} = 104\ ft$$

and
$$H_{LT} = \frac{h_{LT}}{g} = 813\frac{ft\cdot lbf}{slug}\times\frac{s^2}{32.2\,ft}\times\frac{slug\cdot ft}{lbf\cdot s^2} = 25.2\ ft$$

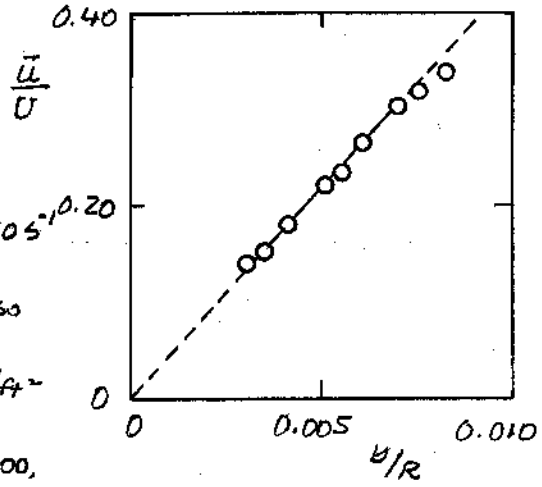
Given: Data measured in fully developed turbulent pipe flow at $Re_U = 50{,}000$ in air:

$\frac{\bar{u}}{U}$	0.343	0.318	0.300	0.264	0.228	0.221	0.179	0.152	0.140
$\frac{y}{R}$	0.0082	0.0075	0.0071	0.0061	0.0055	0.0051	0.0041	0.0034	0.0030

$U = 9.8$ ft/s and $R = 4.86$ in.

Find: (a) Evaluate best-fit value of $d\bar{u}/dy$ from plot.
(b) $\tau_w = \mu\, d\bar{u}/dy$
(c) τ_w calculated from friction factor.

Solution: "Best-fit" slope is given by



$$\frac{d(\bar{u}/U)}{d(y/R)} \approx \frac{\Delta(\bar{u}/U)}{\Delta(y/R)} = \frac{0.40-0}{0.0092-0} = 43.5$$

$$\frac{d\bar{u}}{dy} = \frac{U\, d(\bar{u}/U)}{R\, d(y/R)} = 43.5 \times 9.8\frac{ft}{s} \times \frac{1}{4.86 in.} \times 12\frac{in.}{ft} = 1050\, s^{-1}$$

For standard air, $\mu = 3.72\times10^{-7}$ lbf·s/ft²; so

$$\tau_w = \mu\frac{d\bar{u}}{dy} = 3.72\times10^{-7}\frac{lbf\,s}{ft^2} \times \frac{1050}{s} = 3.91\times10^{-4}\ lbf/ft^2$$

Friction factor is $f = f(Re, \epsilon/D)$. For $Re_U = 50{,}000$, $n = 6.8$ from Eq. 8.23. Then from Eq. 8.24,

$$\frac{\bar{V}}{U} = \frac{2n^2}{(n+1)(2n+1)} = 0.812 \quad \text{and} \quad Re_{\bar{V}} = 0.812\, Re_U = 0.812 \times 50{,}000 = 40{,}600$$

Assuming smooth pipe, $f = 0.021$ from Fig. 8.13.

Balancing forces on a fluid element: $(p+\Delta p)\frac{\pi D^2}{4}$ → ▭ ← $p\frac{\pi D^2}{4}$, $\tau_w \pi DL$

Then $(p+\Delta p)\frac{\pi D^2}{4} - \tau_w \pi DL - p\frac{\pi D^2}{4} = 0$

$$\tau_w = \frac{R}{2}\frac{\Delta p}{L} = \frac{D}{4L} f \frac{1}{D}\rho\frac{\bar{V}^2}{2} = \frac{f}{8}\rho\bar{V}^2 \; ; \quad \bar{V} = 0.812U = 0.812 \times 9.8\frac{ft}{sec} = 7.96\ ft/sec$$

Substituting,

$$\tau_w = \frac{0.021}{8} \times 0.00238\frac{slug}{ft^3} \times (7.96)^2 ft^2/s^2 \times \frac{lbf\,s^2}{slug\cdot ft} = 3.96\times10^{-4}\ lbf/ft^2$$

The result calculated from the friction factor is only 1% higher than that evaluated graphically!

Given: Water flowing in a smooth, horizontal, 3 in. diameter pipe at a mass flowrate of 0.006 slug/s. The pressure drop is observed to be 0.065 lbf/ft² per 100 ft of pipe. According to the Moody chart, the friction factor could be either 0.021 or 0.042. Temperature is 150°F.

Find: Which is correct value for f.

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section, and the definition of h_{LT}.

Basic equations: $\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + g z_2 + h_{LT}$ (1) (2) (1) (2)

$h_{LT} = h_{lm} + f \frac{L}{D} \frac{\bar{V}^2}{2}$ ($h_{lm} = 0$ (1))

Assumptions: (1) Fully developed flow with constant area, so $\bar{V}_1 = \bar{V}_2$ and $h_{lm} = 0$

(2) $z_1 = z_2$

Then

$$\frac{p_1 - p_2}{\rho} = \frac{\Delta p}{\rho} = f \frac{L}{D} \frac{\bar{V}^2}{2} \quad \text{or} \quad f = \frac{p_1 - p_2}{\frac{L}{D} \rho \frac{\bar{V}^2}{2}} = \frac{2D(p_1 - p_2)}{L \rho \bar{V}^2}$$

Under the above assumptions, continuity becomes $\dot{m} = \rho \bar{V} A$, so

$$\bar{V} = \frac{\dot{m}}{\rho A} = 0.006 \frac{\text{slug}}{\text{s}} \times \frac{\text{ft}^3}{1.94 \text{ slug}} \times \frac{4}{\pi} \frac{1}{(0.25)^2 \text{ft}^2} = 0.0630 \frac{\text{ft}}{\text{s}}$$

Thus

$$f = 2(0.25)\text{ft} \times 0.065 \frac{\text{lbf}}{\text{ft}^2} \times \frac{1}{100 \text{ft}} \times \frac{\text{ft}^3}{1.94 \text{ slug}} \times \frac{\text{s}^2}{(0.0630)^2 \text{ft}^2} \times \frac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2}$$

$f = 0.042$ ← f

Therefore flow must be turbulent, at Re ≈ 3500.

Given: Steady flow of water through a smooth, insulated pipe.

$D = 150$ mm and $\bar{V} = 5$ m/s.

Find: Estimate of temperature rise per kilometer of pipe length.

Solution: Apply the definition of head loss.

Computing Equation: $h_{lT} = u_2 - u_1 - \frac{\delta Q}{\delta m}^{=0(4)} = f \frac{L}{D} \frac{\bar{V}^2}{2}$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Fully developed flow
(4) Insulated pipe

Thus $u_2 - u_1 = c_v (T_2 - T_1) = f \frac{L}{D} \frac{\bar{V}^2}{2}$ so $\Delta T = \frac{f}{c_v} \frac{L}{D} \frac{\bar{V}^2}{2}$

Since pipe is smooth, $f = f(Re)$. For water at 20°C, $\nu = 1.00 \times 10^{-6}\ m^2/s$ (Table A.8).

$$Re = \frac{\bar{V} D}{\nu} = 5 \frac{m}{s} \times 0.150\ m \times \frac{s}{1.00 \times 10^{-6} m^2} = 7.50 \times 10^5$$

From Moody diagram (Fig. 8.13) for smooth tube, $f = 0.0123$.

Thus $$\Delta T = \frac{0.0123}{2} \times \frac{kg \cdot K}{4190\ N \cdot m} \times \frac{1000\ m}{0.150\ m} \times (5)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$\Delta T = 0.245\ K \text{ (per kilometer)}$$ ΔT

Given: Small-diameter (i.d. = 0.5 mm) capillary tube made from drawn aluminum is used in place of an expansion valve in a home refrigerator

Find: corresponding relative roughness; with regard to fluid flow, can tube be considered "smooth"?

Solution:

For drawn tubing, from Table 8.1, $e = 0.0015$ mm

Then with $D = 0.5$ mm, $\frac{e}{D} = \frac{0.0015}{0.5} = 0.003$

Looking at the Moody diagram (Fig. 8.13), it is clear that this tube cannot be considered smooth for turbulent flow through the tube.

For laminar flow ($Re < 2300$) the relative roughness has no effect on the flow.

Given: Empirical equation for curves on Moody diagram (Fig. 8.13).

$$\frac{1}{f^{1/2}} = -2.0 \log\left(\frac{e/D}{3.7} + \frac{2.51}{Re\, f^{1/2}}\right) \quad (8.37a)$$

Claim initial guess from $f_0 = 0.25\left[\log\left(\frac{e/D}{3.7} + \frac{5.74}{Re^{0.9}}\right)\right]^{-2}$ (8.37b)

is accurate to ± 1 percent with one calculation.

Find: Validate claim for $Re = 10^4$ and 10^7 with $e/D = 0$ and 0.010.

Solution: Values calculated from these equations are below:

Re	e/D	f_0	f	% Error
1E+04	0	0.0310	0.0309	0.333
1E+07	0	0.00814	0.00810	0.528
1E+04	0.01	0.0440	0.0431	2.24
1E+07	0.01	0.0379	0.0379	0.020

The last column shows the percentage error in the calculated values of f_0, compared to f. At $Re = 10^4$, $e/D = 0.01$, the maximum error is just over 2 percent.

Given: Moody diagram gives Darcy friction factor, f.

Fanning friction factor is $f_F \equiv \dfrac{\tau_w}{\frac{1}{2}\rho \bar{V}^2}$

Find: Relate Darcy and Fanning friction factors for fully developed pipe flow. Show $f = 4f_F$.

Solution: Consider cylindrical CV containing fluid in pipe; apply force balance, definition of f.

Basic equations: $\Sigma F_x = 0$ $\qquad \Delta p = f \dfrac{L}{D} \dfrac{\rho \bar{V}^2}{2}$



From the force balance,

$$(p+\Delta p)\frac{\pi D^2}{4} - \tau_w \pi D L - p\frac{\pi D^2}{4} = 0 \quad \text{or} \quad \tau_w = \frac{D}{4}\frac{\Delta p}{L}$$

Substituting,

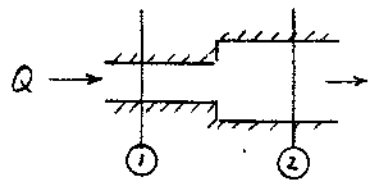
$$\tau_w = \frac{D}{4L} f \frac{L}{D} \frac{\rho \bar{V}^2}{2} = f \frac{\rho \bar{V}^2}{8}$$

But

$$f_F \equiv \frac{\tau_w}{\frac{1}{2}\rho \bar{V}^2} = \frac{f \rho \bar{V}^2}{8} \frac{2}{\rho \bar{V}^2} = \frac{f}{4}$$

f_F

Given: Water flow through sudden enlargement from 1 in. to 2 in. diameter. $Q = 20$ gallons per minute.



Find: Pressure rise across enlargement.
Comparison with value for frictionless flow.

Solution: Apply energy equation for pipe flow.

Computing equation: $\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2 + h_{LT}$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section: $\alpha_1 = \alpha_2 = 1$
(4) Horizontal section

$h_{LT} = K \frac{\bar{V}_1^2}{2}$

Then

$$p_2 - p_1 = \frac{\rho}{2}(\bar{V}_1^2 - \bar{V}_2^2) - \rho h_{LT_{12}}$$

From continuity, $\bar{V}_1 A_1 = \bar{V}_2 A_2$, so $\bar{V}_2 = \bar{V}_1 \frac{A_1}{A_2} = \bar{V}_1 \left(\frac{D_1}{D_2}\right)^2$; $\bar{V}_2^2 = \bar{V}_1^2 \left(\frac{D_1}{D_2}\right)^4$

From Fig. 8.17, at $AR = \left(\frac{D_1}{D_2}\right)^2 = \frac{1}{4}$, $K = 0.56$.

$$\bar{V}_1 = \frac{Q}{A_1} = \frac{4Q}{\pi D_1^2} = \frac{4}{\pi} \times 20 \frac{\text{gal}}{\text{min}} \times \frac{1}{(1)^2 \text{ in.}^2} \times \frac{\text{ft}^3}{7.48 \text{ gal}} \times \frac{\text{min}}{60 \text{ s}} \times 144 \frac{\text{in.}^2}{\text{ft}^2} = 8.17 \text{ ft/s}$$

Substituting,

$$p_2 - p_1 = \frac{\rho \bar{V}_1^2}{2}\left[1 - \left(\frac{D_1}{D_2}\right)^4\right] - K\frac{\rho \bar{V}_1^2}{2} = \frac{1}{2}\rho \bar{V}_1^2 \left[1 - \left(\frac{D_1}{D_2}\right)^4 - K\right]$$

$$= \frac{1}{2} \times 1.94 \frac{\text{slug}}{\text{ft}^3} \times (8.17)^2 \frac{\text{ft}^2}{\text{s}^2} \left[1 - \left(\frac{1}{2}\right)^4 - 0.56\right] \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}}$$

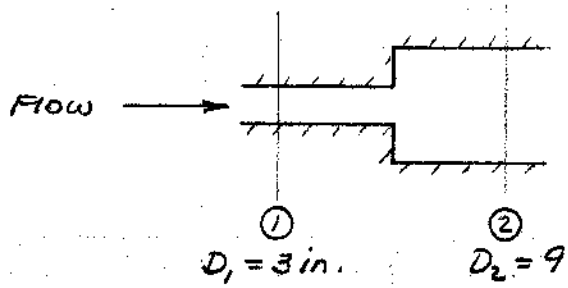
$p_2 - p_1 = 24.4 \text{ lbf/ft}^2$ (0.170 psi) ← Δp_a

For frictionless flow, $K = 0$, and

$p_2 - p_1 = \frac{1}{2}\rho \bar{V}_1^2 \left[1 - \left(\frac{D_1}{D_2}\right)^4\right] = 60.7 \text{ lbf/ft}^2$ (0.422 psi) ← Δp_f

Thus $\frac{\Delta p_{\text{actual}}}{\Delta p_{\text{frictionless}}} = \frac{24.4}{60.7} = 0.402$ or 40.2% ← Ratio

Given: Air flow at standard conditions through a sudden expansion in a circular duct, as shown.



$p_2 - p_1 = 0.25$ in. H_2O

Find: (a) Average velocity of air at inlet
(b) Volume flow rate.

Solution: Apply the energy and continuity equations for steady, incompressible flow that is uniform at each section.

Basic equations: $\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + g\cancel{z_1}^{(1)} = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + g\cancel{z_2}^{(1)} + h_{LT}$

$h_{LT} = \cancel{h_L}^{=0\,(2)} + K\frac{\bar{V}_1^2}{2}$; $\bar{V}_1 A_1 = \bar{V}_2 A_2$

Assumptions: (1) $z_1 = z_2$
(2) $h_L = 0$ between sections ① and ②.

Then

$$\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + K\frac{\bar{V}_1^2}{2}$$

From continuity, $\bar{V}_2 = \bar{V}_1 \frac{A_1}{A_2} = \bar{V}_1 AR$, so

$$\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} = \frac{p_2}{\rho} + \frac{\bar{V}_1^2}{2} AR^2 + K\frac{\bar{V}_1^2}{2}$$

or $\frac{\bar{V}_1^2}{2}(1 - AR^2 - K) = \frac{p_2 - p_1}{\rho}$ so that $\bar{V}_1 = \sqrt{\frac{2(p_2 - p_1)}{\rho(1 - AR^2 - K)}}$

Now $AR = \left(\frac{D_1}{D_2}\right)^2 = 0.11$, so from Fig. 8.15, $K \simeq 0.80$. Also

$$p_2 - p_1 = \gamma_{H_2O}\,\Delta h = 62.4\,\frac{lbf}{ft^3} \times 0.25\,in. \times \frac{ft}{12\,in.} = 1.3\,\frac{lbf}{ft^2}$$

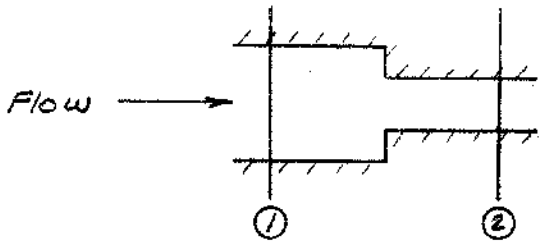
Thus

$$\bar{V}_1 = \left[2 \times 1.3\,\frac{lbf}{ft^2} \times \frac{ft^3}{0.00238\,slug} \times \frac{1}{(1 - (0.11)^2 - 0.8)} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right]^{\frac{1}{2}} = 76.2\ ft/s \quad \bar{V}_1$$

and

$$Q = \bar{V}_1 A_1 = 76.2\,\frac{ft}{s} \times \frac{\pi}{4}(0.25)^2\,ft^2 \times 60\,\frac{s}{min} = 224\ ft^3/min \quad Q$$

Given: Water flow through a circular tube, as shown.



$p_1 - p_2 = 3.4$ kPa

$D_1 = 50$ mm $\quad D_2 = 25$ mm

Find: Flow rate, Q.

Solution: Apply the energy and continuity equations for steady, incompressible flow that is uniform at each section.

Basic equations: $\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + g\cancel{z_1}^{(1)} = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + g\cancel{z_2}^{(1)} + h_{LT}$

$h_{LT} = \cancel{h_L}^{=0\,(2)} + K\frac{\bar{V}_2^2}{2}$; $\bar{V}_1 A_1 = \bar{V}_2 A_2$

Assumptions: (1) $z_1 = z_2$

(2) $h_L = 0$ between sections ① and ②

From Fig. 8.17, at $AR = \frac{A_2}{A_1} = \left(\frac{D_2}{D_1}\right)^2 = \frac{1}{4}$, $K_c = 0.4$. Then

$$\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + K_c\frac{\bar{V}_2^2}{2} \quad \text{and} \quad \frac{\bar{V}_1^2}{2} = \left(\frac{A_2}{A_1}\right)^2\frac{\bar{V}_2^2}{2} = \frac{1}{AR^2}\frac{\bar{V}_2^2}{2}$$

and

$$\frac{p_1}{\rho} + \frac{1}{AR^2}\frac{\bar{V}_2^2}{2} = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + K_c\frac{\bar{V}_2^2}{2}$$

or

$$\frac{p_1 - p_2}{\rho} = \frac{\bar{V}_2^2}{2}\left[1 + K_c - \frac{1}{AR^2}\right] = \frac{\bar{V}_2^2}{2}\left[1 + 0.4 - 0.06\right] = \frac{1.34\bar{V}_2^2}{2}$$

Solving,

$$\bar{V}_2 = \sqrt{\frac{2(p_1 - p_2)}{1.34\rho}} = \left[\frac{2}{1.34} \times 3.4\times10^3 \frac{N}{m^2} \times \frac{m^3}{999\,kg} \times \frac{kg\cdot m}{N\cdot s^2}\right]^{\frac{1}{2}}$$

$$\bar{V}_2 = 2.25 \text{ m/s}$$

Then

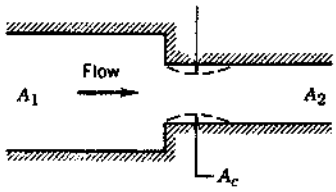
$$Q = \bar{V}_2 A_2 = \frac{\pi \bar{V}_2 D_2^2}{4} = \frac{\pi}{4} \times 2.25\frac{m}{s} \times (0.025)^2 m^2 = 1.10\times10^{-3} \text{ m}^3/\text{s}$$

Q

Given: Flow through a sudden contraction, as shown, with $AR = \frac{A_2}{A_1} = 0.5$.

$$C_c = \frac{A_c}{A_2} = 0.62 + 0.38\left(\frac{A_2}{A_1}\right)^2$$

Only loss is from A_c to A_2.



Find: (a) C_c

(b) Minor loss coefficient, compare with Fig. 8.15.

Solution: For given flow,

$$C_c = 0.62 + 0.38(0.5)^2 = 0.715 \qquad C_c$$

Losses for sudden contractions are given in terms of $\bar{V}_2^2$. For uniform flow at each section,

$$\bar{V}_c A_c = \bar{V}_2 A_2$$

For a sudden expansion, $K_e = f(A_c/A_2) = f[AR = 0.715]$. From Fig. 8.15, $K_e = 0.10$.

$$h_{em} = K_e \frac{\bar{V}_c^2}{2} = 0.10\,\frac{\bar{V}_c^2}{2} = 0.10\,\frac{\bar{V}_2^2}{2}\left(\frac{A_2}{A_c}\right)^2 = 0.10\,\frac{\bar{V}_2^2}{2}\left(\frac{1}{0.715}\right)^2 = 0.196\,\frac{\bar{V}_2^2}{2}$$

The minor loss coefficient for the sudden contraction, based on $\bar{V}_2$, would be

$$K_c = 0.196 \qquad K_c$$

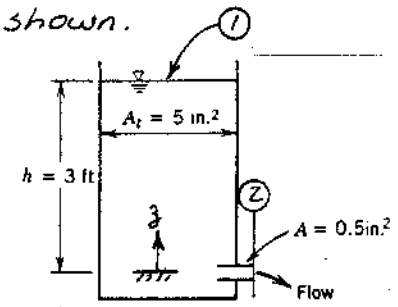
From Fig. 8.15, K_c for a contraction with $AR = 0.5$ is approximately

$$K_c \approx 0.22$$

This is satisfactory agreement (i.e. $K_c \approx 0.2$).

Given: Quasi-steady flow of water from tank shown.

$A_0 = 0.5\ in.^2 = A_2$



Find: (a) Flow rate at instant shown.

(b) How could system be improved?

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation:

$$\left(\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2\right) + h_{LT}$$

$$h_{LT} = h_L + h_{em} = f\frac{L}{D}\frac{\bar{V}^2}{2} + K\frac{\bar{V}^2}{2}$$

Assumptions: (1) $p_1 = p_2 = p_{atm}$
(2) Neglect friction in short tube; $L \approx 0$
(3) Reentrant entrance, $K = 0.78$ (Table 8.2)
(4) Uniform flow at each section, so $\alpha = 1$

Then

$$\frac{\bar{V}_1^2}{2} + gz_1 = \frac{\bar{V}_2^2}{2} + K\frac{\bar{V}_2^2}{2} = (1+K)\frac{\bar{V}_2^2}{2}$$

But from continuity, $\bar{V}_1 A_1 = \bar{V}_2 A_2$, so $\bar{V}_1^2 = \bar{V}_2^2\left(\frac{A_2}{A_1}\right)^2$ and

$$\frac{\bar{V}_2^2}{2}\left[1 + K - \left(\frac{A_2}{A_1}\right)^2\right] = gz_1 \quad \text{or} \quad \bar{V}_2 = \sqrt{\frac{2gz_1}{1+K-(A_2/A_1)^2}}$$

Thus

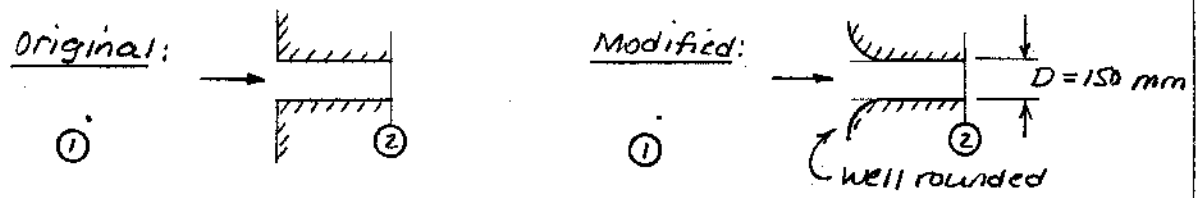
$$\bar{V}_2 = \left[2 \times 32.2\frac{ft}{s^2} \times 3\,ft \times \frac{1}{1+0.78-(0.1)^2}\right]^{1/2} = 10.4\ ft/s$$

and

$$Q = \bar{V}_2 A_2 = 10.4\frac{ft}{s} \times 0.5\,in.^2 \times \frac{ft^2}{144\,in.^2} = 0.0361\ ft^3/s \quad (16.2\ gpm)$$ Q

The flow system could be improved by (1) rounding the entrance and (2) adding a diffuser.

Given: Air flow from a clean room through a duct of 150 mm diameter.



$h_1 - h_2 = 2.5$ mm H_2O

Friction losses negligible, compared to inlet and exit losses.

Find: Increase in volume flow rate for modified duct.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equations: $\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2 + h_{LT}$ ($\frac{\bar{V}_1^2}{2} \approx 0(1)$, $z_1 \approx 0(4)$)

$h_{LT} = h_e + h_{e,m}$; $h_{em} = K_{ent} \frac{\bar{V}_2^2}{2}$; $\Delta p = \rho_{H_2O} g \Delta h$

Assumptions: (1) $\bar{V}_1 \approx 0$ (3) Uniform flow at exit
(2) Neglect elevation changes (4) Neglect frictional losses

Then

$$\frac{\Delta p}{\rho} = \frac{p_1 - p_2}{\rho} = \frac{\bar{V}_2^2}{2} + K_{ent} \frac{\bar{V}_2^2}{2} = \frac{\bar{V}_2^2}{2}(1 + K_{ent}) = \frac{\rho_{H_2O} g \Delta h}{\rho}$$

or

$$\bar{V}_2 = \sqrt{\frac{2(p_1 - p_2)}{\rho(1 + K_{ent})}} = \sqrt{\frac{2 \rho_{H_2O} g \Delta h}{\rho(1 + K_{ent})}}$$

From Table 8.2, $K_{ent} = 0.5$ for square-edged, $K_{ent} = 0.04$ for rounded entrance.

$$\bar{V}_2 = \sqrt{\frac{2}{1.50} \times 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.0025\ m \times \frac{m^3}{1.23\ kg}} = 5.15\ m/s$$

$$\bar{V}_2 (\text{modified}) = \sqrt{\frac{2}{1.04} \times 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.0025\ m \times \frac{m^3}{1.23\ kg}} = 6.19\ m/s$$

Since $Q = \bar{V} A$, then

$$\Delta Q = (\bar{V}_{2m} - \bar{V}_2) A = (6.19 - 5.15) \frac{m}{s} \times \frac{\pi}{4} (0.150)^2 m^2 = 0.0184\ m^3/s \qquad \Delta Q$$

The percentage improvement is

$$\% = \frac{\Delta Q}{Q} \times 100 = \frac{\bar{V}_{2m} - \bar{V}_2}{\bar{V}_2} \times 100 = \frac{6.19 - 5.15}{5.15} \times 100 = 20.2 \text{ percent}$$

Problem 8.91

Given: Consider again flow through the elbow analyzed in Example Problem 4.7. Assume inlet pressure remains constant and the loss coefficient in the nozzle does not change.

Find: (a) Minor loss coefficient for the nozzle using given conditions.
(b) Vertical force as outlet diameter is varied.
(c) Plot of vertical force versus outlet diameter.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2 + h_L + h_{em}$ (8.29)

($p_2 = 0$ (4); $h_L = 0$)

Assume: (1) $\alpha_1 = \alpha_2 \approx 1$ (3) Uniform flow so $\bar{V}_1 A_1 = \bar{V}_2 A_2$ $\bar{V}_2 = \bar{V}_1\frac{A_1}{A_2} = 16$ m/s

(2) Neglect Δz (4) Use gage pressures $A_1 = 0.01\ m^2$; $A_2 = 0.0025\ m^2$

Then $h_{em} = \frac{p_{1g}}{\rho} + \frac{\bar{V}_1^2}{2} - \frac{\bar{V}_2^2}{2} = (120-10.1)10^3\frac{N}{m^2}\times\frac{m^3}{999\,kg}\times\frac{kg\cdot m}{N\cdot s^2} + \frac{1}{2}[(4)^2-(16)^2]\frac{m^2}{s^2} = 0.120\frac{m^2}{s^2}$

But

$h_{em} = K\frac{V_2^2}{2}$; $K = \frac{2h_{em}}{\bar{V}_2^2} = 2\times 0.120\frac{m^2}{s^2}\times\frac{s^2}{(16)^2 m^2} = 9.38\times10^{-4}$ K

For other outlet areas,

$$\frac{p_{1g}}{\rho} + \frac{\bar{V}_1^2}{2} = \frac{\bar{V}_2^2}{2} + K\frac{\bar{V}_2^2}{2} = (1+K)\frac{\bar{V}_2^2}{2} \quad \text{or} \quad \left[1+K-\left(\frac{A_2}{A_1}\right)^2\right]\frac{\bar{V}_2^2}{2} = \frac{p_{1g}}{\rho}$$

$$\bar{V}_2^2 = \frac{2p_{1g}}{\rho}\left[\frac{1}{1+K-(A_2/A_1)^2}\right]$$

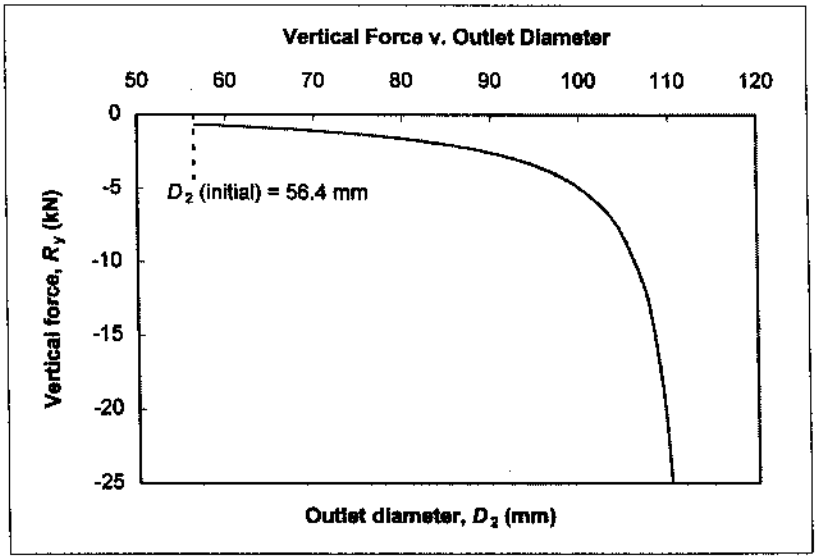
From Example Problem 4.7, $R_y = -\rho V_2^2 A_2 = -2p_{1g}\frac{\pi D_2^2}{4}\left[\frac{1}{1+K-(D_2/D_1)^4}\right]$ R_y

This expression will increase without limit when $1+K = \left(\frac{D_2}{D_1}\right)^4$, or when $D_2 \to D_1$.

Calculated values and a plot are on the next page.

D_1 =	0.113	m	
D_2 =	0.0564	m	Initial value
K =	0.000938	(—)	Nozzle loss coefficient (based on V_2^2)
p_{atm} =	101.0	kPa	
p_1 (abs) =	221.0	kPa	
p_1 (gage) =	120.0	kPa	

x Initial D_2	D_2 (m)	D_2 (mm)	A_2 (m^2)	A_2/A_1	V_2 (m/s)	R_y (kN)
1.00	0.0564	56.4	0.00250	0.250	16.0	-0.639
1.05	0.0592	59.2	0.00276	0.276	16.1	-0.715
1.10	0.0621	62.1	0.00303	0.303	16.3	-0.798
1.15	0.0649	64.9	0.00331	0.331	16.4	-0.890
1.20	0.0677	67.7	0.00360	0.360	16.6	-0.992
1.25	0.0705	70.5	0.00391	0.391	16.8	-1.11
1.30	0.0733	73.3	0.00423	0.423	17.1	-1.23
1.35	0.0762	76.2	0.00456	0.456	17.4	-1.38
1.40	0.0790	79.0	0.00490	0.490	17.8	-1.55
1.45	0.0818	81.8	0.00526	0.526	18.2	-1.74
1.50	0.0846	84.6	0.00563	0.563	18.7	-1.97
1.55	0.0874	87.4	0.00601	0.601	19.4	-2.25
1.60	0.0903	90.3	0.00640	0.640	20.2	-2.60
1.65	0.0931	93.1	0.00681	0.681	21.1	-3.04
1.70	0.0959	95.9	0.00723	0.723	22.4	-3.62
1.75	0.0987	98.7	0.00766	0.766	24.1	-4.43
1.80	0.102	102	0.00810	0.810	26.4	-5.64
1.85	0.104	104	0.00856	0.856	29.9	-7.64
1.90	0.107	107	0.00903	0.903	35.9	-11.6
1.92	0.108	108	0.00922	0.922	39.8	-14.6
1.94	0.109	109	0.00941	0.941	45.6	-19.5
1.95	0.110	110	0.00951	0.951	49.7	-23.5
1.953	0.110	110	0.00954	0.954	51.2	-25.0



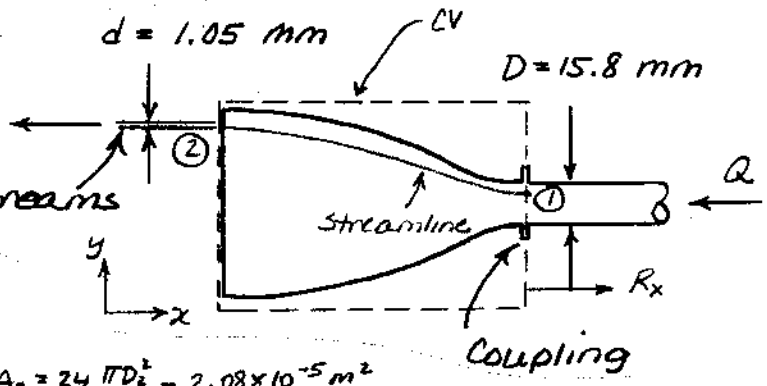
Problem 8.92

Given: Water at 45°C enters a shower head through a circular tube with 15.8 mm inside diameter. The water leaves in 24 streams, each of 1.05 mm diameter. The volume flow rate is 5.67 L/min.

Find: (a) Estimate of the minimum water pressure needed at the inlet to the shower head.
(b) Force needed to hold the shower head onto the end of the circular tube, indicating clearly whether this is a compression or a tension force.

Solution: Apply the energy equation for steady, incompressible pipe flow, and the x component of momentum, using the CV shown.

Assume: (1) Steady flow
(2) Incompressible flow
(3) Neglect changes in z
(4) Uniform flow: $\alpha_1 = \alpha_2 \approx 1$
(5) Use gage pressures



Then

$$\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1 - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2 \right) = h_{LT} = h_e + h_{em}$$

($p_2 = 0$ (gage); $z_1 = z_2$; $h_e = 0$)

$$A_2 = 24 \frac{\pi D_1^2}{4} = 2.08 \times 10^{-5}\ m^2 \qquad A_1 = \frac{\pi D_1^2}{4} = 1.96 \times 10^{-4}\ m^2$$

$$\bar{V}_1 = \frac{Q}{A_1} = 5.67 \frac{L}{min} \times \frac{1}{1.96 \times 10^{-4} m^2} \times \frac{m^3}{1000 L} \times \frac{min}{60 s} = 0.487\ m/s$$

$$\bar{V}_2 = \bar{V}_1 \frac{A_1}{A_2} = 0.487 \frac{m}{s} \times \frac{1.96 \times 10^{-4} m^2}{2.08 \times 10^{-5} m^2} = 4.59\ m/s$$

Use $K = 0.5$, for a square-edged orifice, $\rho = 990\ kg/m^3$ (Table A.8). Then

$$p_1 = \frac{\rho}{2}\left(\bar{V}_2^2 + K \bar{V}_2^2 - \bar{V}_1^2 \right) = \frac{\rho}{2}\left[(1+K) \bar{V}_2^2 - \bar{V}_1^2 \right]$$

$$p_1 = \frac{1}{2} \times 990 \frac{kg}{m^3} \left[(1+0.5)(4.59)^2 - (0.487)^2 \right] \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 15.5\ kPa\ (gage)$$

p_1

Use momentum to find force:

Basic equation: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t} \int_{CV} u \rho\, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$

($F_{Bx} = 0$ (6); $\frac{\partial}{\partial t} = 0$ (1))

Assume: (6) $F_{Bx} = 0$

Then $R_x - p_{1g} A_1 = u_1 \{-\rho Q\} + u_2 \{+\rho Q\} = -V_1 \{-\rho Q\} + (-V_2)\{+\rho Q\} = \rho Q (V_1 - V_2)$

(Step ①)

Step ②: $u_1 = -V_1 \qquad u_2 = -V_2$

$$R_x = p_{1g} A_1 + \rho Q (V_1 - V_2) = 15.5 \times 10^3 \frac{N}{m^2} \times 1.96 \times 10^{-4} m^2 + 990 \frac{kg}{m^3} \times 5.67 \frac{L}{min} \times (0.487 - 4.59) \frac{m}{s} \times \frac{m^3}{1000 L} \times \frac{min}{60 s}$$

$R_x = 2.65\ N$ (in direction shown, i.e., tension)

R_x

Given: Flow through a sudden expansion in a pipe system. Let $AR = \frac{A_1}{A_2}$.

Find: (a) Expression for minor head loss coefficient and plot
(b) Comparison with data of Fig. 8.15.

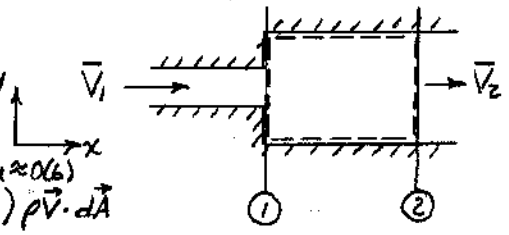
Solution: Apply continuity, x momentum, and energy using CV shown.

Basic Equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} = \{-\rho \bar{V}_1 A_1\} + \{+\rho \bar{V}_2 A_2\}; \quad \bar{V}_1 A_1 = \bar{V}_2 A_2$$

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int u\rho \vec{V}\cdot d\vec{A}$$

$$\dot{Q} + \dot{W}_S = \frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall + \int_{CS}\left(u + \frac{p}{\rho} + \frac{\bar{V}^2}{2} + gz\right)\rho \vec{V}\cdot d\vec{A}$$



Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) Horizontal; $F_{Bx} = 0$
(5) No shaft work; $\dot{W}_S = 0$
(6) Neglect Δz

From momentum $F_{Sx} = p_1 A_2 - p_2 A_2 = \bar{V}_1\{-\rho \bar{V}_1 A_1\} + \bar{V}_2\{+\rho \bar{V}_2 A_2\} = \rho \bar{V}_1 A_1 (\bar{V}_2 - \bar{V}_1)$ ($\bar{V}_2$ = by cont)

From energy $\dot{Q} = \left(u_1 + \frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2}\right)\{-\rho \bar{V}_1 A_1\} + \left(u_2 + \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2}\right)\{+\rho \bar{V}_2 A_2\}$

or $$h_{em} = u_2 - u_1 - \frac{\dot{Q}}{\dot{m}} = \frac{\bar{V}_1^2 - \bar{V}_2^2}{2} + \frac{p_1 - p_2}{\rho} = \frac{\bar{V}_1^2}{2}\left(1 - \frac{\bar{V}_2^2}{\bar{V}_1^2}\right) + \frac{p_1 - p_2}{\rho} \qquad (1)$$

From continuity, $\frac{\bar{V}_2}{\bar{V}_1} = \frac{A_1}{A_2} = AR$ and $\frac{\bar{V}_2^2}{\bar{V}_1^2} = \left(\frac{A_1}{A_2}\right)^2 = AR^2$ (2)

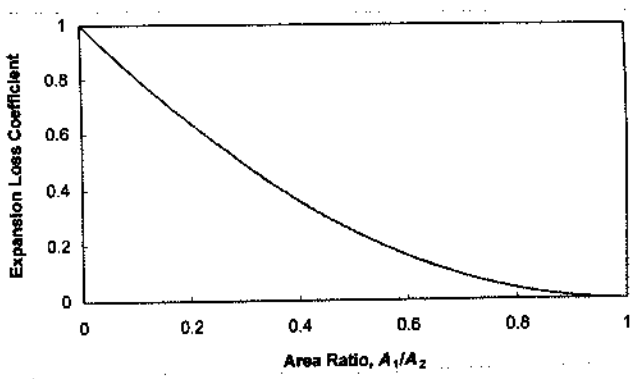
Dividing momentum by ρA_2, $\frac{p_1 - p_2}{\rho} = \bar{V}_1 \frac{A_1}{A_2}(\bar{V}_2 - \bar{V}_1) = \bar{V}_1^2 \frac{A_1}{A_2}\left(\frac{\bar{V}_2}{\bar{V}_1} - 1\right) = \bar{V}_1^2 AR\,(AR - 1)$

Substituting into Eq. 1 and using continuity

$$h_{em} = \frac{\bar{V}_1^2}{2}(1 - AR^2) + \bar{V}_1^2 AR(AR - 1) = \frac{\bar{V}_1^2}{2}(1 - AR^2 + 2AR^2 - 2AR) = \frac{\bar{V}_1^2}{2}(1 - AR)^2$$

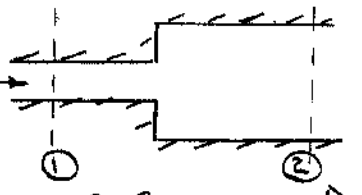
Thus $K_{expansion} = (1 - AR)^2$ Kexp

The plot is



Agreement with Fig. 8.15 is excellent.

Given: Air flow at standard conditions through a sudden expansion in a circular duct, as shown



(reference P 8.86)

$D_1 = 3$ in $\quad D_2 = 9$ in

$p_2 - p_1 = 0.25$ in. H_2O

(a) Would the predicted flow rate be higher or lower if friction were neglected?

(b) Would the pressure rise for the same flow rate be larger or smaller than the given pressure rise if friction were neglected?

Solution:

To answer these questions we need to develop an expression for $\bar{V}_1$ in terms of Δp and the minor loss coefficient K for the sudden expansion.

Basic equation: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + gz_2\right) = h_{LT}$

$h_{LT} = K \frac{\bar{V}_2^2}{2}$ (1) (2) (3) (4)

Assumptions: (1) steady, incompressible flow

(2) $z_1 = z_2$

(3) $h_{LT} = 0$ between (1) & (2) (only loss is due to expansion)

(4) $\alpha_1 = \alpha_2 = 1$

Then $\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + K\frac{\bar{V}_2^2}{2}$

From continuity, $A_2\bar{V}_2 = A_1\bar{V}_1$, so $\bar{V}_2 = \bar{V}_1 \frac{A_1}{A_2} = \bar{V}_1 AR$, and

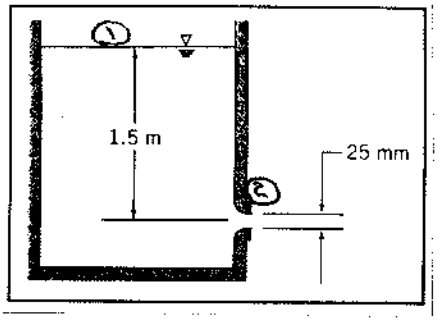
$\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} = \frac{p_2}{\rho} + \frac{\bar{V}_1^2}{2}AR^2 + K\frac{\bar{V}_1^2}{2}$ or $\frac{p_2 - p_1}{\rho} = \frac{\bar{V}_1^2}{2}\left[1 - AR^2 - K\right]$

and $\bar{V}_1 = \sqrt{\frac{2(p_2 - p_1)}{\rho[1 - AR^2 - K]}}$ ——— (1)

(a) If the flow were frictionless (no losses) then $K = 0$. From Eq. 1, the average velocity $\bar{V}_1$ would be less for frictionless flow and hence the predicted flow rate would be lower if friction were neglected.

(b) For the same flow rate $\bar{V}_1$ is the same both frictionless and frictional flow. From Eq. 1, the pressure rise for frictionless flow ($K = 0$) would be greater than for the frictional case.

Given: Water discharges to atmosphere from a large reservoir through a moderately round horizontal nozzle as shown. A short section of 50 mm pipe is attached to the nozzle to form a sudden expansion.



Find: (a) the change in flow rate when the short section is added
(b) magnitude of the minimum pressure

Question: (a) If the flow were frictionless (with the sudden expansion in place) would the minimum pressure be higher, lower, or the same as in (b) above?
(b) Would the flow rate be higher, lower, or the same?

Solution:

Basic equations: $\left(\frac{p_1}{\rho}+\alpha_1\frac{\bar{V}_1^2}{2}+gz_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\frac{\bar{V}_2^2}{2}+gz_2\right)=h_{LT}$ (8.29)

$h_{LT}=\cancel{h_e}+K\frac{\bar{V}_2^2}{2}$ (α_2)

Assumptions: (1) steady, incompressible flow.
(2) $h_e=0$, $K_{nozzle}=0.28$ (Table 8.2)
(3) $\bar{V}_1=0$, $\alpha_2=1.0$
(4) $p_1=p_2=p_{atm}$

Applying Eq. 8.29 between ① and ② gives

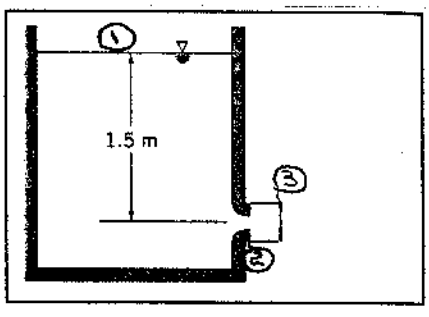
$$g(z_1-z_2)=K_{nozzle}\frac{\bar{V}_2^2}{2}+\frac{\bar{V}_2^2}{2}=\frac{\bar{V}_2^2}{2}(K_{nozzle}+1)\text{, and}$$

$$\bar{V}_2=\left[\frac{2g(z_1-z_2)}{K_{noz}+1}\right]^{1/2}=\left[\frac{2}{(0.28+1)}\times 9.81\frac{m}{s}\times 1.5\,m\right]^{1/2}=4.8\ m/s$$

Add the short section of pipe as shown

$A_3/A_2=(D_3/D_2)^2=(2)^2=4$

From Fig. 8.15 with $A_2/A_3=0.25$, $K_e=0.6$



Applying Eq. 8.29 between ① and ③ with $p_1=p_3=p_{atm}$ and $\alpha_3=1.0$ gives

$$g(z_1-z_3)=K_{nozzle}\frac{\bar{V}_2^2}{2}+K_e\frac{\bar{V}_2^2}{2}+\frac{\bar{V}_3^2}{2}$$

From continuity $A_2\bar{V}_2=A_3\bar{V}_3$

and $$g(z_1-z_3)=\frac{\bar{V}_2^2}{2}\left[K_{noz}+K_e+AR^2\right]\quad\text{where } AR=0.25$$

Then $$\bar{V}_2=\left[\frac{2g(z_1-z_2)}{(K_{noz}+K_e+AR^2)}\right]^{1/2}\quad\text{---------- (1)}$$

Solving for $\bar{V}_2$

$$\bar{V}_2 = \left[\frac{2}{(0.28+0.16)(0.25)^2}\ 9.81\frac{m}{s^2} \times 1.5\,m\right]^{1/2} = 5.59\ m/s$$

The percent increase in flow rate is

$$\%\ \text{increase} = \frac{(\bar{V}_2)_{s.e} - \bar{V}_2}{\bar{V}_2} = \frac{5.59-4.8}{4.8} = 16.5\% \quad \text{Ans (a)}$$

The pressure will be a minimum at section 2. Apply Eq. 8.29 between ① and ② with $\alpha_2 = 1.0$, $p_1 = 0$ gage, $\bar{V}_1 = 0$

$$\left(\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2\right) = K_{nozzle}\frac{\bar{V}_2^2}{2}$$

$$\frac{p_2}{\rho} = g(z_1 - z_2) - \frac{\bar{V}_2^2}{2} - K_{nozzle}\frac{\bar{V}_2^2}{2}$$

$$p_2 = \rho\left[g(z_1-z_2) - \frac{\bar{V}_2^2}{2}(K_{nozzle}+1)\right] \qquad (2)$$

$$= 999\frac{kg}{m^3}\left[9.81\frac{m}{s^2}(1.5\,m) - \left(5.59\frac{m}{s}\right)^2 \times (0.28+1)\right] \cdot \frac{N\cdot s^2}{kg\cdot m}$$

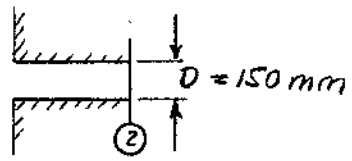
$$p_2 = -5280\ Pa\ (gage) \quad \text{Ans } p_{min}$$

If the flow were frictionless (with the sudden expansion in place), then K_{nozzle} and K_e would both be zero. From Eq. 1 we see that the velocity $\bar{V}_2$ (and hence the flow rate) would be greater for frictionless flow.

If the value of $\bar{V}_2$ is larger, then p_2 (the minimum pressure) would be lower.

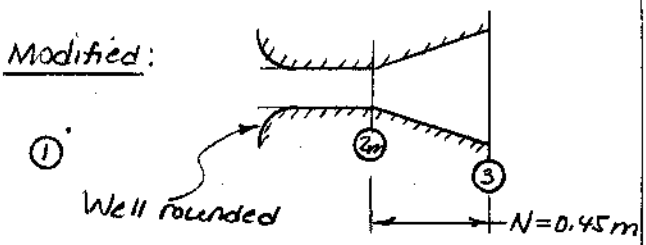
Given: Air flow from a clean room through a duct of 150 mm diameter.

Original:



$h_1 - h_2 = 2.5$ mm H_2O

Modified:



$h_1 - h_3 = 2.5$ mm H_2O

Neglect friction losses compared to "minor" losses.

Find: (a) Area ratio and angle for optimum conical diffuser.
(b) Flow rate for modified system.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equations:

$$\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2 + h_{eT} \quad \text{(or to section 3)}$$

($\alpha_1 \approx O(1)$, $z_1 \approx O(4)$)

$$h_{eT} = h_e + h_{em}; \quad h_{em} = K_{ent} + h_{ediffuser}; \quad \Delta p = \rho_{H_2O} g \Delta h$$

From Eq. 8.42, $h_{ediffuser} = \frac{\bar{V}_2^2}{2}\left[1 - \frac{1}{AR^2} - C_p\right]$

Assumptions: (1) $\bar{V}_1 \approx 0$ (3) Uniform flow at each section
(2) Neglect Δz (4) Neglect frictional losses

For the original system, $\frac{p_1 - p_2}{\rho} = \frac{\bar{V}_2^2}{2} + K_{ent}\frac{\bar{V}_2^2}{2} = 1.5\frac{\bar{V}_2^2}{2} = \frac{\rho_{H_2O} g \Delta h}{\rho}$ ($K_{ent} = 0.5$)

Thus

$$\bar{V}_2 = \sqrt{\frac{2}{1.5}\frac{\rho_{H_2O} g \Delta h}{\rho}} = \sqrt{\frac{2}{1.5} \times 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 0.0025\, m \times \frac{m^3}{1.23\, kg}} = 5.15 \text{ m/s}$$

For the modified system, $\frac{p_1 - p_3}{\rho} = \frac{\bar{V}_3^2}{2} + K_{ent}\frac{\bar{V}_2^2}{2} + \frac{\bar{V}_2^2}{2}\left[1 - \frac{1}{AR^2} - C_p\right] = \frac{\bar{V}_2^2}{2}\left[1 + K_{ent} - C_p\right]$

since $\bar{V}_3^2 = \bar{V}_2^2 \frac{1}{AR^2}$. Thus the best diffuser has the highest C_p.

From Fig. 8.16, $C_p = f(N/R_1, AR)$. $N/R_1 = 2N/D_1 = 2 \times \frac{0.45\,m}{0.15\,m} = 6$. From the figure, the best diffuser is

$C_p \approx 0.62$ at $AR \approx 2.7$ and $2\phi \approx 12$ deg ← AR,

For the modified system,

$$\bar{V}_2 = \sqrt{\frac{2}{1 + K_{ent} - C_p}\frac{\rho_{H_2O} g \Delta h}{\rho}} = \sqrt{\frac{2}{1 + 0.04 - 0.62} \times 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 0.0025\, m \times \frac{m^3}{1.23\, kg}} = 9.74 \text{ m/s}$$

and

$$Q = \bar{V}_2 A_2 = 9.74\frac{m^3}{s} \times \frac{\pi}{4}(0.15)^2 m^2 = 0.172\ m^3/s$$ ← Q

{ The improvement is $\frac{Q_m - Q}{Q} \times 100 = \frac{\bar{V}_m - \bar{V}}{\bar{V}} \times 100 = \frac{9.74 - 5.15}{5.15} \times 100 = 89.1$ percent more }

Given: Steady flow of water from a large tank through a length of smooth plastic tubing, with $D = 3.18$ mm and $L = 15.3$ m.

Find: (a) Maximum volume flow rate for laminar flow.
(b) Estimate maximum water level in tank for laminar flow ($\alpha = 2$ and $K_{ent} = 1.4$)

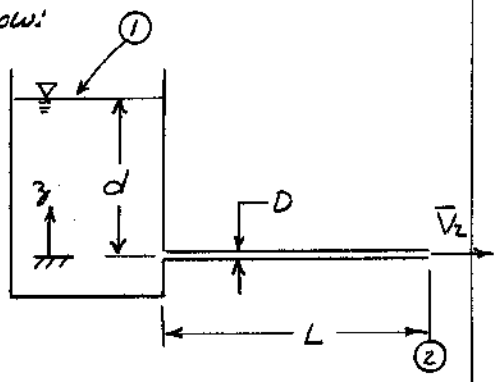
Solution: Assume water at 20°C. From Table A.8, $\rho = 998\ kg/m^3$; $\nu = 1.00 \times 10^{-6}\ m^2/s$.

$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu} \le 2300 \; ; \; \bar{V}_{max} = \frac{2300\nu}{D} = 2300 \times 1.00 \times 10^{-6} \frac{m^2}{s} \times \frac{1}{0.00318\ m} = 0.723\ m/s$$ $\bar{V}$

$$Q = \bar{V} A \; ; \; A = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.00318)^2 m^2 = 7.94 \times 10^{-6}\ m^2$$

$$Q = 0.723 \frac{m}{s} \times 7.94 \times 10^{-6}\ m^2 = 5.74 \times 10^{-6} \frac{m^3}{s} \times 10^3 \frac{L}{m^3} \times 60 \frac{s}{min} = 0.345\ L/min$$ Q

Apply energy equation for steady, ρ = constant pipe flow:



Computing Equation: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{LT}$ ($\bar{V}_1 \approx 0$, $z_2 = 0$)

$$h_{LT} = h_{Lm} + h_L$$

Assumptions: (1) $p_1 = p_2 = p_{atm}$
(2) $\bar{V}_1 \approx 0$
(3) $K_{ent} = 1.4$ (given)

Then $g d = \alpha_2 \frac{\bar{V}_2^2}{2} + K_{ent} \frac{\bar{V}_2^2}{2} + f \frac{L}{D} \frac{\bar{V}_2^2}{2}$ or $d = \frac{\bar{V}_2^2}{2g}\left(\alpha_2 + K_{ent} + f \frac{L}{D}\right)$

For laminar flow, $f = \frac{64}{Re} = \frac{64}{2300} = 0.0278$. Substituting

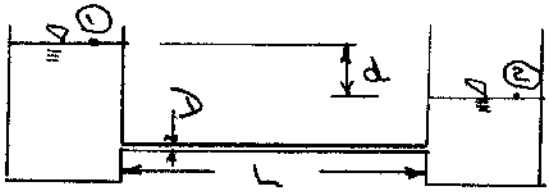
$$d = \frac{1}{2} \times (0.723)^2 \frac{m^2}{s^2} \times \frac{s^2}{9.81\ m}\left(2.0 + 1.4 + 0.0278 \frac{15.3\ m}{0.00318\ m}\right)$$

$$d = 3.65\ m$$ d

Given: System for measuring pressure drop for water flow in smooth tube as shown

$D = 15.9$ mm $L = 3.56$ m

square-edged entrance to pipe



Find: (a) Volume flow rate needed for turbulent flow in pipe
(b) reservoir height differential needed for turbulent pipe flow

Solution:

Flow will be turbulent for $Re_D > 2300$

$$Re_D = \frac{\rho V D}{\mu} = \frac{V D}{\nu} = \frac{Q D}{A \nu} = \frac{Q D}{\frac{\pi}{4} D^2 \nu} = \frac{4Q}{\pi D \nu} \quad \text{so} \quad Q = \frac{\pi D \nu Re}{4}$$

Assume $T = 20°C$, $\nu = 1.00 \times 10^{-6}\ m^2/s$ (Table A.8)

For $Re = 2300$,

$$Q = \frac{\pi}{4} \times 1.0 \times 10^{-6} \frac{m^2}{s} \times 15.9 \times 10^{-3}\ m \times 2300 = 2.87 \times 10^{-5}\ m^3/s$$ ——— Q

Basic equations: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{e_T}$ (8.29)

$h_{e_T} = h_e + h_{en}$ $h_e = f \frac{L}{D} \frac{\bar{V}^2}{2}$, $h_{en} = K \frac{\bar{V}^2}{2}$

Assumptions: (1) $P_1 = P_2 = P_{atm}$ (2) $\bar{V}_1 = \bar{V}_2 = 0$
(3) $K_{ent} = 0.5$ (Table 8.2), $K_{exit} = 1.0$

Then, $z_1 - z_2 = \frac{\bar{V}^2}{2g}\left[f \frac{L}{D} + K_{ent} + K_{exit}\right]$ ——— (1)

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 2.87 \times 10^{-5} \frac{m^3}{s} \times \left(\frac{1}{15.9 \times 10^{-3}\ m}\right)^2 = 0.145\ m/s$$

For turbulent flow in a smooth pipe at $Re = 2300$,

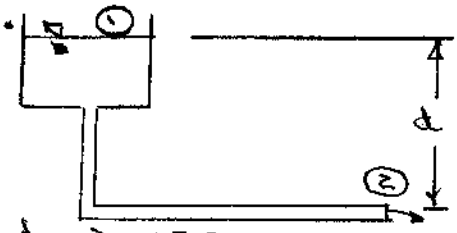
$f = 0.05$ (Fig 8.13)

From Eq. 1

$$d = z_1 - z_2 = \frac{(0.145)^2}{2} \frac{s^2}{m^2} + 9.81 \frac{m}{s^2}\left[0.05 \times \frac{3.56 \times 10^3}{15.9} + 0.5 + 1.0\right]$$

$d = 0.0136\ m$ or 13.6 mm ——— d

Given: System for measuring pressure drop for water flow in smooth pipe supplies water from an overhead constant-head tank.



System includes:
- square-edged entrance
- two 45° standard elbows
- two 90° standard elbows
- fully open gate valve
- pipe length $L = 9.8$ m, diameter $D = 15.9$ mm

Find: elevation of water surface in supply tank above pipe discharge needed to achieve $Re_D = 10^5$.

Solution:

$$Re = \frac{\rho D V}{\mu} = \frac{D V}{\nu}$$ Assume $T = 20°C$, $\nu = 1.00 \times 10^{-6}\ m^2/s$ (Table A.8)

For $Re = 10^5$, $V = \frac{Re\ \nu}{D} = 10^5 \times 1.0 \times 10^{-6}\ \frac{m^2}{s} \times \frac{1}{15.9 \times 10^{-3}}\ m = 6.29\ m/s$

Basic equations: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{V_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{V_2^2}{2} + g z_2\right) = h_{LT}$ (8.29)

$$h_{LT} = h_L + h_{Lm}, \quad h_L = f \frac{L}{D} \frac{V^2}{2}, \quad h_{Lm} = f \frac{L}{D} \frac{V^2}{2} + K_{ent} \frac{V^2}{2}$$

Assumptions: (1) $p_1 = p_2 = p_{atm}$ (2) $V_1 = 0$ (3) $\alpha_2 = 1.0$

Then

$$g(z_1 - z_2) = \frac{V_2^2}{2} + f \frac{L}{D} \frac{V_2^2}{2} + f \frac{V_2^2}{2} \sum \left(\frac{L}{D}\right) + K_{ent} \frac{V_2^2}{2}$$

$$d = (z_1 - z_2) = \frac{V_2^2}{2g}\left[1 + f \frac{L}{D} + 2f\left(\frac{L}{D}\right)_{45°\,el} + 2f\left(\frac{L}{D}\right)_{90°\,el} + f\left(\frac{L}{D}\right)_{gv} + K_{ent}\right]$$

From Table 8.2 $K_{ent} = 0.5$

From Table 8.4 $\left(\frac{L}{D}\right)_{45°\,el} = 16$, $\left(\frac{L}{D}\right)_{90°\,el} = 30$, $\left(\frac{L}{D}\right)_{gv} = 8$

For $Re = 10^5$ in smooth pipe, $f = 0.018$ (Fig. 8.13)

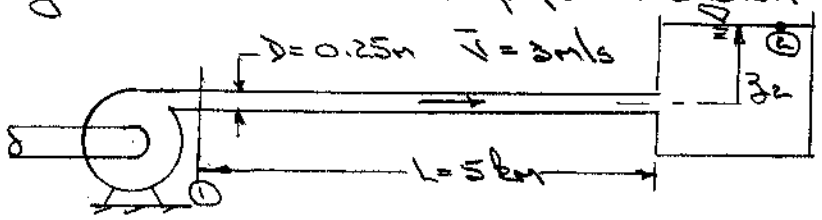
Thus

$$d = \frac{1}{2g} \times (6.29)^2 \frac{s^2}{s^2} \times 9.8 \frac{s^2}{m}\left[1 + 0.018 \times \frac{9.8 \times 10^3}{15.9} + 2(0.018)16 + 2(0.018)30 + 0.018(8) + 0.5\right]$$

$$d = 29.0\ m$$ ← d

This value of d indicates that it will not be possible to obtain a value of $Re = 10^5$ in the flow system. The maximum value of Re will be considerably less than 10^5.

Given: Water flow from a pump to an open reservoir through commercial steel pipe as shown.



Find: the pressure at the pump discharge

Solution:

Apply the energy equation for steady, incompressible flow that is uniform at each section.

Basic equations: $\left(\frac{p_1}{\rho}+\alpha_1\frac{\bar{V}_1^2}{2}+gz_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\frac{\bar{V}_2^2}{2}+gz_2\right)=h_{lT}$ (8.29)

$h_{lT}=h_l+h_{lm}$, $h_l=f\frac{L}{D}\frac{\bar{V}^2}{2}$, $h_{lm}=K_{ent}\frac{\bar{V}^2}{2}$

Assumptions: (1) $z_1=0$ (2) $p_2=p_{atm}=0$ gage.
(3) $\bar{V}_2=0$, $\alpha_1=1.0$
(4) $T=20^\circ C$, $\nu=1.00\times10^{-6}\ m^2/s$ (Table A.8)

$$Re=\frac{\rho\bar{V}D}{\mu}=\frac{\bar{V}D}{\nu}=0.25m\times\frac{3m}{s}\times1.00\times10^{6}\frac{s}{m^2}=7.50\times10^5$$

For commercial steel pipe, $\varepsilon=0.046\ mm$

$$\therefore \varepsilon/D=\frac{0.046}{250}=0.000184$$

From Fig. 8.13 $f=0.015$. Also $K_{ent}=1.0$

Then

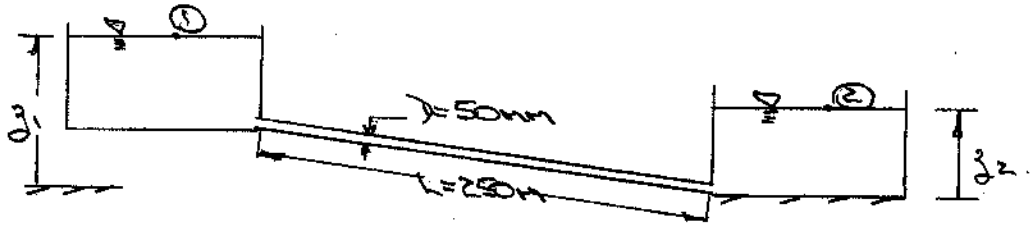
$$p_1=\rho\left[gz_2-\frac{\bar{V}_2^2}{2}+f\frac{L}{D}\frac{\bar{V}^2}{2}+K_{ent}\frac{\bar{V}^2}{2}\right]$$

$$p_1=\rho\left[gz_2+f\frac{L}{D}\frac{\bar{V}^2}{2}\right]$$

$$p_1=998\frac{kg}{m^3}\left[9.81\frac{m}{s^2}\times7m+0.015\times\frac{5\times10^3}{0.25}\times\frac{1}{2}\times3^2\frac{m^2}{s^2}\right]\times\frac{N\cdot s^2}{kg\cdot m}$$

$$p_1=1.42\ MPa\ (gage)$$ ← p_1

Given: Water flow by gravity between two reservoirs through straight galvanized iron pipe. Required flow rate is $Q = 0.007\ m^3/s$



Find: (a) $z_1 - z_2$ required to maintain the flow
(b) Fraction due to minor losses.

Plot: (a) z_1 and (b) minor loss/total loss versus Q.

Solution:

Apply the energy equation for steady incompressible flow between sections ① and ②.

Basic equations: $\left(\frac{p_1}{\rho} + \alpha_1\frac{V_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{V_2^2}{2} + gz_2\right) = h_{l_T}$ (8.29)

$h_{l_T} = h_l + h_{l_m}$; $h_l = f\frac{L}{D}\frac{V^2}{2}$; $h_{l_m} = K_{ent}\frac{V^2}{2} + K_{exit}\frac{V^2}{2}$

Assumptions: (1) $p_1 = p_2 = p_{atm}$ (given)
(2) $V_1 = V_2 \approx 0$
(3) square edged entrance

For square edged entrance (Table 8.2) $K_{ent} = 0.5$; also $K_{exit} = 1.0$

For water at 20°C, $\nu = 1.00 \times 10^{-6}\ m^2/s$ (Table A.8)

$$Re = \frac{\rho VD}{\mu} = \frac{VD}{\nu} = \frac{Q}{A}\frac{D}{\nu} = \frac{Q}{\frac{\pi}{4}D^2}\frac{D}{\nu} = \frac{4Q}{\pi D\nu} \quad \text{---------- (1)}$$

For required Q,

$$Re = \frac{4}{\pi} \times 0.007\ m^3/s \times 1.00\times10^{6}\ \frac{s}{m^2} \times \frac{1}{50\times10^{-3}\ m} = 1.78 \times 10^5$$

For galvanized iron pipe $e = 0.15\ mm$ (Table 8.1), $\frac{e}{D} = 0.003$

Then from Figure 8.13, $f = 0.027$

From Eq. 8.29,

$$z_1 - z_2 = \frac{h_{l_T}}{g} = f\frac{L}{D}\frac{V^2}{2g} + K_{ent}\frac{V^2}{2g} + K_{exit}\frac{V^2}{2g} = \frac{V^2}{2g}\left[f\frac{L}{D} + K_{ent} + K_{exit}\right]$$

$$V = \frac{Q}{A} = \frac{Q}{\frac{\pi}{4}D^2} = \frac{4}{\pi}\frac{Q}{D^2} = \frac{4}{\pi} \times 0.007\ m^3/s \times (0.050\ m)^{-2} = 3.57\ m/s$$

$$z_1 - z_2 = \frac{(3.57)^2}{2}\ \frac{s^2}{m} \times 9.81\ \frac{1}{m}\left[0.027 \times \frac{250}{50} + 0.5 + 1.0\right]$$

$$z_1 - z_2 = 88.7\ m.$$

$$\frac{\text{minor loss}}{\text{total loss}} = \frac{K_{ent} + K_{exit}}{f\frac{L}{D} + K_{ent} + K_{exit}} = \frac{1.5}{135 + 1.5} = 0.011 = 1.1\% \quad \frac{h_{en}}{h_{ET}}$$

To plot Δz vs Q

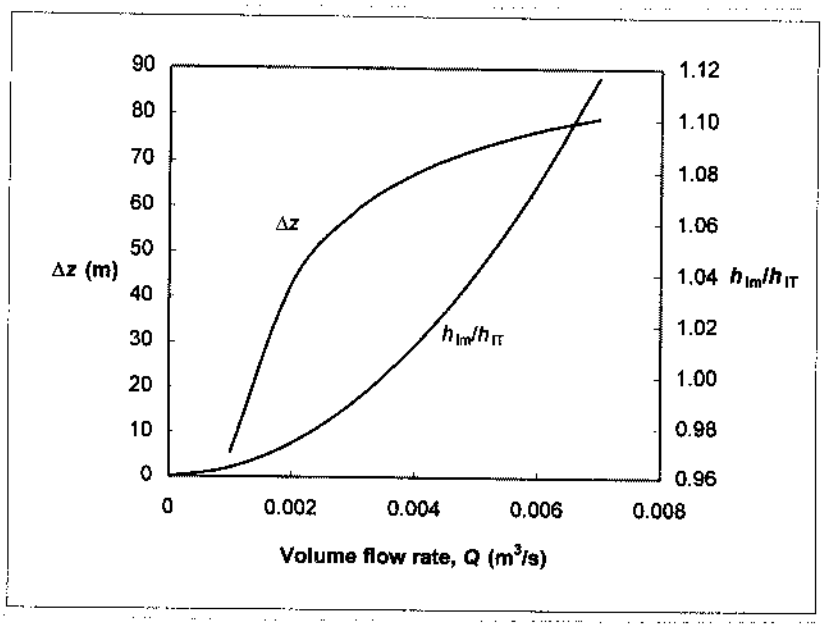
$$V = \frac{Q}{A} = \frac{Q}{\pi D^2/4} = 509\,Q\ (m^3/s)$$

$$\Delta z = \frac{V^2}{2g}\left[K_{ent} + K_{exit} + f\frac{L}{D}\right] = \frac{V^2}{2g}\left[1.5 + 5000f\right]$$

where $f = f(Re, \frac{\varepsilon}{D} = 0.003)$

$$\frac{h_{en}}{h_{ET}} = \frac{K_{ent} + K_{exit}}{K_{ent} + K_{exit} + f\frac{L}{D}} = \frac{1.5}{1.5 + 5000f}$$

Re ratio h_{en}/h_{ET} increases with increasing Re because f decreases with increasing Re.



Given: Air at a flowrate of 35 m^3/min at standard conditions in a smooth duct 0.3 m square.

Find: Pressure drop in mm H_2O per 30 m of horizontal duct.

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section. Use hydraulic diameter.

Basic equation: $\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + g z_2 + f \frac{L}{D_h} \frac{\bar{V}^2}{2} + h_{em}$; $D_h = \frac{4A}{P_w}$

Assumptions: (1) $\bar{V}_1 = \bar{V}_2$
(2) Horizontal
(3) $h_{em} = 0$

Then

$$\Delta p = p_1 - p_2 = f \frac{L}{D_h} \rho \frac{\bar{V}^2}{2}$$

From continuity, $\bar{V} = \frac{Q}{A} = 35 \frac{m^3}{min} \times \frac{1}{(0.3)^2 m^2} \times \frac{min}{60 sec} = 6.48 \, m/s$

$$D_h = \frac{4A}{P_w} = 4 \times (0.3)^2 m^2 \times \frac{1}{4(0.3) m} = 0.3 \, m \; ; \; \nu = 1.45 \times 10^{-5} \, m^2/s \text{ (Table A.10)}$$

$$Re = \frac{\bar{V} D_h}{\nu} = 6.48 \frac{m}{s} \times 0.3 m \times \frac{s}{1.46 \times 10^{-5} m^2} = 1.33 \times 10^5$$

$f = 0.017$ (Fig. 8.13)

Then $\Delta p = \frac{0.017}{2} \times \frac{30 m}{0.3 m} \times 1.23 \frac{kg}{m^3} \times (6.48)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 43.9 \, N/m^2$ ← Δp

For a manometer, $\Delta p = \rho_{H_2O} g \Delta h$

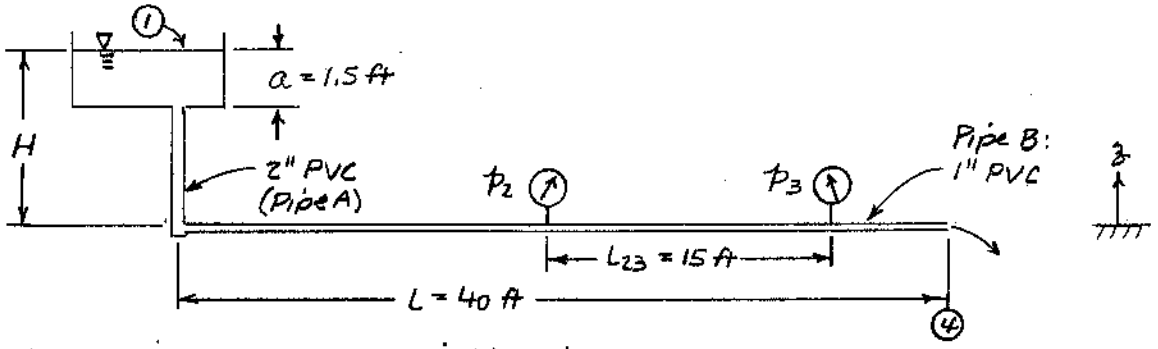
$$\Delta h = \frac{\Delta p}{\rho_{H_2O} g} = 43.9 \frac{N}{m^2} \times \frac{m^3}{999 kg} \times \frac{s^2}{9.81 m} \times \frac{kg \cdot m}{N \cdot s^2} = 0.00448 \, m$$

Thus

$\Delta h = 4.48$ mm H_2O (per 30 m of duct) ← Δh

(This is Δp expressed in mm of water.)

Given: Pipe friction experiment, using water, to reach $Re = 100{,}000$.



Find: (a) Required average speed in 1 in. pipe.
(b) Feasibility of using a constant-head tank.
(c) Pressure difference between taps 2 and 3.

Solution: Assume water at 68 F. From Table A.7, $\rho = 1.94$ slug/ft^3, $\nu = 1.08\times10^{-5}$ ft^2/s.

$$Re = \frac{\bar{V}D}{\nu} = 100{,}000; \quad \bar{V} = \frac{100{,}000\nu}{D_B} = 10^5 \times 1.08\times10^{-5}\frac{ft^2}{s}\times\frac{1}{1\ in.}\times\frac{12\ in.}{ft} = 13.0\ ft/s \qquad \bar{V}$$

Apply energy equation for steady, incompressible pipe flow:

Computing equation: $\left(\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_4}{\rho} + \alpha_4\frac{\bar{V}_4^2}{2} + gz_4\right) = h_{LT} = h_L + h_{Lm}$ ($\bar{V}_1 \approx 0$, $z_4 = 0$)

Assumptions: (1) $p_1 = p_4 = p_{atm}$
(2) $\bar{V}_1 \approx 0$
(3) $\alpha_4 \approx 1$
(4) Neglect minor losses
(5) Smooth pipes

$\bar{V}_A = \bar{V}_B\left(\frac{D_B}{D_A}\right)^2 = \frac{\bar{V}_B}{4} = 3.25$ ft/s

$Re_A = \frac{\bar{V}_A D_A}{\nu} = 50{,}000$; $f_A = 0.021$; $f_B = 0.018$ {From Moody chart (Fig. 8.13)}

Then

$$gH - \alpha_4\frac{\bar{V}_4^2}{2} = f_A\frac{L_A}{D_A}\frac{\bar{V}_A^2}{2} + f_B\frac{L_B}{D_B}\frac{\bar{V}_B^2}{2}$$

or, since $L_A = H - A$,

$$H = \alpha_4\frac{\bar{V}_B^2}{2g} + f_A\frac{L_A}{D_A}\frac{\bar{V}_B^2}{2g}\left(\frac{D_A}{D_B}\right)^4 + f_B\frac{L_B}{D_B}\frac{\bar{V}_B^2}{2g} = \left[\alpha_4 + f_A\underbrace{\frac{H-A}{D_A}\left(\frac{D_A}{D_B}\right)^4}_{\text{neglect}} + f_B\frac{L}{D_B}\right]\frac{\bar{V}_B^2}{2g}$$

$$H \approx \left[1.0 + 0.018\,\frac{40\ ft}{(1/12)\ ft}\right]\frac{1}{2}\times(13.0)^2\frac{ft^2}{s^2}\times\frac{s^2}{32.2\ ft} = 25.3\ ft \qquad H$$

Most ceilings are 12 feet high or less. A constant-head tank is impractical.

Between ② and ③ $\frac{p_2 - p_3}{\rho} = f_B\frac{L_{23}}{D_B}\frac{\bar{V}_B^2}{2}$ or $\Delta p = f\frac{L}{D}\rho\frac{\bar{V}^2}{2}$

$$\Delta p = 0.018\times\frac{15\ ft}{(1/12)\ ft}\times\frac{1}{2}\times 1.94\frac{slug}{ft^3}\times(13.0)^2\frac{ft^2}{s^2}\times\frac{lbf\cdot s^2}{slug\cdot ft}\times\frac{ft^2}{144\ in.^2} = 3.69\ psi \qquad \Delta p$$

Since $\Delta p = \rho g\,\Delta h$, then

$$\Delta h = \frac{\Delta p}{\rho g} = 3.69\frac{lbf}{in.^2}\times 144\frac{in.^2}{ft^2}\times\frac{ft^3}{62.4\ lbf} = 8.51\ ft\ \ (102\ in.)\ of\ water \qquad \Delta h$$

Given: Flow of standard air at 35 m^3/min, in smooth ducts of area, $A = 0.1\ m^2$.

Find: Compare pressure drop per unit length of a round duct with that for rectangular ducts of aspect ratio 1, 2 and 3.

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section. Use hydraulic diameter.

Basic equation: $\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + g z_2 + f\frac{L}{D_h}\frac{\bar{V}^2}{2} + h_{lm}$; $D_h = \frac{4A}{P_w}$

Assumptions: (1) $\bar{V}_1 = \bar{V}_2$
(2) $z_1 = z_2$
(3) $h_{lm} = 0$

Then

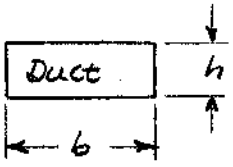
$$\Delta p = p_1 - p_2 = f\frac{L}{D_h}\rho\frac{\bar{V}^2}{2} \quad \text{or} \quad \frac{\Delta p}{L} = \frac{f}{D_h}\rho\frac{\bar{V}^2}{2}$$

But $\bar{V} = \frac{Q}{A} = 35\frac{m^3}{min}\times\frac{1}{0.1\ m^2}\times\frac{min}{60\ sec} = 5.83\ m/s$; $\nu = 1.46\times10^{-5}\ m^2/s$ (Table A.10)

$$Re = \frac{\bar{V}D_h}{\nu} = 5.83\frac{m}{s}\times D_h(m)\times\frac{s}{1.46\times10^{-5}\ m^2} = 3.99\times10^5 D_h(m)\ ;\ \frac{\rho\bar{V}^2}{2} = 20.9\ N/m^2$$

For a round duct, $D_h = D = \left(\frac{4A}{\pi}\right)^{\frac{1}{2}} = \left[\frac{4}{\pi}\times0.1\ m^2\right]^{\frac{1}{2}} = 0.357\ m$

For a rectangular duct, $D_h = \frac{4A}{P_w} = \frac{4bh}{2(b+h)} = \frac{2har}{1+ar}$ where $ar = \frac{b}{h}$.



But $h = \frac{b}{ar}$, so $h^2 = \frac{bh}{ar} = \frac{A}{ar}$, or $h = \sqrt{\frac{A}{ar}}$ and $D_h = \frac{2ar^{1/2}}{1+ar}A^{1/2}$

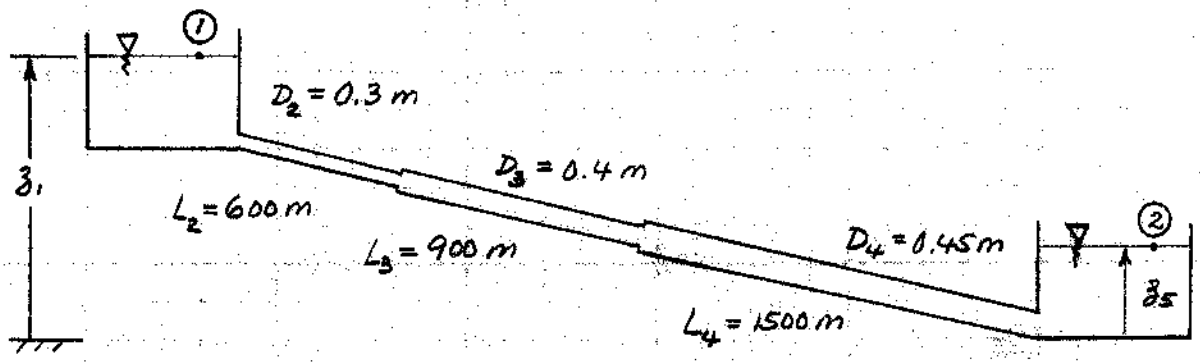
For smooth ducts, use Fig. 8.13 (or Blasius correlation, $f = \frac{0.316}{Re^{1/4}}$) to find f.

Tabulate results:

Duct Section	D_h (m)	Sketch	Re (-)	f (-)	$\Delta p/L$ (N/m^3)	Percent Increase
Round	0.357	○	1.43×10^5	0.0162	0.948	—
Square (ar=1)	0.316	□	1.26×10^5	0.0167	1.11	14.6
ar = 2	0.298	▭	1.19×10^5	0.0170	1.19	20.3
ar = 3	0.274	▭	1.09×10^5	0.0173	1.32	28.2

{Note that f varies only about 7 percent. The large change in $\Delta p/L$ is due primarily to the factor $\frac{f}{D_h}$.}

Given: Reservoirs connected by three clean, cast iron pipes in series. The flow is water at $0.11\ m^3/s$ and $15^\circ C$.



Find: Elevation difference, $z_1 - z_2$

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section.

Basic equations: $\frac{p_1}{\rho}^{(1)} + \frac{\bar{V}_1^2}{2}^{(2)} + gz_1 = \frac{p_2}{\rho}^{(1)} + \frac{\bar{V}_2^2}{2}^{(2)} + gz_2 + h_{eT}$

$$h_{eT} = \sum f\frac{L}{D}\frac{\bar{V}^2}{2} + h_{em}\ ;\ h_{em} = K_{ent}\frac{\bar{V}_2^2}{2} + \sum h_{e_{exp}}^{\approx 0(3)} + K_{exit}\frac{\bar{V}_4^2}{2}$$

Assumptions: (1) $p_1 = p_2 = p_{atm}$
(2) $\bar{V}_1 = \bar{V}_2 \simeq 0$
(3) Neglect $h_{e_{exp}}$ at pipe joints (note all minor losses are probably small due to long lengths of straight pipe sections, but we will check)

For non-smooth pipe, $f = f(Re, e/D)$. $\mu = 1.1 \times 10^{-3}\ N\cdot s/m^2$ from Table A.8.

Section ②: $e/D_2 = 0.26\ mm/300\ mm = 0.00087$ (for cast iron, $e = 0.26\ mm$, Table 8.1)

$$\bar{V}_2 = \frac{Q}{A_2} = 0.11\frac{m^3}{s} \times \frac{4}{\pi}\frac{1}{(0.3)^2 m^2} = 1.56\ m/s$$

$$Re_2 = \frac{\rho \bar{V}_2 D_2}{\mu} = 999\frac{kg}{m^3} \times 1.56\frac{m}{s} \times 0.3\ m \times \frac{m^2}{1.14\times 10^{-3} N\cdot s} \times \frac{N\cdot s^2}{kg\cdot m} = 4.10 \times 10^5$$

From Fig. 8.13, $f_2 = 0.020$

Section ③: $e/D_3 = 0.00065$

$$\bar{V}_3 = \frac{Q}{A_3} = 0.11\frac{m^3}{s} \times \frac{4}{\pi}\frac{1}{(0.4)^2 m^2} = 0.875\ m/s$$

$$Re_3 = \frac{\rho \bar{V}_3 D_3}{\mu} = 999\frac{kg}{m^3} \times 0.875\frac{m}{s} \times 0.4\ m \times \frac{m^2}{1.14\times 10^{-3} N\cdot s} \times \frac{N\cdot s^2}{kg\cdot m} = 3.07 \times 10^5$$

From Fig. 8.13, $f_3 = 0.019$

Section ④: $e/D_4 = 0.00058$

$$\bar{V}_4 = \frac{Q}{A_4} = 0.11 \frac{m^3}{s} \times \frac{4}{\pi} \frac{1}{(0.45)^2 m^2} = 0.692 \text{ m/s}$$

$$Re_4 = \frac{\rho \bar{V}_4 D_4}{\mu} = 999 \frac{kg}{m^3} \times 0.692 \frac{m}{s} \times 0.45 m \times \frac{m^2}{1.14 \times 10^{-3} N \cdot s} \times \frac{N \cdot s^2}{kg \cdot m} = 2.73 \times 10^5$$

From Fig. 8.13, $f_4 = 0.0185$

Then $\Sigma f \frac{L}{D} \frac{\bar{V}^2}{2} = 0.020 \times \frac{600 m}{0.3 m} \times \frac{1}{2} (1.56)^2 \frac{m^2}{s^2} + 0.019 \times \frac{900 m}{0.4 m} \times \frac{1}{2} (0.875)^2 \frac{m^2}{s^2}$

$$+ 0.0185 \times \frac{1500 m}{0.45 m} \times \frac{1}{2} (0.692)^2 \frac{m^2}{s^2} = 79.8 \text{ m}^2/\text{s}^2$$

The minor loss coefficients are $K_{ent} = 0.5$ (Table 8.2) and $K_{exit} = 1.0$. Thus,

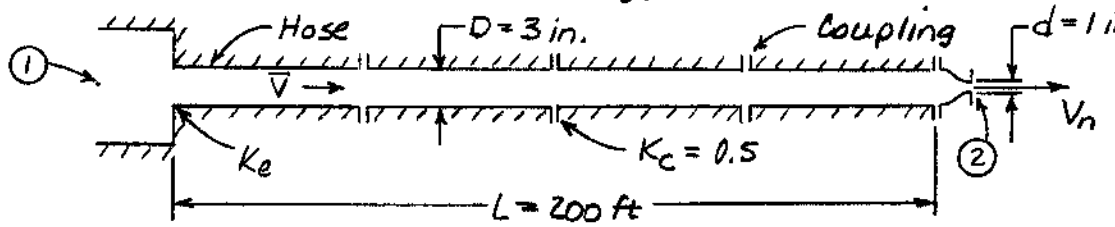
$$h_{em} = K_{ent} \frac{\bar{V}_2^2}{2} + K_{exit} \frac{\bar{V}_4^2}{2}$$

$$h_{em} = 0.5 \times \frac{1}{2} \times (1.56)^2 \frac{m^2}{s^2} + 1.0 \times \frac{1}{2} \times (0.692)^2 \frac{m^2}{s^2} = 0.848 \text{ m}^2/\text{s}^2$$

Therefore minor losses are roughly 1 percent of the frictional losses, so they may be neglected. Thus from the energy equation

$$z_1 - z_5 = \Sigma f \frac{L}{D} \frac{\bar{V}^2}{2g} = 79.8 \frac{m^2}{s^2} \times \frac{s^2}{9.81 m} = 8.13 \text{ m}$$ ← $z_1 - z_5$

Given: Water at $Q = 300$ gpm in hose and nozzle assembly.



$e/D = 0.004$

Find: Supply pressure required.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2 + h_{LT}$ (terms: $\bar{V}_1 \approx 0$ (1); z_1 (2); $p_2 = 0$ (4); z_2 (2))

Assumptions: (1) $\bar{V}_1 \approx 0$, (2) $\Delta z = 0$, (3) $\alpha_2 \simeq 1.0$, (4) $p_2 = p_{atm}$, so $p_{2\,gage} = 0$

(5) $h_{LT} = h_L + h_{em} = f \frac{L}{D} \frac{\bar{V}^2}{2} + (K_e + 4K_c) \frac{\bar{V}^2}{2} + K_n \frac{\bar{V}_n^2}{2}$

From continuity, $\bar{V}_n = \left(\frac{D}{d}\right)^2 \bar{V}$, so $\bar{V}_n^2 = \left(\frac{D}{d}\right)^4 \bar{V}^2$. Thus

$$p_1 = \rho \frac{\bar{V}^2}{2} \left[f \frac{L}{D} + 4K_c + K_e + \left(\frac{D}{d}\right)^4 (1 + K_n) \right]$$

To find f, we need Re. From continuity

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 300 \frac{gal}{min} \times \frac{1}{(0.25)^2 ft^2} \times \frac{ft^3}{7.48 gal} \times \frac{min}{60 s} = 13.6 \text{ ft/s}$$

Assuming $T = 60°F$, $\nu = 1.21 \times 10^{-5}$ ft^2/s (Table A.7), so

$$Re = \frac{\bar{V} D}{\nu} = 13.6 \frac{ft}{s} \times 0.25 ft \times \frac{s}{1.21 \times 10^{-5} ft^2} = 2.69 \times 10^5$$

From the Moody chart, with $e/D = 0.004$, $f = 0.029$. Substituting,

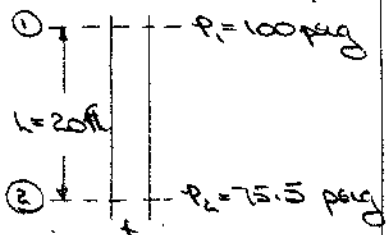
$$p_1 = \frac{1}{2} \times 1.94 \frac{slug}{ft^3} \times (13.6)^2 \frac{ft^2}{s^2} \left[0.029 \times \frac{200 ft}{0.25 ft} + 4(0.5) + 0.5 + \left(\frac{3}{1}\right)^4 (1.02) \right] \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$p_1 = 19{,}400 \frac{lbf}{ft^2} \times \frac{ft^2}{144 in^2} = 135 \text{ psig}$$

p_1

{ Note in this case, friction accounts for 23.2 out of [] = 108 of head loss. Thus it is not necessary to estimate f more precisely. }

Given: Water flow, $Q = 0.11$ ft³/s, through a corroded section of galvanized 1.0 in i.d. pipe with pressure readings as shown



Find: (a) estimate of relative roughness in the pipe section
(b) percent savings in pumping power if ϵ/D value were that for clean pipe.

Solution: Apply the energy equation for steady, incompressible pipe flow

Computing equation:

$$\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{l_T} \quad \text{---(1)}$$

$$h_{l_T} = h_l + h_{l_m} = f \frac{L}{D} \frac{\bar{V}_2^2}{2} + K \frac{\bar{V}_2^2}{2} \quad \text{---(2)} \qquad K = 0 \text{ (4)}$$

Assumptions: (1) $\bar{V}_1 = \bar{V}_2$ from continuity
(2) $\alpha_1 = \alpha_2$
(3) $z_1 - z_2 = 20$ ft
(4) No minor losses

Since $f = f(\epsilon/D, Re)$, solve for f from eqs (1) & (2), calculate Re, and then determine ϵ/D from Fig. 8.13

From eq's (1) & (2)

$$\frac{p_1 - p_2}{\rho} + g(z_1 - z_2) = f \frac{L}{D} \frac{\bar{V}^2}{2} \qquad \therefore f = \frac{D}{L} \frac{2}{\bar{V}^2} \left[\frac{p_1 - p_2}{\rho} + g(z_1 - z_2)\right]$$

$$\bar{V} = \frac{Q}{A} = \frac{Q}{\pi D^2/4} = \frac{4}{\pi} \times 0.11 \frac{\text{ft}^3}{\text{s}} \times \left(\frac{12}{1 \text{ ft}}\right)^2 = 20.2 \text{ ft/s}. \quad \text{Then,}$$

$$f = 2 \times \frac{1}{12} \text{ft} \times \frac{1}{20 \text{ft}} \times \left(\frac{s}{20.2 \text{ft}}\right)^2 \left[(100 - 75.5) \frac{\text{lbf}}{\text{in}^2} \times 144 \frac{\text{in}^2}{\text{ft}^2} \times \frac{\text{ft}^3}{1.94 \text{ slug}} \times \frac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2} + 32.2 \frac{\text{ft}}{\text{s}^2} \times 20 \text{ft}\right] = 0.050$$

Assume $T = 70°$ F, then $\nu = 1.05 \times 10^{-5}$ ft²/s (Table A.7)

$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu} = \frac{1}{12} \text{ft} \times 20.2 \frac{\text{ft}}{\text{s}} \times 1.05 \times 10^{-5} \frac{\text{s}}{\text{ft}^2} = 1.60 \times 10^5$$

For $f = 0.050$ and $Re = 1.60 \times 10^5$, from Fig. 8.13, $\frac{\epsilon}{D} = 0.021$ ← ϵ/D

For a 1.0 in. diameter clean galvanized iron pipe, $\frac{\epsilon}{D} = 0.006$ (Table 8.1)

Then, from Fig. 8.13 $f = 0.0325$ and for the clean pipe

$$h_{l\,clean} = f \frac{L}{D} \frac{\bar{V}^2}{2} = 0.0325 \times \frac{20 \text{ft}}{2 \times 1 \text{ in}} \times \frac{12 \text{ in}}{\text{ft}} \times (20.2)^2 \frac{\text{ft}^2}{\text{s}^2} = 1590 \frac{\text{ft}^2}{\text{s}^2}$$

$$(p_1 - p_2)_{clean} = \rho \left[h_{l\,clean} + g(z_2 - z_1)\right] = 1.94 \frac{\text{slug}}{\text{ft}^3} \left[1590 \frac{\text{ft}}{\text{s}^2} + 32.2 \frac{\text{ft}}{\text{s}^2} (-20 \text{ft})\right] \frac{\text{lbf} \cdot \text{s}^2}{\text{ft} \cdot \text{slug}} \frac{\text{ft}^2}{144 \text{ in}^2}$$

$$\Delta p_{clean} = 12.7 \text{ lbf/in}^2$$

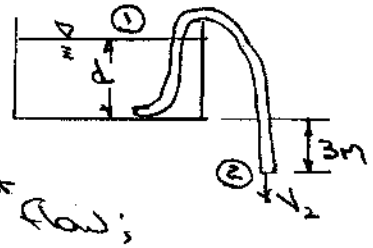
$$\% \text{ Saving in pump power} = \frac{\Delta p_{dirty} - \Delta p_{clean}}{\Delta p_{dirty}} = \frac{24.5 - 12.7}{24.5} = 48.2\% \quad \text{Power Savings}$$

Given: Small swimming pool is drained using a garden hose.

Hose: $D = 20\,mm$, $L = 30\,m$

$e = 0.2\,mm$

$\bar{V}_2 = 1.2\,m/s$



Find: Water depth at instant shown for (a) frictional flow, (b) frictionless flow; explain the results

Solution:

Apply the energy equation for steady incompressible flow between sections (1) and (2)

Basic equations: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + gz_2\right) = h_{LT}$ (8.29)

$h_{LT} = h_L + h_{LM}$; $h_L = f\frac{L}{D}\frac{\bar{V}_1^2}{2}$; $h_{LM} = K_{ent}\frac{\bar{V}_1^2}{2}$

Assumptions: (1) $p_1 = p_2 = p_{atm}$

(2) $\bar{V}_1 = 0$, $\alpha_2 = 1.0$

(3) square edged entrance

Then

$z_1 - z_2 = d + 3m = f\frac{L}{D}\frac{\bar{V}_1^2}{2g} + K_{ent}\frac{\bar{V}_1^2}{2g} + \frac{\bar{V}_2^2}{2g} = \frac{\bar{V}_1^2}{2g}\left[f\frac{L}{D} + K_{ent} + 1\right]$

$\therefore d = \frac{\bar{V}_1^2}{2g}\left[f\frac{L}{D} + K_{ent} + 1\right] - 3m$ ——————— (1)

For square edged entrance (Table 8.2) $K_{ent} = 0.5$

$Re = \frac{\rho \bar{V}_1 D}{\mu} = 0.020m \times 1.2\frac{m}{s} \times 1.00\times10^{-6}\frac{s}{m^2} = 2.4\times10^4$ { assume $T = 20°C$ Table A.8 }

$e/D = 0.2/20 = 0.01$. From Fig. 8.13, $f = 0.04$

Then from Eq. 1

$d = \frac{(1.2)^2}{2}\frac{m^2}{s^2} \times \frac{1}{9.81}\frac{s^2}{m}\left[0.04 \times \frac{30}{0.02} + 0.5 + 1\right] - 3m = 1.51\,m$ ←

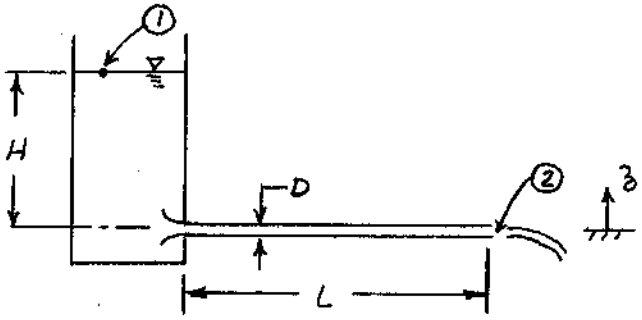
For frictionless flow, $h_{LT} = f\frac{L}{D}\frac{\bar{V}_1^2}{2} + K_{ent}\frac{\bar{V}_1^2}{2} = 0$ and Eq. 1 gives $d = \frac{\bar{V}_2^2}{2g} - 3m = -2.94\,m$

This answer is not physically possible. Hence the assumption of frictionless flow cannot be satisfied. For the stated conditions frictional effects must be present.

Problem 8.109

Open-Ended Problem Statement: In a certain range of Reynolds number, flow in a circular tube may be either laminar or turbulent. The friction factor for the two flow regimes is very different. It has been suggested that this can be demonstrated in the laboratory using water flow from a tank, through a horizontal tube, discharging as a free jet. Show that this flow may alternate between the laminar and turbulent regimes. Design a suitable laboratory-scale setup to demonstrate the alternating-flow phenomenon.

Analysis: Apply the energy equation for steady, incompressible pipe flow:



Computing equation: $\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + gz_1 - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + gz_2\right) = h_{LT} = h_e + h_{em}$ ($\bar{V}_1 \approx 0$)

Assume: (1) $p_1 = p_2 = p_{atm}$
(2) $\bar{V}_1 \approx 0$

Then $gH = \alpha_2 \frac{\bar{V}_2^2}{2} + f\frac{L}{D}\frac{\bar{V}_2^2}{2} + K_{ent}\frac{\bar{V}_2^2}{2} = \left(\alpha_2 + f\frac{L}{D} + K_{ent}\right)\frac{\bar{V}^2}{2}$

so $H = \left(\alpha_2 + f\frac{L}{D} + K_{ent}\right)\frac{\bar{V}_2^2}{2g}$

or $\bar{V}_2 = \left[\frac{2gH}{\alpha_2 + f\frac{L}{D} + K_{ent}}\right]^{\frac{1}{2}}$

For either flow, $Re = \frac{\bar{V}D}{\nu}$; $\bar{V} = \frac{\nu Re}{D}$

For laminar flow, $f = 64/Re$, $\alpha_2 = 2$, and $K_{ent} = 1$ (guess)

For turbulent flow $f = 0.316/Re^{1/4}$ (Blasius), $\alpha_2 \approx 1$, and $K_{ent} = 0.04$ (Table 8.2)

Calculations and a plot are shown on the next page; L and D are reasonable.

Discussion: Starting from rest flow initially will be in the laminar regime.

The friction factor for laminar flow is low, so the flow will accelerate. Transition to the turbulent flow regime will occur as the threshold Reynolds number for turbulent flow is reached. The friction factor will suddenly become higher and flow will decelerate. As the flow decelerates, it slows below the threshold Reynolds number for laminar flow and becomes laminar again.

Once the flow is again in the laminar regime, the cycle will repeat indefinitely.

The appearance of the free jet as it discharges from the horizontal tube makes it easy to tell whether flow in the tube is laminar or turbulent (see the discussion of the characteristic free discharge patterns for laminar and turbulent flow in Problem 8.131).

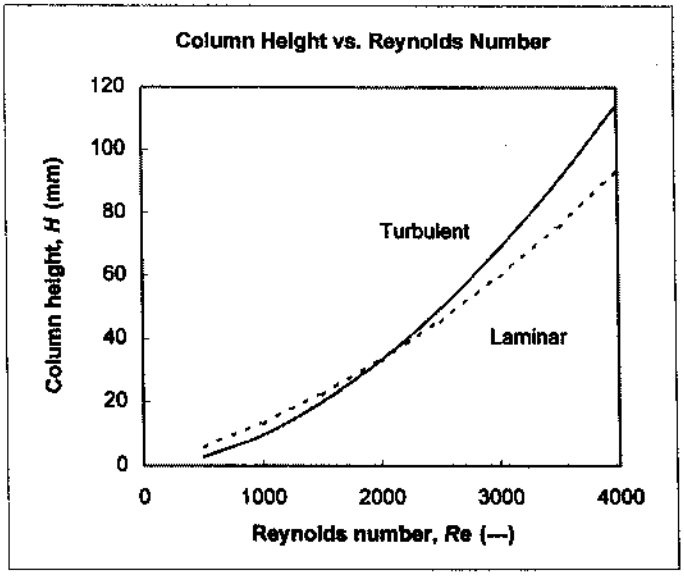
D = 6 mm
L = 0.8 m
T = 25 °C
ν = 8.93E-07 m^2/s (Table A.8)

(a) Laminar flow K_{ent} = 1 (---)
α_2 = 2 (—)

Re	V_{bar}	*f*	*fL/D*	*H* (mm)
500	0.074	0.128	17.1	5.66
1000	0.149	0.0640	8.53	13.0
1500	0.223	0.0427	5.69	22.1
2000	0.298	0.0320	4.27	32.8
2500	0.372	0.0256	3.41	45.3
3000	0.447	0.0213	2.84	59.4
3500	0.521	0.0183	2.44	75.2
4000	0.595	0.0160	2.13	92.7

(b) Turbulent flow K_{ent} = 0.04 (—)
α_2 = 1 (—)

Re	V_{bar}	*f*	*fL/D*	*H* (mm)
500	0.074	0.0668	8.91	2.81
1000	0.149	0.0562	7.49	9.63
1500	0.223	0.0508	6.77	19.8
2000	0.298	0.0473	6.30	33.1
2500	0.372	0.0447	5.96	49.4
3000	0.447	0.0427	5.69	68.4
3500	0.521	0.0411	5.48	90.1
4000	0.595	0.0397	5.30	114

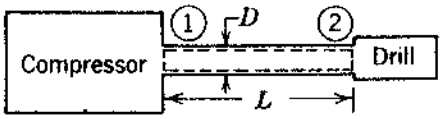


Given: Air flow through a line, of length L and diameter $D = 40$ mm

$p_1 = 670$ kPa(g) $p_2 = 650$ kPa(g)

$T_1 = 40°C$ $\dot{m} = 0.25$ kg/s

ρ = constant



Find: Allowable length of hose

Solution:

Computing equation: $\left(\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2\right) = h_L = h_f + h_{l_m}$ (8.29)

where $h_f = f\frac{L}{D}\frac{\bar{V}^2}{2}$ $\quad h_{l_m} = K\frac{\bar{V}^2}{2}$

For $\rho = C$, then $\bar{V}_1 = \bar{V}_2$, since $A_1 = A_2$. Since p_1 and p_2 are given, neglect minor losses. Assume $\alpha_1 = \alpha_2$ and neglect elevation changes. Then Eq. 8.29 can be written as

$$\frac{p_1 - p_2}{\rho} = f\frac{L}{D}\frac{\bar{V}^2}{2} \quad \text{or} \quad L = \frac{(p_1 - p_2)}{\rho}\frac{2D}{f\bar{V}^2}$$

The density is

$$\rho = \rho_1 = \frac{p_1}{RT_1} = 7.91\times10^5 \frac{N}{m^2}\times\frac{kg\cdot K}{287 N\cdot m}\times\frac{1}{313 K} = 8.81 \text{ kg/m}^3$$

From continuity

$$\bar{V} = \frac{\dot{m}}{\rho A} = \frac{4\dot{m}}{\pi\rho D^2} = \frac{4}{\pi}\times 0.25\frac{kg}{sec}\times\frac{m^3}{8.81 kg}\times\frac{1}{(0.04)^2 m^2} = 22.6 \text{ m/sec}$$

For air at 40°C, $\mu = 1.91\times10^{-5}$ kg/m·s (Table A.6), so

$$Re = \frac{\rho\bar{V}D}{\mu} = 8.81\frac{kg}{m^3}\times 22.6\frac{m}{sec}\times 0.04 m\times\frac{m\cdot sec}{1.91\times10^{-5} kg} = 4.17\times10^5$$

Assume smooth pipe; then from Fig. 8.13, $f = 0.0134$

Substituting gives

$$L = \frac{(p_1 - p_2)}{\rho}\frac{2D}{f\bar{V}^2}$$

$$= 20\times10^3\frac{N}{m^2}\times 2\times 0.04 m\times\frac{m^3}{8.81 kg}\times\frac{1}{0.0134}\times\frac{sec^2}{(22.6)^2 m^2}\times\frac{kg\cdot m}{N\cdot sec^2}$$

$$L = 26.5 \text{ m}$$

Given: Gasoline flow in a horizontal pipeline at 15°C. The distance and pressure drop between pumping stations are 13 km and 1.4 MPa, respectively. The pipe is 0.6 m in diameter. Its roughness corresponds to galvanized iron.

Find: Volume flow rate.

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section.

Basic equation: $\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + g z_2 + h_{LT}$; $h_{LT} = f\frac{L}{D}\frac{\bar{V}^2}{2} + h_{lm}$

($\bar{V}_1^2$: $=O(1)$; $g z_1$: $=O(z)$; $\bar{V}_2^2$: $=O(1)$; $g z_2$: $=O(z)$; h_{lm}: $=O(1)$)

Assumptions: (1) Constant area pipe, so $\bar{V}_1 = \bar{V}_2$, $h_{lm} = 0$
(2) Level, so $z_1 = z_2$

Thus

$$\frac{p_1 - p_2}{\rho} = f\frac{L}{D}\frac{\bar{V}^2}{2} \quad \text{or} \quad \bar{V} = \left[\frac{2D(p_1 - p_2)}{\rho f L}\right]^{\frac{1}{2}}$$

But $f = f(Re, \varepsilon/D)$, and the Reynolds number is not known. Therefore iteration is required. Choose f in the fully-rough zone. From Table 8.1, $\varepsilon = 0.15$ mm; $\varepsilon/D = 0.00025$. Then from Fig. 8.13, $f \approx 0.014$. Substituting,

$$\bar{V} = \left[2 \times 0.6\,m \times 1.4 \times 10^6 \frac{N}{m^2} \times \frac{m^3}{(0.72)1000\,kg} \times \frac{1}{0.014} \times \frac{1}{13 \times 10^3\,m} \times \frac{kg \cdot m}{N \cdot s^2}\right]^{\frac{1}{2}}$$

{SG = 0.72, Table A.2}

$$\bar{V} = 3.58\ m/s$$

Now compute Re and check on guess for f. Choose $\mu = 5 \times 10^{-4}\ N \cdot s/m^2$ (Fig. A.2).*

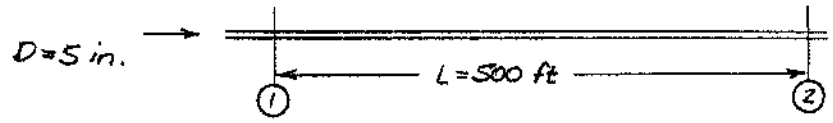
$$Re = \frac{\rho \bar{V} D}{\mu} = (0.72)1000\frac{kg}{m^3} \times 3.58\frac{m}{s} \times 0.6\,m \times \frac{m^2}{5 \times 10^{-4} N \cdot s} \times \frac{N \cdot s^2}{kg \cdot m} = 3.09 \times 10^6$$

Checking on Fig. 8.13, flow is essentially in the fully-rough zone, and initial guess for f was okay. Thus

$$Q = \bar{V}A = 3.58\frac{m}{s} \times \frac{\pi}{4}(0.6)^2 m^2 = 1.01\ m^3/s \qquad Q$$

* Note gasoline is between heptane and octane.

Given: Steady flow of water in 5 in. diameter, horizontal, cast-iron pipe.



Find: Volume flow rate.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation:

$$\left(\frac{p_1}{\rho}+\alpha_1\frac{\bar{V}_1^2}{2}+g\cancel{z_1}\right)-\left(\frac{p_2}{\rho}+\alpha_2\frac{\bar{V}_2^2}{2}+g\cancel{z_2}\right)+h_{LT}$$

$$h_{LT}=h_L+h_{LM}=f\frac{L}{D}\frac{\bar{V}^2}{2}+K\frac{\bar{V}^2}{2}$$

Assumptions: (1) Fully developed flow: $\alpha_1\bar{V}_1^2=\alpha_2\bar{V}_2^2$
(2) Horizontal: $z_1=z_2$
(3) Constant area, so $K=0$

Then

$$\frac{\Delta p}{\rho}=h_{LT}=f\frac{L}{D}\frac{\bar{V}^2}{2} \quad \text{so} \quad \bar{V}=\sqrt{\frac{2\Delta p D}{\rho f L}}$$

Since flow rate (hence Re and f) are unknown, must iterate. Guess a trial value of f in the fully rough zone. From Table 8.1, $e = 0.00085$ ft.

Then $e/D=\frac{0.00085}{5/12}=0.002$. Then from Fig. 8.13, $f \simeq 0.024$ for $Re \geq 6\times10^5$

$$\bar{V}=\left[2\times 23\ \frac{lbf}{in.^2}\times 5\ in.\times\frac{ft^3}{1.94\ slug}\times\frac{1}{0.024}\times\frac{1}{500\ ft}\times 12\frac{in.}{ft}\times\frac{slug\cdot ft}{lbf\cdot s^2}\right]^{1/2}=10.9\ ft/s$$

and, checking Re, with $\nu = 1.21\times10^{-5}\ ft^2/s$ at $T=60°F$ (Table A.7),

$$Re=\frac{\bar{V}D}{\nu}=10.9\frac{ft}{sec}\times 5\ in.\times\frac{s}{1.21\times10^{-5}\ ft^2}\times\frac{ft}{12\ in.}=3.75\times10^5$$

The friction factor at this Re is still $f \simeq 0.024$, so convergence is ok.

$$Q=\bar{V}A=10.9\frac{ft}{s}\times\frac{\pi}{4}\times\left(\frac{5}{12}\right)^2 ft^2=1.49\ ft^3/s$$ Q

$$(Q=1.49\frac{ft^3}{s}\times 7.48\frac{gal}{ft^3}\times 60\frac{s}{min}=669\ gpm)$$

Given: Steady flow of water through a cast iron pipe of diameter $D = 5$ in. The pressure drop over a length of pipe, $L = 500$ ft is $p_1 - p_2 = 23$ psi. Section 2 is located 30 ft above section 1.

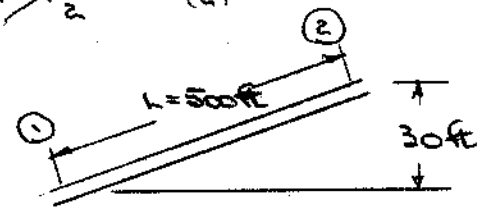
Find: the volume flow rate, Q.

Solution: Apply the energy equation for steady, incompressible pipe flow

Computing equation:

$$\left(\frac{p_1}{\rho} + \alpha\frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha\frac{\bar{V}_2^2}{2} + gz_2\right) = h_{l_T} \quad (1)$$

$$h_{l_T} = h_l + h_{l_m} = f\frac{L}{D}\frac{\bar{V}_2^2}{2} + K\frac{\bar{V}_2^2}{2} \quad (2)$$



Assumptions: (1) $\bar{V}_1 = \bar{V}_2$ from continuity
(2) $\alpha_1 = \alpha_2$
(3) $z_2 - z_1 = 30$ ft
(4) neglect minor losses

For cast iron pipe with $D = 5$ in, $\varepsilon/D = 0.002$ ($\varepsilon = 0.00085$ ft, Table 8.1).

Since $f = f(Re)$ and $\bar{V}$ is unknown, iteration will be required

From Eqs (1) and (2)

$$\left(\frac{p_1}{\rho} + gz_1\right) - \left(\frac{p_2}{\rho} + gz_2\right) = f\frac{L}{D}\frac{\bar{V}_2^2}{2}$$

Then

$$f\bar{V}^2 = \frac{2D}{L}\left[\frac{(p_1 - p_2)}{\rho} + g(z_1 - z_2)\right]$$

$$f\bar{V}^2 = \frac{2 \times 5\text{ in}}{500\text{ ft}} \times \frac{1\text{ ft}}{12\text{ in}}\left[\frac{23\text{ lb}}{\text{in}^2} \times \frac{\text{ft}^3}{1.94\text{ slug}} \times \frac{144\text{ in}^2}{\text{ft}^2} \times \frac{\text{slug-ft}}{\text{lb-s}^2} + 32.2\frac{\text{ft}}{\text{s}^2}(-30\text{ ft})\right] = 1.24\frac{\text{ft}^2}{\text{s}^2}$$

$$f\bar{V}^2 = 1.24\ \text{ft}^2/\text{s}^2.$$

Assume flow is fully rough region, $f = 0.0235$, then $\bar{V} = 7.26$ ft/s

Check Re. Assume $T = 70°F$, $\nu = 1.05 \times 10^{-5}$ ft^2/sec (Table A.7).

Then $Re = \frac{D\bar{V}}{\nu} = \frac{\frac{5}{12}\text{ ft} \times 7.26\frac{\text{ft}}{\text{s}}}{1.05 \times 10^{-5}\text{ ft}^2} = 2.88 \times 10^5$

From Fig. 8.13 with $Re = 2.88 \times 10^5$, $\varepsilon/D = 0.002$, then

$f = 0.024$ and $\bar{V} = 7.19$ ft/s

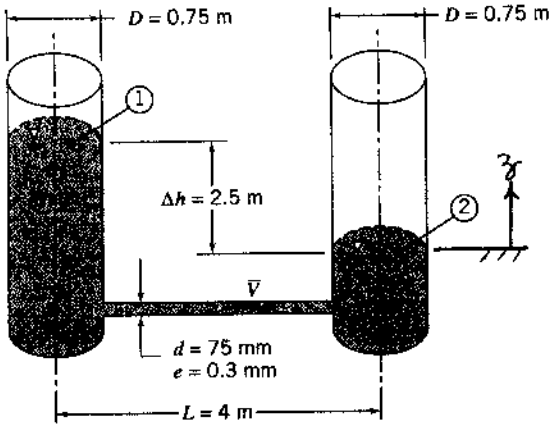
Check $Re = 2.78 \times 10^5$ ✓

Then

$$Q = A\bar{V} = \frac{\pi}{4}D^2\bar{V} = \frac{\pi}{4}\left(\frac{5}{12}\right)^2\text{ft}^2 \times 7.19\frac{\text{ft}}{\text{s}} = 0.980\ \text{ft}^3/\text{s} \quad \leftarrow Q$$

Given: Two open standpipes shown. Water flows by gravity.

Find: Estimate of rate of change of water level in left standpipe.



Solution: Apply the energy equation for quasi-steady, incompressible pipe flow.

Computing equation:

$$\left(\frac{p_1}{\rho}+\alpha_1\frac{\bar{V}_1^2}{2}+gz_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\frac{\bar{V}_2^2}{2}+gz_2\right)=h_{lT};\quad h_{lT}=h_l+h_{em}=\left[f\left(\frac{L-D}{d}\right)+K_{ent}+K_{exit}\right]\frac{\bar{V}^2}{2}$$

From continuity, $A_1V_1=A_p\bar{V}$

Assumptions: (1) Neglect unsteady effects
(2) Incompressible flow
(3) $p_1=p_2=p_{atm}$
(4) $\bar{V}_1=\bar{V}_2$ since diameters are equal

Then $g\Delta h=h_{lT}=\left[f\left(\frac{L-D}{d}\right)+K_{ent}+K_{exit}\right]\frac{\bar{V}^2}{2}$

Flow rate (hence speed) is unknown, so assume flow is in fully rough zone.

$\frac{e}{D}=\frac{0.3}{75}=0.004$, so $f\approx 0.028$ from Moody diagram (Fig. 8.13)

From Table 8.2, $K_{ent}=0.5$; from Fig. 8.15, $K_{exit}=1$. Then

$$\bar{V}=\left[\frac{2g\Delta h}{f\left(\frac{L-D}{d}\right)+K_{ent}+K_{exit}}\right]^{\frac{1}{2}}=\left[\frac{2\times 9.81\frac{m}{s^2}\times 2.5\,m}{0.028\left(\frac{4-0.75}{0.075}\right)+0.5+1.0}\right]^{\frac{1}{2}}=4.25\ m/s$$

Check Re and f. For water at 20°C, $\nu=1.00\times10^{-6}\ m^2/s$ (Table A.8)

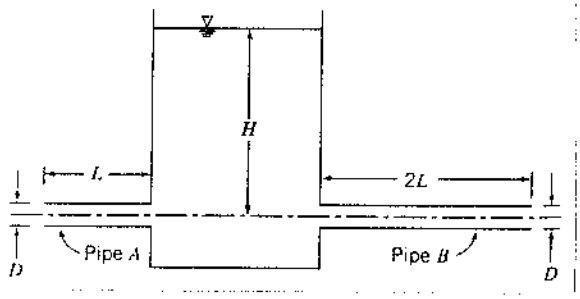
$$Re=\frac{\bar{V}d}{\nu}=4.25\frac{m}{s}\times 0.075\,m\times\frac{s}{1.00\times10^{-6}m^2}=3.19\times10^5$$

From the Moody diagram $f\approx 0.029$, so this is satisfactory agreement.

$$V_1=\frac{A_p}{A_1}V_p=\left(\frac{d}{D}\right)^2V_p=\left(\frac{0.075}{0.75}\right)^2\times 4.25\frac{m}{s}=0.0425\ m/s\ (down)$$

The water level in the left tank <u>falls</u> at about 42.5 mm/s

Given: Two galvanized iron pipes connected to large water reservoir as shown.



Determine: (a) which pipe will pass the larger flow rate (without calculations); justify
(b) the larger flow rate if $H = 12\,m$, $D = 50\,mm$, $L = 50\,m$

Solution:

Flow through each pipe is governed by the energy equation for steady incompressible flow.

Basic equations: $\left(\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2\right) = h_{lt}$ (8.29)

$$h_{lt} = h_l + h_{lm} = f\frac{L}{D}\frac{\bar{V}_1^2}{2} + K_{ent}\frac{\bar{V}_1^2}{2}$$

Assumptions: (1) $p_1 = p_2 = p_3 = p_{atm}$
(2) $\bar{V}_1 = 0$, $\alpha_2 = \alpha_3 = 1.0$

Then

$$g(z_1 - z_2) = h_{lt} + \frac{\bar{V}_2^2}{2} = \frac{\bar{V}_2^2}{2}\left[f\frac{L}{D} + K_{ent} + 1\right] \quad \text{(pipe A)}$$

$$g(z_1 - z_3) = h_{lt} + \frac{\bar{V}_3^2}{2} = \frac{\bar{V}_3^2}{2}\left[f\frac{2L}{D} + K_{ent} + 1\right] \quad \text{(pipe B)}$$

Since $z_1 - z_2 = z_1 - z_3$, then $\bar{V}_2 > \bar{V}_3$ and $Q_A > Q_B$ ←

Through pipe A $\quad gH = \frac{\bar{V}_2^2}{2}\left[f\frac{L}{D} + K_{ent} + 1\right]$

From Table 8.1 $e = 0.15\,m$ $\therefore e/D = 0.15/50 = 0.003$

Assume water at 20°C, $\nu = 1.00\times10^{-6}\,m^2/s$ (Table A.8)

Choose friction factor $f = 0.026$ (in fully rough region, Fig. 8.13)

Then $\bar{V}_2 = \left\{\frac{2gH}{\left[f\frac{L}{D} + K_{ent} + 1\right]}\right\}^{1/2} = \left\{2\times9.81\frac{m}{s^2}\times12\,m\times\frac{1}{\left[0.026\times\frac{50}{0.05} + 0.5 + 1.0\right]}\right\}$

$$\bar{V}_2 = 2.93\,m/s$$

Check $Re = \frac{D\bar{V}_2}{\nu} = 0.05\,m\times2.93\frac{m}{s}\times\frac{1}{1.00\times10^{-6}}\frac{s}{m^2} = 1.47\times10^5$

At this Re, $f = 0.0265$ and $\bar{V}_2 = 2.90\,m/s$

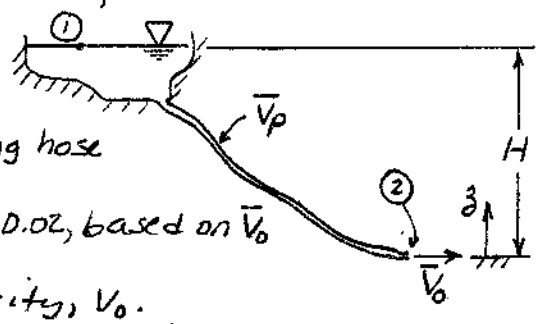
$$Q = A\bar{V} = \frac{\pi}{4}D^2\bar{V} = \frac{\pi}{4}(0.05\,m)^2\times2.90\frac{m}{s} = 5.69\times10^{-3}\,m^3/s \quad Q_A$$ ←

Given: Site for hydraulic mining, $H = 300\ m$, $L = 900\ m$.

Hose with $D = 75\ mm$, $e/D = 0.01$.

Couplings, $\frac{L_e}{D} = 20$, every 10 m along hose

Nozzle diameter, $d = 25\ mm$; $K = 0.02$, based on $\bar{V}_0$



Find: (a) Estimate maximum outlet velocity, V_0.
(b) Determine maximum force of jet on rock face.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\left(\frac{p_1}{\rho}^{=0(1)} + \alpha_1 \frac{\bar{V}_1^2}{2}^{=0(2)} + g z_1\right) - \left(\frac{p_2}{\rho}^{=0(3)} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{LT}$

Assume: (1) $p_1 = 0$; (2) $\bar{V}_1 = 0$; (3) $p_2 = 0$; (4) $\alpha_2 = 1$; (5) $z_2 = 0$; (6) Fully-rough zone

Then $gH = h_{LT} + \frac{\bar{V}_2^2}{2} = f\frac{L}{D}\frac{\bar{V}_p^2}{2} + f_{x90}\frac{L_e}{D}\frac{\bar{V}_p^2}{2} + K\frac{\bar{V}_0^2}{2} + \frac{\bar{V}_e^2}{2}$

From continuity $\bar{V}_p A_p = \bar{V}_0 A_0$; $\bar{V}_2 = \bar{V}_0 \frac{A_0}{A_2}$; $\bar{V}_2^2 = \bar{V}_0^2\left(\frac{A_0}{A_2}\right)^2 = \bar{V}_0^2\left(\frac{d}{D}\right)^4$

Substituting, $gH = \left[f\left(\frac{L}{D} + 90\frac{L_e}{D}\right)\left(\frac{d}{D}\right)^4 + 1 + K\right]\frac{\bar{V}_0^2}{2}$

$$\bar{V}_0 = \left[\frac{2gH}{f\left(\frac{L}{D} + 90\frac{L_e}{D}\right)\left(\frac{d}{D}\right)^4 + 1 + K}\right]^{1/2}$$; in fully-rough zone $\left(\frac{\epsilon}{D} = 0.01\right)$, $f = 0.038$

$$\bar{V}_0 = \left[2 \times 9.81\frac{m}{s^2} \times 300\ m \times \frac{1}{0.038\left(\frac{900\ m}{0.075\ m} + 90\,(20)\right)\left(\frac{0.025}{0.075}\right)^4 + 1 + 0.02}\right]^{1/2} = 28.0\ m/s\ (est.)$$

Check for fully-rough flow zone:

$Re = \frac{\bar{V}_p D}{\nu}$; $\bar{V}_p = \bar{V}_0\left(\frac{d}{D}\right)^4 = 28.0\frac{m}{s}\left(\frac{1}{3}\right)^4 = 0.346\ m/s$ {Assume $T = 20°C$}

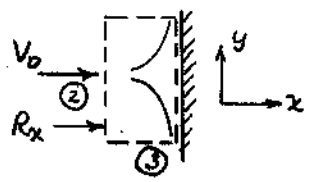
$Re = 0.346\frac{m}{sec} \times 0.075\ m \times \frac{1}{1 \times 10^{-6}\ m^2} = 2.60 \times 10^4$; at $\frac{\epsilon}{D} = 0.01$, $f \approx 0.040$

The new estimate is

$$\bar{V}_0 = \sqrt{\frac{0.038}{0.040}}\,\bar{V}_0\,(est) = \sqrt{\frac{0.038}{0.040}}\,28.0\frac{m}{s} = 27.3\ m/s.$$

$\bar{V}_0$

Apply momentum to find force: CV is shown.



$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

Assumptions: (1) No pressure forces
(2) $F_{Bx} = 0$
(3) Steady flow

(4) Uniform flow at each cross-section

Then

$$R_x = u_2\{-\rho \bar{V}_0 A_0\} + u_3\{+\rho \bar{V}_0 A_0\}$$

$$u_2 = \bar{V}_0 \qquad u_3 = 0$$

$$R_x = -\rho \bar{V}_0^2 A_0$$

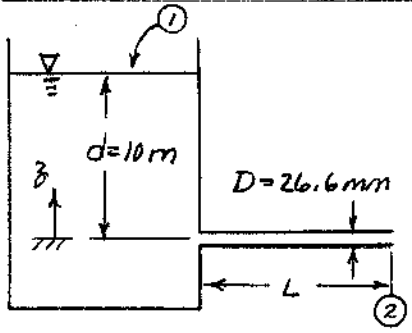
The force <u>on</u> the rock face is

$$K_x = -R_x = \rho \bar{V}_0^2 A_0$$

$$= 999 \frac{kg}{m^3} \times (27.3)^2 \frac{m^2}{s^2} \times \frac{\pi}{4}(0.025)^2 m^2 \times \frac{N \cdot s^2}{kg \cdot m}$$

$K_x = 365$ N (to right) K_x

Given: Flow configuration of Example Problem 8.5, but with a well rounded inlet, $d = 10\ m$, and $D = 26.6\ mm$.



Find: (a) Volume flow rate if $L = 170\ m$.
(b) Length for laminar flow.
(c) What happens to flow rate when flow changes from laminar to turbulent.
(d) Length to reduce flow rate to 1 gal/hr.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{LT} = \left(f\frac{L}{D} + K_{ent}\right)\frac{\bar{V}^2}{2}$ ($\bar{V}_1 \approx 0$ (2), $z_2 = 0$)

Assumptions: (1) $p_1 = p_2 = p_{atm}$
(2) Reservoir large, so $\bar{V}_1 \approx 0$
(3) $\alpha_2 \approx 1$

$K_{ent} \approx 0.04$ (Table 8.2)
$\nu = 1 \times 10^{-6}\ m^2/s$ ($T = 20°C$, Table A.8)

Then $g d = \left(f\frac{L}{D} + K_{ent} + 1\right)\frac{\bar{V}^2}{2}$ or $\bar{V} = \left[\frac{2gd}{f\frac{L}{D} + K_{ent} + 1}\right]^{\frac{1}{2}}$

For smooth pipe, $f = f(Re)$, but $\bar{V}$ is not known. Guess $Re = 10^5$, so $f \approx 0.018$ (Fig. 8.13),

$$\bar{V} \approx \left[\frac{2 \times 9.81 \frac{m}{s^2} \times 10\ m}{0.018\frac{170}{0.0266} + 0.04 + 1}\right]^{\frac{1}{2}} = 1.30\ m/s\ ;\quad Re = 1.30\frac{m}{s} \times 0.0266\ m \times \frac{s}{1\times10^{-6} m^2} = 3.46 \times 10^4$$

Try again with $f \approx 0.023$

$$\bar{V} \approx \left[\frac{196\ m^2/s^2}{0.023\frac{170}{0.0266} + 1.04}\right]^{\frac{1}{2}} = 1.15\ m/s\ ;\ Re = 3.06 \times 10^4\ ;\ f = 0.023\ \checkmark\checkmark \text{ (okay)}$$

Thus $Q = \bar{V}A = 1.15\frac{m}{s} \times \frac{\pi}{4}(0.0266)^2 m^2 \times 1000\frac{L}{m^3} \times 60\frac{s}{min} = 38.3\ L/min$ ($L = 170\ m$) Q

For laminar flow, $f = 64/Re = 64/2300 = 0.0278$; $Re = \frac{\bar{V}D}{\nu}$; $\bar{V} = \frac{\nu Re}{D} = \frac{(1\times10^{-6})(2300)}{0.0266} = 0.00865\ m/s$.

From above, $f\frac{L}{D} = \frac{2gd}{\bar{V}^2} - K_{ent} - 1 = 196\frac{m^2}{s^2} \times \frac{s^2}{(0.00865)^2 m^2} - 1.04 = 2.62 \times 10^4$

$$L = \frac{2.62\times10^4 D}{f} = 2.62\times10^4 \times 0.0266\ m \times \frac{1}{0.0278} = 2.51 \times 10^4\ m = 25.1\ km$$ L

Flow rate would decrease at transition because friction factor increases. Q

At $Q = 1\ gal/hr$, $\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 1\frac{gal}{hr} \times \frac{1}{(0.0266)^2 m^2} \times 3.78\frac{L}{gal} \times \frac{m^3}{1000\ L} \times \frac{hr}{3600\ s} = 1.89\times10^{-3}\ m/s$

$$Re = \frac{\bar{V}D}{\nu} = \frac{(1.89\times10^{-3})(0.0266)}{1\times10^{-6}} = 50.3\ ;\ f = \frac{64}{Re} = \frac{64}{50.3} = 1.27$$

$$f\frac{L}{D} = \frac{2gd}{\bar{V}^2} - K_{ent} - 1 = 196\frac{m^2}{s^2} \times \frac{s^2}{(1.89\times10^{-3})^2 m^2} - 1.04 = 5.49\times10^7$$

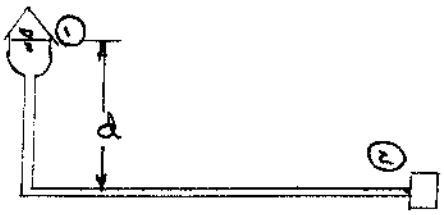
$$L = 5.49\times10^7 \frac{D}{f} = 5.49\times10^7 \times 0.0266\ m \times \frac{1}{1.27} = 1.15\times10^6\ m \text{ or } 1,150\ km$$ L

Given: Fire protection system as shown

pipe: cast iron $D = 6$ in

total length $L = 6000$ ft

For no flow, $p_2 = 90$ psig



Find: (a) elevation d.

(b) maximum flow rate for hydrant wide open ($p_2 = p_{atm}$) assuming $h_{en} = 0.10\, h_e$.

(c) pressure at (2) with fire hose attached and flow rate of 200 gpm.

Solution:

For no flow condition (static fluid) p_2 gage $= \gamma d$

$$d = \frac{p_2}{\gamma} = 90 \frac{lb}{in^2} \times 144 \frac{in^2}{ft^2} \times 1.94 \frac{slug}{ft^3} \times 32.2 \frac{ft}{s^2} \times \frac{1\, lb \cdot s^2}{slug \cdot ft} = 207\ ft$$ ← d

Apply the energy equation for steady, incompressible flow.

Basic equations: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{V_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{V_2^2}{2} + g z_2\right) = h_{et}$ (8.29)

$h_{et} = h_e + h_{en}$, $h_e = f \frac{L}{D} \frac{V_2^2}{2}$, also $h_{en} = 0.1\, h_e$

Assumptions: (1) $p_1 = p_2 = p_{atm}$

(2) $V_1 = 0$, $\alpha_2 = 1$

Then $g(z_1 - z_2) = g d = h_{et} + \frac{V_2^2}{2} = f \frac{L}{D} \frac{V_2^2}{2} + 0.1 f \frac{L}{D} \frac{V_2^2}{2} + \frac{V_2^2}{2}$

or $g d = \frac{V_2^2}{2} \left(1.1 f \frac{L}{D} + 1\right)$

For cast iron pipe, $\varepsilon = 0.00085$ ft (Table 8.1), $D = 6.065$ in (Table B.5)

$$\varepsilon/D = 0.00085\ ft \times \frac{12\ in}{ft} \times \frac{1}{6.065\ in} = 0.00168$$

From Fig 8.13 guess $f = 0.022$ (fully rough region) then

$$V_2 = \left[\frac{2gd}{1.1 f \frac{L}{D} + 1}\right]^{1/2} = \left[2 \times 32.2 \frac{ft}{s^2} \times 207\ ft \times \frac{1}{\left(1.1 \times 0.022 \times \frac{6000 \times 12}{6.065} + 1\right)}\right]^{1/2} = 6.80\ ft/s$$

Check R_e assuming $T = 68°F$, $\nu = 1.08 \times 10^{-5}\ ft^2/s$

$$R_e = \frac{D V_2}{\nu} = \frac{6.065}{12}\ ft \times 6.80 \frac{ft}{s} \times 1.08 \times 10^{-5} \frac{s}{ft^2} = 3.18 \times 10^5$$

From Fig. 8.13 $f = 0.022$ is okay. Thus

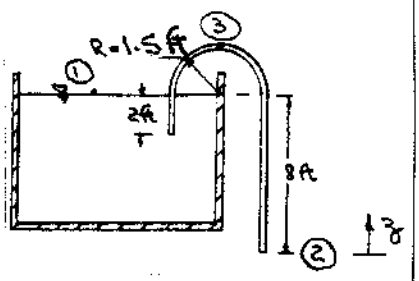
$$Q = A V_2 = \frac{\pi}{4} D^2 V_2 = \frac{\pi}{4} (6.065)^2\ in^2 \times \frac{ft^2}{144\ in^2} \times 6.80 \frac{ft}{s} = 1.36\ ft^3/s$$ ← Q

For $Q = 200$ gpm $V_2 = \frac{Q}{A} = 200 \frac{gal}{min} \times \frac{1\ min}{60\ s} \times 7.48 \frac{ft^3}{gal} \times \frac{\pi}{4} \frac{(144)}{(6.065)^2} \frac{(1\ ft^3/s)}{(646\ gpm)} = 2.22\ ft/s$

$$p_2 = \rho \left[g(z_1 - z_2) - \frac{V_2^2}{2} \left(1.1 f \frac{L}{D} + 1\right)\right] \qquad R_e = \frac{D V_2}{\nu} = 1.04 \times 10^5 \qquad f = 0.024$$

$$p_2 = 1.94 \frac{slug}{ft^3} \left[32.2 \frac{ft}{s^2} \times 207\ ft - \frac{1}{2} (2.22)^2 \frac{ft^2}{s^2} \left(1.1 \times 0.024 \times \frac{6000\ ft}{6.065} + 1\right)\right] \frac{lb \cdot s^2}{slug \cdot ft} = 79.4\ psig$$ ← p_2

Given: Siphon shown is fabricated from 2 in. i.d. drawn aluminum. The liquid is water at 60°F.



Find: Compute the volume flow rate. Estimate p_{min} inside the tube

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section.

Basic equations: $\frac{p_1}{\rho}+\alpha_1\frac{\bar V_1^2}{2}+gz_1=\frac{p_2}{\rho}+\alpha_2\frac{\bar V_2^2}{2}+gz_2+h_{lT}$

$h_{lT}=f\frac{L}{D}\frac{\bar V^2}{2}+h_{lm}$; $h_{lm}=K_{ent}\frac{\bar V^2}{2}+f\left(\frac{L_e}{D}\right)_{bend}\frac{\bar V^2}{2}$

Assumptions: (1) $p_1=p_2=p_{atm}$
(2) $V_1\approx 0$
(3) uniform flow at ②, $\alpha_2=1.0$
(4) reentrant entrace

Then $gz_1=\frac{\bar V_2^2}{2}+f\frac{L}{D}\frac{\bar V_2^2}{2}+K_{ent}\frac{\bar V_2^2}{2}+f\left(\frac{L_e}{D}\right)_{bend}\frac{\bar V_2^2}{2}$

$2gz_1=\bar V_2^2\left[1+K_{ent}+f\left\{\left(\frac{L_e}{D}\right)_{bend}+\frac{L}{D}\right\}\right]$

An iterative solution for $\bar V$ is required

From Table 8.2 for reentrant entrace $K_{ent}=0.78$

For bend $R/d=1.5\,ft/0.167\,ft=9$

- from Fig. 8.17, $L_e/D=28$ for 90° bend
- as first approximation assume $L_e/D=56$ for 180° bend

For straight pipe $L=10\,ft$, $L/D=60$

Then

$$2\times 32.2\frac{ft}{s^2}\times 8ft=\bar V_2^2\left[1+0.78+f\{56+60\}\right]=\bar V_2^2\left[1.78+116f\right]$$

For 2" drawn aluminum tubing, $\epsilon=5\times10^{-6}\,ft$ (Table 8.1), $\epsilon/D=0.00003$

Assume $Re=5\times10^5$, then $f=0.0138$, and $\bar V_2=12.3\,ft/s$

Then $Re=\frac{DV\rho}{\mu}=\frac{\frac{1}{6}ft}{}\times 12.3\frac{ft}{s}\times 1.2\times10^{-5}\frac{lb}{ft^2}=1.71\times10^5$

With $Re=1.71\times10^5$, then $f=0.016$, and $\bar V_2=11.9\,ft/s$

Then $Re=\frac{DV\rho}{\mu}=\frac{ft}{6}\times 11.9\frac{ft}{s}\times\frac{s}{1.2\times10^{-5}\,ft}=1.65\times10^5\Rightarrow f=0.016$ ✓

$\therefore Q=A\bar V=\frac{\pi}{4}D^2\bar V=\frac{\pi}{4}\left(\frac{1}{6}\right)^2ft^2\times 11.9\frac{ft}{s}=0.260\,ft^3/s$ ← Q

The minimum pressure occurs at point ③ in the pipe; $V_3=V_2$

$\frac{p_1}{\rho}+\alpha_1\frac{\bar V_1^2}{2}+gz_1=\frac{p_3}{\rho}+\alpha_3\frac{\bar V_3^2}{2}+gz_3+K_{ent}\frac{\bar V_3^2}{2}+f\left(\frac{L_e}{D}\right)_{bend}\frac{\bar V_3^2}{2}+f\frac{L}{D}\frac{\bar V_3^2}{2}$

$p_3=\rho\left[g(z_1-z_3)-\frac{\bar V_3^2}{2}\left(1+K_{ent}+f\left\{\left(\frac{L_e}{D}\right)_{bend}+\left(\frac{L_e}{D}\right)_{pipe}\right\}\right)\right]$

$=1.94\frac{slug}{ft^3}\left[32.2\frac{ft}{s^2}\times(-1.5ft)-\frac{(11.9)^2}{2}\frac{ft^2}{s^2}\left(1+0.78+0.016\{28+12\}\right)\right]$

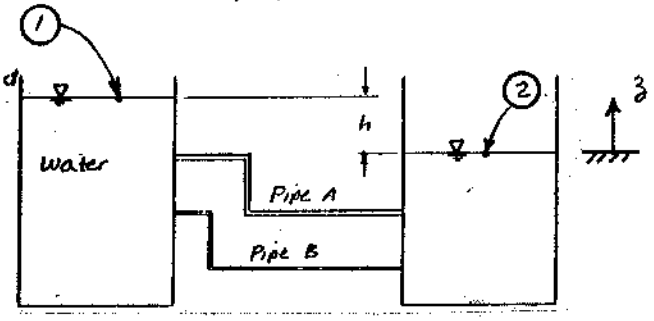
$p_3=-2.96\,psig$ ← p_{min}

Given: Water reservoirs connected by two pipes, as shown.

$D_A = 75$ mm, $D_B = 50$ mm } galvanized iron

$L = 100$ m

$h = 10.5$ m



Find: Comparison of head losses in pipes A and B.
Volume flow rates

Solution: Apply the energy equation for steady, incompressible flow

Computing equation: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{LT}$ ($\bar{V}_1 \approx 0$, $\bar{V}_2 \approx 0$, $z_2 = 0$)

Assumptions: (1) $p_1 = p_2 = p_{atm}$ (3) $T = 20°C$
(2) Large reservoirs so $\bar{V}_1 = \bar{V}_2 \approx 0$

Then

$$h_{LT} = g z_1 = g h = 9.81 \frac{m}{s^2} \times 10.5\ m = 103\ m^2/s^2$$ h_{LT}

for both pipes.

For each pipe, $h_{LT} = \left(f \frac{L}{D} + f\,2\left(\frac{L_e}{D}\right)_{elbow} + K_{ent} + K_{exit}\right) \frac{\bar{V}^2}{2}$

Assume: (3) Minimum bend loss, $L_e/D \approx 12$ (Fig. 8.17)
(4) Square-edged entrance, $K_{ent} = 0.5$ (Table 8.2)
(5) $\alpha \approx 1$ so $K_{exit} \approx 1$

Then

$$g h = \left[f\left(\frac{L}{D} + 24\right) + 1.5\right] \frac{\bar{V}^2}{2} \quad \text{so} \quad \bar{V} = \left[\frac{2gh}{f\left(\frac{L}{D} + 24\right) + 1.5}\right]^{1/2}$$

Flow rate is unknown, so guess f from fully rough zone.

Pipe A: $D = 25$ mm; $\frac{L}{D} = \frac{100\ m}{0.075\ m} = 1330$; $\frac{e}{D} = 0.002$ ($e = 0.15$ mm, Table 8.1)

Guess $f = 0.0235$ (Fig. 8.13)

$$\bar{V} = \left[2 \times 9.81 \frac{m}{s^2} \times 10.5\ m \times \frac{1}{0.0235(1330 + 24) + 1.5}\right]^{1/2} = 2.49\ m/s$$

$$Re = \frac{\bar{V} D}{\nu} = 2.49 \frac{m}{s} \times 0.025\ m \times \frac{s}{1 \times 10^{-6} m^2} = 1.86 \times 10^5, \text{ so } f \approx 0.0245 \text{ (Fig. 8.13)}$$

Thus a better estimate is

$$\bar{V} = \left[2 \times 9.81 \frac{m}{s^2} \times 10.5\ m \times \frac{1}{0.0245(1330 + 24) + 1.5}\right]^{1/2} = 2.44\ m/s$$

$$Q = \bar{V} A = 2.44 \frac{m}{s} \times \frac{\pi}{4} (0.025)^2 m^2 = 0.0108\ m^3/s$$ Q_A

Pipe B: $D = 50\,mm$; $\frac{L}{D} = \frac{100\,m}{0.05\,m} = 2000$; $\frac{e}{D} = 0.003$ (Fig. 8.15)

Guess $f \simeq 0.026$ (Fig. 8.13)

$$\bar{V} = \left[2 \times 9.81 \frac{m}{s^2} \times 10.5\,m \times \frac{1}{0.026(2000+24)+1.5}\right]^{1/2} = 1.95\ m/s$$

$$Re = \frac{\bar{V}D}{\nu} = 1.95 \frac{m}{s} \times 0.050\,m \times \frac{s}{1\times10^{-6}\,m^2} = 9.75 \times 10^4; \text{ so } f \simeq 0.0225 \text{ (Fig. 8.13)}$$

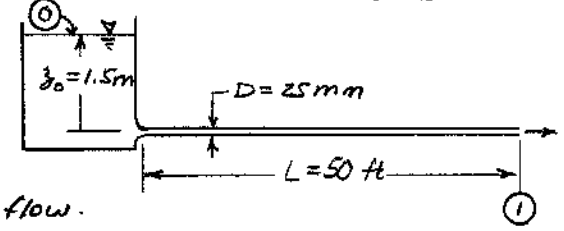
Thus a better estimate for $\bar{V}$ is

$$\bar{V} = \left[2 \times 9.81 \frac{m}{s^2} \times 10.5\,m \times \frac{1}{0.0225(2000+24)+1.5}\right]^{1/2} = 1.90\ m/s$$

$$Q = \bar{V}A = 1.90 \frac{m}{s} \times \frac{\pi}{4}(0.05)^2 m^2 = 0.00373\ m^3/s \quad Q_B$$

Thus $\frac{Q_B}{Q_A} = \frac{0.00373}{0.0108} = 0.345$; the flowrate for Pipe B is lower because (a) $D_B^2 < D_A^2$, and (b) $\left(\frac{L}{D}\right)_B > \left(\frac{L}{D}\right)_A$.

Given: Roman water supply system from Example Problem 8.10, but with 50 foot length of straight pipe with $D = 25$ mm, $\epsilon/D = 0.01$.



Find: (a) Flow rate delivered.
(b) Effect of adding a diffuser.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_0}{\rho} + \alpha_0\frac{\bar{V}_0^2}{2} + gz_0 = \frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1 + h_{LT}$; $h_{LT} = \left(f\frac{L}{D} + K_{ent}\right)\frac{\bar{V}_1^2}{2}$

Assumptions: (1) $p_0 = p_1 = p_{atm}$ (3) $\alpha_1 \approx 1$
(2) $\bar{V}_1 \approx 0$ (4) $K_{ent} = 0.04$

Then

$$gz_0 = \frac{\bar{V}_1^2}{2} + \left(f\frac{L}{D} + K_{ent}\right)\frac{\bar{V}_1^2}{2} \quad \text{or} \quad \bar{V}_1 = \sqrt{\frac{2gz_0}{1 + f\frac{L}{D} + K}}$$

For $\epsilon/D = 0.01$, $f \approx 0.038$ from Fig. 8.13, so

$$\bar{V}_1 = \left[2 \times 9.81\frac{m}{s^2} \times 1.5\,m \times \frac{1}{1 + 0.038 \times 50\,ft \times \frac{1}{25\,mm} \times 304.8\frac{mm}{ft} + 0.04}\right]^{1/2} = 1.10\ m/s$$

Checking, assuming $T = 20°C$,

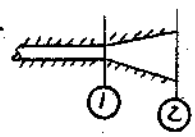
$$Re = \frac{\bar{V}D}{\nu} = 1.10\frac{m}{sec} \times 0.025\,m \times \frac{sec}{1.0 \times 10^{-6}\,m^2} = 2.75 \times 10^4 \text{ ; from Fig. 8.13, } f \approx 0.040\text{, so}$$

$$\bar{V}_1 = \left[2 \times 9.81\frac{m}{s^2} \times 1.5\,m \times \frac{1}{1 + 0.040 \times 50\,ft \times \frac{1}{25\,mm} \times 304.8\frac{mm}{ft} + 0.04}\right]^{1/2} = 1.08\ m/s$$

$$Q = \bar{V}_1 A = 1.08\frac{m}{s} \times \frac{\pi}{4}(0.025)^2 m^2 = 5.30 \times 10^{-4}\ m^3/s \quad \text{(no diffuser)} \qquad Q$$

The diffuser would increase head loss by $K_{diffuser} = 0.3$ (see Example 8.10), but would reduce $\bar{V}_2$ to $\frac{1}{2}\bar{V}_1$. The energy equation would be

$$gz_0 = \frac{\bar{V}_2^2}{2} + \left(f\frac{L}{D} + K_{ent} + K_{diff}\right)\frac{\bar{V}_1^2}{2} = \left(\frac{1}{4} + f\frac{L}{D} + K_{ent} + K_d\right)\frac{\bar{V}_1^2}{2}$$



$\frac{N}{R_1} = 3.0$
$AR = 2.0$

Thus

$$\bar{V}_1 = \sqrt{\frac{2gz_0}{0.25 + f\frac{L}{D} + K_{ent} + K_{diff}}}$$

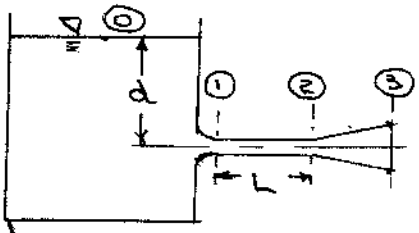
$$\bar{V}_1 = \left[2 \times 9.81\frac{m}{s^2} \times 1.5\,m \times \frac{1}{0.25 + 0.040 \times 50\,ft \times \frac{1}{25\,mm} \times 304.8\frac{mm}{ft} + 0.04 + 0.3}\right]^{1/2} = 1.09\ m/s$$

and

$$Q = \bar{V}_1 A = 1.09\frac{m}{s} \times \frac{\pi}{4}(0.025)^2 m^2 = 5.35 \times 10^{-4}\ m^3/s \quad \text{(with diffuser)} \qquad Q$$

{The diffuser increases flow rate only slightly (~1 percent), because loss is dominated by fL/D.}

Given: Pipe of length L inserted between the nozzle (attached to the water main) and diffuser of Example Problem 8.10.



$D_1 = 25\,mm$, $K_{ent} = 0.04$, $d = 1.5\,m$

Diffuser: $N/R_2 = 3.0$, $AR = 2.0$

$K_{diff} = 0.3$

Flow with nozzle alone:

$Q_i = 2.61 \times 10^{-3}\ m^3/s$, $\bar{V}_1 = 5.32\ m/s$

Flow with nozzle and diffuser ($L = 0$) $Q_d = 3.47 \times 10^{-3}\ m^3/s$

Find: Length (L) of pipe with $e/D = 0.01$ required to give flow rate Q_i with diffuser in place; compare with commissioner's requirement of $L = 50\,ft$ (15.2 m).

Plot: Q/Q_i vs L/D

Solution:

Apply the energy equation for steady, incompressible flow between the water surface and the diffuser discharge.

Basic equations: $\left(\frac{p_0}{\rho} + \alpha_0 \frac{\bar{V}_0^2}{2} + g z_0\right) - \left(\frac{p_3}{\rho} + \alpha_3 \frac{\bar{V}_3^2}{2} + g z_3\right) = h_{LT}$ (8.29)

$h_{LT} = h_L + h_{LM}$; $h_L = f \frac{L}{D} \frac{\bar{V}_2^2}{2}$; $h_{LM} = (K_{ent} + K_{diff}) \frac{\bar{V}_2^2}{2}$

Assumptions: (1) $p_0 = p_3 = p_{atm}$

(2) $\bar{V}_0 = 0$, $\alpha_3 = 1.0$

(3) water @ 60°C, $\nu = 1.00 \times 10^{-6}\ m^2/s$

Then,

$$g(z_0 - z_3) = gd = f \frac{L}{D} \frac{\bar{V}_2^2}{2} + (K_{ent} + K_{diff}) \frac{\bar{V}_2^2}{2} + \frac{\bar{V}_3^2}{2}$$

From continuity $A_2 \bar{V}_2 = A_3 \bar{V}_3$ $\therefore \bar{V}_3 = \frac{\bar{V}_2}{AR}$

$$gd = f \frac{L}{D} \frac{\bar{V}_2^2}{2} + \left(K_{ent} + K_{diff} + \frac{1}{AR^2}\right) \frac{\bar{V}_2^2}{2} = f \frac{L}{D} \frac{\bar{V}_2^2}{2} + 0.59 \frac{\bar{V}_2^2}{2} \quad (1)$$

and

$$L = \frac{D}{f}\left[\frac{2gd}{\bar{V}_2^2} - 0.590\right] \quad (2)$$

For $\bar{V}_1 = 5.32\ m/s$, $Re = \frac{D\bar{V}}{\nu} = 0.025\,m \times 5.32\,\frac{m}{s} \times 1.00 \times 10^{-6}\,\frac{s}{m^2} = 1.33 \times 10^5$

with $e/D = 0.01$, $f = 0.038$ (Fig. 8.13) and

$$L = \frac{0.025\,m}{0.038}\left[2 \times 9.81\,\frac{m}{s^2} \times 1.5\,m \times (5.32)^2\,\frac{s^2}{m^2} - 0.590\right] = 0.296\,m$$

(0.971 ft, $L/D = 11.8$)

This is significantly less than the 50 ft required by the water commissioner. He was extremely conservative.

Note that $Q/Q_i = \bar{V}/\bar{V}_i$ where $\bar{V}_i = 5.32\ m/s$

Increasing L reduces $\bar{V}$

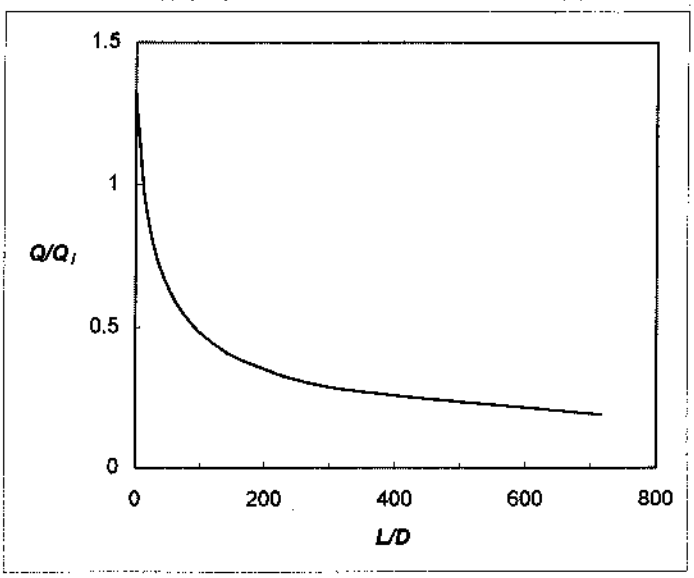
With $L = 0$ $\quad Q/Q_i = 1.33$

$L = 0.226$ m $\quad (L/D = 11.8)$ $\quad Q/Q_i = 1.00$

As L is increased $\bar{V}_2$ (and hence Re) will decrease; the friction factor will increase slightly from 0.038.

The plot of Q/Q_i ($\bar{V}/\bar{V}_i$) is best done by assuming values of $\bar{V}$, and solving Eq. 2 for L.

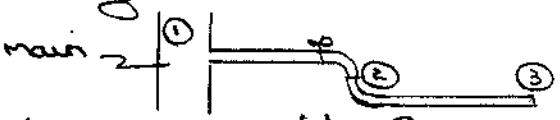
V_{avg} (m/s)	Re (—)	f (—)	L/D (—)	V/V_i (—)
7.06	1.77E+05	0.0382	0.0	1.33
5.32	1.33E+05	0.0384	11.7	1.00
5.0	1.25E+05	0.0384	15.3	0.940
4.5	1.13E+05	0.0384	22.5	0.846
4.0	1.00E+05	0.0385	32.4	0.752
3.5	8.75E+04	0.0386	47.0	0.658
3.0	7.50E+04	0.0387	69.3	0.564
2.5	6.25E+04	0.0388	106	0.470
2.0	5.00E+04	0.0391	173	0.376
1.5	3.75E+04	0.0394	317	0.282
1.0	2.50E+04	0.0402	718	0.188



Given: Water flow from spigot (at 60°F) through an old hose with $D = 0.75$ in and $e = 0.022$ in. Pressure at main remains constant at 50 psig; pressure at spigot varies as square of flow rate. One 50 ft length of hose delivers 15 gpm

Find: (a) pressure at spigot (psig) for this case.
(b) delivery with two 50-ft lengths of hose connected.

Solution:



Apply the energy equation for steady, incompressible flow between the spigot ② and the hose discharge ③

Basic equations: $\left(\frac{p_2}{\rho} + \alpha_2 \frac{V_2^2}{2} + g z_2\right) - \left(\frac{p_3}{\rho} + \alpha_3 \frac{V_3^2}{2} + g z_3\right) = h_{l_T}$ (8.29)

$h_{l_T} = h_l + h_m$, $h_l = f \frac{L}{D} \frac{V_2^2}{2}$

Assumptions: (1) $p_3 = p_{atm}$
(2) $V_2 = V_3$, $\alpha_2 = \alpha_3 = 1.0$
(3) $z_2 = z_3$

Then $p_2 = p f \frac{L}{D} \frac{V_2^2}{2}$ ---------- (1)

$V = \frac{Q}{A} = \frac{15 \text{ gal/min}}{\pi/4 \left(\frac{0.75}{12}\right)^2 \text{ ft}^2} \times \frac{1 \text{ min}}{60 \text{ s}} \times \frac{1 \text{ ft}^3}{7.48 \text{ gal}} = 10.9 \text{ ft/s}$

$Re = \frac{VD}{\nu} = \frac{0.75}{12} \text{ ft} \times 10.9 \frac{\text{ft}}{\text{s}} \times 1.21 \times 10^{-5} \frac{\text{ft}^2}{\text{s}} = 5.63 \times 10^4$ {ν from Table A.7}

$e/D = 0.022 / 0.75 = 0.0293$

From Fig. 8.13 $f = 0.056$. From Eq. 1,

$p_2 = 1.94 \frac{\text{slug}}{\text{ft}^3} \times 0.056 \times \frac{50 \text{ ft}}{0.75 \text{ in}} \times \frac{12 \text{ in}}{\text{ft}} \times \frac{1}{2} \times (10.9)^2 \frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} \times \frac{\text{ft}^2}{144 \text{ in}^2}$

$p_2 = 35.9$ psi gage. p_2

The pressure drop from the main ① to the spigot ② is proportional to the square of the flow rate. Obtain the loss coefficient using the energy equation between ① and ②.

$\left(\frac{p_1}{\rho} + \alpha_1 \frac{V_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{V_2^2}{2} + g z_2\right) = K \frac{V_2^2}{2}$

Assumptions: (4) $V_1 = 0$ (5) $z_1 = z_2$

$p_1 - p_2 = \rho \left[K \frac{V_2^2}{2} + \frac{V_2^2}{2}\right] = \rho \frac{V_2^2}{2} [K + 1]$

$K = \frac{\Delta p}{\frac{1}{2} \rho V_2^2} - 1 = \frac{2(50 - 35.9) \frac{\text{lbf}}{\text{in}^2} \times 144 \frac{\text{in}^2}{\text{ft}^2} \times \frac{\text{ft}^3}{1.94 \text{ slug}}}{(10.9)^2 \frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2}} - 1$

$K = 16.6$

To find the delivery with two hoses, again apply the energy equation from the drain ① to the end ④ of the second hose

$$\left(\frac{p_1}{\rho} + \alpha_1 \frac{V_1^2}{2} + g z_1\right) - \left(\frac{p_4}{\rho} + \alpha_4 \frac{V_4^2}{2} + g z_4\right) = f \frac{L_{24}}{D_{24}} \frac{V_{24}^2}{2} + K \frac{V_{24}^2}{2}$$

$p_4 = p_{atm}$, $z_1 = z_4$, $V_1 = 0$, $\alpha_4 = 1$.

$$\frac{p_1 - p_{atm}}{\rho} = \frac{p_{1g}}{\rho} = \frac{V_{24}^2}{2}\left(f \frac{L_{24}}{D_{24}} + K + 1\right) \quad \text{and}$$

$$V_4 = \left[\frac{2 p_{1g}}{\rho\left(2f \frac{L_{24}}{D_{24}} + K + 1\right)}\right]^{1/2}$$

Delivery will be reduced somewhat with two lengths of hose, but f will not change much. Assume $f = 0.05$ and check

$$V_4 = \left[2 \times 50 \frac{lb}{in^2} \times 144 \frac{in^2}{ft^2} \times 1.94 \frac{slug}{ft^3} \times \frac{1}{\left(0.05 \times \frac{160 ft}{0.25 ft} + 16.6 + 1\right)} \times \frac{ft}{slug \cdot ft/lb}\right]^{1/2}$$

$$V_4 = 8.32 \text{ ft/s}.$$

Checking,

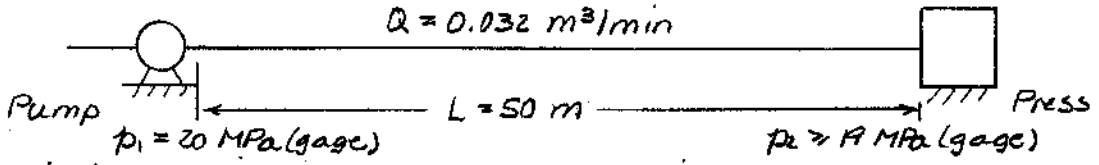
$$Re = \frac{V_{24} D_{24}}{\nu} = 0.\frac{25}{12} ft \times 8.32 \frac{ft}{s} \times 1.21 \times 10^{-5} \frac{s}{ft^2} = 4.30 \times 10^4, \text{ so } f \approx 0.05$$

Two with two hoses,

$$Q = V A = 8.32 \frac{ft}{s} \times \frac{\pi}{4} \times \left(\frac{0.25}{12}\right)^2 ft^2 \times 7.48 \frac{gal}{ft^3} \times \frac{60 s}{min} = 11.5 \text{ gpm} \quad \text{Ans}$$

{Similar calculations could be performed using any desired number of hose lengths.}

Given: Hydraulic press powered by remote high-pressure pump.



Find: Minimum diameter drawn steel tubing for SAE 10W oil at 40°C.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho}+\alpha_1\frac{\bar{V}_1^2}{2}+gz_1=\frac{p_2}{\rho}+\alpha_2\frac{\bar{V}_2^2}{2}+gz_2+h_{LT}$; $h_{LT}=\left[f\left(\frac{L}{D}+\frac{L}{D}\right)+K\right]\frac{\bar{V}^2}{2}$ ($p_2=0$ (3), $z_2=0$ (3))

Assumptions: (1) Fully developed flow, $\alpha_1\bar{V}_1^2=\alpha_2\bar{V}_2^2$, (2) $z_1=z_2$, (3) No minor losses.

Then $\Delta p = f\frac{L}{D}\rho\frac{\bar{V}^2}{2}$

D is not known, so we cannot compute $\bar{V}$ and Re to find f. Q is small, so try laminar flow. For fully developed laminar flow, from Eq. 8.13c,

$$\Delta p = \frac{128\mu QL}{\pi D^4} \quad \text{so} \quad D=\left[\frac{128\mu QL}{\pi\Delta p}\right]^{\frac{1}{4}}$$

For SAE 10W oil at 40°C, $\mu = 3.3\times10^{-2}\ N\cdot sec/m^2$ (Fig. A.2)

$$D=\left[128\times3.3\times10^{-2}\frac{N\cdot s}{m^2}\times0.032\frac{m^3}{min}\times50\ m\times\frac{m^2}{\pi(1\times10^6)N}\times\frac{min}{60\ s}\right]^{\frac{1}{4}}=0.0138\ m$$

Check Re to assure flow is laminar:

$$\bar{V}=\frac{Q}{A}=\frac{4Q}{\pi D^2}=\frac{4}{\pi}\times0.032\frac{m^3}{min}\times\frac{1}{(0.0138)^2m^2}\times\frac{min}{60\ s}=3.57\ m/s$$

$$Re=\frac{\bar{V}D}{\nu}=\frac{\rho\bar{V}D}{\mu}$$

For SAE 10W oil, SG = 0.92 (Table A.2), so

$$Re=(0.92)1000\frac{kg}{m^3}\times3.57\frac{m}{s}\times0.0138\ m\times\frac{m^2}{3.3\times10^{-2}N\cdot s}\times\frac{N\cdot s^2}{kg\cdot m}=1370$$

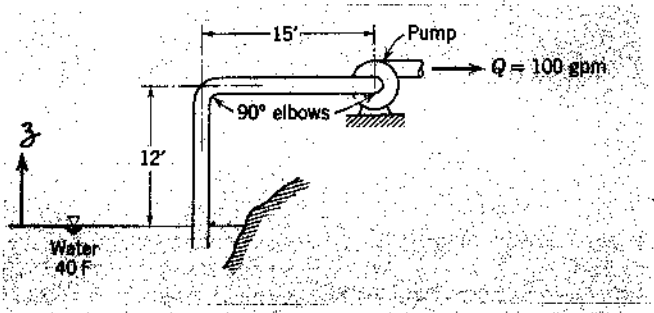
Therefore flow is laminar since $Re < 2300$.

The minimum allowable tubing diameter is $D = 13.8$ mm. D

The next largest standard size should be chosen.

Given: Pump drawing water from reservoir as shown. For satisfactory operation, the suction head (p_2/γ) must not be less than -20 feet of water.

Section (1) is at reservoir surface.



Section (2) is at pump inlet.

Find: Smallest standard commercial steel pipe that will give the required performance.

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section.

Basic equation: $\frac{p_1}{\rho}+\frac{\bar{V}_1^2}{2}+gz_1=\frac{p_2}{\rho}+\frac{\bar{V}_2^2}{2}+gz_2+f\frac{L}{D}\frac{\bar{V}^2}{2}+h_{em}$ ($p_1=0$ (1), $\bar{V}_1\approx 0$ (2), $z_1=0$ (3))

Assumptions: (1) $p_1 = 0$ psig
(2) $\bar{V}_1 \approx 0$
(3) $z_1 = 0$
(4) $h_{em} = \left(K_{ent} + 2f\frac{L_{elbow}}{D}\right)\frac{\bar{V}^2}{2}$, $K_{ent} = 0.78$ for reentrant configuration (Table 8.2), $L_{elbow}/D \approx 12$ (Fig. 8.17)

Then

$$h_2 = \frac{p_2}{\gamma} = -z_2 - \left(1 + f\frac{L}{D} + K_{ent} + 2f\frac{L_e}{D}\right)\frac{\bar{V}_2^2}{2g} = -z_2 - \left[1.78 + f\left(\frac{L}{D}+24\right)\right]\frac{\bar{V}_2^2}{2g}$$

Since D is unknown, iteration is required. Set up calculating equations:

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi}\times\frac{1}{D^2\,in.^2}\times\frac{144\,in.^2}{ft^2}\times 100\frac{gal}{min}\times\frac{ft^3}{7.48\,gal}\times\frac{min}{60\,s} = \frac{40.9}{D^2}\ ft/s$$

$$Re = \frac{\bar{V}D}{\nu} = \frac{4Q}{\pi\nu D} = \frac{4}{\pi}\times 100\frac{gal}{min}\times\frac{min}{60\,s}\times\frac{ft^3}{7.48\,gal}\times\frac{s^2}{1.44\times10^{-5}\,ft^2}\times\frac{1}{D\,in.}\times\frac{12\,in.}{ft} = \frac{208{,}000}{D}$$

$\epsilon = 0.00015$ ft (Table 8.1), f from Fig. 8.13. $L = 27$ ft. D (in.) from Table 8.5.

D (nominal)	D (in.)	$\bar{V}$ (ft/s)	Re	ϵ/D	f	L/D	h_2 (ft)
3	3.068	4.34	67,700	0.0006	0.0226	106	-13.4
$2\frac{1}{2}$	2.469	6.71	84,100	0.0007	0.0220	131	-15.6
2	2.067	9.57	100,000	0.0009	0.0215	157	-20.1

Recognizing that pipe friction calculations are only good to ± 10 percent, recommend

$D = 2\frac{1}{2}$ in. (nominal) pipe ← Dmin

Given: Flow of standard air at 80 m^3/min through a smooth duct of aspect ratio 2.

Find: Minimum size duct for a head loss of 30 mm of water per 30 m of length.

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section. Use hydraulic diameter.

Basic equation: $\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + g z_2 + f \frac{L}{D_h} \frac{\bar{V}^2}{2} + h_{lm}$; $D_h = \frac{4A}{P_w}$

Assumptions: (1) $\bar{V}_1 = \bar{V}_2$
(2) $z_1 = z_2$
(3) $h_{lm} = 0$

Then

$$\Delta p = p_1 - p_2 = f \frac{L}{D_h} \frac{\rho \bar{V}^2}{2} = \frac{f}{2} \frac{L \rho}{D_h} \left(\frac{Q}{A}\right)^2 = \frac{f L Q^2 \rho}{2 D_h A^2}$$

For a rectangular duct, $A = bh = h^2 \left(\frac{b}{h}\right) = h^2 ar$, and

$$D_h = \frac{4bh}{2(b+h)} = \frac{2h^2 ar}{h(1+ar)} = \frac{2har}{1+ar}$$

Substituting,

$$\Delta p = \frac{f L Q^2 \rho}{2} \frac{1+ar}{2har} \frac{1}{h^4 ar^2} = \frac{f \rho L Q^2}{4} \frac{1+ar}{ar^3} \frac{1}{h^5}$$

or

$$h = \left[\frac{f \rho L Q^2}{4 \Delta p} \frac{1+ar}{ar^3}\right]^{\frac{1}{5}} ; \Delta p = \rho_{H_2O} g \Delta h = 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.03 m \times \frac{N \cdot s^2}{kg \cdot m}$$

Thus $\Delta p = 294 \; N/m^2$

$$h = (f)^{1/5} \left[\frac{1}{4} \times 1.23 \frac{kg}{m^3} \times 30 m \times (80)^2 \frac{m^6}{min^2} \times \frac{m^2}{294 N} \times \frac{1+2}{(2)^3} \times \frac{min^2}{3600 s^2} \times \frac{N \cdot s^2}{kg \cdot m}\right]^{1/5}$$

$$h = (f)^{1/5} 0.461 m$$

Guess $f = 0.01$, then $h = 0.184 m$. $\bar{V} = \frac{Q}{A} = \frac{Q}{h^2 ar} = 19.7 m/s$, and $D_h = \frac{2har}{1+ar} = \frac{4}{3} h = 0.245 m$ and $Re = 3.33 \times 10^5$. For a smooth duct at this Reynolds number, $f \simeq 0.013$ (Fig. 8.13).

With $f = 0.013$, then $h = 0.193 m$. $\bar{V} = \frac{Q}{h^2 ar} = 17.9 m/s$, and $D_h = \frac{4}{3} h = 0.257 m$ and $Re = 3.17 \times 10^5$. This value of Re gives $f \simeq 0.0133$.

With $f = 0.0133$, then $h = 0.194 m$, and $b = arh = (2)\, 0.194 m = 0.388 m$ ← b, h

Check: $\bar{V} = \frac{Q}{h^2 ar} = 17.7 m/s$

$$\Delta h = \frac{\Delta p}{\rho_{H_2O} g} = f \frac{L}{D_h} \frac{\rho_a}{\rho_{H_2O}} \frac{\bar{V}^2}{2g} = 0.0303 m \text{ or } 30.3 mm \; \checkmark$$

Given: New industrial plant requires water supply of 5.7 m^3/min. The gage pressure at the main, 50 m from the plant, is 800 kPa. The supply line will have 4 elbows in a total length of 65 m. Pressure in the plant must be at least 500 kPa (gage).

Find: Minimum line size of galvanized iron to install.

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section ($\alpha \approx 1$).

Basic equation: $\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2}^{(2)} + g\cancel{z_1}^{(3)} = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2}^{(2)} + g\cancel{z_2}^{(3)} + f\frac{L}{D}\frac{\bar{V}^2}{2} + h_{em}$

Assumptions: (1) $p_1 - p_2 \le 300\ kPa = \Delta p$

(2) Fully developed flow in constant-area pipe, $\bar{V}_1 = \bar{V}_2 = \bar{V}$

(3) $z_1 = z_2$

(4) $h_{em} = 4\left(\frac{L_e}{D}\right)_{elbow}\frac{\bar{V}^2}{2} = 120\frac{\bar{V}^2}{2}$ $\left(\frac{L_e}{D} = 30, \text{from Table 8.5}\right)$

Then

$$\frac{\Delta p}{\rho} = f\left(\frac{L}{D} + 120\right)\frac{\bar{V}^2}{2} \quad \text{or} \quad \Delta p = \rho f\left(\frac{L}{D} + 120\right)\frac{\bar{V}^2}{2}$$

Since D is unknown, iteration is required. The calculating equations are:

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 5.7\frac{m^3}{min} \times \frac{1}{D^2 m^2} \times \frac{min}{60\,s} = \frac{0.121}{D^2}\ (m/s)$$

$$Re = \frac{\bar{V}D}{\nu} = \frac{4Q}{\pi\nu D} = \frac{4}{\pi} \times 5.7\frac{m^3}{min} \times \frac{s}{1.14\times10^{-6} m^2} \times \frac{1}{D\,m} \times \frac{min}{60\,s} = \frac{1.06\times10^5}{D} \quad (T = 15°C)$$

$e = 0.15$ mm (Table 8.1), f from Fig. 8.13. $L = 65$ m. D from Table 8.5.

D (nom.)	D (m)	$\bar{V}$ (m/s)	Re (-)	e/D (-)	f (-)	L/D (-)	Δp (kPa)
3	0.0779	19.9	1.36×10^6	0.0019	0.024	834	4530
5	0.128	7.39	8.29×10^5	0.0012	0.021	508	360
6	0.154	5.10	6.89×10^5	0.001	0.020	422	141

Pipe friction calculations are accurate only within about ± 10 percent. Line resistance (and consequently Δp) will increase with age.

Recommend installation of 6 in. (nominal) line. D

Problem 8.128

Open-Ended Problem Statement: The chilled-water pipeline system that provides air conditioning for the Purdue University campus is described in Problem 8.140. The pipe diameter is selected to minimize total cost (capital cost plus operating cost). Annualized costs are compared, since capital cost occurs once and operating cost continues for the life of the system. The optimum diameter depends on both cost factors and operating conditions; the analysis must be repeated when these variables change. Perform a pipeline optimization analysis. Solve Problem 8.140 arranging your calculations to study the effect of pipe diameter on annual pumping cost. (Assume friction factor remains constant.) Obtain an expression for total annual cost per unit delivery (e.g., dollars per cubic meter), assuming construction cost varies as the square of pipe diameter. Obtain an analytic relation for the pipe diameter that yields minimum total cost per unit delivery. Assume the present chilled-water pipeline was optimized for a 20-year life with 5 percent annual interest. Repeat the optimization for a design to operate at 30 percent larger flow rate. Plot the annual cost for electrical energy for pumping and the capital cost, using the flow conditions of Problem 8.140, with pipe diameter varied from 300 to 900 mm. Show how the diameter may be chosen to minimize total cost. How sensitive are the results to interest rate?

(From Problem 8.140: The pipe makes a loop 3 miles in length. The pipe diameter is 2 ft and the material is steel. The maximum design volume flow rate is 11,200 gpm. The circulating pump is driven by an electric motor. The efficiencies of pump and motor are $\eta_p = 0.80$ and $\eta_m = 0.90$, respectively. Electricity cost is \$0.067/(kW·hr).)

Analysis: From Problem 8.140, the electrical energy for pumping costs \$174,000 per year for 11,200 gallons per minute circulation. The present line, with D = 24 in., is optimized for this flow rate. $\dot{W} = Q\Delta p$, so $\dot{W}/Q = \Delta p$.

The optimum pipe diameter minimizes total annualized cost, for construction and operation of the pipeline, $C_t = C_c + C_p$. Construction cost C_c is a one-time cost. Annualized pumping cost C_p is computed by summing the present worth of each annual pumping cost over the lifetime of the pipeline. For 20 years at 5 percent per year, spwf = 13.1 (see spreadsheet). Costs may be expressed in terms of diameter as

$$C_t = C_c + C_p = K_c D^2 + \frac{K_p}{D^5} \qquad (1)$$

For the optimum diameter, $dC_t/dD = 2K_c D - 5K_p D^{-6} = 0$, so

$$K_c = \frac{5K_p}{2D^7} = \frac{5C_p}{2D^2} = \frac{5}{2}\times(13.1)\$174{,}000\times\frac{1}{(24)^2\,in.^2} = \$9890/in.^2 \quad \leftarrow K_c$$

From Eq. 1,

$$K_p = C_p D^5 = (13.1)\$174{,}000\times(24)^5\,in.^5 = 1.81\times10^{13}\,\$\cdot in.^5 \quad \leftarrow K_p$$

Calculations with these values are shown on the spreadsheet.

To optimize at a new, larger flow rate, note $C_p \sim \Delta p \sim f\frac{L}{D}\rho\frac{\bar{V}^2}{2} = f\frac{L}{D}\frac{\rho}{2}\left(\frac{Q}{A}\right)^2 \sim f\frac{Q^2}{D^5}$

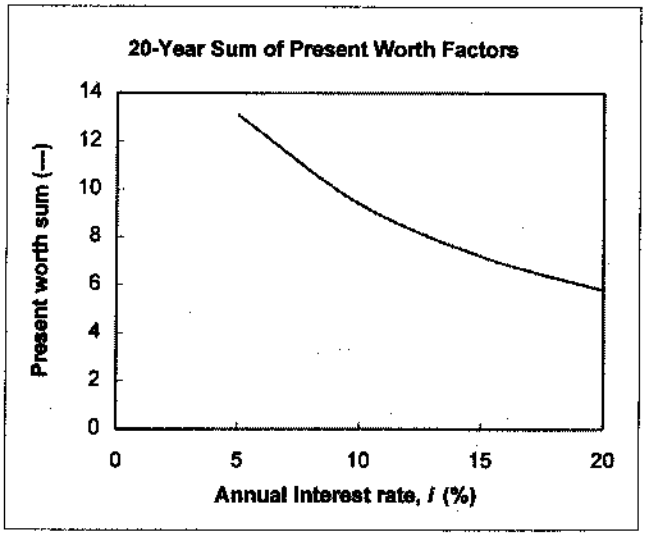
Thus

$$K_p(new) = K_p(old)\left(\frac{Q_{new}}{Q_{old}}\right)^2 = (1.3)^2 K_p(old) = 3.06\times10^{13}\,\$\cdot in.^5$$

The new optimum is at D = 25.9 in., as shown on the second plot.

Results are not too sensitive to interest rate; only K_p varies. $D_{opt} \to 25$ in. for $i = 15\%$.

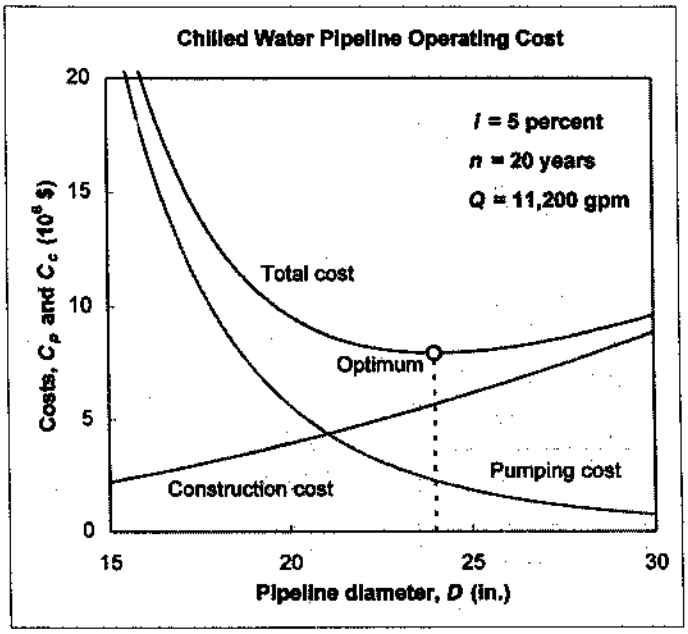
	Annual interest rate (%)			
i =	5	10	15	20
Year	*pwf*	*pwf*	*pwf*	*pwf*
1	1.00	1.00	1.00	1.00
2	0.952	0.909	0.870	0.833
3	0.907	0.826	0.756	0.694
4	0.864	0.751	0.658	0.579
5	0.823	0.683	0.572	0.482
6	0.784	0.621	0.497	0.402
7	0.746	0.564	0.432	0.335
8	0.711	0.513	0.376	0.279
9	0.677	0.467	0.327	0.233
10	0.645	0.424	0.284	0.194
11	0.614	0.386	0.247	0.162
12	0.585	0.350	0.215	0.135
13	0.557	0.319	0.187	0.112
14	0.530	0.290	0.163	0.0935
15	0.505	0.263	0.141	0.0779
16	0.481	0.239	0.123	0.0649
17	0.458	0.218	0.107	0.0541
18	0.436	0.198	0.0929	0.0451
19	0.416	0.180	0.0808	0.0376
20	0.396	0.164	0.0703	0.0313
Sum:	**13.1**	**9.4**	**7.2**	**5.8**



K_c = 9,890 $/in.^{2} Cost of construction per diameter squared

K_p = 1.81E+13 $*in.^{5} Present worth 20-yr cost of pumping 11,200 gpm

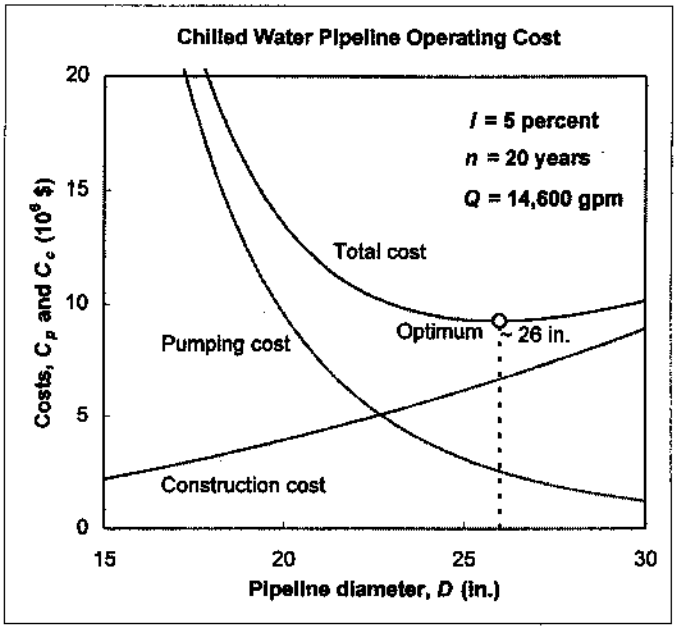
Pipe Diameter, D (in.)	Cost of Pumping, C_p ($10^6 $)	Cost to Construct, C_c ($10^6 $)	Total Cost, C_t ($10^6 $)
15	23.9	2.23	26.1
16	17.3	2.53	19.8
17	12.8	2.86	15.6
18	9.59	3.20	12.8
19	7.32	3.57	10.9
20	5.67	3.96	9.62
21	4.44	4.36	8.80
22	3.52	4.79	8.30
23	2.82	5.23	8.05
24	2.28	5.70	7.97
25	1.86	6.18	8.04
26	1.53	6.69	8.21
27	1.26	7.21	8.47
28	1.05	7.75	8.81
29	0.884	8.32	9.20
30	0.746	8.90	9.65



K_c = 9,890 $/in.^{2} Cost of construction per diameter squared

K_p = 3.06E+13 $*in.^{5} Present worth 20-yr cost of pumping 14,600 gpm

Pipe Diameter, D (in.)	Cost of Pumping, C_p ($10^6 $)	Cost to Construct, C_c ($10^6 $)	Total Cost, C_t ($10^6 $)
15	40.3	2.23	42.6
16	29.2	2.53	31.8
17	21.6	2.86	24.4
18	16.2	3.20	19.4
19	12.4	3.57	15.9
20	9.57	3.96	13.53
21	7.50	4.36	11.86
22	5.95	4.79	10.73
23	4.76	5.23	9.99
24	3.85	5.70	9.54
25	3.14	6.18	9.32
26	2.58	6.69	9.26
27	2.14	7.21	9.35
28	1.78	7.75	9.53
29	1.49	8.32	9.81
30	1.26	8.90	10.2



13-782 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-3882 200 SHEETS EYE-EASE® 5 SQUARE
42-382 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: An air-pipe friction experiment utilizes smooth brass tube, $D = 63.5$ mm, $L = 1.52$ m.
At one flow condition $\Delta p = 12.3$ mm mercury red oil, $U_c = 23.1$ m/s.

Find: (a) Re_D
(b) Friction factor f; compare with value for Fig. 8.13.

Solution:

Apply the energy equation for steady, incompressible flow along the pipe

Basic equation: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{L_T}$ (8.29)

$h_L = f \frac{L}{D} \frac{\bar{V}_1^2}{2}$

Computing equation: $\frac{\bar{V}_1}{U_c} = \frac{2n^2}{(n+1)(2n+1)}$ (8.24)

Assumptions: (1) power law profile, $n = 7$
(2) $\alpha_1 = \alpha_2$, $z_1 = z_2$
(3) air at $T = 15°C$ $\nu = 1.46 \times 10^{-5}$ m²/s (Table A.6)

From Eq. 8.24 with $n = 7$

$\frac{\bar{V}_1}{U_c} = \frac{2(7)^2}{(8)(15)} = 0.817$

$Re_D = \frac{\bar{V}_1 D}{\nu} = 0.0635\,m \times 0.817 \times 23.1 \frac{m}{s} \times 1.45 \times 10^{-5} \frac{s}{m^2} = 8.26 \times 10^4$ Re_D

From Eq. 8.29 $\Delta p / \rho = f \frac{L}{D} \frac{\bar{V}_1^2}{2}$

and

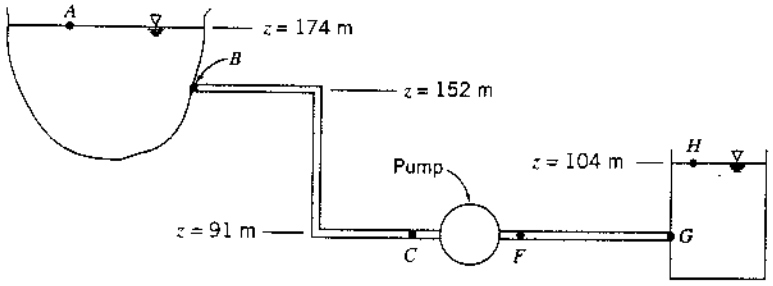
$f = \frac{\Delta p}{\rho \frac{L}{D} \frac{\bar{V}_1^2}{2}} = \frac{\rho_{air} g \Delta h}{\rho_{air} \frac{L}{D} \frac{\bar{V}_1^2}{2}} = \frac{2 \rho_{H_2O} SG_{oil} g \Delta h}{\rho_{air} \frac{L}{D} \bar{V}_1^2}$ {SG = 0.827 Table A.1}

$f = 2 \times \frac{10^3}{1.23} \times 0.827 \times 9.81 \frac{m}{s^2} \times 0.0123\,m \times \frac{0.0635\,m}{1.52\,m} \times (0.817 \times 23.1)^2 \frac{s^2}{m^2}$

$f = 0.0190$ f

From Fig. 8.13 at $Re = 8.26 \times 10^4$ for smooth tube, $f = 0.0187$

Given: Portion of water supply system designed to provide $Q = 130$ L/s at $T = 20°C$.



System B→C
- square edged entrance
- 3 gate valves
- 4 45° elbows
- 2 90° elbows
- 760 m pipe
- $p_C = 197$ kPa gage

System F→G
- 760 m pipe
- 2 gate valves
- 4 90° elbows

All pipe is cast iron, $D = 508$ mm

Find: (a) average velocity in pipeline
(b) gage pressure p_F
(c) shear stress on pipe centerline at C
(d) power input to pump if efficiency $\eta = 80\%$
(e) wall shear stress at G.

Solution:

Since $Q = AV$, $V = \frac{Q}{A} = \frac{Q}{\pi D^2/4} = \frac{4}{\pi} \times \frac{130 \text{ L}}{\text{s}} \times \frac{1}{(0.508 \text{ m})^2} \times 10^{-3} \frac{\text{m}^3}{\text{L}} = 6.46 \text{ m/s}$ $\bar{V}$

To determine the pressure at point F, apply the energy equation for steady, incompressible flow between F and G.

Basic equation: $\left(\frac{p_F}{\rho} + \alpha \frac{\bar{V}_F^2}{2} + g z_F\right) - \left(\frac{p_G}{\rho} + \alpha \frac{\bar{V}_G^2}{2} + g z_G\right) = h_{eT}$ (8.29)

$h_{eT} = h_e + h_{en}$, $h_e = f \frac{L}{D} \frac{\bar{V}^2}{2}$, $h_{en} = \frac{\bar{V}^2}{2} \sum f \frac{L}{D} + \frac{\bar{V}^2}{2} K_{exit}$

Assume: (1) $\bar{V}_H = 0$ (large storage tank) (2) $p_H = p_{atm}$
(3) $\alpha_F \approx 1.0$.

Then $\frac{p_F}{\rho} = h_{eT} + g(z_H - z_F) - \frac{\bar{V}_F^2}{2}$

$$\frac{p_F}{\rho} = h_{e_{F-G}} + 2h_{e_{gv}} + 4h_{e_{90°el}} + h_{e_{exit}} + g(z_H - z_F) - \frac{\bar{V}_F^2}{2}$$

$$\frac{p_F}{\rho} = f\frac{L}{D}\frac{\bar{V}^2}{2} + 2f\left(\frac{L}{D}\right)_{gv}\frac{\bar{V}^2}{2} + 4f\left(\frac{L}{D}\right)_{90°el}\frac{\bar{V}^2}{2} + K_{exit}\frac{\bar{V}^2}{2} + g(z_H - z_F) - \frac{\bar{V}^2}{2} \quad \text{----(1)}$$

From Table 8.4 $\left(\frac{L}{D}\right)_{gv} = 8$, $\left(\frac{L}{D}\right)_{90°el} = 30$; also $K_{exit} = 1$

$Re = \frac{\rho \bar{V} D}{\mu} = 0.508 \text{ m} \times 6.46 \frac{\text{m}}{\text{s}} \times \frac{\text{s}}{1.00 \times 10^{-6} \text{ m}^2} = 3.28 \times 10^6$ (ν from Table A.8)

From Table 8.1, $e = 0.26$ mm $\therefore e/D = 0.00051$

From Fig. 8.13, $f = 0.017$

From Eq (1)

$$\frac{p_F}{\rho} = f\frac{\bar{V}^2}{2}\left[\frac{L}{D} + 2\left(\frac{L}{D}\right)_{gv} + 4\left(\frac{L}{D}\right)_{90°el}\right] + g(z_H - z_F)$$

$$\frac{p_F}{\rho} = f\frac{L}{D}\frac{V^2}{2}\left[\frac{\frac{L}{D}}{0.508} + 2(8) + 4(30)\right] + g(z_A - z_F) = f\frac{V^2}{2}(1630) + g(z_A - z_F)$$

$$p_F = \rho\left[1630 f\frac{V^2}{2} + g(z_A - z_F)\right]$$

$$= 999\ \frac{kg}{m^3}\left[\frac{1630}{2} \times 0.017 \times \left(6.4\frac{ft}{s}\right)^2 \times \frac{m^2}{s^2} + 9.81\frac{m}{s^2}(104-91)m\right] \times \frac{N}{kg\cdot m}$$

$p_F = 705$ kPa (gage) ← p_F

For fully developed flow in a pipe $\tau = \frac{r}{2}\frac{\partial p}{\partial x}$ (8.15)

At the pipe centerline, $\tau = 0$ ← τ_0

To determine the power input to the fluid apply the energy equation across the pump. Assuming 100% efficiency

$$\dot{W}_{pump} = \left(\frac{p}{\rho} + \frac{V^2}{2} + gz\right)_{discharge} - \left(\frac{p}{\rho} + \frac{V^2}{2} + gz\right)_{suction} \quad (8.47)$$

$$\dot{W}_{pump} = \left(\frac{p_F}{\rho} - \frac{p_c}{\rho}\right)\rho AV = (p_F - p_c)Q$$

$$\dot{W}_{pump} = (705 - 197)\times 10^3\ \frac{N}{m^2} \times 1310\frac{L}{s} \times 10^{-3}\frac{m^3}{L} = 6.65\times 10^5\ \frac{N\cdot m}{s}$$

The actual pump input, $\dot{W}_{pump,actual} = \dot{W}_{pump,ideal}/\eta$

$\dot{W}_{pump,actual} = 8.32\times 10^5$ N·m/s = 832 kW ← $\dot{W}_{actual}$

From Eq. 8.15 $\tau_w = \frac{R}{2}\frac{\partial p}{\partial x}$

Along the pipe from F to G $\frac{\partial p}{\partial x} = f\frac{\rho}{D}\frac{V^2}{2}$

$$\therefore \frac{\partial p}{\partial x} = \frac{\Delta p}{L} = \rho f\frac{V^2}{2D} = 999\frac{kg}{m^3} \times \frac{0.017}{0.508\,m} \times \frac{ft}{2} \times \left(6.4\frac{ft}{s}\right)^2 \times \frac{m^2}{ft^2} \times \frac{N}{kg\cdot m/s^2}$$

$$\frac{\partial p}{\partial x} = 698\ N/m^2/m$$

$$\therefore \tau_w = \frac{R}{2}\frac{\partial p}{\partial x} = \frac{0.254\,m}{2} \times 698\frac{N}{m^3} = 88.6\ N/m^2$$ ← τ_w

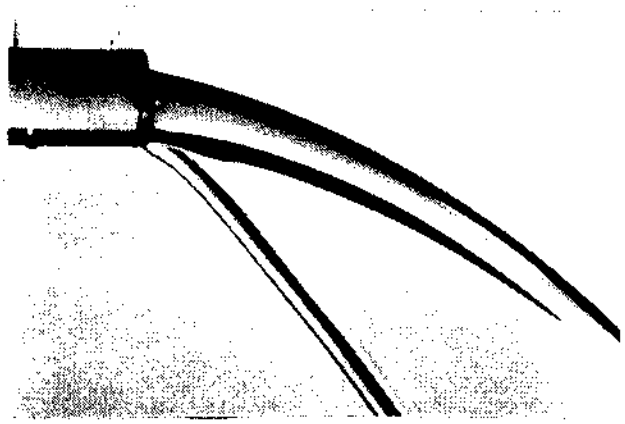
8-81 m [4]

Open-Ended Problem Statement: You are asked to compare the behavior of fully developed laminar flow and fully developed turbulent flow in a horizontal pipe under different conditions. For the same flow rate, which will have the larger centerline velocity? Why? If the pipe discharges to atmosphere what would you expect the trajectory of the discharge stream to look like (for the same flow rate)? Sketch your expectations for each case. For the same flow rate, which flow would give the larger wall shear stress? Why? Sketch the shear stress distribution τ/τ_w as a function of radius for each flow. For the same Reynolds number, which flow would have the larger pressure drop per unit length? Why? For a given imposed pressure differential, which flow would have the larger flow rate? Why?

Discussion: In the following fully developed laminar flow and fully developed turbulent flow in a pipe are compared:

(a) For the same flow rate, laminar flow has the higher maximum velocity, because the turbulent velocity profile is more blunt.

(b) The trajectory of the discharge stream spreads out for laminar flow because of the large variation in velocity across the pipe exit. For turbulent flow the exit profile is more nearly uniform (except for the region adjacent to the wall) and hence the trajectory is more uniform. Since centerline velocity is larger for laminar flow, liquid travels the greatest horizontal distance. Trajectories for the two flow cases are shown below:



(i) Laminar flow



(ii) Turbulent flow

(c) For the same flow rate (same mean velocity), turbulent flow has larger wall shear stress because of the larger velocity gradient at the pipe wall. For fully developed flow the pressure force driving the flow is balanced by the shear force at the wall.

(d) Shear stress varies linearly with radius for both flow cases, from its maximum value at the wall to zero at the pipe centerline.

(e) For the same Reynolds number, turbulent flow has a larger pressure drop per unit length because the friction factor is larger.

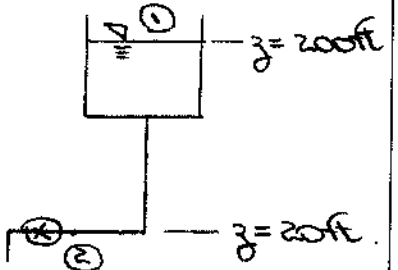
(f) For a given pressure drop (per unit length), laminar flow has the larger flow rate (larger mean velocity), because it has the smaller friction factor.

The two flow cases are compared in the NCFMF video *Turbulence*, in which R. W. Stewart uses a clever experimental setup to contrast the two flow regimes at constant volume flow rate by varying the liquid viscosity. The trajectories of the liquid streams leaving the end of the pipe are particularly well shown.

13-732 500 SHEETS, FILLER 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-383 200 SHEETS EYE-EASE® 5 SQUARE
42-389 100 RECYCLED WHITE 5 SQUARE
42-394 200 RECYCLED WHITE 5 SQUARE
42-396 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Given: Oil flowing from a large tank on a hill to a tanker at the wharf. In stopping the flow, valve on wharf at such a rate that $p_2 = 150 psig$ is maintained in the line immediately upstream of the valve. Assume:

Length of line from tank to valve	10,000 ft
Inside diameter of line	8.0 in.
Elevation of oil surface in tank	200 ft
Elevation of valve on wharf	20 ft
Instantaneous flow rate	1.5 ft^3/s
Head loss in line (exclusive of valve being closed) at this rate of flow	75 ft of oil
Specific gravity of oil	0.88



Find: the initial instantaneous rate of change of volume flow rate.

Solution: For unsteady flow with friction, we modify the unsteady Bernoulli equation (Eq. 6.21) to include a head loss term.

Computing equation: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V_s}{\partial t} ds + h_e$

Assume: (1) $V_1 \approx 0$ (2) $p_1 = p_{atm}$ (3) ρ = constant

Then $\int_1^2 \frac{\partial V_s}{\partial t} ds = \frac{p_1 - p_2}{\rho} + g(z_1 - z_2) - h_e - \frac{V_2^2}{2}$

If we neglect velocity in the tank except for small region near the inlet to the pipe, then

$\int_1^2 \frac{\partial V_s}{\partial t} ds \approx \int_0^2 \frac{\partial V_s}{\partial t} ds$. Since $V_s = V_2$ everywhere, then

$\int_0^2 \frac{\partial V_s}{\partial t} ds = L \frac{dV_2}{dt}$ and

$\frac{dV_2}{dt} = \frac{1}{L}\left[\frac{p_1 - p_2}{\rho} + g(z_1 - z_2) - h_e - \frac{V_2^2}{2}\right]$, $V_2 = \frac{Q}{A} = \frac{4Q}{\pi D^2}$

Note $h_e = h_e(V)$ and hence this result can only be used to obtain the initial instantaneous rate of change of flow velocity

$$\left(\frac{dV_2}{dt}\right)_{initial} = \frac{1}{10^4 ft}\left[-150\frac{lbf}{in^2} \times \frac{144 in^2}{ft^2} \times \frac{1}{0.88 \times 1.94 \frac{slug}{ft^3}} \times \frac{ft \cdot slug}{lbf \cdot s^2} + 32.2\frac{ft}{s^2} \times 180 ft - 75 ft \times 32.2\frac{ft}{s^2} - \frac{1}{2}\left[\frac{4}{\pi} \times 1.5 \times \left(\frac{1}{2}\right)^2\right]^2 \frac{ft^2}{s^2}\right]$$

$$\left(\frac{dV_2}{dt}\right)_{initial} = -0.928 \ ft/s/s$$

The instantaneous rate of change of volume flow rate is

$$dQ/dt = \frac{d}{dt}(AV) = A\frac{dV}{dt} = \frac{\pi D^2}{4}\frac{dV}{dt}$$

$$dQ/dt = \frac{\pi}{4}\left(\frac{2}{3}\right)^2 ft^2 \times (-0.928)\frac{ft}{sec} = -0.324 \ ft^3/s/s \leftarrow dQ/dt$$

Problem *8.133

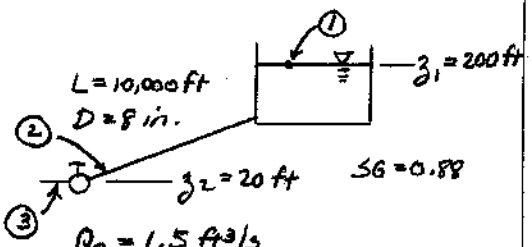
Given: Problem 8.132 describes a situation in which flow in a long pipeline from a hilltop tank is slowed gradually to avoid a large pressure rise.

Find: Expansion of this analysis to predict and plot the closing schedule (valve loss coefficient versus time) needed to maintain the maximum pressure at the valve at or below a given value throughout the process of stopping the flow from the tank.

Solution: Apply the unsteady Bernoulli equation with a head loss term added.

Computing equation:

$$\frac{p_1}{\rho}^{\nearrow 0} + \frac{V_1^2}{2}^{\approx 0(1)} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 + \int_1^2 \frac{\partial V}{\partial t} ds + h_{LT}$$



Assume: (1) $V_1 \approx 0$ (3) ρ = constant

(2) $p_1 = p_{atm}$

At the initial condition, $V = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times \left(\frac{12}{8}\right)^2 \frac{1}{ft^2} \times 1.5 \frac{ft^3}{s} = 4.30$ ft/s

$$H_{LT} = 75 \text{ ft} = \frac{h_{LT}}{g} = f \frac{L}{D} \frac{V^2}{2g}; \quad f \frac{L}{D} = H_{LT} \frac{2g}{V^2} = 2 \times 75 \text{ ft} \times 32.2 \frac{ft}{s^2} \times \frac{s^2}{(4.30)^2 ft^2} = 261$$

Neglecting velocity in tank, $\int_1^2 \frac{\partial V}{\partial t} ds \approx \int_0^L \frac{\partial V}{\partial t} ds = \frac{dV}{dt} L$

Thus $$\frac{dV}{dt} = \frac{1}{L}\left[-\frac{p_2}{\rho} + g(z_1 - z_2) - f\frac{L}{D}\frac{V^2}{2} - \frac{V^2}{2}\right]$$

Substituting values,

$$\frac{dV}{dt} = \frac{1}{10{,}000 \text{ ft}}\left[-150 \frac{lbf}{in.^2} \times \frac{ft^3}{(0.88)\,1.94 \text{ slug}} \times 144 \frac{in.^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2} + 32.2 \frac{ft}{s^2}(200 - 20) ft - (261+1)\frac{V^2}{2}\right]$$

$$\frac{dV}{dt} = -0.686 \frac{ft}{s^2} - 0.0131 V^2 = -(a^2 + b^2 V^2); \quad a = \sqrt{0.686} = 0.828; \quad b = \sqrt{0.0131} = 0.114$$

V in ft/s

Separating variables and integrating

$$\int_{V_0}^{V} \frac{dV}{a^2 + b^2 t} = \frac{1}{ab} \tan^{-1} \frac{bV}{a}\Big]_{V_0}^{V} = \frac{1}{ab}\left[\tan^{-1}\frac{bV}{a} - \tan^{-1}\frac{bV_0}{a}\right] = -\int_0^t dt = -t$$

Thus

$$\tan^{-1}\frac{bV}{a} = -abt + \tan^{-1}\frac{bV_0}{a} \quad \text{or} \quad V = \frac{a}{b}\tan\left[\tan^{-1}\frac{bV_0}{a} - abt\right]$$ ← V(t)

The pressure must drop across the valve:

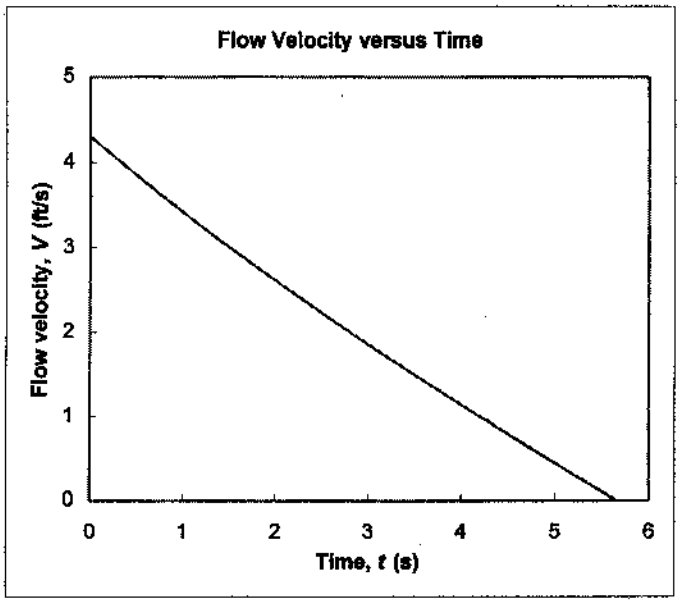
$$\frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 - \left(\frac{p_3}{\rho} + \frac{V_3^2}{2} + g z_3\right) = h_{LT} = K_V \frac{V^2}{2} \quad \text{or} \quad K_V = \frac{2(p_2 - p_3)}{\rho V^2} \approx \frac{2p_2}{\rho V^2}$$

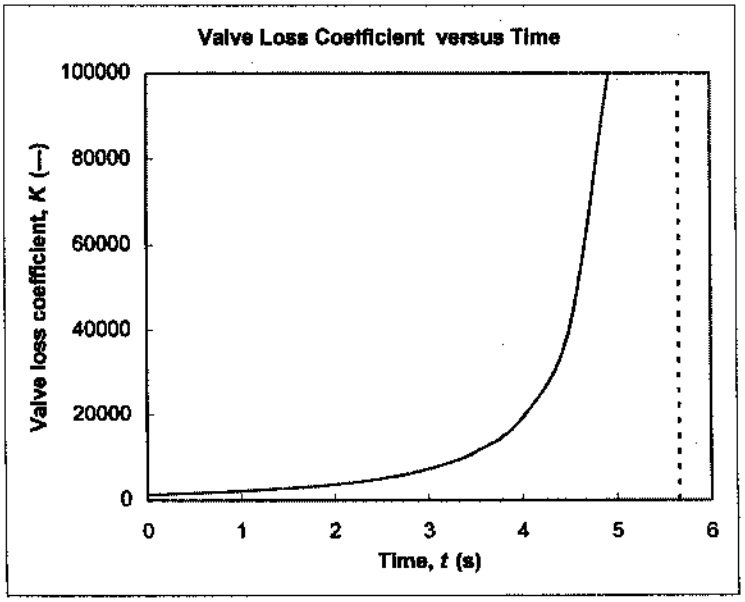
At $t = 0$, $$K_V = 2 \times 150 \frac{lbf}{in.^2} \times \frac{ft^3}{(0.88)1.94 \text{ slug}} \times \frac{s^2}{(4.3)^2 ft^2} \times 144 \frac{in.^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2} = 1{,}370 \quad (t = 0)$$ ← $K_V(0)$

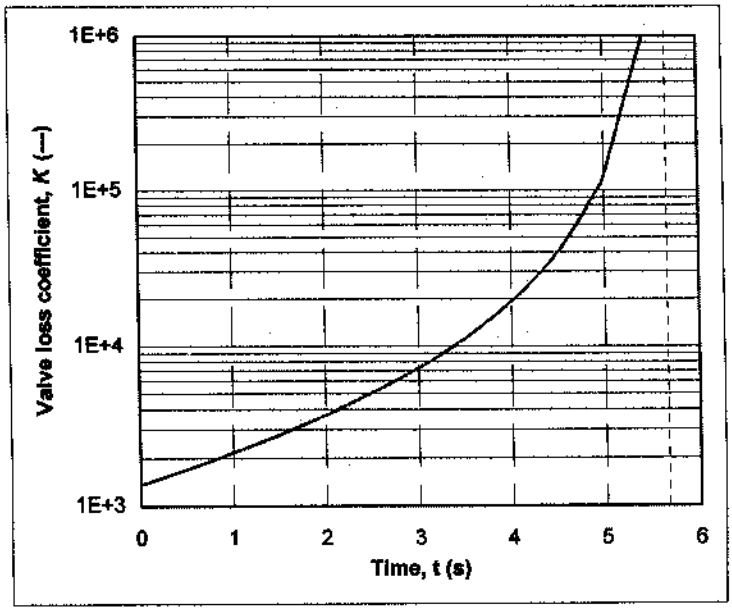
Calculations and plots are shown on the spreadsheet, next page.

	a =	0.828	(---)
	b =	0.114	(---)
	V_0 =	4.3	(ft/s)

t (s)	V (ft/s)	K (---)
0.0	4.300	1.37E+3
0.5	3.849	1.71E+3
1.0	3.421	2.16E+3
1.5	3.011	2.79E+3
2.0	2.616	3.70E+3
2.5	2.235	5.07E+3
3.0	1.865	7.28E+3
3.5	1.504	1.12E+4
4.0	1.150	1.92E+4
4.5	0.801	3.95E+4
4.93	0.503	1.00E+5
5.0	0.455	1.22E+5
5.431	0.159	1.00E+6
5.5	0.112	2.03E+6
5.663	0.000	2.92E+13







Given: The pressure rise, Δp, across a water pump is 9.5 psi when the volume flow rate is $Q = 300$ gpm. The pump efficiency is $\eta = 0.80$.

Find: The power input to the pump.

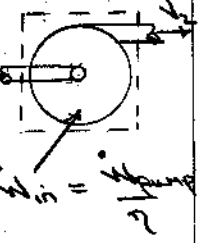
Solution:

Apply the first law of thermodynamics across the pump.

Basic equation: $\dot{W}_{pump} = \dot{m}\left[\left(\frac{p}{\rho} + \frac{V^2}{2} + gz\right)_{discharge} - \left(\frac{p}{\rho} + \frac{V^2}{2} + gz\right)_{suction}\right]$ (8.47)

where $\dot{W}_{pump}$ is energy added to fluid by the pump

Assume: (1) ρ = constant (2) $z_1 = z_2$
(3) uniform properties at inlet, outlet
(4) $A_1 = A_2$, $\therefore V_1 = V_2$



Then $\dot{W}_{pump} = \dot{m}\frac{\Delta p}{\rho} = \rho A V \frac{\Delta p}{\rho} = Q\,\Delta p$

$$\dot{W}_{pump} = 300\ \frac{gal}{min} \times \frac{ft^3}{7.48\ gal} \times \frac{min}{60\ s} \times 9.5\ \frac{lbf}{in^2} \times \frac{144\ in^2}{ft^2} \times \frac{hp \cdot s}{550\ ft \cdot lbf}$$

$$\dot{W}_{pump} = 1.66\ hp$$

The pump efficiency, η, is defined as

$$\eta = \frac{\dot{W}_{pump}}{\dot{W}_{in}} \quad \therefore \dot{W}_{in} = \frac{\dot{W}_{pump}}{\eta} = 2.08\ hp$$

Given: Pump moves $\dot{m} = 20$ lbm/s through a piping system

$p_{discharge} = 46$ psig, $p_{suction} = -3$ psig

$D_{suction} = 3$ in., $D_{discharge} = 2$ in., $\eta_{pump} = 0.70$

Find: Power required to drive the pump.

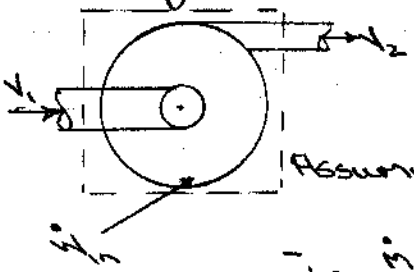
Solution:

Apply the first law of thermodynamics across the pump.

Basic equation:

$$\frac{\dot{W}_{shaft}}{\dot{m}} = \left(\frac{p}{\rho} + \frac{V^2}{2} + gz\right)_{discharge} - \left(\frac{p}{\rho} + \frac{V^2}{2} + gz\right)_{suction} \quad (8.47)$$

$$\eta_{pump} = \frac{\dot{W}_{shaft}/\dot{m}_{pump}}{\dot{W}_{in}/\dot{m}}$$



Assume: (1) p = constant (2) $z_1 = z_2$
(3) uniform properties at inlet + outlet.

$$V = \frac{\dot{V}}{A} = \frac{\dot{m}/\rho}{\pi D^2/4}$$

$$V_1 = \frac{4}{\pi} \times 20\frac{lbm}{s} \times 1.94\frac{slug}{ft^3} \times \frac{ft^3}{32.2\,lbm} \times \frac{1}{(0.25ft)^2} = 6.52\ ft/s$$

$$V_2 = \frac{A_1}{A_2} V_1 = \left(\frac{D_1}{D_2}\right)^2 V_1 = \left(\frac{2in}{3in}\right)^2 \times 6.52\ ft/s = 14.7\ ft/s$$

From Eq. 8.47

$$\dot{W}_{pump} = \dot{m}\left[\frac{p_2 - p_1}{\rho} + \frac{V_2^2 - V_1^2}{2}\right]$$

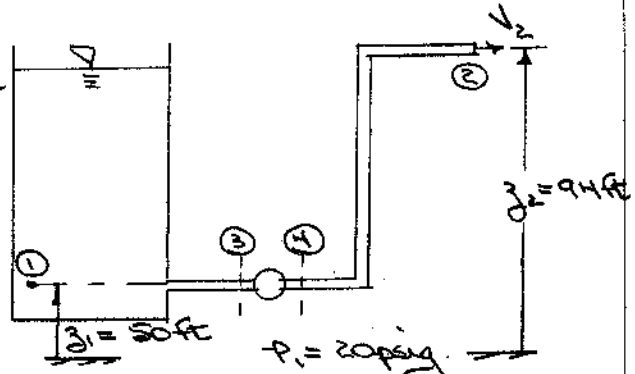
$$\dot{W}_{pump} = 20\frac{lbm}{s}\left[\frac{49\frac{lbf}{in^2} \times 1.94\frac{slug}{ft^3} \times 144\frac{in^2}{ft^2} \times \frac{ft \cdot slug}{lbf \cdot s^2}}{2} + \frac{(14.7)^2 - (6.52)^2\ \frac{ft^2}{s^2}}{2}\right] \frac{slug}{32.2\,lbm} \times \frac{lbf \cdot s^2}{ft \cdot slug}$$

$$\dot{W}_{pump} = 2310\frac{ft \cdot lbf}{s} \times \frac{hp \cdot s}{550\ ft \cdot lbf} = 4.21\ hp$$

$$\dot{W}_{in} = \frac{\dot{W}_{pump}}{\eta_{pump}} = \frac{4.21}{0.7} = 6.01\ hp \quad \dot{W}_{in}$$

Given: Pump in piping system shown moves $Q = 0.439\ ft^3/s$ of water.

System includes:
- $L = 290\ ft$ galvanized pipe, $D = 2.5\ in.$ (nominal)
- 2 gate valves (open)
- 1 angle valve (open)
- 7 standard 90° elbows
- 1 square edge entrance
- 1 free discharge



Find: pressure rise, $p_4 - p_3$, across pump.

Solution:

Computing equation: $\left(\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2\right) + \Delta h_{pump} = h_{e_T}$ (8.148)

$h_{e_T} = h_e + h_{em}$, $h_e = f\frac{L}{D}\frac{\bar{V}_2^2}{2}$, $h_{em} = \frac{\bar{V}_2^2}{2}\left\{\Sigma f\left(\frac{L_e}{D}\right) + \Sigma K\right\}$

Assumptions: (1) $\bar{V}_1 = 0$ (2) $p_2 = p_{atm}$ (3) $\alpha_2 = 1.0$ (4) $T = 60°F$

Then,

$$\Delta h_{pump} = h_{e_T} + g(z_2 - z_1) + \frac{\bar{V}_2^2}{2} - \frac{p_1}{\rho} \quad \text{.....(1)}$$

$$h_{e_T} = \frac{\bar{V}_2^2}{2}\left[f\frac{L}{D} + 2f\left(\frac{L_e}{D}\right)_{gv} + f\left(\frac{L_e}{D}\right)_{av} + 7f\left(\frac{L_e}{D}\right)_{90°el} + K_{ent}\right] \quad \text{.....(2)}$$

From Table 8.4 $(L_e/D)_{gv} = 8$, $(L_e/D)_{av} = 150$, $(L_e/D)_{90°el} = 30$

From Table 8.2 $K_e = 0.5$. From Table 8.5 $D = 2.47\ in$

From Table 8.1 $e = 0.0005\ ft$ $\therefore e/D = 0.0005 \times \frac{12}{2.47} = 0.0024$

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 0.439\frac{ft^3}{s} \times \left(\frac{12}{2.47\ ft}\right)^2 = 13.2\ ft/s$$

$$Re = \frac{\bar{V}D}{\nu} = \frac{2.47}{12}ft \times 13.2\frac{ft}{s} \times 1.21\times10^{-5}\frac{ft^2}{s} = 2.25\times10^5 \quad \{\nu \text{ from Table A.7}\}$$

From Fig. 8.13, $f = 0.025$.

From Eqn. 2

$$h_{e_T} = \frac{1}{2}(13.2)^2\frac{ft^2}{s^2}\left[0.025 \times \frac{290\times12}{2.47} + 2(0.025)(8) + (0.025)(150) + 7(0.025)(30) + 0.5\right]$$

$h_{e_T} = 3930\ ft^2/s^2$. Then from Eq. 1

$$\Delta h_{pump} = 3930\frac{ft^2}{s^2} + 32.2\frac{ft}{s^2}(144ft) + \frac{1}{2}(13.2)^2\frac{ft^2}{s^2} - 20\frac{lbf}{in^2} \times 1.94\frac{slug}{ft^3} \times \frac{144\ in^2}{ft^2} \times \frac{slug\cdot ft}{lbf\cdot s^2}$$

$\Delta h_{pump} = 3950\ ft^2/s^2$

Apply the energy equation across the pump

$$\Delta h_{pump} = \left(\frac{p}{\rho} + \frac{\bar{V}^2}{2} + gz\right)_{discharge} - \left(\frac{p}{\rho} + \frac{\bar{V}^2}{2} + gz\right)_{suction} \quad (8.147)$$

$$\Delta p = \rho\,\Delta h_{pump} = 1.94\frac{slug}{ft^3} \times 3950\frac{ft^2}{s^2} \times \frac{lbf\cdot s^2}{ft\cdot slug} \times \frac{ft^2}{144\ in^2} = 53.2\ lbf/in^2 \quad (p_4 - p_3)$$

Given: Water supply system requires Q = 100 gpm pumped to reservoir at elevation of 340 m. Water pressure at pump inlet (street level) is 400 kPa (gage). Piping is to be commercial steel; $V_{max} = 3.5$ m/s

Find: (a) Minimum pipe diameter
(b) pressure rise across the pump
(c) minimum power needed to drive the pump

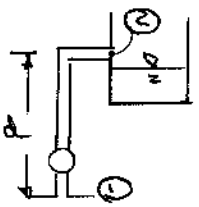
Solution:

$$V = \frac{Q}{A} = \frac{4Q}{\pi D^2} \quad \therefore D = \left[\frac{4Q}{\pi V}\right]^{1/2} = \left[\frac{4}{\pi} \times 100\,\frac{\text{gal}}{\text{min}} \times \frac{\text{ft}^3}{7.48\,\text{gal}} \times \frac{\text{m}^3}{60\,\text{s}} \times \left(\frac{0.305\,\text{m}}{\text{ft}}\right)^3 \times \frac{1}{3.5\,\text{m/s}}\right]^{1/2}$$

$$D = 0.048\,\text{m} = 48\,\text{mm} \longleftarrow D$$

Governing equations: $\left(\frac{p_1}{\rho} + \alpha_1\frac{V_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{V_2^2}{2} + gz_2\right) + \Delta h_{pump} = h_{e_T}$ (8.48)

$h_{e_T} = h_e + h_{em}$, $h_e = f\frac{L}{D}\frac{V^2}{2}$.



Assume: (1) $\alpha_1 = \alpha_2$ (2) $p_2 = p_{atm}$
(3) minor losses are negligible.
(4) water at 20°C ($\nu = 1.00 \times 10^{-6}$ m²/s, Table A.8).

Then

$$\Delta h_{pump} = h_e - \frac{p_1}{\rho} + g(z_2 - z_1) = f\frac{L}{D}\frac{V^2}{2} - \frac{p_1}{\rho} + gd \quad \text{.....(1)}$$

From Table 8.1, $e = 0.046$ mm $\therefore e/D = 0.046/48 = 0.0006$

$$Re = \frac{VD}{\nu} = 0.048\,\text{m} \times 3.5\,\frac{\text{m}}{\text{s}} \times 1.0\times10^{-6}\,\frac{\text{s}}{\text{m}^2} = 1.68\times10^5$$

From Fig. 8.13, $f = 0.021$. Then from Eq. 1

$$\Delta h_{pump} = 0.021 \times \frac{340}{0.048} \times \frac{1}{2}\,\frac{(3.5)^2\,\text{m}^2}{\text{s}^2} - 400\times10^3\,\frac{\text{N}}{\text{m}^2} \times 998\,\frac{\text{kg}}{\text{m}^3} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} + 9.81\,\frac{\text{m}}{\text{s}^2} \times 340\,\text{m}$$

$$\Delta h_{pump} = 3{,}850\ \text{m}^2/\text{s}^2 \quad \text{(This is head added to fluid)}$$

$$\Delta h_{pump} = \frac{\dot{W}_{pump}}{\dot{m}} = \left(\frac{p}{\rho} + \frac{V^2}{2} + gz\right)_{discharge} - \left(\frac{p}{\rho} + \frac{V^2}{2} + gz\right)_{suction} \quad (8.47)$$

Assume: (5) $V_{disch} = V_{suct}$; $z_{discharge} \approx z_{suction}$

$$\Delta p = \rho\,\Delta h = 998\,\frac{\text{kg}}{\text{m}^3} \times 3850\,\frac{\text{m}^2}{\text{s}^2} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} = 3840\,\text{kPa} \longleftarrow \Delta p$$

Also from Eq. 8.47

$$\dot{W}_{pump} = \dot{m}\,\Delta h_{pump} = \rho Q\,\Delta h_{pump}$$

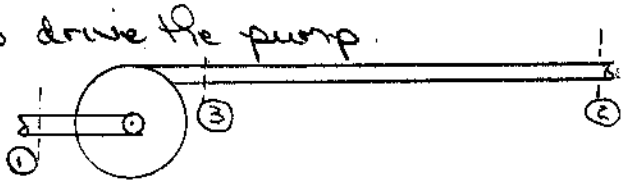
$$\dot{W}_{pump} = 998\,\frac{\text{kg}}{\text{m}^3} \times 100\,\frac{\text{gal}}{\text{min}} \times \frac{\text{ft}^3}{7.48\,\text{gal}} \times \left(\frac{0.305\,\text{m}}{\text{ft}}\right)^3 \times \frac{\text{m}}{60\,\text{s}} \times 3850\,\frac{\text{m}^2}{\text{s}^2} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$\dot{W}_{pump} = 24.3\,\text{kW}\ (32.6\,\text{hp}) \longleftarrow \dot{W}_{pump}$$

Given: lightweight crude oil (SG = 0.855) is pumped horizontally through a 2 in. diameter pipe, 1 mile long, $e = 0.01$ in.; $Q = 4500$ gpm. Oil viscosity similar to SAE 30 oil. Pump efficiency $\eta = 0.75$.

Find: power required to drive the pump.

Solution:



Governing equations:

$$\left(\frac{p_1}{\rho} + \alpha_1\frac{V_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\frac{V_2^2}{2} + gz_2\right) + \Delta h_{pump} = h_{l_T} \qquad (8.48)$$

$$h_{l_T} = h_l + \cancel{h_{l_m}}^{\,0\ (3)} \quad ; \quad h_l = f\frac{l}{D}\frac{V^2}{2}$$

Assumptions: (1) $D_1 = D_2 = D_3$, $\alpha_1 = \alpha_2$ (2) $z_2 = z_1$ (3) $h_{l_m} = 0$

(4) $p_2 = p_1$, i.e. pump provides sufficient pressure rise $(p_3 - p_1)$ to overcome pressure drop $(p_3 - p_2)$

Then, $\Delta h_{pump} = f\frac{l}{D}\frac{V^2}{2}$ ——————— (1)

$$V = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 4500\frac{\text{gal}}{\text{min}} \times \frac{\text{ft}^3}{7.48\text{ gal}} \times \frac{\text{min}}{60\text{ s}} \times \frac{1}{\text{ft}^2} = 12.8\ \text{ft/s}$$

At $T = 59^\circ\text{F}$ (15°C) {from Fig A.2}, $\mu = 0.6\ \text{N}\cdot\text{s/m}^2 \times 0.0209\ \frac{\text{lbf}\cdot\text{s/ft}^2}{\text{N}\cdot\text{s/m}^2} = 1.25\times10^{-2}\ \text{lbf}\cdot\text{s/ft}^2$

Then $Re = \frac{\rho D V}{\mu} = 0.855 \times 1.94\frac{\text{slug}}{\text{ft}^3} \times 1\text{ ft} \times 12.8\frac{\text{ft}}{\text{s}} \times 1.25\times10^{-2}\frac{\text{ft}^2}{\text{lbf}\cdot\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}} = 1700$

Hence flow is laminar and $f = 64/Re = 0.038$

From Eq. (1) $\Delta h_{pump} = 0.038 \times \frac{5280\text{ ft}}{1\text{ ft}} \times \frac{1}{2} \times (12.8)^2\frac{\text{ft}^2}{\text{s}^2} = 16.4\times10^3\ \frac{\text{ft}^2}{\text{s}^2}$

The theoretical power input to the pump is given by

$$\Delta h_{pump} = \frac{\dot{W}_{pump}}{\dot{m}} \qquad (8.47)$$

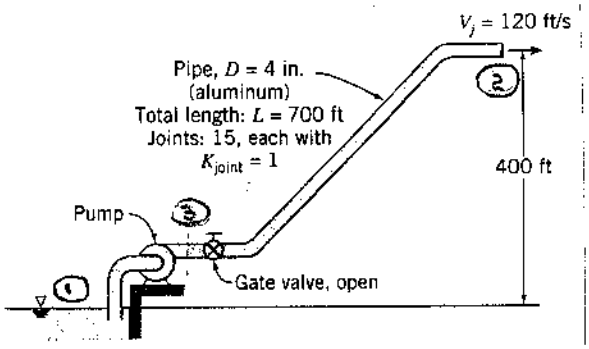
From the definition of efficiency, $\eta = \frac{\dot{W}_{theoretical}}{\dot{W}_{actual}}$, then

$$\dot{W}_{act} = \frac{\dot{W}_{theor}}{\eta} = \frac{\dot{m}\Delta h_{pump}}{\eta} = \frac{\rho Q \Delta h_{pump}}{\eta}$$

$$\dot{W}_{act} = \frac{1}{0.75} \times 0.855 \times 1.94\frac{\text{slug}}{\text{ft}^3} \times 4500\frac{\text{gal}}{\text{min}} \times \frac{\text{ft}^3}{7.48\text{ gal}} \times \frac{\text{min}}{60\text{ s}} \times 16.4\times10^3\frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{hp}\cdot\text{s}}{550\text{ ft}\cdot\text{lb}}$$

$$\dot{W}_{act} = 661\ \text{hp} \longleftarrow \dot{W}_{act}$$

Given: Cooling water supply system
$Q = 600$ gpm
$\eta_{pump} = 0.70$



Find: (a) minimum pressure needed at pump outlet
(b) power requirement

Solution:

Computing equations: $\left(\frac{p_1}{\rho}+\alpha_1\frac{V_1^2}{2}+gz_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\frac{V_2^2}{2}+gz_2\right)+\Delta h_{pump}=h_{L_T}$ (8.48)

$h_{L_T}=h_L+h_{Lm}$, $h_L=f\frac{L}{D}\frac{V^2}{2}$, $h_{Lm}=\frac{V^2}{2}\left(\Sigma K+\Sigma f\left(\frac{L_e}{D}\right)\right)$

Assumptions: (1) $V_1=0$ (2) $\alpha_2=\alpha_3=1$ (3) $p_1=p_2=p_{atm}$

Then
$$\Delta h_{pump}=gz_2+\frac{V_2^2}{2}+f\frac{L}{D}\frac{V^2}{2}+\frac{V^2}{2}\left[K_{ent}+f\left(\frac{L_e}{D}\right)_{gate}+2f\left(\frac{L_e}{D}\right)_{45°el}+15K_j\right] \quad (1)$$

$$V=\frac{Q}{A}=\frac{4Q}{\pi D^2}=\frac{4}{\pi}\times 600\frac{gal}{min}\times 7.48\frac{ft^3}{gal}\times\frac{min}{60sec}\times\left(\frac{1}{0.333ft}\right)^2=15.3\,ft/s$$

$$Re=\frac{\rho VD}{\mu}=\frac{1}{3}ft\times 15.3\frac{ft}{s}\times 1.24\times10^{-5}\frac{ft^2}{s}=4.11\times10^5 \quad \{\nu \text{ at } T=60°F, \text{ Table A.7}\}$$

Table 8.1, $e=5\times10^{-6}$ ft (drawn tubing) $\therefore e/D=1.5\times10^{-5}$

From Fig. 8.13, $f=0.0135$

From Table 8.1, $K_{ent}=0.78$

From Table 8.4, $\left(L_e/D\right)_{gate}=8$, $\left(L_e/D\right)_{90°el}=30$, $\left(L_e/D\right)_{45°el}=16$

Then from Eq. (1)

$$\Delta h_{pump}=32.2\frac{ft}{s^2}\times 400ft+\frac{1}{2}(120)^2\frac{ft^2}{s^2}+0.0135\times\frac{700}{0.333}\times\frac{1}{2}\times(15.3)^2\frac{ft^2}{s^2}$$
$$+\frac{1}{2}(15.3)^2\frac{ft^2}{s^2}\left[0.78+0.0135(30)+2\times0.085(16)+15(1)\right]$$

$$\Delta h_{pump}=2.53\times10^4\,ft^2/s^2$$

The theoretical power input to the pump is given by $\dot{W}_{pump}=\dot{m}\Delta h_{pump}$

From the definition of efficiency, $\eta=\dot{W}_{th}/\dot{W}_{act}$, then

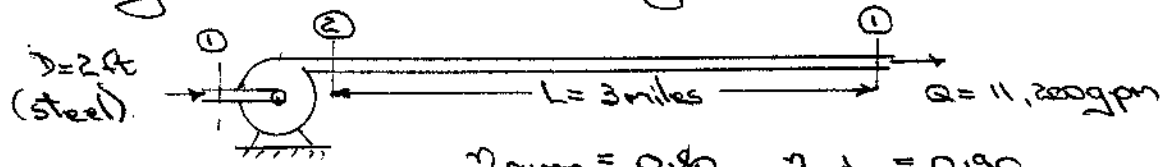
$$\dot{W}_{act}=\frac{\dot{m}\Delta h_{pump}}{\eta}=\frac{\rho Q\Delta h_{pump}}{\eta}$$

$$\dot{W}_{act}=\frac{1.94}{0.7}\frac{slug}{ft^3}\times 600\frac{gal}{min}\times\frac{ft^3}{7.48gal}\times\frac{min}{60s}\times 2.53\times10^4\frac{ft^2}{s^2}\times\frac{lbf\cdot s^2}{ft\cdot slug}\times\frac{hp\cdot s}{550\,ft\cdot lbf}=170\,hp \quad \dot{W}_{in}$$

The discharge pressure from the pump is obtained by applying Eq. 8.48 between sections ① and ③, neglecting losses in the inlet section, elevation change, and kinetic energy at ③

$$p_3-p_A=\rho\Delta h_{pump}=1.94\frac{slug}{ft^3}\times 2.53\times10^4\frac{ft^2}{s^2}\times\frac{lbf\cdot s^2}{ft\cdot slug}\times\frac{ft^2}{144in^2}=341\,psi \quad p_3$$

Given: Chilled-water pipe system for campus air conditioning makes a loop of length L = 3 miles.



$\eta_{pump} = 0.80$, $\eta_{motor} = 0.90$

$c = \$0.067/(kw \cdot hr)$

Find: (a) the pressure drop, $p_2 - p_1$
(b) rate of energy addition to the water
(c) daily cost of electrical energy for pumping

Solution:

Apply energy equation for steady, incompressible pipe flow from pump discharge around loop to pump inlet.

Computing equations: $\left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) - \left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) = h_{e_T}$ (8.29)

$h_{e_T} = h_e + \cancel{h_m}^{\,0}$ $\quad h_e = f \frac{L}{D} \frac{\bar{V}_2^2}{2}$

Assumptions: (1) $\alpha_1 = \alpha_2$, (2) $z_1 = z_2$ (3) neglect minor losses

Then $p_2 - p_1 = f \frac{L}{D} \rho \frac{\bar{V}^2}{2}$, $\bar{V} = \frac{Q}{A} = \frac{11{,}200 \text{ gal/min} \times \pi (1)^2 \text{ ft}^2 \times \text{ft}^3/7.48 \text{ gal} \times \text{min}/60 \text{ s}}{} = 7.94 \text{ ft/s}$

Assume T = 50°F, so $\nu = 1.40 \times 10^{-5} \text{ ft}^2/\text{s}$.

$Re = \frac{D\bar{V}}{\nu} = 2 \text{ ft} \times 7.94 \frac{\text{ft}}{\text{s}} \times 1.40 \times 10^{-5} \frac{\text{s}}{\text{ft}^2} = 1.13 \times 10^6$

From Table 8.1, $\varepsilon = 0.00015$ ft ; $\therefore \varepsilon/D = 0.000075$. Then, from Fig. 8.13, $f = 0.013$, and

$\Delta p = p_2 - p_1 = 0.013 \times \frac{3 \text{ mi}}{2 \text{ ft}} \times \frac{5280 \text{ ft}}{1 \text{ mi}} \times 1.94 \frac{\text{slug}}{\text{ft}^3} \times \frac{1}{2} \times (7.94)^2 \frac{\text{ft}^2}{\text{s}^2} \times \frac{lbf \cdot s^2}{ft \cdot slug} \times \frac{ft^2}{144 in^2}$

$\Delta p = 43.7$ psi. ← Δp

To determine the energy per unit mass applied by the pump:

$\frac{\dot{W}_{pump}}{\dot{m}} = \left(\frac{p}{\rho} + \cancel{\frac{\bar{V}^2}{2}} + \cancel{gz}\right)_{discharge} - \left(\frac{p}{\rho} + \cancel{\frac{\bar{V}^2}{2}} + \cancel{gz}\right)_{suction}$ (8.47)

$\dot{W}_{pump} = \dot{m} \frac{\Delta p}{\rho} = Q \Delta p$

$\dot{W}_{pump} = 11{,}200 \frac{\text{gal}}{\text{min}} \times \frac{\text{ft}^3}{7.48 \text{ gal}} \times \frac{\text{min}}{60 \text{ s}} \times 43.7 \frac{lbf}{in^2} \times \frac{144 in^2}{ft^2} \times \frac{hp \cdot s}{550 ft \cdot lbf} = 286 \text{ hp}$ ← $\dot{W}_{pump}$

The actual energy required to run the pump is

$P = \frac{\dot{W}_{pump}}{\eta_{pump} \eta_{motor}} = 286 \text{ hp} \times \frac{1}{0.80} \times \frac{1}{0.90} = 397 \text{ hp}$.

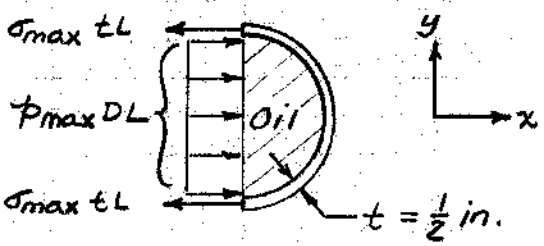
The daily cost is

$C = \frac{\$0.067}{kw \cdot hr} \times 397 \text{ hp} \times 0.746 \frac{kw}{hp} \times 24 \frac{hr}{day} = \$476$

Given: Heavy crude oil (SG = 0.925) pumped through a level pipeline at a rate of 400,000 barrels per day (1 bbl = 42 gal). Pipe is 24 in. in diameter with 1/2 in. wall thickness. Maximum allowable stress in pipe wall is 40,000 psi. Minimum pressure in oil is 75 psia. ($\nu = 1.1 \times 10^{-3}$ ft²/s) Pipeline is steel.

Find: (a) Maximum allowable spacing between pumping stations.
(b) Power added to oil at each pumping station.

Solution: First find the maximum pressure allowable in pipe. Consider a free body diagram of a segment of length, L:



Basic equation: $\Sigma F_x = 0$

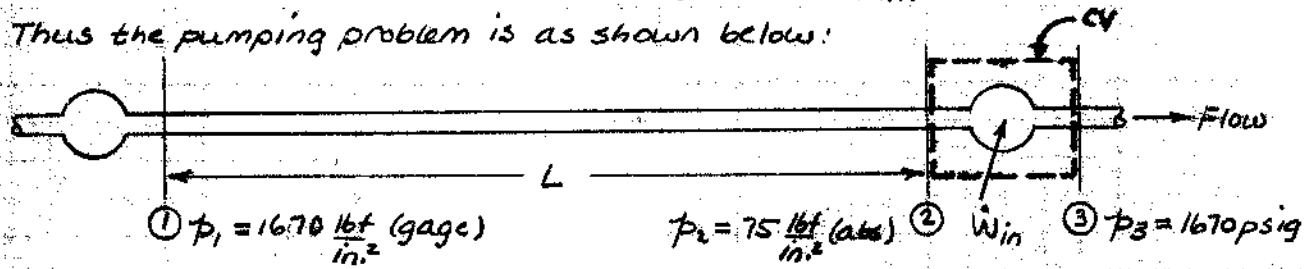
Assumption: Neglect hydrostatic pressure variation, and atmospheric pressure

Then

$$\Sigma F_x = p_{max} DL - 2\sigma_{max} tL = 0$$

$$p_{max} = 2\sigma_{max}\frac{t}{D} = 2 \times 40{,}000 \frac{lbf}{in.^2} \times \frac{\frac{1}{2} in.}{24 in.} = 1670 \frac{lbf}{in.^2} \text{ (gage)}$$

Thus the pumping problem is as shown below:



To find L, apply the energy equation for steady, incompressible flow that is uniform at each section.

Basic equation: $\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + g z_2 + h_{LT}$; $h_{LT} = f\frac{L}{D}\frac{\bar{V}^2}{2} + h_{LM}$ (=0 (3))

Assumptions: (1) $\bar{V}_1 = \bar{V}_2$
(2) $z_1 = z_2$ (level)
(3) $h_{LM} = 0$, since straight, constant area pipe

Then

$$f\frac{L}{D}\frac{\bar{V}^2}{2} = \frac{p_1 - p_2}{\rho} \quad \text{or} \quad L = \frac{D}{f}\left(\frac{p_1 - p_2}{\rho}\right)\frac{2}{\bar{V}^2}$$

$$\bar{V} = \frac{Q}{A} = 4 \times 10^5 \frac{bbl}{day} \times \frac{day}{24 hr} \times \frac{hr}{3600 s} \times 42 \frac{gal}{bbl} \times \frac{ft^3}{7.48 gal} \times \frac{4}{\pi}\frac{1}{(2)^2 ft^2} = 8.27 \text{ ft/s}$$

$f = f(Re, e/D)$. From Table 8.1, $e = 0.000\,15$ ft, so $e/D = 0.000{,}075$. Reynolds number is

$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu} = 8.27 \frac{ft}{sec} \times 2 ft \times \frac{sec}{1.1 \times 10^{-3} ft^2} = 1.50 \times 10^4$$

From Fig. 8.13, $f = 0.028$

Thus

$$L = \frac{2\,ft}{0.028}\left[1670\,\frac{lbf}{in.^2} - (75-14.7)\,\frac{lbf}{in.^2}\right] 144\,\frac{in.^2}{ft^2} \times \frac{ft^3}{(0.925)\,1.94\,slug} \times 2 \times \frac{s^2}{(8.27)^2\,ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2}$$

$L = 270{,}000\ ft$ or 51.1 miles ← L

To find pump power delivered to the oil, apply the energy equation to the CV shown, between sections ② and ③

$$\left(\frac{p}{\rho} + \alpha\frac{\bar{V}^2}{2} + gz\right)_{discharge} - \left(\frac{p}{\rho} + \alpha\frac{\bar{V}^2}{2} + gz\right)_{suction} = \frac{\dot{W}_{pump}}{\dot{m}} = \Delta h_{pump} \qquad (8.47)$$

Since $\bar{V}$ = constant and elevation change is small, this reduces to

$$\Delta h_{pump} = \frac{p_3 - p_2}{\rho}$$

$$= \left[1670 - (75-14.7)\right]\frac{lbf}{in.^2} \times \frac{ft^3}{(0.925)\,1.94\,slug} \times 144\,\frac{in.^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2}$$

$\Delta h_{pump} = 1.29 \times 10^5\ \frac{ft^2}{s^2}$ or $1.29 \times 10^5\ \frac{ft \cdot lbf}{slug}$ added to the oil

The mass flow rate is

$$\dot{m} = \rho Q = (0.925)\,1.94\,\frac{slug}{ft^3} \times 400{,}000\,\frac{bbl}{day} \times 42\,\frac{gal}{bbl} \times \frac{ft^3}{7.48\,gal} \times \frac{day}{24\,hr} \times \frac{hr}{3600\,s}$$

$\dot{m} = 46.6\ slug/s$

The power added to the oil is

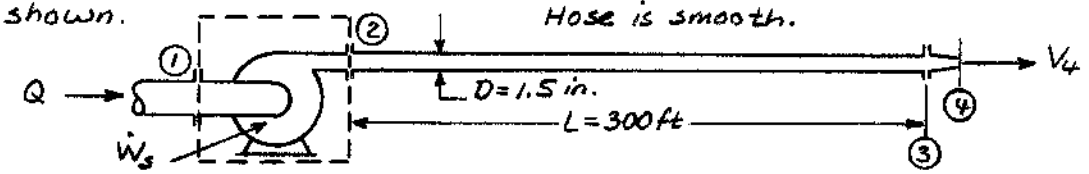
$$\dot{W}_{pump} = \dot{m}\,\Delta h_{pump}$$

$$= 46.6\,\frac{slug}{s} \times 1.29 \times 10^5\,\frac{ft \cdot lbf}{slug} \times \frac{hp \cdot s}{550\,ft \cdot lbf}$$

$\dot{W}_{pump} = 10{,}900\ hp$ ← $\dot{W}_{pu}$

Note pump efficiency does not affect the power that must be added to the oil.

Given: Fire nozzle supplied by booster pump at design conditions as shown. Hose is smooth.



$p_1 = 50$ psig $\qquad \frac{\Delta p}{L} = \frac{33 \text{ psig}}{100 \text{ ft}} \qquad p_3 = 100$ psig

Find: (a) Design flow rate.
(b) Exit velocity, assuming no losses in nozzle.
(c) Power to pump if $\eta = 0.70$.

Solution: Apply the energy equation to fully developed flow in the hose, the Bernoulli equation through the nozzle and the control volume form to the pump.

Basic equations: $\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g\cancel{z_2}^{(3)} = \frac{p_3}{\rho} + \alpha_3 \frac{\bar{V}_3^2}{2} + g\cancel{z_3}^{(3)} + h_{\ell_{T_{23}}}$

$$\frac{p_3}{\rho} + \frac{V_3^2}{2} + g\cancel{z_3}^{(6)} = \frac{p_4}{\rho} + \frac{V_4^2}{2} + g\cancel{z_4}^{(6)}$$

(Eq. 8.47) $$\left(\frac{p}{\rho} + \alpha\frac{\bar{V}^2}{2} + gz\right)_{\text{discharge}} - \left(\frac{p}{\rho} + \alpha\frac{\bar{V}^2}{2} + gz\right)_{\text{suction}} = \frac{\dot{W}_{\text{pump}}}{\dot{m}} = \Delta h_{\text{pump}}$$

Assume water is at $T = 60°F$. Then $\nu = 1.21 \times 10^{-5}$ ft²/s (Table A.7)

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Fully developed flow from ② to ③
(4) Flow along a streamline in nozzle
(5) No friction in nozzle
(6) Neglect elevation changes
(7) Uniform flow at each section across CV
(8) Neglect kinetic energy change across CV

For the hose,

$$\frac{\Delta p}{\rho} = \frac{p_2 - p_3}{\rho} = f\frac{L}{D}\frac{\bar{V}^2}{2} \quad \text{or} \quad \bar{V} = \left[\frac{2\Delta p D}{\rho f L}\right]^{\frac{1}{2}} ; \quad \Delta p = 99 \text{ psi}$$

Since Re is not known, f is not known. Guess $f \simeq 0.015$. Then

$$\bar{V} = \left[2 \times 99 \frac{\text{lbf}}{\text{in.}^2} \times 1.5 \text{ in.} \times \frac{\text{ft}^3}{1.94 \text{ slug}} \times \frac{1}{0.015} \times \frac{1}{300 \text{ ft}} \times 12\frac{\text{in.}}{\text{ft}} \times \frac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2}\right]^{\frac{1}{2}} = 20.2 \text{ ft/s}$$

$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu} = 20.2\frac{\text{ft}}{\text{sec}} \times \frac{1.5}{12} \text{ ft} \times \frac{1}{1.21 \times 10^{-5} \text{ ft}^2} = 2.09 \times 10^5$$

Checking Fig. 8.13 gives $f \simeq 0.016$. Then

$$\bar{V} = \sqrt{\frac{0.015}{0.016}}\ 20.2 \text{ ft/s} = 19.6 \text{ ft/s}, \quad \text{and} \quad Re = \frac{19.6}{20.2} \times 2.09 \times 10^5 = 2.03 \times 10^5$$

This is satisfactory convergence.

$$Q = \bar{V}A = 19.6\ \frac{ft}{sec} \times \frac{\pi}{4}\left(\frac{1.5}{12}\right)^2 ft^2 \times 7.48\ \frac{gal}{ft^3} \times 60\ \frac{sec}{min} = 108\ gpm$$ Q

For the nozzle,

$$\frac{V_4^2}{2} = \frac{p_3 - p_4}{\rho} + \frac{V_3^2}{2} \quad \text{or} \quad V_4 = \sqrt{\frac{2(p_3 - p_4)}{\rho} + V_3^2}$$

Thus

$$V_4 = \left[2 \times 100\ \frac{lbf}{in.^2} \times \frac{ft^3}{1.94\ slug} \times 144\ \frac{in.^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot sec^2} + (19.6)^2\ \frac{ft^2}{sec^2}\right]^{\frac{1}{2}} = 124\ ft/sec$$ V_4

Applying Eq. 8.47 across the pump,

$$\Delta h_{pump} = \frac{p_2 - p_1}{\rho}$$

(This is the head added to the fluid.)

$p_1 = 50$ psig

$p_2 = p_3 + 3 \times 33$ psi

$p_2 = 100 + 3(33) = 199$ psig

The theoretical power input to the pump is $\dot{W}_{pump} = \dot{m}\,\Delta h_{pump}$

The actual power input to the pump is $\mathcal{P}_{act} = \dot{W}_{pump}/\eta = \dot{m}\,\frac{\Delta h_{pump}}{\eta}$

Thus $\mathcal{P}_{act} = \frac{\dot{m}}{\eta}\frac{(p_2 - p_1)}{\rho} = \frac{Q(p_2 - p_1)}{\eta}$

$$= 108\ \frac{gal}{min} \times (199 - 50)\ \frac{lbf}{in.^2} \times \frac{1}{0.7} \times \frac{ft^3}{7.48\ gal} \times 144\ \frac{in.^2}{ft^2} \times \frac{min}{60\ s} \times \frac{hp \cdot s}{550\ ft \cdot lbf}$$

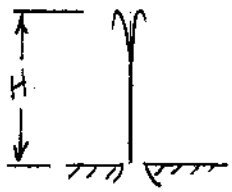
$\mathcal{P}_{act} = 13.4$ hp $\mathcal{P}_{act}$

Given: Fountain on Purdue's Engineering Mall has

$Q = 550$ gpm and $H = 10$ m (32.8 ft)

Find: Estimate of annual cost to operate the fountain.

Solution: Model fountain as a vertical jet (this will give maximum cost).



Computing equations:

$$C(\$/yr) = C_e\left(\frac{\$}{kW\cdot hr}\right) \times \mathcal{P}_{motor}(kW) \times N(hr/yr)$$

Assume $C_e = \$0.067/kW\cdot hr$

$$\mathcal{P}_{motor} = \frac{\mathcal{P}_{hydraulic}}{\eta_{pump}\eta_{motor}} \; ; \; \eta_{motor} = 0.9, \eta_{pump} = 0.8$$

$$\mathcal{P}_{hydraulic} = Q\Delta p$$

$$N = 365\,\frac{days}{yr} \times 24\,\frac{hr}{day} = 8{,}760\ hr/yr$$

The minimum required Δp is $\rho g H$, so

$$\Delta p = 1.94\,\frac{slug}{ft^3} \times 32.2\,\frac{ft}{s^2} \times 32.8\ ft \times \frac{lbf\cdot s^2}{slug\cdot ft} = 2.05\times10^3\ lbf/ft^2$$

Combining,

$$C = \frac{\$0.067}{kW\cdot hr} \times \frac{1}{0.8(0.9)} \times 550\,\frac{gal}{min} \times 2.05\times10^3\,\frac{lbf}{ft^2} \times 8{,}760\,\frac{hr}{yr}$$

$$\times \frac{ft^3}{7.48\ gal} \times \frac{hp\cdot min}{33{,}000\ ft\cdot lbf} \times 0.746\,\frac{kW}{hp}$$

$$C = \$2{,}780/yr$$

C

The fountain does not operate year-round. It might be more fair to say $C \approx \$8$ per day of operation.

Given: Petroleum products transported long distances by pipeline, e.g., the Alaskan pipeline (see Example Problem 8.6).

Find: (a) Estimate of energy needed to pump typical petroleum product, expressed as a fraction of throughput energy carried by pipeline.

(b) Statement and critical assessment of assumptions.

Solution: From Example Problem 8.6, for the Alaskan pipeline, $Q = 1.6 \times 10^6$ bpd.

Thus $$Q = 1.6\times10^6 \frac{bbl}{day} \times 42 \frac{gal}{bbl} \times \frac{ft^3}{7.48\ gal} \times \frac{day}{24\ hr} \times \frac{hr}{3600\ s} = 104\ ft^3/s$$

and $$\dot m = \rho Q = SG\,\rho_{H_2O} Q = 0.93 \times 1.94 \frac{slug}{ft^3} \times 104 \frac{ft^3}{s} = 188\ slug/s$$

The energy content of a typical petroleum product is about 18,000 Btu/lbm, so the throughput energy is

$$\dot E = e\dot m = 18{,}000 \frac{Btu}{lbm} \times 188 \frac{slug}{s} \times 32.2 \frac{lbm}{slug} = 1.09\times10^8\ Btu/s \qquad \dot E$$

From Example Problem 8.6, each pumping station requires 36,800 hp, and they are located $L = 120$ mi apart.

The entire pipeline is about 750 mi long. Thus there must be $N = 750/120$ or about $N = 7$ pumping stations. Thus the total energy required to pump must be

$$\mathcal{P} = N\dot W = 7\ stations \times 36{,}800 \frac{hp}{station} = 258{,}000\ hp \qquad \mathcal{P}$$

Expressed as a fraction of throughput energy

$$\frac{\mathcal{P}}{\dot E} = 258{,}000\ hp \times \frac{s}{1.09\times10^8\ Btu} \times 2545 \frac{Btu}{hp\cdot hr} \times \frac{hr}{3600\ s} = 1.67\times10^{-3} \text{ or } 0.00167$$

Thus about 0.167% of energy is used for transporting petroleum.

The assumptions outlined above appear reasonable. The computed result is probably accurate within ±10%.

A more universal metric would be energy per unit mass and distance, e.g., energy per ton-mile of transport.

$$\frac{E}{M\cdot L} = \frac{E/t}{m/t\cdot L} = \frac{\mathcal{P}}{\dot m L} = 36{,}800\ hp \times \frac{s}{188\ slug} \times \frac{1}{120\ mi} \times 2545 \frac{Btu}{hp\cdot hr} \times \frac{slug}{32.2\ lbm} \times 2000 \frac{lbm}{ton} \times \frac{hr}{3600\ s}$$

Thus $$e = \frac{\mathcal{P}}{\dot m L} = 71.6\ Btu/ton\cdot mi \qquad e$$

This specific metric allows direct comparison with other modes of transport.

Given: Fan with outlet dimensions of 8×16 in. Head vs Capacity curve is approximately

$$H(\text{in. } H_2O) = 30 - 10^{-3}\,[Q\,(\text{ft}^3/\text{min})]^2$$

Find: Air flow rate delivered into a 200 ft. length of straight 8×16 in. duct

Solution:

Basic equation: $\left(\frac{p_1}{\rho g} + \alpha_1\frac{\bar{V}_1^2}{2g} + z_1\right) - \left(\frac{p_2}{\rho g} + \alpha_2\frac{\bar{V}_2^2}{2g} + z_2\right) = H_{L_T}$ (8.30)

$$H_{L_T} = f\frac{l}{D_h}\frac{\bar{V}^2}{2g} + h_{l_m} \quad ; \quad D_h = \frac{4A}{P_w}$$

Assumptions: (1) $\bar{V}_1 = \bar{V}_2$, $\alpha_1 = \alpha_2 \approx 1$
(2) $z_1 = z_2$
(3) $h_{l_m} = 0$



$$A = ab = \frac{8}{12}\text{ ft} \cdot \frac{16}{12}\text{ ft} = 0.889\text{ ft}^2$$

$$D_h = \frac{4A}{P_w} = \frac{4A}{2(a+b)} = \frac{2\times 0.889\text{ ft}^2}{(2/3 + 4/3)\text{ ft}} = 0.889\text{ ft}$$

From Eq. 8.30 $\Delta p = f\frac{l}{D_h}\rho_{air}\frac{\bar{V}^2}{2} = f\frac{l}{D_h}\frac{\rho}{2}\frac{Q^2}{A^2} = \gamma_{H_2O} H_{duct}$

where H_{duct} is the pressure drop in head of water

$$H_{duct} = \frac{f L \rho_{air} Q^2}{2\gamma_{H_2O} D_h A^2} = \frac{f}{2}\cdot\frac{200\text{ft}}{0.889\text{ft}}\times 0.00238\frac{\text{slug}}{\text{ft}^3}\times 62.4\frac{\text{lbf}}{\text{ft}^3}\times\frac{1}{(0.889)^2\text{ft}^4}\times\left[Q\frac{\text{ft}^3}{\text{min}}\times\frac{1}{60}\frac{\text{min}}{\text{s}}\right]^2\times\frac{12\text{ in}}{1\text{ ft}}$$

$$H_{duct} = 1.81\times 10^{-5} f Q^2 \quad (\text{where } H \text{ is in in. } H_2O) \quad \text{.....(1)}$$

For a smooth duct, $f = f(Re)$

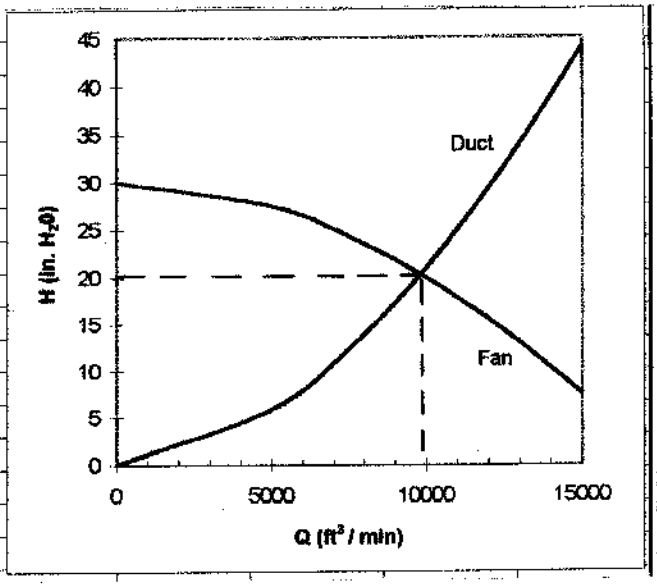
$$Re = \frac{\bar{V} D_h}{\nu} = \frac{D_h Q}{\nu A} \quad \text{For } T = 68^\circ F, \text{ from Table A.9, } \nu = 1.62\times 10^{-4}\text{ ft}^2/\text{s}$$

$$Re = \frac{0.889\text{ ft}}{0.889\text{ft}^2}\times 1.62\times 10^{-4}\frac{\text{s}}{\text{ft}^2}\times Q\frac{\text{ft}^3}{\text{min}}\times\frac{\text{min}}{60\text{s}} = 103\,Q$$

To determine the air flow rate delivered we need to determine the operating point of the fan.
The operating point is at the intersection of the
- fan head capacity curve, and the
- system curve (the head loss in the duct)

This is shown on the plot below.
Note that the friction factor f is determined from the Colebrook equation (8.37a) using Eq. 8.37b for the initial estimate of f.

Q (ft^3/min)	Re (-)	H (fan) ($in.H_2O$)	f_o (-)	f (-)	H (duct) ($in.H_2O$)
0		30			0
5,000	5.15E+05	27.5	0.0130	0.0131	5.9
7,500	7.73E+05	24.4	0.0121	0.0122	12.4
10,000	1.03E+06	20.0	0.0115	0.0116	21.0
12,500	1.29E+06	14.4	0.0111	0.0112	31.6
15,000	1.55E+06	7.5	0.0108	0.0108	44.1



Operating point

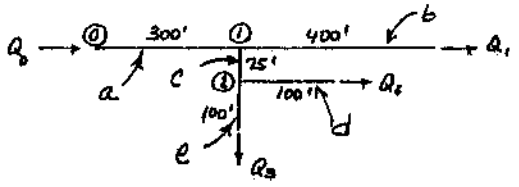
$Q = 10{,}000\ ft^3/min$

$H = 20\ in.H_2O$

Given: Piping system made from 3 in. galvanized iron pipe.

All flows in fully-rough zone.

Neglect minor losses.



Find: (a) p_2 in terms of Q_2

(b) Unknown flow rates as fractions of Q_0.

Solution: Apply energy equation for steady, incompressible pipe flow.

Computing equation: $\left(\frac{p_i}{\rho} + \alpha_i \frac{\bar{V}_i^2}{2} + g z_i\right) - \left(\frac{p_j}{\rho} + \alpha_j \frac{\bar{V}_j^2}{2} + g z_j\right) = h_{ei} + h_{mi} \simeq 0$ (3)

Assumptions: (1) $\bar{V}_i = \bar{V}_j$ in each leg $\quad h_{eij} = f \frac{L}{D} \frac{\bar{V}^2}{2}$

(2) Horizontal, so $z_i = z_j$

(3) Neglect minor losses

(4) Flow in fully rough zone

$e = 0.0005$ ft (Table 8.1), $\frac{e}{D} = 0.002$, and $f \simeq 0.0235$ (Fig. 8.13). From continuity

$$Q = \bar{V}A = \bar{V}\pi\frac{D^2}{4} \quad \text{so} \quad \bar{V} = \frac{4Q}{\pi D^2}$$

Substituting

$$\Delta p_{ij} = \rho f \frac{L_{ij}}{D} \frac{1}{2}\left(\frac{4Q_{ij}}{\pi D}\right)^2 = \frac{8 \rho f L_{ij}}{\pi^2 D^5} Q_{ij}^2$$

Evaluating for water in section d:

$$\Delta p_{ij} = \frac{8}{\pi^2} \times 1.94 \frac{\text{slug}}{\text{ft}^3} \times 0.0235 \times 100 \text{ ft} \times \left(\frac{12}{3}\right)^5 \text{ft}^{-5} Q_d^2 \frac{\text{ft}^6}{\text{s}^2} \times \frac{16 \text{ f} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} \times \frac{\text{ft}^2}{144 \text{ in.}^2}$$

Thus

$$p_2 \text{ (psig)} = 26.3 [Q_2 (\text{ft}^3/\text{s})]^2 \qquad p_2$$

Since legs d and e are the same length, then $Q_2 = Q_3$. Summing flows at point (1): $Q_0 = Q_1 + Q_2 + Q_3 = Q_1 + 2Q_2$.

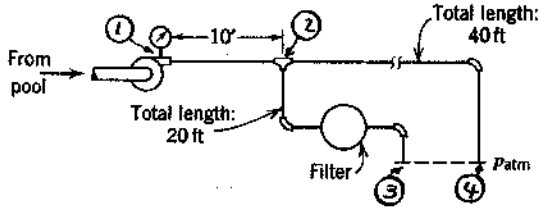
$$p_1 = p_2 + \frac{75}{100}(26.3)(2Q_2)^2 = p_2 + 78.9 Q_2^2 = (26.3 + 78.9) Q_2^2$$

$$p_1 = 105 Q_2^2$$

For flow in leg b, $p_1 = \left(\frac{400}{100}\right) 26.3 Q_1^2 = 105 Q_1^2$

Therefore $Q_1 = Q_2 = Q_3 = \frac{1}{3} Q_0 \qquad Q_i$

Given: Partial-flow filtration system:



Pipes are 3/4 in. nominal PVC (smooth plastic) with $D = 0.824$ in.

Pump delivers 30 gpm at 75°F.

Filter pressure drop is $\Delta p(\text{psi}) = 0.6[Q(\text{gpm})]^2$.

Find: (a) Pressure at pump outlet.
(b) Flow rate through each branch of system.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2 + h_{eT}$; $h_{eT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2}$

Assumptions: (1) $\alpha_1\bar{V}_1^2 = \alpha_2\bar{V}_2^2$; (2) $z_1 = z_2$; (3) $h_{em} = 0$ for $1 \to 2$, (4) Ignore "tee" at ②

The flow rate is $Q_{12} = 30$ gpm $(0.0668\ \text{ft}^3/\text{sec})$, so $\bar{V} = \frac{Q}{A} = 18.0$ ft/sec. Then

$$Re = \frac{\bar{V}D}{\nu} = 18.0\frac{\text{ft}}{\text{sec}} \times \left(\frac{0.824}{12}\right)\text{ft} \times \frac{\text{sec}}{1.0\times10^{-5}\,\text{ft}^2} = 1.24\times10^5, \text{ so } f = 0.017$$

$$\Delta p_{12} = f\frac{L}{D}\rho\frac{\bar{V}^2}{2} = 0.017 \times \frac{10\text{ ft}}{0.824\text{ in.}} \times \frac{1}{2} \times 1.94\frac{\text{slug}}{\text{ft}^3} \times (18.0)^2\frac{\text{ft}^2}{\text{sec}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{ft}}{12\text{ in.}} = 5.40\text{ psi}$$

Branch flow rates are unknown, but flow split must produce the same drop in each branch. Solve by iteration to obtain

$Q_{23} = 5.2$ gpm; $\bar{V}_{23} = 3.12$ ft/s; $Re = 2.15\times10^4$, and $f = 0.025$

$$\Delta p_{23} = f\left(\frac{L}{D} + 2\frac{L_e}{D}\right)\rho\frac{\bar{V}^2}{2} + 0.6Q^2$$

$$\Delta p_{23} = 0.025\left[\frac{240}{0.824} + 2(30)\right]\frac{1}{2} \times 1.94\frac{\text{slug}}{\text{ft}^3} \times (3.12)^2\frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{ft}^2}{144\text{ in.}^2} + 0.6(5.2)^2\frac{\text{lbf}}{\text{in.}^2} = 16.8\text{ psi}$$

$Q_{24} = 24.8$ gpm; $\bar{V}_{24} = 14.9$ ft/s; $Re = 1.03\times10^5$, and $f = 0.018$

$$\Delta p_{24} = f\left(\frac{L}{D} + \frac{L_e}{D}\right)\rho\frac{\bar{V}^2}{2} = 0.018\left(\frac{480}{0.824} + 30\right)\frac{1}{2} \times 1.94\frac{\text{slug}}{\text{ft}^3} \times (14.9)^2\frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{ft}^2}{144\text{ in.}^2} = 16.5\text{ psi}$$

The pump outlet pressure is

$$\Delta p_{pump} = \Delta p_{12} + \Delta p_{23} = (5.4 + 16.8)\text{psi} = 22.2\text{ psi}$$ Δp

The branch flow rates are

$Q_{23} \approx 5.2$ gpm Q_{23}

$Q_{24} \approx 24.8$ gpm Q_{24}

Problem 8.148

Open-Ended Problem Statement: Why does the shower temperature change when a toilet is flushed? Sketch pressure curves for the hot and cold water supply systems to explain what happens.

Discussion: Assume the pressure in the water main servicing the dwelling remains constant. The hot and cold water flow rates reaching the shower are controlled by valve(s) in the shower. Assuming a water heater temperature of 140°F, a cold water temperature of 60°F, and a shower water temperature of 100°F, the hot and cold flow rates must be equal. The two water streams mix before reaching the shower head, then spray out into the shower itself at 100°F.

Supply curves and system curves for the hot and cold water streams are shown below. Diagram *a* is the cold water system and diagram *b* is the hot water system. The numerical values are representative of an actual system.

In general the supply curves for the hot and cold streams are not the same. The difference is caused by the two systems having different pipe lengths and different fittings.

Each stream operates at the flow rate where the curves intersect. An equal flow split is accomplished by adjusting the shower valves to vary their resistances.

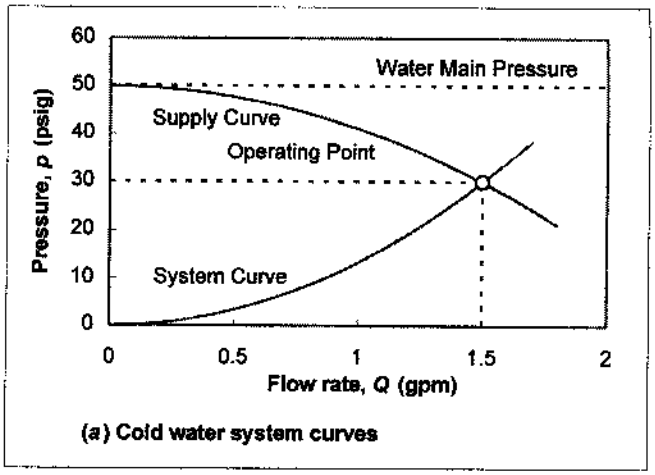
Flushing the toilet temporarily increases the flow rate of cold water to the bathroom. This reduces the cold water supply pressure reaching the shower. The system curves do not change because the valve settings stay the same. Therefore the flow rate of cold water must decrease to again match the supply and system curves (diagram *c*).

When the flow rate of cold water decreases the shower temperature increases, as experience testifies!

13-782 500 SHEETS, FILLER 9 SQUARE
42-381 500 SHEETS, EYE EASE 5 SQUARE
42-382 100 SHEETS EYE EASE 5 SQUARE
42-389 200 SHEETS EYE EASE 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-389 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

(*a*) Cold water system:

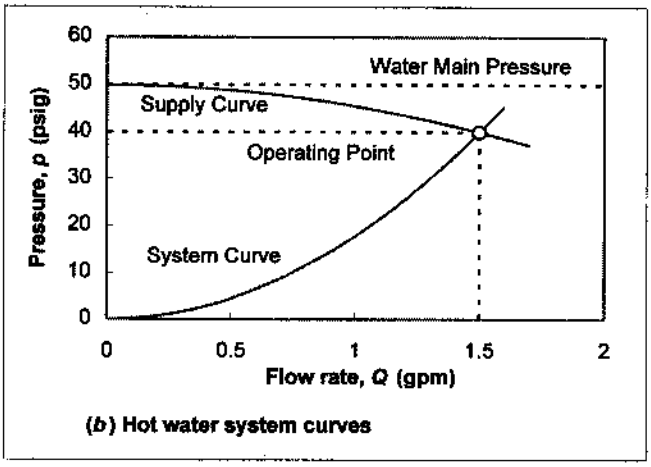
Q (gpm)	System Curve *p* (psig)	Supply Curve *p* (psig)
0	0.00	50.0
0.2	0.53	49.6
0.4	2.13	48.6
0.6	4.80	46.8
0.8	8.53	44.3
1.0	13.3	41.1
1.2	19.2	37.2
1.4	26.1	32.6
1.6	34.1	27.2
1.7	38.5	24.3
1.8		21.2



(*a*) Cold water system curves

(*b*) Hot water system:

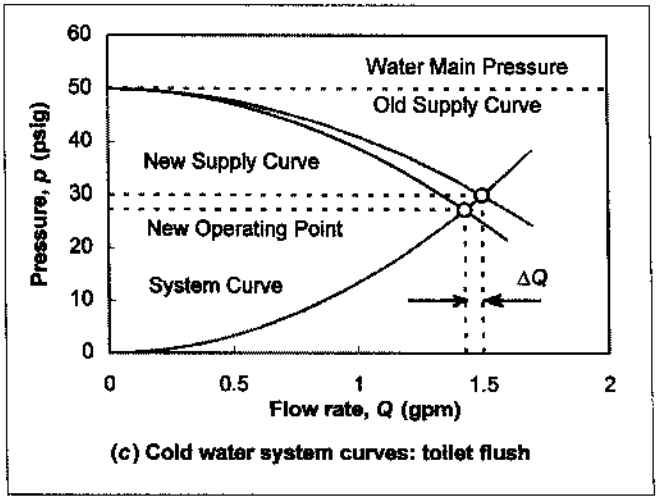
Q (gpm)	System Curve *p* (psig)	Supply Curve *p* (psig)
0	0.00	50.0
0.2	0.71	49.8
0.4	2.84	49.3
0.6	6.40	48.4
0.8	11.38	47.2
1.0	17.78	45.6
1.2	25.60	43.6
1.4	34.84	41.3
1.6	45.51	38.6
1.7		37.2
1.8		



(*b*) Hot water system curves

(*c*) Cold water system: toilet flush

Q (gpm)	System Curve *p* (psig)	Old Supply Curve *p* (psig)	New Supply Curve *p* (psig)
0	0.00	50.0	50.0
0.2	0.53	49.6	49.6
0.4	2.13	48.6	48.2
0.6	4.80	46.8	46.0
0.8	8.53	44.3	42.9
1.0	13.3	41.1	38.9
1.2	19.2	37.2	34.0
1.4	26.1	32.6	28.2
1.430	27.27	31.8	27.28
1.6	34.1	27.2	21.6
1.7	38.5	24.3	
1.8			



(*c*) Cold water system curves: toilet flush

Given: Water flow at 300 gpm (150°F) through a 3 in. diameter orifice installed in a 6 in. i.d. pipe.

Find: Pressure drop across corner taps.

Solution: Apply analysis of Section 8-10; data from Fig. 8.21 apply.

Computing equation:

$$\dot{m}_{actual} = KA_t\sqrt{2\rho(p_1 - p_2)} \qquad (8.51)$$

Flow coefficient is $K = K(Re_{D_1}, \frac{D_t}{D_1})$. At 150°F, $\nu = 4.69 \times 10^{-6}\ ft^2/s$ (Table A.7). Thus,

$$\bar{V}_1 = \frac{Q}{A} = 300\frac{gal}{min} \times \frac{4}{\pi(0.5)^2 ft^2} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} = 3.40\ ft/s$$

$$Re_{D_1} = \frac{\bar{V}D_1}{\nu} = 3.40\frac{ft}{s} \times 0.5\,ft \times \frac{s}{4.69\times10^{-6}\,ft^2} = 3.62 \times 10^5$$

$$\beta = \frac{D_t}{D_1} = \frac{3\ in.}{6\ in.} = 0.5$$

From Fig. 8.21, $K = 0.624$. Then, from Eq. 8.51,

$$\Delta p = \left(\frac{\dot{m}}{KA_t}\right)^2 \frac{1}{2\rho} = \left(\frac{\rho Q}{KA_t}\right)^2 \frac{1}{2\rho} = \frac{\rho}{2}\left(\frac{Q}{KA_t}\right)^2$$

$$= \frac{1}{2} \times 1.94\frac{slug}{ft^3}\left[300\frac{gal}{min} \times \frac{1}{0.624} \times \frac{4}{\pi(0.25)^2 ft^2} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s}\right]^2 \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$\Delta p = 462\ lbf/ft^2\ (3.21\ psi) \qquad \Delta p$$

Given: Square-edged orifice, $d_t = 4$ in., used to meter air flow in a 6 in. I.D. line. The pressure upstream of the orifice is $p_1 = 90.0$ psig. The pressure drop across the orifice is $\Delta p = 30$ in H_2O. The air temperature is 80°F.

Find: The volume flow rate of air in the line.

Solution: Apply analysis of section 8-10; data from Fig. 8.23 apply

Computing equation: $\dot{m}_{actual} = K A_t \sqrt{2\rho(p_1 - p_2)}$ (8.54).

Since $\dot{m} = \rho Q$, then

$$Q = \frac{\dot{m}}{\rho} = K A_t \sqrt{\frac{2(p_1 - p_2)}{\rho}}$$

$$p_1 - p_2 = 30 \text{ in } H_2O = \rho g \Big|_{H_2O} = 1.94 \frac{\text{slug}}{\text{ft}^3} \times 32.2 \frac{\text{ft}}{\text{s}^2} \times 30 \text{ in} \times \frac{\text{ft}}{12 \text{ in}} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{ft} \cdot \text{slug}} = 156 \frac{\text{lb}}{\text{ft}^2}$$

For this small Δp, the assumption of incompressible flow is certainly valid

$$\rho = \frac{p_1}{RT} = 104.7 \frac{\text{lbf}}{\text{in}^2} \times \frac{\text{lbm} \cdot \text{R}}{53.33 \text{ ft} \cdot \text{lbf}} \times \frac{1}{540 \text{ R}} \times \frac{144 \text{ in}^2}{\text{ft}^2} \times \frac{\text{slug}}{32.2 \text{ lbm}} = 0.0163 \text{ slug/ft}^3$$

The flow coefficient $K = K\left(Re_{D_1}, \frac{D_t}{D_1}\right)$

Assume $Re > 2 \times 10^5$. For $\beta = \frac{D_t}{D_1} = \frac{2}{3}$, from Fig. 8.21, $K = 0.615$

$$Q = K A_t \sqrt{\frac{2(p_1 - p_2)}{\rho}} = 0.615 \frac{\pi}{4} \left(\frac{4}{12} \text{ ft}\right)^2 \left[2 \times 156 \frac{\text{lbf}}{\text{ft}^2} \times \frac{\text{ft}^3}{0.0163 \text{ slug}} \times \frac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2}\right]^{1/2}$$

$$Q = 8.15 \text{ ft}^3/\text{s} \quad \text{Q}$$

Check Re. At $T = 80°F$, $\mu = 3.86 \times 10^{-7}$ lbf·s/ft² (Table A.9).

$$Re = \frac{\rho V D}{\mu} = \frac{\rho \frac{Q}{A} D}{\mu} = \frac{\rho Q \frac{4}{\pi D^2}}{\mu} = \frac{4 \rho Q}{\pi D \mu}$$

$$Re = \frac{4}{\pi} \times 0.0163 \frac{\text{slug}}{\text{ft}^3} \times 8.15 \frac{\text{ft}^3}{\text{s}} \times \frac{12 \text{ in}}{\text{ft}} \times 4 \text{ in} \times 3.86 \times 10^{-7} \frac{\text{ft}^2}{\text{lbf} \cdot \text{s}} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{ft} \cdot \text{slug}}$$

$Re = 1.31 \times 10^6$ ✓ assumption is valid

Given: Venturi meter with 75 mm throat, installed in 150 mm diameter line carrying water at 25°C. Pressure drop between upstream and throat taps is 300 mm Hg.

Find: Flow rate of water.

Solution: Apply analysis of Section 8-10.3.

Computing equation:

$$\dot{m}_{actual} = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{2\rho(p_1 - p_2)} \qquad (8.52)$$

For $Re_{D_1} > 2\times10^5$, $0.980 < C < 0.995$. Assume $C = 0.99$, then check Re.

$$\beta = \frac{D_t}{D_1} = \frac{75\ mm}{150\ mm} = 0.5$$

$$\Delta p = p_1 - p_2 = \rho_{Hg} g \Delta h = SG \rho_{H_2O} g \Delta h$$

$\dot{m} = \rho Q$, so

$$Q = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{2\, SG\, g\, \Delta h}$$

$$Q = \frac{0.99}{\sqrt{1-(0.5)^4}} \frac{\pi}{4}(0.075)^2 m^2 \sqrt{2 \times 13.6 \times 9.81 \frac{m}{s^2} \times 0.3\ m} = 0.0404\ m^3/s$$

Thus

$$\bar{V} = \frac{Q}{A_1} = 0.0404 \frac{m^3}{s} \times \frac{4}{\pi(0.15)^2 m^2} = 2.29\ m/s$$

$$Re_{D_1} = \frac{\bar{V} D_1}{\nu} = 2.29 \frac{m}{s} \times 0.15\ m \times \frac{s}{8.93\times10^{-7} m^2} = 3.85 \times 10^5 \quad (\nu \text{ from Table A.8})$$

Thus $Re_{D_1} > 2\times10^5$. The volume flow rate of water is

$$Q = 0.0404\ m^3/s$$ Q

Given: Flow of gasoline through a venturi meter.

$$SG = 0.73,\ D_1 = 2.0\ in.,\ D_t = 1.0\ in.,\ \Delta h = 380\ mm\ Hg$$

Find: Volume flow rate of gasoline.

Solution: Apply the analysis of Section 8-10.3.

Computing equations:

$$\dot{m}_{actual} = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{2\rho(p_1 - p_2)} \qquad (8.52)$$

$$C = 0.99 \quad for \quad Re_{D_1} > 2 \times 10^5$$

For the manometer, $\Delta p = \rho_{Hg} g \Delta h = SG_{Hg} \rho_{H_2O} g \Delta h$

Then

$$Q = \frac{\dot{m}}{\rho} = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{\frac{2\Delta p}{\rho}} = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{\frac{2 SG_{Hg} \rho_{H_2O} g \Delta h}{SG_{gas} \rho_{H_2O}}} = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{\frac{2 SG_{Hg} g \Delta h}{SG_{gas}}}$$

$$Q = \frac{0.99}{\sqrt{1-(0.5)^4}} \frac{\pi}{4} (0.0254)^2 m^2 \sqrt{2 \times \frac{13.6}{0.73} \times 9.81 \frac{m}{s^2} \times 0.38\ m} = 0.00611\ m^3/s$$

Now check Reynolds number:

$$\bar{V}_1 = \frac{Q}{A_1} = 0.00611 \frac{m^3}{s} \times \frac{4}{\pi (0.0508)^2 m^2} = 3.01\ m/s$$

Assume viscosity midway between octane and heptane at 20°C. From Fig. A.1,

$$\mu \approx 5.0 \times 10^{-4}\ N \cdot s / m^2$$

$$Re_{D_1} = \frac{\rho \bar{V}_1 D_1}{\mu} = (0.73)\, 1000 \frac{kg}{m^3} \times 3.01 \frac{m}{s} \times 0.0508\ m \times \frac{m^2}{5.0 \times 10^{-4} N \cdot s} \times \frac{N \cdot s^2}{kg \cdot m} = 2.23 \times 10^5$$

Thus assumption that $C = 0.99$ is okay

$$Q = 0.00611\ m^3/s$$

Q

Given: Flow of water through venturi meter.

$D_1 = 2$ in. $D_t = 1$ in. $\Delta p = 20$ psi

Find: Volume flow rate of water.

Solution: Apply the analysis of Section 8-10.3.

Computing equations:

$$\dot{m}_{actual} = \frac{CA_t}{\sqrt{1-\beta^4}}\sqrt{2\rho(p_1-p_2)} \qquad (8.52)$$

$C = 0.99$ for $Re_{D_1} > 2\times10^5$

Then

$$Q = \frac{\dot{m}}{\rho} = \frac{CA_t}{\sqrt{1-\beta^4}}\sqrt{\frac{2\Delta p}{\rho}}$$

$$Q = \frac{0.99}{\sqrt{1-(0.5)^4}}\frac{\pi}{4}\left(\frac{1}{12}\right)^2 ft^2\sqrt{2\times20\,\frac{lbf}{in.^2}\times\frac{ft^3}{1.94\,slug}\times144\,\frac{in.^2}{ft^2}\times\frac{slug\cdot ft}{lbf\cdot s^2}}\times7.48\,\frac{gal}{ft^3}\times60\,\frac{s}{min}$$

$Q = 136$ gal/min Q

The Reynolds number (with ν from Table A.7) is

$$Re_{D_1} = \frac{\bar{V}_1 D_1}{\nu} = \frac{Q}{A}\frac{D_1}{\nu} = \frac{4Q}{\pi D_1^2}\frac{D_1}{\nu} = \frac{4Q}{\pi\nu D_1}$$

$$= \frac{4}{\pi}\times136\,\frac{gal}{min}\times\frac{s}{1.08\times10^{-5}\,ft^2}\times\frac{1}{(2/12)\,ft}\times\frac{ft^3}{7.48\,gal}\times\frac{min}{60\,s}$$

$Re_{D_1} = 2.14\times10^5$

Therefore $C = 0.99$ may be used.

Given: Test of 1.6 ℓ internal combustion engine at 6000 rpm. Meter air with flow nozzle, $\Delta h \le 0.25$ m. Manometer reads to ± 0.5 mm of water.

Find: (a) Flow nozzle diameter required.
(b) Minimum rate of air flow that can be measured ± 2 percent.

Solution: Apply computing equation for flow nozzle.

Computing equation: $$\dot{m} = K A_t \sqrt{2\rho(p_1 - p_2)} \qquad (8.54)$$

Assumptions: (1) $K = 0.97$ (Section 8-10.2b.)
(2) $\beta = 0$ (nozzle inlet is from atmosphere)
(3) Four-stroke cycle engine with 100 percent volumetric efficiency (ℓ/rev = displacement/2)
(4) Standard air

Then
$$\dot{m} = \rho Q = 1.23 \frac{kg}{m^3} \times \frac{1.6\,\ell}{2\,rev} \times 6000 \frac{rev}{min} \times \frac{m^3}{1000\,\ell} \times \frac{min}{60\,s} = 0.0984 \text{ kg/s}$$

Solving for A_t,
$$A_t = \frac{\dot{m}}{K\sqrt{2\rho\Delta p}} = \frac{\dot{m}}{K\sqrt{2\rho\rho_{H_2O} g \Delta h}}$$

$$A_t = 0.0984 \frac{kg}{s} \times \frac{1}{0.97}\left[\frac{1}{2} \times \frac{m^3}{1.23\,kg} \times \frac{m^3}{999\,kg} \times \frac{s^2}{9.81\,m} \times \frac{1}{0.25\,m}\right]^{\frac{1}{2}} = 1.31 \times 10^{-3} m^2$$

$$A_t = \frac{\pi D_t^2}{4}\ ;\ D_t = \sqrt{\frac{4A_t}{\pi}} = 40.8 \text{ mm} \qquad D_t$$

The allowable error is ± 2 percent, or ± 0.02. As discussed in Appendix E, the square-root relationship halves the experimental uncertainty. Thus

$$e = \pm 0.02 \text{ when } e_{\Delta h} = \pm 0.04\ ;\ \Delta h_{min} = \frac{\pm 0.5 \text{ mm}}{\pm 0.04} = 12.5 \text{ mm}$$

$$\dot{m}_{min} \simeq \dot{m}\sqrt{\frac{\Delta h_{min}}{\Delta h}} = 0.0984 \frac{kg}{s}\sqrt{\frac{12.5 mm}{250 mm}} = 0.0220 \text{ kg/s} \qquad \dot{m}_{min}$$

The air flow rate could be measured with ± 2 percent accuracy down to about

$$\omega = 6000 \text{ rpm} \frac{0.0220}{0.0984} = 1340 \text{ rpm}$$

with this setup.

Given: Venturi meter with 75mm diameter throat installed in a 150mm diameter line. Upstream air pressure is 400 kPa and the temperature is 20°C.

Find: (a) Maximum mass flow rate for incompressible assumption.
(b) Corresponding pressure drop on mercury manometer.

Solution: Use analysis of Section 8-10.3.

Computing equation: $\dot{m}_{actual} = \frac{CA_t}{\sqrt{1-\beta^4}}\sqrt{2\rho(p_1-p_2)}$ (8.52)

Assumptions: (1) Neglect change in density
(2) Ideal gas

Then

$$\rho = \frac{p}{RT} = \frac{4.0\times10^5 \frac{N}{m^2}}{287 \frac{N\cdot m}{kg\cdot K}\times 293 K} = 4.76 \text{ kg/m}^3$$

For incompressible flow, $V \leq 100$ m/s at the throat. Thus

$$\dot{m} = \rho V_2 A_t = 4.76 \frac{kg}{m^3}\times 100\frac{m}{s}\times\frac{\pi}{4}(0.075)^2 m^2$$

$$\dot{m} = 2.10 \text{ kg/s} \quad \text{(maximum mass flow rate)} \qquad \dot{m}$$

The pressure drop may be calculated by solving Eq. 8.52:

$$\Delta p = p_1 - p_2 = \rho_{Hg} g\,\Delta h = \frac{1}{2\rho}\left(\frac{\dot{m}}{CA_t}\right)^2(1-\beta^4)$$

Thus

$$\Delta h = \frac{\Delta p}{\rho_{Hg} g} = \frac{1}{2\rho\rho_{Hg} g}\left(\frac{\dot{m}}{CA_t}\right)^2(1-\beta^4)$$

For $Re_{D_1} \geq 2\times10^5$, $C = 0.99$ may be used. Substituting,

$$\Delta h = \frac{1}{2}\times\frac{m^3}{4.76 kg}\times\frac{m^3}{(13.6)1000 kg}\times\frac{s^2}{9.81 m}\left[2.10\frac{kg}{s}\times\frac{1}{0.99}\times\frac{4}{\pi(0.075)^2 m^2}\right]^2\left[1-(0.5)^4\right]$$

$$\Delta h = 0.170 \text{ m} \quad (170 \text{ mm Hg}) \qquad \Delta h$$

Checking the Reynolds number ($\mu = 1.81\times10^{-5}$ N·s/m², Table A.10

$$Re_{D_1} = \frac{\bar{V}D_1}{\nu} = \frac{\rho\bar{V}D_1}{\mu} = \rho\bar{V}\frac{\pi D_1^2}{4}\frac{4}{\pi\mu D_1} = \frac{4\dot{m}}{\pi\mu D_1}$$

$$Re_{D_1} = \frac{4}{\pi}\times 2.10\frac{kg}{s}\times\frac{m^2}{1.81\times10^{-5} N\cdot s}\times\frac{1}{0.15 m}\times\frac{N\cdot s^2}{kg\cdot m} = 9.85\times10^5 > 2\times10^5$$

Therefore use of $C = 0.99$ is appropriate.

Given: Flow of kerosine at 40°C through a 0.3 m diameter line in a refinery. Flow rate is expected to be ≤ 120 kg/sec. Manometer with range of 1 m of water is available for use with an orifice meter.

Find: (a) Recommended orifice size.
(b) Minimum flow rate measurable with 10 percent accuracy if least count is 1 mm.

Solution: Assume taps are located so data from Fig. 8.21 apply.

Computing equation: $\dot{m} = K A_t \sqrt{2\rho(p_1 - p_2)}$ (8.54)

$$K = K\left(Re_{D_1}, \frac{D_t}{D_1}\right) \quad \text{(Fig. 8.21)}$$

The problem is to find A_t. First find the Reynolds number:

$$Re_{D_1} = \frac{\rho \bar{V} D_1}{\mu} = \frac{4\dot{m}}{\pi \mu D_1} = \frac{4}{\pi} \times 120\ \frac{kg}{s} \times \frac{m \cdot s}{1.1 \times 10^{-3} kg} \times \frac{1}{0.3\ m} = 4.63 \times 10^5$$

This value is large enough that $K = K(D_t/D_1) = K(\beta)$, but β is not known. Iteration is needed. Solving for Δp

$$\Delta p = \frac{1}{2\rho}\left(\frac{\dot{m}}{K A_t}\right)^2 = \frac{1}{2\rho}\left(\frac{\dot{m}}{K\beta^2 A_1}\right)^2 = \frac{\dot{m}^2}{2\rho A_1^2}\left(\frac{1}{K\beta^2}\right)^2$$

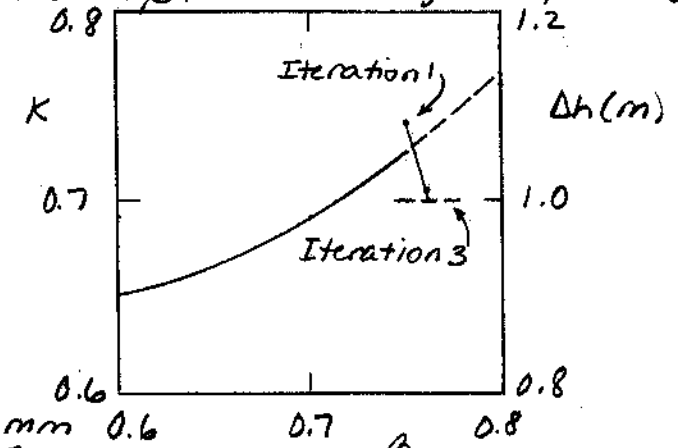
For $\beta = 0.75$, $K = 0.723$ and

$$\Delta p = \frac{1}{2} \times (120)^2 \frac{kg^2}{s^2} \times \frac{m^3}{(0.82)1000\ kg}\left(\frac{4}{\pi(0.3)^2}\right)^2 m^2 \left[\frac{1}{0.723(0.75)^2}\right]^2 \frac{N \cdot s^2}{kg \cdot m}$$

$$\Delta p = 10.6\ kPa \text{ so } \Delta h = \frac{\Delta p}{\rho_w g} = 10.6 \times 10^3 \frac{N}{m^2} \times \frac{m^3}{999\ kg} \times \frac{s^2}{9.81\ m} \times \frac{kg \cdot m}{N \cdot s^2} = 1.08\ m$$

This is too large, so we must increase β. Tabulating and plotting:

Guess β	K	Δh (m)
0.75	0.723	1.08
0.80	0.765	0.748
0.76	0.730	1.01



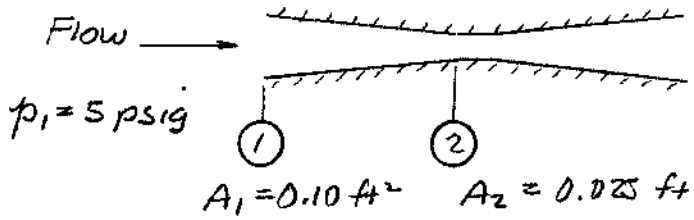
From the plot, the pressure drop is 1 m when $\beta \approx 0.762$.

Thus $D_t = 0.762 D_1 = 0.762 \times 0.3\ m = 229\ mm$ ← D_t

The measurement error is approximately $\pm\frac{1}{2}$ the least count. Thus ±10 percent occurs when $\Delta h = 5$ mm. Since $\dot{m} \sim \sqrt{\Delta h}$,

$\dot{m}_{min} = \dot{m}_{max}[5\ mm/1\ m]^{1/2} = 120\ kg/s \times \sqrt{1/200} = 8.49\ kg/s$ ← $\dot{m}_{min}$

Given: Water at 70°F flows through a Venturi.



Find: Estimate the maximum flow rate with no cavitation. (Express answer in cfs.)

Solution: Apply flowmeter equation.

Computing equation: $\dot{m} = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{2\rho(p_1 - p_2)}$; $\beta^2 = A_t/A_1$

Assume $C = 0.99$ for $Re_{D_1} \geq 2 \times 10^5$

Cavitation occurs when $p_2 \leq p_v$. From Steam table, $p_v = 0.363$ psia at 70 F. Thus

$$p_1 - p_2 = (14.7 + 5.0) - 0.363 = 19.3 \text{ psi}$$

and

$$\dot{m} = 0.99 \times 0.025\ ft^2 \times \frac{1}{\sqrt{1-(0.025/0.1)^2}} \left[2 \times 1.94 \frac{slug}{ft^3} \times 19.3 \frac{lbf}{in.^2} \times 144 \frac{in.^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2} \right]^{1/2}$$

$$\dot{m} = 2.65 \text{ slug/s}$$

But $\dot{m} = \rho \bar{V} A = \rho Q$, so

$$Q = \frac{\dot{m}}{\rho} = 2.65 \frac{slug}{sec} \times \frac{ft^3}{1.94\ slug} = 1.37\ ft^3/s$$ Q

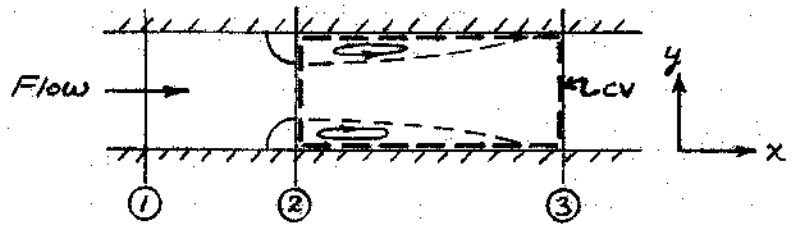
{Note $Q = 1.37 \frac{ft^3}{s} \times 7.48 \frac{gal}{ft^3} \times 60 \frac{s}{min} = 613$ gpm.}

At 70 F, $\nu = 1.05 \times 10^{-5}\ ft^2/s$ (Table A.7). $Re_{D_1} = \frac{\bar{V}_1 D_1}{\nu}$ $A_1 = \frac{\pi D_1^2}{4}$; so

$$D_1 = \sqrt{\frac{4A_1}{\pi}} = \sqrt{\frac{4}{\pi} \times 0.1\ ft^2} = 0.357\ ft\ (4.28\ in.)\ ;\ \bar{V}_1 = \frac{Q}{A_1} = 1.37 \frac{ft^3}{s} \times \frac{1}{0.1\ ft^2} = 13.7\ ft/s$$

Then $Re_{D_1} = 13.7 \frac{ft}{s} \times 0.357\ ft \times \frac{s}{1.05 \times 10^{-5}\ ft^2} = 4.66 \times 10^5$, so $C = 0.99$ is okay. vv

Given: Flow nozzle installation in pipe as shown.



Find: Head loss between sections ① and ③, expressed in coefficient form, $C_L = \frac{p_1 - p_3}{\frac{\rho}{2}V_1^2}$. Show $C_L = \frac{1 - A_2/A_1}{1 + A_2/A_1}$

Plot: C_L vs. D_2/D_1

Solution: Apply the Bernoulli, continuity, momentum and energy equations, using the CV shown.

Basic equations: $\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + g\cancel{z_1}^{(4)} = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + g\cancel{z_2}^{(4)}$

$$\cancel{0}^{=0(1)} = \cancel{\frac{\partial}{\partial t}}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$\cancel{F_{Sx}}^{=0(5)} + \cancel{F_{Bx}}^{=0(6)} = \cancel{\frac{\partial}{\partial t}}^{=0(1)}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

$$\dot{Q} + \cancel{\dot{W}_S}^{=0(6)} = \cancel{\frac{\partial}{\partial t}}^{=0(1)}\int_{CV} e\rho\, d\forall + \int_{CS} \left(u + \frac{\bar{V}^2}{2} + g\cancel{z}^{(4)} + \frac{p}{\rho}\right)\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) No friction between ① and ②
(4) Neglect elevation terms
(5) $F_{Bx} = 0$
(6) $\dot{W}_S = 0$
(7) Uniform flow at each section

From continuity,

$$Q = \bar{V}_1 A_1 = \bar{V}_2 A_2 = \bar{V}_3 A_3$$

Apply Bernoulli along a streamline from ① to ②, noting $A_1 = A_3$,

$$\frac{p_1 - p_2}{\rho} = \frac{\bar{V}_2^2 - \bar{V}_1^2}{2} = \frac{\bar{V}_2^2}{2}\left[1 - \left(\frac{A_2}{A_1}\right)^2\right] = \frac{\bar{V}_2^2}{2}\left[1 - \left(\frac{A_2}{A_3}\right)^2\right]$$

From momentum, and using continuity,

$$F_{Sx} = p_2 A_1 - p_3 A_3 = \bar{V}_2\{-|\rho \bar{V}_2 A_2|\} + \bar{V}_3\{+|\rho \bar{V}_3 A_3|\} = (\bar{V}_3 - \bar{V}_2)\rho \bar{V}_3 A_3$$

or $$\frac{p_3 - p_2}{\rho} = \bar{V}_3(\bar{V}_2 - \bar{V}_3) = \bar{V}_2 \frac{A_2}{A_3}\left[\bar{V}_2 - \bar{V}_2 \frac{A_2}{A_3}\right] = \bar{V}_2^2 \frac{A_2}{A_3}\left(1 - \frac{A_2}{A_3}\right)$$

From energy,

$$\dot{Q} = \left(u_2 + \frac{\bar{V}_2^2}{2} + \frac{p_2}{\rho}\right)\{-|\rho \bar{V}_2 A_2|\} + \left(u_3 + \frac{\bar{V}_3^2}{2} + \frac{p_3}{\rho}\right)\{|\rho \bar{V}_3 A_3|\}$$

or $h_{\ell_{23}} = u_3 - u_2 - \frac{\dot{Q}}{\dot{m}} = \frac{\bar{V}_2^2 - \bar{V}_3^2}{2} - \frac{p_3 - p_2}{\rho} = \frac{\bar{V}_2^2}{2}\left[1 - \left(\frac{A_2}{A_3}\right)^2\right] - \frac{p_3 - p_2}{\rho}$

But $h_{\ell_{12}} \simeq 0$ by assumption (3), so $h_{\ell_{13}} \simeq h_{\ell_{23}}$ and using momentum

$$h_{\ell_{13}} \simeq \frac{\bar{V}_2^2}{2}\left[1 - \left(\frac{A_2}{A_3}\right)^2\right] - \bar{V}_2^2 \frac{A_2}{A_3}\left(1 - \frac{A_2}{A_3}\right)$$

After a little algebra, this may be written

$$h_{\ell_{13}} \simeq \frac{\bar{V}_2^2}{2}\left(1 - \frac{A_2}{A_3}\right)^2$$

Dividing by $(p_1 - p_2)/\rho$, a loss coefficient is derived as

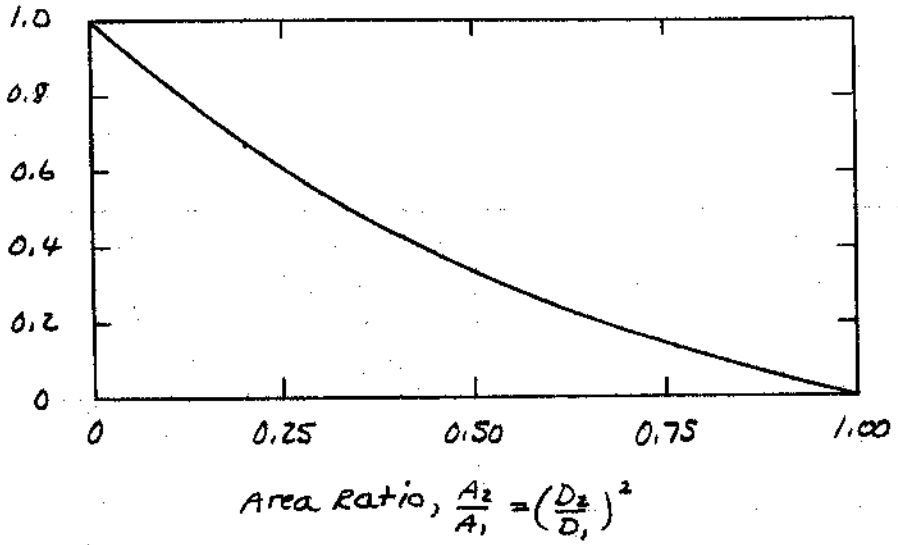
$$C_\ell = \frac{h_{\ell_{13}}}{(p_1 - p_2)/\rho} \simeq \frac{\frac{\bar{V}_2^2}{2}\left(1 - \frac{A_2}{A_3}\right)^2}{\frac{\bar{V}_2^2}{2}\left[1 - \left(\frac{A_2}{A_3}\right)^2\right]} = \frac{\left(1 - A_2/A_3\right)^2}{\left[1 - (A_2/A_3)^2\right]}$$

But $1 - \left(\frac{A_2}{A_3}\right)^2 = \left(1 + \frac{A_2}{A_3}\right)\left(1 - \frac{A_2}{A_3}\right)$, so

$$C_\ell = \frac{h_{\ell_{13}}}{(p_1 - p_2)/\rho} = \frac{1 - A_2/A_3}{1 + A_2/A_3}$$

C_ℓ

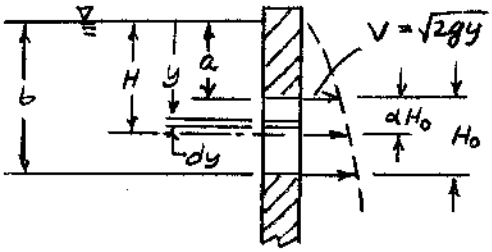
Plotting:



Problem 8.159

Open-Ended Problem Statement: In some western states, water for mining and irrigation was sold by the "miner's inch," the rate at which water flows through an opening in a vertical plank of 1 in.^{2} area, up to 4 in. tall, under a head of 6 to 9 in. Develop an equation to predict the flow rate through such an orifice. Specify clearly the aspect ratio of the opening, thickness of the plank, and datum level for measurement of head (top, bottom, or middle of the opening). Show that the unit of measure varies from 38.4 (in Colorado) to 50 (in Arizona, Idaho, Nevada, and Utah) miner's inches equal to 1 ft^3/s.

Analysis: The geometry of the opening in a vertical plank is shown. The analysis includes the effect on flow speed of the variation in water depth vertically across the opening.



Aspect ratio, $ar = \frac{H_0}{w}$; area, $A = wH_0 = 1\ in.^2$

$\alpha = 0, \frac{1}{2}, 1$

$$Q_{geom} = \int V\,dA = \int_a^b \sqrt{2gy}\; w\,dy = w\sqrt{2g}\left[\tfrac{2}{3}y^{3/2}\right]_a^b = \tfrac{2}{3}wa\sqrt{2ga}\left[\left(\tfrac{b}{a}\right)^{3/2} - 1\right]$$

For $ar = 1$, $\alpha = 0$, $a = H = 9\ in.$, $b = 10.0\ in.$, $w = 1.0\ in.$

$$Q_{geom} = \tfrac{2}{3} \times 1.0\ in. \times 9\ in. \left[2 \times 32.2 \tfrac{ft}{s^2} \times 9\ in. \times \tfrac{ft}{12\ in.}\right]^{\frac{1}{2}} \left[\left(\tfrac{10}{9}\right)^{3/2} - 1\right] \tfrac{ft^2}{144\ in.^2} = 0.0496\ \tfrac{ft^3}{s}$$

$Q_{actual} = 0.6\, Q_{geom} = 0.0297\ ft^3/s$; thus $1/0.0297 = 33.6\ MI = 1\ cfs$ MI

Numerical results are presented in the spread sheet on the next page.

Discussion: All results assume a *vena contracta* in the liquid jet leaving the opening, reducing the effective flow area to 60 percent of the geometric area of the opening.

The calculated unit of measure varies from 31.3 to 52.4 miner's inch per cubic foot of water flow per second. This range encompasses the 38.4 and 50 values given in the problem statement.

Trends may be summarized as follows. The largest flow rate occurs when datum H is measured to the top of the opening in the vertical plank. This gives the deepest submergence and thus the highest flow speeds through the opening.

When $ar = 1$, the opening is square; when $ar = 16$, the opening is 4 inches tall and 1/4 inch wide. Increasing ar from 1 to 16 increases the flow rate through the opening when H is measured to the top of the opening, because it increases the submergence of the lower portion of the opening, thus increasing the flow speeds. When H is measured to the center of the opening ar has almost no effect on flow rate. When H is measured to the bottom of the opening, increasing ar reduces the flow rate. For this case, the depth of the opening decreases as ar becomes larger.

Plank thickness does not affect calculated flow rates since a *vena contracta* is assumed. In this flow model, water separates from the interior edges of the opening in the vertical plank. Only if the plank were several inches thick might the stream reattach and affect the flow rate.

The actual relationship between Q_{flow} and Q_{geom} might be a weak function of aspect ratio. The flow separates from all four edges of the opening in the vertical plank. At large ar, contraction on the narrow ends of the stream has a relatively small effect on flow area. As ar approaches 1 the effect becomes more pronounced, but would need to be measured experimentally. Assuming a constant 60 percent area fraction certainly gives reasonable trends.

Computation of "Miner's Inch" in Engineering Units:

a =	depth to top of opening	(in.)
ar =	aspect ratio of opening	(—)
A =	area of opening	1 in.^{2}
b =	depth to bottom of opening	(in.)
H =	nominal head	(in.)
H_0 =	height of opening	(in.)
MI =	"miner's inch"	(mixed)
Q =	volume flow rate	(ft^3/s)
w =	width of opening	(in.)

Assume $Q_{flow} = 0.6 \times Q_{geometric}$ to account for contraction of the stream leaving the opening.

(a) Measure H to top of opening:

H	ar	H_0	a	b	w	Q_{geom}	Q_{flow}	MI/cfs
9	1	1.00	9.00	10.0	1.00	0.0496	0.0297	33.6
9	2	1.41	9.00	10.4	0.707	0.0501	0.0301	33.3
9	4	2.00	9.00	11.0	0.500	0.0509	0.0305	32.8
9	8	2.83	9.00	11.8	0.354	0.0519	0.0311	32.1
9	16	4.00	9.00	13.0	0.250	0.0533	0.0320	31.3
6	1	1.00	6.00	7.00	1.00	0.0410	0.0246	40.6
6	2	1.41	6.00	7.41	0.707	0.0416	0.0250	40.0
6	4	2.00	6.00	8.00	0.500	0.0425	0.0255	39.2
6	8	2.83	6.00	8.83	0.354	0.0437	0.0262	38.1
6	16	4.00	6.00	10.0	0.250	0.0454	0.0272	36.7

(b) Measure H to middle of opening:

H	ar	H_0	a	b	w	Q_{geom}	Q_{flow}	MI/cfs
9	1	1.00	8.50	9.50	1.00	0.0483	0.0290	34.5
9	2	1.41	8.29	9.71	0.707	0.0483	0.0290	34.5
9	4	2.00	8.00	10.0	0.500	0.0482	0.0289	34.6
9	8	2.83	7.59	10.4	0.354	0.0482	0.0289	34.6
9	16	4.00	7.00	11.0	0.250	0.0482	0.0289	34.6
6	1	1.00	5.50	6.50	1.00	0.0394	0.0236	42.3
6	2	1.41	5.29	6.71	0.707	0.0394	0.0236	42.3
6	4	2.00	5.00	7.00	0.500	0.0394	0.0236	42.3
6	8	2.83	4.59	7.41	0.354	0.0393	0.0236	42.4
6	16	4.00	4.00	8.00	0.250	0.0392	0.0235	42.5

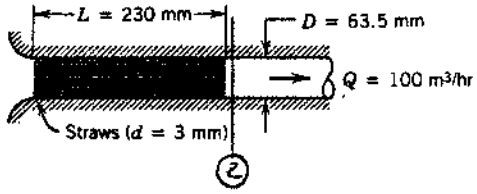
(c) Measure H to bottom of opening:

H	ar	H_0	a	b	w	Q_{geom}	Q_{flow}	MI/cfs
9	1	1.00	8.00	9.00	1.00	0.0469	0.0281	35.5
9	2	1.41	7.59	9.00	0.707	0.0463	0.0278	36.0
9	4	2.00	7.00	9.00	0.500	0.0455	0.0273	36.7
9	8	2.83	6.17	9.00	0.354	0.0442	0.0265	37.7
9	16	4.00	5.00	9.00	0.250	0.0424	0.0254	39.3
6	1	1.00	5.00	6.00	1.00	0.0377	0.0226	44.2
6	2	1.41	4.59	6.00	0.707	0.0370	0.0222	45.1
6	4	2.00	4.00	6.00	0.500	0.0359	0.0215	46.4
6	8	2.83	3.17	6.00	0.354	0.0343	0.0206	48.6
6	16	4.00	2.00	6.00	0.250	0.0318	0.0191	52.4

13-762 500 SHEETS FILLER 5 SQUARE
42-381 60 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-399 200 RECYCLED WHITE 5 SQUARE
Made in U. S. A.
National® Brand

Given: Pipe-flow experiment with flow straightener made from straws.

Find: (a) Reynolds number for flow in each straw. $K_{ent} = 1.4$
(b) Friction factor for flow in each straw. $\alpha = 2.0$
(c) Gage pressure at exit from straws.



Solution: Apply energy equation for steady, incompressible pipe flow.

Computing equation: ≈ 0 (1)

$$\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2 + h_{LT}$$

$$h_{LT} = h_L + h_{em} = f \frac{L}{D} \frac{\bar{V}^2}{2} + K_{ent} \frac{\bar{V}^2}{2} = \left(f \frac{L}{D} + K_{ent} \right) \frac{\bar{V}^2}{2}$$

Assumptions: (1) Flow from atmosphere: $p_1 = p_{atm}$, $\bar{V}_1 \approx 0$
(2) Horizontal
(3) Neglect thickness of straws

Then
$$\bar{V}_2 = \frac{Q}{A} = 100 \frac{m^3}{hr} \times \frac{4}{\pi (0.0635)^2 m^2} \times \frac{hr}{3600 s} = 8.77 \text{ m/s}$$

$$Re_d = \frac{\bar{V}_2 d}{\nu} = 8.77 \frac{m}{sec} \times 0.003 \text{ m} \times \frac{sec}{1.46 \times 10^{-5} m^2} = 1800$$ Re_d

For laminar flow,

$$f = \frac{64}{Re} = \frac{64}{1800} = 0.0356$$ f

The gage pressure at (2) is

$$p_{2g} = -\rho \frac{\bar{V}_2^2}{2} \left(\alpha_2 + K_{ent} + f \frac{L}{D} \right)$$

$$= -\frac{1}{2} \times 1.23 \frac{kg}{m^3} \times (8.77)^2 \frac{m^2}{s^2} \left(2.0 + 1.4 + 0.0356 \times \frac{230 \text{ mm}}{3 \text{ mm}} \right) \frac{N \cdot s^2}{kg \cdot m}$$

$$p_{2g} = -290 \text{ N/m}^2 \text{ (gage)}$$ p_{2g}

This pressure drop is equivalent to

$$\Delta h = \frac{\Delta p}{\rho_{H_2O} g} = 290 \frac{N}{m^2} \times \frac{m^3}{999 kg} \times \frac{s^2}{9.81 m} \times \frac{kg \cdot m}{N \cdot s^2} = 29.6 \text{ mm } H_2O$$

Comments: (1) This pressure drop is large enough to measure readily. The straws could be used as a flowmeter.
(2) Straws would eliminate any swirl from the flow.

Problem 8.161

Given: Volume flow rate in a circular duct is to be measured using a "Pitot traverse," by measuring the velocity in each of several area segments across the duct, then summing.

Find: Comment on the way the traverse should be set up. Quantify and plot the expected error in measurement of flow rate as a function of the number of radial locations used in the traverse.

Solution: First divide the duct cross section into segments of equal area. Then measure velocity at the mean area of each segment.

Assume flow is turbulent, and that the velocity profile is well represented by the 1/7-power profile. From Eq. 8.24 the ratio of average flow velocity to centerline velocity is 0.817.

Distinguish two cases, depending on whether velocity is measured at the centerline.

Case 1: Measure velocity at the duct centerline, plus at (k - 1) other locations.

For $k = 1$, the sole measurement is at the duct centerline. This measures the centerline velocity U, which is 1/0.817 = 1.22 times the average flow velocity $\bar{}$. Thus the volume flow rate estimated by this 1-point measurement is 22 percent larger than the true value.

For $k = 2$, the duct is divided into two segments of equal area. The centerline velocity is measured and assigned the half of the duct area surrounding the centerline. The second measurement point is located at the midpoint of the remaining half of the duct area. Thus this point is located at the radius that encloses 3/4 of the duct area, or $r_2/R = (3/4)^{1/2} = 0.866$, as shown on the attached spreadsheet. The velocity ratio at this point is $\bar{u}/U = 0.92$. Averaging the segmental flow rates gives (1.22 + 0.92)/2 = 1.07. Thus the volume flow rate estimated by this 2-point measurement is 7 percent high.

For $k = 3$, the duct is divided into three portions of equal area. The centerline velocity is measured and assigned the one-third of the duct area surrounding the centerline. The second measurement point is located at the midpoint of the second one-third of the duct area. This point is located at the radius that encloses half the duct area, or at $r_2/R = (1/2)^{1/2} = 0.707$. The third measurement point is located at the midpoint of the third one-third of the duct area. This point is located at the radius enclosing 5/6 of the duct area, or at $r_3/R = (5/6)^{1/2} = 0.913$.

Results of calculations for $k = 4$ and 5 are also given on the spreadsheet.

Case 2: Measure velocity at k locations, not including the centerline.

For $k = 1$, the radius is chosen at half the duct area. Thus $r_1/R = (1/2)^{1/2} = 0.707$, $\bar{u}/U = 0.839$, and $\bar{u}/\bar{} = 1.03$, or about 3 percent too high, as shown on the spreadsheet.

For $k = 2$, the duct is divided into two equal areas. The first measurement is made at the midpoint of the inner area, where the radius includes one fourth of the total area. The second is made at the midpoint of the outer area, where the radius includes three fourths of the total duct area. The results are shown; the flow rate estimate is high by about 1.4 percent.

For $k = 3$, the duct is divided into three equal areas. The first measurement is made at the midpoint of the inner 1/3 of the duct area, where the radius includes 1/6 of the total area. The second is made at the midpoint of the second 1/3 of the duct area, where the radius includes 1/2 of the total duct area. The third is made at the midpoint of the third 1/3 of the duct area, where the radius includes 5/6 of the total duct area. The results are shown; the flow rate estimate is high by about 0.9 percent.

Results of calculations for $k = 4$ and 5 also are given on the spreadsheet.

Remarkably, Case 2 gives less than 2 percent error for any number of locations.

V_{bar}/U = 0.817 n = 7 k = Number of measurement points

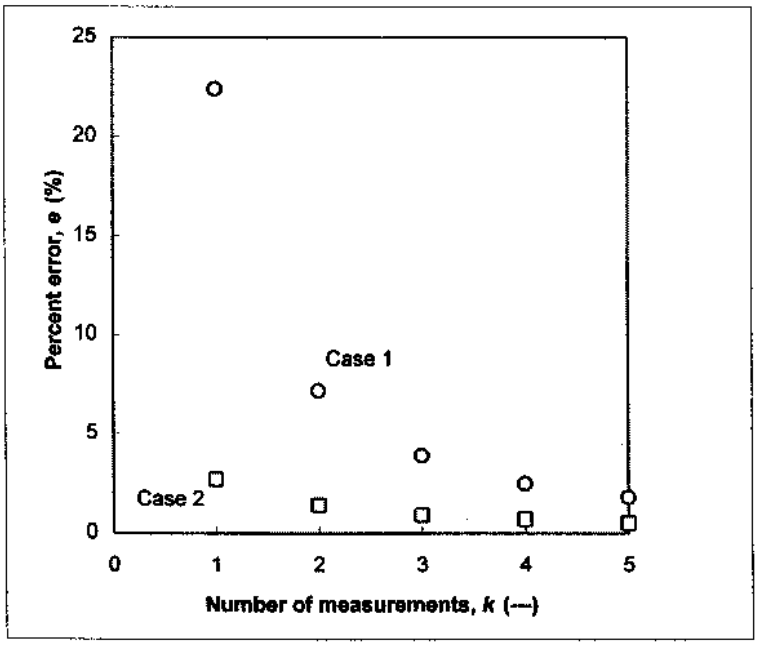
Case 1: Measure at centerline plus at (k - 1) other locations

k	i	r_i/R	u/U	u/V_{bar}	(%) Error
1	1	0.000	1.000	1.22	22.4
2	1	0.000	1.000	1.22	
	2	0.866	0.750	0.92	
				1.07	7.2
3	1	0.000	1.000	1.22	
	2	0.707	0.839	1.03	
	3	0.913	0.706	0.864	
				1.04	3.9
4	1	0.000	1.000	1.22	
	2	0.612	0.873	1.07	
	3	0.791	0.800	0.98	
	4	0.935	0.676	0.828	
				1.03	2.5
5	1	0.000	1.000	1.22	
	2	0.548	0.893	1.09	
	3	0.707	0.839	1.03	
	4	0.837	0.772	0.945	
	5	0.949	0.654	0.801	
				1.02	1.8

Case 2: Measure at k locations not including the centerline

k	i	r_i/R	u/U	u/V_{bar}	(%) Error
1	1	0.707	0.839	1.03	2.7
2	1	0.500	0.906	1.11	
	2	0.866	0.750	0.92	
				1.01	1.4
3	1	0.408	0.928	1.14	
	2	0.707	0.839	1.03	
	3	0.913	0.706	0.86	
				1.01	0.9
4	1	0.354	0.940	1.15	
	2	0.612	0.873	1.07	
	3	0.791	0.800	0.98	
	4	0.935	0.676	0.83	
				1.01	0.7
5	1	0.316	0.947	1.16	
	2	0.548	0.893	1.09	
	3	0.707	0.839	1.03	
	4	0.837	0.772	0.95	
	5	0.949	0.654	0.80	
				1.01	0.5

k	Case 1 e (%)	Case 2 e (%)
1	22.4	2.7
2	7.2	1.4
3	3.9	0.9
4	2.5	0.7
5	1.8	0.5



Given: Model of river towboat to be tested at 1:13.5 scale.
Boat designed to travel at 8 mph in fresh water at 10°C.

Find: (a) Estimate distance from bow where transition occurs.
(b) Where should transition be stimulated on the model?

Solution: From empirical results quoted in Section 9-1, transition occurs at $Re_x \simeq 5 \times 10^5$.

Computing equation: $Re_x = \frac{\rho U x}{\mu} = \frac{U x}{\nu}$

For water at 10°C, $\nu \simeq 1.30 \times 10^{-6}\ m^2/s$ (Table A.8).

$$U = 8\ \frac{mi}{hr} \times 5280\ \frac{ft}{mi} \times 0.3048\ \frac{m}{ft} \times \frac{hr}{3600\ s} = 3.58\ m/s$$

Thus

$$x = \frac{\nu Re}{U} = 1.30 \times 10^{-6}\ \frac{m^2}{s} \times 5 \times 10^5 \times \frac{s}{3.58\ m} = 0.182\ m\ (182\ mm)$$ x

For the model,

$$x_m = \frac{x_p}{13.5} = \frac{182\ mm}{13.5} = 13.5\ mm$$ x_m

Given: Aircraft cruising at 300 knots at 10 km altitude on a standard day. Assume boundary layers on wing surfaces behave as on a flat plate.

Find: Expected extent of laminar boundary-layer flow.

Solution: From empirical results quoted in Section 9-1, transition occurs at $Re_x \simeq 5 \times 10^5$.

Computing equation: $Re_x = \frac{\rho U x}{\mu}$

At 10 km altitude on a standard day, $T = 223\ K$, $\rho/\rho_{SL} = 0.338$ } Table A.3

For air, $\mu = \frac{bT^{3/2}}{S+T}$ (Section A-3.1)

$b = 1.46 \times 10^{-6}\ kg/m \cdot s \cdot K^{1/2}$ and $S = 110\ K$

Thus $\mu = \frac{1.46 \times 10^{-6}\ kg}{m \cdot s \cdot K^{1/2}} \times (223)^{3/2} K^{3/2} \times \frac{1}{(110+223)K} = 1.46 \times 10^{-5}\ kg/m \cdot s$

$U = 300 \frac{nm}{hr} \times 6080 \frac{ft}{nm} \times 0.3048 \frac{m}{ft} \times \frac{hr}{3600\ s} = 154\ m/s$

Then

$$x = \frac{\mu Re}{\rho U} = 1.46 \times 10^{-5} \frac{kg}{m \cdot s} \times 5 \times 10^5 \times \frac{m^3}{(0.338)1.23\ kg} \times \frac{s}{154\ m} = 0.114\ m$$ x

Given: Reynolds number for boundary-layer development on a flat plate,

$$Re_x = \frac{\rho U x}{\mu} = \frac{Ux}{\nu}$$

Find: Prepare a log-log plot of velocity versus distance for water. Show lines for which $Re_x = 5 \times 10^5$ and 1×10^6.

Solution: For water at 20°C, $\nu = 1.00 \times 10^{-6}\ m^2/s$ (Table A.8). Then

$$Ux = \nu Re = \text{constant}$$

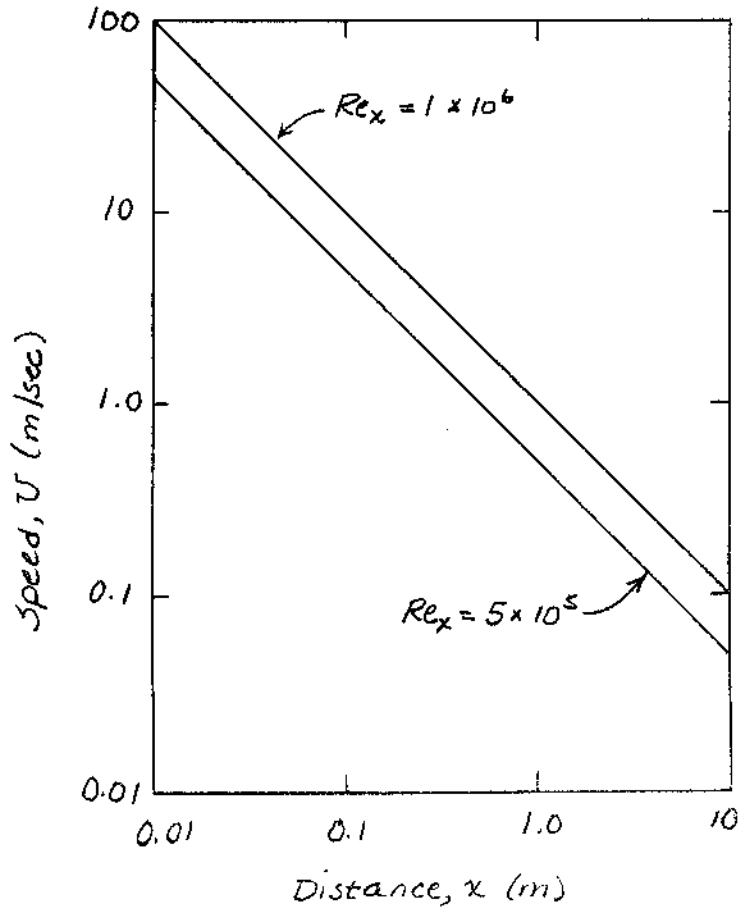
For $Re = 5 \times 10^5$, $U = 1 \times 10^{-6} \frac{m^2}{sec} \times 5 \times 10^5 \times \frac{1}{0.01\ m} = 50\ m/s$ at $x = 0.01\ m$

$U = 1 \times 10^{-6} \frac{m^2}{sec} \times 5 \times 10^5 \times \frac{1}{10\ m} = 0.05\ m/s$ at $x = 10\ m$

For $Re = 1 \times 10^6$, $U = 1 \times 10^{-6} \frac{m^2}{sec} \times 1 \times 10^6 \times \frac{1}{0.01\ m} = 100\ m/s$ at $x = 0.01\ m$

$U = 1 \times 10^{-6} \frac{m^2}{sec} \times 1 \times 10^6 \times \frac{1}{10\ m} = 0.1\ m/s$ at $x = 10\ m$

The plot is



Given: Reynolds number for boundary-layer development on a flat plate,

$$Re_x = \frac{\rho U x}{\mu} = \frac{Ux}{\nu}$$

Find: Prepare a log-log plot of velocity versus distance for standard air. Show lines for $Re_x = 5 \times 10^5$ and 1×10^6 for sea level and 10 km altitude.

Solution: Use Sutherland correlation for μ (Section A-3.1), data from Tables A.3 and A.10

Sea level: $\nu = \frac{\mu}{\rho} = 1.79 \times 10^{-5} \frac{kg}{m \cdot sec} \times \frac{m^3}{1.23\ kg} = 1.46 \times 10^{-5}\ m^2/sec$ (Table A.10)

10 km altitude: $T = 223\ K$, $\rho/\rho_{SL} = 0.338$ (Table A.3)

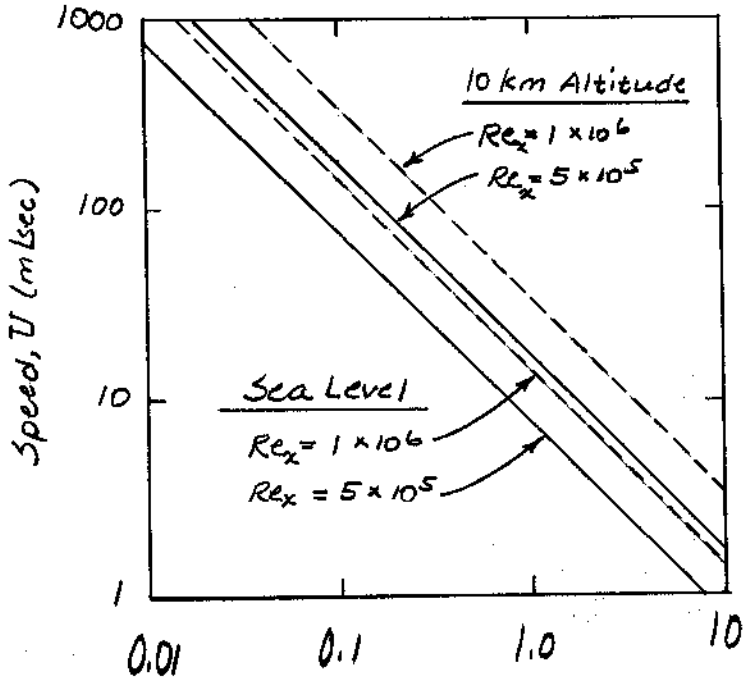
$$\mu = 1.46 \times 10^{-6} \frac{kg}{m \cdot s \cdot K^{1/2}} \times (223)^{0/2} K^{3/2} \times \frac{1}{(110+223)K} = 1.46 \times 10^{-5}\ kg/m \cdot s$$

$$\nu = 1.46 \times 10^{-5} \frac{kg}{m \cdot s} \times \frac{m^3}{(0.338)1.23\ kg} = 3.51 \times 10^{-5}\ m^2/s$$

At sea level, $Ux = \nu Re_x = 1.46 \times 10^{-5} \frac{m^2}{s} \times 5 \times 10^5 = 7.30$ at $Re_x = 5 \times 10^5$

$= 14.6$ at $Re_x = 1 \times 10^6$

At 10 km altitude, $Ux = \nu Re_x = 17.5$ at $Re_x = 5 \times 10^5$ and 35.0 at $Re_x = 1 \times 10^6$.



Given: Aircraft and missiles in high altitude flight may have regions of laminar flow that become turbulent at lower altitudes at the same speed.

Find: Explain a possible mechanism for this effect. Support answer with calculations.

Solution: Transition occurs when the local length Reynolds number exceeds $Re_{x,tr}$. The actual value of $Re_{x,tr}$ depends on pressure gradient, roughness and freestream turbulence level.

Thus, since $Re_x = \frac{\rho U x}{\mu} = \frac{U x}{\nu}$, then $x_{tr} = \frac{\nu Re_{x,tr}}{U}$. For fixed speed, the location of transition depends on the value of ν. Using Standard Atmosphere data (Table A.3) and the Sutherland correlation for viscosity (Eq. A.1):

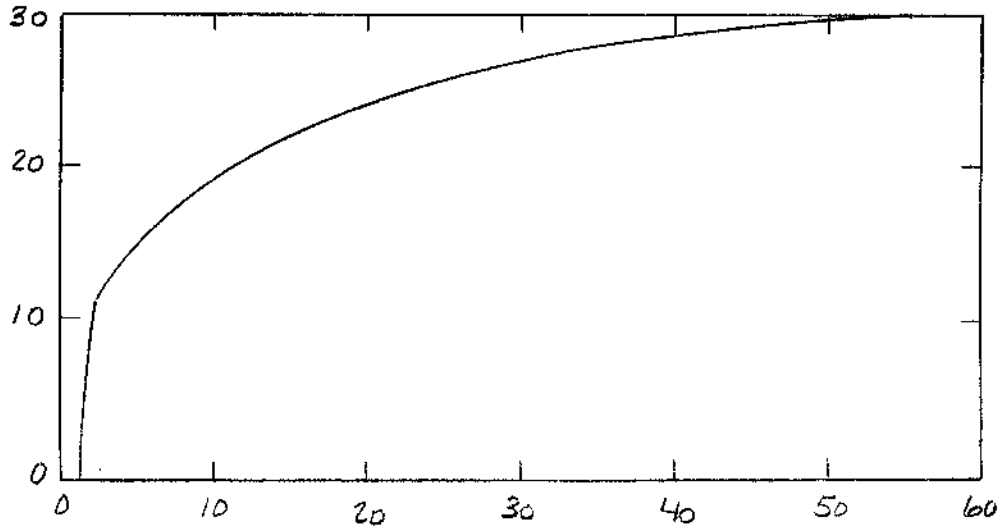
$$\mu\left(\frac{kg \cdot m}{s}\right) = \frac{1.46 \times 10^{-6}[T(K)]^{3/2}}{110 + T(K)} \quad \text{and} \quad \nu = \frac{\mu}{\rho}$$

For $Re_{x,tr}$ constant, then $x_{tr} = x_{tr,SL}(\nu/\nu_{SL})$

Tabulating results:

Altitude (km)	T (K)	ρ/ρ_{SL} (---)	μ (kg/m·s)	ν (m^2/s)	$\frac{x_{tr}}{x_{tr,SL}} = \frac{\nu}{\nu_{SL}}$
0	288	1.0	1.79×10^{-5}	1.46×10^{-5}	1.00
5	256	0.601	1.63×10^{-5}	2.21×10^{-5}	1.51
10	223	0.338	1.46×10^{-5}	3.51×10^{-5}	2.40
15	217	0.159	1.43×10^{-5}	7.30×10^{-5}	5.00
20	217	0.0726	1.43×10^{-5}	16.0×10^{-5}	11.0
30	227	0.0153	1.53×10^{-5}	81.1×10^{-5}	55.5

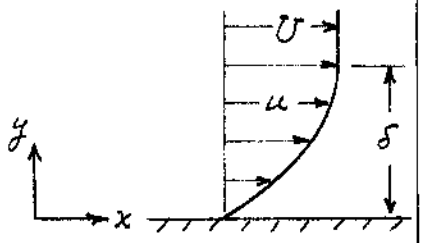
Plotting:



Given: Sinusoidal velocity profile for laminar boundary layer

$$u = A\sin(By) + C$$

Find: (a) State three applicable boundary conditions.
(b) Evaluate A, B and C.



Solution: For the boundary layer, at

(1) $y=0$, $u=0$ (no slip)

(2) $y=\delta$, $u=U$

(3) $\frac{\partial u}{\partial y} = 0$ (no shear stress) B.C.

Applying these boundary conditions,

(1) $u(0) = A\sin(0) + C = 0 \quad \therefore C = 0$

(2) $u(\delta) = A\sin(B\delta) = U$

$$\frac{\partial u}{\partial y} = AB\cos(By)$$

(3) $\left.\frac{\partial u}{\partial y}\right)_{y=\delta} = AB\cos(B\delta) = 0 \quad \therefore B\delta = \frac{\pi}{2}$ or $B = \frac{\pi}{2\delta}$

Then from (2), $A\sin(B\delta) = A\sin(\frac{\pi}{2}) = A = U$, and then

$$u = U\sin(\frac{\pi}{2}\frac{y}{\delta})$$

$A = U$, $B = \frac{\pi}{2\delta}$, $C = 0$ A, B, C

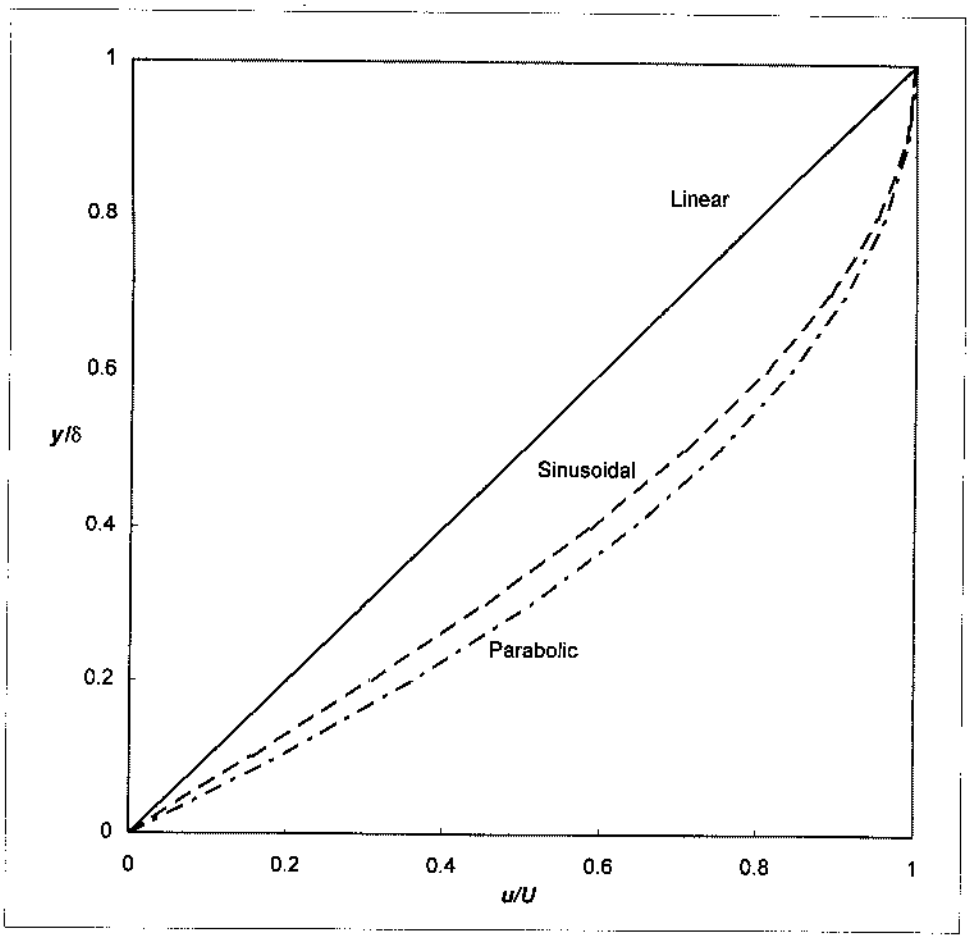
Given: Linear, parabolic, and sinusoidal velocity profiles for laminar boundary layer.

Linear $\frac{u}{U} = \frac{y}{\delta}$

Parabolic $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$ Sinusoidal $\frac{u}{U} = \sin\frac{\pi}{2}\left(\frac{y}{\delta}\right)$

Find: Compare shapes by plotting $\frac{y}{\delta}$ vs. $\frac{u}{U}$.

Solution:

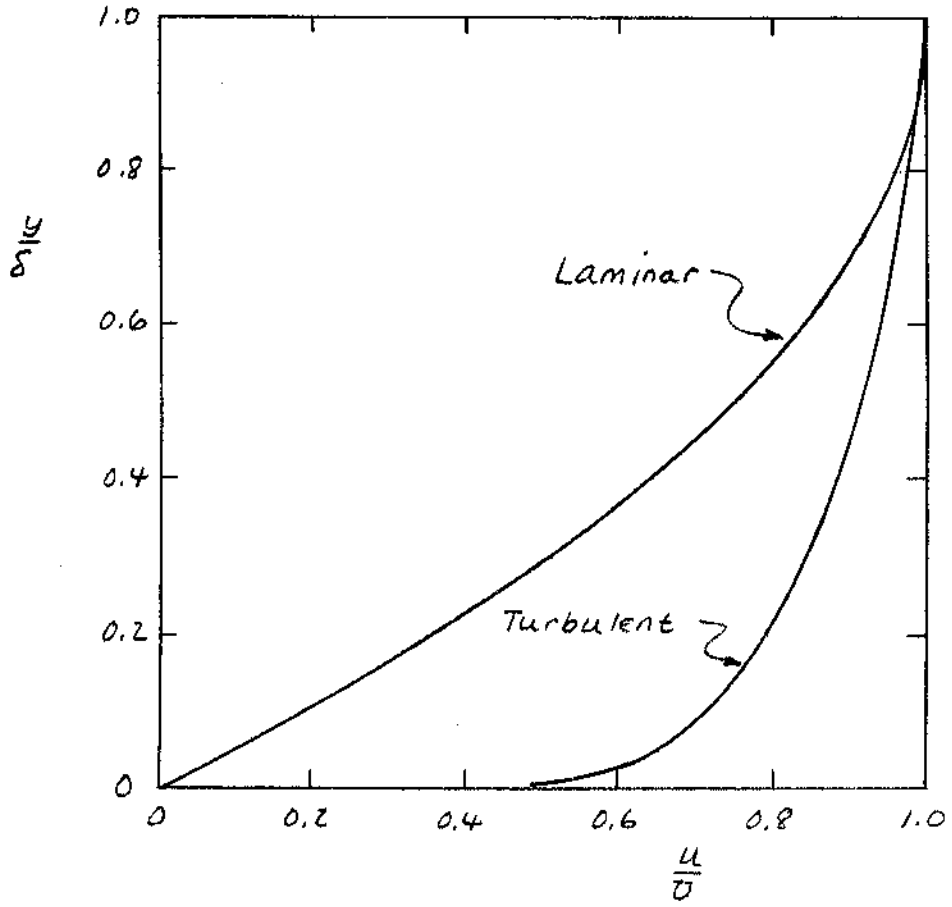


Given: "Power law" velocity profile for turbulent boundary layer and parabolic profile for laminar boundary layer,

"Power law" $\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}$ Parabolic $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$

Find: Compare shapes by plotting $\frac{y}{\delta}$ vs. $\frac{u}{U}$.

Solution:



{Note that the "power law" profile gives an infinite value of du/dy at y=0, since

$$\frac{du}{dy} = \frac{d\left(\frac{u}{U}\right)}{d\left(\frac{y}{\delta}\right)} = \frac{1}{7}\frac{1}{\left(\frac{y}{\delta}\right)^{6/7}} \to \infty \text{ as } \frac{y}{\delta} \to 0.$$

}

Given: Linear, parabolic, and sinusoidal velocity profiles for laminar boundary layer,

(a) linear $u/U = y/\delta$

(b) parabolic $u/U = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$

(c) sinusoidal $u/U = \sin\frac{\pi}{2}\frac{y}{\delta}$

Find: ratio δ^*/δ for each profile

Solution:

Definition: $\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$ (9.1)

For, $\delta/\delta^* = \frac{1}{\delta}\int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^1 \left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right) = \int_0^1 \left(1 - \frac{u}{U}\right) d\eta$

(a) linear profile $u/U = y/\delta = \eta$

$$\delta^*/\delta = \int_0^1 \left(1 - \frac{u}{U}\right) d\eta = \int_0^1 (1-\eta)\, d\eta = \left[\eta - \frac{1}{2}\eta^2\right]_0^1 = \frac{1}{2}$$

(b) Parabolic profile $u/U = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 = 2\eta - \eta^2$

$$\delta^*/\delta = \int_0^1 \left(1 - \frac{u}{U}\right) d\eta = \int_0^1 (1 - 2\eta + \eta^2)\, d\eta = \left[\eta - \eta^2 + \frac{1}{3}\eta^3\right]_0^1 = \frac{1}{3}$$

(c) Sinusoidal profile $u/U = \sin\frac{\pi}{2}\frac{y}{\delta} = \sin\frac{\pi}{2}\eta$

$$\delta^*/\delta = \int_0^1 \left(1 - \frac{u}{U}\right) d\eta = \int_0^1 \left(1 - \sin\frac{\pi}{2}\eta\right) d\eta = \left[\eta + \frac{2}{\pi}\cos\frac{\pi}{2}\eta\right]_0^1 = 1 - \frac{2}{\pi} = 0.363$$

Given: "Power-law" velocity profile for turbulent boundary layer and parabolic profile for laminar boundary layer,

"Power-law" $\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}$ Parabolic $\frac{u}{U} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$

Evaluate: (and compare) ratios δ^*/δ and θ/δ for each profile.

Solution:

Definitions: $\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$ (9.1)

$\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$ (9.2)

Then $\delta^*/\delta = \frac{1}{\delta}\int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^1 \left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right) = \int_0^1 \left(1 - \frac{u}{U}\right) d\eta$

For the "power-law" profile

$\delta^*/\delta = \int_0^1 (1 - \eta^{1/7})\, d\eta = \left[\eta - \frac{7}{8}\eta^{8/7}\right]_0^1 = \frac{1}{8}$

For the parabolic profile

$\delta^*/\delta = \int_0^1 (1 - 2\eta + \eta^2)\, d\eta = \left[\eta - \eta^2 + \frac{\eta^3}{3}\right]_0^1 = \frac{1}{3}$

Thus δ^*/δ (turbulent) $= \frac{3}{8}$ δ^*/δ (laminar) ← δ^*/δ

Also $\theta/\delta = \frac{1}{\delta}\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right) = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\eta$

For the "power-law" profile

$\theta/\delta = \int_0^1 \eta^{1/7}(1 - \eta^{1/7})\, d\eta = \int_0^1 (\eta^{1/7} - \eta^{2/7})\, d\eta = \left[\frac{7}{8}\eta^{8/7} - \frac{7}{9}\eta^{9/7}\right]_0^1 = \frac{7}{72}$

For the parabolic profile

$\theta/\delta = \int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2)\, d\eta = \int_0^1 (2\eta - 5\eta^2 + 4\eta^3 - \eta^4)\, d\eta$

$\theta/\delta = \left[\eta^2 - \frac{5}{3}\eta^3 + \eta^4 - \frac{\eta^5}{5}\right]_0^1 = \left[1 - \frac{5}{3} + 1 - \frac{1}{5}\right] = \frac{2}{15}$

Thus θ/δ (turbulent) $= 0.729$ θ/δ (laminar) ← θ/δ

Profile	δ^*/δ	θ/δ
Power-law	0.125	0.0972
Parabolic	0.333	0.133

Given: Laminar boundary layer on a flat plate with parabolic profile.

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \quad . \quad \text{Also } \delta/x = 5.48/\sqrt{Re_x}$$

Find: expressions for δ^*/x and θ/x.

Solution:

Definitions:

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy \qquad (9.11)$$

$$\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \qquad (9.12)$$

Re

$$\delta^*/\delta = \frac{1}{\delta}\int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^1 \left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right) = \int_0^1 \left(1 - \frac{u}{U}\right) d\eta$$

$$\delta^*/\delta = \int_0^1 (1 - 2\eta + \eta^2)\, d\eta = \left[\eta - \eta^2 + \frac{1}{3}\eta^3\right]_0^1 = \frac{1}{3}$$

$$\delta^* = \frac{1}{3}\delta = \frac{1}{3}\cdot\frac{5.48x}{\sqrt{Re_x}} \quad \text{and} \quad \delta^*/x = 1.83/\sqrt{Re_x} \longleftarrow \delta^*/x$$

Also

$$\theta/\delta = \frac{1}{\delta}\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right) = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\eta$$

$$\theta/\delta = \int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2)\, d\eta = \int_0^1 (2\eta - 5\eta^2 + 4\eta^3 - \eta^4)\, d\eta$$

$$\theta/\delta = \left[\eta^2 - \frac{5}{3}\eta^3 + \eta^4 - \frac{1}{5}\eta^5\right]_0^1 = \left[1 - \frac{5}{3} + 1 - \frac{1}{5}\right] = \frac{2}{15}$$

$$\theta = \frac{2}{15}\delta = \frac{2}{15}\cdot\frac{5.48x}{\sqrt{Re_x}} \quad \text{and} \quad \theta/x = 0.731/\sqrt{Re_x} \longleftarrow \theta/x$$

Given: Linear, parabolic, and sinusoidal profiles used to represent the laminar boundary layer velocity profile.

Evaluate: the ratio θ/δ for each profile.

Solution:

Definition: $\theta = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right) dy$ (9.12)

then, $\theta/\delta = \frac{1}{\delta}\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right) dy = \int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right) d\left(\frac{y}{\delta}\right) = \int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right) d\eta$

Linear profile $\frac{u}{U} = y/\delta = \eta$

$$\theta/\delta = \int_0^1 \eta(1-\eta)\,d\eta = \int_0^1 (\eta-\eta^2)\,d\eta = \left[\tfrac{1}{2}\eta^2 - \tfrac{1}{3}\eta^3\right]_0^1 = \tfrac{1}{6} = 0.167$$

Parabolic profile $\frac{u}{U} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2 = 2\eta - \eta^2$

$$\theta/\delta = \int_0^1 (2\eta-\eta^2)(1-2\eta+\eta^2)\,d\eta = \int_0^1 (2\eta - 5\eta^2 + 4\eta^3 - \eta^4)\,d\eta$$

$$\theta/\delta = \left[\eta^2 - \tfrac{5}{3}\eta^3 + \eta^4 - \tfrac{1}{5}\eta^5\right]_0^1 = \left[1 - \tfrac{5}{3} + 1 - \tfrac{1}{5}\right] = \tfrac{2}{15} = 0.133$$

Sinusoidal profile $\frac{u}{U} = \sin\frac{\pi}{2}\frac{y}{\delta} = \sin\frac{\pi}{2}\eta$

$$\theta/\delta = \int_0^1 \sin\tfrac{\pi}{2}\eta\left(1-\sin\tfrac{\pi}{2}\eta\right) d\eta = \int_0^1 \left(\sin\tfrac{\pi}{2}\eta - \sin^2\tfrac{\pi}{2}\eta\right) d\eta$$

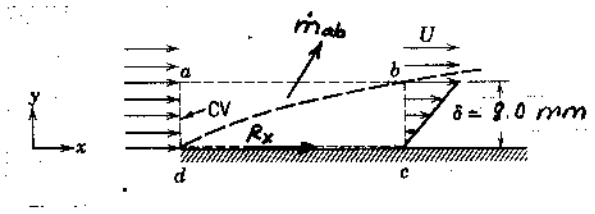
$$\theta/\delta = \left[-\tfrac{2}{\pi}\cos\tfrac{\pi}{2}\eta - \tfrac{2}{\pi}\left\{\tfrac{\pi}{4}\eta - \tfrac{1}{4}\sin\pi\eta\right\}\right]_0^1 = -0-\left(-\tfrac{2}{\pi}\right) - \tfrac{2}{\pi}\left(\tfrac{\pi}{4}\right) - 0$$

$$\theta/\delta = \tfrac{2}{\pi} - \tfrac{1}{2} = 0.137$$

Summarizing:

Profile	Expression	θ/δ
Linear	$u/U = \eta$	0.167
Parabolic	$u/U = 2\eta - \eta^2$	0.133
Sinusoidal	$u/U = \sin\frac{\pi}{2}\eta$	0.137

Given: Air flow over a thin flat plate of width, $b = 0.3\ m$. The flow two-dimensional. Assume that in the boundary layer the velocity profile is linear. (The plate is 1m long.)



At bc, $\frac{u}{U} = \frac{y}{\delta} = \eta$

$U = 2.7\ m/s$

Standard air

Find: (a) Mass flowrate across surface ab.
(b) x component (and direction) of force needed to hold plate.

Solution: Apply the continuity and x component momentum equations.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ (first term $= 0(1)$)

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$ ($F_{Bx} = 0(3)$, first term $= 0(1)$)

Assumptions: (1) Steady flow
(2) No pressure forces
(3) $F_{Bx} = 0$
(4) Uniform flow at da

Then

$$0 = \{-|\rho U b\delta|\} + \int_0^\delta \rho u b\, dy + \dot{m}_{ab} \quad ; \quad \int_0^\delta \rho u b\, dy = \rho U b \delta \int_0^1 \eta\, d\eta = \frac{1}{2}\rho U b \delta$$

or $\dot{m}_{ab} = \frac{1}{2}\rho U b\delta = \frac{1}{2} \times 1.23\frac{kg}{m^3} \times 2.7\frac{m}{s} \times 0.3\,m \times 0.008\,m = 0.00399\ kg/s$ $\dot{m}_{ab}$ ←

From momentum,

$$R_x = u_{da}\{-|\rho U b\delta|\} + u_{ab}\dot{m}_{ab} + \int_0^\delta \rho u^2 b\, dy \quad ; \quad \int_0^\delta \rho u^2 b\, dy = \rho U^2 b\delta \int_0^1 \eta^2 d\eta = \frac{1}{3}\rho U^2 b\delta$$

$u_{da} = U$ $\qquad$ $u_{ab} = U$ (all of ab is outside boundary layer)

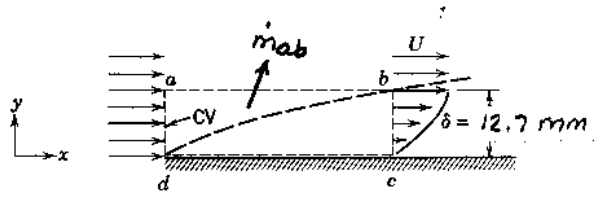
and $R_x = -\rho U^2 b\delta + \frac{1}{2}\rho U^2 b\delta + \frac{1}{3}\rho U^2 b\delta = -\frac{1}{6}\rho U^2 b\delta$

$$R_x = -\frac{1}{6} \times 1.23\frac{kg}{m^3} \times (2.7)^2\frac{m^2}{s^2} \times 0.3\,m \times 0.008\,m \times \frac{N\cdot s^2}{kg\cdot m} = -0.00359\ N$$

This force must be applied to the control volume by the plate. Thus to hold the plate,

$F_x = R_x = -0.00359\ N$ (to the left) F_x ←

Given: Air flow over a thin flat plate of width, $b = 0.3\ m$. Flow is two-dimensional. Assume that in the boundary layer the velocity profile is parabolic. (The plate is 1m long.)



At bc, $\frac{u}{U} = 2\eta - \eta^2$; $\eta = \frac{y}{\delta}$

$U = 2.7\ m/s$

Standard air

Find: (a) Mass flowrate across ab.
(b) x component (and direction) of force needed to hold plate.

Solution: Apply the continuity and x component momentum equations.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, dV + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ ($\frac{\partial}{\partial t} = 0$ (1))

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, dV + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$ ($F_{Bx} = 0$ (3), $\frac{\partial}{\partial t} = 0$ (1))

Assumptions: (1) Steady flow
(2) No pressure forces
(3) $F_{Bx} = 0$
(4) Uniform flow at da

Then

$$0 = \{-|\rho U b \delta|\} + \int_0^\delta \rho u b\, dy + \dot{m}_{ab}$$

But $\int_0^\delta \rho u b\, dy = \rho U b \delta \int_0^1 (2\eta - \eta^2)\, d\eta = \rho U b \delta \left[\eta^2 - \frac{1}{3}\eta^3\right]_0^1 = \frac{2}{3}\rho U b \delta$

Thus $\dot{m}_{ab} = \rho U b \delta - \frac{2}{3}\rho U b \delta = \frac{1}{3}\rho U b \delta$

$$\dot{m}_{ab} = \frac{1}{3} \times 1.23\frac{kg}{m^3} \times 2.7\frac{m}{s} \times 0.3\ m \times 0.00127\ m = 0.0042\ kg/s$$ $\dot{m}_{ab}$

From momentum,

$$R_x = u_{da}\{-|\rho U b \delta|\} + u_{ab}\dot{m}_{ab} + \int_0^\delta u \rho u b\, dy \;;\; u_{da} = u_{ab} = U$$

But $\int_0^\delta u\rho u b\, dy = \rho U^2 b \delta \int_0^1 (2\eta - \eta^2)\, d\eta = \rho U^2 b \delta \left[\frac{4}{3}\eta^3 - \eta^4 + \frac{1}{5}\eta^5\right]_0^1 = \frac{8}{15}\rho U^2 b \delta$

Thus $R_x = -\rho U^2 b \delta + \frac{1}{3}\rho U^2 b \delta + \frac{8}{15}\rho U^2 b \delta = -\frac{2}{15}\rho U^2 b \delta$

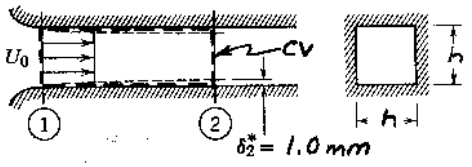
$$R_x = -\frac{2}{15} \times 1.23\frac{kg}{m^3} \times (2.7)^2\frac{m^2}{s^2} \times 0.3\ m \times 0.0127\ m \times \frac{N\cdot s^2}{kg\cdot m} = -0.00456\ N$$

This force must be applied <u>to</u> the control volume by the plate. Thus to hold the plate,

$F_x = R_x = -0.00456\ N$ (to the left) F_x

Given: Flow in the entrance region of a square duct as shown.

$U_0 = 30$ m/s



Fluid is air.

$h = 80$ mm

Find: Pressure change between sections (1) and (2).

Solution: Apply continuity equation to find V_2, then use Bernoulli equation to find pressure change.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ ($\frac{\partial}{\partial t} = 0$ (1))

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 \quad (5)$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) No friction outside boundary layers
(4) Flow along a streamline
(5) $z_1 = z_2$

Then

$$0 = \{-\rho V_1 A_1\} + \{\rho V_2 A_2\} \quad \text{or} \quad V_1 A_1 = V_2 A_2 \quad \text{or} \quad V_2 = V_1 \frac{A_1}{A_{2,eff}}$$

and $$p_1 - p_2 = \frac{\rho}{2}(V_2^2 - V_1^2) = \frac{\rho V_1^2}{2}\left[\left(\frac{A_1}{A_{2,eff}}\right)^2 - 1\right]$$

At section (1), the area is $A_1 = h^2$, but at section (2), the effective flow area is reduced by the wall boundary layers. Using the displacement thickness concept,

$$A_{2,eff} = (h - 2\delta_2^*)^2, \text{ so that } \left(\frac{A_1}{A_{2,eff}}\right)^2 = \left[\frac{h^2}{(h-2\delta_2^*)^2}\right]^2$$

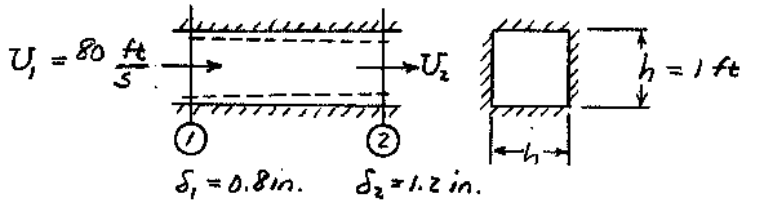
Thus $$p_1 - p_2 = \frac{\rho V_1^2}{2}\left\{\left[\frac{h^2}{(h-2\delta_2^*)^2}\right]^2 - 1\right\}$$

$$= \frac{1}{2} \times 1.23 \frac{kg}{m^3} \times (30)^2 \frac{m^2}{s^2}\left\{\left[\frac{(80)^2 mm^2}{(80-2)^2 mm^2}\right]^2 - 1\right\} \frac{N\cdot s^2}{kg\cdot m}$$

$$p_1 - p_2 = 59.0 \text{ N/m}^2 \quad (59.0 \text{ Pa})$$

$p_1 - p_2$

Given: Laboratory wind tunnels with square test sections, turbulent boundary layers on all test section walls. In boundary layers,



Find: (a) Freestream velocity, U_2, at exit
(b) Change in static pressure, $p_2 - p_1$, along test section.

Solution: Apply the displacement-thickness concept, continuity, and Bernoulli eqs.

Computing equations: $\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$ $\quad U_1 A_{1,effective} = U_2 A_{2,effective}$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g\cancel{z_1} = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g\cancel{z_2}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Flow along a streamline
(4) No friction in core region
(5) Horizontal flow
(6) Thin boundary layers

For the 1/7-power profile

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \delta \int_0^1 \left(1 - \eta^{1/7}\right) d\eta = \delta \left(\eta - \frac{7}{8}\eta^{8/7}\right)\Big|_0^1 = \frac{\delta}{8}$$

For thin boundary layers, $A_{effective} = (h - 2\delta^*)^2$. Thus from continuity

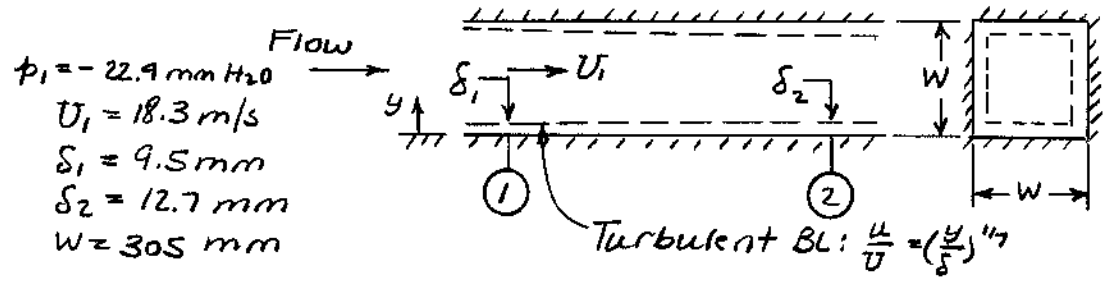
$$U_2 = U_1 \frac{A_{1\,eff}}{A_{2\,eff}} = U_1 \left(\frac{h - 2\delta_1^*}{h - 2\delta_2^*}\right)^2 = 80\ \frac{ft}{s} \left[\frac{1 - 2 \times \frac{0.8\ in.}{8} \times \frac{ft}{12\ in.}}{1 - 2 \times \frac{1.2\ in.}{8} \times \frac{ft}{12\ in.}}\right]^2 = 81.4\ ft/s \quad \leftarrow U_2$$

The pressure change is found from Bernoulli as

$$p_2 - p_1 = \frac{1}{2}\rho U_1^2 - \frac{1}{2}\rho U_2^2 = \frac{1}{2}\rho U_1^2 \left[1 - \left(\frac{U_2}{U_1}\right)^2\right] = \frac{1}{2}\rho U_1^2 \left[1 - \left(\frac{A_1}{A_2}\right)^2\right] = \frac{1}{2}\rho U_1^2 \left[1 - \left(\frac{h - 2\delta_1^*}{h - 2\delta_2^*}\right)^4\right]$$

$$p_2 - p_1 = \frac{1}{2} \times 0.00238 \frac{slug}{ft^3} \times (80)^2 \frac{ft^2}{s^2} \left[1 - \left(\frac{1 - \frac{0.2}{12}}{1 - \frac{0.3}{12}}\right)^4\right] \frac{lbf \cdot s^2}{slug \cdot ft} = -0.264\ lbf/ft^2 \quad \leftarrow p_2 - p_1$$

Given: Air flow in laboratory wind tunnels.



Find: (a) Freestream speed at (2)
(b) Pressure difference, $\Delta p = p_1 - p_2$

Solution: Apply displacement thickness, continuity, Bernoulli equations

Basic equations: $\frac{\delta^*}{\delta} = \int_0^1 \left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right)$; $0 = \underbrace{\frac{\partial}{\partial t}}_{=0(1)} \int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

$$\frac{p_1}{\rho} + \frac{U_1^2}{2} + g\underbrace{z_1}_{(8)} = \frac{p_2}{\rho} + \frac{U_2^2}{2} + g\underbrace{z_2}_{(8)}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) No friction (outside BL)
(4) Along a streamline
(5) Uniform flow (outside BL)
(6) Same BL on four surfaces
(7) Neglect corner effects
(8) Neglect Δz

Then $\frac{\delta^*}{\delta} = \int_0^1 \left(1 - \frac{u}{U}\right) d\lambda = \int_0^1 \left(1 - \lambda^{1/7}\right) d\lambda = \left[\lambda - \frac{7}{8}\lambda^{8/7}\right]_0^1 = \frac{1}{8}$ $(\lambda = y/\delta)$

$$\delta_1^* = \frac{1}{8}\delta_1 = \frac{1}{8} \times 9.5 \text{ mm} = 1.19 \text{ mm}; \quad \delta_2^* = 1.59 \text{ mm}$$

From continuity, $U_1 A_1 = U_1 (L - 2\delta_1^*)^2 = U_2 A_2 = U_2 (L - 2\delta_2^*)^2$

$$U_2 = U_1 \left(\frac{W - 2\delta_1^*}{W - 2\delta_2^*}\right)^2 = 18.3 \frac{m}{s} \left[\frac{305 - 2(1.19)}{305 - 2(1.59)}\right]^2 = 18.4 \text{ m/s} \quad \leftarrow U_2$$

From Bernoulli $\frac{p_1}{\rho} + \frac{U_1^2}{2} = \frac{p_2}{\rho} + \frac{U_2^2}{2}$

$$p_1 - p_2 = \frac{\rho}{2}\left(U_2^2 - U_1^2\right) = \rho \frac{U_1^2}{2}\left[\left(\frac{U_2}{U_1}\right)^2 - 1\right] = \rho \frac{U_1^2}{2}\left[\left(\frac{L - 2\delta_1^*}{L - 2\delta_2^*}\right)^4 - 1\right]$$

$$= \frac{1}{2} \times 1.23 \frac{kg}{m^3} \times (18.3)^2 \frac{m^2}{s^2} \left[\left(\frac{305 - 2(1.19)}{305 - 2(1.59)}\right)^4 - 1\right] \frac{N \cdot s^2}{kg \cdot m}$$

$$p_1 - p_2 = 2.19 \text{ N/m}^2 \quad \leftarrow \Delta p$$

Note since $\Delta p = \rho_w g \Delta h$

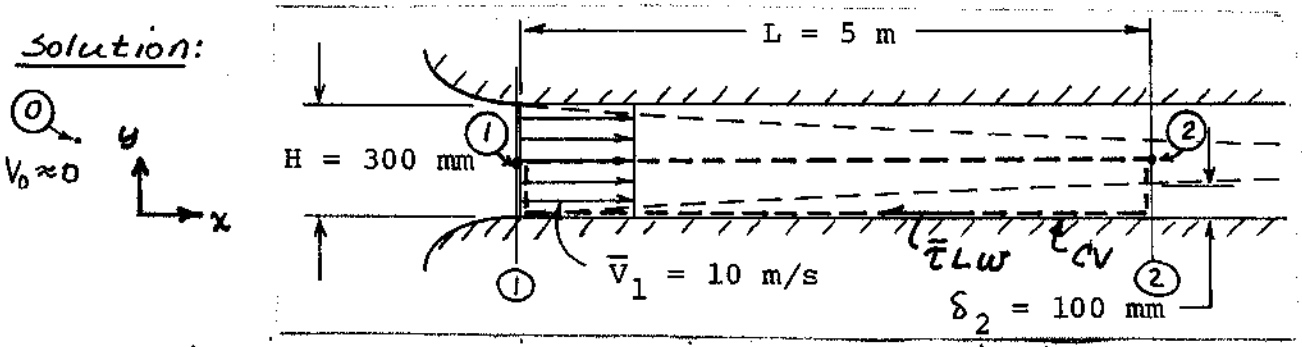
$$\Delta h = \frac{\Delta p}{\rho_w g} = 2.19 \frac{N}{m^2} \times \frac{m^3}{1000 kg} \times \frac{s^2}{9.81 m} \times \frac{kg \cdot m}{N \cdot s^2} \times 1000 \frac{mm}{m} = 0.223 \text{ mm } H_2O$$

Given: Developing flow of air in flat horizontal duct.

Assume $\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}$ in boundary layers.

Find: (a) Show $\delta^*/\delta = 1/8$
(b) Evaluate p_2
(c) Calculate average wall shear stress.

Solution:



Assume flow is steady and incompressible, and that frictional effects are negligible outside the boundary layers.

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \delta \int_0^1 \left(1 - \frac{u}{U}\right) d\eta = \delta \int_0^1 (1 - \eta^{1/7})\, d\eta = \delta \left(\eta - \frac{7}{8}\eta^{8/7}\right)\Big|_0^1 = \delta/8$$ δ^*

From continuity $V_1 A_1 = V_1 W H = V_2 A_2 = V_2 W (H - 2\delta_2^*)$

$$V_2 = V_1 \frac{H}{H - 2\delta_2^*} = 10 \frac{m}{s} \frac{300\ mm}{(300 - 25)\ mm} = 10.9\ m/s$$

From Bernoulli, since z = constant, $\frac{p_0}{\rho} + \frac{V_0^2}{2} = \frac{p}{\rho} + \frac{V^2}{2}$ along a streamline.

$$p_{1g} = p_1 - p_0 = -\frac{1}{2}\rho V_1^2 = -\frac{1}{2} \times 1.23 \frac{kg}{m^3} \times (10)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = -61.5\ Pa$$

$$p_{2g} = p_2 - p_0 = -\frac{1}{2}\rho V_2^2 = -\frac{1}{2} \times 1.23 \frac{kg}{m^3} \times (10.9)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = -73.1\ Pa$$ p_{2g}

Apply momentum using CV shown:

BE $F_{Sx} + \cancel{F_{Bx}}^{=0} = \cancel{\frac{\partial}{\partial t}}^{=0} \int_{CV} u \rho\, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$

$$(p_1 - p_2) W \frac{H}{2} - \bar{\tau} W L = \bar{V}_1 \left\{-\rho \bar{V}_1 \frac{H}{2} W\right\} + \underbrace{\int_0^{\delta_2} u \rho u W\, dy}_{\rho V_2^2 \delta_2 W \int_0^1 \eta^{2/7} d\eta = \rho V_2^2 \left(\frac{7}{9}\delta_2\right) W} + V_2 \left\{+\rho V_2 \left(\frac{H}{2} - \delta_2\right) W\right\}$$

$$\bar{\tau} W L = (p_1 - p_2) W \frac{H}{2} + \rho V_1^2 \frac{H}{2} W - \rho V_2^2 \left(\frac{H}{2} - \frac{2}{9}\delta_2\right) W$$

$$\bar{\tau} = \frac{1}{L}\left[(p_1 - p_2)\frac{H}{2} + \rho V_1^2 \frac{H}{2} - \rho V_2^2 \left(\frac{H}{2} - \frac{2}{9}\delta_2\right)\right]$$

$$= \frac{1}{5\ m}\left[11.6 \frac{N}{m^2} \times 0.15\ m + 1.23 \frac{kg}{m^3} \times (10)^2 \frac{m^2}{s^2} \times 0.15\ m \times \frac{N \cdot s^2}{kg \cdot m} - 1.23 \frac{kg}{m^3} \times (10.9)^2 \frac{m^2}{s^2} (0.15 - 0.022)\ m \times \frac{N \cdot s^2}{kg \cdot m}\right]$$

$$\bar{\tau} = 0.300\ N/m^2$$ $\bar{\tau}$

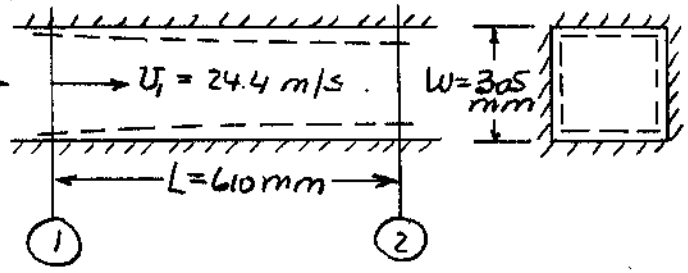
Given: Air flow in laboratory wind tunnel test section.

$h_{L_{0-1}} = -6.5$ mm H_2O

$\delta_1 = 20.3$ mm

$\delta_2 = 25.4$ mm

$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}$ in BLs



Find: (a) Freestream speed at exit, U_2.
(b) Pressure at exit, p_2.

Solution: Apply displacement thickness, continuity, Bernoulli eqs.

Computing equations: $\frac{\delta^*}{\delta} = \int_0^1 \left(1 - \frac{u}{U}\right) dy$; $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ (=0 (1))

$$\frac{p_1}{\rho} + \frac{U_1^2}{2} + g z_1 \overset{(8)}{\cancel{z_1}} = \frac{p_2}{\rho} + \frac{U_2^2}{2} + g \overset{(8)}{\cancel{z_2}}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) No friction (outside BL)
(4) Along a streamline
(5) Uniform flow (outside BL)
(6) Same BL on four walls
(7) Neglect corner effects
(8) Neglect Δz

Then $\frac{\delta^*}{\delta} = \int_0^1 \left(1 - \lambda^{1/7}\right) d\lambda = \left(\lambda - \frac{7}{8}\lambda^{8/7}\right)\Big|_0^1 = 1 - \frac{7}{8} = \frac{1}{8}$ $\left(\lambda \equiv y/\delta\right)$

$\delta_1^* = \frac{1}{8}\delta_1 = \frac{1}{8} \times 20.3$ mm $= 2.54$ mm ; $\delta_2^* = \frac{1}{8}\delta_2 = \frac{1}{8} \times 25.4$ mm $= 3.18$ mm

From continuity, $U_1 A_1 = U_1 (W - 2\delta_1^*)^2 = U_2 A_2 = U_2 (W - 2\delta_2^*)^2$

$$U_2 = U_1 \left(\frac{W - 2\delta_1^*}{W - 2\delta_2^*}\right)^2 = 24.4 \frac{m}{s}\left[\frac{300 - 2(2.54)}{300 - 2(3.18)}\right]^2 = 24.6 \text{ m/s}$$

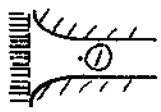
From Bernoulli, $\frac{p_1}{\rho} + \frac{U_1^2}{2} = \frac{p_2}{\rho} + \frac{U_2^2}{2}$

$$p_2 - p_1 = \frac{\rho}{2}\left(U_1^2 - U_2^2\right) = \frac{1}{2} \times 1.23 \frac{kg}{m^3}\left[(24.4)^2 - (24.6)^2\right]\frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = -6.03 \text{ N/m}^2$$

Since $\Delta p = \rho g \Delta h$, then $\Delta h = \Delta p / \rho g$

$$p_2 - p_1 = -6.03 \frac{N}{m^2} \times \frac{m^3}{999 kg} \times \frac{s^2}{9.81 m} \times \frac{kg \cdot m}{N \cdot s^2} = -0.615 \text{ mm } H_2O$$

From ambient to (1): $\left(\frac{p_0}{\rho} \overset{p_0 = p_{atm}}{\cancel{+}} + \frac{U_0^2}{2} \overset{\approx 0}{\cancel{+}}\right) - \left(\frac{p_1}{\rho} + \frac{U_1^2}{2}\right) = h_{LT}$



$$p_1 = -\rho h_{LT} - \frac{1}{2}\rho U_1^2$$

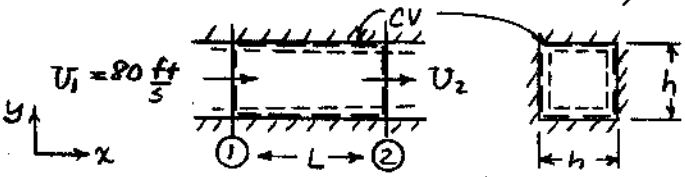
$$p_1 = -6.5 \text{ mm } H_2O - \frac{1}{2} \times 1.23 \frac{kg}{m^3} \times (24.4)^2 \frac{m^2}{s^2} \times \frac{m^3}{999 kg} \times \frac{s^2}{9.81 m} = -43.9 \text{ mm } H_2O$$

$$p_2 = p_1 + (p_2 - p_1) = -43.9 - 0.615 \text{ mm } H_2O = -44.5 \text{ mm } H_2O$$

p_2 ←

Given: Wind tunnel test section of Problem 9.15., turbulent BL flow.

$\frac{u}{U} = (\frac{y}{\delta})^{1/7}$ $U_1 = 80 \frac{ft}{s}$ U_2 $h = 1\ ft$ $L = 2A$



$\delta_1 = 0.8$ in. $\delta_2 = 1.2$ in.

Find: (a) total wall shear force and (b) average wall shear stress.

Solution: Apply x component of momentum equation to CV shown.

Basic equation: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho dV + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$ ($F_{Bx}=0$ (1), $\frac{\partial}{\partial t}=0$ (2))

Assumptions: (1) Horizontal, so $F_{Bx}=0$ (2) Steady flow (3) Uniform flow outside BL (4) No friction outside BL

From continuity, $U_2 = U_1(\frac{h-2\delta_1^*}{h-2\delta_2^*})^2$; from $\frac{u}{U} = (\frac{y}{\delta})^{1/7}$, $\frac{\delta^*}{\delta} = \frac{1}{8}$. Substituting,

$$U_2 = 80.0\frac{ft}{s}\left[\frac{1.0\ ft - 2\times\frac{1}{8}\times 0.8\ in.\times\frac{ft}{12\ in.}}{1.0\ ft - 2\times\frac{1}{8}\times 1.2\ in.\times\frac{ft}{12\ in.}}\right]^2 = 81.4\ ft/s$$

From Bernoulli, for (5) incompressible flow and (6) flow along streamline,

$$\frac{p_1}{\rho}+\frac{U_1^2}{2} = \frac{p_2}{\rho}+\frac{U_2^2}{2};\quad p_1-p_2 = \frac{1}{2}\rho U_2^2 - \frac{1}{2}\rho U_1^2 = \frac{1}{2}\rho U_1^2\left[(\frac{U_2}{U_1})^2-1\right]$$

$$p_1-p_2 = \frac{1}{2}\times 0.00238\frac{slug}{ft^3}\times(80.0)^2\frac{ft^2}{s^2}\times\frac{lbf\cdot s^2}{slug\cdot ft}\left[(\frac{81.4}{80.0})^2-1\right] = 0.269\ lbf/ft^2$$

From momentum, $-F_f + (p_1-p_2)h^2 = \int_{①} u_1\{-\rho u_1 dA\} + \int_{②} u_2\{+\rho u_2 dA\}$ (1)

At each cross-section, $\int_A = \int$ outside BL $+ \int$ inside BL $= \rho U^2(h-2\delta)^2 + \int$ inside BL

$$\int_{inside\ BL} = \int_0^\delta \rho u^2 w dy = \rho U^2\delta w\int_0^1(\frac{u}{U})^2 d(\frac{y}{\delta}) = \rho U^2\delta w\int_0^1 \lambda^{2/7}d\lambda = \rho U^2\delta w\frac{7}{9}$$

Thus $\int_A = \rho U^2(h-2\delta)^2 + \frac{7}{9}\rho U^2\left[h^2-(h-2\delta)^2\right]$, since δw = BL area, or

$$\int_A = \frac{7}{9}\rho U^2 h^2 + \frac{2}{9}\rho U^2(h-2\delta)^2$$

Substituting into Eq. 1,

$$-F_f + (p_1-p_2)h^2 = -\frac{7}{9}\rho U_1^2 h^2 - \frac{2}{9}\rho U_1^2(h-2\delta_1)^2 + \frac{7}{9}\rho U_2^2 h^2 + \frac{7}{9}\rho U_2^2(h-2\delta_2)^2$$

or $$F_f = h^2\left\{(p_1-p_2) + \rho U_1^2\left[\frac{7}{9}+\frac{2}{9}(1-2\frac{\delta_1}{h})^2\right] - \rho U_2^2\left[\frac{7}{9}+\frac{2}{9}(1-2\frac{\delta_2}{h})^2\right]\right\}$$

$$F_f = (1)^2 ft^2\Big\{0.269\frac{lbf}{ft^2} + 0.00238\frac{slug}{ft^3}\Big[(80.0)^2\frac{ft^2}{s^2}\left(\frac{7}{9}+\frac{2}{9}(0.867)^2\right) - (81.4)^2\frac{ft^2}{sec^2}\left(\frac{7}{9}+\frac{2}{9}(0.800)^2\right)\Big]$$
$$\times\frac{lbf\cdot s^2}{slug\cdot ft}\Big\}$$

$F_f = 0.150$ lbf ← F_f

The friction force is

$$F_f = \bar{\tau}_w A_w = \bar{\tau}_w \, 4hL$$

So

$$\bar{\tau}_w = \frac{F_f}{4hL} = \frac{1}{4} \times 0.150 \text{ lbf} \times \frac{1}{1 \text{ ft}} \times \frac{1}{2 \text{ ft}} = 0.0188 \text{ lbf/ft}^2 \leftarrow \bar{\tau}_w$$

42-381 50 SHEETS 5 SQUARE
42-382 100 SHEETS 5 SQUARE
42-389 200 SHEETS 5 SQUARE
NATIONAL

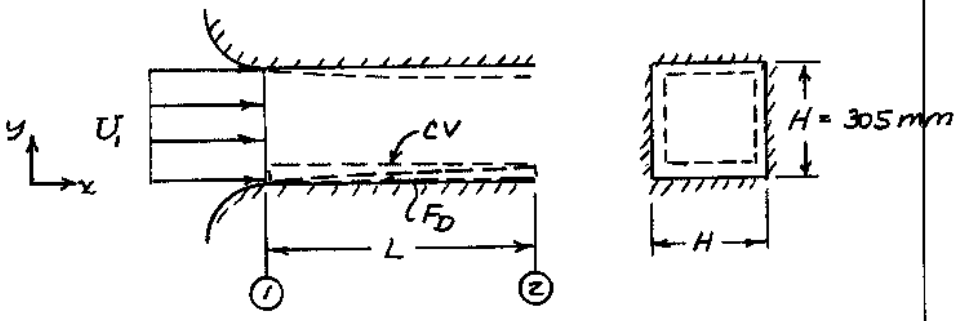
Given: Air flow into wind tunnel contraction and test section as shown.

$U_1 = 50.2$ m/s

$\delta_2 = 20.3$ mm

$H = 305$ mm

$L = 609$ mm



Find: (a) δ_2^*
(b) $p_2 - p_1$
(c) Estimate total drag force caused by friction on each wall.

Solution: Assume turbulent BL with 1/7-power profile, $\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7} = \eta^{1/7}$

By definition $\frac{\delta^*}{\delta} = \int_0^1 \left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right) = \int_0^1 (1 - \eta^{1/7})\, d\eta = \eta - \frac{7}{8}\eta^{8/7} = 1/8$

Thus $\delta_2^* = \frac{1}{8}\delta_2 = \frac{1}{8} \times 20.3 \text{ mm} = 2.54 \text{ mm}$ ← δ_2^*

From continuity, $U_1 A_1 = U_1 H^2 = U_2 A_2 = U_2 (H - 2\delta_2^*)^2$

$$U_2 = U_1 \frac{H^2}{(H - 2\delta_2^*)^2} = 50.2 \frac{m}{s} \times \frac{(305)^2}{(305 - 2 \times 2.54)^2} = 51.9 \text{ m/s}$$

Apply the Bernoulli equation to the steady, incompressible, frictionless flow along a streamline outside the boundary layers:

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 \; ; \; z_2 = z_1, \text{ so } p_2 - p_1 = \frac{\rho}{2}(U_1^2 - U_2^2)$$

So

$$p_2 - p_1 = \frac{1}{2} \times 1.23 \frac{kg}{m^3} \left[(50.2)^2 - (51.9)^2\right] \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = -107 \text{ N/m}^2$$ ← $p_2 - p_1$

Choose the CV shown to evaluate drag caused by friction

Basic equation: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$ ($F_{Bx} = 0$ (1); $\frac{\partial}{\partial t} = 0$ (2))

Assume: (1) Horizontal, so $F_{Bx} = 0$
(2) Steady flow

Then

$$p_1 H \delta_2 - F_D - p_2 H \delta_2 = U_1\{-\rho U_1 H \delta_2\} + \bar{U}\{+\dot{m}_{top}\} + \int_0^{\delta_2} u \rho u H dy$$

or $$F_D = (p_1 - p_2) H \delta_2 + \rho U_1^2 H \delta_2 - \bar{U} \dot{m}_{top} - \int_0^{\delta_2} u \rho u H dy \quad (1)$$

From conservation of mass, $\dot{m}_{top} = \dot{m}_1 - \dot{m}_2 = \rho U_1 H \delta_2 - \int_0^{\delta_2} \rho u H dy$ (2)

Evaluating the integrals, $\int_0^{\delta_2} \rho u H dy = \rho U_2 H \delta_2 \int_0^1 \eta^{1/7} d\eta = \frac{7}{8}\rho U_2 H \delta_2$

and $\int_0^{\delta_2} u \rho u H dy = \rho U_2^2 H \delta_2 \int_0^1 \eta^{2/7} d\eta = \frac{7}{9}\rho U_2^2 H \delta_2$

Thus $\int_0^{\delta_1} \rho u H dy = \frac{7}{8}\rho U_2 H\delta_2 = \frac{7}{8} \times 1.23 \frac{kg}{m^3} \times 51.4 \frac{m}{s} \times 0.305 m \times 0.0203 m = 0.346 \; kg/s$

$\rho U_1 H \delta_2 = 1.23 \frac{kg}{m^3} \times 50.2 \frac{m}{s} \times 0.305 m \times 0.0203 m = 0.382 \; kg/s$

From Eq. 2, $\dot{m}_{top} = (0.382 - 0.346) \frac{kg}{s} = 0.0364 \; kg/s$

Also $\int_0^{\delta_2} u\rho u H dy = \frac{7}{9}\rho U_2^2 H\delta_2 = \frac{7}{9} \times 1.23 \frac{kg}{m^3} \times (51.9)^2 \frac{m^2}{s^2} \times 0.305 m \times 0.0203 m \times \frac{N \cdot s^2}{kg \cdot m} = 16.0 \; N$

and $\bar{U} \approx \frac{1}{2}(U_1 + U_2) = \frac{1}{2}(50.2 + 51.9) \frac{m}{s} = 51.1 \; m/s$

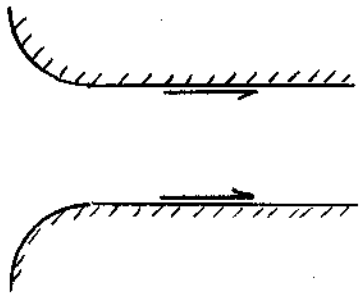
From Eq. 1, $(p_1 - p_2) H\delta_2 = 107 \frac{N}{m^2} \times 0.305 m \times 0.0203 m = 0.662 \; N$

$\rho U_1^2 H \delta_2 = 1.23 \frac{kg}{m^3} \times (50.2)^2 \frac{m^2}{s^2} \times 0.305 m \times 0.0203 m \times \frac{N \cdot s^2}{kg \cdot m} = 19.2 \; N$

Finally, substituting into Eq. 1

$F_D = 0.662 \; N + 19.2 N - 51.1 \frac{m}{s} \times 0.0364 \frac{kg}{s} \times \frac{N \cdot s^2}{kg \cdot m} - 16.0 \; N = 2.00 \; N$ F_D

The viscous drag force acts on the CV in the direction shown. The viscous drag force on the wall of the test section is equal and opposite:

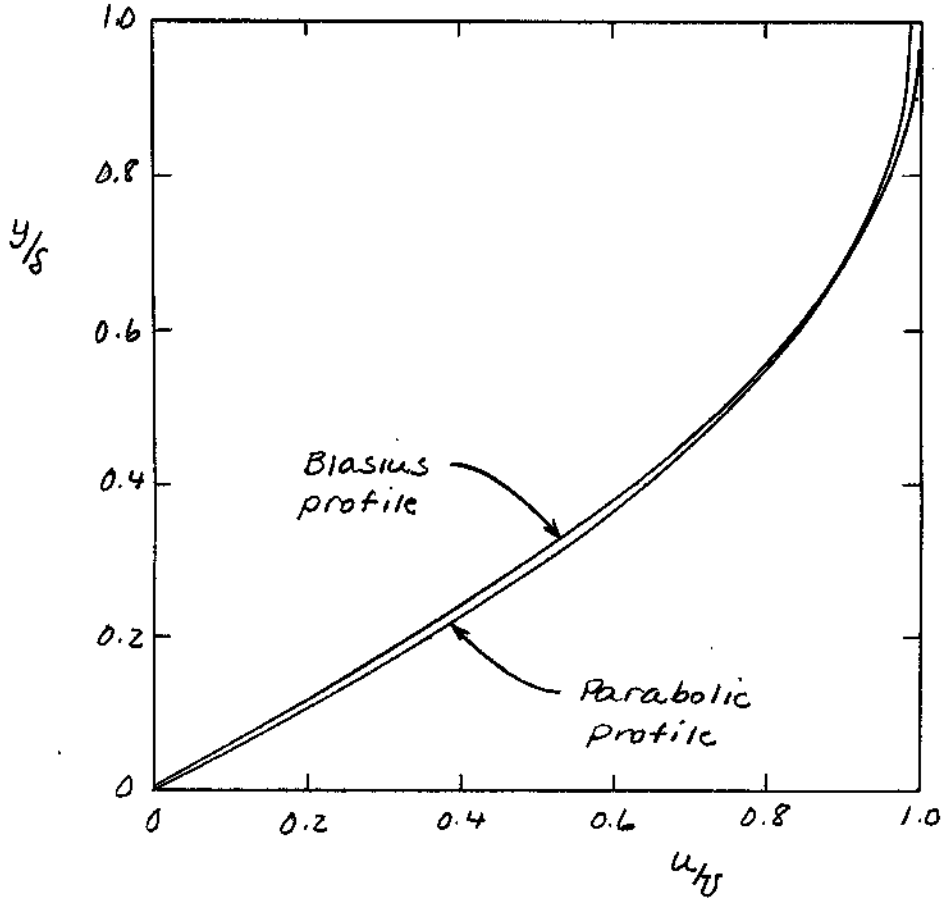


Viscous drag forces on walls of test section

Given: Blasius exact solution for laminar boundary-layer flow.

Find: Plot and compare to parabolic profile, $\frac{u}{U} = 2\eta - \eta^2$.

Solution: The Blasius solution is given in Table 9.1; it is plotted below.



Given: Numerical results of Blasius for laminar boundary-layer flow.

Find: (a) Evaluate τ distribution.
(b) Plot τ/τ_w versus y/δ.
(c) Compare results from the parabolic profile, $\frac{u}{U} = 2\eta - \eta^2$

Solution: For the Blasius solution, $u = U\frac{df}{d\eta} = Uf'(\eta)$, and $\eta = y\sqrt{\frac{U}{\nu x}}$

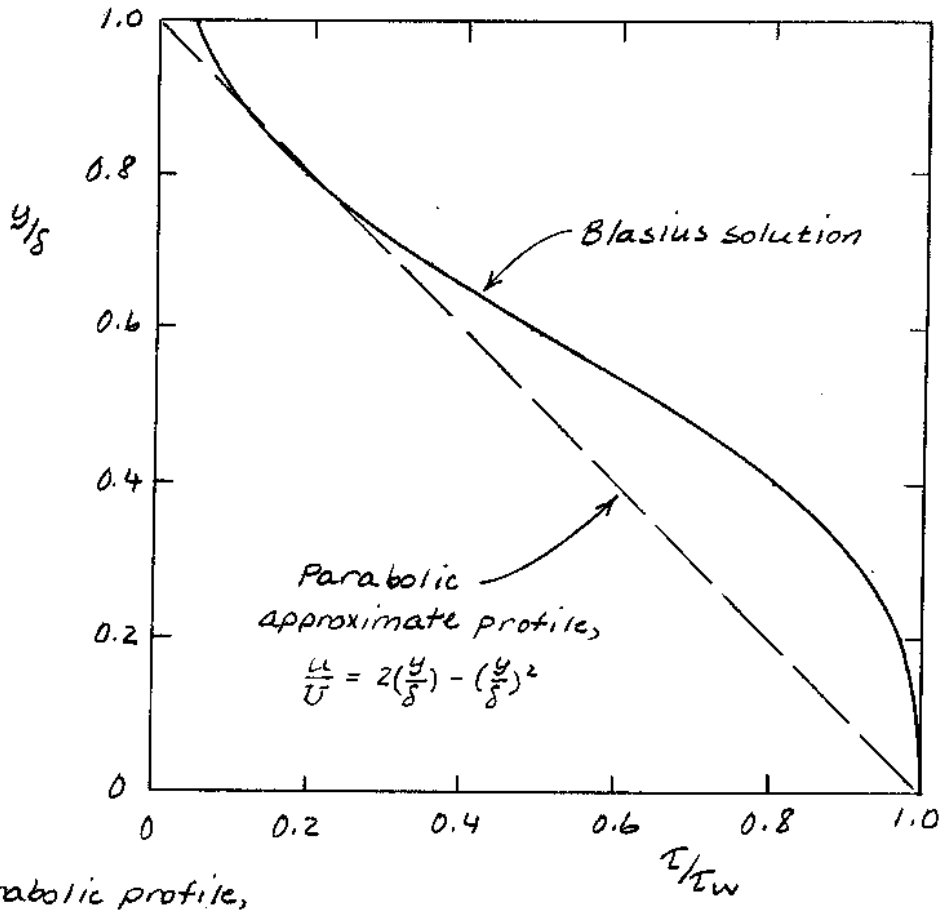
Computing equation: $\tau = \mu\frac{\partial u}{\partial y}$

For the Blasius profile, $\tau = \mu\frac{\partial}{\partial y}\left[U\frac{df}{d\eta}\right] = \mu U\frac{d^2f}{d\eta^2}\frac{\partial \eta}{\partial y} = \mu U f''(\eta)\sqrt{\frac{U}{\nu x}}$

Thus $\frac{\tau}{\rho U^2} = \frac{\mu}{\rho U} f''(\eta)\sqrt{\frac{U}{\nu x}} = \frac{f''(\eta)}{\sqrt{Re_x}}$; $\tau \sim f''(\eta)$ ← τ

From above equation, $\frac{\tau}{\tau_w} = \frac{f''(\eta)}{f''(0)} = \frac{f''(\eta)}{0.33206}$

Since $y = \delta$ at $\eta = 5$, then $y/\delta = \eta/5$. Plotting:



For the parabolic profile,

$$\tau = \mu\frac{\partial u}{\partial y} = \frac{\mu U}{\delta}\frac{d(u/U)}{d(y/\delta)} = \frac{\mu U}{\delta}\left[2 - 2\left(\frac{y}{\delta}\right)\right] \; ; \; \tau_w = \frac{\mu U}{\delta}[2]$$

Thus

$$\tau/\tau_w = 1 - \frac{y}{\delta} \quad \text{(parabolic profile)}$$ ← τ/τ_w

Given: Numerical results of Blasius for laminar boundary-layer flow.

Find: (a) Evaluate τ distribution.
(b) Plot τ/τ_w versus y/δ.
(c) Compare results from the sinusoidal profile, $\frac{u}{U} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$

Solution: For the Blasius solution, $u = U\frac{df}{d\eta} = Uf'(\eta)$, and $\eta = y\sqrt{\frac{U}{\nu x}}$

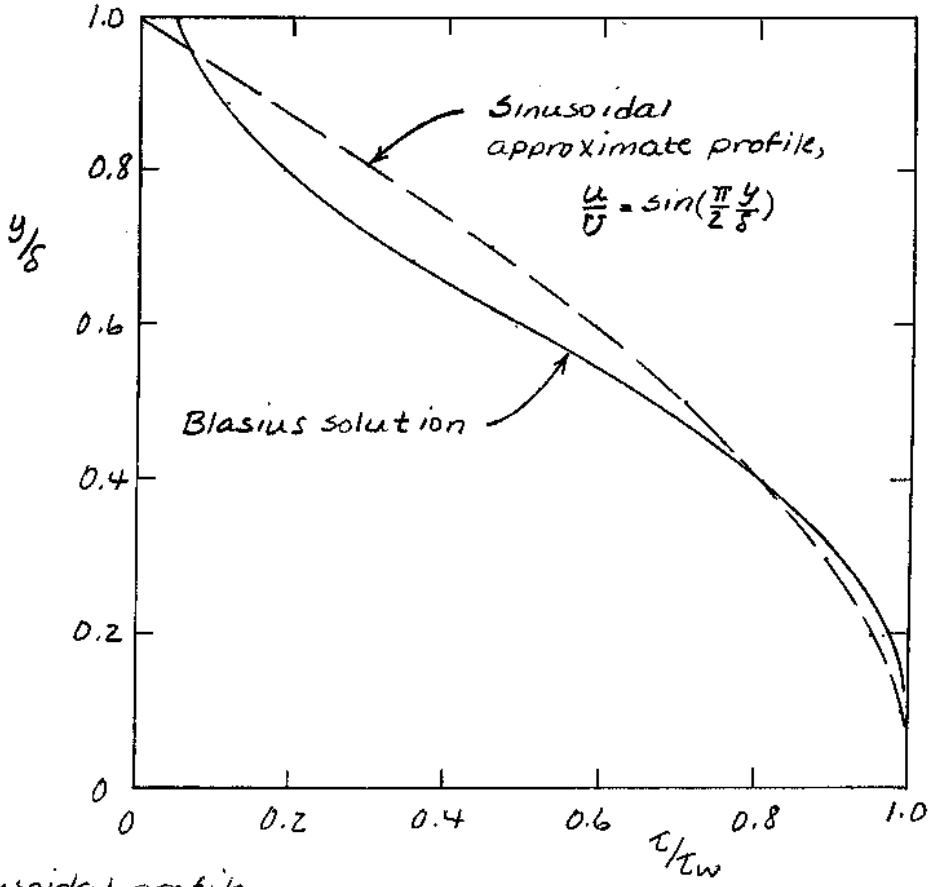
Computing equation: $\tau = \mu\frac{\partial u}{\partial y}$

For the Blasius profile, $\tau = \mu\frac{\partial}{\partial y}\left[U\frac{df}{d\eta}\right] = \mu U\frac{d^2f}{d\eta^2}\frac{\partial\eta}{\partial y} = \mu Uf''(\eta)\sqrt{\frac{U}{\nu x}}$

Thus $\frac{\tau}{\rho U^2} = \frac{\mu}{\rho U}f''(\eta)\sqrt{\frac{U}{\nu x}} = \frac{f''(\eta)}{\sqrt{Re_x}}$; $\tau \sim f''(\eta)$ ← τ

From the above equation, $\frac{\tau}{\tau_w} = \frac{f''(\eta)}{f''(0)} = \frac{f''(\eta)}{0.33206}$

Since $y = \delta$ at $\eta = 5$, then $y/\delta = \eta/5$. Plotting:



For the sinusoidal profile,

$$\tau = \mu\frac{\partial u}{\partial y} = \frac{\mu U}{\delta}\frac{d(u/U)}{d(y/\delta)} = \frac{\mu U}{\delta}\frac{\pi}{2}\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) ; \quad \tau_w = \frac{\mu U}{\delta}\frac{\pi}{2}$$

Thus

$$\tau/\tau_w = \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) \quad \text{(sinusoidal profile)}$$ ← τ/τ_w

Given: Numerical results of Blasius for laminar boundary-layer flow.

Find: Plot v/U versus y/δ for $Re_x = 10^5$.

Solution: For the Blasius solution, $\psi = \sqrt{U\nu x}\, f(\eta)$ and $\eta = y\sqrt{\frac{U}{\nu x}}$

From the stream function, $v = -\frac{\partial \psi}{\partial x} = -\left[\frac{1}{2}\sqrt{\frac{\nu U}{x}} f(\eta) + \sqrt{U\nu x}\,\frac{df}{d\eta}\frac{\partial \eta}{\partial x}\right]$

But $\frac{\partial \eta}{\partial x} = -\frac{1}{2}\frac{y}{x}\sqrt{\frac{U}{\nu x}} = -\frac{1}{2}\frac{\eta}{x}$

Thus $v = -\frac{1}{2}\sqrt{\frac{U\nu}{x}} f(\eta) - \sqrt{U\nu x}\,\frac{df}{d\eta}\left(-\frac{1}{2}\frac{\eta}{x}\right) = \frac{1}{2}\sqrt{\frac{U\nu}{x}}\left[\eta f'(\eta) - f(\eta)\right]$

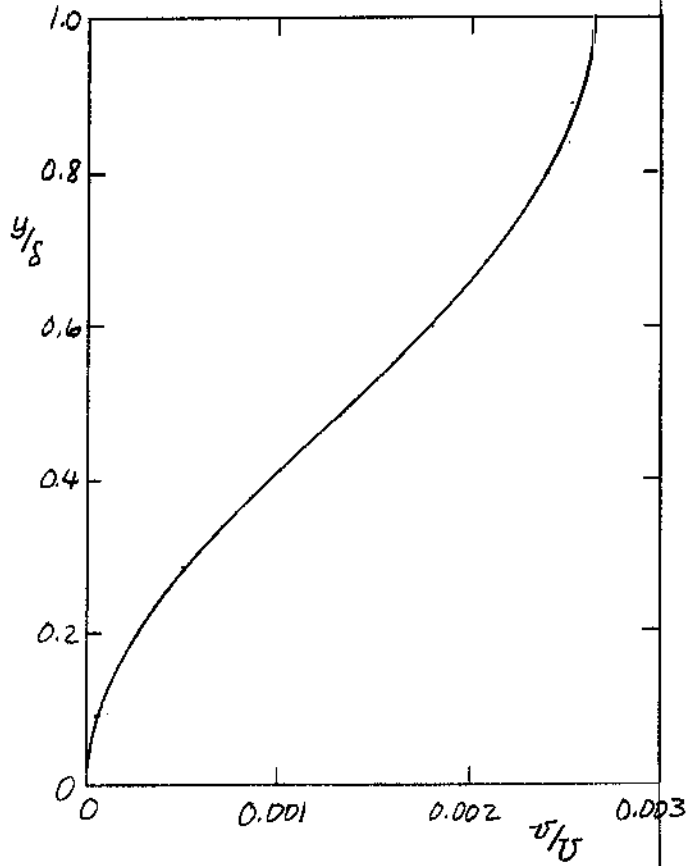
and $\frac{v}{U} = \frac{1}{2}\sqrt{\frac{\nu}{Ux}}\left[\eta f'(\eta) - f(\eta)\right] = \frac{\eta f'(\eta) - f(\eta)}{2\sqrt{Re_x}}$ ← $\frac{v}{U}$

Also $\frac{y}{\delta} = \frac{y}{5\sqrt{\frac{\nu x}{U}}} = \frac{\eta}{5}$

Tabulate from Table 9.1:

η	$\eta f'(\eta) - f(\eta)$	v/U
0	0	0
0.4	0.0265	4.20×10^{-5}
1.0	0.164	2.60×10^{-4}
1.4	0.316	4.99×10^{-4}
2.0	0.610	9.64×10^{-4}
2.4	0.827	1.31×10^{-3}
3.0	1.14	1.80×10^{-3}
3.4	1.32	2.09×10^{-3}
4.0	1.52	2.40×10^{-3}
4.4	1.60	2.53×10^{-3}
5.0	1.67	2.65×10^{-3}

Plot:



Given: Blasius solution to boundary layer equations gives

$$v = \frac{1}{2}\sqrt{\frac{\nu U}{x}}(\eta f' - f)$$

where $\psi = f(\eta)\sqrt{\nu x U}$ and $\eta = y\sqrt{\frac{U}{\nu x}}$

Find: (a) Verify expression for v
(b) Obtain an expression for a_x

Plot: a_x vs η to determine maximum a_x for given x.

Solution: From the definition of ψ, $v = -\frac{\partial \psi}{\partial x}$

$$v = -\frac{\partial \psi}{\partial x} = -\left[\sqrt{\nu U}\frac{\partial f}{\partial x} + f\frac{1}{2}\sqrt{\frac{\nu U}{x}}\right] = -\left[\sqrt{\nu U}\frac{df}{d\eta}\frac{\partial \eta}{\partial x} + f\frac{1}{2}\sqrt{\frac{\nu U}{x}}\right]$$

$$\frac{\partial \eta}{\partial x} = y\sqrt{\frac{U}{\nu}}\left(-\frac{1}{2}x^{-3/2}\right) = -\frac{y}{2x}\sqrt{\frac{U}{\nu x}} = -\frac{\eta}{2x}$$

$$\therefore v = -\left[\sqrt{\nu U}\, f'\left(-\frac{\eta}{2x}\right) + f\frac{1}{2}\sqrt{\frac{\nu U}{x}}\right] = \frac{1}{2}\sqrt{\frac{\nu U}{x}}(\eta f' - f) \quad \leftarrow v$$

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \qquad \text{where } u = Uf'$$

$$\frac{\partial u}{\partial x} = U\frac{df'}{d\eta}\frac{\partial \eta}{\partial x} = Uf''\left(-\frac{\eta}{2x}\right) = -\frac{1}{2}\frac{\eta U f''}{x}$$

$$\frac{\partial u}{\partial y} = U\frac{df'}{d\eta}\frac{\partial \eta}{\partial y} = Uf''\sqrt{\frac{U}{\nu x}}$$

Then

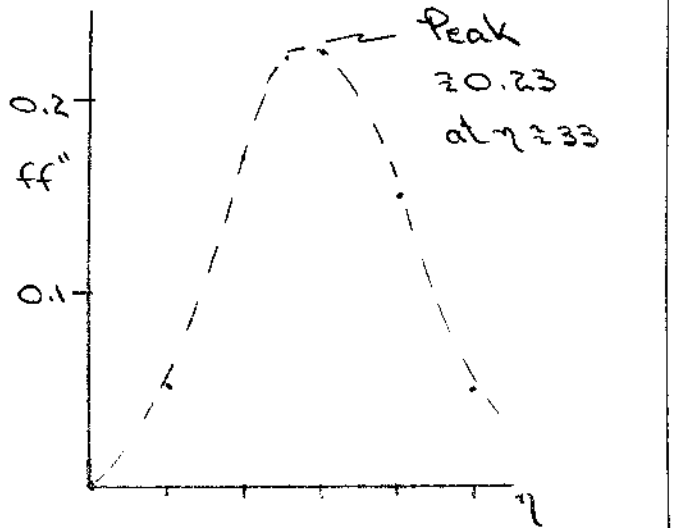
$$a_x = Uf'\left(-\frac{1}{2}\frac{\eta U f''}{x}\right) + \frac{1}{2}\sqrt{\frac{\nu U}{x}}(\eta f' - f)Uf''\sqrt{\frac{U}{\nu x}}$$

$$= -\frac{1}{2}\frac{U^2}{x}\eta f'f'' + \frac{1}{2}\frac{U^2}{x}(\eta f'f'' - ff'')$$

$$a_x = -\frac{U^2}{2x}ff'' \quad \leftarrow a_x$$

For given x, $a_{x\,max}$ for max ff''

η	f	f''	ff''
0	0	0.3321	0
1	0.1656	0.3230	0.053
2	0.6500	0.2668	0.173
3	1.3968	0.1614	0.225
4	2.3057	0.0642	0.148
5	3.2833	0.0159	0.052

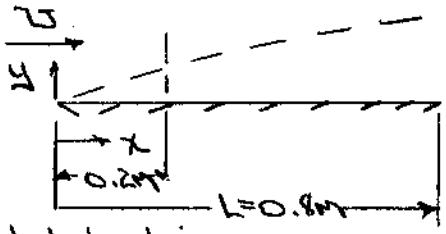


From plot $ff''|_{max} \approx 0.23$

$$\therefore a_{x\,max} \approx -0.115\frac{U^2}{x} \quad (at\ \eta \approx 3) \quad \leftarrow a_{max}$$

Given: Steady, incompressible flow of standard air over a flat plate with $U = 4.3$ m/s.

Find: (a) At $x = 0.2$m, estimate y for $u = 0.95U$
(b) Evaluate streamline slope at this point.
(c) Obtain expression for local skin friction, $\tau_w(x)$
(d) Obtain expression for the total skin friction drag force on the plate
(e) Evaluate the momentum thickness at $L = 0.8$m



Solution: Use results of Blasius solution (Table 9.1)

$\frac{u}{U} = 0.95$ at $\eta = y\sqrt{\frac{U}{\nu x}} = 4$ $\therefore y = 4\sqrt{\frac{\nu x}{U}}$

For standard air $\nu = 1.45 \times 10^{-5}$ m²/s (Table 3.1)

$$\therefore y = 4\left[1.45\times10^{-5}\ \frac{m^2}{sec} \times 0.2m \times \frac{1}{4.3}\frac{sec}{m}\right]^{1/2} = 3.28\times10^{-3}\,m = 3.28\,mm$$ y

The slope of the streamline is given by

$$\left(\frac{dy}{dx}\right)_{st} = \frac{v}{u} \quad \text{where } u = Uf' \text{ and } v = \frac{1}{2}\sqrt{\frac{\nu U}{x}}\,(\eta f' - f) \quad (9.10)$$

$$\left(\frac{dy}{dx}\right)_{st} = \frac{1}{2}\sqrt{\frac{\nu U}{x}}\,(\eta f' - f)\,\frac{1}{Uf'} = \frac{1}{2}\sqrt{\frac{\nu}{Ux}}\,\frac{(\eta f' - f)}{f} = \frac{1}{2\sqrt{Re_x}}\,\frac{(\eta f' - f)}{f'}$$

$$Re_x = \frac{Ux}{\nu} = 4.3\frac{m}{sec} \times 0.2m \times 1.45\times10^{-5}\frac{sec}{m^2} = 5.93\times10^{4}$$

$$\therefore \left(\frac{dy}{dx}\right)_{streamline} = \frac{1}{2}\times\frac{1}{\sqrt{5.93\times10^{4}}}\,\frac{[4\times(0.9555) - 2.3057]}{0.9555}$$

$$\left(\frac{dy}{dx}\right)_{st} = 0.00326$$ $\left(\frac{dy}{dx}\right)_{st}$

$$\tau_w = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)_{y=0} = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \quad \{v = 0 \text{ at } y = 0 \text{ for all } x\}$$

$$\tau_w = \mu U\sqrt{\frac{U}{\nu x}}\left[\frac{d^2f}{d\eta^2}\right]_{\eta=0} = 0.332U\sqrt{\frac{\rho U\mu}{x}}$$

$$\tau_w = 0.332\,\rho U^2\sqrt{\frac{\mu}{\rho U x}} = 0.332\,\frac{\rho U^2}{\sqrt{Re_x}}$$ $\tau_w(x)$

(This result is given by Eq. 9.14)

The friction drag is given by

$$F_D = \int \tau_w \, dA = \int_0^L \tau_w \, b \, dx \qquad \text{where } b \text{ is the plate width}$$

$$F_D = \int_0^L 0.332 \frac{\rho U^2}{\sqrt{Re_x}} b \, dx = 0.332 \rho U^2 \sqrt{\frac{\nu}{U}} b \int_0^L \frac{dx}{x^{1/2}}$$

$$F_D = 0.332 \rho U^2 \sqrt{\frac{\nu}{U}} b \, 2 x^{1/2} \Big]_0^L = 0.664 \rho U^2 (bL) \sqrt{\frac{\nu}{UL}}$$

$$F_D = \rho U^2 (bL) \frac{0.664}{\sqrt{Re_L}} \qquad F_D$$

From the momentum integral equation (Section 9-4)

$$\frac{\tau_w}{\rho U^2} = \frac{d\theta}{dx} \qquad \text{or} \qquad d\theta = \frac{\tau_w}{\rho U^2} dx$$

Then

$$\theta_L = \int_0^L d\theta = \frac{1}{\rho U^2} \int_0^L \tau_w \, dx = \frac{1}{\rho U^2} \frac{F_D}{b} = \frac{0.664 L}{\sqrt{Re_L}}$$

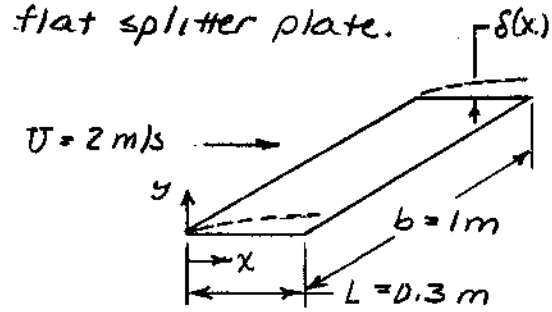
$$\theta_L = 0.664 \times 0.8 \text{ m} \times \frac{1}{\sqrt{2.37 \times 10^5}} = 1.09 \times 10^{-3} \text{ m}$$

$$\theta_L = 1.09 \text{ mm} \qquad \theta_L$$

Given: Laminar boundary layer on flat splitter plate.

Water, $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$

$$\frac{\delta}{x} = \frac{5.48}{\sqrt{Re_x}}$$



Find: Plot δ, δ^* and τ_w vs. x/L.

Solution: Parabolic profile was solved in section 9-5.1.

Computing equation: $\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$ $(T = 20°C)$

For this flow, $Re_L = \frac{UL}{\nu} = 2\frac{m}{s} \times 0.3\,m \times \frac{s}{1\times10^{-6} m^2} = 6\times10^5$; $\frac{\delta_L}{L} = 0.00707$

$\delta_L = 0.3\,m \times 0.00707 = 2.12\,mm$; $\delta = \delta_L\left(x/L\right)^{1/2} = \left(\frac{x}{L}\right)^{1/2} \times 2.12\,mm$ ← δ

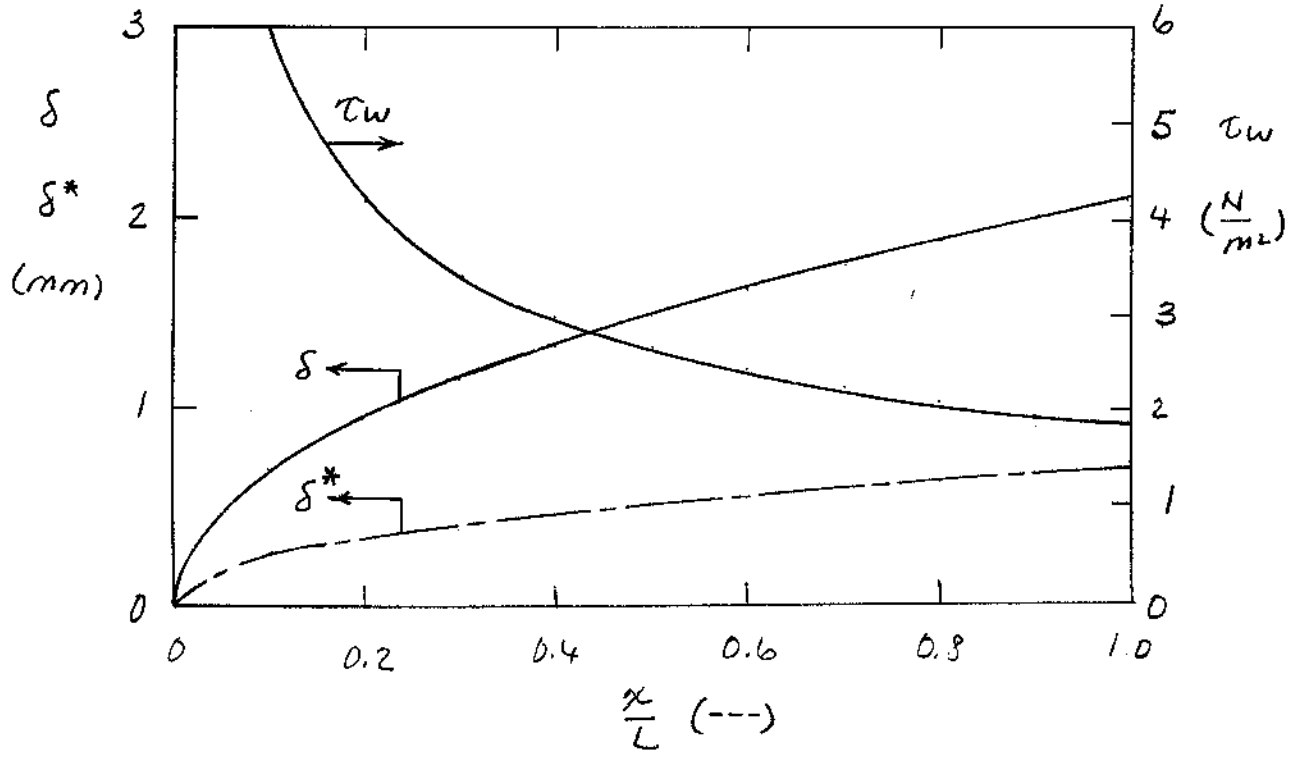
$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \delta\int_0^1 \left(1 - \frac{u}{U}\right) d(y/\delta) = \delta\int_0^1 (1 - 2\eta + \eta^2) d\eta = \eta - \eta^2 + \frac{1}{3}\eta^3\Big]_0^1 = \frac{1}{3}$

$\delta^* = \delta_L^*\left(\frac{x}{L}\right)^{1/2} = \frac{1}{3} \times 2.12\,mm\left(\frac{x}{L}\right)^{1/2} = \left(\frac{x}{L}\right)^{1/2} \times 0.707\,mm$ ← δ^*

From Eq. 9.23, $C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.730}{\sqrt{Re_x}}$ so $\tau_w \sim \frac{1}{x^{1/2}}$

$\tau_{wL} = \frac{1}{2}\rho U^2 \frac{0.730}{\sqrt{Re_L}} = \frac{1}{2} \times 999\frac{kg}{m^3} \times (2)^2\frac{m^2}{s^2} \cdot \frac{0.730}{\sqrt{6\times10^5}} \times \frac{N\cdot s^2}{kg\cdot m} = 1.88\,N/m^2$

$\tau_w = \tau_{wL}\left(\frac{L}{x}\right)^{1/2} = \left(\frac{L}{x}\right)^{1/2} 1.88\,N/m^2$ ← τ_w

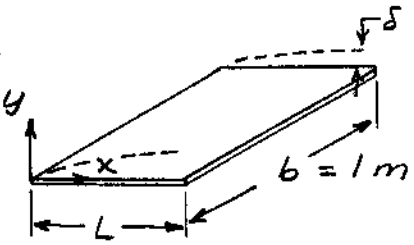


Given: Thin flat plate in water tunnel

$U = 1.6$ m/s →

$L = 0.3$ m

Velocity profile is $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$



Find: Total drag force on plate due to skin friction.

Solution: Check $Re_L = \frac{UL}{\nu} = 1.6\frac{m}{s} \times 0.3\ m \times \frac{s}{10^{-6} m^2} = 4.8 \times 10^5$, so laminar.

Viscous drag for two sides of plate is (Assume T = 20°C.)

$$\text{Drag} = 2\int_0^L \tau_w\, b\, dx \qquad (1)$$

From the definition, $\tau_w = \mu \left.\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\mu U}{\delta} \left.\frac{d(u/U)}{d(y/\delta)}\right]_{\eta=0} = \frac{\mu U}{\delta}(2 - 2\eta)\Big]_{\eta=0} = \frac{2\mu U}{\delta}$

Since $\frac{\delta}{x} = \frac{5.48}{\sqrt{Re_x}}$, $\delta = \frac{5.48\,x}{\sqrt{Re_x}} = \frac{5.48}{\sqrt{U/\nu}} x^{1/2}$ $\qquad \mu = \rho\nu$

Substituting into Eq. 1,

$$F_D = 2\int_0^L \frac{2\mu U}{\delta} b\, dx = \frac{4}{5.48} b\mu U \sqrt{\frac{U}{\nu}} \int_0^L \frac{dx}{x^{1/2}} = \frac{4b\mu U}{5.48}\sqrt{\frac{U}{\nu}}\left[2x^{1/2}\right]_0^L = \frac{8b\mu U}{5.48}\sqrt{\frac{UL}{\nu}}$$

$$= \frac{8}{5.48} \times 1.0\ m \times 999\frac{kg}{m^3} \times 10^{-6}\frac{m^2}{s} \times 1.6\frac{m}{s} \times \left[1.6\frac{m}{s} \times 0.3\ m \times \frac{s}{10^{-6} m^2}\right]^{1/2} \frac{N \cdot s^2}{kg \cdot m}$$

$F_D = 1.62$ N ← F_D

Given: Water flow over a flat plate of length, L, and width, b.

Find: (a) Show that the total drag force on one side of the plate may be given by

$$F_D = \rho U^2 \theta_L b$$

(b) Evaluate θ_L and F_D given the following conditions
- laminar boundary layer
$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$ $\qquad \frac{\delta}{x} = \frac{5.48}{\sqrt{Re_x}}$
- $L = 0.3\,m$, $b = 1.0\,m$, $U = 2.0\,m/s$

Solution:

Assumptions: (1) flat plate flow, so $\frac{\partial p}{\partial x} = 0$ and U = constant
(2) δ is a function of x only and $\delta = 0$ at $x = 0$
(3) incompressible flow

The momentum integral equation can be written as

$$\frac{\tau_w}{\rho U^2} = \frac{d}{dx}(U^2\theta) + \delta^* U \frac{dU}{dx} \quad \text{(0)} \qquad \text{where } \theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

Since U = constant, then $\tau_w = \rho U^2 \frac{d\theta}{dx}$

The drag force on a flat plate parallel to the flow is

$$F_D = \int \tau_w \, dA = \int_0^L \tau_w b \, dx = \int_0^L \rho U^2 \frac{d\theta}{dx} b \, dx = \rho U^2 b \int_0^{\theta_L} d\theta$$

$$F_D = \rho U^2 b \theta_L \longleftarrow F_D$$

For the given velocity profile

$$\frac{\theta}{\delta} = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\eta = \int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2)\, d\eta = \int_0^1 (2\eta - 5\eta^2 + 4\eta^3 - \eta^4)\, d\eta$$

$$\frac{\theta}{\delta} = \eta^2 - \frac{5}{3}\eta^3 + \eta^4 - \frac{2}{5}\eta^5 \Big]_0^1 = 1 - \frac{5}{3} + 1 - \frac{1}{5} = \frac{15 - 25 + 15 - 3}{15} = \frac{2}{15}$$

Then

$$\theta = \frac{2}{15}\delta \quad \text{and} \quad \theta_L = \frac{2}{15}\delta_L = \frac{2}{15} \times \frac{5.48L}{\sqrt{Re_L}} = \frac{0.731L}{\sqrt{Re_L}}$$

Assume $T = 20°C$, $\nu = 1 \times 10^{-6}\ m^2/s$ (Table A.8)

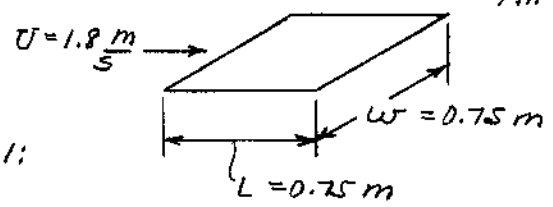
$$\theta_L = \frac{0.731L}{\left[\frac{UL}{\nu}\right]^{1/2}} = \left[1 \times 10^{-6} \frac{s}{m^2} \times 2\frac{m}{s} \times \frac{1}{0.3m}\right]^{1/2} 0.731 \times 0.30m \times 10^3 \frac{mm}{m} = 0.283\,mm \longleftarrow \theta_L$$

$$F_D = \rho U^2 b \theta_L = 999 \frac{kg}{m^3} \times (2)^2 \frac{m^2}{sec^2} \times 1.0m \times 0.283 \times 10^{-3} m \times \frac{N \cdot sec^2}{kg \cdot m}$$

$$F_D = 1.13\ N \longleftarrow F_D$$

Given: Flat plate aligned with flow.

Find: Drag force.



Solution: Apply results of section 9-5.1:

Check the Reynolds number (assume $T = 15°C$, Table A.10)

$$Re_L = \frac{\rho U L}{\mu} = \frac{UL}{\nu} = 1.8\frac{m}{s} \times 0.75 m \times \frac{s}{1.46 \times 10^{-5} m^2} = 9.25 \times 10^4$$

Flow is laminar since $Re_L < 5 \times 10^5$. Therefore (assuming a parabolic velocity profile),

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.730}{\sqrt{Re_x}} \qquad (9.23)$$

The drag force due to viscous shear, on one side of the surface, is

$$F_D = \int_0^L \tau_w w\, dx = \frac{1}{2}\rho U^2 w \int_0^L \frac{0.730}{\sqrt{Re_x}}\, dx = \frac{1}{2}\rho U^2 w \frac{0.730}{\sqrt{U/\nu}} 2\sqrt{L} = \frac{1}{2}\rho U^2 wL \frac{1.46}{\sqrt{Re_L}}$$

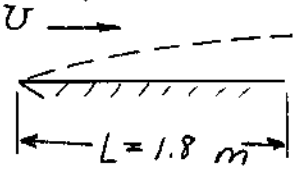
$$= \frac{1}{2} \times 1.23 \frac{kg}{m^3} \times (1.8)^2 \frac{m^2}{s^2} \times 0.75 m \times 0.75 m \times \frac{1.46}{\sqrt{9.25 \times 10^4}} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_D = 0.00538\ N \quad (5.38\ mN)\{\text{per side}\}$$

The total drag force is 10.8 mN.

F_D

Given: Laminar boundary layer in std. air, sinusoidal profile.
Conditions: $U = 3.2$ m/s, $L = 1.8$ m, $b = 0.9$ m.



Find: Plot δ, δ^*, and τ_w vs. x/L.

Solution: Apply momentum integral equation for steady, incompressible flow.

Computing equations: $\tau_w = \rho U^2 \frac{d\theta}{dx}$, $\frac{\delta^*}{\delta} = \int_0^1 (1 - \frac{u}{U}) d\lambda$, $\frac{\theta}{\delta} = \int_0^1 \frac{u}{U}(1 - \frac{u}{U}) d\lambda$

For the sinusoidal profile,

$$\frac{\theta}{\delta} = \int_0^1 \sin\tfrac{\pi}{2}\lambda\,(1 - \sin\tfrac{\pi}{2}\lambda)\,d\lambda = \int_0^1 (\sin\tfrac{\pi}{2}\lambda - \sin^2\tfrac{\pi}{2}\lambda)\,d\lambda$$

$$= \int_0^1 (\sin\tfrac{\pi}{2}\lambda - \frac{1-\cos\pi\lambda}{2})\,d\lambda = \left[-\frac{2}{\pi}\cos\tfrac{\pi}{2}\lambda - \frac{\lambda}{2} + \frac{1}{2\pi}\sin\pi\lambda\right]_0^1$$

$$\frac{\theta}{\delta} = -\frac{2}{\pi}(0-1) - \frac{1}{2} + \frac{1}{2\pi}(0-0) = \frac{2}{\pi} - \frac{1}{2} = \frac{4-\pi}{2\pi} = 0.137$$

Also $\tau_w = \mu \left.\frac{du}{dy}\right)_{y=0} = \mu U \left.\frac{d}{dy}\sin(\frac{\pi}{2}\frac{y}{\delta})\right]_{y=0} = \frac{\pi\mu U}{2\delta}\left.\left(\cos\frac{\pi}{2}\frac{y}{\delta}\right)\right]_{y=0} = \frac{\pi\mu U}{2\delta}$

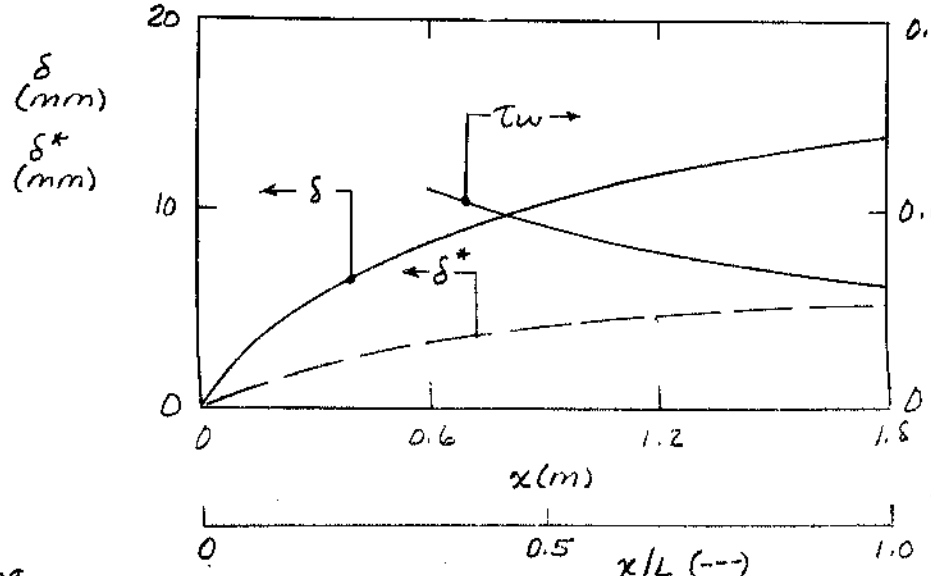
Thus $\tau_w = \rho U^2 \frac{d\theta}{dx} = \frac{\pi\mu U}{2\delta} = \rho U^2 (\frac{4-\pi}{2\pi})\frac{d\delta}{dx}$ or $\delta\,d\delta = \frac{\pi^2}{4-\pi}\frac{\mu}{\rho U}dx$

Integrating, $\frac{\delta^2}{2} = \frac{\pi^2}{4-\pi}\frac{\mu}{\rho U}x$ so $\frac{\delta}{x} = \sqrt{\frac{2\pi^2}{4-\pi}}\sqrt{\frac{\mu}{\rho U x}} = \frac{4.80}{\sqrt{Re_x}}$ ← δ/x

Also $\frac{\delta^*}{\delta} = \int_0^1 (1 - \frac{u}{U})\,d\lambda = \int_0^1 (1 - \sin\frac{\pi}{2}\lambda)\,d\lambda = (\lambda + \frac{2}{\pi}\cos\frac{\pi}{2}\lambda)\Big|_0^1 = 1 - \frac{2}{\pi} = 0.363$ ← δ^*/δ

Tabulate: $Re_x = \frac{Ux}{\nu} = 3.2\frac{m}{s}, x\,m \times \frac{s}{1.46\times10^{-5}m^2} = 2.19\times10^5\,x$

x (m)	Re_x (---)	$1/\sqrt{Re_x}$ (---)	δ (mm)	δ^* (mm)	τ_w (N/m²)
0.6	1.31×10^5	0.00276	7.95	2.89	0.0114
1.2	2.63×10^5	0.00195	11.2	4.07	0.00804
1.8	3.94×10^5	0.00159	13.7	4.97	0.00656



$\tau_w = \frac{\pi}{2}\frac{\mu}{\rho U \delta}\rho U^2$

$= \frac{1}{2}\rho U^2 \frac{\mu}{\rho U x}\frac{x}{\delta}\pi$

$\tau_w = \frac{\pi}{4.80}\frac{1}{\sqrt{Re_x}}\frac{1}{2}\rho U^2$

Plot ←

Given: Laminar boundary layer flow with velocity profile, $\frac{u}{U} = \frac{y}{\delta} = \eta$.

Find: Expressions for δ/x, C_f, using the momentum integral equation.

Solution: The momentum integral equation is

Computing equation: $-\delta \underbrace{\frac{\partial p}{\partial x}}_{=0(1)} - \tau_w = \frac{\partial}{\partial x}\int_0^\delta \rho u\,dy - U\frac{\partial}{\partial x}\int_0^\delta \rho u\,dy$

Assumptions: (1) Flat plate, so U = constant and $\frac{\partial p}{\partial x} = 0$
(2) δ is a function of x only, and $\delta = 0$ at $x = 0$
(3) Incompressible flow

Then

$$\tau_w = U\frac{\partial}{\partial x}\int_0^\delta \rho u\,dy - \frac{\partial}{\partial x}\int_0^\delta \rho u u\,dy = \frac{\partial}{\partial x}\int_0^\delta \rho u(U-u)\,dy$$

or $$\tau_w = \rho U^2 \frac{\partial \delta}{\partial x}\int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right)d\left(\frac{y}{\delta}\right) = \rho U^2 \frac{d\delta}{dx}\int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right)d\eta = \rho U^2 \beta \frac{d\delta}{dx}$$

Now use the given velocity profile:

$$\int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right)d\eta = \int_0^1 \eta(1-\eta)\,d\eta = \left[\tfrac{1}{2}\eta^2 - \tfrac{1}{3}\eta^3\right]_0^1 = \frac{1}{6} = \beta$$

$$\tau_w = \mu\left.\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\mu U}{\delta}\left.\frac{\partial(u/U)}{\partial(y/\delta)}\right]_{y/\delta=0} = \frac{\mu U}{\delta}\left.\frac{d(u/U)}{d\eta}\right]_{\eta=0} = \frac{\mu U}{\delta}$$

Substituting for β and τ_w,

$$\frac{\mu U}{\delta} = \rho U^2 \frac{d\delta}{dx}\left(\frac{1}{6}\right) \quad \text{or} \quad \delta\,d\delta = \frac{6\mu}{\rho U}\,dx$$

Integrating, $\frac{\delta^2}{2} = \frac{6\mu}{\rho U}x + C$, but $C = 0$ since $\delta = 0$ at $x = 0$. Thus

$$\delta = \sqrt{\frac{12\mu}{\rho U}x}$$

or

$$\frac{\delta}{x} = \sqrt{\frac{12\mu}{\rho U x}} = \frac{3.46}{\sqrt{Re_x}}$$

δ/x

Also

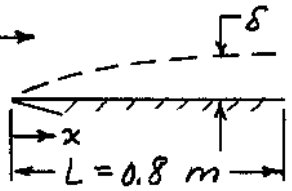
$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{\mu\frac{U}{\delta}}{\frac{1}{2}\rho U^2} = \frac{2\mu}{\rho U \delta} = 2\times\frac{\mu}{\rho U x}\frac{x}{\delta} = \frac{2}{Re_x}\frac{\sqrt{Re_x}}{3.46}$$

$$C_f = \frac{0.577}{\sqrt{Re_x}}$$

C_f

Given: Laminar boundary-layer flow of air, conditions as shown.

Assume $\frac{u}{U} = \frac{y}{\delta}$ $U = 5.3$ m/s → Assume STP.



Find: Plot δ, δ^*, and τ_w vs. x/L.

Solution: Apply the momentum integral equation, Eq. 9.19.

Computing equations: $\tau_w = \rho U^2 \frac{d\delta}{dx}\beta$ $\tau_w = \mu \left.\frac{\partial u}{\partial y}\right)_{y=0}$

Assume: (1) steady flow, (2) incompressible flow, (3) zero pressure gradient

Thus $\tau_w = \mu \left.\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\mu U}{\delta}\left.\frac{d(u/U)}{d(y/\delta)}\right]_{y=0} = \frac{\mu U}{\delta}(1) = \frac{\mu U}{\delta}$; $\frac{\delta^*}{\delta} = \int_0^1 (1-\lambda)\,d\lambda = \frac{1}{2}$

$$\beta = \frac{\theta}{\delta} = \int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right)d\left(\frac{y}{\delta}\right) = \int_0^1 \lambda(1-\lambda)\,d\lambda = \left[\frac{\lambda^2}{2} - \frac{\lambda^3}{3}\right]_0^1 = \frac{1}{6}$$

Substituting into the MIE, $\frac{\mu U}{\delta} = \rho U^2 \frac{d\delta}{dx}\left(\frac{1}{6}\right)$ so $\delta\,d\delta = 6\frac{\mu}{\rho U}dx$

Integrating, $\delta^2 = 12\frac{\mu}{\rho U}x + C$ so $\frac{\delta}{x} = \frac{\sqrt{12}}{\sqrt{Re_x}} = \frac{3.46}{\sqrt{Re_x}}$; $\frac{\delta^*}{x} = \frac{1.73}{\sqrt{Re_x}}$

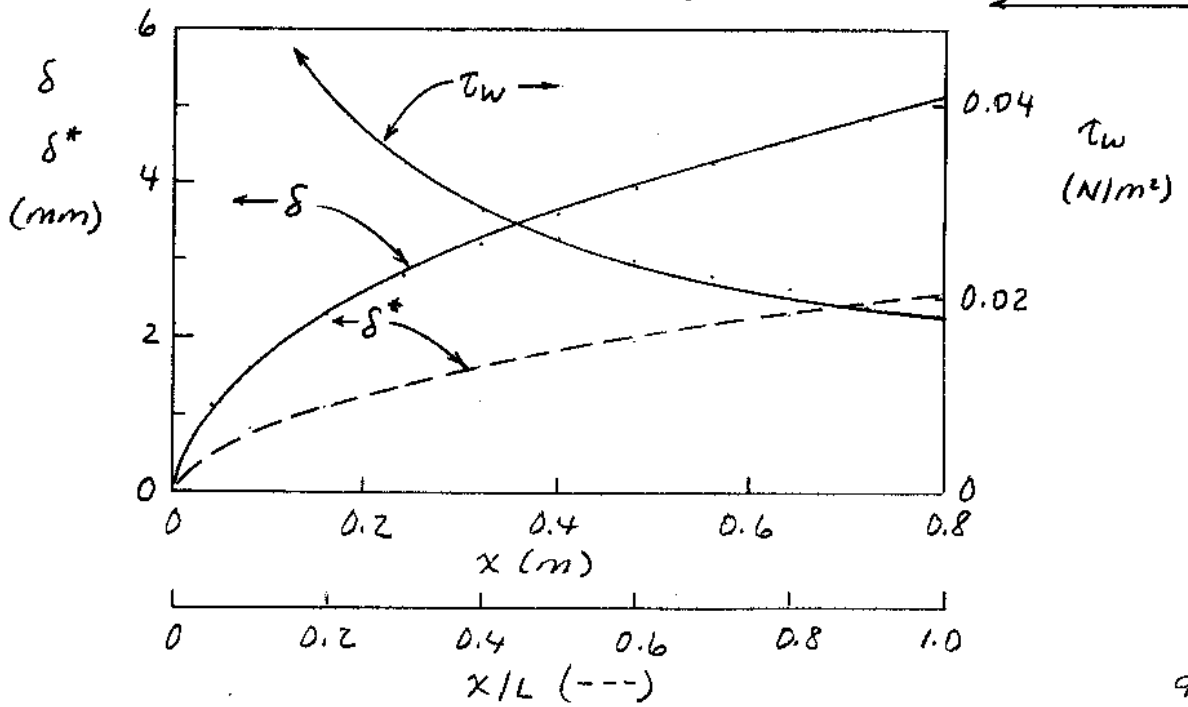
$$\tau_w = \frac{\mu U}{\delta} = \mu U \frac{\sqrt{Re_x}}{3.46x} = \frac{\rho\nu U}{x}\frac{\sqrt{Re_x}}{3.46}$$; $C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.578}{\sqrt{Re_x}}$

At $x = L$, $Re_L = \frac{UL}{\nu} = 5.3\frac{m}{s} \times 0.8\,m \times \frac{s}{1.46\times10^{-5}\,m^2} = 2.90\times10^5$, so laminar

$\frac{\delta}{L} = \frac{3.46}{\sqrt{2.92\times10^5}} = 0.00643$; $\delta_L = 0.00643L = 0.00643 \times 800\,mm = 5.14\,mm$ ← δ_L

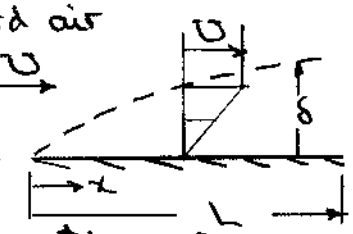
$\delta^* = \frac{1}{2}\delta$, so $\delta^*_L = \frac{1}{2} \times 5.14\,mm = 2.57\,mm$ ← δ^*_L

$$\tau_w(x=L) = \frac{0.578}{\sqrt{2.90\times10^5}} \times \frac{1}{2} \times 1.23\frac{kg}{m^3}(5.3)^2\frac{m^2}{sec^2} \times \frac{N\cdot sec^2}{kg\cdot m} = 0.0185\,N/m^2$$ ← τ_{wL}



Given: Laminar boundary layer forms on flat plate of length $L = 0.8\ m$ and width $b = 1.9\ m$. Free stream velocity, $U = 5.3\ m/s$. Velocity profile in the boundary layer is linear. Standard air

Find: (a) algebraic expression for $\tau_w(x)$
(b) algebraic expression for F_D
(c) magnitude of F_D.



Solution: Apply the momentum integral equation, Eq 9.19

Computing eqs. $\tau_w = \rho U^2 \frac{d\theta}{dx}$, $\theta = \delta \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right)$

For a linear profile $\theta = \delta \int_0^1 \eta(1-\eta)\,d\eta = \delta\left[\frac{\eta^2}{2} - \frac{\eta^3}{3}\right]_0^1 = \frac{1}{6}\delta$

$$\therefore \tau_w = \mu \left.\frac{du}{dy}\right|_{y=0} = \mu \frac{U}{\delta} \left.\frac{d(u/U)}{d\eta}\right|_{\eta=0} = \frac{\mu U}{\delta} = \rho U^2 \frac{1}{6}\frac{d\delta}{dx}$$

Separating variables and integrating gives

$$\delta\, d\delta = 6\frac{\mu}{\rho U}\, dx$$

Then $\frac{1}{2}\delta^2 = 6\frac{\mu}{\rho U}x + C$. Since $\delta = 0$ at $x = 0$ then

$$\delta = \left[\frac{12\mu x}{\rho U}\right]^{1/2} = 3.46\left[\frac{\mu x}{\rho U}\right]^{1/2} \quad \text{or} \quad \frac{\delta}{x} = \frac{3.46}{\sqrt{Re_x}}$$

$$\tau_w = \mu\frac{U}{\delta} = \frac{1}{3.46}\frac{\mu^{1/2}\rho^{1/2}U^{3/2}}{x^{1/2}} = 0.289\, U\frac{\mu}{x}\sqrt{Re_x} \longleftarrow \tau_w(x)$$

The drag force is given by

$$F_D = \int \tau_w\, dA = \int_0^L \tau_w\, b\, dx = b\int_0^L \rho U^2 \frac{d\theta}{dx}\, dx = b\int_0^{\theta_L} \rho U^2\, d\theta$$

$$F_D = \rho U^2 b\,\theta_L \longleftarrow F_D$$

where $\theta_L = \frac{1}{6}\delta_L$ and $\delta_L = \frac{3.46\,L}{\sqrt{Re_L}}$

For given conditions

$$Re_L = \frac{UL}{\nu} = 5.3\frac{m}{s} \times 0.8\,m \times \frac{s}{1.46\times10^{-5}\,m^2} = 2.90\times10^5$$

$$\delta_L = \frac{3.46L}{\sqrt{Re_L}} = 3.46\times0.8\,m \times \frac{1}{(2.90\times10^5)^{1/2}} = 5.14\,mm$$

$$\theta_L = \frac{1}{6}\delta_L = 0.857\,mm$$

$$F_D = \rho U^2 b\theta_L = 1.23\frac{kg}{m^3} \times (5.3)^2\frac{m^2}{s^2} \times 1.9m \times 0.854\times10^{-3}\,m \times \frac{N\cdot s^2}{kg\cdot m}$$

$$F_D = 5.63\times10^{-2}\ N \longleftarrow F_D$$

Given: Water at 15°C flows over a flat plate at $U = 1$ m/s. The plate is $L = 0.4$ m long and $b = 1$ m wide. The boundary layer on each surface of the plate is laminar. The velocity profile is approximated as linear, $u/U = y/\delta = \eta$.

Find: The drag force on the plate

Solution: There are no pressure forces on a flat plate. The drag is due to skin friction, acting on both sides of the plate,

$$F_D = 2\int_0^L \tau_w \, b\,dx \qquad (1)$$

Apply the momentum integral equation to find $\tau_w(x)$.

Computing equation: $\tau_w = \rho U^2 \frac{d\delta}{dx}\int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right)d\eta = \rho U^2 \frac{d\delta}{dx}\beta \qquad (9.19)$

Assumptions: (1) Steady flow, (2) Incompressible flow, (3) $\frac{\partial p}{\partial x} = 0$

Then

$$\beta = \int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right)d\eta = \int_0^1 \eta(1-\eta)\,d\eta = \int_0^1 (\eta-\eta^2)\,d\eta = \frac{\eta^2}{2} - \frac{\eta^3}{3}\Big]_0^1 = \frac{1}{6}$$

and

$$\tau_w = \mu\left.\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\mu U}{\delta}\left.\frac{d(u/U)}{d\eta}\right]_{\eta=0} = \frac{\mu U}{\delta}(1) = \frac{\mu U}{\delta}$$

Substituting in Eq. 9.19,

$$\tau_w = \frac{\mu U}{\delta} = \rho U^2 \frac{d\delta}{dx}\beta \quad \text{or} \quad \delta\,d\delta = \frac{\mu}{\rho U \beta}dx = \frac{6\mu}{\rho U}dx$$

Integrating, $\frac{\delta^2}{2} = \frac{6\mu}{\rho U}x + C$, but $C = 0$ since $\delta = 0$ at $x = 0$. Thus

$$\frac{\delta}{x} = \sqrt{\frac{12\mu}{\rho U x}} = \frac{\sqrt{12}}{\sqrt{Re_x}} = \frac{3.46}{\sqrt{Re_x}}$$

and $\tau_w = \frac{\mu U}{\delta} = \mu U \frac{\sqrt{Re_x}}{3.46x} = \frac{\mu U}{3.46}\sqrt{\frac{U}{\nu}}\,x^{-1/2}$. Substituting into Eq. 1,

$$F_D = 2\int_0^L \frac{\mu U}{3.46}\sqrt{\frac{U}{\nu}}\,x^{-1/2}\,b\,dx = \frac{2\mu U}{3.46}\sqrt{\frac{U}{\nu}}\int_0^L x^{-1/2}b\,dx = \frac{2\mu U}{3.46}\sqrt{\frac{U}{\nu}}\left[2x^{1/2}\right]_0^L b$$

$$F_D = \frac{4\mu U b}{3.46}\sqrt{\frac{U}{\nu}}L^{1/2} = 1.16\mu U b\sqrt{Re_L}$$ (Table A.8)

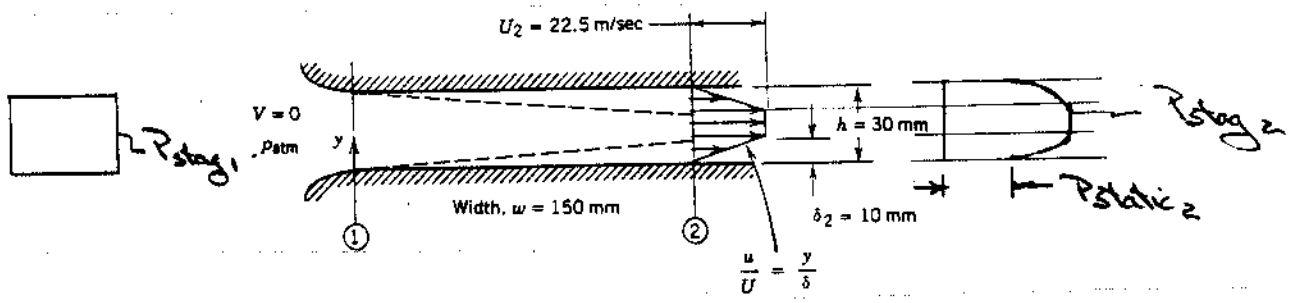
From the given data, $Re_L = \frac{UL}{\nu} = \frac{1\ m}{s} \times 0.4\ m \times \frac{s}{1.14\times10^{-6}\ m^2} = 3.51\times10^5$

Thus

$$F_D = 1.16 \times 1.14\times10^{-3}\ \frac{N\cdot s}{m^2} \times 1\ \frac{m}{s} \times 1\ m\sqrt{3.51\times10^5} = 0.783\ N$$ F_D

{Note flow is laminar because $Re_L = 3.51\times10^5 < 5\times10^5$.}

Given: Standard air flows from the atmosphere into the wide, flat channel shown. Laminar boundary layers form on the top and bottom walls of the channel (neglect boundary-layer effects on the side walls). Assume flat plate boundary layer behavior with linear velocity profile. Static pressure is uniform across the channel at each section.



Compare: (a) static pressure at section 1 to atmospheric pressure.
(b) stagnation pressure at section 1 to atmospheric pressure.

Plot: stagnation pressure across channel at section 1 (assuming uniform flow) and section 2

Comment: on use of Bernoulli equation in this flow field.

Determine: displacement thickness at section 2 and indicate how to determine the static pressure at section 1.

Solution:

The static pressure at section ① is less than atmospheric and the stagnation pressure is equal to atmospheric pressure.

The Bernoulli equation can be applied along the center streamline. Thus

$$p_{atm} = p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2$$

The displacement thickness at section 2 is determined from the definition of δ^*

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \delta \int_0^1 \left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right) = \delta \int_0^1 \left(1 - \frac{u}{U}\right) d\eta$$

$$\delta^* = \delta \int_0^1 (1 - \eta)\, d\eta = \delta \left[\eta - \frac{\eta^2}{2}\right]_0^1 = \frac{\delta}{2} = 5 \text{ mm} \quad \leftarrow \delta^*$$

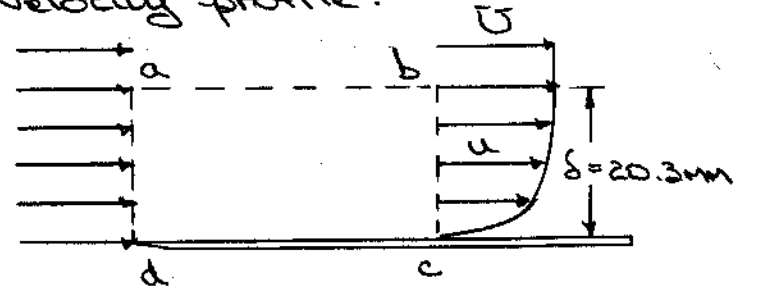
To determine the static pressure at section ①

- use δ_2^* to calculate the effective area at ②
  $A = (h - 2\delta_2^*)b$ where b is the channel depth
- apply continuity between sections ① & ② to find V_1
  $V_1 h b = V_2 (h - 2\delta_2^*) b$
- apply Bernoulli equation between atmosphere and section ① to find p_1

The stagnation pressure at section ① is uniform and equal to p_{atm}

The stagnation pressure at section ② is given by $p_0 = p_2 + \frac{1}{2}\rho u^2$ where $u = u(y)$ and $p_2 < p_1$

Given: Standard air flow over a flat plate of width $b = 3.2\,m$. Boundary layer is turbulent with a "1/7-power" velocity profile.



$U = 50.2\,m/s$

At bc

$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7} = \eta^{1/7}$

Find: (a) mass flow rate across surface ad
(b) mass flux across ab
(c) x-momentum flux across bc
(d) drag force on plate between d and c
(e) estimate of distance from leading edge at which transition may be expected

Solution:

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ (4.13) (=0 (1))

$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u(\rho \vec{V}\cdot d\vec{A})$ (4.19a) (0 (3); =0 (1))

Assumptions: (1) steady flow (2) no pressure forces
(3) $F_{Bx} = 0$ (4) uniform flow at da.

The mass flow rate across ad is $\dot{m}_{ad} = \int \rho \vec{V}\cdot d\vec{A} = -\rho U A = -\rho U \delta b$

$\dot{m}_{ad} = -1.23\,\frac{kg}{m^3} \times 50.2\,\frac{m}{s} \times 20.3\times10^{-3}\,m \times 3.2\,m = -4.01\,kg/s$ $\dot{m}_{ad}$

$\dot{m}_{ab} = \rho \int_0^\delta u\, b\, dy = \rho b U \int_0^\delta \frac{u}{U}\, dy = \rho b U \delta \int_0^1 \frac{u}{U}\, d\left(\frac{y}{\delta}\right) = \rho b U \delta \int_0^1 \eta^{1/7}\, d\eta$

$\dot{m}_{bc} = \rho b U \delta \frac{7}{8} = \frac{7}{8} \times 1.23\,\frac{kg}{m^3} \times 3.2\,m \times 50.2\,\frac{m}{s} \times 20.3\times10^{-3}\,m = 3.51\,kg/s$

From continuity, $0 = \dot{m}_{ad} + \dot{m}_{ab} + \dot{m}_{bc} = -4.01\,kg/s + \dot{m}_{ab} + 3.51\,kg/s$

$\therefore \dot{m}_{ab} = 0.50\,kg/s$ (out) $\dot{m}_{ab}$

The x-momentum flux across bc is

$mf_{bc} = \int_0^\delta u\, \rho \vec{V}\cdot d\vec{A} = \int_0^\delta \rho u^2 b\, dy = \rho b U^2 \delta \int_0^1 \left(\frac{u}{U}\right)^{2/7} d\left(\frac{y}{\delta}\right) = \rho b U^2 \delta \int_0^1 \eta^{2/7}\, d\eta$

$mf_{bc} = \frac{7}{9}\rho b U^2 \delta = \frac{7}{9} \times 1.23\,\frac{kg}{m^3} \times 3.2\,m \times (50.2)^2\,\frac{m^2}{s^2} \times 20.3\times10^{-3}\,m = 157\,kg\cdot m/s^2$ mf_{bc}

From the momentum equation,

$-F_D = \int u \rho \vec{V}\cdot d\vec{A} = -\rho U^2 b\delta + \dot{m}_{ab} U + mf_{bc}$

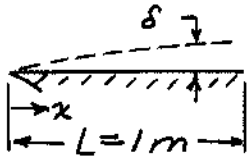
$-F_D = -1.23\,\frac{kg}{m^3} \times (50.2)^2\,\frac{m^2}{s^2} \times 3.2\,m \times 20.3\times10^{-3}\,m + 0.50\,\frac{kg}{s} \times 50.2\,\frac{m}{s} + 157\,kg\cdot m/s^2$

$F_D = 19.3 \frac{kg}{s^2 \cdot m} \cdot \frac{N \cdot s^2}{kg \cdot m} = 19.3\ N$ ← F_D

Transition occurs at $Re = \frac{\rho V L}{\mu} = 5 \times 10^5$ $\quad \nu = 1.46 \times 10^{-5}\ m^2/s$ (Table A.10)

$\therefore L = 5 \times 10^5 \frac{\mu}{\rho V} = 5 \times 10^5 \times 1.46 \times 10^{-5} \frac{s}{m^2} \cdot \frac{s}{80.2\ m} = 0.145\ m$ ← L_{cr}

Given: Turbulent boundary-layer flow of water, conditions of Example Problem 9.4. $U = 1\ m/s$ → Assume $\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}$



Find: Plot δ, δ^*, and τ_w versus distance.

Solution: Apply the results of Example Problem 9.4.

Computing equations: $\frac{\delta}{x} = \frac{0.382}{(Re_x)^{1/5}}$; $\frac{\delta^*}{\delta} = \frac{1}{8}$; $C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.0594}{(Re_x)^{1/5}}$

Assume: (1) BL turbulent from $x = 0$ (i.e. tripped)

For conditions given: $Re_L = \frac{UL}{\nu} = \frac{1\ m}{s} \times 1\ m \times \frac{s}{1 \times 10^{-6} m^2} = 10^6$

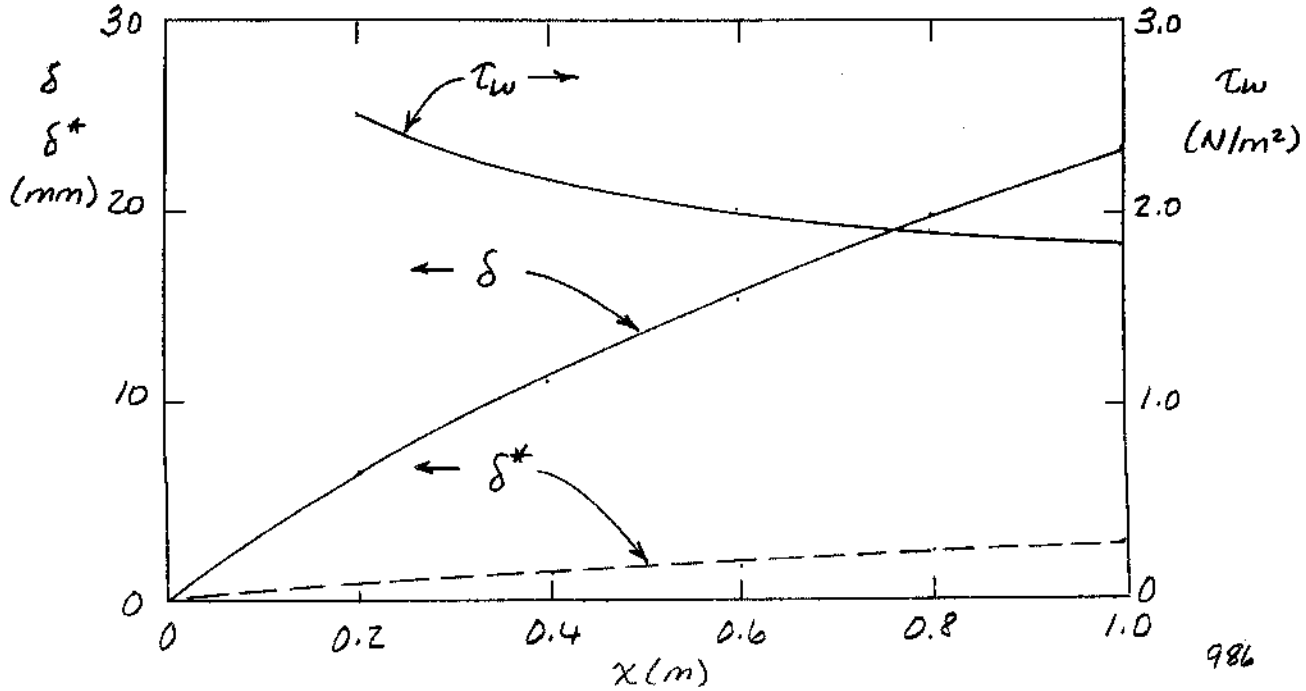
$$q = \frac{1}{2}\rho U^2 = \frac{1}{2} \times 999 \frac{kg}{m^3} \times (1)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 500\ N/m^2$$

$$\tau_w = 0.0594 \times 500 \frac{N}{m^2} \times \frac{1}{(Re_x)^{1/5}} = \frac{29.7}{(Re_x)^{1/5}}$$

Tabulate:

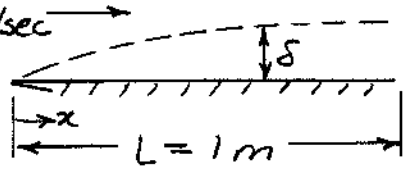
x (m)	Re_x (---)	$Re_x^{1/5}$ (---)	δ (mm)	δ^* (mm)	τ_w (N/m^2)
0.2	2×10^5	11.5	6.64	0.830	2.58
0.4	4×10^5	13.2	11.6	1.45	2.25
0.6	6×10^5	14.3	16.0	2.00	2.08
0.8	8×10^5	15.2	20.1	2.51	1.95
1.0	10×10^5	15.8	24.2	3.03	1.88

Plot:



Given: Turbulent boundary-layer flow of water, $\frac{1}{7}$-power profile.

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}$$



{ Conditions of Example Problem 9.4. }

Find: (a) Expression for wall shear stress, τ_w.
(b) Integrate to obtain expression for drag force, F_D.
(c) Evaluate for the conditions shown.

Solution: Apply results from the momentum integral equation.

Computing equation: $C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.0594}{(Re_x)^{1/5}}$ (9.27)

Solving for τ_w,

$$\tau_w = \frac{1}{2}\rho U^2 \frac{0.0594}{(Re_x)^{1/5}} = 0.0594\left(\frac{1}{2}\rho U^2\right)\left(\frac{U}{\nu}\right)^{-1/5} x^{-1/5}$$ $\tau_w(x)$

Integrating to obtain F_D,

$$F_D = \int_0^L \tau_w b\,dx = \int_0^L \underbrace{0.0594\left(\frac{1}{2}\rho U^2\right)\left(\frac{U}{\nu}\right)^{-1/5}}_{C} x^{-1/5} b\,dx = Cb\int_0^L x^{-1/5}dx$$

$$= Cb\,\frac{5}{4}x^{4/5}\Big]_0^L = \frac{5}{4}Cb\,L^{4/5} = \frac{5}{4}CbL\,L^{-1/5}$$

$$F_D = \frac{5}{4}(0.0594)\frac{1}{2}\rho U^2 bL\left(\frac{U}{\nu}\right)^{-1/5}L^{-1/5} = \frac{1}{2}\rho U^2 bL\,\frac{0.0721}{(Re_L)^{1/5}}$$ F_D

Evaluating, with $b = 1$ m,

$$Re_L = \frac{UL}{\nu} = 1\frac{m}{s} \times 1m \times \frac{s}{1\times10^{-6}m^2} = 1.00\times10^6 \quad (T = 20°C, \text{Table A.8})$$

$$F_D = \frac{1}{2}\times 999\frac{kg}{m^3}\times(1)^2\frac{m^2}{s^2}\times 1m\times 1m\times\frac{0.0743}{(10^6)^{1/5}}\times\frac{N\cdot s^2}{kg\cdot m} = 2.34\,N$$ F_D

Given: Turbulent boundary layer with velocity profile, $\frac{u}{U} = \eta^{1/6}$; $\eta = \frac{y}{\delta}$.

Find: Expressions for δ/x, C_f, using momentum integral equation. Compare with results of "1/7-power" profile, Section 9-5.2.

Solution: The momentum integral equation is

Computing equation: $-\delta \frac{\partial p}{\partial x}^{=0(1)} - \tau_w = \frac{\partial}{\partial x}\int_0^\delta \rho u(U-u)dy - U\frac{\partial}{\partial x}\int_0^\delta \rho(U-u)dy$

Assumptions: (1) Flat plate, so U = constant and $\frac{\partial p}{\partial x} = 0$
(2) δ is a function of x only; $\delta = 0$ at $x = 0$
(3) Incompressible flow
(4) $\tau_w = 0.0233\,\rho U^2 (\nu/U\delta)^{1/4}$

Then

$$\tau_w = U\frac{\partial}{\partial x}\int_0^\delta \rho u dy - \int_0^\delta \rho u^2 dy = \frac{\partial}{\partial x}\int_0^\delta \rho u(U-u)dy$$

or

$$\tau_w = \rho U^2 \frac{d\delta}{dx}\int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right)d\left(\frac{y}{\delta}\right) = \rho U^2 \frac{d\delta}{dx}\beta$$

Evaluating β,

$$\beta = \int_0^1 \eta^{1/6}(1-\eta^{1/6})d\eta = \left[\frac{6}{7}\eta^{7/6} - \frac{6}{8}\eta^{8/6}\right]_0^1 = \frac{6}{56}$$

Substituting

$$0.0233\,\rho U^2\left(\frac{\nu}{U\delta}\right)^{\frac{1}{4}} = \rho U^2 \frac{d\delta}{dx}\beta \quad \text{or} \quad \delta^{1/4}d\delta = \frac{0.0233}{\beta}\left(\frac{\nu}{U}\right)^{\frac{1}{4}}dx$$

Integrating $\frac{4}{5}\delta^{5/4} = \frac{0.0225}{\beta}\left(\frac{\nu}{U}\right)^{\frac{1}{4}}x + C$, but $C = 0$, since $\delta = 0$ at $x = 0$.

Thus $$\delta = \left[\frac{5}{4}\frac{0.0233}{\beta}\left(\frac{\nu}{U}\right)^{\frac{1}{4}}x\right]^{4/5} = 0.353\left(\frac{\nu}{U}\right)^{\frac{1}{5}}x^{\frac{4}{5}}$$

and

$$\frac{\delta}{x} = 0.353\left(\frac{\nu}{Ux}\right)^{\frac{1}{5}} = \frac{0.353}{(Re_x)^{1/5}} \qquad \frac{\delta}{x}$$

Also

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.0233\,\rho U^2\left(\frac{\nu}{U\delta}\right)^{\frac{1}{4}}}{\frac{1}{2}\rho U^2} = 0.0466\left(\frac{\nu}{Ux}\right)^{\frac{1}{4}}\left(\frac{x}{\delta}\right)^{\frac{1}{4}}$$

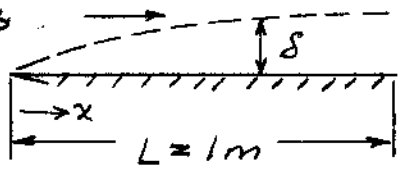
$$= 0.0466\left(\frac{\nu}{Ux}\right)^{\frac{1}{4}}\left(\frac{1}{0.353}\right)^{\frac{1}{4}}\left[\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}\right]^{\frac{1}{4}}$$

$$C_f = \frac{0.0605}{(Re_x)^{1/5}} \qquad C_f$$

Comparing:	$\frac{\delta/x\,(Re_x)^{1/5}}{}$	$\frac{C_f\,(Re_x)^{1/5}}{}$
1/6-power	0.353	0.0605
1/7-power	0.382	0.0594

Given: Turbulent boundary-layer flow of water, $\frac{1}{6}$-power profile.

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/6} \qquad U = 1 \text{ m/s}$$



{Flow conditions of Example Problem 9.4.}

Find: (a) Expression for the ratio, δ/x
(b) Expression for the skin friction coefficient, C_f.
(c) Evaluate F_D for the conditions shown.

Solution: Apply the momentum integral equation.

Computing equation: $\tau_w = \rho U^2 \frac{d\theta}{dx} = \rho U^2 \frac{d\delta}{dx}\beta$ (9.19)

Assumptions: (1) Flat plate, so U = constant and $\frac{\partial p}{\partial x} = 0$
(2) $\delta = \delta(x)$ only; $\delta = 0$ at $x = 0$
(3) Incompressible flow
(4) Wall shear correlation, $\tau_w = 0.0233 \rho U^2 \left(\frac{\nu}{U\delta}\right)^{\frac{1}{4}}$ (9.25)

Now $\beta = \frac{\theta}{\delta} = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right) = \int_0^1 \lambda^{1/6}(1 - \lambda^{1/6})\, d\lambda = \int_0^1 (\lambda^{1/6} - \lambda^{2/6})\, d\lambda \; ; \; \lambda = \frac{y}{\delta}$

$$\beta = \frac{6}{7}\lambda^{7/6} - \frac{6}{8}\lambda^{8/6}\Big]_0^1 = \frac{6(8) - 6(7)}{7(8)} = \frac{6}{56} = 0.107$$

Substituting into the MIE,

$$0.0233 \rho U^2 \left(\frac{\nu}{U\delta}\right)^{\frac{1}{4}} = \rho U^2 \frac{d\delta}{dx}(0.107) \quad \text{or} \quad \delta^{1/4}\, d\delta = 0.218 \left(\frac{\nu}{U}\right)^{\frac{1}{4}} dx$$

Integrating, $\frac{4}{5}\delta^{5/4} = 0.218\left(\frac{\nu}{U}\right)^{\frac{1}{4}} x + C$, but $C = 0$, since $\delta(0) = 0$.

Thus $\delta = \left[0.218 \left(\frac{5}{4}\right)\left(\frac{\nu}{U}\right)^{1/4} x\right]^{4/5} = 0.353 \left(\frac{\nu}{U}\right)^{1/5} x^{4/5}$

and $\frac{\delta}{x} = 0.353 \left(\frac{\nu}{U}\right)^{1/5} x^{-1/5} = \frac{0.353}{(Re_x)^{1/5}}$ δ/x ←

From the wall shear correlation,

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.0233 \rho U^2 \left(\frac{\nu}{U\delta}\right)^{1/4}}{\frac{1}{2}\rho U^2} = 0.0466 \left(\frac{\nu}{U\delta}\right)^{1/4} = 0.0466 \left(\frac{\nu}{Ux}\right)^{1/4}\left(\frac{x}{\delta}\right)^{1/4}$$

$$C_f = 0.0466 \left(\frac{\nu}{Ux}\right)^{1/4}\left[\frac{(Ux/\nu)^{1/5}}{0.353}\right]^{1/4} = \frac{0.0612}{(Re_x)^{1/5}}$$ C_f ←

The drag force is $F_D = \int_0^L \tau_w\, b\, dx = \int_0^L 0.0612\, \frac{1}{2}\rho U^2 \left(\frac{\nu}{U}\right)^{1/5} x^{-1/5} b\, dx$

$$F_D = 0.0612\, \frac{1}{2}\rho U^2 \left(\frac{\nu}{U}\right)^{1/5} b \int_0^L x^{-1/5} dx = 0.0765\, \frac{1}{2}\rho U^2 bL \left(\frac{\nu}{UL}\right)^{1/5}$$

$$F_D = \frac{0.0765}{2} \times 999 \frac{\text{kg}}{\text{m}^3} (1)^2 \frac{\text{m}^2}{\text{s}^2} \times 1\text{m} \times 1\text{m} \times \left(\frac{1}{10^6}\right)^{1/5} \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} = 2.41 \text{ N}$$ F_D ←

Given: Turbulent boundary layer with velocity profile, $\frac{u}{U} = \eta^{1/8}$; $\eta = \frac{y}{\delta}$.

Find: Expressions for δ/x, C_f, using momentum integral equation. Compare with results of "1/7-power" profile, Section 9-5.2.

Solution: The momentum integral equation is

Computing equation: $$-\delta \frac{\partial p}{\partial x} - \tau_w = \frac{\partial}{\partial x}\int_0^\delta u\rho u\,dy - U\frac{\partial}{\partial x}\int_0^\delta \rho u\,dy$$

Assumptions: (1) Flat plate, so U = constant and $\frac{\partial p}{\partial x} = 0$
(2) δ is a function of x only; $\delta = 0$ at $x = 0$
(3) Incompressible flow
(4) $\tau_w = 0.0233\,\rho U^2 (\nu/U\delta)^{1/4}$

Then
$$\tau_w = U\frac{\partial}{\partial x}\int_0^\delta \rho u\,dy - \int_0^\delta u\rho u\,dy = \rho U^2 \frac{d\delta}{dx}\int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right)d\eta = \rho U^2 \frac{d\delta}{dx}\beta$$

Evaluating β,
$$\beta = \int_0^1 \eta^{1/8}(1-\eta^{1/8})\,d\eta = \left[\frac{8}{9}\eta^{9/8} - \frac{8}{10}\eta^{10/8}\right]_0^1 = \frac{8}{90}$$

Substituting,
$$0.0233\,\rho U^2\left(\frac{\nu}{U\delta}\right)^{\frac{1}{4}} = \rho U^2 \frac{d\delta}{dx}\beta \quad \text{or} \quad \delta^{1/4}\,d\delta = \frac{0.0233}{\beta}\left(\frac{\nu}{U}\right)^{\frac{1}{4}} dx$$

Integrating $\frac{4}{5}\delta^{5/4} = \frac{0.0225}{\beta}\left(\frac{\nu}{U}\right)^{\frac{1}{4}} x + C$, but $C = 0$, since $\delta = 0$ at $x = 0$.

Thus
$$\delta = \left[\frac{5}{4}\,\frac{0.0233}{\beta}\left(\frac{\nu}{U}\right)^{\frac{1}{4}} x\right]^{4/5} = 0.410\left(\frac{\nu}{U}\right)^{\frac{1}{5}} x^{4/5}$$

and
$$\frac{\delta}{x} = 0.410\left(\frac{\nu}{Ux}\right)^{1/5} = \frac{0.410}{(Re_x)^{1/5}} \qquad \frac{\delta}{x}$$

Also
$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.0233\,\rho U^2\left(\frac{\nu}{U\delta}\right)^{1/4}}{\frac{1}{2}\rho U^2} = 0.0466\left(\frac{\nu}{Ux}\right)^{\frac{1}{4}}\left(\frac{x}{\delta}\right)^{\frac{1}{4}}$$

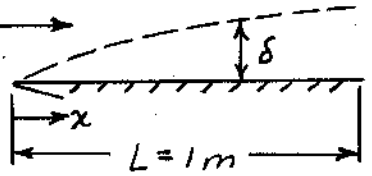
$$C_f = 0.0466\left(\frac{\nu}{Ux}\right)^{\frac{1}{4}}\left(\frac{1}{0.410}\right)^{\frac{1}{4}}\left[\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}\right]^{\frac{1}{4}} = \frac{0.0582}{(Re_x)^{1/5}} \qquad C_f$$

Comparing:

Profile	$\frac{\delta}{x}(Re_x)^{1/5}$	$C_f (Re_x)^{1/5}$
1/8-power	0.410	0.0582
1/7-power	0.382	0.0594

Given: Turbulent boundary-layer flow of water, $\frac{1}{8}$-power profile.

$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{8}}$ $U = 1$ m/s



{Flow conditions of Example Prob 9.4}

Find: (a) Expression for $\tau_w(x)$; (b) Expression for F_D; (c) Evaluate.

Solution: Apply the momentum integral equation.

Computing equation: $\tau_w = \rho U^2 \frac{d\theta}{dx} = \rho U^2 \frac{d\delta}{dx} \beta$ (9.19)

Assumptions: (1) Flat plate, so U = constant and $\frac{\partial p}{\partial x} = 0$
(2) $\delta = \delta(x)$ only; $\delta = 0$ at $x = 0$
(3) Incompressible, steady flow
(4) Blasius correlation, $\tau_w = 0.0332\, \rho \bar{V}^2 \left[\frac{\nu}{R\bar{V}}\right]^{\frac{1}{4}}$ (8.39)

Now $\frac{\bar{V}}{U} = \frac{2n^2}{(n+1)(2n+1)} = \frac{2(8)^2}{(8+1)(16+1)} = 0.837$; $\bar{V} = 0.837U$, so $\tau_w = 0.0243 \rho U^2 \left(\frac{\nu}{U\delta}\right)^{\frac{1}{4}}$

$$\beta = \frac{\theta}{\delta} = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right) = \int_0^1 \lambda^{1/8}(1-\lambda^{1/8})\, d\lambda = \frac{8}{9}\lambda^{9/8} - \frac{8}{10}\lambda^{10/8}\Big]_0^1 = \frac{8}{90}$$

Substituting into the MIE,

$$0.0243 \rho U^2 \left(\frac{\nu}{U\delta}\right)^{\frac{1}{4}} = \rho U^2 \frac{d\delta}{dx}\frac{8}{90} \quad \text{or} \quad \delta^{1/4} d\delta = 0.274 \left(\frac{\nu}{U}\right)^{\frac{1}{4}} dx$$

Integrating, $\frac{4}{5}\delta^{5/4} = 0.274 \left(\frac{\nu}{U}\right)^{\frac{1}{4}} x + C$, but $C = 0$ since $\delta = 0$ at $x = 0$.

Thus $\frac{\delta}{x} = \left[\frac{5}{4}(0.274)\left(\frac{\nu}{Ux}\right)^{1/4}\right]^{4/5} = \frac{0.424}{Re_x^{1/5}}$, and from the Blasius correlation,

$$\tau_w(x) = 0.0243 \rho U^2 \left(\frac{\nu}{Ux}\frac{Re_x^{1/5}}{0.424}\right)^{1/4} = 0.0301\, \rho U^2\, Re_x^{-1/5}$$ $\tau_w(x)$

The drag force is (b = width of plate),

$$dF_D = \tau_w b\, dx \quad \text{or} \quad F_D = \int_0^L \tau_w b\, dx = \int_0^L 0.0301 \rho U^2 \left(\frac{\nu}{Ux}\right)^{1/5} b\, dx$$

Thus $F_D = 0.0301 \rho U^2 \left(\frac{\nu}{U}\right)^{1/5} b \int_0^L x^{-1/5} dx = 0.0301 \rho U^2 \left(\frac{\nu}{U}\right)^{1/5} b \left[\frac{5}{4}x^{4/5}\right]_0^L$

$$F_D = \rho U^2 bL \frac{0.0376}{Re_L^{1/5}}$$ F_D

Evaluating, for $b = 1$ m, $Re_L = \frac{UL}{\nu} = \frac{1\,m}{s} \times 1\,m \times \frac{s}{1\times10^{-6}\,m^2} = 10^6$

$$F_D = 999\,\frac{kg}{m^3} \times (1)^2 \frac{m^2}{s^2} \times 1\,m \times 1\,m \times \frac{0.0376}{(10^6)^{1/5}} \times \frac{N\cdot s^2}{kg\cdot m} = 2.37\ N$$ F_D

Comparing:

Profile	$\delta(x)\, Re_x^{1/5}$	$\tau_w(x)\, Re_x^{1/5}/\rho U^2$
1/8-power	0.424	0.0301
1/7-power	0.382	0.0297

Given: Air at standard conditions flows at 15 m/s over a flat plate.

Find: δ and τ_w at a point 1 m from leading edge for (a) completely laminar flow (parabolic velocity profile) (b) completely turbulent flow ("1/7-power velocity profile")

Solution.

Computing equations:

	Laminar Flow	Turbulent Flow
$\frac{\delta}{x}$ =	$\frac{5.48}{\sqrt{Re_x}}$	$\frac{0.382}{Re_x^{1/5}}$
c_f =	$\frac{0.730}{\sqrt{Re_x}}$	$\frac{0.0594}{Re_x^{1/5}}$

For standard air, $\rho = 1.23\ kg/m^3$, $\nu = 1.46\times10^{-5}\ m^2/s$ (Table A.6)

The Reynolds number is

$$Re_x = \frac{Ux}{\nu} = 15\frac{m}{s}\times 1m \times 1.46\times10^{-5}\frac{s}{m^2} = 1.03\times10^6$$

For laminar flow

$$\delta = \frac{5.48x}{\sqrt{Re_x}} = 5.48\times1m\times\frac{1}{(1.03\times10^6)^{1/2}} = 5.40\ mm \quad \leftarrow \delta_{lam}$$

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.730}{\sqrt{Re_x}} \quad \therefore \tau_w = \frac{1}{2}\rho U^2 \frac{0.730}{\sqrt{Re_x}}$$

$$\tau_w = \frac{1}{2}\times1.23\frac{kg}{m^3}\times(15)^2\frac{m^2}{s^2}\times\frac{0.730}{(1.03\times10^6)^{1/2}}\times\frac{N\cdot s^2}{kg\cdot m} = 0.0995\ N/m^2 \quad \leftarrow \tau_{lam}$$

For turbulent flow

$$\delta = \frac{0.382x}{Re_x^{1/5}} = 0.382\times1m\times\frac{1}{(1.03\times10^6)^{0.2}} = 24.0mm \quad \leftarrow \delta_{turb}$$

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.0594}{Re_x^{1/5}} \quad \therefore \tau_w = \frac{1}{2}\rho U^2 \frac{0.0594}{Re_x^{1/5}}$$

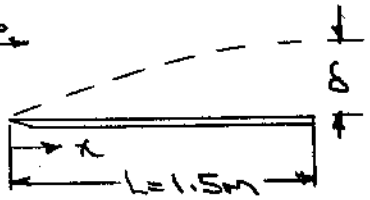
$$\tau_w = \frac{1}{2}\times1.23\frac{kg}{m^3}\times(15)^2\frac{m^2}{s^2}\times\frac{0.0594}{(1.03\times10^6)^{0.2}}\times\frac{N\cdot s^2}{kg\cdot m} = 0.516\ N/m^2 \quad \leftarrow \tau_{turb}$$

Comparing,

$$\frac{\delta_{turb}}{\delta_{lam}} = 4.4 \quad \text{and} \quad \frac{\tau_{w\,turb}}{\tau_{w,lam}} = 5.2$$

Thus the turbulent boundary layer has a much larger skin friction which causes it to grow more rapidly

Given: Air flow over smooth flat plate as shown; width $b = 0.8\,m$.
BL tripped, so turbulent $U = 14.5\,m/s$
Velocity profile is 1/7-power.



Find: (a) δ at $x = L$, (b) τ_w at $x = L$
(c) Drag on portion $0.5m < x < L$.

Solution:

Computing equations: $\frac{\delta}{x} = \frac{0.382}{Re_x^{1/5}}$ $\quad C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.0594}{Re_x^{1/5}}$

Assumptions: (1) steady flow, (2) incompressible flow
(3) zero pressure gradient.
(4) standard air ($T = 15°C$)

$$Re_L = \frac{UL}{\nu} = 14.5\,\frac{m}{s} \times 1.5m \times 1.46\times10^{-5}\,\frac{s}{m^2} = 1.49\times10^6$$

$$\delta_L = \frac{0.382\,L}{Re_L^{1/5}} = 0.382 \times 1.5m \times \frac{1}{(1.49\times10^6)^{0.2}} \cong 33.4\,mm \quad \delta_L$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.0594}{Re_x^{1/5}} \quad \therefore \tau_w = \frac{1}{2}\rho U^2 \frac{0.0594}{Re_x^{1/5}}$$

$$\tau_{w_L} = \frac{1}{2} \times 1.23\,\frac{kg}{m^3} \times (14.5)^2\,\frac{m^2}{s^2} \times \frac{0.0594}{(1.49\times10^6)^{0.2}} \times \frac{N\cdot s^2}{kg\cdot m} = 0.447\,N/m^2 \quad \tau_{w_L}$$

The drag force is given by.

$$F_D = \int_{x_1}^{L} \tau_w b\,dx = \frac{1}{2}\rho U^2 b \int_{x_1}^{L} \frac{0.0594}{Re_x^{1/5}}\,dx = \frac{1}{2}\rho U^2 b \frac{0.0594}{(U/\nu)^{1/5}} \int_{x_1}^{L} \frac{dx}{x^{1/5}}$$

$$= \frac{1}{2}\rho U^2 b \frac{0.0594}{(U/\nu)^{1/5}} \left[\frac{5}{4} x^{4/5}\right]_{x_1}^{L} = \frac{1}{2}\rho U^2 b \frac{0.0594}{(U/\nu)^{1/5}} \frac{5}{4} \left[x^{4/5}\right]_{x_1}^{L}$$

$$F_D = \frac{1}{2}\rho U^2 b \frac{5}{4}\left[L\,C_{f_L} - x_1 C_{f_{x_1}}\right]$$

For $x_1 = 0.5m$, $Re_{x_1} = 0.497\times10^6$, and $C_{f_{x_1}} = 0.00431$
For $x = L = 1.5m$, $Re_L = 1.49\times10^6$, and $C_{f_L} = 0.00346$

Substituting.

$$F_D = \frac{1}{2} \times 1.23\,\frac{kg}{m^3} \times (14.5)^2\,\frac{m^2}{s^2} \times 0.8m \times \frac{5}{4} \times \frac{N\cdot s^2}{kg\cdot m}\left[1.5m \times 0.00346 - 0.5m \times 0.00431\right]$$

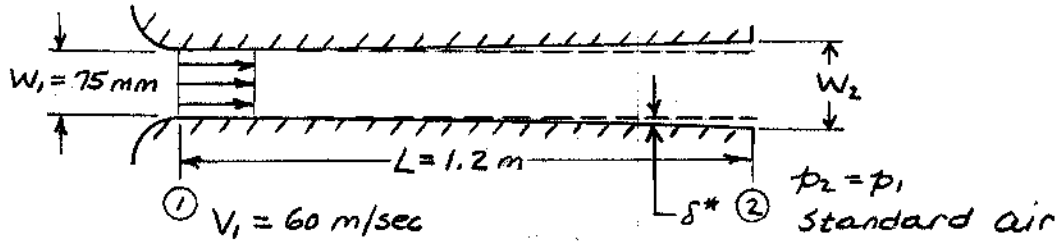
$$F_D = 0.392\,N \quad F_D$$

Alternate solution: $F_D = \rho U^2 b(\theta_L - \theta_{x_1})$

For 1/7-power, $\theta = \frac{7}{72}\delta$. $\theta_L = 3.25mm$; $\theta_{x_1} = 1.35m$

$$F_D = 1.23\,\frac{kg}{m^3} \times (14.5)^2\,\frac{m^2}{s^2} \times 0.8m \times \frac{N\cdot s^2}{kg\cdot m}(0.00325 - 0.00135)m = 0.393\,N \checkmark$$

Given: Incompressible flow of air through a plane-wall diffuser. Diffuser walls diverge slightly to accommodate the boundary layer development, so there is no pressure gradient. Assume flat plate boundary layer development.



Find: (a) Explain why Bernoulli equation is applicable to this flow.
(b) Exit width, W_2.

Solution: The Bernoulli equation may be applied along a streamline in any steady, incompressible flow in the absence of friction. The given flow is steady and incompressible. Frictional effects are confined to the thin wall boundary layers. Therefore the Bernoulli equation may be applied along any streamline in the core flow outside the boundary layers. (Since there is no streamline curvature, the pressure is uniform across sections ① and ②)

Basic equations: $0 = \underbrace{\frac{\partial}{\partial t}\int_{CV}\rho\,dV}_{=0(1)} + \int_{CS}\rho\vec{V}\cdot d\vec{A}$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g\underbrace{z_1}_{(3)} = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g\underbrace{z_2}_{(3)}$$

Assumptions: (1) Steady flow
(2) Turbulent, "1/7-power" boundary layer from entrance
(3) $z_1 = z_2$
(4) $p_1 = p_2$

Then from Bernoulli, $V_1 = V_2$, and from continuity,

$$0 = \{-|\rho V_1 A_1|\} + \{|\rho V_2 A_{2,eff}|\} \quad \text{or} \quad A_{2,eff} = (W_2 - 2\delta_2^*)b = W_1 b$$

Use the analysis of Section 9-5.2: $\frac{\delta}{x} = \frac{0.382}{Re_x^{1/5}}$ or $\frac{\delta}{L} = \frac{0.382}{Re_L^{1/5}}$

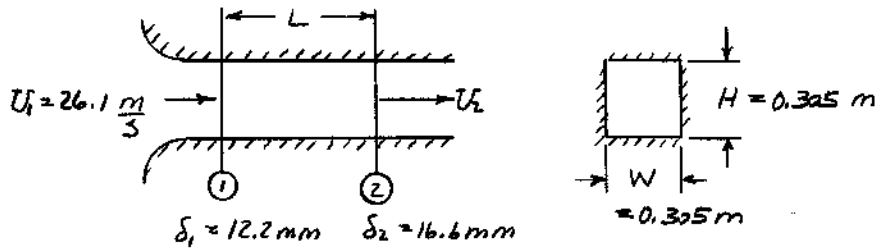
$$Re_L = \frac{\rho V L}{\mu} = \frac{VL}{\nu} = 60\,\frac{m}{s} \times 1.2\,m \times \frac{s}{1.46\times10^{-5}\,m^2} = 4.93\times10^6; \quad \frac{\delta_2}{L} = 0.0175$$

or $\delta_2 = 0.017\,L = (0.0171)\,1.2\,m = 0.0205\,m$, or 20.5 mm. $\delta^* = \int_0^{\delta}(1-\frac{u}{U})\,dy$, so

$$\delta_2^* = \delta_2\int_0^1(1-\tfrac{u}{U})\,d(\tfrac{y}{\delta}) = \delta\int_0^1(1-\eta^{1/7})\,d\eta = \delta\left[\eta - \tfrac{7}{8}\eta^{8/7}\right]_0^1 = \frac{1}{8}\delta = 2.63\,mm$$

Then $W_2 = W_1 + 2\delta_2^* = 75\,mm + 2(2.63\,mm) = 80.3\,mm$ ← W_2

Given: Small laboratory wind tunnels with square test sections.



Boundary layers are turbulent with 1/7-power velocity profiles.

Find: (a) Change in static pressure between ① and ②.
(b) Estimate of distance L.

Solution: For a 1/7-power profile in the turbulent boundary layer, $\delta^* = \delta/8$.
Therefore $\delta_1^* = 12.2/8 = 1.53$ mm and $\delta_2^* = 16.6/8 = 2.08$ mm.

Apply conservation of mass: $U_1 A_1 = U_2 A_2$; $A = W - 2\delta^*$

$$U_2 = U_1\left[\frac{305 - 2(1.53)}{305 - 2(2.08)}\right]^2 = 1.00733\,U_1$$

From Bernoulli (for steady, incompressible, inviscid flow along a streamline),

$$\frac{p_1}{\rho} + \frac{U_1^2}{2} = \frac{p_2}{\rho} + \frac{U_2^2}{2}\;;\quad p_1 - p_2 = \frac{\rho}{2}(U_2^2 - U_1^2) = \frac{\rho U_1^2}{2}\left[(1.00733)^2 - 1\right]$$

$$p_1 - p_2 = \frac{1}{2} \times 1.23 \frac{kg}{m^3} \times (26.1)^2 \frac{m^2}{s^2}\left[(1.00733)^2 - 1\right] \frac{N \cdot s^2}{kg \cdot m} = 6.16\ N/m^2 \qquad p_1 - p_2$$

For a turbulent boundary layer, $\frac{\delta}{x} = \frac{0.382}{Re_x^{1/5}} = \frac{0.382}{\left(\frac{Ux}{\nu}\right)^{1/5}} = 0.382\left(\frac{\nu}{U}\right)^{1/5} \frac{1}{x^{1/5}}$

Thus $\delta = 0.382\left(\frac{\nu}{U}\right)^{1/5} x^{4/5}$ or $x = \left(\frac{\delta}{0.382}\right)^{5/4}\left(\frac{U}{\nu}\right)^{1/4}$

For standard air $\nu = 1.46 \times 10^{-5}\ m^2/s$ (Table A.10). Thus

$$x_1 = \left(\frac{0.0122\ m}{0.382}\right)^{5/4}\left(26.1 \frac{m}{s} \times \frac{s}{1.46 \times 10^{-5} m^2}\right)^{1/4} = 0.494\ m$$

$$x_2 = \left(\frac{0.0166\ m}{0.382}\right)^{5/4}\left(26.3 \frac{m}{s} \times \frac{s}{1.46 \times 10^{-5} m^2}\right)^{1/4} = 0.727\ m$$

Then

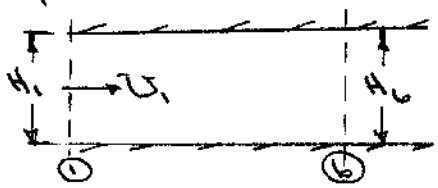
$$L_{12} = x_2 - x_1 = (0.727 - 0.494)\,m = 0.233\ m \text{ or } 233\ mm \qquad L_{12}$$

{Note that 6 significant figures were carried in the factor 1.00733 so that 3 would remain in 1.00733 − 1 = 0.733.}

Given: Wind tunnel with flexible upper wall and constant width $W_1 = 305$ mm. Tunnel height adjusted to give zero pressure gradient. Wall boundary layers represented by 1/7 power profiles. At two sections in the tunnel

① $H_1 = 305$ mm, $\delta_1 = 12.2$ mm, $U_1 = 26.5$ m/s

② $\delta_6 = 16.6$ mm



Find: (a) the height H_6
(b) equivalent length of flat plate to give $\delta_1 = 12.2$ mm
(c) estimate the distance between sections ① and ⑥

Solution: To determine the height H_6 use the continuity equation and the concept of δ^*

From continuity, $A_1 U_1 = A_6 U_6$ where A is the effective flow area. Since $U_1 = U_6$ for zero pressure gradient, $A_1 = A_6$

$$\therefore (W - \delta_1^*)(H_1 - \delta_1^*) = (W - \delta_6^*)(H_6 - \delta_6^*)$$

and $$H_6 = \frac{(W - \delta_1^*)(H_1 - \delta_1^*)}{(W - \delta_6^*)} + \delta_6^*$$

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \delta \int_0^1 \left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right) = \delta \int_0^1 \left(1 - \frac{u}{U}\right) d\eta$$

$$\delta^* = \delta \int_0^1 (1 - \eta^{1/7})\, d\eta = \delta \left[\eta - \frac{7}{8}\eta^{8/7}\right]_0^1 = \frac{1}{8}\delta$$

Substituting into the expression for H_6

$$H_6 = \frac{\left(305 - \frac{1}{8}(12.2)\right)\left(305 - \frac{1}{8}(12.2)\right) \text{mm}}{\left(305 - \frac{1}{8}(16.6)\right)} + 16.6 \text{mm} = 321 \text{ mm} \quad \leftarrow H_6$$

For flat plate turbulent boundary with 1/7 power law profile

$$\frac{\delta}{x} = \frac{0.370}{Re_x^{1/5}} \quad (9.26) \qquad \therefore \delta = 0.370 \left(\frac{\nu}{U}\right)^{1/5} x^{4/5}$$

Then $$x = \left[\frac{\delta}{0.370}\right]^{5/4} \left(\frac{U}{\nu}\right)^{1/4}$$

At section ①, $\delta = 12.2$ mm

$$x_1 = \left[\frac{0.0122 \text{m}}{0.370}\right]^{1.25} \left(\frac{26.5 \frac{\text{m}}{\text{s}}}{1.45 \times 10^{-5} \frac{\text{s}}{\text{m}^2}}\right)^{0.25} = 0.517 \text{ m} \quad \leftarrow L_{eq}$$

At section ⑥, $\delta = 16.6$ mm

$$x_6 = \left[\frac{0.0166 \text{m}}{0.370}\right]^{1.25} \left(\frac{26.5 \frac{\text{m}}{\text{s}}}{1.45 \times 10^{-5} \frac{\text{s}}{\text{m}^2}}\right)^{0.25} = 0.759 \text{ m}$$

Approximate distance $L_6 = x_6 - x_1$

$= 0.759 \text{ m} - 0.517 \text{ m} = 0.242 \text{ m}$ $\quad \leftarrow L_6$

Open-Ended Problem Statement: For flow over a flat plate with zero pressure gradient, will the shear stress increase, decrease, or remain constant along the plate? Justify your answer. Does the momentum flux increase, decrease, or remain constant as the flow proceeds along the plate? Justify your answer. Compare the behavior of laminar flow and turbulent flow (both from the leading edge) over a flat plate. At a given distance from the leading edge, which flow will have the larger boundary-layer thickness? Does your answer depend on the distance along the plate? How would you justify your answer?

Discussion: Shear stress decreases along the plate because the freestream flow speed remains constant while the boundary-layer thickness increases.

The momentum flux decreases as the flow proceeds along the plate. Momentum thickness θ (actually proportional to the defect in momentum within the boundary layer) increases, showing that momentum flux decreases. The force that must be applied to hold the plate stationary reduces the momentum flux of the stream and boundary layer.

The laminar boundary layer has less shear stress than the turbulent boundary layer. Therefore laminar boundary-layer flow from the leading edge produces a thinner boundary layer and less shear stress everywhere along the plate than a turbulent boundary layer from the leading edge.

Since both boundary layers continue to grow with increasing distance from the leading edge, and the turbulent boundary layer continues to grow more rapidly because of its higher shear stress, this comparison will be the same no matter the distance from the leading edge.

500 SHEETS, FILLED 5 SQUARE
500 SHEETS EYE-EASE® 5 SQUARE
100 SHEETS EYE-EASE® 5 SQUARE
200 SHEETS EYE-EASE® 5 SQUARE
100 RECYCLED WHITE 5 SQUARE
200 RECYCLED WHITE 5 SQUARE
12-782
42-381
42-391
42-398
42-392
42-399
Made in U.S.A.
National® Brand

Open-Ended Problem Statement: Large tankers are used to transport petroleum over long distances by sea. Analyze the energy used in transporting petroleum. Express the result as a fraction of the energy contained in the transported oil.

Analysis: Consider the supertanker analyzed in Example Problem 9.5:

The total weight of the ship is equal to its displacement (for seawater, SG = 1.025, Table A.2):

$$W = SG\rho g\forall = (1.025)\,62.4\ \frac{lbf}{ft^3} \times 360\,m \times 70\,m \times 25\,m \times \frac{ft^3}{(0.305)^3\,m^3}$$

$$W = 1.42\times10^9\ lbf \text{ or } 710{,}000 \text{ tons}$$

Assume 75% of the displaced weight is cargo (oil). Then

$$W_{oil} = 0.75 \times 710{,}000 \text{ tons} = 533{,}000 \text{ tons}$$

From the Example Problem, the power required to overcome skin friction drag is $\mathcal{P} = 13{,}000$ hp.

If propeller efficiency is 70%, the shaft power is $\mathcal{P} = 13{,}000/0.7 = 20{,}700$ hp.

Assume an additional 20% is required to overcome wave drag and to power auxiliaries. Then the total shaft power requirement is

$$\mathcal{P}_{shaft} = 20{,}700\ hp \times 1.20 = 24{,}900\ hp$$

The thermal efficiency of the propulsion system may be 40%, so

$$\dot{Q}_{input} = \frac{\mathcal{P}_{shaft}}{\eta_t} = 24{,}900\ hp \times \frac{1}{0.40} \times 2545\ \frac{Btu}{hp\cdot hr} = 1.58\times10^8\ Btu/hr$$

For a 1,000 mile trip, $t = 1000\ mi \times \frac{hr}{13\ nm} \times \frac{nm}{6076\ ft} \times 5280\ \frac{ft}{hr} = 66.8\ hr$

The energy input is $Q = \dot{Q}t = 1.58\times10^8\ \frac{Btu}{hr} \times 66.8\ hr = 1.06\times10^{10}\ Btu$

But oil contains 18,000 Btu/lbm (approx.), so $M = 1.06\times10^{10}\ Btu \times \frac{lbm}{18{,}000\ Btu}$ or 293 tons

Thus $\frac{M_{burned}}{M_{aboard}} = \frac{293 \text{ tons}}{533{,}000 \text{ tons}} = 5.50\times10^{-4}$ or 0.0550 % ← %

Per ton mile, $e = \frac{\dot{Q}}{WV} = 1.58\times10^8\ \frac{Btu}{hr} \times \frac{1}{533{,}000 \text{ tons}} \times \frac{hr}{15.0\ mi} = 19.8\ \frac{Btu}{ton\text{-}mile}$ ← e

Discussion: The energy required for transport may be expressed as a fraction of the total energy contained in the oil. However, this metric is unique to the specific voyage taken by the ship. Even though the fraction of total energy consumed for transport is small, it is difficult to compare tanker transport with other modes using this measure.

A metric that can be used to compare any mode of transportation is energy per ton·mile. The calculation also is shown above. Transport by ship has the lowest specific energy consumption of any mode of transport. (For comparison, the Alaskan Pipeline requires about 120 Btu of energy input per ton·mile for transport.)

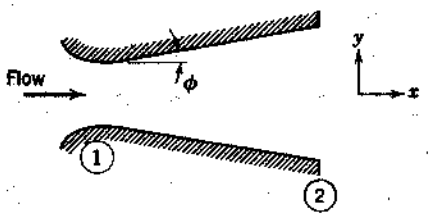
Given: Flow through plane-wall diffuser, as shown in sketch.
We wish to compare the behavior of ideal and real fluids.

Find: (a) For $\phi = 0$ (straight channel)

(i) compare pressure gradients
(ii) which gives higher p_2

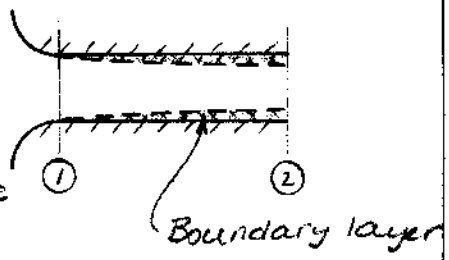
(b) For $\phi > 0$, but no separation

(i) compare pressure gradients
(ii) which gives higher p_2.



Solution: For $\phi = 0$, there will be no change in velocity, and hence no pressure gradient with the ideal fluid. Thus for the ideal fluid, $p_{2i} = p_1$.

With the real fluid, boundary layers will form on the channel walls, reducing the effective flow area. Thus, to satisfy continuity for incompressible flow, $V_2 > V_1$. The velocity can increase only when the pressure falls. Thus for the real fluid, $p_2 < p_1$.

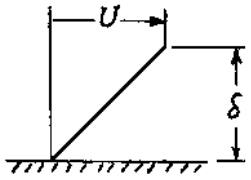


For $\phi = 0$, $p_2 < p_{2i}$ ← $\phi = 0$

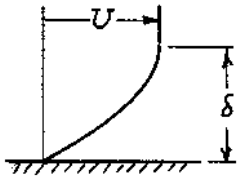
For $\phi > 0$, with no separation, the situation is qualitatively similar. The pressure gradient with the real fluid is reduced by boundary layer development. The ideal fluid gives a larger value of p_2 again.

For $\phi > 0$ (no separation), $p_2 < p_{2i}$ ← $\phi > 0$

Given: Two hypothetical laminar boundary-layer velocity profiles:

(a) 

$\frac{u}{U} = \frac{y}{\delta}$ Linear

(b) 

$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$ Parabolic

Find: (a) Momentum flux for each profile.
(b) Which would be more likely to separate first under the same pressure gradient conditions? Why?

Solution: For a boundary layer of width, w, the momentum flux is

$$\text{Momentum flux} = mf = \int_0^\delta \rho u (u w dy) = \rho U^2 w \delta \int_0^1 \left(\frac{u}{U}\right)^2 d\eta \quad \text{where } \eta = \frac{y}{\delta}$$

For Profile (a), $\frac{u}{U} = \eta$ and

$$mf(a) = \rho U^2 w \delta \int_0^1 \eta^2 d\eta = \rho U^2 w \delta \left[\frac{\eta^3}{3}\right]_0^1 = \frac{1}{3}\rho U^2 w \delta \qquad mf(a)$$

For Profile (b), $\frac{u}{U} = 2\eta - \eta^2$, and

$$mf(b) = \rho U^2 w \delta \int_0^1 (2\eta - \eta^2)^2 d\eta = \rho U^2 w \delta \int_0^1 (4\eta^2 - 4\eta^3 + \eta^4)\, d\eta$$

$$= \rho U^2 w \delta \left(\frac{4}{3}\eta^3 - \eta^4 + \frac{1}{5}\eta^5\right)\Big]_0^1 = \rho U^2 w \delta \left(\frac{4}{3} - 1 + \frac{1}{5}\right) = \frac{8}{15}\rho U^2 w \delta \qquad mf(b)$$

Thus $\frac{mf(b)}{mf(a)} = \frac{8/15}{1/3} = \frac{24}{15} = \frac{8}{5}$

Profile (b) has 60 percent more momentum flux than Profile (a). Therefore Profile (b) could withstand an adverse pressure gradient better than Profile (a).

Given: Laminar boundary layer with velocity profile

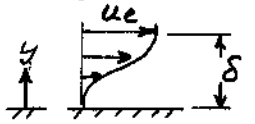
$$\frac{u}{u_e} = a + b\eta + c\eta^2 + d\eta^3 \; ; \; \eta = \frac{y}{\delta}$$

Separation occurs when τ_w becomes zero.

Find: (a) Four boundary conditions for this laminar velocity profile.
(b) Evaluate constants a, b, c, and d.
(c) Calculate shape parameter, H, at separation.

Plot: Profile and compare with parabolic.

Solution: The profile shape will be:



Boundary conditions are:

$y = 0$: $u = 0$ and $\tau = \mu \frac{du}{dy} = 0$; $y = \delta$: $u = u_e$ and $\tau = \mu \frac{du}{dy} = 0$ BC

Applying boundary conditions:

$y = 0 : \eta = 0 \quad \frac{u}{u_e} = 0 = (a + b\eta + c\eta^2 + d\eta^3)_{\eta=0} = a \quad a = 0$ a

$$\frac{du}{dy} = \frac{u_e}{\delta}\frac{d(u/u_e)}{d\eta} = 0 = (b + 2c\eta + 3c\eta^2)_{\eta=0} = b \quad b = 0$$ b

$y = \delta : \eta = 1 \quad \frac{u}{u_e} = 1 = (c\eta^2 + d\eta^3)_{\eta=1} \; ; \; c + d = 1$

$$\frac{d(u/u_e)}{d\eta} = 0 = (2c\eta + 3d\eta^2)_{\eta=1} \; ; \; 2c + 3d = 0$$

Solving

$d = 1 - c \; ; \; 2c + 3(1-c) = 2c + 3 - 3c = 3 - c = 0 \quad c = 3$ c

$d = 1 - c = 1 - 3 = -2 \quad d = -2$ d

The velocity profile is $\frac{u}{u_e} = 3\eta^2 - 2\eta^3$. $H \equiv \delta^*/\theta = \frac{\delta^*}{\delta}\frac{\delta}{\theta}$, so

$$\frac{\delta^*}{\delta} = \int_0^1 \left(1 - \frac{u}{u_e}\right) d\eta = \int_0^1 (1 - 3\eta^2 + 2\eta^3) d\eta = \left[\eta - \eta^3 + \frac{2}{4}\eta^4\right]_0^1 = \frac{1}{2}$$

$$\frac{\theta}{\delta} = \int_0^1 \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) d\eta = \int_0^1 (3\eta^2 - 2\eta^3)(1 - 3\eta^2 + 2\eta^3) d\eta$$

$$= \int_0^1 (3\eta^2 - 2\eta^3 - 9\eta^4 + 12\eta^5 - 4\eta^6) d\eta$$

$$\frac{\theta}{\delta} = \left[\eta^3 - \frac{2}{4}\eta^4 - \frac{9}{5}\eta^5 + \frac{12}{6}\eta^6 - \frac{4}{7}\eta^7\right]_0^1 = \frac{9}{70}$$

Thus $H = \frac{1}{2} \times \frac{70}{9} = \frac{70}{18} = \frac{35}{9} = 3.89$ H

Separating:

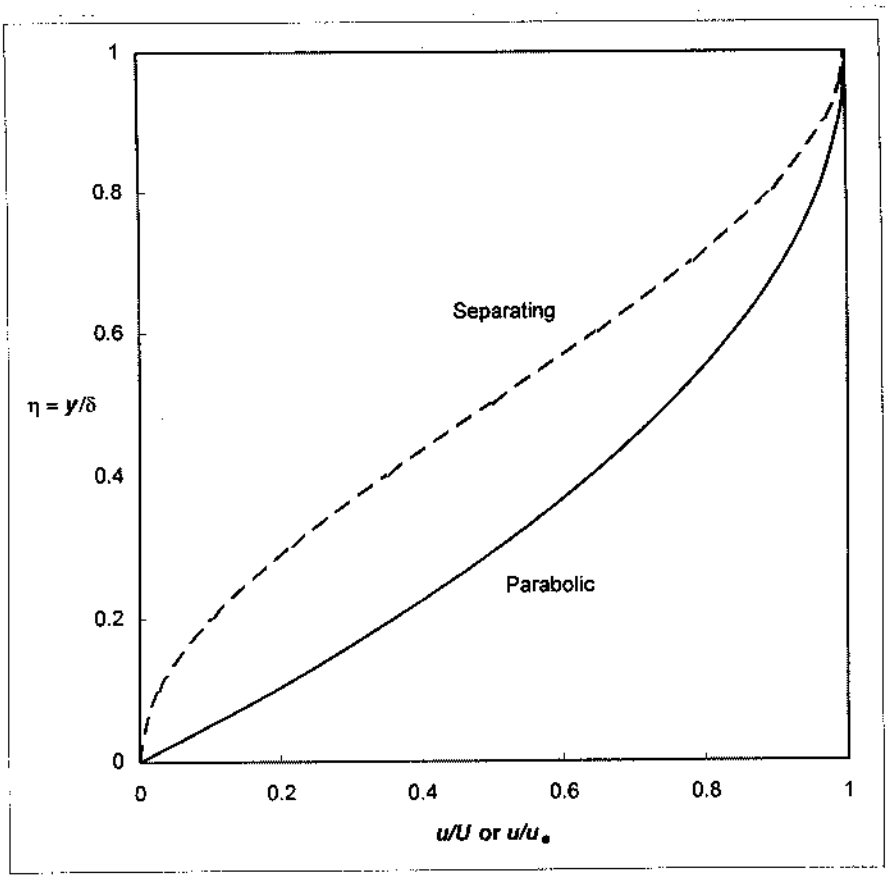
$\frac{u}{u_e} = 3\eta^2 - 2\eta^3$

η (---)	$\frac{u}{u_e}$ (---)
1.0	1.00
0.9	0.972
0.8	0.896
0.7	0.784
0.6	0.648
0.5	0.500
0.4	0.352
0.3	0.216
0.2	0.104
0.1	0.028

Flat plate:

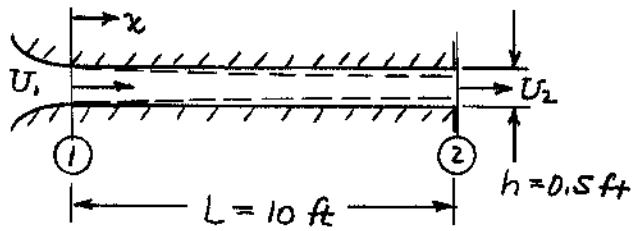
$\frac{u}{U} = 2\eta - \eta^2$

$\frac{u}{U}$ (---)
1.00
0.990
0.960
0.910
0.840
0.750
0.640
0.510
0.360
0.190



Given: Flow of air through wide, flat channel, with laminar boundary layers on the top and bottom walls.

Velocity profiles in BL are parabolic.



Find: (a) Estimate maximum inlet speed for laminar BL.
(b) Evaluate pressure drop, $p_1 - p_2$.

Solution: Assume flat-plate boundary-layer behavior, check later.

Computing equations: $Re_{x_t} = 500{,}000 \qquad \frac{\delta}{x} = \frac{5.48}{\sqrt{Re_x}}$

Assumptions: (1) Steady flow, (2) Incompressible flow, (3) Zero p.g.

Then $Re_{x_t} = \frac{U_{max} L}{\nu} \le 500{,}000$

$$U_{max} = \frac{500{,}000\,\nu}{L} = 500{,}000 \times 1.56 \times 10^{-4} \frac{ft^2}{s} \times \frac{1}{10\ ft} = 7.82\ ft/s \qquad U_{max}$$

For $Re_L = 500{,}000$, $\frac{\delta}{L} = \frac{5.48}{\sqrt{500{,}000}} = 0.00775$; $\delta = 0.00775 \times 10\ ft = 0.0775\ ft$

For a parabolic profile, $\frac{\delta^*}{\delta} = \int_0^1 (1 - \frac{u}{U}) d\lambda = \int_0^1 (1 - 2\lambda + \lambda^2) d\lambda = \frac{1}{3}$, and

$$\delta_2^* = \frac{1}{3}\delta_2 = \frac{1}{3} \times 0.0775\ ft = 0.0258\ ft$$

From continuity $U_1 w h = U_2 w (h - 2\delta_2^*)$ so $U_2 = U_1 \left(\frac{h}{h - 2\delta_2^*}\right)$

$$U_2 = 7.82 \frac{ft}{s} \times \frac{0.5\ ft}{(0.5 - 2 \times 0.0258)\ ft} = 8.72\ ft/s$$

Since the BLs do not merge ($\delta_2 < h/2$), then Bernoulli applies

Computing equation: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + g\cancel{z_1} = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g\cancel{z_2}$

Assumptions: (4) No friction (outside BL), (5) Along a streamline, (6) $z_1 = z_2$.

$$p_1 - p_2 = \frac{1}{2}\rho U_2^2 - \frac{1}{2}\rho U_1^2 = \frac{1}{2}\rho U_1^2 \left[\left(\frac{U_2}{U_1}\right)^2 - 1\right]$$

$$p_1 - p_2 = \frac{1}{2} \times 0.00238 \frac{slug}{ft^3} (7.82)^2 \frac{ft^2}{s^2} \left[\left(\frac{8.72}{7.82}\right)^2 - 1\right] \frac{lbf \cdot s^2}{slug \cdot ft} = 0.0177\ lbf/ft^2$$

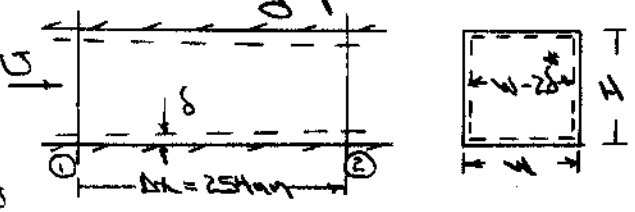
But from fluid statics, $\Delta p = \rho g \Delta h = \gamma \Delta h$

$$\Delta h = \frac{p_1 - p_2}{\gamma} = 0.0177 \frac{lbf}{ft^2} \times \frac{ft^3}{62.4\ lbf} \times 12 \frac{in.}{ft} = 0.00340\ in.\ of\ water \qquad \Delta h$$

{Thus Δp is small, so flat-plate behavior is reasonable.}

Given: Wind tunnel with square cross section, $W_1 = H_1 = 305$ mm.
At inlet section ①, $U_1 = 24.5$ m/s, $\delta_1 = 9.75$ mm
with a "1/7-power" turbulent velocity profile

$dp/dx)_1 = -0.035 \frac{mm\,H_2O}{mm}$



Find: (a) reduction in flow area at ① caused by boundary layer
(b) $d\theta/dx$ at ①
(c) estimate of θ_2

Solution: Apply continuity and the momentum integral eqs.

Computing eqs.: $\frac{d\theta}{dx} = \frac{\tau_w}{\rho U^2} - (H+2)\frac{\theta}{U}\frac{dU}{dx}$; $U/U_e = (y/\delta)^{1/7} = \eta^{1/7}$

$H = \frac{\delta^*}{\theta}$, $\frac{\tau_w}{\rho U^2} = 0.0233\left(\frac{\nu}{U\delta}\right)^{1/4}$

Assumptions: (1) steady flow
(2) uniform flow outside boundary layer.
(3) standard air

% reduction in flow area $= \frac{A_{eff} - A}{A} = \frac{(W-2\delta^*)(H-2\delta^*) - WH}{WH}$

$\frac{\delta^*}{\delta} = \int_0^1 \left(1 - \frac{U}{U_e}\right) d\left(\frac{y}{\delta}\right) = \int_0^1 (1-\eta^{1/7})\,d\eta = \left[\eta - \frac{7}{8}\eta^{8/7}\right]_0^1 = \frac{1}{8}$

$\therefore \delta_1^* = \frac{1}{8}\delta_1 = \frac{1}{8} \times 9.75\text{mm} = 1.22$ mm

area reduction $= \left(1 - \frac{2\delta^*}{W}\right)\left(1 - \frac{2\delta^*}{H}\right) - 1 = \left[1 - \frac{2\times 1.22}{305}\right]\left[1 - \frac{2\times 1.22}{305}\right] - 1$

area reduction $= -0.0159$ (-1.59%) ← %

$\frac{d\theta}{dx} = \frac{\tau_w}{\rho U^2} - (H+2)\frac{\theta}{U}\frac{dU}{dx}$

$\frac{\tau_w}{\rho U^2} = 0.0233\left(\frac{\nu}{U\delta}\right)^{1/4} = 0.0233\left[1.45\times 10^{-5}\frac{m^2}{s} \times 24.5\frac{s}{m} \times 9.75\times 10^{-3} m\right]^{0.25}$

$\frac{\tau_w}{\rho U^2} = 0.00206$ ←

$\frac{\theta}{\delta} = \int_0^1 \frac{U}{U_e}\left(1 - \frac{U}{U_e}\right) d\left(\frac{y}{\delta}\right) = \int_0^1 \eta^{1/7}(1-\eta^{1/7})\,d\eta = \left[\frac{7}{8}\eta^{8/7} - \frac{7}{9}\eta^{9/7}\right]_0^1 = \frac{7}{72}$

$\theta = \frac{7}{72}\delta = \frac{7}{72} \times 9.75\text{mm} = 0.948$ mm ←

$H = \frac{\delta^*}{\theta} = \frac{\frac{1}{8}\delta}{\frac{7}{72}\delta} = \frac{9}{7} = 1.29$ ←

Outside the boundary layer, $p + \frac{1}{2}\rho U^2 = \text{constant}$

Then, $\frac{dp}{dx} = -\rho U \frac{dU}{dx}$ and $\frac{1}{U}\frac{dU}{dx} = -\frac{1}{\rho U^2}\frac{dp}{dx} = -\frac{1}{\rho U^2} d\frac{\rho_w g h}{dx}$

$$\frac{1}{U}\frac{dU}{dx} = -\frac{1}{1.23 \frac{kg}{m^3}} \times (24.5)^2 \frac{m^2}{s^2} \times 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times (-0.035) \frac{mm}{mm}$$

$$\frac{1}{U}\frac{dU}{dx} = 0.465\ m^{-1}$$

Substituting

$$\frac{d\theta}{dx} = \frac{\tau_w}{\rho U^2} - (H+2)\frac{\theta}{U}\frac{dU}{dx}$$

$$= 0.00206 - (1.29+2)\, 0.948\ mm \times 0.465 \frac{1}{m} \times \frac{1\ m}{10^3\ mm}$$

$$\frac{d\theta}{dx} = 0.00206 - 0.00145 = 0.00061\ mm/mm$$

$d\theta/dx = 0.61\ mm/m$ ___ $d\theta/dx$

$$\theta_2 \approx \theta_1 + \left(\frac{d\theta}{dx}\right) \Delta x$$

$$= 0.948 + 0.61 \frac{mm}{m} \times 0.254\ m$$

$\theta_2 \approx 1.10\ mm$ ___ θ_2

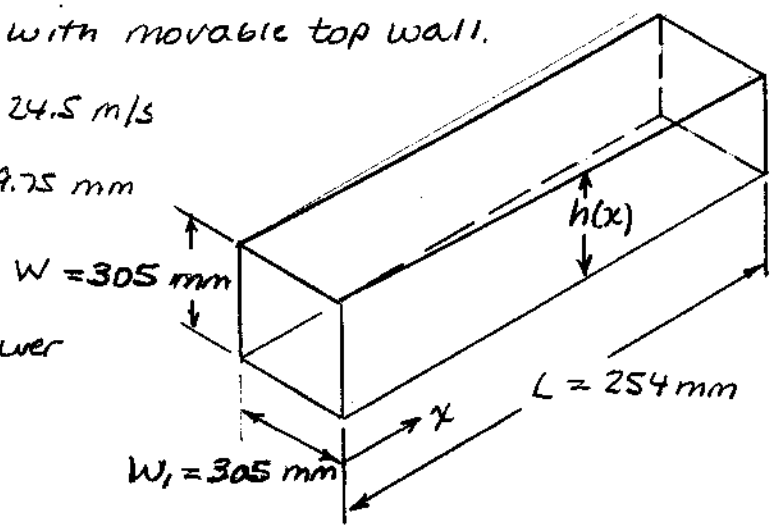
Given: Wind tunnel with movable top wall.

At inlet: $U_1 = 24.5$ m/s

$\delta_1 = 9.75$ mm

Assume 1/7-power turbulent BL development,

δ = constant



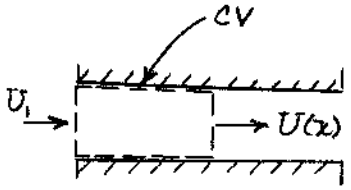
Find: (a) Velocity distribution to make δ = const, (b) $h(x)$ from 0 to L.

Solution: Calculate required pressure gradient ($dp/dx = -\rho U dU/dx$) from the momentum integral equation, then integrate to find $U(x)$, use continuity to solve for $h(x)$:

Computing equations: $\dfrac{\tau_w}{\rho U^2} = \dfrac{d\theta}{dx} + (H+2)\dfrac{\theta}{U}\dfrac{dU}{dx}$ (9.27) ($d\theta/dx$ = O(4))

$0 = \dfrac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$ ($\partial/\partial t$ = O(1))

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow outside BL
(4) Constant BL thickness:



$\delta^* = \frac{1}{8}\delta$ and $\theta = \frac{7}{72}\delta$; $H = \dfrac{\delta^*}{\theta} = \dfrac{72}{(7)8} = 1.29$ for 1/7-power profile

Then from the momentum integral equation

$$\frac{\tau_w}{\rho U^2} = (H+2)\frac{\theta}{U}\frac{dU}{dx}$$

To integrate, we must make an assumption about τ_w.

Case 1: Assume τ_w = constant, and rearrange

Then $U\,dU = \dfrac{\tau_w}{\rho\theta(H+2)}\,dx$

Integrating, $\frac{1}{2}U^2\Big]_1^2 = \dfrac{\tau_w}{\rho\theta(H+2)}x$ or $\frac{1}{2}U_2^2 - \frac{1}{2}U_1^2 = \dfrac{\tau_w}{\rho\theta(H+2)}x$

so $\dfrac{U}{U_1} = \left[1 + 2\dfrac{\tau_w}{\rho U_1^2}\dfrac{x}{\theta(H+2)}\right]^{\frac{1}{2}} = \left[1 + \dfrac{C_f}{\theta(H+2)}x\right]^{\frac{1}{2}}$ ← U(x)

Case 2: Assume $\tau_w \neq$ constant

Then $\tau_w = 0.0233 \rho U^2 \left(\frac{\nu}{U\delta}\right)^{1/4}$ (Eq. 9.25)

Substituting and rearranging

$$\frac{\tau_w}{\rho U^2} = 0.0233 \left(\frac{\nu}{\delta}\right)^{1/4} \frac{1}{U^{1/4}} = (H+z)\theta \frac{dU}{dx}$$

or $$\frac{dU}{U^{0.75}} = 0.0233 \left(\frac{\nu}{\delta}\right)^{1/4} \frac{dx}{(H+z)\theta}$$

Integrating, $4U^{1/4}\Big]_1^2 = 0.0233 \left(\frac{\nu}{\delta}\right)^{1/4} \frac{x}{(H+z)\theta}$

or $$\frac{U}{U_1} = \left[1 + 0.00583 \left(\frac{\nu}{U_1\delta}\right)^{\frac{1}{4}} \frac{x}{(H+z)\theta}\right]^4 \quad U(x)$$

From continuity $U_1 A_1 = UA = U_1 (W_1 - 2\delta_1^*)(H_1 - 2\delta_1^*) = U(W_1 - 2\delta_1^*)(h - 2\delta_1^*)$

Thus $A/A_1 = (U/U_1)^{-1}$ and $\frac{h-2\delta_1^*}{H-2\delta_1^*} = \frac{U_1}{U} = \frac{h/H - 2\delta_1^*/H}{1 - 2\delta_1^*/H}$

$$\frac{h}{W} = \left(1 - 2\frac{\delta_1^*}{h_1}\right)\frac{U_1}{U} + 2\frac{\delta_1^*}{h_1} \quad h(x)$$

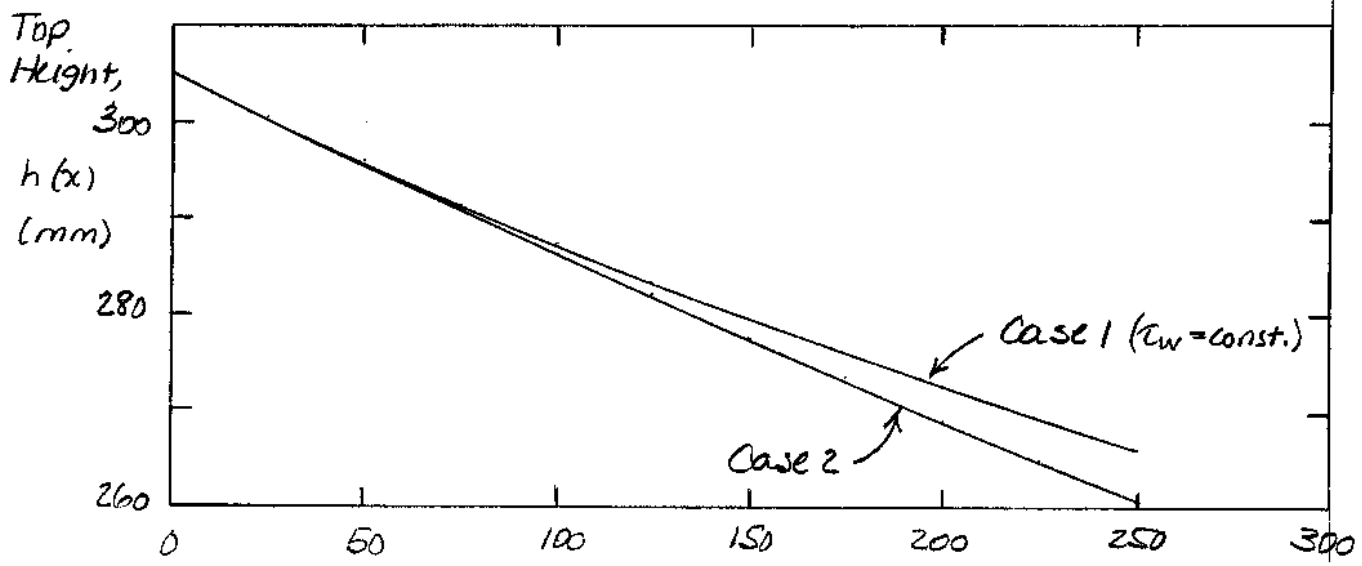
Evaluating: $\delta_1 = 9.75\ mm$, $\delta_1^* = \frac{1}{8}\delta_1 = 1.22\ mm$, $\theta_1 = \frac{7}{72}\delta_1 = 0.948\ mm$

$$\frac{1}{2}\rho U_1^2 = \frac{1}{2} \times 1.23 \frac{kg}{m^3} \times (24.5)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 369\ N/m^2$$

$$Re_{\delta_1} = \frac{U_1 \delta_1}{\nu} = 24.5 \frac{m}{s} \cdot 0.00975\ m \times \frac{s}{1.46 \times 10^{-5} m^2} = 1.64 \times 10^4$$

$$C_f = 0.0466\ Re_{\delta_1}^{-1/4} = 0.0466(0.0883) = 0.00411$$

Plot results:



Given: Flat-bottomed barge, 25 m long and 10 m wide, submerged to a depth of 1.5 m, pushed in a river at 8 kph. Water is at 15°C.

Find: Estimate power required to overcome skin friction.

Solution: First compute Reynolds number based on length, ν (Table A.8),

$$Re_L = \frac{VL}{\nu} = 8000\frac{m}{hr} \times 25\,m \times \frac{s}{1.14\times10^{-6}\,m^2} \times \frac{hr}{3600\,s} = 4.87\times10^7$$

Since transition occurs at $Re_x \lesssim 4\times10^6$, most of the boundary layer is turbulent. The power will be given by drag force times velocity, or

$$\mathcal{P} = F_D\,V$$

Using Eq. 9.34, $C_D = 0.0742/Re_L^{1/5}$, then

$$F_D = C_D\,\tfrac{1}{2}\rho V^2 A = \frac{0.0742}{(Re_L)^{1/5}}\,\tfrac{1}{2}\rho V^2 A = \frac{0.0371}{(Re_L)^{1/5}}\rho V^2 A$$

The area in contact with water is the bottom plus two sides, so

$$A = L(b+2d) = 25\,m\,(10\,m + (2)\,1.5\,m) = 325\,m^2$$

Then

$$\mathcal{P} = \frac{0.037}{(Re_L)^{1/5}}\rho V^3 A$$

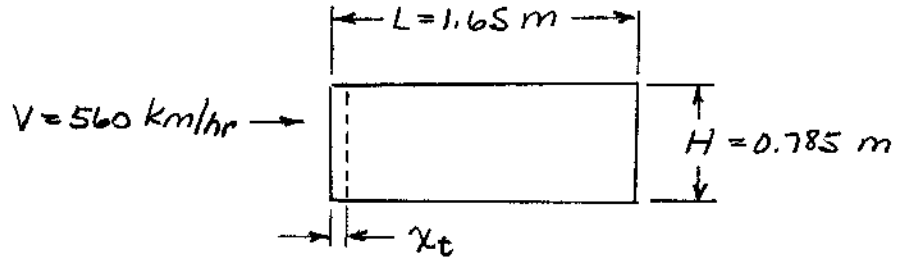
$$= \frac{0.037}{(4.87\times10^7)^{1/5}}\,999\,\frac{kg}{m^3}\left[8000\frac{m}{hr}\times\frac{hr}{3600\,s}\right]^3 325\,m^2 \times \frac{N\cdot s^2}{kg\cdot m}\times\frac{W\cdot s}{N\cdot m}$$

$$\mathcal{P} = 3.82\,kW \qquad \mathcal{P}$$

Note that this result does not include wave resistance, nor the effect of river surface slope.

Given: Stabilizing fin on Bonneville land speed record auto.

$z = 1{,}340$ m



Find: (a) Evaluate Re_L
(b) Location of x_t
(c) Power to overcome skin friction drag on fin.

Solution: Assume standard atmosphere, so $T = 279$ K, $\rho/\rho_0 = 0.877$ (Table A.3); $\mu = 1.79 \times 10^{-5}$ kg/m·s (Table A.7). Then

$$Re_L = \frac{\rho V L}{\mu} = (0.877)\,1.23\,\frac{kg}{m^3} \times 560 \times 10^3 \frac{m}{hr} \times 1.65\,m \times \frac{m \cdot s}{1.79 \times 10^{-5} kg} \times \frac{hr}{3600 s}$$

$$Re_L = 1.55 \times 10^7$$ Re_L

Assume transition occurs at $Re_x = 500{,}000$. Then

$$\frac{x_t}{L} = \frac{Re_{xt}}{Re_L} = \frac{500{,}000}{1.55 \times 10^7} = 0.0323$$

$$x_t = 0.0323\,L = 0.0323 \times 1.65\,m = 0.0532\,m = 53.2\,mm$$ x_t

Calculate drag force using C_D from Fig. 9.8: $F_D = C_D A \frac{1}{2}\rho V^2$

$C_D = 0.0029$ (Fig. 9.8); $A = 2LH = 2 \times 1.65\,m \times 0.785\,m = 2.59\,m^2$ (2 sides)

$$\frac{1}{2}\rho V^2 = \frac{1}{2} \times (0.877)\,1.23\,\frac{kg}{m^3}\left(560 \times 10^3 \frac{m}{hr} \times \frac{hr}{3600 s}\right)^2 \frac{N \cdot s^2}{kg \cdot m} = 1.31 \times 10^4\,N/m^2$$

$$F_D = 0.0029 \times 2.59\,m^2 \times 1.31 \times 10^4 \frac{N}{m^2} = 98.4\,N \text{ (skin friction drag on fin)}$$

The power required is

$$P = F_D V = 98.4\,N \times 560 \times 10^3 \frac{m}{hr} \times \frac{hr}{3600 s} \times \frac{W \cdot s}{N \cdot m} = 15.3\,kW$$ P

Check using Eq. 9.37b:

$$C_D = \frac{0.455}{(\log Re_L)^{2.58}} - \frac{1610}{Re_L} = 0.00270$$

This is slightly less than from the graph, but reasonable agreement.

Given: Jet transport cruising at $V = 820$ km/hr at $z = 12$ km.

Model fuselage as circular cylinder: $D = 4$ m, $L = 40$ m.

Neglect compressibility effects.

Find: (a) Estimate skin-friction drag force.
(b) Evaluate power needed to overcome skin-friction drag.

Solution: From Table A.3 at $z = 12$ km: $T = 216.7$ K ($-56.5°C$)
$\rho/\rho_0 = 0.2546$

$$V = 820\,\frac{km}{hr} \times 1000\,\frac{m}{km} \times \frac{hr}{3600\,sec} = 228\ m/s$$

$$\mu = \frac{bT^{1/2}}{1 + S/T} = 1.458 \times 10^{-6}\,\frac{kg}{m \cdot sec \cdot K^{1/2}} \times (216.7)^{1/2} K^{1/2} \times \frac{1}{1 + 110.4\,K/216.7\,K}$$

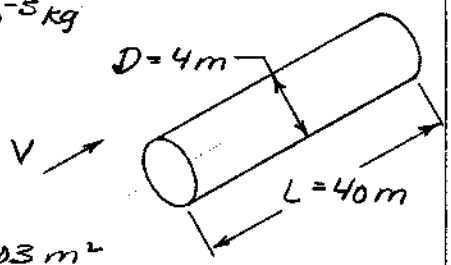
$$\mu = 1.42 \times 10^{-5}\ kg/m \cdot sec$$

$$\rho = 0.2546\,\rho_0 = 0.2546 \times 1.2250\ kg/m^3 = 0.312\ kg/m^3$$

Then $Re_L = \frac{\rho V L}{\mu} = 0.312\,\frac{kg}{m^3} \times 228\,\frac{m}{s} \times 40\ m \times \frac{m \cdot s}{1.42 \times 10^{-5}\,kg} = 2.00 \times 10^8$

Use Eq. 9.35 for $10^7 < Re_L < 10^9$:

$$C_D = \frac{0.455}{(\log_{10} Re_L)^{2.58}} = 0.00193$$



$$F_D = C_D \tfrac{1}{2} \rho V^2 A\ ;\quad A = \pi D L = \pi \times 4\ m \times 40\ m = 503\ m^2$$

$$\tfrac{1}{2}\rho V^2 = \tfrac{1}{2} \times 0.312\,\frac{kg}{m^3} \times (228)^2\,\frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 8.11 \times 10^3\ N/m^2$$

$$F_D = 0.00193 \times 8.11 \times 10^3\,\frac{N}{m^2} \times 503\ m^2 = 7{,}870\ N \qquad F_D$$

This is the drag force caused by skin friction only; the power is

$$P = F_D V = 7870\ N \times 228\,\frac{m}{sec} \times \frac{W \cdot sec}{N \cdot m} = 1.79\ MW \qquad P$$

Given: Towboat model at 1:13.5 scale to be tested in towing tank.
Dimensions are:

Length, 11.1 ft; Beam, 3.11 ft; Draft, 0.62 ft

Model displacement in fresh water is 1200 lbf.

Find: (a) Estimate average length of wetted surface on hull.
(b) Skin friction drag force on prototype at V = 8 mph.

Solution: Represent the towboat as a rectangular solid of length, $\bar{L}$, with the displacement of the boat. From buoyancy,

$$W = \rho g \forall = \rho g \bar{L} B D \quad \text{or} \quad \bar{L} = \frac{W}{\rho g B D}$$

$$\bar{L} = 1200\ \text{lbf} \times \frac{\text{ft}^3}{1.94\ \text{slug}} \times \frac{\text{sec}^2}{32.2\ \text{ft}} \times \frac{1}{3.11\ \text{ft}} \times \frac{1}{0.62\ \text{ft}} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{sec}^2} = 9.96\ \text{ft} \qquad \bar{L}$$

For the prototype,

$$\bar{L}_p = 13.5\,\bar{L}_m = 13.5 \times 9.96\ \text{ft} = 134\ \text{ft}$$

The Reynolds number is (Table A.7)

$$Re_L = \frac{VL}{\nu} = 8\ \frac{\text{mi}}{\text{hr}} \times 134\ \text{ft} \times \frac{\text{s}}{1.08\times10^{-5}\ \text{ft}^2} \times 5280\ \frac{\text{ft}}{\text{mi}} \times \frac{\text{hr}}{3600\ \text{s}} = 1.46\times10^8 \qquad (T = 68^\circ\text{F})$$

Thus flow is predominantly turbulent. Apply turbulent flow analysis.

Computing equations: $F_D = C_D A \frac{1}{2}\rho V^2$ $\qquad C_D = \frac{0.455}{(\log_{10} Re_L)^{2.58}} - \frac{1610}{Re_L}$ (9.37b)

The wetted area is $A = \bar{L}(B + 2D)$.

$$C_D = \frac{0.455}{(\log_{10} 1.46\times10^8)^{2.58}} - \frac{1610}{1.46\times10^8} = 0.00214$$

Substituting,

$$F_D = 0.00214 \times 134\ \text{ft} \times 13.5(3.11 + 2\times0.62)\ \text{ft} \times \frac{1}{2} \times 1.94\ \frac{\text{slug}}{\text{ft}^3} \left(8\ \frac{\text{mi}}{\text{hr}} \times 5280\ \frac{\text{ft}}{\text{mi}} \times \frac{\text{hr}}{3600\ \text{s}}\right)^2 \frac{16\text{f.s}^2}{\text{slug}\cdot\text{ft}}$$

$$F_D = 225\ \text{lbf} \quad \text{(skin friction only)} \qquad F_D$$

Given: Model barge tested at 1:13.5 scale; prototype speed, $V_p = 8$ mph.

Dimensions are: Length, $L = 22$ ft
Beam, $B = 4.0$ ft
Draft, $D = 0.667$ ft

Find: (a) Model test speed, V_m
(b) Boundary layers laminar or turbulent on prototype?
(c) Where position BL trips on model?
(d) Estimate skin friction drag for model and prototype.

Solution: Test should be run so $Fr_m = Fr_p = V_m/\sqrt{gL_m} = V_p/\sqrt{gL_p}$

Thus $V_m = V_p\sqrt{L_m/L_p} = 8\text{mph}\sqrt{1/13.5} = 2.18$ mph ← V_m

$$Re_p = \frac{V_p L_p}{\nu} = (8)(1.47)\frac{ft}{s} \times (22)(13.5)ft \times \frac{s}{1.08\times10^{-5}ft^2} = 3.24\times10^8 \quad (T = 68°F, \text{Table A.7})$$

Therefore boundary-layer flow is <u>turbulent</u>. Transition would occur at $Re_{x_t} \approx 5\times10^5$, so

$$x_t/L = \frac{5\times10^5}{3.24\times10^8} = 0.00154; \quad x_t = 0.00154\,L_m = 0.0339 \text{ ft from front}$$ ←

The wetted area is $A = L(B+2D)$. Assume for turbulent BL flow

$$C_D = 1.25\,C_f = 1.25 \times \frac{0.0594}{(Re_L)^{1/5}} = \frac{0.0743}{(Re_L)^{1/5}}$$

For the model, $L = 22$ ft, and

$$Re_m = \frac{V_m L_m}{\nu} = (2.18)(1.47)\frac{ft}{sec} \times 22\,ft \times \frac{s}{1.08\times10^{-5}ft^2} = 6.53\times10^6$$

$$C_{Dm} = \frac{0.0743}{(6.53\times10^6)^{1/5}} = 0.00322$$

For the model

$$F_{Dm} = C_{Dm}\tfrac{1}{2}\rho V_m^2 A_m = \frac{0.00322}{2} \times 1.94\frac{slug}{ft^3} \times [2.18(1.47)]^2\frac{ft^2}{s^2} \times [22(4+2(0.667))]ft^2 \times \frac{lbf\,s^2}{slug\cdot ft}$$

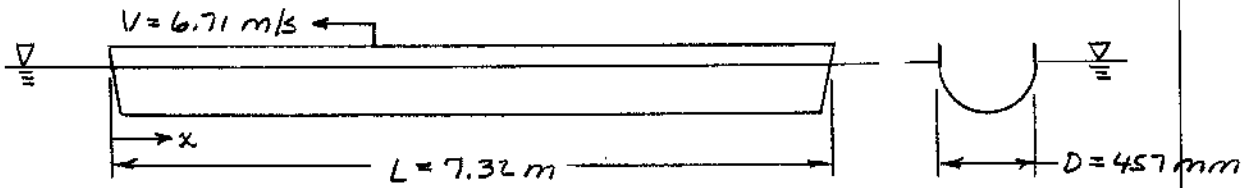
$F_{Dm} = 3.77$ lbf ← F_{Dm}

For the prototype,

$Re_p = 3.24\times10^8$, $C_{Dp} = 0.00148$, $A_p = (13.5)^2 A_m = 21{,}400\ ft^2$

$$F_{Dp} = C_{Dp}\tfrac{1}{2}\rho V_p^2 A_p = \frac{0.00148}{2} \times 1.94\frac{slug}{ft^3}[(8)(1.47)]^2\frac{ft^2}{s^2} \times 21{,}400\,ft^2 \times \frac{lbf\,s^2}{slug\,ft} = 4{,}250 \text{ lbf}$$ ← F_{Dp}

Given: Racing shell of Purdue crew, approximated as half a cylinder.



Find: (a) Location of transition in boundary layers on hull.
(b) Thickness of TBL at rear of hull.
(c) Total skin friction drag on hull.

Solution: Assume flow behaves as on a flat plate, with $Re_{x,t} = 500{,}000$.

$$Re_{x,t} = \frac{V x_t}{\nu} = 500{,}000 \; ; \; x_t = \frac{500{,}000\,\nu}{V} = 5\times10^5 \times 1\times10^{-6}\,\frac{m^2}{s}\times\frac{s}{6.71\,m} = 0.0745\,m \quad \leftarrow x_t$$

($\nu = 1\times10^{-6}\,m^2/s$ for water at 20°C, Table A.8). Thus LBL is only 1% of L.

For the turbulent boundary layer $\frac{\delta}{x} = \frac{0.382}{Re_x^{1/5}}$, so $\delta = \frac{0.382}{Re_L^{1/5}}\,L$

$$Re_L = \frac{VL}{\nu} = 6.71\,\frac{m}{s}\times 7.32\,m \times \frac{s}{1.00\times10^{-6}\,m^2} = 4.91\times10^7$$

$$\delta = 0.382\times\frac{1}{(4.91\times10^7)^{1/5}}\times 7.32\,m = 0.0810\,m \quad \leftarrow \delta$$

The drag force is $F_D = C_D\,A\,\frac{1}{2}\rho V^2$.

$$A \approx wL = \frac{\pi D}{2}\,L = \frac{\pi}{2}\times 0.457\,m \times 7.32\,m = 5.25\,m^2$$

Since $10^7 \le Re_L < 10^9$, then $C_D = \frac{0.455}{(\log Re_L)^{2.58}} = 0.00237$

Then

$$F_D = 0.00237\times 5.25\,m^2\times\frac{1}{2}\times 999\,\frac{kg}{m^3}\times(6.71)^2\,\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} = 280\,N \quad \leftarrow F_D$$

Note the rowers must produce an average power of

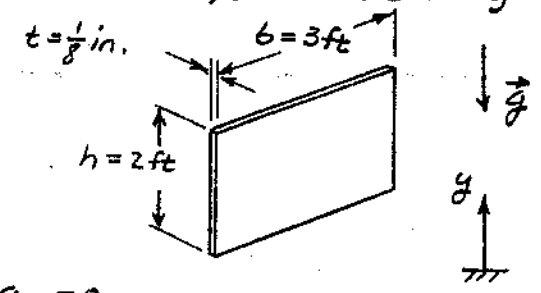
$$\mathcal{P} = F_D V = 280\,N\times 6.71\,\frac{m}{s}\times\frac{W\cdot s}{N\cdot m} = 1.88\,kW$$

to move the shell at this speed.

Given: A sheet of plastic material, SG = 1.5, and $\frac{3}{8}$ in. thick, falls vertically in a tank of water. The sheet is 2 ft high, 3 ft wide. Assume boundary layers are turbulent.

Find: Estimate the terminal velocity of the sheet.

Solution: Apply Newton's second law of motion, evaluate drag due to skin friction.



Basic equations: $\Sigma F_y = m a_y$

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$$

Assumptions: (1) At terminal speed, $a_y = 0$

(2) $C_D = \frac{0.0742}{(Re_L)^{1/5}}$ for a turbulent BL (Section 9-7.1)

Then $\Sigma F_y = F_D + F_{buoyancy} - W = 0$ or $F_D = W - F_{buoyancy}$

Using the definition of C_D, noting that $A = 2bh$,

$$F_D = C_D \tfrac{1}{2}\rho V^2 A = \frac{0.0742}{\left(\frac{\rho V h}{\mu}\right)^{1/5}} \times \tfrac{1}{2}\rho V^2 (2bh) = (\gamma_{plastic} - \gamma_{H_2O})\forall = \gamma_{H_2O}(SG-1)tbh$$

or $0.0742\left(\frac{\nu}{h}\right)^{\frac{1}{5}}\rho V^{\frac{9}{5}} = \gamma_{H_2O}(SG-1)t = \rho g (SG-1)t$

Solving, $$V = \left[\frac{g(SG-1)t}{0.0742}\left(\frac{h}{\nu}\right)^{\frac{1}{5}}\right]^{5/9}$$

For water at 60°F, $\nu = 1.21 \times 10^{-5}\ ft^2/s$, so

$$V = \left[32.2\frac{ft}{s^2} \times \frac{0.5}{0.074} \times \frac{3}{8}\,in. \times \frac{ft}{12\,in.}\left(2ft \times \frac{s}{1.21\times10^{-5}ft^2}\right)^{\frac{1}{5}}\right]^{\frac{5}{9}} = 11.0\ ft/s$$ V

Check length Reynolds number:

$$Re_L = \frac{Vh}{\nu} = 11.0\frac{ft}{s} \times 2\,ft \times \frac{s}{1.21\times10^{-5}ft^2} = 1.82 \times 10^6$$

Thus turbulent boundary-layer flow is a reasonable assumption.

{Note that the sheet would not be likely to fall in a stable vertical position}

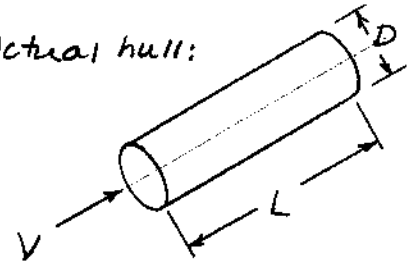
Given: Nuclear submarine, cruising submerged at $V = 27$ kt.

Assume hull is a circular cylinder, $D = 11.0$ m, and $L = 107$ m.

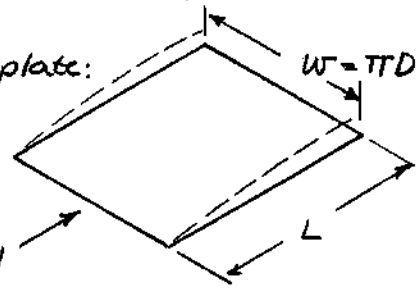
Find: (a) Estimate percentage of hull length with laminar BL.
(b) Calculate drag due to skin friction.
(c) Estimate deceleration rate if all power removed suddenly.

Solution: Treat hull as a flat plate with same wetted area.

Actual hull:



Model plate:



Computing equations: $Re_{x_t} = 500{,}000$ $\quad C_D = \dfrac{0.455}{(\log_{10} Re_L)^{2.58}}$

For seawater, $\nu = 1.05 \times 10^{-6}\ m^2/sec$ (Table A.2), so $\quad (T = 20°C)$

$$Re_L = \frac{VL}{\nu} = 27\frac{nm}{hr} \times 6076\frac{ft}{nm} \times 0.305\frac{m}{ft} \times \frac{hr}{3600\ s} \times 107\ m \times \frac{s}{1.05\times10^{-6}\ m^2} = 1.42\times10^9$$

Thus $\dfrac{x_t}{L} = \dfrac{Re_{x_t}}{Re_L} = \dfrac{500{,}000}{1.42\times10^9} = 3.52\times10^{-4}$ or $x_t = 0.0352\%$ of L $\quad \%L$ ←

Neglect laminar BL; assume flow is completely turbulent.

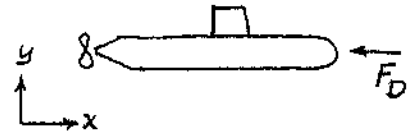
$$C_D = \frac{0.455}{(\log_{10} Re_L)^{2.58}} = \frac{0.455}{(9.15)^{2.58}} = 0.00150 \ ; \ A = wL = 34.6\ m \times 107\ m = 3.70\times10^3\ m^2$$

$$q = \tfrac{1}{2}\rho V^2 = \tfrac{1}{2} \times 1025\frac{kg}{m^3}\left(\frac{27(6076)(0.305)}{3600}\right)^2 \frac{m^2}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} = 99.0\ kPa$$

$$F_D = C_D\, q\, A = 0.00150 \times 99.0\times10^3 \frac{N}{m^2} \times 3.70\times10^3 m^2 = 5.49\times10^5\ N \qquad F_D$$ ←

Assume neutrally buoyant. Then $m = \rho\forall = \rho\,\dfrac{\pi D^2}{4} L$

$$m = \frac{\pi}{4} \times 1025\frac{kg}{m^3} \times (11.0)^2 m^2 \times 107\ m = 1.04\times10^7\ kg$$



Summing forces in the x direction,

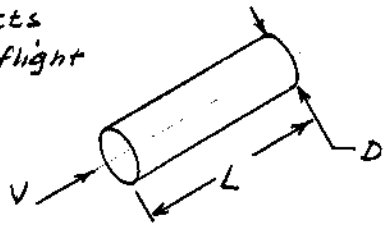
$$\Sigma F_x = -F_D = ma_x \ ; \ a_x = -\frac{F_D}{m} = -\frac{5.49\times10^5 N}{1.04\times10^7\ kg} \times \frac{kg\cdot m}{N\cdot s^2} = -0.0528\ m/s^2 \qquad a_x$$ ←

{ Deceleration due to skin friction drag is small (~ −0.005 g). Pressure drag and propeller drag would increase deceleration rate somewhat. }

Given: 600-seat jet transport proposed by Airbus Industrie. Fuselage has length $L = 70$ m and diameter $D = 7.5$ m. Aircraft operates 14 hr per day, 6 days per week, cruising at $V = 257$ m/s ($M = 0.87$) at $z = 12$ km. The thrust specific fuel consumption (TSFC) is 0.06 kg/N·hr.

Find: (a) Estimate of skin friction drag on fuselage.
(b) Annual fuel saved by 1% reduction in drag by modifying surface.

Solution: Assume: (1) BL behaves as though on flat plate, $A = \pi DL = 1650\ m^2$
(2) Neglect compressibility effects
(3) All fuel consumed in cruise flight



Need Reynolds number

$$Re_L = \frac{\rho V L}{\mu}$$

From Table A.3, $T = 216.7$ K and $\rho/\rho_{SL} = 0.2546$; $\rho = 0.2546 \times 1.23 \frac{kg}{m^3} = 0.313\ kg/m^3$

From Eq. A.1, $\mu = 1.458 \times 10^{-6} \frac{kg}{m \cdot s \cdot K^{1/2}} \times (216.7)^{3/2} K^{3/2} \times \frac{1}{(110.4 + 216.7)K} = 1.42 \times 10^{-5} \frac{kg}{m \cdot s}$

Thus

$$Re_L = 0.313 \frac{kg}{m^3} \times 257 \frac{m}{s} \times 70\ m \times \frac{m \cdot s}{1.42 \times 10^{-5} kg} = 3.97 \times 10^8$$

From Eq. 9.35 ($Re_L < 10^9$),

$$C_D = \frac{0.455}{(\log Re_L)^{2.58}} = \frac{0.455}{(\log 3.97 \times 10^8)^{2.58}} = 0.0177$$

and

$$F_D = C_D A \tfrac{1}{2} \rho V^2 = \tfrac{1}{2} \times 0.00177 \times 1650\ m^2 \times 0.313 \frac{kg}{m^3} \times (257)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 3.02 \times 10^4\ N$$ ← F_D

Then $\Delta F_D = 0.01\ F_D = 3.02 \times 10^2\ N = 302\ N$

and $\Delta F_C = \Delta F_D \times TSFC \times t$

$$\Delta F_C = 302\ N \times 0.06 \frac{kg}{N \cdot hr} \times 365 \frac{day}{yr} \times 14 \frac{hr}{day} \times \frac{6}{7} = 7.94 \times 10^4\ kg/yr$$ ← ΔF_C

The specific gravity of jet fuel (kerosine) is about 0.82 (Table A.2). Thus

$$\Delta \forall = \frac{\Delta F_C}{\rho} = \frac{\Delta F_C}{SG\ \rho_{H_2O}}$$

$$\Delta \forall = 7.94 \times 10^4 \frac{kg}{yr} \times \frac{m^3}{(0.82)\,1000\ kg} \times \frac{ft^3}{(0.305)^3 m^3} \times 7.48 \frac{gal}{ft^3} = 2.55 \times 10^4\ gal/yr$$

$$\Delta C \approx 2.55 \times 10^4 \frac{gal}{yr} \times \frac{\$1}{gal} \approx \$25{,}000$$

This is a substantial saving per aircraft. The cost saving for a fleet would be impressive.

Given: Ship model test data presented in Figs. 7.1 and 7.2; $L_p = 409$ ft, $A_p = 19{,}500$

Find: (a) Skin friction drag coefficients and corresponding forces for model and prototype at $Fr = 0.5$.
(b) Verify the coefficients presented in Fig. 7.2 for $Fr = 0.5$.

Solution: The following data may be read (approximately) from the figures:

Data	C_f	C_w	C_t	L (ft)	A (ft²)
Model	0.0035	0.0037	0.0072	5.11	3.05
Prototype	0.0015	0.0037	0.0052	409	19,500

Calculating equations: $Fr \equiv V/\sqrt{gL}$ $\quad C_f = \dfrac{F_f}{\frac{1}{2}\rho V^2 A}$ $\quad C_w = \dfrac{F_w}{\frac{1}{2}\rho V^2 A}$

$C_t = C_f + C_w$ $\quad C_f \sim (\log_{10} Re_L)^{-2.58}$, $Re > 10^7$

Model Data $V = Fr\sqrt{gL} = 0.5\sqrt{32.2\,\frac{ft}{s^2} \times 5.11\,ft} = 6.41$ ft/s

$\frac{1}{2}\rho V^2 A = \frac{1}{2} \times 1.94\,\frac{slug}{ft^3} \times (6.41)^2\,\frac{ft^2}{s^2} \times 3.05\,ft^2 \times \frac{lbf \cdot s^2}{slug \cdot ft} = 122$ lbf

So $F_f = C_f A \frac{1}{2}\rho V^2 = 0.0035 \times 122$ lbf $= 0.427$ lbf

$F_w = C_w A \frac{1}{2}\rho V^2 = 0.0037 \times 122$ lbf $= 0.451$ lbf

$F_t = F_f + F_w = 0.427 + 0.451$ lbf $= 0.878$ lbf ← Model

and $Re = \dfrac{VL}{\nu} = 6.41\,\frac{ft}{s} \times 5.11\,ft \times \dfrac{s}{1.08 \times 10^{-5}\,ft^2} = 3.03 \times 10^6$ ($T = 68°F$)

Prototype Data $V = Fr\sqrt{gL} = 0.5\sqrt{32.2\,\frac{ft}{s^2} \times 409\,ft} = 57.4$ ft/s (34.0 kt)

$\frac{1}{2}\rho V^2 A = \frac{1}{2} \times (1.025)\,1.94\,\frac{slug}{ft^3} \times (57.4)^2\,\frac{ft^2}{s^2} \times 19{,}500\,ft^2 \times \frac{lbf \cdot s^2}{slug \cdot ft} = 6.39 \times 10^7$ lbf

So $F_f = C_f A \frac{1}{2}\rho V^2 = 0.0015 \times 6.39 \times 10^7$ lbf $= 9.59 \times 10^4$ lbf

$F_w = C_w A \frac{1}{2}\rho V^2 = 0.0037 \times 6.39 \times 10^7$ lbf $= 2.36 \times 10^5$ lbf

$F_t = F_f + F_w = 95{,}900 + 236{,}000$ lbf $= 332{,}000$ lbf ← Prototype

and $Re = \dfrac{VL}{\nu} = 57.4\,\frac{ft}{s} \times 409\,ft \times \dfrac{s}{1.72 \times 10^{-5}\,ft^2} = 1.36 \times 10^9$ (Seawater, 40°F)

Comparison: $C_w = f(Fr)$, so $C_w)_m = C_w)_p$

$C_f = f(Re) \sim \dfrac{1}{Re^{1.15}}$ $\quad \dfrac{C_f)_m}{C_f)_p} = \dfrac{0.0035}{0.0015} = 2.33$; $\left[\dfrac{\log_{10} Re_p}{\log_{10} Re_m}\right]^{2.58} = \left(\dfrac{9.13}{6.48}\right)^{2.58} = 2.42$

For both model and prototype, $C_t = C_f + C_w$; agreement acceptable.

Given: Supertanker with 600,000 metric ton displacement.
Length, $L = 300$ m; Beam, $b = 80$ m; Draft, $D = 25$ m
Ship steams at 14 kt in seawater at 4°C

Estimate: (a) BL thickness at stern of ship.
(b) total skin-friction drag.
(c) power required to overcome skin-friction drag.

Solution:

Apply results of momentum integral analysis (Section 9-5.2) and correlations for drag coefficients (Section 9-7.1)

Computing equations:

$$\frac{\delta}{L} = \frac{0.382}{Re_L^{.2}} \qquad (9.26)$$

$$C_D = \frac{0.455}{(\log Re_L)^{2.58}} - \frac{1610}{Re_L} \qquad (9.37b)$$

Assumptions: (1) boundary layers behave as on a flat plate
(2) 1/7-th power turbulent velocity profiles

For seawater at 4°C, $SG = 1.025$, $\nu = 1.05\,\nu_{water}$ (Table A.2)
At 4°C, $\nu_{water} = 1.55 \times 10^{-6}$ m²/s (Table A.8).

$$U = 14 \frac{nm}{hr} \times \frac{1852 m}{1 nm} \times \frac{hr}{3600 s} = 7.20 \text{ m/s}$$

$$Re_L = \frac{UL}{\nu} = 7.20 \frac{m}{s} \times 300 m \times \frac{1}{1.05 \times 1.55 \times 10^{-6}} \frac{s}{m^2} = 1.33 \times 10^9$$

Thus

$$\delta_L = \frac{0.382 \times 300 m}{(1.33 \times 10^9)^{0.2}} = 1.72 \text{ m} \quad \leftarrow \delta_L$$

Model drag area as in Example Problem 9.5

$$A = (b+2D)L = [80 + 2(25)] m \times 300 m = 3.90 \times 10^4 \text{ m}^2$$

$$C_D = \frac{0.455}{(\log 1.33 \times 10^9)^{2.58}} - \frac{1610}{Re_L} = 0.00151$$

$$F_D = C_D A \tfrac{1}{2} \rho U^2 = 0.00151 \times 3.90 \times 10^4 m^2 \times \frac{1}{2} \times (1.025) \times 10^3 \frac{kg}{m^3} \times (7.20)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_D = 1.56 \text{ MN} \quad \leftarrow F_D$$

(This is skin-friction drag; wave drag cannot be estimated.)

The power is

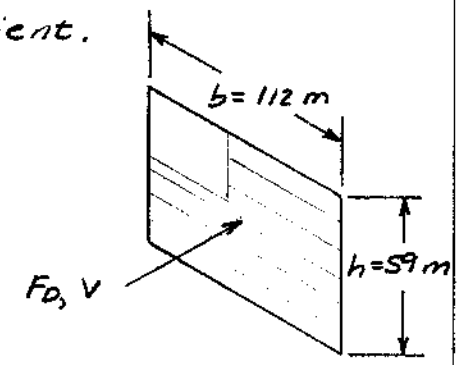
$$P = F_D U = 1.56 \times 10^6 N \times 7.20 \frac{m}{s} \times \frac{N \cdot s}{N \cdot m} = 11.2 \text{ MW} \quad \leftarrow P$$

Given: Flag, 59 m high and 112 m wide, mounted vertically.

Find: Force on flag in 16 km/hr wind. Was failure a surprise?

Solution: Apply definition of drag coefficient.

Computing equation: $F_D = C_D \frac{1}{2} \rho V^2 A$



Assumptions: (1) Flag acts as flat plate
(2) Standard air

The aspect ratio is $\frac{b}{h} = \frac{112\,m}{59\,m} = 1.9$.

From Fig. 9.10, $C_D \simeq 1.15$. Then

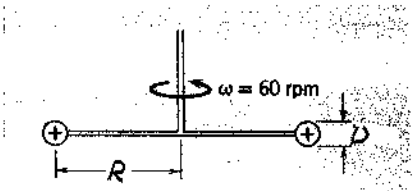
$$F_D = 1.15 \times \frac{1}{2} \times 1.23 \frac{kg}{m^3} \left(16 \times 10^3 \frac{m}{hr} \times \frac{hr}{3600\,s}\right)^2 112\,m \times 59\,m \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_D = 92.3\ kN \longleftarrow F_D$$

The flag failure should have been expected. This is a large force.

Given: Rotary mixer, constructed as shown:
The mixer is rotated in brine, SG = 1.1

Neglect motion of liquid and drag on rods.



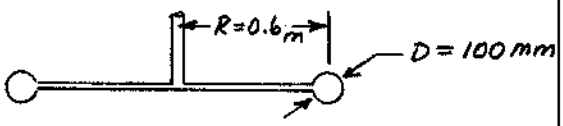
Find: (a) Torque
(b) Horsepower required to drive mixer.

Solution: Use drag coefficient data from Table 9.2, and

Basic equations: $T = 2RF_D$

$$\mathcal{P} = T\omega$$

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$$



Thus $V = R\omega = 0.6\,m \times 60\frac{rev}{min} \times 2\pi\frac{rad}{rev} \times \frac{min}{60\,s} = 3.77\ m/s$

From Table 9.2, $C_D = 1.17$ for a disk, so neglecting drag of rods,

$$F_D = \frac{1}{2}\rho V^2 \frac{\pi D^2}{4} C_D = \frac{\pi}{8} \times (1.1)999\frac{kg}{m^3} \times (3.77)^2\frac{m^2}{s^2} \times (0.1)^2 m^2 \times 1.17 \times \frac{N\cdot s^2}{kg\cdot m} = 71.8\ N$$

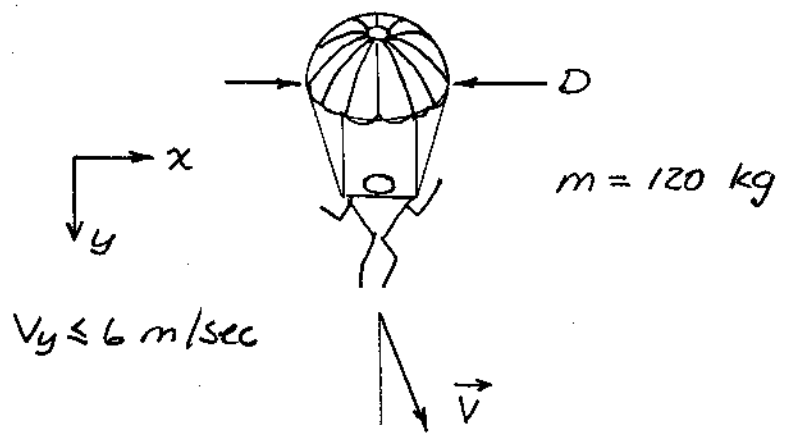
Then

$$T = 2RF_D = 2 \times 0.6\,m \times 71.8\ N = 86.2\ N\cdot m \longleftarrow T$$

and

$$\mathcal{P} = T\omega = 86.2\ N\cdot m \times 60\frac{rev}{min} \times 2\pi\frac{rad}{rev} \times \frac{min}{60\,s} \times \frac{W\cdot s}{N\cdot m} = 542\ W \longleftarrow \mathcal{P}$$

Given: Parachute and man with total mass, $m = 120$ kg.



Find: Minimum diameter, D.

Solution: Apply Newton's second law of motion, definition of drag coefficient.

Computing equations: $\Sigma F_y = mg - F_D = ma_y$; $C_D = \dfrac{F_D}{\frac{1}{2}\rho V^2 A}$

Assumptions: (1) Standard air
(2) Parachute behaves as open hemisphere
(3) V_y = constant; $a_y = 0$

Then $\Sigma F_y = mg - F_D = 0$ or $F_D = C_D A \frac{1}{2}\rho V_y^2 = mg$

$$A = \frac{\pi D^2}{4} \quad \text{so} \quad mg = C_D \frac{\pi D^2}{8} \rho V_y^2$$

$$D^2 = \frac{8mg}{\pi C_D \rho V_y^2} \quad \text{and} \quad D = \sqrt{\frac{8mg}{\pi C_D \rho V_y^2}}$$

From Table 9.3, $C_D = 1.42$ for an open hemisphere.

$$D = \left[\frac{8}{\pi} \times 120\ \text{kg} \times 9.81 \frac{\text{m}}{\text{s}^2} \times \frac{1}{1.42} \times \frac{\text{m}^3}{1.23\ \text{kg}} \times \frac{\text{s}^2}{(6)^2 \text{m}^2}\right]^{1/2}$$

$$D = 6.90\ \text{m}$$

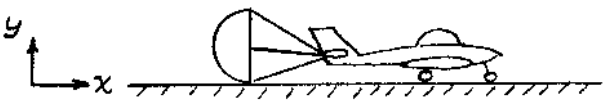
D

Given: Military aircraft with $M = 8000$ kg, lands at 350 km/hr, and is slowed by a parachute with $A = 10\ m^2$ area.

Find: (a) Estimate of time needed to slow to 200 km/hr.
(b) Maximum deceleration rate.

Solution: Model the 'chute as an open hemisphere. $C_D = 1.42$ (Table 9.3)

Apply Newton's second law:



(Neglect drag of aircraft.)

$$\Sigma F_x = ma = m\frac{dV}{dt}$$

$$\Sigma F_x = -F_D = -C_D A \tfrac{1}{2}\rho V^2 = m\frac{dV}{dt} \quad \text{or} \quad -\frac{dV}{V^2} = \frac{\rho A C_D}{2m}\,dt = k\,dt$$

where $k = \frac{1}{2} \times 1.23\frac{kg}{m^3} \times 10\ m^2 \times 1.42 \times \frac{1}{8000\ kg} = 1.09 \times 10^{-3}\ m^{-1}$

Integrating

$$\int_{V_0}^{V_f} -\frac{dV}{V^2} = \frac{1}{V}\Big]_{V_0}^{V_f} = \frac{1}{V_f} - \frac{1}{V_0} = kt \quad \text{or} \quad t = \frac{1}{k}\left[\frac{1}{V_f} - \frac{1}{V_0}\right]$$

$$V_0 = 350\frac{km}{hr} \times 1000\frac{m}{km} \times \frac{hr}{3600\,s} = 97.2\ m/s$$

$$V_f = 200\frac{km}{hr} \times 1000\frac{m}{km} \times \frac{hr}{3600\,s} = 55.6\ m/s$$

Thus

$$t = \frac{m}{1.09 \times 10^{-3}}\left[\frac{1}{55.6} - \frac{1}{97.2}\right]\frac{s}{m} = 7.06\ s$$

t

Maximum deceleration occurs at V_0:

$$\frac{dV}{dt} = -\frac{C_D A \frac{1}{2}\rho V^2}{m}$$

$$= -\frac{1}{2} \times 1.42 \times 10\ m^2 \times 1.23\frac{kg}{m^3} \times (97.2)^2\frac{m^2}{s^2} \times \frac{1}{8000\ kg}$$

$$\left(\frac{dV}{dt}\right)_{max} = -10.3\ m/s^2$$

$\left(\frac{dV}{dt}\right)_{max}$

Given: Ballistic data for 44 magnum revolver bullet:

$V_i = 250$ m/s, $V_f = 210$ m/s } over $\Delta x = 150$ m

$D = 11.2$ mm

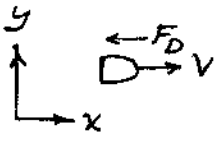
$m = 15.6$ g

Find: Evaluate average drag coefficient.

Solution: Apply Newton's second law, definition of drag coefficient.

Computing equation: $F_D = \bar{C}_D A \frac{1}{2} \rho V^2$

Basic equation: $\Sigma F_x = m a_x = m \frac{dV}{dt} = m V \frac{dV}{dx}$



From the free-body diagram, $\Sigma F_x = -F_D$, so

$$m V \frac{dV}{dx} = -F_D = -\bar{C}_D A \frac{1}{2} \rho V^2$$

Thus

$$\frac{dV}{V} = -\frac{\bar{C}_D A \rho}{2m} dx$$

Integrating

$$\int_{V_i}^{V_f} \frac{dV}{V} = \ln V \Big]_{V_i}^{V_f} = \ln \frac{V_f}{V_i} = -\frac{\bar{C}_D A \rho}{2m} \Delta x$$

Solving, using density of standard air,

$$\bar{C}_D = -\frac{2m}{\rho A \Delta x} \ln \frac{V_f}{V_i}$$

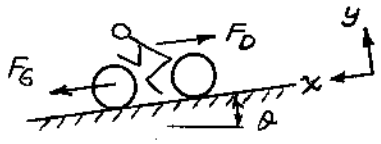
$$= -2 \times 0.0156 \text{ kg} \times \frac{\text{m}^3}{1.23 \text{ kg}} \times \frac{4}{\pi (0.0112)^2 \text{ m}^2} \times \frac{1}{150 \text{ m}} \ln\left(\frac{210}{250}\right)$$

$$\bar{C}_D = 0.299$$

$\bar{C}_D$

Given: Bike and rider with $M = 100$ kg, $A = 0.46$ m^2, and negligible rolling resistance, has terminal speed, $V_t = 15$ m/s, on a hill with 8 percent grade. Drag coefficient estimated as $C_D = 1.2$.

Find: (a) Verify this calculation of drag coefficient.
(b) Distance for bike and rider to slow from 15 to 10 m/sec after reaching level road.



Solution: Treat the bike and rider as a system. From a free-body diagram,

$$\Sigma F_x = F_G - F_D = mg\sin\theta - C_D A \tfrac{1}{2}\rho V^2 = ma_x \qquad \theta = \tan^{-1}(0.08) = 4.57^\circ$$

At terminal speed, $a_x = 0$. Then $mg\sin\theta = C_D A \frac{1}{2}\rho V_t^2$, so

$$C_D = \frac{2mg\sin\theta}{A\rho V_t^2} = 2 \times 100\,\text{kg} \times 9.81\,\frac{\text{m}}{\text{s}^2} \times \sin 4.57^\circ \times \frac{1}{0.46\,\text{m}^2} \times \frac{\text{m}^3}{1.23\,\text{kg}} \times \frac{\text{s}^2}{(15)^2\,\text{m}^2} = 1.23 \qquad C_D$$

Thus $C_D \approx 1.2$ is correct.

On a flat surface, $\Sigma F_x = -F_D = -C_D A \frac{1}{2}\rho V^2 = m\frac{dV}{dt} = mV\frac{dV}{ds}$

Thus $mV\frac{dV}{ds} = -C_D A \frac{1}{2}\rho V^2$

or

$$ds = -\frac{2m}{C_D \rho A}\frac{dV}{V}$$

Integrating

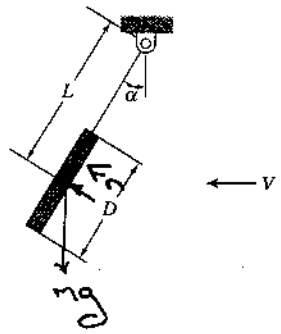
$$s - \cancel{s_0}^{\,0} = \int_{s_0}^{s} ds = -\frac{2m}{C_D\rho A}\int_{V_0}^{V}\frac{dV}{V} = -\frac{2m}{C_D\rho A}\ln V\Big]_{V_0}^{V} = -\frac{2m}{C_D\rho A}\ln\left(\frac{V}{V_0}\right)$$

$$s = -2 \times 100\,\text{kg} \times \frac{1}{1.23} \times \frac{\text{m}^3}{1.23\,\text{kg}} \times \frac{1}{0.46\,\text{m}^2} \ln\frac{10}{15} = 117\,\text{m} \qquad s$$

Given: Disk hung in air stream from a pivoted strut. For $D = 1$ in., experiment shows $\alpha = 10°$ for $V = 50$ ft/s

Find: mass of the disk.

Plot: theoretical curve of $\alpha = \alpha(V)$



Solution:

Sum moments about the pivot

$$\Sigma M_o = 0 = mgL\sin\alpha - F_n L \quad \text{.....(1)}$$

$$F_n = \tfrac{1}{2}\rho V_n^2 A C_D \quad \text{where } V_n = V\cos\alpha$$

Assume: (1) C_D is valid with $V_n = V\cos\alpha$ (given)
(2) drag on strut is negligible (given)
(3) friction in pivot is negligible (given)

Then from Eq. 1

$$m = \frac{F_n}{g\sin\alpha} = \frac{\rho V^2\cos^2\alpha\, A C_D}{2g\sin\alpha} = \frac{\pi}{8}\frac{\rho V^2\cos^2\alpha\, D^2 C_D}{g\sin\alpha} \quad \text{.....(2)}$$

From Table 9.3, $C_D = 1.17$

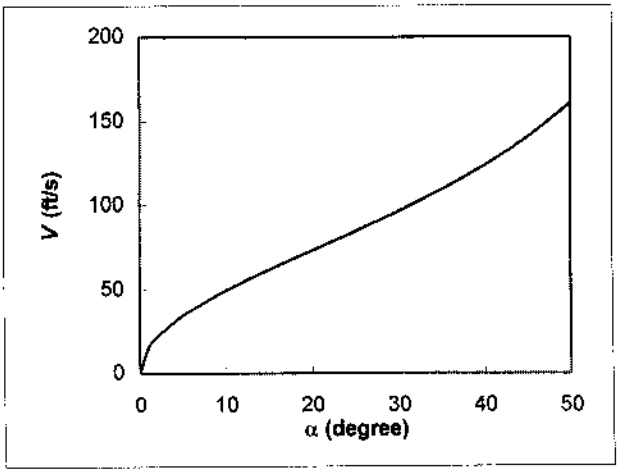
$$m = \frac{\pi}{8}\times 0.00238\ \frac{\text{slug}}{\text{ft}^3}\times (50)^2\frac{\text{ft}^2}{\text{s}^2}\times(\cos 10°)^2\times\left(\frac{1}{12}\right)^2\text{ft}^2\times 1.17\times\frac{\text{s}^2}{32.2\ \text{ft}}\times\frac{1}{\sin 10°} = 3.29\times10^{-3}\ \text{slug} \quad \text{Ans}$$

Plot α vs V

From Eq. 2, $\dfrac{\sin\alpha}{\cos^2\alpha} = \dfrac{\tan\alpha}{\cos\alpha} = \dfrac{\pi}{8}\left(\dfrac{\rho V^2 D^2 C_D}{mg}\right)$ Ans

$$V = \left[\frac{8}{\pi}\frac{mg}{\rho D^2 C_D}\right]^{1/2}\left[\frac{\tan\alpha}{\cos\alpha}\right]^{1/2} = 118\left[\frac{\tan\alpha}{\cos\alpha}\right]^{1/2}\ \text{(ft/s)}$$

α (deg)	V (ft/s)
0	0
1	15.6
2	22.1
5	35.0
10	49.9
15	62.1
20	73.4
25	84.6
30	96.3
35	109
40	123
45	140
50	161



Given: Windmills to be made from surplus 55-gal oil drums.

For a drum, $D = 24$ in., $H = 29$ in.

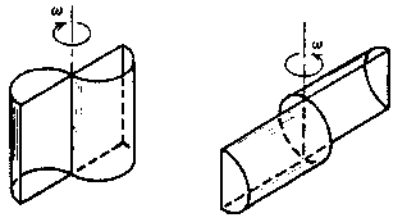
Find: Which configuration would be better, Why, and by how much?

Solution: Sum moments about pivot, neglecting friction, interference.

Configuration A:

$$\Sigma M = \frac{D}{2} F_u - \frac{D}{2} F_d = \frac{D}{2}(F_u - F_d)$$

$$\Sigma M = \frac{D}{2}(C_{Du} - C_{Dd}) A \tfrac{1}{2} \rho V^2$$



Configuration (A) (B)

Configuration B:

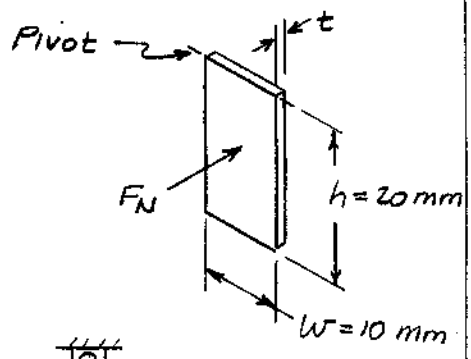
$$\Sigma M = \frac{H}{2} F_u - \frac{H}{2} F_d = \frac{H}{2}(F_u - F_d)$$

$$\Sigma M = \frac{H}{2}(C_{Du} - C_{Dd}) A \tfrac{1}{2} \rho V^2$$

Performance of B will be better because $H > D$.

$$\frac{H-D}{D} = \frac{29-24}{24} = 0.208 \text{ or } 20.8 \text{ percent improvement!}$$

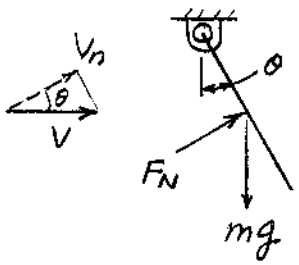
Given: Single-vane anemometer made from brass plate, t thick, with h = 20 mm and w = 10 mm.



Find: (a) Relationship for wind speed as a function of deflection angle, θ.
(b) Plate thickness to give θ = 30° at V = 10 m/s.

Solution: Sum moments about pivot.

$$\Sigma M = F_N \frac{h}{2} - mg \frac{h}{2} \sin\theta = 0$$



$$F_N = C_D A \tfrac{1}{2} \rho V_n^2 = mg\sin\theta$$

$$C_D A \tfrac{1}{2} \rho V^2 \cos^2\theta = mg\sin\theta \qquad (1)$$

$$V = \left[\frac{2 mg \sin\theta}{C_D A \rho \cos^2\theta} \right]^{1/2}$$

V

From plate geometry, $m = \rho wht = SG\rho_{H_2O} wht$. From Eq. 1,

$$SG\rho_{H_2O} g wht \sin\theta = C_D A \tfrac{1}{2} \rho V^2 \cos^2\theta \qquad \{\text{From Table A.1, } SG = 8.55 \text{ for brass.}\}$$

$$t = \frac{C_D A \rho V^2 \cos^2\theta}{2 SG \rho_{H_2O} w h \sin\theta g} = \frac{C_D \rho V^2 \cos^2\theta}{2 SG \rho_{H_2O} \sin\theta g} \qquad \text{since } A = wh$$

From Fig. 9.10, $C_D = 1.2$ at $b/h = 2.0$, so

$$t = \frac{1.2}{8.55} \times \frac{1.23}{2} \frac{kg}{m^3} \times (10)^2 \frac{m^2}{s^2} \times \cos^2(30°) \times \frac{m^3}{999\ kg} \times \frac{1}{\sin(30°)} \times \frac{s^2}{9.81\ m} \times 1000 \frac{mm}{m}$$

$$t = 1.30\ mm$$

t

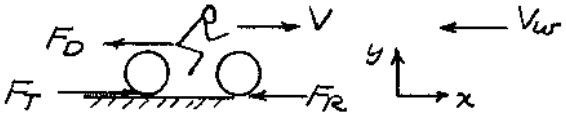
Given: Bike rider can maintain $V = 37$ km/hr on a calm day.

$$M = 80 \text{ kg},\ F_R = 4 \text{ N},\ C_D = 1.2,\ A = 0.25 \text{ m}^2$$

Bet made that can ride at $V = 30$ km/hr into $V_w = 10$ km/hr.

Find: (a) Power output capability of rider.
(b) Prospects to win bet.

Solution: Consider free-body diagram of bike & rider:



$$\Sigma F_x = F_T - F_R - F_D = 0$$

$$F_T = F_R + F_D\ ;\quad F_D = C_D A \tfrac{1}{2}\rho (V + V_w)^2$$

Power is $P = F_T V$. Thus for calm conditions, $V = 37$ km/hr $= 10.3$ m/s,

$$P_0 = V(F_R + C_D A \tfrac{1}{2}\rho V^2)$$

$$P_0 = 10.3\frac{m}{s}\left(4N + \frac{1.2}{2} \times 0.25 m^2 \times 1.23\frac{kg}{m^3} \times (10.3)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}\right)\frac{W \cdot s}{N \cdot m} = 243 \text{ W} \quad P_0$$

With a headwind, $V = 30$ km/hr $= 8.33$ m/s

$$V_w = 10 \text{ km/hr} = 2.78 \text{ m/s}$$

$$P_w = 8.33\frac{m}{s}\left(4N + \frac{1.2}{2} \times 0.25 m^2 \times 1.23\frac{kg}{m^3} \times (8.33 + 2.78)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}\right)\frac{W \cdot s}{N \cdot m} = 223 \text{ W} \quad P_w$$

The power required to travel at 30 km/hr into the wind is less than that needed to travel 37 km/hr on a calm day. Rider should be able to win bet!

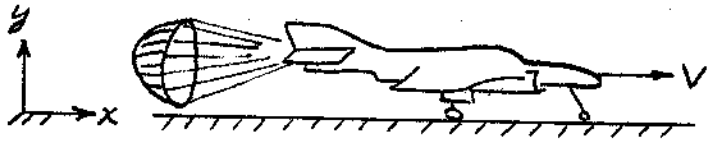
Given: F-4 aircraft slowed by dual parachutes, each 12 ft in diameter. Craft weighs 32,000 lbf, lands at 160 kt. Neglect drag of aircraft; brakes not applied.

Find: Time required to decelerate to 100 kt.

Solution: Apply Newton's second law of motion, definition of C_D.

Basic equations: $\Sigma F_x = ma_x$

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$$



Then

$$\Sigma F_x = -2F_D = -C_D \rho V^2 A = ma_x = \frac{W}{g}\frac{dV}{dt} \qquad (1)$$

or $$\frac{dV}{V^2} = -\frac{C_D \rho g A}{W}\,dt$$

Integrating,

$$\int_{V_i}^{V_f} \frac{dV}{V^2} = -\frac{1}{V}\Big]_{V_i}^{V_f} = \frac{1}{V_i} - \frac{1}{V_f} = \int_0^t -\frac{C_D \rho g A}{W}\,dt = -\frac{C_D \rho g A}{W}\,t$$

or

$$t = \frac{W}{C_D \rho g A}\left[\frac{1}{V_f} - \frac{1}{V_i}\right]$$

Since two chutes (assume hemispheres),

$$A = 2\left(\frac{\pi D^2}{4}\right) = \frac{\pi D^2}{2}$$

From Table 9.3, $C_D = 1.42$ for hemisphere facing stream. For standard air, $\rho g = \gamma \approx 0.075$ lbf/ft^3, and

$$t = 32{,}000\ \text{lbf} \times \frac{1}{1.42} \times \frac{\text{ft}^3}{0.075\ \text{lbf}} \times \frac{2}{\pi} \times \frac{1}{(12)^2\ \text{ft}^2}\left[\frac{1}{100} - \frac{1}{160}\right]\frac{\text{hr}}{\text{nm}} \times 3600\,\frac{\text{s}}{\text{hr}} \times \frac{\text{nm}}{6080\ \text{ft}}$$

or

$$t = 2.45\ \text{s} \longleftarrow t$$

To find distance, set $a_x = \frac{dV}{dt} = V\frac{dV}{dx}$. Then, from Eq. 1,

$$-2C_D \rho \frac{V^2}{2} A = \frac{W}{g} V \frac{dV}{dx}$$

and $$\frac{dV}{V} = -\frac{C_D A \rho g}{W}\,dx$$

Integrating,

$$\int_{V_i}^{V_f} \frac{dV}{V} = \ln V\Big]_{V_i}^{V_f} = \ln \frac{V_f}{V_i} = -\frac{C_D A \rho g}{W}\,x \quad \text{or} \quad x = -\frac{W}{C_D A \rho g}\ln\frac{V_f}{V_i}$$

Thus

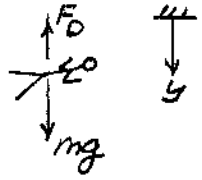
$$x = -\frac{1}{1.42} \times 32{,}000\ \text{lbf} \times \frac{2}{\pi (12)^2\ \text{ft}^2} \times \frac{\text{ft}^3}{0.075\ \text{lbf}} \times \ln\left(\frac{100}{160}\right) = 624\ \text{ft} \longleftarrow x$$

Given: Experimental data for a sky diver with $M = 75$ kg:

Prone, spread-eagled	$C_D A = 0.85\ m^2$
Vertical fall	$C_D A = 0.11\ m^2$

Find: Estimate time and distance needed to reach 95 percent of terminal speed at 3000 m altitude on a standard day.

Solution: From Table A.3, $\rho/\rho_{SL} = 0.7423$ at 3000 m altitude. Consider free-body diagram of sky diver:



$$\Sigma F_y = mg - F_D = mg - C_D A \tfrac{1}{2}\rho V^2 = ma_y = m\frac{dV}{dt} = mV\frac{dV}{dy}$$

At terminal speed, $a_y = 0$. Then $mg = C_D A \tfrac{1}{2}\rho V_t^2$

so $V_t^2 = \dfrac{2mg}{C_D A \rho}$. From above, $\dfrac{1}{g}\dfrac{dV}{dt} = \dfrac{V_t}{g}\dfrac{d(V/V_t)}{dt} = 1 - \dfrac{C_D A \rho V^2}{2mg} = 1 - \left(\dfrac{V}{V_t}\right)^2$

Thus $$\frac{d(V/V_t)}{1-(V/V_t)^2} = \frac{g}{V_t}\,dt$$

Integrating $$\int_0^{0.95} \frac{d(V/V_t)}{1-(V/V_t)^2} = \tanh^{-1}\left(\frac{V}{V_t}\right)\Big]_0^{0.95} = 1.83 = \frac{gt}{V_t}\ ;\ t = \frac{1.83\,V_t}{g}$$ t ←

Also $$1-\left(\frac{V}{V_t}\right)^2 = \frac{V}{g}\frac{dV}{dy} = \frac{V_t^2}{g}\left(\frac{V}{V_t}\right)\frac{d(V/V_t)}{dy} \quad \text{or} \quad \frac{(V/V_t)\,d(V/V_t)}{1-(V/V_t)^2} = \frac{g}{V_t^2}\,dy$$

Integrating, $$\int_0^{0.95} \frac{(V/V_t)\,d(V/V_t)}{1-(V/V_t)^2} = -\frac{1}{2}\ln\left[1-(V/V_t)^2\right]_0^{0.95} = 1.16 = \frac{g}{V_t^2}\,y\ ;\ y = \frac{1.16\,V_t^2}{g}$$ y ←

Calculating for $C_D A = 0.85\ m^2$:

$$V_t = \left[2 \times 75\ kg \times 9.81\ \frac{m}{s^2} \times \frac{1}{0.85\ m^2 \times (0.7423) 1.23\ kg} \times \frac{m^3}{}\right]^{1/2} = 43.5\ m/s$$

$$t = \frac{1.83\,V_t}{g} = 1.83 \times 43.5\ \frac{m}{s} \times \frac{s^2}{9.81\ m} = 8.11\ s$$

$$y = 1.16\frac{V_t^2}{g} = 1.16\,(43.5)^2\ \frac{m^2}{s^2} \times \frac{s}{9.81\ m} = 224\ m$$

Tabulating:

Position	V_t (m/s)	t (s)	y (m)
Prone	43.5	8.11	224
Vertical*	121	22.6	1,730

* These are estimates; density would vary significantly during this fall.

Given: Land speed record vehicle at Bonneville Salt Flats, elevation 4400 ft. Engine power, $\mathcal{P} = 500$ hp, frontal area, $A = 15\ ft^2$, and $C_D = 0.15$.

Find: Theoretical maximum speed (a) in still air, (b) 20 mph head wind.

Solution: Apply definitions of power, drag coefficient.

Computing equations: $\mathcal{P} = F_D V$, $C_D = \dfrac{F_D}{\frac{1}{2}\rho(V+V_w)^2 A}$

Assumptions: (1) Neglect rolling drag
(2) $\rho \approx 0.878\,\rho_0$ (Table A.3)

For no wind case, $V_w = 0$, and

$$\mathcal{P} = F_D V = C_D \tfrac{1}{2}\rho V^2 A V = C_D \tfrac{1}{2}\rho V^3 A$$

$$V = \left[\frac{2\mathcal{P}}{\rho C_D A}\right]^{1/3} = \left[2 \times 500\ hp \times \frac{ft^3}{(0.878)0.00238\ slug} \times \frac{1}{0.15} \times \frac{1}{15\ ft^2} \times \frac{550\ ft\cdot lbf}{hp\cdot s} \times \frac{slug\cdot ft}{lbf\cdot s^2}\right]^{1/3}$$

$V = 489$ ft/s (333 mph) ← V

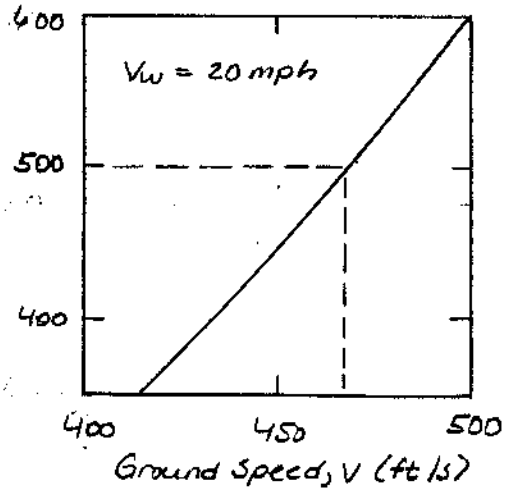
With a head wind,

$$\mathcal{P} = F_D V = C_D \tfrac{1}{2}\rho (V+V_w)^2 A V \quad \text{or} \quad \mathcal{P}(hp) = 4.27 \times 10^{-6}(V+V_w)^2 V\ (ft^3/s^3)$$

This can be solved by iteration. Using $V_w = 20$ mph or 29.3 ft/s,

V (ft/s)	$\mathcal{P}$ (hp)
400	315
470	501
500	599

Plotting:



From the plot, $V \simeq 468$ ft/s (319 mph) ← V

{Note that the maximum speed is not reduced by 20 mph when wind is present, because drag is nonlinear.}

Given: 1970s-size American car. Frontal area, $A = 23.4\ ft^2$; weight, $W = 4500\ lbf$; $C_D = 0.5$; $F_R = 0.015\,W$

Plot: Power required vs. road speed

Find: (a) Speed at which aerodynamic drag exceeds rolling resistance
(b) Power required to cruise at 55 and at 70 mph.

Solution: Apply definition of drag coefficient, $C_D = \dfrac{F_D}{\frac{1}{2}\rho V^2 A}$

$$F_T = F_R + F_D \quad ; \quad \mathcal{P} = F_T V = F_R V + C_D \tfrac{1}{2}\rho V^3 A$$

Assume standard air, $\rho = 0.00238\ slug/ft^3$.

$$\mathcal{P}_D = C_D \tfrac{1}{2}\rho V^3 A = 0.5 \times \tfrac{1}{2} \times 0.00238 \frac{slug}{ft^3} \left[V \frac{mi}{hr} \times 5280 \frac{ft}{mi} \times \frac{hr}{3600 s} \right]^3 23.4\ ft^2 \times \frac{16f\text{-}s^2}{slug\text{-}ft} \times \frac{hp\text{-}s}{550\ ft\text{-}lbf}$$

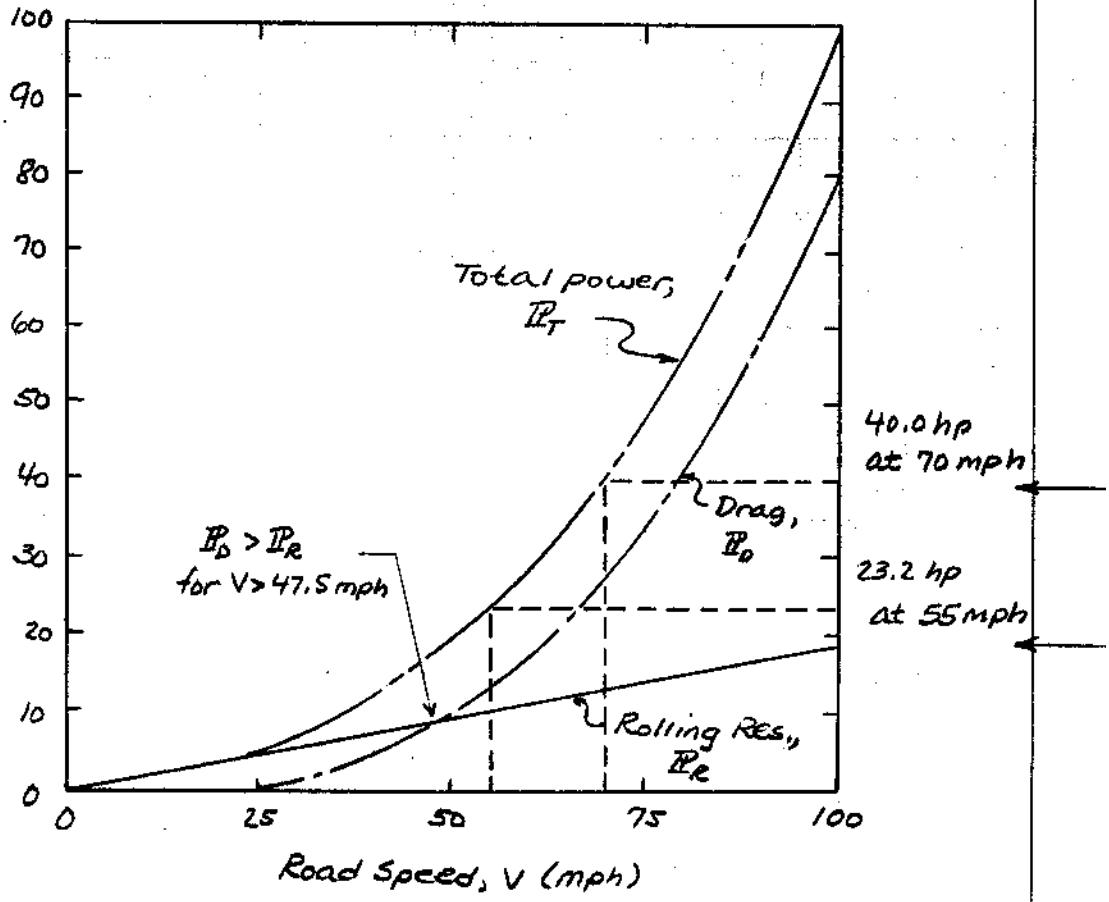
Thus $\mathcal{P}_D(hp) = 7.99 \times 10^{-5} [V(mph)]^3$

$$\mathcal{P}_R = F_R V = 0.015\,WV = 0.015 \times 4500\ lbf \times V \frac{mi}{hr} \times 5280 \frac{ft}{mi} \times \frac{hr}{3600 s} \times \frac{hp\text{-}s}{550\ ft\text{-}lbf}$$

$$\mathcal{P}_R(hp) = 0.180\,V(mph)$$

and $\mathcal{P}_T = \mathcal{P}_R + \mathcal{P}_D = 0.18\,V + 7.99 \times 10^{-5} V^3$ (V in mph)

Plotting:



$\mathcal{P}_D > \mathcal{P}_R$ when $7.99 \times 10^{-5} V^3 > 0.18\,V$, or $V \geq \sqrt{\dfrac{0.18}{7.99 \times 10^{-5}}} = 47.5$ mph

1032

Given: Tractor-trailer rig, with $A = 102\ ft^2$, $C_D = 0.9$. Rolling resistance is 6 lbf per 1000 lbf; $W = 72{,}000$ lbf. BSFC is 0.34 lbm/hp·hr, $\eta_d = 0.92$, and $\rho = 6.9$ lbm/gal. Truck travels 120,000 mi/yr.

Find: (a) Estimate fuel economy at 55 mph.
(b) Fuel saved by air deflector that reduces C_D by 15 percent.

Solution: Tractive force is $F_T = F_R + F_D$ ← aerodynamic force
↖ rolling resistance force

Engine power is $P_e = \frac{P_T}{\eta_d} = \frac{F_T V}{\eta_d}$

Thus

$$F_R = C_R W = 0.006 \times 72{,}000\ lbf = 432\ lbf$$

$$F_D = C_D A \tfrac{1}{2}\rho V^2 \qquad V = 55\frac{mi}{hr} \times 5280\frac{ft}{mi} \times \frac{hr}{3600\ s} = 80.7\ ft/s$$

$$F_D = 0.9 \times 102\ ft^2 \times \frac{1}{2} \times 0.00238\frac{slug}{ft^3} \times (80.7)^2\frac{ft^2}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} = 711\ lbf$$

so

$$F_T = F_R + F_D = 432 + 711 = 1140\ lbf$$

$$P_e = 1140\ lbf \times 80.7\frac{ft}{s} \times \frac{1}{0.92} \times \frac{hp \cdot s}{550\ ft \cdot lbf} = 182\ hp$$

Finally

$$FE = \frac{\rho V}{P_e\ BSFC} = 6.9\frac{lbm}{gal} \times 55\frac{mi}{hr} \times \frac{1}{182\ hp} \times \frac{hp \cdot hr}{0.34\ lbm} = 6.13\ mi/gal$$ ← FE

With the air deflector,

$$F_D = (1 - 0.15)\ 711\ lbf = 604\ lbf$$

$$F_T = F_R + F_D = 432 + 604 = 1040\ lbf$$

$$P_e = 1040\ lbf \times 80.7\frac{ft}{s} \times \frac{1}{0.92} \times \frac{hp \cdot s}{550\ ft \cdot lbf} = 166\ hp$$

and $$FE = 6.9\frac{lbm}{gal} \times 55\frac{mi}{hr} \times \frac{1}{166\ hp} \times \frac{hp \cdot hr}{0.34\ lbm} = 6.72\ mi/gal$$

The fuel saving would be

$$\Delta Q = \left(\frac{1}{FE}\right)_{without} - \frac{1}{FE_{with}}\Big) mileage = \left(\frac{1}{6.13} - \frac{1}{6.72}\right)\frac{gal}{mi} \times 120{,}000\frac{mi}{yr} = 1720\ gal/yr$$ ← Q

The percentage saving would be

$$\frac{\Delta Q}{Q} = \frac{\left(\frac{1}{FE_{without}} - \frac{1}{FE_{with}}\right)}{\left(\frac{1}{FE}\right)_{without}} = 0.0878 \text{ or } 8.78 \text{ percent savings}$$

Given: Porsche automobile : $C_D = 0.35$ $m = 1{,}360\ kg$

$A = 1.83\ m^2$

$P = 143\ bhp$

Power to cruise at 55 mph is 13.9 hp.

Find: (a) Maximum acceleration capability at 55 mph (24.6 m/sec)
(b) Maximum speed if $F_R = 0.01\,mg$.

Solution: At cruise condition $F_T = F_R + F_D = 0.01\,mg + C_D A \frac{1}{2}\rho V^2$

$$F_D = (0.35)1.83\,m^2 \times \frac{1}{2} \times 1.23\,\frac{kg}{m^3} \times (24.6)^2\,\frac{m^2}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} = 238\ N\ ;\ P_D = F_D V = 7.85\ hp$$

$$F_R = 0.01 \times 1360\ kg \times 9.81\,\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} = 133\ N\ ;\ P_R = F_R V = 4.40\ hp$$

The driveline efficiency is

$$\eta_D = \frac{P_R + P_D}{P_E} = \frac{(4.40 + 7.85)\,hp}{13.9\ hp} = 0.881$$

The excess power available to accelerate at 55 mph is

$$P_a = \eta_D (P_{max} - P_{cruise}) = 0.881(143 - 13.9)\,hp = 114\ hp\ ;\ F_a = 3450\ N$$

The acceleration capability is

$$F_a = ma\ ;\ a = \frac{F_a}{m} = 3450\ N \times \frac{1}{1360\ kg} \times \frac{kg\cdot m}{N\cdot s^2} = 2.54\ m/s^2\ (0.259\,g's)$$ a_{max}

At maximum speed,

$$\eta_D P_{max} = 0.01\,mg\,V + C_D A \tfrac{1}{2}\rho V^2 V = 133\,V + 0.392\,V^3\ (V \text{ in } m/s)$$

$$\eta_D P_{max} = 0.881 \times 143\,hp \times 746\,\frac{N\cdot m}{hp\cdot s} = 9.40 \times 10^4$$

By iteration,

$V_{max} = 60.3\ m/s$ (135 mph) V_{max}

Given: Ford "Probe" driven along level highway at 100 km/hr in standard air. Frontal area, $A = 1.8\ m^2$, and $C_D = 0.31$

Find: (a) power required to overcome aerodynamic drag
(b) estimate maximum speed if engine is rated at 145 hp.

Solution: Power is $\mathcal{P} = F_D V$

Governing equation: $F_D = C_D \frac{1}{2}\rho V^2 A$

$$\mathcal{P} = F_D V = C_D \frac{1}{2}\rho V^3 A$$

$$= 0.31 \times \frac{1}{2} \times 1.23 \frac{kg}{m^3} \times \left[100 \frac{km}{hr} \times 10^3 \frac{m}{km} \times \frac{hr}{3600\ s}\right]^3 \times 1.8\ m^2 \times \frac{N \cdot s^2}{kg \cdot m}$$

$$\mathcal{P} = 7.36 \times 10^3 \frac{N \cdot m}{s} \times \frac{hp}{746\ W} \times \frac{W \cdot s}{N \cdot m} = 9.86\ hp \qquad \mathcal{P}$$

Since $\mathcal{P} = C_D \frac{1}{2}\rho V^3 A$, then

$$V = \left[\frac{2\mathcal{P}}{C_D \rho A}\right]^{1/3}$$

If we assume 100% drive train efficiency and neglect rolling resistance then the 145 hp overcomes aerodynamic drag at maximum speed.

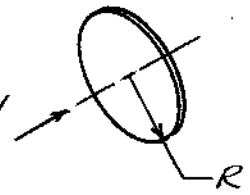
$$V_{max} = \left[\frac{2}{0.31} \times 145\ hp \times \frac{746\ N \cdot m}{hp \cdot s} \times 1.23 \frac{kg}{m^3} \times \frac{1}{1.8\ m^2} \times \frac{N \cdot s^2}{kg \cdot m}\right]^{1/3}$$

$$V_{max} = 68.1 \frac{m}{s} \quad (245\ km/hr,\ 152\ mph) \qquad V_{max}$$

Given: Round, thin disk of radius, R, with pressure data:

$C_p = 1-\left(\frac{r}{R}\right)^6$ (front)

$C_p = -0.42$ (rear)



Find: Calculate the drag coefficient, C_D.

Solution: Computing equations are

$$C_p = \frac{p-p_\infty}{\frac{1}{2}\rho V^2} \qquad C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} \qquad A = \pi R^2$$

Assumptions: (1) Steady, incompressible flow
(2) Neglect skin friction drag (disk thin, edge area small)

Then $$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} = \frac{\int_A (p_f - p_r)\,dA}{\frac{1}{2}\rho V^2 \pi R^2} = \frac{\int_0^R (p_f - p_r)\,2\pi r\,dr}{\frac{1}{2}\rho V^2 \pi R^2}$$

From defn of C_p, $p_f = p_\infty + C_{pf}\frac{1}{2}\rho V^2$, $p_r = p_\infty + C_{pr}\frac{1}{2}\rho V^2$

and $$p_f - p_r = (C_{pf} - C_{pr})\frac{1}{2}\rho V^2$$

Substituting,

$$C_D = \frac{\frac{1}{2}\rho V^2 \int_0^R (C_{pf} - C_{pr})\,2\pi r\,dr}{\frac{1}{2}\rho V^2 \pi R^2} = \frac{2}{R^2}\int_0^R \left[1-\left(\frac{r}{R}\right)^6 + 0.42\right] r\,dr$$

$$= 2\int_0^1 \left[1.42-\left(\frac{r}{R}\right)^7\right]\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) = 2\left[\frac{1.42}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{8}\left(\frac{r}{R}\right)^8\right]_0^1$$

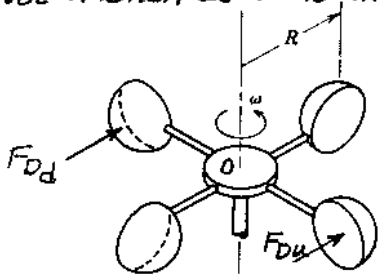
$$= 2\,(0.710 - 0.125)$$

$$C_D = 1.17$$

C_D

Given: Cup anemometer with $D = 50$ mm, $R = 75$ mm, which is to start rotating when wind speed is above $V = 1$ km/hr.

Find: (a) Relation between ω and V if friction is neglected.
(b) Maximum friction torque that can be present in the pivot.
(c) Error in speed measurement caused by pivot friction at $V = 10$ km/hr.



Solution: Sum moments about pivot: (+)

At start, $\omega = 0$

$$\Sigma M_O = RF_{Du} - RF_{Dd} - T_f = R(C_{Du} - C_{Dd}) A \tfrac{1}{2}\rho v^2 - T_f = 0$$

$$T_f = 0.075\,m\,(1.42 - 0.38)\,\frac{\pi(0.050)^2 m^2}{4} \times 1.23\,\frac{kg}{m^3} \times \left(1\,\frac{km}{hr} \times \frac{1000\,m}{km} \times \frac{hr}{3600\,s}\right)^2 \frac{N \cdot s^2}{kg \cdot m}$$

$$= 7.27 \times 10^{-6}\ N \cdot m \quad T_f$$

At $V = 10$ mph, $\omega \neq 0$

$$\Sigma M = RF_{Du} - RF_{Dd} - T_f = RC_{Du} A \tfrac{1}{2}\rho (V - R\omega)^2 - RC_{Dd} A \tfrac{1}{2}\rho (V + R\omega)^2 - T_f = 0 \qquad (1)$$

If friction torque were zero, could solve for $R\omega$:

$$\frac{V - R\omega}{V + R\omega} = \frac{1 - \frac{R\omega}{V}}{1 + \frac{R\omega}{V}} = \left(\frac{C_{Dd}}{C_{Du}}\right)^{1/2} \quad \text{or} \quad 1 - \frac{R\omega}{V} = \left(\frac{C_{Dd}}{C_{Du}}\right)^{1/2}\left(1 + \frac{R\omega}{V}\right)$$

$$1 - \left(\frac{C_{Dd}}{C_{Du}}\right)^{1/2} = \frac{R\omega}{V}\left[1 + \left(\frac{C_{Dd}}{C_{Du}}\right)^{1/2}\right] \quad \text{or} \quad \frac{R\omega}{V} = \frac{1 - (C_{Dd}/C_{Du})^{1/2}}{1 + (C_{Dd}/C_{Du})^{1/2}} = 0.3181 \quad \frac{R\omega}{V}$$

Equation 1 can be solved by trial and error. The solution is $\frac{R\omega}{V} = 0.3145$.

Thus the effect of friction is to reduce $R\omega$ by

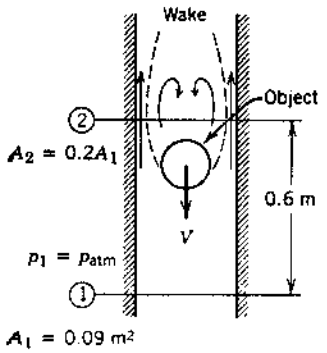
$$\frac{\Delta R\omega}{R\omega} = \frac{0.3145 - 0.3181}{0.3181} = -0.0113 \text{ or } -1.13 \text{ percent}$$

However, note the device runs at $R\omega = 0.315\,V$ at best, with no friction!

Given: Object falling in air down chute.

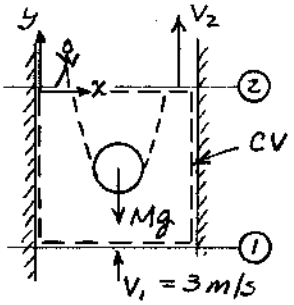
$V = 3$ m/s

Frictional effects negligible.



Find: (a) Flow speed, V_2, relative to object.
(b) Static pressure, p_2.
(c) Mass of object.

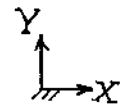
Solution: Choose a CV moving with object. Apply continuity, Bernoulli, and y momentum.



Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV}\rho\,d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$ ($\frac{\partial}{\partial t}$ = 0 (1))

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g\cancel{z_1} = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g\cancel{z_2}$$

$$F_{sy} + F_{by} = \frac{\partial}{\partial t}\int_{CV} v\rho\,d\forall + \int_{CS} v\rho\vec{V}\cdot d\vec{A}$$ ($\frac{\partial}{\partial t}$ = 0 (1))



Assumptions: (1) Steady flow relative to CV
(2) Incompressible flow
(3) Neglect friction
(4) Flow along a streamline
(5) Neglect Δz
(6) Uniform flow and pressures at ① and ②.
(7) No net flow in wake

Then from continuity,

$$0 = \{-\rho V_1 A_1\} + \{+\rho V_2 A_2\} \quad \text{so} \quad V_2 = V_1\frac{A_1}{A_2} = 3\frac{m}{s}\times\frac{1}{0.2} = 15\ m/s$$ V_2

From Bernoulli,

$$p_2 = p_1 + \tfrac{1}{2}\rho V_1^2 - \tfrac{1}{2}\rho V_2^2 = p_1 + \tfrac{1}{2}\rho V_1^2\left[1 - \left(\frac{V_2}{V_1}\right)^2\right]$$

$$p_2(\text{gage}) = \frac{1}{2}\times 1.23\frac{kg}{m^3}\times(3)^2\frac{m^2}{s^2}\left[1-\left(\frac{1}{0.2}\right)^2\right]\frac{N\cdot s^2}{kg\cdot m} = -133\ N/m^2\ (\text{gage})$$ p_2

From momentum

$$p_1A_1 - p_2A_2 - Mg = v_1\{-\rho V_1A_1\} + v_2\{+\rho V_2A_2\} = \rho V_1A_1(V_2 - V_1) = \rho V_1^2A_1\left(\frac{V_2}{V_1} - 1\right)$$

$$v_1 = V_1 \qquad v_2 = V_2$$

Thus

$$M = \frac{A_1}{g}\left[p_1 - p_2 - \rho V_1^2\left(\frac{V_2}{V_1} - 1\right)\right]$$

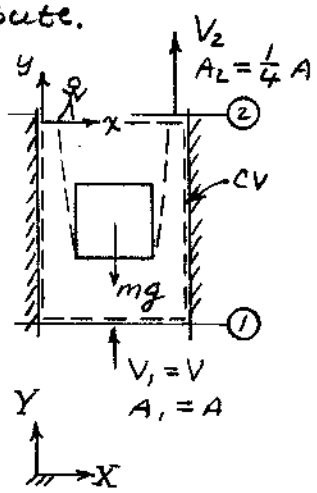
$$M = 0.09\ m^2\times\frac{s^2}{9.81\ m}\left[133\frac{N}{m^2}\times\frac{kg\cdot m}{N\cdot s^2} - 1.23\frac{kg}{m^3}\times(3)^2\frac{m^2}{s^2}\left(\frac{1}{0.2} - 1\right)\right] = 0.814\ kg$$ M

Given: Object of mass, m, falling in air down mail chute.

Motion is steady.

Wake area at ② is 3/4 of chute area.

Pressure is constant in wake.



Find: Expression for terminal speed of object.

Solution: Choose a CV moving with the object.
Apply continuity, Bernoulli, and y momentum.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$ ($\frac{\partial}{\partial t} = 0$ (1))

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 \quad (z_1 = z_2)$$

$$F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall + \int_{CS} v\rho \vec{V}\cdot d\vec{A} \quad (\frac{\partial}{\partial t} = 0 \text{ (1)})$$

Assumptions: (1) Steady flow relative to CV
(2) Incompressible flow
(3) Neglect friction
(4) Flow along a streamline
(5) Uniform flow and pressures at ① and ②.
(6) Neglect Δz
(7) No net flow in wake

From continuity,

$$0 = \{-\rho V_1 A_1\} + \{\rho V_2 A_2\} \quad \text{so} \quad V_2 = V_1 \frac{A_1}{A_2} = V \frac{A}{A_2}$$

From Bernoulli

$$p_1 - p_2 = \frac{1}{2}\rho V_2^2 - \frac{1}{2}\rho V_1^2 = \frac{1}{2}\rho V^2\left[\left(\frac{V_2}{V}\right)^2 - 1\right] = \frac{1}{2}\rho V^2\left[\left(\frac{A}{A_2}\right)^2 - 1\right]$$

From momentum

$$p_1 A - p_2 A - mg = v_1\{-\rho V_1 A_1\} + v_2\{+\rho V_2 A_2\} = \rho V A (V_2 - V) = \rho V^2 A\left(\frac{V_2}{V} - 1\right)$$

$$v_1 = V \qquad v_2 = V_2$$

or

$$(p_1 - p_2)A - mg = \rho V^2 A\left(\frac{A}{A_2} - 1\right)$$

Substituting for $p_1 - p_2$

$$\frac{1}{2}\rho V^2 A\left[\left(\frac{A}{A_2}\right)^2 - 1\right] - mg = \rho V^2 A\left(\frac{A}{A_2} - 1\right) \quad \text{or} \quad mg = \frac{1}{2}\rho V^2 A\left[\left(\frac{A}{A_2}\right)^2 - 2\left(\frac{A}{A_2}\right) + 1\right]$$

Thus

$$V = \left[\frac{2mg}{\rho A}\,\frac{1}{\left(\frac{A}{A_2}\right)^2 - 2\left(\frac{A}{A_2}\right) + 1}\right]^{1/2}$$

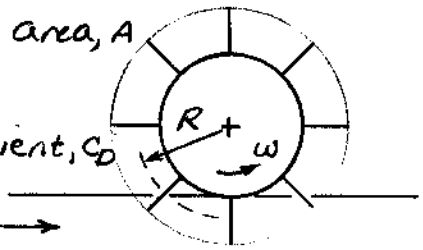
Where A_2 is the net flow area at Section ②.

Given: Paddle wheel immersed in river current as shown.

Assume only one paddle equivalent is immersed at a time.

Paddle area, A

Drag coefficient, C_D



Find: Expressions for (a) force, (b) torque, and (c) power produced by the wheel, and find optimum angular speed.

Solution: Computing equations $F_D = C_D A \frac{1}{2}\rho V_{rel}^2$

$$T = F_D R \quad , \quad P = T\omega$$

Assumptions: (1) Neglect air resistance, since $\rho_{air} \ll \rho_{water}$
(2) Use velocity relative to the paddle

Thus $V_{rel} = V - U = V - R\omega$

$$F_D = C_D A \tfrac{1}{2}\rho V_{rel}^2 = C_D A \tfrac{1}{2}\rho (V-U)^2 \qquad F_D$$

The torque is

$$T = F_D R = C_D A \tfrac{1}{2}\rho (V-U)^2 R \qquad T$$

The power is

$$P = T\omega = C_D A \tfrac{1}{2}\rho (V-U)^2 R\omega = C_D A \tfrac{1}{2}\rho (V-U)^2 U \qquad P$$

To optimize power, set $\frac{dP}{dU} = 0$

$$\frac{dP}{dU} = C_D A \tfrac{1}{2}\rho \left[2(V-U)U(-1) + (V-U)^2 \right] = 0$$

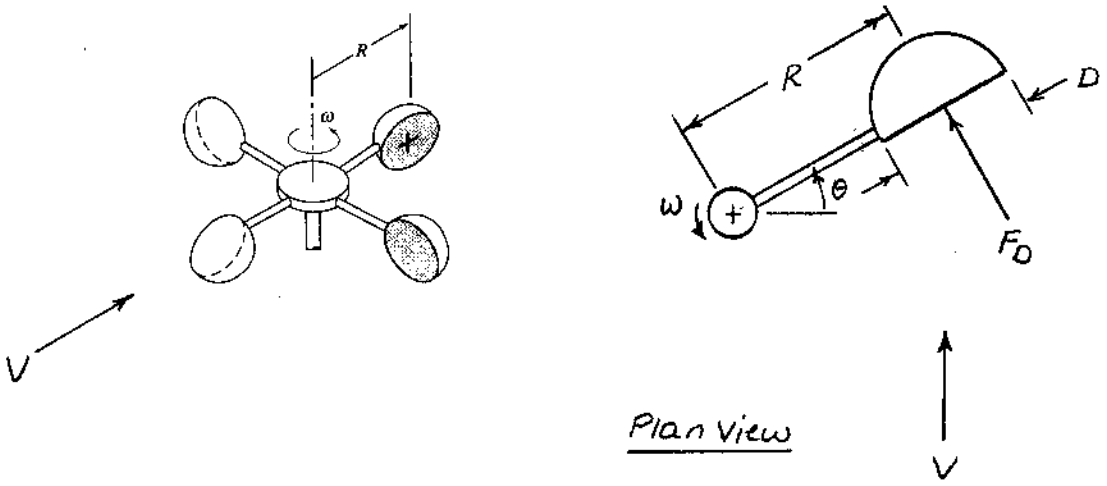
Cancelling a factor $(V-U)$ gives $-2U + V - U = 0$ or $V - 3U = 0$

Thus $U = R\omega = \frac{V}{3}$ so

$$\omega_{opt} = \frac{V}{3R} \qquad \omega_{opt}$$

Open-Ended Problem Statement: A rotating-cup anemometer to measure wind speed (see Problem 9.87) is located atop a water sculpture. It is impractical to calibrate the anemometer in place because it is 11 m above the ground. Instead, it is proposed that the anemometer be analyzed to predict its rotation speed as a function of wind speed. Model the angular speed of the anemometer as wind speed is varied in standard air. Plot a "calibration curve" predicted by the analytical model using 125 mm cup center radius, 125 mm cup diameter, and 10^{-5} N·m retarding torque in the pivot. Discuss the limitations (if any) on your theoretical analysis. Discuss how your assumptions affect the predicted performance of the anemometer. Would the actual performance of the anemometer be better or worse than predicted by your analysis? Why? What factors account for the difference(s)?

Analysis: The geometry and notation used in the model are shown below:



See the following pages for the analysis, calculations, and plot.

Discussion: Several assumptions made to model this problem analytically were:

(1) Aerodynamic drag on the hemispherical cups was modeled using the component of relative velocity normal to the open face of the cup and constant drag coefficients C_{Du} for upstream facing and C_{Dd} for downstream facing hemispheres. During rotation the hemispherical cups actually experience angles between 0 and 180° relative to the wind. Data for the drag coefficient of a hemisphere over the complete range of angles would be needed to improve the model.

(2) Each cup was assumed to operate in an undisturbed stream with no influence from its neighbors. The "masking" effect of wakes from upstream cups was ignored.

(3) Any swirl present was ignored. The rotating hemispherical cups would add swirl to the air flow, disturbing it from the uniform, parallel flow assumed in the analysis.

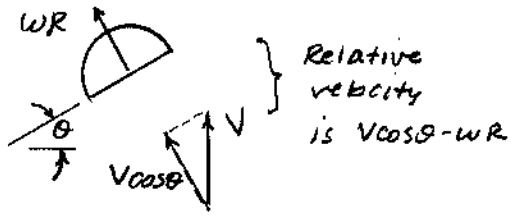
(4) The interference between the flow around the struts and the flow around each hemispherical cup was ignored.

(5) Drag of the struts was not considered. Because of the difference in relative velocities as the strut approaches the wind direction or travels away from it, a non-zero net drag force and torque caused by the struts would tend to oppose rotation of the anemometer.

The actual performance of the anemometer would be worse than predicted by the simplified model: it would rotate more slowly at any wind speed than predicted by the model.

Referring to the sketch on the first page, the aerodynamic torque exerted by the cup in the position shown is

$$T = RF_D = R C_D A \tfrac{1}{2}\rho V_n^2$$



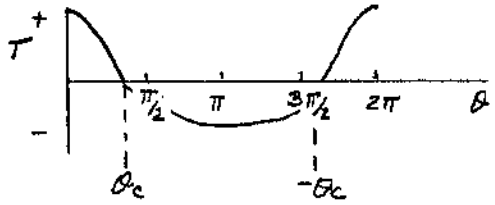
From geometry, $V_n = V\cos\theta - \omega R$

If $V_n > 0$, then $\cos\theta > \frac{\omega R}{V}$, and C_{Du} is used.

If $V_n < 0$, then $\cos\theta < \frac{\omega R}{V}$, and C_{Dd} is used.

Thus V_n changes from positive to negative when $\theta = \theta_c = \cos^{-1}(\frac{\omega R}{V})$.

Schematically the torque vs. θ looks like



The average torque for a complete revolution is given by

$$\bar{T} = \frac{1}{2\pi}\int_0^{2\pi} T\,d\theta$$

By symmetry this may be written $\bar{T} = \frac{1}{2\pi}\,2\int_0^{\pi} T\,d\theta = \frac{1}{\pi}\int_0^{\pi} T\,d\theta$

Substituting,

$$\bar{T} = \frac{1}{\pi}\int_0^{\theta_c} R C_{Du} A \tfrac{1}{2}\rho (V\cos\theta - \omega R)^2 d\theta + \frac{1}{\pi}\int_{\theta_c}^{\pi} R C_{Dd} A \tfrac{1}{2}\rho (V\cos\theta - \omega R)^2 d\theta$$

(C_{Dd} must be negative; use $-C_{Dd}$ or $(\,)|(\,)|$)

$$\bar{T} = \frac{RA}{\pi}\tfrac{1}{2}\rho V^2\left[\int_0^{\theta_c} C_{Du}(\cos\theta - \omega R/V)^2 d\theta - \int_{\theta_c}^{\pi} C_{Dd}(\cos\theta - \omega R/V)^2 d\theta\right]$$

But $(\cos\theta - \omega R/V)^2 = \cos^2\theta - 2\frac{\omega R}{V}\cos\theta + (\frac{\omega R}{V})^2$, which integrates to

$$\int(\cos\theta - \tfrac{\omega R}{V})^2 d\theta = \left[\frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2} - 2(\tfrac{\omega R}{V})\sin\theta + (\tfrac{\omega R}{V})^2\theta\right] = f(\theta)$$

Thus $$\bar{T} = \frac{RA}{\pi}\tfrac{1}{2}\rho V^2\left\{C_{Du}[f(\theta_c) - \cancel{f(0)}^{\,0}] - C_{Dd}[f(\pi) - f(\theta_c)]\right\} \qquad (1)$$

In Eq. 1 the term with C_{Du} is positive, the term with C_{Dd} negative. The net aerodynamic torque must be large enough to overcome friction. This will determine the operating speed ω for any wind speed V.

The results show the anemometer will not spin steadily until the wind speed exceeds about 3 m/s.

Problem 9.91 (cont'd.)

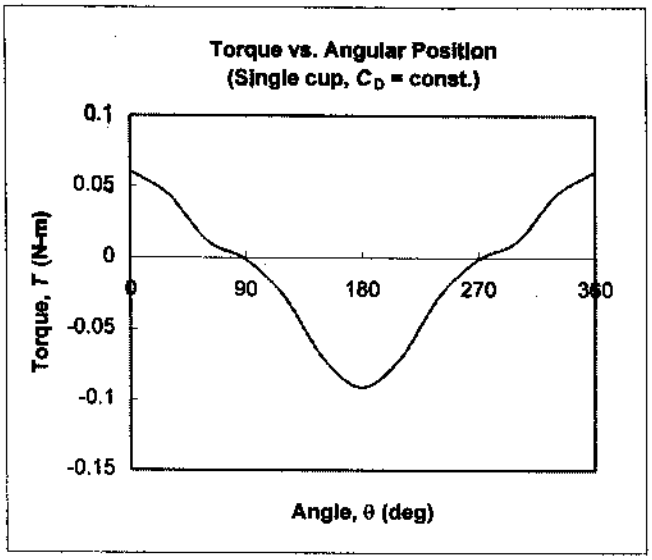
Input data:

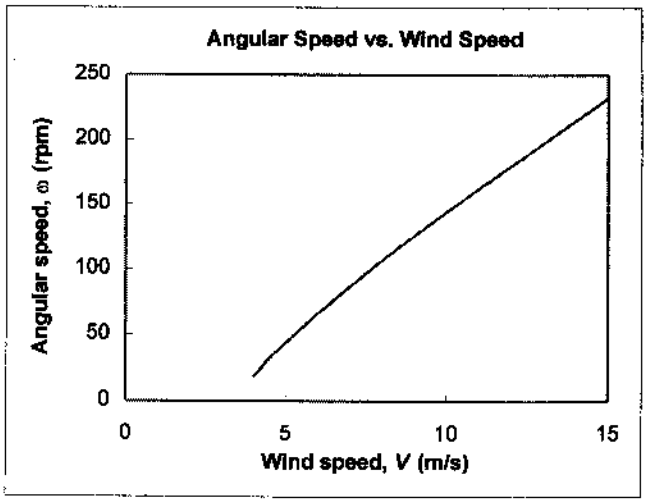
C_{Du} =	1.42		Drag coefficient facing flow
C_{Dd} =	0.38		Drag coefficient away from flow
D =	125	mm	Cup diameter
R =	125	mm	Radius to cup center
V =	7.5	m/s	Wind speed
ω =	6	rad/s	Angular speed of rotor

Calculated results:

A =	0.0123	m^2	Cup frontal area
q =	34.6	N/m^2	Dynamic pressure
$\omega R/V$ =	0.100		Angular speed ratio
θ_c =	84.3	deg	Critical angle

Angle	Angle	Normal Velocity		Torque
θ (deg)	θ (rad)	V_n (m/s)	V_n*[V_n]	T (N-m)
0	0	6.75	45.6	0.0610
30	0.524	5.75	33.0	0.0442
60	1.05	3.00	9.00	0.0121
90	1.57	-0.750	-0.563	0.0
120	2.09	-4.50	-20.3	-0.0271
150	2.62	-7.25	-52.5	-0.0703
180	3.14	-8.25	-68.1	-0.0912
210	3.67	-7.25	-52.5	-0.0703
240	4.19	-4.50	-20.3	-0.0271
270	4.71	-0.750	-0.563	0.0
300	5.24	3.00	9.00	0.0121
330	5.76	5.75	33.0	0.0442
360	6.28	6.75	45.6	0.0610





					Single Cup		4 Cups			
$\omega R/V$	θ_c (rad)	θ_c (deg)	f(θ_c)	f(π)	T_{aero} (norm)	T_{aero} (N-m)	T_{aero} (N-m)	T_{net} (N-m)	V (m/s)	$\omega R/V$ (@ T = 0)
0	1.57	90.0	0.785	1.57	0.817	0.0138	0.0552	0.0452		
0.05	1.52	87.1	0.689	1.58	0.641	0.0108	0.0433	0.0333		
0.10	1.47	84.3	0.601	1.60	0.473	0.00798	0.0319	0.0219		
0.15	1.42	81.4	0.520	1.64	0.312	0.00526	0.0210	0.0110	4.0	0.0610
0.20	1.37	78.5	0.446	1.70	0.157	0.00266	0.0106	0.000631	5.0	0.114
0.25	1.32	75.5	0.378	1.77	0.010	0.000161	0.000643	-0.00936	7.5	0.169
0.30	1.27	72.5	0.318	1.85	-0.132	-0.00224	-0.00895	-0.0189	10.0	0.188
0.35	1.21	69.5	0.263	1.96	-0.269	-0.00454	-0.0182	-0.0282	12.5	0.197
0.40	1.16	66.4	0.215	2.07	-0.401	-0.00677	-0.0271	-0.0371	15.0	0.203
0.45	1.10	63.3	0.173	2.21	-0.528	-0.0089	-0.0356	-0.0456		
0.50	1.05	60.0	0.136	2.36	-0.651	-0.0110	-0.0440	-0.0540		

500 SHEETS, FILLER, 5 SQUARE
500 SHEETS EYE-EASE® 5 SQUARE
100 SHEETS EYE-EASE® 5 SQUARE
200 SHEETS EYE-EASE® 5 SQUARE
100 RECYCLED WHITE 5 SQUARE
200 RECYCLED WHITE 5 SQUARE
National® Brand

Given: Pitot-static probe with stem diameter, $d = 6\,mm$, inserted distance, $L = 300\,mm$, into wind-tunnel flow with $U = 25\,m/s$.

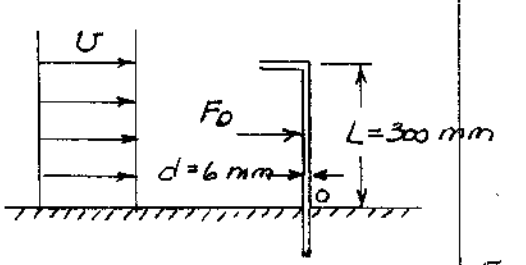
Find: (a) Drag force on probe.
(b) Bending moment on probe.

Solution: Apply definition of drag coefficient, $C_D = \dfrac{F_D}{\frac{1}{2}\rho U^2 A}$

Find C_D for cylinder from Fig. 9.13:

$$Re = \frac{\rho U D}{\mu} = \frac{U D}{\nu} = 25\frac{m}{s} \times 0.006\,m \times \frac{s}{1.46\times10^{-5}\,m^2} = 1.03\times10^{4} \quad \text{(standard air)}$$

Thus, $C_D \approx 1.2$



$$F_D = C_D A \tfrac{1}{2}\rho U^2 = C_D d L \tfrac{1}{2}\rho U^2$$

$$= 1.2 \times 0.006\,m \times 0.300\,m \times \frac{1}{2} \times 1.23\frac{kg}{m^3} \times (25)^2\frac{m^2}{s^2} \times \frac{N\cdot s^2}{kg\cdot m}$$

$$F_D = 0.830\,N$$

F_D ←

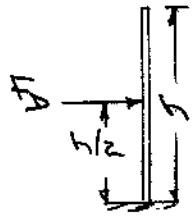
For uniform flow, F_D acts at $L/2$ from wall. Therefore moment about O is

$$M_O = F_D \frac{L}{2} = 0.830\,N \times \frac{0.300}{2}\,m = 0.125\,N\cdot m \quad \text{(clockwise)}$$

M_O ←

Given: Auto antenna, $L = 2\,m$, $D = 8\,mm$, moving through standard air at $V = 125\,km/hr$.

Find: Bending moment tending to snap it off.



Solution:

Basic equation: $C_D = \dfrac{F_D}{\frac{1}{2}\rho V^2 A}$; $A = LD$

$C_D = C_D(Re)$. For standard air, $\nu = 1.46\times10^{-5}\,m^2/s$ (Table A.1d).

$$Re = \frac{DV}{\nu} = 0.008\,m \times 125\times10^3\,\frac{m}{hr} \times 1.46\times10^{-5}\,\frac{s}{m^2} \times \frac{hr}{3600\,s} = 19{,}000$$

Re from Fig. 9.13, $C_D = 1.0$ and the bending moment is

$$M = F_D\frac{L}{2} = C_D\frac{1}{2}\rho V^2 A\frac{L}{2} = C_D\frac{1}{4}\rho V^2 D L^2$$

$$M = \frac{1.0}{4} \times 1.23\,\frac{kg}{m^3} \times \left(125\times10^3\,\frac{m}{hr} \times \frac{hr}{3600\,s}\right)^2 \times (0.008\,m) \times (2\,m)^2 \times 1.5^2\,\frac{N\cdot s^2}{kg\cdot m}$$

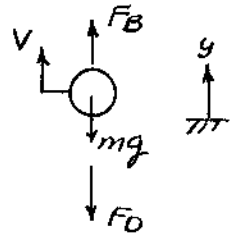
$$M = 11.9\ N\cdot m \qquad M$$

Given: Small droplets of oil (SG = 0.85) rising in water.

Find: (a) Relationship for terminal speed, V_t (m/s), as a function of droplet diameter, D (in mm).
(b) Range of D for which Stokes flow is a reasonable assumption.

Solution: Draw free-body diagram of droplet, apply Newton's second law.

Basic equation: $\Sigma F_y = -mg + F_B - F_D = ma_y$



Assume: Stokes' drag law, $F_D = 3\pi\mu V D$, for $Re < 10$

Then $-\rho \forall g + \rho_{H_2O} \forall g - 3\pi\mu V_t D = 0$ at terminal speed, V_t.

Solving, $V_t = \dfrac{(\rho_{H_2O} - \rho_0)\forall g}{3\pi\mu D} = \rho_{H_2O}(1 - SG_0)\dfrac{\pi D^3}{6}\dfrac{g}{3\pi\mu D} = \dfrac{(1 - SG_0)D^2 g}{18\nu}$

Evaluating,

$$V_t(m/s) = \frac{(1-0.85)}{18} \times D^2\, mm^2 \times 9.81\,\frac{m}{s} \times \frac{s}{1.00\times10^{-6} m^2} \times \frac{m^2}{10^6\, mm^2} \qquad (T = 20^\circ C)$$

$$V_t(m/s) = 0.0818\,[D(mm)]^2 \longleftarrow V_t$$

For Stokes flow, $Re < 1$, so

$$Re = \frac{\rho V_t D}{\mu} = \frac{V_t D}{\nu} = \frac{(1 - SG_0)D^3 g}{18\nu^2} \le 1$$

Thus

$$D^3 \le \frac{18\,\nu^2}{(1 - SG_0)g} \quad \text{or} \quad D \le \left[\frac{18\,\nu^2}{(1 - SG_0)g}\right]^{1/3}$$

Evaluating,

$$D \le \left[\frac{18}{(1-0.85)}(1.00\times10^{-6})^2 m^4 \times \frac{s^2}{9.81\, m}\right]^{1/3} = 2.31\times10^{-4} m \;(0.231\ mm) \longleftarrow D$$

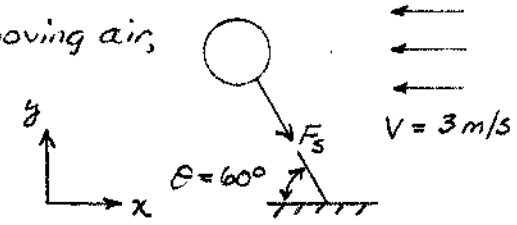
Thus Stokes' flow will be a valid assumption for $D < 0.231$ mm.

Given: Spherical hydrogen-filled balloon, $D = 0.6$ m, in standard air.

In stationary air,



In moving air,



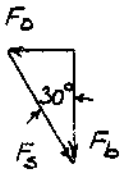
Find: Drag coefficient of balloon.

Solution: Apply Newton's second law of motion, definition of C_D.

Basic equations: $\Sigma F_x = m\cancel{a_x}^{=0(1)}$, $C_D = \dfrac{F_D}{\frac{1}{2}\rho V^2 A}$

Assumptions: (1) Balloon is stationary
(2) F_b (buoyancy of balloon) = 1.3 N

Then a free body diagram gives



From the diagram, $F_D = F_s \tan 30°$

$$F_D = 1.3\,N \times 0.577 = 0.750\,N$$

For standard air, $\rho = 1.23$ kg/m^3, $A = \pi D^2/4$, so

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} = \frac{8}{\pi}\frac{F_D}{\rho V^2 D^2}$$

$$C_D = \frac{8}{\pi} \times 0.750\,N \times \frac{m^3}{1.23\,kg} \times \frac{s^2}{(3)^2 m^2} \times \frac{1}{(0.6)^2 m^2} \times \frac{kg \cdot m}{N \cdot s^2} = 0.479$$

C_D

Given: Wind tunnel with standard air drawn in.

Sphere with $D = 30$ mm on a force balance.

Static pressure in tunnel, $p = -40$ mm (oil, SG = 0.85)

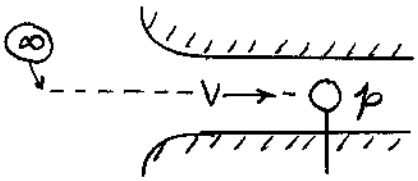
Find: (a) Freestream air speed

(b) Reynolds number for flow over sphere

(c) Drag force on sphere.

Solution: Apply Bernoulli

$$p_\infty + \tfrac{1}{2}\rho \underbrace{V_\infty^2}_{\approx 0 \,(6)} + \rho g \underbrace{z_\infty}_{(5)} = p + \tfrac{1}{2}\rho V^2 + \rho g \underbrace{z}_{(5)}$$



Assume: (1) Steady flow
(2) Incompressible flow
(3) Flow along a streamline
(4) No friction (neglect honeycomb and/or screens
(5) Neglect z
(6) $V_\infty \approx 0$

Then $p_\infty = p + \tfrac{1}{2}\rho V^2$ or $V = \sqrt{\dfrac{2(p_\infty - p)}{\rho}}$

But $p_\infty - p = -\rho_{oil} g \Delta h = -SG\,\rho_{H_2O}\, g \Delta h$

$$V = \left[\frac{-2\,SG\,\rho_{H_2O}\, g\, \Delta h}{\rho}\right]^{\frac{1}{2}}$$

$$V = \left[-2(0.85) \times 1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times (-0.04\ m) \times \frac{m^3}{1.23\ kg}\right]^{\frac{1}{2}} = 23.3\ m/s$$ V ←

and

$$Re = \frac{VD}{\nu} = 23.3 \frac{m}{s} \times 0.03\ m \times \frac{s}{1.45 \times 10^{-5} m^2} = 48{,}200$$ Re ←

Re is subcritical; BLs are laminar, and $C_D = 0.47$

$$F_D = C_D A \tfrac{1}{2}\rho V^2 \qquad A = \frac{\pi D^2}{4} = 7.07 \times 10^{-4}\ m^2$$

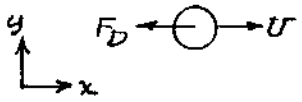
$$= 0.47 \times 7.07 \times 10^{-4} m^2 \times \frac{1}{2} \times 1.23 \frac{kg}{m^3} \times (23.3)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_D = 0.111\ N$$ F_D ←

Given: Field hockey ball with $D = 73$ mm and $m = 160$ g, leaving stick at $U_0 = 50$ m/s. Ball is smooth sphere.

Find: Estimate distance traveled in horizontal flight to reduce speed of ball 10 percent.

Solution: Apply Newton's second law of motion:



Basic equation: $\Sigma F_x = ma_x = m\frac{dU}{dt} = mU\frac{dU}{dx}$

Thus $-F_D = -C_D A \frac{1}{2}\rho U^2 = mU\frac{dU}{dx}$ or $dx = -\frac{2m}{C_D A \rho}\frac{dU}{U}$

Check Re to find C_D (use ν at $T = 15°C$ from Table A.10):

$$Re \le \frac{U_0 D}{\nu} = 50\frac{m}{s} \times 0.075\,m \times \frac{s}{1.46\times10^{-5}\,m^2} = 2.57\times10^5 \qquad \text{(standard air)}$$

From Fig. 9.11, flow is subcritical and $C_D \simeq 0.47 = \text{constant}$.

Thus
$$x = \int_0^x dx = \int_{U_0}^{U} -\frac{2m}{C_D A\rho}\frac{dU}{U} = -\frac{2m}{C_D A\rho}\ln U\Big]_{U_0}^{U} = -\frac{2m}{C_D A\rho}\ln\left(\frac{U}{U_0}\right)$$

or
$$x = -2\times 0.160\,kg \times \frac{1}{0.47}\times\frac{4}{\pi(0.073)^2 m^2}\times\frac{m^3}{1.23\,kg}\ln(0.9) = 13.9\,m \longleftarrow x$$

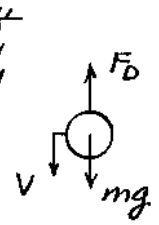
Given: Hailstones with $D = 10$ mm, falling in standard air, if spherical.

Find: Terminal speed.

Solution: Apply Newton's second law of motion, definition of C_D.

Computing equations:

$$\Sigma F_y = ma_y \qquad C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$$



At terminal speed, $V = V_t$, then $a_y = 0$; neglect buoyancy.

Summing forces, $mg - F_D = mg - C_D \frac{1}{2}\rho V_t^2 A = 0$

or $$V_t = \sqrt{\frac{2mg}{C_D \rho A}}$$

But $mg = \rho_{ice} \forall g = SG_{ice}\rho_{H_2O}\frac{\pi D^3}{6}g$. From Table A.1, $SG_{ice} = 0.917$. Also, $A = \frac{\pi D^2}{4}$

Thus $$V_t = \sqrt{\frac{2\, SG_{ice}\rho_{H_2O}\frac{\pi D^3}{6}g}{C_D \rho \frac{\pi D^2}{4}}} = \sqrt{\frac{4}{3}\frac{SG_{ice}\rho_{H_2O} Dg}{C_D \rho}}$$

$$V_t = \left[\frac{4}{3} \times 0.917 \times 999\frac{kg}{m^3} \times 0.01\, m \times 9.81\frac{m}{s^2} \times \frac{1}{C_D} \times \frac{m^3}{1.23\, kg}\right]^{1/2} = \frac{9.87}{\sqrt{C_D}}\ (m/s)$$

Assume C_D is in flat region of C_D versus Re curve: $C_D \simeq 0.47$. Then

$$V_t = \frac{9.87}{\sqrt{0.47}}\ m/s = 14.4\ m/s$$

Check Reynolds number, using ν for standard air ($T = 15°C$, Table A.10):

$$Re = \frac{\rho V_t D}{\mu} = \frac{V_t D}{\nu} = 14.4\frac{m}{s} \times 0.01\, m \times \frac{s}{1.46 \times 10^{-5} m^2} = 9.86 \times 10^3$$

At this Re, $C_D \approx 0.41$, so

$$V_t = \frac{9.87}{\sqrt{0.41}}\ m/s \simeq 15.4\ m/s \qquad V_t$$

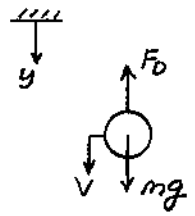
Actual hailstones are not spherical, but this estimate probably is a reasonable one.

Given: Raindrop (assume spherical) with $D = \frac{1}{8}$ in., falling in standard air.

Find: Terminal speed.

Solution: Apply Newton's second law of motion, definition of C_D.

Computing equations: $\Sigma F_y = ma_y \qquad C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$



At terminal speed, $V = V_t$, and $a_y = 0$; neglect buoyancy.

Summing forces, $mg - F_D = mg - C_D A \frac{1}{2}\rho V_t^2 = 0$

or $V_t = \sqrt{\frac{2mg}{C_D \rho A}}$

But $m = \rho_{H_2O} \forall = \rho_{H_2O} \frac{\pi D^3}{6}$, and $A = \frac{\pi D^2}{4}$. Thus

$$V_t = \sqrt{\frac{2\rho_{H_2O}\frac{\pi D^3}{6} g}{C_D \rho \frac{\pi D^2}{4}}} = \sqrt{\frac{4}{3}\frac{\rho_{H_2O} D g}{C_D \rho}}$$

$$V_t = \left[\frac{4}{3} \times 1.94 \frac{\text{slug}}{\text{ft}^3} \times \frac{1}{8}\text{ in.} \times 32.2 \frac{\text{ft}}{\text{s}^2} \times \frac{1}{C_D} \times \frac{\text{ft}^3}{0.00238\ \text{slug}} \times \frac{\text{ft}}{12\text{ in.}}\right]^{1/2} = \frac{19.1}{\sqrt{C_D}}\ \text{ft/s}$$

Assume C_D is in flat region of C_D versus Re curve: $C_D \simeq 0.47$. Then

$$V_t = \frac{19.1}{\sqrt{0.47}}\ \text{ft/sec} = 27.9\ \text{ft/s}$$

Check Reynolds number, using T = 15°C for standard air (ν from Table A.9):

$$Re = \frac{\rho V_t D}{\mu} = \frac{V_t D}{\nu} = 27.9 \frac{\text{ft}}{\text{s}} \times \frac{1}{8}\text{ in.} \times \frac{\text{s}}{1.57 \times 10^{-4}\ \text{ft}^2} \times \frac{\text{ft}}{12\text{ in.}} = 1.85 \times 10^3$$

At this Re, $C_D \approx 0.41$, so

$$V_t = \frac{19.1}{\sqrt{0.41}}\ \text{ft/s} \approx 29.8\ \text{ft/s}$$

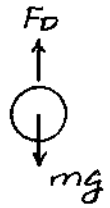
V_t

Actual raindrops are not quite spherical, so this estimate probably is reasonably close.

Problem 9.100

Open-Ended Problem Statement: The drag coefficient for a sphere at sonic speed is approximately 1.3. Is it possible for a steel sphere to reach $M = 1$ in free fall at sea level? Is this a practical possibility? Reconsider for free fall at high altitude.

Analysis: At terminal speed aerodynamic drag is equal to weight of the sphere. Thus



$$F_D = C_D A \tfrac{1}{2}\rho V^2 = mg = \rho \forall g = SG\,\rho_{H_2O}\frac{\pi D^3}{6} g$$

Substituting $A = \frac{\pi D^2}{4}$, $C_D \frac{\pi D^2}{8}\rho V^2 = SG\,\rho_{H_2O}\frac{\pi D^3}{6} g$

or $$D = \frac{3}{4}\frac{C_D \rho V^2}{SG\,\rho_{H_2O} g}$$



At $M = 1$, $V = C = 340$ m/s at sea level (Example Problem 11.3); $SG = 7.83$ (Table A.1).

$$D = \frac{3}{4} \times 1.3 \times 1.23 \frac{kg}{m^3} \times (340)^2 \frac{m^2}{s^2} \times \frac{m^3}{(7.83)1000\,kg} \times \frac{s^2}{9.81\,m} = 1.80\ m$$ D

The corresponding mass of the steel sphere is

$$m = SG\,\rho_{H_2O}\frac{\pi D^3}{6} = \frac{\pi}{6} \times (7.83)1000 \frac{kg}{m^3} \times (1.80)^3 m^3 = 23{,}900\ kg$$ m

This sphere is immense!

Note $D \sim \rho_{air}$, so at altitude $m \sim D^3 \sim \rho/\rho_{SL}{}^3$. (Sonic speed varies with altitude, but choose the sea level value.) Then from Table A.3,

Altitude (m)	ρ/ρ_{SL} (---)	D (m)	m (kg)
0	1.000	1.80	23,900
10,000	0.3376	0.608	920
20,000	0.07258	0.131	9.14
30,000	0.01503	0.0271	0.0811

At $z = 10{,}000$ m the size and mass still are outrageous!

At $z = 20{,}000$ m the size and mass approximate an olympic shot put.

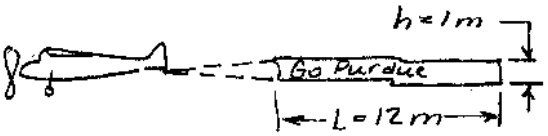
At $z = 30{,}000$ m the size approximates a golf ball but the mass is larger.

Discussion: The immense size of the steel sphere required to achieve $M = 1$ in free fall at sea level makes this feat theoretically possible, but realistically impractical.

At high altitude the required speed can be reached with a more reasonable size sphere (at $z =$ 30,000 m the required sphere would be slightly smaller than a golf ball). However, the practicality still might be addressed by the questions "How would the sphere get to these altitudes?" and "Why would anyone want to do this?"

Given: Light plane towing banner with $h = 1$ m and $L = 12$ m.

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 L h} = 0.05\left(\frac{L}{h}\right)$$



Find: (a) Power required to tow at $V = 90$ km/hr.
(b) Compare with drag of a rigid flat plate.
(c) Why is drag larger for banner?

Solution: Apply the definitions of drag coefficient and power.

Computing equations: $C_D = \dfrac{F_D}{\frac{1}{2}\rho V^2 L h}$ $\qquad P = F_D V$

Assume banner not located in aircraft wake.

$$C_D = 0.05\,\frac{L}{h} = 0.05 \times 12\,m \times \frac{1}{1\,m} = 0.60$$

$$F_D = C_D L h \tfrac{1}{2}\rho V^2 = 0.60 \times 12\,m \times 1\,m \times \frac{1}{2} \times 1.23\,\frac{kg}{m^3}\left[90\,\frac{km}{hr} \times 1000\,\frac{m}{km} \times \frac{hr}{3600\,s}\right]^2 \frac{N \cdot s^2}{kg \cdot m}$$

$$F_D = 2.77\ kN$$

$$P = F_D V = 2.77 \times 10^3\,N \times 90\,\frac{km}{hr} \times 1000\,\frac{m}{km} \times \frac{hr}{3600\,s} = 69.3\ kW$$

P

For flat plate, check Re (assume standard air, $T = 15°C$, ν from Table A.10):

$$Re = \frac{\rho V L}{\mu} = \frac{VL}{\nu} = 90\,\frac{km}{hr} \times 12\,m \times \frac{s}{1.46 \times 10^{-5}\,m^2} \times 1000\,\frac{m}{km} \times \frac{hr}{3600\,s} = 2.05 \times 10^7$$

Therefore flow is completely turbulent. Use Eq. 9.37b:

$$C_D = \frac{0.455}{(\log_{10} Re_L)^{2.58}} - \frac{1610}{Re_L} = 0.00261$$

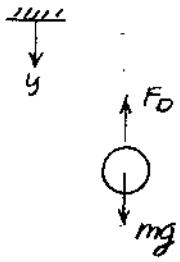
$$F_D = C_D L h \tfrac{1}{2}\rho V^2 = \frac{0.00261}{0.60} \times 2.77\ kN = 12.0\ N\ (\text{per side})$$

$$F_D\,(\text{total}) = 2F_D = 24.0\ N$$

F_D plate

Drag of the banner is increased markedly by pressure forces in folds, waves, and "flutters." The flat plate correlations only account for viscous drag.

Given: Tennis ball, with $D = 64$ mm and $m = 57$ g, dropped in standard sea level air.



Find: (a) Terminal speed of ball, V_t.
(b) Time to reach 95 percent of V_t.
(c) Distance to reach 95 percent of V_t.

Solution: Apply Newton's second law of motion.

Basic equation: $\Sigma F_y = mg - F_D = ma_y = m\frac{dV}{dt} = mV\frac{dV}{dy}$

At terminal speed, $V = V_t$, and $a_y = 0$. Then $mg = F_D = C_D A \frac{1}{2}\rho V_t^2$, so

$$V_t = \sqrt{\frac{2mg}{C_D \rho A}}$$

Assume $C_D = 0.42$ from flat part of curve in Fig. 9.11. Then

$$V_t = \left[2 \times 0.057\,\text{kg} \times 9.81\frac{\text{m}}{\text{s}^2} \times \frac{1}{0.42} \times \frac{\text{m}^3}{1.23\,\text{kg}} \times \frac{4}{\pi (0.064)^2 \text{m}^2}\right]^{1/2} = 25.9 \text{ m/s} \quad \leftarrow V_t$$

Check: $Re = \frac{V_t D}{\nu} = 25.9\frac{\text{m}}{\text{s}} \times 0.064\,\text{m} \times \frac{\text{s}}{1.46 \times 10^{-5}\,\text{m}^2} = 1.14 \times 10^5 \rightarrow C_D = 0.42$, so V_t okay!

To find time, solve $mg - C_D A \frac{1}{2}\rho V^2 = m\frac{dV}{dt} \rightarrow 1 - \left(\frac{V}{V_t}\right)^2 = \frac{V_t}{g}\frac{dV/V_t}{dt} \rightarrow 1 - v^2 = \frac{V_t}{g}\frac{dv}{dt}$

Separating, $dt = \frac{V_t}{g}\frac{dv}{1-v^2}$. Integrating, $t = \frac{V_t}{g}\int_0^{0.95}\frac{dv}{1-v^2} = \frac{V_t}{g}\tanh^{-1}(0.95)$

Thus, since $\tanh^{-1}(0.95) = 1.83$

$$t = 1.83 \times 25.9\frac{\text{m}}{\text{s}} \times \frac{\text{s}^2}{9.81\,\text{m}} = 4.83 \text{ s} \quad (\text{to } V = 0.95 V_t) \quad \leftarrow t$$

To find distance, solve $mg - C_D A \frac{1}{2}\rho V^2 = mV\frac{dV}{dy} \rightarrow 1 - \left(\frac{V}{V_t}\right)^2 = \frac{V_t^2}{g}\frac{dV/V_t}{dy} \rightarrow 1 - v^2 = \frac{V_t^2}{g}\frac{dv}{dy}$

Separating, $dy = \frac{V_t^2}{g}\frac{v\,dv}{1-v^2}$. Integrating, $y = \frac{V_t^2}{g}\int_0^{0.95}\frac{v\,dv}{1-v^2} = -\frac{V_t^2}{2g}\ln(1-v^2)\Big]_0^{0.95}$

Thus

$$y = -\frac{V_t^2}{2g}\ln[1-(0.95)^2] = -\frac{1}{2} \times (25.9)^2 \frac{\text{m}^2}{\text{s}^2} \times \frac{\text{s}^2}{9.81\,\text{m}} \ln[1-(0.95)^2] = 79.6 \text{ m} \quad \leftarrow y$$

Given: Small sphere falling through castor oil at 20 C; D = 6 mm.

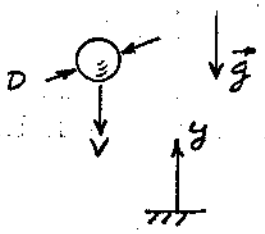
Terminal speed is 60 mm/s.

Find: (a) Compute C_D for given sphere.
(b) Density of sphere.
(c) Compare terminal speed in water.

Solution: Apply Newton's second law of motion, definition of C_D.

Basic equations: $\Sigma F_y = ma_y$

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$$



Assume $Re < 1$, so Stokes flow, $F_D = 3\pi\mu DV$.

From the definition, noting A is the frontal area, $A = \frac{\pi D^2}{4}$,

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} = \frac{3\pi\mu DV}{\frac{1}{2}\rho V^2 \frac{\pi D^2}{4}} = \frac{24\mu}{\rho VD} = \frac{24}{Re}$$

For castor oil at 20 C, $\mu = 0.9\ N\cdot s/m^2$ (Fig. A.2) and $SG = 0.97$ (Table A.2).

$$Re = \frac{SG\,\rho_{H_2O} VD}{\mu} = (0.97)\,999\frac{kg}{m^3} \times 0.06\frac{m}{s} \times 0.006\,m \times \frac{m^2}{0.9\,N\cdot s} \times \frac{N\cdot s^2}{kg\cdot m} = 0.388 < 1 \quad \checkmark\checkmark$$

$$C_D = \frac{24}{Re} = \frac{24}{0.388} = 61.9 \quad \leftarrow C_D$$

From Newton's second law, $F_D + F_{buoyancy} - mg = 0$ or $F_D = mg - F_b$. Thus

$$F_D = C_D \tfrac{1}{2}\rho V^2 A = mg - F_b = (SG_s - SG_0)\rho_{H_2O} g\forall \text{ or } SG_s = SG_0 + \frac{C_D \frac{1}{2}\rho V^2 A}{\rho_{H_2O} g\forall} = SG_0\left(1 + \frac{C_D V^2 A}{2g\forall}\right)$$

But $\frac{A}{\forall} = \frac{\pi r^2}{\frac{4}{3}\pi r^3} = \frac{3}{4r} = \frac{3}{2D}$, so $SG_s = SG_0\left(1 + \frac{3}{4}\frac{C_D V^2}{gD}\right)$

$$SG_s = 0.97\left(1 + \frac{3}{4} \times 61.9 \times (0.06)^2 m^2 \frac{s^2}{9.81\,m} \times \frac{1}{0.006\,m}\right) = 3.72, \text{ so } \rho_s = 3720\frac{kg}{m^3} \quad \leftarrow \rho_s$$

In water, the net weight and drag must balance, or

$$F_D = \tfrac{1}{2}\rho V^2 C_D A = W - F_b = \gamma_{H_2O}(SG_s - 1)\forall \quad \text{or} \quad V = \left[\frac{4(SG-1)gD}{3C_D}\right]^{\frac{1}{2}}$$

However, C_D is a function of Re, so iteration is needed. From Fig. 9.11, $C_D \approx 0.4$ over a range of Re. Using $C_D = 0.4$,

$$V = \left[\frac{4}{3}\left(\frac{2.72}{0.4}\right) 9.81\frac{m}{s^2} \times 0.006\,m\right]^{\frac{1}{2}} = 0.731\ m/s \quad \leftarrow V$$

Check Re:

$$Re = \frac{VD}{\nu} = 0.731\frac{m}{s} \times 0.006\,m \times \frac{s}{1.1\times 10^{-6} m^2} = 3990 \quad \text{ok!}$$

Given: Curve fit for drag coefficient of a sphere versus Re:

$C_D = 24/Re$ $\quad Re \le 1$

$C_D = 24/Re^{0.646}$ $\quad 1 < Re \le 400$

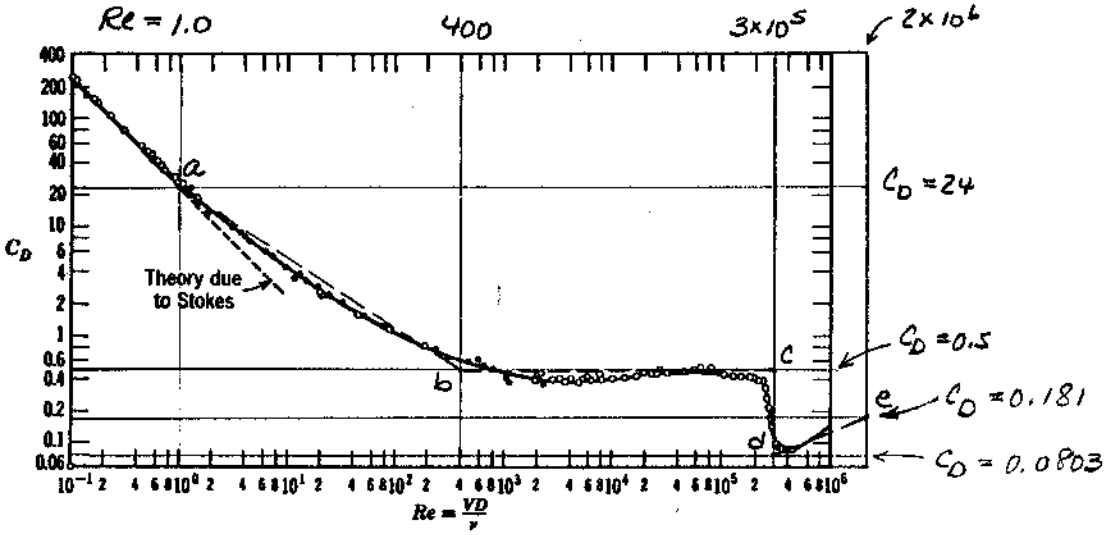
$C_D = 0.5$ $\quad 400 < Re \le 3 \times 10^5$

$C_D = 0.000366\, Re^{0.4275}$ $\quad 3 \times 10^5 < Re \le 2 \times 10^6$

$C_D = 0.18$ $\quad Re > 2 \times 10^6$

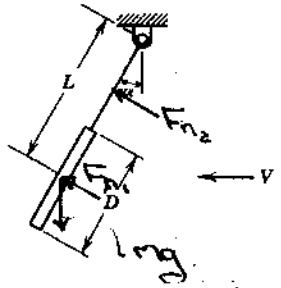
Find: Use data from Fig. 9.11 to evaluate the maximum error between the curve fit and experimental data.

Solution: The curve-fit segments are plotted on Fig. 9.11 below:



The maximum significant error occurs in the region where C_D is modeled as equal to the constant value, $C_D = 0.5$. The curve fit appears to be about 10 percent high in the region from $Re \approx 10^3$ to $Re \simeq 10^4$.

Given: Disk hung in air stream from pivoted strut, of diameter $d = 3$ mm, $L = 40$ mm. For $D = 1$ in. experiments show that $\alpha = 10°$ for $V = 50$ ft/s.



Find: (a) mass of the disk
(b) Plot theoretical curve of $\alpha = \alpha(V)$

Solution: Sum moments about pivot

$$\Sigma M_O = 0 = mgL\sin\alpha - F_{n_1} L - \tfrac{1}{2}(L - \tfrac{D}{2}) F_{n_2} \quad \text{...(1)}$$

$$C_D = \frac{F_n}{\frac{1}{2}\rho V_n^2 A} \quad \text{where } V_n = V\cos\alpha$$

$V = 50 \frac{ft}{sec} = 15.2$ m/s

$D = 1$ in $= 25.4$ mm

Assume: (a) friction in pivot is negligible
(b) C_D valid with $V_n = V\cos\alpha$

$$F_{n_1} = \tfrac{1}{2}\rho V_n^2 A C_{D_1} = \tfrac{1}{2}\rho V^2 \cos^2\alpha \tfrac{\pi}{4}D^2 C_{D_1}$$

$$F_{n_2} = \tfrac{1}{2}\rho V_n^2 A C_{D_2} = \tfrac{1}{2}\rho V^2 \cos^2\alpha (L - \tfrac{D}{2}) d\, C_{D_2}$$

From Table 9.3 $C_{D_1} = 1.17$

For standard air $Re_d = \frac{dV}{\nu} = 3\times10^{-3}\,m \times 15.2\,\frac{m}{s} \times 1.45\times10^{-5}\,\frac{lb}{m^2} = 3140$

From Fig. 9.13 $C_{D_2} = 0.9$

From Eq. (1)

$$m = \frac{1}{g\sin\alpha}\left[\tfrac{1}{2}\rho V^2\cos^2\alpha \tfrac{\pi}{4}D^2 C_{D_1} + \tfrac{1}{2}(1 - \tfrac{D}{2L})\tfrac{1}{2}\rho V^2\cos^2\alpha (L - \tfrac{D}{2}) d\, C_{D_2}\right]$$

$$m = \frac{\rho V^2\cos^2\alpha}{4g\sin\alpha}\left[\tfrac{1}{2}\pi D^2 C_{D_1} + (1 - \tfrac{D}{2L})(L - \tfrac{D}{2}) d\, C_{D_2}\right]$$

$$m = 1.23\,\tfrac{kg}{m^3} \times (15.2)^2\,\tfrac{m^2}{s^2} \times \cos^2 10° \times \tfrac{1}{4\times 9.81\,m/s^2} \times \sin 10° \times \tfrac{1}{10^6\,m^2}$$

$$\left[\tfrac{\pi}{2}\times(25.4)^2\,mm \times 1.17 + (1 - \tfrac{1}{2\times 40}\tfrac{25.4}{1})(40 - \tfrac{25.4}{2})(3)(0.9)\right]$$

$$m = 4.04\times10^{-5}\,\tfrac{kg}{m^2}\left[1190 + 50.3\right] m^2$$

$$m = 0.0501\,kg \quad \text{m}$$

This is equivalent to 3.32×10^{-3} slug (1.77 oz)

Note that the effect of drag on the strut is negligible compared to the drag on the disk.

Neglecting the last term in eq. (1) we can write

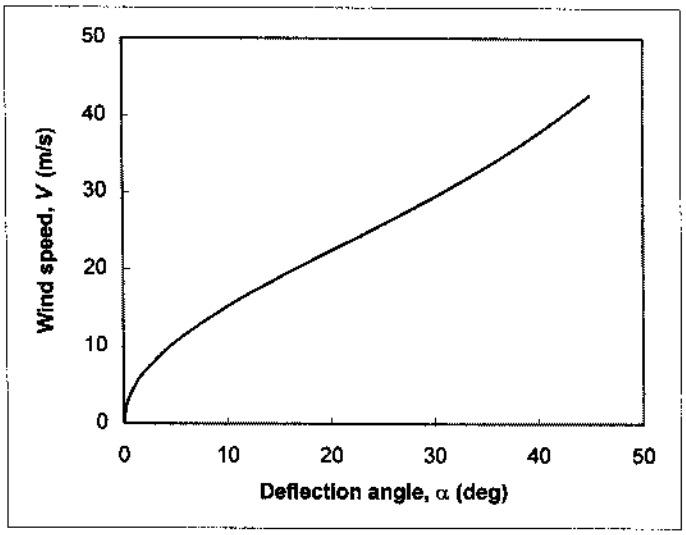
$$mgL\sin\alpha = F_{n_1} L = \tfrac{1}{2}\rho V^2\cos^2\alpha \tfrac{\pi}{4}D^2 C_{D_1} L$$

Then,

$$\frac{\sin\alpha}{\cos^2\alpha} = \frac{\pi}{8}\frac{\rho V^2 d^2 G}{mg} = \frac{\pi}{8} \times 1.23\,\frac{kg}{m^3} \times (0.025\,m)^2 \times \frac{1}{0.048\,kg} \times 1.17 \times \frac{sec^2}{9.81\,m} \times V^2$$

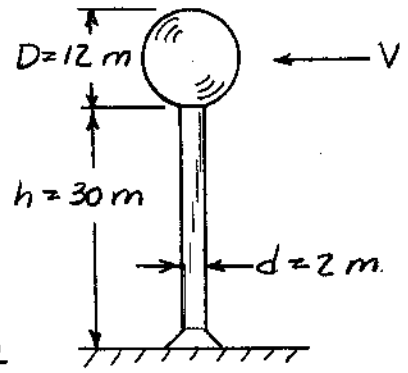
$$\frac{\tan\alpha}{\cos\alpha} = 7.74 \times 10^{-4}\, V^2 \qquad (\text{with } V \text{ in } m/sec)$$

α (deg)	V (m/s)
0	0
0.25	2.37
0.5	3.36
1	4.75
2	6.72
5	10.7
10	15.2
15	18.9
20	22.4
25	25.8
30	29.3
35	33.2
40	37.6
45	42.7



Given: Water tower as shown.

Standard day.



Find: Estimate bending moment at base of tower.

Solution: Apply definition of C_D, sum moments about base.

Computing equations: $C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$; $\overset{\curvearrowleft +}{\Sigma M} = \Sigma FL$

Assumptions: (1) F_{DS} acts at center of sphere; F_{DC} at center of cylinder
(2) Neglect interference between sphere and cylinder.

Then $M = F_{DS}\left(h + \frac{D}{2}\right) + F_{DC}\left(\frac{h}{2}\right)$

$A_S = \frac{\pi D^2}{4} = \frac{\pi}{4}(12)^2 m^2 = 113 m^2$

$A_C = hd = 30 m \times 2 m = 60 m^2$

$V = 100 \frac{km}{hr} \times 1000 \frac{m}{km} \times \frac{hr}{3600 s} = 27.8 m/s$

$q = \frac{1}{2}\rho V^2 = \frac{1}{2} \times 1.23 \frac{kg}{m^3} \times (27.8)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 475 N/m^2$

$C_D = C_D(Re)$. For standard air (Table A.10), $\nu = 1.46 \times 10^{-5} m^2/s$, so

$Re_S = \frac{VD}{\nu} = 27.8 \frac{m}{s} \times 12 m \times \frac{s}{1.46 \times 10^{-5} m^2} = 2.28 \times 10^7$

This Re is too large for Fig. 9.11. Thus guess $C_{DS} = 0.18$ (Problem 9.125).

$F_{DS} = C_{DS} q A_S = 0.18 \times 475 \frac{N}{m^2} \times 113 m^2 = 9.66 kN$

$Re_C = \frac{Vd}{\nu} = 27.8 \frac{m}{s} \times 2.0 m \times \frac{s}{1.46 \times 10^{-5} m^2} = 3.81 \times 10^6$

This Re is too large for Fig. 9.13. Thus guess $C_{DC} \approx 0.4$.

$F_{DC} = C_{DC} q A_C = 0.4 \times 475 \frac{N}{m^2} \times 60 m^2 = 11.4 kN$

The moment is

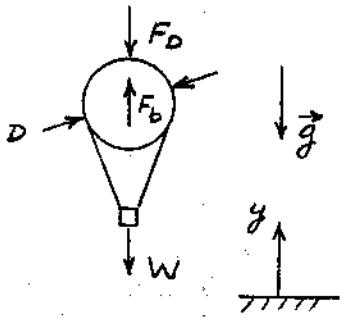
$M = 9.66 kN\left(30 m + \frac{12 m}{2}\right) + 11.4 kN\left(\frac{30 m}{2}\right) = 519 kN \cdot m$ M

Given: Spherical balloon ascending at $V = 3$ m/s through standard air. Balloon and load have mass of 150 kg.

Find: Required diameter.

Solution: Apply Newton's second law of motion, definition of C_D

Basic equations: $\Sigma F_y = ma_y$; $C_D \equiv \frac{F_D}{\frac{1}{2}\rho V^2 A}$



Assumption: Balloon at terminal speed; $a_y = 0$

Summing forces, $F_b - F_D - mg = 0$

$$F_b = \rho g \forall = \rho g \frac{\pi}{6} D^3 \; ; \; F_D = C_D \frac{1}{2}\rho V^2 A = \frac{\pi}{8} C_D \rho V^2 D^2$$

Substituting, $\frac{\pi}{6}\rho g D^3 - \frac{\pi}{8} C_D \rho V^2 D^2 - mg = 0$, or $D^3 - \frac{3}{4}\frac{C_D V^2}{g} D^2 - \frac{6m}{\pi\rho} = 0$

But C_D is a function of Reynolds number. For standard air, $\nu = 1.46 \times 10^{-5}\ m^2/s$ (Table A.10). With $D \simeq 5$ m,

$$Re = \frac{VD}{\nu} \simeq \frac{3\ m}{s} \times 5\ m \times \frac{s}{1.46 \times 10^{-5} m^2} = 1.03 \times 10^6$$

Then from Fig. 9.11, C_D is between 0.1 and 0.2. Estimate $C_D \approx 0.16$. Then

$$\frac{3}{4}\frac{C_D V^2}{g} = \frac{3}{4} \times 0.16 \times (3)^2 \frac{m^2}{s^2} \times \frac{s^2}{9.81\ m} = 0.110\ m$$

$$\frac{6m}{\pi\rho} = \frac{6}{\pi} \times 150\ kg \times \frac{m^3}{1.23\ kg} = 233\ m^3$$

so

$$D^3 - 0.110\, D^2 - 233 = 0 \qquad (D \text{ in meters})$$

Iterating gives:

D (m)	$D^3 - 0.110\, D^2 - 233$ (m^3)
5.0	−111
6.0	−22.0
6.1	−10.1
6.19	−0.04

Thus $D \simeq 6.2$ m

D

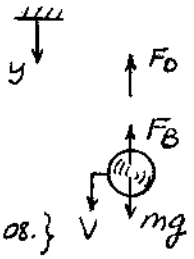
Given: Cast-iron "12-pounder" ($m = 12$ lbm) cannon ball rolls off ship and sinks in ocean where depth is $d = 1000$ m.

Find: Estimate time elapsed before cannon ball hits sea bottom.

Solution: Apply Newton's second law of motion, definition of C_D.

Computing equations:

$$\Sigma F_y = ma_y \qquad C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$$



First find diameter of ball. In air,

$$W = mg = \rho \forall g = SG\rho_{H_2O}\frac{\pi D^3}{6}g = 12 \text{ lbf} \quad \{\text{From Table A.1, } SG = 7.08.\}$$

Thus

$$D = \left[\frac{6W}{\pi SG \rho_{H_2O} g}\right]^{1/3} = \left[\frac{6}{\pi} \times 12 \text{ lbf} \times \frac{1}{7.08} \times \frac{ft^3}{1.94 \text{ slug}} \times \frac{s^2}{32.2 \text{ ft}} \times \frac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot s^2}\right]^{1/3} = 0.373 \text{ ft}$$

or

$$D = 0.373 \text{ ft} \times 0.3048 \frac{m}{ft} = 0.114 \text{ m} \qquad D$$

At terminal speed, $V = V_t$, and $a_y = 0$. Summing forces,

$$mg - F_B - F_D = SG\rho_{H_2O}\forall g - SG_{sw}\rho_{H_2O}\forall g - C_D A \tfrac{1}{2}\rho V_t^2 = 0$$

or

$$V_t = \left[\frac{2(SG_{ci} - SG_{sw})\rho_{H_2O}\forall g}{C_D SG_{sw}\rho_{H_2O} A}\right]^{1/2}$$

Introducing $\forall = \frac{\pi D^3}{6}$ and $A = \frac{\pi D^2}{4}$, then

$$V_t = \left[\frac{4}{3}\frac{(SG_{ci}/SG_{sw} - 1)Dg}{C_D}\right]^{1/2} = \left[\frac{4}{3}(7.08/1.025 - 1)0.114 \text{ m} \times 9.81 \frac{m}{s^2} \times \frac{1}{C_D}\right]^{1/2} = \frac{2.97}{\sqrt{C_D}} \text{ m/s}$$

Choose $C_D = 0.47$ from flat range of curve:

$$V_t = \frac{2.97}{\sqrt{0.47}} \text{ m/sec} = 4.33 \text{ m/s} \qquad \{\text{At } T = 20°C, \nu_{sw} = 1.05\nu_{H_2O} \text{ (Table A.2).}\}$$

Then $Re = \frac{\rho V_t D}{\mu} = \frac{V_t D}{\nu_{sw}} = 4.33 \frac{m}{s} \times 0.114 \text{ m} \times \frac{s}{1.05 \times 10^{-6} m^2} = 4.70 \times 10^5$

This is a supercritical Re, so choose $C_D \approx 0.09$ (Fig. 9.11). Then

$$V_t = \frac{2.97}{\sqrt{0.09}} \text{ m/s} = 9.90 \text{ m/s} \qquad V_t$$

Then $Re = 9.90 \frac{m}{s} \times 0.114 \text{ m} \times \frac{s}{1.05 \times 10^{-6} m^2} = 1.07 \times 10^6$

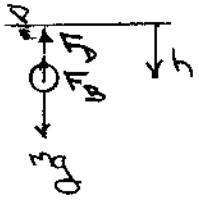
From Fig. 9.11, $C_D \approx 0.14$. Therefore $V_t = \frac{2.97}{\sqrt{0.14}} = 7.94$ m/s

$$t = \frac{d}{V_t} = 1000 \text{ m} \times \frac{s}{7.94 \text{ m}} = 126 \text{ s} \qquad t$$

Given: American golf ball (mass = 1.62 oz, diameter = 1.68 in.) dropped from rest in fresh water.

Estimate: (a) terminal speed of ball
(b) distance required to reach $0.95V_t$.

Solution:



$$\sum F_y = mg - F_B - F_D = m\frac{dV}{dt} = mV\frac{dV}{dh} \quad \text{.....(1)}$$

$$F_D = C_D \tfrac{1}{2}\rho V^2 A$$

$$mg = 1.62\,oz \times \frac{1\,lbm}{16\,oz} \times \frac{slug}{32.2\,lbm} \times 32.2\frac{ft}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} = 0.1013\,lbf$$

$$F_B = \rho V g = 1.94\frac{slug}{ft^3} \times \frac{\pi}{6}\left(\frac{1.68}{12}\right)^3 ft^3 \times 32.2\frac{ft}{s^2} \times \frac{lbf \cdot s^2}{ft \cdot slug} = 0.0898\,lbf$$

$$F_{net} = mg - F_B = 0.0115\,lbf$$

At terminal speed, $F_{net} - F_D = 0$ $\therefore F_{net} = C_D \tfrac{1}{2}\rho V^2 A$

$$V_t = \left[\frac{2 \times F_{net}}{C_D \rho A}\right]^{1/2} \quad \text{where } A = \frac{\pi}{4}D^2 = \frac{\pi}{4}\left(\frac{1.68}{12}ft\right)^2 = 1.539 \times 10^{-2}\,ft^2$$

Assume $C_D = 0.5$ (Fig. 9.11)

$$V_t = \left[2 \times 0.0115\,lbf \times \frac{1}{0.5} \times 1.94\frac{slug}{ft^3} \times \frac{1}{1.539\,ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right]^{1/2} = 1.24\,ft/s \quad V_t$$

Check Re, assuming $T = 60°F$ ($\nu = 1.21 \times 10^{-5}\,ft^2/s$, Table A.7)

$$Re = \frac{DV}{\nu} = \left(\frac{1.68}{12}\right)ft \times 1.24\frac{ft}{s} \times 1.21 \times 10^{-5}\frac{s}{ft^2} = 1.43 \times 10^4 \quad \text{OK by } C_D = 0.5$$

From Eq (1) $mg - F_B - F_D = F_{net} - F_D = mV\frac{dV}{dh}$

$$F_{net} - C_D \tfrac{1}{2}\rho A V^2 = mV\frac{dV}{dh} \quad \text{or} \quad \int_0^h dh = \int_0^{0.95V_t} \frac{mV\,dV}{F_{net} - C_D \frac{1}{2}\rho A V^2}$$

$$h = -\frac{m}{C_D \frac{1}{2}\rho A}\ln\left[F_{net} - C_D \tfrac{1}{2}\rho A V^2\right]_0^{0.95V_t} = \frac{2m}{C_D \rho A}\ln\frac{F_{net}}{F_{net} - C_D \frac{1}{2}\rho A (0.95V_t)^2}$$

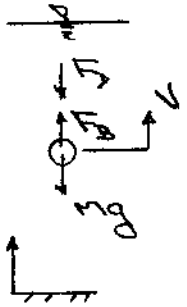
$$h = \frac{2m}{C_D \rho A}\ln\frac{1}{1 - \frac{C_D \rho A (0.95V_t)^2}{2F_{net}}} = \frac{2m}{C_D \rho A}\ln\frac{1}{1-(0.95)^2}$$

$$h = 2 \times \left(\frac{1.62}{16}\right)lbm \times \frac{slug}{32.2\,lbm} \times \frac{1}{0.5} \times 1.94\frac{slug}{ft^3} \times \frac{1}{1.539\,ft^2} \times 2.32$$

$$h = 0.982\,ft \quad \underline{\qquad\qquad} \quad h$$

Given: Ping-Pong ball (mass = 2.69×10^{-3} kg, diameter 38.1 mm) is released in fresh water

Estimate: (a) terminal speed of ball
(b) distance required to reach $0.95V_t$



Solution:

$$\Sigma F_y = F_b - mg - F_D = ma_y = m\frac{dV}{dt} = mV\frac{dV}{dy} \quad \text{---(1)}$$

$$A = \frac{\pi D^2}{4} = \frac{\pi}{4}\times(0.0381\,m)^2 = 1.14\times10^{-3}\,m^2$$

$$\forall = \frac{\pi D^3}{6} = \frac{\pi}{6}\times(0.0381\,m)^3 = 2.896\times10^{-5}\,m^3$$

$$F_b = \rho g\forall = 999\,\frac{kg}{m^3}\times 2.896\times10^{-5}\,m^3\times 9.81\,\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} = 0.284\,N$$

$$mg = 2.69\times10^{-3}\,kg\times 9.81\,\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} = 0.0264\,N$$

$$F_D = C_D\tfrac{1}{2}\rho V^2 A \quad ; \quad F_{net} = F_b - mg = 0.258\,N$$

At terminal speed, $a_y = 0$ and $F_b - F_{mg} = F_{net} = F_D = C_D\tfrac{1}{2}\rho V_t^2 A$

$$V_t = \left[\frac{2F_{net}}{C_D\rho A}\right]^{1/2} \qquad \text{Assume } C_D = 0.5 \quad (\text{Fig. } 9.11)$$

$$V_t = \left[2\times0.258N\times\frac{1}{0.5}\times 999\frac{kg}{m^3}\times 1.14\times10^{-3}\,m^2\times\frac{N\cdot s^2}{kg\cdot m}\right]^{1/2} = 0.952\,m/s \quad \leftarrow V_t$$

Check Re, assuming $T = 15^\circ C$ ($\nu = 1.14\times10^{-6}\,m^2/s$, Table A.8)

$$Re = \frac{DV_t}{\nu} = 0.0381\,m\times 0.952\,\frac{m}{s}\times 1.14\times10^{6}\,\frac{s}{m^2} = 3.18\times10^4$$

okay $C_D \approx 0.5$

From Eq. 1

$$F_b - mg - F_D = F_{net} - F_D = F_{net} - C_D\tfrac{1}{2}\rho V^2 A = mV\frac{dV}{dy}$$

$$\int_0^{0.95V_t}\frac{mV\,dV}{F_{net} - C_D\frac{1}{2}\rho V^2 A} = \int_{y_i}^{y_f} dy \qquad \text{and} \qquad \Delta y = \frac{-m}{C_D\frac{1}{2}\rho A}\ln\left(F_{net} - C_D\tfrac{1}{2}\rho V^2 A\right)\Big|_0^{0.95V_t}$$

$$\Delta y = \frac{2m}{C_D\rho A}\ln\frac{F_{net}}{F_{net} - C_D\frac{1}{2}\rho(0.95V_t)^2} = \frac{2m}{C_D\rho A}\ln\frac{1}{1 - \frac{C_D A\rho(0.95)^2 V_t^2}{F_{net}}}$$

$$\Delta y = \frac{2m}{C_D\rho A}\ln\frac{1}{1-(0.95)^2}$$

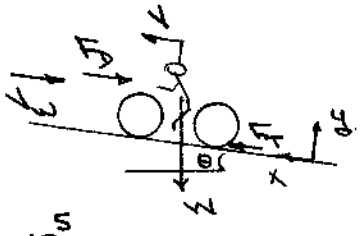
$$\Delta y = 2\times2.69\times10^{-3}\,kg\times\frac{1}{0.5}\times 999\frac{kg}{m^3}\times\frac{1}{1.14\times10^{-3}\,m^2}\times 2.33$$

$$\Delta y = 0.0220\,m \quad \leftarrow \Delta y$$

Given: Rider, $m_r = 70$ kg, to ride bike of mass $m_b = 10$ kg up a 6° incline at steady speed of 4.1 m/s into head wind of 0.9 m/s. Frontal area = $0.3\ m^2$. Neglect rolling resistance.

Estimate: (a) drag force on rider and bike
(b) power required to climb the incline
(c) speed on level ground with no wind

Solution:



$$F_D = C_D \tfrac{1}{2} \rho V^2 A$$

Model rider as a cylinder ($D = 0.4$ m)

$$Re = \frac{DV}{\nu} = 0.4\,m \times \frac{5.0\,m}{s} \times 1.46\times10^{-5}\,\frac{s}{m^2} = 1.37\times10^{5}$$

From Fig. 9.13, $C_D = 1.2$

$$F_D = 1.2 \times \tfrac{1}{2} \times 1.23\,\frac{kg}{m^3} \times (4.1+0.9)^2\,\frac{m^2}{s^2} \times 0.3\,m^2 \times \frac{N\cdot s^2}{kg\cdot m} = 5.54\ N \quad \underline{\text{Ans } F_D}$$

Consider a free body diagram of bike and rider moving at constant speed.

$$\Sigma F_x = F_T - F_D - W\sin\theta = 0, \quad \text{so} \quad F_T = F_D + W\sin\theta$$

Power, $\mathcal{P} = F_T V = (F_D + W\sin\theta)V = (F_D + (m_r + m_b)g\sin\theta)V$

$$\mathcal{P} = \left[5.54\,N + (70+10)\,kg \times 9.81\,\frac{m}{s^2} \times \sin 6^\circ \times \frac{N\cdot s^2}{kg\cdot m}\right] 4.1\,\frac{m}{s}$$

$$\mathcal{P} = 359\ W \quad (0.481\ hp) \quad \underline{\text{Ans } \mathcal{P}}$$

If prof develops same power on level ground through still air

$$\mathcal{P} = F_D V = C_D \tfrac{1}{2} \rho V^2 A\, V = C_D \frac{\rho A}{2} V^3$$

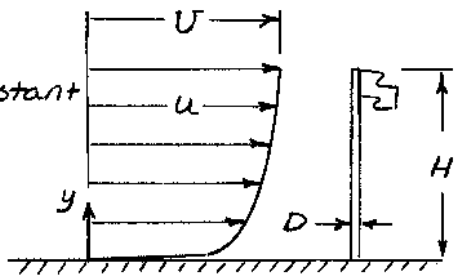
$$V = \left[\frac{2\mathcal{P}}{C_D \rho A}\right]^{1/3}$$

Assume same C_D and frontal area

$$V = \left[\frac{2}{1.2} \times 359\,\frac{N\cdot m}{s} \times 1.23\,\frac{kg}{m^3} \times \frac{1}{0.3\,m^2} \times \frac{kg\cdot m}{N\cdot s^2}\right]^{1/3} = 11.7\ m/s \quad \underline{\text{Ans } V}$$

Given: Cylindrical flag pole of height, H.

Wind-speed profile, $\frac{u}{U} = \left(\frac{y}{H}\right)^{1/7}$; C_D constant



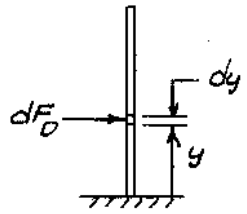
Find: (a) Drag force
(b) Bending moment
(c) Compare with values for uniform profile, U.

Solution: Apply definition of drag coefficient, $C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$

Assume: (1) C_D = constant
(2) C_D same as circular cylinder

On an element of the pole,

$$dF_D = C_D \tfrac{1}{2}\rho u^2 dA = C_D \tfrac{1}{2}\rho U^2 \left(\frac{y}{H}\right)^{2/7} D\,dy$$



Thus

$$F_D = \int_0^H dF_D = \int_0^H C_D \tfrac{1}{2}\rho U^2 \left(\frac{y}{H}\right)^{2/7} D\,dy$$

$$F_D = C_D \tfrac{1}{2}\rho U^2 DH \int_0^1 \left(\frac{y}{H}\right)^{2/7} d\left(\frac{y}{H}\right) = C_D \tfrac{1}{2}\rho U^2 DH \left[\frac{7}{9}\left(\frac{y}{H}\right)^{9/7}\right]_0^1 = \frac{7}{9} C_D \tfrac{1}{2}\rho U^2 DH \quad F_D$$

On an element of the pole,

$$dM = y\,dF_D = y\,C_D \tfrac{1}{2}\rho u^2 dA = y\,C_D \tfrac{1}{2}\rho u^2 D\,dy$$

Thus

$$M = \int_0^H dM = \int_0^H y\,C_D \tfrac{1}{2}\rho U^2 \left(\frac{y}{H}\right)^{2/7} D\,dy = C_D \tfrac{1}{2}\rho U^2 DH^2 \int_0^1 \left(\frac{y}{H}\right)\left(\frac{y}{H}\right)^{2/7} d\left(\frac{y}{H}\right)$$

$$M = C_D \tfrac{1}{2}\rho U^2 DH^2 \int_0^1 \left(\frac{y}{H}\right)^{9/7} d\left(\frac{y}{H}\right) = C_D \tfrac{1}{2}\rho U^2 DH^2 \left[\frac{7}{16}\left(\frac{y}{H}\right)^{16/7}\right]_0^1 = \frac{7}{16} C_D \tfrac{1}{2}\rho U^2 DH^2 \quad M$$

Comparing,

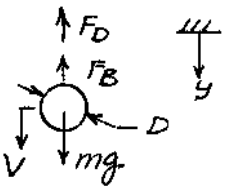
$$\frac{F_D(1/7\text{-profile})}{F_D(\text{uniform})} = \frac{\frac{7}{9} C_D \frac{1}{2}\rho U^2 DH}{C_D \frac{1}{2}\rho U^2 DH} = \frac{7}{9}$$

$$\frac{M(1/7\text{-profile})}{M(\text{uniform})} = \frac{\frac{7}{16} C_D \frac{1}{2}\rho U^2 DH^2}{C_D \frac{1}{2}\rho U^2 DH \frac{H}{2}} = \frac{7/16}{1/2} = \frac{7}{8} \quad \frac{M}{M_U}$$

Given: Stokes drag law for smooth spheres, $F_D = 3\pi\mu VD$, to be verified experimentally by dropping steel balls in glycerin.

Find: (a) Largest steel ball for which $Re < 1$.
(b) Height of glycerine column needed to reach 95 percent of terminal speed.

Solution: Draw free-body diagram of ball, apply Newton's second law.



Basic equation: $\Sigma F_y = mg - F_B - F_D = m\frac{dV}{dt} = mV\frac{dV}{dy}$ (1)

$$\forall = \frac{\pi D^3}{6} \qquad m = \rho_s \forall \qquad F_B = \rho_g \forall g \qquad F_D = 3\pi\mu VD$$

At terminal speed, V_t, acceleration is zero. Thus

$$\rho_s \forall g - \rho_g \forall g - 3\pi\mu V_t D = 0 \quad \text{or} \quad V_t = \frac{(\rho_s - \rho_g)\frac{\pi D^3}{6} g}{3\pi\mu D} = \frac{(\rho_s - \rho_g) D^2 g}{18\mu}$$

$$\text{or} \quad V_t = \frac{(\rho_s/\rho_g - 1)\rho_g D^2 g}{18\mu} = \frac{(SG_s/SG_g - 1) D^2 g}{18\nu} = \frac{(7.8/1.26 - 1) D^2 g}{18\nu} = 0.288\, D^2 g/\nu$$

(from Table A.2, $SG_g = 1.26$). Stokes' drag law holds for $Re < 1$. Thus

$$Re = \frac{\rho V_t D}{\mu} = \frac{V_t D}{\nu} = \frac{0.288 D^3 g}{\nu^2} < 1 \quad \text{or} \quad D^3 < \frac{1}{0.288}\frac{\nu^2}{g} \quad \text{or} \quad D < \left[\frac{3.47\nu^2}{g}\right]^{1/3}$$

Assuming $T = 20^\circ C$, then from Fig. A.3, $\nu = 0.0012\ m^2/s$, so

$$D < \left[3.47 \times (0.0012)^2 m^4 \times \frac{s^2}{9.81\ m}\right]^{1/3} = 0.00799\ m \quad (7.99\ mm)$$ D

From Eq. 1,

$$\rho_s \forall g - \rho_g \forall g - 3\pi\mu VD = \rho_s \forall V\frac{dV}{dy}$$

Dividing by $(\rho_s - \rho_g)\forall g$ gives

$$1 - \frac{3\pi\mu D}{(\rho_s - \rho_g)\frac{\pi D^3}{6} g} V = 1 - \frac{V}{V_t} = \frac{\rho_s \forall}{(\rho_s - \rho_g)\forall g} V\frac{dV}{dy} = \left(\frac{\rho_s}{\rho_s - \rho_g}\right)\frac{V}{g}\frac{dV}{dy} = \left(\frac{\rho_s}{\rho_s - \rho_g}\right)\frac{V_t^2}{g}\frac{V}{V_t}\frac{d(V/V_t)}{dy}$$

Separating variables, $dy = \left(\frac{\rho_s}{\rho_s - \rho_g}\right)\frac{V_t^2}{g}\frac{(V/V_t)\,d(V/V_t)}{1 - V/V_t} = \left(\frac{\rho_s}{\rho_s - \rho_g}\right)\frac{V_t^2}{g}\frac{r\,dr}{1-r}$

Integrating, $\int_0^{0.95}\frac{r\,dr}{1-r} = \int_1^{0.05}\frac{(1-x)(-dx)}{x} = \int_1^{0.05} dx - \int_1^{0.05}\frac{dx}{x} = x - \ln x\Big]_1^{0.05} = -0.95 - \ln(0.05)$

Thus

$$y = 2.05\left(\frac{\rho_s}{\rho_s - \rho_g}\right)\frac{V_t^2}{g} = 2.05\left(\frac{SG_s}{SG_s - SG_g}\right)\frac{V_t^2}{g}$$

But $V_t = 0.288\frac{D^2 g}{\nu} = 0.288 \times (0.0172)^2 m^2 \times 9.81\frac{m}{s^2} \times \frac{s}{0.0012\ m^2} = 0.697\ m/s$

so

$$y = 2.05 \times \frac{7.8}{(7.8 - 1.26)} \times (0.697)^2 \frac{m^2}{s^2} \times \frac{s^2}{9.81\ m} = 0.121\ m \quad (121\ mm)$$ y

Given: Hydrogen bubbles immersed in water, $0.01 < D < 3$ mm.
Correlation for drag coefficient of smooth spheres

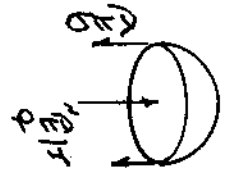
$C_D = 24/Re$ for $Re \le 1$; $C_D = 24/Re^{0.646}$ for $1 \le Re \le 400$

$C_D = 0.5$ for $400 \le Re \le 3\times10^5$

Consider: effect of surface tension on pressure in bubble.

Plot: Terminal speed of bubbles in water vs diameter.

Solution:



Consider a free-body diagram of cut bubble

$\Sigma F_z = 0 = p\frac{\pi D^2}{4} - \sigma\pi D \quad \therefore \quad p = \frac{4\sigma}{D}$

Assume: (1) surface tension as in water-air interface, $\sigma = 73$ mN/m (Table A.4)

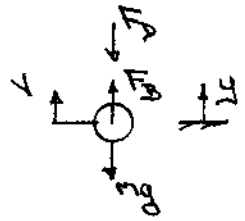
(2) ideal gas so $R = 4124$ J/(kg·K) (Table A.6), $T = 20°C$

Then $p = \frac{4\sigma}{D} = 4\times 0.073 \frac{N}{m}\times 0.01\times10^{-3} m = 29.2$ kPa

and $\rho = \frac{p}{RT} = (101+29.2)10^3 \frac{N}{m^2}\times\frac{kg\cdot K}{4124 N\cdot m}\times\frac{1}{(273+20)K} = 0.108 \frac{kg}{m^3}$

Thus $\rho_{H_2} << \rho_{H_2O}$, so it may be ignored.

Apply Newton's 2nd law to bubble



$\Sigma F_y = F_B - F_D - \cancel{mg}^{\approx 0} = ma_y$

At terminal speed, $a_y = 0$, so $F_B = F_D$

$F_B = \rho_{H_2O} g \forall$ where $\forall = \frac{\pi}{6}D^3 \quad \therefore F_B = \rho_{H_2O} g \frac{\pi}{6} D^3$

$F_D = C_D \frac{1}{2}\rho A V_t^2 \quad \therefore \rho g \frac{\pi}{6}D^3 = C_D \rho \frac{\pi}{8} D^2 V_t^2$ and $V_t^2 = \frac{4}{3}\frac{gD}{C_D}$ ... (1)

Assume water at $T = 20°C$, $\nu = 1\times10^{-6}$ m²/s

$Re \le 1$

$C_D = \frac{24}{Re} = \frac{24\nu}{V_t D} \quad \therefore V_t^2 = \frac{4}{3}\frac{gD}{24\nu}\times D V_t \quad$ or $\quad V_t = \frac{gD^2}{18\nu}$ ... (2)

$V_t = \frac{9.81}{18}\frac{m}{s^2}\times 1\times10^6 \frac{s}{m^2}\times[D(m)]^2\times\frac{10^6 mm^2}{m^2}$

$V_t = 0.545 D^2 \quad \{V \text{ in m/s}, D \text{ in mm}\}$

$Re \le 1$ means $VD < 1\times10^{-6}$ m²/s.

Thus Eq. 2 is valid for $D \le 0.122$ mm

$400 \le Re \le 3\times10^5$

$$V_t^2 = \frac{4}{3}\frac{D g}{C_D} = \frac{4}{3}\times 9.81\ \frac{m}{s^2}\times\frac{1}{0.5}\times D = 26.16\,D \quad \text{or} \quad V_t = 5.115\,D^{1/2} \quad \text{---(3)}$$

$Re = 400$ means $VD = 400\nu = 4\times10^{-4}\ m^2/s$.

then $D = 1.83$ mm

$Re = 3\times10^5$ means $VD = 0.30\ m^2/s$

then $D = 151$ mm.

$1 \le Re \le 400$ $\qquad C_D = 24/Re^{0.646}$, $\qquad 0.122 \le D \le 1.83$ mm

$$V_t = \left[\frac{4}{3}\frac{Dg}{C_D}\right]^{1/2} = \left[\frac{4}{3}\frac{Dg}{24}\left\{\frac{D}{\nu}\right\}^{0.646}\right]^{1/2}$$

$$V_t = \left(\frac{2g}{18}\right)^{1/2}\frac{D^{0.823}}{\nu^{0.323}} \quad \text{or} \quad V_t^{0.677} = \left[\frac{2g}{18}\right]^{1/2}\frac{1}{\nu^{0.323}}D^{0.823}$$

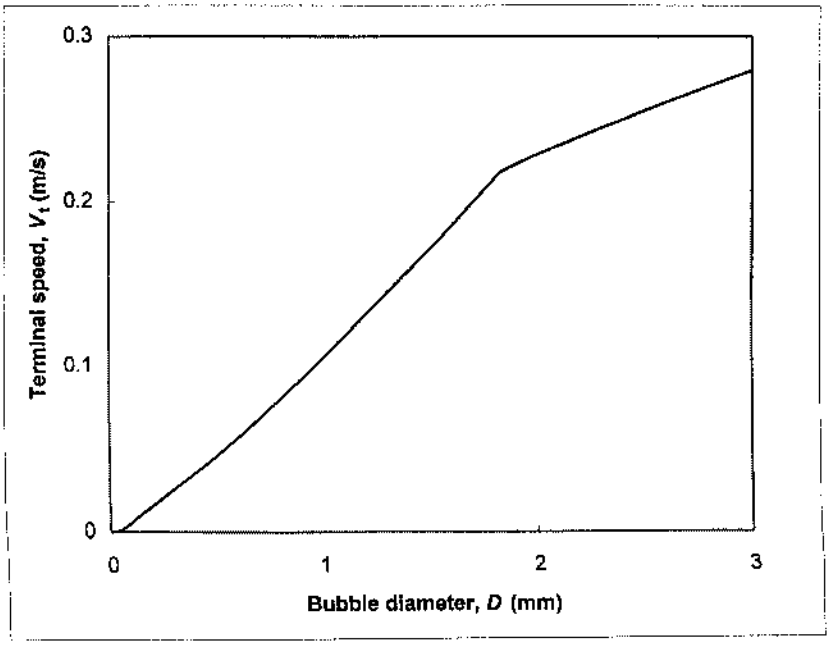
$$V_t = \left[\left(\frac{2g}{18}\right)^{1/2}\right]^{1.477}\times\frac{D^{1.216}}{\nu^{0.477}} = \left(\frac{2g}{18}\right)^{0.739}\times\frac{D^{1.216}}{\nu^{0.477}}$$

$$V_t = 465\,D^{1.216}$$

Summarizing

$Re < 1$	$0 \le D \le 0.122$ mm	$V_t = 0.545\,D^2$	{V in m/s; D in mm}
$1 \le Re \le 400$	$0.122 < D < 1.83$ mm	$V_t = 465\,D^{1.216}$	{V in m/s; D in m}
$400 \le Re \le 3\times10^5$	$1.83 < D < 151$ mm	$V_t = 5.115\,D^{1/2}$	{V in m/s; D in m}

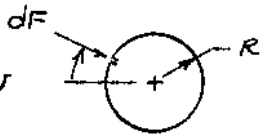
D (mm)	V_t (m/s)
0	0
0.01	5.45E-5
0.05	1.36E-3
0.10	5.45E-3
0.122	8.11E-3
0.123	8.18E-3
0.50	4.50E-2
1.00	1.05E-1
1.50	1.71E-1
1.829	2.18E-1
1.831	2.18E-1
2.00	2.29E-1
2.50	2.56E-1
3.00	2.80E-1



Given: Measured data for pressure difference versus angle for flow around a circular cylinder at $Re = 80{,}000$.

Find: (a) Estimate C_D for this flow.
(b) Compare with data from Fig. 9.13; explain any difference.

Solution: Consider the geometry sketched. Apply the definition of drag coefficient.



Computing equation: $C_D = \dfrac{F_D}{\frac{1}{2}\rho U^2 A}$

Assumption: Neglect viscous force; $dF_D = dF\cos\theta = -p\,dA\cos\theta = p_w R\,d\theta\cos\theta$

Then $F_D = \int_A dF_D = \int_0^{2\pi} p_w R\,d\theta\cos\theta = 2\int_0^{\pi} p_w R\,d\theta\cos\theta = \int_0^{\pi} p\cos\theta\,(w\,2R)\,d\theta$

Since $\int_0^{\pi} p_\infty \cos\theta\,d\theta = 0$, then $F_D = \int_0^{\pi}(p - p_\infty)\cos\theta\,(w\,2R)\,d\theta$

The stagnation pressure is $p(0) - p_\infty = \frac{1}{2}\rho U^2$, so

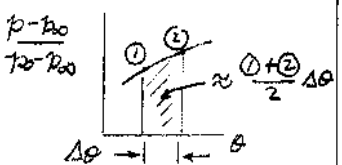
$$C_D = \frac{F_D}{\frac{1}{2}\rho U^2 A} = \frac{\int_0^{\pi}(p-p_\infty)\cos\theta\,(w\,2R)\,d\theta}{(p_0 - p_\infty)(w\,2R)} = \int_0^{\pi}\left(\frac{p-p_\infty}{p_0-p_\infty}\right)\cos\theta\,d\theta$$

Tabulating:

θ (deg)	$p - p_\infty$ (in. H_2O)	$\frac{p-p_\infty}{p_0-p_\infty}$	$\left(\frac{p-p_\infty}{p_0-p_\infty}\right)\cos\theta$
0	0.52	1.00	1.00
10	0.46	0.885	0.871
20	0.33	0.635	0.596
30	0.13	0.250	0.217
40	-0.13	-0.250	-0.192
50	-0.40	-0.769	-0.494
60	-0.57	-1.10	-0.548
70	-0.60	-1.15	-0.395
80	-0.55	-1.06	-0.184
90	-0.52	-1.00	0.00
100	-0.51	-0.981	0.170
110	-0.52	-1.00	0.342
120	-0.52	-1.00	0.500
130	-0.52	-1.00	0.643
140	-0.52	-1.00	0.766
150	-0.52	-1.00	0.866
160	-0.54	-1.04	0.976
170	-0.54	-1.04	1.02
180	-0.54	-1.04	1.04

$\Sigma = 7.194$

Trapezoidal rule:



Half of the end points are subtracted to avoid double counting.

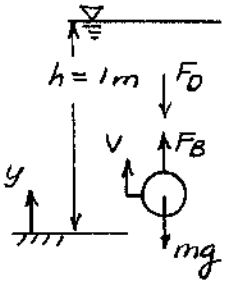
$$C_D \approx \left\{\Sigma\left(\frac{p-p_\infty}{p_0-p_\infty}\right)\cos\theta - \frac{1}{2}()_0 - \frac{1}{2}()_{180}\right\}\Delta\theta = \left\{7.194 - \frac{1}{2}(1.00 + 1.04)\right\} 10\ \text{deg} \times \frac{\pi\ \text{rad}}{180\ \text{deg}} = 1.08 \quad C_D$$

From Fig. 9.13, $C_D \approx 1.2$. The difference is due to skin friction effects.

Given: Thin-walled plastic sphere with $D = 10$ mm and $m = 0.05$ g, immersed in glycerin bath at depth, $h = 1$ m, is released and rises to the surface.

Find: Time needed to reach the surface.

Solution: Apply Newton's second law of motion, definition of C_D.



Computing equations:

$$\Sigma F_y = ma_y \qquad C_D = \frac{24}{Re} \quad \text{for } Re < 1$$

Assume: $V = V_t$, so $a_y = 0$; terminal speed reached quickly

Summing forces, $F_B - mg - F_D = 0$

But $F_B = \rho \forall g = \rho \frac{\pi D^3}{6} g$ and $F_D = C_D \frac{1}{2}\rho V_t^2 A = \frac{24}{\frac{V_t D}{\nu}} \frac{1}{2}\rho V_t^2 \frac{\pi D^2}{4} = 3\pi \mu V_t D$

Substituting,

$$\rho \frac{\pi D^3}{6} g - mg - 3\pi \mu V_t D = 0$$

So

$$V_t = \frac{g}{3\pi \mu D}\left(\rho \frac{\pi D^3}{6} - m\right) \qquad \left\{\begin{array}{l} SG = 1.26 \text{ for glycerin (Table A.2)} \\ \mu = 1.4 \text{ kg/m·s (Fig. A.2)} \end{array}\right\}$$

$$= \frac{1}{3\pi} \times 9.81 \frac{m}{sec^2} \times \frac{m \cdot s}{1.4\ kg} \times \frac{1}{0.01\ m}\left[(1.26)\,999 \frac{kg}{m^3} \times \frac{\pi}{6}(0.01)^3 m^3 - 0.05 \times 10^{-3} kg\right]$$

$$V_t = 0.0453 \text{ m/s}$$

Then

$$t = \frac{h}{V_t} = 1 m \times \frac{s}{0.0453\ m} = 22.1 \text{ s}$$

t

Verify Stokes flow assumption:

$$Re = \frac{\rho V_t D}{\mu} = (1.26)\,999 \frac{kg}{m^3} \times 0.0453 \frac{m}{s} \times 0.01\ m \times \frac{m \cdot s}{1.4\ kg} = 0.407$$

Since $Re < 1$, Stokes flow is a reasonable assumption.

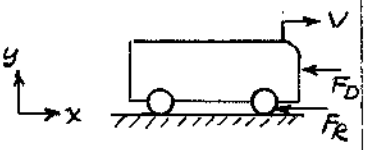
Given: Coastdown test data from level road, calm day, measured for vehicle with $W = 25{,}000$ lbf and $A = 79$ ft^2. $F_D(5\text{ mph}) \ll F_D(55\text{ mph})$.

V (mph)	5	55
$\frac{dV}{dt}$ (mph/s)	−0.150	−0.475

Find: Aerodynamic drag coefficient for this vehicle.

Solution: Apply Newton's second law of motion, definition of C_D.

Computing equations: $\Sigma F_x = ma_x$ $\quad C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$



Summing forces, $-F_R - F_D = ma_x$

At 55 mph $\quad -F_R - F_{D55} = ma_{x55}$

At 5 mph $\quad -F_R - \overset{\approx 0}{F_{D5}} = ma_{x5}$

Subtracting, obtain $-F_{D55} = m(a_{x55} - a_{x5}) = m\left[\left(\frac{dV}{dt}\right)_{55} - \left(\frac{dV}{dt}\right)_5\right] = -C_D A \rho \frac{V^2}{2}$

Thus $$C_D = \frac{m\left[\left(\frac{dV}{dt}\right)_5 - \left(\frac{dV}{dt}\right)_{55}\right]}{\frac{1}{2}\rho V^2 A} = \frac{2W\left[\left(\frac{dV}{dt}\right)_5 - \left(\frac{dV}{dt}\right)_{55}\right]}{\rho g V^2 A}$$

Evaluating, assuming standard air, with $\rho g = 0.0765$ lbf/ft^3

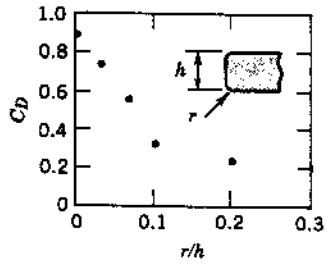
$$C_D = 2 \times 25000\text{ lbf} \times \frac{\text{ft}^3}{0.0765\text{ lbf}} \times \frac{\text{hr}^2}{(55)^2\text{ mi}^2} \times \frac{1}{79\text{ ft}^2} \times \left[-0.150 - (-0.475)\right]\frac{\text{mi}}{\text{hr}\cdot\text{s}} \times \frac{\text{mi}}{5280\text{ ft}} \times 3600\frac{\text{s}}{\text{hr}}$$

$$C_D = 0.606$$

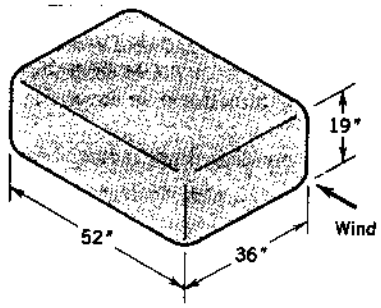
C_D

Given: Rooftop carrier dimensions and drag coefficient data shown:

$r = 4$ in.



Drag coefficient v. radius ratio [34]



Auto travels at

$V = 65$ mph, BSFC $= 0.46$ lbm/hp·hr, $\eta_d = 0.85$, FE $= 30$ mpg

(without the carrier).

Find: (a) Estimate drag force on carrier.
(b) Calculate additional rate of fuel consumption.
(c) Compute the effect on fuel economy.

Solution: Apply definition of drag coefficient.

Computing equation: $F_D = C_D A \frac{1}{2}\rho V^2$ $\qquad V = 65$ mph $(95.3$ ft/s$)$

From the diagram, $\frac{r}{h} = \frac{4}{19} = 0.211$, so $C_D \approx 0.24$; $A = \frac{(19)(36)}{144} = 4.75$ ft²

$$F_D = 0.24 \times 4.75\,\text{ft}^2 \times \frac{1}{2} \times 0.00238 \frac{\text{slug}}{\text{ft}^3} \times (95.3)^2 \frac{\text{ft}^2}{\text{sec}^2} \times \frac{\text{lbf}\cdot\text{sec}^2}{\text{slug}\cdot\text{ft}} = 12.3\ \text{lbf}$$ F_D

Additional power is $\Delta P = \frac{F_D V}{\eta_d} = 12.3\,\text{lbf} \times 95.3 \frac{\text{ft}}{\text{s}} \times \frac{1}{0.85} \times \frac{\text{hp·s}}{550\,\text{ft·lbf}} = 2.51$ hp

Additional fuel consumption is

$$\Delta FC = \text{BSFC}\,\Delta P = 0.46 \frac{\text{lbm}}{\text{hp·hr}} \times 2.51\,\text{hp} = 1.15\ \text{lbm/hr}$$ ΔFC

Fuel consumption for car only is ($SG_{gas} = 0.72$, Table A.2):

$$FC = \frac{V}{FE} = 65 \frac{\text{mi}}{\text{hr}} \times \frac{\text{gal}}{30\,\text{mi}} \times \frac{\text{ft}^3}{7.48\,\text{gal}} \times (0.72)\,62.4 \frac{\text{lbm}}{\text{ft}^3} = 13.0\ \text{lbm/hr}$$

Total fuel consumption (with the carrier) is

$$FC_t = FC + \Delta FC = 13.0 \frac{\text{lbm}}{\text{hr}} + 1.15 \frac{\text{lbm}}{\text{hr}} = 14.2 \frac{\text{lbm}}{\text{hr}}$$

Fuel economy with the carrier is

$$FE = \frac{V}{FC} = 65 \frac{\text{mi}}{\text{hr}} \times \frac{\text{hr}}{14.2\,\text{lbm}} \times (0.72)\,62.4 \frac{\text{lbm}}{\text{ft}^3} \times \frac{\text{ft}^3}{7.48\,\text{gal}} = 27.5\ \text{mpg}$$ FE

Notes: (1) Area was computed conservatively as W×H.
(2) No account was taken of velocity increase over the roof of the car, Fig. 9.25.

Given: Spherical sonar transducer with $D = 0.375$ m, to be towed in seawater, fully submerged, at $V = 31$ kt.

To avoid cavitation, minimum pressure on transducer surface must be > 30 kPa (abs).

Find: (a) Hydrodynamic drag force on transducer.
(b) Minimum depth of submergence.

Solution: $V = 31.1 \frac{nm}{hr} \times 1852 \frac{m}{nm} \times \frac{hr}{3600 s} = 16.0 \ m/s$

$$q = \frac{1}{2}\rho V^2 = \frac{1}{2} \times (1.025)\, 1000 \frac{kg}{m^3} \times (16.0)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} \times \frac{Pa \cdot m^2}{N} = 131 \ kPa$$

$$Re = \frac{VD}{\nu} = 16.0 \frac{m}{s} \times 0.375 m \times \frac{1}{(1.08)1 \times 10^{-6} m^2} = 5.56 \times 10^6$$

Therefore flow over sphere is supercritical; from Fig. 9.11, $C_D \approx 0.18$

$$F_D = C_D A \frac{1}{2}\rho V^2 \qquad A = \frac{\pi D^2}{4} = \frac{\pi}{4} \times (0.375)^2 m^2 = 0.110 \ m^2$$

$$F_D = 0.18 \times 0.110 \ m^2 \times 131 \times 10^3 \frac{N}{m^2} = 2.59 \ kN \longleftarrow F_D$$

From Fig. 9.12, the minimum pressure on a sphere with supercritical flow is $C_p \approx -1.2$

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho V^2} = \frac{p - p_\infty}{q} = -1.2$$



or $p_\infty = p - C_p q$

$= 30 \ kPa(abs) - (-1.2)\, 131 \ kPa$

$p_\infty (abs) = 187 \ kPa(abs)$

Thus

$$p_\infty (gage) = p_\infty (abs) - p_{atm} = (187 - 101) kPa = 86.2 \ kPa$$

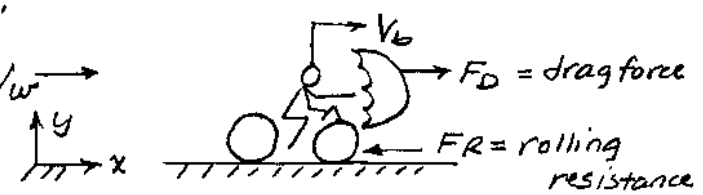
But $p_\infty (gage) = \rho g h$, so

$$h = \frac{p_\infty (gage)}{\rho g} = 86.2 \times 10^3 \frac{N}{m^2} \times \frac{m^3}{(1.025)1000 \ kg} \times \frac{s^2}{9.81 \ m} \times \frac{kg \cdot m}{N \cdot s^2} = 8.57 \ m \longleftarrow h$$

Problem 9.120

Open-Ended Problem Statement: While walking across campus one windy day, Floyd Fluids speculates about using an umbrella as a "sail" to propel a bicycle along the sidewalk. Develop an algebraic expression for the speed a bike could reach on level ground with the umbrella "propulsion system." The frontal area of the bike and rider is estimated as 0.3 m^2. Evaluate the bike speed that could be achieved with an umbrella 1.22 m in diameter in a wind that blows at 24 km/hr. Discuss the practicality of this propulsion system.

Analysis: *Draw a free-body diagram:*



Sum forces in x direction:

$$\Sigma F_x = F_D - F_R = 0$$

But $F_D = (C_{Du} A_u + C_{Db} A_b) \frac{1}{2} \rho (V_w - V_b)^2$

$F_R = C_R mg$

$A_u = \frac{\pi D_u^2}{4} = \frac{\pi}{4}(1.22)^2 m^2 = 1.17 m^2$

Choose $C_{Du} = 1.42$ (Table 9.3), $C_{Db} \approx 1.2$ (for a circular cylinder, Fig. 9.13)

Assume $C_R = 0.75\%$, $m = 75$ kg, so $F_R = 0.0075 \times 75 kg \times 9.81 \frac{m}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 5.52$ N

Then $V_b = V_w - \left[\frac{2F_R}{\rho (C_{Du} A_u + C_{Db} A_b)}\right]^{\frac{1}{2}}$ ← V_b

But

$$V_w = 24 \frac{km}{hr} \times 1000 \frac{m}{km} \times \frac{hr}{3600 s} = 6.67 \text{ m/s}$$

$$V_b = 6.67 \frac{m}{s} - \left[2 \times 5.52 N \times \frac{m^3}{1.23 kg} \frac{1}{(1.42)1.17 m^2 + (1.20)0.3 m^2} \times \frac{kg \cdot m}{N \cdot s^2}\right]^{\frac{1}{2}}$$

$$V_b = 6.67 \frac{m}{s} - 2.11 \frac{m}{s} = 4.56 \frac{m}{s} \text{ or } 16.4 \frac{km}{hr}$$ ← V_b

Thus Floyd's bicycle (with the umbrella propelling it) travels at 68.3% wind speed.

{Without the umbrella, $V_b = 1.68 \frac{m}{s}$ or $6.04 \frac{km}{hr}$, by setting $C_{Du} = 0$ above.}

Discussion: Floyd is confused about his fluid mechanics principles if he thinks he can exceed the wind speed. It is impossible to obtain a propulsive force from aerodynamic drag unless the bicycle is moving more slowly than the wind. The drag force must be sufficient to overcome the rolling resistance of the bike and rider. At equilibrium speed the drag force and rolling resistance force must be equal and opposite.

The only benefit could be achieved by adding drag force more rapidly than rolling resistance. An umbrella, with its relatively high drag and low weight, is ideal for this purpose.

However, one would somehow have to hold the umbrella perpendicular to the wind while riding the bike. This would be dangerous at best, especially if the bike had hand-activated brakes.

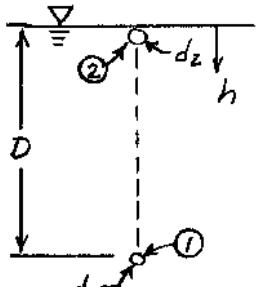
Since the umbrella must be held perpendicular to the wind, it would be very effective at blocking the rider's view of the road ahead!

In summary, this "system" of propulsion appears quite impractical.

Given: Air bubble rising in seawater (Problem 3.7), $d_1 = 10$ mm at $h = 30$ m below sea surface. $T = 30°C$.

Find: Estimate of time interval for bubble to reach the surface.

Solution: Bubble will reach terminal speed quickly. It will expand as pressure decreases.



Assume bubble is isothermal. Then $pV = \text{constant}$. Thus

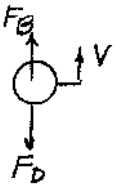
$$V = V_1 \frac{p_1}{p} = \frac{\pi d_1^3}{6}\left(\frac{p_{atm} + \rho g D}{p_{atm} + \rho g h}\right); \quad V_1 = \frac{\pi d_1^3}{6} = 5.24 \times 10^{-7}\ m^3$$

$$V_2 = V_1 \frac{p_1}{p} = V_1\left(\frac{p_{atm} + \rho g D}{p_{atm}}\right). \quad \text{But } V \sim d^3, \text{ so } d = d_1\left(\frac{p_{atm} + \rho g D}{p_{atm}}\right)^{1/3}$$

$$d_2 = 10\ mm\left(\frac{101.3 \times 10^3 \frac{N}{m^2} + (1.025)1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 30\ m \times \frac{N \cdot s^2}{kg \cdot m}}{101.3 \frac{N}{m^2}}\right)^{1/3} = 15.8\ mm$$

At terminal speed, $F_B = \rho_{seawater} g V = F_D = C_D A \frac{1}{2}\rho V^2$

Free-body diagram:



(Neglect mass of air in bubble.) Then $V = \left[\frac{2gV}{C_D A}\right]^{\frac{1}{2}}$

But $V = \frac{\pi d^3}{6}$ and $A = \frac{\pi d^2}{4}$, so $V = \left[\frac{4gd}{3C_D}\right]^{\frac{1}{2}}$

Choose $C_D \approx 0.5$ for a sphere (Fig. 9.11). Then

$$V_1 = \left[\frac{4}{3} \times 9.81 \frac{m}{s^2} \times 0.010\ m \times \frac{1}{0.5}\right]^{\frac{1}{2}} = 0.511\ m/s$$

Check the Reynolds number. ($\nu_{seawater} = 1.05\,\nu_{water}$, from Table A.2; at $T = 30°C$ $\nu_{water} = 8.01 \times 10^{-7}\ m^2/s$ from Table A.8).

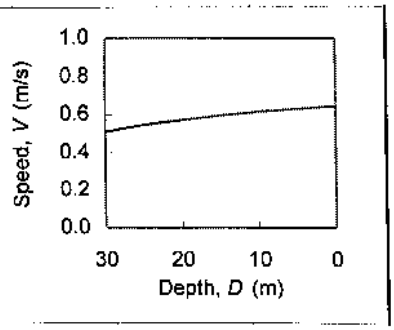
$$Re = \frac{Vd}{\nu} = 0.511 \frac{m}{s} \times 0.010\ m \times \frac{s}{(1.05)8.01 \times 10^{-7}\ m^2} = 6.08 \times 10^3$$

Thus from Fig. 9.11, $C_D = 0.5$ is reasonable,

$$V_2 = \left[\frac{4}{3} \times 9.81 \frac{m}{s^2} \times 0.0158\ m \times \frac{1}{0.5}\right]^{\frac{1}{2}} = 0.643\ m/s$$

Thus V varies slightly. Using a spreadsheet, with $\Delta t = \frac{\Delta h}{V}$, $t \approx 51$ s

Depth	Diameter	Speed	Time	Total Time
h (m)	d (m)	V (m/s)	Δt (s)	t (s)
30	0.0100	0.511		
25	0.0114	0.547	9.45	
20	0.0126	0.574	8.92	
15	0.0136	0.595	8.55	
10	0.0144	0.614	8.27	
5	0.0152	0.630	8.04	
0	0.0158	0.644	7.85	51.1

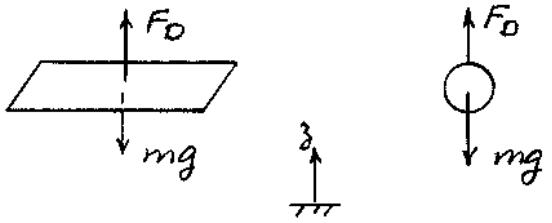


Problem 9.122

Open-Ended Problem Statement: Hold a flat sheet of paper parallel to the ground. Drop the sheet. Notice that it falls more slowly than if it were wadded into a ball. Explain.

Analysis: When dropped, the sheet accelerates to its terminal speed, at which the aerodynamic drag force on the sheet is equal to the gravitational force on the sheet.

Draw a free-body diagram:



Sum forces: $\Sigma F_z = F_D - mg = ma_z$

At terminal speed $a_z = 0$, so

$$F_D = C_D A \tfrac{1}{2}\rho V_t^2 = mg$$

or

$$V_t = \left[\frac{2mg}{\rho C_D A}\right]^{\frac{1}{2}}$$

For the flat sheet,

$$A = (8\tfrac{1}{2} \times 11)\,in.^2 \times (0.0254)^2 \frac{m^2}{in.^2} = 0.0603\ m^2$$

$C_D \approx 1.2$ (Table 9.2), so $C_D A \approx 1.2 \times 0.0603\ m^2 = 0.0724\ m^2$

For the wadded sheet (assume $D \approx 50$ mm),

$$A = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.05)^2 m^2 = 0.00196\ m^2$$

$C_D \approx 0.47$ (Fig. 9.11), so $C_D A \approx 0.47 \times 0.00196\ m^2 = 0.000923\ m^2$

Then, since the mass of both paper sheets is the same,

$$\frac{V_t(ball)}{V_t(flat)} = \left[\frac{2mg}{\rho C_D A(ball)}\right]^{\frac{1}{2}}\left[\frac{\rho C_D A(flat)}{2mg}\right]^{\frac{1}{2}} = \left[\frac{C_D A(flat)}{C_D A(ball)}\right]^{\frac{1}{2}}$$

Substituting,

$$\frac{V_t(ball)}{V_t(flat)} = \left[\frac{0.0724\ m^2}{0.000923\ m^2}\right]^{\frac{1}{2}} = 8.86 \approx 9$$

V_t

Thus the balled up sheet falls nearly 10 times as fast as the flat sheet!

Discussion: Both the flat paper and the wadded paper have the same mass.

The flat sheet has the higher drag-area product. The surface area is that of the whole sheet and the drag coefficient is approximately that for a disk normal to the flow, about 1.2 (Table 9.3).

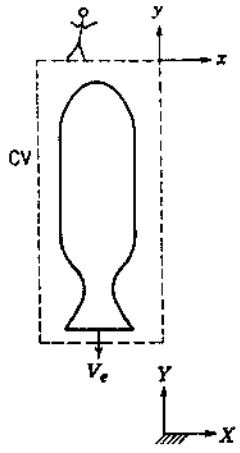
The wadded sheet has the lower drag-area product. The projected area of the balled paper is a small fraction of the area of the entire sheet. The drag coefficient is approximately that for a sphere, about 0.47 (Fig. 9.11).

Since both the drag coefficient and the area are smaller, the wadded sheet falls at a faster speed than the flat sheet.

Given: Rocket of Example Problem 4.12, launched vertically from rest.

$M_0 = 400$ kg
$V_e = 3500$ m/s
$\dot{m} = 5$ kg/s

$C_D = 0.3$
$D = 700$ mm



Find: Rocket speed versus time; compare with results from Example Problem 4.12; Plot

Solution: Apply y momentum equation to linearly accelerating CV shown.

Basic equation:

$$F_{Sy} + F_{By} - \int_{CV} a_{rfy}\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\rho\, d\forall + \int_{CS} v_{xyz}\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

($\frac{\partial}{\partial t} \approx 0(1)$)

Assumptions: (1) Neglect v; $\frac{\partial}{\partial t}$ in CV
(2) $F_D = C_D A \frac{1}{2}\rho V^2$; ρ = constant

Then

$$-F_D - Mg - M\frac{dV}{dt} = v_e\{+\dot{m}_e\} = -\dot{m}V_e \quad \text{and} \quad M = M_0 - \dot{m}t$$

$$v_e = -V_e$$

or

$$\frac{dV}{dt} = \frac{V_e\dot{m}}{M_0 - \dot{m}t} - \frac{F_D}{M_0 - \dot{m}t} - g = \frac{V_e\dot{m} - C_D A\frac{1}{2}\rho V^2}{M_0 - \dot{m}t} - g$$

Evaluating terms

$$V_e\dot{m} = 3500\frac{m}{s} \times 5\frac{kg}{s} \times \frac{N\cdot s^2}{kg\cdot m} = 1.75\times 10^4\ N$$

$$C_D A\frac{1}{2}\rho V^2 = \frac{0.3}{2}\times\frac{\pi}{4}(0.700)^2 m^2 \times 1.23\frac{kg}{m^3}\times V^2\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} = 0.0710[V(m/s)]^2\ N$$

$$M_0 - \dot{m}t = 400\ kg - 5\frac{kg}{s}\times t(s) = 400 - 5t(s)\ \ kg$$

Substituting,

$$\frac{dV}{dt}(m/s^2) = \frac{1.75\times 10^4 - 0.0710[V(m/s)]^2}{400 - 5t(s)} - 9.81$$

$\frac{dV}{dt}$

Numerical integration (using 4th-order Runge-Kutta) gives

t(s)	0	2	4	6	8	10	
V(m/s)	0	68.4	136	199	255	302	V
V(m/s)	0	69.0	140	214	290	369	($C_D = 0$; Example Problem 4.12)

The effect of drag becomes more pronounced as speed increases!

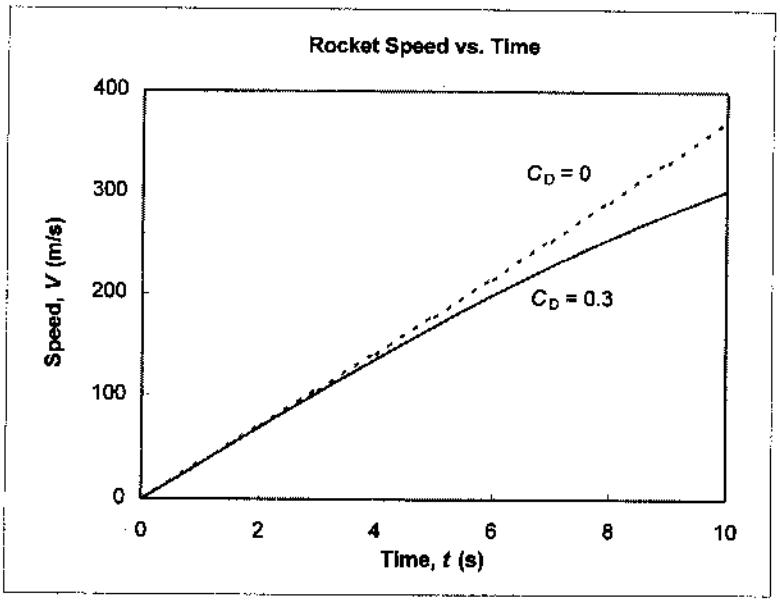
Input data:

C_D =	0	none	Drag coefficient
D =	700	mm	Diameter of rocket
g =	9.81	m/s^2	Acceleration of gravity
h =	1	s	Time interval
M_0 =	400	kg	Initial mass
m(dot) =	5	kg/s	Mass flow rate
V_e =	3500	m/s	Exhaust gas speed
V_0 =	0	m/s	Initial speed of rocket
ρ =	1.23	kg/m^3	Air density

Calculated results:

$\rho C_D A$ = 0.0000

Interval Number	Time t (s)	Old Speed V (m/s)	Accel a (m/s^2)	Runge-Kutta Constants k_1	k_2	k_3	k_4	New Speed V (m/s)
0	0	0	33.9	33.94	34.22	34.22	34.49	34.2
1	1	34.2	34.5	34.49	34.78	34.78	35.06	69.0
2	2	69.0	35.1	35.06	35.35	35.35	35.64	104
3	3	104	35.6	35.64	35.94	35.94	36.24	140
4	4	140	36.2	36.24	36.55	36.55	36.86	177
5	5	177	36.9	36.86	37.17	37.17	37.49	214
6	6	214	37.5	37.49	37.81	37.81	38.14	252
7	7	252	38.1	38.14	38.47	38.47	38.80	290
8	8	290	38.8	38.80	39.14	39.14	39.49	329
9	9	329	39.5	39.49	39.84	39.84	40.19	369
10	10	369	40.2	40.19	40.55	40.55	40.91	410



9-7.3[5]

Open-Ended Problem Statement: Wiffle™ balls made from light plastic with numerous holes are used to practice baseball and golf. Explain the purpose of the holes and why they work. Explain how you could test your hypothesis experimentally.

Discussion: The basic concept of the Wiffle ball is a low-mass, high-drag configuration that can be hit or struck with full force, but will not fly fast or far. Thus the Wiffle ball can be used for practice in a limited space.

The low mass is achieved by making the ball of relatively thin plastic material. This gives it low mass for its size, and is a step toward making the drag force relatively high compared to the weight of the ball.

Even higher drag force is achieved by perforating the surface of the Wiffle ball with numerous large holes. These holes further reduce the mass of the Wiffle ball.

In the sub-critical flow regime (below $Re_D \approx 2 \times 10^5$) skin friction drag accounts for less than 5 percent of the total drag of a sphere. The holes increase the skin friction drag of the ball by allowing boundary-layer fluid to escape into the interior of the ball. Each new bit of surface then sees essentially a new boundary layer developing, with attendant high shear stress.

Pressure drag accounts for the majority of the drag of a sphere at any Reynolds number above about 1000. The holes disrupt the flow pattern around the ball and probably trigger early separation. This ensures that the ball remains in the high-drag sub-critical flow regime no matter what its actual Reynolds number.

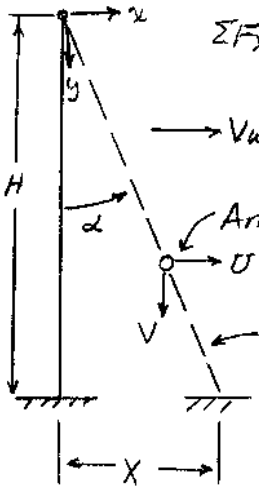
This hypothesis could be tested experimentally by comparing the performance of two balls, one with holes and one without. (The balls should have nearly the same mass and diameter.) With the help of an assistant, drop the balls from some height (for example, down a stairwell). After each ball has reached terminal speed, measure the time required for it to fall through a fixed distance. Then calculate and compare the drag coefficients for the two balls. If the drag coefficient for the ball with holes is significantly larger than for the ball without holes, the hypothesis is confirmed.

Several balls of each type might be evaluated experimentally to obtain an idea of the Reynolds number dependence of the results.

Problem 9.125

Open-Ended Problem Statement: Towers for television transmitters may be up to 500 m in height. In the winter, ice forms on structural members. When the ice thaws, chunks break off and fall to the ground. How far from the base of a tower would you recommend placing a fence to limit danger to pedestrians from falling ice chunks?

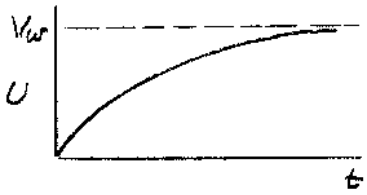
Analysis: An ice chunk detaching from a tower starts at rest, falls by gravity, and simultaneously is blown sideways by wind. Because drag is proportional to relative speed squared, it may be treated in separate x and y components.



$$\Sigma F_x = F_D = m\frac{dU}{dt} = C_D A \tfrac{1}{2}\rho (V_w - U)^2 = m\frac{dU}{dt} = -m\frac{d(V_w - U)}{dt} \qquad (1)$$

This equation is solved in Example Problem 4.11. The result is

$$\frac{U}{V_w} = \frac{bt}{1+bt}$$



$$b = \frac{\rho C_D A}{2m} V_w$$

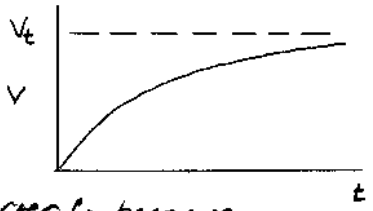
U quickly approaches V_w

$$\Sigma F_y = mg - F_D = m\frac{dV}{dt} = mV\frac{dV}{dt} = mg - C_D A \tfrac{1}{2}\rho V^2 = mg - kV^2 \qquad (2)$$

This equation is solved in Example Problem 1.2. The result is

$$V = \left\{\frac{mg}{k}\left(1 - e^{-\frac{2k}{m}y}\right)\right\}^{1/2} \quad \text{or} \quad \frac{V}{V_t} = 1 - e^{-2\frac{k}{m}y}$$

$$V_t = \left[\frac{mg}{k}\right]^{1/2}$$



Model the chunk as a geometric sphere, but with larger C_D because of jagged edges. Both b and V_t depend on diameter; $V_t = [8.11\, D(\text{mm})]^{1/2}$, or 11.0 m/s for D = 15 mm.

Thus a reasonable approximation is falling at V_t and moving sideways at V_w. For V_w = 10 mph (4.47 m/s), and V_t = 11.0 m/s, then $\alpha = \tan^{-1}(4.47/11.0) = 22.1°$; and $X = H \tan 22.1° = 203$ m.

To more precisely compute distance, obtain x(t) and y(t) by solving Eqs. 1 and 2 numerically, and plot the particle path.

Discussion: Because towers may be very tall, ice chunks can travel long distances from the base even in moderate winds. Considerable area around the base of a tower must be fenced to keep personnel on the ground safe from falling ice.

The analysis in this problem would be accurate if the drag-area product $C_D A$ for an ice chunk were known precisely. However, the size of the structural members and the thickness of the ice coating are both unknown. Therefore it is difficult to choose the most probable drag-area product. We recommend you bracket the sizes of known ice chunks, pick a reasonable range of drag coefficients, and then use the analysis to develop guidelines for the safety of personnel.

Problem 9.126

Open-Ended Problem Statement: The "shot tower," used to produce spherical lead shot, has been recognized as a mechanical engineering landmark. In a shot tower, molten lead is dropped from a high tower; as the lead solidifies, surface tension pulls each shot into a spherical shape. Discuss the possibility of increasing the "hang time," or of using a shorter tower, by dropping molten lead into an air stream that is moving upward. Support your discussion with appropriate calculations.

Analysis: This problem may be analyzed parametrically, in terms of shot diameter. Consider the range from "bird shot" of about 1 mm to musket balls of about 15 mm diameter to illustrate the results.

Analysis with still air: Terminal speed is reached when aerodynamic drag force exactly equals the weight of the shot. The first plot shows terminal speed versus diameter of lead shot.

The solution for shot speed versus distance traveled, with no upward air movement, parallels the solution of Example Problem 1.2, which gives the fraction of terminal speed reached in a tower of specified height. For any tower height (choose 50 m to illustrate the results) the fraction of terminal speed reached decreases with increasing shot diameter (see the second plot).

The solution for hang time versus diameter is shown in the third plot.

Analysis with upward flow of air: The solution for shot speed versus distance traveled is more complex when air in the shot tower flows upward. Introducing upward air flow in the tower increases drag force compared to shot weight. Therefore the shot accelerate more slowly in the upward flow. It is possible to obtain an analytical solution, but the result is so complex that it is difficult to interpret. Results can be obtained for specific cases by integrating the differential equations numerically.

The solution for shot speed versus time also is more complex when air flows upward. Again numerical integration can be used to obtain results for specific cases.

Outline of Procedure: Derive a differential equation for shot acceleration from a free-body diagram. Integrate once to obtain shot velocity as a function of time. Integrate again to obtain shot position as a function of time.

From the results of the second integration, identify the "hang time" when the shot reaches the bottom of the tower. Plot hang time versus diameter and compare with results for the case with no upward air flow.

Set the upward air flow velocity to zero and compare numerical results with the analytical results for the case without flow to validate your model.

Discussion: The terminal speed reached by small shot in still air is quite low. Therefore, the "hang time" of small shot can be increased significantly by providing upward flow of air at reasonable speed in the tower.

Larger shot have higher terminal speeds. However, the higher terminal speed does not reduce hang time much because the large shot reach only a smaller fraction of their terminal speed in the 50 m tower height.

Introducing upward flow of air in the shot tower increases the drag force and results in slower acceleration of the shot. Therefore the hang time is increased. The increase in hang time allows more time for cooling, and should result in the production of more nearly spherical shot.

Input data:

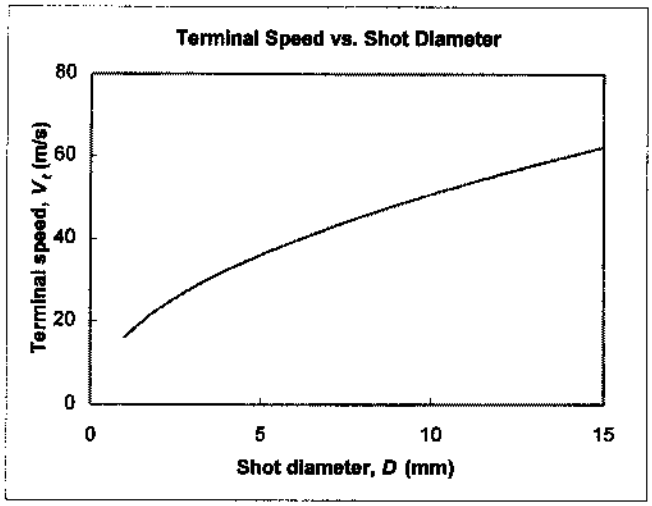
C_D =	0.47	(---)	Drag coefficient of sphere
SG_s =	11.4	(---)	Specific gravity of (lead) shot
Δz =	50	m	Height of shot tower
ρ_{air} =	1.23	kg/m^3	Density of air

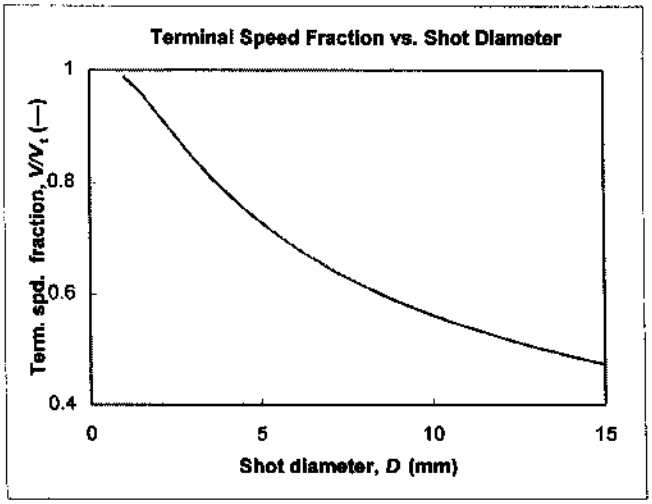
Calculated parameters:

k/D^2 =	2.27E-07	kg/m-mm^2	Drag factor, $F_D = kV^2$
m/D^3 =	5.969E-06	kg/mm^3	Mass of shot

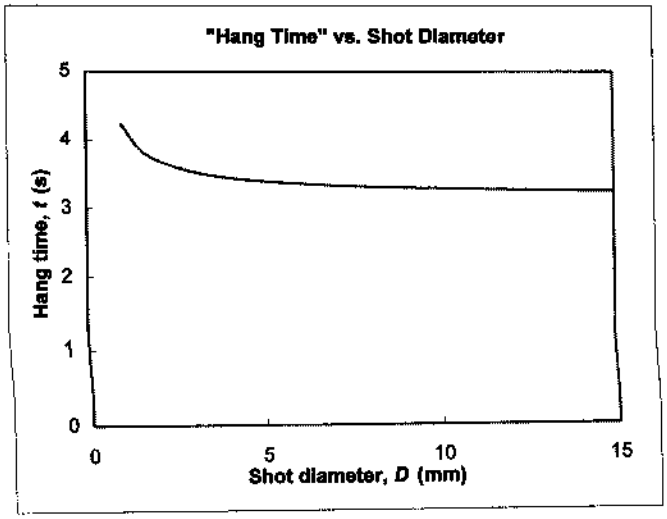
(1) Shot falling in still air:

D (mm)	V_t (m/s)	V/V_t (---)
1	16.1	0.989
1.5	19.7	0.960
2	22.7	0.922
3	27.8	0.848
4	32.1	0.783
5	35.9	0.730
6	39.3	0.685
7	42.5	0.647
8	45.4	0.615
9	48.2	0.587
10	50.8	0.562
11	53.3	0.541
12	55.6	0.521
13	57.9	0.504
14	60.1	0.488
15	62.2	0.473





D (mm)	V_t (m/s)	V/V_t (---)	t (s)
1	16.1	0.989	4.24
1.5	19.7	0.960	3.89
2	22.7	0.922	3.71
3	27.8	0.848	3.54
4	32.1	0.783	3.45
5	35.9	0.730	3.40
6	39.3	0.685	3.36
7	42.5	0.647	3.34
8	45.4	0.615	3.32
9	48.2	0.587	3.31
10	50.8	0.562	3.29
11	53.3	0.541	3.29
12	55.6	0.521	3.28
13	57.9	0.504	3.27
14	60.1	0.488	3.27
15	62.2	0.473	3.26



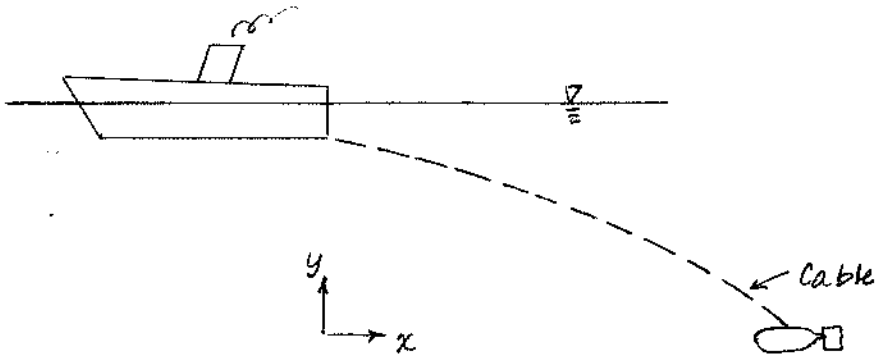
Open-Ended Problem Statement: Objects must be towed underwater for many reasons, from finding fish to locating submarines. Analyze the problem of towing an object under the sea surface. Develop a chart to predict the length of tow cable needed and the force that must be exerted to tow the cable and a typical submerged object.

Analysis: As shown in the sketch below, the tow cable curves continuously from the towing ship to the object being towed. The solution for the cable shape is obtained by numerically integrating several coupled differential equations. The equations are obtained by considering an infinitesimal section of cable, then by balancing forces in both the horizontal and vertical directions. The local rate of change of curvature of the cable is quite small. Therefore any stiffness inherent in the cable structure is neglected; the cable is assumed to be completely flexible.

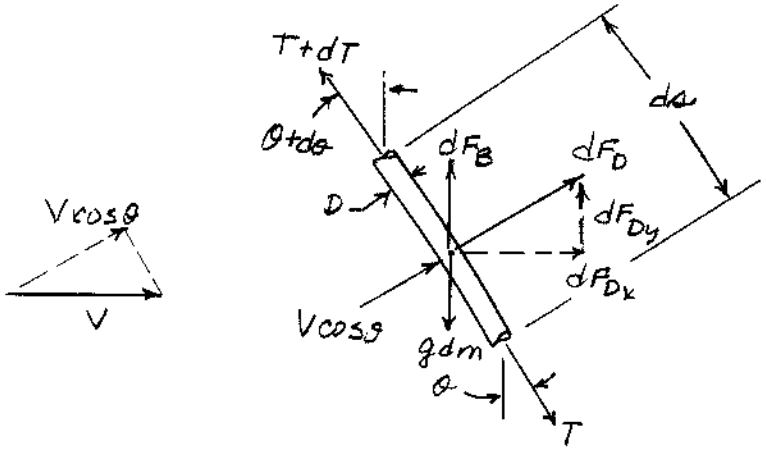
The problem is solved for an example situation. The discussion below details how to use the results to predict the cable length required, and to predict how far behind and below the ship the towed object is positioned during steady operation.

To use the solution, one begins with the net weight, drag characteristics, and desired depth of the object to be towed, the properties of the cable, and the towing speed. The properties of the object are used to determine the initial conditions (tension and angle) where the cable connects to the underwater object. The numerical solution is used to predict the total length of tow cable needed, and the force required to tow the combination of the cable and the underwater object.

The towing vessel, tow cable, and underwater object are sketched below:



Draw a free-body diagram of an element ds of the cable:



$$\Sigma F_x = dF_{Dx} + T\cancel{\sin\theta} - \left(T\cancel{\sin\theta} + \frac{d}{ds}T\sin\theta\, ds\right) = 0 \qquad (1)$$

$$\Sigma F_y = dF_B + dF_{Dy} - T\cancel{\cos\theta} + \left(T\cancel{\cos\theta} + \frac{d}{ds}T\cos\theta\, ds\right) - g\,dm = 0 \qquad (2)$$

$$dF_D = C_D A \tfrac{1}{2}\rho(V\cos\theta)^2 = C_D A \tfrac{1}{2}\rho V^2\cos^2\theta = C_D A q\cos^2\theta$$

$$\left.\begin{aligned} dF_{Dx} &= dF_D\cos\theta = C_D A q\cos^3\theta \\ dF_{Dy} &= dF_D\sin\theta = C_D A q\cos^2\theta\sin\theta \end{aligned}\right\} A = D\,ds$$

$$dm = \rho_c\, dV = \rho_c A\, ds \quad \{A = \tfrac{\pi D^2}{4}\} \quad \{\rho_c = \text{density of cable}\}$$

$$g\,dm = \rho_c g \tfrac{\pi D^2}{4} ds$$

$$dF_B = \rho g\, dV = \rho g \tfrac{\pi D^2}{4} ds \quad \{\rho = \text{density of seawater}\}$$

Substituting into Eq. 1,

$$C_D D\cancel{ds}\, q\cos^2\theta + \frac{dT}{ds}\sin\theta\cancel{ds} + T\cos\theta\frac{d\theta}{ds}\cancel{ds} = 0$$

or

$$-\tan\theta\frac{dT}{ds} - T\frac{d\theta}{ds} + C_D D q\cos^2\theta = 0 \qquad (3)$$

Substituting into Eq. 2,

$$C_D D \cancel{ds} q \cos^2\theta \sin\theta + g \frac{\pi D^2}{4}(\rho - \rho_c)\cancel{ds} + \frac{d}{ds} T \cos\theta \cancel{ds} = 0$$

$$\cos\theta \frac{dT}{ds} - T \sin\theta \frac{d\theta}{ds} + C_D D q \cos^2\theta \sin\theta + g \frac{\pi D^2}{4}(\rho - \rho_c) = 0$$

$$\frac{dT}{ds} - T \tan\theta \frac{d\theta}{ds} + C_D D q \cos\theta \sin\theta + g \frac{\pi D^2}{4\cos\theta}(\rho - \rho_c) = 0 \qquad (4)$$

Now solve for $\frac{dT}{ds}$ and $\frac{d\theta}{ds}$. Multiply Eq. 3 by $\tan\theta$ and subtract from Eq. 4:

$$(1 + \tan^2\theta)\frac{dT}{ds} + \qquad + g \frac{\pi D^2}{4\cos\theta}(\rho - \rho_c) = 0 \qquad (5)$$

Multiply Eq. 4 by $\tan\theta$ and add to Eq. 3:

$$-T(1 + \tan^2\theta)\frac{d\theta}{ds} + C_D D q \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + g \frac{\pi D^2 \sin\theta}{4\cos^2\theta}(\rho - \rho_c) = 0 \qquad (6)$$

Also $\frac{dy}{ds} = \cos\theta$, so $dy = \cos\theta\, ds$ (7)

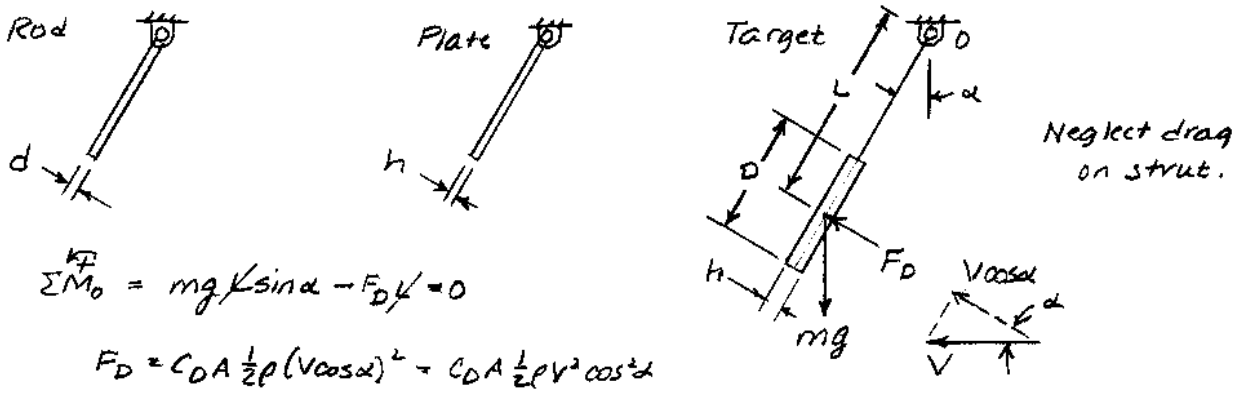
$\frac{dx}{ds} = \sin\theta$, so $dx = \sin\theta\, ds$ (8)

Solve Eqs. 5 through 8 numerically, simultaneously and plot results.

Problem 9.128

Open-Ended Problem Statement: Design a wind anemometer that uses aerodynamic drag to move or deflect a member or linkage, producing an output that can be related to wind speed, for the range from 1 to 10 m/s in standard air. Consider three alternative design concepts. Select the best concept and prepare a detailed design. Specify the shape, size, and material for each component. Quantify the relation between wind speed and anemometer output. Present results as a "calibration curve" of anemometer output versus wind speed. Discuss reasons why you rejected the alternative designs and chose your final design concept.

Analysis: The "target" concept was chosen for analysis. The drag force acting on the target is calculated, then moments are summed about the pivot (see Problem 9.105). The results are:



$$\curvearrowleft \Sigma M_O = mgL\sin\alpha - F_D L = 0$$

$$F_D = C_D A \tfrac{1}{2}\rho (V\cos\alpha)^2 = C_D A \tfrac{1}{2}\rho V^2 \cos^2\alpha$$

So $$C_D A \tfrac{1}{2}\rho V^2 \cos^2\alpha = mg\sin\alpha = \rho_m \forall_m g \sin\alpha = SG\rho_{H_2O} A h g \sin\alpha \qquad (1)$$

Assume $\alpha = 60°$ at highest wind speed (when $V = 10$ m/s; $q = \tfrac{1}{2}\rho V^2 = 61.5\ N/m^2$).

Then $$\frac{\sin\alpha}{\cos^2\alpha} = \frac{C_D A q}{SG\rho_{H_2O} A g h} = \frac{C_D q}{SG\rho_{H_2O} g h} = f(\alpha) \qquad \text{At } \alpha = 60°,\ f(\alpha) = 3.46$$

$$h = \frac{C_D q}{SG\rho_{H_2O} g f(\alpha)} = 1.2 \times 61.5 \frac{N}{m^2} \times \frac{m^3}{(SG)1000\,kg} \times \frac{s^2}{9.81\,m} \times \frac{1}{3.46} \times \frac{kg\cdot m}{N\cdot s^2} = \frac{0.00217\,m}{SG} = \frac{2.17\ mm}{SG}$$

Choose aluminum, with SG = 2.64 (Table A.1). Then

$$h = \frac{2.17\ mm}{2.64} = 0.822\ mm$$

h

Solve Eq. 1 for velocity

$$V = \left[\frac{2SG\rho_{H_2O} hg}{\rho C_D} \times f(\alpha)\right]^{\frac{1}{2}} \quad \text{(see plot)}$$

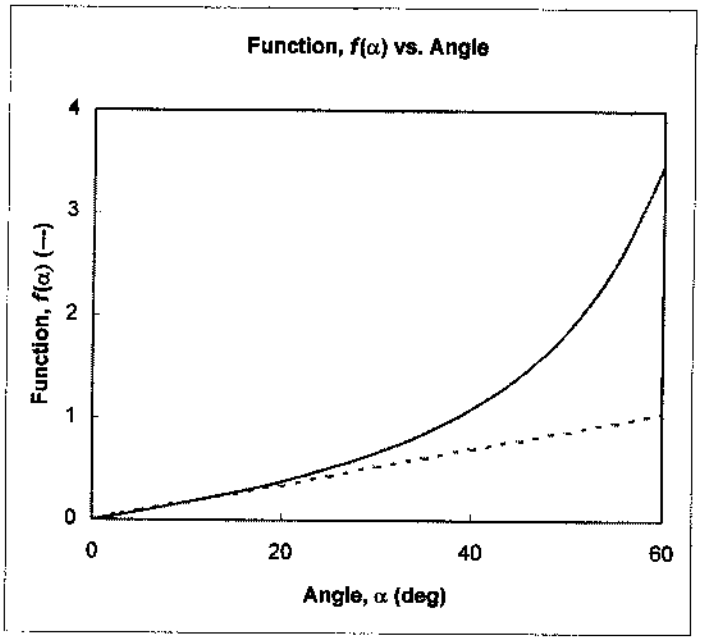
V

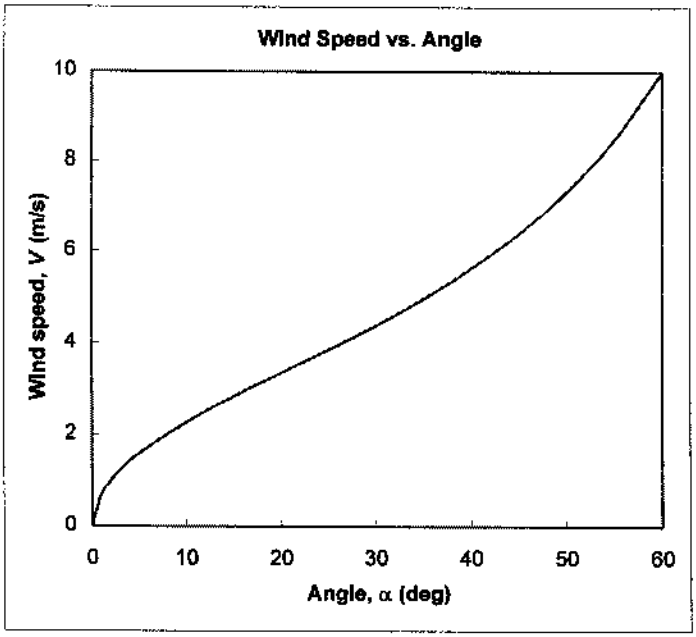
Discussion: Concepts considered included a manometer that sensed stagnation pressure, a parallelogram linkage supporting a vertical target, and bending of a thin member in the air stream. The three final concepts chosen were variations on the theme of a single hanging member supported from a single pivot, and were chosen for their simplicity.

The major advantage of the target concept is that different materials can be used for the rod and target; this concept can be tailored to give the largest deflection angle for a given wind speed. Therefore this device should be capable of the most accurate indication at low wind speeds.

Drag force on the target is assumed to depend on the component of wind velocity acting normal to the target. This model could be improved by using actual experimental data for the drag coefficient of a disk at angle of attack.

α (deg)	α (rad)	$f(\alpha)$ (—)	V (m/s)
0	0	0	0
1	0.0175	0.0175	0.71
2	0.0349	0.0349	1.00
3	0.0524	0.0525	1.23
5	0.0873	0.0878	1.59
10	0.175	0.179	2.27
15	0.262	0.277	2.83
20	0.349	0.387	3.34
25	0.436	0.515	3.85
30	0.524	0.667	4.39
35	0.611	0.855	4.97
40	0.698	1.10	5.62
45	0.785	1.41	6.39
50	0.873	1.85	7.31
55	0.960	2.49	8.47
60	1.05	3.46	10.0





Given: Aircraft with 60 m of 6 mm diameter guy wires flies 150 kph in standard air near sea level. Assume flow around the wires is two-dimensional.

Find: Maximum power saving from fairing (streamlining) wires.

Solution: Apply definitions of drag coefficient and power.

Basic equations: $C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$, $\mathcal{P} = F_D V$

Then

$$\mathcal{P} = C_D A \tfrac{1}{2}\rho V^2 V = C_D A \tfrac{1}{2}\rho V^3$$

But C_D is a function of Reynolds number. For standard air, $\nu = 1.46 \times 10^{-5}\ m^2/s$ (Table A.10), Thus

$$Re = \frac{VD}{\nu} = 150 \times 10^3 \frac{m}{hr} \times 0.006\,m \times \frac{s}{1.46 \times 10^{-5} m^2} \times \frac{hr}{3600\,s} = 17{,}100$$

From Fig. 9.13, $C_D \simeq 1.2$ for this Reynolds number. $A = DL$, so

$$\mathcal{P} = \frac{1.2}{2} \times 0.006\,m \times 60\,m \times 1.23\,\frac{kg}{m^3}\left(150 \times 10^3 \frac{m}{hr} \times \frac{hr}{3600\,s}\right)^3 \frac{N \cdot s^2}{kg \cdot m} \times \frac{W \cdot s}{N \cdot m}$$

$$\mathcal{P} = 19.2\ kW \text{ (with standard wires)}$$

Data in Fig. 9.14 suggest that the drag coefficient could be reduced to $C_D \simeq 0.06$ by fairing wires. Thus

$$\mathcal{P}_{faired} = \frac{C_{D\,faired}}{C_D}\mathcal{P}$$

and the power saving is

$$\Delta\mathcal{P} = \mathcal{P} - \mathcal{P}_{faired} = \left(1 - \frac{C_{D\,faired}}{C_D}\right)\mathcal{P} = \left(1 - \frac{0.06}{1.2}\right) 19.2\,kW$$

$$\Delta\mathcal{P} = 18.2\ kW \longleftarrow \Delta\mathcal{P}$$

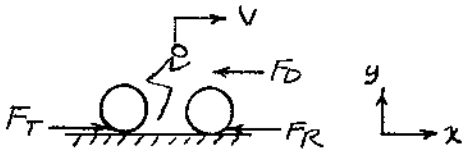
Thus $\frac{\Delta\mathcal{P}}{\mathcal{P}} = \frac{18.2}{19.2} = 0.948$ or 94.8%

Given: Bicycle of Problem 9.78: $F_R = 4\ N$, $C_D = 1.2$, $A = 0.25\ m^2$, $V_{max} = 37\ kph$.

Fairing installed, so $C_D = 0.90$, but $A = 0.30\ m^2$

Find: Top speed with fairing installed.

Solution: Consider free-body diagram of bike and rider; apply Newton's second law of motion.



Computing equations: $\Sigma F_x = ma_x$

$$F_D = C_D A \tfrac{1}{2}\rho V^2 \qquad P = F_T V$$

Power output for original conditions, $V = 37\ km/hr\ (= 10.3\ m/s)$, and

$$P = F_T V_{max} = (F_R + F_D)V_{max} = \left(F_R + C_D A \tfrac{1}{2}\rho V^2\right)V \tag{1}$$

$$P = \left[4\ N + 1.2 \times 0.25\ m^2 \times \frac{1}{2} \times 1.23\ \frac{kg}{m^3} \times (10.3)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}\right] 10.3 \frac{m}{s} \times \frac{W \cdot s}{N \cdot m} = 243\ W$$

For new conditions, assume the same power, solve for V. From Eq. 1,

$$V = \left[\frac{2}{C_D A \rho}\left(\frac{P}{V} - F_R\right)\right]^{1/2} \tag{2}$$

(The equation is cubic in V, so it cannot be solved directly. However, Eq. 2 can be used to solve iteratively.) Evaluating,

$$V = \left[2 \times \frac{1}{0.9} \times \frac{1}{0.30\ m^2} \times \frac{m^3}{1.23\ kg}\left(243\ \frac{N \cdot m}{s} \times \frac{1}{V} - 4\ N \times \frac{kg \cdot m}{N \cdot s^2}\right)\right]^{1/2} = \left[6.02\left(\frac{243}{V} - 4\right)\right]^{1/2}$$

Tabulate:

Guess V (m/s)	Calculated V (m/s)	
10.3	10.9	
10.4	10.8	
10.5	10.7	
10.6	10.7	← (new speed here)
10.7	10.6	

V

The improvement is $\frac{V_{fair} - V}{V} \times 100 \simeq \frac{10.6 - 10.3}{10.3} \times 100 \simeq 2.9$ percent!

9-7.4 [4]

Open-Ended Problem Statement: Why does an arrow have feathers? Explain how feathers affect the arrow's flight.

Discussion: The purpose of the feathers is to stabilize the arrow in flight. The feathers add little drag because they are almost aligned with the flow (most are attached at a slight angle to impart rotation to the arrow shaft). The stability comes from having the area of the feathers located far back along the arrow.

When disturbed in flight, e.g., by a wind gust, the feathers caused the center of aerodynamic pressure to act behind the center of gravity of the arrow. This causes the arrow to turn slightly in the direction from which the gust came. This reaction stabilizes the flight of the arrow and produces a predictable and accurate trajectory.

(The same principle is used to make "weather vanes" to cause them to point into the wind. It also is used on conventional aircraft: the vertical tail area helps to provide lateral stability in flight of the aircraft.)

13-782 500 SHEETS, FILLER, 5 SQUARE
42-381 500 SHEETS, EYE-EASE, 5 SQUARE
42-389 100 SHEETS EYE-EASE 5 SQUARE
42-389 200 SHEETS EYE-EASE 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42-398 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

9-7.4 [4]

Open-Ended Problem Statement: Why do modern guns have rifled barrels?

Discussion: Almost all projectiles fired by modern guns have smoothly rounded noses and abruptly tapered ("boat-tailed") or square rear ends. The minimum drag for these shapes is obtained when the projectile travels with its axis parallel to the direction of motion and its nose pointed forward.

Rifling in a gun barrel imparts spin about the longitudinal axis of the projectile. This rotation about the longitudinal axis causes the projectile to act as a gyroscope and stabilizes it during flight to keep its nose pointed in the direction of motion.

Early smoothbore guns primarily used ball projectiles. The balls were spherical and molded from lead. Since the ball shape was spherical and had no preferred orientation, no benefit would have been achieved from rifling that caused spin. Therefore the gun barrels were bored smooth, i.e., without rifling grooves, hence these guns were called "smoothbore" guns.

13 782 500 SHEETS, FILLER, 5 SQUARE
42-381 50 SHEETS EYE-EASE® 5 SQUARE
42-382 100 SHEETS EYE-EASE® 5 SQUARE
42-389 200 SHEETS EYE-EASE® 5 SQUARE
42-392 100 RECYCLED WHITE 5 SQUARE
42 399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

9-7.4 [4]

Open-Ended Problem Statement: Why is it possible to kick a football farther in a spiral motion than in an end-over-end tumbling motion?

Discussion: A football has a prolate spheroid shape. It is almost circular when viewed from the front (parallel to the major axis), and longer and more elliptical when viewed from the side (along a minor axis). The football has more frontal area when traveling with the major axis perpendicular to the motion that when it is "spiraling" with the major axis parallel to the direction of travel.

The drag coefficient of the ball when parallel to the flow in spiral motion undoubtedly is less than when perpendicular to the flow. As a rough approximation, the perpendicular drag coefficient might be similar to that of a cylinder ($C_D = 1.2$), whereas the spiral drag coefficient probably is less (perhaps $C_D \approx 0.2$ - 0.3) than that of a sphere ($C_D = 0.5$). Thus the drag coefficient when traveling with the long axis perpendicular to the flow may be 4 to 6 times as large as when traveling in spiral motion with the long axis parallel to the flow. The difference in the drag-area product C_DA will be even larger.

In tumbling motion the drag-area product varies cyclically between the two extremes we have discussed. On average the drag-area product for the tumbling ball is considerably larger, perhaps 2 to 3 times as large, as when the ball is in spiral motion. Therefore the maximum range (travel distance) that can be achieved with tumbling motion is much less than that for spiral motion.

Also, a well kicked or thrown spiral is a thing of beauty. Perhaps function follows form here!

Open-Ended Problem Statement: How do cab-mounted wind deflectors for tractor-trailer trucks work? Explain using diagrams of the flow pattern around the truck and pressure distribution on the surface of the truck.

Discussion: Consider both the cab and the trailer flow patterns and pressure distributions in no-wind and crosswind situations.

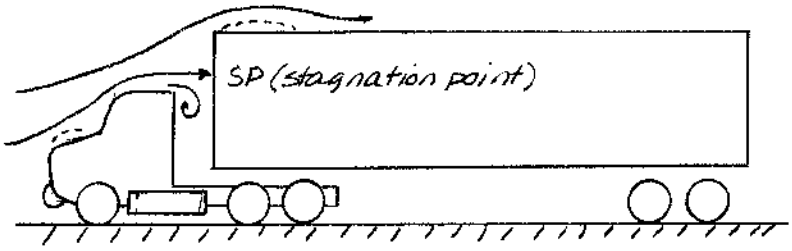
No-wind situation: Without the deflector, flow separation from the roof and sides of the tractor creates a low-pressure wake and high drag force on the tractor. (Flow patterns and pressure distributions on the tractor and the front of the trailer are sketched below.) The cab-mounted deflector reduces the pressures on the front of the tractor, thus reducing the aerodynamic drag force on the tractor.

Without the deflector, high-speed air separates from the roof of the tractor and impinges on the vertical front face of the trailer. The cab-mounted deflector reduces the amount of high-speed air hitting the front of the trailer, reducing the net aerodynamic drag force on the trailer. (Ideally air from the deflector flows smoothly along the top and sides of the trailer.)

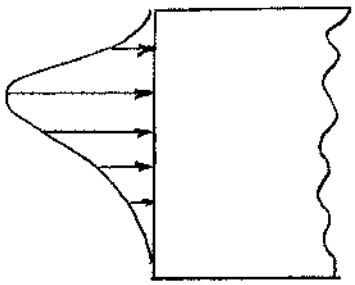
Crosswind situation: Without the deflector, flow separates from the lee side of the tractor, altering the pressure field and increasing the drag on the tractor. The cab-mounted deflector, especially in combination with side seals, minimizes the increase in drag by reducing the amount of separation around the tractor.

Without the deflector, the front face of the trailer is impacted by the high-speed air from the freestream flow. Massive separation occurs on the lee side of the trailer, thus altering the pressure field and increasing the drag on the trailer. With the cab-mounted deflector, the amount of high-speed air impacting the trailer is markedly reduced. This alters the flow pattern and minimizes the increase in drag caused by the crosswind.

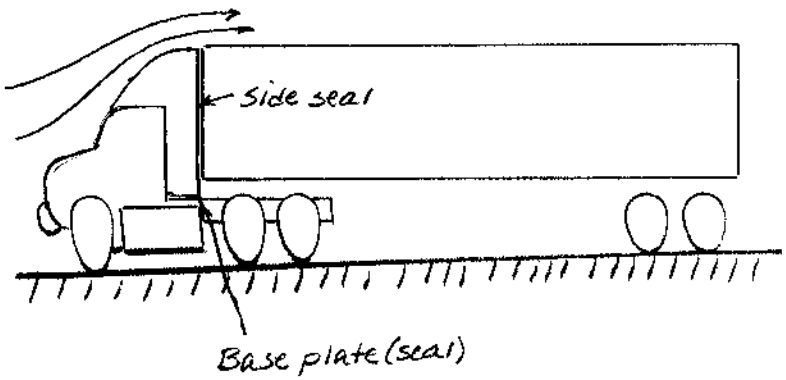
Without cab-mounted wind deflector:

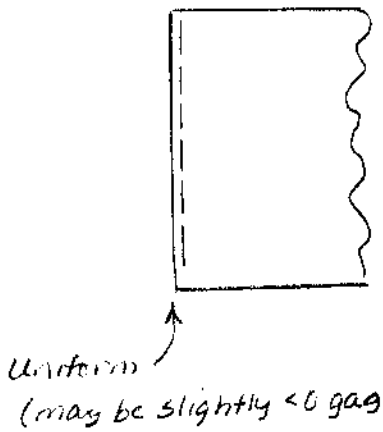


Trailer face:



With cab-mounted wind deflector:





Given: Aircraft in level flight at a speed of 250 km/hr in standard air. The aircraft mass is 850 kg, and has $C_L = 0.4$ and $C_D = 0.065$.

Find: Effective lift area of aircraft.

Solution: Apply definition of lift coefficient.

Basic equation: $C_L \equiv \frac{F_L}{\frac{1}{2}\rho V^2 A_p}$

Assume: Lift equals weight in level flight.

Then $F_L = mg = C_L \frac{1}{2}\rho V^2 A_p$ or $A_p = \frac{mg}{C_L \frac{1}{2}\rho V^2} = \frac{2mg}{C_L \rho V^2}$

$$A_p = \frac{2}{0.4} \times 850\,kg \times 9.81\,\frac{m}{s^2} \times \frac{m^3}{1.23\,kg} \left(\frac{hr}{250 \times 10^3\,m} \times 3600\,\frac{s}{hr}\right)^2 = 7.03\,m^2 \quad A_p$$

Given: Aircraft with NACA 23012 section airfoils and effective lift area, $A = 25\ m^2$. Maximum flap setting corresponds to condition ② in Fig. 9.23. Takeoff speed is 150 kph. Neglect added lift due to ground effect.

Find: Maximum gross mass at takeoff, speed in Denver ($z = 1.61$ km).

Solution: Apply definition of lift coefficient.

Basic equation: $C_L = \dfrac{F_L}{\frac{1}{2}\rho V^2 A_p}$

Assumption: Lift force must equal gravity force at takeoff.

$$F_L = mg = C_D A_p \tfrac{1}{2}\rho V^2$$

For maximum mass, need maximum lift, so use $C_{L,max}$:

$$m_{max} = \frac{C_{L,max} A_p \rho V^2}{2g}$$

From Fig. 9.23, $C_{L,max} = 2.67$ for condition ②. Then for std. air,

$$m_{max} = \frac{2.67}{2} \times 25\ m^2 \times 1.23 \frac{kg}{m3} \left(150 \times 10^3 \frac{m}{hr} \times \frac{hr}{3600\ s} \right)^2 \frac{s^2}{9.81\ m}$$

$$m_{max} = 7260\ kg \longleftarrow m_{max}$$

{This represents the maximum mass theoretically possible when the aircraft is on the verge of stalling. To attempt takeoff at such a large mass would be ill-advised.}

In Denver, $z = 1.61$ km. From Table A.3, at $z = 1.61$ km, $\rho/\rho_0 = 0.855$.

At the same gross mass, the lift force remains the same. Thus

$$F_{L_0} = C_D A \tfrac{1}{2}\rho_0 V_0^2 = F_{L_D} = C_D A \tfrac{1}{2}\rho_D V_D^2 \quad \text{or} \quad \rho_0 V_0^2 = \rho_D V_D^2$$

and

$$V_D = V_0 \left(\frac{\rho_0}{\rho_D} \right)^{1/2} = 150\ kph \left(\frac{1}{0.855} \right)^{1/2} = 162\ kph \longleftarrow V_D$$

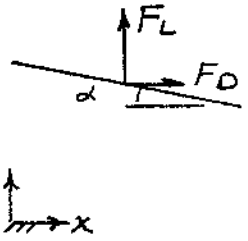
{The takeoff speed must increase about 8 percent.}

Given: Rectangular flat plate, $L = 3$ ft and $b = 4$ ft, moving in standard air at $U = 44.5$ ft/s, with $\alpha = 12°$. For these conditions, $C_L = 0.72$ and $C_D = 0.17$, based on projected area of plate.

Find: (a) Resultant force vector.
(b) Power required to keep plate moving.

Solution: Apply definitions of lift and drag coefficients.

Computing equations: $C_L = \dfrac{F_L}{\frac{1}{2}\rho U^2 A}$ $\qquad C_D = \dfrac{F_D}{\frac{1}{2}\rho U^2 A}$



Assumptions: (1) Standard air, $\rho = 0.0765$ lbm/ft^3
(2) Neglect induced drag

Then $A = bL = 4\text{ ft} \times 3\text{ ft} = 12.0\text{ ft}^2$

$$q = \frac{1}{2}\rho U^2 = \frac{1}{2} \times 0.0765 \frac{\text{lbm}}{\text{ft}^3} \times (44.5)^2 \frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{slug}}{32.2\text{ lbm}} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

$$q = 2.35 \text{ lbf/ft}^2$$

So

$$F_L = C_L A q = 0.72 \times 12\text{ ft}^2 \times 2.35 \frac{\text{lbf}}{\text{ft}^2} = 20.3\text{ lbf}$$

$$F_D = C_D A q = \frac{C_D}{C_L} F_L = \frac{0.17}{0.72} \times 20.3\text{ lbf} = 4.79\text{ lbf}$$

Thus $\vec{F}_R = F_D \hat{i} + F_L \hat{j} = 4.79\hat{i} + 20.3\hat{j}$ lbf $\qquad \vec{F}_R$

The power is $P = F_D U$

$$P = 4.79\text{ lbf} \times 44.5 \frac{\text{ft}}{\text{s}} \times \frac{\text{hp}\cdot\text{s}}{550\text{ ft}\cdot\text{lbf}} = 0.388\text{ hp} \qquad P$$

The induced drag coefficient would be approximately

$$C_{D_i} = \frac{C_L^2}{\pi a r} = \frac{1}{\pi} \times (0.72)^2 \times \frac{1}{1.33} = 0.124$$

This contribution would be significant if not already included in the $C_D = 0.17$.

Given: Hydrofoil craft with effective foil area, $A = 0.7\ m^2$ and mass, $m = 1800\ kg$. Foils have $C_L = 1.6$ and $C_D = 0.5$. Neglect induced drag.

Find: (a) Minimum speed to support craft on foils.
(b) Power required at this speed.
(c) Maximum speed if 110 kW is available.

Solution: Apply definitions of lift, drag coefficients and power.

Computing equations: $C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A}$; $C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$; $\mathcal{P} = F_D V$

Assumptions: (1) Lift force equals gravity force.
(2) Neglect induced drag.

Then
$$F_L = mg = C_L A \tfrac{1}{2}\rho V^2 \quad \text{so} \quad V = \left[\frac{2mg}{C_L \rho A}\right]^{1/2}$$

Minimum speed is

$$V_{min} = \left[\frac{2}{1.6} \times 1800\ kg \times 9.81 \frac{m}{s^2} \times \frac{m^3}{999\ kg} \times \frac{1}{0.7\ m^2}\right] = 5.62\ m/s \quad (10.9\ kt) \quad V_{min}$$

The drag force at any speed is

$$F_D = C_D A \tfrac{1}{2}\rho V^2 \quad \text{so} \quad F_D = \frac{C_D}{C_L} F_L = \frac{C_D}{C_L} mg$$

and
$$\mathcal{P} = F_D V = \frac{C_D}{C_L} mg V$$

The minimum power is

$$\mathcal{P}_{min} = \frac{C_D}{C_L} mg V_{min} = \frac{0.5}{1.6} \times 1800\ kg \times 9.81 \frac{m}{s^2} \times 5.62 \frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$\mathcal{P}_{min} = 31.0 \times 10^3 \frac{N \cdot m}{s} \times \frac{W \cdot s}{N \cdot m} = 31.0\ kW \quad \mathcal{P}_{min}$$

As speed increases, the craft will ride higher in the water, decreasing the lifting area such that $F_L = mg$. Thus

$$\mathcal{P}_{max} = \frac{C_D}{C_L} mg V_{max} \quad \text{or} \quad V_{max} = \frac{C_L}{C_D} \frac{\mathcal{P}_{max}}{mg}$$

Assuming C_D/C_L remains constant,

$$V_{max} = \frac{1.6}{0.5} \times 110 \times 10^3\ W \times \frac{1}{1800\ kg} \times \frac{s^2}{9.81\ m} \times \frac{N \cdot m}{W \cdot s} \times \frac{kg \cdot m}{N \cdot s^2} = 19.9\ m/s \quad (38.7\ kt) \quad V_{max}$$

Given: Light plane, $M = 2000$ lbm, has wing planform area, $A_p = 100\ ft^2$. Top speed in level flight is $V_{max} = 180$ mph. Airfoil on craft has NACA 23015 section. Plane is safe for aerobatic maneuvers provided $a_{vertical} < 5g$.

Find: Is it possible for the pilot to exceed an instantaneous vertical acceleration of 5g from level flight

Solution:

Definition: $C_L \equiv \dfrac{F_L}{\frac{1}{2}\rho V^2 A_p}$

$$V_{max} = 180 \frac{mi}{hr} \times \frac{hr}{3600 s} \times 5280 \frac{ft}{mi} = 264\ ft/s$$

Assume plane is level flight at V_{max}; pilot increases angle of attack to give C_{Lmax}.

From Fig. 9.17, $C_{Lmax} = 1.72$ at $\alpha \approx 18°$

Then

$$F_{Lmax} = \frac{1}{2}\rho V^2 A_p C_{Lmax}$$

$$= \frac{1}{2} \times 0.002377 \frac{slug}{ft^3} \times (264)^2 \frac{ft^2}{s^2} \times 100 ft^2 \times 1.72 \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$F_{Lmax} = 14{,}200\ lbf$$

For the plane $F_y = ma_y$

$$a_y = \frac{F_L}{m} = \frac{14{,}200\ lbf}{2000\ lbm} \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}$$

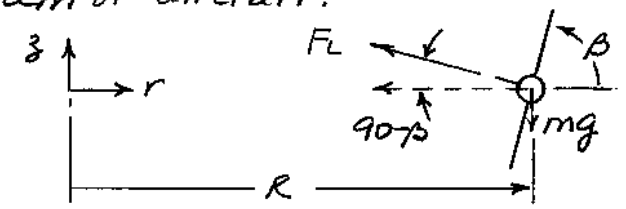
$$a_y = 229\ ft/s^2 = 7.1\ g$$

This is the initial instantaneous vertical acceleration. The increased drag would slow the plane below V_{max}.

Given: USAF F-16 with $A_p = 27.9\ m^2$ and $C_{L,max} = 1.6$, at maximum gross mass of $M = 11{,}600\ kg$. Turn flown level with aircraft banked.

Find: (a) Minimum speed in standard air for $a_t = 5g$.
(b) Corresponding radius.
(c) Discuss effect of altitude.

Solution: Draw free-body diagram of aircraft:



Computing equations:

$$C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A_p} \qquad a_r = -\frac{V^2}{R}$$

Assume: (1) Standard air, $\rho = 1.23\ kg/m^3$
(2) Pilot feels a_t along F_L

Then $F_L = ma_t = C_L A \frac{1}{2}\rho V^2$ or $V = \sqrt{\frac{2ma_t}{C_L A \rho}}$ is minimum at $C_{L,max}$.

$$V_{min} = \left[2 \times 11{,}600\ kg \times (5)\, 9.81 \frac{m}{s^2} \times \frac{1}{1.6} \times \frac{1}{27.9\ m^2} \times \frac{m^3}{1.23\ kg}\right]^{1/2}$$

$V_{min} = 144\ m/s$ (minimum speed) ← V_{min}

Need a_r to find V. Sum forces vertically

$$\Sigma F_z = F_L \sin(90-\beta) - mg = ma_z = 0$$

$$\sin(90-\beta) = \cos\beta = \frac{mg}{F_L} = \frac{mg}{5mg} = \frac{1}{5}\ ;\ \beta = \cos^{-1}\left(\frac{1}{5}\right) = 78.5^\circ$$

Sum forces radially:

$$\Sigma F_r = -F_L \cos(90-\beta) = ma_r = m\left(-\frac{V^2}{R}\right)$$

$$R = \frac{mV^2}{F_L \sin\beta} = \frac{mV^2}{5mg\sin\beta} = \frac{V^2}{5g\sin\beta}$$

$$= \frac{1}{5} \times (144)^2 \frac{m^2}{s^2} \times \frac{s^2}{9.81\ m} \times \frac{1}{\sin 78.5^\circ}$$

$R = 431\ m$ ← R

As altitude increases, density decreases, and V is raised. This also increases R. At $z = 15\ km$, $\rho/\rho_0 = 0.159$. Thus

$$\frac{V}{V_0} = \sqrt{\frac{\rho_0}{\rho}} = 2.51 \quad \text{and} \quad \frac{R}{R_0} = \frac{V^2}{V_0^2} = 6.29$$

Given: Light plane with NACA 23015 airfoil, $S = 10\ m$, $C = 1.8\ m$, cruises at $V = 225\ km/hr$ near sea level on a standard day.

Find: Determine cruise speed with NACA 66$_2$-215 section airfoil.

Solution: Apply definitions of coefficients, use data from Fig. 9.19.

Computing equations: $C_D = C_{D,\infty} + C_{D,i} = C_{D,\infty} + \frac{C_L^2}{\pi ar}$

From a free-body diagram $F_L = W$, $P = F_D V/\eta_p$

From Fig. 9.19, recognize airfoils should operate near design lift coefficients. Thus assume:

Section	C_L	$C_{D,\infty}$
23015	0.3	0.0062
66$_2$-215	0.2	0.0031

$ar = \frac{S}{C} = \frac{10\ m}{1.8\ m} = 5.56$

Thus $C_{D_{old}} \approx 0.0062 + \frac{(0.3)^2}{\pi(5.56)} = 0.0062 + 0.00515 = 0.0114$

$$C_{D_{new}} \approx 0.0031 + \frac{(0.2)^2}{\pi(5.56)} = 0.0031 + 0.00229 = 0.00539$$

Since for level flight,

$$P = F_D V/\eta_p = \frac{C_D A \frac{1}{2}\rho V^2 V}{\eta_p}, \text{ then } V = \left[\frac{2\eta_p P}{C_D A}\right]^{\frac{1}{3}}$$

Assuming $\eta_p P$ remains constant, then $V_{new} = V_{old}\left[\frac{C_{D,old}}{C_{D,new}}\right]^{\frac{1}{3}}$

$$V_{new} \approx 225\ \frac{km}{hr}\left[\frac{0.0114}{0.00539}\right]^{\frac{1}{3}} = 289\ km/hr$$ V_{new}

Check assumption on C_L: since $F_L = W = C_L A \frac{1}{2}\rho V^2$, then

$$C_{L_{new}} \approx C_{L_{old}}\left[\frac{V_{old}}{V_{new}}\right]^2 = 0.3\left[\frac{225}{289}\right]^2 = 0.182$$

Therefore the above estimate for new cruise speed is probably conservative.

Given: Boeing 727 aircraft, with NACA 2301c section, $A_p = 1600\ ft^2$, and effective aspect ratio, $ar = 6.5$. Aircraft flies at $V = 450$ kt, with $W = 175{,}000$ lbf.

Find: Estimate thrust needed to maintain steady, level flight.

Solution: For steady, level flight, thrust equals drag and lift equals weight.

Computing equations: $F_L = W = C_L \frac{1}{2}\rho V^2 A$ (1)

$$F_D = T = C_D \frac{1}{2}\rho V^2 A \quad (2)$$

$$C_D = C_{D,\infty} + C_{D,i} = C_{D,0} + \frac{C_L^2}{\pi ar}$$

Assumptions: (1) Standard air
(2) Data from Fig. 9.23 apply

$$V = 150 \frac{nm}{hr} \times 6076 \frac{ft}{nm} \times \frac{hr}{3600\ sec} = 253\ ft/sec$$

$$q = \frac{1}{2}\rho V^2 = \frac{1}{2} \times 0.00238 \frac{slug}{ft^3} \times (253)^2 \frac{ft^2}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} = 76.2\ lbf/ft^2$$

From Eq. 1, $C_L = \frac{W}{qA} = 175000\ lbf \times \frac{ft^2}{76.2\ lbf} \times \frac{1}{1600\ ft^2} = 1.44$

From Fig. 9.23, this corresponds to operation with a single slot open, and $C_{D,0} \cong 0.04$. Thus

$$C_D = C_{D,0} + \frac{C_L^2}{\pi ar} = 0.04 + \frac{(1.44)^2}{\pi(6.5)} = 0.142$$

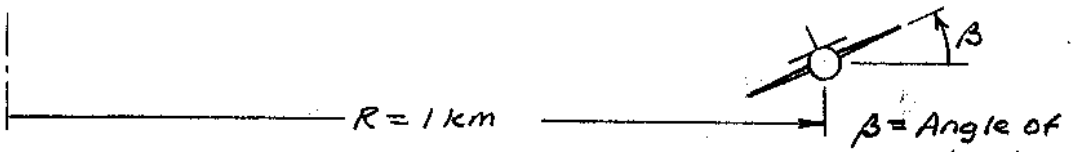
To find thrust, note

$$\frac{T}{F_L} = \frac{C_D}{C_L}\frac{qA}{qA} = \frac{C_D}{C_L} = \frac{0.142}{1.44} = 0.0986$$

Thus

$$T = F_L \frac{C_D}{C_L} = W \frac{C_D}{C_L} = 175{,}000\ lbf \times 0.0986 = 17{,}300\ lbf \quad T \leftarrow$$

Given: Aircraft with gross mass, $m = 4500$ kg, flown in a circular path at 250 kph. The plane has an effective lift area of 22.0 m^2, and is fitted with NACA 23015 section airfoils. $ar = 7$.

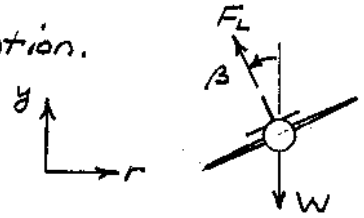


Estimate:

(a) Drag on aircraft.

(b) Power required to maintain level flight.

Solution: Apply Newton's second law of motion.



Basic Equation: $\Sigma \vec{F} = m\vec{a}$

Assumptions: (1) Level flight, so $a_y = 0$

(2) $a_r = -V^2/R$

Using the free-body diagram shown,

$$\Sigma F_y = F_L \cos\beta - mg = 0 \quad \text{or} \quad F_L \cos\beta = mg$$

$$\Sigma F_r = -F_L \sin\beta = ma_r = -m\frac{V^2}{R}, \text{ or } F_L \sin\beta = \frac{mV^2}{R}$$

Thus $$\frac{F_L \sin\beta}{F_L \cos\beta} = \tan\beta = \frac{V^2}{Rg} = \left(250\times10^3 \frac{m}{hr} \times \frac{hr}{3600\,s}\right)^2 \frac{1}{10^3 m} \times \frac{s^2}{9.81\,m} = 0.492; \quad \beta = 26.2°$$

$$F_L = \frac{mg}{\cos\beta} = 4500\,kg \times 9.81\frac{m}{s^2} \times \frac{1}{\cos 26.2°} \times \frac{N\cdot s^2}{kg\cdot m} = 49.2\,kN$$

From the definition, $C_L \equiv \dfrac{F_L}{\frac{1}{2}\rho V^2 A_p} = \dfrac{2F_L}{\rho V^2 A_p}$, so

$$C_L = 2 \times 4.92\times10^4\,N \times \frac{m^3}{1.23\,kg}\left(\frac{hr}{250\times10^3 m} \times 3600\frac{s}{hr}\right)^2 \frac{1}{22\,m^2} \times \frac{kg\cdot m}{N\cdot s^2} = 0.754$$

The drag coefficient is

$$C_D = C_{D,0} + C_{D,i} = C_{D,0} + \frac{C_L^2}{\pi ar}$$

For the section, $C_{D,0} = 0.007$ at $C_L = 0.754$ (Fig. 9.19). Thus

$$C_D = 0.007 + \frac{(0.754)^2}{\pi(7)} = 0.007 + 0.0259 = 0.0329$$

$$F_D = \frac{C_D}{C_L} F_L = \frac{0.0329}{0.754} \times 49.2\,kN = 2.15\,kN \qquad F_D$$

The power is

$$\mathbb{P} = F_D V = 2.15\times10^3 N \times 250\times10^3 \frac{m}{hr} \times \frac{hr}{3600\,s} \times \frac{W\cdot s}{N\cdot m} = 149\,kW \qquad \mathbb{P}$$

{Any additional drag due to skin friction on fuselage or other surfaces, and interference drag, are neglected in this calculation.}

1102

Given: Chaparral 2F with rear-mounted airfoil having span, $s = 6$ ft, and chord, $c = 1$ ft. Lift and drag coefficients same as conventional section in Fig. 9.17. Consider $V = 120$ mph (calm day), $\alpha = -12°$ (down).

Find: (a) Maximum downward force.
(b) Maximum increase in deceleration force.

Solution: Apply definitions of C_L and C_D.

Computing equations: $C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A}$ $C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} = C_{D,0} + \frac{C_L^2}{\pi ar}$ $ar = \frac{s}{c}$

From Fig. 9.17, at $\alpha = 12°$, $C_L = 1.4$ and $C_{D,0} = 0.013$. Thus, since $\alpha = -12°$,

$$F_L = -C_L A \tfrac{1}{2}\rho V^2 \qquad A = sc = 6\ ft^2$$

$$= -1.4 \times 6\ ft^2 \times \frac{1}{2} \times 0.00238 \frac{slug}{ft^3} \times \left(120 \frac{mi}{hr} \times 5280 \frac{ft}{mi} \times \frac{hr}{3600\ s}\right)^2 \frac{lbf \cdot s^2}{slug \cdot ft}$$

$F_L = -310$ lbf (downward force) ← F_L

Then $F_D = F_L \frac{C_D}{C_L} = \frac{0.013 + \frac{(1.4)^2}{\pi(6)}}{1.4} \times 310\ lbf = 25.9\ lbf$

Braking thrust increases as drag increases and as normal force increases tire adhesion (friction). Thus

$$\Delta F_B = \mu_k F_L + F_D$$

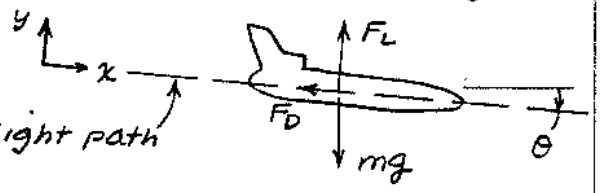
For $\mu_k = 1.0$ (probably conservative for racing tires),

$\Delta F_B = 1.0 \times 310\ lbf + 25.9\ lbf = 336\ lbf$ ← ΔF_B

Given: Unpowered flight with lift, drag, and weight in equilibrium.

Find: (a) Show glide slope angle is $\tan\theta = C_D/C_L$
(b) Evaluate minimum glide slope angle for Boeing 727-200 of Example Problem 9.8.
(c) Glide distance from altitude of 10 km on a standard day.

Solution: Consider free-body diagram:



Sum forces along (x) and normal to (y) flight path:

$$\Sigma F_x = -F_D + mg\sin\theta = 0 \qquad mg\sin\theta = F_D$$
$$\Sigma F_y = F_L - mg\cos\theta = 0 \qquad mg\cos\theta = F_L$$
$$\Rightarrow \tan\theta = \frac{F_D}{F_L} = \frac{C_D}{C_L}$$ tanθ

Use relationships from Section 9-8:

Computing equation: $C_D = C_{D,0} + \frac{C_L^2}{\pi ar} = C_{D,0} + C_{D,i}$

Thus $$\frac{C_D}{C_L} = \frac{C_{D,0}}{C_L} + \frac{C_L}{\pi ar} \qquad (1)$$

To minimize, set $d(C_D/C_L)/dC_L = 0$

$$\frac{d}{dC_L}\left(\frac{C_D}{C_L}\right) = (-1)\frac{C_{D,0}}{C_L^2} + \frac{1}{\pi ar} = 0 \quad \text{when} \quad C_{D,0} = \frac{C_L^2}{\pi ar} = C_{D,i}$$

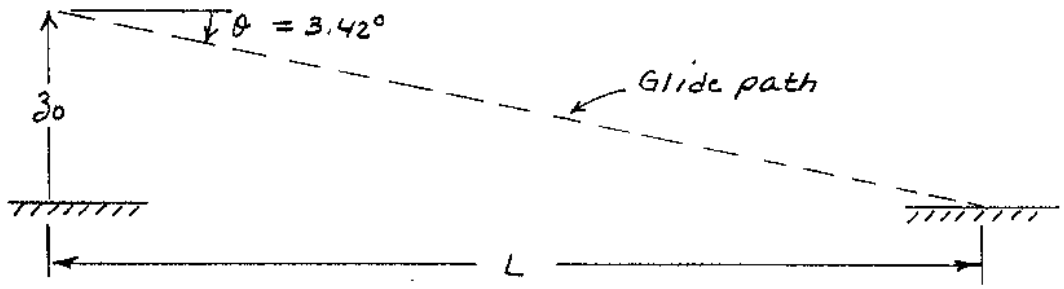
From Example Problem 9.8, $C_{D,0} = 0.0182$ and $ar = 6.5$. Thus optimum is

$$C_L = (\pi ar\, C_{D,0})^{1/2} = [\pi(6.5)\,0.0182]^{1/2} = 0.610$$

and from Eq. 1,

$$\frac{C_D}{C_L} = \frac{0.0182}{0.61} + \frac{0.61}{\pi(6.5)} = 0.0597 = \tan\theta\ ;\ \theta = \tan^{-1}(0.0597) = 3.42^\circ$$ θ

Note θ is independent of atmospheric conditions. Thus θ = constant



$$\frac{z_0}{L} = \tan\theta\ ;\ L = \frac{z_0}{\tan\theta} = \frac{10\text{ km}}{0.0597} = 168\text{ km}$$ L

Problem 9.146

Open-Ended Problem Statement: The hood ornament of a car is decorative, but it increases the aerodynamic drag of the car. Estimate the cost of the increased fuel consumption over the life of a vehicle caused by a typical hood ornament.

Analysis: Assume $C_D = 1.2$ (Table 9.3), $A = wh = 1\text{ in.} \times 2\text{ in.} \times \frac{\text{ft}^2}{144\text{ in.}^2} = 0.0139\text{ ft}^2$; $C_DA = 0.0167\text{ ft}^2$

Assume $V = 50$ mph (73.3 ft/s): $q = \frac{1}{2}\rho V^2 = \frac{1}{2} \times 0.00238 \frac{\text{slug}}{\text{ft}^3} \times (73.3)^2 \frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf-s}^2}{\text{slug-ft}} = 6.40\text{ lbf/ft}^2$

Then $F_D = C_DA\,\frac{1}{2}\rho V^2 = C_DA\,q = 0.0167\text{ ft}^2 \times 6.40 \frac{\text{lbf}}{\text{ft}^2} = 0.107\text{ lbf}$

The energy dissipated by the ornament is

$$E = X F_D = 100{,}000\text{ mi} \times 0.107\text{ lbf} \times 5280 \frac{\text{ft}}{\text{mi}} \times \frac{\text{Btu}}{778\text{ ft-lbf}} = 72{,}600\text{ Btu}$$

Assume the drivetrain thermal efficiency is 20 percent. The fuel energy is

$$Q = \frac{E}{\eta_t} = 72{,}600\text{ Btu} \times \frac{1}{0.20} = 363{,}000\text{ Btu}$$

The fuel volume is

$$\forall = 363{,}000\text{ Btu} \times \frac{\text{lbm}}{18000\text{ Btu}} \times \frac{\text{ft}^3}{(0.72)\,62.4\text{ lbm}} \times 7.48 \frac{\text{gal}}{\text{ft}^3} = 3.36\text{ gal} \quad \text{or} \sim \$3.50$$

$\forall$, C

At 25 mpg, the auto consumes $100{,}000\text{ mi} \times \frac{\text{gal}}{25\text{ mi}} = 4000\text{ gal}$ of fuel

Thus the ornament uses $3.36/4000 = 0.00084$ or 0.084% of fuel.

%

Discussion: The hood ornament is relatively small compared to the frontal area of the typical car. Although it is shaped like a flat plate oriented perpendicular to the flow, and may have a high drag coefficient, its C_DA product is very small compared to that of the entire car (which might have $C_D = 0.3$ and $A = 2.5\text{ m}^2$, for a C_DA product of 0.75 m^2).

The extra fuel consumption that can be attributed to the hood ornament can be calculated, but it is negligible compared to the fuel used to propel the auto.

Beginning with 1998 models, many manufacturers in the U.S. and abroad have begun to omit the hood ornaments on new cars. This is more likely a cost-cutting and/or anti-theft move rather than one intended to enhance fuel economy!

Given: Man-powered aircraft, the Gossamer Condor:

$W/A = 0.4$ lbf/ft^2 $\quad W = 200$ lbf $\quad ar = 17$ $\quad F_D = 6$ lbf at 12 mph

Pilot could sustain 0.39 hp for 2 hr.

Find: (a) Minimum power to fly aircraft.
(b) Compare to pilot output capability.

Solution: Apply relationships from Section 9-8:

Computing equations: $W = F_L = C_L A \frac{1}{2}\rho V^2 \qquad P = V F_D$

$$T = F_D = C_D A \tfrac{1}{2}\rho V^2 \; ; \; C_D = C_{D,0} + \frac{C_L^2}{\pi ar}$$

The task is to find V to minimize P:

$$P = V F_D = V\left(C_D A \tfrac{1}{2}\rho V^2\right) = \left(C_{D,0} + \frac{C_L^2}{\pi ar}\right) A \tfrac{1}{2}\rho V^3 \qquad (1)$$

But C_L varies with aircraft speed:

$$F_L = W = C_L A \tfrac{1}{2}\rho V^2 \; ; \; C_L = \frac{2W}{\rho V^2 A} \; ; \; C_L^2 = \left(\frac{2W}{\rho A}\right)^2 \frac{1}{V^4}$$

Substituting into Eq. 1,

$$P = \left[C_{D,0} + \frac{1}{\pi ar}\left(\frac{2W}{\rho A}\right)^2 \frac{1}{V^4}\right] A \tfrac{1}{2}\rho V^3$$

To minimize power, set $dP/dV = 0$. Then

$$\frac{dP}{dV} = C_{D,0} A \tfrac{1}{2}\rho V^2 (3) + (-1)\frac{1}{\pi ar}\left(\frac{2W}{\rho A}\right)^2 \frac{1}{V^2} A \tfrac{1}{2}\rho = 0$$

Thus at minimum power,

$$3C_{D,0} = \frac{1}{\pi ar}\left(\frac{2W}{\rho A}\right)^2 \frac{1}{V^4} = \frac{C_L^2}{\pi ar} = C_{D,i} \quad \text{or} \quad C_{D,i} = 3C_{D,0} \quad \text{minimum power}$$

From given data, at 12 mph (17.6 ft/s):

$$\tfrac{1}{2}\rho V^2 = \tfrac{1}{2} \times 0.00238 \frac{\text{slug}}{\text{ft}^3} \times (17.6)^2 \text{ft}^2/\text{s}^2 \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} = 0.369 \text{ lbf/ft}^2$$

$$C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A} = \frac{W/A}{\frac{1}{2}\rho V^2} = 0.4 \frac{\text{lbf}}{\text{ft}^2} \times \frac{\text{ft}^2}{0.369 \text{ lbf}} = 1.08 \; ; \; A = \frac{W}{W/A} = 200 \text{ lbf} \times \frac{\text{ft}^2}{0.4 \text{ lbf}} = 500 \text{ ft}^2$$

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} = 6 \text{ lbf} \times \frac{\text{ft}^2}{0.369 \text{ lbf}} \times \frac{1}{500 \text{ ft}^2} = 0.0325$$

$$C_{D,0} = C_D - C_{D,i} = C_D - \frac{C_L^2}{\pi ar} = 0.0325 - \frac{(1.08)^2}{\pi(17)} = 0.0167 \quad (C_{D,0} = \text{constant})$$

At flight speed for minimum power, $C_{D,i} = 3C_{D,0}$

Thus at minimum power

$$C_{D,i} = \frac{C_L^2}{\pi ar} = 3C_{D,0} = 3(0.0107) = 0.0321$$

so

$$C_L = (0.0321\,\pi ar)^{1/2} = 1.31 \quad \text{(minimum power)}$$

since

$$F_L = W = C_L A \tfrac{1}{2}\rho V^2$$

then

$$V = \sqrt{\frac{2W}{C_L \rho A}} \quad \text{and} \quad \frac{V_{min}}{V} = \sqrt{\frac{C_L}{C_{L\,min}}}$$

Thus at minimum power

$$V_{min} = 12 \text{ mph} \sqrt{\frac{1.08}{1.31}} = 10.9 \text{ mph } (16.0 \text{ ft/s}) \quad \text{(minimum power)}$$ V_{min}

The power requirement would be

$$P_{pilot} = \frac{P_{flight}}{\eta_{drive}\,\eta_{prop}}$$

$$P_{flight} = VF_D = V C_D A \tfrac{1}{2}\rho V^2 = (C_{D,0} + 3C_{D,0}) A \tfrac{1}{2}\rho V^3 = 2C_{D,0} A \rho V^3$$

$$= 2(0.0107)\,500 \text{ ft}^2 \times 0.00238 \frac{\text{slug}}{\text{ft}^3} \times (16.0)^3 \frac{\text{ft}^3}{\text{s}^3} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{hp}\cdot\text{s}}{550 \text{ ft}\cdot\text{lbf}}$$

$P_{flight} = 0.190$ hp (power for flight)

If $\eta_{drive} = 0.9$ and $\eta_{prop} = 0.7$,

$$P_{pilot} \approx \frac{0.190}{(0.9)(0.7)} = 0.302 \text{ hp} \quad \text{(minimum power)}$$ P_{pilot}

Thus $P_{pilot} < 0.39$ hp!

Open-Ended Problem Statement: How does a Frisbee™ fly? What causes it to curve left or right? What is the effect of spin on its flight?

Discussion: When viewed from the side, the Frisbee shape has a rounded upper surface and a flat bottom surface. Such a shape is capable of generating lift as it travels through air.

When a Frisbee is not spinning, the lift vector probably acts slightly forward of the maximum thickness on the profile. When spinning, the motion of the surface likely affects the development and separation of the boundary layers. This may displace the center of lift slightly to the right or left of center, depending on the direction of spin.

A Frisbee is not stable when thrown without spin: it will tend to tumble as it moves through the air. Spin is used to stabilize the motion (just as a spinning gyroscope tends to remain upright). The combination of spin and the off-center lift vector cause the Frisbee to precess as a gyroscope. Therefore its spin axis can change from vertical while in flight, causing the flight path to curve right or left.

The Frisbee also can be thrown intentionally to curve right or left. This is done by inclining the spin axis so that it is not vertical at launch. When the spin axis is inclined to the left (as seen by the thrower), the Frisbee drifts to the left along a more-or-less constant radius path. Inclining the spin axis to the right causes the opposite effect.

Open-Ended Problem Statement: An automobile travels down the road with a bicycle attached to a carrier across the rear of the trunk. The bicycle wheels rotate slowly. Explain why and in what direction the rotation occurs.

Discussion: All objects moving in ground effect generate lift (the air flows over the top faster than over the bottom because of the shape of the automobile). Any object that produces lift carries with it a bound vortex that creates circulation about the profile that accompanies lift.

The bound vortex creates two trailing vortices, one on each side of the car, which rotate in opposite directions as they follow in the wake of the automobile. When viewed from the rear of the auto, the left side trailing vortex rotates clockwise and the right side trailing vortex rotates counterclockwise.

The swirl in the trailing vortex motion is responsible for the motion of the bicycle wheels (check it out on your next auto trip during the summer months!). The swirl causes shear stresses that tend to rotate the bicycle wheels in the same senses as the trailing vortices. Again viewing from the rear of the auto, the left wheel rotates clockwise and the right wheel counterclockwise.

(Sometimes the rear wheel of the bicycle cannot freewheel. In this case only the front wheel turns slowly as the car drives down the road.)

13-782 500 SHEETS FILLER 5 SQUARE
42-381 96 SHEETS EYE-EASE® 5 SQUARE
42-389 100 SHEETS EYE-EASE® 5 SQUARE
42-3899 200 SHEETS EYE-EASE® 5 SQUARE
42-3899 500 RECYCLED (50) APRICOT 5 SQUARE
43-3899 500 RECYCLED (50) WHITE 5 SQUARE
43-399 200 RECYCLED WHITE 5 SQUARE
Made in U.S.A.
National® Brand

Open-Ended Problem Statement: Roadside signs tend to oscillate in a twisting motion when a strong wind blows. Discuss the phenomena that must occur to cause this behavior.

Discussion: Many roadside signs are mounted on a single post formed from stamped steel. The post has an open "C" cross-section, which provides little torsional rigidity. Wind gusts can excite oscillations in a sign, which acts as a flat plate at an angle of attack relative to the oncoming wind. When at an angle of attack, a plate develops both a lift force and a moment that tends to twist the sign farther from its equilibrium position. While the sign twists, the post provides a resisting torque as a result of being twisted from its equilibrium position.

As the sign twists, the angle of attack relative to the oncoming air increases. An overshoot phenomenon called dynamic stall allows the flow to remain attached and the angle of attack to grow larger before stall occurs than if the change in angle of attack had been slow and gradual. Once stall occurs, the lift force and moment decrease, and the motion is no longer forced. Then the sign tends to return to its undisturbed position.

The moment of inertia of the sign causes it to overshoot the equilibrium position. The sign continues beyond equilibrium and develops a lift force and a moment tending to move it farther past equilibrium. The process repeats, with growing amplitude, until a more-or-less steady-state oscillation is reached.

The sign and post form a spring-mass-damper mechanical system. The sign is the mass, the post is the spring, and hysteresis and aerodynamic resistance to oscillation provide the damping. The "steady" oscillation occurs near the natural frequency of the system.

At steady state, the rate at which energy is added to the sign by the gusting wind exactly balances the rate at which energy is dissipated by hysteresis in the sign motion and its supporting post. The oscillations can continue almost indefinitely, and with considerable amplitude, as can be observed on a windy day. In some cases the oscillations lead to fatigue failure of the sign post.

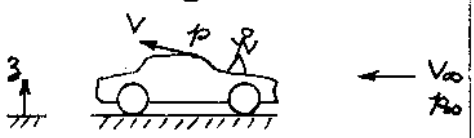
Given: Air moving over automobile, as shown in Fig. 9.25.

Find: (a) Estimate pressure reduction in car when a window is "cracked" while traveling at $V_\infty = 100$ km/hr.
(b) Air speed in freestream near window opening.

Solution: Apply the Bernoulli equation and pressure coefficient definition.

Basic equations: $\frac{p_\infty}{\rho} + \frac{V_\infty^2}{2} + g\cancel{z_\infty} = \frac{p}{\rho} + \frac{V^2}{2} + g\cancel{z}$ $\qquad C_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2}$

Assumptions: (1) Steady flow seen from auto
(2) Incompressible flow
(3) No friction
(4) Flow along a streamline
(5) Neglect changes in elevation



From Fig. 9.25, C_p near driver's window ranges between −1.23 and −0.40.

At $V_\infty = 100$ km/hr,

$$q = \frac{1}{2}\rho V_\infty^2 = \frac{1}{2} \times 1.23 \frac{kg}{m^3} \times \left[100 \frac{km}{hr} \times 1000 \frac{m}{km} \times \frac{hr}{3600 s} \right]^2 \frac{N \cdot s^2}{kg \cdot m} = 475 \, N/m^2$$

Thus the pressures outside may be between

$$p - p_\infty = C_p \frac{1}{2}\rho V_\infty^2 = -1.23 \times 475 \frac{N}{m^2} = -584 \, N/m^2 \text{ (gage)}$$

and $$p - p_\infty = -0.40 \times 475 \frac{N}{m^2} = -190 \, N/m^2 \text{ (gage)} \longleftarrow p$$

From the Bernoulli equation,

$$\frac{V^2}{2} = \frac{V_\infty^2}{2} + \frac{p_\infty - p}{\rho} = \frac{V_\infty^2}{2}\left(1 - \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2}\right) = \frac{V_\infty^2}{2}(1 - C_p)$$

Thus
$$V = V_\infty \sqrt{1 - C_p}$$

The local flow speeds range from

$$V = V_\infty \sqrt{1 - (-1.23)} = 100 \frac{km}{hr} \sqrt{2.23} = 149 \text{ km/hr } (41.5 \text{ m/s})$$

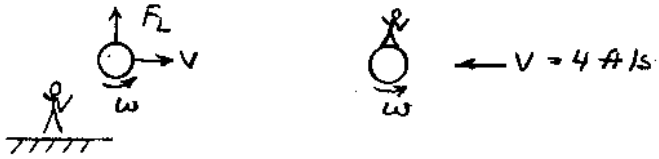
$$V = V_\infty \sqrt{1 - (-0.40)} = 100 \frac{km}{hr} \sqrt{1.40} = 118 \text{ km/hr } (32.9 \text{ m/s}) \longleftarrow V$$

Thus local flow speeds are significantly higher than V_∞.

Given: Classroom demonstration of lift on spinning cylinder.

$L = 10$ in., $D = 2$ in.

$\omega = 300$ rpm



Find: Estimate lift force acting on cylinder.

Solution: Apply definition of lift coefficient, data from Fig. 9.29.

Computing equation: $F_L = C_L A \frac{1}{2}\rho V^2$

From Fig. 9.29, $C_L = C_L(\omega D/2V)$

$$\omega = 300 \frac{\text{rev}}{\text{min}} \times 2\pi \frac{\text{rad}}{\text{rev}} \times \frac{\text{min}}{60\ \text{s}} = 31.4\ \text{rad/s}$$

$$\frac{\omega D}{2V} = \frac{1}{2} \times 31.4 \frac{\text{rad}}{\text{s}} \times 2\ \text{in.} \times \frac{\text{s}}{4\ \text{ft}} \times \frac{\text{ft}}{12\ \text{in.}} = 0.654$$

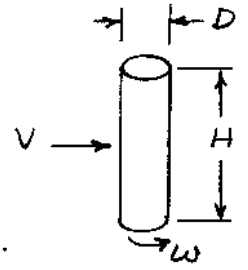
There is a data band in Fig. 9.29. The highest value is $C_L \approx 1.1$

$$A = DL = 2\ \text{in.} \times 10\ \text{in.} \times \frac{\text{ft}^2}{144\ \text{in.}^2} = 0.139\ \text{ft}^2$$

For standard atmosphere conditions:

$$F_L = 1.1 \times 0.139\ \text{ft}^2 \times \frac{1}{2} \times 0.00238 \frac{\text{slug}}{\text{ft}^3} \times (4)^2 \frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} = 0.00291\ \text{lbf}$$ F_L

{This is quite a small force, but the speed is low.}



Given: Rotating cylinders for ship propulsion.

$D = 3\ m,\ H = 15\ m,\ N \le 750\ rpm$

Find: (a) Calculate maximum lift and drag forces on each cylinder at $V = 50\ km/hr$.
(b) Compare to force at optimum L/D.
(c) Estimate power needed to spin rotor at $N = 750\ rpm$.

Solution: Apply definitions of coefficients, data from Fig. 9.29.

Computing equations: $C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A}$ $C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$ $A = DH$

Coefficients are functions of spin ratio, $\omega D/2V$.

$$\omega = 750 \frac{rev}{min} \times 2\pi \frac{rad}{rev} \times \frac{min}{60\ s} = 78.5\ rad/s\ ; \quad V = 50 \frac{km}{hr} \times 1000 \frac{m}{km} \times \frac{hr}{3600\ s} = 13.9 \frac{m}{s}$$

$$\frac{\omega D}{2V} = \frac{1}{2} \times 78.5 \frac{rad}{s} \times 3\ m \times \frac{s}{13.9\ m} = 8.47\ ; \quad q = \frac{1}{2}\rho V^2 = \frac{1}{2} \times 1.23 \frac{kg}{m^3} \times (13.9)^2 \frac{m^2}{s^2} = 119\ N/m^2$$

This is above the range of data shown in Fig. 9.29, so ω could be less.
Choose $C_L \approx 9.5$, $C_D \approx 3.5$. $A = DH = 45\ m^2$

$$F_L = 9.5 \times 45\ m^2 \times 119 \frac{N}{m^2} = 50.9\ kN \qquad F_L$$

$$F_D = 3.5 \times 45\ m^2 \times 119 \frac{N}{m^2} = 18.7\ kN \qquad F_D$$

From inspection of Fig. 9.29, this appears close to the optimum L/D.
Then

$$F = \left[F_L^2 + F_D^2\right]^{\frac{1}{2}} = \left[(50.9)^2 + (18.7)^2\right]^{\frac{1}{2}} kN = 54.2\ kN \qquad F$$

Torque will be product of mean shear stress, $\bar{\tau}$, area, and radius.

$$T = \bar{\tau} A R = \bar{\tau} (\pi D H) R = \frac{\pi \bar{\tau} D^2 H}{2}$$

Estimate (very roughly) $\bar{\tau}$ from flat plate correlation. Assume

$$Re = \frac{(V+U)D}{\nu} = \frac{(V+\omega R)D}{\nu} = (13.9 + 118) \frac{m}{s} \times 3\ m \times \frac{s}{1.45 \times 10^{-5} m^2} = 2.73 \times 10^7$$

From Fig. 9.8, $C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} = 0.003$; $\bar{\tau} = \frac{F_D}{A} = C_D q = 0.003 \times 119 \frac{N}{m^2} = 0.357 \frac{N}{m^2}$

Then

$$T = \frac{\pi}{2} \times 0.357 \frac{N}{m^2} \times (3)^2 m^2 \times 15\ m = 75.7\ N \cdot m$$

and

$$\mathcal{P} = \omega T = 78.5 \frac{rad}{s} \times 75.7\ N \cdot m \times \frac{W/s}{N \cdot m} = 5.94\ kW \qquad \mathcal{P}$$

{ The power estimate is gross only. More specific information is needed to design with confidence. }

Given: Golf ball with mass, $m = 48\,g$, and diameter, $D = 43\,mm$, is hit from a sand trap with speed, $V = 20\,m/sec$, and backspin, $\omega = 2000\,rpm$.

Find: (a) Lift and drag forces acting on ball.
(b) Express as fractions of mg.

Solution: Use data from Fig. 9.28 for lift and drag coefficients.

Computing equations: $F_L = C_L A \frac{1}{2}\rho V^2 \qquad F_D = C_D A \frac{1}{2}\rho V^2$

At $V = 20\,m/sec$, then $q = \frac{1}{2}\rho V^2 = \frac{1}{2} \times 1.23 \frac{kg}{m^3} \times (20)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 246\,N/m^2$

From Fig. 9.28, $C_L = C_L\left(Re, \frac{\omega D}{2V}\right)$

$$Re = \frac{VD}{\nu} = 20\frac{m}{s} \times 0.043\,m \times \frac{s}{1.45 \times 10^{-5} m^2} = 5.93 \times 10^4 \text{ (assume close to } 1.26 \times 10^5)$$

$$\frac{\omega D}{2V} = \frac{1}{2} \times 2000 \frac{rev}{min} \times 0.043\,m \times \frac{s}{20\,m} \times 2\pi \frac{rad}{rev} \times \frac{min}{60\,s} = 0.225$$

Then $C_L \simeq 0.23$ and $F_L = C_L A \frac{1}{2}\rho V^2 = C_L A\, q$

$$F_L = 0.23 \times \frac{\pi (0.043)^2 m^2}{4} \times 246 \frac{N}{m^2} = 0.0822\,N \qquad F_L$$

From Fig. 9.28 $C_D = 0.31$, so

$$F_D = \frac{C_D}{C_L} F_L = \frac{0.31}{0.23} \times 0.0822\,N = 0.111\,N \qquad F_D$$

For the ball, $mg = 0.048\,kg \times 9.81 \frac{m}{sec^2} \times \frac{N \cdot sec^2}{kg \cdot m} = 0.471\,N$

Thus $\frac{F_L}{mg} = \frac{0.0822\,N}{0.471\,N} = 0.175$

$\frac{F_D}{mg} = \frac{0.111\,N}{0.471\,N} = 0.236$ Ratios

Given: American and British golf balls (Problems 1.11 and 1.12), dimensions below. Hit from tee at V = 85 m/sec, with backspin, N = 9000 rpm.

Find: (a) Evaluate lift and drag forces on each ball (express as fractions of body force).
(b) Estimate radius of curvature of trajectory.
(c) Which ball would have the longer range?

Solution: Apply definitions of lift and drag coefficients, data from Fig. 9.28.

Computing equations: $C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A} = \frac{F_L}{qA}$; $C_D = \frac{F_D}{qA}$; $q = \frac{1}{2}\rho V^2$; $A = \frac{\pi D^2}{4}$

The parameters are Re_D and $\omega D/2V$; tabulate results:

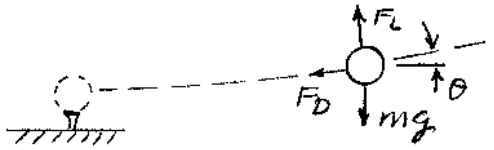
Ball	m (oz)	D (in.)	$\frac{\omega D}{2V}$ (---)	Re (---)	C_L (---)	C_D (---)	F_L (N)	F_D (N)
American	1.62	1.68	0.236	2.50×10^5	0.27	0.32	1.71	2.03
British	1.62	1.62	0.228	2.41×10^5	0.26	0.31	1.53	1.83

Taking ratios for the American ball: British ball:

$F_L/mg = 1.71\,N \times \frac{1}{1.62\,oz} \times \frac{16\,oz}{1\,lb} \times \frac{1\,lb}{4.448\,N} = 3.80$ 3.40 F_L/mg

$F_D/mg = 4.51$ 4.07 F_D/mg

Draw FBD to compute trajectory:



$\Sigma F_{\perp \text{to path}} = F_L - mg\cos\theta = \frac{mV^2}{R}$

Assume θ small, so $\cos\theta \approx 1$. Then

$R \approx \frac{mV^2}{F_L - mg} = \frac{V^2/g}{F_L/mg - 1} = \frac{1}{3.80-1} \times (85)^2 \frac{m^2}{s^2} \times \frac{s^2}{9.81\,m} = 263\ m$ (American)

$= \frac{1}{3.40-1} \times (85)^2 \frac{m^2}{s^2} \times \frac{s^2}{9.81\,m} = 307\ m$ (British) R

(Note because $F_L/mg > 1$, the balls actually rise!)

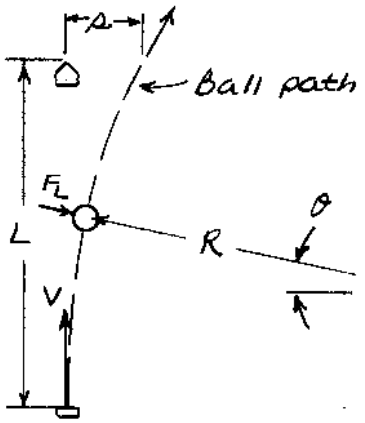
Drag probably is more important than lift in affecting range of a drive. Therefore one probably would expect the British ball to carry farther. Range

Given: Baseball, with $m = 145$ g and $D = 74$ mm, thrown at $V = 90$ km/hr from pitcher's mound to home plate, $L = 18$ m distant.

Find: (a) Spin rate for maximum horizontal deviation from straight path.
(b) How far will ball deviate?

Solution: Ball will curve if spun due to lift. Apply data from Fig. 9.27.

Computing equations: $\Sigma F_r = ma_r$ $\quad C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A}$



Summing forces in the radial direction,

$$\Sigma F_r = -F_L = ma_r = m\left(-\frac{V^2}{R}\right)$$

Thus $F_L = C_L A \frac{1}{2}\rho V^2 = m\frac{V^2}{R}$ or $R = \frac{2m}{C_L A \rho}$ (1)

If C_L stays constant, then R = constant.

From Fig. 9.27, $C_L = C_L\left(Re, \frac{\omega D}{2V}\right)$. For the conditions given,

$$Re = \frac{\rho V D}{\mu} = \frac{VD}{\nu} = 90\frac{km}{hr} \times 0.074\,m \times \frac{s}{1.45\times10^{-5}\,m^2} \times 1000\frac{m}{km} \times \frac{hr}{3600\,s} = 12.8\times10^4$$

This is only slightly greater than for the curve, $Re = 10.7\times10^4$ in the figure.

From the sketch above, $\Delta + R\cos\theta = R$, so $\Delta = R(1-\cos\theta)$ and $L = R\sin\theta$. Then

$$\Delta = R\left(1-\sqrt{1-\sin^2\theta}\right) = R\left(1-\sqrt{1-(L/R)^2}\right) \quad (2)$$

Inspection of this relationship shows Δ increases as R gets smaller. Thus, from Eq. 1, since $R \sim 1/C_L$, we want the maximum value, $C_L \approx 0.4$.

From Fig. 9.27, we want $1.5 < \frac{\omega D}{2V} < 1.8$ to obtain $C_{L\,max}$. Thus

$$\omega = 1.5\times\frac{2V}{D} = 3\times 25\frac{m}{s}\cdot\frac{1}{0.074\,m} = 1010\ \text{rad/s} \quad \text{or} \quad 9680\ \text{rpm}$$

to

$$\omega = 1.8\times\frac{2V}{D} = 3.6\times 25\frac{m}{s}\cdot\frac{1}{0.074\,m} = 1220\ \text{rad/s} \quad \text{or} \quad 11{,}600\ \text{rpm} \qquad \omega$$

Using $C_L = 0.4$, from Eq. 1,

$$R = \frac{2m}{C_L A \rho} = 2\times0.145\,kg\times\frac{1}{0.4}\times\frac{4}{\pi(0.074)^2 m^2}\times\frac{m^3}{1.23\,kg} = 137\,m$$

From Eq. 2,

$$\Delta = 137\,m\left[1-\sqrt{1-\left(\frac{18\,m}{137\,m}\right)^2}\right] = 1.19\,m \qquad \Delta$$

Problem 10.1

Given: Dimensions of a centrifugal pump impeller:
$N = 1150$ rpm

	Inlet ①	Outlet ②
Radius, r (mm)	100	300
Blade width, b (mm)	50	40
Blade angle, β (deg)	70	80

Find: (a) Reoretical head, (b) mechanical power input for water flow rate of $Q = 0.10\ m^3/s$

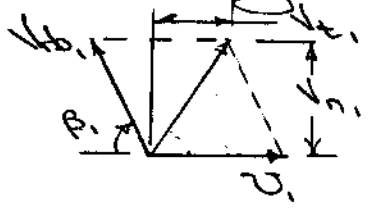
Solution:

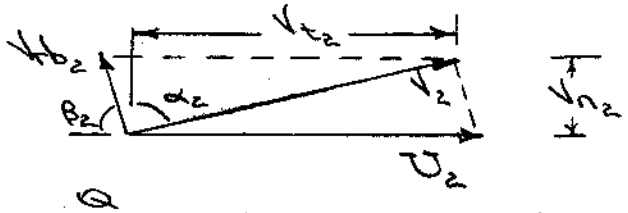
Computing equations: $\dot{W}_s = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m}$ (10.2b)

$H = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1})$ (10.2c)

Assumptions: (1) Uniform flow at blade inlet and outlet
(2) Flow enters & leaves tangent to blade

Draw velocity diagrams:





From continuity, $V_n = \frac{Q}{2\pi r b} = V_{rb}\sin\beta \quad \therefore V_{rb} = \frac{V_n}{\sin\beta}$

From geometry, $V_t = U - V_{rb}\cos\beta = U - \frac{V_n}{\sin\beta}\cos\beta = U - \frac{Q}{2\pi r b}\cot\beta$

Substituting numerical values,

$\omega = 1150 \frac{rev}{min} \times 2\pi \frac{rad}{rev} \times \frac{min}{60s} = 120\ rad/s$

$U_1 = \omega r_1 = 120 \frac{rad}{s} \times 0.1\,m = 12.0\ m/s \quad ; \quad U_2 = 36.0\ m/s$

$V_{t_1} = U_1 - \frac{Q}{2\pi r_1 b_1}\cot\beta_1 = 12.0 \frac{m}{s} - 0.10 \frac{m^3}{s} \times \frac{\cot 70^\circ}{2\pi} \times \frac{1}{0.1m} \times \frac{1}{0.05m} = 10.8 \frac{m}{s}$

$V_{t_2} = 36.0 \frac{m}{s} - 0.10 \frac{m^3}{s} \times \frac{\cot 80^\circ}{2\pi} \times \frac{1}{0.3m} \times \frac{1}{0.04m} = 35.8\ m/s$

$H = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1}) = \frac{1}{9.81\,m/s^2}\left(36.0 \frac{m}{s} \times 35.8 \frac{m}{s} - 12.0 \frac{m}{s} \times 10.8 \frac{m}{s}\right) = 118\,m$ ← H

$\dot{W}_s = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m} = gH\rho Q = 9.81 \frac{m}{s^2} \times 118m \times 999 \frac{kg}{m^3} \times 0.1 \frac{m^3}{s} \times \frac{N \cdot s^2}{kg \cdot m} \times \frac{W \cdot s}{N \cdot m}$ ← $\dot{W}_s$

$\dot{W}_s = 116\ kW$ ←

Problem 10.2

Given: Dimensions of a centrifugal pump impeller:
$N = 850$ rpm

	Inlet ①	Outlet ②
Radius, r (mm)	200	600
Blade width, b (mm)	60	40
Blade angle, β (deg)	50	70

Find: (a) theoretical head, (b) mechanical power input for water flow rate of $Q = 0.50\ m^3/s$

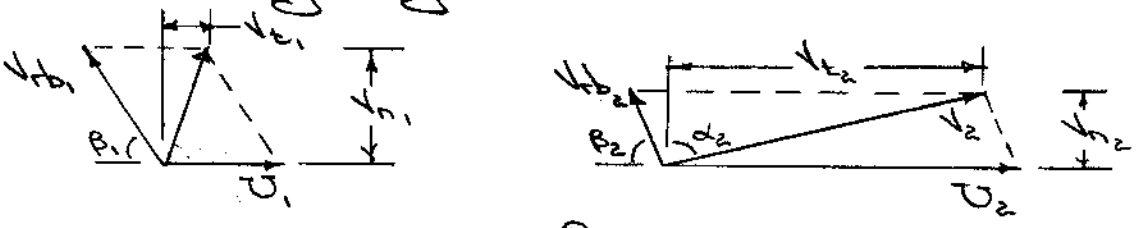
Solution:

Computing equations: $\dot{W}_s = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m}$ (10.26)

$H = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1})$ (10.2c)

Assume: (1) uniform flow at blade inlet and outlet,
(2) flow enters and leaves tangent to the blade

Draw velocity diagrams:



From continuity, $V_n = \frac{Q}{2\pi r b} = V_{rb}\sin\beta \quad \therefore V_{rb} = \frac{V_n}{\sin\beta}$

From geometry, $V_t = U - V_{rb}\cos\beta = U - \frac{V_n}{\sin\beta}\cos\beta = U - \frac{Q}{2\pi r b}\cot\beta$

Substituting numerical values,

$\omega = 850 \frac{rev}{min} \times 2\pi \frac{rad}{rev} \times \frac{min}{60\,s} = 89.0\ rad/s$

$U_1 = \omega r_1 = 89.0 \frac{rad}{s} \times 0.200\,m = 17.8\ m/s \;;\; U_2 = 53.4\ m/s$

$V_{t_1} = U_1 - \frac{Q}{2\pi r_1 b_1}\cot\beta_1 = 17.8 \frac{m}{s} - \frac{0.50}{2\pi} \frac{m^3}{s} \times \frac{1}{0.200\,m} \times \frac{\cot 50^\circ}{0.060\,m} = 12.2\ m/s$

$V_{t_2} = 53.4 \frac{m}{s} - \frac{0.50}{2\pi} \frac{m^3}{s} \times \frac{1}{0.600\,m} \times \frac{\cot 70^\circ}{0.040\,m} = 52.2\ m/s$

$H = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1}) = \frac{s^2}{9.81\,m}\left(53.4 \frac{m}{s} \times 52.2 \frac{m}{s} - 17.8 \frac{m}{s} \times 12.2 \frac{m}{s}\right) = 262\,m$ H

$\dot{W}_s = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m} = gHp\dot{Q} = 9.81 \frac{m}{s^2} \times 262\,m \times 999 \frac{kg}{m^3} \times 0.50 \frac{m^3}{s} \times \frac{N/s^2}{kg\cdot m} \times \frac{W}{N\cdot m/s}$ $\dot{W}_s$

$\dot{W}_s = 1280\ kW$

Problem 10.3

Given: Dimensions of a centrifugal pump impeller driven at $N = 575$ rpm

	Inlet ①	Outlet ②
Radius, r (mm)	400	1200
Blade width, b (mm)	120	80
Blade angle, β (deg)	40	60

Find: (a) theoretical head, (b) mechanical power input for water flow rate of $Q = 5.00\ m^3/s$

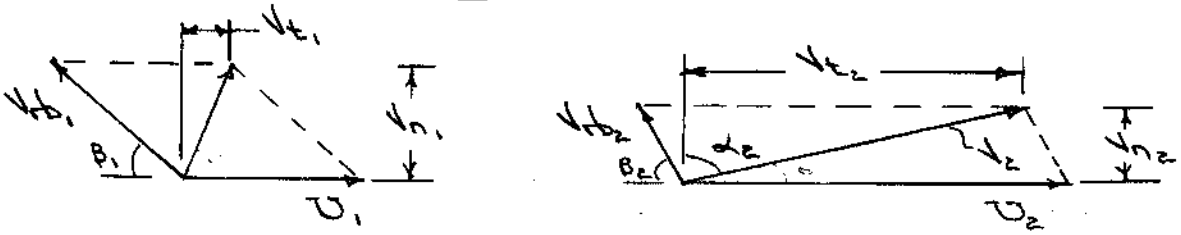
Solution:

Computing equations: $\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m}$ (10.2b)

$$H = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1}) \quad (10.2c)$$

Assume: (1) uniform flow at blade inlet and outlet
(2) flow enters and leaves tangent to blade

Draw velocity diagrams:



From continuity, $V_n = \frac{Q}{2\pi r b} = V_b \sin\beta \quad \therefore V_b = \frac{V_n}{\sin\beta}$

From geometry, $V_t = U - V_b\cos\beta = U - \frac{V_n}{\sin\beta}\cos\beta = U - \frac{Q}{2\pi r b}\cot\beta$

Substituting numerical values

$$U_1 = \omega r_1 = 575\ \frac{\text{rev}}{\text{min}} \times 2\pi\ \frac{\text{rad}}{\text{rev}} \times \frac{\text{min}}{60\text{s}} \times 0.400\text{m} = 24.1\ \text{m/s}\ ;\ U_2 = 72.3\ \text{m/s}$$

$$V_{t_1} = U_1 - \frac{Q}{2\pi r_1 b_1}\cot\beta_1 = 24.1\ \frac{\text{m}}{\text{s}} - \frac{5.00}{2\pi}\ \frac{\text{m}^3}{\text{s}} \times \frac{1}{0.400\text{m}} \times \frac{\cot 40^\circ}{0.120\text{m}} = 4.34\ \text{m/s}$$

$$V_{t_2} = 72.3\ \frac{\text{m}}{\text{s}} - \frac{5.00}{2\pi}\ \frac{\text{m}^3}{\text{s}} \times \frac{1}{1.20\text{m}} \times \frac{\cot 60^\circ}{0.080\text{m}} = 67.5\ \text{m/s}$$

$$H = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1}) = \frac{\text{s}^2}{9.81\text{m}}\left(72.3\ \frac{\text{m}}{\text{s}} \times 67.5\ \frac{\text{m}}{\text{s}} - 24.1\ \frac{\text{m}}{\text{s}} \times 4.34\ \frac{\text{m}}{\text{s}}\right) = 487\text{m} \quad \leftarrow H$$

$$\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m} = gHpQ = 9.81\ \frac{\text{m}}{\text{s}^2} \times 487\text{m} \times 999\ \frac{\text{kg}}{\text{m}^3} \times 5.00\ \frac{\text{m}^3}{\text{s}} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} \times \frac{\text{W}\cdot\text{s}}{\text{N}\cdot\text{m}}$$

$$\dot{W}_m = 23.9\ \text{MW} \quad \leftarrow \dot{W}_m$$

Problem 10.4

Given: Impeller dimensions of Example Problem 10.1:

	Inlet ①	Outlet ②
Radius, r (in.)		2.0
Blade width, b (in.)		0.383
Blade angle, β (deg)		60°

$Q = 150$ gpm
$N = 3450$ rpm

Construct: velocity diagram for exit flow leaving tangent to the blade.

Find: (a) ideal head rise, (b) mechanical power input

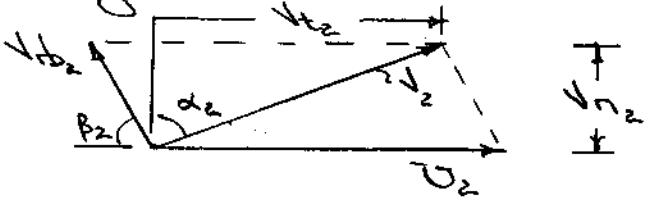
Solution:

Computing equations: $\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m}$ (10.25) ($U_1 = 0$ at ①)

$H = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1})$ (10.22) ($U_1 = 0$ at ①)

Assumptions: (1) axial inlet flow (given) so $V_{t_1} = 0$
(2) at blade outlet, flow is uniform and leaves tangent to blade

Exit velocity diagram:



From continuity, $V_{n_2} = \frac{Q}{2\pi r_2 b_2} = V_{tb_2} \sin\beta_2 \quad \therefore V_{tb_2} = \frac{V_{n_2}}{\sin\beta_2}$

From geometry, $V_{t_2} = U_2 - V_{tb_2}\cos\beta_2 = U_2 - \frac{V_{n_2}}{\sin\beta_2}\cos\beta_2 = U_2 - V_{n_2}\cot\beta_2$

Substituting numerical values,

$$U_2 = \omega r_2 = 3450 \frac{\text{rev}}{\text{min}} \times 2\pi \frac{\text{rad}}{\text{rev}} \times \frac{\text{min}}{60\text{s}} \times 2.0\,\text{in} \times \frac{\text{ft}}{12\,\text{in}} = 60.2\ \text{ft/s}$$

$$V_{n_2} = \frac{Q}{2\pi r_2 b_2} = \frac{1}{2\pi} \times 150\frac{\text{gal}}{\text{min}} \times \frac{\text{min}}{60\text{s}} \times \frac{\text{ft}^3}{7.48\,\text{gal}} \times \frac{1}{2.0\,\text{in}} \times \frac{1}{0.383\,\text{in}} \times \frac{144\,\text{in}^2}{\text{ft}^2} = 10.0\ \text{ft/s}$$

$$V_{t_2} = U_2 - V_{n_2}\cot\beta_2 = 60.2\frac{\text{ft}}{\text{s}} - 10.0\frac{\text{ft}}{\text{s}}\cot 60^\circ = 54.4\ \text{ft/s}$$

$$\dot{m} = \rho Q = 1.94\frac{\text{slug}}{\text{ft}^3} \times 150\frac{\text{gal}}{\text{min}} \times \frac{\text{min}}{60\text{s}} \times \frac{\text{ft}^3}{7.48\,\text{gal}} = 0.648\ \text{slug/s}$$

$$H = \frac{1}{g} U_2 V_{t_2} = \frac{\text{s}^2}{32.2\,\text{ft}} \times 60.2\frac{\text{ft}}{\text{s}} \times 54.4\frac{\text{ft}}{\text{s}} = 102\ \text{ft} \quad \leftarrow H$$

$$\dot{W}_m = \dot{m} U_2 V_{t_2} = \dot{m} g H = 0.648\frac{\text{slug}}{\text{s}} \times 32.2\frac{\text{ft}}{\text{s}^2} \times 102\,\text{ft} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{hp}\cdot\text{s}}{550\,\text{ft}\cdot\text{lbf}}$$

$$\dot{W}_m = 3.87\ \text{hp} \quad \leftarrow \dot{W}_m$$

Given: Centrifugal pump $R_1 = 37.5$ mm, $R_2 = 150$ mm, $b_2 = 12.7$ mm, $N = 1750$ rpm, $\beta_2 = 60°$, $Q = 42.5$ L/s (design)

(a) Draw inlet and outlet velocity diagrams, assuming b = constant.
(b) Calculate inlet vane angle for shockless entry at the design flow rate.
(c) Evaluate the theoretical power input to the pump at design flow.

Solution: Apply continuity and the Euler turbomachine equation.

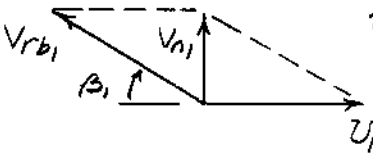
Computing equations: $V_n = \frac{Q}{2\pi r b}$ $\quad \dot{W}_m = \rho Q (U_2 V_{t2} - U_1 V_{t1})$

$\omega = 1750 \frac{rev}{min} \times 2\pi \frac{rad}{rev} \times \frac{min}{60 s} = 183$ rad/s; $U_1 = 6.86$ m/s, $U_2 = 27.5$ m/s

$V_{n2} = \frac{1}{2\pi} \times 0.0425 \frac{m^3}{s} \times \frac{1}{0.150 m} \times \frac{1}{0.0127 m} = 3.55$ m/s; Assume $V_{n1} = \frac{R_2}{R_1} V_{n2} = 14.2$ m/s

at the design flow rate. The velocity diagrams are:

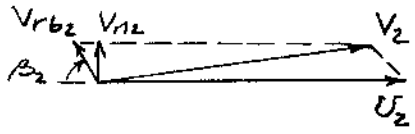
Inlet:



$\tan\beta_1 = \frac{V_{n1}}{U_1}$

$\beta_1 = 64.2°$ (shockless entry) $\longleftarrow \beta_1$

Outlet:



From the outlet velocity diagram, $V_{t2} = U_2 - V_{n2} \cos\beta_2$

$V_{t2} = 27.5 \frac{m}{s} - 3.55 \frac{m}{s} \times \cos 60° = 25.7$ m/s; $V_{t1} = 6.86 \frac{m}{s} - 14.2 \frac{m}{s} \times \cos 64.2° = 0.680$ m/s

Then

$$\dot{W}_m = 999 \frac{kg}{m^3} \times 0.0425 \frac{m^3}{s} \left(27.5 \frac{m}{s} \times 25.7 \frac{m}{s} - 6.86 \frac{m}{s} \times 0.680 \frac{m}{s} \right) \frac{N \cdot s^2}{kg \cdot m} \times \frac{W \cdot s}{N \cdot m}$$

$\dot{W}_m = 29.8$ kW $\longleftarrow \dot{W}_m$

Problem 10.6

Given: Dimensions of a centrifugal pump impeller:

	Inlet ①	Outlet ②
Radius, r (mm)	100	300
Blade width, b (mm)	50	40
Blade angle, β (deg)	60	80

Find: (a) rotational speed for which $V_{t_1} = 0$, (b) theoretical head, and (c) mechanical power input, if water flow rate is $\mathcal{Q} = 0.5\ m^3/s$.

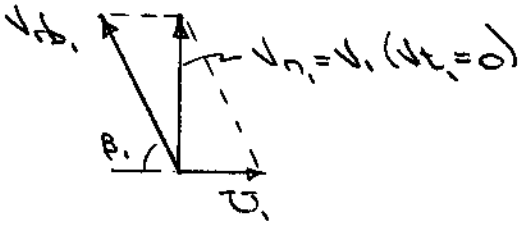
Solution:

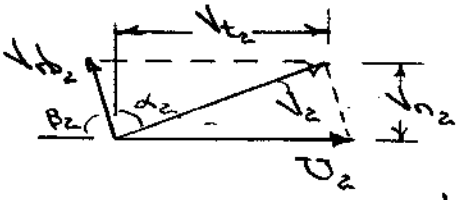
Computing equations: $\dot{W}_s = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m}$ (10.28) (V_{t_1} o(3))

$H = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1})$ (10.29) (V_{t_1} o(3))

Assumptions: (1) Uniform flow at blade inlet and outlet
(2) Flow enters & leaves tangent to blade
(3) $V_{t_1} = 0$ (given)

Draw velocity diagrams.





From continuity, $V_n = \frac{\mathcal{Q}}{2\pi r b} = V_{rb} \sin\beta$ $\therefore V_{rb} = \frac{V_n}{\sin\beta}$

From geometry, $V_t = U - V_{rb}\cos\beta = \omega r - \frac{V_n}{\sin\beta}\cos\beta = \omega r - \frac{\mathcal{Q}}{2\pi r b}\cot\beta$

For $V_{t_1} = 0$, then $\omega r_1 - \frac{\mathcal{Q}}{2\pi r_1 b_1}\cot\beta_1 = 0$ & $\omega = \frac{\mathcal{Q}}{2\pi b_1 r_1^2}\cot\beta$

$$\omega = \frac{0.5}{2\pi}\frac{m^3}{s}\times\frac{1}{0.05m}\times\frac{1}{(0.10)^2 m^2}\cot 60^\circ = 57.9\ rad/s\ (553\ rpm) \leftarrow \omega$$

Substituting numerical values

$$U_2 = \omega r_2 = 57.9\frac{rad}{s}\times 0.3m = 17.4\ m/s$$

$$V_{t_2} = U_2 - \frac{\mathcal{Q}}{2\pi r_2 b_2}\cot\beta_2 = 17.4\frac{m}{s} - \frac{0.5}{2\pi}\frac{m^3}{s}\times\frac{1}{0.3m}\times\frac{\cot 80^\circ}{0.04m} = 16.2\ m/s$$

$$H = \frac{1}{g}U_2 V_{t_2} = \frac{s^2}{9.81m}\times 17.4\frac{m}{s}\times 16.2\frac{m}{s} = 28.7\ m \leftarrow H$$

$$\dot{W}_s = U_2 V_{t_2}\dot{m} = gH\rho\mathcal{Q} = 9.81\frac{m}{s^2}\times 28.7m\times 999\frac{kg}{m^3}\times 0.5\frac{m^3}{s}\times\frac{N\cdot s^2}{kg\cdot m}\times\frac{W\cdot s}{N\cdot m}$$

$$\dot{W}_s = 141\ kW \leftarrow \dot{W}_s$$

Problem 10.7

Given: Dimensions of a centrifugal pump impeller:
$N = 850$ rpm

	Inlet ①	Outlet ②
Radius, r (mm)	200	600
Blade width, b (mm)	60	40
Blade angle, β (deg)	50	70

Find: (a) volume flow rate Q for which $V_{t_1} = 0$, (b) theoretical head, and (c) mechanical power input

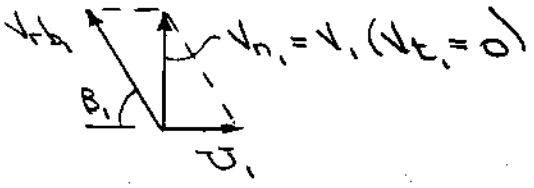
Solution:

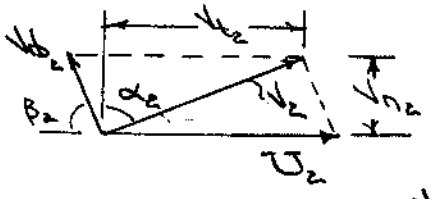
Computing equations: $\dot{W}_s = (U_2 V_{t_2} - \cancel{U_1 V_{t_1}}^{\,0(3)})\dot{m}$ (10.2b)

$H = \frac{1}{g}(U_2 V_{t_2} - \cancel{U_1 V_{t_1}}^{\,0(3)})$ (10.2c)

Assume: (1) uniform flow at blade inlet and outlet
(2) flow enters and leaves tangent to the blade
(3) $V_{t_1} = 0$ (given)

Draw velocity diagrams:





From continuity, $V_n = \frac{Q}{2\pi r b} = V_{rb}\sin\beta \quad \therefore V_{rb} = \frac{V_n}{\sin\beta}$

From geometry, $V_t = U - V_{rb}\cos\beta = U - \frac{V_n}{\sin\beta}\cos\beta = U - \frac{Q}{2\pi r b}\cot\beta$

For $V_{t_1} = 0$, then $U_1 - \frac{Q}{2\pi r_1 b_1}\cot\beta_1 = 0$ and $Q = \frac{2\pi r_1 b_1 U_1}{\cot\beta_1}$

Substituting numerical values

$U_1 = \omega_1 r_1 = 850\ \frac{\text{rev}}{\text{min}} \times 2\pi\ \frac{\text{rad}}{\text{rev}} \times \frac{\text{min}}{60\text{ s}} \times 0.200\text{ m} = 17.8\text{ m/s}$; $U_2 = 53.4\text{ m/s}$

$Q = \frac{2\pi}{\cot 50^\circ} \times 0.200\text{ m} \times 0.060\text{ m} \times 17.8\ \frac{\text{m}}{\text{s}} = 1.60\ \text{m}^3/\text{s}$ ← Q

$V_{t_2} = U_2 - \frac{Q}{2\pi r_2 b_2}\cot\beta_2 = 53.4\ \frac{\text{m}}{\text{s}} - 1.60\ \frac{\text{m}^3}{\text{s}} \times \frac{1}{2\pi} \times \frac{1}{0.600\text{ m}} \times \frac{\cot 70^\circ}{0.040\text{ m}} = 49.5\text{ m/s}$

$H = \frac{1}{g} U_2 V_{t_2} = \frac{\text{s}^2}{9.81\text{ m}} \times 53.4\ \frac{\text{m}}{\text{s}} \times 49.5\ \frac{\text{m}}{\text{s}} = 269\text{ m}$ ← H

$\dot{W}_s = \dot{m} U_2 V_{t_2} = \rho Q g H = 999\ \frac{\text{kg}}{\text{m}^3} \times 1.60\ \frac{\text{m}^3}{\text{s}} \times 9.81\ \frac{\text{m}}{\text{s}^2} \times 269\text{ m} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} \times \frac{\text{W}\cdot\text{s}}{\text{N}\cdot\text{m}}$

$\dot{W}_s = 4{,}200\text{ kW}$ ←

Problem 10.8

Given: Dimensions of a centrifugal pump impeller driven at N = 575 rpm with a water flow rate $Q = 8.00 m^3/s$

	Inlet ①	Outlet ②
Radius, r (mm)	400	1200
Blade width, b (mm)	120	80
Blade angle, β (deg)	?	60

Find: (a) the inlet blade angle β_1 for which $V_{t_1} = 0$, (b) theoretical head, and (c) mechanical power input

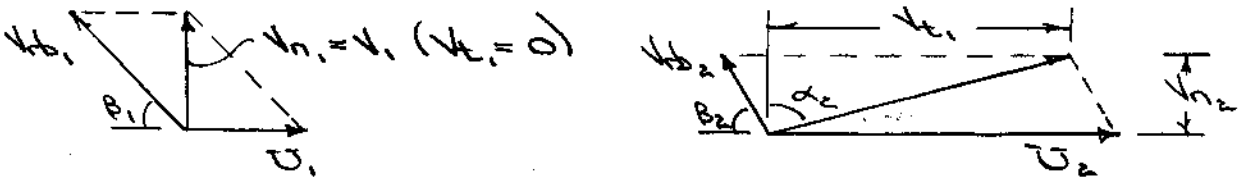
Solution:

Computing equations: $\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m}$ (10.26)

$H = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1})$ (10.27)

Assume: (1) uniform flow at blade inlet and outlet
(2) flow enters and leaves tangent to the blade
(3) $V_{t_1} = 0$ (given)

Draw velocity diagrams:



From continuity, $V_n = \frac{Q}{2\pi r b} = V_{rb}\sin\beta$ $\therefore V_{rb} = \frac{V_n}{\sin\beta}$

From geometry, $V_t = U - V_{rb}\cos\beta = U - \frac{V_n}{\sin\beta}\cos\beta = U - \frac{Q}{2\pi r b}\cot\beta$

For $V_{t_1} = 0$, then $U_1 - \frac{Q}{2\pi r_1 b_1}\cot\beta = 0$ and $\cot\beta_1 = \frac{2\pi r_1 b_1 U_1}{Q}$

Substituting numerical values

$U_1 = \omega_1 r_1 = 575\ \frac{rev}{min} \times 2\pi\ \frac{rad}{rev} \times \frac{min}{60s} \times 0.400m = 24.1 m/s$; $U_2 = 72.3 m/s$

$\cot\beta_1 = 2\pi \times 0.400m \times 0.120m \times 24.1\frac{m}{s} \times \frac{s}{8.00m^3} = 0.909$ $\beta_1 = 47.7°$ ← β_1

$V_{t_2} = U_2 - \frac{Q}{2\pi r_2 b_2}\cot\beta_2 = 72.3\frac{m}{s} - \frac{8.00}{2\pi}\frac{m^3}{s} \times \frac{1}{1.200m} \times \frac{\cot 60°}{0.080m} = 64.6 m/s$

$H = \frac{1}{g} U_2 V_{t_2} = \frac{s}{9.81m^2} \times 72.3\frac{m}{s} \times 64.6\frac{m}{s} = 476 m$ ← H

$\dot{W}_m = \dot{m} U_2 V_{t_2} = \rho Q g H = 999\frac{kg}{m^3} \times 8.00\frac{m^3}{s} \times 9.81\frac{m}{s^2} \times 476m \times \frac{N \cdot s^2}{kg \cdot m} \times \frac{W \cdot s}{N \cdot m}$

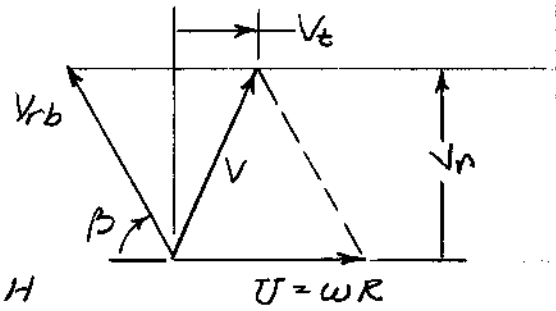
$\dot{W}_m = 37.3 MW$ ← $\dot{W}_m$

Given: Centrifugal water pump

	Inlet ①	Outlet ②
$Q = 1{,}000$ gpm	R (in.)	3.0
$N = 1{,}750$ rpm	b (in.)	0.75
Axial inlet flow	β (deg)	65

Find: (a) Theoretical head rise
(b) Theoretical power input

Velocity diagram:



Solution: Use velocity diagram, Euler turbomachine eq.

Computing equation:

$$H = \frac{1}{g}(U_2 V_{t2} - U_1 V_{t1}) \quad ; \quad \mathcal{P} = \rho Q g H$$

($V_{t1} = 0$ (1))

Assumptions: (1) Axial inlet flow (given), so $V_{t1} = 0$

$$U_2 = \omega R_2 = 1750 \frac{\text{rev}}{\text{min}} \times 2\pi \frac{\text{rad}}{\text{rev}} \times \frac{\text{min}}{60\,\text{s}} \times 0.25\,\text{ft} = 45.8\ \text{ft/s}$$

From continuity

$$V_{n2} = \frac{Q}{2\pi R_2 b_2} = \frac{1}{2\pi} \times 1000 \frac{\text{gal}}{\text{min}} \times \frac{\text{ft}^3}{7.48\,\text{gal}} \times \frac{\text{min}}{60\,\text{s}} \times \frac{1}{0.25\,\text{ft} * (0.75/12)\,\text{ft}} = 22.7 \frac{\text{ft}}{\text{s}}$$

From velocity diagram,

$$V_{t2} = U_2 - V_{rb_2}\cos\beta = U_2 - \frac{V_{n2}}{\sin\beta}\cos\beta = U_2 - V_{n2}\cot\beta$$

$$V_{t2} = 45.8 \frac{\text{ft}}{\text{s}} - 22.7 \frac{\text{ft}}{\text{s}} \times \cot 65^\circ = 35.2\ \text{ft/s}$$

Substituting,

$$H = \frac{1}{g} U_2 V_{t2} = \frac{\text{s}^2}{32.2\,\text{ft}} \times 45.8 \frac{\text{ft}}{\text{s}} \times 35.2 \frac{\text{ft}}{\text{s}} = 50.1\ \text{ft}$$ ← H

The power requirement is

$$\mathcal{P} = \rho Q g H$$

$$= 1.94 \frac{\text{slug}}{\text{ft}^3} \times 1000 \frac{\text{gal}}{\text{min}} \times 32.2 \frac{\text{ft}}{\text{s}^2} \times 50.1\,\text{ft} \times \frac{\text{ft}^3}{7.48\,\text{gal}} \times \frac{\text{min}}{60\,\text{s}} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{hp}\cdot\text{s}}{550\,\text{ft}\cdot\text{lbf}}$$

$$\mathcal{P} = 12.7\ \text{hp}$$ ← $\mathcal{P}$

Given: Impeller dimensions of Example Problem 11.2: $Q = 150$ gpm

$D_1 = 1.25$ in.

Find: Construct velocity diagram for shockless flow at the impeller inlet. $b = 0.383$ in.

Investigate effects on inlet flow angle of: $N = 3450$ rpm
(a) variations in impeller width
(b) variations in inlet swirl velocity

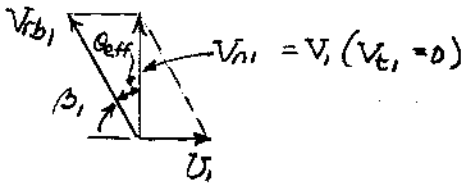
Solution: $Q = 150 \frac{gal}{min} \times \frac{ft^3}{7.48\ gal} \times \frac{min}{60\ s} = 0.334\ ft^3/sec$; $r_1 = 0.0521$ ft

$b = 0.0319$ ft; $\omega = 3450 \frac{rev}{min} \times 2\pi \frac{rad}{rev} \times \frac{min}{60\ s} = 361$ rad/s

From continuity, $V_{n1} = \frac{Q}{2\pi r_1 b_1} = \frac{1}{2\pi} \times 0.334 \frac{ft^3}{s} \times \frac{1}{0.0521\ ft} \times \frac{1}{0.0319\ ft} = 32.0$ ft/s

$U_1 = \omega r_1 = 361 \frac{rad}{s} \times 0.0521\ ft = 18.8$ ft/s

$\beta_1 = \tan^{-1} \frac{V_{n1}}{U_1} = \tan^{-1}\left(\frac{32.0}{18.8}\right) = 59.6°$



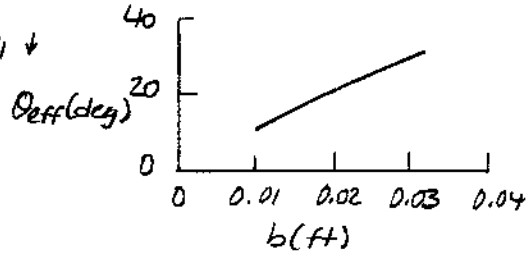
Thus for radial vanes,

$\theta_{eff} = \frac{\pi}{2} - \beta_1 = 90° - 59.6° = 30.4°$ ← θ_{eff}

To change θ_{eff}: (a) Vary b with no inlet swirl: $V_1 = V_{n1} = \frac{Q}{2\pi r_1 b_1}$

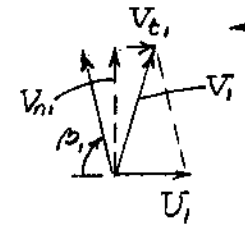
$\beta_1 = \tan^{-1} \frac{Q}{2\pi r_1 b_1 U_1}$ so $\beta_1 \uparrow$ as $b_1 \downarrow$

$\theta_{eff} = 90° - \beta_1$



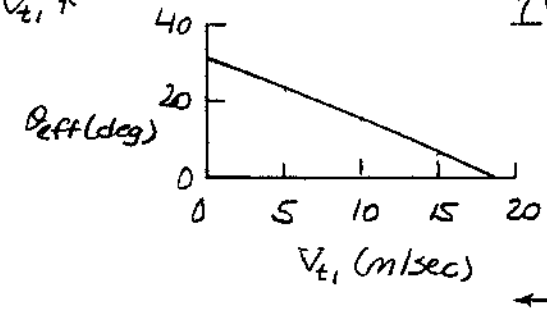
← b

(b) Vary inlet swirl (V_{t_1}) with $b = 0.0319$ ft:



$\beta_1 = \tan^{-1} \frac{V_{n1}}{U_1 - V_{t_1}}$ so $\beta_1 \uparrow$ as $V_{t_1} \uparrow$

$\theta_{eff} = 90° - \beta_1$



← V_{t_1}

Given: Centrifugal pump $R_1 = 37.5$ mm, $N = 1750$ rpm, $R_2 = 150$ mm, $\beta_2 = 60°$, $b_2 = 12.7$ mm, $Q = 42.5$ L/s (design)

Assume ideal behavior with 100 percent efficiency.

Find: (a) Shutoff head.
(b) Absolute and relative discharge velocities, total head, and power required at design flow rate, Q.

Solution: Apply the Euler turbomachine equation.

Computing equation: $H = \frac{1}{g}(U_2 V_{t2} - U_1 V_{t1})$; $\dot{W}_m = \rho Q g H$

At shutoff, $V_{t2} = U_2 = \omega R_2$, so $H_t = \frac{1}{g}(\omega R_2 \omega R_2) = \frac{(\omega R_2)^2}{g} = \frac{U_2^2}{g}$

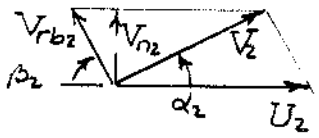
$\omega = 1750 \frac{rev}{min} \times 2\pi \frac{rad}{rev} \times \frac{min}{60\,s} = 183$ rad/s ; $U_2 = 183 \frac{rad}{s} \times 0.150\,m = 27.5$ m/s

$H_0 = (27.5)^2 \frac{m^2}{s^2} \times \frac{s^2}{9.81\,m} = 77.1$ m (Shutoff head) ← H

From continuity, $V_{n2} = \frac{Q}{2\pi r_2 b_2}$; $Q = 42.5 \frac{L}{s} \times \frac{m^3}{10^3 L} = 4.25 \times 10^{-2}$ m³/s

$V_{n2} = \frac{1}{2\pi} \times 4.25 \times 10^{-2} \frac{m^3}{s} \times \frac{1}{0.150\,m} \times \frac{1}{0.0127\,m} = 3.55$ m/s

The velocity diagram (assuming flow leaves tangent to blade) is



$V_{n2} = V_{rb2} \sin\beta$; $V_{rb2} = \frac{V_{n2}}{\sin\beta} = 4.10$ m/s ← V_{rb2}

$V_2 = \left[V_{n2}^2 + V_{t2}^2\right]^{\frac{1}{2}}$; $V_{t2} = U_2 - V_{n2} \tan\left(\frac{\pi}{2} - \beta_2\right)$

$V_{t2} = U_2 - V_{n2} \cot\beta_2 = 27.5 \frac{m}{s} - 3.55 \frac{m}{s} \cdot \cot 60° = 25.5$ m/s ; $\alpha_2 = \tan^{-1}\left(\frac{V_{n2}}{V_{t2}}\right)$

$V_2 = \left[(3.55)^2 + (25.5)^2\right]^{\frac{1}{2}} \frac{m}{s} = 25.7$ m/s ($\alpha_2 = 7.93°$) ← V_2

Finally, neglecting V_{t1},

$H = \frac{s^2}{9.81\,m} \times 27.5 \frac{m}{s} \times 25.7 \frac{m}{s} = 72.0$ m ← H

and

$\dot{W}_m = 999 \frac{kg}{m^3} \times 0.0425 \frac{m^3}{s} \times 9.81 \frac{m}{s^2} \times 72.0\,m \times \frac{N \cdot s^2}{kg \cdot m} \times \frac{W \cdot s}{N \cdot m} = 30.0$ kW ← $\dot{W}_m$

{ The theoretical power required would be reduced by any swirl in the inlet flow. }

Given: Dimensions of a centrifugal pump designed to deliver water at $Q = 460$ gpm.

	Inlet	Outlet
Radius, r (in.)	3.0	6.0
Blade width, b (in.)	0.3	0.25
Blade angle, β (deg)	25	40

Draw: inlet and outlet velocity diagrams

Find: (a) Design speed, given $V_{t_1} = 0$
(b) Outlet absolute flow angle
(c) Theoretical head developed by the pump
(d) Minimum mechanical power delivered to pump

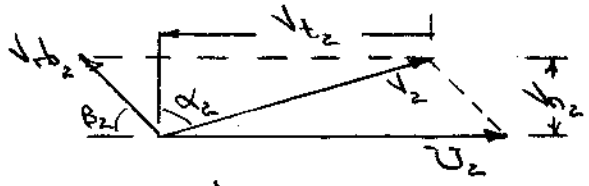
Solution:

Computing equations: $\dot{W}_s = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m}$ (10.2b) ($V_{t_1} = 0$ given)

$H = \frac{g_c}{g}(U_2 V_{t_2} - U_1 V_{t_1})$ (10.2c)

Velocity diagrams:





From continuity, $V_n = \frac{Q}{2\pi r b} = V_t \sin\beta$ $\therefore V_t = \frac{V_n}{\sin\beta}$; $\frac{V_{n_2}}{V_{n_1}} = \frac{r_1 b_1}{r_2 b_2}$

From geometry, $V_t = U - V_t \cos\beta = \omega r - \frac{V_n}{\sin\beta}\cos\beta = \omega r - \frac{Q}{2\pi r b}\cot\beta$

$\alpha_2 = \tan^{-1} V_{n_2}/V_{t_2}$

For $V_{t_1} = 0$, then $\omega r_1 - \frac{Q}{2\pi r_1 b_1}\cot\beta = 0$ and $\omega = \frac{Q}{2\pi b_1 r_1^2}\cot\beta$

$$\omega = \frac{460}{2\pi}\frac{\text{gal}}{\text{s}}\times\frac{\text{min}}{60\text{s}}\times\frac{\text{ft}^3}{7.48\text{gal}}\times\frac{1}{0.3\text{in}}\times\frac{1}{(3.0\text{in})^2}\times\frac{1728\text{in}^3}{\text{ft}^3}\cot 25^\circ = 224\ \frac{\text{rad}}{\text{s}}$$ ← ω

$U_1 = \omega r_1 = 56$ ft/s, $U_2 = \omega r_2 = 112$ ft/s

$$V_{n_2} = \frac{Q}{2\pi r_2 b_2} = \frac{460}{2\pi}\frac{\text{gal}}{\text{s}}\times\frac{\text{min}}{60\text{s}}\times\frac{\text{ft}^3}{7.48\text{gal}}\times\frac{1}{6.0\text{in}}\times\frac{1}{0.25\text{in}}\times\frac{144\text{in}^2}{\text{ft}^2} = 15.7\ \text{ft/s}$$

$V_{t_2} = U_2 - V_{n_2}\cot\beta = 112\ \text{ft/s} - 15.7\ \text{ft}\cot 40^\circ = 93.3\ \text{ft/s}$

Outlet angle, $\alpha_2 = \tan^{-1}\frac{V_{t_2}}{V_{n_2}} = \tan^{-1}\frac{93.3}{15.7} = 80.4^\circ$ ← α_2

$$H = \frac{g_c}{g} U_2 V_{t_2} = \frac{1}{32.2}\frac{\text{s}^2}{\text{ft}}\times 112\frac{\text{ft}}{\text{s}}\times 93.3\frac{\text{ft}}{\text{s}} = 325\ \text{ft}$$ ← H

$$\dot{W}_s = \dot{m} U_2 V_{t_2} = \rho Q g H = 1.94\frac{\text{slug}}{\text{ft}^3}\times 460\times\frac{1}{60}\times\frac{1}{7.48}\frac{\text{ft}^3}{\text{s}}\times 32.2\frac{\text{ft}}{\text{s}^2}\times 325\,\text{ft}$$
$$\times\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}\times\frac{\text{hp}\cdot\text{s}}{550\,\text{ft}\cdot\text{lbf}}$$

$\dot{W}_s = 37.8$ hp ← $\dot{W}_s$

Given: Centrifugal pump operating on gasoline.

$$Q = 375 \text{ gpm}, \mathcal{P} = 19.3 \text{ hp}, \eta = 0.812$$

Find: Pressure rise produced by pump, expressed as (a) ft of water and (b) ft of gasoline.

Solution: Apply definition of efficiency

Computing equation: $\eta = \dfrac{\rho Q g H}{\dot{W}_m}$ (10.3a)

Solving, $H = \dfrac{\eta \dot{W}_m}{\rho Q g}$

$$H = 0.812 \times 19.3 \text{ hp} \times \frac{\text{ft}^3}{1.94 \text{ slug}} \times \frac{\text{min}}{375 \text{ gal}} \times \frac{\text{s}^2}{32.2 \text{ ft}} \times 550 \frac{\text{ft} \cdot \text{lbf}}{\text{hp} \cdot \text{s}} \times 60 \frac{\text{s}}{\text{min}}$$

$$\times 7.48 \frac{\text{gal}}{\text{ft}^3} \times \frac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2}$$

$H = 165$ ft of water ← H

From Table A.2, SG = 0.72 for gasoline. Thus

$$H_g = H_w \frac{\rho_w}{\rho_g} = \frac{H_w}{SG}$$

$$H_g = 165 \text{ ft} \times \frac{1}{0.72} = 229 \text{ ft of gasoline} \leftarrow H$$

Given: Centrifugal water pump designed for $N = 1160$ rpm, $\eta = 0.75$.

$r_1 = 3.5$ in. $\quad r_2 = 6.5$ in.
$b_1 = 0.4$ in. $\quad b_2 = 0.3$ in.
$Q = 1$ ft³/s $\quad \beta_2 = 40$ deg

(a) Draw the velocity diagrams.
(b) Find inlet blade angle so $V_{t1} = 0$.
(c) Determine the outlet absolute flow angle (measured w.r.t. to normal).
(d) Evaluate hydraulic power and head.

Solution: Apply continuity and the Euler turbomachine equation.

Computing equations: $V_n = \frac{Q}{2\pi r b} \qquad \dot{W}_m = \rho Q (U_2 V_{t2} - U_1 V_{t1})$

$$\omega = 1160 \frac{\text{rev}}{\text{min}} \times 2\pi \frac{\text{rad}}{\text{rev}} \times \frac{\text{min}}{60\,\text{s}} = 121 \text{ rad/s}; \quad U_1 = 35.3 \text{ ft/s}; \quad U_2 = 65.5 \text{ ft/s}$$

$$V_{n1} = \frac{1}{2\pi} \times 1 \frac{\text{ft}^3}{\text{s}} \times \frac{1}{3.5 \text{ in.}} \times \frac{1}{0.4 \text{ in.}} \times 144 \frac{\text{in.}^2}{\text{ft}^2} = 16.4 \text{ ft/s}; \quad V_{n2} = \frac{r_1 b_1}{r_2 b_2} V_{n1} = 11.8 \text{ ft/s}$$

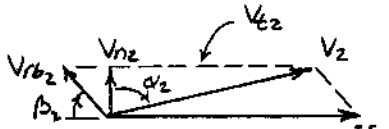
Inlet:



$$\tan\beta_1 = \frac{V_{n1}}{U_1}$$

$$\beta_1 = 24.9°$$

Outlet:



β_1

From the diagram $V_{t2} = U_2 - V_{n2}\cot\beta_2 = U_2 - \frac{V_{n2}}{\tan\beta_2} = 65.5 - \frac{11.8}{\tan 40°} = 55.6$ ft/s

$$\alpha_2 = \tan^{-1}\frac{V_{t2}}{V_{n2}} = \tan^{-1}\left(\frac{55.6}{11.8}\right) = 78.0°$$

α_2

$$\dot{W}_m = 1.94 \frac{\text{slug}}{\text{ft}^3} \times 1 \frac{\text{ft}^3}{\text{s}} \left(65.5 \frac{\text{ft}}{\text{s}} \times 55.6 \frac{\text{ft}}{\text{s}} - 0\right) \frac{\text{lbf·s}^2}{\text{slug·ft}} \times \frac{\text{hp·s}}{550 \text{ ft·lbf}} = 12.8 \text{ hp}$$

$\dot{W}_m$

$$H_p = \frac{U_2 V_{t2} - U_1 V_{t1}}{g} = \frac{\dot{W}_m}{\rho Q g} = 65.5 \times 55.6 \frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{s}^2}{32.2 \text{ ft}} = 113 \text{ ft}$$

H_p

Given: Centrifugal water pump, $D_2 = 6$ in., axial inlet flow, $N = 1750$ rpm, $\beta_2 = 65°$, and $b_2 = 0.175$ in.

The pump casing converts 60 percent of the absolute velocity head to static pressure rise. Head loss through the pump suction and discharge is 0.25 times the radial component of velocity head leaving the impeller.

Find: Estimate the volume flow rate, head rise, power input, and pump efficiency at the maximum efficiency point, if the torque to overcome seal and bearing friction is 10 percent of the ideal torque at 1000 gpm.

Solution: Apply the Euler turbomachine equation and continuity.

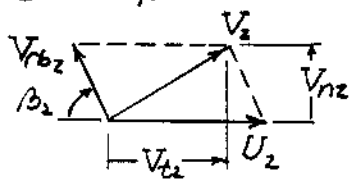
Computing equations: $H = \frac{1}{g}(U_2 V_{t2} - U_1 \cancel{V_{t1}}^{=0(1)})$; $V_n = \frac{Q}{2\pi r b}$; $\eta = \frac{H}{H_i + H_f}$

Assume: (1) Axial inlet flow, (2) Uniform flow at sections.

$$\omega = 1750 \frac{rev}{min} \times 2\pi \frac{rad}{rev} \times \frac{min}{60\,s} = 183 \text{ rad/s} \; ; \quad U_2 = \omega R_2 = 183 \frac{rad}{s} \cdot 0.25 \text{ ft} = 45.8 \text{ ft/s}$$

Illustrate the procedure using $Q = 1000$ gpm $(2.23 \text{ ft}^3/\text{s})$:

$$V_{n2} = \frac{1}{2\pi} \times 1000 \frac{gal}{min} \times \frac{1}{0.125 \text{ ft}} \times \frac{12}{0.25 \text{ ft}} \times \frac{ft^3}{7.48 \text{ gal}} \times \frac{min}{60\,s} = 22.7 \text{ ft/s}$$



$$V_{t2} = U_2 - V_{n2} \tan\left(\frac{\pi}{2} - \beta_2\right) = U_2 - V_{n2} \cot \beta_2$$

$$V_{t2} = 45.8 - 22.7 \cot 65° = 35.2 \text{ ft/s}$$

$$V_2 = \left[V_{n2}^2 + V_{t2}^2\right]^{\frac{1}{2}} = \left[(45.8)^2 + (35.2)^2\right]^{\frac{1}{2}} = 41.9 \text{ ft/s}$$

$$H_{ideal} = \frac{1}{g} U_2 V_{t2} = \frac{s^2}{32.2 \text{ ft}} \times 45.8 \frac{ft}{s} \times 35.2 \frac{ft}{s} = 50.1 \text{ ft}$$

$$T_f = 0.1 \frac{\dot{W}_m}{\omega} = 0.1 \frac{\rho Q g H_{ideal}}{\omega} = 0.1 \frac{\dot{w} H_{ideal}}{\omega}$$

$$\dot{w} = 1.94 \frac{slug}{ft^3} \times 2.23 \frac{ft^3}{s} \times 32.2 \frac{ft}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} = 139 \text{ lbf/s}$$

$$T_f = 0.1 \times 139 \frac{lbf}{s} \times 50.1 \text{ ft} \times \frac{s}{187 \text{ rad}} = 3.72 \text{ ft} \cdot \text{lbf}$$

$$H_{actual} = 0.6 \frac{V_2^2}{2g} - 0.75 \frac{V_{n2}^2}{2g} = \left[\frac{0.6}{2} \times (41.9)^2 \frac{ft^2}{s^2} - \frac{0.75}{2} \times (22.7)^2 \frac{ft^2}{s^2}\right] \frac{s^2}{32.2 \text{ ft}} = 10.3 \text{ ft}$$

$$\eta = \frac{\dot{W}_h}{\dot{W}_m + \omega T_f} = \frac{\dot{w} H_{actual}}{\dot{w} H_{ideal} + \omega T_f} = \frac{139(10.3) \text{ ft} \cdot \text{lbf/s}}{\left[139(50.1) + 187(3.72)\right] \text{ft} \cdot \text{lbf/s}} = 0.188$$

Similar computations for various flow rates are tabulated and plotted on the next page. From the plot, peak efficiency is $\eta = 0.222$ at about $Q = 400$ gpm, with $H_{actual} = 15.9$ ft.

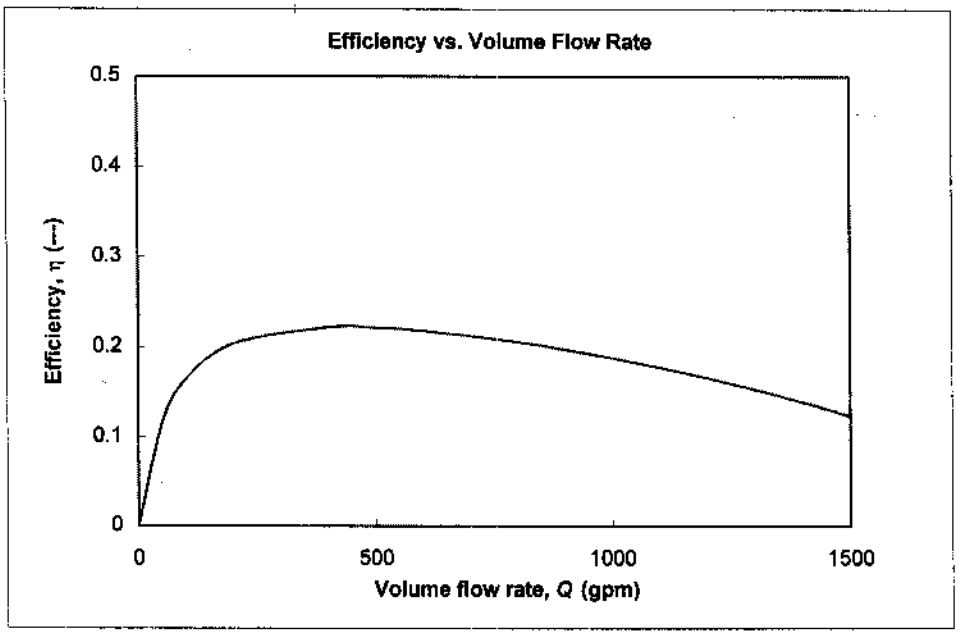
η, Q, H

The required power input at 1000 gpm is

$$\dot{W}_m = \rho Q H_i g + \omega T_f = \left(139 \frac{lbf}{s} \times 50.1 ft + 38.0 ft \cdot lbf \times 187 \frac{rad}{s}\right) \frac{hp \cdot s}{550 ft \cdot lbf} = 25.6 hp$$

This is equivalent to 19.1 kW.

Tabulating and plotting:



Q (gpm)	V_{n2} (ft/s)	V_{t2} (ft/s)	V_2 (ft/s)	H_{ideal} (ft)	H_{actual} (ft)	m dot (lbm/s)	η (---)
0	0	45.8	45.8	65.2	19.6	0	0.000
50	1.1	45.3	45.3	64.4	19.1	7	0.116
100	2.3	44.8	44.8	63.7	18.7	14	0.164
200	4.5	43.7	43.9	62.2	17.7	28	0.203
400	9.1	41.6	42.6	59.2	15.9	56	0.222
500	11.3	40.5	42.1	57.7	15.0	70	0.222
600	13.6	39.5	41.7	56.2	14.1	83	0.218
800	18.2	37.3	41.5	53.1	12.2	111	0.206
1000	22.7	35.2	41.9	50.1	10.4	139	0.188
1200	27.2	33.1	42.9	47.1	8.5	167	0.165
1400	31.8	31.0	44.4	44.1	6.6	195	0.138
1500	34.0	29.9	45.3	42.6	5.7	209	0.123

Given: Centrifugal pump operating with water, at shutoff.

Actual head rise is 70 percent of theoretical.

Find: (a) Prepare log-log plot of impeller radius versus theoretical head rise, with standard motor speeds as parameters.
(b) Explain how this plot might be used for preliminary design.

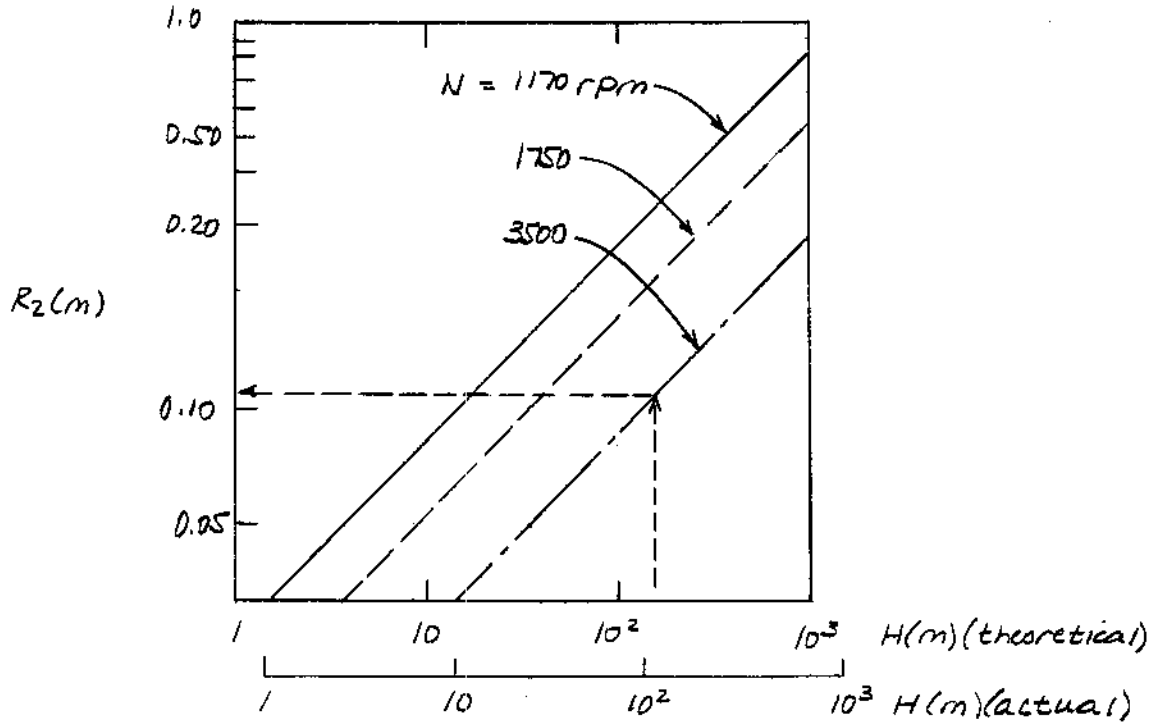
Solution: Apply the Euler turbomachine equation.

Computing equation: $H = \frac{1}{g}(U_2 V_{t2} - U_1 \overbrace{V_{t1}}^{\approx 0 (2)})$

Assumptions: (1) No through flow, (2) Neglect V_{t1}

Then $H = \frac{1}{g}(\omega R_2 \omega R_2) = \frac{\omega^2 R_2^2}{g}$ or $\log H = 2\log\omega + 2\log R_2 - \log g$

These will be straight lines on a plot of $\log R_2$ vs. $\log H$ (at constant ω):



For a given application, enter the abscissa with the desired head, move up to the desired driver speed, then move left to the ordinate and read the required impeller radius. The example (--- line) illustrates.

Given: Performance curves (Appendix D) for Peerless 4AE12 pump at 1750 and 3550 nominal rpm, with a 12.12 in. impeller.

Find: Obtain and plot curve-fits for total head vs. delivery at each speed for this pump.

Solution: Tabulate data from Figs. D.4 (1750 rpm) and D.5 (3550 rpm):

1750 rpm:	Q (gpm)	0	200	400	600
	H (ft)	155	150	137	106

Curve-fit: $\hat{H}(ft) = 156 - 1.36 \times 10^{-4}[Q(gpm)]^2$; $r^2 = 0.994$

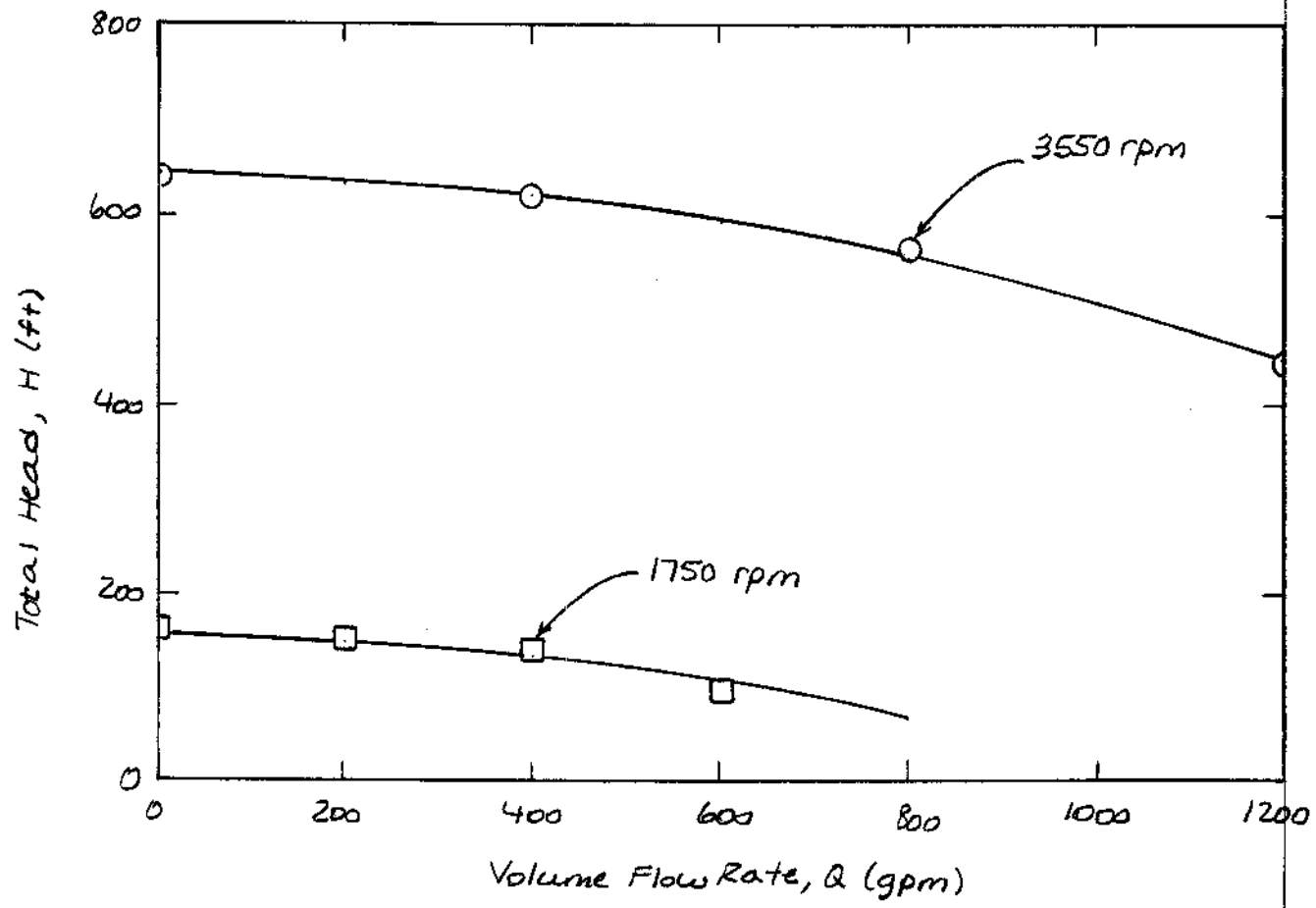
	$\hat{H}$ (ft)	156	151	134	107

3550 rpm:	Q (gpm)	0	400	800	1200
	H (ft)	635	620	565	445

Curve-fit: $\hat{H}(ft) = 641 - 1.33 \times 10^{-4}[Q(gpm)]^2$; $r^2 = 0.994$

	$\hat{H}$ (ft)	641	619	556	449

Plot:



Given: Performance curves (Appendix D) for Peerless 16A 18B pump at 705 and 880 nominal rpm, with an 18.0 in. diameter impeller.

Find: Obtain and plot curve-fits for total head versus delivery at each speed for this pump.

Solution: Tabulate data from Figs. D.9 (705 rpm) and D.10 (880 rpm):

705 rpm:	Q (gpm)	0	2000	4000	6000	8000
	H (ft)	59	56	50	43	32

Curve-fit: $\hat{H}(ft) = 57.8 - 4.09 \times 10^{-7}[Q(gpm)]^2$; $r^2 = 0.994$

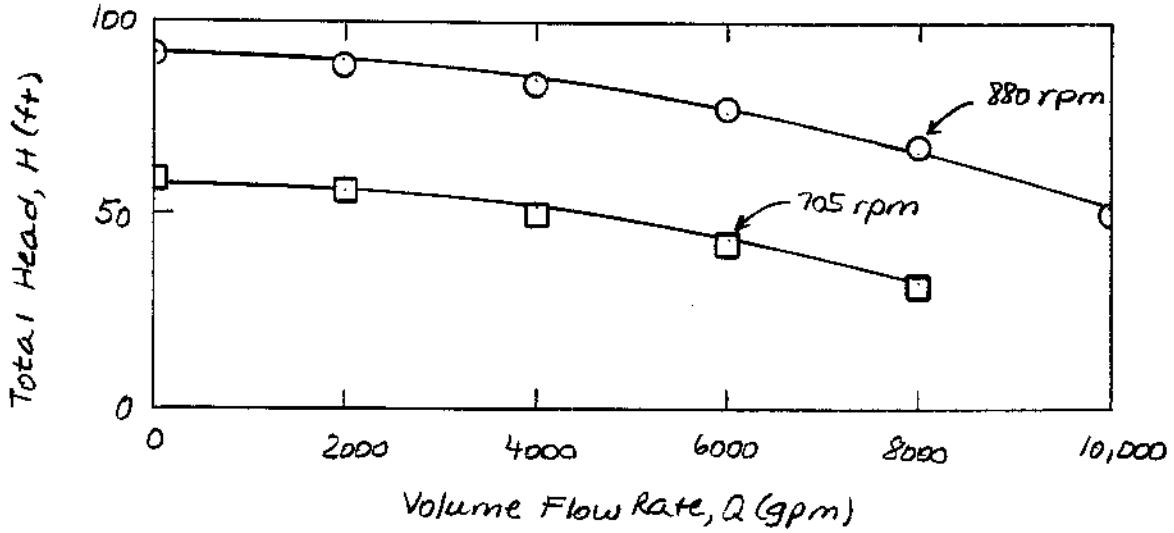
	$\hat{H}$ (ft)	57.8	56.2	51.3	43.1	31.6

880 rpm:	Q (gpm)	0	2000	4000	6000	8000	10,000
	H (ft)	92	89	84	78	68	50

Curve-fit: $\hat{H}(ft) = 91.5 - 4.01 \times 10^{-7}[Q(gpm)]^2$; $r^2 = 0.992$

	$\hat{H}$ (ft)	91.5	89.9	85.1	77.1	65.9	51.5

Plot:



Given: Data from tests of a Peerless Type 1430 pump operated at 1750 rpm with a 14.0 in. diameter impeller are:

Flow rate, Q (gpm)	270	420	610	720	1000
Total head, H (ft)	198	195	178	165	123
Power input, $\mathcal{P}$ (hp)	25	30	35	40	45

Find: (a) Plot the performance curves for this pump; include a curve of efficiency versus volume flow rate.
(b) Locate the best efficiency point and specify the pump rating at this point.

Solution: Apply the definition of efficiency.

Computing equations: $\eta_p = \frac{\dot{W}_h}{\dot{W}_m}$; $\dot{W}_h = \rho Q g H$

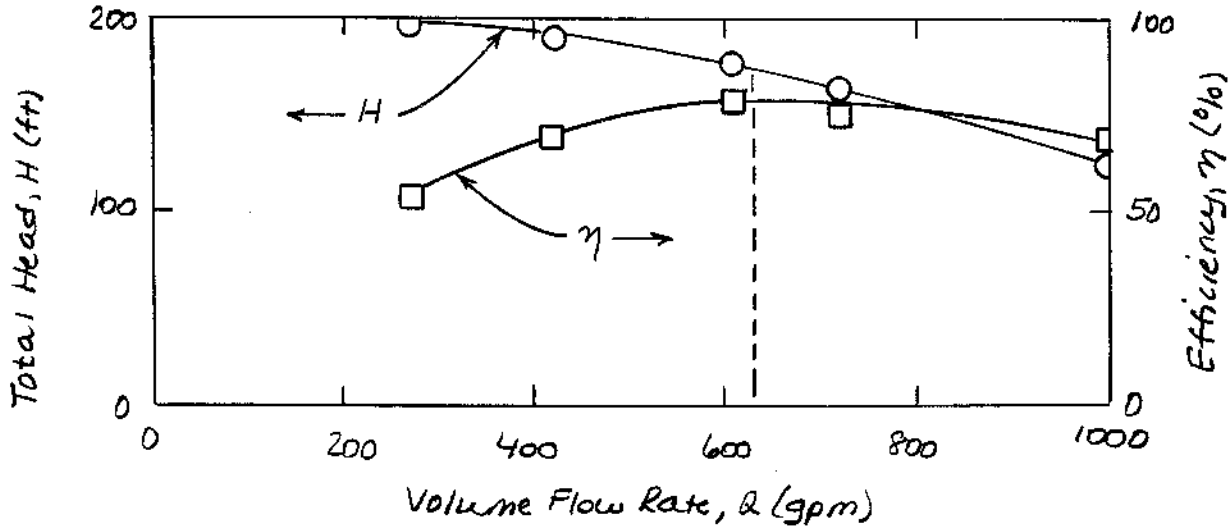
$$\dot{W}_h = 1.94 \frac{slug}{ft^3} \times Q \frac{gal}{min} \times 32.2 \frac{ft}{s^2} \times H\,ft \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \frac{ft^3}{7.48\,gal} \times \frac{hp \cdot min}{33000\,ft \cdot lbf}$$

$$\dot{W}_h(hp) = 2.53 \times 10^{-4} Q(gpm)\, H(ft)$$

Thus

Q (gpm)	270	420	610	720	1000
η (---)	0.541	0.690	0.784	0.751	0.691

Plotting:



The approximate best efficiency point (BEP) is shown by the dashed line.

At the BEP, $\eta \approx 79$ percent

$Q = 630$ gpm

$H = 176$ ft ← BEP

Given: Data from tests of a Peerless Type 1440 pump operated at 1750 rpm with a 14.0 in. diameter impeller are:

Flow rate, Q (gpm)	290	440	550	790	920	1280
Total head, H (ft)	204	203	200	187	175	135
Power input, $\mathcal{P}$ (hp)	30	35	40	45	50	60

Find: (a) Plot the performance curves for this pump; include a curve of efficiency versus volume flow rate.
(b) Locate the best efficiency point and specify the pump rating at this point.

Solution: Apply the definition of pump efficiency.

Computing equations: $\eta_p = \dfrac{\dot{W}_h}{\dot{W}_m}$; $\dot{W}_h = \rho Q g H$

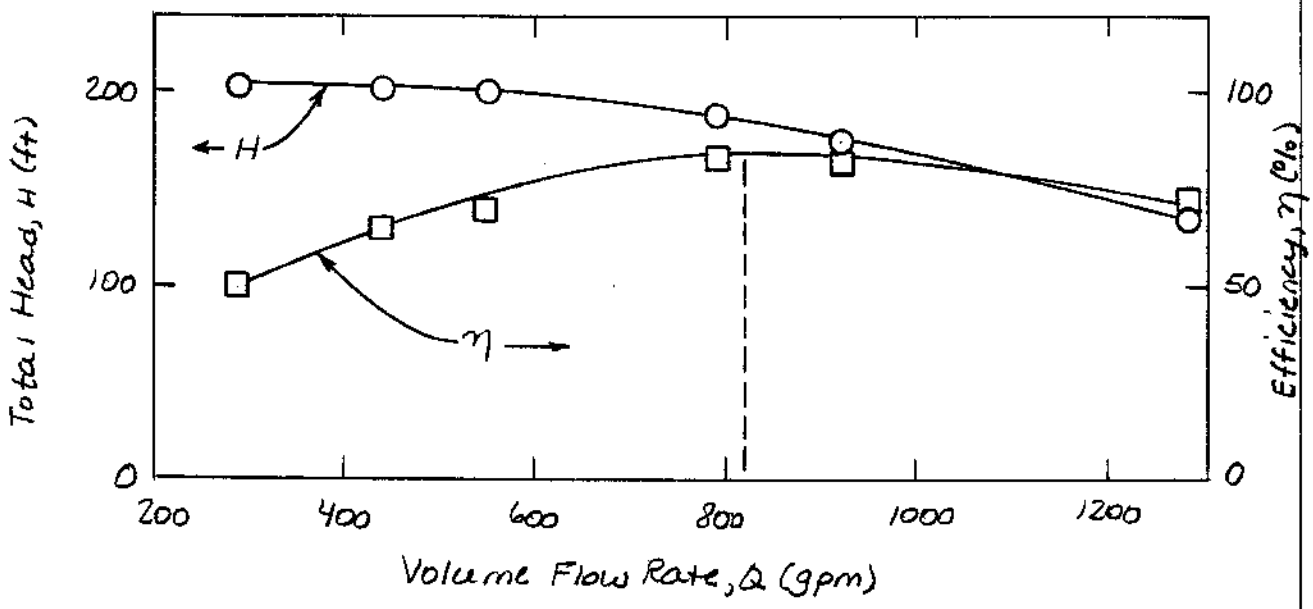
$$\dot{W}_h = 1.94 \frac{slug}{ft^3} \times Q\, gal/min \times 32.2 \frac{ft}{s^2} \times H\, ft \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \frac{ft^3}{7.48 gal} \times \frac{hp \cdot min}{33000\, ft \cdot lbf}$$

$$\dot{W}_h(hp) = 2.53 \times 10^{-4} Q(gpm)\, H(ft)$$

Thus

Q (gpm)	290	440	550	790	920	1280
η (---)	0.499	0.645	0.695	0.829	0.814	0.728

Plotting:



The approximate best efficiency point (BEP) is shown by the dashed line.

At the BEP, $\eta \approx 0.83$

$Q = 820$ gpm

$H = 185$ ft

BEP

Given: Data measured during tests of a centrifugal pump at 3500 rpm:

$Q = 11.5\ m^3/hr$

$T = 3.68\ N \cdot m$

Parameter	Inlet, Section ①	Outlet, Section ②
Gage pressure, p (kPa)	95.2	412
Elevation above datum, z (m)	1.25	2.75
Average speed of flow, $\bar{V}$ (m/sec)	2.35	3.62

Find: (a) Total dynamic heads at inlet and outlet.
(b) Hydraulic power input to the fluid.
(c) Pump efficiency.
(d) Specify motor size needed.
(e) Calculate electric power required if $\eta_m = 0.85$.

Solution: Total dynamic head is $H_t = \frac{p}{\rho g} + \frac{\bar{V}^2}{2g} + z$. Evaluating,

$$H_{t1} = 95.2\times10^3 \frac{N}{m^2}\times\frac{m^3}{999\ kg}\times\frac{s^2}{9.81\ m}\times\frac{kg\cdot m}{N\cdot s^2} + \frac{(2.35)^2}{2}\frac{m^2}{s^2}\times\frac{s^2}{9.81\ m} + 1.25\ m = 13.2\ m \quad H_{t1}$$

$$H_{t2} = 412\times10^3 \frac{N}{m^2}\times\frac{m^3}{999\ kg}\times\frac{s^2}{9.81\ m}\times\frac{kg\cdot m}{N\cdot s^2} + \frac{(3.62)^2}{2}\frac{m^2}{s^2}\ \frac{s^2}{9.81\ m} + 2.75\ m = 45.5\ m \quad H_{t2}$$

$$\dot{W}_h = \rho Q g\,\Delta H = 999\frac{kg}{m^3}\times 11.5\frac{m^3}{hr}\times 9.81\frac{m}{s^2}\times(45.5-13.2)m\times\frac{N\cdot s^2}{kg\cdot m}\times\frac{hr}{3600\ s}\times\frac{W\cdot s}{N\cdot m}$$

$$\dot{W}_h = 1.01\times10^3\ W = 1.01\ kW \quad \dot{W}_h$$

The mechanical power input is

$$\dot{W}_m = \omega T = 3500\frac{rev}{min}\times 3.68\ N\cdot m\times 2\pi\frac{rad}{rev}\times\frac{min}{60\ s}\times\frac{W\cdot s}{N\cdot m} = 1.35\ kW$$

$$\eta_p = \frac{\dot{W}_h}{\dot{W}_m} = \frac{1.01}{1.35} = 0.748 \approx 75\ \text{percent} \quad \eta_p$$

The motor size needed is

$$\mathcal{P}_m = 1.35\ kW\times\frac{hp}{0.746\ kW} = 1.81\ hp, \text{ so specify } \mathcal{P}_m = 2\ hp \quad \mathcal{P}_m$$

The input power requirement is

$$\mathcal{P}_e = \frac{\mathcal{P}_m}{\eta_m} = \frac{1.35\ kW}{0.85} = 1.59\ kW \quad \mathcal{P}_e$$

Write the pump specific speed in terms of the flow coefficient and the head coefficient

Solution:

$$N_s = \frac{\omega Q^{1/2}}{h^{3/4}} \qquad \text{.....(1.6a)}$$

Flow coefficient $\pi_1 = \frac{Q}{\omega D^3}$

Head coefficient $\pi_2 = \frac{h}{\omega^2 D^2}$

Then

$$N_s = \left[\frac{Q}{\omega D^3}\right]^{1/2} \cdot \left[\frac{\omega^2 D^2}{h}\right]^{3/4} = \frac{Q^{1/2}}{\omega^{1/2} D^{3/2}} \cdot \frac{\omega^{3/2} D^{3/2}}{h^{3/4}} = \frac{\omega Q^{1/2}}{h^{3/4}}$$

$$N_s = \pi_1^{1/2} / \pi_2^{3/4} \longleftarrow N_s$$

Write the turbine specific speed in terms of the power coefficient and the head coefficient

Solution:

$$N_s = \omega Q^{1/2} / g^{1/2} h^{3/4} \quad \text{----- 10.18 a}$$

Power coefficient $\quad \pi_3 = \dfrac{Q}{\rho \omega^3 D^5}$

Head coefficient $\quad \pi_2 = \dfrac{h}{\omega^2 D^2}$

Then

$$N_s = \left[\frac{Q}{\rho \omega^3 D^5}\right]^{1/2} \left[\frac{\omega^2 D^2}{h}\right]^{5/4} = \frac{Q^{1/2}}{\rho^{1/2} \omega^{3/2} D^{5/2}} \times \frac{\omega^{5/2} D^{5/2}}{h^{5/4}} = \frac{Q^{1/2} \omega}{\rho^{1/2} h^{5/4}}$$

$$N_s = \pi_3^{1/2} / \pi_2^{5/4} \qquad N_s$$

Given: Kilogram force ≡ force exerted on 1 kg in standard gravity.

Metric horsepower (hpm) ≡ 75 m·kgf/s.

Find: (a) Develop a conversion relating hpm to U.S. hp.
(b) Relate specific speed for a hydraulic turbine -- expressed in units of rpm, hpm, and m -- to the specific speed calculated in U.S. customary units.

Solution:

$$1\ hp(U.S.) = 550\frac{ft\cdot lbf}{s} \times 0.305\frac{m}{ft} \times 0.4536\frac{kgf}{lbf} \times \frac{hpm\cdot s}{75\ m\cdot kgf} = 1.01\ hpm$$ ← hpm

$$N_{s\,cu} = \frac{N(\mathcal{P})^{1/2}}{(h)^{5/4}} = \frac{N(rpm)[\mathcal{P}(hpUS)]^{1/2}}{[h(ft)]^{5/4}}$$

$$= N(rpm)\frac{[\mathcal{P}(hpUS)]^{1/2}}{[\mathcal{P}(hpm)]^{1/2}} \times [\mathcal{P}(hpm)]^{\frac{1}{2}} \times \frac{[h(m)]^{5/4}}{[h(ft)]^{5/4}} \times \frac{1}{[h(m)]^{5/4}}$$

$$= N(rpm)\frac{[\mathcal{P}(hpm)]^{1/2}}{[h(m)]^{5/4}} \times \left[\frac{\mathcal{P}(hpUS)}{\mathcal{P}(hpm)}\right]^{1/2} \times \left[\frac{h(m)}{h(ft)}\right]^{5/4}$$

$$= N_s(rpm, hpm, m) \times (1.01)^{1/2} \times (0.305)^{5/4}$$

$$N_{s\,cu} = 0.228\ N_s(rpm, hpm, m)$$ ← Ns

Check: $N = 1\ rpm$, $\mathcal{P} = 1\ hp$, $h = 1\ ft$; $N_s(USCS) = 1$

$N = 1\ rpm$, $\mathcal{P} = 1\ hpm$, $h = 1\ m$; $N_s = \frac{(1)(\frac{1}{1.01})^{1/2}}{(1.305)^{5/4}} = 4.39$

$$\frac{N_s(USCS)}{N_s(rpm, hpm, m)} = \frac{1}{4.39} = 0.228\ \checkmark\checkmark$$

Given: Small centrifugal pump: $N = 2875$ rpm with water

$Q = 252$ gpm, $H = 138$ ft at BEP

$\eta = 0.76$

Find: (a) Determine specific speed at this test condition.
(b) Sketch the impeller shape you expect.
(c) Compute the required power input.

Solution:

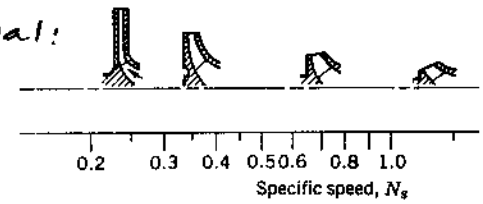
Computing equations: $N_S = \dfrac{N(\text{rpm})[Q(\text{gpm})]^{\frac{1}{2}}}{[H(\text{ft})]^{\frac{3}{4}}}$; $\dot{W}_m = \dfrac{\dot{W}_h}{\eta} = \dfrac{\rho Q g H}{\eta}$

$$N_{S_{US}} = \frac{2875 \text{ rpm}\,(252 \text{ gpm})^{1/2}}{(138 \text{ ft})^{3/4}} = 1130$$ N_S

The dimensionless specific speed is

$$N_S(\text{n.d.}) = \frac{N_S(\text{US})}{2733} = \frac{1130}{2733} = 0.413$$

From Fig. 10.14, impeller will be centrifugal:

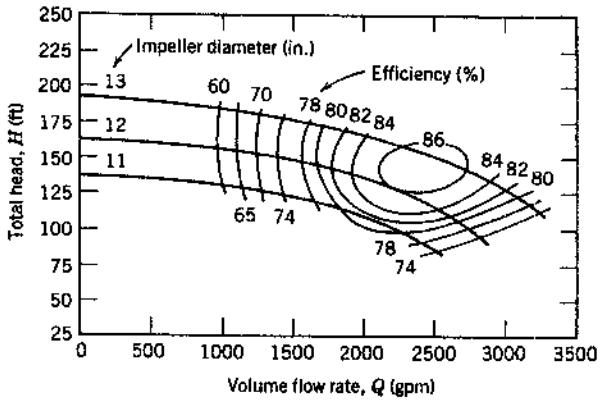


The power input is $\dot{W}_m = \dfrac{\dot{W}_h}{\eta_p} = \dfrac{\rho Q g H}{\eta}$

$$\dot{W}_m = \frac{1}{0.76} \times 1.94 \frac{\text{slug}}{\text{ft}^3} \times 252 \frac{\text{gal}}{\text{min}} \times 32.2 \frac{\text{ft}}{\text{s}^2} \cdot 138 \text{ ft} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} \times \frac{\text{ft}^3}{7.48 \text{ gal}} \times \frac{\text{hp} \cdot \text{min}}{33000 \text{ ft} \cdot \text{lbf}}$$

$\dot{W}_m = 11.6$ hp $\dot{W}_m$

Given: Typical performance curves for a centrifugal pump, tested with three different impeller diameters:



Find: (a) Specify the flow rate and head at the best efficiency point (BEP) with a 12 in. diameter impeller.
(b) Scale these data to predict the BEP for 11 in. and 13 in. diameter impellers.
(c) Comment on the accuracy of the scaling procedure.

Solution: From the graph, BEP occurs for the 12 in. impeller at

$Q \simeq 2200$ gpm and $H \simeq 130$ ft ← BEP_{12}

From Section 10-4.3, scaling rules are

$$Q_2 = Q_1\left(\frac{D_2}{D_1}\right)^3 ; \quad Q_{11} = 2200 \text{ gpm}\left(\frac{11}{12}\right)^3 = 1690 \text{ gpm}$$

$$Q_{13} = 2200 \text{ gpm}\left(\frac{13}{12}\right)^3 = 2800 \text{ gpm}$$

$$H_2 = H_1\left(\frac{D_2}{D_1}\right)^2 ; \quad H_{11} = 130 \text{ ft}\left(\frac{11}{12}\right)^2 = 109 \text{ ft}$$

$$H_{13} = 130 \text{ ft}\left(\frac{13}{12}\right)^2 = 153 \text{ ft}$$

Thus BEP_{11} is at $Q = 1690$ gpm, $H = 109$ ft ← BEP_{11}

BEP_{13} is at $Q = 2800$ gpm, $H = 153$ ft ← BEP_{13}

The complete scaling rules tend to move the volume flow rate too far. Accuracy would be improved using $Q_2 = Q_1(D_2/D_1)^2$, since the impeller width does not change, and $H_2 = H_1(D_2/D_1)^4$, since $H \simeq V^2$. With these modified rules (page 526),

$(Q_{11}, H_{11}) = 1850$ gpm, 109 ft and $(Q_{13}, H_{13}) = 2580$ gpm, 153 ft

These modified scaling points are closer to the measured BEPs.

Given: Mixed-flow pump at BEP ($\eta = 0.82$) has $D = 500$ mm, and delivers $Q = 0.175\ m^3/s$ at $H = 39$ m when operating at $N = 1170$ rpm.

Find: (a) Calculate the specific speed of this pump.
(b) Evaluate the required power input.
(c) Determine the curve-fit parameters using BEP and shutoff points.
(d) Scale the performance curve to estimate the flow, head, efficiency, and power required at 860 rpm.

Solution: In SI units, $N = 123$ rad/s, $Q = 0.175\ m^3/s$, and $h = gH = 383\ m^2/s^2$.

$$N_S = \frac{\omega(Q)^{1/2}}{h^{3/4}} = 123\frac{rad}{s}\times(0.175)^{1/2}\frac{m^{3/2}}{s^{1/2}}\times\frac{s^{3/2}}{(383)^{3/4}\,m^{3/2}} = 1.23$$ ← N_S

$$\dot{W}_m = \frac{\dot{W}_h}{\eta} = \frac{\rho Q g H}{\eta}$$

$$\dot{W}_m = \frac{999}{0.82}\frac{kg}{m^3}\times 0.175\frac{m^3}{s}\times 383\frac{m^2}{s^2}\cdot\frac{N\cdot s^2}{kg\cdot m}\times\frac{W\cdot s}{N\cdot m} = 350\ kW$$ ← $\dot{W}_m$

At shutoff, $V_{t2} = U_2$, so $H_0 = \frac{U_2^2}{g} = \frac{(\omega R_2)^2}{g}$

$$H_0 = \left(123\frac{rad}{s}\times\frac{0.500}{2}\,m\right)^2\frac{s^2}{9.81\,m} = 96.4\ m$$

Thus $H = H_0 - AQ^2$ or $A = \frac{H_0 - H}{Q^2}$

$$A = (96.4 - 39.0)\,m\times\frac{s^2}{(0.75)^2\,m^6} = 102\ m^{-5}s^2$$

or

$$H(m) = 96.4 - 102\left[Q(m^3/s)\right]^2 \quad (1170\ rpm)$$ ← H

At 860 rpm, $H_0' = \left(\frac{860}{1170}\right)^2 H_0 = 52.1$ m, and $A' = A$

Thus

$$H'(m) = 52.1 - 102\left[Q'(m^3/s)\right]^2 \quad (860\ rpm)$$ ← H

At BEP,

$$Q' = Q\left(\frac{\omega'}{\omega}\right) = 0.175\frac{m^3}{s}\left(\frac{860}{1170}\right) = 0.551\ m^3/s$$

$$H' = H\left(\frac{\omega'}{\omega}\right)^2 = 39\,m\left(\frac{860}{1170}\right)^2 = 21.1\ m$$

$$\eta' = \eta = 0.82$$

$$\mathcal{P}' = \mathcal{P}\left(\frac{\omega'}{\omega}\right)^3 = 350\ kW\left(\frac{860}{1170}\right)^3 = 139\ kW$$ ← 860 rpm

Given: Wastewater treatment facility, $Q = 30$ mgd (million gallons per day) at $H = 30$ ft.

Pumps must have non-clogging impellers with estimated efficiency, $\eta = 0.65$, electric motors of 50 hp or less.

Find: (a) Determine number of motor-pump units needed.
(b) Recommend operating speed.

Solution: The total power to the fluid is

$$\dot{W}_h = \rho Q g H = \gamma Q H = 62.4 \frac{lbf}{ft^3} \times 30 \times 10^6 \frac{gal}{day} \times 30\ ft \times \frac{ft^3}{7.48\ gal} \times \frac{day}{24\ hr} \times \frac{hr}{60\ min}$$

$$\times \frac{hp \cdot min}{33000\ ft \cdot lbf} = 158\ hp$$

The required mechanical power input is

$$\dot{W}_m = \frac{\dot{W}_h}{\eta} = \frac{158\ hp}{0.65} = 243\ hp$$

Therefore 5 pump-motor units at 50 hp are needed. ← n

The flowrate per pump is

$$Q = \frac{1}{5} \times 30 \times 10^6 \frac{gal}{day} \times \frac{day}{24\ hr} \times \frac{hr}{60\ min} = 4170\ gpm$$

From Fig. 10.15, peak efficiency tends to occur with $N_S \approx 2500$.

Since $N_S = \frac{N\sqrt{Q}}{H^{3/4}}$, then

$$N = N_S \frac{H^{3/4}}{Q^{1/2}} = 2500 \times \frac{(30)^{3/4}}{(4170)^{1/2}} = 496\ rpm$$

The nearest standard speed to 496 rpm should be used. ← N

{Note: The actual installation probably would be built with $n = 6$ pumps so one could be shut down for service while 5 carry the load.}

Given: Centrifugal water pump operating at 1750 rpm. Impeller has backward-curved blades with $\beta_2 = 60°$ and $b_2 = 0.50$ in. At $Q = 350$ gpm, $V_{n2} = 11.7$ ft/sec.

Find: Estimate head this pump could deliver at 1170 rpm.

Solution: From continuity, $Q = 2\pi R_2 b_2 V_{n2}$, so

$$R_2 = \frac{Q}{2\pi b_2 V_{n2}} = \frac{1}{2\pi} \times 350 \frac{gal}{min} \times \frac{1}{0.50\,in.} \times \frac{s}{11.7\,ft} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} \times 144 \frac{in^2}{ft^2} = 3.06\,in.$$

At the new operating speed,

$$Q' = Q\frac{\omega'}{\omega} = 350\,gpm \times \frac{1170\,rpm}{1750\,rpm} = 234\,gpm$$

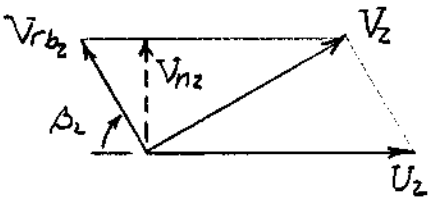
so

$$V_{n2}' = V_{n2}\frac{Q'}{Q} = 11.7\frac{ft}{s} \times \frac{1170\,rpm}{1750\,rpm} = 7.82\,ft/s$$

Also

$$U_2' = \omega' R_2 = 1170\frac{rev}{min} \times 3.06\,in. \times 2\pi\frac{rad}{rev} \times \frac{min}{60\,s} \times \frac{ft}{12\,in.} = 31.2\,ft/s$$

Constructing the velocity diagram,



$$V_{t2} = U_2 - V_{n2}\cot\beta_2$$

$$V_{t2} = 31.2\frac{ft}{s} - 7.82\frac{ft}{s} \times \cot 60° = 26.7\,ft/s$$

and

$$H = \frac{U_2 V_{t2}}{g} = 31.2\frac{ft}{s} \cdot 26.7\frac{ft}{s} \times \frac{s^2}{32.2\,ft} = 25.9\,ft$$ H

This is the theoretical head developed. It may be an over-estimate of the actual head produced by the machine.

Given: Area bound curves for pump model selection and performance curves for individual pump models, Appendix D.

Find: Use these data to verify the similarity rules for a Peerless Type 4AE12 pump operated at 1750 and 3550 nominal rpm, with 11.00 in. impeller.

Solution: From Figs. D.4 and D.5, at the best efficiency point (BEP):

N (rpm)	Q (gpm)	H (ft)	W_m (hp)	η (%)
1750	470	104	17	73+
3550	970	430	135	74+

The similarity rules are

$$\frac{Q_1}{\omega_1 D_1^3} = \frac{Q_2}{\omega_2 D_2^3}, \quad \frac{H_1}{\omega_1^2 D_1^2} = \frac{H_2}{\omega_2^2 D_2^2}, \quad \frac{P_1}{\omega_1^3 D_1^5} = \frac{P_2}{\omega_2^3 D_2^5}, \text{ and } \eta_1 = \eta_2$$

Evaluating, with $D_1 = D_2$,

$$Q_1 = Q_2 \frac{\omega_1}{\omega_2} = 970 \text{ gpm} \frac{1750 \text{ rpm}}{3550 \text{ rpm}} = 478 \text{ gpm}$$

$$H_1 = H_2 \left(\frac{\omega_1}{\omega_2}\right)^2 = 430 \text{ ft} \left(\frac{1750 \text{ rpm}}{3550 \text{ rpm}}\right)^2 = 104 \text{ ft}$$

$$P_1 = P_2 \left(\frac{\omega_1}{\omega_2}\right)^3 = 135 \text{ hp} \left(\frac{1750 \text{ rpm}}{3550 \text{ rpm}}\right)^3 = 16.2 \text{ hp}$$

$$\eta_1 = \eta_2 = 0.74^+$$

Comparing shows excellent agreement.

Given: Area bound curves for pump model selection and performance curves for individual pump models, Appendix D.

Find: Use these data and the similarity rules to predict performance of a Type 10AE12 pump operated at 1150 nominal rpm, with D = 12.0 in. impeller.

Solution: With D = 12 in. diameter impeller, from Fig. D.8, at 1760 rpm,

Q (gpm)	1500	2000	2500	3000	3500	4000
H (ft)	148	141	133	123	110	95

The similarity rules are

$$\frac{Q_1}{\omega_1 D_1^3} = \frac{Q_2}{\omega_2 D_2^3}, \quad \frac{H_1}{\omega_1^2 D_1^2} = \frac{H_2}{\omega_2^2 D_2^2}$$

Evaluating, with $D_1 = D_2$,

$$Q_2 = Q_1 \frac{\omega_2}{\omega_1} = Q_1\left(\frac{1150}{1760}\right) = 0.653\, Q_1$$

$$H_2 = H_1 \left(\frac{\omega_2}{\omega_1}\right)^2 = H_1\left(\frac{1150}{1760}\right)^2 = 0.427\, H_1$$

Therefore the expected performance at 1150 rpm is:

Q (gpm)	980	1310	1630	1960	2290	2610
H (ft)	63.2	60.2	56.8	52.5	47.0	40.6

Q, H

{Note: Use of 3 significant figures probably is not justified, since the data for 1760 rpm were read from a graph.}

Given: Data from Appendix D for Peerless Type 4AE12 pump.

Find: Verify the similarity rules for the effects of diameter change at 1750 and 3550 nominal rpm.

Solution: From Figs. D.4 and D.5 at the best efficiency point (BEP):

N (rpm)	D (in.)	Q (gpm)	H (ft)	$\dot{W}_m$ (hp)	η (%)
1750	10.00	455	87	13	73+
	11.00	470	104	17	73+
	12.12	500	123	22	73+
3550	10.00	930	360	115	74+
	11.00	970	430	135	74+
	12.12	1030	500	180	74+

The similarity rules are:

$$\frac{Q_1}{\omega_1 D_1^3} = \frac{Q_2}{\omega_2 D_2^3}\ ;\ \frac{H_1}{\omega_1^2 D_1^2} = \frac{H_2}{\omega_2^2 D_2^2}\ ,\ \frac{P_1}{\omega_1^3 D_1^5} = \frac{P_2}{\omega_2^3 D_2^5}\ ,\ \text{and } \eta_1 = \eta_2 .$$

Evaluating, with $\omega_1 = \omega_2 = 1750$ rpm,

$Q_{10} = Q_{11}\left(\frac{10}{11}\right)^3 = 353$; $Q_{12} = Q_{11}\left(\frac{12.12}{11}\right)^3 = 629$; $H_{10} = 86.0$ ft, $H_{12} = 126$ ft

$P_{10} = 10.6$ hp, $P_{12} = 27.6$ hp ; η = constant

Evaluating, with $\omega_1 = \omega_2 = 3550$ rpm,

$Q_{10} = 729$ gpm, $Q_{12} = 1300$ gpm; $H_{10} = 355$ ft, $H_{12} = 522$ ft ; $P_{10} = 83.8$ hp,

$P_{12} = 219$ hp ; η = constant

Comparing results with data shows:

(1) flow rate is scaled poorly

(2) head is scaled well

(3) power is scaled poorly (because flow rate is scaled poorly)

Better results are obtained using the modified scaling rules (see p. 526); then

$Q \sim D^2$ so $Q_{10} = 388$ gpm and $P \sim D^4$ so $P_{10} = 11.6$ hp at 1750 rpm

and $Q_{10} = 802$ gpm and $P_{10} = 92.2$ hp at 3550 rpm.

Given: Data in Appendix D for Peerless Type 16A18B pump.

Find: Verify the similarity rules for (a) impeller diameter change and (b) speed change.

Solution: From Figs. D.9 and D.10 at the best efficiency point (BEP):

N (rpm)	D (in.)	Q (gpm)	H (ft)	Wm (hp)	η (%)
705	18.0	6250	42	76	86+
	17.0	5850	37	63	86+
	16.0	5600	32	54	86
880	18.0	7900	69	155	87+
	17.0	7400	59	125	87
	16.0	7100	50	105	85

The similarity rules are:

$$\frac{Q_1}{\omega_1 D_1^3} = \frac{Q_2}{\omega_2 D_2^3}, \quad \frac{H_1}{\omega_1^2 D_1^2} = \frac{H_2}{\omega_2^2 D_2^2}, \quad \frac{P_1}{\omega_1^3 D_1^5} = \frac{P_2}{\omega_2^3 D_2^5}, \text{ and } \eta_1 = \eta_2$$

Evaluating with $\omega_1 = \omega_2 = 705$ rpm,

$Q_{17} = Q_{18}\left(\frac{17}{18}\right)^3 = 5270$ gpm, $Q_{16} = 4390$ gpm; $H_{17} = H_{18}\left(\frac{17}{18}\right)^2 = 37.5$ ft,

$H_{16} = 33.2$ ft; $P_{17} = P_{18}\left(\frac{17}{18}\right)^5 = 57.1$ hp, $P_{16} = 42.2$ hp; η = constant

At 880 rpm, $Q_{17} = 6660$ gpm, $Q_{16} = 5550$ gpm; $H_{17} = 61.5$ ft, $H_{16} = 54.5$ ft;

$P_{17} = 116$ hp, $P_{16} = 86.0$ hp; η = constant

Evaluating with $D_1 = D_2 = 18.0$ in.,

$Q_{705} = Q_{880}\left(\frac{705}{880}\right) = 6330$ gpm; $H_{705} = H_{880}\left(\frac{705}{880}\right)^2 = 44.3$ ft;

$P_{705} = P_{880}\left(\frac{705}{880}\right)^3 = 79.7$ hp; η = constant

Comparing results with data shows at <u>constant speed</u>:

(1) flow rate scales poorly, (2) head scales well, (3) power scales poorly with changes in diameter.

Comparing results with data shows at <u>constant diameter</u>:

all quantities scale well with changes in speed.

Flow rate scaling may be improved using the modified procedure discussed on page 526, in which $Q \sim D^2$ and $P \sim D^4$.

Given: Performance curves for Peerless Type 4AE12 pump, Appendix D. with 12.12 in. diameter impeller.

Find: (a) Develop and plot curve-fits for 1750 and 3550 nominal rpm.
(b) Verify the effect of pump speed on scaling pump curves using the procedure of Example Problem 11.7.

Solution: Tabulate performance data and curve-fits:

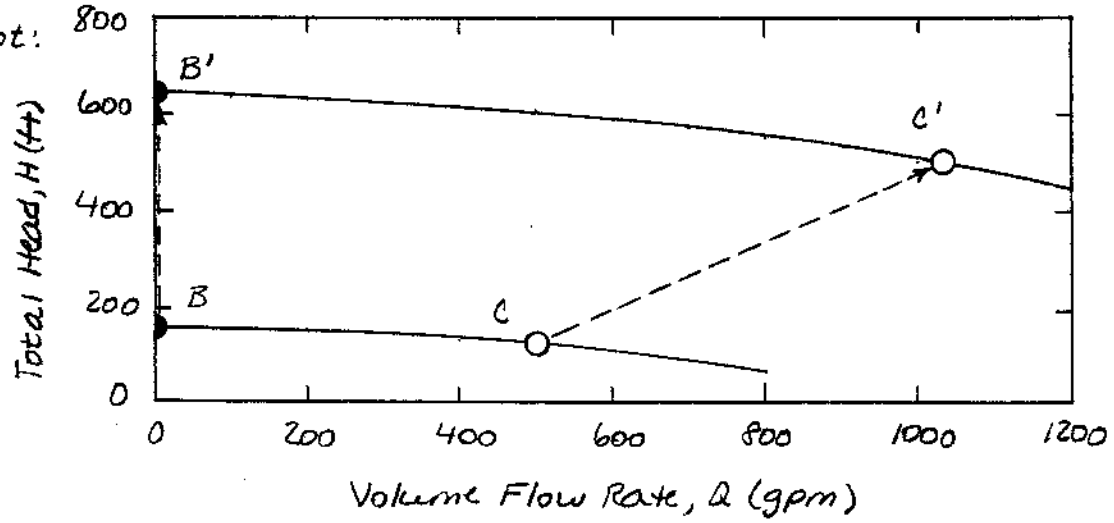
1750 rpm:	Q(gpm)	0	200	400	600	BEP:	500
	H(ft)	155	150	137	106		123

Curve-fit: $\hat{H}(ft) = 156 - 1.36 \times 10^{-4} [Q(gpm)]^2$; $r^2 = 0.994$

3550 rpm:	Q(gpm)	0	400	800	1200	1030
	H(ft)	635	620	565	445	500

Curve-fit: $\hat{H}(ft) = 641 - 1.33 \times 10^{-4} [Q(gpm)]^2$; $r^2 = 0.994$

Plot:



Using the procedure of Example Problem 11.7:

$Q_{B'} = Q_B = 0$; $H_{B'} = H_B \left(\frac{\omega'}{\omega}\right)^2 = 156 \text{ ft} \left(\frac{3550}{1750}\right)^2 = 642 \text{ ft}$

$Q_{C'} = Q_C \left(\frac{\omega'}{\omega}\right) = 500 \text{ gpm} \left(\frac{3550}{1750}\right) = 1010 \text{ gpm}$

$H_{C'} = H_C \left(\frac{\omega'}{\omega}\right)^2 = 123 \text{ ft} \left(\frac{3550}{1750}\right)^2 = 506 \text{ ft}$

Comparing with the curve-fit parameters shows good agreement:

$\hat{H} = H_0 - A Q^2$

$H_0 = 641$ ft compared to $H_0' = 642$ ft

$A' = 1.33 \times 10^{-4}$ ft/(gpm)^2

$A = 1.36 \times 10^{-4}$ ft/(gpm)^2

} within 2.3%

Given: Performance curves for Peerless Type 10AE12 pump, Appendix D.

Find: (a) Develop and plot a curve-fit for 1760 nominal rpm.
(b) Scale the curve-fit to a pump speed of 1150 nominal rpm, using the procedure of Example Problem 10.7.

Solution: Tabulate performance data and curve-fit, for D = 12 in. diameter impeller at 1760 nominal rpm:

Q(gpm)	1500	2000	2500	3000	3500	4000
H(ft)	148	141	133	123	110	95
$\hat{H}$(ft)	148	141	133	122	110	95.5

Curve-fit: $\hat{H}(ft) = 157 - 3.83\times10^{-6}[Q(gpm)]^2$; $r^2 = 0.999$

or $\hat{H} = H_0 - AQ^2$

The similarity rules are

$$\frac{Q_1}{\omega_1 D_1^3} = \frac{Q_2}{\omega_2 D_2^3} \; ; \; \frac{H_1}{\omega_1^2 D_1^2} = \frac{H_2}{\omega_2^2 D_2^2}$$

The pump diameter stays constant, so

$$Q_2 = Q_1\left(\frac{\omega_2}{\omega_1}\right) \text{ and } H_2 = H_1\left(\frac{\omega_2}{\omega_1}\right)^2 = H_1\left(\frac{1150}{1760}\right)^2 = 0.427\,H_1$$

Following the procedure of Example Problem 10.7, then at 1150 rpm,

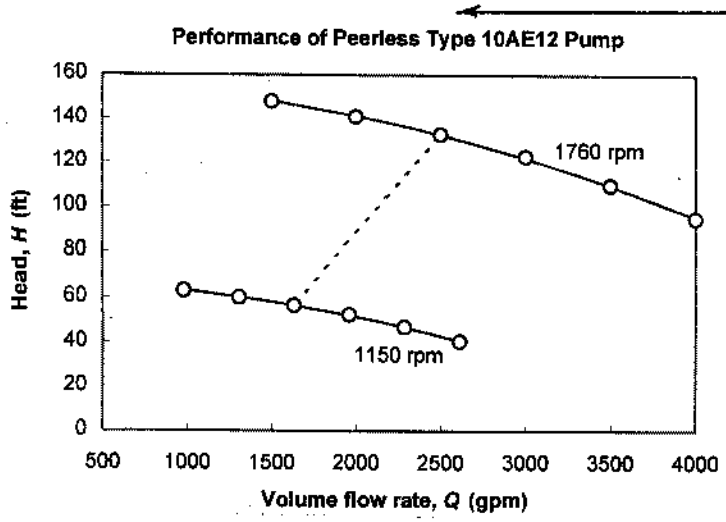
$$\hat{H}(ft) = 0.427\,H_0 - AQ^2$$

$$= (0.427)\,157\,ft - 3.83\times10^{-6}[Q(gpm)]^2$$

$$\hat{H}(ft) = 67.0\,ft - 3.83\times10^{-6}[Q(gpm)]^2$$

$\hat{H}$(ft) (1150 rpm)

The plot is:



Given: Performance curves for Peerless Type 16A 18B pump, Appendix D. (with D = 18.0 in. impeller).

Find: (a) Develop and plot curve-fits for 705 and 880 nominal rpm.
(b) Verify the effect of pump speed on scaling pump curves using the procedure of Example Problem 10.7.

Solution: Tabulate performance data and curve-fits:

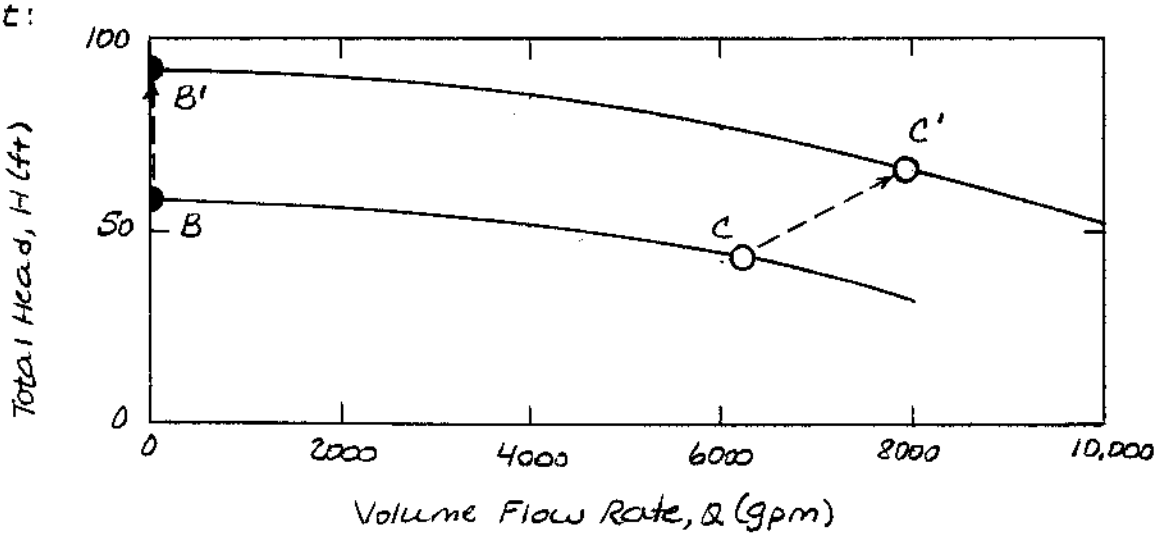
705 rpm:	Q (gpm)	0	2000	4000	6000	8000	BEP:	6250
	H (ft)	59	56	50	43	32		42

Curve-fit: $\hat{H}(ft) = 57.8 - 4.09\times10^{-7}[Q(gpm)]^2$; $r^2 = 0.994$

880 rpm:	Q (gpm)	0	2000	4000	6000	8000	BEP:	7900
	H (ft)	92	89	84	78	68		69

Curve-fit: $\hat{H}(ft) = 91.5 - 4.01\times10^{-7}[Q(gpm)]^2$; $r^2 = 0.992$

Plot:



Using the procedure of Example Problem 10.7:

$$Q_{B'} = Q_B = 0;\quad H_{B'} = H_B\left(\frac{\omega'}{\omega}\right)^2 = 59\ ft\left(\frac{880}{705}\right)^2 = 91.9\ ft$$

$$Q_{C'} = Q_C\left(\frac{\omega'}{\omega}\right) = 6250\ gpm\left(\frac{880}{705}\right) = 7800\ gpm$$

$$H_{C'} = H_C\left(\frac{\omega'}{\omega}\right)^2 = 42\ ft\left(\frac{880}{705}\right)^2 = 65.4\ ft$$

Comparing the curve-fit parameters shows good agreement:

$\hat{H} = H_0 - AQ^2$ $\quad H_0 = 91.5$ ft compared to $H_{B'} = 91.9$ ft

$A' = 4.01\times10^{-7}$ ft/(gpm)^2
$A' = 4.09\times10^{-7}$ ft/(gpm)^2
} within 2.0%

Open-Ended Problem Statement: Problem 10.16 suggests that pump head at best efficiency is typically about 70 percent of shutoff head. Use pump data from Appendix D to evaluate this suggestion. A further suggestion in Section 10-4.3 is that the appropriate scaling for tests of a pump casing with different impeller diameters is $Q \propto D^2$. Use pump data to evaluate this suggestion.

Discussion: Data selected from pump performance curves in Appendix D is tabulated and plotted on the next page. Data were selected at the maximum efficiency point for the largest (D_{max}) and smallest (D_{min}) diameter impellers with which each pump was tested.

The head at the best efficiency point with the largest impeller was selected to compare with the shutoff head for the same impeller. These data are shown in the first graph, where they are compared to the average ratio, $H_{BEP} = 0.766\ H_0$. There is some scatter, but the trend of agreement is fairly clear. The actual values suggest a higher ratio than the 0.7 mentioned in Problem 10.16.

The flow rate ratio Q_{max}/Q_{min} was compared with the square of the impeller diameter ratio $(D_{max}/D_{min})^2$. These data ratios are shown in the second graph, where they are compared to the correlation line. Agreement is not perfect, but the trend supports a positive correlation of 0.751. The predicted relationship between diameter and flow rate is $Q_{max}/Q_{min} = 0.751\ (D_{max}/D_{min})^2$.

(Use of three significant figures probably is not justified in this problem. The data are read from small graphs in the Appendix that have already been smoothed by the manufacturer. Also there is some uncertainty in selecting the best efficiency point on each curve.)

Sample	Fig.	Model	Speed (rpm)	H_0 (ft)	H_{BEP} (ft)	H_{BEP}/H_0 (—)
1	D.3	4AE11	1750	113	95	0.84
2	D.5	4AE12	3550	636	500	0.79
3	D.6	6AE14	1750	209	160	0.77
4	D.7	8AE20G	1770	430	365	0.85
5	D.8	10AE12	1760	170	112	0.66
6	D.9	16A18B	705	59	42	0.71
7	D.10	16A18B	880	92	69	0.75
					Average:	0.766

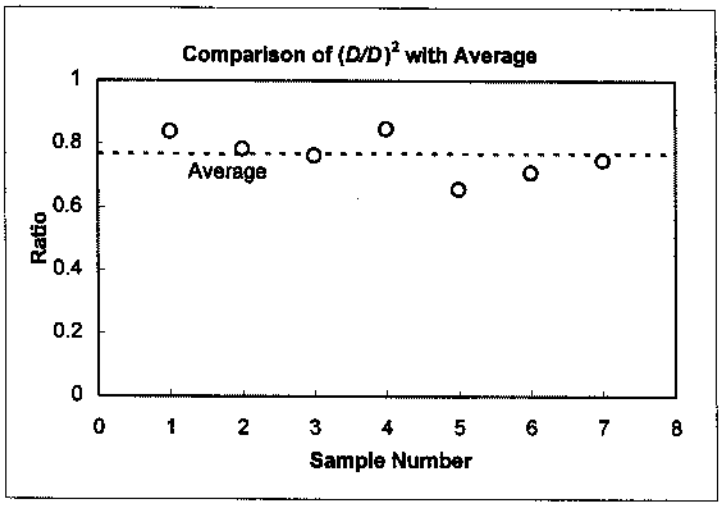
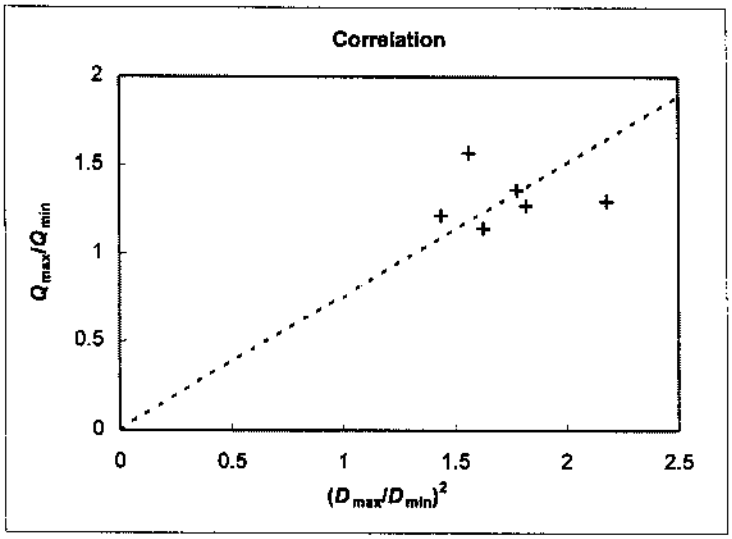


Fig.	Model	Speed (rpm)	D_{min} (in.)	Q_{BEP} (gpm)	D_{max} (in.)	Q_{BEP} (gpm)	$(D_{max}/D_{min})^2$ (—)	Q_{max}/Q_{min} (—)
D.3	4AE11	1750	7.62	740	11.25	960	2.2	1.3
D.5	4AE12	3550	9.5	910	12.12	1040	1.6	1.1
D.6	6AE14	1750	10.38	1375	14.0	1750	1.8	1.3
D.7	8AE20G	1770	16.0	2200	20.0	3450	1.6	1.6
D.8	10AE12	1760	9.0	2500	12.0	3400	1.8	1.4
D.9	16A18B	705	15.0	5100	18.0	6200	1.4	1.2
D.10	16A18B	880	15.0	6500	18.0	7900	1.4	1.2
						Correlation:		0.751



Given: Pump to operate at $Q = 250$ cfs, $H = 400$ ft, and $N = 870$ rpm.

Model test to be run in facility where $Q \le 5$ cfs and a 300 hp dynamometer is available. Assume model and prototype efficiencies are comparable.

Find: Appropriate model test speed and scale ratio.

Solution: To obtain homologous operating points, run model test at same specific speed as prototype.

$$Q = 250\,\frac{ft^3}{sec} \times 7.48\,\frac{gal}{ft^3} \times 60\,\frac{sec}{min} = 112{,}000\ gpm$$

$$Ns_{cu} = \frac{NQ^{1/2}}{H^{3/4}} = \frac{870\ rpm\,(112{,}000\ gpm)^{1/2}}{(400\ ft)^{3/4}} = 3260$$

(The dimensionless specific speed is $Ns,nd = 3260/2733 = 1.19$. Figure 10.14 indicates a mixed-flow geometry.) Figure 10.15 indicates $\eta = 0.92$ at this Ns. Thus

$$\dot{W}_p = \frac{\dot{W}_h}{\eta} = \frac{\rho Q g H}{\eta} = \frac{1}{0.92} \times 62.4\,\frac{lbf}{ft^3} \times 250\,\frac{ft^3}{s} \times 400\ ft \times \frac{hp \cdot s}{550\ ft \cdot lbf} = 12{,}300\ hp$$

For the model,

$$H_m = \eta \dot{W}_m / \rho Q g = 0.92 \times 300\ hp \times \frac{ft^3}{62.4\ lbf} \times \frac{s}{5\ ft^3} \times 550\,\frac{ft \cdot lbf}{hp \cdot s} = 487\ ft$$

To match specific speeds, then

$$N_{m_{cu}} = Ns_{cu}\frac{H_m^{3/4}}{Q_m^{1/2}} = 3260\,\frac{(487\ ft)^{3/4}}{(2240\ gpm)^{1/2}} = 7140\ rpm \longleftarrow N_m$$

The scale ratio may be obtained from the scaling laws. For example, since

$$\frac{Q_m}{\omega_m D_m^3} = \frac{Q_p}{\omega_p D_p^3} \qquad \frac{D_m}{D_p} = \left[\frac{Q_m}{Q_p}\,\frac{\omega_p}{\omega_m}\right]^{1/3} = \left[\frac{1}{50} \times \frac{870}{7140}\right]^{1/3} = 0.135 \longleftarrow D_m/D_p$$

Thus $D_m = 0.135\,D_p$ (scale ratio is $1/0.135 = 7.43$ to 1)

Check using the head ratio, $\frac{H_m}{\omega_m^2 D_m^2} = \frac{H_p}{\omega_p^2 D_p^2}$

$$H_m = H_p\left(\frac{\omega_m}{\omega_p}\right)^2\left(\frac{D_m}{D_p}\right)^2 = 400\ ft\left(\frac{7140}{870}\right)^2(0.135)^2 = 491\ ft \approx 487\ ft$$

This is acceptable agreement, considering roundoff error.

{Great care would be needed to avoid cavitation in the model pump at speeds above 7000 rpm.}

Given: Catalog data for centrifugal pump at design conditions:

$Q = 250$ gpm $\quad \Delta p = 18.6$ psi $\quad N = 1750$ rpm

Laboratory flume requires $Q_f = 200$ gpm at $H_f = 32$ ft; the only available motor develops 3 hp at 1750 rpm.

Find: (a) Is motor suitable?
(b) How might the pump/motor match be improved?

Solution: To obtain efficiency and pump power requirement, find specific speed.

$$H = \frac{\Delta p}{\rho g} = 18.6 \frac{lbf}{in.^2} \times \frac{ft^3}{62.4\ lbf} \times 144 \frac{in.^2}{ft^2} = 42.9\ ft \quad ; Q = 250 \frac{gal}{min} = 0.557\ cfs$$

$$N_{SCU} = \frac{N Q^{1/2}}{H^{3/4}} = \frac{1750\ rpm\ (250\ gpm)^{1/2}}{(42.9\ ft)^{3/4}} = 1650$$

From Fig. 10.15, $\eta \approx 0.73$. Thus

$$\dot{W}_m = \frac{\dot{W}_h}{\eta} = \frac{\rho Q g H}{\eta} = \frac{1}{0.73} \times 62.4 \frac{lbf}{ft^3} \times 0.557 \frac{ft^3}{s} \times 42.9\ ft \times \frac{hp \cdot s}{550\ ft \cdot lbf} = 3.71\ hp$$

The motor is <u>not</u> suitable to drive the pump directly. ←

The pump at 1750 rpm produces more head and flow than necessary. It may be run at reduced speed, e.g., by using a belt drive.

To produce $Q_f = 200$ gpm, solve $\frac{Q_p}{\omega_p D_p^3} = \frac{Q_f}{\omega_f D_f^3}$; $\omega_f = \frac{200}{250} \times 1750 = 1400$ rpm

To produce $H_f = 32$ ft, solve $\frac{H_p}{\omega_p^2 D_p^2} = \frac{H_f}{\omega_f^2 D_f^2}$; $\omega_f = \sqrt{\frac{H_f}{H_p}}\,\omega_p = \sqrt{\frac{32}{42.9}} \times 1750 = 1510$ rpm

At 1510 rpm the power requirement will be given by $\frac{\mathcal{P}_p}{\omega_p^3 D_p^5} = \frac{\mathcal{P}_f}{\omega_f^3 D_f^5}$, so

$$\mathcal{P}_f = \mathcal{P}_p \left(\frac{\omega_f}{\omega_p}\right)^3 = 3.71\ hp \left(\frac{1510}{1750}\right)^3 = 2.38\ hp$$

This is well within the capability of the 3 hp motor. Therefore run pump at 1510 rpm. ←

Given: A reaction turbine is designed to produce 25,000 hp at 90 rpm under 150 ft of head.

Laboratory facilities are available to provide 25 ft of head and to absorb 50 hp from the model turbine. (Assume comparable efficiencies for model and prototype turbines.)

Find: Appropriate model test speed, scale ratio, and volume flow rate.

Solution: Duplicate specific speed for dynamic similarity:

$$N_{s_{cu}} = \frac{N\mathcal{P}^{1/2}}{H^{5/4}} = \frac{(90\ rpm)(25{,}000\ hp)^{1/2}}{(150\ ft)^{5/4}} = 27.1$$

Solving for model test speed,

$$N = \frac{N_{s_{cu}} H^{5/4}}{\mathcal{P}^{1/2}} = \frac{27.1\ (25\ ft)^{5/4}}{(50\ hp)^{1/2}} = 214\ rpm \longleftarrow N$$

Under dynamically similar conditions $\frac{H_p}{\omega_p^2 D_p^2} = \frac{H_m}{\omega_m^2 D_m^2}$

$$\frac{D_m}{D_p} = \left[\frac{H_m}{H_p}\frac{\omega_p^2}{\omega_m^2}\right]^{\frac{1}{2}} = \left[\frac{H_m}{H_p}\right]^{\frac{1}{2}}\frac{\omega_p}{\omega_m} = \left(\frac{25}{150}\right)^{\frac{1}{2}}\left(\frac{90}{214}\right) = 0.172 \longleftarrow D_m/D_p$$

Also $\frac{Q_p}{\omega_p D_p^3} = \frac{Q_m}{\omega_m D_m^3}$ so $Q_m = Q_p \frac{\omega_m}{\omega_p}\left(\frac{D_m}{D_p}\right)^3$

To find Q_p, need efficiency. From Fig. 10.17, $\eta_p = 0.93$. By definition,

$$\eta = \frac{\dot{W}_m}{\dot{W}_h} = \frac{\dot{W}_m}{\rho Q g H} \; ; \; Q_p = \frac{\dot{W}_m}{\eta \rho g H}$$

$$Q_p = \frac{1}{0.93} \times 25{,}000\ hp \times \frac{ft^3}{62.4\ lbf} \times \frac{1}{150\ ft} \times 550\ \frac{ft \cdot lbf}{hp \cdot s} = 1580\ ft^3/s$$

and

$$Q_m = 1580\ \frac{ft^3}{s}\left(\frac{214\ rpm}{90\ rpm}\right)(0.172)^3 = 19.1\ ft^3/s \longleftarrow Q_m$$

Given: Model efficiency using curve-fit, $\eta = aQ - bQ^3$, where a and b are constants.

Find: (a) Describe a procedure to evaluate a and b from data.
(b) Evaluate using data for Peerless Type 10AE12 pump, with impeller diameter D = 12.0 in., operating at 1760 rpm.

Solution: From Fig. D.8, data are:

η (%)	70	75	80	84	86	86	84
Q (gpm)	1850	2100	2400	2780	3100	3700	4075

Two equations are needed to solve for constants a and b directly. A second equation may be obtained by differentiating. At peak efficiency,

$$\frac{d\eta}{dQ} = a - 3bQ^2 = 0$$

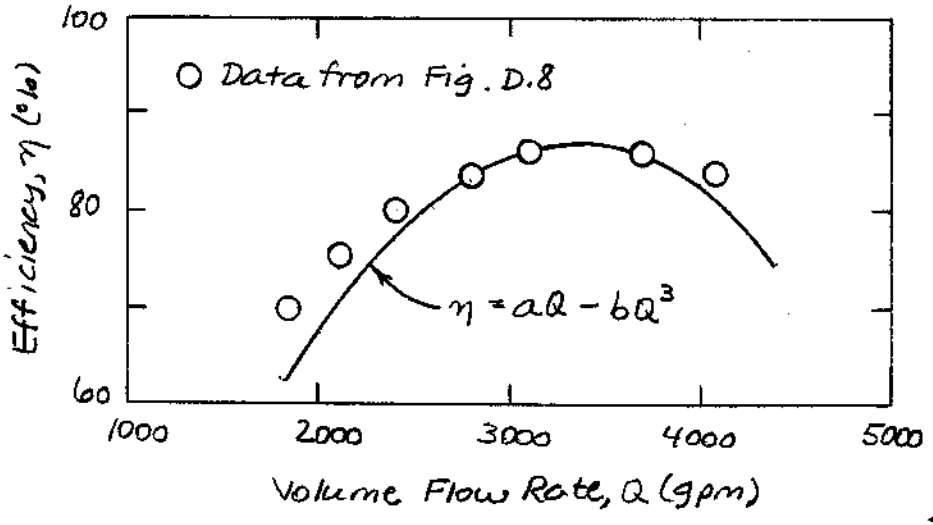
Assume peak efficiency is 87 percent at 3400 gpm. Then

$$\eta_{max} = aQ - bQ^3$$
$$0 = a - 3bQ^2$$

Substituting from the second equation into the first gives

$$\eta_{max} = 3bQ^3 - bQ^3 = 2bQ^3 \; ; \; b = \frac{\eta_{max}}{2Q^3} = 1.11 \times 10^{-9} \; ; \; a = 3bQ^2 = 0.0384$$ a, b

Plotting:



The curve-fit does a good job near peak efficiency, but tends to underestimate the measured data elsewhere.

{ An alternative curve-fit procedure is to plot η/Q versus $a - bQ^2$, then do a least-squares fit (using all the data) to obtain a and b. Then $a = 0.0426\ (gpm)^{-1}$, $b = -1.56 \times 10^{-9}\ (gpm)^{-2}$; $r^2 = 0.996$. This underestimates η at $Q > 3500$ gpm. }

Given: Model pump has $Q = 1.25$ L/s of water at 15°C, $H = 18.6$ m, at $N = 3500$ rpm.

Find: (a) Water temperature to obtain dynamically similar operation at 1750 rpm.
(b) Estimate volume flow rate and head at 1750 rpm.
(c) Comment on NPSH requirements for the tests.

Solution: For dynamic similarity, match Reynolds numbers. Thus

$$Re_1 = \frac{VD}{\nu} = \frac{\omega D^2}{\nu}\Big)_1 = Re_2 = \frac{\omega D^2}{\nu}\Big)_2$$

D = constant, so $\frac{\omega_1}{\nu_1} = \frac{\omega_2}{\nu_2}$; $\nu_2 = \nu_1 \frac{\omega_2}{\omega_1} = \nu_1 \frac{1750}{3500} = 0.5\,\nu_1$

From Table A.8, $\nu_1 = 1.27 \times 10^{-6}$ m²/s at 15°C. Thus $\nu_2 = 0.64 \times 10^{-5}$ m²/s.

From Table A.8, $\nu = 0.64 \times 10^{-5}$ m²/s when $T \approx 49$°C ← T

For similar operation, $\frac{Q_1}{\omega_1 D_1^3} = \frac{Q_2}{\omega_2 D_2^3}$ and $\frac{H_1}{\omega_1^2 D_1^2} = \frac{H_2}{\omega_2^2 D_2^2}$. Thus

$$Q_2 = Q_1 \frac{\omega_2}{\omega_1} = Q_1 \frac{1750}{3500} = 0.5\,Q_1 = 0.625 \text{ L/s}$$ ← Q_2

$$H_2 = H_1\left(\frac{\omega_2}{\omega_1}\right)^2 = H_1\left(\frac{1750}{3500}\right)^2 = 0.25\,H_1 = 4.65 \text{ m}$$ ← H_2

The liquid at $T_2 = 49$°C would be much closer to boiling. The inlet pressure would have to be raised to avoid cavitation.

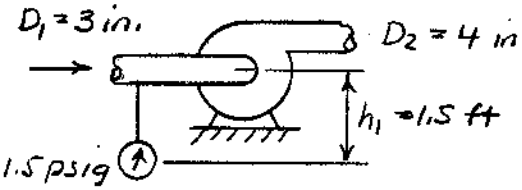
The increase would be

$$\Delta p = p_{v_2} - p_{v_1} = (0.0118 - 0.0017)\,10^6 \frac{N}{m^2} = 10.1 \text{ kPa}$$

The equivalent water column height is

$$\Delta H = \frac{\Delta p}{\rho g} = 8.90 \times 10^3 \frac{N}{m^2} \times \frac{m^3}{(0.989)\,1000 \text{ kg}} \times \frac{s^2}{9.81 \text{ m}} \times \frac{\text{kg} \cdot \text{m}}{N \cdot s^2} = 1.04 \text{ m}$$ ← ΔH

Given: Four-stage boiler feed pump:



$Q = 400$ gpm, $H = 400$ ft
$N = 3500$ rpm, $T = 240°F$

Pump to be factory certified using water at 80°F.

Find: (a) Calculate the NPSHA in the field condition.
(b) Evaluate suction head required in lab to duplicate.

Solution: NPSHA is defined as

$$NPSHA = p_t - p_v = p_g + p_{atm} + \frac{1}{2}\rho V^2 - p_v \quad \text{(expressed in feet)}$$

$$h_g = \frac{p_g}{\rho g} + z_1 = 21.5\ \frac{lbf}{in.^2} \times \frac{ft^3}{62.4\ lbf} \times 144\ \frac{in.^2}{ft^2} - 1.5\ ft = 48.1\ ft$$

$$h_{atm} = 14.7\ \frac{lbf}{in.^2} \times \frac{ft^3}{62.4\ lbf} \times 144\ \frac{in.^2}{ft^2} = 33.9\ ft$$

$$h_v = \frac{p_v}{\rho g} = 26.3\ \frac{lbf}{in.^2} \times \frac{ft^3}{62.4\ lbf} \times 144\ \frac{in.^2}{ft^2} = 60.7\ ft \quad \text{(Moran \& Shapiro)}$$

$$h_V = \frac{1}{2}\rho V^2 \frac{1}{\rho g} = \frac{V^2}{2g};\ V = \frac{Q}{A} = 400\ \frac{gal}{min} \times \frac{4}{\pi (0.25)^2 ft^2} \times \frac{ft^3}{7.48\ gal} \times \frac{min}{60\ s} = 18.2\ ft/s$$

$$h_V = \frac{1}{2} \times (18.2)^2\ \frac{ft^2}{sec^2} \times \frac{sec^2}{32.2\ ft} = 5.14\ ft$$

$$NPSHA = 48.1 + 33.9 + 5.14 - 60.7 = 26.4\ ft$$ ← Field NPSHA

In the lab, NPSHA must be the same. Thus, since $h_v = 1.17$ ft,

$$h_g = NPSHA - h_{atm} - h_V + h_v \quad \text{(from Table A.7)}$$

$$= 26.4\ ft - 33.9\ ft - 5.14\ ft + 1.17\ ft$$

$$h_g = -11.5\ ft$$

$$h_{abs} = h_g + h_{atm} = -11.5 + 33.9 = 22.5\ ft\ H_2O \text{ absolute}$$ ← Lab NPSH

Expressed in psi,

$$p_{abs} = \rho g\, h_{abs} = 62.4\ \frac{lbf}{ft^3} \times 22.5\ ft \times \frac{ft^2}{144\ in.^2} = 9.73\ psia$$ ← Lab

Given: Test data for Peerless Type 1430 pump operated at 1750 rpm:

Flow Rate, Q (gpm)	200	400	600	800	1000
Net positive suction head required, *NPSHR* (ft)	7.8	9.8	13.6	19.2	28.7

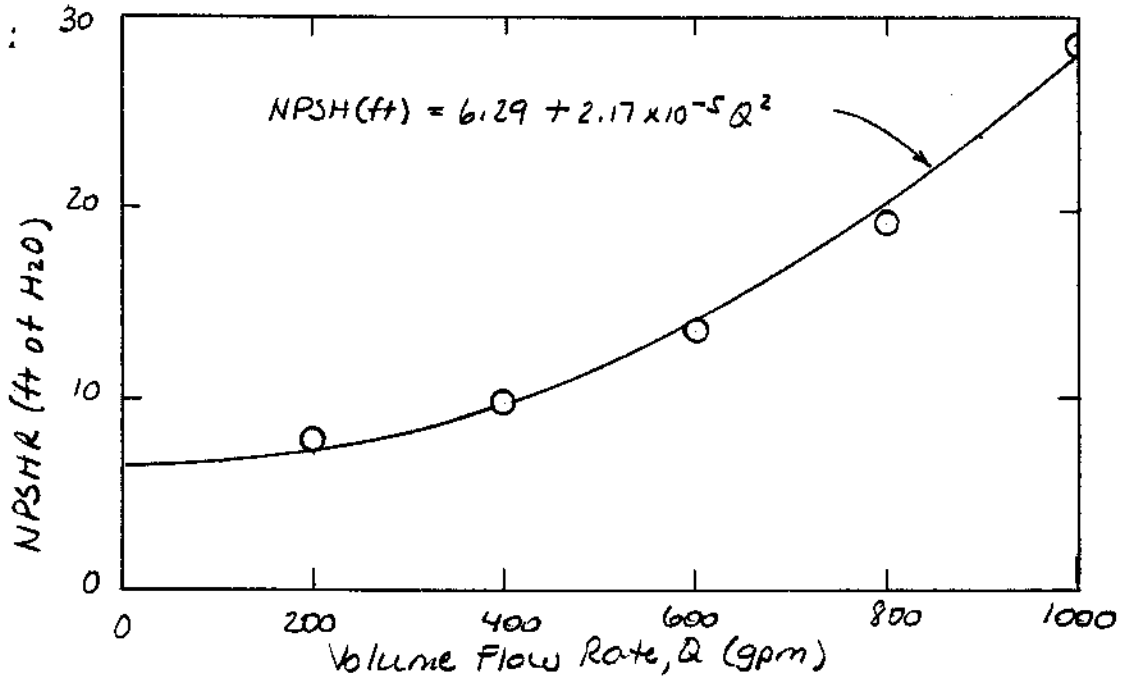
Find: (a) Develop and plot a curve-fit equation for NPSHR versus volume flow rate in the form $NPSHR = a + bQ^2$.
(b) Compare with measured data.

Solution: The curve-fit obtained by linear regression of NPSHR versus Q^2 is

$$NPSHR(ft) = 6.29 + 2.17\times10^{-5}[Q(gpm)]^2 \; ; \; r^2 = 0.992$$

Thus $a = 6.29$ ft and $b = 2.17\times10^{-5}$ ft/(gpm)^2 ← a, b

Plotting:



Given: Test data for Peerless Type 1440 pump, operated at 1750 rpm:

Flow Rate, Q (gpm)	200	400	600	800	1000	1200	1300
Net positive suction head required, *NPSHR* (ft)	5.5	6.3	8.5	11.3	15.8	22.0	27.0

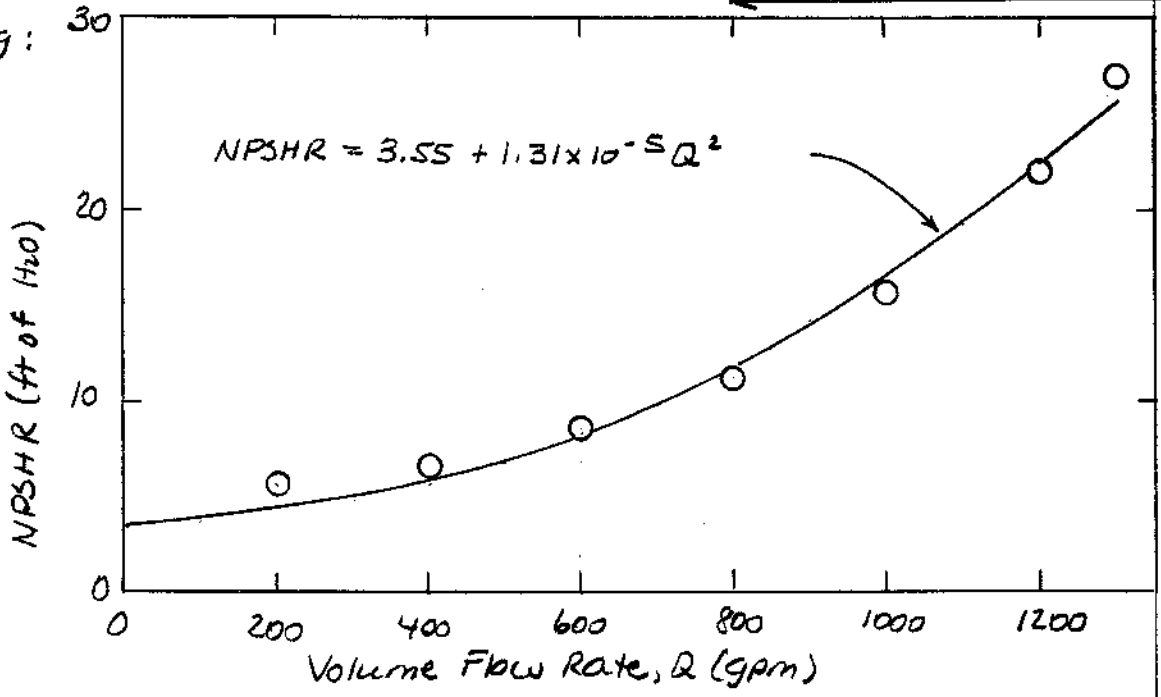
Find: (a) Develop and plot a curve-fit equation for NPSHR versus volume flow rate in the form $NPSHR = a + bQ^2$.
(b) Compare with measured data.

Solution: The curve-fit obtained by linear regression of NPSHR versus Q^2 is

$$NPSHR(ft) = 3.55 + 1.31 \times 10^{-5}[Q(gpm)]^2; \quad r^2 = 0.980$$

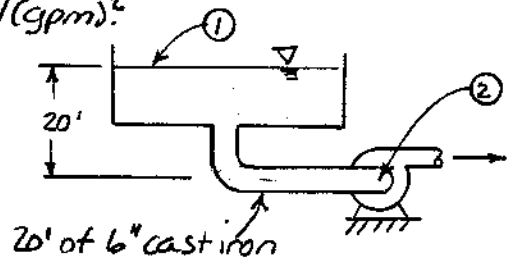
Thus $a = 3.55$ ft and $b = 1.31 \times 10^{-5}$ ft/(gpm)^2 a, b

Plotting:



Given: NPSHR by a pump on lab water is $H_r = H_0 + AQ^2$, where $H_0 = 10$ ft and $A = 4.1 \times 10^{-5}$ ft/(gpm)^2.

Supply pipe system:



Find: Maximum volume flow rate for operation without cavitation.

Solution: Apply the energy equation for pipe flow and definition of NPSH.

Computing equations: $\frac{p_1}{\rho g} + \alpha\frac{\bar{V}_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \alpha_2\frac{\bar{V}_2^2}{2g} + z_2 + \frac{h_{LT}}{g}$; $h_{LT} = f\frac{L}{D}\frac{\bar{V}^2}{2} + K + f\frac{L_e}{D}$

($\alpha_1 = 0(1)$, $\bar{V}_1 \approx 0(2)$, $z_2 = 0(4)$)

$$NPSHA = \frac{p_{t,abs} - p_v}{\rho g} = \frac{p_2}{\rho g} + \frac{p_{atm}}{\rho g} + \frac{\bar{V}_2^2}{2g} - \frac{p_v}{\rho g}$$

$K_{ent} = 0.5$

$\frac{L_e}{D}$(elbow) = 12

Assumptions: (1) $p_1 = 0$ gage, (2) $\bar{V}_1 \approx 0$, (3) $\alpha_2 \approx 1$, (4) $z_2 = 0$

Then $p_2 = \rho g z_1 - \rho\frac{\bar{V}_2^2}{2} - \rho\left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}_2^2}{2} = \rho g\left\{z_1 - \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K + 1\right]\right\}\frac{\bar{V}_2^2}{2g}$ (1)

We can obtain an expression for p_2 in the form $p_2 = \rho g(C_1 - C_2 Q^2)$.

Evaluating for $Q = 500$ gpm (1.11 ft^3/s) at $T = 68°F$ ($p_v = 0.339$ psia),

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 1.11\frac{ft^3}{s} \times \frac{1}{(0.5)^2 ft^2} = 5.65 \text{ ft/s} \quad ; A = 0.196 \text{ ft}^2$$

$$Re = \frac{\bar{V}D}{\nu} = 5.65\frac{ft}{s} \times 0.5 ft \times \frac{s}{1.10\times10^{-5} ft^2} = 2.57 \times 10^5 ; \frac{e}{D} = 0.00170; f = 0.023$$

Then $C_1 = 20.0$ ft and $C_2 = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K + 1\right]\frac{1}{2gA^2}$ ($e = 0.00085$ ft, Table 8.1)

$$C_2 = \left[0.023\left(\frac{20}{0.5} + 12\right) + 0.5 + 1\right]\frac{1}{2} \times \frac{s^2}{32.2 ft} \times \frac{1}{(0.196)^2 ft^4} \times \frac{ft^6}{(7.48)^2 gal^2} \times \frac{min^2}{3600 s^2} = 5.41\times10^{-6}\frac{ft}{(gpm)^2}$$

Expressing NPSHA as a pressure, we obtain for cold water

$$NPSHA = p_2 + p_{atm} + \rho\frac{\bar{V}_2^2}{2} - p_v = (7.32 + 14.7 + 0.215 - 0.339)\text{psia} = 21.9 \text{ psia}$$

This is equivalent to 56 ft of lab water. Performing similar calculations at other flow rates yields

$$NPSHA(\text{psia}) = 23.0 - 4.42\times10^{-6}[Q(\text{gpm})]^2 \quad (68°F \text{ water})$$

$$\text{or } NPSHA(\text{ft}) = 56.0 - 1.02\times10^{-5}[Q(\text{gpm})]^2 \quad (68°F \text{ water})$$

The system operating point can be at NPSHA = NPSHR, or

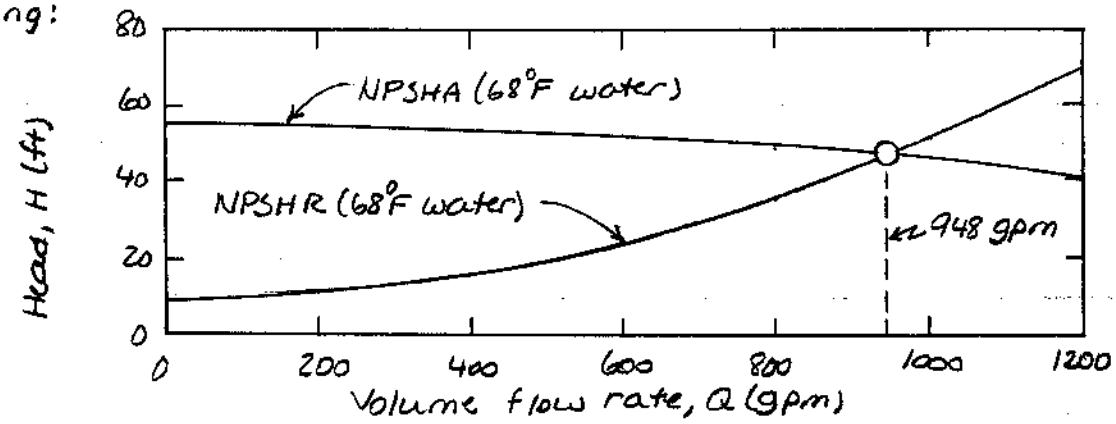
$$NPSHR = H_0 + AQ^2 = NPSHA = H_a - BQ^2$$

Solving for Q,

$$Q = \left[\frac{H_a - H_0}{A+B}\right]^{1/2} = \left[(56.0 - 10.0)\,\text{ft} \times \frac{(\text{gpm})^2}{(1.02 + 4.10)10^{-5}\,\text{ft}}\right]^{\frac{1}{2}} = 948\ \text{gpm}$$

Q_{max}

Plotting:



Given: Flow system and pump of Problem 10.46.

Find: Maximum flow rate for hot water and plot vs. temperature, including the density variation.

Solution: From the statement of Problem 10.46, the required NPSH is

$$H_r = H_{0r} + AQ^2 \quad \text{where } H_{0r} = 10 \text{ ft of water (68°F)}$$

$$A = 4.1 \times 10^{-5} \text{ ft/(gpm)}^2$$

From the analysis of Problem 10.46, the available NPSH is

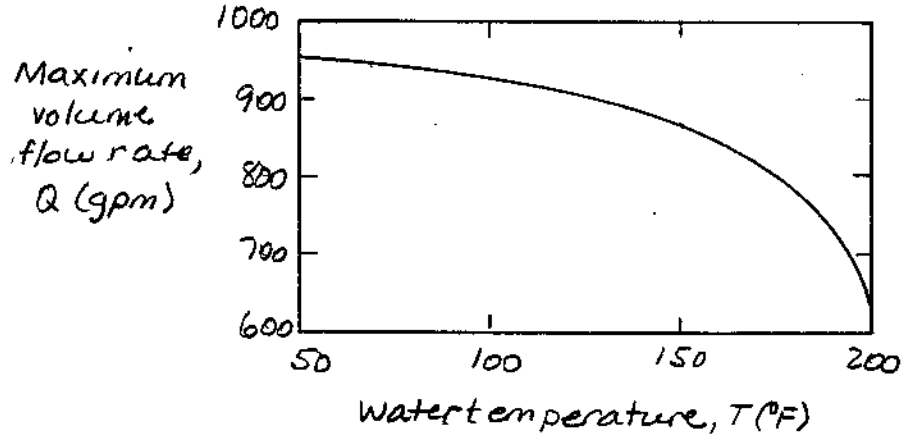
$$H_a = H_{0a} - BQ^2 \quad \text{where } H_{0a} = 56.0 \text{ ft of water (68°F)}$$

$$B = 1.02 \times 10^{-5} \text{ ft/(gpm)}^2 \quad (68°F)$$

Increasing water temperature raises NPSHR because the vapor pressure goes up. Increasing water temperature lowers NPSHA for two reasons: (1) density decreases, lowering the static head, and (2) vapor pressure goes up. Thus the crossover point moves toward lower volume flow rate. Results may be tabulated as follows:

T (°F)	p_v (psia)	ρ (slug/ft³)	ν (ft²/sec)	B (ft/(gpm)²)	H_0 (ft)	Q_{max} (gpm)
50	0.178	1.940	1.41×10^{-5}	1.02×10^{-5}	9.6	952
68	0.339	1.940	1.07	1.02	10.0	948
100	0.950	1.927	0.739	1.01	11.4	934
150	3.72	1.902	0.478	1.00	17.8	865
200	11.5	1.868	0.341	0.982	35.8	631

Plotting versus water temperature:



Open-Ended Problem Statement: A large deep fryer at a snack-food plant contains hot oil that is circulated through a heat exchanger by pumps. Solid particles and water droplets coming from the food product are observed in the flowing oil. What special factors must be considered in specifying the operating conditions for the pumps?

Discussion: Any solid particles must be able to pass through the pumps without clogging. If the particles are large, this may require larger than normal clearances within the pumps.

If the water droplets flashed to steam, they would form local pockets of water vapor. The pockets of water vapor would disrupt the flow patterns in the pumps in the same way as cavitation in a homogeneous liquid. To prevent this "cavitation" from occurring, static pressure everywhere in the flow circuit must be maintained above the saturation pressure of the water droplets at the temperature of the flowing oil.

The net positive suction head at the pump inlets must be sufficiently high to prevent any problems from occurring within the pumps themselves.

The solid particles may act as nucleation sites, which would foster the development of vapor pockets in the flow. This might increase the net positive suction head required by the pump above that measured in tests using water. The system must be sized to maintain a large net positive suction head at the design flow rate.

Finally, the viscosity of the oil must be considered. If viscosity is high, pump performance will be degraded compared to pumping water. Then a larger pump must be specified to handle the flow requirement of the hot oil circulation system.

Given: Centrifugal pump, operating at N = 1750 rpm, lifts water between two reservoirs connected by two cast-iron pipes in series.

$L_1 = 200$ ft, $D_1 = 4$ in.; $L_2 = 200$ ft, $D_2 = 3$ in.; $z_3 = 10$ ft.

Find: (a) head requirement, (b) power need, and (c) hourly cost of electrical energy, for $Q = 150$ gpm

Solution:

Apply the energy equation to the total system for steady, incompressible flow using ③ and ④ at reservoir surfaces

Computing eq.: $\cancel{\frac{p_3}{\rho g}} + \alpha_3 \frac{V_3^2}{2g} + z_3 + H_a = \cancel{\frac{p_4}{\rho g}} + \alpha_4 \frac{V_4^2}{2g} + z_4 + \frac{h_{LT}}{g}$ (10.24b)

$$h_{LT} = f_1 \frac{L_1}{D_1} \frac{V_1^2}{2} + f_2 \frac{L_2}{D_2} \frac{V_2^2}{2}$$

Assumptions: (1) $p_3 = p_4 = p_{atm}$, $V_3 = V_4 \approx 0$
(2) neglect minor losses

Then

$$H_a = z_4 - z_3 + f_1 \frac{L_1}{D_1} \frac{V_1^2}{2g} + f_2 \frac{L_2}{D_2} \frac{V_2^2}{2g} \quad \text{.....(1)}$$

$$V_1 = \frac{Q}{A_1} = \frac{150 \text{ gal}}{\text{min}} \times \frac{\text{ft}^3}{7.48 \text{ gal}} \times \frac{\text{min}}{60 \text{ s}} \times \pi \left(\frac{1}{3}\right)^2 \text{ft}^2 = 3.83 \text{ ft/s}$$

$V_2 = 6.81$ ft/s. For water at 59°F, $\nu = 1.23 \times 10^{-5}$ ft²/s (Table A.7)

$$Re_1 = \frac{V_1 D_1}{\nu} = 3.83 \frac{\text{ft}}{\text{s}} \times \frac{1}{3} \text{ft} \times 1.23 \times 10^{-5} \frac{\text{s}}{\text{ft}^2} = 1.04 \times 10^5$$

$Re_2 = 1.38 \times 10^5$. From Table 8.1 for cast iron, $e = 0.00085$ ft

$e/D_1 = 0.0026$, $e/D_2 = 0.0034$

From Fig. 8.13, $f_1 = 0.026$, $f_2 = 0.028$. Substituting into (1)

$$H_a = 10 \text{ ft} + 0.026 \times \frac{200 \times 12}{4} \times \frac{(3.83)^2 \text{ft}^2}{2 \text{ s}^2} \times \frac{\text{s}^2}{32.2 \text{ ft}} + 0.028 \times \frac{200 \times 12}{3} \times \frac{(6.81)^2 \text{ft}^2}{2 \text{ s}^2} \times \frac{\text{s}^2}{32.2 \text{ ft}}$$

$H_a = 29.7$ ft ← H_a

Specific speed. $N_{s_a} = \frac{N Q^{1/2}}{H^{3/4}} = \frac{1750 \times (150)^{1/2}}{(29.7)^{3/4}} = 1680$

From Fig 10.15, $\eta_p \approx 0.70$. Then $P_s = \frac{\dot{m} g H}{\eta_p} = \frac{\rho Q g H}{\eta_p}$

$$P_s = \frac{1}{0.7} \times \frac{150 \text{ gal}}{\text{min}} \times \frac{\text{ft}^3}{7.48 \text{ gal}} \times \frac{\text{min}}{60 \text{ s}} \times 62.4 \frac{\text{lb}}{\text{ft}^3} \times 29.7 \text{ ft} \times \frac{\text{hp} \cdot \text{s}}{550 \text{ ft} \cdot \text{lb}} = 1.61 \text{ hp}$$ ← P_s

Cost = $c P_e$. Assume $c = \$0.08$/kw·hr, $\eta_m = 0.85$

Then $P_e = \frac{P_s}{\eta_m}$ and

$$\text{Cost} = c \frac{P_s}{\eta_m} = \frac{\$0.08}{\text{kw} \cdot \text{hr}} \times \frac{1.61 \text{ hp}}{0.85} \times 0.746 \frac{\text{kw}}{\text{hp}} = \$1.13/\text{hr}$$ ← C

Given: Water supply for Grand Canyon National Park,
$L = 13{,}200$ ft, $Q = 600$ gpm
Location 1: Colorado River, $z_1 = 3734$ ft
Location 2: South Rim, $z_2 = 7022$ ft in storage tank.
Head loss due to friction is $h_{LT}/g = 290$ ft.

Find: (a) estimate diameter of commercial steel pipe.
(b) pumping power if $\eta_p = 0.61$.

Solution:

Apply the energy equation to the total system for steady incompressible flow using ① and ② at inlet and reservoir surface respectively.

Governing eq.: $\frac{p_1}{\rho g} + \alpha_1 \frac{\bar{V}_1^2}{2g} + z_1 + H_a = \frac{p_2}{\rho g} + \alpha_2 \frac{\bar{V}_2^2}{2g} + z_2 + \frac{h_{LT}}{g}$ (10.24b)

$h_{LT} = f \frac{l}{D} \frac{\bar{V}^2}{2g}$

Assumptions: (1) $p_1 = p_2 = p_{atm}$, $\bar{V}_1 \approx \bar{V}_2 \approx 0$
(2) neglect minor losses.

Then $H_a = z_2 - z_1 + \frac{h_{LT}}{g}$ ..... (1) and $\frac{h_{LT}}{g} = f \frac{l}{D} \frac{\bar{V}^2}{2g} = 290$ ft

Since $f = f(Re, \epsilon/D)$ and D is unknown, we must iterate.

For commercial steel, $\epsilon = 0.00015$ ft

The procedure is
- assume D, calculate $\bar{V}$, Re; determine f (Eqs 8.37a,b); calculate h_{LT}/g and compare to value of 290 ft.

D (in.)	V (ft/s)	Re	f_0	$f^{0.5}$	f	h_{LT}/g (ft)
12	1.70	1.46E+06	0.0139	8.517	0.0138	8.2
10	2.45	1.75E+06	0.0141	8.433	0.0141	20.8
8	3.83	2.19E+06	0.0146	8.306	0.0145	65.3
6	6.81	2.92E+06	0.0153	8.111	0.0152	289

Use $D = 6.0$ in ←

The total pump head is $H_a = 7022 - 3734 + 290 = 3578$ ft.

The pump power is $\mathcal{P}_{in} = \frac{\dot{m} g H}{\eta_p} = \frac{\rho g Q H}{\eta_p}$

$\mathcal{P}_{in} = \frac{1}{0.61} \times 62.4 \frac{lbf}{ft^3} \times 600 \frac{gal}{min} \times \frac{ft^3}{7.48 gal} \times \frac{min}{60 s} \times 3578 ft \times \frac{hp \cdot s}{550 ft \cdot lbf}$

$\mathcal{P}_{in} = 890$ hp. ← $\mathcal{P}_{in}$

Given: Peerless horizontal split-case 4AE12 pump with 11" impeller, operating at 1750 rpm, lifts water between two reservoirs connected by two cast-iron pipes in series.

$L_1 = 200$ ft, $D_1 = 4$ in.; $L_2 = 200$ ft, $D_2 = 3$ in.; $\Delta z = 10$ ft

Plot the system head curve and determine the pump operating point.

Solution:

Apply the energy equation to the total system for steady, incompressible flow using ③ and ④ at reservoir surfaces.

Governing eq.: $\cancel{\frac{p_3}{\rho g}} + \cancel{\alpha_3 \frac{V_3^2}{2g}} + z_3 + H_a = \cancel{\frac{p_4}{\rho g}} + \cancel{\alpha_4 \frac{V_4^2}{2g}} + z_4 + h_{LT}$ (10.24b)

$h_{LT} = f_1 \frac{L_1}{D_1} \frac{V_1^2}{2g} + f_2 \frac{L_2}{D_2} \frac{V_2^2}{2g}$

Assumptions: (1) $p_3 = p_4 = p_{atm}$, $V_3 = V_4 \approx 0$

(2) Neglect minor losses

Then

$$H_a = z_4 - z_3 + f_1 \frac{L_1}{D_1} \frac{V_1^2}{2g} + f_2 \frac{L_2}{D_2} \frac{V_2^2}{2g} \quad \text{.....(1)}$$

Express V as a function of Q.

$$V_1 = \frac{Q}{A_1} = \frac{4Q}{\pi D_1^2} = \frac{4}{\pi}\left(\frac{12}{4}\right)^2 \text{ft}^2 \left(\frac{gal}{min}\right) \frac{ft^3}{7.48 gal} \frac{1 min}{60 s} = 0.0255\, Q\,(gpm)$$

$$V_2 = 0.0454\, Q\,(gpm)$$

The friction factor is determined from the Colebrook eq.

$$\frac{1}{f^{0.5}} = -2.0 \log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{Re\, f^{0.5}}\right) \quad (8.37a)$$

using the equation of Miller for the original estimate

$$f_o = 0.25\left[\log\left(\frac{\varepsilon/D}{3.7} + \frac{5.74}{Re^{0.9}}\right)\right]^{-2} \quad (8.37b)$$

Assuming $T = 59°F$, $\nu = 1.23 \times 10^{-5}$ ft^2/s (Table A.1)

$$Re_1 = \frac{V_1 D_1}{\nu} = \frac{4/12}{1.23} \times 10^5 V_1 = 2.71 \times 10^4 V_1, \quad Re_2 = 2.03 \times 10^4 V_2$$

For cast iron, $\varepsilon = 0.00085$ ft (Table 8.1)

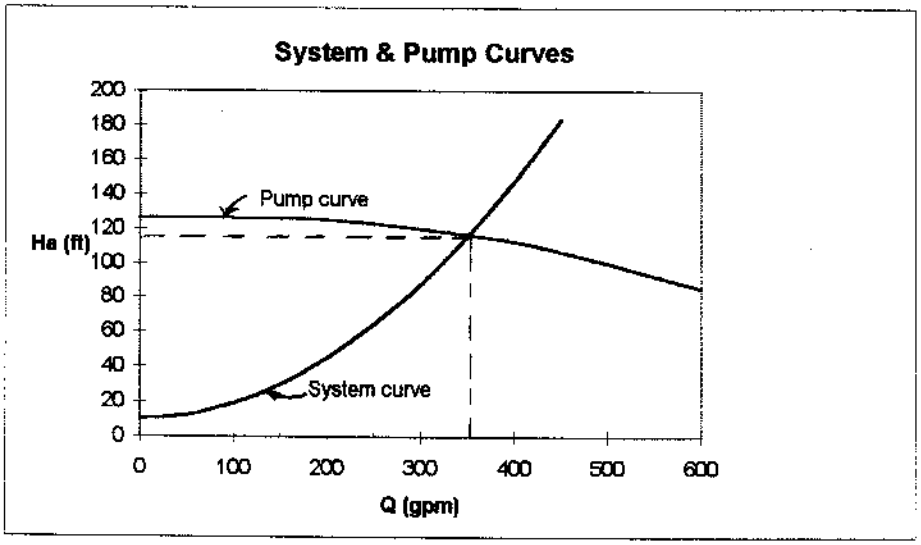
$\varepsilon/D_1 = 0.00255$ $\quad$ $\varepsilon/D_2 = 0.00340$

$$H_a = 10 \text{ ft} + f_1 \times 9.317\, V_1^2 + f_2 \times 12.42\, V_2^2$$

The pump curve is obtained from Fig. D.4

Q (gpm)	V_1 (ft/s)	Re_1	$(f_0)_1$	$(f^{-0.5})_1$	f_1	V_2 (ft/s)	Re_2	$(f_0)_2$	$(f^{-0.5})_2$	f_2	H_a (ft)
0	0.0	0.00E+00				0					10.0
50	1.3	3.46E+04	0.0290	5.905	0.0287	2.3	4.61E+04	0.0299	5.817	0.0295	12
100	2.6	6.91E+04	0.0273	6.063	0.0270	4.5	9.22E+04	0.0287	5.933	0.0284	19
150	3.8	1.04E+05	0.0266	6.154	0.0264	6.8	1.38E+05	0.0282	5.977	0.0280	30
200	5.1	1.38E+05	0.0263	6.193	0.0261	9.1	1.84E+05	0.0280	6.000	0.0278	45
250	6.4	1.73E+05	0.0261	6.217	0.0259	11.4	2.30E+05	0.0278	6.014	0.0277	64
300	7.7	2.07E+05	0.0259	6.233	0.0257	13.6	2.76E+05	0.0277	6.023	0.0276	88
350	8.9	2.42E+05	0.0258	6.245	0.0256	15.9	3.23E+05	0.0276	6.030	0.0275	115
400	10.2	2.76E+05	0.0257	6.255	0.0256	18.2	3.69E+05	0.0276	6.036	0.0275	147
450	11.5	3.11E+05	0.0256	6.262	0.0255	20.4	4.15E+05	0.0275	6.040	0.0274	183

Q (gpm)	H_p (ft)
0	126
100	126
200	125
300	120
400	113
500	100
600	85



Pump operates at Q = 350 gpm.
H = 115 ft

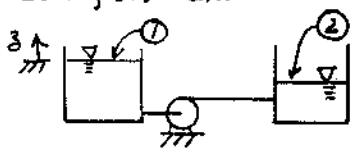
Given: Centrifugal pump installed in system with $L = 1000$ ft of $D = 16$ in. cast-iron pipe, with downstream reservoir $z = 50$ ft below level of upstream reservoir.

Find: (a) Determine and plot the system head curve.
(b) Find the volume flow rate (magnitude and direction) with the pump not operating.
(c) Estimate friction loss, power requirement, and hourly energy cost to pump water at $Q = 14{,}600$ gpm thru system.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \alpha_1 \frac{\bar{V}_1^2}{2g} + z_1 + H_p = \frac{p_2}{\rho g} + \alpha_2 \frac{\bar{V}_2^2}{2} + z_2 + \frac{h_{LT}}{g}$; $h_{LT} = \left(f\frac{L}{D} + K\right)\frac{\bar{V}^2}{2}$ ($\bar{V}_1 \approx 0$ (1), $\bar{V}_2 \approx 0$ (2))

Assumptions: (1) $p_1 = p_2 = p_{atm}$, (2) $\bar{V}_1 = \bar{V}_2 \approx 0$, (3) $z_1 = 0$, $z_2 = -50$ ft, (4) Kexit = 1



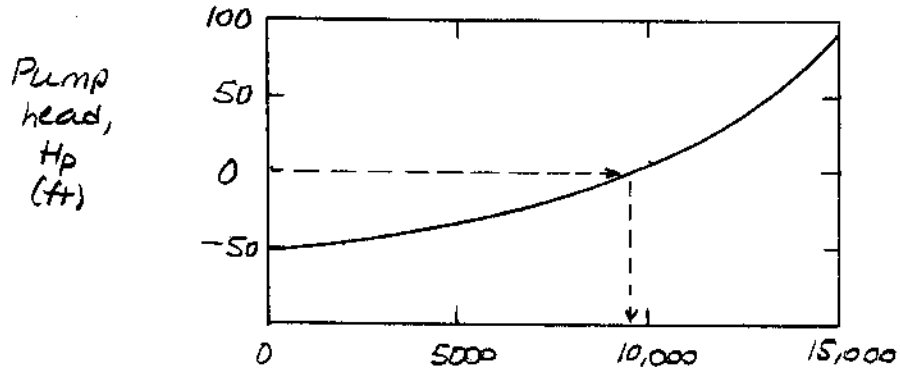
Sample calculation at $Q = 10{,}000$ gpm $(22.3\ ft^3/s)$

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 22.3 \frac{ft^3}{s} \times \frac{1}{(1.33)^2 ft^2} = 16.1\ ft/s$$

$$Re = \frac{\bar{V}D}{\nu} = 16.1 \frac{ft}{sec} \times 1.33 ft \times \frac{s}{1.05 \times 10^{-5} ft^2} = 2.00 \times 10^6 ;\ \frac{\varepsilon}{D} = 0.000638 ;\ f = 0.018 \quad (\varepsilon = 0.00085\ ft,\ \text{Table 8.1})$$

$$H_p = z_2 - z_1 + \left(f\frac{L}{D} + K\right)\frac{\bar{V}^2}{2g} = -50 ft + \left(0.018 \times \frac{1000\ ft}{1.33\ ft} + 1\right)\frac{1}{2} \times (16.1)^2 ft^2 \times \frac{s^2}{32.2 ft} = 8.5\ ft$$

Plotting results of similar calculations: ($T = 70°F$)



Plot

At zero input power, $H_p \approx 0$, and $Q \approx 9500$ gpm (dashed lines)

At $Q = 14{,}600$ gpm, $H_p = 75.4$ ft. Assuming optimum N_s (≈ 3000), from Fig. 10.15, $\eta_p \approx 0.92$. Then

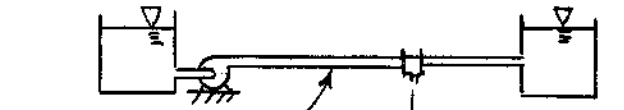
$$\dot{W}_m = \frac{\dot{W}_h}{\eta_p} = \frac{\rho Q g H}{\eta_p} = \frac{1}{0.92} \times 62.4 \frac{lbf}{ft^3} \times 32.6 \frac{ft^3}{s} \times 75.4\ ft \times \frac{hp \cdot s}{550\ ft \cdot lbf} = 303\ hp$$

$\dot{W}_m$

$$C = \frac{\dot{W}_m}{\eta_m} = \frac{1}{0.90} \times 303\ hp \times 0.746 \frac{kW}{hp} \times \$\frac{0.08}{kW \cdot hr} = \$\ 20.10/hr$$

Cost

Given: Piping system with tap at constant flow rate, $Q_3 = 300$ gpm.



$L_1 = 1500$ ft, $D_1 = 10$ in. $\quad Q_3 = 300$ gpm

$L_2 = 500$ ft, $D_2 = 6$ in.

Find: (a) Obtain and plot system head versus flow rate curve.
(b) Find delivery with pump of Fig. 10.18, operating at 1750 rpm.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \alpha_1 \frac{\bar{V}_1^2}{2g} + z_1 + H_p = \frac{p_2}{\rho g} + \alpha_2 \frac{\bar{V}_2^2}{2g} + z_2 + \frac{h_{LT}}{g}$; $h_{LT} = \left[f\left(\frac{L_1}{D_1} + \frac{L_2}{D_2}\right) + K \right] \frac{\bar{V}_2^2}{2}$

Assumptions: (1) $p_1 = p_2 = p_{atm}$, (2) $\bar{V}_1 = \bar{V}_2 \approx 0$, (3) $z_1 = z_2$, (4) $K_{exit} = 1$

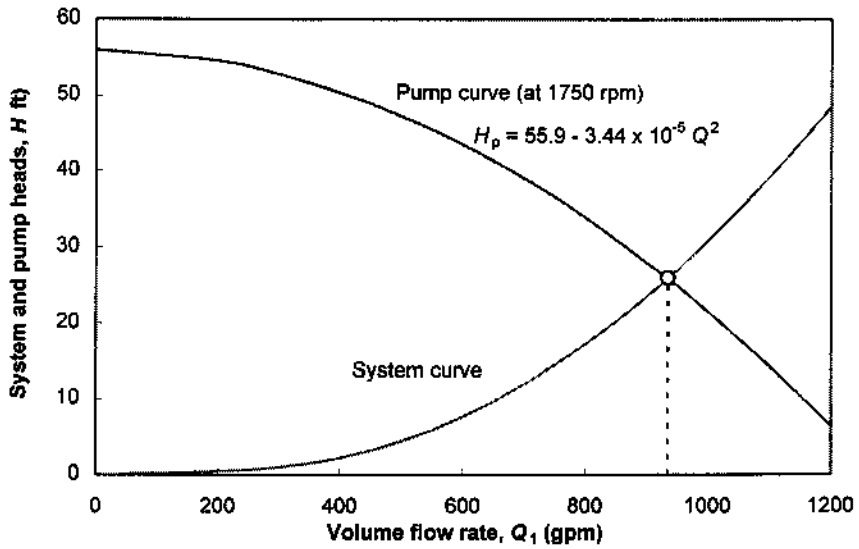
Sample calculation at $Q_1 = 500$ gpm $(1.11$ ft^3/s$)$, $Q_2 = 200$ gpm $(0.446$ ft^3/s$)$

$\bar{V}_1 = \frac{Q_1}{A_1} = \frac{4Q_1}{\pi D_1^2} = 2.04$ ft/s. $\quad \bar{V}_2 = \frac{Q_2}{A_2} = \frac{4Q_2}{\pi D_2^2} = 2.27$ ft/s. $\quad e = 0.00085$ ft (Table 8.1)

$Re_1 = 1.59 \times 10^5$, $e/D_1 = 0.00102$, $f_1 = 0.021$; $Re_2 = 1.06 \times 10^5$, $e/D_2 = 0.00170$, $f_2 = 0.024$

$H_p = f_1 \frac{L_1}{D_1} \frac{\bar{V}_1^2}{2g} + \left(f_2 \frac{L_2}{D_2} + K\right) \frac{\bar{V}_2^2}{2} = (2.44 + 2.38)$ ft $= 4.82$ ft

Plotting results of similar calculations:



Plot

The operating point may be found graphically, as shown above. Thus

$Q_p = 930$ gpm (so $Q_1 = 930$ gpm and $Q_2 = 630$ gpm) at $H_p = 26$ ft

Q_p, H_p

Given: Performance data for a pump:

H (ft)	148	140	130	115	100	75	50
Q (gpm)	0	800	1200	1600	2000	2400	2800

The system is $L = 1200$ ft of commercial steel pipe, with $D = 12$ in. and two 90° elbows and an open gate valve, connecting two open reservoirs, with an elevation increase of 50 ft.

Find: (a) Estimate the delivery with the gate valve open.
(b) Determine the gate valve loss coefficient to cut Q in half.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \frac{\alpha_1 \bar{V}_1^2}{2g} + z_1 + H_p = \frac{p_2}{\rho g} + \frac{\alpha_2 \bar{V}_2^2}{2g} + z_2 + \frac{h_{LT}}{g}$; $h_{LT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2}$

Assumptions: (1) $p_1 = p_2 = p_{atm}$, (2) $\bar{V}_1 = \bar{V}_2 \approx 0$, (3) $z_1 = 0$, $z_2 = 50$ ft, (4) $K_{exit} = 1$, $K_{ent} = 0.5$

Sample calculation at 2000 gpm (4.46 ft³/s), $T = 70°F$, so $\nu = 1.05 \times 10^{-5}$ ft²/s (Table A.7)

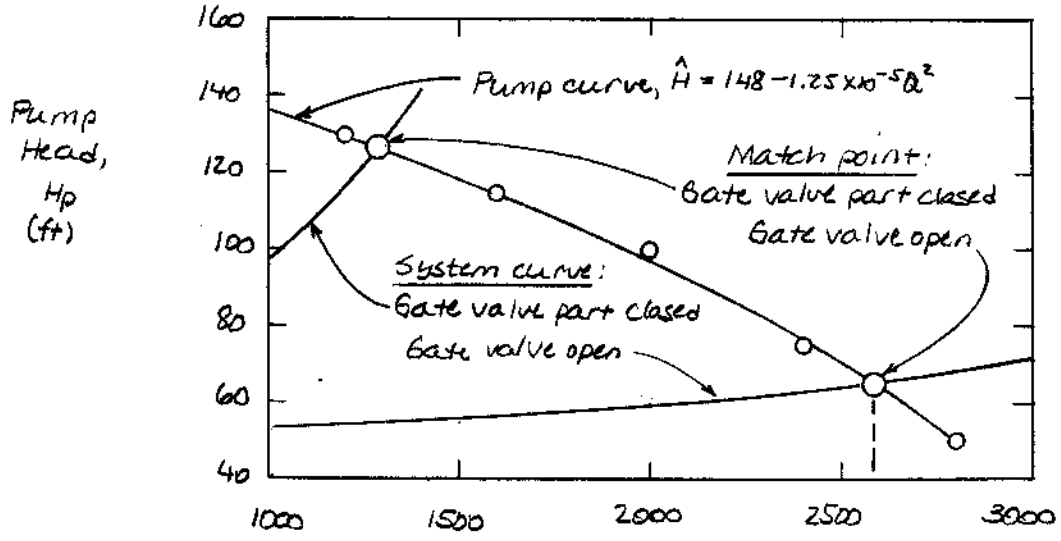
$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 4.46 \frac{ft^3}{sec} \times \frac{1}{(1)^2 ft^2} = 5.68 \text{ ft/s}; \quad Re = \frac{\bar{V}D}{\nu} = 5.41 \times 10^5$$

$e = 0.00015$ ft (Table 8.1), $e/D = 0.00015$, $f = 0.015$

$$H_p = z_2 + \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2g} = 50 \text{ ft} + \left[0.015\left\{\frac{1200}{1} + 2(12) + 8\right\} + 1 + 0.5\right]\frac{1}{2} \times \frac{(5.68)^2 ft^2}{s^2} \times \frac{s^2}{32.2 ft} = 60.2 \text{ ft}$$

(Elbow, Gate valve)

Plotting results of similar computations:



Volume flow rate with gate valve open: $Q = 2560$ gpm ← Q

Volume flow rate with gate valve partially shut: $Q = 1290$ gpm, so $H_p = 128$ ft.
By iteration, $L_e/D = 22{,}400$ ← L_e/D

Given: Performance data for two pumps of Problem 10.54 in parallel (add volume flow rates at given head):

H (ft)	148	140	130	115	100	75	50
Q (gpm)	0	1600	2400	3200	4000	4800	5600

Two open reservoirs, $\Delta z = 50$ ft, $L = 1200$ ft, $D = 12$ in., commercial steel, two 90° elbows and open gate valve.

Find: (a) Estimate delivery with gate valve open
(b) Determine gate valve loss coefficient to cut flow by half.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \frac{\alpha_1 \bar{V}_1^2}{2g}^{\approx 0\,(2)} + z_1^{=0\,(3)} + H_p = \frac{p_2}{\rho g} + \frac{\alpha_2 \bar{V}_2^2}{2g}^{\approx 0\,(2)} + z_2 + \frac{h_{LT}}{g}$; $h_{LT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2}$

Assumptions: (1) $p_1 = p_2 = p_{atm}$, (2) $\bar{V}_1 = \bar{V}_2 \approx 0$, (3) $z_1 = 0$, $z_2 = 50$ ft, (4) $K_{exit} = 1$, $K_{ent} = 0.5$

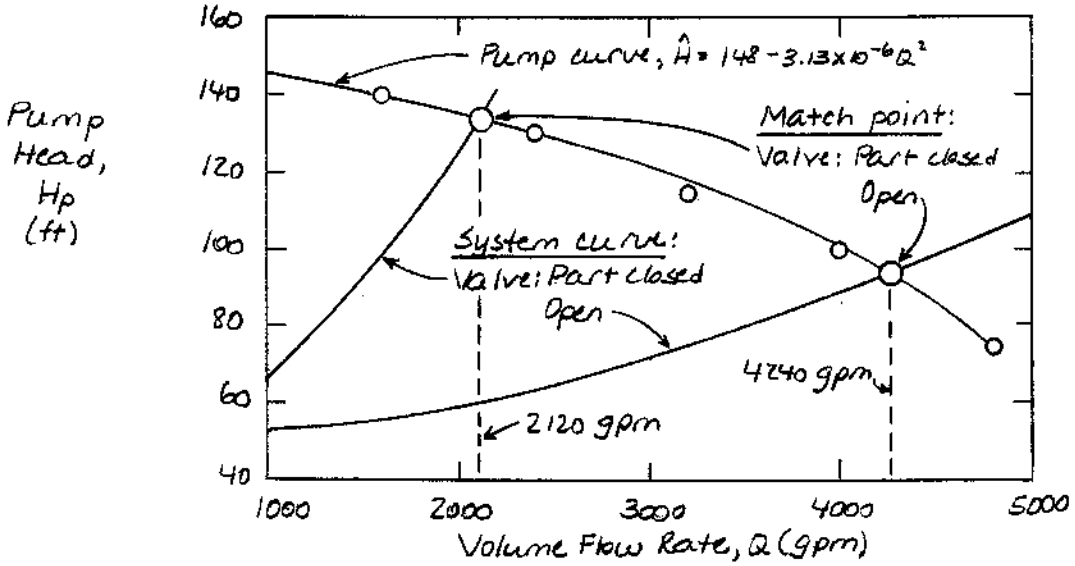
Sample calculation at $Q = 2000$ gpm ($Q = 4.46$ ft³/sec):

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 4.46 \frac{ft^3}{s} \times \frac{1}{(1)^2 ft^2} = 5.68 \text{ ft/s} \; ; \; Re = \frac{\bar{V}D}{\nu} = 5.41 \times 10^5 ; \; e/D = 0.00015 ; \; f = 0.015$$

$$H_p = z_2 + \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2g} = 50 \text{ ft} + \left[0.015\left\{\frac{1200}{1} + 2(12) + 8\right\} + 1 + 0.5\right]\frac{1}{2} \times \frac{(5.68)^2 ft^2}{s^2} \times \frac{s^2}{32.2 ft} = 60.2 \text{ ft}$$

(Elbow; Gate valve)

Plotting results of similar computations:



Volume flow rate with gate valve open: $Q = 4300$ gpm ← Q

Volume flow rate with gate valve part closed: $Q = 2150$ gpm, so $H = 134$ ft.
Finally, $e/D = 8400$ ← L_e/D

Given: Performance data for two pumps of Problem 10.54 in series (double head at each volume flow rate):

H (ft)	296	280	260	230	200	150	100
Q (gpm)	0	800	1200	1600	2000	2400	2800

Two open reservoirs, $\Delta z = 50$ ft, $L = 1200$ ft, $D = 12$ in. commercial steel, two 90° elbows and open gate valve.

Find: (a) Estimate delivery with gate valve open.
(b) Determine gate valve loss coefficient to cut flow by half.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \frac{\alpha_1 \bar{V}_1^2}{2g}^{\approx 0(2)} + z_1 + H_p = \frac{p_2}{\rho g} + \frac{\alpha_2 \bar{V}_2^2}{2g}^{\approx 0(2)} + z_2 + \frac{h_{LT}}{g}$; $h_{LT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2}$

Assumptions: (1) $p_1 = p_2 = p_{atm}$, (2) $\bar{V}_1 = \bar{V}_2 \approx 0$, (3) $z_1 = 0$, $z_2 = 50$ ft, (4) $K_{ent} = 0.5$, $K_{exit} = 1$

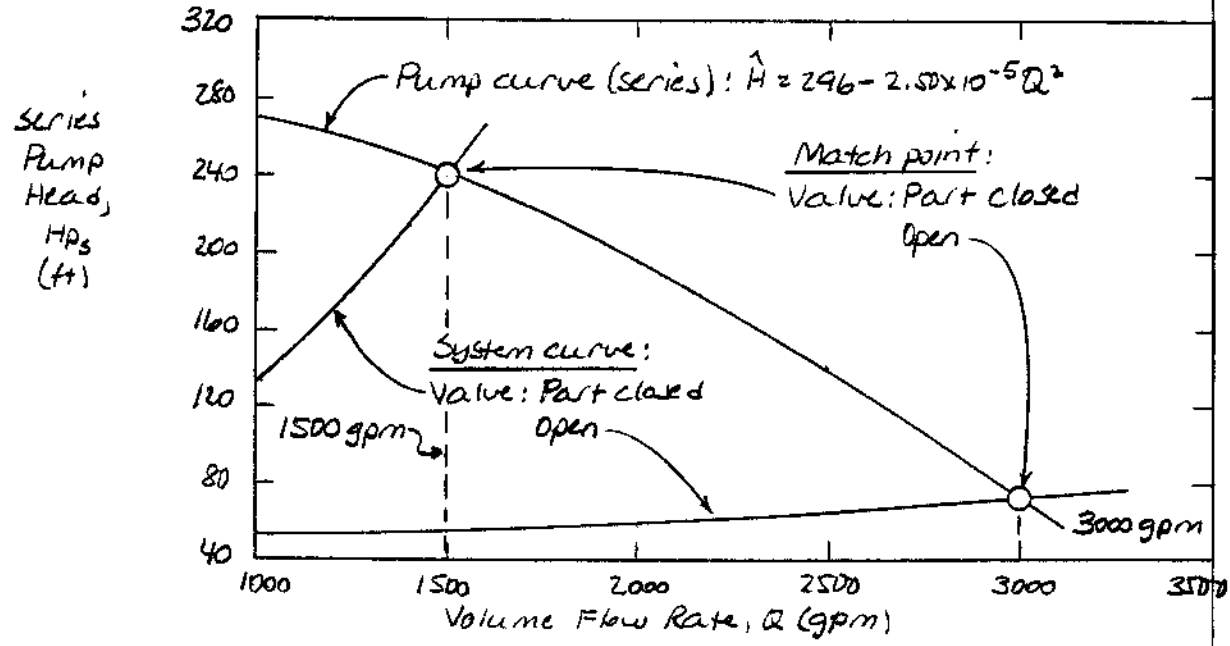
Sample calculation at $Q = 2000$ gpm ($Q = 4.46$ ft³/sec):

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times \frac{4.46 \text{ ft}^3}{\text{s}} \times \frac{1}{(1)^2 \text{ ft}^2} = 5.68 \text{ ft/s}; \quad Re = \frac{\bar{V}D}{\nu} = 5.41 \times 10^5; \quad \varepsilon/D = 0.0015; \quad f = 0.015$$

$$H_p = z_2 + \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2g} = 50 \text{ ft} + \left[0.015\left\{\frac{1200}{1} + 2(12) + 8\right\} + 1.5\right]\frac{1}{2} \times (5.68)^2 \frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{s}^2}{32.2 \text{ ft}} = 60.2 \text{ ft}$$

(Elbow (2), Valve)

Plotting results of similar computations:



Volume flow rate with gate valve open: $Q = 3000$ gpm → Q

Volume flow rate with gate valve part closed: $Q = 1500$ gpm, so $H = 240$ ft.
Finally, $\varepsilon/D = 42{,}000$ → L_e/D

Given: Performance data for a pump:

H(ft)	148	140	130	115	100	75	50
Q(gpm)	0	800	1200	1600	2000	2400	2800

Pipe Age	Multiplier on f
20	2.0
40	2.4

Two open reservoirs, $\Delta z = 50$ ft, $L = 1200$ ft, $D = 12$ in. commercial steel, two 90° elbows, and open gate valve.

Find: (a) Estimate the percentage reductions in flow after 20 and 40 years.
(b) Repeat if pump head reduced 10 percent in 20, 25 percent in 40 years.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \frac{\alpha_1 \bar{V}_1^2}{2g} + z_1 + H_p = \frac{p_2}{\rho g} + \frac{\alpha_2 \bar{V}_2^2}{2g} + z_2 + \frac{h_{LT}}{g}$; $h_{LT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K \right] \frac{\bar{V}^2}{2}$

Assumptions: (1) $p_1 = p_2 = p_{atm}$, (2) $\bar{V}_1 = \bar{V}_2 \approx 0$, (3) $z_1 = 0$, $z_2 = 50$ ft, (4) $K_{ent} = 0.5$, $K_{exit} = 1$

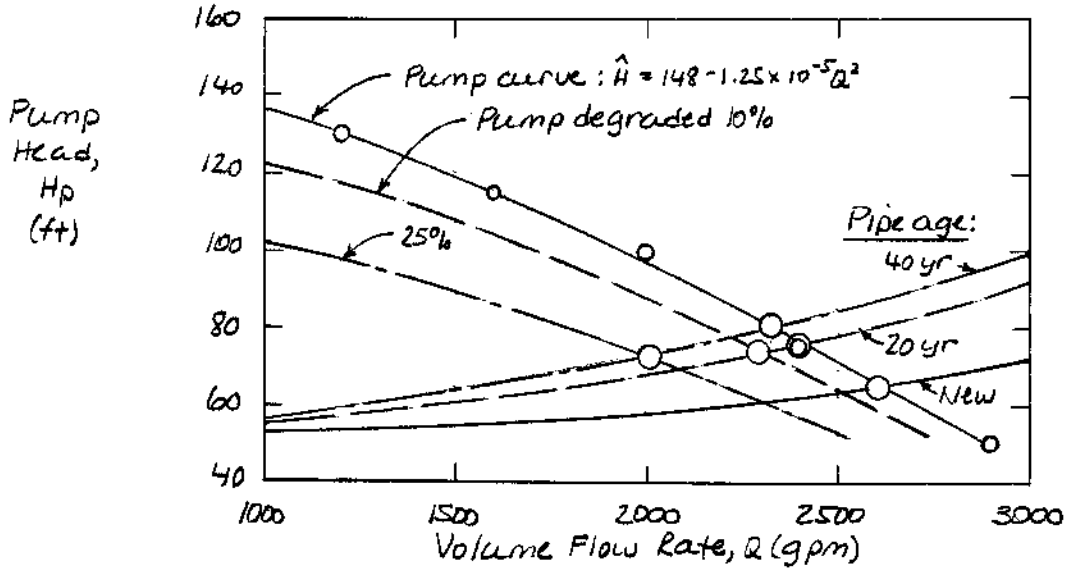
Sample calculation at $Q = 2000$ gpm ($Q = 4.46$ ft³/s):

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 4.46 \frac{ft^3}{s} \times \frac{1}{(1)^2 ft^2} = 5.68 \text{ ft/s}; \quad Re = \frac{\bar{V} D}{\nu} = 5.41 \times 10^5; \quad \frac{e}{D} = 0.00015; \quad f = 0.015$$

$$H_p = z_2 + \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K \right] \frac{\bar{V}^2}{2g} = 50 \text{ ft} + \left[0.015 \left\{ \frac{1200}{1} + 2(12) + 8 \right\} + 1.5 \right] \frac{1}{2} \times \frac{(5.68)^2 ft^2}{s^2} \times \frac{s^2}{32.2 ft} = 60.2 \text{ ft}$$

(Elbow (2), valve (8))

Plotting results of similar calculations:



Results (in gpm) are:

		Pipe Age		
		New	20 yr	40 yr
Pump Performance (%)	100	2570	2390	2330
	90		2280	
	75			2020

Flow rate reductions are −7.0% (20 yr) and −9.3% (40 yr) with constant pump performance and −11.3% (20 yr) and −21.4% (40 yr) with degraded pump. ΔQ

Given: Pumps of Problem 10.55 in parallel (double Q at same H), pipe age:

H (ft)	148	140	130	115	100	75	50
Q (gpm)	0	1600	2400	3200	4000	4800	5600

Age (yr)	20	40
x_f	2.0	2.4

Two open reservoirs, $\Delta z = 50$ ft, $L = 1200$ ft, $D = 12$ in., commercial steel, two 90° elbows and open gate valve.

Find: (a) Estimate percentage reductions in flow after 20 and 40 years.
(b) Repeat if pump head reduced 10 percent in 20, 25 percent in 40 years.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \frac{\alpha_1 \bar{V}_1^2}{2g} + z_1 + H_p = \frac{p_2}{\rho g} + \frac{\alpha_2 \bar{V}_2^2}{2g} + z_2 + \frac{h_{LT}}{g}$; $h_{LT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2}$

Assumptions: (1) $p_1 = p_2 = p_{atm}$, (2) $\bar{V}_1 = \bar{V}_2 \approx 0$, (3) $z_1 = 0$, $z_2 = 50$ ft, (4) $K_{ent} = 0.5$, $K_{exit} = 1$

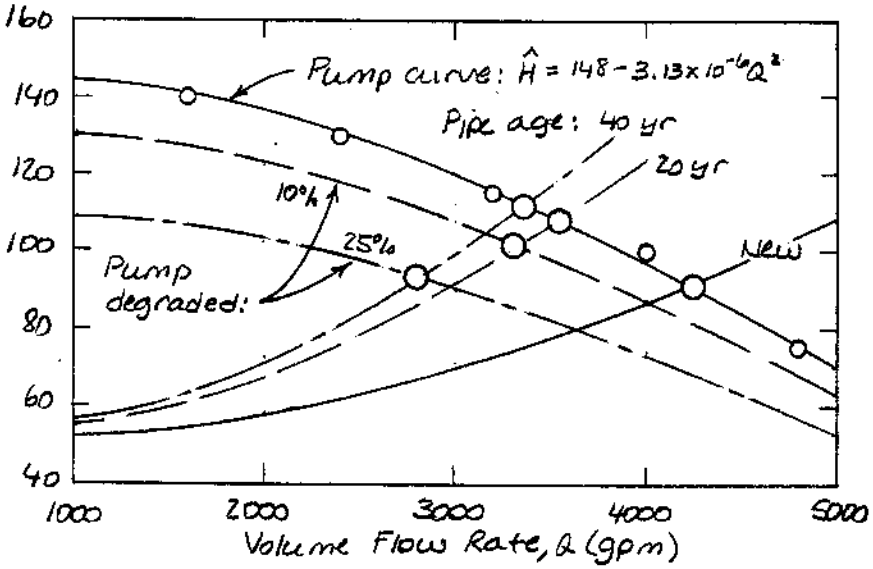
Sample calculation at $Q = 2000$ gpm ($Q = 4.46$ ft³/sec):

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 4.46 \frac{ft^3}{s} \times \frac{1}{(1)^2 ft^2} = 5.68 \text{ ft/s}; \quad Re = \frac{\bar{V}D}{\nu} = 5.41 \times 10^5; \quad \varepsilon/D = 0.00015; \quad f = 0.015$$

$$H_p = z_2 + \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2g} = 50 \text{ ft} + \left\{0.015\left\{\frac{1200}{1} + 2(12) + 8\right\} + 1.5\right\}\frac{1}{2} \times \frac{(5.68)^2 ft^2}{s^2} \times \frac{s^2}{32.2 ft} = 60.2 \text{ ft}$$

(Elbow (2), Valve)

Plotting results:



Results (in gpm) are:

		Pipe Age		
		New	20 yr	40 yr
Pump Performance (%)	100	4290	3560	3370
	90		3270	
	75			2790

Flow rate reductions are −17.0% (20 yr) and −21.4% (40 yr) with constant pump performance and −23.8% (20 yr) and −35.0% (40 yr) with degraded pump. ΔQ

Given: Pumps of Problem 10.46 in series (double H at each Q), pipe aging data:

H (ft)	296	280	260	230	200	150	100
Q (gpm)	0	800	1200	1600	2000	2400	2800

Age (yr)	20	40
x f	2.0	2.4

Two open reservoirs, $\Delta z = 50$ ft, $L = 1200$ ft, $D = 12$ in., commercial steel, two 90° elbows and open gate valve.

Find: (a) Estimate percentage reductions in flow after 20 and 40 years.
(b) Repeat if pump head reduced 10 percent in 20, 25 percent in 40 years.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \frac{\alpha_1 \bar{V}_1^2}{2g} + z_1 + H_p = \frac{p_2}{\rho g} + \frac{\alpha_2 \bar{V}_2^2}{2g} + z_2 + \frac{h_{eT}}{g}$; $h_{eT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2}$

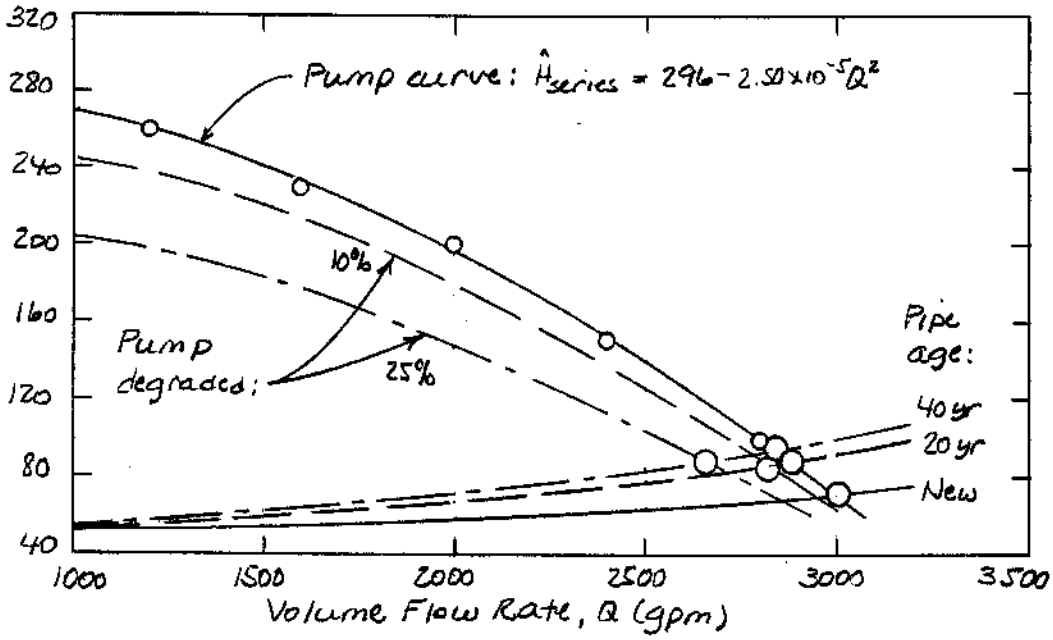
Assumptions: (1) $p_1 = p_2 = p_{atm}$, (2) $\bar{V}_1 = \bar{V}_2 \approx 0$, (3) $z_1 = 0$, $z_2 = 50$ ft, (4) $K_{ent} = 0.5$, $K_{exit} = 1$

Sample calculation at $Q = 2000$ gpm ($Q = 4.46$ ft^3/sec):

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 4.46 \frac{ft^3}{s} \times \frac{1}{(1)^2 ft^2} = 5.68 \text{ ft/s}; \quad Re = \frac{\bar{V}D}{\nu} = 5.41 \times 10^5; \quad \frac{\varepsilon}{D} = 0.00045; \quad f = 0.015$$

$$H_p = z_2 + \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2g} = 50 \text{ ft} + \left[0.015\left\{\frac{1200}{1} + 2(12) + 8\right\} + 1.5\right]\frac{1}{2} \times \frac{(5.68)^2 ft^2}{s^2} \times \frac{s^2}{32.2 ft} = 60.2 \text{ ft}$$

Plotting results:



Results (in gpm) are:

		Pipe Age		
		New	20 yr	40 yr
Pump Performance (%)	100	3000	2880	2840
	90		2820	
	75			2660

Flow rate reductions are −4.0% (20 yr) and −5.3% (40 yr) with constant pump performance and −6.0% (20 yr) and −11.3% (40 yr) with degraded pump. ΔQ

Given: Water supply for Englewood, Colorado; L = 5800 ft, D = 27 in., steel.

Section 1: North Platte River at $z_1 = 5280$ ft.

Section 2: Reservoir at $z_2 = 5310$ ft

Design flow rates: Q = 31 cfs (initially), 38 cfs (ultimately).

Find: (a) Calculate and plot the system resistance curve.
(b) Specify an appropriate pumping system.
(c) Estimate power required for steady-state operation at both flow rates.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \frac{\alpha_1 \bar{V}_1^2}{2g} + z_1 + H_p = \frac{p_2}{\rho g} + \frac{\alpha_2 \bar{V}_2^2}{2g} + z_2 + \frac{h_{LT}}{g}$; $h_{LT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K \right] \frac{\bar{V}^2}{2g}$

($\alpha_1 \approx 1.0$ (2); $\alpha_2 \approx 1.0$ (2); $\frac{L_e}{D} = 0$ (4))

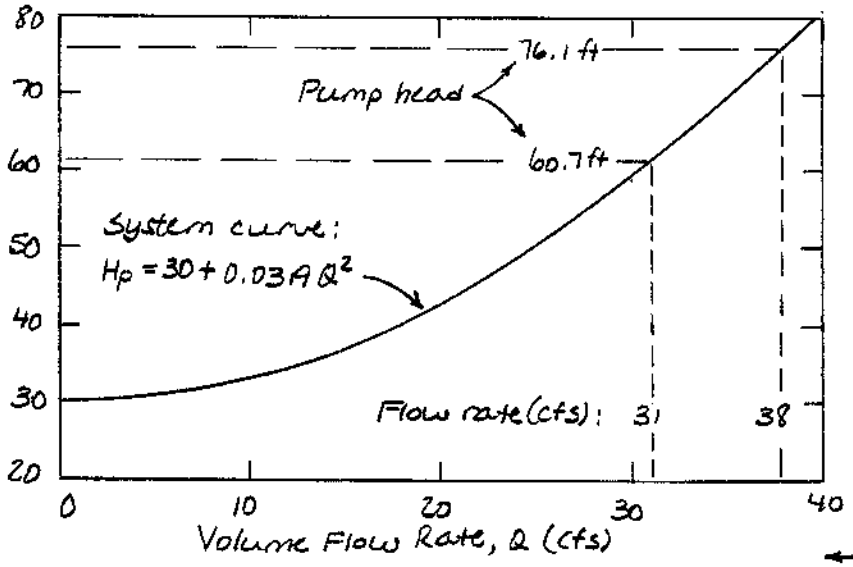
Assumptions: (1) $p_1 = p_2 = p_{atm}$, (2) $\bar{V}_1 \approx \bar{V}_2 \approx 0$, (3) $z_1 = 5280$ ft, $z_2 = 5310$ ft, (4) $K_{ent} = 0$, $K_{exit} = 1$, $\frac{L_e}{D} = 0$

Sample calculation at Q = 31 cfs (13,900 gpm), with T = 70°F, e = 0.00015 ft (Table 8.1)

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 31 \frac{ft^3}{s} \times \left(\frac{12}{27}\right)^2 \frac{1}{ft^2} = 7.80 \text{ ft/s}; \quad Re = \frac{\bar{V}D}{\nu} = 1.64 \times 10^6; \quad \frac{e}{D} = 6.67 \times 10^{-5}; \quad f = 0.0124$$

$$H_p = z_2 - z_1 + \left(f\frac{L}{D} + K\right)\frac{\bar{V}^2}{2g} = (5{,}310 - 5{,}280)\text{ ft} + \left(0.0124 \frac{5800}{(27)/12} + 1\right)\frac{1}{2} \times (7.80)^2 \frac{ft^2}{s^2} \times \frac{s^2}{32.2 ft} = 61.1 \text{ ft}$$

Plotting results:



Plot

The maximum flow rate is $Q_{max} = 38 \frac{ft^3}{s} \times 7.48 \frac{gal}{ft^3} \times 60 \frac{s}{min} = 17{,}100$ gpm.

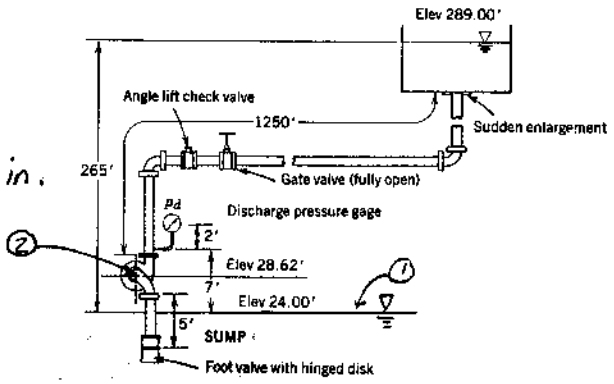
Could choose two Peerless 16A 18B pumps at 880+ rpm (Fig. D.10) or three 10AE 14(6) pumps at 1750 rpm (Fig. D.2). Efficiency (Fig. 11.15) might be $\eta_p = 0.91$. — n, η

Pump power is $\dot{W}_m = \frac{\dot{W}_n}{\eta_p} = \frac{\rho Q g H}{\eta_p} = 361$ hp (at Q = 38 cfs), 235 hp (at 31 cfs) — $\dot{W}_m$

Given: Flow system shown.

Design flow rate: $Q = 200$ gpm

Pipe is commercial steel, $D = 4$ in.



Find: (a) Calculate the NPSHA.
(b) Select a suitable pump.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \alpha_1 \frac{\bar{V}_1^2}{2g} + z_1 + H_p = \frac{p_2}{\rho g} + \alpha_2 \frac{\bar{V}_2^2}{2g} + z_2 + \frac{h_{eT}}{g}$; $h_{eT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2}$ ($\bar{V}_1 \approx 0$ (2), $\bar{V}_2 \approx 0$ (2))

Assumptions: (1) Reservoirs open to atmosphere, (2) Reservoir velocities negligible, (3) $T = 70°F$ ($p_v = 0.363$ psia, Table A.7)

At $Q = 200$ gpm $(0.446\ ft^3/s)$, so $\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 0.1446 \frac{ft^3}{s} \times \left(\frac{12}{4.026}\right)^2 \frac{1}{ft^2} = 5.04$ ft/s; $\epsilon = 0.00015$ ft

Thus $Re = \frac{\bar{V}D}{\nu} = 1.58 \times 10^5$; $\frac{\epsilon}{D} = 0.000447$; $f = 0.0190$; $\frac{\bar{V}^2}{2g} = \frac{1}{2} \times (5.04)^2 \frac{ft^2}{s^2} \times \frac{s^2}{32.2\ ft} = 0.394$ ft

For the inlet, $\frac{p_2}{\rho g} + \alpha_2 \frac{\bar{V}_2^2}{2g} = z_1 - z_2 - \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2g} = -5.60$ ft, since $\bar{V}_1 \approx 0$

(K: foot valve, 75; entrance, 0.78)

Then $NPSHA = \frac{p_{t,abs} - p_v}{\rho g} = \frac{p_t}{\rho g} + \frac{p_{atm}}{\rho g} - \frac{p_v}{\rho g} = (-5.60 + 33.9 - 0.838)\ ft = 27.5$ ft — NPSHA

For the complete system, $L/D = (1250 + 5)\ ft \times \left(\frac{12}{4.026}\right)\frac{1}{ft} = 3740$, $\frac{L_e}{D} = 25 + 55 + 8 + 30 = 138$ (check valve, gate valve)

$H_p = z_2 - z_1 + \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2g} = (289 - 24)\ ft + \left[0.019(3740 + 138) + 0.78 + 1\right] 0.394\ ft$

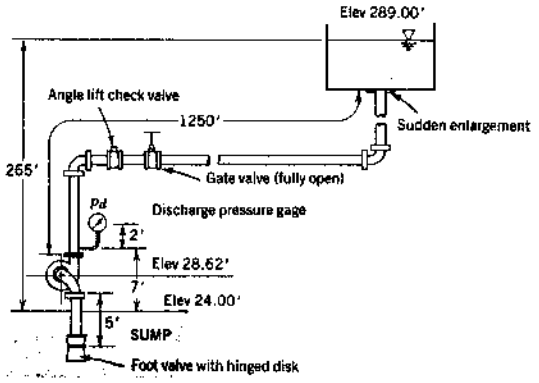
$H_p = 295$ ft at $Q = 200$ gpm — H_p

From Fig. D.1, a 4AE12 pump (4 in. discharge line) would fit the application. This pump could produce the required head at a speed between 1750 and 3550 rpm (Figs. D.4 and D.5), but the efficiency may not be acceptable.

{Consult a complete catalog to make a better selection.}

Given: Flow system and data of Problem 10.61.

Data for pipe aging from Problem 10.57.



Find: (a) Select pumps that will maintain system flow rate for 10 and 20 years.
(b) Compare delivery to that with pump sized for new pipes only.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \alpha_1\frac{\bar{V}_1^2}{2g} + z_1 + H_p = \frac{p_2}{\rho g} + \alpha_2\frac{\bar{V}_2^2}{2g} + z_2 + \frac{h_{LT}}{g}$; $h_{LT} = \left[f\left(\frac{L}{D}+\frac{L_e}{D}\right)+K\right]\frac{\bar{V}^2}{2}$ ($\bar{V}_1^2 \approx 0$ (2), $\bar{V}_2^2 \approx 0$ (3))

Assumptions: (1) $p_1 = p_2 = p_{atm}$, (2) $\bar{V}_1 = \bar{V}_2 \approx 0$, (3) Minor losses as in Problem 10.61

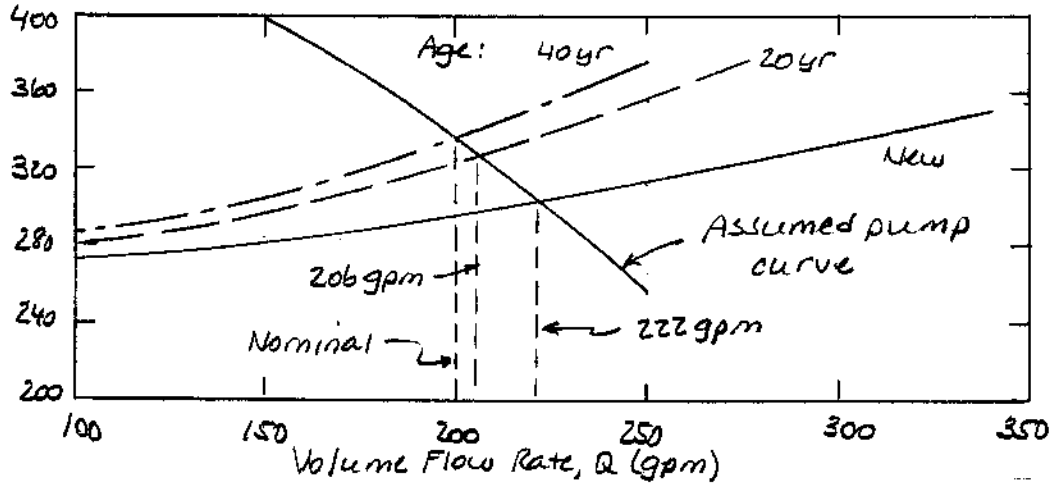
Sample calculations at $Q = 200$ gpm $(0.446\ ft^3/s)$ for new pipe, $\varepsilon = 0.00015$ ft (Table 8.1):

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 0.446\frac{ft^3}{s} \times \left(\frac{12}{4.026}\right)^2 \frac{1}{ft^2} = 5.04\ ft/s\ ;\ Re = 1.58\times10^5;\ \frac{\varepsilon}{D} = 0.000447;\ f = 0.0190$$

For the complete system, $\frac{L}{D} = (1250+5)ft \times \left(\frac{12}{4.026}\right)\frac{1}{ft} = 3740$; $\frac{L_e}{D} = 138$; $K = 1.28$

$$H_p = z_2 - z_1 + \left[f\left(\frac{L}{D}+\frac{L_e}{D}\right)+K\right]\frac{\bar{V}^2}{2g} = (289-24)ft + \left[0.019(3740+138)+1.28\right]0.394\ ft = 295\ ft$$

Plotting results:



Assume H at 200 gpm is 70% of H_0. Then $H_{200} = 336$ ft, $H_0 = \frac{1}{0.7}336\ ft = 480\ ft$ and $H = H_0 - AQ^2$; $A = (H_0 - H)/Q^2 = 0.3\,H_0/Q^2 = 0.3 \times 480\ ft/(200\ gpm)^2 = 3.60\times10^{-3}\ ft/(gpm)^2$.

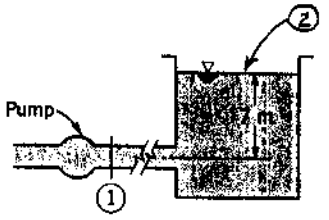
Sizing the pump for 200 gpm at 40 years would (assuming no change in pump characteristics) produce 206 gpm at 20 years and 222 gpm in the new system.

The extra head (336 ft, compared to 295 ft) at 200 gpm could be obtained by increasing impeller diameter about 7-10% compared to the pump of Problem 10.61.

Given: Flow system of Problem 8.100.

$D = 0.25\ m,\ L = 5\ km,\ \bar{V} = 3\ m/s$

Minimum NPSHR = 4.5 m



Data on pipe aging from Problem 10.57; commercial steel.

Find: (a) Select a suitable pump.
(b) Determine and compare system flow rate after 10 years.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \frac{\alpha_1 \bar{V}_1^2}{2g} + z_1 + H_p = \frac{p_2}{\rho g} + \alpha_2 \frac{\bar{V}_2^2}{2g} + z_2 + \frac{h_{LT}}{g}$ ($\bar{V}_2 \approx 0$ (2)); $h_{LT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K \right]$

Assumptions: (1) $\frac{p_1}{\rho g} = 4.5\ m$, $p_2 = 0$, (2) $\bar{V}_2 \approx 0$ in reservoir, (3) $\alpha_1 \approx K_{exit} = 1$

$$H_p = z_2 - z_1 - \frac{p_1}{\rho g} - \frac{\alpha_1 \bar{V}_1^2}{2g} + \left(f\frac{L}{D} + K_{exit}\right)\frac{\bar{V}^2}{2g} = 7\ m - 4.5\ m + \left(f\frac{L}{D} + K_e - 1\right)\frac{\bar{V}^2}{2g} = 2.5\ m + f\frac{L}{D}\frac{\bar{V}^2}{2g}$$

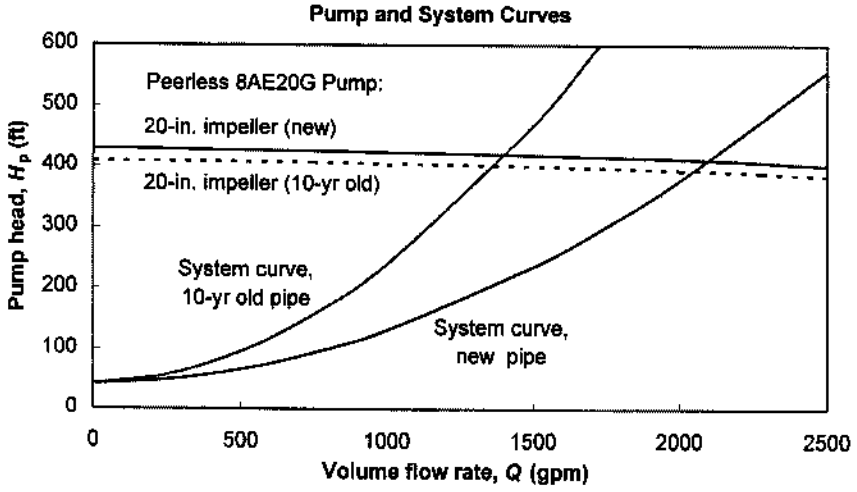
$\bar{V} = 3.0\ m/s$, so $Re = \frac{\bar{V}D}{\nu} = 7.50 \times 10^5$; $\frac{\varepsilon}{D} = \frac{0.00015\ ft}{0.25\ m} \times 0.305\frac{m}{ft} = 0.00197$; $f = 0.023$

$$H_p = 2.5\ m + 0.023 \times \frac{5000\ m}{0.25\ m} \times \frac{1}{2} \times (3.0)^2 \frac{m^2}{s^2} \times \frac{s^2}{9.81\ m} = 113\ m \text{ or } 369\ ft$$

$$Q = \bar{V}A = \bar{V}\frac{\pi D^2}{4} = 3.0\frac{m}{s} \times \frac{\pi}{4}(0.25)^2 m^2 \times \frac{ft^3}{(0.305)^3 m^3} \times 7.48\frac{gal}{ft^3} \times 60\frac{s}{min} = 2330\ gpm$$

From Fig. D.7, choose a Peerless Type 8AE20G pump, with D = 20 in. impeller, operating at 1750 rpm. This pump will require about 280 hp.

After 10 years, the friction factor will increase by a factor of 2.20 (Problem 10.57). Assume pump performance degrades 5% in 10 yr. Plotting:



From the intersection, the 10 yr flow rate is $Q_{old} \approx 1400$ gpm

Q_{old}

Given: Flow system of Problem 8.136:

Location	Elevation	Pressure
Entrance	50.0 ft	20 psig
Discharge	94.0 ft	0 psig

$D = 2.5$ in. (nominal), $L = 290$ ft

2 open gate valves, 1 open angle valve, 7 standard 90° elbows
1 square-edged entrance from a reservoir, 1 free discharge

$Q = 0.439$ ft³/s ($Q = 197$ gpm) Galvanized pipe

Find: (a) Select an appropriate pump.
(b) Check the NPSHR vs. the NPSHA for this system.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \frac{\alpha_1 \bar{V}_1^2}{2g} + z_1 + H_p = \frac{p_2}{\rho g} + \frac{\alpha_2 \bar{V}_2^2}{2g} + z_2 + \frac{h_{LT}}{g}$; $h_{LT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2}$ ($\alpha_1 \approx 0$ (1))

Assumptions: (1) $\bar{V}_1 \approx 0$, (2) $K_{ent} = 0.5$, (3) $\frac{L_e}{D} = 2(8) + 1(150) + 7(30) = 376$; $D = 2.47$ in.

(4) Galvanized pipe, $e = 0.0005$ ft; $\frac{e}{D} = \frac{0.0005 \text{ ft}}{2.47 \text{ in.}} \times 12 \frac{\text{in.}}{\text{ft}} = 0.00243$

Then

$$H_p = \frac{p_2 - p_1}{\rho g} + \frac{\alpha_2 \bar{V}_2^2}{2g} + z_2 - z_1 + \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2g}; \quad \bar{V} = \frac{Q}{A} = 13.2 \text{ ft/s}; \quad Re = \frac{\bar{V}D}{\nu} = 2.54 \times 10^5; \quad f = 0.02$$

$$H_p = (0 - 20)\frac{\text{lbf}}{\text{in.}^2} \times 144 \frac{\text{in.}^2}{\text{ft}^2} \times \frac{\text{ft}^3}{62.4 \text{ lbf}} + \frac{1}{2} \times (13.2)^2 \frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{s}^2}{32.2 \text{ ft}} + (94.0 - 50.0) \text{ ft}$$

$$+ \left[0.025\left(\frac{290}{2.47/12} + 376\right) + 0.5\right]\frac{1}{2} \times (13.2)^2 \frac{\text{ft}^2}{\text{s}^2} \cdot \frac{\text{s}^2}{32.2 \text{ ft}} = 123 \text{ ft} \quad H_p$$

The pump requirement is $Q = 197$ gpm at $H = 123$ ft. This could be supplied by a Peerless Type 4AE12 pump, with impeller $D = 11$ in., operating at 1750 rpm.

(This pump may be slightly too large, since this operating point is at a flow rate below that for best efficiency.)

The NPSHR for this pump at $Q = 197$ gpm is about 5 ft.

The NPSHA is $\frac{p_1}{\rho g} + \frac{p_{atm}}{\rho g} + \frac{\alpha_1 \bar{V}_1^2}{2g} - p_v = 46.2 + 33.9 + 2.71 - 0.782 \text{ ft} = 82.0 \text{ ft}$

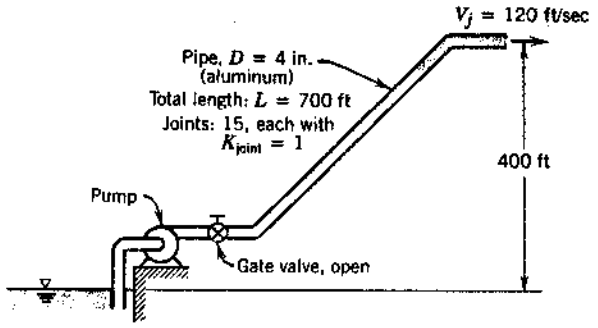
Thus NPSHA >> NPSHR NPSH

Cavitation-free operation is assured.

Given: Flow system of Problem 8.139:

$Q = 600$ gpm

$\eta_p \approx 0.70$



Find: (a) Select an appropriate pump.
(b) Compare pump efficiency with estimate in problem.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \alpha_1 \frac{\bar{V}_1^2}{2g} + z_1 + H_p = \frac{p_2}{\rho g} + \alpha_2 \frac{\bar{V}_2^2}{2g} + z_2 + \frac{h_{LT}}{g}$ ($\bar{V}_1 \approx 0$ (2)); $h_{LT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K \right] \frac{\bar{V}^2}{2}$

Assumptions: (1) $p_1 = p_2 = p_{atm}$, (2) $\bar{V}_1 \approx 0$, (3) Neglect elbow and nozzle losses, (4) $\frac{L_e}{D}$(valve) = 8

$$\bar{V} = \frac{Q}{A} = 600 \frac{gal}{min} \times \frac{4}{\pi}\left(\frac{12}{4}\right)^2 \frac{1}{ft^2} \times \frac{ft^3}{7.48\ gal} \times \frac{min}{60\ s} = 15.3 \frac{ft}{s}; \quad \frac{\bar{V}^2}{2g} = \frac{1}{2} \times (15.3)^2 \frac{ft^2}{s^2} \times \frac{s^2}{32.2\ ft} = 3.63\ ft$$

$$Re = \frac{\bar{V}D}{\nu} = 15.3 \frac{ft}{s}\left(\frac{4}{12}\right) ft \times \frac{s}{1.08 \times 10^{-5}\ ft^2} = 4.77 \times 10^5; \text{ smooth}; \ f = 0.013 \quad (T = 68°F)$$

$$H_p = z_2 - z_1 + \alpha_2 \frac{\bar{V}_2^2}{2g} + \frac{h_{LT}}{g} = z_2 - z_1 + \frac{\bar{V}_2^2}{2g} + \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + 15 K_j \right] \frac{\bar{V}^2}{2g}$$

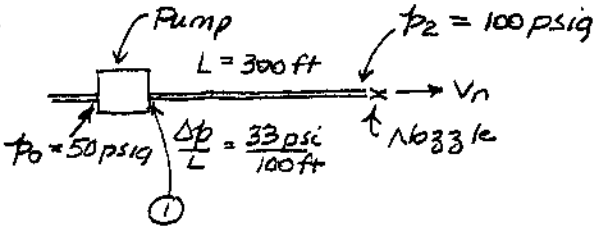
$$H_p = 400\ ft + \frac{1}{2} \times (120)^2 \frac{ft^2}{s^2} \cdot \frac{s^2}{32.2\ ft} + \left[0.013\left(\frac{700}{4/12} + 8\right) + 15 \right] 3.63\ ft = 778\ ft$$

Thus the pump requirement is $H_p = 778$ ft at $Q = 600$ gpm. This head is too great to be developed by a single-stage pump (Fig. D.1). From Fig. D.12, the flow could be supplied by a 5TUT168 3-stage pump, driven at 1750 rpm.

Single-stage pumps have peak efficiencies of 86 percent at 1750 rpm (Fig. D.8). Thus 70 percent efficiency might be reasonable for a 3-stage pump, since $(0.86)^3 = 0.636$.

Given: Fire hose and nozzle as shown.

Canvas hose: $D = 3$ in.
$e = 0.001$ ft



Find: (a) Design flow rate
(b) Maximum nozzle exit speed
(c) Pump selection
(d) Efficiency and power

Solution: Apply the energy equation for pipe flow:

Computing equation: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{LT} = h_L + h_{Lm}$

$h_L = f \frac{L}{D} \frac{\bar{V}^2}{2}$

Assumptions: (1) $\bar{V}_1 = \bar{V}_2$; (2) $z_1 = z_2$, (3) $h_{Lm} = 0$

Then $\frac{p_1}{\rho} - \frac{p_2}{\rho} = \frac{\Delta p}{\rho} = f \frac{L}{D} \frac{\bar{V}^2}{2}$; $\bar{V} = \left[\frac{2 \Delta p}{f \rho L}\right]^{\frac{1}{2}}$; $\frac{e}{D} = \frac{0.001 \text{ ft}}{3 \text{ in.}} \times 12 \frac{\text{in.}}{\text{ft}} = 0.004$; $f = 0.028$ (fully rough)

$$\bar{V} = \left[\frac{2}{0.028} \times 3 \text{ in.} \times 3(33) \frac{\text{lbf}}{\text{in.}^2} \times \frac{\text{ft}^3}{1.94 \text{ slug}} \times \frac{1}{300 \text{ ft}} \times 12 \frac{\text{in.}}{\text{ft}} \times \frac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2}\right]^{\frac{1}{2}} = 20.9 \text{ ft/s}$$

$Q = \bar{V} A$; $A = \frac{\pi D^2}{4} = \frac{\pi}{4}\left(\frac{3}{12}\right)^2 \text{ft}^2 = 0.0491 \text{ ft}^2$

$$Q = 20.9 \frac{\text{ft}}{\text{s}} \times 0.0491 \text{ ft}^2 \times 7.48 \frac{\text{gal}}{\text{ft}^3} \times \frac{60 \text{ s}}{\text{min}} = 461 \text{ gpm (design flow rate)} \quad Q$$

Apply Bernoulli to nozzle.

$\frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 = \frac{p_{atm}}{\rho} + \frac{V_n^2}{2} + g z_n$; $V_n = \left[\frac{2 p_2}{\rho} + V_2^2\right]^{\frac{1}{2}}$

$$V_n = \left[2 \times 100 \frac{\text{lbf}}{\text{in.}^2} \times \frac{\text{ft}^3}{1.94 \text{ slug}} \times 144 \frac{\text{in.}^2}{\text{ft}^2} + (20.9)^2 \frac{\text{ft}^2}{\text{s}^2}\right]^{\frac{1}{2}} = 124 \text{ ft/s} \quad V_n$$

The pump head requirement (neglecting V and z) will be

$$H_p = \frac{p_1 - p_0}{\rho g} = [100 + 3(33) - 50] \frac{\text{lbf}}{\text{in.}^2} \times \frac{\text{ft}^3}{62.4 \text{ lbf}} \times 144 \frac{\text{in.}}{\text{ft}} = 344 \text{ ft} \quad H_p$$

From the pump selector chart (Fig. D.1) choose 3AE96 or 4AE10 pump, 3500 rpm.

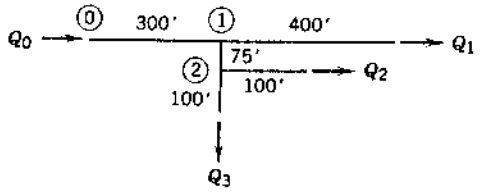
Based on 4AE12 at 3550 rpm (Fig. D.5), expect $\eta \approx 0.75$ η

$$P = \frac{Q \Delta p}{\eta} = \frac{Q \rho g \Delta h}{\eta} = \frac{1}{0.75} \times 20.9 \frac{\text{ft}}{\text{s}} \times 0.0491 \text{ ft}^2 \times 62.4 \frac{\text{lbf}}{\text{ft}^3} \times 344 \text{ ft} \times \frac{\text{hp} \cdot \text{s}}{550 \text{ ft} \cdot \text{lbf}} = 53.4 \text{ hp} \quad P$$

Given: Pipe network of Problem 8.146:

Pipe is 3 in. galvanized.

Consider flow in fully rough zone, neglect minor losses.



Find: Select a pump to deliver a total flow rate, $Q = 300$ gpm, through this pipe network.

Solution: Friction factor is constant when flow is in the fully rough zone; Δp depends only upon Q and L, $\Delta p = f\frac{L}{D}\frac{\rho}{2}\left(\frac{Q}{A}\right)^2$.

Legs with Q_1 and Q_2 are the same length, so $Q_1 = Q_2$.

Flow from ① to ② is at $Q_{12} = Q_1 + Q_2 = 2Q_2$. Δp_{12} is 4 times that with Q_2, so it is equivalent to Q_2 in $L = 300$ ft of pipe.

Thus p_1 is equivalent to that from Q_2 in $L = 400$ ft of pipe, and $Q_1 = Q_2 = Q_3 = 100$ gpm.

$$p_0 = p_0 - p_1 + p_1 - p_{atm} = \Delta p(Q = 300\text{ gpm}, L = 300') + \Delta p(Q = 100, L = 400)$$

The actual diameter (Table 8.4) is $D = 3.068$ in. ≈ 3.07 in. From 0 to 1,

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 300\frac{gal}{min} \times \left(\frac{12}{3.07}\right)^2 \frac{1}{ft^2} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} = 13.0\text{ ft/s}$$

$$\rho\frac{\bar{V}^2}{2} = \frac{1}{2} \times 1.94\frac{slug}{ft^3} \times (13.0)^2\frac{ft^2}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} = 164\text{ lbf/ft}^2$$

$$\frac{e}{D} = \frac{0.0005\text{ ft}}{3.07\text{ in.}} \times \frac{12\text{ in.}}{ft} = 0.00195;\ \text{In fully rough zone, } f = 0.023$$

$$\Delta p_{01} = 0.023 \times \frac{300\text{ ft}}{3.07\text{ in.}} \times 164\frac{lbf}{ft^2} \times \frac{ft}{12\text{ in.}} = 30.7\text{ lbf/in.}^2$$

From 1 to atmosphere, $\bar{V} = 4.33$ ft/s, $\frac{\rho\bar{V}^2}{2} = 18.2$ lbf/ft^2

$$\Delta p_{1a} = 0.023 \times \frac{400\text{ ft}}{3.07\text{ in.}} \times 18.2\frac{lbf}{ft^2} \times \frac{ft}{12\text{ in.}} = 4.55\text{ lbf/in.}^2$$

$$p_0 = \Delta p_{01} + \Delta p_{1a} = 30.7 + 4.55\text{ psi} = 35.3\text{ psig}$$

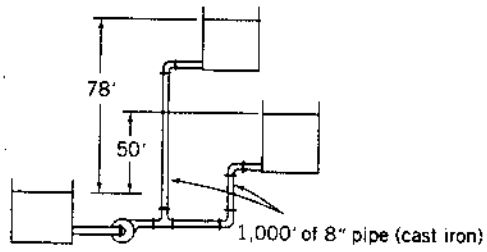
The required head is

$$H_p = \frac{p_0}{\rho g} = 35.3\frac{lbf}{in.^2} \times \frac{ft^3}{62.4\,lbf} \times 144\frac{in.^2}{ft^2} = 81.5\text{ ft at } Q = 300\text{ gpm.}$$

From Fig. D.4, choose Peerless Type 4AE12 pump, with impeller $D = 9.5$ in, operating at 1750 rpm. Pump

Given: Flow system with two static lifts
Each line is 1000 ft of 8 in. pipe

(a) Plot the system head vs flow curve.
(b) Calculate flow rate when $H_a = 88$ ft
(c) Explain what happens when $H_a < 78$ ft



Solution:

Apply the energy equation to the system for steady, incompressible flow between reservoir surfaces ① and ② and ① and ③

Computing eq.: $\cancel{\frac{p_1}{\rho g}} + \cancel{\alpha_1 \frac{\bar{V}_1^2}{2g}} + z_1 + H_a = \cancel{\frac{p_2}{\rho g}} + \cancel{\alpha_2 \frac{\bar{V}_2^2}{2g}} + z_2 + \frac{h_{eT}}{g}$ (10.24b)

$$h_{eT} = \left[f\left(\frac{L}{D}\right) + K \right] \frac{\bar{V}_2^2}{2}$$

Assumptions: (1) $p_1 = p_2 = p_3 = p_{atm}$ (2) $\bar{V}_1 = \bar{V}_2 = \bar{V}_3 \approx 0$
(3) $z_1 = 0$ (4) $K_{ent} = 1$

Express V as a function of Q

$$V = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi}\left(\frac{12}{8}\right)^2 \text{ft}^2 \cdot Q\left(\frac{\text{gal}}{\text{min}}\right) \cdot \frac{1 \text{ ft}^3}{7.48 \text{ gal}} \cdot \frac{1 \text{ min}}{60 \text{ s}} = 0.00638\, Q(\text{gpm})$$

The friction factor is determined from the Colebrook eq.

$$\frac{1}{f^{0.5}} = -2.0 \log\left(\frac{e/D}{3.7} + \frac{2.51}{Re\, f^{0.5}}\right) \quad (8.37a)$$

using the Miller equation for the original estimate

$$f_o = 0.25\left[\log\left(\frac{e/D}{3.7} + \frac{5.74}{Re^{0.9}}\right)\right]^{-2} \quad (8.37b)$$

Assuming $T = 59°F$, $\nu = 1.23 \times 10^{-5}$ ft²/s (Table A.7)

$$Re = \frac{VD}{\nu} = \frac{\frac{8}{12}}{1.23} \times 10^5 V = 5.42 \times 10^4 V$$

For cast iron, $e = 0.00085$ ft (Table 8.1) $\therefore e/D = 0.00128$

For the 78 ft lift. (leg 1)

$$L_e/D = L_e/D)_{\text{tee,branch}} + L_e/D)_{90°\text{el}} = 60 + 30 = 90$$

$$K = 1.0 \qquad L/D = 1500$$

$$H_a = 78 \text{ ft} + \left[1590 f + 1\right] \frac{V^2}{2g}$$

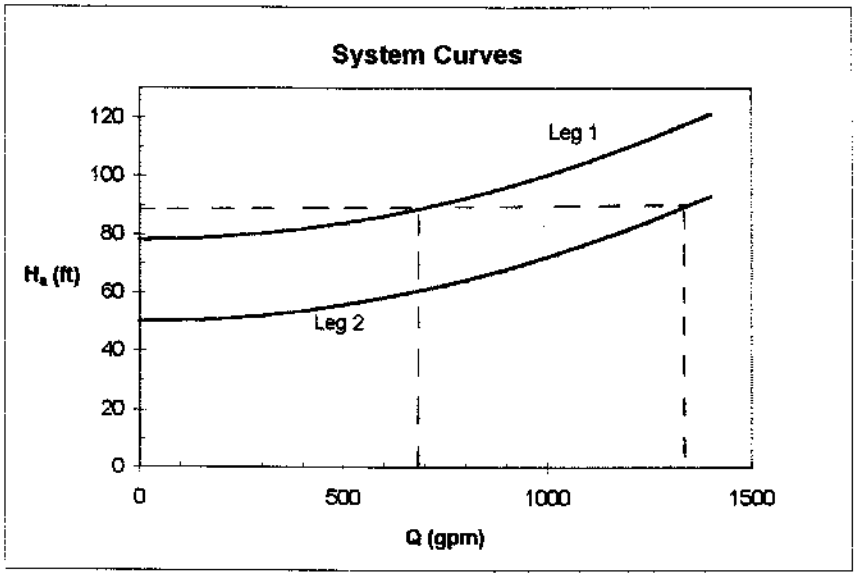
For the 50 ft lift (leg 2)

$$L_e/D = L_e/D)_{\text{tee,run}} + 2 L_e/D)_{90°\text{el}} = 20 + 2 \times 30 = 80$$

$K = 1$ $\quad L/D = 1500$

$$H_a = 50\,ft + \left[1580 f + 1\right] V^2/2g$$

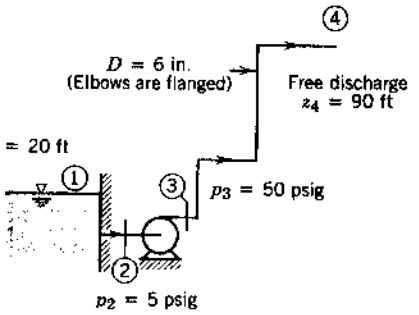
Q (gpm)	V (m/s)	Re	f_0	$f^{-0.5}$	f	H_a (ft) (Leg 1)	H_a (ft) (Leg 2)
0	0.00	0.00E+00				78	50
200	1.28	6.92E+04	0.0241	6.473	0.0239	79	51
400	2.55	1.38E+05	0.0227	6.663	0.0225	82	54
600	3.83	2.07E+05	0.0222	6.739	0.0220	86	58
800	5.10	2.77E+05	0.0219	6.780	0.0218	92	64
1000	6.38	3.46E+05	0.0217	6.806	0.0216	100	72
1200	7.66	4.15E+05	0.0216	6.824	0.0215	110	82
1400	8.93	4.84E+05	0.0215	6.837	0.0214	121	93



Given: Flow system of Problem 8.78.

$Q = 2\ ft^3/s$ (898 gpm)

Pipes are commercial steel, $D = 6.065$ in. $e = 0.00015$ ft, Table 8.1



Find: (a) NPSHA at pump inlet.
(b) Select a suitable pump.
(c) Use data from Problem 10.57 to estimate reduction in flow rate after 10 years' operation.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \alpha_1 \frac{\bar{V}_1^2}{2g} + z_1 + H_p = \frac{p_4}{\rho g} + \alpha_4 \frac{\bar{V}_4^2}{2g} + z_4 + \frac{h_{cr}}{g}$; $h_{cr} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2}$ ($\alpha \approx 1.0$ (2))

Assumptions: (1) $p_1 = p_4 = p_{atm}$, (2) $\bar{V}_1 \approx 0$, (3) $\frac{L_e}{D} = 4(12) = 48$ (Fig. 8.17), $K_{ent} = 0.5$; (4) $T = 68°F$ ($\nu = 1.08 \times 10^{-5}\ ft^2/s$)

Data are given to find NPSHA: $Q = 2\ ft^3/s$ ($Q = 898$ gpm); $\bar{V} = \frac{Q}{A} = 9.98\ ft/s$.

$$NPSHA = \frac{p_2 + \frac{1}{2}\rho \bar{V}_2^2 + p_{atm} - p_v}{\rho g} = (11.5 + 1.55 + 33.9 - 0.782)\ ft = 46.2\ ft$$ ← NPSHA

The pump must supply $Q = 898$ gpm at $H_p = \Delta p/\rho g$, or

$$H_p = (50-5)\frac{lbf}{in.^2} \times \frac{ft^3}{62.4\ lbf} \times 144\frac{in.^2}{ft^2} = 104\ ft$$

From Fig. D.3, select a Peerless Type 4AE11 with $D \approx 10.5$ in. (at 1750 rpm) ← Pump

For the system, $Re = \frac{\bar{V}D}{\nu} = 4.67 \times 10^5$; $\frac{e}{D} = 0.000297$; $f = 0.0163$. Solving for L,

$$f\frac{L}{D} = \left[(z_1 + H_p - z_4)\frac{2g}{\bar{V}^2} - \left(f\frac{L_e}{D} + K + 1\right)\right] = \left[(20 + 104 - 90)\frac{1}{1.55\ ft} - (0.0163 \times 48 + 0.5 + 1)\right]$$

$$f\frac{L}{D} = 19.7;\quad \frac{L}{D} = \frac{19.7}{0.0163};\quad L = 1210 \times 6.065\ in. \times \frac{ft}{12\ in.} = 612\ ft \text{ of pipe}$$ ← New Pipe

The system head curve for the "as-new" condition is

$$H_S = z_4 - z_1 + \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K + \alpha\right]\frac{\bar{V}^2}{2g} \approx 70 + 34.1\frac{[Q(gpm)]^2}{(898)^2} = 70 + 4.23 \times 10^{-5}[Q(gpm)]^2$$ ← H_S New

The head curve for the new pump is

$$H_p = H_0 - BQ^2;\quad B = \frac{H_0 - H_p}{Q^2} = \frac{(120-104)\ ft}{(898)^2(gpm)^2} = 1.98 \times 10^{-5}\ ft/(gpm)^2$$

$$H_p[ft] = 120 - 1.98 \times 10^{-5}[Q(gpm)]^2$$ ← H_p New

From Problem 10.57, the age factor for 10-yr old pipe is 2.20. Thus

$$H_S = z_4 - z_1 + \left[2.2f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K + \alpha\right]\frac{\bar{V}^2}{2g} \approx 70 + \frac{72.2Q^2}{(898)^2} = 70 + 8.96 \times 10^{-5}[Q(gpm)]^2$$ ← H_S Old

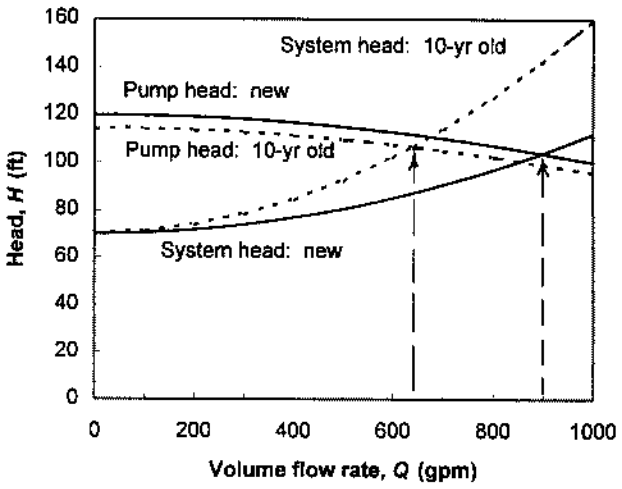
The 10-yr old pump produces 95% of the head of a new pump. Thus

$$H_p(old) = 0.95\, H_p(new) = 114 - 1.88\times10^{-5}[Q(gpm)]^2$$

Hp Old

These results are tabulated and plotted below:

Q	System Head		Pump Head	
	Hs (new)	Hs (old)	Hp (new)	Hp (old)
(gpm)	(ft)	(ft)	(ft)	(ft)
0	70.0	70.0	120	114
100	70.4	70.9	120	114
200	71.7	73.5	119	113
300	73.8	78.0	118	112
400	76.8	84.2	117	111
500	80.6	92.2	115	109
600	85.2	102	113	107
640	87.3	106.3	112	106.3
700	90.7	113	110	105
800	97.1	127	107	102
898	104.1	141	104.1	98.9
900	104	142	104	98.8
1000	112	159	100	95.3



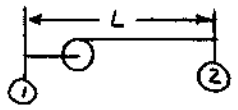
The results confirm that the system operates at 898 gpm when new.

The volume flow rate decreases to only about 640 gpm after 10 years.

The reduction is $\frac{\Delta Q}{Q} = \frac{640-898}{898}\times 100 = -28.7$ percent

ΔQ / Qnew

Given: Chilled water circulation system of Problem 8.101:



$L = 3$ mi $(15{,}800$ ft$)$, $D = 2$ ft (steel), $Q = 11{,}200$ gpm, Loop configuration

Find: (a) Select suitable pumps for parallel operation.
(b) Calculate power for 3 pumps in parallel.
(c) Calculate volume flow rate and power if 1 or 2 pumps operate.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \alpha_1\frac{\bar{V}_1^2}{2g} + z_1 + H_a = \frac{p_2}{\rho g} + \alpha_2\frac{\bar{V}_2^2}{2g} + z_2 + \frac{h_{LT}}{g}$; $h_{LT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2}$

Assumptions: (1) $p_1 = p_2$, (2) $\alpha_1\bar{V}_1^2 = \alpha_2\bar{V}_2^2$, (3) $z_1 = z_2$, (4) Neglect minor losses, $\frac{L_e}{D} \approx 0$, $K \approx 0$

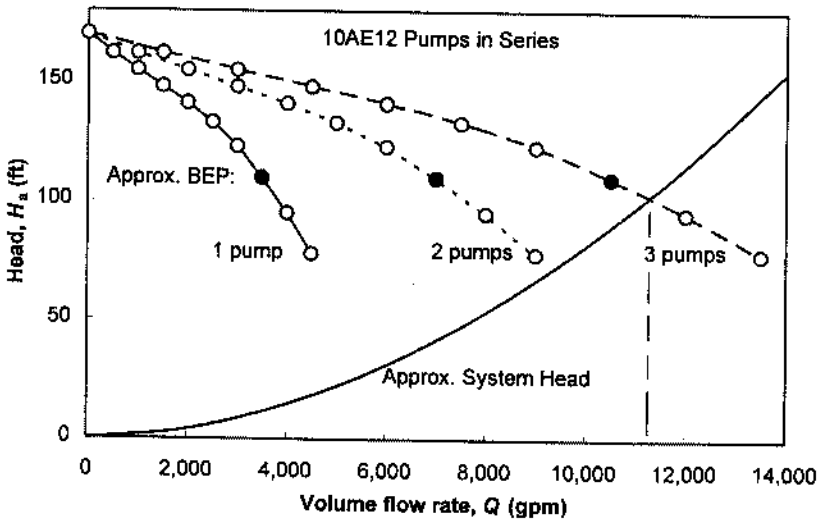
$$\bar{V} = \frac{Q}{A} = 11{,}200\ \frac{\text{gal}}{\text{min}} \times \frac{4}{\pi(2)^2\ \text{ft}^2} \times \frac{\text{ft}^3}{7.48\ \text{gal}} \times \frac{\text{min}}{60\ \text{sec}} = 7.94\ \text{ft/s};\quad \frac{\bar{V}^2}{2g} = \frac{1}{2} \times (7.94)^2\frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{s}^2}{32.2\ \text{ft}} = 0.979\ \text{ft}$$

Assume $T = 40$ F, so $\nu = 1.64 \times 10^{-5}$ ft^2/s; $Re = \frac{\bar{V}D}{\nu} = 9.68 \times 10^5$; $\frac{e}{D} = 7.5 \times 10^{-5}$; $f = 0.013$

$$H_a = f\frac{L}{D}\frac{\bar{V}^2}{2g} = 0.013 \times \frac{3(5280)\ \text{ft}}{2\ \text{ft}} \times 0.979\ \text{ft} = 101\ \text{ft}$$ H_p

For three pumps in parallel, each will operate at $Q/3 = 3730$ gpm. The requirement for each pump is $H = 101$ ft at $Q = 3730$ gpm. This can be supplied by Peerless Type 10AE12 pumps with impellers of $D = 12$ in. diameter, operating at $N = 1760$ nominal rpm. The efficiency at this operating point is $\eta \approx 0.85$. 3 pumps

Find operating points graphically for 1, 2, and 3 pumps:

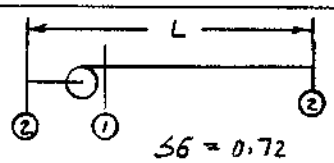


The graphical solution is shown

Q_1 = not satisfactory, $Q_2 = 9400$ gpm (marginal), $Q_3 = 11{,}200$ gpm (OK) Q

Assuming $\eta_p \approx 0.7$, then $\dot{W}_{m_1} = \frac{\rho Q g H}{\eta} \approx 78$ hp, $\dot{W}_{m_2} \approx 241$ hp, and $\dot{W}_{m_3} \approx 409$ hp

Given: Gasoline pipeline of Problem 8.111:



$L = 13$ km, $D = 0.6$ m, $\Delta p_{12} = 1.4$ MPa $SG = 0.72$

Roughness equivalent to galvanized iron ($e = 0.15$ mm $= 0.00015$ m)

Find: (a) Select suitable pumps for parallel operation.
(b) Calculate the power required for 4 pumps in parallel.
(c) Calculate volume flow rate and power with 1, 2, and 3 pumps.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Governing equation: $\frac{p_1}{\rho g} + \frac{\alpha_1 \bar{V}_1^2}{2g} + z_1 + H_p = \frac{p_2}{\rho g} + \frac{\alpha_2 \bar{V}_2^2}{2g} + z_2 + \frac{h_{LT}}{g}$; $h_{LT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K \right] \frac{\bar{V}^2}{2}$

Assumptions: (1) $\bar{V}_1 = \bar{V}_2$, $\alpha_1 = \alpha_2$, (2) $z_1 = z_2$, (3) Neglect minor losses, $\frac{L_e}{D} \approx 0$, $K \approx 0$

Find flow rate to size pump. From 1 to 2, $H_p = 0$, so $D = 0.6 \text{ m} \times \frac{\text{ft}}{0.305 \text{ m}} = 1.97$ ft

$\frac{p_1}{\rho g} = \frac{p_2}{\rho g} + \frac{h_{LT}}{g}$; $\Delta p = \rho h_{LT} = \rho f \frac{L}{D} \frac{\bar{V}^2}{2}$; $\bar{V} = \left[\frac{2 \Delta p \, D}{\rho f L} \right]^{\frac{1}{2}}$ $\frac{e}{D} = \frac{0.00015 \text{ m}}{0.6 \text{ m}} = 2.5 \times 10^{-4}$

But $f = f(Re, e/D)$; Re is not known. Choose f from fully-rough region, $f = 0.014$.

$\Delta p = 1.4 \times 10^6 \text{ Pa} \times \frac{14.7 \text{ psi}}{101 \times 10^3 \text{ Pa}} = 204$ psi; $\rho = 0.72 \rho_{H_2O} = 0.72 \times 1.94 \frac{\text{slug}}{\text{ft}^3} = 1.40$ slug/ft^3

$\bar{V} = \left[2 \times 204 \frac{\text{lbf}}{\text{in.}^2} \times 1.97 \text{ ft} \times \frac{\text{ft}^3}{1.40 \text{ slug}} \times \frac{1}{0.014} \times \frac{1}{42{,}600 \text{ ft}} \times \frac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2} \times 144 \frac{\text{in.}^2}{\text{ft}^2} \right]^{\frac{1}{2}} = 11.8$ ft/s

Check: $Re = \frac{\bar{V} D}{\nu} = 11.8 \frac{\text{ft}}{\text{s}} \times 1.97 \text{ ft} \times \frac{\text{s}}{8.6 \times 10^{-6} \text{ ft}^2} = 2.70 \times 10^6$; $f = 0.0146$ (see Note below.)

$\bar{V} = \left[\frac{0.014}{0.0146} \right]^{1/2} 11.8 \frac{\text{ft}}{\text{s}} = 11.6$ ft/s, $Q = \bar{V} A = 35.3 \frac{\text{ft}^3}{\text{s}} = 15{,}700$ gpm Q

For parallel operation with four pumps, each must supply $\frac{Q}{4} = 3930$ gpm.

The head requirement is $H_p = \frac{p_1 - p_2}{\rho g} = 204 \frac{\text{lbf}}{\text{in.}^2} \times \frac{\text{ft}^3}{(0.72) 62.4 \text{ lbf}} \times 144 \frac{\text{in.}^2}{\text{ft}^2} = 654$ ft (gasoline)

This combination of head and flow rate cannot be supplied by a single-stage pump. From Fig. D.11, the two-stage Peerless Type 10 TU 22C pump may be chosen. Pump

The input power requirement is $\dot{W}_m = \frac{\rho Q g H}{\eta_p}$. Assuming $\eta_p = 0.65$,

$\dot{W}_m = \frac{1}{0.65} \times 15{,}700 \frac{\text{gal}}{\text{min}} \times 204 \frac{\text{lbf}}{\text{in.}^2} \times \frac{\text{ft}^3}{7.48 \text{ gal}} \times \frac{\text{min}}{60 \text{ s}} \times 144 \frac{\text{in.}^2}{\text{ft}^2} \times \frac{\text{hp} \cdot \text{s}}{550 \text{ ft} \cdot \text{lbf}} = 2870$ hp (total) $\dot{W}_m$

Flow rates with fewer pumps operating may be found from a plot. The pump characteristic may be approximated as (assume $H = 0.7 H_0$):

$\hat{H}_p \approx 934 - 1.81 \times 10^{-5} Q^2$ $\{ H_0 = \frac{654}{0.7} = 934; B = \frac{H_0 - H}{Q^2} = \frac{(0.3) 934}{(3930)^2} = 1.81 \times 10^{-5} \}$ 1193

{Note: gasoline is between octane and heptane, Fig. A.3. For $T = 15°C$, $\nu \approx 8 \times 10^{-7} \frac{\text{m}^2}{\text{s}} = 8.6 \times 10^{-6} \frac{\text{ft}^2}{\text{s}}$.}

For two pumps in parallel, $\hat{H} = H_0 - A\left(\frac{Q}{2}\right)^2 \approx 934 - 4.54\times10^{-6} Q^2$

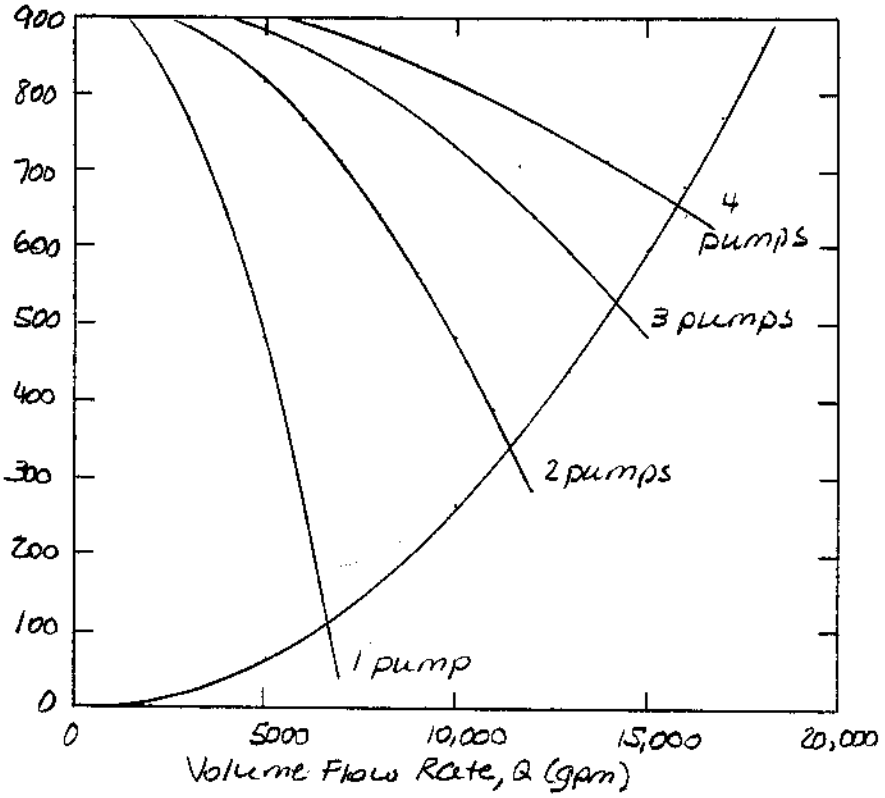
For three pumps in parallel, $\hat{H} = H_0 - A\left(\frac{Q}{3}\right)^2 \approx 934 - 2.02\times10^{-6} Q^2$

For four pumps in parallel, $\hat{H} = H_0 - A\left(\frac{Q}{4}\right)^2 \approx 934 - 1.13\times10^{-6} Q^2$

The pipe system characteristic is approximately given by

$\hat{H}_s \approx BQ^2$; $B = \frac{\hat{H}_s}{Q^2} = \frac{654}{(15{,}700)^2} = 2.65\times10^{-6}$ ft/gpm^2, so $\hat{H}_s = 2.65\times10^{-6} Q^2$

Plotting results:

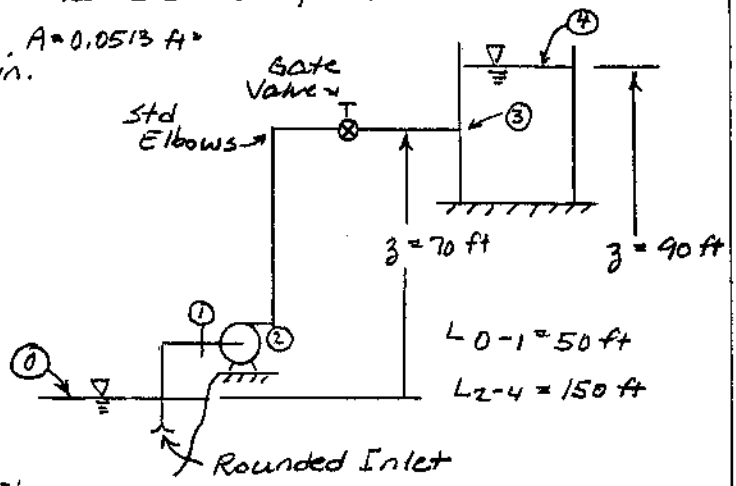


The approximate volume flow rates, heads, and power requirements (assuming $\eta_p = 0.65$) are:

Number of Pumps	1	2	3	4
Flow Rate (gpm)	6710	11,400	14,200	15,700
Head (ft gasoline)	119	345	531	654
Power (hp)	224	1100	2110	2880

Given: Pump and piping system at lakeside home, as shown.

Galvanized iron pipe. $A = 0.0513\ ft^2$
$D = 3$ in. (nominal) $= 3.068$ in.



Find: (a) System head-flow curve
(b) System operating point
(c) Power input if $\eta_p = 0.8$
(d) Sketch of system curve when $z_4 = 90$ ft
(e) Sketch of system curve when $z_3 = 75$ ft, valve part-closed, and $Q = 0.1\ ft^3/s$.
(f) Which case has higher pump efficiency?

Solution: Apply the energy equation for pipe flow. The pump must overcome the gravity lift plus the head losses in the pipe and fittings.

Assume: (1) Nominal speed is $\bar{V} = 12$ ft/s, $T = 60°F$, $\nu = 1.21\times10^{-5}\ ft^2/s$ (Table A.7)
(2) Flow in fully rough zone ($\varepsilon = 0.00005$ ft (Table 8.1), $\varepsilon/D = 0.002$, $f \approx 0.024$)
(3) Cases: (1) Water in tank below ③ } Valve open
(2) Water in tank at $z = 90$ ft }
(3) Valve closed so $Q = 0.1\ ft^3/s$, valve part closed

Then

$$H_{LT} = \frac{h_{LT}}{g} = \left[K_{ent} + f\left(\frac{L}{D} + 3\frac{L_e}{D}(\text{elbow}) + \frac{L_e}{D}(\text{gate valve})\right) + K_{exit}\right]\frac{\bar{V}^2}{2g}$$

$$H_{LT} = \left[0.04 + 0.024\left(\frac{200\ ft}{3.068\ in.}\times\frac{12\ in.}{ft} + 3(30) + 8\right) + 1\right]\frac{\bar{V}^2}{2g} = 22.2\,\frac{\bar{V}^2}{2g}$$

and $H_s = z_{end} + 22.2\dfrac{\bar{V}^2}{2g}$ → H_s

Assume $\bar{V} = 12$ ft/s, $Q = 276$ gpm, $\dfrac{\bar{V}^2}{2g} = 2.24$ ft, $H_s = 70 + 22.2(2.24) = 120$ ft

Case 1: $z_{end} = 70$ ft Operating point: $Q = 276$ gpm, $H_p = H_s = 120$ ft ← Oper. Pt.

$$P = \frac{\rho g Q H}{\eta_p} = 62.4\frac{lbf}{ft^3}\times 12.0\frac{ft}{s}\times 0.0513\ ft^2\times 120\ ft\times\frac{1}{0.8}\times\frac{hp\cdot s}{550\ ft\cdot lbf} = 10.5\ hp$$ ← P

Case 2: $z_{end} = 90$ ft; $H_s = 90 + 22.2(2.24) = 140$ ft

$$H_s = 90 + 50.0\frac{[Q(gpm)]^2}{(276)^2(gpm)^2} = 90 + 6.56\times10^{-4}[Q(gpm)]^2$$ → H_s

Case 3: $Q = 0.1\ ft^3/s = 44.9$ gpm; $H_s = H_p$; Assume $H_{BEP} = 0.7\ H_0$

$$H_p = H_0 + \frac{(H_0 - H_{op})}{Q_{op}^2}Q^2 = \frac{120}{0.7} - \frac{(120/0.7 - 120)}{(276)^2}Q^2 = 169 - 6.75\times10^{-4}Q^2$$ → 1195

$H_p = 168$ ft at $Q = 44.9$ gpm → H_p

over

Water pumped from lake to storage tank on bluff:

Input Data:

Friction factor:	f =	0.024	(---)
Pipe diameter:	D =	3.068	in.

Calculated Results:

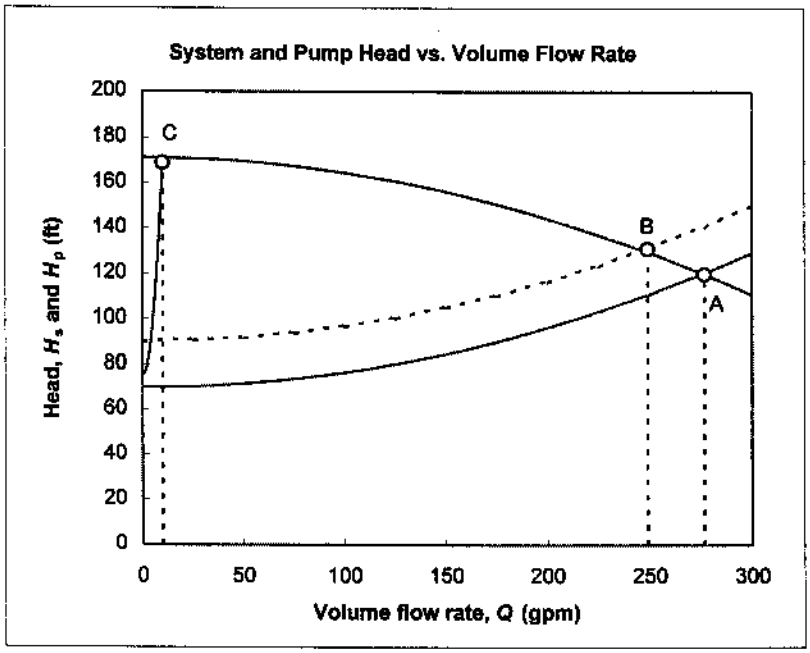
Pipe area:	A =	0.0513	ft^2

System Curves for Various Conditions:

			Case 1:	Case 2:	Case 3: Valve partially closed
			H_s	H_s	H_s
Q	V	$V^2/2g$	(z_3 = 70 ft)	(z_3 = 90 ft)	Q (z_3 = 75 ft)
(gpm)	(ft/s)	(ft)	(ft)	(ft)	(gpm) (ft)
0	0	0.00	70.0	90.0	0 75.0
25	1.09	0.02	70.4	90.4	2 78.8
50	2.17	0.07	71.7	91.7	4 90.0
75	3.26	0.16	73.7	93.7	6 109
100	4.34	0.29	76.6	96.6	8 135
125	5.43	0.46	80.3	100	10 169
150	6.51	0.66	84.9	105	
175	7.60	0.90	90.2	110	
200	8.68	1.17	96.4	116	
225	9.77	1.48	103	123	
249.3	10.8	1.82	111	131	
277	12.0	2.24	121	141	
300	13.0	2.63	129	149	

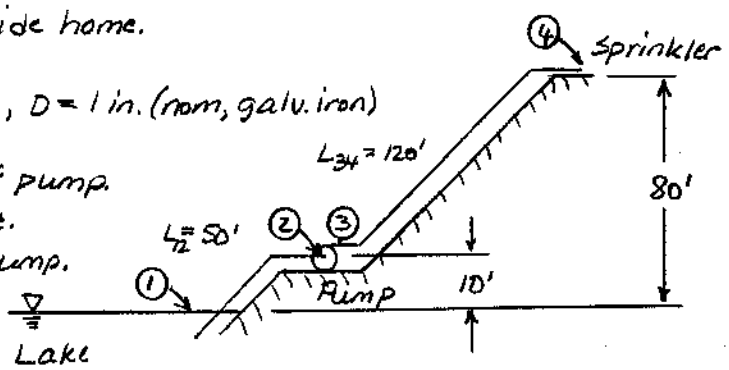
Pump Head Curve:

Q	H_p
(gpm)	(ft)
0	171
50	170
100	165
150	156
200	145
250	130
277	120
300	111



Given: Sprinkler system for lakeside home.

$Q = 10$ gpm, $p_4 = 50$ psig, $D = 1$ in. (nom, galv. iron)



Find: (a) Head loss on suction side of pump.
(b) Gage pressure at pump inlet.
(c) Hydraulic power req'mt for pump.
(d) Change to $D = 1.5$ in.?
(e) Pump halfway up hill?

Solution: Assume $T = 68°F$, so $\nu = 1.08 \times 10^{-5}$ ft²/s (Table A.7).

For 1 in. (nominal) pipe, $D = 1.049$ in. (Table 8.5). $\bar{V} = \frac{Q}{A}$; $A = \frac{\pi}{4}D^2 = \frac{\pi}{4}\left(\frac{1.049}{12}\right)^2 \text{ft}^2 = 0.00600$ ft²

$$\bar{V} = \frac{Q}{A} = \frac{10 \text{ gal}}{\text{min}} \times \frac{1}{0.00600 \text{ ft}^2} \times \frac{\text{ft}^3}{7.48 \text{ gal}} \times \frac{\text{min}}{60 \text{ s}} = 3.71 \text{ ft/s}; \quad \frac{\bar{V}^2}{2} = \frac{1}{2} \times (3.71)^2 \frac{\text{ft}^2}{\text{s}^2} = 6.90 \text{ ft}^2/\text{s}^2$$

$$Re = \frac{\bar{V}D}{\nu} = 3.71 \frac{\text{ft}}{\text{s}} \times 1.049 \text{ in.} \times \frac{\text{s}}{1.08 \times 10^{-5} \text{ ft}^2} \times \frac{\text{ft}}{12 \text{ in.}} = 3.00 \times 10^4; \quad e = 0.0005 \text{ ft (Table 8.1)}$$

Therefore $\frac{e}{D} = 0.0005 \text{ ft} \times \frac{1}{1.049 \text{ in.}} \times 12 \frac{\text{in.}}{\text{ft}} = 0.00572$ and $f \approx 0.034$ (Fig. 8.13).

Apply energy equation for steady incompressible pipe flow:

$$\left(\frac{p_1}{\rho g} + \alpha_1 \frac{\bar{V}_1^2}{2g} + z_1\right) - \left(\frac{p_2}{\rho g} + \alpha_2 \frac{\bar{V}_2^2}{2g} + z_2\right) = \frac{h_{LT_{12}}}{g} = \left(f\frac{L_{12}}{D} + K_{ent} + f\frac{L_e}{D_{45°}} + f\frac{L_e}{D_{90°}}\right)\frac{\bar{V}^2}{2g}$$

($p_1 = 0$ (1); $\bar{V}_1 = 0$ (2); $z_1 = 0$ (3))

$= 0.78$ (Table 8.2) $\quad = 16 \quad = 30$ (Table 8.4)

Assumptions: (1) $p_1 = p_{atm} = 0$ (gage)
(2) $V_1 = 0$
(3) $z_1 = 0$
(4) $\alpha_2 \approx 1$

$$\frac{L_{12}}{D} = \frac{50 \text{ ft}}{1.049 \text{ in.}} \times 12 \frac{\text{in.}}{\text{ft}} = 572$$

$$p_2 \text{ (gage)} = -\rho g\left(\frac{\bar{V}_2^2}{2g} + z_2 + H_{LT}\right)$$

$$H_{LT_{12}} = \left[0.034(572 + 30 + 16) + 0.78\right] 6.88 \frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{s}^2}{32.2 \text{ ft}}$$

$$H_{LT_{12}} = 4.67 \text{ ft} \quad \leftarrow h_{LT_{12}}$$

$$p_2 \text{ (gage)} = -62.4 \frac{\text{lbf}}{\text{ft}^3}\left(6.88 \frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{s}^2}{32.2 \text{ ft}} + 10 \text{ ft} + 4.67 \text{ ft}\right) \frac{\text{ft}^2}{144 \text{ in.}^2} = -6.45 \text{ psig} \quad \leftarrow p_2$$

To find hydraulic power, must know p_3. Apply energy equation:

$$\left(\frac{p_3}{\rho g} + \alpha_3 \frac{\bar{V}_3^2}{2g} + z_3\right) - \left(\frac{p_4}{\rho g} + \alpha_4 \frac{\bar{V}_4^2}{2g} + z_4\right) = h_{LT_{34}} = \left(f\frac{L_{34}}{D} + 2K_{45°}\right)\frac{\bar{V}^2}{2g}$$

Assumptions: (1) $\bar{V}_3 = \bar{V}_4 = \bar{V} = \bar{V}_2$

$$\frac{L_{34}}{D} = \frac{120 \text{ ft}}{1.049 \text{ in.}} \times 12 \frac{\text{in.}}{\text{ft}} = 1,370$$

$$p_3 = p_4 + \rho g\left[z_4 - z_3 + \left(f\frac{L_{34}}{D} + 2f\frac{L_e}{D_{45°}}\right)\frac{\bar{V}^2}{2g}\right]$$

$$p_3 = 50 \text{ psig} + 62.4 \frac{\text{lbf}}{\text{ft}^3}\left[(80 - 10)\text{ft} + \left(0.034 \times (1370 + 2 \times 16)\right) 6.90 \frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{s}^2}{32.2 \text{ ft}}\right] \frac{\text{ft}^2}{144 \text{ in.}^2} = 84.8 \text{ psig}$$

Thus the pump head is $H_p = \frac{p_3 - p_2}{\rho g} = [84.8 - (-6.45)] \frac{\text{lbf}}{\text{in.}^2} \times \frac{\text{ft}^3}{62.4 \text{ lbf}} \times 144 \frac{\text{in.}^2}{\text{ft}^2} = 211 \text{ ft}$

since $\bar{V}_2 = \bar{V}_3$ and the elevations are corrected to the pump centerline ($z_2 = z_3$).

From Eq. 10.8a, $\dot{W}_h = \rho Q g H_p$. Thus $\dot{W}_h = \rho g Q H_p$

$$\dot{W}_h = 62.4\frac{lbf}{ft^3} \times 10\frac{gal}{min} \times 211\,ft \times \frac{ft^3}{7.48\,gal} \times \frac{hp \cdot min}{33{,}000\,ft \cdot lbf} = 0.533\ hp$$ $\dot{W}_h$

Changing to D = 1.5 in. (nominal) pipe would reduce the mean velocity and hence the head loss and the minor loss. For this pipe D = 1.610 in. (Table 8.4).

$$A = \frac{\pi D^2}{4} = \frac{\pi}{4}\left(\frac{1.610}{12}\right)^2 ft^2 = 0.0141\ ft^2;\ \text{so}\ \bar{V} = \frac{Q}{A} = 10\frac{gal}{min} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} \times \frac{1}{0.0141\,ft^2} = 1.58\ ft/s$$

$$Re = \frac{\bar{V}D}{\nu} = 1.58\frac{ft}{s} \times \frac{1.61}{12}ft \times \frac{s}{1.08\times10^{-5}\,ft^2} = 1.96\times10^4;\ \frac{e}{D} = 0.00373;\ f \approx 0.032\ \text{(Fig. 8.13)}$$

Thus $\frac{\bar{V}_2^2}{2g} = \frac{1}{2}(1.58)^2\frac{ft^2}{s^2} \times \frac{s^2}{32.2\,ft} = 0.0388\ ft$

$$H_{LT_{12}} = \left(\left(0.031 \times \frac{50}{(1.61/12)} + 30 + 16\right) + 0.78\right)0.0388\ ft = 0.534\ ft$$

$$H_{LT_{34}} = \left(0.031 \times \left(\frac{120}{(1.61/12)} + 2\times16\right)\right)0.0388\ ft = 1.11\ ft$$

$$p_2(gage) = -\rho g\left(\frac{\bar{V}_2^2}{2g} + z_2 + H_{LT_{12}}\right)$$

$$p_2(gage) = -62.4\frac{lbf}{ft^3}(0.0388 + 10 + 0.534)ft \times \frac{ft^2}{144\,in.^2} = -4.58\ psig$$

and

$$p_3(gage) = p_4 + \rho g\left(z_4 - z_3 + H_{LT_{34}}\right)$$

$$p_3(gage) = 50\ psig + 62.4\frac{lbf}{ft^3}(80 - 10 + 1.11)ft \times \frac{ft^2}{144\,in.^2} = 80.8\ psig$$

The pump head is $H_p = \frac{p_3 - p_2}{\rho g} = (80.8 - (-4.58))\frac{lbf}{in.^2} \times \frac{ft^3}{62.4\,lbf} \times 144\frac{in.^2}{ft^2} = 197\ ft$

Thus the pump power is $\dot{W}_h = \rho g Q H_p = 0.498\ hp$

The power reduction is $\Delta\dot{W}_h = \frac{\dot{W}_h(1.5\,in.) - \dot{W}_h(1\,in.)}{\dot{W}_h(1\,in.)} = -6.7\ percent$ $\Delta\dot{W}_h$

The pump should <u>not</u> be moved up the hill. The NPSHA now is

$$NPSHA = \frac{1}{\rho g}\left[p(abs) + \rho\frac{\bar{V}_2^2}{2} - p_v\right] = \frac{1}{\rho g}\left[p_{atm} + p(gage) + \gamma\frac{\bar{V}_2^2}{2} - p_v\right]$$

At $T = 68°F$, $p_v = 0.339$ psia (Table A.7).

$$NPSHA = \frac{ft^3}{62.4\,lbf}\left[14.7\frac{lbf}{in.^2} + (-6.45)\frac{lbf}{in.^2} + 1.94\frac{slug}{ft^3} \times 6.90\frac{ft^2}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \frac{ft^2}{144\,in.^2} - 0.339\frac{lbf}{in.^2}\right]144\frac{in.^2}{ft^2}$$

$NPSHA = 18.5\ ft$ (1 in. pipe) $\quad 22.6\ ft$ (1.5 in. pipe) NPSHA

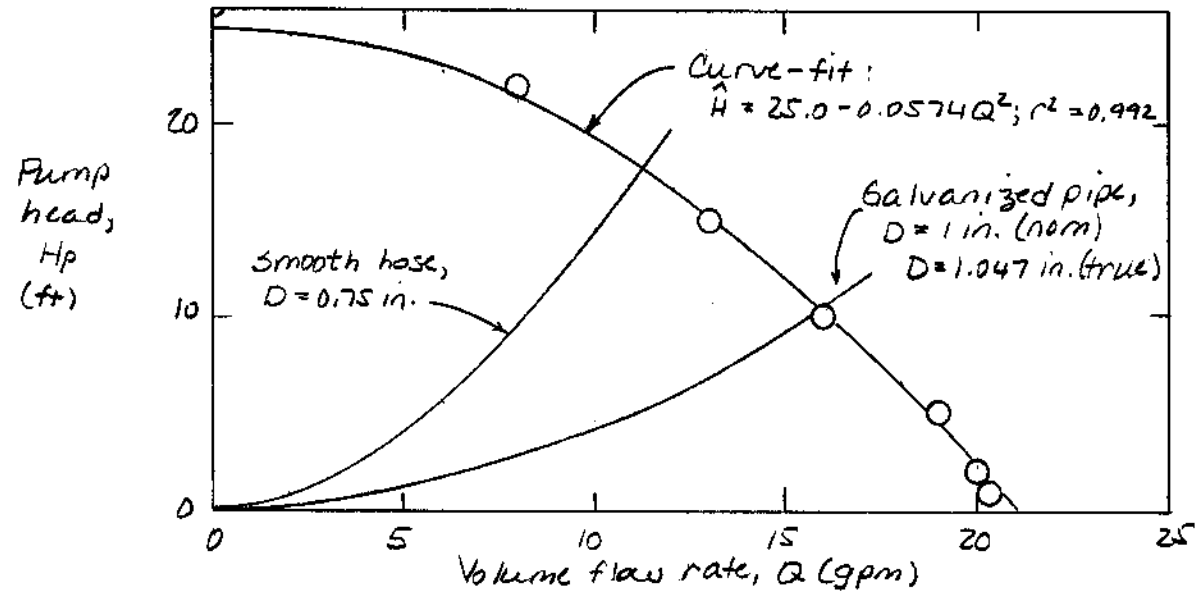
This should not be reduced further by raising the pump. If anything, the pump should be lowered to increase NPSHA.

Given: Manufacturer data for submersible pump:

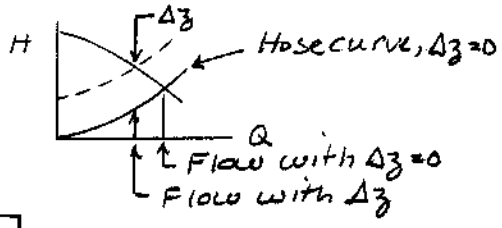
Discharge Height (ft)	1	2	5	10	15	20	26.3
Water Flow Rate (gpm)	20.4	20	19	16	13	8	0

Find: (a) Plot a performance curve for this pump.
(b) Develop and show a curve-fit to the data.
(c) Calculate and plot the pump delivery versus discharge height for (1) 50' of 3/4 in. garden hose, (2) 50' of 1 in. pipe.

Solution: The performance data and curve-fit are plotted below:

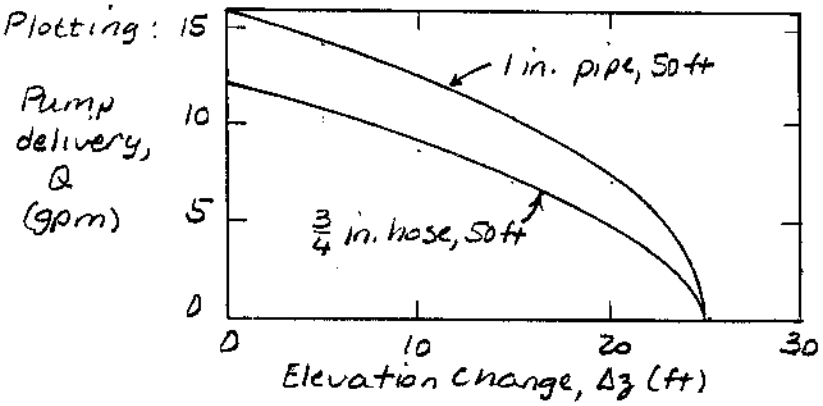


The energy equation for steady, incompressible pipe flow was used to develop the system curves shown above, for $\Delta z = 0$. To obtain delivery versus height, add the elevation change to the head for $\Delta z = 0$ to find the new intersection with the pump curve:



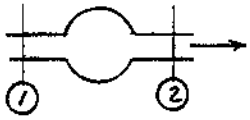
The same result is obtained when the difference between the pump curve and the hose, at a given flow rate, equals Δz.

Plotting:



Given: Fire hose and nozzle of Problem 8.142: L = 300 ft, D = 1.5 in., smooth hose. At design flow rate, Δp = 33 psi per 100 ft.

Pump conditions are: p_1 = 50 psig
p_2 = 199 psig



Find: (a) Discuss pump selection for this system.
(b) Would a bypass improve system operation?
(c) Specify an appropriate pump and impeller diameter.
(d) Calculate the pump power input.

Solution: Need head and volume flow rate to select pump.

$$H = \frac{p_2 - p_1}{\rho g} = (199-50)\frac{lbf}{in.^2} \times \frac{ft^3}{62.4\ lbf} \times 144\frac{in.^2}{ft^2} = 344\ ft$$

Find Q from pressure drop: using FM, Q = 0.2466 ft³/s (111 gpm).

From Fig. D.1, this combination of high head and low flow rate could be supplied by a Peerless Type 2AE11 pump operating at 3500 rpm.* ← Pump

This pump would operate at higher efficiency at this head in the 300-400 gpm range of flow rate. Thus several hoses could be operated at once or a bypass could be used.

Pump power is $\dot{W}_m = \frac{\rho Q g H}{\eta_p}$. At Q = 300 gpm (0.668 ft³/sec), with $\eta_p \approx 0.6$

$$\dot{W}_m = \frac{1}{0.6} \times 0.668\frac{ft^3}{s} \times 62.4\frac{lbf}{ft^3} \times 344\ ft \times \frac{hp \cdot s}{550\ ft \cdot lbf} = 43.5\ hp$$ ← $\dot{W}_m$

The hydraulic power developed is $\dot{W}_h = \rho Q g H$ = 26.1 hp.

Pump efficiency can be estimated from Fig. 10.15 by finding N_{Scu}.

$$N_{Scu} = \frac{N(rpm)[Q(gpm)]^{1/2}}{[H(ft)]^{3/4}} = \frac{3500(300)^{1/2}}{(344)^{3/4}} = 759$$

From Fig. 10.15, one might expect $\eta \approx 0.6$-0.65. A smaller pump would be worse.

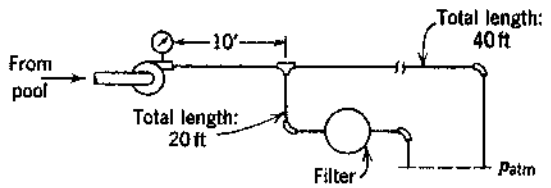
*Impeller diameter could be chosen from a pump performance curve.

Given: Swimming pool filtration system of Problem 8.147.

Assume pipe used is 3/4 in. (nominal) smooth PVC plastic.

Pump delivers $Q = 30$ gpm of water at 75°F.

Filter pressure drop is $\Delta p = 0.6 Q^2$, where Δp in psi and Q in gpm.



Find: (a) Specify speed and impeller diameter of suitable pump.
(b) Estimate pump efficiency.

Solution: Need head to specify pump. Using FM, with $Q = 30$ gpm (0.0668 cfs) and $D = 0.824$ in. (0.0687 ft), $\Delta p_{12} = 5.56$ psi.

Flow must split to give same pressure drop in each branch. Assuming L_e/D for each elbow is 30, iteration gives:

$Q_{24} = 24.8$ gpm (0.0553 cfs) and $\Delta p = 16.69$ psi

$Q_{23} = 5.2$ gpm (0.0116 cfs) and $\Delta p = 16.2 + 0.6023 = 16.8$ psi

Neglecting any pressure at the pump inlet, the pump must supply

$$\Delta p_{pump} = \Delta p_{12} + \Delta p_{23} = 5.56 + 16.7 = 22.3 \text{ psi}$$

The pump head is $H = \dfrac{\Delta p}{\rho g} = 22.3 \dfrac{lbf}{in.^2} \times \dfrac{ft^3}{62.4\ lbf} \times 144 \dfrac{in.^2}{ft^2} = 51.5$ ft → H

This pump is too small to be found in Fig. D.1. Therefore approximate its characteristics using N_s and H_o. Assume $N = 3500$ rpm:

$$N_{s_{CU}} = \frac{N Q^{1/2}}{H^{3/4}} = \frac{3500 (30)^{1/2}}{(51.5)^{3/4}} = 997$$

From Fig. 10.15, $\eta \approx 0.62$ or less. Assuming $H = 0.7\, H_o$, then

$$H = 0.7 H_o = 0.7 \frac{(\omega R)^2}{g}; \quad R = \frac{1}{\omega}\left(\frac{gH}{0.7}\right)^{\frac{1}{2}} = \frac{5}{367 \text{ rad}}\left[32.2 \frac{ft}{s^2} \times 51.5 \text{ ft} \times \frac{1}{0.7}\right]^{\frac{1}{2}} = 0.133 \text{ ft}$$

The impeller diameter is approximately

$$D = 2R = 2 \times 0.133 \text{ ft} \times 12 \frac{in.}{ft} = 3.18 \text{ in.}$$ → D

The pump power requirement is

$$\dot{W}_m = \frac{\rho Q g H}{\eta_p} = \frac{1}{0.6} \times 0.0668 \frac{ft^3}{s} \times 62.4 \frac{lbf}{ft^3} \times 51.5 \text{ ft} \times \frac{hp \cdot s}{550 \text{ ft} \cdot lbf} = 0.651 \text{ hp}$$

A 3/4 horsepower motor should be used. → $\dot{W}_m$

Given: Performance data for Buffalo Forge Type BL centrifugal fan ($D = 36.5$ in.), tested at 600 rpm:

Volume flow rate, Q (cfm)	6000	8000	10,000	12,000	14,000	16,000
Static pressure rise, Δp (in. H_2O)	2.10	2.00	1.76	1.37	0.92	0.42
Power input, $\mathcal{P}$ (hp)	2.75	3.18	3.48	3.51	3.50	3.22

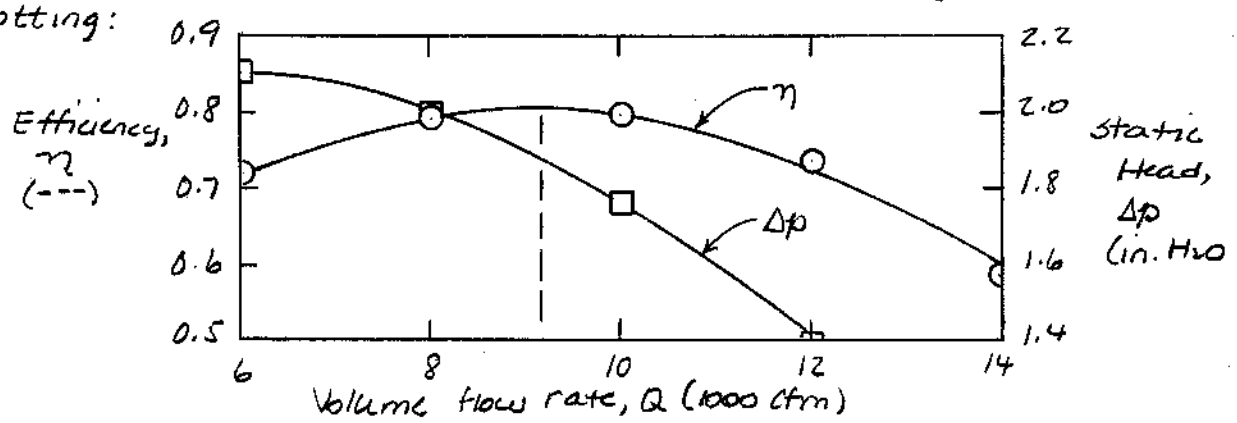
Find: (a) Plot performance versus volume flow rate.
(b) Calculate static efficiency and show on plot.
(c) Locate the best efficiency point (BEP).
(d) Specify the fan rating at the BEP.

Solution: Apply the definition of efficiency, $\eta \equiv \frac{\dot{W}_h}{\dot{W}_m} = \frac{Q\,\Delta p}{\dot{W}_m}$. At $Q = 10{,}000$ cfm,

$$\Delta p = \rho g \Delta h = 62.4 \frac{\text{lbf}}{\text{ft}^3} \times 1.76 \text{ in.}(H_2O) \times \frac{\text{ft}}{12 \text{ in.}} = 9.15 \text{ lbf/ft}^2$$

$$\eta = \frac{10{,}000 \text{ ft}^3}{\text{min}} \times 9.15 \frac{\text{lbf}}{\text{ft}^2} \times \frac{1}{3.48 \text{ hp}} \times \frac{\text{min}}{60 \text{ s}} \times \frac{\text{hp} \cdot \text{s}}{550 \text{ ft} \cdot \text{lbf}} = 0.797$$ η

Plotting:



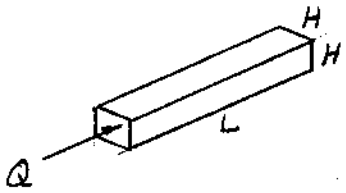
The BEP is at $Q \approx 9200$ cfm, where $\eta \approx 0.80$ and $\Delta p \approx 1.87$ in. H_2O. BEP

Given: Fan of Problem 10.77 (see data table, next page).

Find: (a) Minimum size square sheet metal duct that will carry Q = 12000 cfm over a distance L = 50 ft.
(b) Estimate of increase in delivery if speed increased to N = 800 rpm

Solution: Assume smooth, and T = 70°F, so $\nu = 1.05\times10^{-5}$ ft²/s (Table A.7).

$$D_h = \frac{4A}{P} = \frac{4H^2}{4H} = H$$



Sample calculation: H = 1.5 ft

$$\bar{V} = \frac{Q}{A} = \frac{Q}{H^2} = 12000\,\frac{ft^3}{min}\times\frac{1}{(1.5)^2\,ft^2}\times\frac{min}{60\,s} = 88.9\ ft/s$$

$$Re = \frac{\bar{V}D_h}{\nu} = \frac{\bar{V}H}{\nu} = 88.9\,\frac{ft}{s}\times 1.5\,ft\times\frac{s}{1.05\times10^{-5}\,ft^2} = 1.27\times10^7$$

Smooth, so $e \approx 0$, $e/D = 0$

$$f_0 = 0.25\left[\log\left(\frac{5.74}{Re^{0.9}}\right)\right]^{-2} = 0.00787 \qquad \text{Eq. 8.37b}$$

$$\frac{1}{f^{0.5}} = -2.0\log\left(\frac{2.51}{Re\,f^{0.5}}\right) = 11.3 \qquad \text{Eq. 8.37a}$$

$$f = \left(\frac{1}{11.3}\right)^2 = 0.00783$$

Then from the energy equation,

$$\Delta h = \frac{\Delta p}{\rho_w g} = \frac{1}{\rho_w g} f\frac{L}{D_h}\frac{\rho_a V^2}{2} = f\frac{L}{D_h}\frac{V^2}{2g}\frac{\rho_a}{\rho_w}$$

$$L/D_h = 50\ ft/1.5\ ft = 33.3$$

$$\Delta h = 0.00783\times 33.3\times\frac{1}{2}\times(88.9)^2\,\frac{ft^2}{s^2}\times\frac{s^2}{32.2\,ft}\times\frac{1.23}{999}\times 12\,\frac{in.}{ft}$$

$$\Delta h = 0.473\ in.\ H_2O \longleftarrow \Delta h$$

Calculations are tabulated and plotted on the next page. Matching occurs with

$$H = 1.205\ ft\ (14.5\ in.) \longleftarrow H$$

Fan performance is scaled to 800 rpm using the "fan laws." Thus

$$Q' = \frac{\omega'}{\omega}Q = \frac{800\ rpm}{600\ rpm}\times 6000\ cfm = 8000\ cfm$$

$$\Delta h' = \left(\frac{\omega'}{\omega}\right)^2\Delta h = \left(\frac{800}{600}\right)^2 2.10\ in.\ H_2O = 3.73\ in.\ H_2O$$

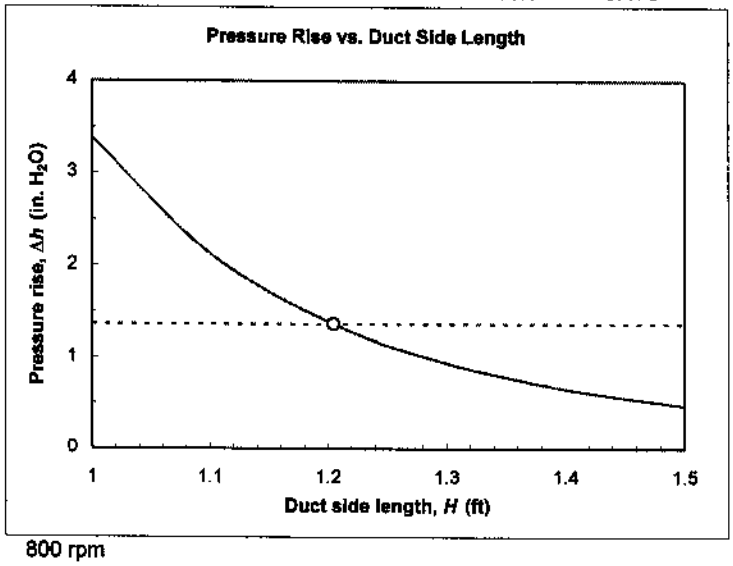
At 800 rpm, $Q' \approx 16000$ cfm $\longleftarrow Q'$

Input Data:

Duct length:	L =	50	ft
Kinematic viscosity:	ν =	1.05E-05	ft^2/s
Temperature:	T =	70	°F
Volume flow rate:	Q =	12,000	ft^3/min

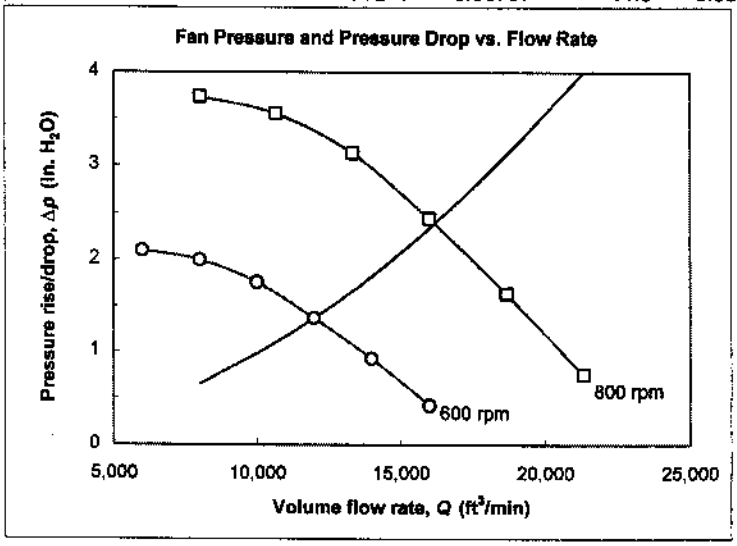
Calculated Results:

H (ft)	V (ft/s)	Re (---)	f_0 (---)	$1/f^{0.5}$ (---)	f (---)	L/H (---)	Δh (in. H_2O)
1.0	200	1.90E+7	0.00745	11.6	0.00739	50.0	3.39
1.1	165	1.73E+7	0.00755	11.6	0.00749	45.5	2.13
1.205	138	1.58E+7	0.00764	11.5	0.00759	41.5	1.37
1.3	118	1.47E+7	0.00772	11.4	0.00767	38.5	0.948
1.4	102	1.36E+7	0.00780	11.4	0.00775	35.7	0.661
1.5	88.9	1.27E+7	0.00787	11.3	0.00783	33.3	0.473



Change fan speed:

600 rpm		800 rpm								
Q (cfm)	Δh_{fan} (in. H_2O)	Q (cfm)	Δh_{fan} (in. H_2O)	V (ft/s)	Re (---)	f_0 (---)	$1/f^{0.5}$ (---)	f (---)	L/H (---)	Δh (in. H_2O)
6,000	2.10	8,000	3.73	91.8	1.05E+7	0.00808	11.2	0.00804	41.5	0.65
8,000	2.00	10,667	3.56	122.4	1.41E+7	0.00777	11.4	0.00771	41.5	1.10
10,000	1.76	13,333	3.13	153.0	1.76E+7	0.00753	11.6	0.00747	41.5	1.67
12,000	1.37	16,000	2.44	183.7	2.11E+7	0.00735	11.7	0.00729	41.5	2.34
14,000	0.92	18,667	1.64	214.3	2.46E+7	0.00720	11.8	0.00713	41.5	3.12
16,000	0.42	21,333	0.75	244.9	2.81E+7	0.00707	11.9	0.00701	41.5	4.00



Given: Performance data for Buffalo Forge Type BL centrifugal fan ($D = 36.5$ in.), tested at 600 rpm (Problem 10.77):

Volume flow rate, Q (cfm)	6000	8000	10,000	12,000	14,000	16,000
Static pressure rise, Δp (in. H_2O)	2.10	2.00	1.76	1.37	0.92	0.42
Power input, $\mathcal{P}$ (hp)	2.75	3.18	3.48	3.51	3.50	3.22

At $Q = 12{,}000$ cfm, dynamic head is $h_d = 0.16$ in. H_2O.

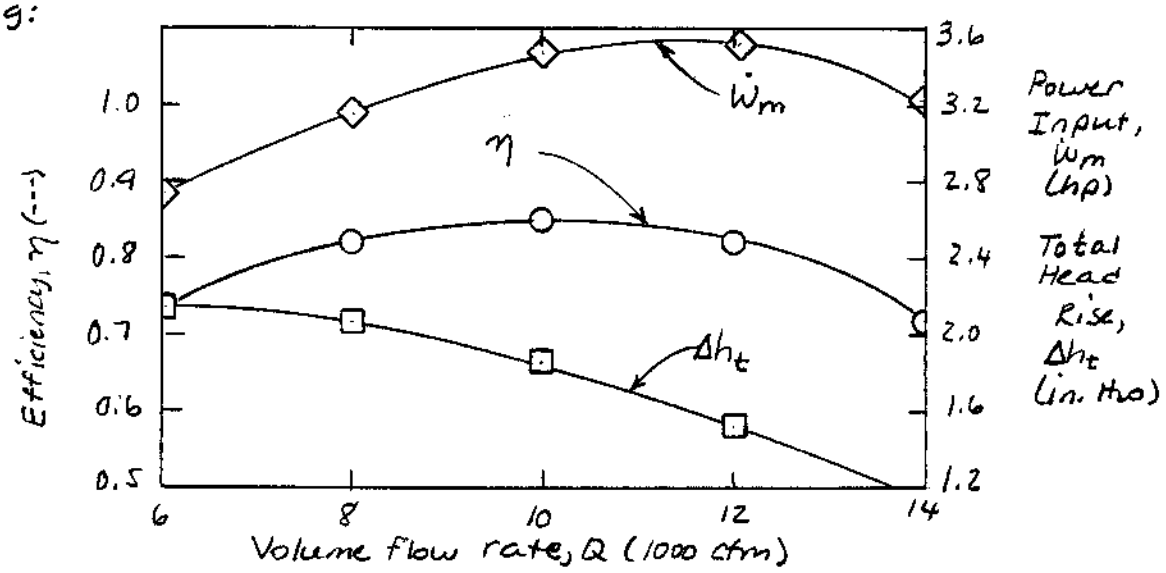
Find: (a) Evaluate fan outlet area.
(b) Plot total pressure rise and input horsepower vs. flow rate.
(c) Calculate fan total efficiency and show on plot.
(d) Find the BEP and specify the fan rating there.

Solution: Apply the definition of efficiency, $\eta = \frac{\dot{W}_h}{\dot{W}_m} = \frac{Q\,\Delta p_t}{\dot{W}_m}$. At $Q = 10{,}000$ cfm,

$$\Delta p_t = \rho g\,\Delta h_t = 62.4\,\frac{lbf}{ft^3} \times \left[1.76\text{ in.} + 0.16\text{ in.}\left(\frac{10{,}000}{12{,}000}\right)^2\right]\frac{ft}{12\text{ in.}} = 9.72\ lbf/ft^2$$

$$\eta = 10{,}000\,\frac{ft^3}{min} \times 9.72\,\frac{lbf}{ft^2} \times \frac{1}{3.48\ hp} \times \frac{min}{60\ s} \times \frac{hp \cdot s}{550\ ft \cdot lbf} = 0.846$$ η

Plotting:



The best efficiency point (BEP) is at $Q \approx 10{,}000$ cfm, where $\eta_t \approx 0.85$ and $\Delta h_t \approx 1.87$ in. H_2O. BEP

Also

$$p_d = \frac{1}{2}\rho \bar{V}^2 = \rho_w g\, h_d\ ;\ \bar{V} = \sqrt{\frac{2\rho_w g\, h_d}{\rho}} = 26.5\ ft/s$$

$$Q = \bar{V}A\ ;\ A = \frac{Q}{\bar{V}} = 10{,}000\,\frac{ft^3}{min} \times \frac{s}{26.5\ ft} \times \frac{min}{60\ s} = 6.29\ ft^2$$ A_{out}

Given: Performance data of Problem 10.77 for Buffalo Forge Type BL centrifugal fan (D = 36.5 in., tested at 600 rpm:

Volume flow rate, Q (cfm)	6000	8000	10,000	12,000	14,000	16,000
Static pressure rise, Δp (in. H_2O)	2.10	2.00	1.76	1.37	0.92	0.42
Power input, $\mathcal{P}$ (hp)	2.75	3.18	3.48	3.51	3.50	3.22

This fan also is manufactured with 40.3, 44.5, 49.0, and 54.3 in. diameter wheels.

Find: (a) Pick a fan to deliver Q = 30,000 cfm against a static pressure rise of 5.0 in. H_2O.
(b) Determine the fan speed and input power required.

Solution: Scaling laws and specific speed for fans apply.

Computing equations: $\frac{Q'}{Q} = \left(\frac{\omega'}{\omega}\right)\left(\frac{D'}{D}\right)^3$, $\frac{h'}{h} = \left(\frac{\omega'}{\omega}\right)^2\left(\frac{D'}{D}\right)^2$, $\frac{\mathcal{P}'}{\mathcal{P}} = \left(\frac{\omega'}{\omega}\right)^3\left(\frac{D'}{D}\right)^5$

There are two unknowns, D and ω. To obtain a preliminary estimate, match specific speeds. From the solution to Problem 10.77, the BEP is at Q = 9200 cfm, h = 1.87 in. H_2O. To match, set

$$N_S = \frac{N(Q)^{1/2}}{(h)^{3/4}} = \frac{600\ \text{rpm}(9200\ \text{cfm})^{1/2}}{(1.87\ \text{in.})^{3/4}} = N_S' = \frac{N'(30{,}000\ \text{cfm})^{1/2}}{(5.00\ \text{in.})^{3/4}} = 36{,}000$$

$$N' = 600\ \text{rpm}\left(\frac{9200}{30{,}000}\right)^{\frac{1}{2}}\left(\frac{5.00}{1.87}\right)^{\frac{3}{4}} = 695\ \text{rpm} = \omega'$$

From the Q scaling law, $\frac{D'}{D} = \left[\frac{Q'}{Q}\frac{\omega}{\omega'}\right]^{\frac{1}{3}} = \left[\frac{30000}{9200}\ \frac{600}{695}\right]^{\frac{1}{3}} = 1.41$; D' = 51.5 in.

This is between the two largest diameters, so check both:

D' = 49.0 in.: $\omega' = \omega\ \frac{Q'}{Q}\left(\frac{D}{D'}\right)^3 = 863$ rpm; $N_S' = 44{,}700$; this is far to the right of BEP

D' = 54.3 in.: $\omega' = \omega\ \frac{Q'}{Q}\left(\frac{D}{D'}\right)^3 = 594$ rpm; $N_S = 30{,}800$; this is just to the left of BEP

Then $\Delta p' = \Delta p\left(\frac{\omega'}{\omega}\right)^2\left(\frac{D'}{D}\right)^2 = 1.87\ \text{in. } H_2O\left(\frac{594}{600}\right)^2\left(\frac{54.3}{36.5}\right)^2 = 4.06$ in.

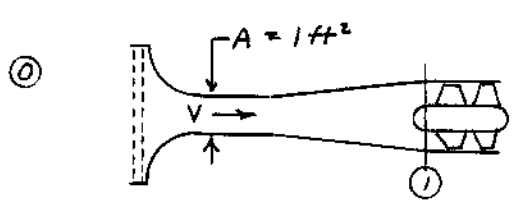
Therefore must operate faster to make the desired pressure rise.

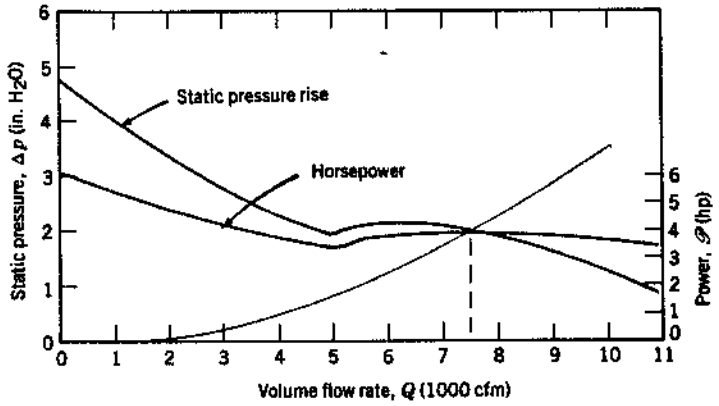
$$\omega'' = \omega\left(\frac{\Delta p''}{\Delta p'}\right)^{\frac{1}{2}} = 594\ \text{rpm}\left(\frac{5.00}{4.06}\right)^{\frac{1}{2}} = 659\ \text{rpm}$$ ← ω'

The power will be

$$\mathcal{P}' = \mathcal{P}\left(\frac{\omega''}{\omega}\right)^3\left(\frac{D'}{D}\right)^5 = 3.4\ \text{hp}\left(\frac{659}{600}\right)^3\left(\frac{54.3}{36.5}\right)^5 = 32.8\ \text{hp}$$ ← $\mathcal{P}'$

Given: Wind tunnel, with 1 foot square test section, powered by fan below. Tunnel contains two screens, each with $K = 0.12$, and a diffuser between the test section and the 24 in. ϕ fan inlet.





Find: (a) Calculate and plot Δp versus Q.
(b) Estimate the maximum air speed.

Solution: Apply energy

Computing equation: $\frac{p_0}{\rho} + \frac{V_0^2}{2} + g z_0 = \frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1 + h_{LT}$; $h_{LT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K \right] \frac{V^2}{2}$ ($V_0 \approx 0$ (2))

Assumptions: (1) $p_0 = p_{atm}$, (2) $V_0 \approx 0$, $\alpha_1 \approx 1$, (3) $z_0 = z_1$, (4) Losses in diffuser, screens

$$\frac{\Delta p_{fan}}{\rho} = \frac{p_{atm} - p_1}{\rho} = \frac{V_1^2}{2} + h_{LT} = \frac{V_1^2}{2} + (2K_{screen} + K_{diffuser})\frac{V_1^2}{2} = \left[2K_s + K_d + \left(\frac{A}{A_1}\right)^2\right]\frac{Q^2}{2A^2}$$

From continuity, $V_1 A_1 = VA$; $V_1^2 = V^2\left(\frac{A}{A_1}\right)^2$; $V = \frac{Q}{A}$; $V^2 = \frac{Q^2}{A^2}$; $A = 1\ ft^2$; $A_1 = \frac{\pi}{4} D_1^2 = 3.14\ ft^2$

From Fig. 8.19, $K_d = C_{p_i} - C_p = 1 - \left(\frac{1}{AR}\right)^2 - 0.70 = 1 - \left(\frac{1}{3.14}\right)^2 - 0.70 = 0.199$

$$\Delta p_{fan} = [2(0.12) + 0.199 + 0.101]\frac{1}{2} \times Q^2 \frac{ft^6}{min^2} \times \frac{1}{(1)^2 ft^4} \times 0.00238 \frac{slug}{ft^3} \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \frac{min^2}{3600\ s^2}$$

$$\Delta h_{fan} = \frac{\Delta p}{\rho g} = 1.79 \times 10^{-7} [Q(cfm)^2] \frac{lbf}{ft^2} \times \frac{ft^3}{62.4\ lbf} \times 12 \frac{in.}{ft} = 3.43 \times 10^{-8} [Q(cfm)^2]$$

The resulting curve is plotted above; computed values are tabulated below.

The system will operate where the fan curve and system curve cross. The approximate operating point is $Q = 7400$ cfm at $h = 1.9$ in. H_2O.

The test section speed is

$$V = \frac{Q}{A} = 7400 \frac{ft^3}{min} \times \frac{1}{1\ ft^2} \times \frac{min}{60\ s} = 123\ ft/s$$ V

Q (1000 cfm)	1	2	3	4	5	6	7	8	9	10
Δp (in. H_2O)	0.03	0.14	0.31	0.55	0.86	1.24	1.68	2.20	2.78	3.43

Given: Axial-flow fan and wind tunnel of Problem 10.81.

Find: (a) Scale performance of fan as it varies with operating speed.
(b) Develop and plot a "calibration curve" showing test section flow speed (m/sec) versus fan speed (rpm).

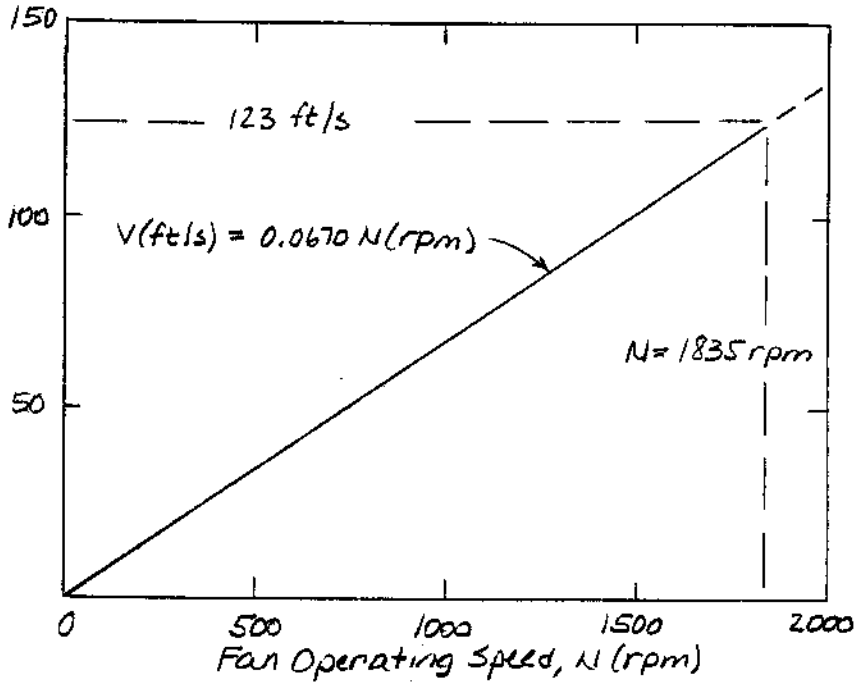
Solution: From the solution to Problem 10.81, $\Delta h_{fan} \propto Q^2 \propto V^2$.

The scaling laws for varying fan speed suggest $Q \propto \omega$ and $p \propto \omega^2$.

Thus $\Delta h_{fan} \propto Q^2 \propto p \propto \omega^2$ or $Q \propto \omega$. The volume flow rate (and the test section flow speed) should vary directly with ω.

From the results of Problem 10.81, $V = 123$ ft/s when $N = 1835$ rpm.

Plotting:



The slope of the linear relationship is $\frac{V}{N} = 123 \frac{ft}{s} \times \frac{1}{1835\ rpm} = 0.0670$ ft/s/rpm.

Thus $V(ft/s) = 0.0670\ N(rpm)$

V

Given: Experimental test data for aircraft engine fuel pump:

Pump supplies fuel at 450 pph, 150 psig to the controller.

Pump Speed (rpm)	Back Pressure (psig)	Fuel Flow (pph*)	Pump Speed (rpm)	Back Pressure (psig)	Fuel Flow (pph)	Pump Speed (rpm)	Back Pressure (psig)	Fuel Flow (pph)
	200	1810		200	1730		200	89
4536	300	1810	4355	300	1750	453	250	73
(100%)	400	1810	(96%)	400	1735	(10%)	300	58.5
	500	1790		500	1720		350	45
	900	1720		900	1635		400	30

* Fuel flow rate measured in pounds per hour (pph).

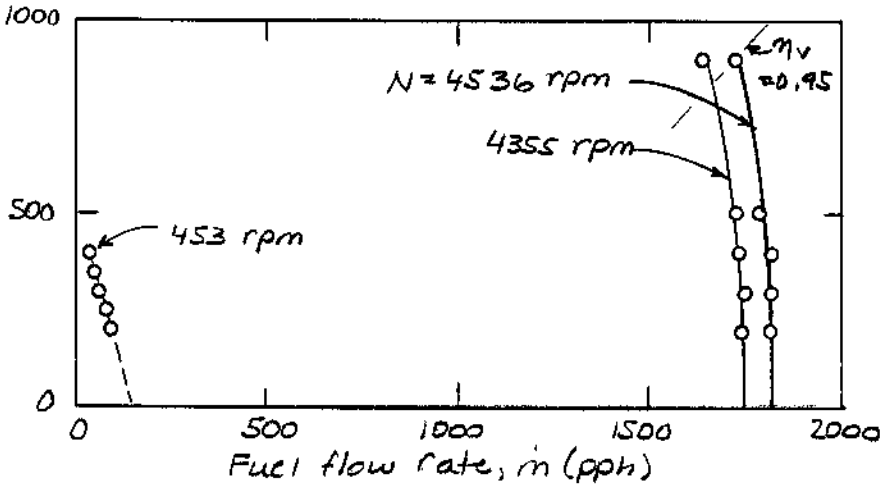
Find: (a) Plot curves of fuel pressure versus delivery at the three constant speeds.

(b) Estimate the pump displacement volume per revolution.

(c) Calculate volumetric efficiency at each point, sketch η_v contours.

(d) Evaluate energy loss due to throttling at 100% speed, full delivery.

Solution:



For the pump, $\dot{m} = \rho \forall N$, so $\forall = \frac{\dot{m}}{\rho N}$. Analyzing the 4536 rpm case,

$$\forall \approx 1810 \frac{lbm}{hr} \times \frac{gal}{6.8\ lbm} \times \frac{min}{4536\ rev} \times \frac{ft^3}{7.48\ gal} \times 1728 \frac{in.^3}{ft^3} \times \frac{hr}{60\ min} = 0.226\ in.^3/rev$$ $\forall$

At constant speed, $\eta_v = \frac{\forall_{actual}}{\forall_{geometric}} = \frac{\dot{m}}{\dot{m}(p=0)}$. Calculation shows η_v decreases as speed is reduced, see below.

Energy loss is $\dot{W}_L = \left(\frac{\dot{m}_p - \dot{m}_L}{\rho}\right) p_L = (1810 - 450) \frac{lbm}{hr} \times \frac{gal}{6.8\ lbm} \times 150 \frac{lbf}{in.^2} \times \frac{ft^3}{7.48\ gal}$

$$\times 144 \frac{in.^2}{ft^2} \times \frac{hp \cdot s}{550\ ft \cdot lbf} \times \frac{hr}{3600\ s}$$

$\dot{W}_L = 0.292$ hp $\dot{W}_L$

At 453 rpm, the best volumetric efficiency is

$$\eta_v \approx \frac{\dot{m}}{\dot{m}(p=0)} \times \frac{4536}{453} \approx \frac{89\ pph}{1810\ pph} \times \frac{4536}{453} = 0.0492, \text{ or about } 5\%$$ η_v

At 4355 rpm,

$$\eta_v \approx \frac{1730\ pph}{1810\ pph} \times \frac{4536}{4355} = 0.996, \text{ or more than } 99\% \text{ (this is doubtful).}$$ η_v

Given: Air boat with propeller having $D = 5$ ft, driven at maximum speed, $N = 1800$ rpm, by a 160 hp engine.

Find: Estimate the maximum thrust produced by the propeller at (a) standstill and (b) $V = 30$ mph.

Solution: Assume the aircraft propeller coefficients in Fig. 10.40 are applicable to this propeller, and air at standard conditions.

At $V = 0$, $J = 0$. Etrapolating gives $C_F = 0.80$ and $C_P = 0.39$.

The engine power sounds low. Check to see if the prop can be spun at $N = 1800$ rpm ($n = 30$ rps):

$$\mathcal{P} = C_P \rho n^3 D^5 = 0.39 \times 0.00238 \frac{\text{slug}}{\text{ft}^3} \times (30)^3 \frac{\text{rev}^3}{\text{s}^3} \times (5)^5 \text{ft}^5 \times \frac{\text{lbf}\cdot\text{sec}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{hp}\cdot\text{s}}{650\ \text{ft}\cdot\text{lbf}} = 142 \text{ hp}$$

This appears reasonable. If the engine produces 160 hp, since $\mathcal{P} \propto n^3$, then

$$n_{max} = n_{1800}\left(\frac{160}{142}\right)^{1/3} = 1870 \text{ rpm } (31.2 \text{ rev/s})$$

The thrust force at standstill ($J = 0$) is found from $C_F \approx 0.16$ (Fig. 10.40)

$$F_T = C_F \rho n^2 D^4 = 0.16 \times 0.00238 \frac{\text{slug}}{\text{ft}^3} \times (30.0)^2 \frac{\text{rev}^2}{\text{s}^2} \times (5)^4 \text{ft}^4 \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} = 214 \text{ lbf}$$

F_T ($V = 0$) ←

At $V = 30$ mph (44 ft/s), $J \approx \frac{V}{nD} = \frac{44 \text{ ft}}{\text{s}} \times \frac{\text{s}}{30.0 \text{ rev}} \times \frac{1}{5 \text{ ft}} = 0.293$, so $C_P \approx 0.44$.

Thus $\mathcal{P} = \mathcal{P}(V=0) \times \frac{0.44}{0.39} = 142 \text{ hp} \times \frac{0.44}{0.39} = 160$ hp (at 1,800 rpm).

At $J = 0.293$, Fig. 10.40 gives $C_F = 0.145$, lower than at $J = 0$.

The thrust force at 30 mph is

$$F_T = 0.145 \times 0.00238 \frac{\text{slug}}{\text{ft}^3} \times (30.0)^2 \frac{\text{rev}^2}{\text{s}^2} \times (5)^4 \text{ft}^4 \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} = 194 \text{ lbf}$$

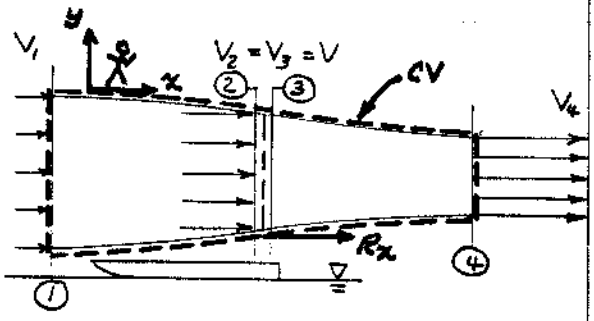
F_T ($V = 30$ mph) ←

The engine probably cannot produce 160 hp at the low speed, 1800 rpm. More information would be needed to match the actual power output at the lower engine speeds. A simpler solution might be to cut down the propeller to absorb full power at 1800 rpm. This would increase F_T.

Given: Air boat propeller

$\dot{m} = 40$ kg/s = constant

At rest, $V_4 = 40$ m/s



Find: (a) Propeller diameter.
(b) Thrust produced at rest.
(c) Thrust when boat moving at 10 m/s.

Solution: Apply continuity and x momentum equations using CV moving with boat, as shown.

Basic equation: $F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,dV + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$ ($F_{Bx}=0$ (2); $\frac{\partial}{\partial t}=0$ (3))

Assumptions: (1) Atmospheric pressure surrounds CV
(2) Horizontal; $F_{Bx}=0$
(3) Steady flow w.r.t. CV
(4) Uniform flow at each section
(5) Use velocities relative to CV

Then

$$T = R_x = u_1\{-\dot{m}\} + u_4\{+\dot{m}\} = (V_4 - V_1)\dot{m} \qquad (1)$$

$u_1 = V_1 \quad u_4 = V_4$

It can be shown (see Example Prob. 10.13) that $V = \frac{1}{2}(V_4 + V_1)$. Thus for the static case ($V_1 \approx 0$)

$$V = \tfrac{1}{2}\left(40\,\tfrac{m}{sec} + 0\right) = 20 \text{ m/s}$$

From continuity, $\dot{m} = \rho V A = \rho V \frac{\pi D^2}{4}$ or $D = \left[\frac{4\dot{m}}{\pi\rho V}\right]^{1/2}$

$$D = \left[\frac{4}{\pi} \times 40\,\tfrac{kg}{s} \times \frac{m^3}{1.23\,kg} \times \frac{s}{20\,m}\right]^{1/2} = 1.44 \text{ m} \quad \text{D} \longleftarrow$$

For $V_1 = 0$,

$$T = \dot{m}(V_4 - \cancel{V_1}) = 40\,\tfrac{kg}{s} \times 40\,\tfrac{m}{s} \times \frac{N\cdot s^2}{kg\cdot m} = 1.60 \text{ kN (static case)} \quad \text{T} \longleftarrow$$

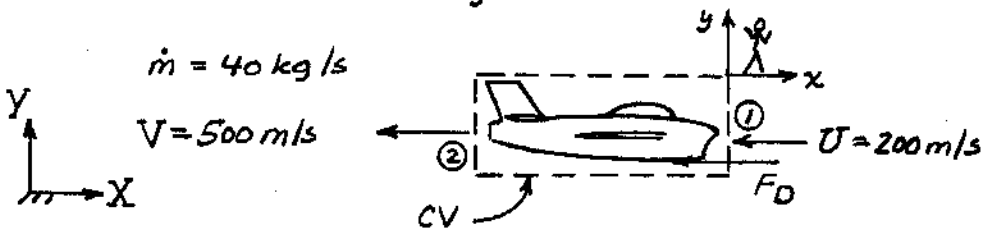
Since $V = \frac{1}{2}(V_1 + V_4)$, then

$$V_4 = 2V - V_1 = 2 \times 20\,\tfrac{m}{s} - 10\,\tfrac{m}{s} = 30 \text{ m/s}$$

For $V_1 = 10$ m/s

$$T = \dot{m}(V_4 - V_1) = 40\,\tfrac{kg}{s}(30-10)\,\tfrac{m}{s} \times \frac{N\cdot s^2}{kg\cdot m} = 800 \text{ N } (V_1 = 10 \text{ m/s}) \quad \text{T} \longleftarrow$$

Given: Jet-propelled aircraft travelling as shown.



Find: Propulsive efficiency, $\eta \equiv \dfrac{\text{useful work output}}{\text{mechanical energy input to fluid}}$

Solution: Apply x component of momentum equation and first law of thermodynamics using moving CV shown.

Basic equations:

$$F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

(F_{Bx} = 0 (2); $\frac{\partial}{\partial t}$ = 0 (3))

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\,\rho\,d\forall + \int_{CS}(e + pv)\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

($\dot{W}_{shear}$ = 0 (3))

Assumptions: (1) No net pressure force
(2) $F_{Bx} = 0$
(3) Steady flow
(4) Uniform flow at each section

$$e = u + \frac{V_{xyz}^2}{2} + gz$$

Then from momentum

$$-F_D = u_1\{-\dot{m}\} + u_2\{\dot{m}\} = (-U)(-\dot{m}) + (-V)(\dot{m}) \quad \text{or} \quad F_D = \dot{m}(V-U)$$

The useful work rate is $F_D U$.

Mechanical energy is added at the rate

$$-\dot{W} = \left(\frac{U^2}{2}\right)\{-\dot{m}\} + \frac{V^2}{2}\{\dot{m}\} = \frac{\dot{m}}{2}(V^2 - U^2)$$

Thus

$$\eta = \frac{\dot{m}(V-U)U}{\frac{\dot{m}}{2}(V^2-U^2)} = \frac{2(V-U)U}{(V^2-U^2)} = \frac{2U}{V+U} = \frac{2}{1+V/U}$$

$$\eta = \frac{2}{1+500/200} = 0.571 \text{ or } 57.1 \text{ percent} \quad \eta$$

Given: Ship drag data (Figs. 7.1 and 7.2) and dimensions (Problem 9.67).

Performance characteristics of marine propeller (Fig. 10.40). Propeller operates at maximum efficiency when ship steams at maximum speed, $V = 37.6$ kt.

Find: Calculate size, operating speed, and power input for a single propeller to propel this vessel.

Solution: From Problem 9.67, $L = 409$ ft and $A = 19,500$ ft^2. At maximum speed,

$$V = 37.6 \frac{nm}{hr} \times 6076 \frac{ft}{nm} \times \frac{hr}{3600\ s} = 63.5\ ft/s$$

The Froude number is

$$Fr = \frac{V}{\sqrt{gL}} = 63.5 \frac{ft}{s} \times \left[\frac{s^2}{32.2\ ft} \times \frac{1}{409\ ft}\right]^{\frac{1}{2}} = 0.553$$

From Fig. 7.1, $C_D \approx 0.0054$. The definition is $C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$, so

$$F_D = C_D A \tfrac{1}{2}\rho V^2 \; ; \; \tfrac{1}{2}\rho V^2 = \tfrac{1}{2} \times (1.025)\, 1.94 \frac{slug}{ft^3} \times (63.5)^2 \frac{ft^2}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} = 4010\ lbf/ft^2$$

$$F_D = 0.0054 \times 19,500\ ft^2 \times 4010 \frac{lbf}{ft^2} = 422,000\ lbf$$

From Fig. 10.40(a), the maximum efficiency is $\eta = 0.67$ at $J = 0.85$. Then

$$nD = \frac{V}{J} = 63.5 \frac{ft}{s} \times \frac{1}{0.85} = 74.7\ ft/s \qquad (1)$$

Since $C_F = 0.11 = \frac{F_D}{\rho n^2 D^4} = \frac{F_D}{\rho (n^2 D^2) D^2} = \frac{F_D}{\rho (nD)^2 D^2}$, then

$$D = \left[\frac{F_D}{\rho (nD)^2 C_F}\right]^{\frac{1}{2}} = \left[422,000\ lbf \times \frac{ft^3}{(1.025)\,1.94\ slug} \times \frac{s^2}{(74.7)^2 ft^2} \times \frac{1}{0.11} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right]^{\frac{1}{2}} = 18.6\ ft \qquad D$$

From Eq. 1,

$$n = \frac{nD}{D} = 74.7 \frac{ft}{s} \times \frac{1}{18.6\ ft} = 4.02\ rev/s \quad (241\ rpm) \qquad n$$

The input power would be

$$\mathcal{P}_{in} = \frac{\mathcal{P}_{out}}{\eta} = \frac{F_D V}{\eta} = 422,000\ lbf \times 63.5 \frac{ft}{s} \times \frac{1}{0.67} \times \frac{hp \cdot s}{550\ ft \cdot lbf} = 72,700\ hp \qquad \mathcal{P}_{in}$$

Given: Propeller for Gossamer Condor human-powered aircraft has $D = 12$ ft and rotates at $N = 107$ rpm.

Additional information on the aircraft in Problem 9.147:
$W = 200$ lbf, $W/A = 0.4$ lbf/ft^2, $ar = 17$, $P_{pilot} = 0.39$ hp, $F_D = 6$ lbf at $V = 12$ mph

Find: Estimate the dimensionless performance characteristics and efficiency of this propeller at cruise conditions.

Solution: From the solution to Problem 9.147, minimum power to propel the aircraft occurs at $V = 10.7$ mph (16.0 ft/s). Assume this is the cruise condition.

From the given data, at 12 mph (17.6 ft/s), $F_D = 6$ lbf

$$\frac{1}{2}\rho V^2 = \frac{1}{2} \times 0.00238 \frac{\text{slug}}{\text{ft}^3} \times (17.6)^2 \text{ft}^2/\text{s}^2 \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} = 0.369 \text{ lbf/ft}^2$$

$$C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A} = \frac{W}{qA} = \frac{W/A}{q} = \frac{0.4 \text{ lbf/ft}^2}{0.369 \text{ lbf/ft}^2} = 1.08$$

$$C_D = C_L \frac{F_D}{F_L} = 1.08 \frac{6 \text{ lbf}}{200 \text{ lbf}} = 0.0324$$

$$C_{D,0} = C_D - C_{D,i} = C_D - \frac{C_L^2}{\pi ar} = 0.0324 - \frac{(1.08)^2}{\pi(17)} = 0.0106$$

At $V = 10.7$ mph (16.0 ft/s), $q = 0.305$ lbf/ft^2

$$C_L = \frac{W}{qA} = \frac{W/A}{q} = 0.4 \frac{\text{lbf}}{\text{ft}^2} \times \frac{\text{ft}^2}{0.305 \text{ lbf}} = 1.31 \; ; \; C_{D,i} = \frac{C_L^2}{\pi ar} = 0.0321$$

$$C_D = C_{D,0} + C_{D,i} = 0.0106 + 0.0321 = 0.0427 \; ; \; F_D = F_L \frac{C_D}{C_L} = 200 \text{ lbf} \frac{0.0427}{1.31} = 6.52 \text{ lbf}$$

For the propeller,

$$J = \frac{V}{nD} = 16.0 \frac{\text{ft}}{\text{s}} \times \left(\frac{60}{107}\right) \frac{\text{s}}{\text{rev}} \times \frac{1}{12.0 \text{ ft}} = 0.748 \qquad J$$

$$C_F = \frac{F_D}{\rho n^2 D^4} = 6.52 \text{ lbf} \times \frac{\text{ft}^3}{0.00238 \text{ slug}} \times \left(\frac{60}{107}\right)^2 \frac{\text{s}^2}{\text{rev}^2} \times \frac{1}{(12.0)^4 \text{ft}^4} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2} = 0.0415 \qquad C_F$$

Assume a 30 percent reserve for climbing and maneuvers. Then if $\eta_d = 0.9$,

$$\eta_p = \frac{P_{out}}{P_{in}\eta_d} = \frac{F_D V}{(0.7)0.39 \text{ hp}\,(0.9)} = \frac{1}{0.246 \text{ hp}} \times 6.52 \text{ lbf} \times \frac{16.0 \text{ ft}}{\text{s}} \times \frac{\text{hp}\cdot\text{s}}{550 \text{ ft}\cdot\text{lbf}} = 0.771 \qquad \eta$$

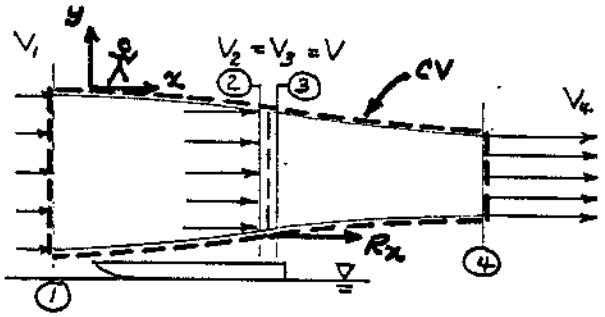
Finally, $P_{prop} = \eta_d P_{in} = 0.246 \text{ hp} = \omega T \; ; \; T = \frac{0.246 \text{ hp}}{\omega} = 12.1 \text{ ft}\cdot\text{lbf}$

$$C_T = \frac{T}{\rho n^2 D^5} = 12.1 \text{ ft}\cdot\text{lbf} \times \frac{\text{ft}^3}{0.00238 \text{ slug}} \times \left(\frac{60}{107}\right)^2 \frac{\text{s}^2}{\text{rev}^2} \times \frac{1}{(12.0)^5 \text{ft}^5} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2} = 0.00642 \qquad C_T$$

$$C_p = \frac{C_T}{\eta} = 0.00642 \times \frac{60}{107} = 0.0036 \qquad C_p$$

Given: Propulsive efficiency, $\eta = \dfrac{\text{useful work produced}}{\text{mechanical energy input}}$ for propeller

Find: (a) η for moving air boat of Problem 10.85.
(b) η if boat not moving



Solution: Consider moving air boat:

From analysis of Example Prob. 10.3, thrust is

$$T = \dot{m}(V_4 - V_1)$$

At $V_1 = 10$ m/sec, $V = 20$ m/s

$$V_4 = 30 \text{ m/s}$$

all relative to moving air boat, as shown in the sketch above.

Applying the energy equation to steady, incompressible, uniform flow through the moving CV gives the minimum power input requirement

$$\dot{W}_{s,min} = \dot{m}\left(\frac{V_4^2}{2} - \frac{V_1^2}{2}\right)$$

Useful work is done at the rate of

$$\dot{W}_{useful} = V_1 \times \text{Thrust} = V_1 \dot{m}(V_4 - V_1)$$

Combining these expressions,

$$\eta = \frac{\dot{m} V_1 (V_4 - V_1)}{\dot{m}\left(\frac{V_4^2}{2} - \frac{V_1^2}{2}\right)} = \frac{V_1 (V_4 - V_1)}{\frac{1}{2}(V_4 - V_1)(V_4 + V_1)} = \frac{2V_1}{V_1 + V_4} = \frac{2}{1 + V_4/V_1}$$

For the moving air boat

$$\eta = \frac{2}{1 + \frac{30 \text{ m}}{3} \times \frac{5}{10 \text{ m}}} = 0.500 \text{ or } 50 \text{ percent} \qquad \eta$$

For the stationary air boat, $V_1 = 0$, and no useful work is done. For the stationary boat

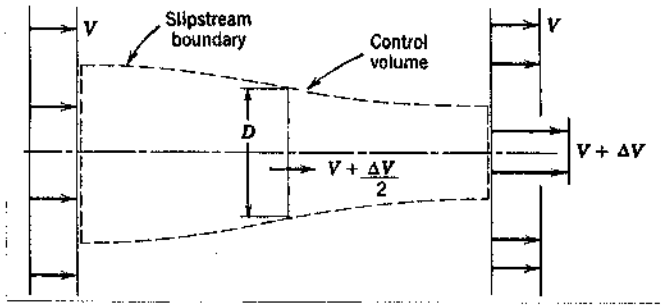
$$\eta_0 = \frac{2(0)}{0 + V_4} = 0 \qquad \eta_0$$

Given: Equations for thrust, power, and efficiency of propulsion devices derived in Section 10-5.1d.

Find: (a) Show that for constant thrust, $\eta = \dfrac{2}{1+\left(1+\dfrac{F_T}{\rho\frac{V^2}{2}\frac{\pi D^2}{2}}\right)^{\frac{1}{2}}}$

(b) Interpret physically.

Solution: Apply 1-D forms of momentum and energy to CV of Fig. 10.38:



Computing equations:

Continuity $\dot{m} = \rho\left(V+\frac{\Delta V}{2}\right)\frac{\pi D^2}{4}$

Momentum $F_T = \dot{m}\,\Delta V$ (10.28)

Energy $\mathcal{P}_{in} = \dot{m}V\Delta V\left(1+\frac{\Delta V}{2V}\right)$ (10.29)

Assumptions: (1) Steady flow, (2) Incompressible flow, (3) Uniform flow, (4) Frictionless flow

Propulsion efficiency is $\eta_p = \dfrac{\mathcal{P}_{out}}{\mathcal{P}_{in}} = \dfrac{F_T V}{\dot{m}V\Delta V\left(1+\frac{\Delta V}{2V}\right)} = \dfrac{\dot{m}V\Delta V}{\dot{m}V\Delta V\left(1+\frac{\Delta V}{2V}\right)} = \dfrac{1}{1+\frac{\Delta V}{2V}}$ (1)

F_T may be written using continuity as

$$F_T = \dot{m}\,\Delta V = \rho\left(V+\frac{\Delta V}{2}\right)\frac{\pi D^2}{4}\Delta V = 2\rho V^2\frac{\pi D^2}{4}\left(1+\frac{\Delta V}{2V}\right)\left(\frac{\Delta V}{2V}\right) = 2\rho V^2\frac{\pi D^2}{4}(1+\lambda)\lambda$$

where $\lambda = \frac{\Delta V}{2V}$. For constant F_T,

$$\lambda^2 + \lambda - \frac{F_T}{2\rho V^2\frac{\pi D^2}{4}} = 0$$

Solving via the quadratic equation, and choosing the positive root

$$\lambda = \frac{-1 \pm \sqrt{1+4\dfrac{F_T}{2\rho V^2\frac{\pi D^2}{4}}}}{2} = \frac{1}{2}\left\{-1+\sqrt{1+\frac{F_T}{\rho\frac{V^2}{2}\frac{\pi D^2}{4}}}\right\}$$

From Eq. 1, $\eta_p = \dfrac{1}{1+\lambda} = \dfrac{1}{1+\frac{1}{2}\{\ \}} = \dfrac{2}{2+\{\ \}} = \dfrac{2}{1+\left(1+\frac{F_T}{\rho\frac{V^2}{2}\frac{\pi D^2}{4}}\right)^{\frac{1}{2}}}$ η_p

The ratio, $F_T/\frac{\pi D^2}{4}$, may be interpreted as the disk loading: the force developed per unit area of the actuator disk. Note $\eta_p \to 1$ as $\frac{\pi D^2}{4}$ increases.

Given: Preliminary calculations for a hydroelectric power generation site show a net head, $H = 2350$ ft, is available at water flow rate, $Q = 75$ ft^3/s.

Find: Compare the geometry and efficiency of Pelton wheels designed to run at (a) 450 rpm and (b) 600 rpm.

Solution: Apply specific speed equation to classify performance.

Computing equation: $Ns_{cu} = \dfrac{N P^{1/2}}{H^{5/4}}$ (rpm, hp, and ft units)

From Fig. 10.17, $\eta_{max} \approx 0.89$ at $Ns_{cu} = 5$. The output power (used to define Ns_{cu}) is

$$P_{out} = \eta \rho Q g H = 0.89 \times 62.4 \frac{lbf}{ft^3} \times 75 \frac{ft^3}{s} \times 2350 \text{ ft} \times \frac{hp \cdot s}{550 \text{ ft} \cdot lbf} = 17{,}800 \text{ hp}$$

At $N = 450$ rpm

$$Ns_{cu} = \frac{450 \text{ rpm} (17{,}800 \text{ hp})^{\frac{1}{2}}}{(2350)^{5/4}} = 3.67, \text{ so } \eta \approx 0.88$$

Neglect nozzle losses and elevation above the tailrace. Then

$$V_j \approx \sqrt{2gH} = \left[2 \times 32.2 \frac{ft}{s^2} \times 2350 \text{ ft}\right]^{\frac{1}{2}} = 389 \text{ ft/s}$$

From Fig. 10.10, $U = R\omega \approx 0.47 V_j = 183$ ft/s. Thus

$$D = 2R = \frac{2(0.47 V_j)}{\omega} = 2 \times 183 \frac{ft}{s} \times \frac{s}{47.1 \text{ rad}} = 7.77 \text{ ft}$$

The jet diameter is found from $Q = V_j A_j = \pi V_j D_j^2 / 4$, so

$$D_j = \sqrt{\frac{4Q}{\pi V_j}} = \left[\frac{4}{\pi} \times 75 \frac{ft^3}{s} \times \frac{s}{389 \text{ ft}}\right]^{\frac{1}{2}} = 0.495 \text{ ft } (5.95 \text{ in.})$$

The ratio of jet diameter to wheel diameter is

$$r = \frac{D_j}{D} = \frac{0.495 \text{ ft}}{7.77 \text{ ft}} = 0.0637 \quad \text{or } 1:15.7 \quad \text{(this is reasonable)}$$

Results from similar computations at $N = 600$ rpm are:

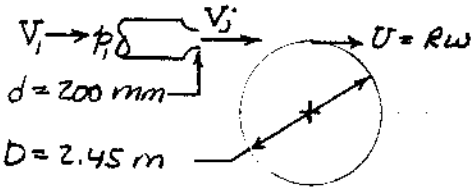
N (rpm)	Ns_{cu} (USCS)	D (ft)	D_j/D (---)	η (---)
450	3.67	7.77	1:15.7	0.88
600	4.89	5.83	1:11.8	0.89

The unit operating at 600 rpm is closer to $Ns_{cu} = 5$, where peak hydraulic efficiency is expected.

Given: Conditions at the inlet to the nozzle of a Pelton wheel:

$p_1 = 4.81$ MPa (gage), $V_1 = 6.10$ m/s

$K_{nozzle} = 0.04$, $\eta = 0.86$



Find: (a) Power output
(b) Normal operating speed
(c) Approximate runaway speed
(d) Torque at normal operating speed
(e) Torque at zero speed

Solution: Apply the energy equation for steady, incompressible flow, results from Example Problem 10.5, and Fig. 10.10.

Computing equations: $\frac{p_1}{\rho g} + \frac{\alpha_1 \bar{V}_1^2}{2g} + z_1 = \frac{p_j}{\rho g} + \frac{\alpha_j V_j^2}{2g} + z_2 + \frac{h_{LT}}{g}$; $h_{LT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2}$

$T = \rho Q R (V_j - U)(1 - \cos\theta)$ (Example Problem 11.5)

Assumptions: (1) $p_j = p_{atm}$, (2) $\alpha_1 = \alpha_j = 1$, (3) $z_1 = z_2$, (4) $\frac{L}{D} + \frac{L_e}{D} = 0$; K based on $V_j^2/2$,
(5) $U_{opt} = 0.47\, V_j$, (6) $(1 - \cos\theta) = 1.97$

Then

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{V_j^2}{2g} + K\frac{V_j^2}{2g} = (1+K)\frac{V_j^2}{2g}\ ;\ V_j = \sqrt{\frac{2\left(\frac{p_1}{\rho} + \frac{V_1^2}{2}\right)}{1+K}}$$

$$V_j = \left[\left(2 \times 4.81\times10^6 \frac{N}{m^2} \times \frac{m^3}{999\ kg} \times \frac{kg\cdot m}{N\cdot s^2} + (6.10)^2 \frac{m^2}{s^2}\right)\frac{1}{1+0.04}\right]^{\frac{1}{2}} = 96.4\ m/s$$

$$Q = V_j A_j = V_j \frac{\pi D_j^2}{4} = \frac{\pi}{4} \times 96.4 \frac{m}{s} \times (0.200)^2 m^2 = 3.03\ m^3/s \quad (107\ ft^3/s)$$

$$\mathcal{P} = \eta \rho Q g H\ ;\ H = \frac{p_1}{\rho g} + \frac{V_1^2}{2g} = 493\ m$$

$$\mathcal{P} = 0.86 \times 999 \frac{kg}{m^3} \times 3.03 \frac{m^3}{s} \times 9.81 \frac{m}{s^2} \times 493\ m \times \frac{N\cdot s^2}{kg\cdot m} \times \frac{W\cdot s}{N\cdot m} = 12.6\ MW\ (16{,}900\ hp)$$ $\mathcal{P}$

From Fig. 10.10, normal operating speed is $U = 0.47 V_j$

$$U = 0.47 \times 96.4 \frac{m}{s} = 45.3\ m/s, \quad \omega = \frac{U}{R} = \frac{2U}{D} = 2 \times 45.3 \frac{m}{s} \cdot \frac{1}{2.45\ m} = 37.0\ rad/s$$ ω
(353 rpm)

At runaway, $U \approx V_j = 96.4$ m/s

$$\omega_{runaway} = \frac{V_j}{0.47 V_j}\omega = \frac{\omega}{0.47} = 78.7\ rad/s$$ ω_{run}

From Example Problem 10.5, torque is $\eta\, T_{ideal} = \eta \rho Q R (V_j - U)(1 - \cos\theta)$, so

$$T = 0.86 \times 999 \frac{kg}{m^3} \times 3.03 \frac{m^3}{s} \times 1.23\ m \times (0.53) 96.4 \frac{m}{s} (1.97) \times \frac{N\cdot s^2}{kg\cdot m} = 3.22\times10^5\ N\cdot m$$ T_{oper}

and stall torque occurs at $U = 0$, so

$$T_{stall} = T\left(\frac{1}{0.53}\right) = 6.08\times10^5\ N\cdot m$$ T_{stall}

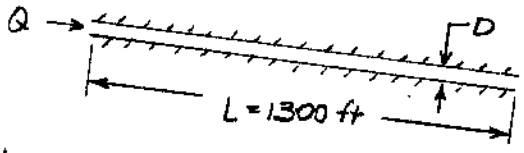
{To check, $T = \frac{\mathcal{P}}{\omega} = 12.6\times10^6\ W \times \frac{s}{37.0\ rad} \times \frac{N\cdot m}{W\cdot s} = 3.41\times10^5\ N\cdot m$, which is close!}

Given: Francis (reaction) turbines at Niagara Falls:

$D_0 = 176$ in., $\mathcal{P} = 72{,}500$ hp, $N = 107$ rpm, $\eta = 0.938$, $H = 214$ ft (net)

Penstock has $L = 1300$ ft; $H_{net} = 0.85\, H_{gross}$.

Find: (a) Calculate specific speed.
(b) Evaluate volume flow rate.
(c) Estimate penstock size.



Solution: $N_{S_{CU}} = \dfrac{N\mathcal{P}^{\frac{1}{2}}}{H^{\frac{5}{4}}} = \dfrac{107\text{ rpm}\,(72{,}500\text{ hp})^{\frac{1}{2}}}{(214\text{ ft})^{\frac{5}{4}}} = 35.1$ ← N_S

Efficiency is defined as $\eta = \mathcal{P}/\rho g H$, so

$$Q = \frac{\mathcal{P}}{\eta \rho g H} = \frac{1}{0.938} \times 72{,}500\text{ hp} \times \frac{\text{ft}^3}{62.4\text{ lbf}} \times \frac{1}{214\text{ ft}} \times \frac{550\text{ ft}\cdot\text{lbf}}{\text{hp}\cdot\text{s}} = 31800\text{ ft}^3/\text{s}$$ ← Q

Apply the definition of head loss: $h_{LT} = g H_{LT} = f \dfrac{L}{D} \dfrac{\bar{V}^2}{2}$

$H_{LT} = 0.15\, H_{gross} = 0.15 \dfrac{H_{net}}{0.85} = 0.176\, H_{net} = 0.176 \times 214\text{ ft} = 37.7\text{ ft}$

$\bar{V} = \dfrac{Q}{A} = \dfrac{4Q}{\pi D^2}$; $\bar{V}^2 = \dfrac{16Q^2}{\pi^2 D^4}$; $g\Delta H = f \dfrac{L}{D} \dfrac{\bar{V}^2}{2} = f \dfrac{L}{D} \dfrac{8Q^2}{\pi^2 D^4} = \dfrac{8 f L Q^2}{\pi^2 D^5}$

Solving, $D = \left[\dfrac{8 f L Q^2}{\pi^2 g \Delta H}\right]^{\frac{1}{5}}$ (Assume T = 50°F)

This system is very large, so it is difficult to estimate f. Assuming concrete-lined penstocks, $e = 0.01$ ft (Table 8.1). Start with $f = 0.01$ to get by iteration (see below),

$$D = 26.8\text{ ft}$$

The flow properties are

$$\bar{V} = 56.8\text{ ft/sec},\ Re = 1.18 \times 10^8,\ \text{and}\ f = 0.0157$$

Flow is in the fully rough zone with $e/D = 0.01/26.8 = 0.00037$.

Using $e = 0.02$ gives $D = 28.1$ ft, so $D \approx 27$–28 ft ← D

The iterations are:

Assumed D (ft)	V (ft/s)	Re	f_0	$f^{-0.5}$	Calculated f	Calculated D (ft)
					0.01	24.5
24.5	67.5	1.18E+08	0.0160	7.91	0.0160	26.9
26.9	56.0	1.08E+08	0.0157	7.99	0.0156	26.8
26.80	56.4	1.08E+08	0.0157	7.99	0.0157	26.77
26.77	56.5	1.08E+08	0.0157	7.99	0.0157	26.77
					0.02	28.1

Given: Francis turbine Units 19, 20, and 21 at Grand Coulee Dam.

Rated conditions: $P = 820{,}000$ hp, $N = 72$ rpm, $H = 285$ ft, $\eta = 0.95$

Turbines operate from $220 < H < 355$ ft.

Find: (a) Calculate specific speed at rated conditions.
(b) Estimate maximum water flow rate through each turbine.

Solution: Apply definitions of specific speed and efficiency.

Computing equations: $N_{s_{cu}} = \dfrac{N P^{1/2}}{H^{5/4}}$ $\eta = \dfrac{P}{\rho Q g H}$

Thus

$$N_{s_{cu}} = \frac{72 \text{ rpm } (820{,}000 \text{ hp})^{\frac{1}{2}}}{(285 \text{ ft})^{5/4}} = 55.7 \quad \leftarrow N_{s_{cu}}$$

From η,

$$Q = \frac{P}{\eta \rho g H}$$

so Q is maximum at minimum head. Assuming $\eta = 0.95$, then

$$Q \approx \frac{1}{0.95} \times 820{,}000 \text{ hp} \times \frac{\text{ft}^3}{62.4 \text{ lbf}} \times \frac{1}{220 \text{ ft}} \times 550 \frac{\text{ft} \cdot \text{lbf}}{\text{hp} \cdot \text{s}} = 34{,}600 \text{ ft}^3/\text{s} \quad \text{(max)} \quad \leftarrow Q$$

{ This is an estimate because η may not be constant, nor may it be possible to develop full power at $H = 220$ ft. }

Given: Data in Fig. 10.11 for Pelton wheel in PG&E Tiger Creek plant.

Rating is $P = 361{,}000$ hp at $N = 225$ rpm under $H = 1190$ ft (net)

Assume reasonable flow angles and nozzle loss coefficient.

(a) Determine rotor diameter.
(b) Estimate jet diameter.
(c) Compute volume flow rate of water.

Solution: From Bernoulli, the ideal jet velocity would be $V_i = \sqrt{2gH}$. Assuming $C_v = 0.98$ (4 percent loss in nozzle), then

$$V_j = C_v V_i = C_v\sqrt{2gH} = 0.98\left[2 \times 32.2 \frac{ft}{s} \times 1190\,ft\right]^{\frac{1}{2}} = 271\ ft/s$$

From Fig. 10.10, $U = R\omega = 0.47\,V_j$ at optimum conditions. Then

$$D = 2R = \frac{2(0.47)V_j}{\omega} = 0.94 \times 271\frac{ft}{s} \times \frac{s}{23.6\ rad} = 10.8\ ft \quad \text{D}$$

From Fig. 11.11, $\eta = 0.86$ at full load. Thus

$$\eta = \frac{P}{\rho q g H}\ ;\ Q = \frac{P}{\eta \rho g H}\ ;\ Q = V_j A_j = V_j \frac{\pi D_j^2}{4}$$

$$Q = \frac{1}{0.86} \times 361{,}000\ hp \times \frac{ft^3}{62.4\ lbf} \times \frac{1}{1190\ ft} \times 550\frac{ft \cdot lbf}{hp \cdot s} = 310\ ft^3/s \quad \text{Q}$$

$$D_j = \left[\frac{4Q}{\pi V_j}\right]^{\frac{1}{2}} = \left[\frac{4}{\pi} \times 310\frac{ft}{s} \times \frac{s}{271\ ft}\right]^{\frac{1}{2}} = 1.21\ ft\ (14.5\ in.) \quad \text{Dj}$$

Note $D_j/D = 1.21/10.8 = 0.112\ (1:8.93)$.

Given: Measured data for reaction turbines at Shasta Dam, Fig. 10.13.

Each turbine is rated at $\mathcal{P} = 103{,}000$ hp at $N = 138.6$ rpm, under a net head of $H = 380$ ft.

Find: (a) Specific speed at rated conditions.
(b) Shaft torque at rated conditions.
(c) Calculate the water flow rate per turbine needed to produce rated output power; plot versus head.

Solution: Apply the definitions of specific speed and efficiency, use data from Fig. 11.13:

Computing equations: $N_{S_{cu}} = \dfrac{N\mathcal{P}^{\frac{1}{2}}}{H^{\frac{5}{4}}}$ $\eta = \dfrac{\mathcal{P}}{\rho Q g H}$ $\mathcal{P} = \omega T$

At rated conditions, $N_{S_{cu}} = \dfrac{(138.6\ \text{rpm})(103{,}000\ \text{hp})^{\frac{1}{2}}}{(380\ \text{ft})^{\frac{5}{4}}} = 26.5$ N_S ←

$$T = \frac{\mathcal{P}}{\omega} = 103{,}000\ \text{hp} \times \frac{s}{14.5\ \text{rad}} \times 550\ \frac{\text{ft}\cdot\text{lbf}}{\text{hp}\cdot\text{s}} = 3.91 \times 10^6\ \text{ft}\cdot\text{lbf}$$ ←

Find Q from definition of η; at rated conditions, $\eta \approx 0.93$ (Fig. 11.13):

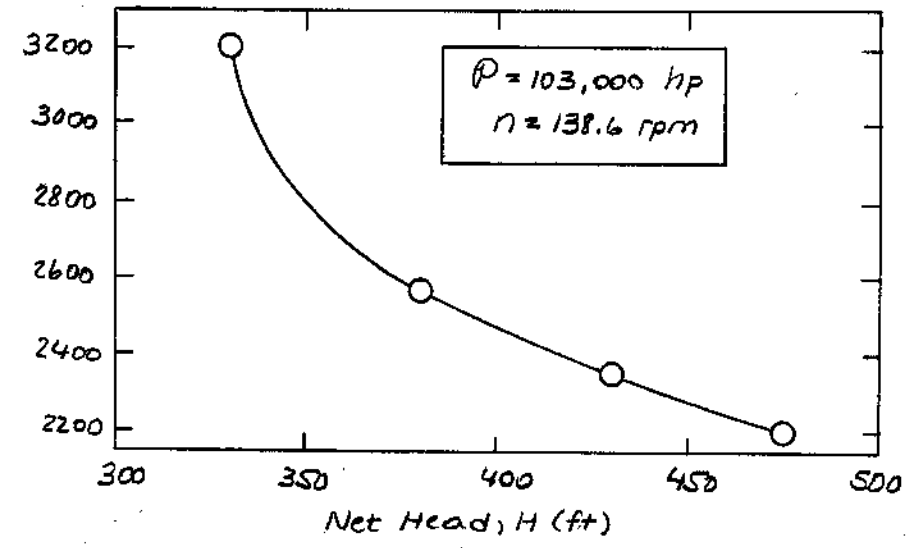
$$Q = \frac{\mathcal{P}}{\eta \rho g H} = \frac{1}{0.93} \times 103{,}000\ \text{hp} \times \frac{\text{ft}^3}{62.4\ \text{lbf}} \times \frac{1}{380\ \text{ft}} \times 550\ \frac{\text{ft}\cdot\text{lbf}}{\text{hp}\cdot\text{s}} = 2570\ \text{ft}^3/\text{s}$$

Tabulating similar calculations:

H (ft)	$\mathcal{P}$ (hp)	η (---)	Q (ft³/s)
238	*	–	–
280	*	–	–
330	103,000	0.86	3200
380	103,000	0.93	2570
430	103,000	0.90	2350
475	103,000	0.87	2200

* cannot produce rated power at this head

Plotting:



Given: Single-wheel impulse turbine to develop $\mathcal{P} = 20{,}000$ hp at a location where $H = 1120$ ft.

Find: (a) Determine appropriate speed, wheel diameter, and jet diameter for single- and multiple-jet operation; compare double overhung.
(b) Estimate the water consumption.

Solution: Model the jet as ideal, use $U = 0.47 V_j$ (Fig. 10.10), $N_S = 5$, and $\eta = 0.89$ (Fig. 10.17).

Computing equations: $V_j = \sqrt{2gH}$ $\quad N_{Scu} = \dfrac{N(\mathcal{P})^{\frac{1}{2}}}{H^{\frac{5}{4}}}$ $\quad \eta = \dfrac{\mathcal{P}}{\rho Q g H}$ $\quad Q = V_j A_j$

The jet and wheel peripheral speeds are ($U = R\omega$),

$$V_j = \left[2 \times 32.2 \frac{ft}{s^2} \times 1120\ ft\right]^{\frac{1}{2}} = 269\ ft/s\ ; \quad U = 0.47 \times 269 \frac{ft}{s} = 126\ ft/s$$

For a single jet, $N = \dfrac{N_S H^{5/4}}{\mathcal{P}^{1/2}} = \dfrac{(5)(1120)^{5/4}}{(20000)^{1/2}} = 229$ rpm

For multiple jets, use the power per jet. For $n = 2$, $N = \sqrt{2}(229) = 324$ rpm, etc.

The wheel diameter is $D = 2R = 2U/\omega$. The water consumption is

$$Q = \frac{\mathcal{P}}{\eta \rho g H} = \frac{1}{0.89} \times 20{,}000\ hp \times \frac{ft^3}{62.4\ lbf} \times \frac{1}{1120\ ft} \times 550 \frac{ft \cdot lbf}{hp \cdot s} = 177\ ft^3/s$$ Q

The water must be divided among n jets, $Q = V_j\, n A_j = V_j\, n \dfrac{\pi D_j^2}{4}$, so

$$D_j = \left[\frac{4Q}{\pi n V_j}\right]^{\frac{1}{2}} = \frac{1}{\sqrt{n}} \left[\frac{4}{\pi} \times 177 \frac{ft^3}{s} \times \frac{s}{269\ ft}\right]^{\frac{1}{2}} = \frac{0.915\ ft}{\sqrt{n}} = \frac{11.0\ in.}{\sqrt{n}}$$

Tabulating results for a single wheel:

Number of Jets (---)	Wheel Speed (rpm)	Wheel Diameter (ft)	Jet Diameter (in.)
1	229	10.5	11.0
2	324	7.43	7.78
3	397	6.06	6.35
4	458	5.25	5.50
5	512	4.70	4.92

Results for a double-overhung wheel are the same as for a single wheel with two jets.

Given: Test data from model impulse turbine.

$H = 65.5$ ft

Wheel Speed (rpm)	No-Load Discharge (cfs)	Net Brake Scale Reading (lbf) ($R = 5.25$ ft)					
275	0.110	6.8	14.9	22.0	28.9	40.0	48.0
300	0.125	5.9	12.9	19.8	25.5	36.0	43.8
Discharge (cfs)		0.397	0.773	1.114	1.414	1.896	2.315

Find: Calculate and plot power output and efficiency versus water flow rate.

Solution: Apply definitions of power and efficiency

Computing equations: $T = FR \quad \mathcal{P} = \omega T \quad \eta = \dfrac{\mathcal{P}}{\rho Q g H}$

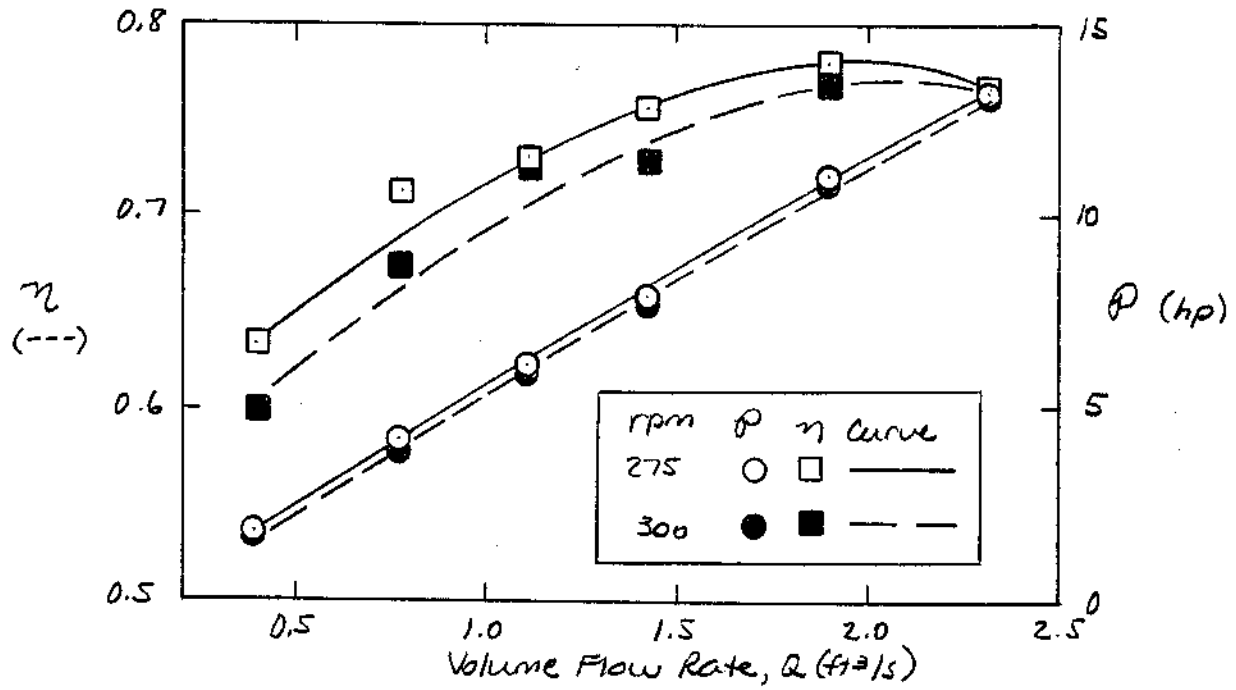
Sample computation at $N = 300$ rpm, $Q = 0.397$ cfs:

$$T = FR = 5.9 \text{ lbf} \times 5.25 \text{ ft} = 31.0 \text{ ft·lbf}; \quad \omega = 300 \frac{\text{rev}}{\text{min}} \times 2\pi \frac{\text{rad}}{\text{rev}} \times \frac{\text{min}}{60 \text{ s}} = 31.4 \text{ rad/s}$$

$$\mathcal{P} = \omega T = 31.4 \frac{\text{rad}}{\text{s}} \times 31.0 \text{ ft·lbf} \times \frac{\text{hp·s}}{550 \text{ ft·lbf}} = 1.77 \text{ hp} \quad \leftarrow \mathcal{P}$$

$$\eta = \frac{\mathcal{P}}{\rho Q g H} = 1.77 \text{ hp} \times \frac{\text{ft}^3}{62.4 \text{ lbf}} \times \frac{\text{s}}{0.397 \text{ ft}^3} \times \frac{1}{65.5 \text{ ft}} \times 550 \frac{\text{ft·lbf}}{\text{hp·s}} = 0.600 \quad \leftarrow \eta$$

Plotting similar results versus flow rate:



Given: Published data for PG&E Tiger Creek Power Plant:

$H_{gross} = 1219$ ft, $Q = 750$ ft^3/s, $P = 58$ MW (rating is 60 MW)

Plant is claimed to produce 968 kW·hr/acre·ft of water and 336.4×10^6 kW·hr/yr of operation.

Find: (a) Estimate the net head at the site, the turbine specific speed, and the turbine efficiency.
(b) Comment on the internal consistency of the published data.

Solution: Apply definitions of specific speed and efficiency.

Computing equations: $N_{s_{cu}} = \dfrac{N P^{1/2}}{H^{5/4}}$ $\qquad \eta = \dfrac{P}{\rho Q g H_{net}}$

Use the operating point to estimate net head. From Fig. 10.11, assume $\eta = 0.87$: ← η

$$H_{net} = \frac{P}{\eta \rho Q g} = \frac{1}{0.87} \times 58 \times 10^6 \text{ W} \times \frac{\text{ft}^3}{62.4 \text{ lbf}} \times \frac{\text{s}}{750 \text{ ft}^3} \times \frac{\text{hp}}{746 \text{ W}} \times \frac{550 \text{ ft·lbf}}{\text{hp·s}} = 1050 \text{ ft}$$ ← H_{net}

Thus

$H_{net} / H_{gross} = 1050/1219 = 0.861$ or 86.1 percent (Reasonable)

The specific speed should be $N_{s_{cu}} \approx 5$. Checking, ← N_s

$$N = \frac{N_s H^{5/4}}{P^{1/2}} = \frac{5\,(1050 \text{ ft})^{5/4}}{(77{,}700 \text{ hp})^{1/2}} = 107$$

This is too low, so the plant must have several turbines. Reducing P to the output per turbine would raise N.

Check data consistency:

$$58 \times 10^6 \text{ W} \times 24 \frac{\text{hr}}{\text{day}} \times 365 \frac{\text{day}}{\text{yr}} = 508 \times 10^6 \text{ kW·hr/yr}$$

$$60 \times 10^6 \text{ W} \qquad = 526 \times 10^6 \text{ kW·hr/yr}$$

} Both values are ~50% higher than quoted.

$$58 \times 10^6 \text{ W} \times \frac{\text{s}}{750 \text{ ft}^3} \times \frac{\text{hr}}{3600 \text{ s}} \times \frac{43{,}600 \text{ ft}^3}{\text{acre·ft}} = 937 \text{ kW·hr/acre·ft}$$

$$60 \times 10^6 \text{ W} \qquad = 969 \text{ kW·hr/acre·ft}$$

} Excellent agreement

Given: Definition of specific speed for a hydraulic turbine in U.S. units:

$$Ns_{cu} = \frac{N(rpm)[\mathcal{P}(hp)]^{\frac{1}{2}}}{[h(ft)]^{5/4}}$$

Impulse turbine: $N = 400$ rpm, $H = 1190$ ft, $\eta = 0.86$, with $D_j = 6$ in.

Find: (a) Develop a conversion from Ns_{cu} to a dimensionless Ns in SI units.
(b) Evaluate Ns of turbine in U.S. and S.I. units.
(c) Estimate the wheel diameter.

Solution: Apply definitions of specific speed and efficiency.

Computing equations: $Ns_{cu} = \frac{N\mathcal{P}^{1/2}}{H^{5/4}}$ $\eta \equiv \frac{\mathcal{P}}{\rho Q g H}$ $Ns = \frac{\omega \mathcal{P}^{1/2}}{\rho^{1/2}(gH)^{5/4}}$

From dimensional analysis, recall that for pumps, $Ns = \frac{NQ^{1/2}}{H^{3/4}}$ was dimensionless only when H is expressed as gH, energy per unit mass. Thus the dimensions are

$$[Ns] = \left[\frac{NQ^{1/2}}{(gH)^{3/4}}\right] = \frac{1}{t}\left(\frac{L^3}{t}\right)^{\frac{1}{2}}\left(\frac{t^2}{L^2}\right)^{\frac{3}{4}} = \frac{1}{t}\frac{L^{3/2}}{t^{1/2}}\frac{t^{3/2}}{L^{3/2}} = 1 \quad \checkmark\checkmark$$

To form the Ns for turbines, Q must be multiplied by $\rho g H$ to obtain power. Thus for turbines, the dimensionless specific speed is

$$[Ns] = \left[\frac{NQ^{1/2}}{(gH)^{3/4}} \times \frac{(\rho g H)^{1/2}}{(\rho g H)^{1/2}}\right] = \left[\frac{N\mathcal{P}^{1/2}}{\rho^{1/2}(gH)^{5/4}}\right] = \frac{1}{t}\left(\frac{FL}{t} \times \frac{ML}{Ft^2}\right)^{\frac{1}{2}}\left(\frac{L^3}{M}\right)^{\frac{1}{2}}\left(\frac{t^2}{L^2}\right)^{\frac{5}{4}} = \frac{M^{\frac{1}{2}}L^{\frac{5}{2}}t^{\frac{5}{2}}}{M^{\frac{1}{2}}L^{5/2}t^{5/2}} = 1 \quad \checkmark\checkmark$$

The simplest way to convert is to evaluate each specific speed, then take the ratio.

The jet speed will be approximately $V_j = \sqrt{2gH} = \left[2 \times 32.2\frac{ft}{s^2} \times 1190\,ft\right]^{\frac{1}{2}} = 277$ ft/s

and $Q = V_j \frac{\pi D_j^2}{4} = 277\frac{ft}{s} \times \frac{\pi}{4}\left(\frac{1}{2}\right)^2 ft^2 = 54.4\ ft^3/s$. Thus

$$\mathcal{P} = \eta \rho Q g H = 0.86 \times 62.4\frac{lbf}{ft^3} \times 54.4\frac{ft^3}{s} \times 1190\,ft \times \frac{hp \cdot s}{550\,ft \cdot lbf} = 6320\ hp\ (4.71\ MW)$$

For the wheel, $U = 0.47 V_j = R\omega$; $R = 3.10$ ft; $D = 6.20$ ft ← D

In U.S. units, $Ns_{cu} = \frac{400\ rpm\ (6320\ hp)^{\frac{1}{2}}}{(1190\ ft)^{5/4}} = 4.55$ ← Ns

In S.I. units, $\omega = 41.9$ rad/s and $gH = 3560\ m^2/s^2$, so

$$Ns = \frac{\omega \mathcal{P}^{1/2}}{\rho^{1/2}(gH)^{5/4}} = 41.9\frac{rad}{s} \times (4.71 \times 10^6 W)^{\frac{1}{2}} \times \left(\frac{m^3}{999\ kg}\right)^{\frac{1}{2}}\left(\frac{s^2}{3560\ m^2}\right)^{\frac{5}{4}} = 0.105$$ ← Ns

The conversion is $\frac{Ns_{cu}}{Ns} = \frac{4.55}{0.105} = 43.5$ ← Ratio

Given: Hydraulic turbine site with reservoir 300 m above, Turbine efficiency is 0.8, and it must produce 25 kW of mechanical power.

Find: (a) Minimum size standard pipe
(b) Required volume flow rate
(c) Discussion of turbine efficiency, roughness, and a diffuser.

Solution: From the results of Example Problem 10.15, want (approximately) $H_e = \Delta z/3$. (Assume $f = 0.02$.)

Thus $H_L = \frac{h_e}{g} = f\frac{L}{D}\frac{\bar{V}^2}{2g} = \frac{\Delta z}{3}$

As in Fig. 10.41, assume $L = 2\Delta z$. Then $f\frac{2\Delta z}{D}\frac{\bar{V}^2}{2g} = \frac{\Delta z}{3}$ or $f\frac{\bar{V}^2}{gD} = \frac{1}{3}$ or

$$\bar{V} = \sqrt{\frac{gD}{3f}} = \left[\frac{1}{3}\times 32.2\frac{ft}{s^2}\times D(in.)\times\frac{ft}{12\,in.}\times\frac{1}{0.02}\right]^{\frac{1}{2}} = 6.69\,[D(in.)]^{\frac{1}{2}}$$

Then

$$Q = \bar{V}A = \bar{V}\frac{\pi D^2}{4}\ ;\quad \dot{m} = \rho Q\ ;\quad P_h = \dot{m}\frac{\bar{V}^2}{2}\ ;\quad P_m = \eta_T P_h$$

Some values are:

D (in.)	V (bar) (ft/s)	Q (ft³/s)	m (dot) (slug/s)	P_h (hp)	P_m (hp)	P_m (kW)
10	21.2	11.5	22.4	9.11	7.29	5.44
12	23.2	18.2	35.3	17.2	13.8	10.3
14	25.0	26.8	51.9	29.6	23.7	17.6
15.46	26.3	34.3	66.5	41.8	33.5	25.0
16	26.8	37.4	72.5	47.2	37.8	28.2
18	28.4	50.2	97.3	71.3	57.0	42.5

The required output power requires $D \approx 15.5$ in. The smallest standard pipe size is

$D = 16$ in. ← D

$\Delta z = 300$ m suggests a Pelton wheel. From Fig. 10.17, the optimum efficiency of a Pelton wheel occurs near $N_{Scu} = 5$. Thus

$$N_{Scu} = \frac{N(rpm)[P(hp)]^{\frac{1}{2}}}{[\Delta z(ft)]^{5/4}} = N(rpm)\frac{(33.5\,hp)^{\frac{1}{2}}}{(984\,ft)^{5/4}} = 0.00105\,N(rpm) = \frac{1.05}{1000\,rpm}$$

At synchronous speed (3600 rpm), $N_{Scu} = 3.78$ ← N_{Scu}

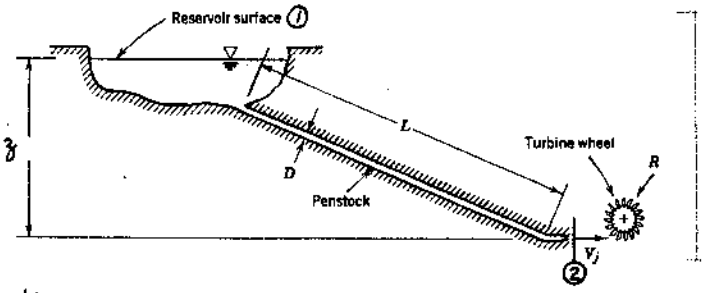
Discussion: Turbine efficiency varies with specific speed (Fig. 10.17). However, efficiency remains relatively high (probably above 0.85) over the range of values of N_{Scu} from 4 to 6.

Roughness appears to the 1/2 power, so it has a secondary effect on pipeline velocity. A 20 percent error in estimating friction factor should affect the required water speed in the penstock less than 10 percent and the power estimate only about 30 percent.

A Pelton wheel is an impulse turbine that does not flow full of water; it directs the liquid stream with open buckets. A diffuser cannot be installed in this flow system.

Given: Small hydraulic impulse turbine installation as shown.

$H = 300$ ft
$L = 1000$ ft (welded steel)
$D = 6$ in., $d_j = 2$ in.
$K_{ent} = 0.5$
$K_{nozzle} = 0.04$



(a) Find V_j, Q, and $\mathcal{P}_h$
(b) Plot $\mathcal{P}_h$ vs. d_j to find optimum d_j.
(c) Then explore effects of varying loss coefficients and pipe roughness.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \alpha_1\frac{\bar V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \alpha_2\frac{\bar V_2^2}{2g} + z_2 + \frac{h_{LT}}{g}$; $h_{LT} = \left[f\left(\frac{L}{D}+\frac{L_e}{D}\right)+K\right]\frac{\bar V^2}{2}$ ($\bar V_1 \approx 0$ (2))

Assumptions: (1) $p_1 = p_2 = p_{atm}$, (2) $\bar V_1 \approx 0$, $\alpha_2 = 1$, (3) $L_e/D = 0$, (4) K_{nozzle} based on V_j^2

Then

$$H = \frac{V_j^2}{2g} + \left(f\frac{L}{D} + K_{entrance}\right)\frac{\bar V^2}{2g} + K_{nozzle}\frac{V_j^2}{2g}$$

From continuity, $\bar V A = V_j A_j$, so $\bar V = V_j A_j/A = V_j(d/D)^2$; $\bar V^2 = V_j^2(d/D)^4$, and

$$H = \left[\left(f\frac{L}{D} + K_{ent}\right)\left(\frac{d}{D}\right)^4 + 1 + K_{nozzle}\right]\frac{V_j^2}{2g} \quad ; \quad V_j = \left[\frac{2gH}{\left(f\frac{L}{D} + K_{ent}\right)\left(\frac{d}{D}\right)^4 + 1 + K_n}\right]^{\frac{1}{2}}$$

Assume $e = 0.00015$ ft (Table 8.1), so $e/D = 0.0003$. From Fig. 8.13, in the fully rough zone, $f = 0.015$. Then for $d_j = 2$ in.

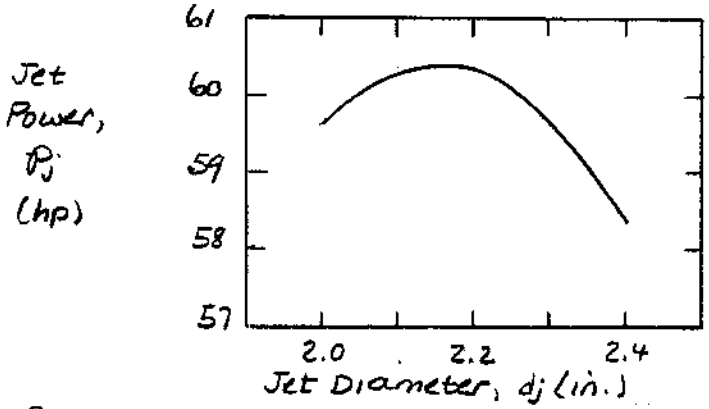
$$V_j = \left[2 \times 32.2\frac{ft}{s^2} \times 300\,ft \times \frac{1}{\left(0.015\frac{1000\,ft}{0.5\,ft} + 0.5\right)\left(\frac{2}{6}\right)^4 + 1 + 0.04}\right]^{\frac{1}{2}} = 117\ ft/sec$$

($\bar V = 13.0$ ft/s, $Re = \bar V D/\nu = 6.05 \times 10^5$, so $f = 0.016$, which makes $V_j = 116$ ft/s.)

The jet flow rate is $Q = V_j A_j = 116\frac{ft}{s} \times \frac{\pi}{4}\left(\frac{2}{12}\right)^2 ft^2 = 2.53\ ft^3/s$, and the jet power is

$$\mathcal{P}_h = \rho Q\frac{V_j^2}{2} = \frac{1}{2} \times 1.94\frac{slug}{ft^3} \times (116)^2\frac{ft^2}{s^2} \times 2.53\frac{ft^3}{s} \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \frac{hp \cdot s}{550\,ft \cdot lbf} = 60.0\ hp$$

Repeating these calculations using a computer program gives:



Peak power, $\mathcal{P}_j \approx 60.3$ hp, occurs for $2.15 < d < 2.20$ in. ← Opt

Loss coefficients have a minor effect. Making both K_{ent} and K_n zero increases $\mathcal{P}_j$ by 4.8 percent.

Pipe roughness causes larger changes; $\mathcal{P}_j$ increased 12.8 percent with $e = 0$ (smooth).

Given: NASA-DOE wind turbine generator at Plum Brook, Ohio.

Two blades, $D = 38$ m; delivers $P = 100$ kW when $V \geq 29$ km/hr, operating at 40 rpm, with powertrain efficiency, $\eta = 0.75$.

Find: For the maximum power condition, estimate the rotor tip speed and power coefficient.

Solution: Apply definitions

Computing equations: $U = \omega R \quad X = \omega R / V \quad C_p = \dfrac{P}{\frac{1}{2}\rho V^3 \pi R^2}$

At $N = 40$ rpm,

$$\omega = 40 \frac{rev}{min} \times 2\pi \frac{rad}{rev} \times \frac{min}{60\,s} = 4.19 \text{ rad/s}$$

$$U = 4.19 \frac{rad}{s} \times \left(\frac{38}{2}\right) m = 79.6 \text{ m/s} \quad \leftarrow U$$

$$V = 29 \frac{km}{hr} \times 1000 \frac{m}{km} \times \frac{hr}{3600\,s} = 8.06 \text{ m/s}$$

$$X = U/V = 79.6/8.06 = 9.88$$

(Obviously X decreases as wind speed goes up.)

$$P_m = \frac{P_e}{0.75} = \frac{1}{0.75} \times 100 \text{ kW} = 133 \text{ kW}$$

$$\frac{1}{2}\rho V^3 \pi R^2 = \frac{\pi}{2} \times 1.23 \frac{kg}{m^3} \times (8.06)^3 \frac{m^3}{s^3} \times \left(\frac{38}{2}\right)^2 m^2 \times \frac{N \cdot s^2}{kg \cdot m} \times \frac{W \cdot s}{N \cdot m} = 365 \text{ kW}$$

$$C_p = \frac{P_m}{\frac{1}{2}\rho V^3 \pi R^2} = \frac{133 \text{ kW}}{365 \text{ kW}} = 0.364 \quad \leftarrow C_p$$

Given: Display model of American multiblade farm windmill to be built.

The model, with D = 2 ft, is to develop full power at V = 22 mph wind speed.

Find: (a) Calculate the angular speed of the model for optimum power generation.
(b) Estimate the power output.

Solution: Assume efficiency trends shown in Fig. 10.45.

Computing equations: $C_p = \frac{P}{\frac{1}{2}\rho V^3 \pi R^2}$ $\quad \chi = \omega R/V$

From Fig. 10.45, C_p max ≈ 0.3 at $\chi = 0.8$. Thus for V = 22 mph (32.3 ft/s),

$$\omega = \frac{\chi V}{R} = 0.8 \times 32.3 \frac{ft}{s} \frac{1}{1\,ft} = 25.8 \text{ rad/s} \qquad \omega$$

$$N = 25.8 \frac{rad}{s} \times \frac{rev}{2\pi\, rad} \times 60 \frac{s}{min} = 246 \text{ rpm} \qquad N$$

Also

$$P = C_p \frac{1}{2}\rho V^3 \pi R^2$$

$$= \frac{\pi}{2} \times 0.30 \times 0.00238 \frac{slug}{ft^3} \times (32.3)^3 \frac{ft^3}{s^3} \times (1)^2 ft^2 \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \frac{hp \cdot s}{550\, ft \cdot lbf}$$

$$P = 0.0687 \text{ hp} \quad (51.3 \text{ W}) \qquad P$$

Given: Largest known Darrieus vertical-axis wind turbine built by DOE near Sandia, New Mexico, is 60 ft tall and 30 ft diameter; the rotor swept area is $A \approx 1200\ ft^2$.

Find: Estimate the maximum power this windmill can produce in a wind with $V = 20$ mph (29.3 ft/s).

Solution: Assume the efficiency trends shown in Fig. 10.45.

Computing equations: $C_P = \dfrac{P}{\frac{1}{2}\rho V^3 \pi R^2}$ $\quad X = \omega R / V$

From Fig. 10.45, $C_{P_{max}} \approx 0.34$ at $X = 5.3$. Use swept area in place of πR^2:

$$P = \frac{1}{2} \times 0.34 \times 0.00238 \frac{slug}{ft^3} \times (29.3)^3 \frac{ft^3}{s^3} \times 1200\ ft^2 \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \frac{hp \cdot s}{550\ ft \cdot lbf} = 22.2\ hp \qquad P$$

To generate maximum power, the windmill must rotate at

$$\omega = \frac{XV}{R} = 5.3 \times 29.3 \frac{ft}{s} \times \frac{1}{15\ ft} = 10.4\ rad/s \quad (98.9\ rpm) \qquad \omega$$

Given: Typical American multiblade farm windmill, D = 7 ft, designed to produce maximum power in winds with V = 15 mph.

Find: Estimate the rate of water delivery, as a function of the height to which the water is pumped, for this windmill.

Solution: Assume the efficiency trends shown in Fig. 10.45.

Computing equations: $C_p = \dfrac{\mathcal{P}}{\frac{1}{2}\rho V^3 \pi R^2}$ $\qquad X = \omega R/V$

From Fig. 10.45, $C_{p\,max} \approx 0.3$ at $X = 0.8$. V = 15 mph (22.0 ft/s). Then the power developed is

$$\mathcal{P} = \frac{\pi}{2} \times 0.3 \times 0.00238 \frac{slug}{ft^3} \times (22.0)^3 \frac{ft^3}{s^3} \times \left(\frac{7}{2}\right)^2 ft^2 \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \frac{hp \cdot s}{550\ ft \cdot lbf} = 0.266\ hp$$

Converting this mechanical power to pumping gives hydraulic power as

$$\mathcal{P}_h = \rho Q g h = \eta \mathcal{P}_m$$

Thus $Q_h = \dfrac{\eta \mathcal{P}_m}{\rho g} = 0.7 \times 0.266\ hp \times \dfrac{ft^3}{62.4\ lbf} \times 550 \dfrac{ft \cdot lbf}{hp \cdot s} \times 7.48 \dfrac{gal}{ft^3} \times 60 \dfrac{s}{min}$

$$Q_h = 737\ gpm \cdot ft$$

Q_h

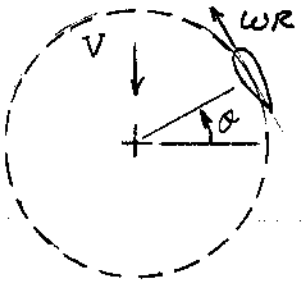
Q varies inversely with the distance lifted, h. The volume flow rate actually delivered would be less, due to suction lift, pipe friction, and minor losses.

Given: Section lift and drag coefficient data for a NACA 0012 section, tested at $Re = 6 \times 10^6$ with standard roughness:

Angle of attack, α (deg)	0	2	4	6	8	10	12
Lift coefficient, C_L (—)	0	0.23	0.45	0.68	0.82	0.94	1.02
Drag coefficient, C_D (—)	0.0098	0.0100	0.0119	0.0147	0.0194	—	—

Find: (a) Analyze air flow relative to a blade element in a Darrieus rotor.
(b) Develop a numerical model for a blade element.
(c) Calculate the power coefficient as a function of tip speed ratio.
(d) Compare with the trend shown in Fig. 10.45.

Solution: Consider plan view of rotor element, absolute velocities:



Computing equations: $F_L = C_L \frac{1}{2}\rho V_r^2 A_p$, V_r = relative velocity

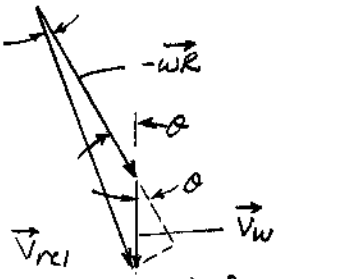
$F_D = C_D \frac{1}{2}\rho V_r^2 A_p$ A_p = planform

A_s = swept

$C_p = \dfrac{P}{\frac{1}{2}\rho V^3 A_s}$ V = wind velocity

Resolve to relative velocity, for position shown:

$\vec{V}_{abs} = \vec{V}_{blade} + \vec{V}_{rel}$; $\vec{V}_{rel} = \vec{V}_{abs} - \vec{V}_{blade}$



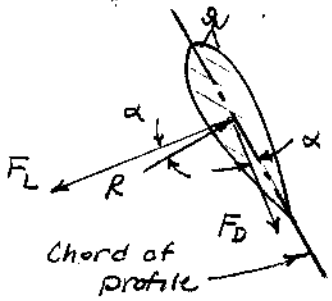
To compute V_{rel}, resolve into components along (a) and transverse (t) to the airfoil chord:

$V_{rel})_a = \omega R + V_w \cos\theta$

$V_{rel})_t = V_w \sin\theta$

$V_{rel} = \left[(V_{rel})_a^2 + (V_{rel})_t^2 \right]^{\frac{1}{2}}$

$\alpha = \tan^{-1}\left[(V_{rel})_t / (V_{rel})_a \right]$



Lift force (F_L) is normal to $\vec{V}_{rel}$ and drag force (F_D) is parallel to $\vec{V}_{rel}$. Thus

$T = R(F_L \sin\alpha - F_D \cos\alpha)$ (torque, $T > 0$ when $F_L/F_D > \cot\alpha$)

Both C_L and C_D must be modeled as functions of angle of attack, α. From a graph of C_L and C_D versus α, a satisfactory representation is

$C_L = 0.12\alpha - 0.0026|\alpha|\alpha$, $-12 < \alpha < 12$ degrees ; $C_L = 0$, $|\alpha| > 12$ degrees

$C_D = 0.00952 + 1.52 \times 10^{-4}\alpha^2$; $-12 < \alpha < 12$ degrees; $C_D = 0.0314$, $|\alpha| > 12$ degrees

(The models in the stalled region, $|\alpha| > 12$ degrees, obviously are crude.)

Sample calculation: Choose $R = 10$ ft, $c = 0.5$ ft, $w = 1$ ft, $X = 5$, $V_w = 20$ mph

At $\theta = 30°$, with $V_w = 20$ mph (29.3 ft/s)

$X = \omega R / V_w$; $\omega = \frac{X V_w}{R} = 5 \times 29.3 \frac{ft}{s} \times \frac{1}{10\,ft} = 14.7$ rad/s (N = 140 rpm)

$\omega R = 14.7 \frac{rad}{s} \times 10\,ft = 147$ ft/s

$V_{rel})_a = \omega R + V_w \cos\theta = 147 + 29.3 \cos 30° = 172$ ft/s

$V_{rel})_t = V_w \sin\theta = 29.3 \sin\theta = 14.7$ ft/s

$V_{rel} = \left[V_{rel})_a^2 + V_{rel})_t^2 \right]^{\frac{1}{2}} = \left[(172)^2 + (14.7)^2 \right]^{\frac{1}{2}} = 173$ ft/s

$\alpha = \tan^{-1}\left[V_{rel})_t / V_{rel})_a \right] = \tan^{-1}(14.7/172) = 4.88$ degrees

$q = \frac{1}{2}\rho V_{rel}^2 = \frac{1}{2} \times 0.00238 \frac{slug}{ft^3} \times (173)^2 \frac{ft^2}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} = 35.6$ lbf/ft²

A_p (projected area of airfoil sectio..) $= cw = 0.5\,ft \times 1\,ft = 0.5$ ft²

$C_L = 0.12\alpha - 0.0026|\alpha|\alpha = 0.12 \times 4.88 - 0.0026|4.88|4.88 = 0.524$

$C_D = 0.00952 + 1.52 \times 10^{-4} \alpha^2 = 0.00952 + 1.52 \times 10^{-4} (4.88)^2 = 0.0131$

$F_L = C_L q A_p = 0.524 \times 35.6 \frac{lbf}{ft^2} \times 0.5\,ft^2 = 9.33$ lbf

$F_D = C_D q A_p = 0.0131 \times 35.6 \frac{lbf}{ft^2} \times 0.5\,ft^2 = 0.233$ lbf

$F_L/F_D = 40.0$

$T = R(F_L \sin\alpha - F_D \cos\alpha) = 10\,ft\,(9.33 \sin(4.88°) - 0.233 \cos(4.88°))\,lbf = 5.62$ ft·lbf

$\mathcal{P} = \omega T = 14.7 \frac{rad}{s} \cdot 5.62\,ft \cdot lbf = 82.6 \frac{ft \cdot lbf}{s}$ (0.150 hp)

$C_p = \frac{\mathcal{P}}{\frac{1}{2}\rho V_w^3 A_s}$; A_s = area swept by element $= 2Rw = 2 \times 10\,ft \times 1\,ft = 20$ ft²

$C_p = 82.6 \frac{ft \cdot lbf}{s} \times \frac{ft^3}{(\frac{1}{2})\,0.00238\,slug} \times \frac{s^3}{(29.3)^3\,ft^3} \times \frac{1}{20\,ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2} = 0.138$ (at $\theta = 15°$)

Obtain $\bar{C}_p$ for a complete rotor revolution by integrating numerically. Such results are presented on the next page, and plotted versus tip speed ratio, $X = \omega R / V_w$.

From the plot, $\bar{C}_p$ is small at low X. It increases as X is raised, then peaks and decreases again. Comparison with Fig. 10.45 shows the trends are similar, but the model predicts useful power at larger X than observed experimentally. Blade elements at smaller radii on the rotor would produce less power, since ω = constant along rotor. $\bar{C}_p$ at large X is also sensitive to C_D.

Low $\bar{C}_p$ at small X occurs because the airfoil is stalled.

Computed results:

```
**********************************************************
*                                                        *
*  Problem 10.107: Performance of Darrieus Turbine       *
*                  (Blade Element Model)                 *
*                                                        *
*     Alan T. McDonald                                   *
*     School of Mechanical Engineering                   *
*     PURDUE UNIVERSITY                                  *
*     West Lafayette, Indiana 47907                      *
*                                                        *
*     May 30, 1992                                       *
*                                                        *
**********************************************************

 Airfoil:        NACA 0012 Section; Chord, c = 6 in.

 Blade element:  Span, w = 1 ft; Radius, R = 10 ft

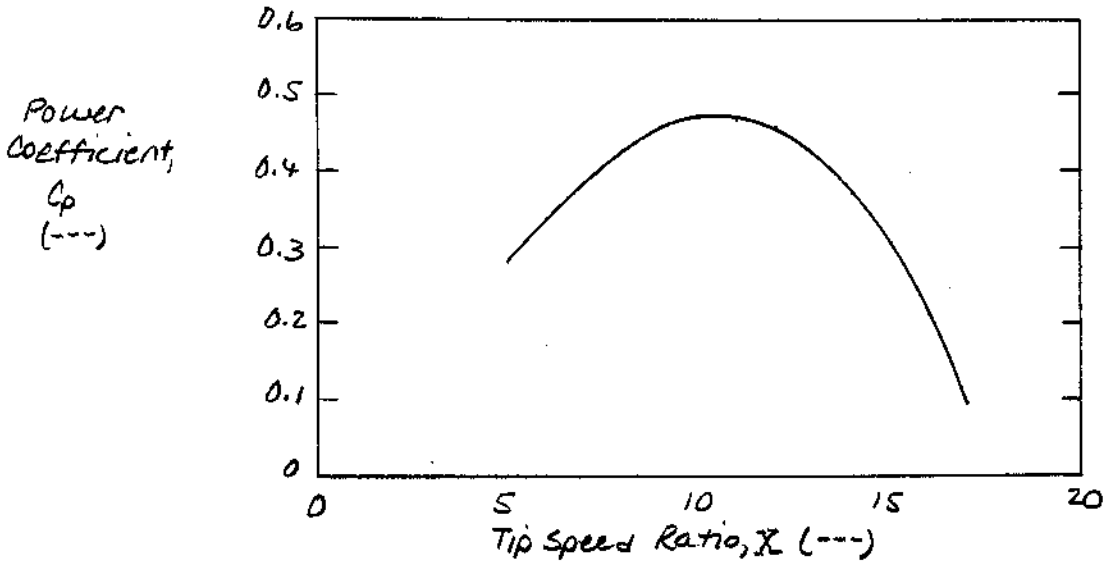
 Input data:     Tip speed ratio, X =  5.0 (---)
                 Wind speed, Vw = 20 mph (29.3 ft/sec)

 Calculated:     Rotor speed, omega = 140.4 rpm
```

theta (deg)	Vrel (ft/s)	alpha (deg)	Cl (--)	Cd (--)	Fl (lbf)	Fd (lbf)	T (ft-lbf)	Cp (--)
0	176	0.00	0.00	0.010	0.0	0.176	-1.8	-0.043
30	173	4.87	0.52	0.013	9.3	0.233	5.6	0.136
60	164	8.95	0.87	0.022	13.8	0.343	18.1	0.439
90	150	11.31	1.02	0.029	13.7	0.384	23.1	0.562
120	135	10.89	1.00	0.027	10.8	0.295	17.5	0.425
150	122	6.90	0.70	0.017	6.3	0.148	6.1	0.147
180	118	-0.00	-0.00	0.010	-0.0	0.078	-0.8	-0.019
210	122	-6.90	-0.70	0.017	-6.3	0.148	6.1	0.147
240	135	-10.89	-1.00	0.027	-10.8	0.295	17.5	0.425
270	150	-11.31	-1.02	0.029	-13.7	0.384	23.1	0.562
300	164	-8.95	-0.87	0.022	-13.8	0.343	18.1	0.439
330	173	-4.87	-0.52	0.013	-9.3	0.233	5.6	0.136
360	176	0.00	0.00	0.010	0.0	0.176	-1.8	-0.043

Average power coefficient for complete revolution: Cp,bar = 0.280

Plotting results of similar calculations at various tip speed ratios give:

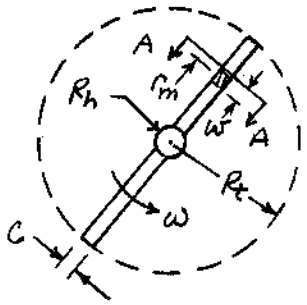


Given: Lift and drag data for NACA 23015 airfoil section, Fig. 9.17.

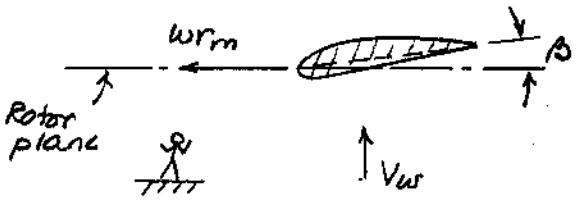
Consider two-blade, horizontal-axis wind turbine with this section.

Find: (a) Analyze air flow relative to a blade element in rotating turbine.
(b) Develop numerical model for blade element.
(c) Calculate power coefficient as a function of tip speed ratio.
(d) Compare with the trend shown in Fig. 11.45.

Solution: Front view of rotor; blade element shown cross-hatched:

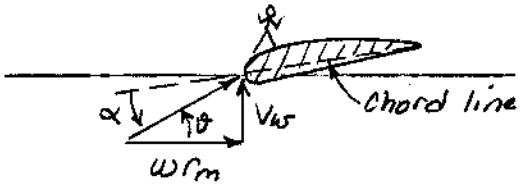


Section A-A:
(absolute velocities)



β = blade twist angle

(relative velocities)

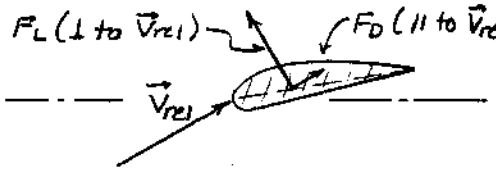


Resolve to relative velocity:

$\vec{V}_{abs} = \vec{V}_{blade\ element} + \vec{V}_{rel}$

$\vec{V}_{rel} = \vec{V}_w - \vec{V}_{blade} = \vec{V}_w + (-\vec{V}_{blade})$

α = angle of attack



Computing equations:

$F_L = C_L \frac{1}{2}\rho V_r^2 A_p$; V_r = relative velocity

$F_D = C_D \frac{1}{2}\rho V_r^2 A_p$; A_p = planform area

{Note: β > 0 is required for windmill to self start.}

$T = n_b r_m (F_L \sin\theta - F_D \cos\theta)$; n_b = number of blades (2)

$V_{rel} = [(\omega r_m)^2 + V_w^2]^{\frac{1}{2}}$; $\theta = \tan^{-1}(V_w/\omega r_m)$; $\alpha = \theta - \beta$

Both C_L and C_D must be modeled as functions of angle of attack, α. From Fig. 9.17, satisfactory representations are:

$0 \le \alpha < 12°$ $\quad C_L = 0.12 + 0.107\alpha$ $\qquad$ $0 \le \alpha < 4°$ $\quad C_D = 0.0065 + 9.55\times10^{-5}\alpha$

$12° \le \alpha < 18°$ $\quad C_L = 0.12 + 0.107\alpha - 0.00852(\alpha - 12)^2$ $\qquad$ $4° \le \alpha < 16°$ $\quad C_D$ = above $+ 7.72\times10^{-5}(\alpha - 4)^2$

$18° \le \alpha$ $\quad C_L = 0.2$ $\qquad$ $16° \le \alpha$ $\quad C_D = 0.02$

{Obviously both models are crude for C_L ($\alpha > 18°$), C_D ($\alpha > 16°$).} Choose R_t = 10 ft, r_m = 5.5 ft, C = 6 in., ω = 1 ft, V_w = 20 mph (29.4 ft/s), X = 5, and β = 5°. Then

$\omega = \frac{X V_w}{R_t} = 5.0 \times 29.4 \frac{ft}{s} \times \frac{1}{10\ ft} = 14.7$ rad/s (140 rpm); $\omega r_m = 80.9$ ft/s

$V_{rel} = [(80.9)^2 + (29.4)^2]^{\frac{1}{2}} = 86.1$ ft/s; $q = \frac{1}{2}\rho V_{rel}^2 = 8.82$ lbf/ft²; $A_p = wc = 0.5$ ft²

$\theta = \tan^{-1}(29.4/80.9) = 20.0°$; $\alpha = \theta - \beta = 20.0 - 5.0 = 15.0°$; $C_L = 1.65$; $C_D = 0.017$

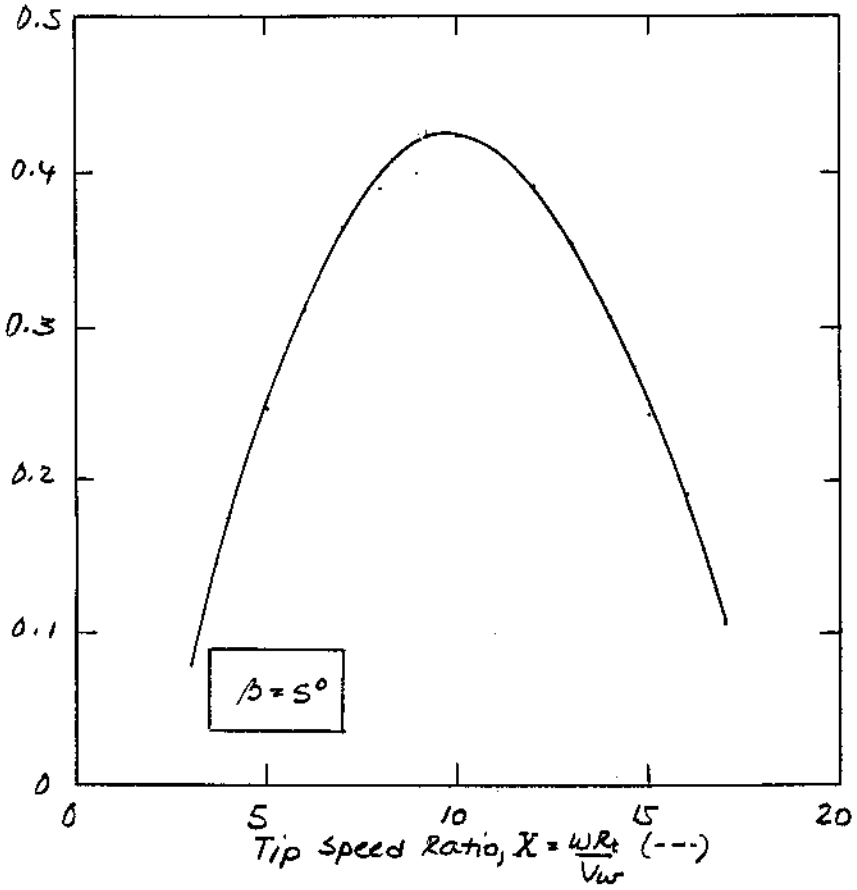
$F_L = 1.65 \times 8.82 \frac{\text{lbf}}{\text{ft}^2} \times 0.5 \text{ ft}^2 = 7.28$ lbf; $F_D = 0.017 \times 8.82 \frac{\text{lbf}}{\text{ft}^2} \times 0.5 \text{ ft}^2 = 0.075$ lbf

$T = 2 \times 5.5 \text{ ft} (7.28 \times \sin 20° - 0.075 \cos 20°)$ lbf $= 26.6$ ft·lbf

Similar calculations for the other blade elements show that torque for the complete propeller is $T_p = 159$ ft·lbf. The power coefficient is

$$C_p = \frac{P}{\frac{1}{2}\rho V_w^3 \pi R_t^2} = \frac{\omega T_p}{\frac{1}{2}\rho V_w^3 \pi R_t^2} = \frac{2}{\pi} \times 14.7 \frac{\text{rad}}{\text{s}} \times 159 \text{ ft·lbf} \times \frac{\text{ft}^3}{0.00238 \text{ slug}} \times \frac{\text{s}^3}{(29.4)^3 \text{ft}^3} \times \frac{1}{(10)^2 \text{ft}^2} \times \frac{\text{slug·ft}}{\text{lbf·s}^2} = 0.246$$

Calculated results are tabulated on the next page, plotted and discussed below:



Trends shown are similar to Fig. 10.45. At small X, the blade is entirely stalled so useful output is low. At large X, α becomes negative near the tips, reducing output.

This model <u>does not</u> include: (1) axial interference that reduces normal velocity below V_w as loading increases, or (2) swirl introduce by blade drag. Both these effects reduce performance. For more details, see Division L, Section XI of [30].

Computed results:

```
*********************************************************
*                                                       *
*  Problem 10.108: Performance of Horizontal-Axis       *
*                  Wind Turbine                         *
*          (Blade Element Model, 2-Blade Propeller)     *
*                                                       *
*    Alan T. McDonald                                   *
*    School of Mechanical Engineering                   *
*    PURDUE UNIVERSITY                                  *
*    West Lafayette, Indiana 47907                      *
*                                                       *
*    June 1, 1992                                       *
*                                                       *
*********************************************************
```

Airfoil: NACA 23015 Section; Chord, c = 6 in.
Tip radius, Rt = 10 ft
Twist angle, beta = 5 degrees

Blade element: Delta r, dr = 1.00 ft

Input data: Tip speed ratio, X = 5.0 (---)
Wind speed, Vw = 20 mph (29.4 ft/s)

Calculated: Rotor speed, omega = 140.4 rpm

Rm (ft)	Vrel (ft/s)	alpha (deg)	Cl (--)	Cd (--)	Fl (lbf)	Fd (lbf)	T (ft-lbf)
1.50	22	48.13	0.20	0.020	0.06	0.006	0.13
2.50	37	33.66	0.20	0.020	0.16	0.016	0.44
3.50	51	24.74	0.20	0.020	0.32	0.032	0.90
4.50	66	18.96	0.20	0.020	0.52	0.052	1.48
5.50	81	14.98	1.65	0.017	6.41	0.067	23.39
6.50	96	12.10	1.41	0.013	7.69	0.069	28.53
7.50	110	9.93	1.18	0.010	8.55	0.074	31.99
8.50	125	8.24	1.00	0.009	9.31	0.081	34.90
9.50	140	6.89	0.86	0.008	9.95	0.091	37.25

Torque for complete propeller: T = 159.0 ft-lbf

Power coefficient for windmill: Cp = 0.246 (---)

Given: Steady flow through a turbine. Air expands from $T_1 = 1300°C$, $p_1 = 2.0$ MPa(abs) to $T_2 = 500°C$, $p_2 = 101$ kPa.

Find: (a) $u_2 - u_1$ (b) $h_2 - h_1$ (c) $s_2 - s_1$
(d) Show process on a Ts diagram

Solution:

$$\Delta u = u_2 - u_1 = c_v (T_2 - T_1) = 717.4 \frac{J}{kg \cdot K} (500 - 1300) K = -574 \text{ kJ/kg} \quad \leftarrow u_2 - u_1$$

$$\Delta h = h_2 - h_1 = c_p (T_2 - T_1) = 1004 \frac{J}{kg \cdot K} (-800 K) = -803 \text{ kJ/kg} \quad \leftarrow h_2 - h_1$$

To calculate the entropy change, we use the Tds equation

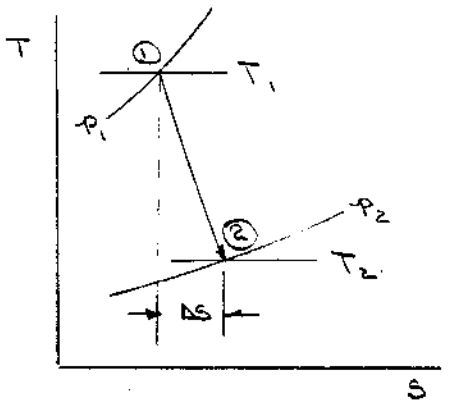
$$T ds = dh - v\,dp = c_p dT - RT \frac{dp}{p}$$

$$\therefore ds = c_p \frac{dT}{T} - R \frac{dp}{p}$$

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

$$= 1004 \frac{J}{kg \cdot K} \ln \frac{500+273}{1300+273} - 286.9 \frac{J}{kg \cdot K} \ln \frac{0.101}{2.0}$$

$$s_2 - s_1 = (-713.3 + 856.6) J/kg \cdot K = 143 \text{ J/kg} \cdot K \quad \leftarrow s_2 - s_1$$



Given: Air at state ①: $p_1 = 300$ kPa (abs), $T_1 = 60°C$.

expands to state ②: $p_2 = 150$ kPa (abs), $T_2 = 27°C$

Find: Is an adiabatic process possible? Justify.

Solution:

Basic equations $Tds \geq \delta Q$

$$Tds = dh - v\,dp$$

Assumptions: (1) Ideal gas
(2) Constant specific heats

From the second law, $Tds \geq 0$ for an adiabatic process. Thus process is possible if $s_2 \geq s_1$.

From the Tds equation,

$$ds = \frac{1}{T}dh - \frac{1}{\rho T}dp = c_p \frac{dT}{T} - R\frac{dp}{p}$$

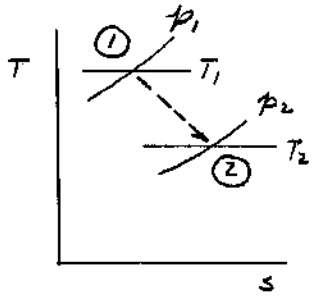
Integrating,

$$s_2 - s_1 = \int_{s_1}^{s_2} ds = c_p \int_{T_1}^{T_2} \frac{dT}{T} - R\int_{p_1}^{p_2} \frac{dp}{p} = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1}$$

$$= 1004\frac{J}{kg\cdot K}\ln\frac{(273+27)}{(273+60)} - 287\frac{J}{kg\cdot K}\ln\frac{150}{300}$$

$$s_2 - s_1 = 94.2\ J/kg\cdot K > 0$$

Process is possible



Given: Air, $m = 10$ lbm, in a closed tank is cooled from state ①, $T_1 = 500°F$, $P_1 = 400$ psia to state ②, $T_2 = 100°F$

Find: (a) $S_2 - S_1$ (b) $U_2 - U_1$ (c) $H_2 - H_1$

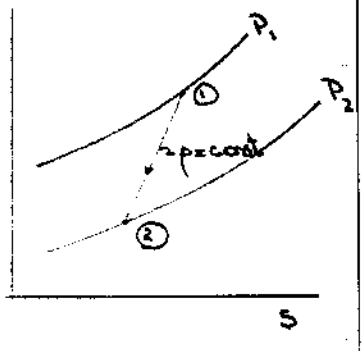
Solution:

Computing equations: $P = \rho RT$ $\quad du = c_v\,dT$ $\quad dh = c_p\,dT$

$$Tds = dh - v\,dP$$

Assumptions: (1) air behaves as an ideal gas.
(2) specific heats are constant.

The process is a constant volume process. Thus



$$\frac{P_2}{P_1} = \frac{T_2}{T_1} \quad \text{and}$$

$$P_2 = P_1 \frac{T_2}{T_1} = 400 \text{ psia} \times \left(\frac{560}{960}\right) = 233 \text{ psia}$$

Since $Tds = dh - v\,dP$

$$ds = \frac{1}{T} dh - \frac{v}{T} dP = c_p \frac{dT}{T} - R \frac{dP}{P}$$

For constant c_p,

$$s_2 - s_1 = \int_{s_1}^{s_2} ds = \int_{T_1}^{T_2} c_p \frac{dT}{T} - \int_{P_1}^{P_2} R \frac{dP}{P} = c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1}$$

$$= 0.240 \frac{\text{Btu}}{\text{lbm}\cdot°R} \ln \frac{560}{960} - 53.3 \frac{\text{ft lbf}}{\text{lbm}\cdot°R} \times \frac{\text{Btu}}{778 \text{ ft lbf}} \ln \frac{233}{400}$$

$$s_2 - s_1 = -0.0923 \text{ Btu/lbm}\cdot°R$$

$$S_2 - S_1 = m(s_2 - s_1) = 10 \text{ lbm} \times -0.0923 \frac{\text{Btu}}{\text{lbm}\cdot°R} = -0.923 \text{ Btu/}°R \quad \longleftarrow S_2 - S_1$$

$dU = m\,du = mc_v\,dT$. For constant c_v

$$U_2 - U_1 = \int_{U_1}^{U_2} dU = mc_v \int_{T_1}^{T_2} dT = mc_v (T_2 - T_1)$$

$$U_2 - U_1 = 10 \text{ lbm} \times 0.171 \frac{\text{Btu}}{\text{lbm}\cdot°R} \times (560 - 960)°R = -684 \text{ Btu} \quad \longleftarrow U_2 - U_1$$

$dH = m\,dh = mc_p\,dT$ For constant c_p

$$H_2 - H_1 = \int_{H_1}^{H_2} dH = mc_p \int_{T_1}^{T_2} dT = mc_p (T_2 - T_1).$$

$$H_2 - H_1 = 10 \text{ lbm} \times 0.240 \frac{\text{Btu}}{\text{lbm}\cdot°R} \times (560 - 960)°R = -960 \text{ Btu} \quad \longleftarrow H_2 - H_1$$

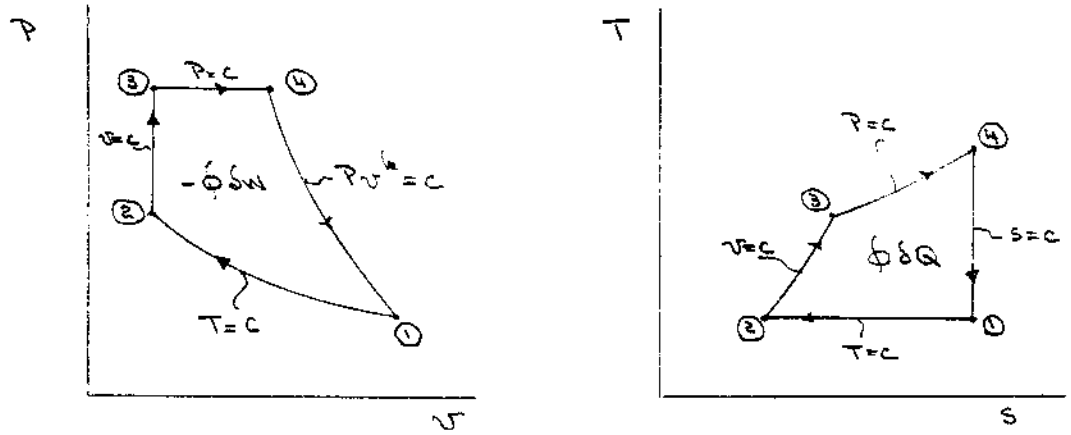
Given: Gas in a closed system undergoes a cycle made up of the following processes

1 → 2	$T = c$	reversible compression
2 → 3	$v = c$	reversible heat addition
3 → 4	$P = c$	reversible expansion
4 → 1		reversible adiabatic expansion

Sketch: Pv and Ts diagrams for cycle

Find: signs for $\oint \delta W$, $\oint \delta Q$, $\oint ds$, $\oint dU$, $\oint dH$

Solution:



Basic equation $\delta W + \delta Q = dU$

$$\oint \delta W = -\oint \delta Q + \oint dU$$

For a cycle

$\oint dU = 0$ $\oint dH = 0$ $\oint ds = 0$

Since all processes are reversible, $\delta W = -p\,dv$

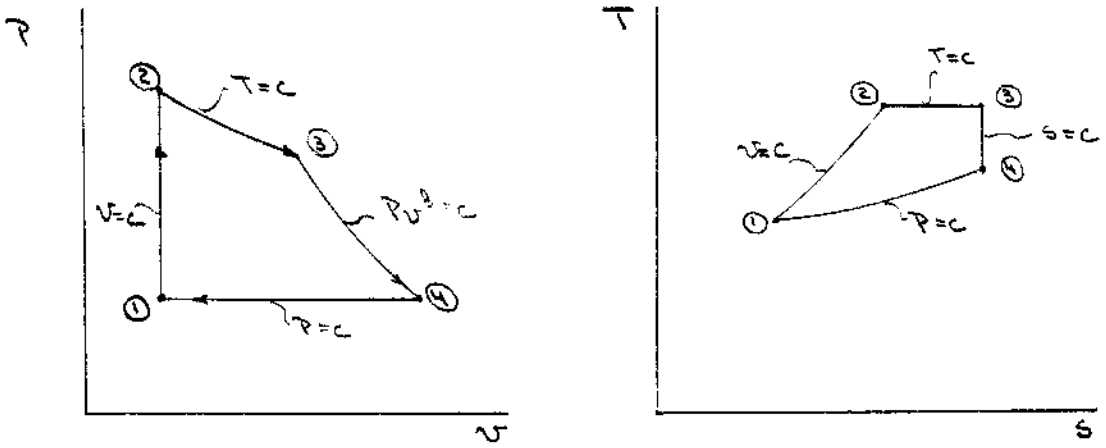
$-\oint \delta W$ = area enclosed by curve on Pv plane $\delta W < 0$ → $\oint \delta W$

Then $\oint \delta Q > 0$ → $\oint \delta Q$

Given: Ideal gas undergoes a series of reversible processes as follows
- 1→2 V = const heat addition
- 2→3 T = const expansion
- 3→4 Adiabatic expansion to $P_4 = P_1$
- 4→1 P = const

Sketch: Pv and Ts diagrams of the cycle
Find: Signs for $\oint \delta Q$, $\oint \delta W$, $\oint du$, $\oint ds$, $\oint dh$

Solution:



Basic equation: $\delta W + \delta Q = dU$

$$\oint \delta W = -\oint \delta Q + \oint dU$$

For a cycle

$$\oint du = 0 \qquad \oint ds = 0 \qquad \oint dh = 0$$

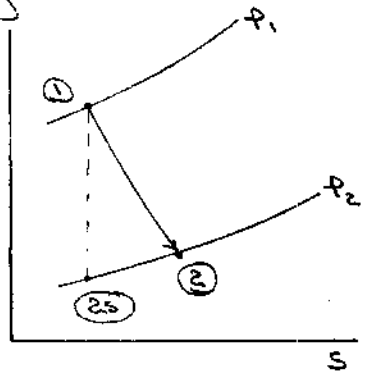
Since all processes are reversible, $\delta W = -p\,dv$

$-\oint \delta W$ = area enclosed by curve on Pv plane $\therefore \delta W < 0$

and $\oint \delta Q > 0$

Given: Steady flow of air, $\dot{m} = 0.5$ kg/s, through a turbine.
At inlet, $V_1 \approx 0$, $T_1 = 1300°C$, $p_1 = 2.0$ MPa (abs)
At outlet, $V_2 = 200$ m/s, $p_2 = 101$ kPa, $T_2 = 500°C$

Find: (a) power produced by the turbine
(b) label state points on a Ts diagram



Solution:

For an isentropic expansion through the turbine,

$$T_{2s=c} = T_1 \left(\frac{p_2}{p_1}\right)^{(k-1)/k}$$

$$T_{2s=c} = (1300+273)\text{K} \left(\frac{0.101\ \text{MPa}}{2.0\ \text{MPa}}\right)^{\frac{1.4-1}{1.4}}$$

$$T_{2s=c} = 670\,\text{K}\ (397°C)$$

Writing the first law of thermodynamics between the turbine inlet and outlet

$$\dot{W} + \dot{Q} = \dot{m}\left[\left(h_2 + \frac{V_2^2}{2}\right) - \left(h_1 + \frac{V_1^2}{2}\right)\right]$$

(Assume $\dot{Q} = 0$)

For an ideal gas with constant specific heats, $h_2 - h_1 = c_p(T_2 - T_1)$

$$\therefore \dot{W} = \dot{m}\left[c_p(T_2 - T_1) - \frac{V_2^2}{2}\right]$$

$$= 0.5\ \frac{\text{kg}}{\text{s}}\left[1004\ \frac{\text{N·m}}{\text{kg·K}}\ 773-1573\text{K} - \frac{1}{2}(200)^2\ \frac{\text{m}^2}{\text{s}^2} * \frac{\text{N·s}^2}{\text{kg·m}}\right]$$

$$\dot{W} = -392 \times 10^3\ \frac{\text{N·m}}{\text{s}}$$ (negative sign indicates work out).

$$\therefore \dot{W}_{out} = 392\ \text{kW}$$

$\dot{W}_{out}$

Given: Air compressed from standard conditions to fill tank with $\forall = 10\ m^3$, $p = 4.5$ MPa (gage). Ideal gas. Reversible.

Find: (a) Energy for isothermal compression
(b) Energy for isentropic compression
(c) Energy removed by cooling from (b) to (a)
(d) Sketch Ts diagram

Solution: Apply ideal gas, energy, isentropic process equations.

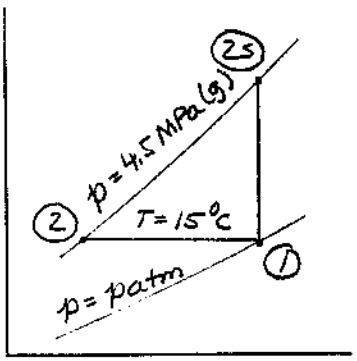
Computing equations: $Q - W = \Delta E = m(u_2 - u_1)$, $W = -\int p\,d\forall$

$p/\rho^k = \text{constant}$, $p = \rho RT$

The tank contains $m = \rho\forall$

$$\rho = \frac{p}{RT} = (4.5 + 0.101)10^6 \frac{N}{m^2} \times \frac{kg \cdot K}{287\ N \cdot m} \times \frac{1}{(273+15)\ K} = 55.7 \frac{kg}{m^3}$$

$$m = \rho\forall = 55.7 \frac{kg}{m^3} \times 10\ m^3 = 557\ kg$$



For process 1 → 2,

$$W = -\int p\,d\forall = -m\int p\,dv = -m\int RT\frac{dp}{p} = -mRT\ln p_2/p_1$$

$$W_{12} = -557\ kg \times 287 \frac{J}{kg \cdot K} \times (273+15)\ K \times \ln\left(\frac{4.5+0.101}{0.101}\right) = 176\ MJ \quad \leftarrow W_{12}$$

For process 2 → 2s, $\frac{p_{2s}}{p_1} = \left(\frac{\rho_{2s}}{\rho_1}\right)^k \rightarrow \frac{T_{2s}}{T_1} = \left(\frac{p_{2s}}{p_1}\right)^{\frac{k-1}{k}} = \left(\frac{4.601}{0.101}\right)^{0.286} = 2.98$

$$T_{2s} = 2.98\ T_1 = 2.98(273+15)K = 858\ K \quad \leftarrow T_{2s}$$

$$W_{12s} = m(u_{2s} - u_1) - \cancel{Q_{12s}}^{0} = mC_v(T_{2s} - T_1)$$

$$W_{12s} = 557\ kg \times 0.717 \frac{kJ}{kg \cdot K}(858 - 288)K = 228\ MJ \quad \leftarrow W_{12s}$$

Process 2s → 2 is at constant pressure

$$Q_{2s2} = m(u_2 - u_{2s}) - W_{2s2} = m(u_2 - u_{2s}) - mp(v_2 - v_{2s}) = m(h_2 - h_{2s})$$

$$= mC_p(T_2 - T_{2s})$$

$$Q_{2s2} = 557\ kg \times 1.00 \frac{kJ}{kg \cdot K} \times (288 - 858)K = -317\ MJ \quad \text{(out of air)} \quad \leftarrow Q_{2s2}$$

Given: Natural gas (thermodynamic properties of methane) flows in a pipe of diameter, $D = 0.6\,m$. At compressor inlet: $T_1 = 13^{\circ}C$, $V_1 = 32\ m/s$, $p_1 = 0.5\ MPa$ (gage). At compressor outlet, $p_2 = 8.0\ MPa$ (gage). The compressor efficiency $\eta_c = 0.85$.

Find: (a) $\dot{m}$ (b) T_2, V_2 (c) $\dot{W}_{in}$ d) label state points on Ts diagram

Solution:

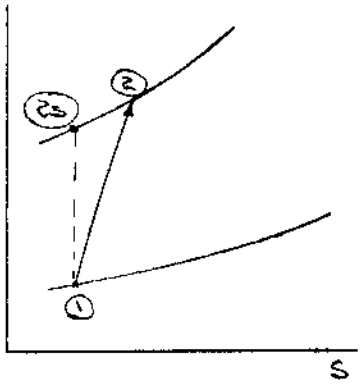
The mass flow rate is given by $\dot{m} = \rho V A$ where $\rho = \frac{p}{RT}$

$$\therefore \dot{m} = \frac{p_1}{RT_1} V_1 \frac{\pi D^2}{4} = (500+101)\times 10^3 \frac{N}{m^2} \times \frac{kg\cdot K}{518.3\ N\cdot m} \times \frac{1}{286\ K} \times 32\frac{m}{s} \times \frac{\pi}{4}(0.6)^2 m^2$$

$$\dot{m} = 36.7\ kg/s \qquad \dot{m}$$

For an isentropic compression

$$T_{2s=c} = T_1\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} = 286K\left(\frac{8.101\ MPa}{0.1601\ MPa}\right)^{\frac{1.31-1}{1.31}} = 529\ K$$



$$\eta_c = \frac{T_{2s} - T_1}{T_2 - T_1} \qquad \therefore T_2 - T_1 = \frac{T_{2s} - T_1}{\eta_c}$$

$$T_2 = T_1 + \frac{T_{2s} - T_1}{\eta_c} = 286K + \frac{(529-286)K}{0.85}$$

$$T_2 = 572\ K \qquad T_2$$

From continuity, $\dot{m} = \rho_1 V_1 A_1 = \rho_2 V_2 A_2$. Assuming $A_1 = A_2$, then

$$V_2 = \frac{\rho_1}{\rho_2} V_1 = \frac{p_1}{p_2}\frac{T_2}{T_1} V_1 = \frac{0.1601}{8.101} \times \frac{572}{286} \times 32\frac{m}{s} = 4.75\ m/s \qquad V_2$$

Writing the first law of thermodynamics between compressor inlet + outlet

$$\dot{W} + \cancel{\dot{Q}}^{\,\approx 0} = \dot{m}\left[\left(h_2 + \frac{V_2^2}{2}\right) - \left(h_1 + \frac{V_1^2}{2}\right)\right] \qquad (\text{Assume } \dot{Q} = 0)$$

$$\dot{W} = \dot{m}\left[(h_2 - h_1) + \frac{1}{2}(V_2^2 - V_1^2)\right] = \dot{m}\left[c_p(T_2 - T_1) + \frac{1}{2}(V_2^2 - V_1^2)\right]$$

$$\dot{W} = 36.7\frac{kg}{s}\left[2190\frac{N\cdot m}{kg\cdot K}(572-286)K + \frac{1}{2}\left\{(4.75)^2 - (32)^2\ \frac{m^2}{s^2} \times \frac{N\cdot s^2}{kg\cdot m}\right\}\right]$$

$$\dot{W} = 36.7\left[626\times 10^3 - 501\right]\frac{N\cdot m}{s}$$

$$\dot{W} = 23\ MW \qquad \dot{W}_{in}$$

Given: Balloon inflated isothermally from $r = 5$ to $r = 7$ in.

Flow is $Q = 0.10$ cfm of standard air (59F, 14.7 psia)

Balloon skin tension is $\sigma = kA$, where $k = 200$ lbf/ft^3, and A = surface area of balloon.

Find: Time required.

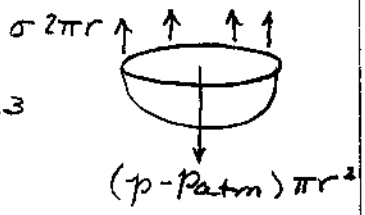
Solution: The mass flow rate is $\dot{m} = \rho_{std} Q$ = constant, so

Computing equation: $\Delta t = \dfrac{\Delta m}{\dot{m}}$ $\qquad p = \rho RT$

Assume: (1) Standard air, $\rho = 0.0765$ lbm/ft^3; (2) Ideal gas

Then $\dot{m} = \rho Q = 0.0765 \dfrac{\text{lbm}}{\text{ft}^3} \times 0.10 \dfrac{\text{ft}^3}{\text{min}} \times \dfrac{\text{min}}{60\ \text{s}} = 1.28 \times 10^{-4}$ lbm/s

From a force balance on the balloon:



$$(p - p_{atm})\pi r^2 = \sigma\, 2\pi r = k(4\pi r^2)\, 2\pi r = 8\pi^2 k r^3$$

or $p = p_{atm} + 8\pi k r$

For $r = 5$ in., $p = 14.7 + 8\pi \times 200 \dfrac{\text{lbf}}{\text{ft}^3} \times 5\ \text{in.} \times \dfrac{\text{ft}^3}{1728\ \text{in.}^3} = 29.2$ psia

$$\rho = \frac{p}{RT} = 29.2 \frac{\text{lbf}}{\text{in.}^2} \times \frac{\text{lbm} \cdot {}^\circ\text{R}}{53.3\ \text{ft} \cdot \text{lbf}} \times \frac{1}{519{}^\circ\text{R}} \times 144 \frac{\text{in.}^2}{\text{ft}^2} = 0.152 \ \text{lbm/ft}^3$$

$$\forall = \frac{4}{3}\pi r^3 = \frac{4\pi}{3} \times (5)^3\ \text{in.}^3 \times \frac{\text{ft}^3}{1728\ \text{in.}^3} = 0.303\ \text{ft}^3$$

$$m = \rho \forall = 0.152 \ \frac{\text{lbm}}{\text{ft}^3} \times 0.303\ \text{ft}^3 = 0.0461\ \text{lbm}$$

For $r = 7$ in., $p = 14.7 \dfrac{\text{lbf}}{\text{in.}^2} + 8\pi \times 200 \dfrac{\text{lbf}}{\text{ft}^3} \times 7\ \text{in.} \times \dfrac{\text{ft}^3}{1728\ \text{in.}^3} = 35.1$ psia

Tabulating,

r (in.)	p (psia)	ρ (lbm/ft^3)	$\forall$ (ft^3)	m (lbm)
5	29.2	0.152	0.303	0.0461
7	35.1	0.183	0.831	0.152

Then $\Delta m = m_7 - m_5 = 0.152 - 0.0461$ lbm $= 0.106$ lbm

and $$\Delta t = 0.106\ \text{lbm} \times \frac{\text{s}}{1.28 \times 10^{-4}\ \text{lbm}} = 828\ \text{s} \quad (\simeq 14\ \text{min})$$

Δt ←

Given: Aircraft flying at two elevations in standard atmosphere.

$z_1 = 500\,m$ $V_1 = 180\,m/s$

$z_2 = 15{,}000\,m$ $V_2 = 320\,m/s$

Find: (a) M_1 (b) M_2

Solution: From Table A.3, $T_1 = 285\,K$, $T_2 = 217\,K$

Definition $M = \frac{V}{c}$

Computing equation: $c = \sqrt{kRT}$

Assumptions: (1) air behaves as an ideal gas

At $z_1 = 500\,m$, $c_1 = (kRT_1)^{1/2} = \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 285\,K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 338\,m/s$

$M_1 = \frac{V_1}{c_1} = \frac{180}{338} = 0.533$ ← M_1

At $z_2 = 15{,}000\,m$, $c_2 = (kRT_2)^{1/2} = \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 217\,K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 295\,m/s$

$M_2 = \frac{V_2}{c_2} = \frac{320}{295} = 1.08$ ← M_2

Given: Boeing 727 cruises at 520 mi/hr at an altitude of 33,000 ft on a standard day

Find: (a) cruise Mach number of the aircraft
(b) flight speed corresponding to $M_{max} = 0.9$

Solution:

At 33,000 ft, $z = 10.06$ km. From Table A.3, $T = 223$ K.

For air, $c = \sqrt{kRT} = \left[1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 223 K \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 299$ m/s

$$V = 520 \frac{mi}{hr} \times 5280 \frac{ft}{mi} \times \frac{hr}{3600 s} \times 0.3048 \frac{m}{ft} = 232 \text{ m/s}$$

$$M = \frac{V}{c} = \frac{232 \text{ m/s}}{299 \text{ m/s}} = 0.776 \quad \leftarrow M_{cr}$$

At $M = 0.90$

$$V = Mc = 0.90 \times 299 \text{ m/s} = 269 \text{ m/s} \quad (603 \text{ mph}) \quad \leftarrow V$$

Given: The Lockheed SR-71 aircraft is thought to cruise at $M = 3.3$ at altitude $z = 85{,}000$ ft.

Find: (a) speed of sound and flight speed for these conditions.
(b) Compare speed to muzzle speed (700 m/sec) of a 30-06 rifle bullet.

Solution:

At altitude, $z = 85{,}000 \text{ ft} \times 0.3048 \frac{\text{m}}{\text{ft}} = 25.9 \text{ km}$

From Table A.3, $T = 222 \text{ K}$

$$\therefore c = \sqrt{kRT} = \left[1.4 \times 287 \frac{\text{N·m}}{\text{kg·K}} \times 222 \text{K} \times \frac{\text{kg}}{\text{N/m·s}^2}\right]^{1/2} = 299 \text{ m/s} \quad \text{c}$$

$$V = Mc = 3.3 \times 299 \text{ m/s} = 987 \text{ m/s} \quad \text{V}$$

$$\frac{V}{V_{bullet}} = \frac{987}{700} = 1.41$$

Problem 11.13

Open-Ended Problem Statement: Lightning strikes and you see the distant flash. A few seconds later you hear the thunder clap. Explain how you could estimate the distance to the lightning strike.

Discussion: The speed of sound in air is approximately 300 m/s (1000 ft/s). Count off the time between seeing the flash (the image reaches your eye almost instantaneously) and hearing the thunder. For example, use the "one thousand one, one thousand two, etc." method to estimate the time. Multiply the time in seconds by the speed of sound.

If the interval between the flash and the thunder sound is three seconds, the distance is 900 m, or approximately one kilometer. For an interval of five seconds, the distance is 5,000 ft, or about one mile.

Given: Specific volume data from steam table.

Find: Calculate and plot sonic speed in water, 32-400°F.

Solution: Use compressed liquid table.*

Computing equation: $c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$; $\rho = \frac{1}{v}$

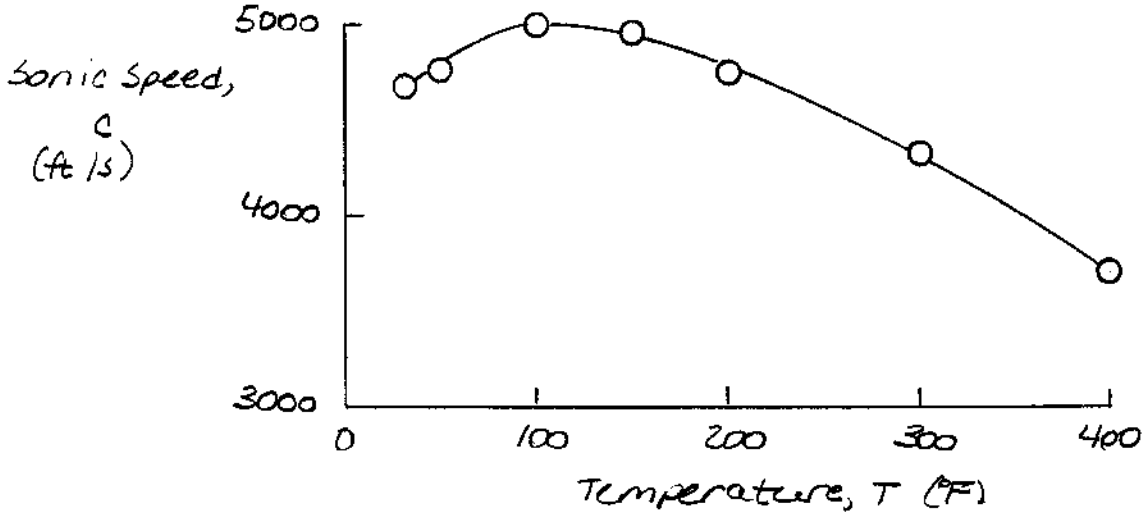
Assumption: $\left(\frac{\partial p}{\partial \rho}\right)_s \simeq \left(\frac{\Delta p}{\Delta \rho}\right)_T$

Then $c = \sqrt{\left(\frac{\Delta p}{\Delta \rho}\right)_T} = \sqrt{\left(\frac{p_2 - p_1}{\rho_2 - \rho_1}\right)_T} = \sqrt{\left(\frac{p_2 - p_1}{\frac{1}{v_2} - \frac{1}{v_1}}\right)_T}$

Tabulate:

	500 psia	1000 psia	
T (°F)	v (ft³/lbm)	v (ft³/lbm)	c (ft/s)
32	0.015994	0.015967	4,680
50	0.015998	0.015972	4,770
100	0.016106	0.016082	5,000
150	0.016318	0.016293	4,970
200	0.016608	0.016580	4,770
300	0.017416	0.017379	4,350
400	0.018608	0.018550	3,710

Plot:



Calculation:

$$c = \left[\Delta p \frac{lbf}{in.^2} \times \frac{ft^3}{\Delta p\ lbm} \times 32.2 \frac{lbm}{slug} \times 144 \frac{in.^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right]^{\frac{1}{2}} = (\ )\ ft/s$$

* Moran, M. J. and Shapiro, H. N., Fundamentals of Engineering Thermodynamics, 3rd ed. New York: Wiley, 1995.

Given: Published data indicate that at an altitude $z = 30{,}000$ ft, the F-5G is capable of making sustained horizontal turns at a rate of:

(i) $6°/s$ at $M = 0.7$

(ii) $3.5°/s$ at $M = 1.6$

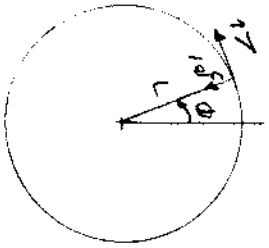
Find: the radius of curvature and normal acceleration produced by these turns

Solution:

At altitude, $z = 30{,}000 \text{ ft} \times 0.3048 \frac{m}{ft} = 9.14 \text{ km}$

From Table A.3, $T = 229.7 + 0.14(223.3 - 229.7) = 229 \text{ K}$

$$c = \sqrt{kRT} = \left[1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 229 \text{ K} \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 303 \text{ m/s}$$



For motion in a circular path

$$V = r\omega = r\frac{d\theta}{dt} \qquad \therefore r = \frac{V}{d\theta/dt}$$

$$a_n = \frac{V^2}{r} = r\omega^2$$

Case (i)

$$r = \frac{V}{d\theta/dt} = \frac{Mc}{d\theta/dt} = 0.7 \times 303 \frac{m}{s} \times \frac{6 \text{ deg}}{s} \times \frac{\pi \text{ rad}}{180 \text{ deg}} = 2030 \text{ m} \quad \leftarrow r$$

$$a_n = \frac{V^2}{r} = \frac{M^2c^2}{r} = (0.7)^2 \times (303)^2 \frac{m^2}{s^2} \times \frac{1}{2030 \text{ m}} = 22.2 \text{ m/s}^2 \quad \leftarrow a_n$$

Case (ii)

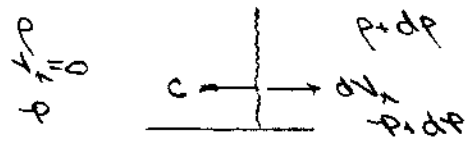
$$r = \frac{V}{d\theta/dt} = \frac{Mc}{d\theta/dt} = 1.6 \times 303 \frac{m}{s} \times \frac{3.5 \text{ deg}}{s} \times \frac{\pi \text{ rad}}{180 \text{ deg}} = 7940 \text{ m} \quad \leftarrow r$$

$$a_n = \frac{V^2}{r} = \frac{M^2c^2}{r} = (1.6)^2 \times (303)^2 \frac{m^2}{s^2} \times \frac{1}{7940 \text{ m}} = 29.6 \text{ m/s}^2 \quad \leftarrow a_n$$

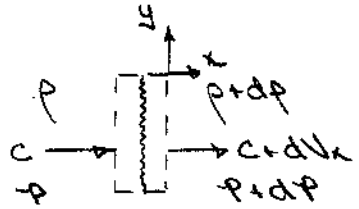
Given: Derivation of sonic speed (Eq. 11.14, Section 11-2.1).

Rederive assuming direction of fluid motion behind the wave is dV_x to the right.

Solution:



(a) Propagating Wave



(b) Inertial CV moving with wave

Apply continuity to CV (b)

$$0 = \{-|\rho c A|\} + \{|(\rho+d\rho)(c+dV_x)A|\}$$

$$0 = -\rho cA + \rho cA + \rho dV_x A + cA d\rho + d\rho dV_x \overset{\to 0}{}$$

then

$$\rho dV_x + c\,d\rho = 0 \quad \text{or} \quad dV_x = -\frac{c}{\rho}d\rho \qquad \text{.....(1)}$$

Applying x-momentum equation to same CV gives

$$\Sigma F_{sx} = pA - (p+dp)A = \int_{CS} V_x \rho \vec{V}\cdot d\vec{A} = c\{-|\rho Ac|\} + (c+dV_x)\{|\rho Ac|\}$$

$\{\dot{m}_{in} = \dot{m}_{out}\}$

$$-A\,dp = \rho Ac\,dV_x \quad \text{, and}$$

$$dV_x = -\frac{dp}{\rho c} \qquad \text{.....(2)}$$

Combining equations (1) and (2) we obtain

$$-\frac{c}{\rho}dp = -\frac{dp}{\rho c}$$

or

$$c^2 = \frac{dp}{d\rho}$$

This is the same result as obtained in the derivation of Section 11-2.1 with the direction of the fluid motion behind the wave to the left.

Given: The temperature varies linearly from sea level to approximately 10 km altitude in the standard atmosphere

Find: a) the lapse rate, i.e. dT/dz
b) Derive an expression for the rate of change of sonic speed with altitude
c) Evaluate dc/dz at sea level and at $z = 10$ km.

Solution:

For a linear temperature variation, $T = T_0 + mz$

$$\frac{dT}{dz} = m = \frac{T - T_0}{z} = \frac{(223.3 - 288.2)\,K}{10{,}000\,m} = -6.49 \times 10^{-3}\ \frac{K}{m}$$ ← $\frac{dT}{dz}$

For an ideal gas, $c = \sqrt{kRT} = [kR(T_0 + mz)]^{1/2}$

$$\frac{dc}{dz} = \frac{1}{2}\,\frac{mkR}{[kR(T_0 + mz)]^{1/2}} = \frac{mkR}{2c}$$ ← $\frac{dc}{dz}$

At sea level, $T = 288.2\,K$

$$c = [kRT]^{1/2} = \left[1.4 \times 286.9\ \frac{N\cdot m}{kg\cdot K} \times 288.2\,K \times \frac{kg\cdot m}{N\cdot s^2}\right]^{1/2} = 340\ m/s$$

$$\frac{dc}{dz} = \frac{mkR}{2c} = -6.49 \times 10^{-3}\ \frac{K}{m} \times \frac{1.4}{2} \times 286.9\ \frac{N\cdot m}{kg\cdot K} \times \frac{s}{340\,m} \times \frac{kg\cdot m}{N\cdot s^2}$$

$$\frac{dc}{dz} = -3.83 \times 10^{-3}\ s^{-1}$$ ← $\left.\frac{dc}{dz}\right|_{z=0}$

At $z = 10$ km, $T = 223.3\,K$

$$c = [kRT]^{1/2} = \left[1.4 \times 286.9\ \frac{N\cdot m}{kg\cdot K} \times 223.3\,K \times \frac{kg\cdot m}{N\cdot s^2}\right]^{1/2} = 299\ m/s$$

$$\frac{dc}{dz} = \frac{mkR}{2c} = -6.49 \times 10^{-3}\ \frac{K}{m} \times \frac{1.4}{2} \times 286.9\ \frac{N\cdot m}{kg\cdot K} \times \frac{s}{299\,m} \times \frac{kg\cdot m}{N\cdot s^2}$$

$$\frac{dc}{dz} = -4.36 \times 10^{-3}\ s^{-1}$$ ← $\left.\frac{dc}{dz}\right|_{z=10\,km}$

Problem 11.18

Open-Ended Problem Statement: How could you measure the approximate speed of sound in air?

Discussion: A source of sound and a measured distance are needed.

Many things could be used for a sound source. For example, a firecracker or portable boat horn could be used to make a loud noise.

The observer must be located a known distance from the sound source. She must time the interval between when the noise is made and when it is heard at her location.

The observer starts the timer when the sound is made. (The observer should be able to see the puff of smoke when a firecracker explodes; a light could be turned on to identify the instant when a boat horn is sounded.) The observer stops the timer when she hears the sound.

To calculate the speed of sound, divide the known distance by the time measured for the sound to travel the distance. For example, if the measured distance is one kilometer and the travel time is 3 seconds, the speed of sound is 1000 m/3 s = 333 m/s, or about 300 m/s.

Given: Photo of bullet moving through standard air shows Mach angle, $\alpha = 28°$

Find: Speed of bullet

Solution:

Computing equations, $\sin\alpha = \frac{1}{M}$ $c = \sqrt{kRT}$

Assumptions: (1) air behaves as an ideal gas
(2) constant specific heats

$M = \frac{1}{\sin\alpha}$, $M = \frac{V}{c}$ $\therefore V = cM = \frac{c}{\sin\alpha}$

Since $c = \sqrt{kRT}$, then

$$V = \frac{1}{\sin\alpha}(kRT)^{1/2} = \frac{1}{\sin 28°}\left(1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 288\,K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2}$$

$V = 725\ m/s \quad (2380\ ft/s)$

Given: Air at 25C flowing at $M = 1.9$

Find: (a) air speed
(b) Mach angle

Solution:

Definitions: $M = \frac{V}{c}$, $\sin\alpha = \frac{1}{M}$ where α = Mach angle

Computing equation: $c = \sqrt{kRT}$

Assumptions: 1) air behaves as an ideal gas
2) constant specific heats

$$V = Mc = M(kRT)^{1/2} = 1.9\left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 298C \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2}$$

$V = 657$ m/s ——— V

$\sin\alpha = \frac{1}{M} = \frac{1}{1.9} = 0.526$

$\alpha = 31.8°$ ——— α

Given: Gas at $P = 50$ psia $\rho = 0.27$ lbm/ft^3
Projectile fired into gas. Total angle of Mach cone is 20°

Find: speed of projectile relative to gas

Solution:

Basic equation: $P = \rho RT$
Definitions: $\sin\alpha = \frac{1}{M}$ $M = \frac{V}{c}$
Computing eq: $c = \sqrt{kRT}$
Assumption: ideal gas

$\sin\alpha = \frac{1}{M}$ $M = \frac{1}{\sin\alpha} = \frac{1}{\sin 6°} = 5.76$

$c = [kRT]^{1/2} = \left[k\frac{P}{\rho}\right]^{1/2}$

$= \left[1.4 \times 50 \frac{lbf}{in^2} \times \frac{ft^3}{0.27 lbm} \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2} \times \frac{144 in^2}{ft^2}\right]^{1/2}$

$c = 1100$ ft/s

$M = \frac{V}{c}$ $\therefore$ $V = Mc = 5.76 \times 1100$ ft/s $= 6320$ ft/s V

Given: An aircraft flies at $M = 1.4$ at elevation $z = 200\,m$
The air temperature is $35^\circ C$

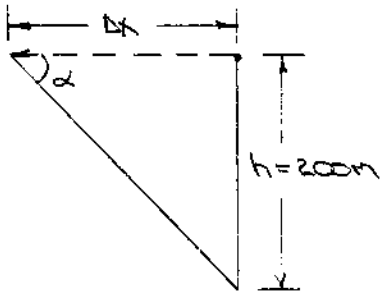
Find: (a) air speed of aircraft
(b) time between instant when aircraft passes directly overhead and instant when Mach cone passes a point on the ground.

Solution: Assume T = constant over 200m elevation.

$T = 35\,C = 308\,K$

$$c = \sqrt{kRT} = \left[1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 308\,K \times \frac{kg\cdot m}{N\cdot s^2}\right]^{1/2} = 352\ m/s$$

$$V = Mc = 1.4 \times 352\ m/s = 493\ m/s \longleftarrow V$$



From the instant the aircraft is directly overhead until the Mach cone reaches the ground, the plane travels a distance Δx at speed $V = 493\ m/s$

$$\sin\alpha = \frac{1}{M} = \frac{1}{1.4} = 0.7143$$

$$\alpha = 45.6^\circ$$

$$\frac{h}{\Delta x} = \tan\alpha \quad \therefore \Delta x = \frac{h}{\tan\alpha} = \frac{200\,m}{\tan 45.6^\circ} = 196\ m$$

Since the plane moves at constant speed V

$$\Delta x = V\,\Delta t \quad \text{and} \quad \Delta t = \frac{\Delta x}{V} = \frac{196\,m}{493\,m/s}$$

$$\Delta t = 0.398\ s \longleftarrow \Delta t$$

Given: A schlieren photograph taken in the NTF shows a Mach angle, $\alpha = 57°$, at a location where $T = -270°F$ and $p = 1.3$ psia.

Find: (a) the local Mach number and flow speed
(b) the unit Reynolds number for the flow

Solution:

$$\sin\alpha = \frac{1}{M} \quad \therefore M = \frac{1}{\sin\alpha} = \frac{1}{\sin 57°} = 1.19 \qquad M$$

$$c = \sqrt{kRT} = \left[1.4 \times 53.3 \frac{ft\cdot lbf}{lbm\cdot °R} \times 32.2 \frac{lbm}{slug} \times 190°R \times \frac{slug\cdot ft}{lbf\cdot s^2}\right]^{1/2} = 676 \text{ ft/s}$$

$$V = Mc = 1.19(676 \text{ ft/s}) = 804 \text{ ft/s} \qquad V$$

$$Re_x = \frac{\rho V x}{\mu}$$

$$\rho = \frac{p}{RT} = 1.3 \frac{lbf}{in^2} \times 53.3 \frac{lbm\cdot °R}{ft\cdot lbf} \times \frac{1}{190°R} \times \frac{144 in^2}{ft^2} = 0.0185 \text{ lbm/ft}^3$$

From Eq. A.1 (Appendix A)

$$\mu = \frac{bT^{1/2}}{1 + S/T}$$

$b = 1.458 \times 10^{-6}$ kg/m·s·K$^{1/2}$
$S = 110.4$ K
T in K

$T = -270F = -167C = 106$ K.

$$\mu = 1.458 \times 10^{-6} \frac{kg}{m\cdot s\cdot K^{1/2}} (106K)^{1/2} \times \frac{1}{1 + \frac{110.4}{106}} = 7.35 \times 10^{-6} \text{ kg/m·s}$$

$$\mu = 7.35 \times 10^{-6} \frac{kg}{m\cdot s} \times \frac{N\cdot s^2}{kg\cdot m} \times \frac{2.089 \times 10^{-2} lbf\cdot s/ft^2}{1 N\cdot s}$$

$$\mu = 1.54 \times 10^{-7} \text{ lbf·s/ft}^2$$

$$\therefore \frac{Re}{x} = \frac{\rho V}{\mu} = 0.0185 \frac{lbm}{ft^3} \times 804 \frac{ft}{s} \times \frac{ft^2}{1.54 \times 10^{-7} lbf\cdot s} \times \frac{slug}{32.2 lbm} \times \frac{lbf\cdot s^2}{slug\cdot ft}$$

$$\frac{Re}{x} = 3.00 \times 10^6 \text{ ft}^{-1} = 9.84 \times 10^6 \text{ m}^{-1} \qquad Re/x$$

Given: Aircraft passes overhead at an altitude of 3 km, travelling at $M = 1.35$. The air temperature is constant at $T = 30°C$ and a head wind blows at $v_{air} = 10$ m/s.

Find: (a) the airspeed of the aircraft
(b) time between instant when aircraft passes directly overhead and instant when sound reaches the ground.

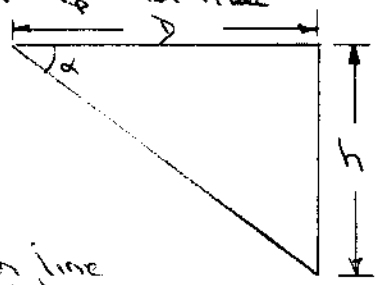
Solution:

T = constant $= 30°C = 303$ K

$$c = (kRT)^{1/2} = \left(1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 303\ K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 349\ m/s$$

$$V = Mc = 1.35 \times 349 \frac{m}{sec} = 471\ m/s \longleftarrow \text{airspeed}$$

The airspeed is the velocity of the plane relative to the air. The ground speed is then $\vec{V}_p = \vec{v}_{air} + \vec{V}_{p/a}$ or $V_p = 461$ m/sec



From the instant the aircraft is directly overhead until the Mach cone reaches the ground, the plane travels a distance, ℓ, at speed $V_p = 461$ m/s.

The value of the time, t, is then $t = \ell / V_p$

Since the air temperature is constant, the Mach line is straight and $\ell = h/\tan\alpha$, where $\alpha = \sin^{-1}(1/M)$

$$\alpha = \sin^{-1}\left(\frac{1}{M}\right) = \sin^{-1}\left(\frac{1}{1.35}\right) = 47.8°$$

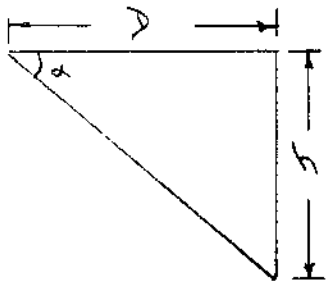
Then,

$$t = \frac{\ell}{V_p} = \frac{h}{\tan\alpha}\frac{1}{V_p} = \frac{3000\ m}{\tan 47.8°} \cdot \frac{s}{461\ m} = 5.90\ s \longleftarrow t$$

Given: Supersonic aircraft with $V = 3000$ ft/s at altitude, $h = 10{,}000$ ft on a standard day. At $t=0$, plane is directly overhead.

Find: value of t when aircraft is first heard

Solution:



From the instant the aircraft is directly overhead until the Mach cone reaches the ground, the plane travels a distance, d, at speed $V = 3000$ ft/s.
The value of the time, t, is then $t = \frac{d}{V}$

If the Mach line is straight then $d = \frac{h}{\tan\alpha}$
where $\alpha = \sin^{-1}\left(\frac{1}{M}\right)$

Since the temperature is not constant over the elevation, the Mach line will not be straight.
We can find a range of t by considering the range of temperature over the altitude

At $h = 10{,}000$ ft (3048 m) $T = 268.4$ K. Then assuming T = constant

$$c = (kRT)^{1/2} = \left(1.4 \times 286.9 \frac{N\cdot m}{kg\cdot K} \times 268.4\,K \times \frac{kg\cdot m}{N\cdot sec^2}\right)^{1/2} = 328 \text{ m/s}$$

$$M = \frac{V}{c} = \frac{3000 \text{ ft}}{s} \times 0.3048 \frac{m}{ft} \times \frac{s}{328 m} = 2.79$$

$$\alpha = \sin^{-1}\left(\frac{1}{M}\right) = \sin^{-1}\left(\frac{1}{2.79}\right) = 21°$$

$$t = \frac{d}{V} = \frac{h}{\tan\alpha}\frac{1}{V} = \frac{10{,}000 \text{ ft}}{\tan 21°} \times \frac{s}{3000 \text{ ft}} = 8.68 \text{ s}$$

At $h = 0$, $T = 288.2$ K Then assuming T = constant.

$$c = (kRT)^{1/2} = \left(1.4 \times 286.9 \frac{N\cdot m}{kg\cdot K} \times 288.2\,K \times \frac{kg\cdot m}{N\cdot sec^2}\right)^{1/2} = 340 \text{ m/s}$$

$$M = \frac{V}{c} = \frac{3000 \text{ ft}}{s} \times 0.3048 \frac{m}{ft} \times \frac{s}{340 m} = 2.69$$

$$\alpha = \sin^{-1}\left(\frac{1}{M}\right) = \sin^{-1}\left(\frac{1}{2.69}\right) = 21.8°$$

$$t = \frac{d}{V} = \frac{h}{\tan\alpha}\frac{1}{V} = \frac{10{,}000 \text{ ft}}{\tan 21.8°} \times \frac{s}{3000 \text{ ft}} = 8.33 \text{ s}$$

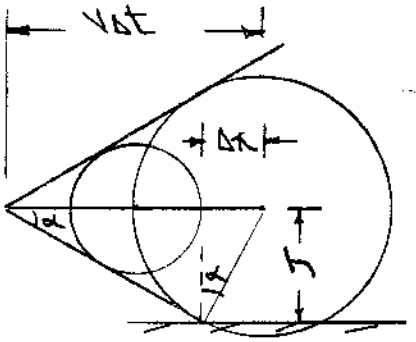
Thus we concluded, $8.33 \text{ s} < t < 8.68 \text{ s}$
Since T varies linearly with elevation, it is reasonable to take the average value

$$t = 8.51 \text{ s}$$

Given: Supersonic aircraft with $V = 3000$ ft/s at altitude $h = 10{,}000$ ft on a standard day. At $t=0$, plane is directly overhead.

Find: the location at which the sound wave that first strikes the observer was emitted.

Solution:



To find the origin of the sound wave that first strikes the observer, we need to consider the current location (at the instant the observer is within the Mach cone) of sound waves emitted earlier in the plane's flight.

These are shown in the diagram assuming the craft moves through stationary air. For this case, the distance we seek is Δx. From the geometry of the diagram, $\Delta x = h \tan\alpha$

At $h = 10{,}000$ ft (3048 km), $T = 268$ K (Table A.3)

Assuming T = constant = 268 K over the range $0 \le h \le 10{,}000$ ft

$$c = (kRT)^{1/2} = \left(1.4 \times 286.9 \frac{N \cdot m}{kg \cdot K} \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 328 \text{ m/s}$$

$$M = \frac{V}{c} = 3000 \frac{ft}{s} \times 0.3048 \frac{m}{ft} \times \frac{s}{328 m} = 2.79$$

$$\alpha = \sin^{-1}\frac{1}{M} = \sin^{-1}\frac{1}{2.79} = 21^\circ \text{ and } \Delta x = 10{,}000 \text{ ft} \cdot \tan 21^\circ = 3840 \text{ ft}$$

Note that at $h = 0$, $T = 288$ K.

Assuming T = constant = 288 K over the range of $0 \le h \le 10{,}000$ ft:

$c = 340$ m/s, $M = 2.69$, $\alpha = 21.8^\circ$, and $\Delta x = 4000$ ft

Since T varies linearly with elevation over the range $0 \le h \le 10{,}000$ ft, it is reasonable to take the average value.

$$\Delta x = 3920 \text{ ft}$$

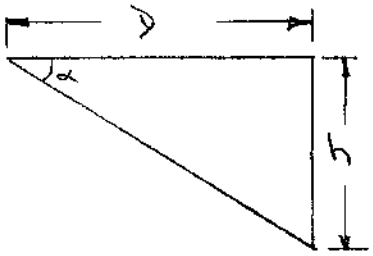
Given: Concorde supersonic transport cruises at M = 2.2 at an altitude, h = 17 km on a standard day.
At t = 0, plane is directly overhead.

Find: value of t when aircraft is first heard.

Solution:

At altitude, T = 216.7 K $c = (kRT)^{1/2} = \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 216.7 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 295 \text{ m/s}$

$V = Mc = 2.2 \times 295 \text{ m/s} = 649 \text{ m/s}$.



If the speed of sound were constant all the way to the ground the Mach line would remain straight the Mach angle, α, would be constant with

$\alpha = \sin^{-1}\left(\frac{1}{M}\right) = \sin^{-1}\left(\frac{1}{2.2}\right) = 27°$

Then, from the diagram $\ell = \frac{h}{\tan\alpha}$

and $t = \frac{\ell}{V} = \frac{h}{\tan\alpha V} = \frac{17000 \text{ m}}{\tan 27°} \cdot \frac{1}{649 \frac{m}{sec}} = 51.4 \text{ s}$

However, the speed of sound varies over the altitude because the temperature varies with altitude.

At sea level T = 288.2 K

$c = (kRT)^{1/2} = \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 288.2 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 340 \text{ m/s}$

The corresponding value of Mach number for V = 649 m/s is

$M = \frac{V}{c} = \frac{649}{340} = 1.91$

$\alpha = \sin^{-1}\left(\frac{1}{M}\right) = \sin^{-1}\left(\frac{1}{1.91}\right) = 31.6°$

Thus, if the speed of sound were constant (at the sea level) value over the entire altitude, then

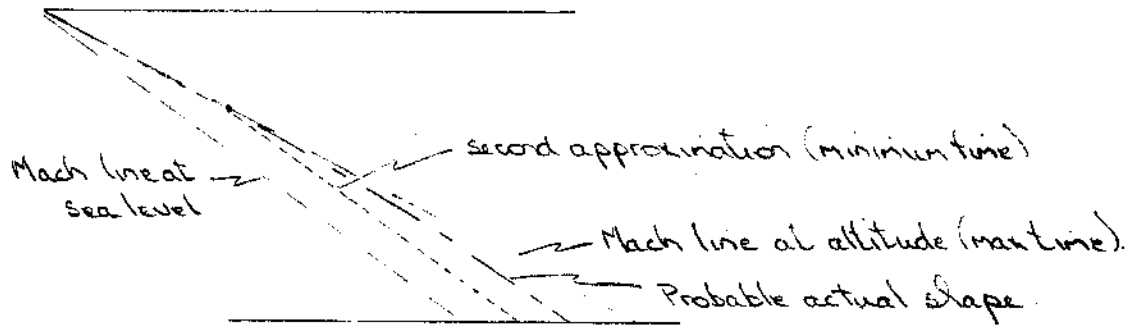
$t = \frac{\ell}{V} = \frac{h}{\tan\alpha V} = \frac{17000 \text{ m}}{\tan 31.6} \times \frac{s}{649 \text{ m}} = 42.6 \text{ s}$

We can obtain a better approximate by considering the variation of temperature with altitude.
From Table A.3

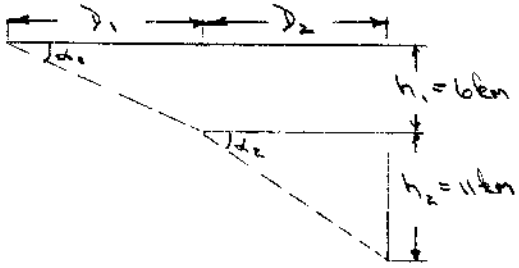
$11 \text{ km} \le y < 20 \text{ km}$ $\quad T = 216.7 \text{ K}$

$0 < y \le 11 \text{ km}$ $\quad T$ varies linearly with y

$T = T_0 - by$ $\quad T_0 = 288.2 \text{ K}$, $b = 6.50 \text{ K/km}$



Since T is constant for $y > y_0 = 11$ km, the second approximation which assumes the Mach line at sea level for $0 < y \leq 11$ km gives the minimum time



$$\ell_1 = \frac{h_1}{\tan\alpha_1} = \frac{6\text{ km}}{\tan 27^\circ} = 11.77\text{ km}$$

$$\ell_2 = \frac{h_2}{\tan\alpha_2} = \frac{11\text{ km}}{\tan 31.6^\circ} = 17.88\text{ km}$$

$$\ell = \ell_1 + \ell_2$$

$$t = \frac{\ell}{v} = \frac{29.65\text{ km}}{649\text{ m/s}} = 45.7\text{ s}$$

Consequently, $45.7 \leq t \leq 51.4$ s

Since the two values are reasonably close, it is appropriate to take the average value and say $t = 48.5$ s ← t

Problem 11.28

Open-Ended Problem Statement: Opponents of supersonic transport aircraft claim that sound waves can be refracted in the upper atmosphere and that, as a result, sonic booms can be heard several hundred miles away from the ground track of the aircraft. Explain the phenomenon of sound wave refraction.

Discussion: A sound wave is refracted when the speed of sound varies with altitude in the atmosphere. (The variation in sound speed is caused by temperature variations in the atmosphere, as shown in Fig. 3.3.)

Imagine a plane wave front that initially is vertical. When the wave encounters a region where the temperature increases with altitude (such as between 20.1 km and 47.3 km altitude in Fig. 3.3), the sound speed increases with elevation. Therefore the upper portion of the wave travels faster than the lower portion. The wave front turns gradually and the sound wave follows a curved path through the atmosphere. Thus a wave that initially is horizontal bends and follows a curved path, tending to reach the ground some distance from the source.

The curvature and the path of the sound wave could be calculated for any specific temperature variation in the atmosphere. However, the required analysis is beyond the scope of this text.

Given: The maximum density in a compressible flow field occurs at stagnation conditions.

Find: The Mach number values for flow of air at which p_0 and p differ by (a) 2% (b) 5%.

Solution:

Assume air is an ideal gas with $k = 1.4$

Computing equation: $\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{\frac{1}{k-1}}$

Solving for M,

$$M = \left\{ \frac{2}{k-1} \left[\left(\frac{p_0}{p} \right)^{k-1} - 1 \right] \right\}^{1/2}$$

$$M = \left\{ 5 \left[\left(\frac{p_0}{p} \right)^{0.4} - 1 \right] \right\}^{1/2}$$

For $p_0/p = 1.02$, $M = 0.199$ M

$p_0/p = 1.05$, $M = 0.314$ M

Given: Aircraft in flight
Stagnation pressure at nose is $p_o = 44.3$ kPa (abs)
Local undisturbed pressure and temperature are
$p = 27.6$ kPa (abs) and $T = -50°C$

Find: (a) M (b) air speed, of the craft

Solution:

Assume air is an ideal gas, with $k = 1.4$

Computing equations: $\frac{p_o}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$, $c = (kRT)^{1/2}$, $M = \frac{V}{c}$

Solving for M,

$$M = \left\{\frac{2}{k-1}\left[\left(\frac{p_o}{p}\right)^{\frac{k-1}{k}} - 1\right]\right\}^{1/2} = \left\{\frac{2}{1.4-1}\left[\left(\frac{44.3}{27.6}\right)^{\frac{1.4-1}{1.4}} - 1\right]\right\}^{1/2} = 0.851 \quad M$$

$$V = Mc = M(kRT)^{1/2} = 0.851\left[1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 223 K \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 255 \frac{m}{s} \quad V$$

Given: Flow of standard air, $V = 600$ m/s

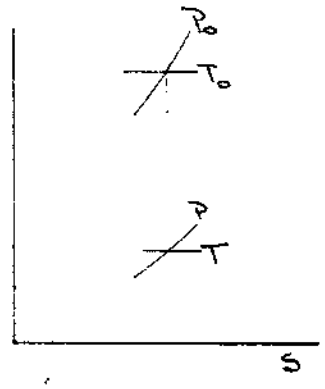
Find: P_o, h_o, T_o

Solution:

Computing equations: $\frac{P_o}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$

$\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2$

$c = \sqrt{kRT}$



Assumption: air behaves as an ideal gas, $k = 1.4$

$c = (kRT)^{1/2} = \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 288 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 340$ m/s

$M = \frac{V}{c} = \frac{600}{340} = 1.76$

$P_o = p\left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)} = 101 \text{ kPa}\left[1 + 0.2(1.76)^2\right]^{3.5} = 546$ kPa ← P_o

$T_o = T\left[1 + \frac{k-1}{2} M^2\right] = 288 K\left[1 + 0.2(1.76)^2\right] = 466$ K ← T_o

$dh = c_p dT$ For c_p = constant

$h_o - h = \int_h^{h_o} dh = \int_T^{T_o} c_p dT = c_p (T_o - T) = 1000 \frac{J}{kg \cdot K} (466 - 288) K$

$h_o - h = 178$ kJ/kg ← $h_o - h$

Given: Body moving through standard air at 200 m/s

Find: Stagnation pressure, assuming
(a) incompressible flow
(b) compressible flow

Solution:

For standard air, $p = 101$ kPa, $T = 15°C$.

Computing equations: $p_0 = p + \frac{1}{2}\rho V^2$ (incompressible)

$\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$ (compressible).

(a) Incompressible flow

$$p_0 = p + \frac{1}{2}\rho V^2 = 101\ \text{kPa} + \frac{1}{2} \times 1.225 \frac{\text{kg}}{\text{m}^3} \times (200)^2 \frac{\text{m}^2}{\text{s}^2} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} \times \frac{\text{kPa}}{10^3 \text{ N/m}^2}$$

$p_0 = 125.5$ kPa $\quad p_{0,\text{inc}}$

(b) Compressible flow

$M = \frac{V}{c}$ $\qquad c = (kRT)^{1/2} = \left(1.4 \times 287 \frac{\text{N}\cdot\text{m}}{\text{kg}\cdot\text{K}} \times 288\text{K} \times \frac{\text{kg}}{\text{N}\cdot\text{s}^2/\text{m}}\right)^{1/2} = 340$ m/s

$M = \frac{200}{340} = 0.588$

$p_0 = p\left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)} = 101\ \text{kPa}\left[1 + 0.2(0.588)^2\right]^{3.5} = 127.6$ kPa $\quad p_{0,\text{com}}$

Given: DC-10 aircraft cruises at altitude, $z = 12$ km on a standard day.

$p_0 = 29.6$ kPa $\quad p = 19.4$ kPa

Find: (a) M (b) V (c) T_0 (on aircraft).

Solution:

Computing equations: $\frac{p_0}{p} = \left(1 + \frac{k-1}{2} M^2\right)^{\frac{k}{k-1}} \qquad \frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$

Solving the first equation for M

$$M = \left\{ \frac{2}{(k-1)} \left[\left(\frac{p_0}{p} \right)^{\frac{k-1}{k}} - 1 \right] \right\} = \left\{ \frac{2}{1.4-1} \left[\left(\frac{29.6}{19.4} \right)^{\frac{1.4-1}{1.4}} - 1 \right] \right\}^{1/2} = 0.801 \quad \leftarrow M$$

At $z = 12$ km, $T = 216.7$ K (Table A.3)

$$c = (kRT)^{1/2} = \left(1.4 \times 286.9 \frac{N \cdot m}{kg \cdot K} \times 216.7 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 295 \text{ m/s}$$

$V = Mc = 0.801 \times 295$ m/s $= 236$ m/s $\quad \leftarrow V$

$$T_0 = T\left(1 + \frac{k-1}{2} M^2\right) = 216.7 K \left[1 + \frac{(1.4-1)}{2} (0.801)^2\right] = 245 K \quad \leftarrow T_0$$

Given: The Anglo-French "Concorde" cruises at $M = 2.2$ at an altitude, $z = 20$ km.

Find: (a) c (b) V (c) α
(d) compare V with muzzle speed ($V_0 = 460$ m/sec) of rifle bullet
(e) T_0 on aircraft

Solution:

At $z = 20$ km, $T = 216.7$ K (Table A.3).

$$c = (kRT)^{1/2} = \left[1.4 \times 286.9 \frac{N\text{-}m}{kg\text{-}K} \times 216.7\,K \times \frac{kg\text{-}m}{N\text{-}s^2}\right]^{1/2} = 295 \text{ m/s} \quad \leftarrow c$$

$$V = Mc = 2.2 \times 295 \text{ m/s} = 649 \text{ m/s} \quad \leftarrow V$$

$$\alpha = \sin^{-1}\left(\frac{1}{M}\right) = \sin^{-1}\left(\frac{1}{2.2}\right) = 27.0^\circ \quad \leftarrow \alpha$$

$$\frac{V}{V_0} = \frac{649}{460} = 1.41 \quad \leftarrow V/V_0$$

$$T_0 = T\left(1 + \frac{k-1}{2} M^2\right) = 216.7\,K \left[1 + \frac{(1.4-1)}{2}(2.2)^2\right] = 426\,K \quad \leftarrow T_0$$

Given: A jet transport cruises at $M = 0.85$ at altitude $z = 12.0$ km on a standard day

Find: (a) stagnation pressure, p_0, sensed by probe on aircraft
(b) speed V calculated by incompressible Bernoulli eq.
(c) percent error of $V_{p=c}$

Solution

At altitude $z = 12.0$ km (from Table A.3)

$T = 216.7$ K $\quad p/p_{sl} = 0.1915 \quad \rho/\rho_{sl} = 0.2546$

$\therefore p = 0.1915\, p_{sl} = 0.1915 \times 101.3$ kPa $= 19.4$ kPa

$\rho = 0.2546\, \rho_{sl} = 0.2546 \times 1.225$ kg/m^3 $= 0.3119$ kg/m^3

$$p_0 = p\left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)} = 19.40 \text{ kPa}\left[1 + \frac{(1.4-1.0)}{2}(0.85)^2\right]^{\frac{1.4}{0.4}} = 31.1 \text{ kPa} \quad \leftarrow p_0$$

For incompressible flow $p_0 = p + \frac{1}{2}\rho V^2$

$$\therefore V = \left[\frac{2(p_0 - p)}{\rho}\right]^{1/2} = \left[2 \times (31.1 - 19.4) \times 10^3 \frac{N}{m^2} \times 0.312 \frac{kg}{m^3} \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2}$$

$$V_{p=c} = 274 \text{ m/s} \quad \leftarrow V_{p=c}$$

The true speed is given by $V = Mc = M(kRT)^{1/2}$

$$V = 0.85\left(1.4 \times 286.9 \frac{N \cdot m}{kg \cdot K} \times 216.7 \text{ K} \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 251 \text{ m/s}$$

The percent error is then

$$\text{Error} = \frac{V_{p=c} - V}{V} \times 100 = \frac{274 - 251}{251} \times 100 = 9.2\% \quad \leftarrow \text{Error}$$

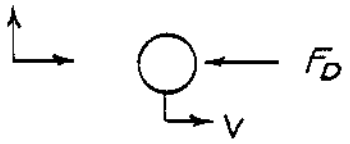
Given: Cannonball with $m = 5.44$ kg, $D = 110$ mm, fired horizontally at sea level on a standard day.

Find: (a) Horizontal acceleration
(b) Maximum pressure
(c) Maximum temperature
} at $M = 1$ $C_D = 1.3$

Solution: Apply Newton's second law of motion:

Basic equation: $\Sigma F_x = ma_x$

$c = \sqrt{kRT}$ $F_D = C_D A \frac{1}{2}\rho V^2$



Then $V = C = \left[1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 288\,K \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 340$ m/s

and $\Sigma F_x = -F_D = -C_D A \frac{1}{2}\rho V^2 = ma_x$ so $a_x = -\frac{C_D A \rho V^2}{2m}$

$A = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.110)^2 m^2 = 9.50 \times 10^{-3} m^2$

$a_x = -\frac{1.3}{2} \times 9.50 \times 10^{-3} m^2 \times 1.23 \frac{kg}{m^3} \times (340)^2 \frac{m^2}{s^2} \times \frac{1}{5.44\,kg} = -161$ m/s^2 a_x

(Cannonball slows because of aerodynamic resistance.)

The maximum pressure and temperature will be local isentropic stagnation conditions as seen from the sphere.

$\frac{p_0}{p} = \left(1 + \frac{k-1}{2} M_\infty^2\right)^{\frac{k}{k-1}} = (1.2)^{3.5} = 1.89$



$p_0 = 1.89 \times 101$ kPa $= 191$ kPa (abs) p_0

$\frac{T_0}{T} = \left(1 + \frac{k-1}{2} M_\infty^2\right) = 1.20$

$T_0 = 1.20 \times 288\,K = 346\,K\ (73°C)$ T_0

Given: Supersonic wind tunnel with $M = 3.0$, $T = 60°F$, $p = 5$ psia, and test section, $A = 2.0$ ft^2.

Find: (a) Stagnation conditions, T_0 and p_0.
(b) Mass flow rate.

Solution: Apply isentropic stagnation

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$ $\qquad \frac{p_0}{p} = \left(1 + \frac{k-1}{2} M^2\right)^{k/k-1}$

Assume: (1) One-dimensional flow (4) Ideal gas
(2) Isentropic flow
(3) Steady flow

Then $T_0 = T\left(1 + \frac{k-1}{2} M^2\right) = (460+60)°R\left[1 + \underbrace{\frac{1.4-1}{2}(3.0)^2}_{2.8}\right] = 1460°R\ (996°F)$ T_0 ←

$p_0 = p\left(1 + \frac{k-1}{2} M^2\right)^{\frac{k}{k-1}} = 5.0 \text{ psia}\,(2.8)^{3.5} = 5.0 \text{ psia}\,(36.7)$

$p_0 = 184$ psia p_0 ←

$\dot{m} = \rho V A$

$\rho = \frac{p}{RT} = 5.0 \frac{lbf}{in^2} \times \frac{lbm \cdot °R}{53.3 ft \cdot lbf} \times \frac{1}{(460+60)°R} \times 144 \frac{in^2}{ft^2} = 0.0260 \text{ lbm/ft}^3$

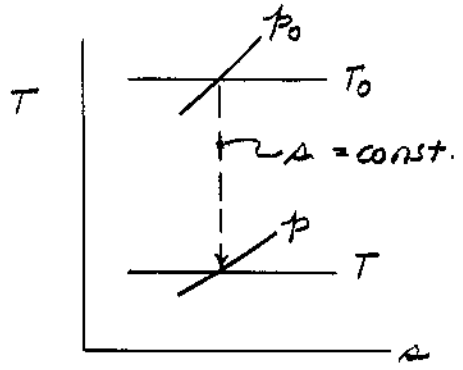
$V = Mc \qquad c = \sqrt{kRT} = \left[1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot °R} \times 520°R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right]^{1/2}$

$c = 1120$ ft/s

$V = 3.0 \times 1120 \frac{ft}{s} = 3350$ ft/s

$\dot{m} = 0.0260 \frac{lbm}{ft^3} \times 3350 \frac{ft}{s} \times 2 ft^2 = 174$ lbm/s $\dot{m}$ ←

The Ts diagram is



Given: Steady air flow through a constant area duct.
Properties change due to friction, but flow is adiabatic.

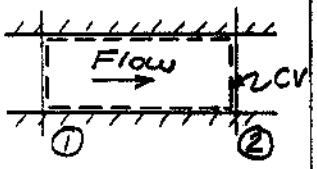
Find: (a) Show that the energy equation reduces to

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} = \text{constant}$$

(b) Show that for adiabatic flow $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$

(c) Effects on T_0, p_0.

Solution: Apply the energy equation to the CV shown:



$$\text{BE} \quad \dot{Q}^{\,=0(1)} + \dot{W}_s^{\,=0(2)} + \dot{W}_{shear}^{\,=0(3)} = \frac{\partial}{\partial t}^{=0(4)} \int_{CV} e\rho\, d\forall + \int_{CS} (e + pv)\rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) $\dot{Q} = 0$ (adiabatic)
(2) $\dot{W}_s = 0$
(3) $\dot{W}_{shear} = 0$
(4) Steady flow
(5) Uniform flow at each section
(6) Neglect Δz

Then

$$0 = \left(u_1 + p_1 v_1 + \frac{V_1^2}{2}\right)\{-|\rho_1 V_1 A|\} + \left(u_2 + p_2 v_2 + \frac{V_2^2}{2}\right)\{|\rho_2 V_2 A|\}$$

But $h \equiv u + pv$, and $|\rho_1 V_1 A| = |\rho_2 V_2 A| = |\rho V A| = \dot{m}$, so

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} = h + \frac{V^2}{2} = h_0 = \text{constant} \qquad h_0$$

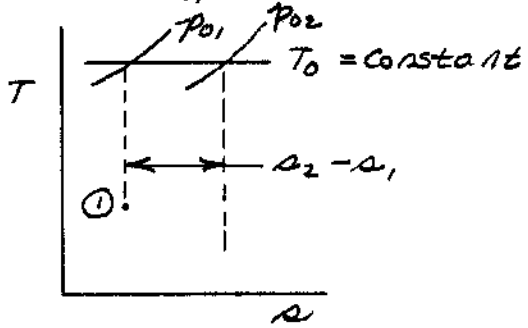
Assumption (7) Ideal gas; $h_0 - h = c_p(T_0 - T)$, $c_p = \frac{kR}{k-1}$, $c^2 = kRT$

Thus $c_p T_0 = c_p T + \frac{V^2}{2}$

$$\frac{T_0}{T} = 1 + \frac{V^2}{2c_p T} = 1 + \frac{(k-1)V^2}{2kRT} = 1 + \frac{k-1}{2}\frac{V^2}{c^2} = 1 + \frac{k-1}{2} M^2 \qquad \frac{T_0}{T}$$

From the energy equation, $T_{01} = T_{02} = T_0 = \text{constant}$ $\qquad T_0$

The Ts diagram is



Since flow is frictional, $s_2 > s_1$. Therefore $p_{02} < p_{01}$ $\qquad p_{02}$

Given: For aircraft flying at supersonic speeds, the lift and drag coefficients are functions of M only.

Aircraft: $s = 75\ m$, $V = 780\ m/s$, $z = 20\ km$ standard atmosphere

Model: $s = 0.9\ m$, $T = 10°C$, $p = 10\ kPa\ (abs)$

Find: (a) air speed for model test
(b) stagnation temperature for model test
(c) stagnation pressure for model test

Solution:

At $z = 20\ km$, $T = 216.7\ K$ (Table A.3).

$$c = (kRT)^{1/2} = \left(1.4 \times 286.9 \frac{N\cdot m}{kg\cdot K} \times 216.7\,K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 295\ m/s$$

Thus for aircraft, $M = \frac{V}{c} = \frac{780}{295} = 2.64$

This Mach number must be duplicated in the model test. In the tunnel,

$$c = (kRT)^{1/2} = \left(1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 283\,K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 337\ m/s$$

$$\therefore V = Mc = 2.64(337\ m/s) = 890\ m/s \quad \longleftarrow V$$

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2 \qquad \therefore T_0 = T\left(1 + \frac{k-1}{2}M^2\right)$$

$$T_0 = 283\,K\left[1 + \frac{(1.4-1)}{2}(2.64)^2\right] = 677\,K \quad \longleftarrow T_0$$

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)} \qquad \therefore p_0 = p\left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)}$$

$$p_0 = 10\ kPa\left[1 + 0.2(2.64)^{1/2}\right]^{3.5} = 212\ kPa \quad \longleftarrow p_0$$

Given: The Lockheed "Blackbird" aircraft is thought to cruise at $M = 3.3$ at altitude, $z = 26$ km.
Because the speed is supersonic, a normal shock occurs in front of a total-head tube. The stagnation pressure decreases by 74.7 percent across the shock.

Find: (a) V (b) p_o (c) p_o on aircraft (d) T_o on aircraft

Solution:

At altitude $z = 26$ km, $T = 222.5$ K, $p/p_{SL} = 0.0216$ (Table A.3).

$$c = (kRT)^{1/2} = \left[1.4 \times 286.9 \frac{N \cdot m}{kg \cdot K} \times 222.5\,K \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 299 \text{ m/s}$$

$$V = Mc = 3.3 \times 299 \text{ m/s} = 987 \text{ m/s} \quad \leftarrow V$$

The stagnation pressure ahead of the shock (designated p_{o_1}) is given by

$$p_{o_1} = p\left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)} = 0.0216\, p_{SL}\left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)}$$

$$p_{o_1} = 0.0216 \times 101.3 \text{ kPa}\left[1 + \frac{(1.4-1)}{2}(3.3)^2\right]^{\frac{1.4}{0.4}} = 125 \text{ kPa} \quad \leftarrow p_{o_1}$$

Designating the stagnation pressure behind the shock as p_{o_2}, then

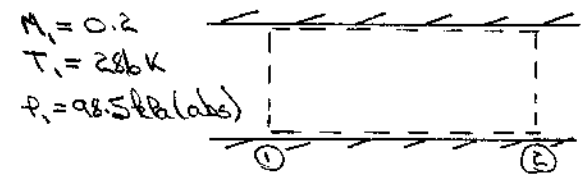
$$\frac{p_{o_1} - p_{o_2}}{p_{o_1}} = 0.747 \quad \text{or} \quad p_{o_2} = p_{o_1} - 0.747\, p_{o_1} = 0.253\, p_{o_1}$$

$$p_{o_2} = 0.253 \times 125 \text{ kPa} = 31.6 \text{ kPa} \quad \leftarrow p_{o_2}$$

The stagnation temperature does not change across a shock.

$$T_o = T\left[1 + \frac{k-1}{2}M^2\right] = 222.5\,K\left[1 + \frac{(1.4-1)}{2}(3.3)^2\right] = 707\,K \quad \leftarrow T_o$$

Given: Steady flow of air through a long insulated duct
Properties at two sections as shown.



Find: $T_{o_1}, p_{o_1}, T_{o_2}, p_{o_2}, s_2 - s_1$

Solution:

Computing equations: $\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2 \qquad \frac{p_o}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$

$$T_{o_1} = T_1\left[1 + \frac{k-1}{2} M_1^2\right] = 286K\left[1 + 0.2(0.2)^2\right] = 288\ K \quad \longleftarrow T_{o_1}$$

$$p_{o_1} = p_1\left[1 + \frac{k-1}{2} M_1^2\right]^{k/(k-1)} = 98.5\,kPa\left[1 + 0.2(0.2)^2\right]^{3.5} = 101\ kPa(abs) \quad \longleftarrow p_{o_1}$$

$$T_{o_2} = T_2\left[1 + \frac{k-1}{2} M_2^2\right] = 268.9K\left[1 + 0.2(0.6)^2\right] = 288\ K \quad \longleftarrow T_{o_2}$$

{For adiabatic flow, the energy equation gives $h_{o_1} = h_{o_2}$
Hence $T_{o_1} = T_{o_2}$}

$$p_{o_2} = p_2\left[1 + \frac{k-1}{2} M_2^2\right]^{k/(k-1)} = 31.3\,kPa\left[1 + 0.2(0.6)^2\right]^{3.5} = 39.9\ kPa \quad \longleftarrow p_{o_2}$$

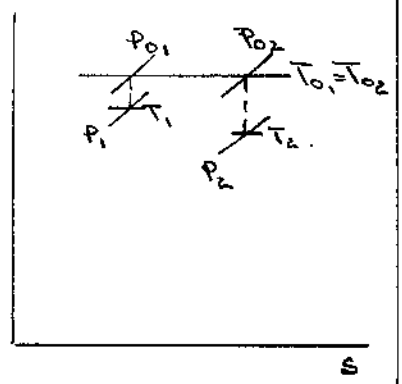
$$Tds = dh - vdp = c_p dT - RT\frac{dp}{p}$$

$$ds = c_p \frac{dT}{T} - R\frac{dp}{p}$$

$$s_2 - s_1 = s_{o_2} - s_{o_1} = -R\ln\frac{p_{o_2}}{p_{o_1}}$$

$$= -287\ \frac{J}{kg\cdot K}\ln\frac{39.92}{101.3}$$

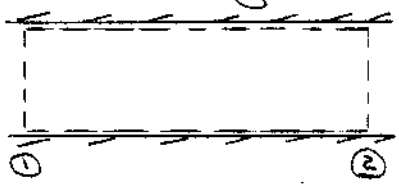
$$s_2 - s_1 = 267\ J/kg\cdot K \quad \longleftarrow s_2 - s_1$$



Conclusions: (1) T_o = constant ∴ same total energy

(2) $p_{o_2} > p_{o_1}$ ∴ reduced available energy

Given: Steady flow of air through a combustion chamber. Heat addition causes properties to change between inlet and outlet as given below.



Find: T_{o1}, p_{o1}, T_{o2}, p_{o2}, $s_2 - s_1$

Solution:

Computing equations: $\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2$ $\qquad \frac{p_o}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$

$$T_{o1} = T_1\left[1 + \frac{k-1}{2} M_1^2\right] = 580K\left[1 + 0.2(0.2)^2\right] = 585\,K$$ ← T_{o1}

$$p_{o1} = p_1\left[1 + \frac{k-1}{2} M_1^2\right]^{k/(k-1)} = 1.0MPa\left[1 + 0.2(0.2)^2\right]^{3.5} = 1.03\,MPa(abs)$$ ← p_{o1}

$$T_{o2} = T_2\left[1 + \frac{k-1}{2} M_2^2\right] = 1727K\left[1 + 0.2(0.4)^2\right] = 1782\,K$$ ← T_{o2}

$$p_{o2} = p_2\left[1 + \frac{k-1}{2} M_2^2\right]^{k/(k-1)} = 862.7\,kPa\left[1 + 0.2(0.4)^2\right]^{3.5} = 0.963\,MPa(abs)$$ ← p_{o2}

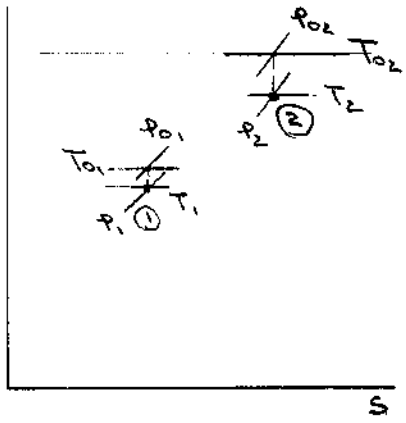
$$T ds = dh - v\,dp = c_p\,dT - RT\frac{dp}{p}$$

$$ds = c_p\frac{dT}{T} - R\frac{dp}{p}$$

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1}$$

$$= 1004\,\frac{J}{kg\cdot K}\ln\frac{1727}{580} - 286.9\,\frac{J}{kg\cdot K}\ln\frac{0.8627}{1.0}$$

$$s_2 - s_1 = 1138\,J/kg\cdot K$$ ← $s_2 - s_1$



Given: Air passes through a normal shock in a supersonic wind tunnel. Conditions upstream (state ①) and downstream (state ②) of the shock are given below.

①	②
$M_1 = 1.8$	$M_2 = 0.6165$
$T_1 = 270\,K$	$T_2 = 413.6\,K$
$p_1 = 10\,kPa\,(abs)$	$p_2 = 36.13\,kPa\,(abs)$

Find: T_{o_1}, p_{o_1}, T_{o_2}, p_{o_2}, $s_2 - s_1$

Solution:

Computing equations: $\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2$ $\qquad \frac{p_o}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/k-1}$

$$T_{o_1} = T_1\left[1 + \frac{k-1}{2} M_1^2\right] = 270K\left[1 + 0.2(1.8)^2\right] = 445\,K \longleftarrow T_{o_1}$$

$$p_{o_1} = p_1\left[1 + \frac{k-1}{2} M_1^2\right]^{k/k-1} = 10.0\,kPa\left[1 + 0.2(1.8)^2\right]^{3.5} = 57.5\,kPa\,(abs) \longleftarrow p_{o_1}$$

$$T_{o_2} = T_2\left[1 + \frac{k-1}{2} M_2^2\right] = 413.6K\left[1 + 0.2(0.6165)^2\right] = 445.K \longleftarrow T_{o_2}$$

(Flow through the shock is adiabatic, $T_{o_2} = T_{o_1}$)

$$p_{o_2} = p_2\left[1 + \frac{k-1}{2} M_2^2\right]^{k/k-1} = 36.13\,kPa\left[1 + 0.2(0.6165)^2\right]^{3.5} = 46.7\,kPa\,(abs) \longleftarrow p_{o_2}$$

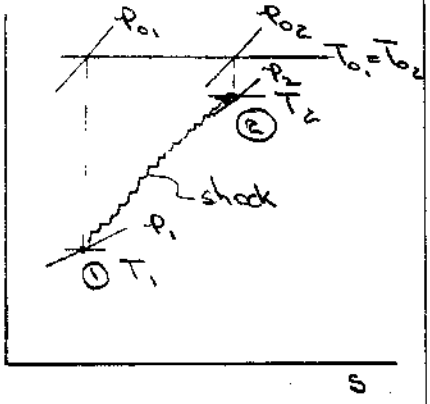
$$Tds = dh - v\,dp = C_p dT - RT\frac{dp}{p}$$

$$ds = C_p\frac{dT}{T} - R\frac{dp}{p}$$

$$s_2 - s_1 = s_{o_2} - s_{o_1} = -R\ln\frac{p_{o_2}}{p_{o_1}}$$

$$= -287\,\frac{J}{kg\cdot K}\ln\frac{46.69}{57.46}$$

$$s_2 - s_1 = 59.6\,J/kg\cdot K \longleftarrow s_2 - s_1$$



Given: Air flow through a turbine. Conditions at inlet (state ①) and outlet (state ②) are:

$M_1 = 0.4$, $T_1 = 2350°F$, $p_1 = 90.0$ psia

$M_2 = 0.8$, $T_2 = 1200°F$, $p_2 = 3.00$ psia

Find: T_{o_1}, p_{o_1}, T_{o_2}, p_{o_2}, $s_2 - s_1$

Solution:

Computing equations: $\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2$ $\qquad \frac{p_o}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$

$T_{o_1} = T_1\left(1 + \frac{k-1}{2} M_1^2\right) = (2350 + 460)°R\left[1 + 0.2(0.4)^2\right] = 2900°R$ ——— T_{o_1}

$p_{o_1} = p_1\left(1 + \frac{k-1}{2} M_1^2\right)^{k/(k-1)} = 90.0 \text{ psia}\left[1 + 0.2(0.4)^2\right]^{3.5} = 100 \text{ psia}$ ——— p_{o_1}

$T_{o_2} = T_2\left(1 + \frac{k-1}{2} M_2^2\right) = (1200 + 460)°R\left[1 + 0.2(0.8)^2\right] = 1870°R$ ——— T_{o_2}

$p_{o_2} = p_2\left(1 + \frac{k-1}{2} M_2^2\right)^{k/(k-1)} = 3.00 \text{ psia}\left[1 + 0.2(0.8)^2\right]^{3.5} = 4.57 \text{ psia}$ ——— p_{o_2}

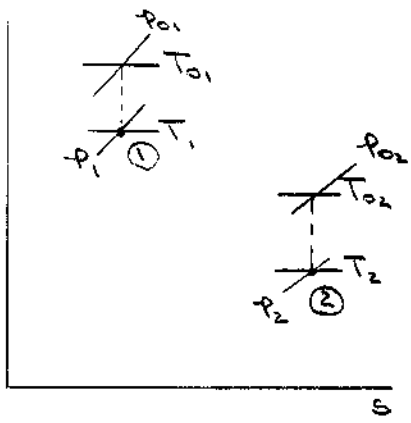
$T ds = dh - v\,dp = c_p\,dT - RT\frac{dp}{p}$

$ds = c_p \frac{dT}{T} - R\frac{dp}{p}$

$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1}$

$= 0.24 \frac{\text{Btu}}{\text{lbm}\cdot°R} \ln\frac{1660}{2810} - 53.3 \frac{\text{ft}\cdot\text{lbf}}{\text{lbm}\cdot°R} \cdot \frac{\text{Btu}}{778\ \text{ft}\cdot\text{lbf}} \ln\frac{13}{90}$

$s_2 - s_1 = 0.107 \frac{\text{Btu}}{\text{lbm}\cdot°R}$ ——— $s_2 - s_1$



Given: A jet transport cruises at altitude $z = 12.0$ km on a standard day

Find: percent error in speed calculated by incompressible Bernoulli equation over the Mach number range $0.1 \le M \le 0.9$

Solution:

At altitude $z = 12.0$ km (From Table A.3)

$T = 216.7$ K $\quad p/p_{SL} = 0.1915 \quad \rho/\rho_{SL} = 0.2546$

$\therefore p = 0.1915\, p_{SL} = 0.1915 \times 101.3$ kPa $= 19.4$ kPa

$\rho = 0.2546\, \rho_{SL} = 0.2546 \times 1.225$ kg/m^3 $= 0.3119$ kg/m^3

$$p_0 = p\left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)} = 19.4\left[1 + 0.2M^2\right]^{3.5} \text{ kPa} \quad \text{.....(1)}$$

$$V_{act} = Mc = M(kRT)^{1/2} = M\left(1.4 \times 2869 \frac{\text{N}}{\text{kg}\cdot\text{K}} \times 216.7\text{K} \times \frac{\text{kg}\cdot\text{m}/\text{s}^2}{\text{N}}\right)^{1/2}$$

$$V_{act} = 295 \cdot M \text{ (m/s)} \quad \text{.....(2)}$$

$$V_{p=c} = \left[\frac{2}{\rho/\rho}(p_0 - p)\right]^{1/2} = \left[\frac{2}{\rho/\rho}\left\{p\left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)} - p\right\}\right]^{1/2}$$

$$V_{p=c} = \left[\frac{2p}{\rho}\right]^{1/2}\left\{\left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)} - 1\right\}^{1/2}$$

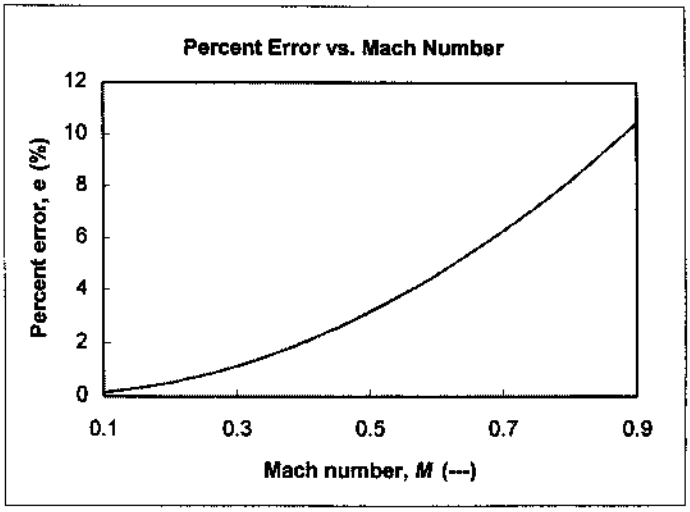
$$= \left[2 \times 19.4 \times 10^3 \frac{\text{N}}{\text{m}^2} \times 0.3119^{-1} \frac{\text{m}^3}{\text{kg}} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right]^{1/2}\left\{\left[1 + 0.2M^2\right]^{3.5} - 1\right\}^{1/2}$$

$$V_{p=c} = 352.7\left\{\left[1 + 0.2M^2\right]^{3.5} - 1\right\}^{1/2} \text{ (m/s)} \quad \text{.....(3)}$$

$$\%\,\text{Error} = \frac{V_{p=c} - V_{act}}{V_{act}} \times 100\% \quad \text{.....(4)}$$

The plot of % error vs Mach number is given below

M (---)	V_{actual} (m/s)	$V_{p=c}$ (m/s)	% Error (---)
0.1	29.5	29.5	0.2
0.2	59.0	59.3	0.5
0.3	88.5	89.5	1.2
0.4	118	120	2.0
0.5	148	152	3.2
0.6	177	185	4.6
0.7	207	219	6.3
0.8	236	255	8.2
0.9	266	293	10.4



Problem 11.46

Given: Compressible flow field.

Find: (a) Plot of percentage change in density $(\rho_0 - \rho)/\rho$ as a function of M.
(b) Values of M for which percentage change is less than 5 percent.

Solution: Assume ideal gas behavior, with $k = 1.4$.

Computing equation: $$\frac{\rho_0}{\rho} = \left[1 + \frac{k-1}{2} M^2\right]^{\frac{1}{k-1}} \qquad (11.17c)$$

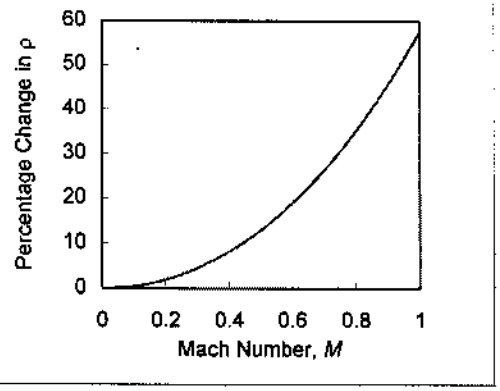
Rearranging, $$\frac{\rho_0 - \rho}{\rho} = \frac{\rho_0}{\rho} - 1 = \left[1 + \frac{k-1}{2} M^2\right]^{\frac{1}{k-1}} - 1$$

With $k = 1.4$, $$\frac{\rho_0}{\rho} - 1 = \left[1 + 0.2 M^2\right]^{2.5} - 1$$

The percentage change is $$\left(\frac{\rho_0}{\rho} - 1\right) 100 = 100\left\{\left[1 + 0.2 M^2\right]^{2.5} - 1\right\}$$

Tabulating and plotting:

Mach Number	Percent Change in ρ
0	0.00
0.1	0.50
0.2	2.01
0.3	4.56
0.314	5.00
0.4	8.19
0.5	13.0
0.6	19.0
0.7	26.3
0.8	35.1
0.9	45.6
1.0	57.7



Plot

By iteration, $\left(\frac{\rho_0}{\rho} - 1\right) \times 100$ is less than 5% for $0 \le M < 0.314$

<5%

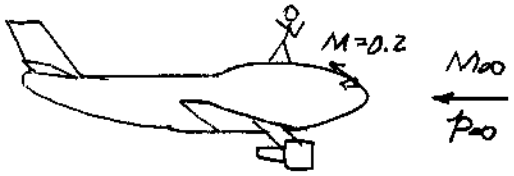
Given: Boeing 747 cruising at $M = 0.87$ at $z = 13$ km, std. day.
Window located where $M = 0.2$ relative to surface.
Cabin pressurized to equivalent of $z = 2.5$ km, std. day.

Find: Pressure difference across window.

Solution: Apply isentropic stagnation relations.

Computing equation: $p_0 = p\left(1 + \frac{k-1}{2}M^2\right)^{k/k-1}$

Assumptions: (1) Ideal gas
(2) Isentropic flow



Consider observer on aircraft: air is decelerated isentropically from $M_\infty = 0.87$ to $M = 0.2$.

From Table A.3:

Altitude (km)	p/p_0 (---)	p (kPa)
2.5	0.7372	74.7
13.0	0.1636	16.6

Calculated: $p = \left(\frac{p}{p_0}\right) p_0$

$p_0 = 101.3$ kPa

For isentropic stagnation:

$$p_0 = p_\infty \left(1 + \frac{k-1}{2}M_\infty^2\right)^{k/k-1} = 16.6 \text{ kPa}\left(1 + \frac{1.4-1}{2}(0.87)^2\right)^{3.5} = 27.2 \text{ kPa (abs)}$$

From stagnation to $M = 0.2$:

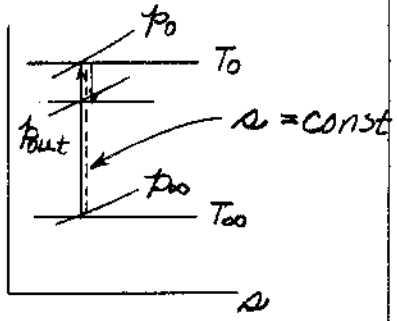
$$p_{out} = \frac{p_0}{\left(1 + \frac{k-1}{2}M^2\right)^{k/k-1}} = \frac{27.2 \text{ kPa}}{\left(1 + 0.2(0.2)^2\right)^{3.5}} = 26.5 \text{ kPa (abs)}$$

Pressure difference across window is:

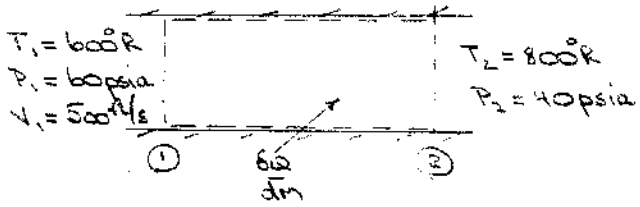
$$\Delta p = p_{in} - p_{out} = (74.7 - 26.5) \text{ kPa} = 48.2 \text{ kPa}$$ Δp

{Inside pressure is higher; window force is toward outside.}

The corresponding Ts diagram is:



Given: Steady flow of air in a constant area duct, with friction and heat transfer, as shown:



Find: $\frac{\delta Q}{dm}$, P_{o_2}

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$; $\rho VA = \text{const}$; $P = \rho RT$

Computing equation: $\frac{P_o}{P} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$

Assumptions: (1) steady flow (4) $\Delta E = 0$
(2) uniform flow at each section (5) ideal gas
(3) $W_s = W_{shear} = 0$

$$\rho_1 = \frac{P_1}{RT_1} = 60 \frac{lbf}{in^2} \times \frac{lbm \cdot °R}{53.3\ ft \cdot lbf} \times \frac{1}{600°R} \times \frac{144\ in^2}{ft^2} = 0.270\ lbm/ft^3$$

$$\rho_2 = \frac{P_2}{RT_2} = 40 \frac{lbf}{in^2} \times \frac{lbm \cdot °R}{53.3\ ft \cdot lbf} \times \frac{1}{800°R} \times \frac{144\ in^2}{ft^2} = 0.135\ lbm/ft^3$$

$\rho_1 V_1 A = \rho_2 V_2 A$ and $V_2 = \frac{\rho_1}{\rho_2} V_1 = \frac{0.270}{0.135} \times 500 \frac{ft}{s} = 1000 \frac{ft}{s}$

$$\frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2} - h_1 - \frac{V_1^2}{2} = c_p (T_2 - T_1) + \frac{V_2^2 - V_1^2}{2}$$

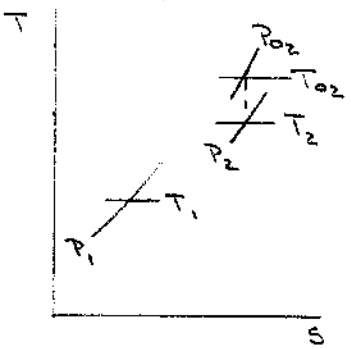
$$= 0.24 \frac{Btu}{lbm \cdot °R} \times (800 - 600)°R + \frac{1}{2}\left[(1000)^2 - (500)^2\right] \frac{ft^2}{s^2} \times \frac{lbf \cdot s^2}{ft \cdot slug} \times \frac{slug}{32.2\ lbm} \times \frac{Btu}{778\ ft \cdot lbf}$$

$$\frac{\delta Q}{dm} = 63.0\ Btu/lbm$$ $\frac{\delta Q}{dm}$

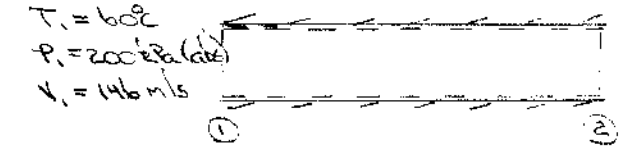
$M_2 = \frac{V_2}{c_2}$; $c_2 = (kRT_2)^{1/2} = (1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot °R} \times 800°R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft/s^2}{lbf})^{1/2} = 1390\ ft/s$

$$M_2 = \frac{1000}{1390} = 0.719$$

$P_o/P = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$; $P_{o_2} = P_2\left[1 + \frac{k-1}{2} M_2^2\right]^{k/(k-1)} = 40\ psia \left[1 + 0.2(0.719)^2\right]^{3.5} = 56.5\ psia$ P_{o_2}



Given: Adiabatic flow through long straight pipe of cross-sectional area, $A = 0.05\ m^2$, as shown. Fluid is air.



Find: $p_{o_1}, p_{o_2}, T_{o_1}, T_{o_2}, s_2 - s_1$

Solution:

Computing equations: $\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2$ $\qquad \frac{p_o}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$

Basic equations: $0 = \frac{\partial}{\partial t}\int_{cv} \rho\, d\forall + \int_{cs} \rho \vec{V}\cdot d\vec{A}$

$\dot{Q} + \dot{W}_s + \dot{W}_{shear} = \frac{\partial}{\partial t}\int_{cv} e\rho\, d\forall + \int_{cs}\left(e + \frac{p}{\rho}\right)\rho \vec{V}\cdot d\vec{A}$

Assumptions: (1) $\dot{Q} = 0$ (adiabatic flow) (5) uniform flow at each section
(2) $\dot{W}_s = 0$ (6) $\Delta z = 0$
(3) $\dot{W}_{shear} = 0$ (7) ideal gas, $k = 1.4$
(4) steady flow (8) $A_1 = A_2 = A$ = constant

$M_1 = \frac{V_1}{c_1}$ $\qquad c_1 = (kRT_1)^{1/2} = \left(1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 333K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 366\ m/s$

$M_1 = \frac{V_1}{c_1} = \frac{146}{366} = 0.399$ $\qquad T_{o_1} = T_1\left[1 + \frac{k-1}{2} M_1^2\right] = 333K\left[1 + 0.2(0.399)^2\right] = 344K$ ← T_{o_1}

$p_{o_1} = p_1\left[1 + \frac{k-1}{2} M_1^2\right]^{k/(k-1)} = 200kPa\left[1 + 0.2(0.399)^2\right]^{3.5} = 223kPa$ ← p_{o_1}

From the energy equation, $0 = \left(u_1 + p_1v_1 + \frac{V_1^2}{2}\right)\{-|\rho_1 V_1 A|\} + \left(u_2 + p_2v_2 + \frac{V_2^2}{2}\right)\{|\rho_2 V_2 A|\}$

From continuity, $0 = -|\rho_1 V_1 A| + |\rho_2 V_2 A|$ or $\rho_1 V_1 = \rho_2 V_2$

Then, using $h = u + pv$,

$h_1 + \frac{V_1^2}{2} = h_{o_1} = h_2 + \frac{V_2^2}{2} = h_{o_2}$ or $h_{o_1} = h_{o_2}$

For an ideal gas with constant specific heats, $T_{o_2} = T_{o_1} = 344\ K$ ← T_{o_2}

From continuity, $\rho_2 = \rho_1 \frac{V_1}{V_2} = \frac{p_1}{RT_1}\frac{V_1}{V_2} = 200\times10^3 \frac{N}{m^2} \times \frac{kg\cdot K}{287\ N\cdot m} \times \frac{1}{333K} \times \frac{146}{280} = 1.09\ kg/m^3$

Then, $T_2 = \frac{p_2}{\rho_2 R} = \frac{95.6\times10^3 \frac{N}{m^2}}{1.09\ kg/m^3} \times \frac{kg\cdot K}{287\ N\cdot m} = 306K\ (33°C)$

$c_2 = (kRT_2)^{1/2} = \left(1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 306K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 351\ m/s$

$M_2 = \frac{V_2}{c_2} = \frac{280}{351} = 0.798$

$p_{o_2} = p_2\left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)} = 95.6kPa\left[1 + 0.2(0.798)^2\right]^{3.5}$

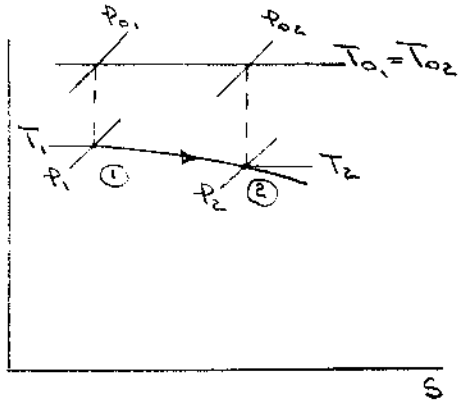
$p_{o_2} = 145\ kPa$ ← p_{o_2}

$T\,ds = dh - v\,dp = c_p\,dT - \frac{RT}{p}\,dp$

$ds = c_p\frac{dT}{T} - R\frac{dp}{p} = ds_o = -R\frac{dp_o}{p_o}$

$s_{o_2} - s_{o_1} = s_2 - s_1 = -R\ln\frac{p_{o_2}}{p_{o_1}}$

$s_2 - s_1 = -287\frac{J}{kg\cdot K}\ln\frac{145}{223} = 0.124\frac{kJ}{kg\cdot K}$ ← $s_2 - s_1$



Given: Aircraft speed is calculated from measured values of dynamic pressure
Assume aircraft flies at altitude $z = 10$ km on a standard day

Find: (a) the subsonic Mach number above which the incompressible form of the Bernoulli equation predicts a speed error of 2 percent compared to the true air speed calculated including compressibility effects
(b) Is the result independent of free stream conditions

Solution:

For incompressible flow $p_0 = p + \frac{1}{2}\rho V^2$

$$\therefore V_i = \left[\frac{2(p_0 - p)}{\rho}\right]^{1/2}$$

For an ideal gas $p = \rho RT$, and

$$V_i = \left[\frac{2RT\,(p_0 - p)}{p}\right]^{1/2} = \left[\frac{2}{k}\,kRT\left(\frac{p_0}{p} - 1\right)\right]^{1/2} = c\left[\frac{2}{k}\left(\frac{p_0}{p} - 1\right)\right]^{1/2}$$

$$M_i = \frac{V_i}{c} = \left[\frac{2}{k}\left(\frac{p_0}{p} - 1\right)\right]^{1/2}$$

Then $$\frac{M_i}{M} = \left[\frac{2}{kM^2}\left(\frac{p_0}{p} - 1\right)\right]^{1/2}$$

But $$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)}$$

$$\therefore \frac{M_i}{M} = \left\{\frac{2}{kM^2}\left[\left(1 + \frac{k-1}{2}M^2\right)^{k/(k-1)} - 1\right]\right\}^{1/2}$$

For $k = 1.4$

$$\frac{M_i}{M} = \left\{\frac{1.4286}{M^2}\left[\left(1 + 0.2M^2\right)^{3.5} - 1\right]\right\}^{1/2}$$

M	M_i/M	
0.1	1.0013	
0.2	1.0050	
0.3	1.0113	← M = 0.3 for speed accuracy of 1%
0.4	1.0201	← M = 0.4 for speed accuracy of 2%

Result is independent of free stream conditions.

Given: A CO_2 cartridge contains 276 mL, $P_c = 45$ MPa (gage) and $T_o = 25°C$

Find: T^*, p^*, V^* that correspond to these stagnation conditions

Solution:

Computing equations: $\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2$ $\qquad \frac{p_o}{p} = \left(1 + \frac{k-1}{2} M^2\right)^{k/(k-1)}$

For CO_2, $k = 1.29$. At critical conditions, $M = 1$

$$\frac{T_o^*}{T^*} = 1 + \frac{k-1}{2} = 1.145 \qquad \therefore T^* = \frac{T_o^*}{1.145} = \frac{298\,K}{1.145} = 260\,K \leftarrow T^*$$

$$\frac{p_o^*}{p^*} = \left[1 + \frac{k-1}{2}\right]^{k/(k-1)} = \left[1.145\right]^{4.444} = 1.826$$

$$\therefore p^* = \frac{p_o^*}{1.826} = \frac{45.101\ MPa}{1.826} = 24.7\ MPa\ (abs) \leftarrow p^*$$

$$V^* = c^* = (kRT^*)^{1/2} = \left(1.29 \times 189 \frac{N \cdot m}{kg \cdot K} \times 260\,K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 252\ m/s \leftarrow V^*$$

Given: A gas storage reservoir contains helium at $T_o = 250°K$ and $p_o = 6.0$ MPa (gage)

Find: T^*, p^*, V^* that correspond to these stagnation conditions.

Solution:

Governing equations: $\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2$ $\quad \frac{p_o}{p} = \left(1 + \frac{k-1}{2} M^2\right)^{k/k-1}$

For helium, $k = 1.66$. At critical conditions, $M = 1$

$$\frac{T_o^*}{T^*} = 1 + \frac{k-1}{2} = 1.33 \qquad \therefore T^* = \frac{T_o^*}{1.33} = \frac{2500K}{1.33} = 1880\ K \quad \leftarrow T^*$$

$$\frac{p_o^*}{p^*} = \left[1 + \frac{k-1}{2}\right]^{k/k-1} = [1.33]^{2.5152} = 2.0488$$

$$\therefore p^* = \frac{p_o^*}{2.0488} = \frac{6.101\ MPa}{2.0488} = 2.98\ MPa\ (abs) \quad \leftarrow p^*$$

$$V^* = c^* = (kRT^*)^{1/2} = \left(1.66 \times 2077 \frac{N \cdot m}{kg \cdot K} \times 1880K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 2550\ m/s \quad \leftarrow V^*$$

Given: Stagnation conditions in a solid-propellant rocket motor are $T_0 = 3500K$ and $p_0 = 40 MPa$ (gage). Assume ideal gas behavior with $R = 323 J/kg \cdot K$, $k = 1.2$. The Mach number is unity at the throat of the nozzle.

Find: T_t, p_t, V_t

Solution:

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$ $\qquad \frac{p_0}{p} = \left(1 + \frac{k-1}{2} M^2\right)^{k/(k-1)}$

Assume flow to throat is isentropic.

At throat, $M = 1$

$$T_{0t} = T_t \left(1 + \frac{k-1}{2}\right) = 1.1\, T_t \quad \therefore T_t = \frac{T_0}{1.1} = \frac{3500K}{1.10} = 3180K \longleftarrow T_t$$

$$\frac{p_{0t}}{p_t} = \left(1 + \frac{k-1}{2}\right)^{k/(k-1)} = (1.1)^{1.2/0.2} = 1.7716$$

$$\therefore p_t = \frac{p_0}{1.7716} = \frac{40.101\ MPa}{1.7716} = 22.6 MPa (abs) \longleftarrow p_t$$

$$V_t = (kRT_t)^{1/2} = \left[1.2 \cdot 323 \frac{N \cdot m}{kg \cdot K} \cdot 3180K \cdot \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 1110 m/s \longleftarrow V_t$$

Given: The hot gas stream (with thermodynamic properties of air) at the turbine inlet of a JT9-1 jet engine is at $T = 2350°F$ and $p = 140\ kPa$ (abs). The Mach number at that location is $M = 0.32$

Find: T^*, p^*, V^* that correspond to these conditions.

Solution:

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$ $\qquad \frac{p_0}{p} = \left(1 + \frac{k-1}{2} M^2\right)^{k/(k-1)}$

We begin by computing the stagnation conditions at the turbine inlet

$T_0 = T\left(1 + \frac{k-1}{2} M^2\right) = 2810°R\left[1 + 0.2(0.32)^2\right] = 2868°R$

$p_0 = p_1\left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)} = 140\ kPa\left[1 + 0.2(0.32)^2\right]^{3.5} = 150\ kPa$

We now find the critical conditions that correspond to these stagnation conditions.

At critical conditions, $M = 1$

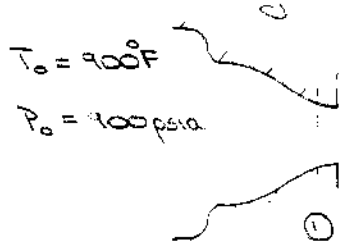
$\frac{T_0^*}{T^*} = 1 + \frac{k-1}{2} = 1.2 \quad \therefore T^* = \frac{T_0^*}{1.2} = \frac{2868°R}{1.2} = 2390°R$ ← T^*

$\frac{p_0^*}{p^*} = \left[1 + \frac{k-1}{2}\right]^{k/(k-1)} = \left[1.2\right]^{3.5} = 1.8929$

$\therefore p^* = \frac{p_0^*}{1.8929} = \frac{150\ kPa}{1.8929} = 79.2\ kPa\ (abs)$ ← p^*

$V^* = c^* = (kRT^*)^{1/2} = \left(1.4 \times 53.3\ \frac{ft\text{-}lb}{lbm°R} \times 2390°R \times 32.2\ \frac{lbm}{slug} \times \frac{slug\text{-}ft/s^2}{lb\text{-}ft}\right)^{1/2} = 2400\ ft/s$ ← V^*

Given: Steam flows steadily and isentropically through a nozzle



Find: V_1, M_1, $\dot{m}$, passage shape

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} = h_2 + \frac{V^2}{2} = h_o = \text{constant}$

$$\rho AV = \text{constant} = \dot{m}$$

Assumptions: (1) steady flow
(2) isentropic flow
(3) uniform flow at a section
(4) superheated steam can be treated as an ideal gas, $R = 85.8$ ft·lbf/lbm·°R, $k = 1.28$
(5) $\Delta z = 0$

From steam tables for superheated vapor with $T_o = 900°F$, $P_o = 900$ psia,
$h_o = 1451.8$ Btu/lbm $\quad s_o = 1.6257$ Btu/lbm·°R

From steam tables for superheated vapor with $s_1 = 1.6257$ Btu/lbm·°R, and $P_1 = 600$ psia,
$T_1 = 780°F \quad h_1 = 1396.5$ Btu/lbm, $v_1 = 1.167$ ft³/lbm

From the first law,

$$V_1 = \left[2(h_o - h_1)\right]^{1/2} = \left[2(1451.8 - 1396.5)\frac{\text{Btu}}{\text{lbm}} \times 778\frac{\text{ft·lbf}}{\text{Btu}} \times 32.2\frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug·ft}}{\text{lbf·s}^2}\right]^{1/2}$$

$$V_1 = 1660 \text{ ft/s} \longleftarrow V_1$$

$$\dot{m} = \rho AV = \frac{1}{v}\pi\left(\frac{D}{4}\right)^2 V = \frac{1}{1.167\text{ ft}^3} \times \frac{\pi}{4} \times \left(\frac{0.188}{12}\text{ ft}\right)^2 \times 1660\frac{\text{ft}}{\text{s}} = 0.274 \text{ lbm/s} \longleftarrow \dot{m}$$

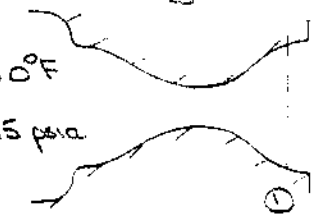
$$M_1 = \frac{V_1}{c_1} \qquad c_1 = (kRT_1)^{1/2} = \left(1.30 \times 85.8\frac{\text{ft·lbf}}{\text{lbm·°R}} \times 1240\text{°R} \times 32.2\frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug·ft}}{\text{lbf·s}^2}\right)^{1/2}$$

$$c_1 = 2110 \text{ ft/s}$$

$$M_1 = \frac{V_1}{c_1} = \frac{1660}{2110} = 0.787 \longleftarrow$$

Since $M_1 < 1.0$, passage converges as shown above

Given: Steam flows steadily and isentropically through a nozzle.



$P_1 = 290$ psia
$D_1 = 0.50$ in
(nozzle shape unspecified)

Find: V_1, M_1, $\dot{m}$, passage shape

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} = h + \frac{V^2}{2} = h_o = \text{constant}$

$\rho AV = \text{constant} = \dot{m}$

Assumptions: (1) steady flow
(2) isentropic flow
(3) uniform flow at a section
(4) $\Delta z = 0$
(5) superheated steam can be treated as an ideal gas, $R = 85.8$ ft·lbf/lbm°R, $k = 1.28$

From steam tables for superheated vapor with $T_o = 880°F$, $P_o = 875$ psia,

$h_o = 1441.4$ Btu/lbm $\quad s_o = 1.6209$ Btu/lbm°R

From steam tables for superheated vapor with $P_1 = 290$ psia, $s_1 = 1.6209$ Btu/lbm°R

$T_1 = 580°F \quad h_1 = 1304.5$ Btu/lbm $\quad v_1 = 2.031$ ft³/lbm

From the first law,

$$V_1 = \left[2(h_o - h_1)\right]^{1/2} = \left[2(1441.4 - 1304.5)\frac{\text{Btu}}{\text{lbm}} \times 778\frac{\text{ft·lbf}}{\text{Btu}} \times 32.2\frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug·ft}}{\text{lbf·s}^2}\right]^{1/2}$$

$V_1 = 2620$ ft/sec ← V_1

From continuity,

$$\dot{m} = \rho AV = \frac{1}{v_1}\pi\left(\frac{D_1}{2}\right)^2 V$$

$$\dot{m} = \frac{\text{lbm}}{2.031\ \text{ft}^3} \times \frac{\pi}{4} \times \left(\frac{0.50}{12}\right)^2 \text{ft}^2 \times 2620\frac{\text{ft}}{\text{s}} = 1.76\ \text{lbm/s}$$ ← $\dot{m}$

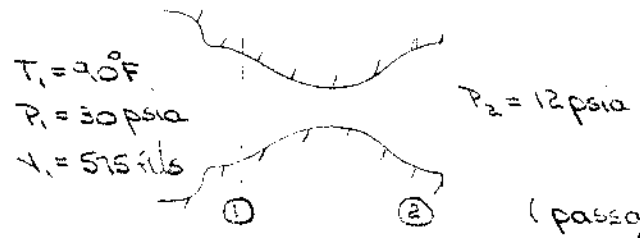
$M_1 = \frac{V_1}{c_1}$ $\qquad c_1 = (RkT_1)^{1/2} = \left(1.30 \times 85.8\frac{\text{ft·lbf}}{\text{lbm°R}} \times 1040°R \times 32.2\frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug·ft}}{\text{lbf·s}^2}\right)^{1/2}$

$c_1 = 1930$ ft/s

$M_1 = \frac{V_1}{c_1} = \frac{2620}{1930} = 1.36$ ← M_1

Since $M_1 > 1.0$, passage is converging-diverging as shown above. /29.5

Given: Isentropic flow of air through a passage



(passage shape unspecified)

Find: M_2, passage shape

Solution

Computing equations: $\frac{P_0}{P} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$ $\qquad c = \sqrt{kRT}$

Assumptions: (1) steady flow
(2) isentropic flow
(3) uniform flow at a section
(4) ideal gas

$M_1 = \frac{V_1}{c_1}$ $\qquad c_1 = (kRT)^{1/2} = \left(1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot °R} \times 550°R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right)^{1/2}$

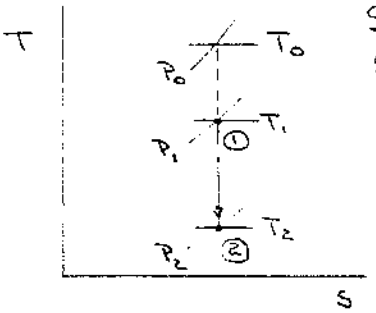
$c_1 = 1150$ ft/s

$M_1 = \frac{V_1}{c_1} = \frac{575}{1150} = 0.5$

$P_{0_1} = P_1 \left[1 + \frac{k-1}{2} M_1^2\right]^{k/(k-1)} = 30 \text{ psia} \left[1 + 0.2(0.5)^2\right]^{3.5} = 35.6$ psia

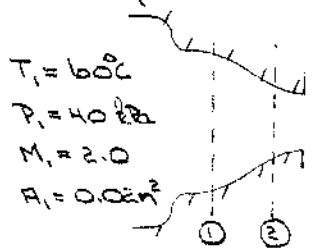
$\frac{P_{0_2}}{P_2} = \left[1 + \frac{k-1}{2} M_2^2\right]^{k/(k-1)}$ $\qquad P_{0_2} = P_{0_1}$

$M_2 = \left\{\frac{2}{k-1}\left[\left(\frac{P_{0_1}}{P_2}\right)^{(k-1)/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{35.6}{12}\right)^{0.286} - 1\right]\right\}^{1/2} = 1.35$ ← M_2



Since $M_2 > 1$, passage must be converging - diverging as shown

Given: Steady, isentropic flow of air through a passage



Find: M_2, shape of passage

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$

Assumptions: (1) steady flow (4) $\Delta g = 0$
(2) isentropic flow (5) ideal gas
(3) uniform flow at a section

$M_2 = \frac{V_2}{c_2}$ where $c_2 = (kRT_2)^{1/2}$. Hence T_2 must be found

$h_2 = h_1 + \frac{1}{2}(V_1^2 - V_2^2)$

$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 2.0 \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 333 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 732$ m/s

$T_2 = T_1 + \frac{1}{2c_p}(V_1^2 - V_2^2)$

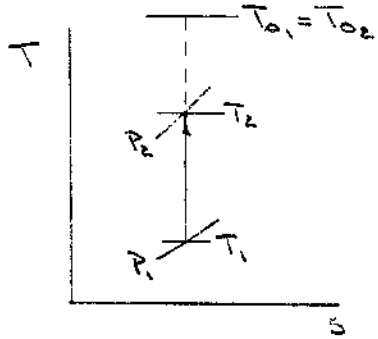
$= 333K + \left[(732)^2 - (519)^2\right] \frac{m^2}{s^2} \times \frac{1}{2} \times \frac{kg \cdot K}{1.0 \times 10^3 J} \times \frac{N \cdot s^2}{kg \cdot m}$

$T_2 = 466$ K

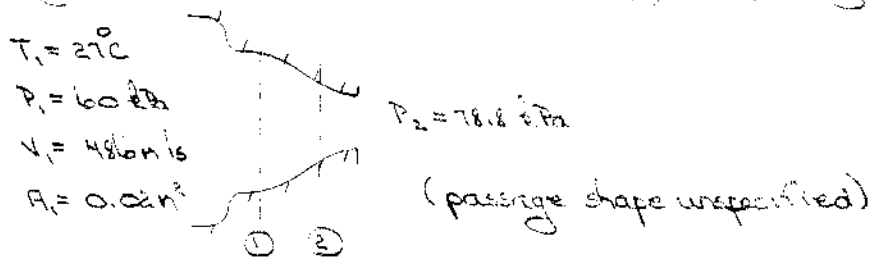
$c_2 = (kRT_2)^{1/2} = \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 466 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 432$ m/s

$M_2 = \frac{V_2}{c_2} = \frac{519}{432} = 1.20$ ← M_2

Since $M_2 < M_1$ and $M_2 > 1.0$, then passage from ① to ② is a supersonic diffuser as shown above



Given: Steady, isentropic flow of air through a passage



Find: M_2

Solution:

Computing equations: $\frac{P_0}{P} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$ $\quad c = \sqrt{kRT}$

Assumptions: 1) Steady flow 3) uniform flow at a section
2) Isentropic flow 4) Ideal gas

For isentropic flow, $P_{0_1} = P_{0_2} = P_0 = \text{constant}$

$M_1 = \frac{V_1}{c_1}$ $\quad c_1 = (kRT_1)^{1/2} = \left(1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 300K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 347$ m/s

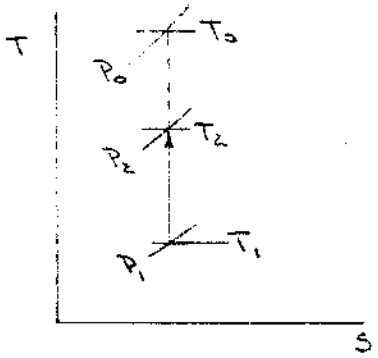
$M_1 = \frac{V_1}{c_1} = \frac{486}{347} = 1.40$ $\quad P_{0_1} = P_1\left[1 + \frac{k-1}{2} M_1^2\right]^{k/(k-1)} = 60 \text{ kPa}\left[1 + 0.2(1.40)^2\right]^{3.5}$

$P_{0_1} = 191$ kPa

$\frac{P_{0_2}}{P_2} = \left[1 + \frac{k-1}{2} M_2^2\right]^{k/(k-1)}$ $\quad P_{0_2} = P_{0_1}$

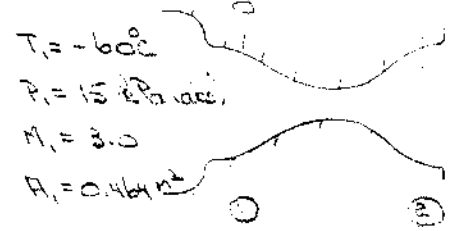
$M_2 = \left\{\frac{2}{k-1}\left[\left(\frac{P_{0_1}}{P_2}\right)^{(k-1)/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{191}{78.8}\right)^{0.286} - 1\right]\right\}^{1/2} = 1.20$ ← M_2

Since $M_2 < M_1$ and $M_2 > 1.0$, then passage from ① to ② is a supersonic diffuser as shown above



Problem 12.6

Given: Steady, isentropic flow of air through a passage at a rate $\dot{m} = 100$ kg/sec



Find: V_2, A_2, passage shape

Solution:

Basic equations: $\dot{m} = \rho V A = \text{constant}$ $\quad P = \rho R T$

Computing equations: $\dfrac{T_0}{T} = 1 + \dfrac{k-1}{2} M^2$ $\quad c = \sqrt{kRT}$

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2} M^2}{1 + \frac{k-1}{2}}\right]^{(k+1)/2(k-1)} \qquad (12.6)$$

Assumptions: 1) steady flow 2) isentropic flow 3) uniform flow at a section 4) ideal gas

$$T_{0_1} = T_1\left(1 + \frac{k-1}{2} M_1^2\right) = 213\ K\,(1 + 0.2(3.0^2)) = 596\ K$$

$$\frac{T_{0_2}}{T_2} = \left(1 + \frac{k-1}{2} M_2^2\right) \qquad T_{0_2} = T_{0_1} \quad \text{Hence}$$

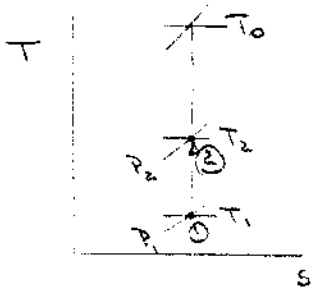
$$M_2 = \left\{\frac{2}{k-1}\left[\frac{T_{0_2}}{T_2} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\frac{596}{111} - 1\right]\right\}^{1/2} = 1.50$$

$$V_2 = M_2 c_2 = M_2 \sqrt{kRT_2} = 1.50 \sqrt{1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 411 K \times \frac{kg\cdot m}{N\cdot s^2}} = 610\ m/sec \quad \leftarrow V_2$$

For $M_1 = 3.0$, from Eq 12.6, $A_1/A^* = 4.235$ $\therefore A^* = 0.1096$

For $M_2 = 1.5$, from Eq 12.6, $A_2/A^* = 1.176$ $\therefore A_2 = 0.129\ m^2$ $\quad \leftarrow A_2$

Since $M_2 > 1.0$, section ② must be ahead of the throat of the supersonic diffuser sketched above



Problem 12.7

Given: Isentropic flow of air through a passage with $\dot{m} = 1.0$ kg/s
Stagnation conditions: $T_o = 5°C$ $P_o = 304$ kPa (abs.)
Exhaust is to atmospheric pressure, $P_e = 101$ kPa

Find: (a) Exit area, A_e
(b) Sketch M and P vs distance along the nozzle

Solution:

Basic equation: $\dot{m} = \rho A V = \text{constant}$, $P = \rho R T$

Computing equations: $\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2$

$\frac{P_o}{P} = \left[1 + \frac{k-1}{2} M^2\right]^{k/k-1}$ $\quad c = \sqrt{kRT}$

Assumptions: 1) steady flow
2) isentropic flow
3) uniform flow at a section
4) ideal gas

$\frac{P_o}{P_e} = \left[1 + \frac{k-1}{2} M_e^2\right]^{k/k-1}$ $\quad M_e = \left\{\frac{2}{k-1}\left[\left(\frac{P_o}{P_e}\right)^{k-1/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{304}{101}\right)^{0.286} - 1\right]\right\}^{1/2}$

$M_e = 1.36$

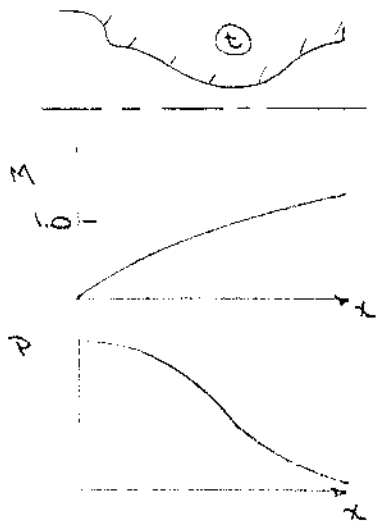
Since $M_e > 1.0$, passage must be converging-diverging nozzle

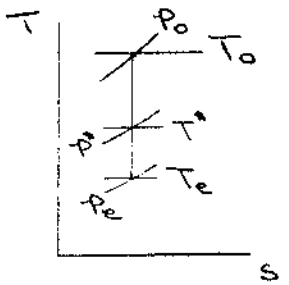
$\frac{T_o}{T_e} = 1 + \frac{k-1}{2} M_e^2$ $\quad T_e = \frac{T_o}{1 + \frac{k-1}{2} M_e^2} = \frac{278\ K}{1 + 0.2(1.36)^2} = 203\ K$

$\rho_e = \frac{P_e}{RT_e} = 101 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287\ N \cdot m} \times \frac{1}{203 K} = 1.73\ kg/m^3$

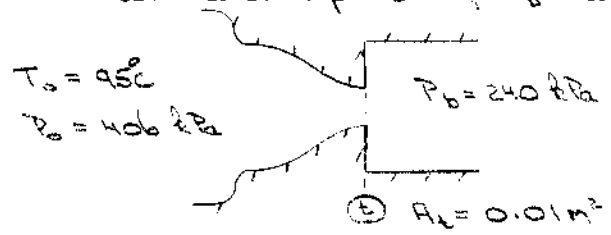
$V_e = M_e C_e = M_e (kRT_e)^{1/2} = 1.36 \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 203 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 388\ m/s$

$\dot{m} = \rho_e V_e A_e$ $\quad A_e = \frac{\dot{m}}{\rho_e V_e} = 1.0 \frac{kg}{s} \times \frac{m^3}{1.73\ kg} \times \frac{s}{388\ m} = 1.49 \times 10^{-3}\ m^2$ ← A_e





Given: Isentropic flow of air through a converging nozzle discharges into a receiver with a back pressure, $P_b = 240$ kPa



Find: $\dot{m}$

Solution

Basic equations: $\dot{m} = \rho V A = \text{const}$ $\quad P = \rho RT$

Computing equations: $\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2$ $\quad \frac{P_o}{P} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$

Assumptions: (1) steady flow (3) uniform flow at a section
(2) isentropic flow (4) ideal gas

Since $P_b/P_o = 0.591 > 0.528$, the nozzle is not choked and $P_2 = P_b = 240$ kPa

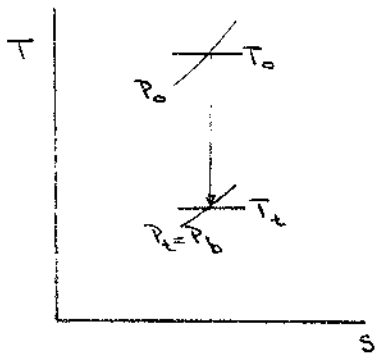
$$\frac{P_o}{P_2} = \left[1 + \frac{k-1}{2} M_2^2\right]^{\frac{k}{k-1}} \qquad M_2 = \left\{\frac{2}{k-1}\left[\left(\frac{P_o}{P_2}\right)^{(k-1)/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{406}{240}\right)^{0.286} - 1\right]\right\}^{1/2} = 0.90$$

$$\frac{T_o}{T_2} = 1 + \frac{k-1}{2} M^2 \qquad T_2 = \frac{T_o}{1 + \frac{k-1}{2} M_2^2} = \frac{368\ K}{1 + 0.2(0.90)^2} = 317\ K$$

$$V_2 = M_2 c_2 = M_2 (kRT_2)^{1/2} = 0.90\left(1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 317 K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 322\ m/s$$

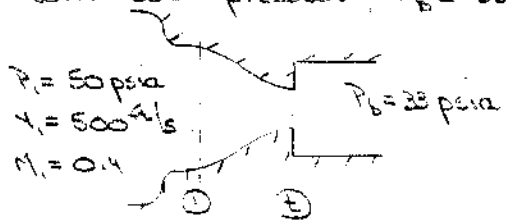
$$\rho_2 = \frac{P_2}{RT_2} = 240 \times 10^3 \frac{N}{m^2} \times \frac{kg\cdot K}{287\ N\cdot m} \times \frac{1}{317 K} = 2.64\ kg/m^3$$

$$\dot{m} = \rho V A = 2.64 \frac{kg}{m^3} \times 322 \frac{m}{s} \times 0.01\ m^2 = 8.50\ kg/s \longleftarrow \dot{m}$$



Problem 12.9

Given: Isentropic flow of air through a converging nozzle discharges into a receiver with back pressure $P_b = 33$ psia



Find: P_t, V_t, M_t

Solution:

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$ $\qquad \frac{P_0}{P} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$

Assumptions: (1) steady flow (3) uniform flow at a section
(2) isentropic flow (4) ideal gas

The nozzle will be choked if $P_b / P_0 \le 0.528$

$\frac{P_0}{P_1} = \left[1 + \frac{k-1}{2} M_1^2\right]^{k/(k-1)}$ $\qquad P_0 = 50 \text{ psia} \left[1 + 0.2(0.4)^2\right]^{3.5} = 55.8$ psia

$\frac{P_b}{P_0} = \frac{33}{55.8} = 0.591 > 0.528$. Hence nozzle not choked. $P_t = P_b = 33$ psia $\qquad P_t$

$\frac{P_0}{P_t} = \left[1 + \frac{k-1}{2} M_t^2\right]^{k/(k-1)}$, $M_t = \left\{\frac{2}{k-1}\left[\left(\frac{P_0}{P_t}\right)^{(k-1)/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{55.8}{33.0}\right)^{0.2857} - 1\right]\right\}^{1/2} = 0.90$ $\qquad M_t$

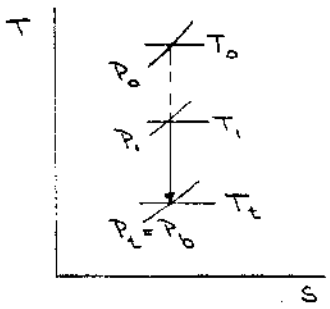
$V = Mc = M\sqrt{kRT}$ and $T = \frac{V^2}{M^2 kR}$

$T_1 = \frac{V_1^2}{M_1^2 kR} = (500)^2 \frac{ft^2}{s^2} \cdot \frac{1}{0.4^2} \cdot \frac{1}{1.4} \cdot \frac{lbm\,°R}{53.3\,ft\,lbf} \cdot \frac{slug}{32.2\,lbm} \cdot \frac{lbf \cdot s^2}{slug \cdot ft} = 650°R$

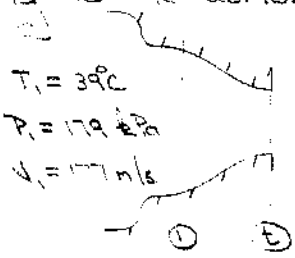
$T_0 = T_1\left(1 + \frac{k-1}{2} M_1^2\right) = 650°R\left[1 + 0.2(0.4)^2\right] = 671°R$ $\qquad T_0$ = constant

$T_t = \frac{T_0}{1 + \frac{k-1}{2} M_t^2} = \frac{671°R}{1 + 0.2(0.90)^2} = 577°R$

$V_t = M_t c_t = M_t\sqrt{kRT_t} = 0.90\left(1.4 \times 53.3 \frac{ft\,lbf}{lbm\,°R} \times 577°R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right)^{1/2} = 1060$ ft/s $\qquad V_t$



Given: Air flows isentropically through a converging nozzle, discharging to the atmosphere



Find: p_t

Solution:

Computing equations: $\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$ $\qquad c = \sqrt{kRT}$

Assumptions: (1) Steady flow
(2) Isentropic flow in the nozzle
(3) Uniform flow at a section
(4) Ideal gas

The nozzle will be choked, i.e. $M_t = 1.0$ if $\frac{p_t}{p_0} \le 0.528$

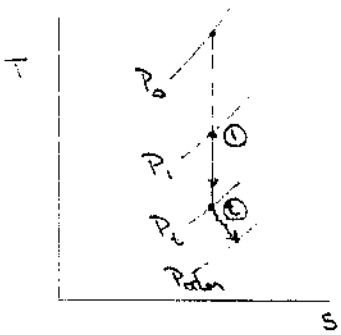
$M_1 = \frac{V_1}{c_1}$ $\qquad c_1 = \sqrt{kRT_1} = \sqrt{1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 312 K \times \frac{kg \cdot m}{N \cdot s^2}} = 354\ m/s$

$M_1 = \frac{V_1}{c_1} = \frac{177}{354} = 0.5$

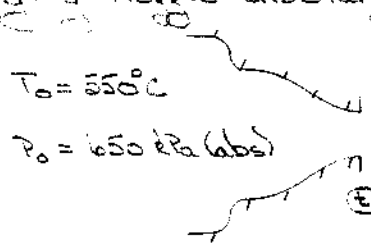
$\frac{p_0}{p_1} = \left[1 + \frac{k-1}{2} M_1^2\right]^{k/(k-1)}$ $\qquad p_0 = p_1 \left[1 + \frac{k-1}{2} M_1^2\right]^{k/(k-1)} = 179\ kPa \left[1 + 0.2(0.5)^2\right]^{3.5} = 212\ kPa$

Then, $\frac{p_{atm}}{p_0} = \frac{101}{212} = 0.476 < 0.528$ $\qquad \therefore M_t = 1.0$

For $M_t = 1.0$, $\frac{p_t}{p_0} = 0.528$ $\qquad p_t = 0.528\, p_0 = 0.528 \cdot 212\ kPa = 112\ kPa$ $\leftarrow p_t$



Given: Isentropic flow of air from a large tank through a converging nozzle discharges to atmosphere



Find: $\dot{m}$

Solution:

Basic equations: $\dot{m} = \rho V A = \text{const.}$ $P = \rho R T$

Computing equations: $\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2$ $\frac{P_o}{P} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$ $c = \sqrt{kRT}$

Assumptions: (1) steady flow (3) uniform flow at a section
(2) isentropic flow in nozzle (4) ideal gas

Since $P_b/P_o = \frac{101}{650} = 0.155 < 0.528$, the nozzle is choked and $M_t = 1.0$

From the continuity equation $\dot{m} = \rho V A$ and hence we need to determine ρ_t and V_t

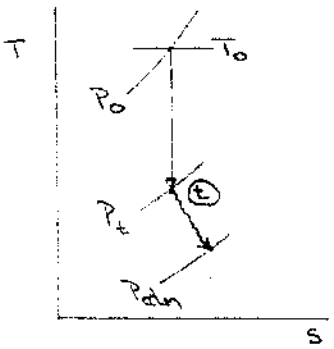
$\frac{T_o}{T_t} = 1 + \frac{k-1}{2} M_t^2$; $T_t = \frac{T_o}{1 + \frac{k-1}{2} M_t^2} = \frac{823\ K}{1 + 0.2(1.0)^2} = 686\ K$

$V_t = M_t c_t = M_t \left(kRT_t\right)^{1/2} = 1.0\left(1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 686 K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 525\ m/s$

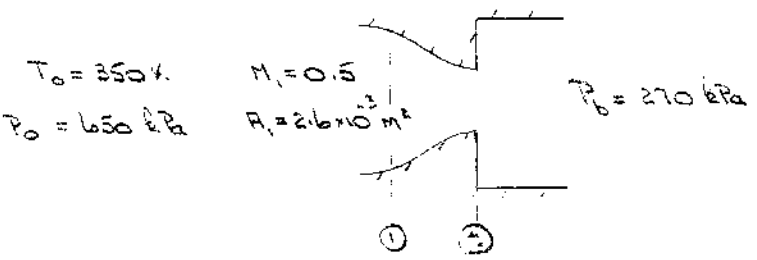
$\frac{P_o}{P_t} = \left[1 + \frac{k-1}{2} M_t^2\right]^{k/(k-1)}$; $P_t = \frac{P_o}{\left[1 + \frac{k-1}{2} M_t^2\right]^{k/(k-1)}} = \frac{650\ kPa}{\left[1 + 0.2(1.0)^2\right]^{3.5}} = 343\ kPa$

$\rho_t = \frac{P_t}{RT_t} = 343 \times 10^3 \frac{N}{m^2} \times \frac{kg\cdot K}{287\ N\cdot m} \times \frac{1}{686 K} = 1.74\ kg/m^3$

Finally $\dot{m} = \rho_t V_t A_t = 1.74 \frac{kg}{m^3} \times 525 \frac{m}{s} \times 6 \times 10^{-4} m^2 = 0.548\ kg/s$ ← $\dot{m}$



Given: Isentropic flow of air through a converging nozzle discharges to a back pressure, P_b.



Find: A_t

Solution

Computing equations:

$$\frac{P_o}{P} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$$

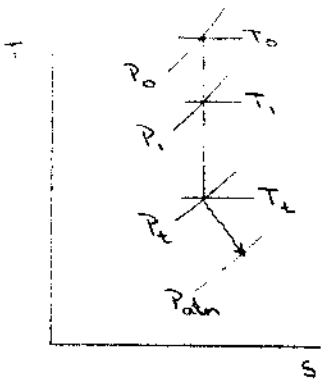
$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2} M^2}{1 + \frac{k-1}{2}}\right]^{(k+1)/2(k-1)} \qquad (12.6)$$

Assumptions: (1) steady flow
(2) isentropic flow in nozzle
(3) uniform flow at a section
(4) ideal gas

Since $P_b/P_o = \frac{270 \text{ kPa}}{650 \text{ kPa}} = 0.415 < 0.528$, the nozzle is choked and $M_t = 1.0$

From Eq 12.6 with $M_1 = 0.5$, $A_1/A^* = 1.340$

Then $A_t = A^* = A_1/1.340 = \frac{2.6 \times 10^{-3} \text{ m}^2}{1.340} = 1.94 \times 10^{-3} \text{ m}^2$ ← A_t



Given: Small engine tested under standard atmospheric conditions consumes air at rate, $\dot{m} = 0.35$ kg/min. Pressure in inlet manifold is $h = 635$ mm Hg vacuum

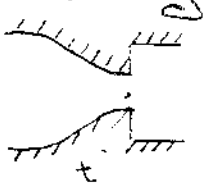
Find: the minimum flow area

Solution:

Assume atmospheric conditions are stagnation values and that flow to the minimum area is isentropic. Treat air as an ideal gas.

Basic equations: $\dot{m} = \rho V A$ $\quad p = \rho R T$

Computing equations: $\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{\frac{k}{k-1}}$, $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$, $c = [kRT]^{1/2}$



Check for choking

$p_0 = \rho_{Hg} g \Delta h_{Hg}$ so $\Delta h_{Hg} = \frac{p_0}{SG_{Hg} \rho_{H_2O} g}$

$$\Delta h = 101.3 \times 10^3 \frac{N}{m^2} \times \frac{m^3}{(13.6)\,999\,kg} \times 9.81 \frac{m}{s^2} \times \frac{kg \cdot m}{N \cdot s^2} \times 10^3 \frac{mm}{m} = 760\ mm\ Hg$$

Then, $\frac{p_{min}}{p_0} = \frac{(760 - 635)\ mm}{760\ mm} = 0.164 < \frac{p^*}{p_0} = 0.528$

$\therefore$ flow is choked, $M_t = 1$

$$T_t = \frac{T_0}{1 + \frac{k-1}{2} M_t^2} = \frac{T_0}{1.2} = \frac{(273+15)K}{1.2} = 240\ K$$

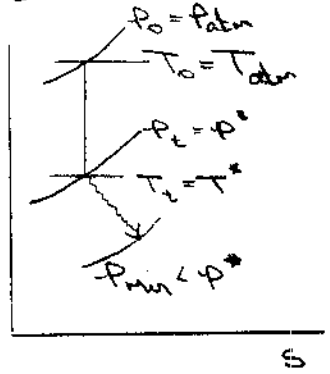
$$p_t = \frac{p_0}{[T_0/T]^{k/(k-1)}} = \frac{p_0}{[1.2]^{3.5}} = (0.528)(101.3)\,kPa = 53.5\ kPa$$

$$\rho_t = \frac{p_t}{RT_t} = 53.5 \times 10^3 \frac{N}{m^2} \times 287 \frac{kg \cdot K}{N \cdot m} \times \frac{1}{240 K} = 0.777\ kg/m^3$$

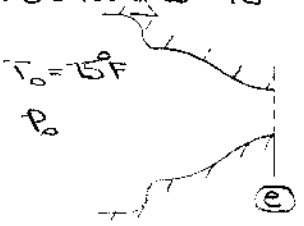
$$V_t = c_t = [kRT_t]^{1/2} = \left[1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 240 K \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 311\ m/s$$

$$A_t = \frac{\dot{m}}{\rho V_t} = 0.35 \frac{kg}{min} \times \frac{m^3}{0.777 kg} \times \frac{s}{311 m} \times \frac{1 min}{60 s} \times 10^6 \frac{mm^2}{m^2}$$

$A_t = 24.1\ mm^2$ $\longleftarrow A_{min}$



Given: Flow of air from stagnation state through a converging nozzle discharges to the atmosphere. $P_e \geq 45$ psig



$P_b = 14.7$ psia

$A_e = 1.5$ in^2

$P_e = 45$ psig $= 59.7$ psia

Find: P_o, $\dot{m}$

Solution:

Basic equations: $\dot{m} = \rho A V$ $\quad P = \rho R T$

Computing equations: $\frac{P_o}{P_e} = \left[1 + \frac{k-1}{2} M_e^2\right]^{\frac{k}{k-1}}$ $\quad \frac{T_o}{T_e} = 1 + \frac{k-1}{2} M_e^2$

Assumptions: (1) steady flow (3) uniform flow at a section
(2) isentropic flow (4) ideal gas behavior

Since $P_e > P_b$, nozzle is choked and $M_e = 1.0$

$$P_o = P_e\left[1 + \frac{k-1}{2} M_e^2\right]^{\frac{k}{k-1}} = 59.1 \text{ psia}\left[1 + 0.2\right]^{3.5} = 113 \text{ psia} \quad \leftarrow P_o$$

$$\frac{T_o}{T_e} = 1 + \frac{k-1}{2} M_e^2 \quad \therefore \quad T_e = \frac{T_o}{1.2} = \frac{535°R}{1.2} = 446°R$$

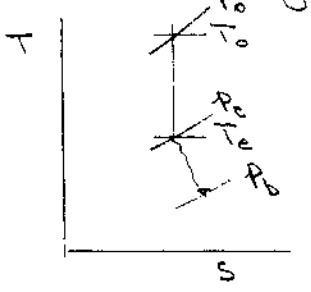
$$V_e = c_e = (kRT_e)^{1/2} = \left[1.4 \times 53.3 \frac{ft\text{-}lbf}{lbm\text{-}°R} \times 446°R \times 32.2 \frac{lbm}{slug} \times \frac{slug\text{-}ft/s^2}{lbf}\right]^{1/2} = 1035 \text{ ft/s}$$

$$\rho_e = \frac{P_e}{RT_e} = 59.1 \frac{lbf}{in^2} \times 144 \frac{in^2}{ft^2} \times \frac{lbm\text{-}°R}{53.3\ ft\text{-}lbf} \times \frac{1}{446°R} = 0.362 \text{ lbm/ft}^3$$

Then:

$$\dot{m} = \rho_e V_e A_e = 0.362 \frac{lbm}{ft^3} \times 1035 \frac{ft}{s} \times 1.5 \text{ in}^2 \times \frac{ft^2}{144 \text{ in}^2} = 3.90 \text{ lbm/s}$$

For steady flow, $\dot{m} = 3.25$ lbm/s must be supplied to the tank $\leftarrow \dot{m}$

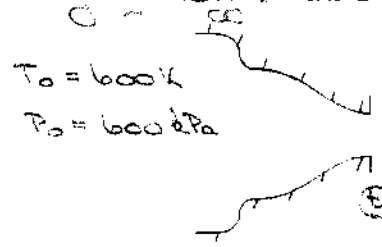


The corresponding volume flow rate of standard air is

$$Q = \frac{\dot{m}}{\rho_{std}} = 3.90 \frac{lbm}{s} \times 0.0765 \frac{ft^3}{lbm} \times 60 \frac{s}{min}$$

$$Q = 3{,}000 \text{ SCFM}$$

Given: Reversible, adiabatic flow of air from a large tank through a converging nozzle discharging to atmosphere



Find: (a) range of tank pressure, P_o, for which $M_2 = 1.0$
(b) $\dot{m}$ for conditions given

Solution:

Basic equations: $\dot{m} = \rho V A = \text{const}$ $\quad P = \rho RT$

Computing equations: $\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2$ $\quad \frac{P_o}{P} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$

Assumptions: (1) steady flow (3) uniform flow at a section
(2) isentropic flow in nozzle (4) ideal gas

The nozzle will be choked, i.e. $M_2 = 1.0$ for $P_b/P_o \le 0.528$

Since $P_b = 101\ kPa$, nozzle is choked for

$$P_o = \frac{P_b}{0.528} = \frac{101\ kPa}{0.528} = 191\ kPa \quad \longleftarrow P_o$$

Thus for $P_o = 600\ kPa$, $M_2 = 1.0$

$$\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2 \quad ; \quad T_2 = \frac{T_o}{1 + \frac{k-1}{2} M_2^2} = \frac{600\ K}{1 + 0.2(1.0)^2} = 500\ K$$

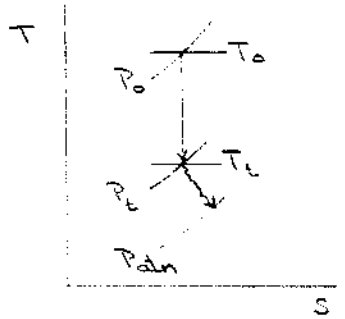
$$V_2 = M_2 c_2 = M_2 (kRT_2)^{1/2} = 1.0 \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 500 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 448\ m/s$$

$$\frac{P_o}{P_2} = \left[1 + \frac{k-1}{2} M_2^2\right]^{k/(k-1)} \quad ; \quad P_2 = \frac{P_o}{\left[1 + \frac{k-1}{2} M_2^2\right]^{k/(k-1)}} = \frac{600\ kPa}{[1 + 0.2(1.0)^2]^{3.5}} = 317\ kPa$$

$$\rho_2 = \frac{P_2}{RT_2} = 317 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287\ N \cdot m} \times \frac{1}{500 K} = 2.21\ kg/m^3$$

Finally,

$$\dot{m} = \rho_2 V_2 A_2 = 2.21 \frac{kg}{m^3} \times 448 \frac{m}{s} \times 1.29 \times 10^{-3}\ m^2 = 1.28\ kg/s \quad \longleftarrow \dot{m}$$



Given: Large tank initially evacuated to 27 in Hg (vacuum).
Ambient conditions are 29.4 in Hg and 70°F
At t=0, orifice of diameter $D = 0.25$ in, is opened in the tank wall. Vena contracta area is 65% of geometric area.

Find: (a) Initial mass flow rate into the tank.
(b) Show process on Ts diagram.
(c) Plot (schematic) mass flow rate as a function of time
(d) Explain the plot.

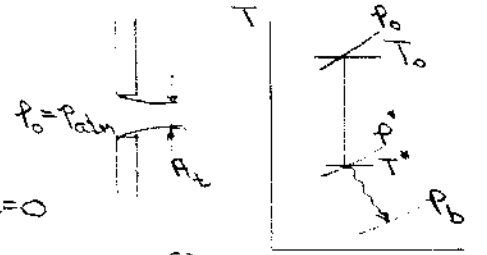
Solution:

Basic equations: $\dot{m} = \rho A V$ $\quad P = \rho R T$

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$ $\quad \frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/k-1}$

$c = (kRT)^{1/2}$

Assumptions: (1) Isentropic flow to "throat" at vena contracta
(2) ideal gas behavior
(3) $A_t = 0.65 A$



$\frac{p_b}{p_0} = \frac{29.4 - 27.0}{29.4} = 0.0816 < 0.528$

$\therefore$ flow choked at t=0

$p_0 = \gamma \Delta h_0 = SG\, \gamma_{H_2O} \Delta h_0 = 13.6 \times 62.4 \frac{lbf}{ft^3} \times 29.4 in \times \frac{ft^3}{1728 in^3} = 14.4$ psia

For $M_t = 1.0$

$T_t = T^* = 0.833 T_0 = 0.833 (537°R) = 441°R$

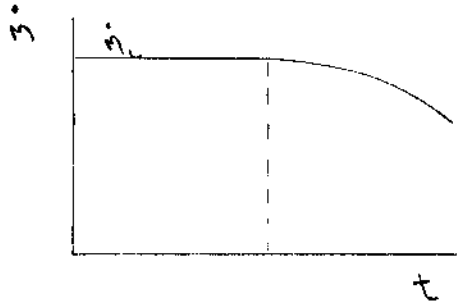
$p_t = p^* = 0.528 p_0 = 0.528 (14.4 psia) = 7.60$ psia

$V_t = V^* = (kRT^*)^{1/2} = \left[1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot °R} \times 441°R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft/s^2}{lbf}\right]^{1/2} = 1030$ ft/s

$\rho_t = \rho^* = \frac{p^*}{RT^*} = 7.60 \frac{lbf}{in^2} \times 144 \frac{in^2}{ft^2} \times \frac{lbm \cdot °R}{53.3 ft \cdot lbf} \times \frac{1}{441°R} = 0.0466$ lbm/ft³

$A_t = 0.65 A = 0.65 \times \frac{\pi D^2}{4} = 0.65 \times \pi \times \left(\frac{0.25}{12}\right)^2 ft^2 \times \frac{1}{4} = 2.22 \times 10^{-4}$ ft²

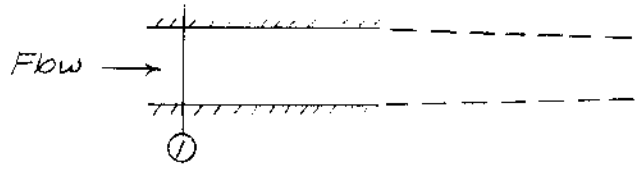
$\therefore \dot{m} = \rho^* V^* A_t = 0.0466 \frac{lbm}{ft^3} \times 1030 \frac{ft}{s} \times 2.22 \times 10^{-4} ft^2 = 0.0107 \frac{lbm}{s}$ $\leftarrow \dot{m}_t$



Mass flow rate will remain at $\dot{m}_t$ so long as $p_b \leq p^*$. It will then decrease since for $p_b > p^*$ flow is no longer choked.

Problem 12.17

Given: Air flow in duct, with $\dot{m} = 0.5$ lbm/s; $A_1 = 1.0$ in.^{2}



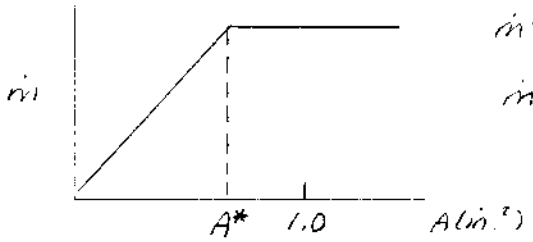
$M_1 = 0.6$

$p_1 = 30$ psia

Find: (a) How would flow rate vary with downstream area, for isentropic flow.
(b) Schematic plot
(c) Smallest allowable downstream area for which flow rate would not be decreased.
(d) Pressure at location of Part c.

Solution: (a) Flow rate would remain constant, provided $A > A^*$ (not choked).
Flow rate would decrease with further reduction in area.

(b) The schematic plot is



(c) To evaluate A^*, set $\dot{m} = \rho V A = \rho^* V^* A^*$. Thus, since $V = Mc = M\sqrt{kRT}$,

$$A^* = A\frac{\rho}{\rho^*}\frac{V}{V^*} = A\frac{\rho}{\rho_0}\frac{\rho_0}{\rho^*}\frac{M}{M^*}\sqrt{\frac{T_1}{T_0}\frac{T_0}{T^*}}$$

$$A^* = 1.0\text{ in.}^2\left[\frac{1+\frac{k-1}{2}}{1+\frac{k-1}{2}M_1^2}\right]^{\frac{1}{k-1}}\frac{M_1}{1}\left[\frac{1}{1+\frac{k-1}{2}M_1^2}\right]^{\left(1+\frac{k-1}{2}\right)^{\frac{1}{2}}} = 0.842\text{ in.}^2 \quad \leftarrow A^*$$

(d) $$p_0 = p_1\left[1+\frac{k-1}{2}M_1^2\right]^{\frac{k}{k-1}} = 30\text{ psia}\left[1+0.2(0.6)^2\right]^{3.5} = 38.3\text{ psia}$$

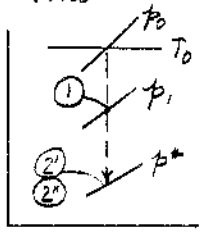
$$p^* = \frac{p_0}{\left[1+\frac{k-1}{2}\right]^{k/k-1}} = \frac{38.3\text{ psia}}{(1+0.2)^{3.5}} = 20.2\text{ psia} \quad \leftarrow p^*$$

Alternatively, from Eq. 12.6,

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1+\frac{k-1}{2}M^2}{1+\frac{k-1}{2}}\right]^{\frac{k+1}{2(k-1)}} = \frac{1}{0.6}\left[\frac{1+0.2(0.6)^2}{1.2}\right]^{\frac{2.4}{0.8}} = 1.118$$

Thus $A^* = A/A/A^* = \frac{1.0\text{ in.}^2}{1.188} = 0.842\text{ in.}^2$ $\leftarrow A^*$

The Ts diagram is



Given: Spherical cavity with $D = 18$ in., initially evacuated, to be filled with air for a combustion experiment. Final pressure is to be $p_f = 5$ psia, measured after temperature equilibrium is reached. Assume inlet valve acts like a converging nozzle with throat diameter, $d = 0.05$ in., and surroundings at STP.

Find: (a) How long valve should be open to fill cavity.
(b) Entropy change for air in tank.

Solution: Since $p_f/p_{atm} < 0.528$, flow is choked, and $\dot{m}$ = constant.

Computing equations: $p = \rho RT$ $\dot{m} = \rho^* V^* A$ $c = \sqrt{kRT}$

$$Tds = dh - vdp \qquad T = T_0/\left(1 + \frac{k-1}{2}M^2\right)$$

$$p = p_0/\left(1 + \frac{k-1}{2}M^2\right)^{\frac{k}{k-1}}$$

Assumptions: (1) Steady flow
(2) One-dimensional flow
(3) Isentropic flow in nozzle
(4) Ideal gas
(5) STP: $T_{atm} = 59°F$, $p_{atm} = 14.7$ psia

At final conditions, $m_f = \rho_f \forall$

$$\rho_f = \frac{p_f}{RT_f} = \frac{p_f}{RT_{atm}} = 5.00\ \frac{lbf}{in.^2} \times \frac{lbm°R}{53.3 ft\cdot lbf} \times \frac{1}{(460+59)°R} \times 144\ \frac{in.^2}{ft^2} = 0.0260\ lbm/ft^3$$

$$\forall = \frac{4\pi R^3}{3} = \frac{\pi D^3}{6} = \frac{\pi}{6}\left(18\ in. \times \frac{ft}{12\ in.}\right)^3 = 1.77\ ft^3$$

So $m_f = 0.0260\ \frac{lbm}{ft^3} \times 1.77\ ft^3 = 0.0460\ lbm$

Since $\dot{m}$ = constant, then $\Delta t = \frac{m_f}{\dot{m}}$. But $\dot{m} = \rho^* V^* A^* = \rho^* C^* A^*$

$$\rho^* = \frac{p^*}{RT^*}\ ;\ T^* = \frac{T_{atm}}{1 + \frac{k-1}{2}(1)^2} = \frac{(460+59)°R}{1.2} = 433°R\ ;\ p^* = \frac{p_{atm}}{(1.2)^{k/k-1}} = \frac{14.7 psia}{1.89} = 7.78\ psia$$

$$\rho^* = 7.78\ \frac{lbf}{in.^2} \times \frac{lbm°R}{53.3\ ft\cdot lbf} \times \frac{1}{433°R} \times 144\ \frac{in.^2}{ft^2} = 0.0485\ lbm/ft^3$$

$$V^* = C^* = \sqrt{kRT^*} = \left[1.4 \times 53.3\ \frac{ft\cdot lbf}{lbm°R} \times 433°R \times 32.2\ \frac{lbm}{slug} \times \frac{slug\cdot ft}{lbf\cdot s^2}\right]^{1/2} = 1020\ ft/s$$

$$A^* = \frac{\pi d^2}{4} = \frac{\pi}{4}\left(0.05\ in. \times \frac{ft}{12\ in.}\right)^2 = 1.36 \times 10^{-5} ft^2$$

Then $\dot{m} = 0.0485\ \frac{lbm}{ft^3} \times 1020\ \frac{ft}{s} \times 1.36 \times 10^{-5} ft^2 = 6.73 \times 10^{-4}\ lbm/s$

$$\Delta t = \frac{m_f}{\dot{m}} = 0.0460\ lbm \times \frac{s}{6.73 \times 10^{-4}\ lbm} = 68.4\ s$$ Δt

Note that the tank contents initially would be cold. When the valve is closed, $\rho = m_f/\forall$ = constant. Then $p = \rho RT$ = (constant)T will be lower than desired initially. After the valve is closed, heat transfer will occur, T will increase, and p will increase as the tank contents reach equilibrium.

over →

From the Tds equation, $Tds = dh - vdp = c_p dT - \frac{RT}{p}dp$, so

$$ds = c_p \frac{dT}{T} - R\frac{dp}{p}$$

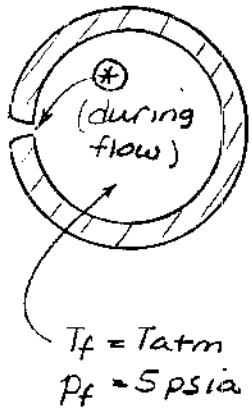
Integrating,

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right)$$

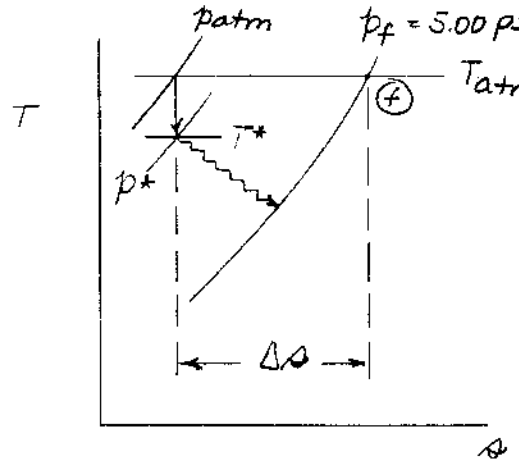
Substituting (note since $T_f = T_{atm}$, $\ln\left(\frac{T_f}{T_a}\right) = 0$):

$$s_2 - s_1 = -53.3\ \frac{ft\cdot lbf}{lbm\cdot °R} \times \ln\left(\frac{5}{14.7}\right) \times \frac{Btu}{778\ ft\cdot lbf}$$

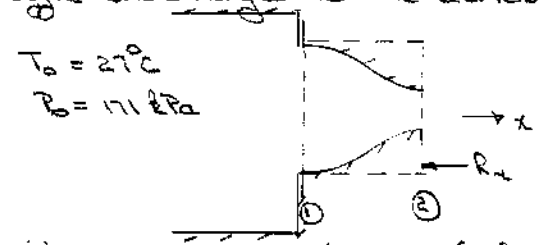
$$\Delta s = 0.0739\ Btu/lbm\cdot °R$$



The corresponding Ts diagram is



Given: Isentropic flow of air from a large tank through a converging nozzle discharges to the atmosphere



Find: Magnitude and direction of force required to keep nozzle in place

Solution:

Basic equations: $F_{Sx} = P_1A_1 - P_2A_2 - P_{atm}(A_1 - A_2) - R_x = \dot{m}(V_2 - V_1)$

$\dot{m} = \rho VA = \text{const}$ $\qquad P = \rho RT$

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$ $\qquad \frac{P_0}{P} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)}$

Assumptions: (1) steady flow (3) uniform flow at a section
(2) isentropic flow (4) ideal gas

$$M_2 = \left\{\frac{2}{k-1}\left[\left(\frac{P_0}{P_2}\right)^{(k-1)/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{171}{101}\right)^{0.286} - 1\right]\right\}^{1/2} = 0.901$$ Hence flow not choked

$$T_2 = T_0 / \left[1 + \frac{k-1}{2}M_2^2\right] = 300\,K / \left[1 + 0.2(0.901)^2\right] = 258\ K$$

$$V_2 = M_2 c_2 = M_2 (kRT_2)^{1/2} = 0.901\left(1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 258\,K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 290\ m/s$$

$$\rho_2 = \frac{P_2}{RT_2} = 101\times10^3 \frac{N}{m^2} \times \frac{kg\cdot K}{287\,N\cdot m} \times \frac{1}{258\,K} = 1.36\ kg/m^3$$

$$\dot{m} = \rho_2 V_2 A_2 = 1.36 \frac{kg}{m^3} \times 290 \frac{m}{s} \times 0.015\,m^2 = 5.92\ kg/s$$

$$T_1 = T_0 / \left[1 + \frac{k-1}{2}M_1^2\right] = 300\,K / \left[1 + 0.2(0.2)^2\right] = 298\ K$$

$$V_1 = M_1 c_1 = M_1 (kRT_1) = 0.2\left(1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 298\,K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 69.2\ m/s$$

$$P_1 = P_0 / \left[1 + \frac{k-1}{2}M_1^2\right]^{k/(k-1)} = 171\ kPa / \left[1 + 0.2(0.2)^2\right]^{3.5} = 166\ kPa$$

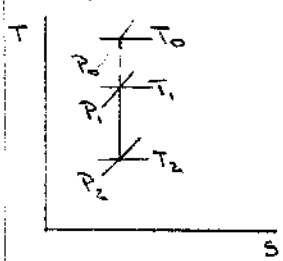
$$\rho_1 = \frac{P_1}{RT_1} = 166\times10^3 \frac{N}{m^2} \times \frac{kg\cdot K}{287\,N\cdot m} \times \frac{1}{298\,K} = 1.94\ kg/m^3$$

$$A_1 = \dot{m}/\rho_1 V_1 = 5.92 \frac{kg}{s} \times 1.94 \frac{m^3}{kg} \times \frac{s}{69.2\,m} = 0.0441\ m^2$$

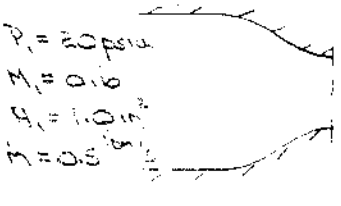
$$R_x = P_1A_1 - P_2A_2 - P_{atm}(A_1 - A_2) - \dot{m}(V_2 - V_1) = P_{1g}A_1 - \cancel{P_{2g}A_2}^{\,=0} - \dot{m}(V_2 - V_1)$$

$$= (166 - 101)\times10^3 \frac{N}{m^2} \times 0.0441\,m^2 - 5.92 \frac{kg}{s}(290 - 69.2)\frac{m}{s} \times \frac{N\cdot s^2}{kg\cdot m}$$

$R_x = 1560$ N (to the left) ← R_x



Given: Steady, isentropic flow of air
duct area to be reduced as much as possible without reducing flowrate



Find: a) T_0 b) $\%\Delta A$ possible c) V_2 and p_2

Solution:

Basic equations: $\dot{m} = \rho V A$ $\quad p = \rho R T$

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$ $\quad \frac{p_0}{p} = \left(1 + \frac{k-1}{2} M^2\right)^{k/(k-1)}$

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2} M^2}{1 + \frac{k-1}{2}}\right]^{(k+1)/2(k-1)} \qquad (12.6)$$

Assumptions: (1) steady flow (3) uniform flow at a section
(2) isentropic flow (4) ideal gas

To determine T_0, we first need to find T_1

$$\dot{m} = \rho_1 V_1 A_1 = \frac{p_1}{RT_1} M_1 c_1 A_1 = \frac{p_1}{RT_1} M_1 (kRT_1)^{1/2} A_1 = p_1 M_1 A_1 \left(\frac{k}{RT_1}\right)^{1/2}$$

Solving for T_1,

$$T_1 = \frac{k}{R}\left(\frac{p_1 M_1 A_1}{\dot{m}}\right)^2 = 1.4 \times \frac{\text{lbm}\cdot\text{R}}{53.3\,\text{ft}\cdot\text{lbf}} \left[20\frac{\text{lbf}}{\text{in}^2} \times 0.6 \times 1.0\,\text{in}^2 \times \frac{\text{s}}{0.5\,\text{lbm}} \times 32.2\frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug}\cdot\text{ft}/\text{s}^2}{\text{lbf}}\right]^2 \frac{\text{slug}}{32.2\,\text{lbm}} \cdot \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}$$

$T_1 = 487°\text{R}$

$$T_{0_1} = T_1\left[1 + \frac{k-1}{2} M_1^2\right] = 487°\text{R}\left[1 + 0.2(0.6)^2\right] = 522°\text{R}$$ $\boxed{T_0}$

Maximum area reduction occurs where $M_2 = 1$

For $M_1 = 0.6$, from Eq 12.6, $A_1/A^* = 1.188$ $\quad \therefore A^* = A_2 = \frac{A_1}{1.188} = 0.842$ in^2

$$\%\,\Delta A = \frac{A_2 - A_1}{A_1} \times 100 = \frac{0.842 - 1.0}{1.0} \times 100 = -15.8\%$$ $\boxed{\%\,\Delta A}$

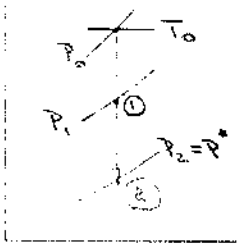
$T_2 = T_0 / \left[1 + \frac{k-1}{2} M_2^2\right] = 522°\text{R} / \left[1 + 0.2(1)^2\right] = 435°\text{R}$

$$V_2 = M_2 c_2 = M_2 (kRT_2)^{1/2} = 1.0\left(1.4 \times 53.3 \frac{\text{ft}\cdot\text{lbf}}{\text{lbm}\cdot°\text{R}} \times 435°\text{R} \times 32.2\frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug}\cdot\text{ft}/\text{s}^2}{\text{lbf}}\right)^{1/2}$$

$V_2 = 1020$ ft/s $\boxed{V_2}$

$$p_{0_1} = p_{0_2} = p_1\left[1 + \frac{k-1}{2} M_1^2\right]^{k/k-1} = 20\text{ psia}\left[1 + 0.2(0.6)^2\right]^{3.5} = 25.5\text{ psia}$$

$$p_2 = p_{0_2} / \left[1 + \frac{k-1}{2} M_2^2\right]^{k/k-1} = 25.5\text{ psia} / \left[1 + 0.2(1)^2\right]^{3.5} = 13.5\text{ psia}$$ $\boxed{p_2}$



Given: "Rocket" cart propelled by compressed air, converging nozzle.

Initially in tank, $p_0 = 1.52$ MPa (abs), $T_0 = 27°C$, $M_0 = 23.2$ kg

$A_e = 31.7$ mm^2; $F_R = 5.2$ N, aerodynamic drag is negligible.

Find: (a) Pressure in exit plane.
(b) Mass flow rate of air through nozzle.
(c) Acceleration of assembly.
(d) Static and stagnation states on Ts diagram.

Solution: Assume steady, one-dimensional flow of an ideal gas.

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$; $\frac{p_0}{p} = \left(1 + \frac{k-1}{2}M^2\right)^{\frac{k}{k-1}}$; $c = \sqrt{kRT}$; $p = \rho RT$

Check for choking: $\frac{p_{atm}}{p_0} = \frac{101 \times 10^3}{1.52 \times 10^6} = 0.0664 < 0.528$, so choked. $M_e = 1$

Thus

$$T_e = \frac{T_0}{1 + \frac{k-1}{2}M_e^2} = \frac{(273+27)K}{1 + \frac{k-1}{2}} = 250\ K$$

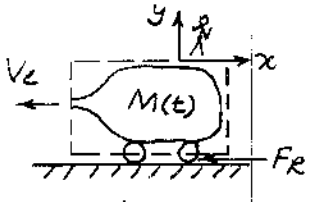
$$p_e = \frac{p_0}{\left(1 + \frac{k-1}{2}M_e^2\right)^{k/k-1}} = \frac{1.52 \times 10^6\ Pa}{(1+\frac{k-1}{2})^{k/k-1}} = \frac{1.52 \times 10^6\ Pa}{(1.2)^{3.5}} = 803\ kPa\ (abs)$$ p_e

$$\dot{m} = \rho_e V_e A_e\ ;\ \rho_e = \frac{p_e}{RT_e} = 803 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287\ N \cdot m} \times \frac{1}{250\ K} = 11.2\ kg/m^3$$

$$V_e = M_e c_e = c_e = \sqrt{kRT_e} = \left[1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 250\ K \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 317\ m/s$$

$$\dot{m} = 11.2 \frac{kg}{m^3} \times 317 \frac{m}{s} \times 31.7 \times 10^{-6} m^2 = 0.113\ kg/s$$ $\dot{m}$

Apply momentum to find acceleration of assembly.



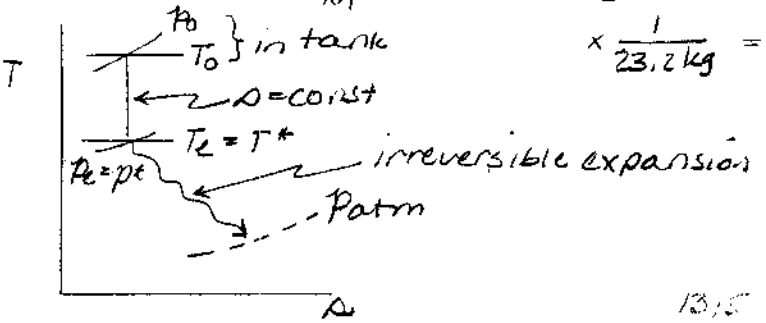
Basic equation: $F_{Sx} + F_{Bx} - \int_{CV} a_{rfx}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} u\rho\,d\forall + \int_{CS} u\rho\,\vec{V}\cdot d\vec{A}$ ($F_{Bx} = 0$ (4); $\frac{\partial}{\partial t} \approx 0$ (5))

Assume: (4) Horizontal; (5) $u \approx 0$ within CV; (6) Uniform flow at exit.

Then $-F_R + (p_e - p_{atm})A_e - M a_{rfx} = u_e\{+\dot{m}\} = -V_e \dot{m}$

$$u_e = -V_e$$

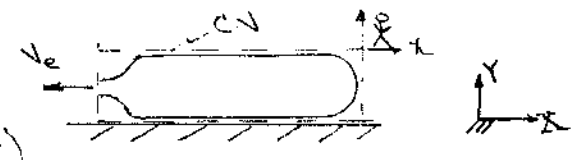
$$a_{rfx} = \frac{V_e \dot{m} - F_R + (p_e - p_{atm})A_e}{M} = \left[317 \frac{m}{s} \times 0.113 \frac{kg}{s} - 5.2\ N + (803 - 101)10^3 \frac{N}{m^2} \times 31.7 \times 10^{-6} m^2\right] \times \frac{1}{23.2\ kg} = 2.28\ m/s^2$$ a_{rfx}



Given: Cylinder of helium at $P = 3000$ psia, $T = 68°F$ is knocked over. Valve is broken off and gas escapes through converging area. Minimum flow area is $0.10\ in^2$; gas flow is uniform there. Mass of cylinder is 125 lb. Cylinder axis is horizontal.

Find: (a) initial $\dot{m}$ out of cylinder
(b) initial acceleration of cylinder
(c) Show states and process path on T-s diagram.

Solution:



Assumptions: (1) Isentropic flow to exit
(2) Choked flow ($p_{atm} << p^*$)
(3) Ideal gas behavior $R = 386$ ft·lbf/lbm·°R, $k = 1.66$

$\dot{m} = \rho_e V_e A_e$ where $A_e = 0.10\ in^2$ $\rho_e = \frac{P_e}{RT_e}$ and $M_e = 1.0$

$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$ $\therefore T_e = T^* = \frac{T_0}{1 + \frac{k-1}{2}} = \frac{528°R}{1.33} = 397°R$

$\frac{P_0}{P} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)}$ $\therefore P_e = P^* = \frac{P_0}{\left[1 + \frac{k-1}{2}\right]^{k/(k-1)}} = \frac{3000\ psia}{(1.33)^{2.515}} = 1464$ psia

$\rho_e = \frac{P_e}{RT_e} = 1464 \times 144 \frac{lbf}{ft^2} \times \frac{lbm°R}{386\ ft·lbf} \times \frac{1}{397°R} = 1.376\ lbm/ft^3$

$V_e = V^* = (kRT_e)^{1/2} = \left[1.66 \times 386 \frac{ft·lbf}{lbm·°R} \times 397°R \times 32.2 \frac{lbm}{slug} \times \frac{slug·ft}{lbf·s^2}\right]^{1/2} = 2862\ ft/s$

$\dot{m} = \rho_e V_e A_e = 1.376 \frac{lbm}{ft^3} \times 2862 \frac{ft}{s} \times 0.10\ in^2 \times \frac{ft^2}{144\ in^2} = 2.73\ lbm/s$ ← $\dot{m}$

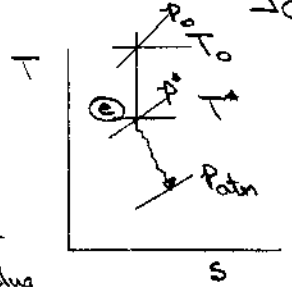
To determine initial acceleration of the cylinder, apply the x-momentum equation to the control volume shown above.

Basic equation: $F_{Sx} + \cancel{F_{Bx}} - \int_{CV} a_{rx}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} V_x\,\rho\,d\forall + \int_{CS} V_x\,\rho\,\vec{V}\cdot d\vec{A}$

Assumptions: (4) neglect friction between cylinder and surface
(5) $F_{Bx} = 0$ (6) V_x zero in cylinder (7) $Wt_{CV} = 125$ lb (including gas)

M_{CV}

$P_e A_e - P_{atm} A_e - M_{CV} a_{rx} = -V_e \{+|\dot{m}_e|\}$



or

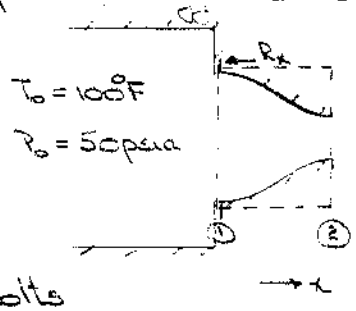
$a_{rx} = \frac{P_{e\,gage} A_e + \dot{m} V_e}{M_{CV}}$

$a_{rx} = \frac{(1464 - 14.7)\frac{lbf}{in^2} \times (0.1)in^2 + 2.73 \frac{lbm}{s} \times 2862 \frac{ft}{s} \times \frac{slug}{32.2\ lbm} \times \frac{lbf\,s^2}{ft\,slug}}{\frac{125\ lb}{32.2\ ft/s^2}}$

$a_{rx} = 99.8\ ft/s^2$ ← a_{rf}

Note: for coefficient of friction $\mu = 0.20$, $a_{rx} = 93.4\ ft/s^2$

Given: Isentropic flow of air through a converging nozzle discharges to atmosphere; nozzle is bolted to a large tank.



Find: Force in bolts

Solution:

Basic equations: $F_{S_x} = P_1A_1 - P_2A_2 - P_{atm}(A_1 - A_2) - R_x = \dot{m}(V_2 - V_1)$

$\dot{m} = \rho VA = const$ $\quad P = \rho RT$

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$ $\quad \frac{P_0}{P} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)}$

Assumptions: (1) steady flow (4) $F_{B_x} = 0$
(2) isentropic flow in nozzle (5) $V_1 \approx 0$
(3) uniform flow at a section

First check for choking

$\frac{P_b}{P_0} = \frac{14.7}{50} = 0.294 < 0.528$ and hence the nozzle is choked

$M_2 = 1.0$ and $P_2 = 0.528\, P_0 = 0.528(50 psia) = 26.4 psia$

$T_0/T = 1 + \frac{k-1}{2}M^2$; $T_2 = T_0/\left[1 + \frac{k-1}{2}M_2^2\right] = 560°R/\left[1 + 0.2(1.0)^2\right] = 467°R$

$\rho_2 = \frac{P_2}{RT_2} = 26.4 \frac{lb}{in^2} \times \frac{lbm°R}{53.34\, ft\cdot lbf} \times \frac{1}{467°R} \times 144 \frac{in^2}{ft^2} = 0.153 \, lbm/ft^3$

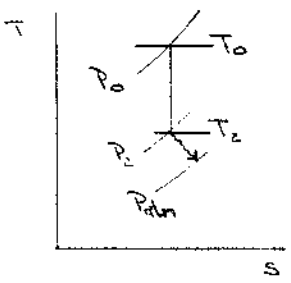
$V_2 = M_2c_2 = M_2(kRT_2)^{1/2} = 1.0\left(1.4 \times 53.3 \frac{ft\cdot lbf}{lbm°R} \times 467°R \times 32.2 \frac{lbm}{slug} \times \frac{slug\cdot ft}{lbf\cdot s^2}\right)^{1/2} = 1060 \, ft/s$

$\dot{m} = \rho_2 V_2 A_2 = 0.153 \frac{lbm}{ft^3} \times 1060 \frac{ft}{s} \times 1.0 in^2 \times \frac{ft^2}{144 in^2} = 1.13 \, lbm/s$

$R_x = P_1A_1 - P_2A_2 - P_{atm}(A_1 - A_2) - \dot{m}(V_2 - \cancel{V_1}^{\approx 0}) = P_{1g}A_1 - P_{2g}A_2 - \dot{m}V_2$

$R_x = (50 - 14.7) \frac{lbf}{in^2} \times 10 in^2 - (26.4 - 14.7) \frac{lbf}{in^2} \times 1.0 in^2 - 1.13 \frac{lbm}{s} \times 1060 \frac{ft}{s} \times \frac{slug}{32.2 lbm} \times \frac{lbf\cdot s^2}{ft\cdot slug}$

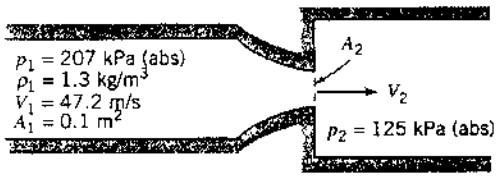
$R_x = 304 lbf$ Since R_x acts to the left on CV, bolts are in tension R_x



Given: Isentropic flow of an ideal gas through converging nozzle

Gas is not air; $k = 1.4$, R is unknown

Flow is steady and uniform at each section



Find: a) nozzle exit area, A_2, (b) exit speed, V_2

Solution:

Computing equations: $M = V/c$, $c = \sqrt{kRT}$, $\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$, $\frac{p_0}{p} = \left(1 + \frac{k-1}{2}M^2\right)^{k/(k-1)}$

$$A/A^* = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2}M^2}{1 + \frac{k-1}{2}}\right]^{(k+1)/2(k-1)} \qquad (12.6)$$

Assumptions: (1) steady flow, (2) 1-D flow, (3) ideal gas, (4) isentropic flow

Check for choking: $c_1 = \sqrt{kRT} = \sqrt{kp/\rho}$

$$c_1 = \left[1.4 \times 207 \times 10^3 \frac{N}{m^2} \times \frac{m^3}{1.30 kg} \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 472 \text{ m/s}; \quad M_1 = \frac{V_1}{c_1} = \frac{47.2}{472} = 0.1$$

$$p_0 = p_1\left[1 + \frac{k-1}{2}M_1^2\right]^{k/(k-1)} = 207 kPa\left[1 + 0.2(0.1)^2\right]^{3.5} = 208 kPa(abs)$$

For $k = 1.4$, $p^*/p_0 = 0.528$, so $p_2/p_0 = 125/208 = 0.601 > 0.528$, so flow is not choked. At state 2

$$M_2 = \left\{\frac{2}{k-1}\left[\left(\frac{p_0}{p_2}\right)^{\frac{k-1}{k}} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{1}{0.601}\right)^{0.286} - 1\right]\right\}^{1/2} = 0.885$$

For $M_1 = 0.1$, from Eq 12.6, $A_1/A^* = 5.822$ $\therefore A^* = 0.0178$

For $M_2 = 0.885$, " " , $A_2/A^* = 1.012$ $\therefore A_2 = 0.0174 m^2$ ← A_2

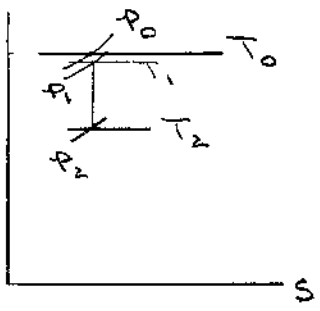
{The corresponding diameter is $D = \left(\frac{4A}{\pi}\right)^{1/2} = 0.149 m$}

Since flow is isentropic $T_{0_1} = T_{0_2}$, and

$$\frac{T_1}{T_2} = \frac{T_1}{T_0} \times \frac{T_0}{T_2} = \frac{1 + \frac{k-1}{2}M_2^2}{1 + \frac{k-1}{2}M_1^2} = \frac{1 + 0.2(0.885)^2}{1 + 0.2(0.1)^2} = 1.154$$

$$c_2 = c_1\left(\frac{T_2}{T_1}\right)^{1/2} = 472 \frac{m}{s} \times \left(\frac{1}{1.154}\right)^{1/2} = 439 \text{ m/s}$$

$$V_2 = M_2 c_2 = 0.885 \times 439 \text{ m/s} = 389 \text{ m/s} \leftarrow V_2$$



Given: Insulated tank of compressed air, $\forall = 107\ ft^3$.

Initial conditions: $T_1 = 800°R$, $p_1 = 400\ psia$

Blowdown: $\Delta m = 64.5\ lbm$ in $\Delta t = 60\ s$.

Find: Estimate the nozzle throat area.

Solution: Apply conservation of mass, isentropic process equations.

Computing equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$

$p/\rho^k = const \qquad \frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$

Assumptions: (1) Large tank, so stagnation properties there
(2) Isentropic (reversible adiabatic) blowdown process
(3) Uniform flow at throat
(4) Choked flow, since $p_{atm}/p_1 \ll 0.528$, so $V_t = c_t$

Compute final density and pressure, and check for choking:

$$\rho_1 = \frac{p_1}{RT_1} = 400\frac{lbf}{in.^2}\times\frac{lbm\cdot°R}{53.3\ ft\cdot lbf}\times\frac{1}{800°R}\times 144\frac{in.^2}{ft^2} = 1.35\ lbm/ft^3$$

$$m_1 = \rho_1 \forall = 1.35\frac{lbm}{ft^3}\times 107\ ft^3 = 144\ lbm$$

$$m_2 = m_1 - \Delta m = (144 - 64.5)\ lbm = 79.5\ lbm$$

$$\rho_2 = \frac{m_2}{\forall} = 79.5\ lbm \times \frac{1}{107\ ft^3} = 0.743\ lbm/ft^3$$

For an isentropic process in the tank, $p_1/\rho_1^k = p_2/\rho_2^k$

$$p_2 = p_1\left(\frac{\rho_2}{\rho_1}\right)^k = 400\ psia\left(\frac{0.743}{1.35}\right)^{1.4} = 173\ psia \quad \therefore \text{still choked}$$

$$T_2 = \frac{p_2}{\rho_2 R} = 173\frac{lbf}{in.^2}\times\frac{ft^3}{0.743\ lbm}\times\frac{lbm\cdot°R}{53.3\ ft\cdot lbf}\times 144\frac{in.^2}{ft^2} = 629°R$$

Estimate mean flow rate at mean properties. From continuity

$$\bar{\dot{m}} = \frac{\Delta m}{\Delta t} = \bar{\rho}_t \bar{V}_t A_t \;;\; A_t = \frac{\Delta m}{\bar{\rho}_t \bar{V}_t \Delta t} \;;\; \bar{T} = \frac{T_1 + T_2}{2} = \frac{800 + 629}{2} = 715°R$$

$$\bar{p} = \frac{p_1 + p_2}{2} = \frac{400 + 173}{2} = 287\ psia$$

so $$\bar{\rho} = \frac{\bar{p}}{R\bar{T}} = 287\frac{lbf}{in.^2}\times\frac{lbm\cdot°R}{53.3\ ft\cdot lbf}\times\frac{1}{715°R}\times 144\frac{in.^2}{ft^2} = 1.08\ lbm/ft^3 \text{ (in tank)}$$

$$\bar{\rho}_t = \frac{\bar{\rho}}{\left(1 + \frac{k-1}{2}M_t^2\right)^{1/k-1}} = 1.08\frac{lbm}{ft^3}\times\frac{1}{(1.2)^{2.5}} = 0.685\ lbm/ft^3$$

$$\bar{T}_t = \frac{\bar{T}}{1 + \frac{k-1}{2}M_t^2} = \frac{715°R}{1.2} = 596°R \;;\; \bar{c}_t = \sqrt{kR\bar{T}_t}$$

Thus $\bar{c}_t = \bar{V}_t = \left[1.4 \times 53.3 \frac{ft \cdot lbf}{lbm\,^\circ R} \times 596^\circ R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right]^{\frac{1}{2}} = 1200\ ft/s$

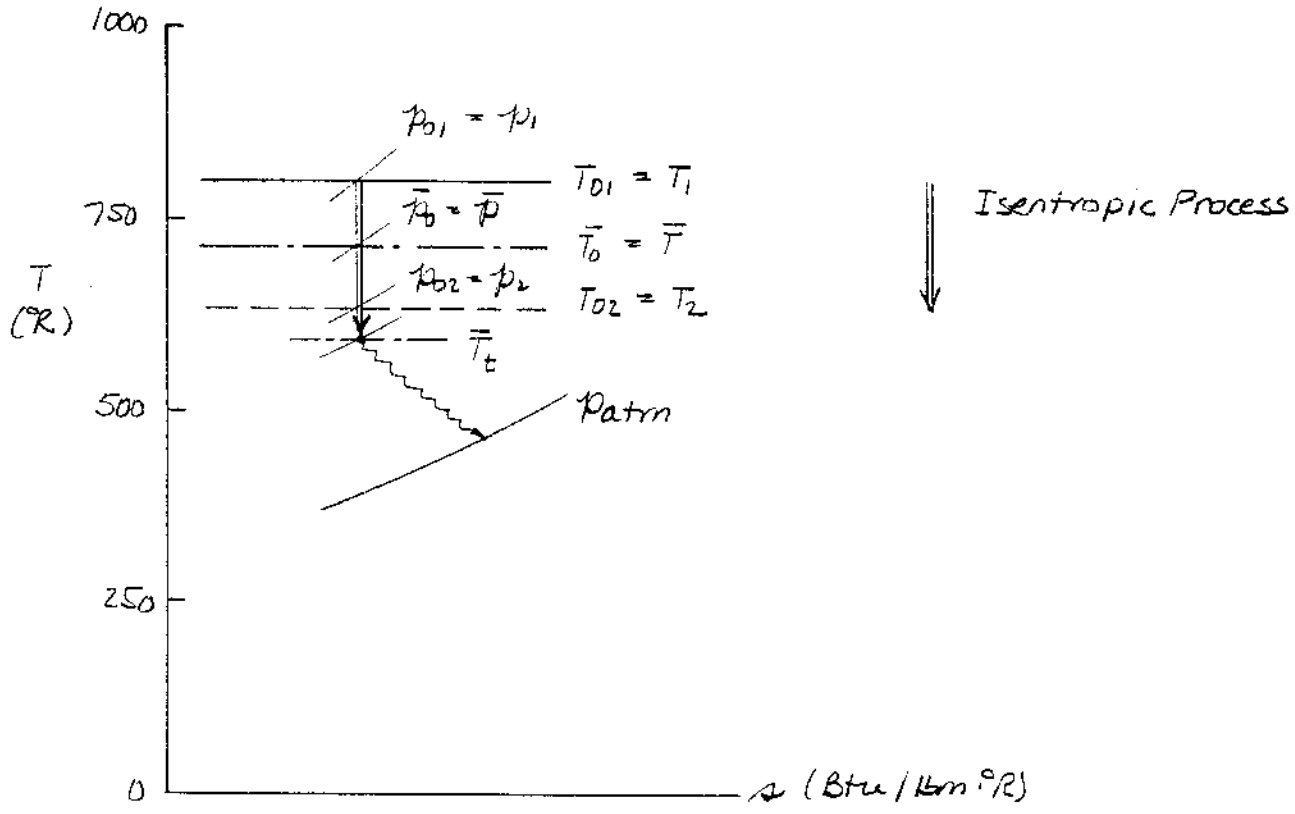
Substituting,

$$A_t = 64.5\ lbm \times \frac{ft^3}{0.685\ lbm} \times \frac{s}{1200\ ft} \times \frac{1}{30\ s} = 0.00262\ ft^2 \text{ or } 0.377\ in.^2$$ A_t

{ If the throat cross-section is circular, $A_t = \frac{\pi D_t^2}{4}$ so the diameter is

$D_t = \sqrt{\frac{4A_t}{\pi}} = \left[\frac{4}{\pi} \times 0.377\ in.^2\right]^{\frac{1}{2}} = 0.693\ in.$ }

The Ts diagram is



Given: Large insulated tank, pressurized to 620 kPa (gage), supplying air to converging nozzle with discharge to atmosphere. Initial temperature in tank is 127°C.

Find: (a) Initial Mach number at nozzle exit plane.
(b) p in exit plane when flow starts.
(c) How exit plane pressure varies with time.
(d) How flow rate varies with time.
(e) Air temperature in tank when flow rate approaches zero.

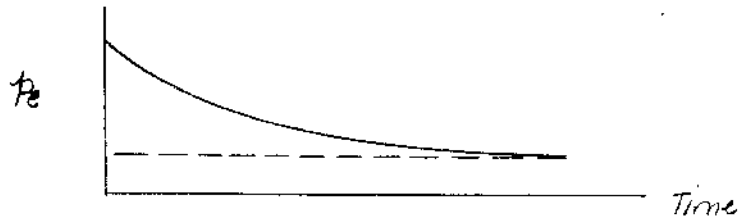
Solution: Assume stagnation conditions in tank, $p_0 = p_{atm}$. Then

$$\frac{p_b}{p_0} = \frac{101.3\ kPa}{(620+101.3)\ kPa\ (abs)} = 0.140 \ll 0.523, \text{ so flow is choked!} \quad M_e = 1$$ ← M_e

At exit plane, $M_e = 1$, so

$$p_e = p_0\left(\frac{1}{1+\frac{k-1}{2}M_e^2}\right)^{\frac{k}{k-1}} = (620+101.3)\ kPa\ (abs)\left(\frac{1}{1.2}\right)^{3.5} = 381\ kPa\ (abs)$$ ← p_e

Exit plane pressure decreases with time, asymptotically approaching p_{atm}.



Flow rate varies similarly.

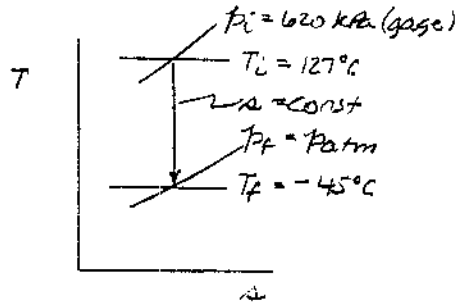
When flow rate approaches zero, $p_0 \to p_{atm}$. Assume tank behaves as a reversible adiabatic process. Thus

$$\frac{p_f}{p_0} = \left(\frac{T_f}{T_0}\right)^{\frac{k}{k-1}}$$

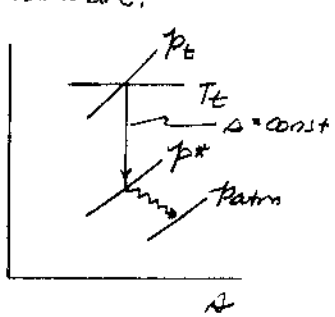
Thus $$T_f = T_0\left(\frac{p_f}{p_0}\right)^{\frac{k-1}{k}} = (273+127)K\left(\frac{101.3}{620+101.3}\right)^{0.286} = 228\ K$$

$$T_f = (228-273)°C = -45°C$$ ← T_f

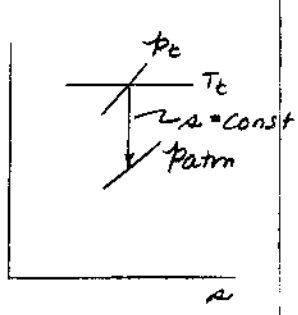
The Ts diagram for air in the tank is:



The Ts diagrams for air flowing from the tank are:



While choked
($p_t > p_{atm}/0.528$)



No longer choked
($p_t \le p_{atm}/0.528$)

Given: Jet transport aircraft cruises at 11 km altitude. Initial cabin conditions are $T_i = 25°C$, P_i = 2.5 km altitude. Cabin volume $\forall = 25 m^3$. Air escapes through a small hole with effective flow area, $A = 0.002 m^2$

Find: Time required for cabin pressure to decrease by 40 percent

Solution:

Basic equations: $\frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$

$\frac{p}{\rho^k} = \text{constant} \qquad p = \rho R T$

Assumptions: (1) model flow as isentropic flow through a converging nozzle
(2) assume uniform properties within the cabin, isentropic expansion
(3) ideal gas behavior

Stagnation conditions within the cabin are

$T_i = 298\ K \qquad P_i = P_{atm}$ at 2.5 km = 74.7 kPa (Table A.3)

$P_t = 0.60 P_i = 44.8$ kPa

Back pressure $P_b = P_{atm}$ at 11 km = 22.7 kPa

Then $P_b/P_i = 0.304$ and $P_b/P_t = 0.507$. Flow is choked

Note: conditions in cabin are stagnation conditions

From continuity

$$\frac{\partial}{\partial t}\int_{CV} \rho d\forall = -\int_{CS} \rho \vec{V} \cdot d\vec{A} = -\rho_e V_e A_e$$

$$+\ \frac{dp}{dt} = -\rho_e V_e A_e.$$

For choked flow, $M_e = 1.0$

$$\frac{p}{p_e} = \left[1 + \frac{k-1}{2} M_e^2\right]^{\frac{k}{k-1}} = (1.20)^{2.5} = 1.5774 \qquad \therefore \rho_e = 0.6339\,\rho$$

$$\frac{T}{T_e} = 1 + \frac{k-1}{2} M_e^2 = 1.2 \qquad \therefore T_e = 0.8333\,T$$

$$V_e = (kRT_e)^{1/2} = (kR)^{1/2}(0.8333T)^{1/2} = 0.9129\,(kR)^{1/2} T^{1/2}$$

Then

$$+\ \frac{dp}{dt} = -\rho_e V_e A_e = -0.6339\rho\,(0.9129)(kR)^{1/2} T^{1/2} A_e$$

$$+\ \frac{dp}{dt} = -0.5787\,(kR)^{1/2} A_e\, \rho T^{1/2}$$

For an isentropic expansion p and T can be related

$$\frac{p}{\rho^k} = \text{const} = \frac{pRT}{\rho^k} \qquad \therefore \quad p^{(1-k)} T = \text{constant}$$

For $\rho^{(1-k)} T = \rho_i^{(1-k)} T_i$ or $T = T_i \left(\frac{\rho}{\rho_i}\right)^{(1-k)}$ and $T^{1/2} = \frac{T_i^{1/2}}{\rho_i^{(k-1)/2}} \rho^{(k-1)/2}$

Substituting we obtain

$$-\frac{d\rho}{dt} = -0.5787 (kR)^{1/2} \frac{A_e}{A} \rho \frac{T_i^{1/2}}{\rho_i^{(k-1)/2}} \rho^{(k-1)/2}$$

$$\frac{d\rho}{dt} = -0.5787 (kR)^{1/2} \frac{A_e}{A} \frac{T_i^{1/2}}{\rho_i^{(k-1)/2}} \rho^{\frac{k+1}{2}} = C_1 \rho^{(k+1)/2}$$

$$\frac{d\rho}{\rho^{(k+1)/2}} = -C_1 dt \qquad \text{where } C_1 = 0.5787 (kR)^{1/2} \frac{A_e}{A} \frac{T_i^{1/2}}{\rho_i^{(k-1)/2}}$$

To integrate, we write

$$-C_1 t = \int_{\rho_i}^{\rho_f} \rho^{-\frac{(k+1)}{2}} d\rho = \frac{1}{1-\frac{(k+1)}{2}} \rho^{1-\frac{(k+1)}{2}} \Big]_{\rho_i}^{\rho_f} = \frac{2}{(1-k)} \rho^{\frac{(1-k)}{2}} \Big]_{\rho_i}^{\rho_f}$$

$$-C_1 t = \frac{2}{(k-1)} \left[\rho_i^{(1-k)/2} - \rho_f^{(1-k)/2} \right] = \frac{2}{(k-1)} \rho_i^{(1-k)/2} \left[1 - \left(\frac{\rho_f}{\rho_i}\right)^{(1-k)/2} \right]$$

$$C_1 t = \frac{2}{(k-1)} \rho_i^{(1-k)/2} \left[\left(\frac{\rho_i}{\rho_f}\right)^{(k-1)/2} - 1 \right]$$

$$0.5787 (kR)^{1/2} \frac{A_e}{A} \frac{T_i^{1/2}}{\rho_i^{(k-1)/2}} t = \frac{2}{(k-1)} \rho_i^{(1-k)/2} \left[\left(\frac{\rho_i}{\rho_f}\right)^{(k-1)/2} - 1 \right]$$

$$0.5787 (kR)^{1/2} \frac{A_e}{A} T_i^{1/2} t = \frac{2}{(k-1)} \left[\left(\frac{\rho_i}{\rho_f}\right)^{\frac{(k-1)}{2}} - 1 \right]$$

Since $P/\rho^k = \text{const}$, $\frac{\rho_i}{\rho_f} = \left(\frac{p_i}{p_f}\right)^{1/k}$.

and $$0.5787 (kR)^{1/2} \frac{A_e}{A} T_i^{1/2} t = \frac{2}{(k-1)} \left[\left(\frac{p_i}{p_f}\right)^{\frac{(k-1)}{2k}} - 1 \right]$$

Substituting numerical values

$$0.5787 \left[1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 298 K \times \frac{kg \cdot m}{N \cdot s^2} \right]^{1/2} 0.002 m^2 \times \frac{1}{25 m^3} t$$

$$= \frac{2}{0.4} \left[\left(\frac{1}{0.6}\right)^{0.1429} - 1 \right]$$

$$0.01602 t = 0.3786$$

$$t = 23.6 \text{ s.}$$

t

Given: Air escapes from high-pressure bicycle tire through hole having $D = 0.254$ mm, $p_1 = 620$ kPa (gage), and $T = 27°C$, which remains constant. Internal volume of tire is $\forall = 4.26 \times 10^{-4}$ m^3, and also is constant.

Find: (a) Time needed for p in tire to drop to 310 kPa (gage).
(b) Entropy change of air in tire during this process.
(c) Sketch a Ts diagram showing states and process paths.

Solution: Apply continuity equation, isentropic relationships.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ $\qquad \frac{T_0}{T} = \left(1 + \frac{k-1}{2} M^2\right)$; $\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{k}{k-1}}$

Check for choking: $\frac{p_{atm}}{p_{min}} = \frac{101}{310+101} = 0.246 < 0.528$ so <u>always</u> choked.

Thus $\dot{m} = \rho^* V^* A^*$. Assume: (1) Uniform density in tire: $\int_{CV} = \rho \forall$
(2) Uniform flow at throat
(3) Isentropic process to throat.

Then

$$0 = \forall \frac{d\rho}{dt} + \rho^* V^* A_e$$

But $\rho^* = \frac{\rho}{\left(1 + \frac{k-1}{2} M_*^2\right)^{1/k-1}} = \frac{\rho}{(1.2)^{2.5}} = 0.634\,\rho$

so $\frac{d\rho}{\rho} = -0.634 \frac{V^* A_e}{\forall} dt$

Integrating, $\ln \frac{\rho_2}{\rho_1} = -0.634 \frac{V^* A_e}{\forall} t = \ln \frac{p_2}{p_1}$ since T = constant

Thus

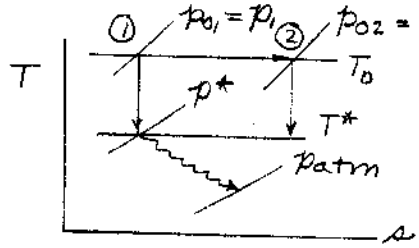
$$t = -\frac{\forall}{0.634\, V^* A_e} \ln \frac{p_2}{p_1}$$

$$V^* = c^* = \sqrt{kRT^*} = \left[1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times \frac{273+27}{1.2} K \times \frac{kg\cdot m}{N\cdot s^2}\right]^{1/2} = 317 \text{ m/s}$$

$$A^* = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.000254)^2 m^2 = 5.07 \times 10^{-8} m^2$$

$$t = -\frac{1}{0.634} \times 4.26 \times 10^{-4} m^3 \times \frac{s}{317\, m} \times \frac{1}{5.07 \times 10^{-8} m^2} \times \ln\left(\frac{310+101}{620+101}\right) = 23.5 \text{ s} \qquad t$$

Ts diagram:



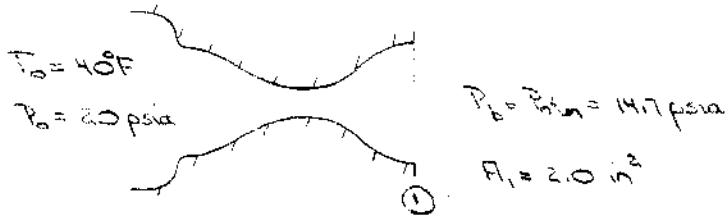
Process ① → ② in tire

Process (isentropic) ① → * (moving to) ② → * in converging passage.

In tire,

$$\Delta s = c_p \ln \frac{T^*}{T_1} - R \ln \frac{p_2}{p_1} = -287 \frac{N\cdot m}{kg\cdot K} \times \ln \frac{(310+101)}{(620+101)} = 161 \text{ J/(kg·K)} \qquad \Delta s$$

Given: Isentropic flow of air from a large tank through a converging-diverging nozzle discharges to atmosphere



Find: $\dot{m}$

Solution:

Basic equations: $\dot{m} = \rho V A$ $\quad P = \rho R T$

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$ $\quad \frac{P_0}{P} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$

Assumptions: 1) steady flow 3) uniform flow at a section
2) isentropic flow 4) ideal gas

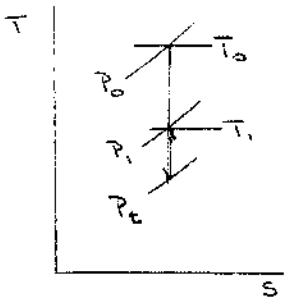
$$\frac{P_b}{P_0} = \left[1 + \frac{k-1}{2} M_1^2\right]^{-k/(k-1)} \quad ; \quad M_1 = \left\{\frac{2}{k-1}\left[\left(\frac{P_0}{P_b}\right)^{(k-1)/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{20}{14.7}\right)^{0.286} - 1\right]\right\}^{1/2} = 0.678$$

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2 \quad ; \quad T_1 = \frac{T_0}{1 + \frac{k-1}{2} M_1^2} = \frac{500°R}{1 + 0.2(0.678)^2} = 458°R$$

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 0.678 \left(1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot °R} \times 458°R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right)^{1/2} = 711 \text{ ft/s}$$

$$\rho_1 = \frac{P_1}{RT_1} = 14.7 \frac{lbf}{in^2} \times \frac{lbm \cdot °R}{53.3 \text{ ft} \cdot lbf} \times \frac{1}{458°R} \times \frac{144 in^2}{ft^2} = 0.0867 \text{ lbm/ft}^3$$

$$\dot{m} = \rho_1 V_1 A_1 = 0.0867 \frac{lbm}{ft^3} \times 711 \frac{ft}{s} \times 2.0 in^2 \times \frac{ft^2}{144 in^2} = 0.856 \text{ lbm/s} \quad \dot{m}$$



Given: Converging-diverging nozzle with $M_{design} = 3.0$ and $A_e = 250\ mm^2$. Nozzle bolted to side of large tank with $p = 4.5\ MPa$ (gage) and $T = 750\ K$. Flow in nozzle is isentropic.

Find: (a) Pressure in nozzle exit plane.
(b) Mass flow rate in nozzle.
(c) Sketch Ts diagram: label tank, throat, exit, and ambient.

Solution: Assume stagnation conditions in tank, isentropic flow in nozzle.

Computing equations: $\frac{p_0}{p} = \left(1 + \frac{k-1}{2}M^2\right)^{\frac{k}{k-1}}$; $\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$; $c = \sqrt{kRT}$

For $M_e = 3.0$, $\frac{p_0}{p_e} = \left[1 + \frac{k-1}{2}(3.0)^2\right]^{\frac{k}{k-1}} = 36.7$

$$p_e = \frac{p_0}{36.7} = \frac{(4.5\times10^6 + 101\times10^3)\,Pa}{36.7} = 125\ kPa\ (abs)\ \text{or}\ 24\ kPa\ (gage)$$ ← p_e

Thus the nozzle is underexpanded. For steady, 1-D flow, $\dot{m} = \rho_e V_e A_e$.

$$T_e = \frac{T_0}{1 + \frac{k-1}{2}(3.0)^2} = \frac{750\ K}{1 + 0.2(3)^2} = 268\ K$$

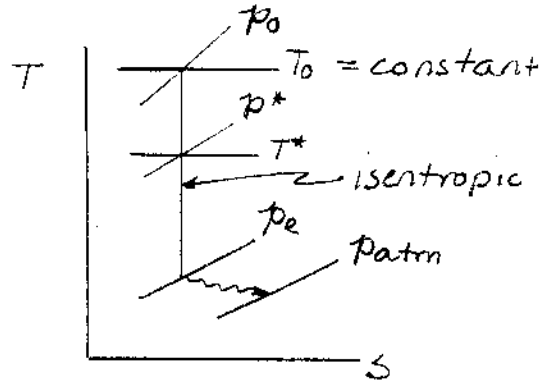
$$c_e = \sqrt{kRT_e} = \left[1.4 \times 287\ \frac{N\cdot m}{kg\cdot K} \times 268\ K \times \frac{kg\cdot m}{N\cdot s^2}\right]^{1/2} = 328\ m/s$$

$$\rho_e = \frac{p_e}{RT_e} = 125\times10^3\ \frac{N}{m^2} \times \frac{kg\cdot K}{287\ N\cdot m} \times \frac{1}{268\ K} = 1.63\ kg/m^3$$

Finally, $V_e = M_e c_e = 3.0 \times 328\ m/s = 984\ m/s$

$$\dot{m} = \rho_e V_e A_e = 1.63\ \frac{kg}{m^3} \times 984\ \frac{m}{s} \times 250\ mm^2 \times \frac{m^2}{10^6\ mm^2} = 0.401\ kg/s$$ ← $\dot{m}$

Ts diagram:

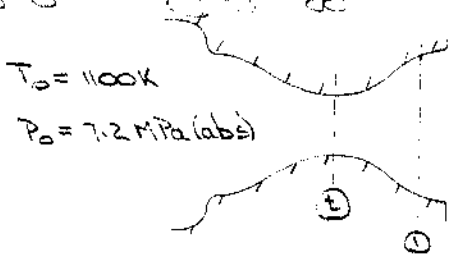


0 = tank conditions

* = throat conditions

e = exit plane conditions

Given: Isentropic flow of air from stagnation state through a converging - diverging nozzle as shown



$A_t = 0.01 m^2$

Find: V_1, $\dot{m}$

Solution:

Basic equations: $\dot{m} = \rho V A$ $\quad P = \rho RT$

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$ $\quad \frac{P_0}{P} = \left[1 + \frac{k-1}{2} M^2\right]^{k/k-1}$

Assumptions: (1) steady flow (3) uniform flow at a section
(2) isentropic flow (4) ideal gas

$\frac{T_0}{T_1} = 1 + \frac{k-1}{2} M_1^2$; $T_1 = \frac{T_0}{1 + \frac{k-1}{2} M_1^2} = \frac{1100K}{1 + 0.2(4.0)^2} = 262 K$.

$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 4.0 \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 262K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 1300 m/s$ ← V_1

Since $M_1 = 4.0$, nozzle must be choked and $M_t = 1.0$

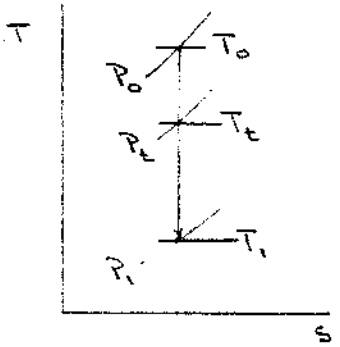
$P_t = \frac{P_0}{\left[1 + \frac{k-1}{2} M_t^2\right]^{k/k-1}} = \frac{7.2 \times 10^6 Pa}{[1 + 0.2(1.0)^2]^{3.5}} = 3.80 MPa$

$T_t = \frac{T_0}{1 + \frac{k-1}{2} M_t^2} = \frac{1100K}{1 + 0.2(1.0)^2} = 917K$

$\rho_t = \frac{P_t}{RT_t} = 3.80 \times 10^6 \frac{N}{m^2} \times \frac{kg \cdot K}{287 N \cdot m} \times \frac{1}{917K} = 14.4 kg/m^3$

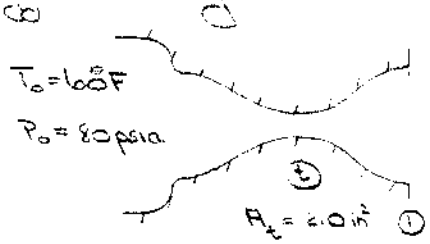
$V_t = M_t c_t = M_t (kRT_t)^{1/2} = 1.0 \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 917K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 607 m/s$

$\dot{m} = \rho_t V_t A_t = 14.4 \frac{kg}{m^3} \times 607 \frac{m}{s} \times 0.01 m^2 = 87.4 kg/s$ ← $\dot{m}$



Problem 12.32

Given: Isentropic flow of air from a large tank through a converging-diverging nozzle discharges to a back pressure P_b



Find: $\dot{m}$, A_1

Solution

Governing equations:

$$\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2 \qquad \frac{P_o}{P} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$$

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2} M^2}{\frac{k+1}{2}}\right]^{(k+1)/2(k-1)} \qquad (12.6)$$

Assumptions: (1) steady flow (2) isentropic flow (3) uniform flow at a section (4) ideal gas

$$P_o/P_1 = \left[1 + \frac{k-1}{2} M_1^2\right]^{k/(k-1)}, \quad M_1 = \left\{\frac{2}{k-1}\left[\left(\frac{P_o}{P_1}\right)^{(k-1)/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{80}{12.9}\right)^{0.286} - 1\right]\right\}^{1/2} = 1.85$$

$$T_o/T = 1 + \frac{k-1}{2} M^2 \quad ; \quad T_1 = \frac{T_o}{1 + \frac{k-1}{2} M_1^2} = \frac{520°R}{1 + 0.2(1.85)^2} = 309°R$$

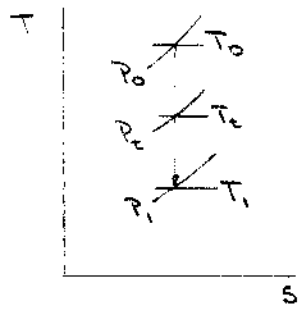
$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 1.85 \left(1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot °R} \times 309°R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft/s^2}{lbf}\right)^{1/2} = 1594 \text{ ft/s}$$

$$\rho_1 = \frac{P_1}{RT_1} = 12.9 \frac{lbf}{in^2} \times \frac{lbm \cdot °R}{53.3 ft \cdot lbf} \times \frac{1}{309°R} \times \frac{144 in^2}{ft^2} = 0.113 \text{ lbm/ft}^3$$

Since $M_1 = 1.85$, nozzle must be choked and $M_2 = 1.0$; $A_2 = A^*$

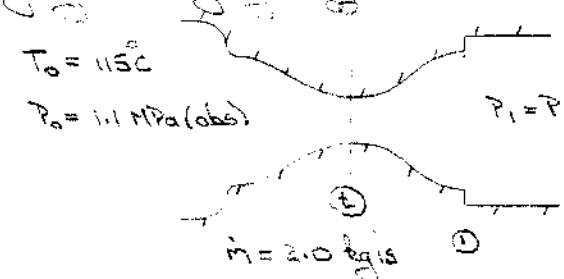
For $M_1 = 1.85$, from Eq. 12.6 (and Fig E.1), $A_1/A^* = 1.496$; $\therefore A_1 = 2.99 in^2$ ← A_1

$$\dot{m} = \rho_1 V_1 A_1 = 0.113 \frac{lbm}{ft^3} \times 1594 \frac{ft}{s} \times 2.99 in^2 \times \frac{ft^2}{144 in^2} = 3.74 \text{ lbm/s}$$ ← $\dot{m}$



Problem 12.33

Given: Isentropic flow of air from stagnation state through a converging-diverging nozzle to a back pressure P_b



Find: A_t, A_1

Solution:

Computing equations:

$$\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2 \qquad \frac{p_o}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$$

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2} M^2}{1 + \frac{k-1}{2}}\right]^{(k+1)/2(k-1)} \qquad (12.6)$$

Assumptions: 1) steady flow 2) isentropic flow 3) uniform flow at a section 4) ideal gas

$$p_o/p_1 = \left[1 + \frac{k-1}{2} M_1^2\right]^{k/(k-1)} \quad ; \quad M_1 = \left\{\frac{2}{k-1}\left[\left(\frac{p_o}{p_1}\right)^{(k-1)/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{1.1}{0.141}\right)^{0.286} - 1\right]\right\}^{1/2} = 2.0$$

$$T_o/T = 1 + \frac{k-1}{2} M^2 \quad ; \quad T_1 = \frac{T_o}{1 + \frac{k-1}{2} M_1^2} = \frac{388\,K}{1 + 0.2(2.0)^2} = 216\,K$$

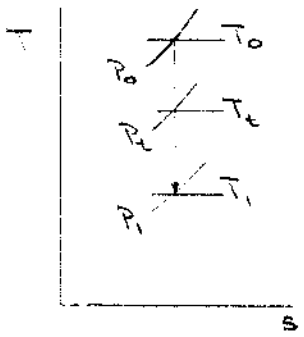
$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 2.0 \left(1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 216\,K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 589\ m/s$$

$$\rho_1 = \frac{p_1}{RT_1} = \frac{141 \times 10^3\ N/m^2}{287\ N\cdot m/kg\cdot K \times 216\,K} = 2.27\ kg/m^3$$

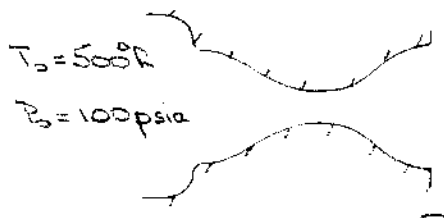
$$A_1 = \frac{\dot{m}}{\rho_1 V_1} = \frac{2.0\ kg/s}{2.27\ kg/m^3 \times 589\ m/s} = 1.50 \times 10^{-3}\ m^2 \qquad A_1$$

For $M_1 = 2.0$, from Eq. 12.6 and Eq. E.1, $A_1/A^* = 1.688$

Then $A_t = A^* = A_1/1.688 = 8.89 \times 10^{-4}\ m^2$ $\qquad A_t$



Given: Isentropic flow of air from a large tank through a converging-diverging nozzle discharging to atmosphere



Find: (a) $\dot{m}$
(b) effect on $\dot{m}$ of raising T_o to 2000°R

Solution:

Basic equations: $\dot{m} = \rho V A$ $\quad P = \rho R T$

Computing equations: $\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2$ $\quad \frac{P_o}{P} = \left[1 + \frac{k-1}{2} M^2\right]^{k/k-1}$

Assumptions: 1) steady flow 3) uniform flow at a section
2) isentropic flow 4) ideal gas

$$\frac{P_o}{P_e} = \left[1 + \frac{k-1}{2} M_e^2\right]^{k/k-1} \quad ; \quad M_e = \left\{\frac{2}{k-1}\left[\left(\frac{P_o}{P_e}\right)^{k-1/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{100}{14.7}\right)^{0.286} - 1\right]\right\}^{1/2} = 1.91$$

$$\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2 \quad ; \quad T_e = \frac{T_o}{1 + \frac{k-1}{2} M_e^2} = \frac{500°R}{1 + 0.2(1.91)^2} = 289°R$$

$$V_e = M_e C_e = M_e (kRT_e)^{1/2} = 1.91\left(1.4 \times 53.3 \frac{ft\,lbf}{lbm°R} \times 289°R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right)^{1/2} = 1592 \text{ ft/s}$$

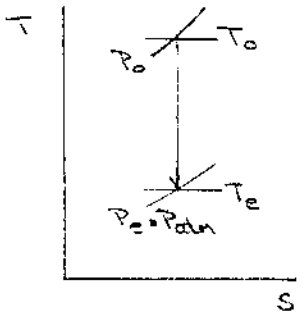
$$\rho_e = \frac{P_e}{RT_e} = 14.7 \frac{lbf}{in^2} \times \frac{lbm°R}{53.3\ ft\,lbf} \times \frac{1}{289°R} \times 144 \frac{in^2}{ft^2} = 0.137 \text{ lbm/ft}^3$$

$$\dot{m} = \rho_e V_e A_e = 0.137 \frac{lbm}{ft^3} \times 1590 \frac{ft}{s} \times 4.0 \text{ in}^2 \times \frac{ft^2}{144 \text{ in}^2} = 6.05 \text{ lbm/s} \quad \leftarrow \dot{m}$$

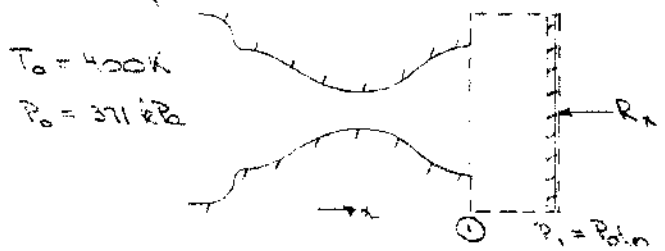
$\dot{m} = \rho_e V_e A_e$ If T_o is increased by a factor of 4 (holding pressures constant):

(1) T_e will increase by a factor of 4 (since $T_o/T = \text{const}$)
(2) V_e " " " " " " 2 (since $V_e \propto T_e^{1/2}$)
(3) ρ_e " decrease " " " " 4 (since $\rho_e \propto 1/T_e$)

Thus the mass flow rate, $\dot{m}$, will decrease by a factor of 2 ←



Given: Flow of nitrogen through a converging-diverging nozzle leaves nozzle at atmospheric pressure, exhaust impinges on vertical flat plate.



Find: Force required to hold the plate

Solution:

Basic equations: $F_{S_x} = -R_x = -\dot{m} V_1$ (momentum eq. for CV shown)

$\dot{m} = \rho V A = const$ $\quad P = \rho R T$

Computing equations: $\frac{T_o}{T_1} = 1 + \frac{k-1}{2} M^2$ $\quad \frac{P_o}{P} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$

Assumptions: (1) steady flow (4) P_{atm} over entire C.S.
(2) isentropic flow in nozzle (5) $F_{B_x} = 0$
(3) uniform flow at a section (6) ideal gas $k = 1.40$

$$\frac{P_o}{P_1} = \left[1 + \frac{k-1}{2} M_1^2\right]^{k/(k-1)} ; \quad M_1 = \left\{\frac{2}{k-1}\left[\left(\frac{P_o}{P_1}\right)^{(k-1)/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{371}{101}\right)^{0.286} - 1\right]\right\}^{1/2} = 1.50$$

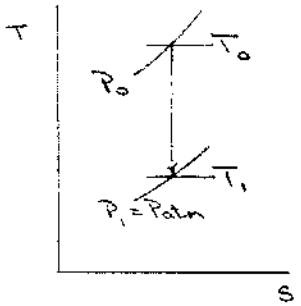
$$\frac{T_o}{T_1} = 1 + \frac{k-1}{2} M_1^2 ; \quad T_1 = \frac{T_o}{1 + \frac{k-1}{2} M_1^2} = \frac{400K}{1 + 0.2(1.50)^2} = 276K$$

$$V_1 = M_1 c_1 = M_1 (kRT)^{1/2} = 1.5 \left(1.4 \times 297 \frac{N \cdot m}{kg \cdot K} \times 276K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 508\ m/s$$

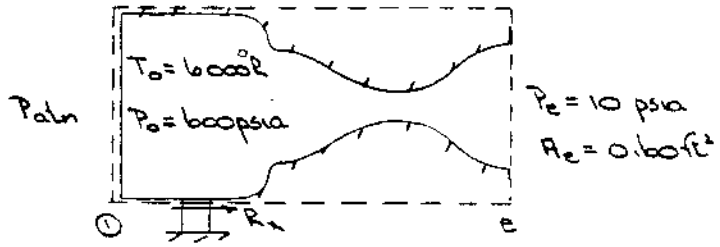
$$\rho_1 = \frac{P_1}{RT_1} = 101 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{297\ N \cdot m} \times \frac{1}{276K} = 1.23\ kg/m^3$$

$$\dot{m} = \rho_1 V_1 A_1 = 1.23 \frac{kg}{m^3} \times 508 \frac{m}{s} \times 0.003 m^2 = 1.87\ kg/s$$

$$R_x = \dot{m} V_1 = 1.87 \frac{kg}{s} \times 508 \frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m} = 950N \leftarrow \text{(to the left as shown). } R_x$$



Given: Small solid fuel rocket motor tested on a thrust stand. Exhaust gases expand isentropically to $p_{atm} = 10$ psia. Treat gas as an ideal gas with $k = 1.2$, $R = 60.0$ ft·lbf/lbm·R.



Find: (a) $\dot{m}$ (b) force exerted against test stand.

Solution:

Basic equations: $F_{sx} = p_1 A_1 - p_e A_e - p_{atm}(A_1 - A_e) + R_x = \dot{m} V_e$

$\dot{m} = \rho V A \qquad p = \rho R T$

Computing equations: $\frac{T}{T_o} = 1 + \frac{k-1}{2} M^2 \qquad \frac{p_o}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$

Assumptions: (1) steady flow (3) uniform flow at a section
(2) isentropic flow (4) ideal gas, $k = 1.2$

$$p_o/p_e = \left[1 + \frac{k-1}{2} M_e^2\right]^{k/(k-1)}; \quad M_e = \left\{\frac{2}{k-1}\left[\left(\frac{p_o}{p_e}\right)^{(k-1)/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.2}\left[\left(\frac{600}{10}\right)^{0.167} - 1\right]\right\}^{1/2} = 3.13$$

$$T_o/T_e = 1 + \frac{k-1}{2} M_e^2 = 1.9786 \qquad \therefore T_e = \frac{T_o}{1.9786} = \frac{6000°R}{1.9786} = 3030°R$$

$$V_e = M_e c_e = M_e (kRT_e)^{1/2} = 3.13\left(1.2 \times 60.0 \frac{ft\cdot lbf}{lbm\cdot R} \times 3030°R \times 32.2 \frac{lbm}{slug} \times \frac{slug\cdot ft/s^2}{lbf}\right)^{1/2} = 8300 \text{ ft/s}$$

$$\rho_e = \frac{p_e}{RT_e} = 10 \frac{lbf}{in^2} \times \frac{lbm\cdot°R}{60 ft\cdot lbf} \times \frac{1}{3030°R} \times 144 \frac{in^2}{ft^2} = 7.92 \times 10^{-3} \text{ lbm/ft}^3$$

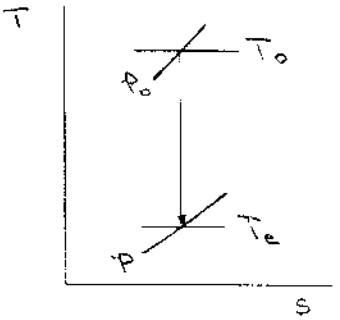
$$\dot{m} = \rho_e V_e A_e = 7.92 \times 10^{-3} \frac{lbm}{ft^3} \times 8300 \frac{ft}{s} \times 0.60 ft^2 = 39.4 \text{ lbm/s} \quad \dot{m}$$

$$R_x = \dot{m} V_e - p_1 A_1 + p_e A_e + p_{atm}(A_1 - A_e) = \dot{m} V_e + (p_e - p_{atm}) A_e + (\cancel{p_{atm}} - p_1)^{\to 0} A_1$$

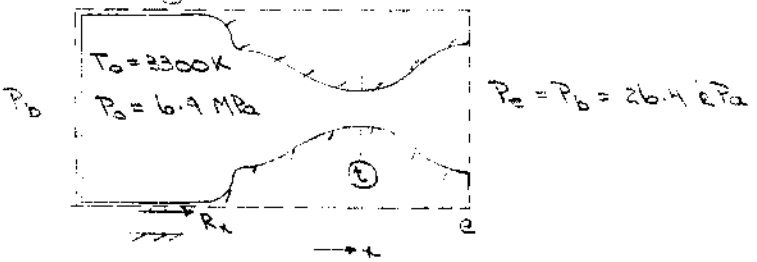
$$= 39.4 \frac{lbm}{s} \times 8300 \frac{ft}{s} + \frac{slug}{32.2 lbm} \times \frac{lbf\cdot s^2}{ft\cdot slug} + (10 - 14.7) \frac{lbf}{in^2} \times 0.60 ft^2 \times 144 \frac{in^2}{ft^2}$$

$R_x = 9750$ lbf (to the right)

Force on the test stand, $K_x = -R_x = -9750$ lbf (to the left) K_x



Given: Liquid rocket motor designed to expand exhaust gases isentropically to design back pressure corresponding to an altitude of 10,000 m. Thrust produced is 100 kN. Exhaust gases treated as water vapor and ideal gas behavior assumed.



Find: a) $\dot{m}$ b) A_e c) A_e/A_t

Solution

Basic equations: $F_{sx} = R_x = \dot{m} V_e$ $\dot{m} = \rho V A$ $P = \rho R T$

Computing equations $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$ $\frac{P_0}{P} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2} M^2}{1 + \frac{k-1}{2}}\right]^{(k+1)/2(k-1)} \qquad (12.6)$$

Assumptions: (1) Steady flow (4) Thrust specified at altitude
(2) Isentropic flow (5) $F_{ex} = 0$
(3) Uniform flow at section (6) Ideal gas, $k = 1.3$, $R = 461 \frac{N \cdot m}{kg \cdot K}$

$P_0/P = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$; $M_e = \left\{\frac{2}{k-1}\left[\left(\frac{P_0}{P}\right)^{(k-1)/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.30}\left[\left(\frac{690}{12.64}\right)^{0.231} - 1\right]\right\}^{1/2} = 4.18$

$T_0/T = 1 + \frac{k-1}{2} M^2$; $T_e = \frac{T_0}{1 + \frac{k-1}{2} M_e^2} = \frac{3300\,K}{1 + 0.15(4.18)^2} = 911\,K$

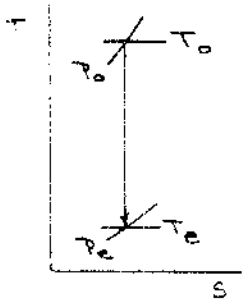
$V_e = M_e C_e = M_e (kRT_e)^{1/2} = 4.18\left[1.3 \times 461 \frac{N}{kg \cdot K} \times 911\,K \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 3090$ m/s

$R_x = \dot{m} V_e$; $\dot{m} = \frac{R_x}{V_e} = 10^5\,N \times \frac{s}{3090\,m} \times \frac{kg \cdot m}{N \cdot s^2} = 32.4$ kg/s ← $\dot{m}$

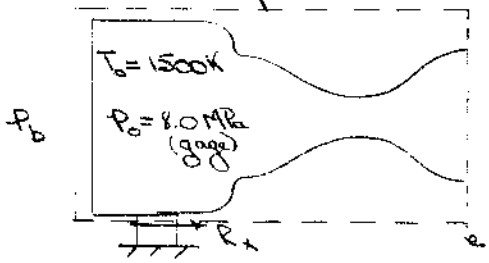
$\rho_e = \frac{P_e}{RT_e} = 26.4 \times 10^3 \frac{N}{m^2} \times \frac{1}{461 \frac{N \cdot m}{kg \cdot K}} \times \frac{1}{911\,K} = 6.29 \times 10^{-2}$ kg/m^3

$\dot{m} = \rho_e V_e A_e$; $A_e = \frac{\dot{m}}{\rho_e V_e} = 32.4\,kg/s \times \frac{m^3}{6.29 \times 10^{-2}\,kg} \times \frac{s}{3090\,m} = 0.167\,m^2$ ← A_e

For $M_e = 4.18$, from Eq 12.6, $A_e/A^* = A_e/A_t = 19.4$ ← A_e/A_t



Given: Small rocket motor, fueled with H_2 and O_2, is tested on thrust stand at a simulated altitude of 10 km. Combustion product is water vapor which may be treated as an ideal gas.



$M_e = 3.5$

$A_e = 700\ mm^2$

p_b = p at 10 km altitude



Find: (a) p_e (b) $\dot{m}$ (c) force on test stand

Solution:

Basic equations: $\dot{m} = \rho V A$, $p = \rho R T$

Assumptions: (1) steady flow (3) ideal gas behavior $k = 1.32$, $R = 461\ J/kg\cdot K$
(2) isentropic flow

At 10 km altitude, $p_b = 26.5$ kPa (Table A.3)

Evaluate design pressure at exit

$$\frac{p_o}{p} = \left[1 + \left(\frac{k-1}{2}\right)M^2\right]^{k/(k-1)} \quad \therefore \quad p_d = \frac{8.10\times10^6\ Pa}{[1 + 0.15(3.5)^2]^{4.333}} = 88.3\ kPa\ (abs)$$

Since $p_b < p_d$, $p_e = p_d = 88.3$ kPa ← p_e

$$\frac{T_o}{T_e} = 1 + \frac{k-1}{2}M_e^2 \quad \therefore T_e = \frac{T_o}{1+(k-1)M^2} = \frac{1500\ K}{1+0.15(3.5)^2} = 529\ K$$

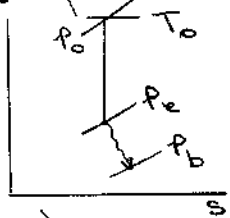
$$\rho_e = \frac{p_e}{RT_o} = 88.3\times10^3\ \frac{N}{m^2}\times\frac{kg\cdot K}{461\ N\cdot m}\times\frac{1}{529\ K} = 0.362\ kg/m^3$$

$$V_e = MC_e = 3.5\left[1.30\times461\ \frac{N\cdot m}{kg\cdot K}\times529\ K\times\frac{kg\cdot m}{N\cdot s^2}\right]^{1/2} = 1970\ m/s$$

$$\dot{m} = \rho_e V_e A_e = 0.362\ \frac{kg}{m^3}\times1970\ \frac{m}{s}\times700\ mm^2\times\frac{m^2}{10^6\ mm^2} = 0.499\ kg/s$$ ← $\dot{m}$

To determine force on test stand apply momentum equation to CV shown

Basic equation: $F_{S_x} + F_{B_x}^{\,0} = \frac{\partial}{\partial t}\int_{CV} V_x\,\rho\,d\forall + \int_{CS} V_x\,\rho\,\vec{V}\cdot d\vec{A}$



$$R_x + p_b A_e - p_e A_e = \dot{m} V_e$$

Force on test stand is $K_x = -R_x$

$$\therefore K_x = -R_x = -\dot{m}V_e - A_e(p_e - p_b)$$

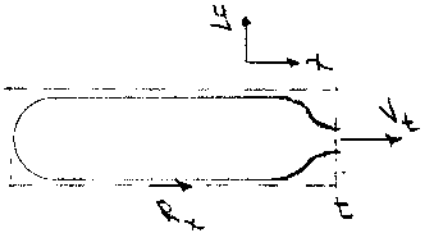
$$K_x = -0.499\ \frac{kg}{s}\times1970\ \frac{m}{s}\times\frac{N\cdot s^2}{kg\cdot m} - 700\ mm^2\times\frac{m^2}{10^6\ mm^2}(88.3-26.5)\times10^3\ \frac{N}{m^2}$$

$$K_x = -1,026\ N\ (\text{to left})$$ ← K_x

Given: Compressed CO_2 at 6000 psig and 70°F in a cartridge is expanded through a converging nozzle ($D_t = 0.020$ in). Back pressure is atmospheric

Find: (a) P_t (b) $\dot{m}$ (c) thrust available
(d) increase in thrust if diverging section were added to nozzle to expand gas to atmospheric pressure

Solution



R_x = force on CV to hold cartridge stationary

Basic equation: $F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} V_x \rho \, d\forall + \int_{CS} V_x \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) isentropic flow
(2) stagnation conditions in cartridge, $V_{CV} \approx 0$
(3) ideal gas $k = 1.29$
$R = 35.11$ ft·lbf/lbm·°R
(4) uniform flow at a section

From isentropic relations,

$$p^* = \frac{p_0}{\left[1 + \frac{k-1}{2} M^{*2}\right]^{k/(k-1)}} = \frac{6,014.7 \text{ psia}}{\left[1 + \frac{0.29}{2}\right]^{1.29/0.29}} = 3290 \text{ psia}$$

Since $p_b \ll p^*$, then $p_t = p^* = 3293$ psia ← p_t

$$\dot{m} = \rho_t V_t A_t$$

$$T_t = \frac{T_0}{1 + \frac{k-1}{2} M_t} = \frac{530°R}{1 + 0.29/2} = 463°R$$

$$V_t = C_t = \left(kRT_t\right)^{1/2} = \left[1.29 \times 35.11 \frac{\text{ft·lbf}}{\text{lbm·°R}} \times 463°R \times 32.2 \frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug·ft}}{\text{lbf·s}^2}\right]^{1/2} = 822 \text{ ft/s}$$

$$\rho_t = \frac{p_t}{RT_t} = 3293 \frac{\text{lbf}}{\text{in}^2} \times 35.11 \frac{\text{lbm·°R}}{\text{ft·lbf}} \times \frac{1}{463°R} \times \frac{144 \text{ in}^2}{\text{ft}^2} = 29.2 \text{ lbm/ft}^3$$

$$\therefore \dot{m} = \rho_t V_t A_t = 29.2 \frac{\text{lbm}}{\text{ft}^3} \times 822 \frac{\text{ft}}{\text{s}} \times \frac{\pi}{4}(0.020)^2 \text{ in}^2 \times \frac{\text{ft}^2}{144 \text{ in}^2} = 0.0524 \frac{\text{lbm}}{\text{s}}$$ ← $\dot{m}$

For one-dimensional flow with $F_{B_x} = 0$ the momentum equation reduces to

$$R_x - p_{t_{gage}} A_t = \dot{m} V_t$$

$$\therefore R_x = \dot{m} V_t + p_{t,g} A_t$$

$$R_x = 0.0524 \frac{\text{lbm}}{\text{s}} \times 822 \frac{\text{ft}}{\text{s}} \times \frac{\text{slug}}{32.2 \text{ lbm}} \times \frac{\text{lbf·s}^2}{\text{slug·ft}} + (3293 - 14.7) \frac{\text{lbf}}{\text{in}^2} \times \frac{\pi}{4}(0.020)^2 \text{ in}^2$$

$$R_x = 2.37 \text{ lbf}$$

Since R_x is force on CV required to hold cartridge stationary,
Thrust = $-R_x = -2.37$ lbf (ie to left) ← Thrust

(a) When a diverging section is added to expand the gas to atmospheric pressure, $p_e = p_t = p_{atm}$ and the exit flow is supersonic

$$\frac{p_0}{p_e} = \left[1 + \frac{k-1}{2} M_e^2\right]^{\frac{k}{k-1}}$$

$$M_e = \left\{\frac{2}{k-1}\left[\left(\frac{p_0}{p_e}\right)^{\frac{k-1}{k}} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.29}\left[\left(\frac{604.7}{14.7}\right)^{0.2248} - 1\right]\right\}^{1/2} = 4.445$$

$$T_e = \frac{T_0}{1 + \frac{k-1}{2} M_e^2} = \frac{530^\circ R}{1 + \frac{0.29}{2}(4.452)^2} = 137^\circ R$$

$$V_e = M_e C_e = M_e (kRT_e)^{1/2} = 4.445\left[1.29 \times 35.11 \frac{ft\,lbf}{lbm\,^\circ R} \times 137^\circ R \times \frac{32.2\,lbm}{slug} \times \frac{slug \cdot ft}{lbf\,s^2}\right]^{1/2}$$

$$V_e = 1987\ ft/s$$

Since conditions at the throat are unchanged by the addition of the diverging section, the mass flow rate does not change $\dot{m} = 0.0524\ lbm/s$

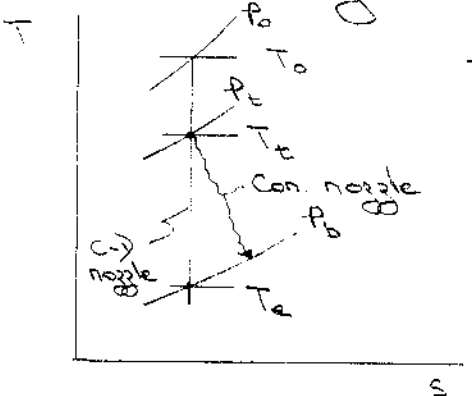
From the momentum equation

$$R_x = \dot{m} V_e = 0.0524 \frac{lbm}{s} \times 1987 \frac{ft}{s} \times \frac{slug}{32.2\,lbm} \times \frac{lbf \cdot s^2}{ft \cdot slug}$$

$$R_x = 3.23\ lbf$$

The thrust is then $T = -3.23\ lbf$ (to the left)

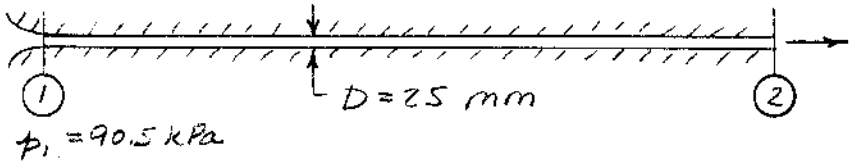
Note: The calculated thrust is the initial (maximum) thrust since stagnation conditions will decrease with time



The percent increase in initial thrust is

$$\frac{T_{C-D} - T_{con}}{T_{con}} = \frac{3.23 - 2.31}{2.31} = 36.3\%$$

Given: Air drawn into insulated duct through smooth nozzle.



Find: (a) Mass flow rate in duct.
(b) Range of duct exit pressures for choked flow at exit.

Solution:

Basic equations: $p = \rho RT$ $\dot{m} = \rho V A$

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$; $\frac{p_0}{p} = \left(1 + \frac{k-1}{2}M^2\right)^{\frac{k}{k-1}}$

Assumptions: (1) Steady flow (4) Ideal gas
(2) Uniform flow at each section
(3) Isentropic flow in nozzle, adiabatic flow in tube

Atmospheric conditions are stagnation conditions since $V \approx 0$.

Flow in nozzle is isentropic so $T_{01} = T_0$, $p_{01} = p_0$

$$M_1 = \left\{\frac{2}{k-1}\left[\left(\frac{p_0}{p_1}\right)^{\frac{k-1}{k}} - 1\right]\right\}^{1/2} = 0.399 \; ; \; T_1 = \frac{T_0}{1 + \frac{k-1}{2}M_1^2} = 291\ K$$

$$\rho_1 = \frac{p_1}{RT_1} = 90.5\times10^3\ \frac{N}{m^2} \times \frac{kg\cdot K}{287\ N\cdot m} \times \frac{1}{291\ K} = 1.08\ kg/m^3 \; ; \; A = \frac{\pi D^2}{4} = 4.91\times10^{-4}\ m^2$$

$$V_1 = M_1 c_1 = M_1\sqrt{kRT_1} = 0.399\sqrt{1.4\times287\ \frac{N\cdot m}{kg\cdot K}\times 291\ K\times\frac{kg\cdot m}{N\cdot s^2}} = 136\ m/s$$

$$\dot{m} = \rho_1 V_1 A = 1.08\ \frac{kg}{m^3}\times 136\ \frac{m}{s}\times 4.91\times10^{-4}\ m^2 = 0.0721\ kg/s$$ $\dot{m}$

When flow is choked, $V_2 = c_2$. Since adiabatic, $T_{02} = T_0$.

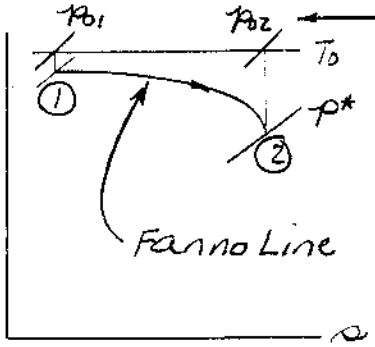
$$T_2 = \frac{T_0}{1 + \frac{k-1}{2}M_2^2} = \frac{300\ K}{1.2} = 250\ K; \; c_2 = \sqrt{kRT_2} = 317\ m/s = V_2$$

From continuity $\rho_1 V_1 = \rho_2 V_2$, so $\rho_2 = \rho_1 V_1 / V_2 = 0.463\ kg/m^3$, and

$$p_2 = \rho_2 R T_2 = 0.463\ \frac{kg}{m^3}\times 287\ \frac{N\cdot m}{kg\cdot K}\times 250\ K = 33.2\ kPa\ (abs)$$ p_2

Flow will choke for any back pressure at or below $p_2 = p^*$.

The Ts diagram is:



* From Fanno line flow functions (Table E.3), $M_1 = 0.399$, $p/p^* = 2.703$

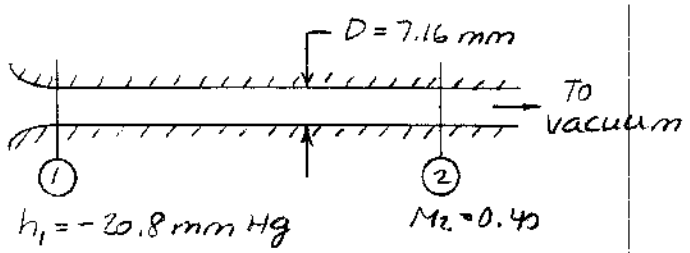
$$p^* = \frac{p_1}{(p/p^*)} = \frac{90.5}{2.703} = 33.5\ kPa$$ vv

Problem 12.41

Given: Fanno line flow apparatus in laboratory, smooth brass tube fed by converging nozzle.

$T = 23°C$

$h_{barometer} = 755.1$ mm Hg



Find: (a) M_1
(b) Mass flow rate in tube
(c) p_2

Solution: Apply equations for steady, 1-D compressible flow:

Computing equations: $T_0 = T\left(1 + \frac{k-1}{2} M^2\right)$ {Entire flow adiabatic}

$p_0 = p\left(1 + \frac{k-1}{2} M^2\right)^{\frac{k}{k-1}}$ {Isentropic in nozzle}

Assume: (1) Stagnation conditions in laboratory
(2) Ideal gas

Then $\frac{p_0}{p_1} = \left(1 + \frac{k-1}{2} M_1^2\right)^{\frac{k}{k-1}} = \frac{755.1 \text{ mm Hg}}{(755.1 - 20.8) \text{ mm Hg}} = 1.03$

$$M_1 = \left\{\frac{2}{k-1}\left[\left(\frac{p_0}{p_1}\right)^{\frac{k-1}{k}} - 1\right]\right\}^{1/2} = \left\{5\left[\left(\frac{755.1}{734.3}\right)^{0.286} - 1\right]\right\}^{1/2} = 0.200 \quad \leftarrow M_1$$

From continuity, $\dot{m} = \rho_1 V_1 A_1$; $\rho_1 = \frac{p_1}{RT_1}$

$p_1 = \rho_{Hg} g h_1 = (13.5)\,1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.734 \text{ m} \times \frac{N \cdot s^2}{kg \cdot m} = 97.2 \text{ kPa (abs)}$

$T_1 = \frac{T_0}{1 + \frac{k-1}{2} M_1^2} = \frac{(273+23)K}{1 + 0.2(0.200)^2} = 294 \text{ K}$; $c_1 = \sqrt{kRT_1} = 344$ m/s

$\rho_1 = 97.2 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287 \text{ N} \cdot \text{m}} \times \frac{1}{294 \text{ K}} = 1.15 \text{ kg/m}^3$

$V_1 = M_1 c_1 = 0.200 \times 344 \text{ m/s} = 68.8 \text{ m/s}$

$A_1 = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.00716)^2 \text{m}^2 = 4.03 \times 10^{-5} \text{m}^2$

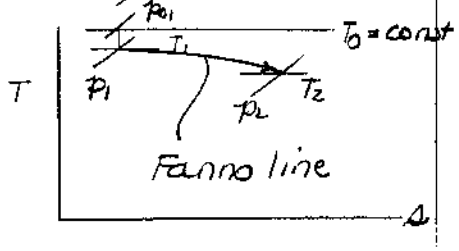
$\dot{m} = 1.15 \frac{kg}{m^3} \times 68.8 \frac{m}{s} \times 4.03 \times 10^{-5} \text{m}^2 = 3.19 \times 10^{-3} \text{ kg/s} \quad \leftarrow \dot{m}$

Since T_0 = constant, $T_2 = T_0 / \left(1 + \frac{k-1}{2} M_2^2\right) = 287$ K ; $c_2 = 340$ m/s ; $V_2 = M_2 c_2 = 136$ m/s

$\rho_2 = \rho_1 \frac{V_1}{V_2} = 1.15 \frac{kg}{m^3} \times \frac{68.8}{136} = 0.582 \text{ kg/m}^3$

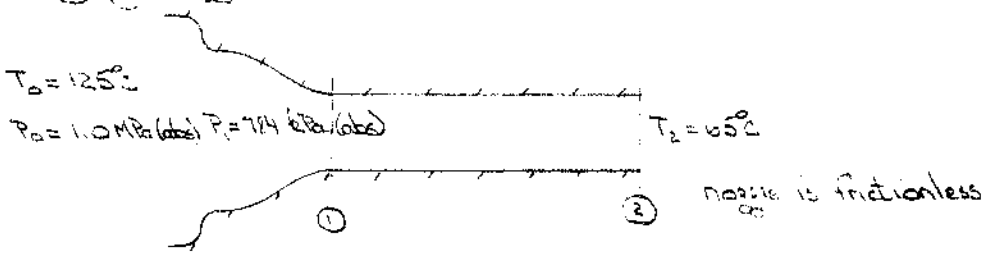
Then $p_2 = \rho_2 R T_2$

$p_2 = 0.582 \frac{kg}{m^3} \times 287 \frac{N \cdot m}{kg \cdot K} \times 287 \text{ K} = 47.9 \text{ kPa (abs)}$



$\leftarrow p_2$

Given: Steady, adiabatic flow of air from a large tank through a converging nozzle connected to a constant area duct



Find: p_2, $s_2 - s_1$

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} = h_0$ $\quad \dot{m} = \rho V A$ $\quad Tds = dh - vdp$

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$ $\quad \frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$

Assumptions: 1) steady flow 5) $z_2 = 0$
2) isentropic flow in nozzle ① 6) uniform flow at a section
3) $z = 0$ in duct 7) ideal gas
4) $\dot{W}_s = \dot{W}_{shear} = 0$

$$p_0/p = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)} \qquad M_1 = \left\{\frac{2}{k-1}\left[\left(\frac{p_0}{p_1}\right)^{(k-1)/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{1.0}{0.784}\right)^{0.286} - 1\right]\right\}^{1/2} = 0.60$$

$$T_0/T = \left[1 + \frac{k-1}{2} M^2\right] \qquad T_1 = \frac{T_0}{1 + \frac{k-1}{2} M_1^2} = \frac{398\ K}{1 + 0.2(0.60)^2} = 371\ K$$

Since T_0 = constant, $\quad M_2 = \left\{\frac{2}{k-1}\left[\frac{T_0}{T_2} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\frac{398}{338} - 1\right]\right\}^{1/2} = 0.942$

$$V_2 = M_2 c_2 = M_2 (kRT_2)^{1/2} = 0.942\left(1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 338\ K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 347\ m/s$$

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 0.60\left(1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 371 K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 232\ m/s$$

$$\rho_1 = \frac{p_1}{RT_1} = 784 \times 10^3 \frac{N}{m^2} \times \frac{kg\cdot K}{287\ N\cdot m} \times \frac{1}{371 K} = 7.36\ kg/m^3$$

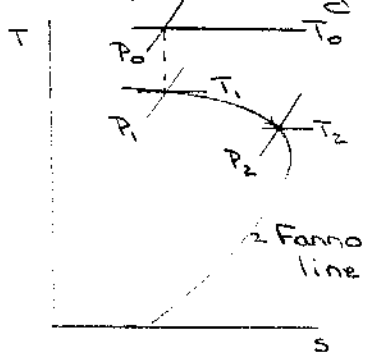
$$\dot{m} = \rho_1 V_1 A = \rho_2 V_2 A \qquad \rho_2 = \frac{V_1}{V_2}\rho_1 = \frac{232}{347} \times 7.36\ kg/m^3 = 4.92\ kg/m^3$$

$$p_2 = \rho_2 R T_2 = 4.92 \frac{kg}{m^3} \times 287 \frac{N\cdot m}{kg\cdot K} \times 338\ K = 477\ kPa$$ p_2

$$Tds = dh - v\,dp = c_p dT - \frac{1}{\rho} dp$$

$$s_2 - s_1 = \int_{s_1}^{s_2} ds = \int_{T_1}^{T_2} c_p dT - \int_{p_1}^{p_2} R \frac{dp}{p} = c_p \ln\frac{T_2}{T_1} - R \ln\frac{p_2}{p_1} = 10^3 \frac{J}{kg\cdot K} \ln\left(\frac{338}{371}\right) - 287 \frac{J}{kg\cdot K} \ln\left(\frac{477}{784}\right)$$

$$s_2 - s_1 = 49.5\ J/kg\cdot K$$ $s_2 - s_1$



Fanno-Line Flow Functions

From Appendix E.2 with $M_1 = 0.60$,

$T_1/T^* = 1.119$ (12.18a) $\quad \therefore T^* = 332\ K$

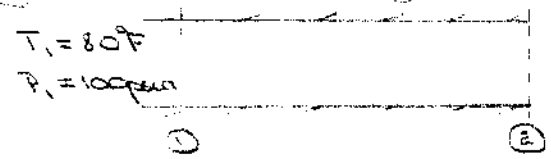
$p_1/p^* = 1.763$ (12.18d) $\quad \therefore p^* = 445\ kPa$

Then, with $T_2/T^* = 1.018$,

$M_2 = 0.945$ (12.18a) and

$p_2/p^* = 1.069$ (12.18d) $\quad \therefore p_2 = 476\ kPa$ ✓

Given: Steady flow of air through an insulated pipe



$\dot{m} = 600 lbm/min$
$D_1 = D_2 = D = 4 in$

Find: (a) P_{min} (b) V_{max} in pipe

Solution:

Basic equations: $\dot{m} = \rho V A$ $P = \rho RT$

Compiling equations: $T_o/T = 1 + \frac{k-1}{2} M^2$ T_o = constant

Assumptions: (1) steady flow (3) uniform flow at a section
(2) adiabatic flow (4) ideal gas

$$\rho_1 = \frac{P_1}{RT_1} = 100 \frac{lbf}{in^2} \times \frac{lbm \cdot R}{53.3 ft \cdot lbf} \times \frac{1}{540R} \times \frac{144 in^2}{ft^2} = 0.500 lbm/ft^3$$

$$\dot{m} = \rho V_1 A \quad ; \quad A = \pi D^2/4$$

$$V_1 = \frac{4\dot{m}}{\pi D^2 \rho_1} = \frac{4}{\pi} \times 600 \frac{lbm}{min} \times \left(\frac{12}{4}\right)^2 \frac{1}{ft^2} \times \frac{ft^3}{0.500 lbm} \times \frac{min}{60 s} = 229 ft/s$$

$$M_1 = \frac{V_1}{c_1} \qquad c_1 = (kRT_1)^{1/2} = \left(1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot R} \times 540R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft/s^2}{lbf}\right)^{1/2} = 1140 ft/s$$

$$M_1 = \frac{229}{1140} = 0.201$$

Since $M_1 < 1.0$, the minimum pressure and max velocity occur for $M_2 = 1.0$

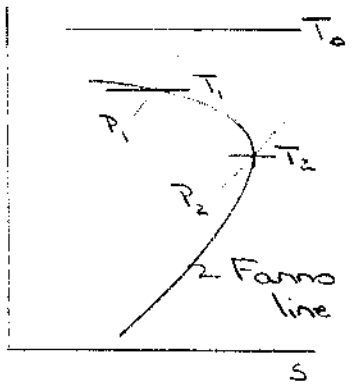
$$T_o/T = 1 + \frac{k-1}{2} M^2 \quad ; \quad \frac{T_1}{T_2} = \frac{1 + \frac{k-1}{2} M_1^2}{1 + \frac{k-1}{2} M_2^2} = \frac{1 + 0.2(0.201)^2}{1 + 0.2(1.0)^2} = 0.840$$

$$T_2 = 0.840 (T_1) = 0.840 (540°R) = 454°R$$

$$V_2 = M_2 c_2 = M_2 (kRT_2)^{1/2} = 1.0 \left(1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot R} \times 454R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft/s^2}{lbf}\right)^{1/2} = 1040 ft/s \leftarrow V_{max}$$

$$\dot{m} = \rho_1 V_1 A = \rho_2 V_2 A \qquad \rho_2 = \frac{V_1}{V_2} \rho_1 = \frac{229}{1040} \times 0.500 \frac{lbm}{ft^3} = 0.110 lbm/ft^3$$

$$P_2 = \rho_2 R T_2 = 0.110 \frac{lbm}{ft^3} \cdot 53.3 \frac{ft \cdot lbf}{lbm \cdot R} \cdot 454R \times \frac{ft^2}{144 in^2} = 18.5 psia \leftarrow P_{min}$$



*Fanno-line Flow Functions

From Appendix E with $M_1 = 0.20$,

$T_1/T^* = 1.191$ (12.18a) $\therefore T_2 = T^* = 453°R$

$P_1/P^* = 5.456$ (12.18d) $\therefore P_2 = P^* = 18.3 psia$

$V_1/V^* = 0.2182$ (12.18c) $\therefore V_2 = V^* = 1050 ft/s$

Given: Compressible flow through long tube (17.16 mm i.d.). Air from atmosphere drawn through by vacuum pump downstream. As back pressure is lowered, pressure distribution along tube changes until p_b = 626 mm Hg (vacuum)

Find: a) $\dot{m}_{max}$ (b) p_{0e} (c) $s_e - s_1$

Solution:

Basic equations: $\dot{m} = \rho V A$ $\quad p = \rho R T$ $\quad T ds = dh - v\,dp$

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$ $\quad \frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{k/(k-1)}$

Assumptions: (1) steady flow (2) ideal gas
(3) adiabatic flow through tube
(4) uniform flow at a section

Since pressure distribution does not change when p_b reaches 626 mm Hg (vacuum), flow is choked, $M_e = 1.0$ and $p_e = p_b$

$$p_e = p_{atm} - \rho g \Delta h = 101 \times 10^3 \frac{N}{m^2} - 13.55 \times 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.626 m \times \frac{N \cdot s^2}{kg \cdot m} = 17.87 \times 10^3 \frac{N}{m^2}$$

$$T_e = \frac{T_0}{1 + \frac{k-1}{2} M_e^2} = \frac{(273+20)K}{1+0.2} = 244\ K$$

$$\rho_e = \frac{p_e}{RT_e} = 17.87 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287\ N \cdot m} \times \frac{1}{244 K} = 0.255\ kg/m^3$$

$$V_e = (kRT_e)^{1/2} = \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 244 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 313\ m/s$$

$$\dot{m}^* = \rho_e V_e A = 0.255 \frac{kg}{m^3} \times 313 \frac{m}{s} \times \frac{\pi}{4}(0.0716 m)^2 = 0.00321\ kg/s \quad \leftarrow \dot{m}^*$$

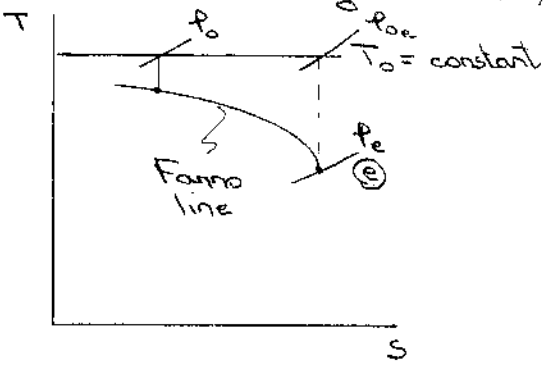
$$p_{0e} = p_e \left(1 + \frac{k-1}{2} M_e^2\right)^{k/(k-1)} = 17.87\ kPa\ (1+0.2)^{3.5} = 33.8\ kPa\ (abs) \quad \leftarrow p_{0e}$$

For an ideal gas, the Tds equation can be written as

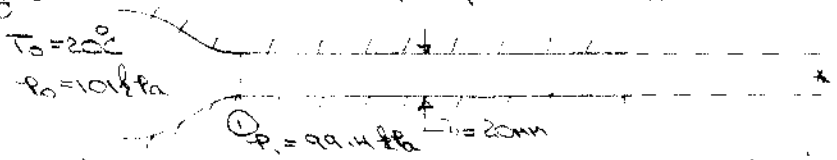
$$T ds = dh - v\,dp = c_p dT - RT \frac{dp}{p} \quad \therefore \quad ds = c_p \frac{dT}{T} - R \frac{dp}{p}$$

Then, $s_e - s_1 = s_{0e} - s_{01} = c_p \ln \frac{T_{0e}}{T_{01}} - R \ln \frac{p_{0e}}{p_{01}}$

$$s_e - s_1 = -287 \frac{J}{kg \cdot K} \ln\left(\frac{33.8}{101}\right) = 314\ J/kg \cdot K \quad \leftarrow s_e - s_1$$



Given: Air, at 20°C and 101 kPa, is drawn through a converging nozzle into a long, 20mm diameter, insulated tube. At the nozzle outlet (tube inlet) $p_1 = 99.4$ kPa



Find: a) $\dot{m}$ b) T^* and p^* for nozzle isentropic flow
c) T^* and p^* for adiabatic tube flow

Solution:

Basic equations: $p = \rho RT$ $\dot{m} = \rho VA$

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$ $\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{k/(k-1)}$

Assumptions: (1) steady flow (2) ideal gas
(3) uniform flow at a section
(4) isentropic flow in nozzle, adiabatic flow in tube.

Since flow in nozzle is isentropic, $T_{0_1} = T_0$ and $p_{0_1} = p_0$

$$\therefore T_1 = \frac{T_{0_1}}{1 + \frac{k-1}{2} M_1^2} = \frac{293\text{K}}{1 + 0.2(0.15)^2} = 291.7\text{K}\ ; \quad M_1 = \left\{\frac{2}{k-1}\left[\left(\frac{p_{0_1}}{p_1}\right)^{(k-1)/k} - 1\right]\right\}^{1/2} = 0.151$$

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 0.151\left[1.4 \times 287 \frac{\text{N·m}}{\text{kg·K}} \times 291.7\text{K} \times \frac{\text{kg·m/s}^2}{\text{N}}\right]^{1/2} = 51.7\ \text{m/s}$$

$$\rho_1 = \frac{p_1}{RT_1} = \frac{99.4 \times 10^3\ \text{N/m}^2}{287\ \text{N·m/kg·K} \times 291.7\text{K}} = 1.187\ \text{kg/m}^3$$

$$\dot{m} = \rho_1 V_1 A_1 = 1.187\ \frac{\text{kg}}{\text{m}^3} \times 51.7\ \frac{\text{m}}{\text{s}} \times \frac{\pi}{4}(0.020\text{m})^2 = 0.0193\ \text{kg/s}$$ $\dot{m}$

(b) For isentropic flow, $\frac{T_0}{T^*} = 1 + \frac{k-1}{2} = 1.20$ $\therefore T^* = \frac{T_0}{1.20} = \frac{293\text{K}}{1.20} = 244\text{K}$ T^*_{isen}

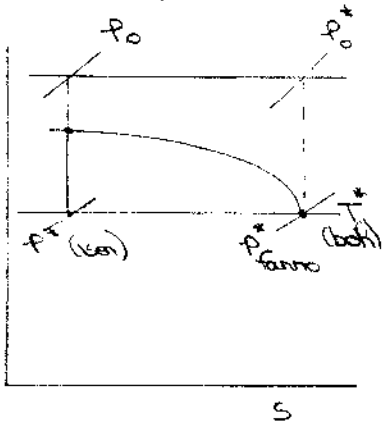
$\frac{p_0}{p^*} = \left(\frac{T_0}{T^*}\right)^{k/(k-1)}$ $\therefore p^* = \frac{p_0}{(T_0/T^*)^{k/(k-1)}} = \frac{101\ \text{kPa}}{(1.2)^{3.5}} = 53.4\ \text{kPa (abs)}$ p^*_{isen}

(c) For Fanno line flow $T_0^* = T_0$ $\therefore T^* = T^*_{\text{isen}} = 244\text{K}$ T^*

At T^*, $M = 1$, $\therefore V = c = (kRT^*)^{1/2} = \left[1.4 \times 287 \frac{\text{N·m}}{\text{kg·K}} \times 244\text{K} \times \frac{\text{kg·m/s}^2}{\text{N}}\right]^{1/2} = 313\ \text{m/s}$

From continuity $\rho_1 A_1 = \rho^* V^*$ $\therefore \rho^* = \frac{V_1}{V^*}\rho_1 = \frac{51.4}{313} \times 1.187\ \frac{\text{kg}}{\text{m}^3} = 0.1949\ \text{kg/m}^3$

$p^* = \rho^* RT^* = 0.1949\ \frac{\text{kg}}{\text{m}^3} \times 287\ \frac{\text{N·m}}{\text{kg·K}} \times 244\text{K} = 13.6\ \text{kPa (abs)}$ p^*

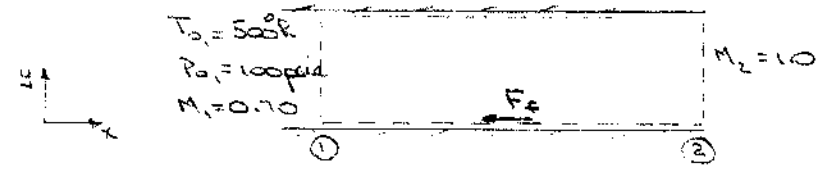


* Fanno-Line Flow Functions (Appendix E.2)

For $M_1 = 0.151$, Eq 12.18d gives $p_1/p^* = 7.238$

$\therefore p^* = 13.7$ kPa. ✓

Given: Adiabatic flow of air in a constant-area duct with friction



Find: Friction force exerted on the fluid by the pipe

Solution:

Basic equations: $F_{S_x} = P_1A - P_2A - F_f = \dot{m}(V_2 - V_1)$ $\quad \dot{m} = \rho VA$

Computing equations: $T_o/T = 1 + \frac{k-1}{2}M^2$ $\quad P_o/P = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)}$

Assumptions: (1) steady flow; (2) adiabatic flow, T_o = const; (3) uniform flow at a section; (4) $F_{B_x} = 0$; (5) ideal gas

$P_o/P = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)}$ $\quad P_1 = \frac{P_{o_1}}{\left[1 + \frac{k-1}{2}M_1^2\right]^{k/(k-1)}} = \frac{100 psia}{[1 + 0.2(0.70)^2]^{3.5}} = 72.1 psia$

$T_o/T = 1 + \frac{k-1}{2}M^2$ $\quad T_1 = \frac{T_o}{1 + \frac{k-1}{2}M_1^2} = \frac{500°R}{1 + 0.2(0.70)^2} = 455°R$

$\rho_1 = \frac{P_1}{RT_1} = 72.1 \frac{lbf}{in^2} \cdot \frac{lbm°R}{53.3 ft.lbf} \cdot \frac{1}{455°R} \cdot 144 \frac{in^2}{ft^2} = 0.428 lbm/ft^3$

$V_1 = M_1C_1 = M_1(kRT_1)^{1/2} = 0.70\left(1.4 \cdot 53.3 \frac{ft.lbf}{lbm°R} \cdot 455°R \cdot 32.2 \frac{lbm}{slug} \cdot \frac{slug.ft/s^2}{lbf}\right)^{1/2} = 732 ft/s$

$\dot{m} = \rho_1V_1A = 0.428 \frac{lbm}{ft^3} \cdot 732 \frac{ft}{s} \cdot 1.0 ft^2 = 313 lbm/s$

$T_o/T = 1 + \frac{k-1}{2}M^2$, $T_{o_2} = T_{o_1}$, $T_2 = \frac{T_o}{1 + \frac{k-1}{2}M_2^2} = \frac{500°R}{1 + 0.2(1.0)^2} = 417°R$

$V_2 = M_2C_2 = M_2(kRT_2)^{1/2} = 1.0\left(1.4 \cdot 53.3 \frac{ft.lbf}{lbm°R} \cdot 417°R \cdot 32.2 \frac{lbm}{slug} \cdot \frac{slug.ft/s^2}{lbf}\right)^{1/2} = 1000 ft/s$

$\dot{m} = \rho_1V_1A = \rho_2V_2A$ $\quad \rho_2 = \frac{V_1}{V_2}\rho_1 = \frac{732}{1000} \cdot 0.428 lbm/ft^3 = 0.313 lbm/ft^3$

$P_2 = \rho_2RT_2 = 0.313 \frac{lbm}{ft^3} \cdot 53.3 \frac{ft.lbf}{lbm°R} \cdot 417°R \cdot \frac{ft^2}{144 in^2} = 48.3 psia$

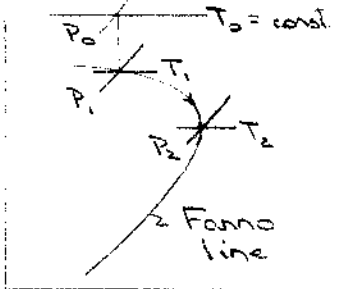
Solving the momentum equation for F_f

$F_f = (P_1 - P_2)A - \dot{m}(V_2 - V_1)$

$= (72.1 - 48.3) \frac{lbf}{in^2} \cdot 1 ft^2 \cdot 144 \frac{in^2}{ft^2} - 313 \frac{lbm}{s}(1000 - 732) \frac{ft}{s} \cdot \frac{slug}{32.2 lbm} \cdot \frac{lbf.s^2}{slug.ft}$

$F_f = 822 lbf$ ← F_f

F_f is the force on the control volume from the surroundings. Consequently, F_f is the force on the fluid from the pipe: F_f opposes the motion



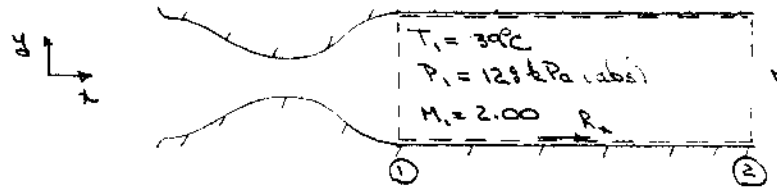
* Fanno-Line Flow Functions

From Appendix E.2 with $M_1 = 0.70$,

$P_1/P^* = 1.493$ (12.18d) $\quad \therefore P^* = P_2 = 48.3 psia$

$V_1/V^* = 0.732$ (12.18c) $\quad \therefore V^* = V_2 = 1000 ft/s$

Given: Flow of air through an insulated constant area duct supplied by a converging-diverging nozzle; flow is shockless



$A_1 = A_2 = A = 650\ mm^2$

Find: T_2, net force of fluid on the pipe, $s_2 - s_1$

Solution:

Basic equations: $F_{S_x} = P_1A - P_2A + R_x = \dot{m}(V_2 - V_1)$; $Tds = dh - vdp$

Computing equation: $T_0/T = 1 + \frac{k-1}{2} M^2$ (T_0 = constant)

Assumptions: (1) steady flow (4) uniform flow at a section
(2) isentropic flow in nozzle (5) $F_{S_x} = 0$
(3) adiabatic flow in duct (6) ideal gas

$T_0/T = 1 + \frac{k-1}{2} M^2$; $\frac{T_2}{T_1} = \frac{\left[1 + \frac{k-1}{2} M_1^2\right]}{\left[1 + \frac{k-1}{2} M_2^2\right]} = \frac{1 + 0.2(2.0)^2}{1 + 0.2(1.0)^2} = 1.50$

$\therefore T_2 = 1.50 T_1 = 1.5 \times 312K = 468\ K$ ← T_2

$\rho_1 = \frac{P_1}{RT_1} = 128 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287\ N \cdot m} \times \frac{1}{312K} = 1.43\ kg/m^3$

$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 2.0 \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 312K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 708\ m/s$

$\dot{m} = \rho_1 V_1 A = 1.43 \frac{kg}{m^3} \times 708 \frac{m}{s} \times 650 \times 10^{-4}\ m^2 = 0.658\ kg/s$

$V_2 = M_2 c_2 = M_2 (kRT_2)^{1/2} = 1.0 \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 468K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 434\ m/s$

$\rho_1 V_1 A = \rho_2 V_2 A$; $\rho_2 = \frac{V_1}{V_2} \rho_1 = \frac{708}{434} \times 1.43\ kg/m^3 = 2.33\ kg/m^3$

$P_2 = \rho RT_2 = 2.33 \frac{kg}{m^3} \times 287 \frac{N \cdot m}{kg \cdot K} \times 468K = 313\ kPa$

Then,
$R_x = (P_2 - P_1)A + \dot{m}(V_2 - V_1) = (313 - 128) \times 10^3 \frac{N}{m^2} \times 650 \times 10^{-4}\ m^2 + 0.658 \frac{kg}{s}(434 - 708) \frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m}$

$R_x = -60\ N$ and force of fluid on pipe, $F_x = -R_x = 60\ N$ (to right) ← R_x

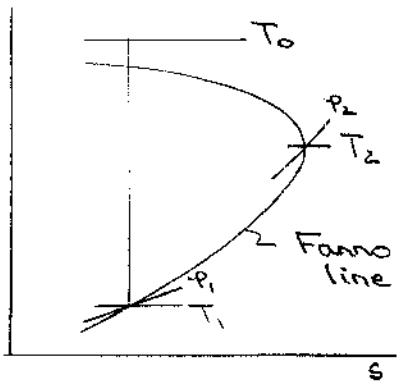
$Tds = dh - vdp = c_p dT - RT \frac{dp}{p}$

$ds = c_p \frac{dT}{T} - R \frac{dp}{p}$

$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1}$

$= 10^3 \frac{J}{kg \cdot K} \ln \frac{468}{312} - 287 \frac{J}{kg \cdot K} \ln \frac{313}{128}$

$s_2 - s_1 = 149\ J/kg \cdot K$ ← $s_2 - s_1$



* Fanno-Line Flow Functions

From Appendix E.2 for $M_1 = 2.00$,

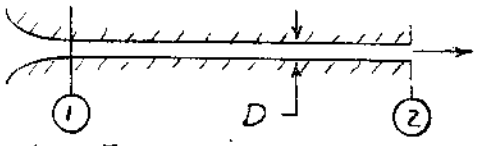
$T_1/T^* = 0.6667$ (12.18a) $\therefore T^* = T_2 = 468\ K$

$P_1/P^* = 0.4083$ (12.18d) $\therefore P^* = P_2 = 313\ kPa$

Given: Air flow through nozzle then duct, from tank at 15°C with variable pressure. The duct exhausts to atmosphere.

When $M_2 = 1$: $T_0 = 15°C$

$D = 0.249$ in.



$p_1 = 53.2$ psia $\quad p_2 = 14.7$ psia

$M_1 = 0.30 \quad M_2 = 1.0$

Find: (a) Tank pressure, p_{0_t}

(b) T_2, p_{02}, and $\dot{m}$

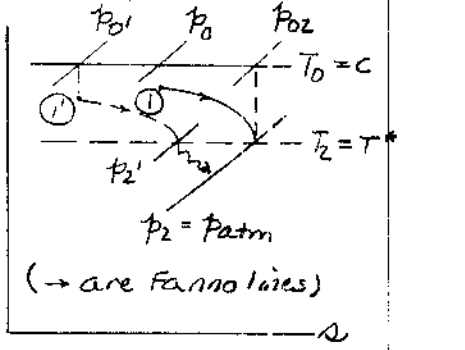
(c) Show on a Ts diagram the effect of raising to $p_0 = 100$ psia.

(d) Plot pressure distribution vs. distance.

Solution: Assume conditions in tank are stagnation properties; flow is isentropic in nozzle, Fanno flow in duct.

Basic equations: $T_0 = T\left(1 + \frac{k-1}{2}M^2\right) = \text{const (energy)}$

$$p_0 = p\left(1 + \frac{k-1}{2}M^2\right)^{\frac{k}{k-1}} \text{ (isentropic)}$$



Assume steady, 1-D compressible flow, with constant stagnation temperature, $T_0 = 15°C$ (59°F)

$$T_0 = (460 + 59)°R = 519°R$$

$$p_{0_t} = p_1\left(1 + \frac{k-1}{2}M_1^2\right)^{\frac{k}{k-1}} = 53.2 \text{ psia}\left(1 + 0.2(0.3)^2\right)^{3.5} = 56.6 \text{ psia} \quad \leftarrow p_{0t}$$

At exit plane, $M_2 = 1$

$$T_2 = \frac{T_0}{\left(1 + \frac{k-1}{2}M_2^2\right)} = \frac{519°R}{1 + 0.2(1)^2} = 433°R \quad \leftarrow T_2$$

$$p_{02} = p_2\left(1 + \frac{k-1}{2}M_2^2\right)^{\frac{k}{k-1}} = 14.7 \text{ psia}\left(1 + 0.2(1)^2\right)^{3.5} = 27.8 \text{ psia} \quad \leftarrow p_{02}$$

At exit plane, $V = V^* = C^* = \sqrt{kRT^*} = 1020$ ft/s

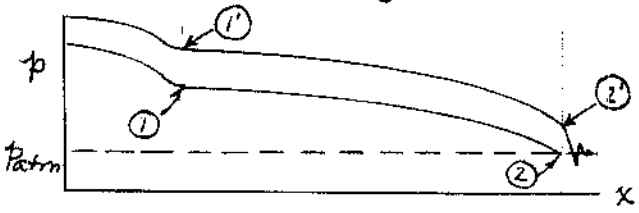
$$\rho_2 = \frac{p_2}{RT_2} = 14.7\frac{\text{lbf}}{\text{in.}^2} \times \frac{1}{53.3 \text{ ft}\cdot\text{lbf}} \times \frac{1}{433°R} \times 144\frac{\text{in.}^2}{\text{ft}^2} = 0.0917 \text{ lbm/ft}^3$$

$$\dot{m} = \rho_2 V_2 A = 0.0917\frac{\text{lbm}}{\text{ft}^3} \times 1020\frac{\text{ft}}{\text{s}} \times \frac{\pi}{4}\left(\frac{0.249}{12}\text{ ft}\right)^2 = 0.0316 \text{ lbm/s} \quad \leftarrow \dot{m}$$

When p_t is increased, density and mass flow rate increase. Flow will shift to a new Fanno line as shown dashed in the Ts diagram above.

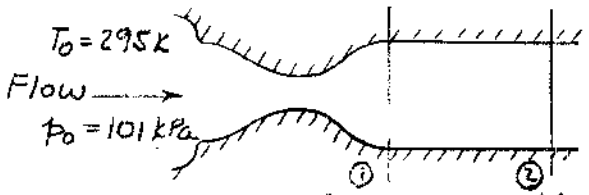
T_0 is fixed, so T^* remains fixed.

{Since ρ and $\dot{m}$ increase, $p_2' > p_2 = p_{atm}$. The plot of p vs. x is as shown.}



Given: Supersonic wind tunnel with conditions shown, no shocks.

System is insulated.



Find: (a) T_2, (b) p_2, (c) $s_2 - s_1$
(d) Ts diagram

Solution: Consider steady, 1-D, comp. flow, ideal gas.

Computing equations: $T_0 = T\left(1 + \frac{k-1}{2} M^2\right)$, $p_0 = p\left(1 + \frac{k-1}{2} M^2\right)^{k/k-1}$ (s = const)

$T ds = dh - v\,dp$ $\rho V A$ = constant

Assumptions: (1) Isentropic flow in nozzle
(2) Insulated, so T_0 = constant

Then $T_2 = \frac{T_0}{1 + \frac{k-1}{2} M_2^2} = \frac{295\ K}{1 + 0.2(1.1)^2} = 238\ K$ T_2

From continuity, $\rho_1 V_1 = \rho_2 V_2$, since $A_1 = A_2$. But $V = Mc = M\sqrt{kRT}$ and $\rho = p/RT$, so

$$\frac{p_1}{RT_1} M_1 \sqrt{kRT_1} = \frac{p_2}{RT_2} M_2 \sqrt{kRT_2} \quad ; \quad \frac{p_1}{\sqrt{T_1}} M_1 = \frac{p_2}{\sqrt{T_2}} M_2 \quad ; \quad p_2 = p_1 \frac{M_1}{M_2} \sqrt{\frac{T_2}{T_1}}$$

From isentropic relations,

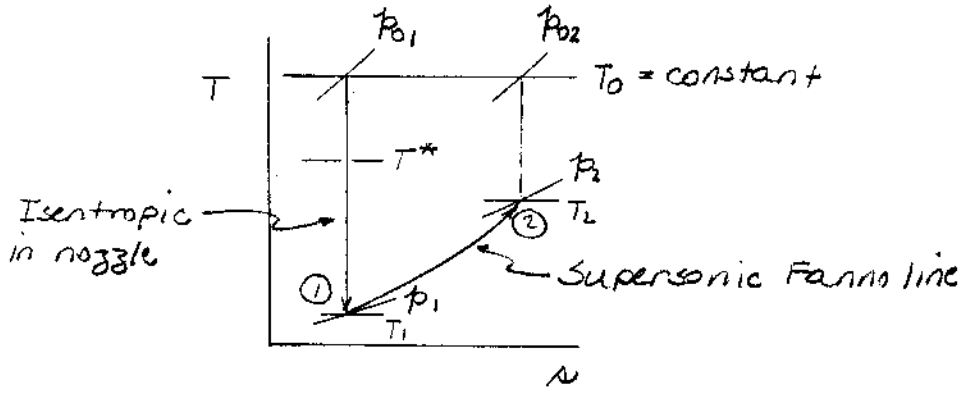
$$T_1 = \frac{T_0}{1 + \frac{k-1}{2} M_1^2} = \frac{295\ K}{1 + 0.2(2.1)^2} = 157\ K \quad ; \quad p_1 = \frac{p_{01}}{\left(1 + \frac{k-1}{2} M_1^2\right)^{k/k-1}} = \frac{101\ kPa}{(1.88)^{3.5}} = 11.1\ kPa$$

Thus $p_2 = \frac{2.1}{1.1} \sqrt{\frac{238\ K}{157\ K}}\ 11.1\ kPa = 26.1\ kPa\ (abs)$ p_2

$$ds = \frac{dh}{T} - v\frac{dp}{T} = c_p \frac{dT}{T} - \frac{RT}{p}\frac{dp}{T} = c_p \frac{dT}{T} - R\frac{dp}{p}$$

$$\Delta s = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} = 1004 \frac{J}{kg \cdot K} \ln\left(\frac{238}{157}\right) - 287 \frac{J}{kg \cdot K} \ln\left(\frac{26.1}{11.1}\right) = 172\ J/kg \cdot K \quad \Delta s$$

The Ts diagram is

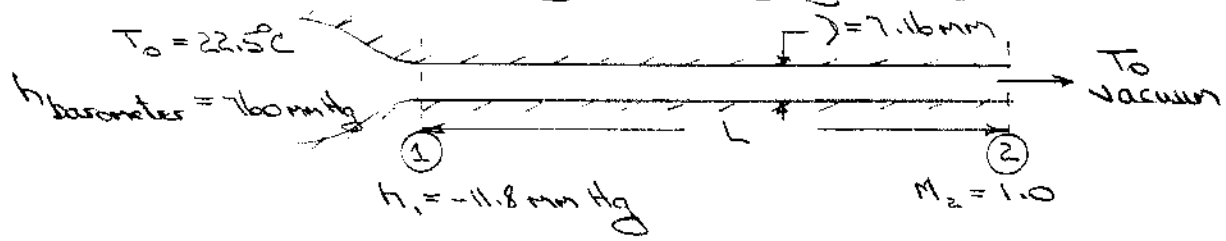


* Fanno-Line Flow Functions (Appendix E.2).

For $M_1 = 2.1$, Eq. 12.18d gives $p_1/p^* = 0.380$, so $p_2 = 26.1$ kPa (abs)
$M_2 = 1.1$ $p_2/p^* = 0.894$

Problem *12.50

Given: Fanno line flow apparatus in laboratory, smooth brass tube fed by converging nozzle.



Find: (a) M_1 (b) L_{12} (c) T_2, p_{o2}

Solution: * Compressible flow functions to be used

Assumptions: (1) Steady flow
(2) Isentropic flow in nozzle
(3) adiabatic " in duct
(4) $W_s = W_{shear} = 0$
(5) $\Delta z = 0$
(6) ideal gas, air
(7) uniform flow at a section

$$\frac{p_1}{p_o} = \frac{p_1}{p_{atm}} = \frac{\rho g \Delta h_1}{\rho g \Delta h_{atm}} = \frac{(760 - 11.8)\text{mm}}{760\text{ mm}} = 0.9845$$

From App. E.1 (Eq. 11.17a) $M_1 = 0.150$ ← M_1

$$p_1 = \rho_{Hg} g h_1 = (13.5)1000\frac{kg}{m^3} \times 9.81\frac{m}{sec^2} \times 0.748\,m \times \frac{N\cdot sec^2}{kg\cdot m} = 99.1\,kPa(abs)$$

From App. E.1 (Eq. 11.17b), $T_1/T_o = 0.9555$ $\therefore T_1 = 294\,K$

From App. E.2, with $M_1 = 0.150$

$p_1/p^* = 7.287$, $T_1/T^* = 1.195$, $\frac{\bar{f} L_{max}}{D} = 27.93$

$p_2 = p^* = 13.60$ and $p_{o2} = \frac{p_2}{0.5283} = 25.7\,kPa$, $T^* = T_2 = 247K$ ← p_{o2}, T_2

$f = f(Re)$, $Re = \frac{\rho V D}{\mu}$

$$\rho_1 = \frac{p_1}{RT_1} = \frac{99.1\times10^3\frac{N}{m^2}}{287\frac{J}{kg\cdot K}\times\frac{N\cdot m}{J}\times 294\,K} = 1.17\,kg/m^3$$

$$V_1 = M_1 c_1 = M_1(kRT_1)^{1/2} = 0.149\left(1.4\times287\frac{N\cdot m}{kg\cdot K}\times294K\times\frac{kg\cdot m}{N\cdot sec^2}\right)^{1/2} = 51.2\,m/s$$

From Table A.10 @ $T_1 = 21°C$, $\mu = 1.82\times10^{-5}\,N\cdot sec/m^2$

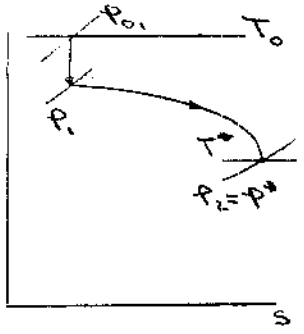
$$Re_1 = \frac{\rho V D}{\mu} = 1.17\frac{kg}{m^3}\times51.2\frac{m}{s}\times7.16\times10^{-3}m\times\frac{m^2}{1.82\times10^{-5}N}\times\frac{N\cdot sec^2}{kg\cdot m}$$

$Re_1 = 2.36\times10^4$

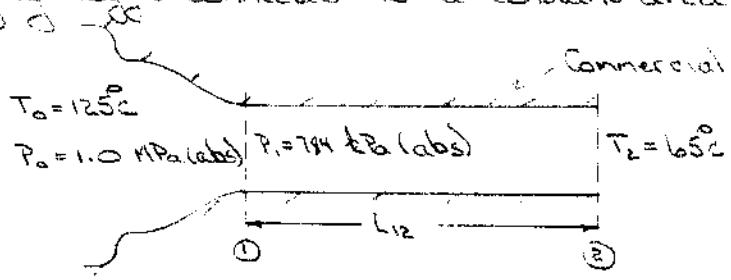
From Fig. 8.13, friction factor $f = 0.0245$

$$\therefore L_{12} = 27.9\frac{D}{f} = 27.9\times7.16\times10^{-3}\,m\cdot\frac{1}{0.0245}$$

$L_{12} = 8.2\,m$ ← L_{12}



Given: Steady, adiabatic flow of air from a large tank through a converging nozzle connected to a constant-area duct



Find: Length, L_{12}

Solution:

* Compressible flow functions to be used in solution

Assumptions: (1) steady flow
(2) isentropic flow in nozzle
(3) $\dot{Q} = 0$ in duct
(4) $W_s = W_{shear} = 0$
(5) $z_1 = z_2$
(6) uniform flow at c.s.
(7) ideal gas

For $P_1/P_0 = \frac{1000}{784} = 1.276$, from App. E.1, $M_1 = 0.160$, $T_1/T_0 = 0.9928$

$\therefore T_1 = 371\,K$

From App. E.2 for $M_1 = 0.160$, $T_1/T^* = 1.119$, $P_1/P^* = 1.762$, $\left[\frac{\bar{f}L_{max}}{D}\right]_1 = 0.4908$

$\therefore T^* = 332\,K$, $P^* = 445$ kPa

$T_2/T^* = \frac{338}{332} = 1.018$. From App. E.2 $M_2 = 0.945$, $\left[\frac{\bar{f}L_{max}}{D}\right]_2 = 0.00400$

$$\frac{\bar{f}L_{12}}{D} = \left[\frac{\bar{f}L_{max}}{D}\right]_1 - \left[\frac{\bar{f}L_{max}}{D}\right]_2 = 0.4908 - 0.00400 = 0.4868$$

$$\rho_1 = \frac{P_1}{RT_1} = 784\times10^3 \frac{N}{m^2} \times \frac{kg\cdot K}{287\,N\cdot m} \times \frac{1}{371\,K} = 7.36\ kg/m^3$$

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 0.160\left(1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 371\,K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 232\ m/s$$

$Re_1 = \frac{\rho V_1 D}{\mu}$ From Table A.6 at $T_1 = 98°C$, $\mu = 2.16\times10^{-5}\ N\cdot s/m^2$

$$Re_1 = 7.36\frac{kg}{m^3} \times 232\frac{m}{s} \times 5\times10^{-2}\,m \times \frac{m^2}{2.16\times10^{-5}\,N\cdot s} \times \frac{N\cdot s^2}{kg\cdot m} = 3.95\times10^6$$

For commercial steel pipe, from Table 8.1, $e = 4.6\times10^{-5}\,m$

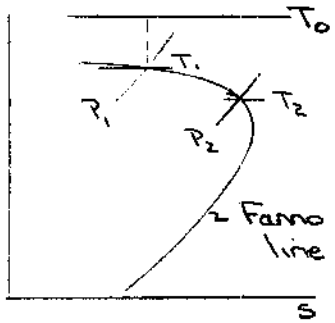
Then $\frac{e}{D} = \frac{4.6\times10^{-5}}{5\times10^{-2}} = .00092$

From Fig 8.13, friction factor, $f = 0.0192$

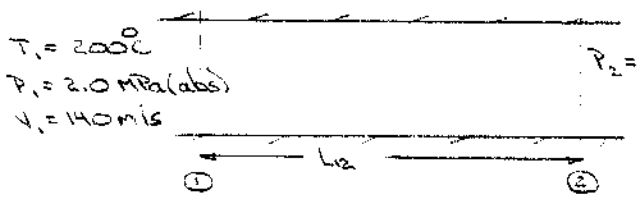
$$\frac{\bar{f}L_{12}}{D} = 0.4868$$

$$L_{12} = \frac{0.4868\,D}{f} = \frac{0.4868 \times 5\times10^{-2}\,m}{0.0192}$$

$$L_{12} = 1.27\ m$$



Given: Adiabatic flow of air in a constant-area duct



Find: T_2, L_{12}

Solution:

* Compressible flow functions to be used in the solution

Assumptions: (1) steady flow (3) uniform flow at a section
(2) adiabatic flow (4) ideal gas

$M_1 = \frac{V_1}{c_1}$ $\quad c_1 = (kRT_1)^{1/2} = \left(1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 473\,K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 436$ m/s

$M_1 = \frac{140}{436} = 0.321$ From Appendix E.2, with $M_1 = 0.321$,

$T_1/T^* = 1.176$ $\quad P_1/P^* = 3.378$ $\quad \left[\frac{f L_{max}}{D}\right]_1 = 1.409$

$\therefore T^* = 402\,K$, $\quad P^* = 0.592$ MPa

For $P_2/P^* = \frac{1.26}{0.592} = 2.128$. From App. E.2, $M_2 = 0.502$, $\frac{T_2}{T^*} = 1.142$, $\left[\frac{f L_{max}}{D}\right]_2 = 1.053$

$T_2 = 1.142\,T^* = 1.142\,(402\,K) = 459\,K$ ← T_2

$f\frac{L_{12}}{D} = f\frac{L_{max}}{D}\Big]_1 - f\frac{L_{max}}{D}\Big]_2 = 1.409 - 1.053 = 3.356$

$\rho_1 = \frac{P_1}{RT_1} = 2.0\times10^6 \frac{N}{m^2} \times \frac{kg\cdot K}{287\,N\cdot m} \times \frac{1}{473\,K} = 14.7$ kg/m³

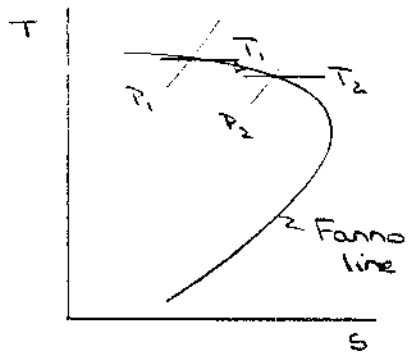
$Re = \frac{\rho_1 V_1 D}{\mu}$. To obtain μ at 200C, use Sutherland correlation (Appendix A)

$\mu = \frac{bT^{1/2}}{1 + S/T} = 1.458\times10^{-6} \frac{kg}{m\cdot s\cdot K^{1/2}} \times (473\,K)^{1/2} \times \frac{1}{\left(1 + \frac{110.4}{473}\right)} \times \frac{N\cdot s^2}{kg\cdot m} = 2.57\times10^{-5}$ N·s/m²

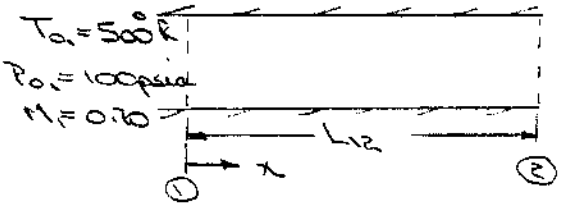
$Re = \frac{\rho_1 V_1 D}{\mu} = 14.7 \frac{kg}{m^3} \times 140 \frac{m}{s} \times 1.5\times10^{-1}\,m \times \frac{m^2}{2.57\times10^{-5}\,N\cdot s} = 1.20\times10^7$

With $e/D = 0.0003$, from Fig. 8.13 friction factor, $f = 0.0147$

$f\frac{L_{12}}{D} = 3.356$, $\quad L_{12} = 3.356\frac{D}{f} = \frac{3.356}{0.0147} \times 0.15\,m = 34.2\,m$ ← L_{12}



Given: Adiabatic flow of air in a constant area duct with friction



circular duct:
A = 1.0 ft²
commercial steel

Find: L_{12}
Plot: $p(x)$, $M(x)$

Solution:

Compressible flow functions to be used in the solution

Assumptions: (1) steady flow (3) uniform flow at a section
(2) adiabatic flow, T_o = const (4) ideal gas

For M_1 = 0.20, from Appendix E.1, $T_o/T_1 = 1.098$ (11.17b) $\therefore T_1 = 455°R$
$p_o/p_1 = 1.387$ (11.17a) $\therefore p_1 = 72.1$ psia

For M_1 = 0.20, from Appendix E.2, $p_1/p^* = 1.493$ (12.18d) $\therefore p^* = 48.3$ psia
$\bar{f}L_{max}/D_h = 0.2081$ (12.17)

For M_2 = 1.0, $\bar{f}L_{max}/D_h = 0$ $\therefore \bar{f}L_{12}/D_h = 0.2081$

$\bar{f} = \bar{f}(Re, e/D)$

$A = \frac{\pi D^2}{4}$ $\therefore D = \left(\frac{4A}{\pi}\right)^{1/2} = \left[\frac{4}{\pi} \times 1.0\,ft^2\right]^{1/2} = 1.13$ ft

For commercial steel from Table 8.1, $e = 0.00015$ ft $\therefore \frac{e}{D} = 0.00013$

$\rho_1 = \frac{p_1}{RT_1} = 72.1 \frac{lbf}{in^2} \times 53.3 \frac{ft \cdot lbf}{lbm \cdot °R} \times \frac{1}{455°R} \times 144 \frac{in^2}{ft^2} = 0.428\ lbm/ft^3$

$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 0.2(1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot °R} \times 455°R \times 32.2 \frac{lbm}{slug} \times \frac{slug}{lbf \cdot s^2/ft})^{1/2} = 132$ ft/s

$T_1 = 455°R = -5°F = -20°C = 253$ K.

From the Sutherland correlation, $\mu = \frac{bT^{1/2}}{1 + S/T}$ (A.1)

$\mu = 1.458 \times 10^{-6} \frac{kg}{m \cdot s \cdot K^{1/2}} \times \frac{(253\,K)^{1/2}}{1 + 110/253} = 1.62 \times 10^{-5} \frac{kg}{m \cdot s} \times \frac{N \cdot s^2/m}{kg \cdot m} = 1.62 \times 10^{-5} \frac{N \cdot s}{m^2}$

$\mu = 1.62 \times 10^{-5} \frac{N \cdot s}{m^2} \times 0.0209 \frac{lbf \cdot s/ft^2}{N \cdot s/m^2} = 3.39 \times 10^{-7} lbf \cdot s/ft^2$

$Re = \frac{\rho V D}{\mu} = 0.428 \frac{lbm}{ft^3} \times 132 \frac{ft}{s} \times 1.13\,ft \times \frac{1}{3.39 \times 10^{-7} lbf \cdot s} \times \frac{slug}{32.2\,lbm} \times \frac{lbf \cdot s^2}{ft \cdot slug}$

$Re = 3.24 \times 10^7$

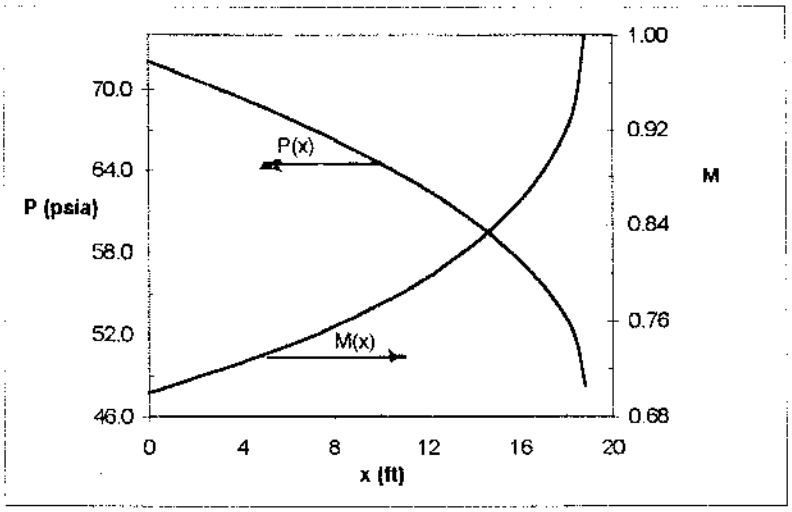
From Fig 8.13, friction factor $f = 0.0125$

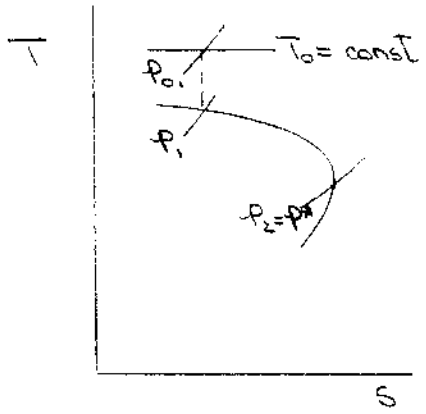
$\bar{f}\frac{L_{12}}{D} = 0.2081$ $\therefore L_{12} = \frac{0.2081\,D}{f} = \frac{0.2081}{0.0125} \times 1.13\,ft = 18.8$ ft $\leftarrow L_{12}$

To plot P(x), M(x)

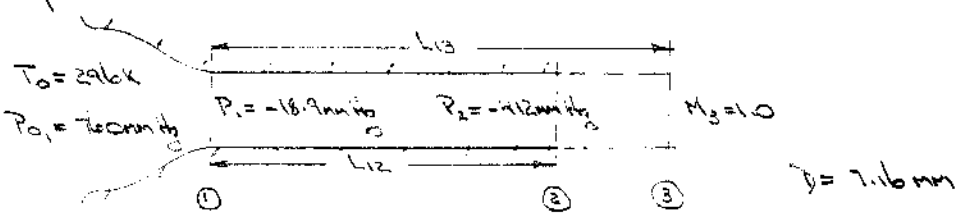
- assume values of M, $0.70 \leq M \leq 1.00$
- calculate corresponding $f(L/D)_M$ from Eq. 12.17
- solve for corresponding $f(\Delta L/D)$ where $\Delta L = x$, assuming constant f
- calculate corresponding P/P^* from Eq. 12.18d

M	$(fL/D)_M$	$\Delta(fL/D)$	x(ft)	P/P*	P(psia)
0.70	0.2081	0	0	1.4935	72.1
0.74	0.1411	0.0670	6.1	1.4054	67.9
0.78	0.0917	0.1165	10.5	1.3261	64.0
0.82	0.0559	0.1522	13.8	1.2542	60.6
0.86	0.0310	0.1772	16.0	1.1889	57.4
0.90	0.0145	0.1936	17.5	1.1291	54.5
0.94	0.0048	0.2033	18.4	1.0743	51.9
1.0	0.0000	0.2081	18.8	1	48.3





Given: Air flow in a smooth, insulated, constant-area tube, as given in Example Problem 12.7



Plot: $P(x)$, $T(x)$, $M(x)$ vs x/D from pipe inlet to choked condition

Solution:

- Compressible flow functions to be used in the solution

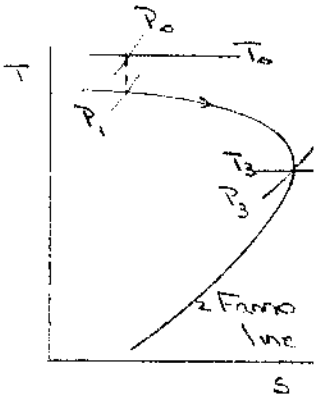
Assumptions: (1) steady flow (4) uniform flow at a section
(2) isentropic flow in nozzle (5) ideal gas
(3) $\dot{Q} = 0$ in duct

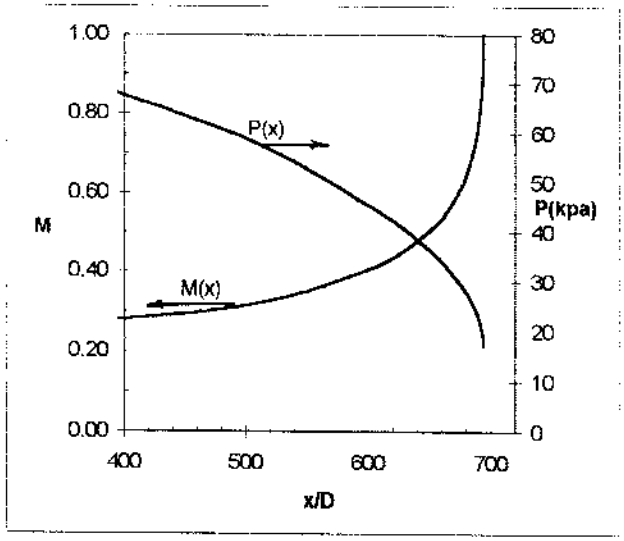
From the solution of Example Problem 12.7 we know that

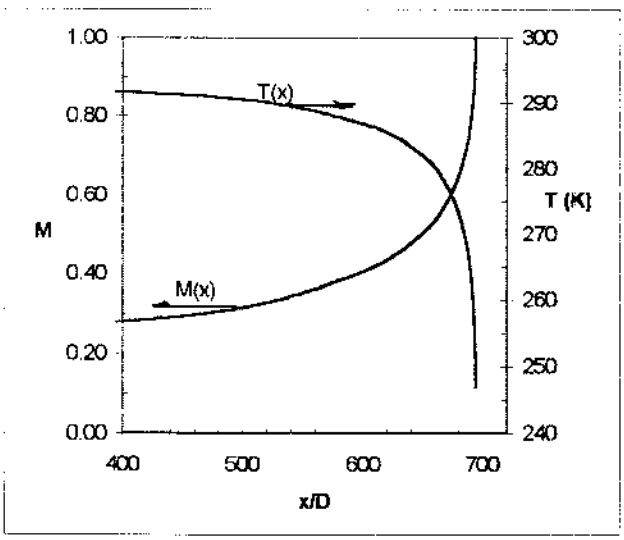
$M_1 = 0.190$ $T_1 = 294 K$, $P_1 = 98.8 kPa$, $P_{o_1} = 101 kPa$, $\bar{f} = 0.0235$

To determine $P(x)$, $T(x)$, $M(x)$, use Appendix E.2

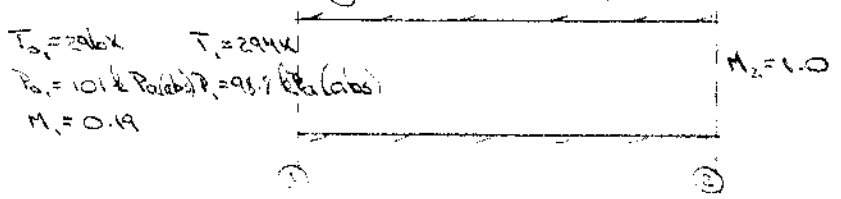
M	$(fL/D)_M$	$\Delta(fL/D)$	x/D	T/T*	T(K)	P/P*	P(kPa)
0.19	16.38	0	0	1.191	294	5.745	98
0.30	5.30	11.08	469	1.179	291	3.619	62
0.40	2.31	14.07	596	1.163	287	2.696	46
0.50	1.07	15.31	649	1.143	282	2.138	37
0.60	0.49	15.88	673	1.119	276	1.763	30
0.70	0.21	16.17	685	1.093	270	1.493	26
0.80	0.07	16.30	691	1.064	263	1.289	22
0.90	0.01	16.36	693	1.033	255	1.129	19
1.00	0.00	16.38	694	1.000	247	1.000	17







Given: Air flow in a smooth, insulated, constant-area tube. Inlet conditions as given in Example Problem 13.8



s^* is the entropy at the condition where $M = 1.0$

Plot the Fanno line $[T/T_o \text{ vs } (s-s^*)/c_p]$ for $0.19 \le M \le 1.0$

Solution:

* Compressible flow functions to be used in the solution

Basic equations: $Tds = dh - v\,dP$, $P = \rho RT$

$$Tds = dh - v\,dP = c_p\,dT - \frac{1}{\rho}\,dP \qquad ds = c_p\frac{dT}{T} - \frac{dP}{\rho T}$$

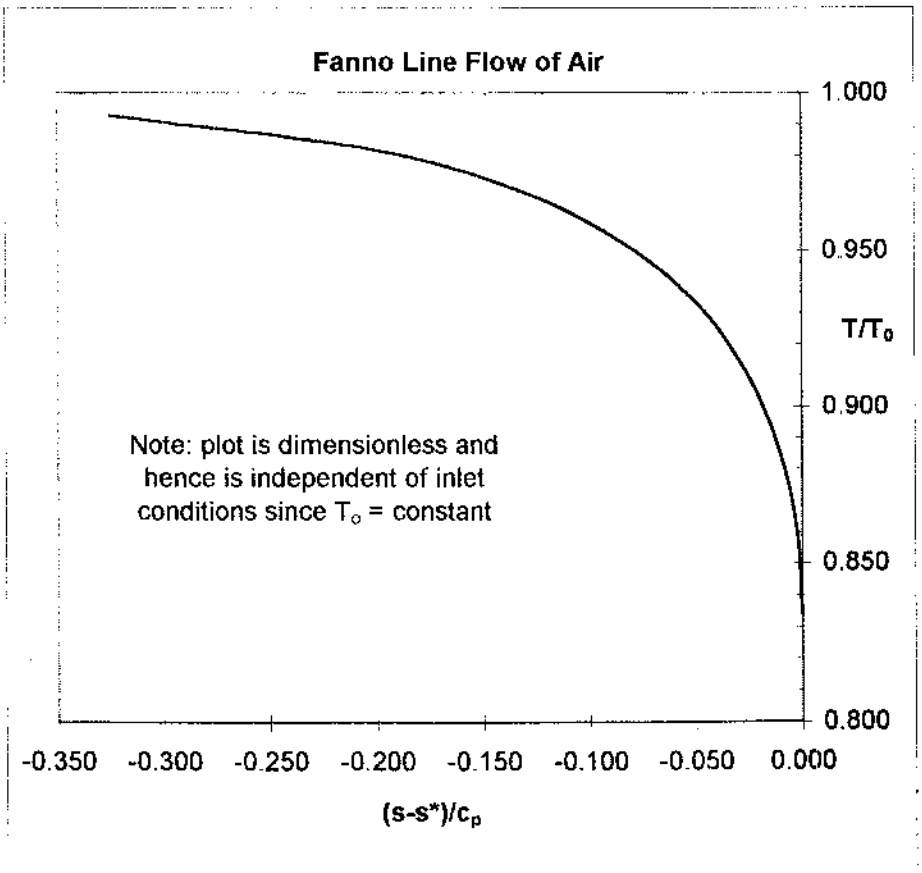
$$s - s^* = s_o - s_o^* = c_p\int_{T_o^*}^{T_o}\frac{dT}{T} - R\int_{P_o^*}^{P_o}\frac{dP}{P} = c_p\ln\frac{T_o}{T_o^*} - R\ln\frac{P_o}{P_o^*} = -(c_p - c_v)\ln\frac{P_o}{P_o^*}$$

$$\frac{s-s^*}{c_p} = -\frac{(c_p - c_v)}{c_p}\ln\frac{P_o}{P_o^*} = -\frac{(k-1)}{k}\ln\frac{P_o}{P_o^*} = -\frac{0.4}{1.4}\ln\frac{P_o}{P_o^*} = -0.286\ln\frac{P_o}{P_o^*}$$

For a given value of M, T/T_o is found from Appendix E.1 (11.17b)

P_o/P_o^* " " " " E.2 (12.18e)

M	T/T_o	P_o/P_o^*	$(s-s^*)/c_p$
0.19	0.993	3.112	-0.325
0.30	0.982	2.035	-0.203
0.40	0.969	1.590	-0.133
0.50	0.952	1.340	-0.084
0.60	0.933	1.188	-0.049
0.70	0.911	1.094	-0.026
0.80	0.887	1.038	-0.011
0.90	0.861	1.009	-0.003
1.00	0.833	1.000	0.000



Given: Fanno line flow for air; s^* is the entropy at the condition where $M = 1.0$.

Plot: Fanno line for Mach numbers in the range $0.1 < M < 3.0$, $\frac{T}{T^*}$ vs $\frac{s-s^*}{c_p}$

Solution:

* Compressible flow functions to be used in the solution

Basic equation $T\,ds = dh - v\,dP$ $\qquad P = \rho RT$

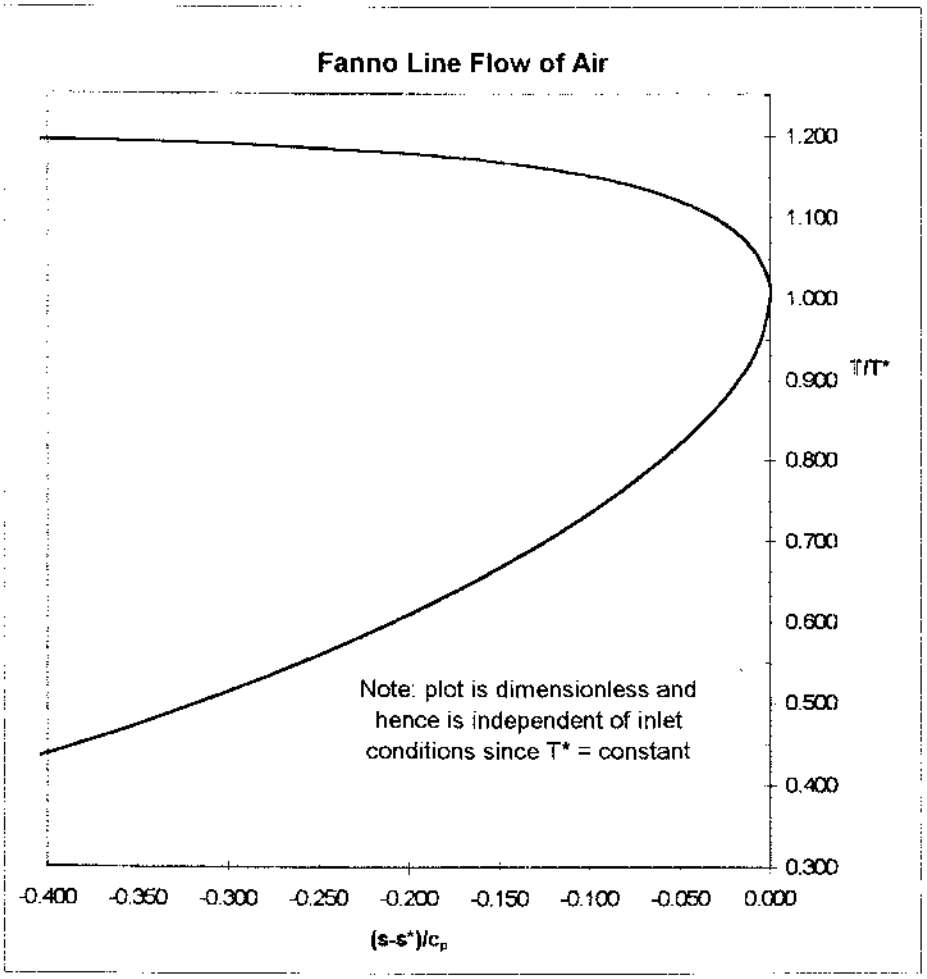
$$T\,ds = dh - v\,dP = c_p\,dT - \frac{1}{\rho}\,dP \qquad ds = c_p\frac{dT}{T} - \frac{dP}{\rho T}$$

$$s - s^* = s_0 - s_0^* = c_p \int_{T_0^*}^{T_0} \frac{dT}{T} - R\left(\int_{P_0^*}^{P_0} \frac{dP}{P}\right)_{T_0 = T_0^*} = c_p \ln\frac{T_0}{T_0^*} - R\ln\frac{P_0}{P_0^*} = -(c_p - c_v)\ln\frac{P_0}{P_0^*}$$

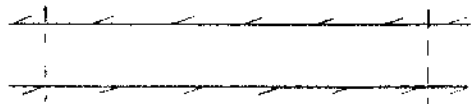
$$\frac{s - s^*}{c_p} = -\frac{(c_p - c_v)}{c_p}\ln\frac{P_0}{P_0^*} = -\frac{k-1}{k}\ln\frac{P_0}{P_0^*} = -\frac{0.4}{1.4}\ln\frac{P_0}{P_0^*} = -0.286\ln\frac{P_0}{P_0^*}$$

For a given value of M, use Appendix E.2 to determine T/T^* (12.18a) and P_0/P_0^* (12.18e).

M	T/T*	P_0/P_0^*	$(s-s^*)/c_p$
0.10	1.198	5.822	-0.504
0.14	1.195	4.182	-0.409
0.20	1.190	2.964	-0.311
0.30	1.179	2.035	-0.203
0.40	1.163	1.590	-0.133
0.50	1.143	1.340	-0.084
0.60	1.119	1.188	-0.049
0.70	1.093	1.094	-0.026
0.80	1.064	1.038	-0.011
0.90	1.033	1.009	-0.003
1.00	1.000	1.000	0.000
1.20	0.932	1.030	-0.009
1.40	0.862	1.115	-0.031
1.60	0.794	1.250	-0.064
1.80	0.728	1.439	-0.104
2.00	0.667	1.688	-0.150
2.20	0.610	2.005	-0.199
2.40	0.558	2.403	-0.251
2.60	0.510	2.896	-0.304
2.80	0.467	3.500	-0.358
3.00	0.429	4.235	-0.413



Given: Air flows through an insulated constant-area duct ($D = 2.12$ ft, $e/D = 0.002$, $L_{12} = 40$ ft) with conditions at sections ① & ② as shown.

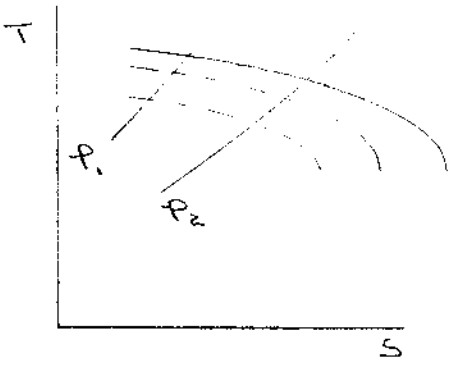


$T_1 = 100F$ ① $P_1 = 17.0$ psia

② $P_2 = 14.7$ psia $M_2 < 1.0$

Find: (a) Is it possible to solve for M_1 and M_2? Prove answer graphically
(b) Find $\dot{m}$ and T_2

Solution:



(a) It is possible to solve for M_1 and M_2. There is a different Fanno line for each different flow rate (or M_1)

- need to find the value of M_1 that gives the pressure drop $P_1 - P_2$ over the length L_{12}

(b) Procedure for trial and error solution is to assume M_1 and calculate P_2

Use Fanno-line flow functions of Appendix E-2

Assume M_1,

- determine P^*, Appendix E (12.18d)
  $\bar{f}L/D$ " " (12.17)
  $\bar{f} = \frac{1}{4} f(Re)$
- calculate $\bar{f}L_{12}/D$ and $\bar{f}(L/D)_2 = \bar{f}(L/D)_1 - \bar{f}L_{12}/D$
- knowing $\bar{f}(L/D)_2$, iterate (12.17) to determine M_2
- determine P_2 from M_2 (12.18d) and check vs known P_2

Repeat with another assumed value of M_1

Additional computing equations:

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = M_1 \left(1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot {}^\circ R} \times 560 {}^\circ R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft/s^2}{lbf}\right)^{1/2} = 1160 M_1$$

$$\rho_1 = \frac{P_1}{RT_1} = \frac{17 lbf}{in^2} \times 144 \frac{in^2}{ft^2} \times \frac{lbm \cdot {}^\circ R}{53.3 ft \cdot lbf} \times \frac{1}{560 {}^\circ R} \times \frac{slug}{32.2 lbm} = 2.55 \times 10^{-3} \frac{slug}{ft^3}$$

From Table A.9, $\mu = 3.97 \times 10^{-7}$ lbf·s/ft²

$$Re = \frac{\rho V D}{\mu} = 2.55 \times 10^{-3} \frac{slug}{ft^3} \times 1160 M_1 \frac{ft}{s} \times 2.12 ft \times 3.97 \times 10^{-7} \frac{ft^2}{lbf \cdot s} \times \frac{lbf \cdot s^2}{ft \cdot slug} = 7.45 \times 10^6 M_1$$

For $M_1 = 0.4$, $Re = 3.0 \times 10^6$. With $e/D = 0.002$, Fig. 8.13 gives $f = 0.023$

Assume f is constant.

Problem* 12.57 (cont'd)

M_1	T_1/T^*	P_1/P^*	P^*(psia)		$(fL_m/D)_1$	V_1 (ft/s)	$Re(10^6)$	f	fL_{12}/D	$(fL_m/D)_2$	M_2	T_2/T^*	P_2/P^*	P_2(psia)
0.400	1.163	2.6958	6.30606	0.186	2.31	464	2.98	0.0230	0.434	1.875	0.427	1.158	2.520	15.9
0.450	1.153	2.3865	7.12347	0.234	1.57	522	3.35	0.0230	0.434	1.132	0.493	1.144	2.170	15.5
0.500	1.143	2.1381	7.95102	0.286	1.07	580	3.73	0.0230	0.434	0.635	0.568	1.127	1.869	14.9
0.510	1.141	2.0942	8.118	0.297	0.99	592	3.80	0.023	0.434	0.556	0.585	1.123	1.812	14.7

Iterate to determine M_2 for known M_1

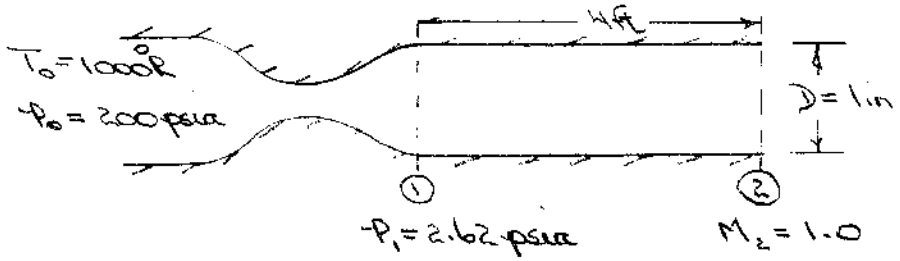
M_1	$(fL_m/D)_2$	$(M_2)_{guess}$		$(fL_m/D)_2$
0.400	1.875	0.427	0.211	1.870
0.450	1.132	0.493	0.278	1.128
0.500	0.635	0.568	0.364	0.633
0.510	0.556	0.585	0.384	0.563

$M_1 = 0.510$ $\quad \frac{T_1}{T^*} = 1.141$, $\frac{T_2}{T^*} = 1.123$, $\therefore \frac{T_2}{T_1} = 0.984$

$T_2 = 551^\circ R$

$\dot{m} = \rho V_1 A = 2.55 \times 10^{-3} \frac{slug}{ft^3} \times 592 \frac{ft}{s} \times \frac{\pi}{4}(2.0)^2 ft^2 = 5.33\ slug/s$

Given: Flow air ($T_o = 1000°R$, $p_o = 200$ psia) through a converging-diverging nozzle to a constant area duct, of length $L = 4$ ft and diameter $D = 1$ in. Properties are as shown



Find: (a) average friction factor, $\bar{f}$ (b) $p_1 - p_2$

Solution: * Compressible flow functions to be used in solution

Assumptions: (1) steady flow (2) uniform flow at a section (3) isentropic flow in nozzle, adiabatic flow in duct (4) ideal gas

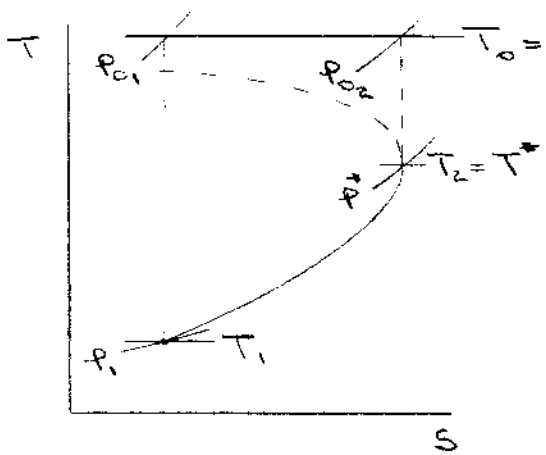
$\frac{p_1}{p_{o_1}} = 0.0131$ From Appendix E.1, $M_1 = 3.50$

Then from Appendix E.2, $\frac{\bar{f} L_{max}}{D} = 0.5864$, $\frac{p_1}{p^*} = 0.1685$

Thus $\bar{f} = 0.5864 \frac{D}{L_{max}} = 0.5864 \cdot \frac{1 \text{ in}}{4 \text{ ft}} \cdot \frac{\text{ft}}{12 \text{ in}} = 0.0122$ ← $\bar{f}$

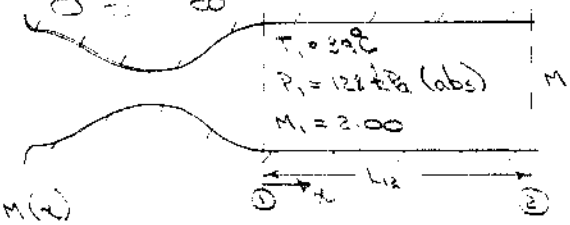
and $p^* = p_2 = \frac{p_1}{0.1685} = \frac{2.62 \text{ psia}}{0.1685} = 15.6$ psia

$p_1 - p_2 = -13.0$ psi ← $p_1 - p_2$



Problem *12.59

Given: Flow of air through a constant-area duct supplied by a converging-diverging nozzle; flow is frictionless



Find: L_{12}, plot P(x), M(x)

Solution:

* Compressible flow functions to be used in the solution

Assumptions: (1) steady flow (3) uniform flow at a section
(2) adiabatic flow (4) ideal gas

For $M_1 = 2.00$, from App E.2, $P_1/P^* = 0.4082$, $[f(L_{max})/D]_1 = 0.3050$

For $M_2 = 1.00$, " " " , $[f(L_{max})/D]_2 = 0$

Then $\frac{f L_{12}}{D} = \left[\frac{f L_{max}}{D}\right]_1 - \left[\frac{f L_{max}}{D}\right]_2 = 0.3050$ and $P^* = P_{2,choked} = 313.5$ psia

$\rho_1 = \frac{P_1}{RT_1} = 126\times10^3 \frac{N}{m^2} \cdot \frac{kg\cdot K}{287\ N\cdot m} \cdot \frac{1}{312K} = 1.42\ kg/m^3$

$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 2.00\left(1.4 \cdot 287 \frac{N\cdot m}{kg\cdot K} \cdot 312K \cdot \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 708\ m/s$

$A = \pi D^2/4$; $D = (4A/\pi)^{1/2} = \left(\frac{4}{\pi} \cdot 650\ mm^2\right)^{1/2} = 28.8\ mm$

$Re_1 = \frac{\rho V_1 D}{\mu}$ From Table A.10 at $T = 39°C$, $\mu = 1.85\times10^{-5}\ N\cdot s/m^2$

$Re_1 = 1.43 \frac{kg}{m^3} \cdot 708 \frac{m}{s} \cdot 2.88\times10^{-2} m \cdot \frac{m^2}{1.85\times10^{-5} N\cdot s} \cdot \frac{N\cdot s^2}{kg\cdot m} = 1.58\times10^6$

For commercial steel from Table 8.1 $e = 4.6\times10^{-5}$ m

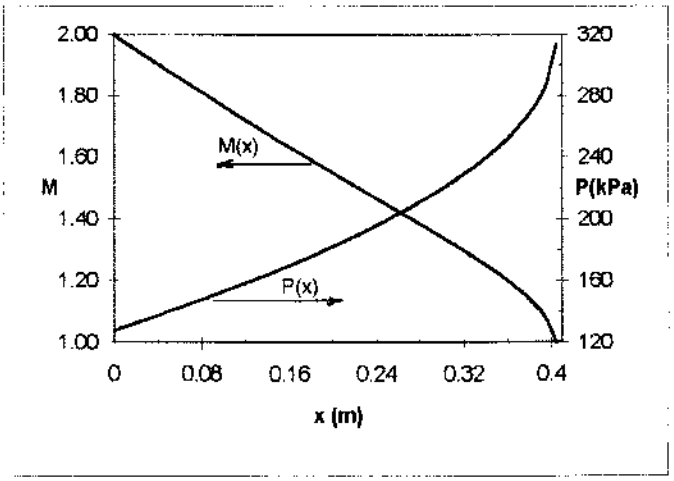
$e/D = 4.6\times10^{-5}/2.88\times10^{-2} = 1.59\times10^{-3} = 0.00159$

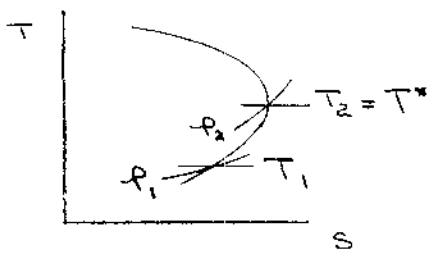
From Fig. 8.13, friction factor, $f = 0.0217$

$\frac{f L_{12}}{D} = 0.3050$ $L_{12} = (.3050)D/f = \frac{0.3050}{0.0217} \cdot 2.88\times10^{-2} m = 0.405\ m$ ← L_{12}

To determine P(x) and M(x) use Appendix E (Eqs 12.17 - 12.18d)

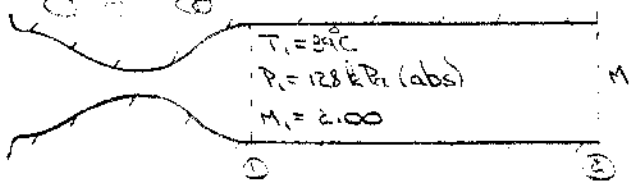
M	fL_m/D	$\Delta(fL_m/D)$	x(m)	T/T*	P/P*	P(kPa)
2.00	0.305	0	0	0.667	0.408	128
1.90	0.274	0.031	0.041	0.697	0.439	138
1.70	0.208	0.097	0.129	0.760	0.513	161
1.60	0.172	0.133	0.176	0.794	0.557	175
1.50	0.136	0.169	0.224	0.828	0.606	190
1.40	0.100	0.205	0.272	0.862	0.663	208
1.30	0.065	0.240	0.319	0.897	0.728	228
1.20	0.034	0.271	0.360	0.932	0.804	252
1.10	0.010	0.295	0.392	0.966	0.894	280
1.00	0.000	0.305	0.405	1.000	1.000	314





Problem * 12.60

Given: Flow of air through a constant-area duct supplied by a converging-diverging nozzle; flow is reversible (Problem 12.47)



s^* is the entropy at the condition where $M = 1.00$

Plot: Subsonic and supersonic portions of the Fanno line [T/T_0 vs. $(s-s^*)/c_p$]

Solution:

* Compressible flow functions to be used in the solution

Basic equation: $Tds = dh - v\,dP$ $\qquad P = \rho RT$

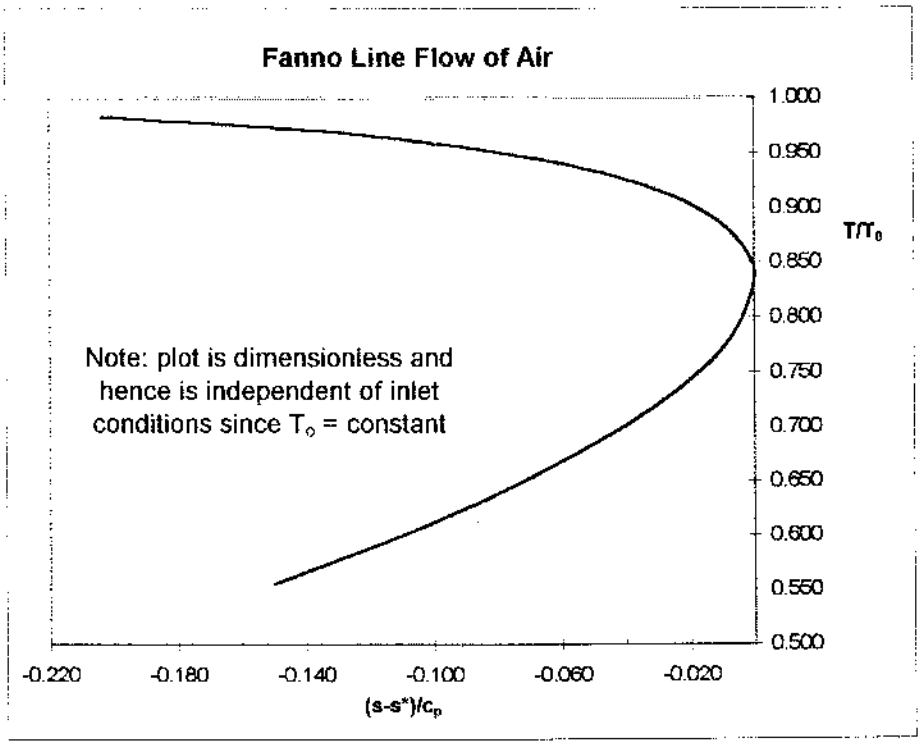
$$Tds = dh - \frac{1}{\rho}dP = c_p\,dT - \frac{1}{\rho}dP \qquad ds = c_p\frac{dT}{T} - \frac{dP}{\rho T}$$

$$s - s^* = s_0 - s_0^* = c_p\int_{T_0^*}^{T_0}\frac{dT}{T} - R\int_{P_0^*}^{P_0}\frac{dP}{P} = c_p\ln\frac{T_0}{T_0^*} - R\ln\frac{P_0}{P_0^*} = -(c_p - c_v)\ln\frac{P_0}{P_0^*}$$

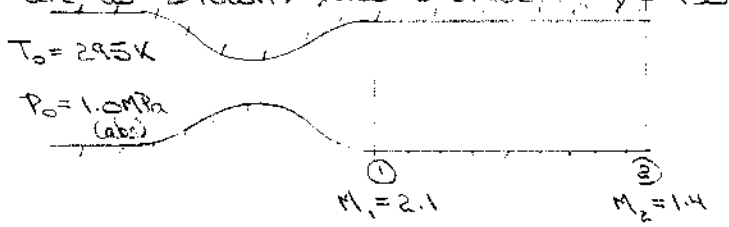
$$\frac{s-s^*}{c_p} = -\frac{c_p - c_v}{c_p}\ln\frac{P_0}{P_0^*} = -\frac{k-1}{k}\ln\frac{P_0}{P_0^*} = -\frac{0.4}{1.4}\ln\frac{P_0}{P_0^*} = -0.286\ln\frac{P_0}{P_0^*}$$

For a given value of M_1, T/T_0 is found from Appendix E.1 (11.17b)
P_0/P_0^* " " " " " E.2 (12.18e)

M	T/T_0	P_0/P_0^*	$(s-s^*)/c_p$
2.00	0.556	1.688	-0.150
1.80	0.607	1.439	-0.104
1.60	0.661	1.250	-0.064
1.40	0.718	1.115	-0.031
1.20	0.776	1.030	-0.009
1.00	0.833	1.000	0.000
0.90	0.861	1.009	-0.003
0.80	0.887	1.038	-0.011
0.70	0.911	1.094	-0.026
0.60	0.933	1.188	-0.049
0.50	0.952	1.340	-0.084
0.40	0.969	1.590	-0.133
0.30	0.982	2.035	-0.203



Given: Flow of air from a large tank (p_o = 1.0 MPa(abs), T_o = 295 K) through a C-D nozzle to a constant area duct. Properties are as shown. Duct is smooth, D = 150 mm.



Find: (a) p_2 (b) L_{1-2} (c) $s_2 - s_1$

Solution: * Compressible flow functions to be used in solution.

Assumptions: (1) steady flow (2) uniform flow at a section
(3) isentropic flow in nozzle, adiabatic flow in duct
(4) ideal gas

(a)

From App. E.1 for $M_1 = 2.1$ $T/T_o = 0.5314$ $\therefore T_1 = 156.7$ K

$p/p_o = 0.1094$ $\therefore p_1 = 109.4$ kPa.

$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 2.1\left[1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 156.7\,K \times \frac{kg\cdot m}{N\cdot s^2}\right]^{1/2} = 527$ m/s

For App. E.2 for $M = 2.1$ $p_o/p_o^* = 1.837$ $\therefore p_o^* = 544.4$ kPa

$p/p^* = 0.3802$ $\therefore p^* = 287.7$ kPa.

for $M_2 = 1.4$ $p_2/p^* = 0.6632$ $\therefore p_2 = 190.8$ kPa ← p_2

$p_{o_2}/p_o^* = 1.115$ $\therefore p_{o_2} = 320.8$ kPa ← —

(b) From App. E.2 for $M_1 = 2.1$, $\left(\bar{f} L_{max}/D\right)_1 = 0.3339$

$M_2 = 1.4$, $\left(\bar{f} L_{max}/D\right)_2 = 0.09974$

$\therefore \left(\bar{f} L_{12}/D\right)_h = 0.3339 - 0.09974 = 0.2342$ ← ---

To obtain L_{12}, we need f; $f = f(Re)$; $Re = \frac{\rho V D}{\mu}$ {use conditions at ①}

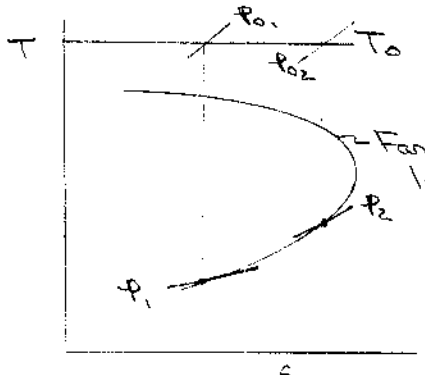
To obtain μ at 156.7 K, use Sutherland correlation (Appendix A)

$$\mu = \frac{bT^{1/2}}{1+S/T} = 1.458\times10^{-6} \frac{kg}{m\cdot s\cdot K^{1/2}} \times (156.7\,K)^{1/2} \times \frac{1}{1+\frac{110.4}{156.7}} = 1.071\times10^{-5} \frac{N\cdot s}{m^2}$$

$$Re = \frac{\rho_1 V_1 D}{\mu} = \frac{p_1 V_1 D}{RT_1 \mu} = 109.4\times10^3 \frac{N}{m^2} \times 527 \frac{m}{s} \times (0.15\,m) \times \frac{kg\cdot K}{287\,N\cdot m} \times 156.7\,K \times 1.071\times10^{-5} \frac{N\cdot s/m^2}{N\cdot s^2/kg\cdot m}$$

$Re = 1.80\times10^7$ For smooth pipe from Fig 8.13 friction factor, $f \approx 0.007$

$\therefore 0.007\, L_{12}/D)_h = 0.2342$ $L_{12} = \frac{0.2342}{0.007}) = 5.02$ m ← L_{12}

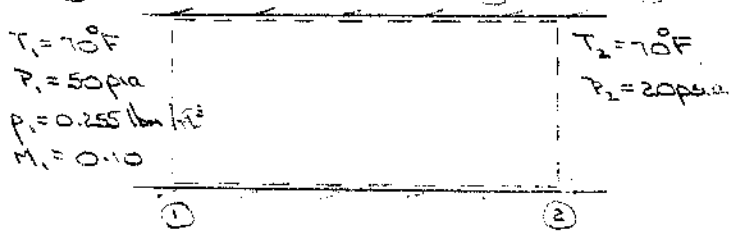


From the Tds eq.

$$s_2 - s_1 = s_{o_2} - s_{o_1} = c_p \ln \frac{T_{o_2}}{T_{o_1}} - R \ln \frac{p_{o_2}}{p_{o_1}}$$

$$s_2 - s_1 = -287 \frac{N\cdot m}{kg\cdot K} \ln \frac{320.8}{1000} = 0.326 \frac{kJ}{kg\cdot K}$$ ← $s_2 - s_1$

Given: Steady, isothermal flow of gas through a constant area pipe



Find: M_2, direction of $\dot{Q}$; plot Ts diagram

Solution:

Basic equation: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\qquad \dot{m} = \rho V A$

Computing equation: $T_0/T = 1 + \frac{k-1}{2} M^2$

Assumptions: (1) steady flow (4) ideal gas
(2) uniform flow at section 5: $\dot{W}_s = \dot{W}_{shear} = 0$
(3) $\Delta z = 0$

From continuity $\dot{m} = \rho_1 V_1 A = \rho_2 V_2 A$ $\therefore \rho_1 V_1 = \rho_2 V_2$

Since $P = \rho RT$, $\rho = \frac{P}{RT}$. Also $M = \frac{V}{c}$ and $V = Mc$

Then $\frac{P_1}{RT_1} M_1 c_1 = \frac{P_2}{RT_2} M_2 c_2$

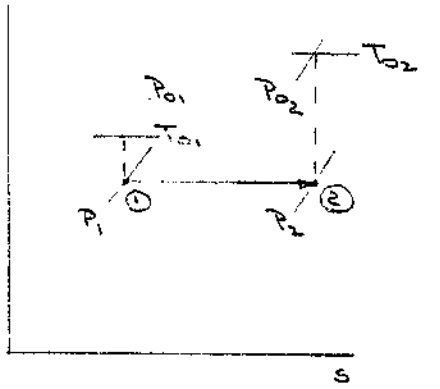
Since $c = \sqrt{kRT}$ and $T_1 = T_2$, then $c_1 = c_2$ and

$P_1 M_1 = P_2 M_2$ $\qquad M_2 = \frac{P_1}{P_2} M_1 = \frac{50}{20} \cdot 0.10 = 0.25$ ← M_2

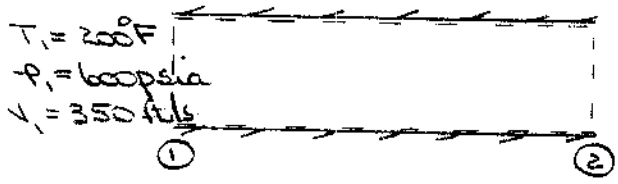
$h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\qquad$ Then $\frac{\delta Q}{dm} = h_{o_2} - h_{o_1} = c_p (T_{o_2} - T_{o_1})$

Now $T_o = T\left[1 + \frac{k-1}{2} M^2\right]$

Since T = const and $M_2 > M_1$, then $T_{o_2} > T_{o_1}$ and $\frac{\delta Q}{dm} > 0$ ← δQ / lbm



Given: Air enters a constant area channel at conditions shown and proceeds to choking under isothermal flow conditions.



Find: limiting pressure, p_2
Compare with p_2 for frictional adiabatic flow.

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\qquad \dot{m} = \rho A V$

Computing equation: $T_0/T = 1 + \frac{k-1}{2} M^2$

Assumptions: (1) steady flow (2) ideal gas
(3) uniform flow at a section (4) $\Rightarrow \dot{W}_{shear} = 0$

$$c_1 = (kRT_1)^{1/2} = \left(1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot °R} \times 660°R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft/s^2}{lbf}\right)^{1/2} = 1260 \text{ ft/s}$$

$$M_1 = \frac{V_1}{c_1} = \frac{350}{1260} = 0.278$$

$$V_2 = M_2 c_2 = M_2 c_1 = \frac{1}{\sqrt{1.4}} \times 1260 \text{ ft/s} = 1060 \text{ ft/s}$$

$$\rho_1 V_1 = \rho_2 V_2 \quad \text{or} \quad \frac{p_1}{RT_1} V_1 = \frac{p_2}{RT_2} V_2 \qquad \text{.....(1)}$$

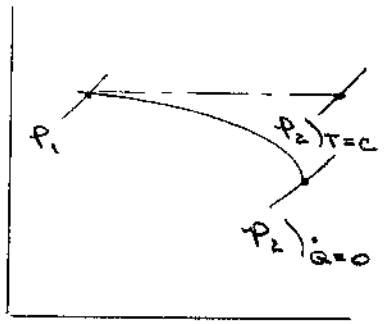
Since $T_1 = T_2$, $p_2 = p_1 \frac{V_1}{V_2} = 600 psia \times \frac{350}{1060} = 198 psia$ $\quad \leftarrow p_2|_{T=C}$

For adiabatic flow, T_0 = constant and $M_2 = 1.0$

$$T_{0_2} = T_{0_1} = T_1\left(1 + \frac{k-1}{2} M_1^2\right) = T_2\left(1 + \frac{k-1}{2} M_2^2\right)$$

$$\frac{T_2}{T_1} = \frac{1 + 0.2(0.278)^2}{1 + 0.2} = 0.846$$

$$T_2 = 558°R$$



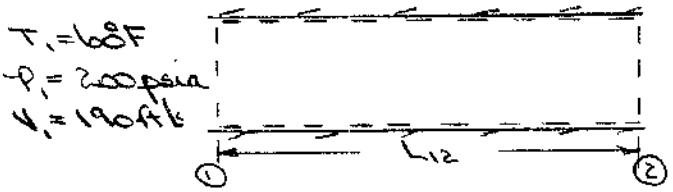
$$V_2 = c_2 = (kRT_2)^{1/2} = (1.4 \times 53.3 \times 558 \times 32.2)^{1/2} = 1160 \text{ ft/s}$$

From continuity (Eq. 1)

$$p_2 = p_1 \frac{V_1}{V_2} \frac{T_2}{T_1} = 600 psia \times \frac{350}{1160} \times 0.846$$

$$p_2 = 153 psia \quad \leftarrow p_2|_{\dot{Q}=0}$$

Given: Air enters a 6 in diameter pipe at the conditions shown. The flow is isothermal and the average friction factor is 0.016. At section ② downstream $p_2 = 75$ psia



Find: (a) M_2 and (b) L_{12}

Solution:

Basic equations: $\dot{m} = \rho V A$ $\quad p = \rho R T$

Computing equation: $f\frac{L_{max}}{D} = \frac{1-kM^2}{kM^2} + \ln kM^2$ (12.26)

From continuity $\rho_1 V_1 = \rho_2 V_2$ or $\frac{p_1}{T_1} V_1 = \frac{p_2}{T_2} V_2$

Since $T_1 = T_2$ and $V = Mc = M(kRT)^{1/2}$

$$M_2 = M_1 \frac{p_1}{p_2}$$

$$c_1 = (kRT_1)^{1/2} = \left(1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot °R} \times 520°R \times 32.2 \frac{lbm}{slug} \cdot \frac{slug \cdot ft/s^2}{lbf}\right)^{1/2} = 1120 \text{ ft/s}$$

$$M_1 = \frac{V_1}{c_1} = \frac{190}{1120} = 0.170$$

$$\therefore M_2 = 0.170 \times \frac{200}{75} = 0.452 \quad \leftarrow M_2$$

At $M_1 = 0.170$,

$$f\left(\frac{L_{max}}{D}\right)_1 = \frac{1-1.4(0.170)^2}{1.4(0.170)^2} + \ln 1.4(0.170)^2 = 20.5$$

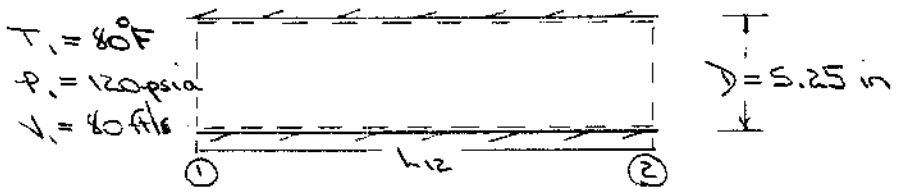
At $M_2 = 0.452$.

$$f\left(\frac{L_{max}}{D}\right)_2 = \frac{1-1.4(0.452)^2}{1.4(0.452)^2} + \ln 1.4(0.452)^2 = 1.24$$

$$\therefore f\frac{L_{12}}{D} = f\left(\frac{L_{max}}{D}\right)_1 - f\left(\frac{L_{max}}{D}\right)_2 = 20.5 - 1.24 = 19.3$$

$$\therefore L_{12} = 19.3 \frac{D}{f} = 19.3 \times 0.5 ft \times \frac{1}{0.016} = 603 \text{ ft} \quad \leftarrow L_{12}$$

Given: Air enters a clean steel pipe of length $L = 950$ ft and diameter $D = 5.25$ in. at the conditions shown.



Find: pressure drop, $p_1 - p_2$, assuming (a) incompressible (b) isothermal, and (c) adiabatic flow

Solution:

Basic equations: $\dot{m} = \rho V A$ $\quad p = \rho R T$

Computing equations: $p_1 - p_2 = \rho f \frac{L}{D} \frac{V^2}{2}$ (ρ = constant).

$$f\frac{L_{max}}{D} = \frac{1-kM^2}{kM^2} + \ln kM^2 \quad (T = \text{constant})$$

$$\frac{f L_{max}}{D} = \frac{1-M^2}{kM^2} + \frac{k+1}{2k} \ln\left[\frac{(k+1)M^2}{2\left(1+\frac{k-1}{2}M^2\right)}\right] \quad (\dot{Q}=0)$$

$$\rho_1 = \frac{p_1}{RT_1} = 120\frac{lbf}{in^2} \times 144\frac{in^2}{ft^2} \times 53.3\frac{lbm\cdot°R}{ft\cdot lbf} \times 540°R = 0.600\ lbm/ft^3$$

From Table A.9 {at $T = 80°F$} $\mu = 3.86\times10^{-7}$ lbf·s/ft²

$$Re_1 = \frac{\rho D V}{\mu} = 0.600\frac{lbm}{ft^3} \times \frac{5.25}{12}ft \times 80\frac{ft}{s} \times 3.86\times10^{-7}\frac{ft^2}{lbf\cdot s} \times \frac{slug}{32.2\,lbm} \times \frac{lbf\cdot s^2}{slug\cdot ft}$$

$Re_1 = 1.69\ 10^6$

For commercial steel (Table 8.1), $\varepsilon = 0.00015$ ft. $\therefore \varepsilon/D = 0.00034$

From Fig. 8.13, $f = 0.0155$

(a) For incompressible flow

$$p_1 - p_2 = \rho f \frac{L}{D}\frac{V^2}{2} = 0.60\frac{lbm}{ft^3} \times 0.0155 \times \frac{950\times12}{5.25} \times \frac{(80)^2}{2}\frac{ft^2}{s^2} \times \frac{slug}{32.2\,lbm} \times \frac{lbf\cdot s^2}{slug\cdot ft} \times \frac{ft^2}{144\,in^2}$$

$p_1 - p_2 = 13.9$ psia $\longleftarrow (p_1 - p_2)_{\rho=c}$

(b) For isothermal flow

$$c_1 = (kRT_1)^{1/2} = \left(1.4\times53.3\frac{ft\cdot lbf}{lbm\cdot°R}\times540°R\times32.2\frac{lbm}{slug}\times\frac{slug\cdot ft}{lbf\cdot s^2}\right)^{1/2} = 1140\ ft/s$$

$$M_1 = \frac{V_1}{c_1} = \frac{80}{1140} = 0.0702$$

At state (1) $\left(f\frac{L_{max}}{D}\right)_1 = \frac{1-kM_1^2}{kM_1^2} + \ln kM_1^2$

$$= \frac{1-1.4(0.0702)^2}{1.4(0.0702)^2} + \ln\left[1.4(0.0702)^2\right]$$

$$\left(f\frac{L_{max}}{D}\right)_1 = 139$$

$$f\frac{L}{D}_{1\to 2} = 0.0155 \times \frac{950 \times 12}{5.25} = 33.6$$

$$\therefore f\frac{L_{max}}{D}_2 = 139 - 33.6 = 105 = \frac{1-kM_2^2}{kM_2^2} + \ln kM_2^2$$

Trial and error solution for M_2

M_2	$f\frac{L_{max}}{D}_2$
0.10	66.2
0.08	106
0.081	103
0.0805	105

$V_2 = M_2 c_2 = M_2 c_1 = 0.0805 \times 1140$ ft/s $= 91.8$ ft/s

$P_1 V_1 = P_2 V_2$ or $\frac{P_1}{T_1} V_1 = \frac{P_2}{T_2} V_2$ ..... (1)

Since $T_2 = T_1$, $P_2 = P_1 \frac{V_1}{V_2} = 120$ psia $\times \frac{80}{91.8} = 105$ psia

$P_1 - P_2 = 15.0$ psia $\quad (P_2 - P_1)_{T=c}$

(c) For adiabatic flow $\quad M_1 = 0.0702$

$$f\frac{L_{max}}{D}_1 = \frac{1-(0.0702)^2}{1.4(0.0702)^2} + \frac{1.4+1}{2(1.4)} \ln\left[\frac{(2.4)(0.0702)^2}{2(1+0.2(0.0702)^2}\right] = 139.8$$

$$\therefore f\frac{L_{max}}{D}_2 = 139.8 - 33.6 = 106.2 = \frac{1-M_2^2}{1.4M_2^2} + \frac{2.4}{2.8} \ln\left[\frac{2.4M_2^2}{2(1+0.2M^2)}\right]$$

Trial and error solution for M_2

M_2	$f\frac{L_{max}}{D}_2$
0.085	94.1
0.080	106.7
0.0802	106.2 ✓

For adiabatic flow, T_0 = constant

$$\therefore \frac{T_2}{T_1} = \frac{1+\frac{k-1}{2}M_1^2}{1+\frac{k-1}{2}M_2^2} = \frac{1+0.2(0.0702)^2}{1+0.2(0.0802)^2} = 1.00$$

$\therefore V_2 = M_2 c_2 = M_2 c_1 = 0.0802 \times 1140$ ft/sec $= 91.4$ ft/s

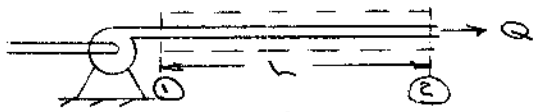
From continuity (Eq. 1)

$$P_2 = P_1 \frac{V_1}{V_2} = 120 \text{ psia} \times \frac{80}{91.4} = 105 \text{ psia}$$

$\therefore P_1 - P_2 = 15.0$ psia $\quad (P_2 - P_1)_{Q=0}$

Note: P_2 is essentially the same for isothermal and adiabatic flows. The value is higher that for incompressible flow

Given: Natural gas (molecular mass $M_m = 18$, $k = 1.3$) is pumped through a constant area pipe ($D = 36$ in) for a distance $L = 40$ miles



$D = 36$ in
$T = 78°F$ = const.

$p_1 \leq 90$ psig $\quad p_2 \geq 10$ psig

Find: Volume flowrate, Q (ft³ day @ 70°F and 1 atm)

Solution:

Basic equations: $\dot{m} = \rho V A \qquad p = \rho RT$

Computing equation: $f\frac{L_{max}}{D} = \frac{1 - kM^2}{kM^2} + \ln kM^2$

Then, $f\frac{L_{1\text{-}2}}{D} = f\frac{L_{max}}{D}\Big)_1 - f\frac{L_{max}}{D}\Big)_2 = \frac{1 - kM_1^2}{kM_1^2} - \frac{1 - kM_2^2}{kM_2^2} + \ln\frac{M_1^2}{M_2^2}$

Since information on pressures is known, we relate p and M

From the ideal gas equation of state, for T = constant $\frac{\rho_1}{\rho_2} = \frac{p_1}{p_2}$

From the continuity equation $\frac{\rho_1}{\rho_2} = \frac{V_2}{V_1}$ and for T = constant, $\frac{V_2}{V_1} = \frac{M_2}{M_1}$

Hence $\frac{p_1}{p_2} = \frac{M_2}{M_1}$ and $M_2 = \frac{p_1}{p_2} M_1$

Substituting for M_2 into the equation for $f\frac{L_{1\text{-}2}}{D_h}$ and rearranging we obtain

$$f\frac{L_{1\text{-}2}}{D_h} = \frac{1 - (p_2/p_1)^2}{kM_1^2} - \ln\left(\frac{p_1}{p_2}\right)^2$$

Solving this equation for M_1, then $M_1 = \left\{\frac{1}{k}\left[\frac{1 - \left(\frac{p_2}{p_1}\right)^2}{f\frac{L_{1\text{-}2}}{D_h} + \ln\left(\frac{p_1}{p_2}\right)^2}\right]\right\}^{1/2}$

$\frac{L}{D_h} = 40 \text{ mi} \times 5280 \frac{\text{ft}}{\text{mi}} \times \frac{1}{3 \text{ ft}} = 70{,}400$

Assume pipe is commercial steel. From Table 8.1, $e = 0.00015$ ft and hence $e/D = 0.00005$. Assume $Re > 3.0 \times 10^5$, then $f = 0.0105$. Solving for M_1,

$$M_1 = \left\{\frac{1}{1.3}\left[\frac{1 - \left(\frac{24.7}{104.7}\right)^2}{0.0105(70{,}400) + \ln\left(\frac{104.7}{24.7}\right)^2}\right]\right\}^{1/2} = 0.0313$$

$R_{\text{natgas}} = \frac{R_u}{M} = 1544 \frac{\text{ft·lbf}}{\text{lbmole·°R}} \times \frac{\text{lbmole}}{18 \text{ lbm}} = 85.8 \frac{\text{ft·lbf}}{\text{lbm·°R}}$

$c_1 = (kRT_1)^{1/2} = \left(1.3 \times 85.8 \frac{\text{ft·lbf}}{\text{lbm·°R}} \times 530°\text{R} \times \frac{32.2 \text{ lbm}}{\text{slug}} \times \frac{\text{slug·ft}}{\text{lbf·s}^2}\right)^{1/2} = 1380 \text{ ft/s}$

$V_1 = M_1 c_1 = 0.0313 \times 1380 \text{ ft/s} = 43.2 \text{ ft/s}$

$\rho_1 = \frac{p_1}{RT_1} = 104.7 \frac{\text{lbf}}{\text{in}^2} \times \frac{\text{lbm·°R}}{85.8 \text{ ft·lbf}} \times \frac{1}{530°\text{R}} \times 144 \frac{\text{in}^2}{\text{ft}^2} = 0.332 \text{ lbm/ft}^3$

Check assumption on Re: from Fig 9.2 (for methane), $\mu = 1.08 \times 10^{-5} \times 2.089 \times 10^{-2} \text{ lbf·s/ft}^2$

$Re = \frac{\rho V D}{\mu} = 0.332 \frac{\text{lbm}}{\text{ft}^3} \times 43.2 \frac{\text{ft}}{\text{s}} \times 3 \text{ft} \times 2.26 \times 10^{-7} \frac{\text{ft}^2}{\text{lbf·s}} \times \frac{\text{slug}}{32.2 \text{ lbm}} \times \frac{\text{lbf·s}^2}{\text{ft·slug}} = 5.91 \times 10^6$

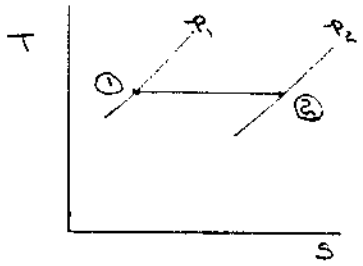
Then $f = 0.011$ and $\frac{fL_{1-2}}{D} = 774$ $M_1 = 0.0205$ and $V_1 = 42.1$ ft/sec

$\dot{m} = \rho_1 V_1 A = 0.332 \frac{lbm}{ft^3} \times 42.1 \frac{ft}{sec} \times \frac{\pi}{4}(3)^2 ft^2 = 98.8$ lbm/sec

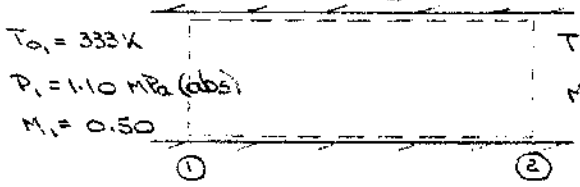
$\rho_{atm} = \frac{P_{atm}}{RT} = 14.7 \frac{lbf}{in^2} \times \frac{lbm \cdot R}{85.8\ ft\ lbf} \times \frac{1}{530R} \times \frac{144\ in^2}{ft^2} = 4.65 \times 10^{-2}$ lbm/ft^3

Volume flowrate at atmospheric pressure

$Q = \frac{\dot{m}}{\rho_{atm}} = 98.8 \frac{lbm}{sec} \times \frac{ft^3}{4.65 \times 10^{-2} lbm} \times \frac{3600\ sec}{hr} \times \frac{24\ hr}{day} = 1.84 \times 10^6$ ft^3/day ← Q



Given: Frictionless flow of air through a constant area duct



Find: $\delta Q/dm$, $P_1 - P_2$

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\qquad P_1A - P_2A = \dot{m}(V_2 - V_1)$

Computing equation: $T_o/T = 1 + \frac{k-1}{2} M^2$

Assumptions: (1) steady flow; (2) frictionless flow; (3) uniform flow at a section; (4) ideal gas; (5) $F_{body} = 0$; (6) $\dot{W}_s = \dot{W}_{shear} = 0$; (7) $\frac{\partial}{\partial} = 0$

$$h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2} \qquad \frac{\delta Q}{dm} = h_{o_2} - h_{o_1} = c_p(T_{o_2} - T_{o_1}) = 10^3 \frac{J}{kg \cdot K}(478 - 333)K = 145 \frac{kJ}{kg} = \frac{\delta Q}{dm}$$

$$P_1A - P_2A = \dot{m}(V_2 - V_1) = \rho V_1 A (V_2 - V_1) \quad \text{and} \quad P_1 - P_2 = \rho V_1 (V_2 - V_1)$$

$$T_o/T = 1 + \frac{k-1}{2} M^2 \qquad T_1 = \frac{T_{o_1}}{1 + \frac{k-1}{2} M_1^2} = \frac{333 K}{1 + 0.2(0.50)^2} = 317 K$$

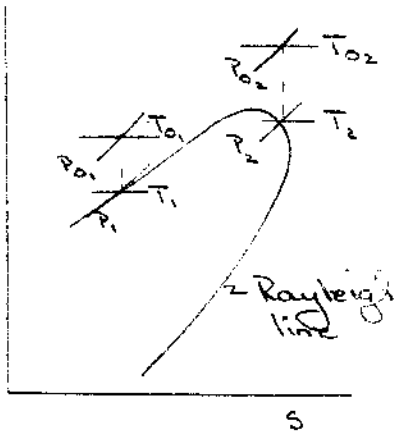
$$T_2 = \frac{T_{o_2}}{1 + \frac{k-1}{2} M_2^2} = \frac{478 K}{1 + 0.2(0.90)^2} = 411 K$$

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 0.50\left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 317 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 178 \ m/s$$

$$V_2 = M_2 c_2 = M_2 (kRT_2)^{1/2} = 0.90\left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 411 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 366 \ m/s$$

$$\rho_1 = \frac{P_1}{RT_1} = 1.10 \times 10^6 \frac{N}{m^2} \times \frac{kg \cdot K}{287 \ N \cdot m} \times \frac{1}{317 K} = 12.1 \ kg/m^3$$

$$P_1 - P_2 = \rho V_1 (V_2 - V_1) = 12.1 \frac{kg}{m^3} \times 178 \frac{m}{s} \times (366 - 178) \frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m} = 405 \ kPa = P_1 - P_2$$

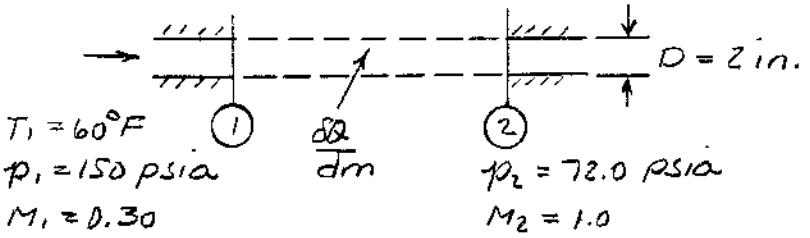


* Rayleigh-Line Flow Functions

From Appendix E.3

- for $M_1 = 0.5$, $P_1/P^* = 1.778$ (12.30a)
  $\therefore P^* = 619$ kPa
- for $M_2 = 0.90$, $P_2/P^* = 1.125$ (12.30a)
  $\therefore P_2 = 696$ kPa $\Rightarrow P_1 - P_2 = 404$ kPa.

Given: Air flow through a constant-area duct. Neglect friction.



Find: Heat transfer per pound of air.

Solution: Basic equations $p = \rho RT$ $\dot{m} = \rho VA$

Computing equations: $\frac{\delta Q}{dm} = c_p (T_{02} - T_{01})$ (energy)

$p_1 - p_2 = \rho_1 V_1 (V_2 - V_1)$ (momentum)

Assumptions: (1) Steady flow (3) Ideal gas
(2) Uniform flow at each section (4) no friction

At section 1

$$\rho_1 = \frac{p_1}{RT_1} = 150 \frac{lbf}{in.^2} \times \frac{lbm \cdot °R}{53.3\ ft \cdot lbf} \times \frac{1}{(460+60)°R} \times 144 \frac{in.^2}{ft^2} = 0.779\ lbm/ft^3$$

From momentum, $c_1 = \sqrt{kRT_1} = \left[1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot °R} \times 520°R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right]^{1/2}$

$V_2 = \frac{p_1 - p_2}{\rho_1 V_1} + V_1$ $\quad c_1 = 1120$ ft/s ; $V_1 = M_1 c_1 = 335$ ft/s

$$V_2 = (150-72) \frac{lbf}{in.^2} \times \frac{ft^3}{0.779\ lbm} \times \frac{s}{335\ ft} \times 144 \frac{in.^2}{ft^2} \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2} + 335 \frac{ft}{s} = 1120\ ft/s$$

From continuity, $\rho_1 V_1 = \rho_2 V_2$ so $\rho_2 = \rho_1 V_1 / V_2 = 0.152$ lbm/ft³. Thus

$$T_2 = \frac{p_2}{\rho_2 R} = 72.0 \frac{lbf}{in.^2} \times \frac{ft^3}{0.152\ lbm} \times \frac{lbm \cdot °R}{53.3\ ft \cdot lbf} \times 144 \frac{in.^2}{ft^2} = 1280°R$$

$T_{02} = T_2 \left(1 + \frac{k-1}{2} M_2^2\right) = 1280\ R\ (1.2) = 1540°R$; $T_{01} = T_1 \left(1 + \frac{k-1}{2} M_1^2\right) = 529°R$

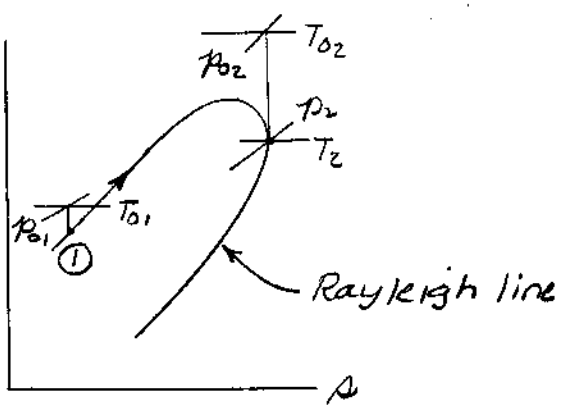
Thus $\frac{\delta Q}{dm} = 0.240 \frac{Btu}{lbm \cdot °R} (1540 - 529)°R = 243$ Btu/lbm $\quad \delta Q/dn$

Flow is along a Rayleigh line: T

* Rayleigh Line Flow Functions (Appendix E.3)!

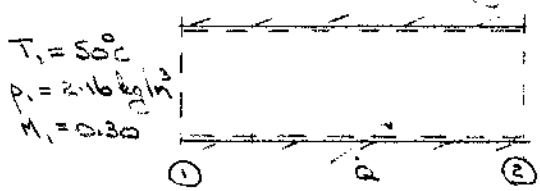
For $M_1 = 0.30$, $T_0/T_0^* = 0.3469$ (12.30d), so

$$T_0^* = \frac{T_{01}}{T_0/T_0^*} = \frac{529}{0.3469} = 1530\ R$$



Problem 12.69

Given: Frictionless flow of air through a constant area duct.



Find: $\delta q/dm$, $s_2 - s_1$

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta q}{dm} = h_2 + \frac{V_2^2}{2}$ $\qquad Tds = dh - vdp$

Computing equations: $T_0/T = 1 + \frac{k-1}{2} M^2$

Assumptions: (1) Steady flow (4) Ideal gas
(2) Frictionless flow (5) $W_s = W_{shear} = 0$
(3) Uniform flow at section (6) $\Delta z = 0$

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 0.30 \times 1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 323K \times \frac{kg\cdot m}{N\cdot s^2} = 108\ m/sec$$

$$p_1 = \rho RT_1 = 2.16 \frac{kg}{m^3} \times 287 \frac{N\cdot m}{kg\cdot K} \times 323K = 200\ kPa$$

$$\rho_1 V_1 = \rho_2 V_2 \qquad \rho_2 = \frac{p_2}{RT_2} \quad ; \quad V_2 = M_2 c_2 = M (kRT_2)^{1/2}$$

$$\therefore \rho_1 V_1 = \frac{p_2}{RT_2} M (kRT_2)^{1/2} = p_2 M_2 \left(\frac{k}{RT_2}\right)^{1/2}. \text{ Solving for } T_2,$$

$$T_2 = \frac{k}{R}\left(\frac{p_2 M_2}{\rho_1 V_1}\right)^2 = 1.4 \times \frac{kg\cdot K}{287 N\cdot m}\left[150\times10^3 \frac{N}{m^2} \times 0.6 \times \frac{1}{2.16 kg/m^3} \times 108 \frac{s}{m} \times \frac{kg\cdot m}{N\cdot s^2}\right]^2 \times \frac{kg\cdot s^2}{kg\cdot m}$$

$$T_2 = 726\ K.$$

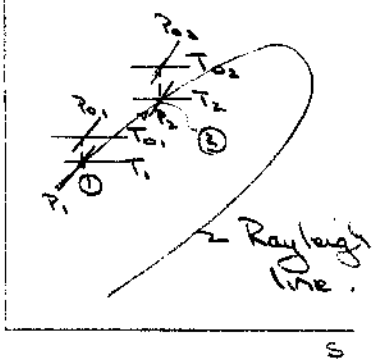
$$s_2 - s_1 = \int_1^2 ds = \int_{T_1}^{T_2} c_p \frac{dT}{T} - \int_{p_1}^{p_2} R \frac{dp}{p} = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

$$s_2 - s_1 = 1.0 \frac{kJ}{kg\cdot K} \ln \frac{726}{323} - 0.287 \frac{kJ}{kg\cdot K} \ln \frac{150}{200} = 0.892\ kJ/kg\cdot K \quad \longleftarrow s_2 - s_1$$

$$T_{o_1} = T_1 \left[1 + \frac{k-1}{2} M_1^2\right] = 323K \left[1 + 0.2(0.3)^2\right] = 329K$$

$$T_{o_2} = T_2 \left[1 + \frac{k-1}{2} M_2^2\right] = 726K \left[1 + 0.2(0.6)^2\right] = 778K$$

$$\frac{\delta q}{dm} = h_{o_2} - h_{o_1} = c_p (T_{o_2} - T_{o_1}) = 1.0 \frac{kJ}{kg\cdot K} (778 - 329) K = 449\ kJ/kg \quad \longleftarrow \frac{\delta q}{dm}$$



* Rayleigh-Line Flow Functions (App E.3)

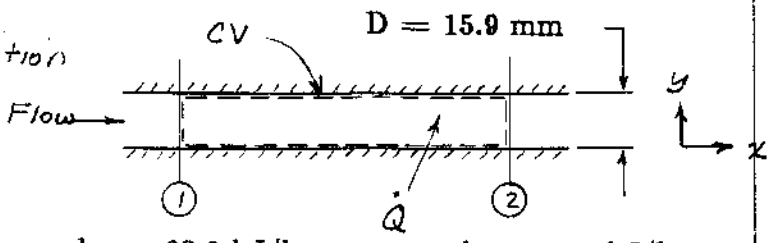
For $M_1 = 0.30$: $T_o/T_o^* = 0.3469$ $\therefore T_o^* = 948K$; $T_1/T^* = 0.4089$ $\therefore T^* = 790K$; $p_1/p^* = 2.131$ $\therefore p^* = 93.9\ kPa$

For $M_2 = 0.60$: $T_{o_2}/T_o^* = 0.8189$ $\therefore T_{o_2} = 776K$; $T_2/T^* = 0.9167$ $\therefore T_2 = 724K$; $p_2/p^* = 1.596$ $\therefore p_2 = 150\ kPa$

Note: In using the flow functions, it is not necessary to know $p_2 = 150\ kPa$.

Given: Freon flowing in constant-area tube boils to vapor. Frictional effects small compared to heat addition.

Find: (a) Rate of heat addition
(b) $p_1 - p_2$



Solution:

$h_1 = 60.9$ kJ/kg $\quad h_2 = 145$ kJ/kg
$\rho_1 = 1620$ kg/m^3 $\quad \rho_2 = 13.4$ kg/m^3
$\dot{m} = 0.835$ kg/s

Basic equation: $\dot{m} = \rho V A$

Computing equations:

$$\dot{Q} = \dot{m}(h_{02} - h_{01}) \; ; \quad h_0 = h + \frac{V^2}{2}$$

$$A = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.0159)^2 m^2 = 1.99 \times 10^{-4} m^2$$

$$p_1 A - p_2 A = \dot{m}(V_2 - V_1)$$

Assumptions: (1) Steady flow
(2) Frictionless flow
(3) Uniform flow at each section
(4) No work
(5) Neglect Δz

$$V_1 = \frac{\dot{m}}{\rho_1 A} = 0.835 \frac{kg}{sec} \times \frac{m^3}{1620\ kg} \times \frac{1}{1.99 \times 10^{-4} m^2} = 2.59 \text{ m/s} \; ; \quad h_{01} \simeq h_1$$

$$V_2 = \frac{\dot{m}}{\rho_2 A} = 0.835 \frac{kg}{sec} \times \frac{m^3}{13.4\ kg} \times \frac{1}{1.99 \times 10^{-4} m^2} = 313 \text{ m/s}$$

$$h_{02} = 145 \times 10^3 \frac{N \cdot m}{kg} + \frac{1}{2}(313)^2 \frac{m^2}{sec^2} \times \frac{N \cdot sec^2}{kg \cdot m} = 194 \text{ kJ/kg}$$

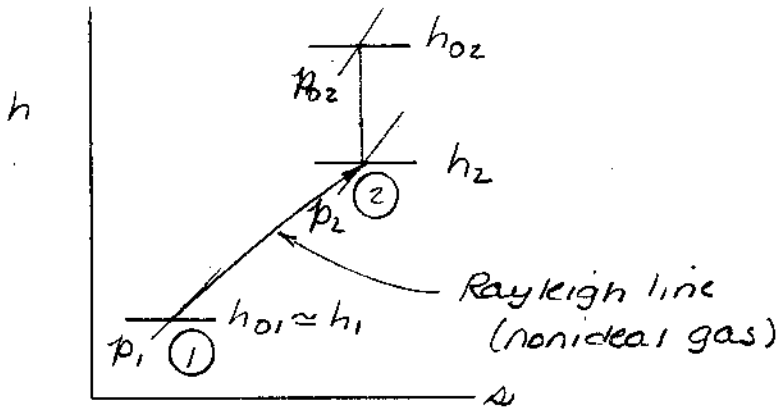
$$\dot{Q} = \dot{m}(h_{02} - h_{01}) \simeq \dot{m}(h_{02} - h_1)$$

$$\dot{Q} = 0.835 \frac{kg}{s}(194 - 60.9) \frac{kJ}{kg} \times \frac{W \cdot s}{J} = 111 \text{ kW}$$ $\dot{Q}$

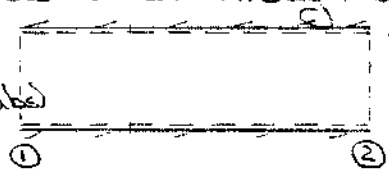
$$p_1 - p_2 = \frac{\dot{m}}{A}(V_2 - V_1) = 0.835 \frac{kg}{s} \times \frac{1}{1.99 \times 10^{-4} m^2}(313 - 2.59) \frac{m}{s} \cdot \frac{N \cdot s^2}{kg \cdot m}$$

$$p_1 - p_2 = 1.30 \text{ MPa}$$ $p_1 - p_2$

The hs diagram is



Given: Frictionless flow of air through a constant area duct.



$T_1 = 52°C$ $P_1 = 60\ kPa\ (abs)$ $T_2 = 45°C$ $M_2 = 1.0$ $\dot{m} = 1.42\ kg/s$ $D = 100\ mm$

Find: $\delta q/dm$, $s_2 - s_1$, $P_{o_1} - P_{o_2}$

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\qquad Tds = dh - vdP$

Computing equations: $T_o/T = 1 + \frac{k-1}{2} M^2$ $\qquad P_o/p = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$

Assumptions: (1) steady flow (4) ideal gas
(2) frictionless flow (5) $\dot{W}_s = \dot{m} w_{shaft} = 0$
(3) uniform flow at a section (6) $\frac{\delta Q}{dm} = 0$

$A = \frac{\pi}{4} D^2 = \frac{\pi}{4}(0.1)^2\ m^2 = 7.85 \times 10^{-3}\ m^2$

$\rho = \rho RT$ $\qquad \rho_1 = \frac{P_1}{RT_1} = 60 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287\ N \cdot m} \times \frac{1}{325\ K} = 0.643\ kg/m^3$

$\dot{m} = \rho V A$ $\qquad V_1 = \frac{\dot{m}}{\rho_1 A} = 1.42 \frac{kg}{s} \times \frac{m^3}{0.643\ kg} \times \frac{1}{7.85 \times 10^{-3}\ m^2} = 281\ m/s$

$V_2 = M_2 C_2 = M_2 (kRT_2)^{1/2} = 1.0 (1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 318\ K \times \frac{kg}{N \cdot s^2})^{1/2} = 357\ m/s$

$\dot{m} = \rho_1 V_1 A = \rho_2 V_2 A$; $\rho_2 = \frac{V_1}{V_2} \rho_1 = \frac{281}{357} \times 0.643\ kg/m^3 = 0.506\ kg/m^3$

$P = \rho RT$ $\qquad P_2 = \rho_2 R T_2 = 0.506 \frac{kg}{m^3} \times 287 \frac{N \cdot m}{kg \cdot K} \times 318\ K = 46.2\ kPa$

$Tds = dh - vdP = c_p dT - \frac{1}{\rho} dP$ $\qquad ds = c_p \frac{dT}{T} - R \frac{dP}{P}$

$s_2 - s_1 = \int_{s_1}^{s_2} ds = \int_{T_1}^{T_2} c_p \frac{dT}{T} - \int_{P_1}^{P_2} R \frac{dP}{P} = c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1} = 1.0 \frac{kJ}{kg \cdot K} \ln \frac{318}{325} - 0.287 \frac{kJ}{kg \cdot K} \ln \frac{46.2}{60}$

$s_2 - s_1 = 0.0522\ kJ/kg \cdot K$ ← $s_2 - s_1$

$M_1 = \frac{V_1}{C_1}$ $\qquad C_1 = (kRT_1)^{1/2} = (1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 325 K \times \frac{kg \cdot m}{N \cdot s^2})^{1/2} = 361\ m/s$, $M_1 = \frac{V_1}{C_1} = \frac{281}{361} = 0.778$

$\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2$ $\qquad T_{o_1} = T_1 \left[1 + \frac{k-1}{2} M_1^2\right] = 325\ K \left[1 + 0.2(0.778)^2\right] = 364\ K$

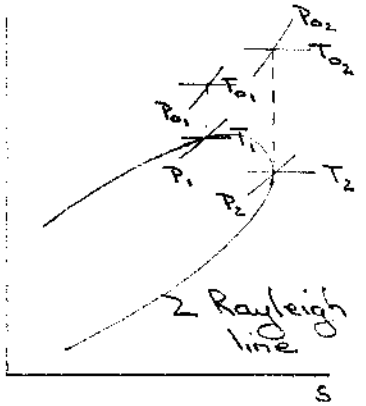
$T_{o_2} = T_2 \left[1 + \frac{k-1}{2} M_2^2\right] = 318\ K \left[1 + 0.2(1.0)^2\right] = 382\ K$

$h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\qquad \frac{\delta Q}{dm} = h_{o_2} - h_{o_1} = c_p (T_{o_2} - T_{o_1}) = 1.0 \frac{kJ}{kg \cdot K} (382 - 364) K = 18\ kJ/kg$ ← $\delta Q/dm$

$\frac{P_o}{P} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$ $\qquad P_{o_1} = P_1 \left[1 + \frac{k-1}{2} M_1^2\right]^{k/(k-1)} = 60\ kPa \left[1 + 0.2(0.778)^2\right]^{3.5} = 89.5\ kPa$

$P_{o_2} = P_2 \left[1 + \frac{k-1}{2} M_2^2\right]^{k/(k-1)} = 46.2\ kPa \left[1 + 0.2(1.0)^2\right]^{3.5} = 87.5\ kPa$

$P_{o_1} - P_{o_2} = (89.5 - 87.5)\ kPa = 2.0\ kPa$ ← $P_{o_1} - P_{o_2}$



* Rayleigh-Line Flow Functions

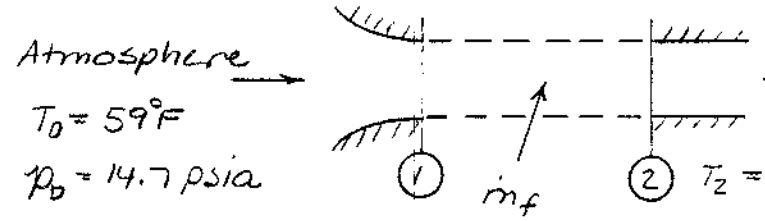
From Appendix E.3 for $M_1 = 0.778$

$T_{o_1}/T_o^* = 0.954$ (12.30d) $\therefore T_o^* = T_{o_2} = 381\ K$

$P_{o_1}/P_o^* = 1.024$ (12.30e) $\therefore P_o^* = P_{o_2} = 87.4\ kPa$

$P_1/P^* = 1.299$ (12.30a) $\therefore P^* = P_2 = 46.2\ kPa$

Given: Gasoline evaporates in air in a short section of duct, where the area is constant and friction is negligible.



$T_1 = 43.6°F$ Latent heat of vaporization: $\lambda = 140$ Btu/lbm

Find: (a) T_{02} and p_2 leaving duct.
(b) Change in stagnation pressure for process.

Solution: Basic equations: $p = \rho RT$, $\dot{m} = \rho VA$

Computing equations: $\frac{\delta Q}{dm} = c_p(T_{02} - T_{01})$ $\quad p_1 - p_2 = \rho_1 V_1 (V_2 - V_1)$

$$T_0 = T\left(1 + \tfrac{k-1}{2} M^2\right) \qquad p_0 = p\left(1 + \tfrac{k-1}{2} M^2\right)^{\frac{k}{k-1}}$$

Assumptions: (1) Steady flow
(2) Uniform flow at each section
(3) Isentropic flow in inlet
(4) Ideal gas
(5) Neglect fuel mass and chemistry

Flow is isentropic through inlet, so $T_0 = T_1\left(1 + \frac{k-1}{2} M_1^2\right)$ or

$$M_1 = \left[\left(\frac{T_0}{T_1} - 1\right)\frac{2}{k-1}\right]^{1/2} = \left[\left(\left(\frac{460+59}{460+43.6}\right) - 1\right)5\right]^{1/2} = 0.391$$

$$p_1 = \frac{p_0}{\left(1 + \frac{k-1}{2} M_1^2\right)^{k/k-1}} = \frac{14.7}{\left[1 + 0.2(0.391)^2\right]^{3.5}} = 13.2 \text{ psia}$$

$$\rho_1 = \frac{p_1}{RT_1} = 13.2 \frac{\text{lbf}}{\text{in.}^2} \times \frac{\text{lbm}\cdot°R}{53.3\ \text{ft}\cdot\text{lbf}} \times \frac{1}{(460+43.6)°R} \times 144 \frac{\text{in.}^2}{\text{ft}^2} = 0.0708 \text{ lbm/ft}^3$$

$$c_1 = \sqrt{kRT_1} = \left[1.4 \times 53.3 \frac{\text{ft}\cdot\text{lbf}}{\text{lbm}\cdot°R} \times (460+43.6)°R \times 32.2 \frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}\right]^{1/2} = 1100 \text{ ft/s}$$

$$V_1 = M_1 c_1 = 0.391 \times 1100 \text{ ft/s} = 430 \text{ ft/s}$$

Energy is removed from the air as the fuel vaporizes. The "heat transfer" is

$$\frac{\delta Q}{dm} = \frac{-\lambda}{AFR} = \frac{-140 \text{ Btu}}{\text{lbm(fuel)}} \times \frac{\text{lbm(fuel)}}{16 \text{ lbm(air)}} = -8.75 \text{ Btu/lbm}$$

Thus
$$T_{02} = T_{01} + \frac{1}{c_p}\frac{\delta Q}{dm} = 519°R + \frac{\text{lbm}\cdot°R}{0.240 \text{ Btu}} \times \frac{-8.75 \text{ Btu}}{\text{lbm}} = 483°R$$ T_{02} ←

For isentropic stagnation $T_{02} = T_2\left(1 + \frac{k-1}{2} M_2^2\right)$ so

$$M_2 = \left[\left(\frac{T_{02}}{T_2} - 1\right)\frac{2}{k-1}\right]^{1/2} = \left[\left(\frac{483}{460+9.6} - 1\right)5\right]^{1/2} = 0.378\,;\ c_2 = \sqrt{kRT_2} = 1070 \text{ ft/s}$$

$$V_2 = M_2 c_2 = 0.378 \times 1070 \text{ ft/s} = 404 \text{ ft/s}$$

From momentum, $p_2 = p_1 + \rho_1 V_1 (V_1 - V_2)$

$$p_2 = 13.2 \text{ psia} + 0.0708 \frac{\text{lbm}}{\text{ft}^3} \times 430 \frac{\text{ft}}{\text{s}} (430 - 404) \frac{\text{ft}}{\text{s}} \times \frac{\text{slug}}{32.2 \text{ lbm}} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} \times \frac{\text{ft}^2}{144 \text{ in}^2}$$

$p_2 = 13.4$ psia ← p_2

For local isentropic stagnation

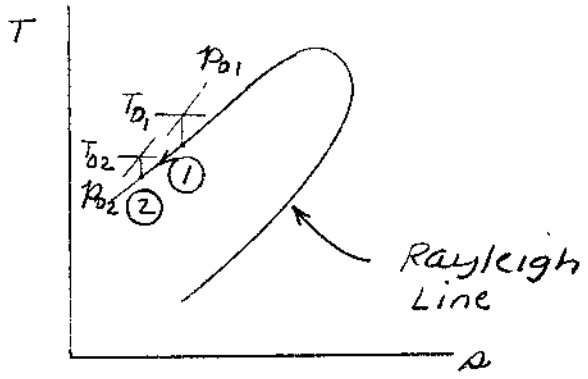
$$p_{02} = p_2 \left(1 + \frac{k-1}{2} M_2^2\right)^{\frac{k}{k-1}} = 13.4 \text{ psia} \left(1 + 0.2(0.378)^2\right)^{3.5} = 14.8 \text{ psia}$$

Thus

$$p_{02} - p_{01} = (14.8 - 14.7) \text{psia} = +0.1 \text{ psia}$$ ← Δp_0

{As expected, the stagnation pressure increases when thermal energy is removed from the air stream.}

The Ts diagram is:



---

* Rayleigh-Line Flow Functions (Appendix E.3):

$M_1 = 0.391$, $T_0/T_0^* = 0.5131$, $T/T^* = 0.5975$; $p_0/p_0^* = 1.169$

(12.30d) (12.30b) (12.30c)

$$T^* = \frac{T_1}{T/T^*} = \frac{(460 + 43.6)^\circ R}{0.5975} = 843^\circ R$$

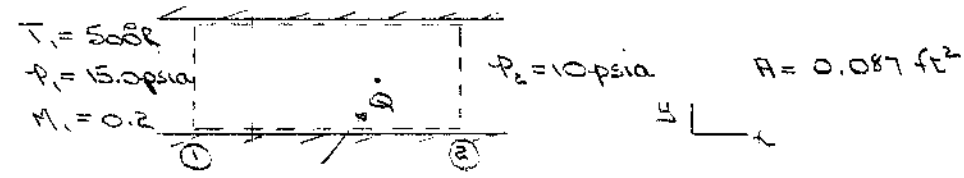
$p_{02} = 14.8$ psia

$T_2 = 9.6^\circ F = (460 + 9.6)^\circ R = 469.6^\circ R$

$\frac{T_2}{T^*} = \frac{469.6}{843} = 0.5571$, so $M_2 = 0.371$ (from Eq. 12.30b)

This is reasonable agreement, considering the numbers.

Given: Frictionless flow of air in a constant-area duct



Find: (a) V_2 and T_2 (b) $\dot{Q}$

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\qquad p_1A - p_2A = \dot{m}(V_2 - V_1)$

Computing equation: $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$

Assumptions: (1) steady flow (2) uniform flow at a section (3) ideal gas (4) frictionless flow (5) $F_{B_x} = 0$ (6) $\dot{W}_s = \dot{W}_{shear} = 0$

$$\rho_1 = \frac{p_1}{RT_1} = \frac{15\ lbf/in^2}{53.3\ ft\cdot lbf/lbm\cdot°R} \times \frac{lbm\cdot°R}{500°R} \times \frac{144\ in^2}{ft^2} = 0.0811\ lbm/ft^3$$

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 0.2\left[1.4 \times 53.3 \frac{ft\cdot lbf}{lbm\cdot°R} \times 500°R \times \frac{32.2\ lbm}{slug} \times \frac{slug\cdot ft}{lbf\cdot s^2}\right]^{1/2} = 219\ ft/s$$

From the momentum eq.: $(p_1 - p_2)A = \dot{m}(V_2 - V_1) = \rho_1 V_1 A (V_2 - V_1)$

$$\therefore V_2 = \frac{(p_1 - p_2)}{\rho_1 V_1} + V_1 = \frac{(15-10)\ lbf/in^2}{0.0811\ lbm/ft^3} \times \frac{ft^3}{lbm} \times \frac{s}{219\ ft} \times \frac{144\ in^2}{ft^2} \times \frac{32.2\ lbm}{slug} \times \frac{slug\cdot ft}{lbf\cdot s^2} + 219\ ft/s$$

$$V_2 = 1520\ ft/s \longleftarrow V_2$$

From continuity $\rho_2 = \frac{V_1}{V_2}\rho_1 = \frac{219}{1520} \times 0.0811\ lbm/ft^3 = 0.0117\ lbm/ft^3$

$$T_2 = \frac{p_2}{\rho_2 R} = \frac{10\ lbf/in^2}{0.0117\ lbm/ft^3} \times \frac{ft^3}{lbm} \times \frac{lbm\cdot°R}{53.3\ ft\cdot lbf} \times \frac{144\ in^2}{ft^2} = 2310°R \longleftarrow T_2$$

$$T_{0_1} = T_1\left(1 + \frac{k-1}{2} M_1^2\right) = 500\left[1 + 0.2(0.2)^2\right] = 504°R$$

$$c_2 = (kRT_2)^{1/2} = \left[1.4 \times 53.3 \frac{ft\cdot lbf}{lbm\cdot°R} \times 2310°R \times \frac{32.2\ lbm}{slug} \times \frac{slug\cdot ft}{lbf\cdot s^2}\right]^{1/2} = 2360\ ft/s$$

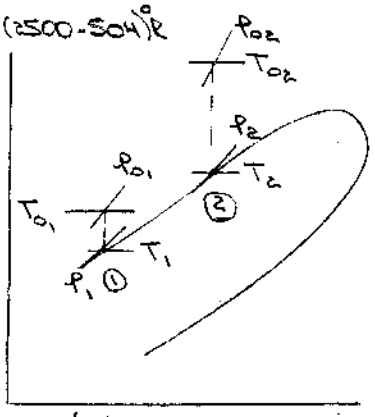
$$M_2 = \frac{V_2}{c_2} = \frac{1520}{2360} = 0.644 \qquad T_{0_2} = T_2\left[1 + \frac{k-1}{2} M_2^2\right] = 2310°R\left[1 + 0.2(0.644)^2\right] = 2500°R$$

Since $h_0 = h + \frac{V^2}{2}$, the energy eq. can be written as $\frac{\delta Q}{dm} = h_{0_2} - h_{0_1}$

$$\therefore \dot{Q} = \dot{m}\frac{\delta Q}{dm} = \dot{m}(h_{0_2} - h_{0_1}) = \dot{m} c_p (T_{0_2} - T_{0_1}) = \rho_1 V_1 A c_p (T_{0_2} - T_{0_1})$$

$$\dot{Q} = 0.0811 \frac{lbm}{ft^3} \times 219 \frac{ft}{s} \times 0.087 ft^2 \times 0.24 \frac{Btu}{lbm\cdot°R} \times (2500 - 504)°R$$

$$\dot{Q} = 740\ Btu/s$$



* Rayleigh-Line Flow Functions (App. E.3)

At $M_1 = 0.2$

$p_1/p^* = 2.273$ (12.30a) $\therefore p^* = 6.60$ psia

$T_1/T^* = 0.2066$ (12.30b) $\therefore T^* = 2420°R$

$T_{0_1}/T_0^* = 0.1736$ (12.30d) $\therefore T_0^* = 2900°R$

At ② $p_2/p^* = 1.515$ $\therefore M_2 = 0.646$

Also $T_{0_2}/T_0^* = 0.8644$ $\therefore T_{0_2} = 2510°R$ and $T_2/T^* = 0.9408$ $\therefore T_2 = 2280°R$

Given: Combustor modeled as frictionless flow through a constant area duct with heat transfer. Air-fuel ratio is large enough so properties are those of air



Heating value of the fuel is 18,000 Btu/lbm.

Find: (a) p_2 (b) $\dot{Q}$ (c) $\dot{m}_{fuel}$

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\qquad p_1 A - p_2 A = \dot{m}(V_2 - V_1)$

Computing equation: $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$

Assumptions: (1) steady flow (2) frictionless flow
(3) ideal gas, properties are those of air
(4) uniform flow at a section
(5) $W_s = W_{shear} = 0$

$$\rho_1 = \frac{p_1}{RT_1} = 235 \frac{lbf}{in^2} \times \frac{lbm°R}{53.3 ft\cdot lbf} \times \frac{1}{1260°R} \times 144 \frac{in^2}{ft^2} = 0.504 \text{ lbm/ft}^3$$

$$c_1 = (kRT_1)^{1/2} = \left[1.4 \times 53.3 \frac{ft\cdot lbf}{lbm\cdot °R} \times 1260°R \times 32.2 \frac{lbm}{slug} \times \frac{slug\cdot ft}{lbf\cdot s^2}\right]^{1/2} = 1740 \text{ ft/s}$$

$$M_1 = V_1/c_1 = 609/1740 = 0.350 \qquad T_{o_1} = T_1\left[1 + \frac{k-1}{2} M_1^2\right] = 1260°R\left[1 + 0.2(0.35)^2\right] = 1290°R$$

$$T_{o_2} = T_2\left[1 + \frac{k-1}{2} M_2^2\right] = 1840°R\left[1 + 0.2(0.476)^2\right] = 1920°R$$

From energy eq. $\dot{Q} = \dot{m}(h_{o_2} - h_{o_1}) = \dot{m} c_p (T_{o_2} - T_{o_1}) = 15 \frac{lbm}{s} \cdot 0.24 \frac{Btu}{lbm\cdot °R} \times (1920 - 1290)°R$

$$\dot{Q} = 2270 \text{ Btu/s} \qquad \dot{Q}$$

But $\dot{Q} = \dot{m}_{fuel} h_{fuel}$ $\therefore \dot{m}_f = \frac{\dot{Q}}{h_f} = 2270 \frac{Btu}{s} \times \frac{lbm}{18{,}000 Btu} = 0.126 \text{ lbm/s}$ $\qquad \dot{m}_f$

$$\dot{m}_f/\dot{m}_{air} = 0.126/15 = 0.0084 \qquad \dot{m}_f/\dot{m}_{air}$$

$$V_2 = M_2 c_2 = M_2 (kRT_2)^{1/2} = 0.476\left[1.4 \times 53.3 \frac{ft\cdot lbf}{lbm°R} \times 1840°R \times 32.2 \frac{lbm}{slug} \times \frac{slug\cdot ft}{lbf\cdot s^2}\right]^{1/2} = 1000 \text{ ft/s}$$

From momentum

$$p_2 = p_1 - \frac{\dot{m}}{A}(V_2 - V_1) = p_1 - \rho V_1 (V_2 - V_1)$$

$$p_2 = 235 \frac{lbf}{ft^2} - 0.504 \frac{lbm}{ft^3} \times 609 \frac{ft}{s} \times (1000 - 609)\frac{ft}{s} \times \frac{slug}{32.2 lbm} \times \frac{lbf\cdot s^2}{ft\cdot slug} \times \frac{ft^2}{144 in^2} = 209 \text{ psia} \qquad p_2$$

* Rayleigh-Line Flow Functions (App. E.3)

For $M_1 = 0.35$,

$T_o/T_o^* = 0.4389$ (12.30d) $\therefore T_o^* = 2940°R$

$T_1/T^* = 0.5141$ (12.30b) $\therefore T^* = 2450°R$

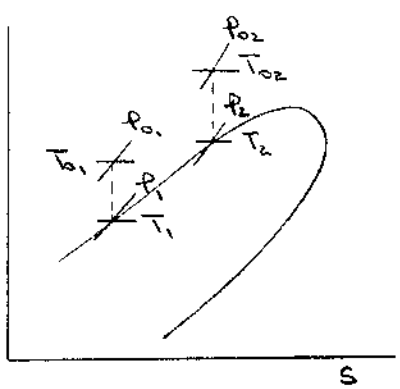
$p_1/p^* = 2.0487$ (12.30a) $\therefore p^* = 114.7$ psia

For $M_2 = 0.476$

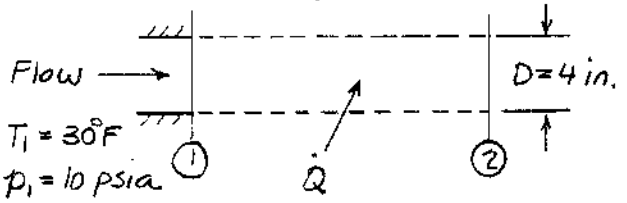
$T_{o_2}/T_o^* = 0.6551$ $\therefore T_{o_2} = 1930°R$

$T_2/T^* = 0.7522$ $\therefore T_2 = 1840°R$

$p_2/p^* = 1.822$ $\therefore p_2 = 209$ psia



Given: Frictionless flow of air in constant-area duct, with heat transfer.



$\dot{m} = 1.2$ lbm/s

Find: (a) Heat addition to choke the flow.
(b) Resulting change in stagnation pressure.
(c) T-s diagram

Solution: Apply equations for 1-D, steady compressible flow of ideal gas.

Computing equations: $T_0 = T\left(1+\frac{k-1}{2}M^2\right)$; $p_0 = p\left(1+\frac{k-1}{2}M^2\right)^{\frac{k}{k-1}}$; $p = \rho RT$

$\dot{m} = \rho V A$; $\frac{\delta Q}{dm} = c_p(T_{02} - T_{01})$; $p_1 - p_2 = \frac{\dot{m}}{A}(V_2 - V_1)$

From continuity, $V_1 = \frac{\dot{m}}{\rho_1 A}$; $\rho_1 = \frac{p_1}{RT_1} = 10.0\frac{lbf}{in^2}\times\frac{lbm°R}{53.3\,ft\cdot lbf}\times\frac{1}{(460+30)°R}\times 144\frac{in^2}{ft^2} = 0.0551\frac{lbm}{ft^3}$

$V_1 = 1.2\frac{lbm}{s}\times\frac{ft^3}{0.0551\,lbm}\times\frac{4}{\pi(4)^2 in^2}\times 144\frac{in^2}{ft^2} = 250$ ft/s

$c_1 = \sqrt{kRT_1} = \left[1.4\times 53.3\frac{ft\cdot lbf}{lbm°R}\times(460+30)°R\times 32.2\frac{lbm}{slug}\times\frac{slug\cdot ft}{lbf\cdot s^2}\right]^{1/2} = 1090$ ft/s

} $M_1 = 0.229$

From momentum, $p_1 - p_2 = \rho_2 V_2^2 - \rho_1 V_1^2$; $p_1(1+kM_1^2) = p_2(1+kM_2^2)$

$p_2 = p_1\left(\frac{1+kM_1^2}{1+k}\right) = 10.0\text{ psia}\ \frac{1+1.4(0.229)^2}{2.4} = 4.47$ psia

Energy cannot be used: $\delta Q/dm$ is not known. From continuity

$\rho_1 V_1 = \frac{p_1}{RT_1}M_1 c_1 = \frac{p_1}{RT_1}M_1\sqrt{kRT_1} = \frac{p_2}{RT_2}M_2\sqrt{kRT_2}$; $\frac{p_1 M_1}{\sqrt{T_1}} = \frac{p_2 M_2}{\sqrt{T_2}}$

$T_2 = T_1\left(\frac{p_2}{p_1}\frac{M_2}{M_1}\right)^2 = (460+30)°R\left(\frac{4.47}{10.0}\times\frac{1.0}{0.229}\right)^2 = 1,870\,°R$

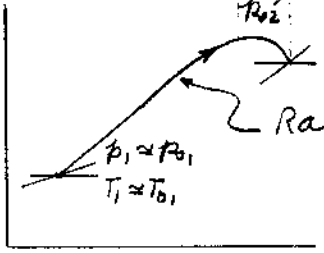
$T_{02} = 1,870°R\,(1+0.2(1)^2) = 2,240°R$; $T_{01} = 490°R\,(1+0.2(0.229)^2) = 495°R$

$\frac{\delta Q}{dm} = 0.240\frac{Btu}{lbm°R}(1,870-495)°R = 330$ Btu/lbm ← $\delta Q/dm$

$p_{01} = 10.0\text{ psia}\,(1+0.2(0.229)^2)^{3.5} = 10.4$ psia; $p_{02} = 4.47\text{ psia}\,(1+0.2(1)^2)^{3.5} = 8.46$ psia

$\Delta p_0 = p_{02} - p_{01} = (8.46 - 10.4)\text{ psia} = -1.94$ psia ← Δp_0

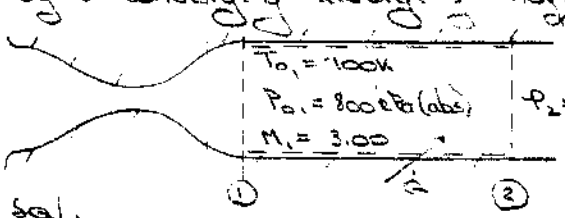
T-s diagram:



Rayleigh-Line Functions (Apps. E.3)
$M_1 = 0.229$ (Eq. 12.30d)
$T_0/T_0^* = 0.22075$; $T_{02} = 2240°R$
Eq. 12.30c
$p_0/p_0^* = 1.2251$; $p_{02} = 8.49$ psia

Problem 12.76

Given: Frictionless flow of air through a constant-area duct supplied by a converging-diverging nozzle



Find: V_2, M_2, $s_2 - s_1$

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{dQ}{dm} = h_2 + \frac{V_2^2}{2}$ $\quad P_1A_1 - P_2A_2 = \dot{m}(V_2 - V_1)$

Computing equations: $T_o/T = 1 + \frac{k-1}{2}M^2$ $\quad P_o/P = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)}$

Assumptions: (1) Steady flow (5) $F_{B_x} = 0$
(2) Frictionless flow (6) $\dot{W}_s = \dot{W}_{shear} = 0$
(3) uniform flow at section (7) $\frac{dQ}{dm} = 0$
(4) ideal gas

$$T_o/T = 1 + \frac{k-1}{2}M^2 \qquad T_1 = \frac{T_{o_1}}{1 + \frac{k-1}{2}M_1^2} = \frac{700K}{1 + 0.2(3.0)^2} = 250\ K$$

$$P_o/P = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)} \qquad P_1 = \frac{P_{o_1}}{\left[1 + \frac{k-1}{2}M_1^2\right]^{k/(k-1)}} = \frac{800\ kPa}{[1 + 0.2(3.0)^2]^{3.5}} = 21.8\ kPa$$

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 3.0\left(1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 250K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 951\ m/s$$

$$P = \rho RT \qquad \rho_1 = \frac{P_1}{RT_1} = 21.8 \times 10^3 \frac{N}{m^2} \times \frac{kg\cdot K}{287\ N\cdot m} \times \frac{1}{250K} = 0.304\ kg/m^3$$

$$(P_1 - P_2)A = \rho_1 V_1 A (V_2 - V_1) \quad \therefore \quad V_2 = V_1 + \frac{(P_1 - P_2)}{\rho_1 V_1}$$ Solving for V_2,

$$V_2 = 951\ m/s + (21.8 - 46.4)10^3 \frac{N}{m^2} \times 0.304 \frac{kg}{m^3} \times 951 \frac{s}{m} \times \frac{kg\cdot m}{N\cdot s^2} = 866\ m/s \quad \leftarrow V_2$$

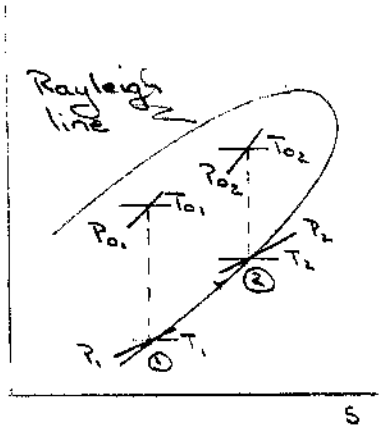
From continuity, $\rho_2 = \rho_1 \frac{V_1}{V_2} = 0.304 \frac{kg}{m^3} \times \frac{951}{866} = 0.334\ kg/m^3$

$$T_2 = \frac{P_2}{\rho_2 R} = \frac{46.4 \times 10^3 \frac{N}{m^2}}{0.334 \frac{kg}{m^3} \times 287 \frac{N\cdot m}{kg\cdot K}} = 484\ K$$

$$c_2 = (kRT_2)^{1/2} = \left[1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 484K \times \frac{kg\cdot m}{N\cdot s^2}\right]^{1/2} = 441\ m/s; \quad M_2 = \frac{V_2}{c_2} = \frac{866}{441} = 1.96 \quad \leftarrow M_2$$

$$T_{o_2} = T_2\left[1 + \frac{k-1}{2}M^2\right] = 484\left[1 + 0.2(1.96)^2\right] = 856K$$

$$s_2 - s_1 = h_{o_2} - h_{o_1} = c_p(T_{o_2} - T_{o_1}) = 1 \frac{kJ}{kg\cdot K}(856K - 700K) = 156\ kJ/kg \quad \leftarrow \frac{156}{s_2 - s_1}$$



* Rayleigh-Line Flow Functions (Appendix E.3)

For $M_1 = 3.0$, $T_{o_1}/T_o^* = 0.6540$, $\therefore T_o^* = 1070K$
$T_1/T^* = 0.2803$ $\therefore T^* = 892K$
$P_1/P^* = 0.1765$ $\therefore P^* = 124\ kPa$
$V_1/V^* = 1.588$ $\therefore V^* = 599\ m/s$

At section (2) for $P_2/P^* = 0.3742$
$M_2 = 1.966$ ✓
$T_{o_2}/T_o^* = 0.800$ $\therefore T_{o_2} = 856K$ ✓
$T_2/T^* = 0.542$ $\therefore T_2 = 483K$
$V_2/V^* = 1.447$ $\therefore V_2 = 867\ m/s$.

Given: Frictionless flow of air through a constant area duct.



Find: M_2, T_2, T_{o_2}, $\dot{Q}$

Solution:

* Compressible flow functions (Appendix E) to be used in solution

Basic equation: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$

Assumptions: 1) steady flow; 2) frictionless flow; 3) uniform flow at sections; 4) ideal gas; 5) $F_{B_x} = 0$; 6) $\dot{W}_s = \dot{W}_{shear} = 0$; 7) $\Delta z = 0$

$$\rho_1 = \frac{P_1}{RT_1} = 126\times10^3 \frac{N}{m^2} \times \frac{kg\cdot K}{287\ J\cdot m} \times \frac{1}{533\ K} = 0.8237\ kg/m^3$$

$$\dot{m} = \rho_1 V_1 A \qquad V_1 = \frac{\dot{m}}{\rho_1 A} = 1.83\ \frac{kg}{s} \cdot \frac{m^3}{0.8237\ kg} \cdot \frac{1}{0.02 m^2} = 111\ m/s$$

$$M_1 = \frac{V_1}{c_1} \qquad c_1 = (kRT_1)^{1/2} = \left(1.4 \times 287\ \frac{N\cdot m}{kg\cdot K} \times 533 K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 463\ m/s$$

$$M_1 = \frac{111}{463} = 0.240$$

From App. E.1, $T_{o_1}/T_1 = 1.012$ (11.17b) $\therefore T_{o_1} = 537$ K

From App E.3, $T_{o_1}/T_o^* = 0.2395$ (12.30d), $T_1/T^* = 0.2841$ (12.30b), $P_1/P^* = 2.221$ (12.30a)

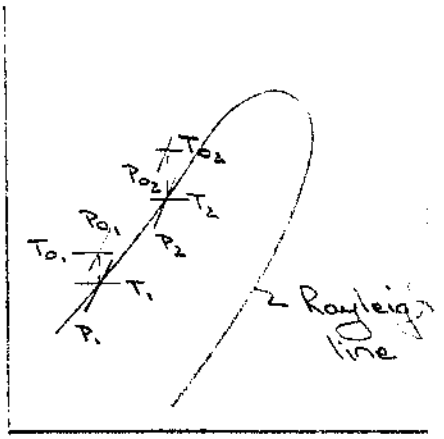
$\therefore T_o^* = 2250$K $\quad T^* = 1876$K $\quad P^* = 56.73$ kPa

At section (2) $\frac{P_2}{P^*} = \frac{101}{56.73} = 1.780$ From App E.3 (Eq 12.30a), $M = 0.50$ ← M_2

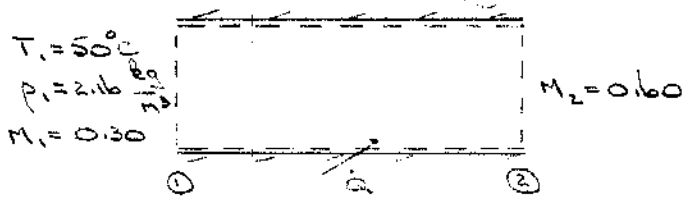
Also, $T_{o_2}/T_o^* = 0.6914$ (12.30d), $T_2/T^* = 0.7901$. Therefore: $T_{o_2} = 1556$K, $T_2 = 1480$K ← T_{o_2}, T_2

$$\dot{Q} = \dot{m}\frac{\delta Q}{dm} = \dot{m}(h_{o_2} - h_{o_1}) = \dot{m}C_p(T_{o_2} - T_{o_1}) = 1.83\ \frac{kg}{s} \cdot 1.0\ \frac{kJ}{kg\cdot K} \cdot (1556 - 537)K$$

$$\dot{Q} = 1.86\ kJ/s$$ ← $\dot{Q}$



Given: Frictionless flow of air through a constant area duct



Find: $\delta q/dm$, $s_2 - s_1$, $P_{o_1} - P_{o_2}$

Solution:

* Compressible flow functions (Appendix E) to be used in solution

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta q}{dm} = h_2 + \frac{V_2^2}{2}$ $\qquad Tds = dh - vdP$

Assumptions: (1) steady flow (4) ideal gas
(2) frictionless flow (5) $\dot{W}_s = \dot{W}_{shear} = 0$
(3) uniform flow at a section (6) $\Delta z = 0$

$P_1 = \rho_1 R T_1 = 2.16 \frac{kg}{m^3} \cdot 287 \frac{N \cdot m}{kg \cdot K} \cdot 323\,K = 200\,kPa$

$M_1 = 0.30$ From App. E.1 $T_1/T_{o_1} = 0.9823$ (11.17b), $P_1/P_{o_1} = 0.9395$ (11.17a)

$\therefore T_{o_1} = 329\,K$ $\qquad P_{o_1} = 213\,kPa$

From App. E.3 $T_{o_1}/T_o^* = 0.3469$ (12.30d), $P_{o_1}/P_o^* = 1.199$ (12.30e)

$\therefore T_o^* = 948\,K$ $\qquad P_o^* = 178\,kPa$

$M_2 = 0.60$ From App. E.3 $T_{o_2}/T_o^* = 0.8189$ (12.30d), $P_{o_2}/P_o^* = 1.075$ (12.30e)

$\therefore T_{o_2} = 776\,K$ $\qquad P_{o_2} = 191\,kPa$

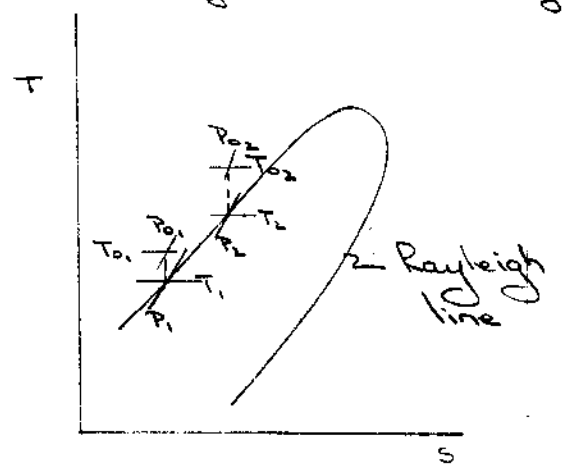
$P_{o_1} - P_{o_2} = (213 - 191)\,kPa = 22\,kPa$ ← $P_{o_1} - P_{o_2}$

$\delta q/dm = h_{o_2} - h_{o_1} = c_p(T_{o_2} - T_{o_1}) = 1.0 \frac{kJ}{kg \cdot K} (776 - 329)K = 447\,kJ/kg$ ← $\delta q/dm$

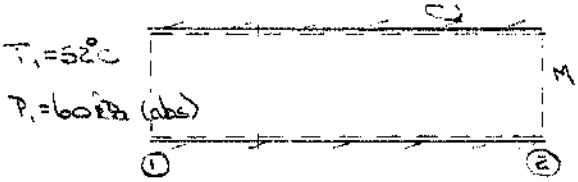
$Tds = dh - vdP = c_p dT - \frac{1}{\rho} dP \qquad ds = c_p \frac{dT}{T} - R \frac{dP}{P}$

$s_2 - s_1 = s_{o_2} - s_{o_1} = \int_{s_{o_1}}^{s_{o_2}} ds = \int_{T_{o_1}}^{T_{o_2}} c_p \frac{dT}{T} - \int_{P_{o_1}}^{P_{o_2}} R \frac{dP}{P} = c_p \ln \frac{T_{o_2}}{T_{o_1}} - R \ln \frac{P_{o_2}}{P_{o_1}}$

$s_2 - s_1 = 1.0 \frac{kJ}{kg \cdot K} \ln \frac{776}{329} - 0.287 \frac{kJ}{kg \cdot K} \ln \frac{191}{213} = 0.889\,kJ/kg \cdot K$ ← $s_2 - s_1$



Given: Frictionless flow of air through a constant area duct



Find: Solution, fluid properties at section (2)

Solution:

* Compressible flow functions (Appendix E) to be used in solution

Basic equation: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$

Assumptions: (1) steady flow (4) ideal gas
(2) frictionless flow (5) $\dot{W}_s = \dot{W}_{shear} = 0$
(3) uniform flow at a section (6) $\Delta z = 0$

$A_1 = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.1)^2 m^2 = 7.85 \times 10^{-3} m^2$

$P = \rho RT$ $\quad \rho_1 = \frac{P_1}{RT_1} = 60 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287 N \cdot m} \times \frac{1}{325 K} = 0.643 kg/m^3$

$\dot{m} = \rho A V$ $\quad V_1 = \frac{\dot{m}}{\rho_1 A} = 1.42 \frac{kg}{s} \times 0.643 kg \times \frac{m^3}{7.85 \times 10^{-3}} = 281 m/s$

$M_1 = \frac{V_1}{c_1}$ $\quad c_1 = (kRT_1)^{1/2} = (1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 325 K \times \frac{kg \cdot m}{N \cdot s^2})^{1/2} = 362 m/s$

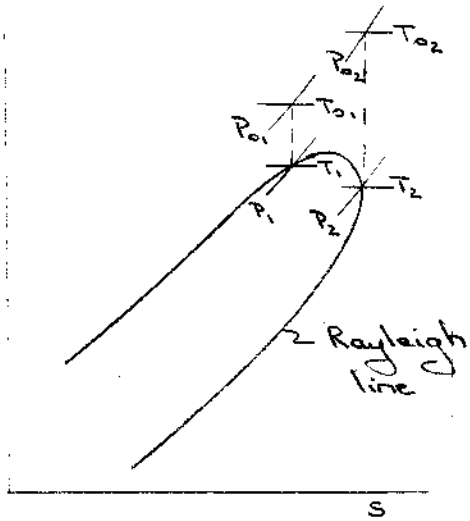
$M_1 = \frac{V_1}{c_1} = \frac{281}{362} = 0.776$

From App E.1, $\frac{T}{T_0}_1 = 0.8925$ $\quad \frac{P}{P_0}_1 = 0.6717$ $\quad \therefore T_{0_1} = 364 K$, $P_{0_1} = 89.3 kPa$

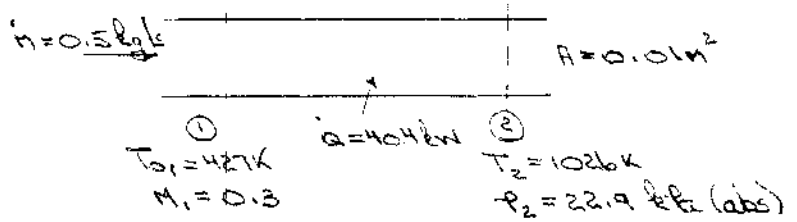
From App E.3, $\frac{T_0}{T_0^*} = 0.9535$ $\quad \frac{P_0}{P_0^*} = 1.024$ $\quad \frac{T}{T^*}_1 = 1.022$ $\quad \frac{P}{P^*}_1 = 1.302$ $\quad \frac{V}{V^*}_1 = \frac{\rho^*}{\rho}_1 = 0.7844$

$\therefore T_{0_2} = 382 K$, $P_{0_2} = 87.2 kPa$, $T_2 = 318 K$, $P_2 = 46.1 kPa$ $\quad \rho_2 = 0.504 kg/m^3$, $V_2 = 358 m/s$ ← (2)

$\frac{\delta Q}{dm} = h_{0_2} - h_{0_1} = c_p(T_{0_2} - T_{0_1}) = 1.0 \frac{kJ}{kg \cdot K}(382 - 364) K = 18.0 kJ/kg$ ← Sol'n



Given: Air enters an engine combustion chamber where heat is added during a frictionless process in a tube with constant area, $A = 0.01\ m^2$. Conditions are as shown.



Find: (a) M_2 (b) p_1 (c) $p_{o_2} - p_{o_1}$

Solution: * Compressible flow functions (Appendix E) to be used in solution

Basic equations: $h_{o_1} + \frac{\delta\phi}{\dot{m}} = h_{o_2}$ $\qquad \dot{m} = \rho V A$ $\qquad \phi = gz$

Assumptions: (1) steady flow (2) frictionless flow
(3) ideal gas (4) uniform flow at a section
(5) $\dot{m}_2 = \dot{m}_{shear} = 0$

From the energy equation, $\dot{Q} = \dot{m}(h_{o_2} - h_{o_1}) = \dot{m} c_p (T_{o_2} - T_{o_1})$

$$\therefore T_{o_2} = T_{o_1} + \frac{\dot{Q}}{\dot{m} c_p} = 427\,K + 404 \times 10^3\,W \times \frac{s}{0.3\,kg} \times \frac{kg \cdot K}{1004\,J} \times \frac{J}{W \cdot s} = 1232\,K$$

At $M_1 = 0.3$, from App. E.3, $T_{o_1}/T_o^* = 0.3469$ $\therefore T_o^* = 1231\ K$

Since $T_{o_2} \approx T_o^*$, then $M_2 = 1.0$ ← M_2

$\therefore p_2 = p^* = 22.9\ kPa$

From Appendix E.3 for $M_1 = 0.3$, $p_1/p^* = 2.131$ $\therefore p_1 = 48.8\ kPa$ ← p_1

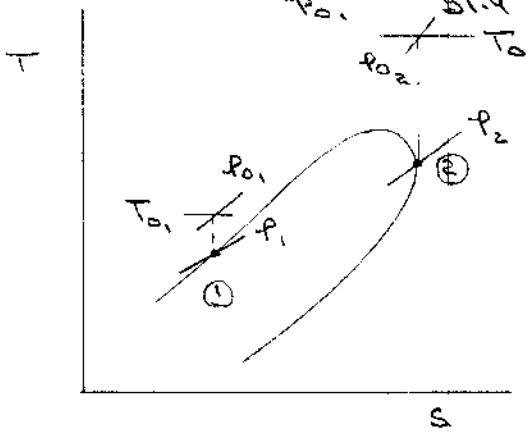
" " E.1 " $M_1 = 0.3$, $p_1/p_{o_1} = 0.9395$ $\therefore p_{o_1} = 51.9\ kPa$

From Appendix E.1 for $M_2 = 1.0$, $p_2/p_{o_2} = 0.5283$ $\therefore p_{o_2} = 43.3\ kPa$

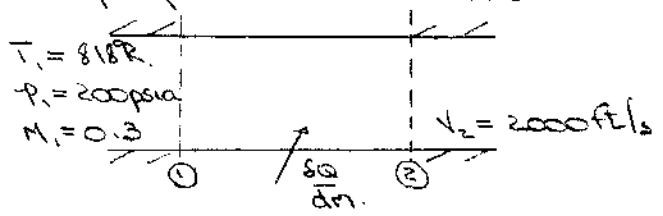
$$\therefore p_{o_2} - p_{o_1} = (43.3 - 51.9)\,kPa = -8.6\ kPa$$

$$\frac{\Delta p_o}{p_{o_1}} = \frac{-8.6}{51.9} = -0.166 \quad (\text{or } -16.6\%)$$

← Δp_o



Given: Combustor modeled as frictionless flow through a constant-area duct with heat addition. Air fuel ratio is large enough so properties are those of air



Find: a) T_2, p_2, ρ_2, M_2 b) $\dot{Q}$

Solution: (using compressible flow functions - Appendix E)

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $p = \rho RT$ $\dot{m} = \rho VA$

Assumptions: (1) steady flow (2) frictionless flow
(3) ideal gas, properties are those of air
(4) uniform flow at a section
(5) $W_s = W_{shear} = 0$

$$\rho_1 = \frac{p_1}{RT_1} = 200 \frac{lbf}{in^2} \times \frac{lbm°R}{53.3 ft\,lbf} \times \frac{1}{818°R} \times 144 \frac{in^2}{ft^2} = 0.6606 \, lbm/ft^3$$

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 0.3 \left[1.4 \times 53.3 \frac{ft\,lbf}{lbm°R} \times 818°R \times 32.2 \frac{lbm}{slug} \times \frac{slug\,ft}{lbf\,s^2}\right]^{1/2} = 421 \, ft/s$$

From App. E.1 $M_1 = 0.3$ $T/T_0 = 0.9823$ $\therefore T_{o_1} = 832.7°R$.
$p/p_0 = 0.9395$ $\therefore p_{o_1} = 212.9 psia$

From continuity, $\rho_1 V_1 = \rho_2 V_2$

$$\therefore \rho_2 = \frac{V_1}{V_2} \rho_1 = \frac{421}{2000} \times 0.6606 \, lbm/ft^3 = 0.139 \, lbm/ft^3 \quad \leftarrow \rho_2$$

From App E.3 $M_1 = 0.3$ $\frac{T_{o_1}}{T_o^*} = 0.3469$ $\frac{p_{o_1}}{p_o^*} = 1.199$ $\frac{T_1}{T^*} = 0.4089$ $\frac{p_1}{p^*} = 2.131$ $\frac{V_1}{V^*} = 0.1918$

$\therefore T_o^* = 2400°R$ $p_o^* = 177.6 psia$ $T^* = 2000°R$ $p^* = 93.85 psia$ $V^* = 2193 \frac{ft}{sec}$

At section ② $V_2/V^* = 2000/2193 = 0.9120$

From App. E.3 $M_2 = 0.90$ $\leftarrow M_2$

$T_{o_2}/T_o^* = 0.9921$, $p_{o_2}/p_o^* = 1.005$, $T_2/T^* = 1.025$, $p_2/p^* = 1.125$

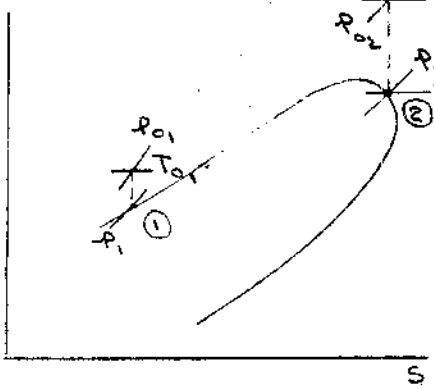
$\therefore T_{o_2} = 2380°R$, $p_{o_2} = 178 psia$, $T_2 = 2050°R$, $p_2 = 106 psia$ $\leftarrow T_{o_2}, p_{o_2}, T_2, p_2$

From the energy equation,

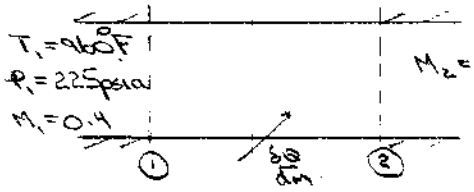
$$\dot{Q} = \dot{m} \frac{\delta Q}{dm} = \rho_1 V_1 A (h_{o_2} - h_{o_1}) = \rho_1 V_1 A c_p (T_{o_2} - T_{o_1})$$

$$\dot{Q} = 0.6606 \frac{lbm}{ft^3} \times 420.6 \frac{ft}{s} \times 0.5 ft^2 \times 0.24 \frac{BTU}{lbm°R} \times (2380 - 832.7)°R$$

$$\dot{Q} = 5.16 \times 10^4 \, Btu/s \quad \leftarrow \dot{Q}$$



Given: Combustor modelled as frictionless flow through a constant-area duct with heat addition. Air fuel ratio is large enough so properties are those of air



Find: (a) $\left(\frac{\delta Q}{dm}\right)_{max}$ (b) properties at section (2) (c) $p_{o_2} - p_{o_1}$

Solution (using compressible flow functions - Appendix E)

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\quad p = \rho RT$ $\quad \dot{m} = \rho VA$

Assumptions: (1) steady flow (2) frictionless flow
(3) ideal gas properties are those of air
(4) uniform flow at a section
(5) $W_s = W_{shear} = 0$

$$\rho_1 = \frac{p_1}{RT_1} = 225 \frac{lbf}{in^2} \times \frac{lbm \cdot °R}{53.3 ft \cdot lbf} \times \frac{1}{1420°R} \times \frac{144 in^2}{ft^2} = 0.4281 \, lbm/ft^3$$

$$V_1 = M_1 c_1 = M_1 \left[kRT_1\right]^{1/2} = 0.4 \left[1.4 \times 53.3 \frac{ft \cdot lbf}{lbm °R} \times 1420°R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right]^{1/2} = 739 \, ft/s$$

From App. E.1, $M_1 = 0.4$ $\quad T_1/T_{o_1} = 0.9690$ $\quad \therefore T_{o_1} = 1465°R$
$p_1/p_{o_1} = 0.8956$ $\quad p_{o_1} = 251.2 psia$

For max $\frac{\delta Q}{dm}$, $M_2 = 1.0$ $\quad \therefore$ Properties at (2) are * $\quad M_2 = 1.0$ ← M_2

From App. E.3, for $M_1 = 0.4$

$T_{o_1}/T_o^* = 0.5290$ $\quad p_{o_1}/p_o^* = 1.157$ $\quad T_1/T^* = 0.6152$ $\quad p_1/p^* = 1.961$ $\quad \rho_1^*/\rho_1 = 0.3137$

$\therefore T_o^* = T_{o_2} = 2770°R$, $\quad p_{o_2} = p_o^* = 217 psia$, $\quad T^* = T_2 = 2310°R$, $\quad p_2 = p^* = 115 psia$, $\quad \rho_2 = 0.134 \frac{lbm}{ft^3}$ ←

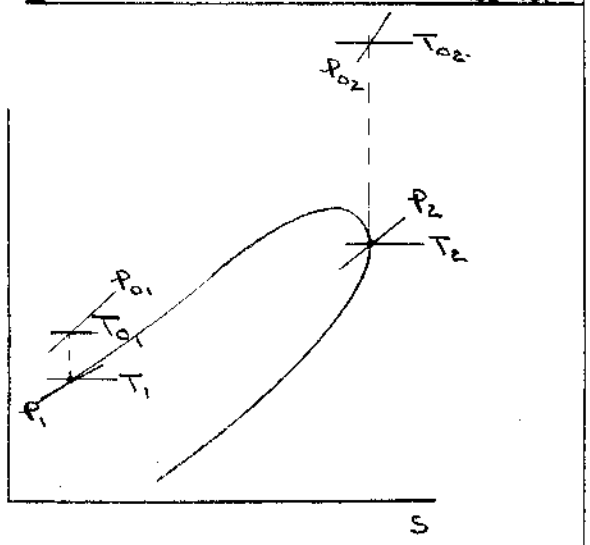
$p_{o_2} - p_{o_1} = (217 - 251) psia = -34 psia$ ← $p_{o_2} - p_{o_1}$

From the energy equation,

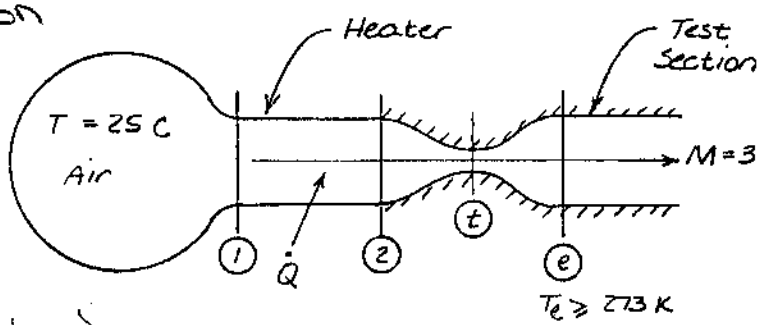
$$\frac{\delta Q}{dm} = h_{o_2} - h_{o_1} = c_p (T_{o_2} - T_{o_1})$$

$$\frac{\delta Q}{dm} = 0.24 \frac{Btu}{lbm °R} \times (2770 - 1465)°R$$

$$\frac{\delta Q}{dm} = 313 \, Btu/lbm$$ ←



Given: Supersonic wind tunnel, with test section Mach number $M = 3$, is supplied from a high-pressure tank of air at 25°C. Air from the tank is heated in a short, constant area section upstream of the converging-diverging nozzle which feeds the test section. The heat addition $\dot{Q} = 10$ kW is sufficient to ensure $T > 0°C$ at the entrance to the test section



Find: (a) T_{o_2}
(b) $\dot{m}_{max}$
(c) A_e/A_t

Solution:

(using compressible flow functions)

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\quad \dot{m} = \rho V A$ $\quad p = \rho R T$

Assumptions: (1) steady flow (2) uniform flow at a section
(3) frictionless flow in the heater
(4) isentropic flow through the nozzle
(5) ideal gas (6) $W_s = \dot{W}_{shear} = 0$

$T_{o_1} = T = 298$ K

From Appendix E.1 at $M = 3$, $T/T_o = 0.3571$, $A_e/A^* = A_e/A_t = 4.23$ ← $\frac{A}{A_t}$

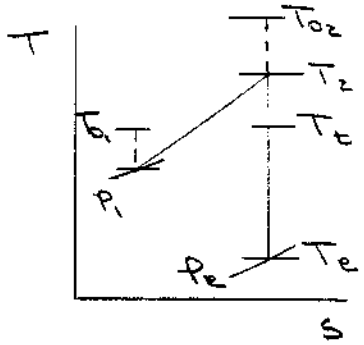
With $T_e = 273$K, $T_{oe} = T_{o_2} = 273\text{K}/0.3571 = 764$ K ← T_{o_2}

From the energy equation $\frac{\delta Q}{dm} = h_{o_2} - h_{o_1} = c_p(T_{o_2} - T_{o_1})$

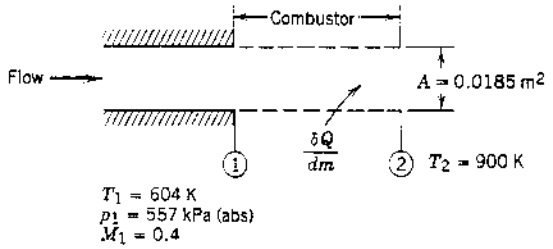
Since $\dot{Q} = \dot{m}\frac{\delta Q}{dm}$, then $\dot{m} = \frac{\dot{Q}}{c_p(T_{o_2} - T_{o_1})}$

and

$$\dot{m} = 10\,\text{kW} \times \frac{10^3\,\text{N·m}}{\text{s·kW}} \times \frac{\text{kg·K}}{1000\,\text{N·m}} \times \frac{1}{(764 - 298)\,\text{K}} = 0.0215\ \text{kg/s}$$ ← $\dot{m}$



Given: Steady flow combustor operating under conditions shown. Assume thermodynamic properties are those of pure air.



Find: (a) T_{o_2}
(b) M_2
(c) $\delta q/dm$, $\dot{Q}$
(d) $\dot{Q}/\dot{Q}_{max}$

Solution:

(using compressible flow functions - Appendix E)

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta q}{dm} = h_2 + \frac{V_2^2}{2}$ $\quad \dot{m} = \rho V A$ $\quad p = \rho R T$

Assumptions: (1) steady flow (2) ideal gas
(3) uniform flow at a section
(4) $\dot{W}_s = \dot{W}_{shear} = 0$

$$\rho_1 = \frac{p_1}{RT_1} = 557\times10^3 \frac{N}{m^2} \times \frac{kg\cdot K}{287 N\cdot m} \times \frac{1}{604\ K} = 3.21\ kg/m^3$$

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 0.4\left(1.4\times287 \frac{N\cdot m}{kg\cdot K}\times 604 K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 197\ m/s$$

$$\dot{m} = \rho_1 V_1 A = 3.21 \frac{kg}{m^3}\times 197 \frac{m}{s}\times 0.0185\ m^2 = 11.7\ kg/s$$

From App. E.1 at $M_1 = 0.4$, $T_1/T_{o_1} = 0.969$ $\quad \therefore T_{o_1} = 623\ K$

From App. E.3 at $M_1 = 0.4$, $T/T^* = 0.6152$, $T_o/T_o^* = 0.5290$

$\therefore T^* = 982$ and $T_o^* = 1180\ K$

At section ② $T_2 = 900K$ $\quad \therefore (T/T^*)_2 = 0.916$

From App. E.3, $M_2 = 0.60$, $T_o/T_o^*)_2 = 0.8189$

$\therefore T_{o_2} = 966\ K$, $\quad M_2 = 0.60$ ← M_2, T_{o_2}

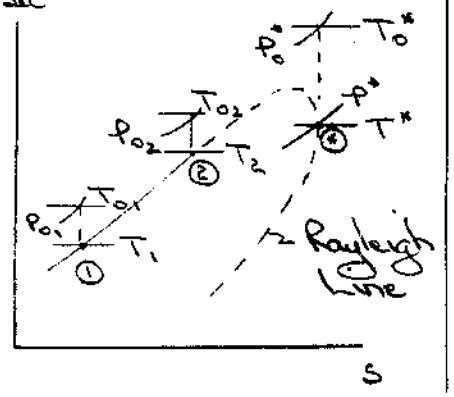
From the energy equation,

$$\frac{\delta q}{dm} = h_{o_2} - h_{o_1} = c_p(T_{o_2} - T_{o_1}) = 1.00 \frac{kJ}{kg\cdot K}(966-623)K = 343 \frac{kJ}{kg}$$ ← $\delta q/dm$

$$\dot{Q} = \dot{m}\frac{\delta q}{dm} = 11.7 \frac{kg}{sec}\times 343 \frac{kJ}{kg} = 406 \frac{kJ}{sec} = 4010\ kW$$ ← $\dot{Q}$

The heat transfer may be expressed as a fraction of the maximum possible heat addition

$$\frac{\delta q/dm}{(\delta q/dm)_{max}} = \frac{T_{o_2} - T_{o_1}}{T_o^* - T_{o_1}} = \frac{966-623}{1180-623} = 0.616$$ ←



Given: Rayleigh line flow for air; s^* is the entropy at the condition where $M = 1.0$.

Plot: Rayleigh line for Mach numbers in the range $0.4 \le M \le 3.0$; use coordinates T/T^* and $(s-s^*)/c_p$

Solution

* Compressible flow functions (Appendix E) to be used in solution

Basic equation: $Tds = dh - v\,dP$, $P = \rho RT$

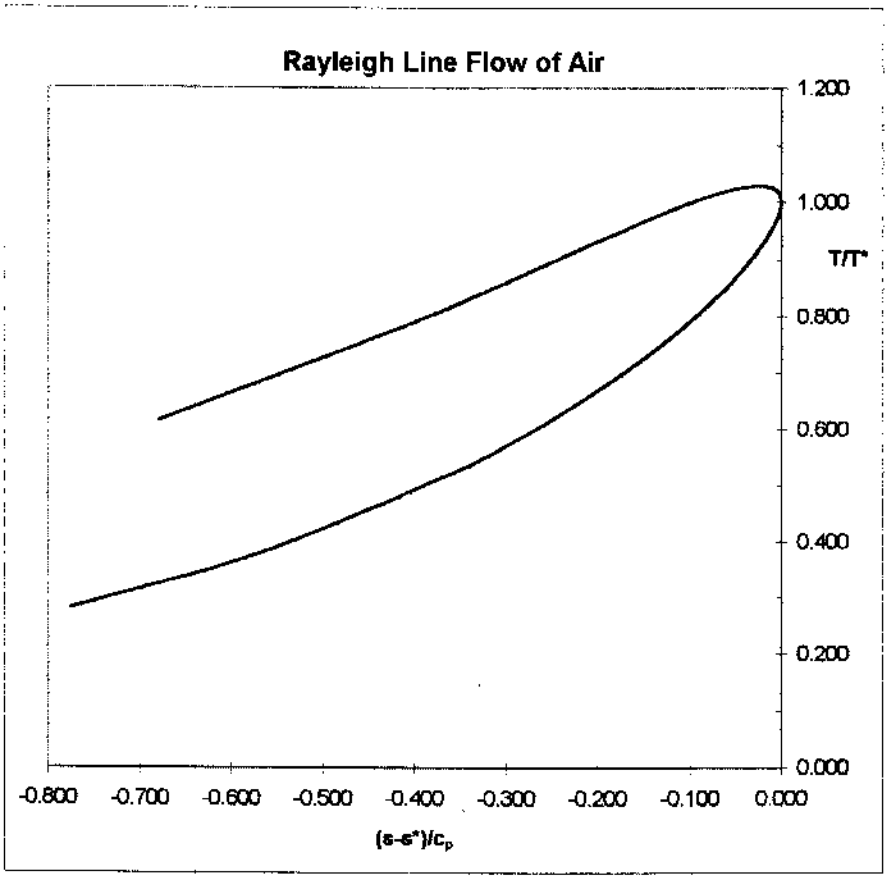
$$Tds = dh - v\,dP = c_p dT - \frac{1}{\rho} dP \qquad ds = c_p \frac{dT}{T} - \frac{dP}{P}$$

$$s - s^* = \int_{T^*}^{T} c_p \frac{dT}{T} - \int_{P^*}^{P} R \frac{dP}{P} = c_p \ln \frac{T}{T^*} - R \ln \frac{P}{P^*} = c_p \ln \frac{T}{T^*} - (c_p - c_v) \ln \frac{P}{P^*}$$

$$\frac{s - s^*}{c_p} = \ln \frac{T}{T^*} - \left(1 - \frac{1}{k}\right) \ln \frac{P}{P^*} = \ln \frac{T}{T^*} - \frac{(k-1)}{k} \ln \frac{P}{P^*} = \ln \frac{T}{T^*} - 0.286 \ln \frac{P}{P^*}$$

Data from Appendix E.3

M	p/p*	$(s-s^*)/c_p$	T/T*
0.40	1.961	-0.678	0.615
0.50	1.778	-0.400	0.790
0.60	1.596	-0.221	0.917
0.70	1.423	-0.108	0.993
0.80	1.266	-0.042	1.025
0.90	1.125	-0.009	1.025
1.00	1.000	0.000	1.000
1.10	0.891	-0.007	0.960
1.20	0.796	-0.027	0.912
1.30	0.713	-0.055	0.859
1.40	0.641	-0.089	0.805
1.50	0.578	-0.128	0.753
1.60	0.524	-0.169	0.702
1.70	0.476	-0.212	0.654
1.80	0.434	-0.257	0.609
1.90	0.396	-0.302	0.567
2.00	0.364	-0.348	0.529
2.50	0.246	-0.570	0.379
3.00	0.176	-0.776	0.280



Given: Rayleigh line flow of air with inlet conditions: $T_1 = 39°C$, $P_1 = 128$ kPa, $M_1 = 2.00$; s^* is the entropy at the condition where $M = 1.0$

Plot: Supersonic and subsonic branches of the Rayleigh line; use coordinates T/T_{o1} and $(s-s^*)/c_p$.

Solution:

Compressible flow functions (Appendix E) to be used in solution

Basic equation: $Tds = dh - v\,dP$, $P = \rho RT$

$$Tds = dh - v\,dP = c_p\,dT - \frac{1}{\rho}dP \qquad ds = c_p\frac{dT}{T} - \frac{dP}{P}$$

$$s - s^* = \int_{T^*}^{T} c_p\,dT - \int_{P^*}^{P} R\frac{dP}{P} = c_p \ln\frac{T}{T^*} - R\ln\frac{P}{P^*} = c_p\ln\frac{T}{T^*} - (c_p - c_v)\ln\frac{P}{P^*}$$

$$\frac{s-s^*}{c_p} = \ln\frac{T}{T^*} - \left(1 - \frac{1}{k}\right)\ln\frac{P}{P^*} = \ln\frac{T}{T^*} - \frac{k-1}{k}\ln\frac{P}{P^*} = \ln\frac{T}{T^*} - 0.2857\ln\frac{P}{P^*}$$

For a given value of M

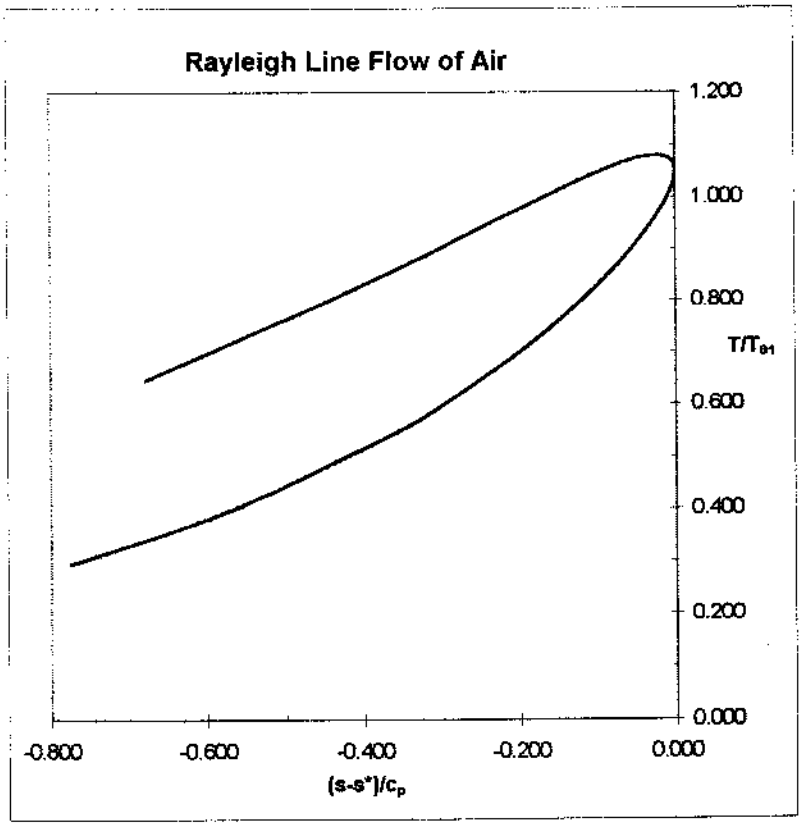
- T/T^* and P/P^* are obtained from Appendix E.3
- $T/T_{o1} = T/T^* \cdot T^*/T_{o1}$

From App E.1 at $M_1 = 2.0$, $T_{o1}/T_1 = 1.80$ (Eq 11.17b) $\therefore T_{o1} = 562$ K

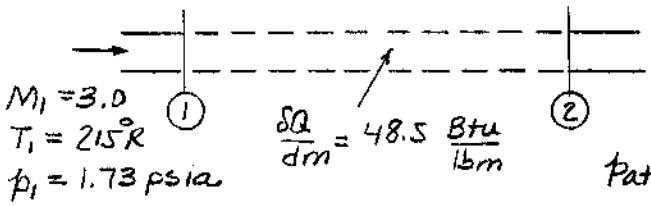
" " E.3 at $M_1 = 2.0$, $T/T^* = 0.5289$ (Eq 12.30b) $\therefore T^* = 580$ K

$$T/T_{o1} = \frac{580}{562}\frac{T}{T^*} = 1.050\frac{T}{T^*}$$

M	T/T*	p/p*	(s-s*)/c_p	T/T_{o1}
0.40	0.615	1.961	-0.678	0.646
0.50	0.790	1.778	-0.400	0.829
0.60	0.917	1.596	-0.221	0.962
0.70	0.993	1.423	-0.108	1.042
0.80	1.025	1.266	-0.042	1.077
0.90	1.025	1.125	-0.009	1.076
1.00	1.000	1.000	0.000	1.050
1.10	0.960	0.891	-0.007	1.008
1.20	0.912	0.796	-0.027	0.957
1.30	0.859	0.713	-0.055	0.902
1.40	0.805	0.641	-0.089	0.846
1.50	0.753	0.578	-0.128	0.790
1.60	0.702	0.524	-0.169	0.737
1.70	0.654	0.476	-0.212	0.686
1.80	0.609	0.434	-0.257	0.639
1.90	0.567	0.396	-0.302	0.596
2.00	0.529	0.364	-0.348	0.555
2.50	0.379	0.246	-0.570	0.398
3.00	0.280	0.176	-0.776	0.294



Given: Frictionless flow of air in a constant-area duct, discharging to atmospheric pressure, with the flow conditions shown.



Find: (a) Compare p_2 with atmospheric pressure.
(b) Is M_2 greater than, equal to, or less than unity?
(c) Sketch p vs. x along the channel.

Solution: Apply equations for Rayleigh line flow of an ideal gas.

Basic equations: $c_p T_{01} + \frac{\delta Q}{dm} = c_p T_{02}$ $\quad p_1 A - p_2 A = \dot{m}(V_2 - V_1)$

Computing equation: $T_0 = T\left(1 + \frac{k-1}{2} M^2\right)$

Assumptions: (1) Steady flow
(2) Frictionless flow
(3) Uniform flow at each section
(4) Ideal gas
(5) $F_{Bx} = 0$
(6) $W_s = W_{shear} = 0$
(7) $\Delta z = 0$

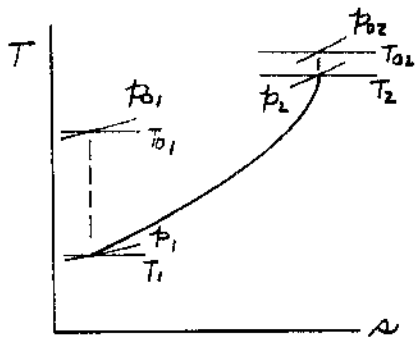
The minimum possible Mach number for supersonic flow with heating is $M_2 = 1$.

$$T_{01} = T_1\left(1 + \frac{k-1}{2} M^2\right) = 215°R\left(1 + 0.2(3.0)^2\right) = 613°R$$

$$T_{02} = T_{01} + \frac{1}{c_p}\frac{\delta Q}{dm} = 613°R + \frac{lbm \cdot °R}{0.240\ Btu} \times 48.5 \frac{Btu}{lbm} = 815°R$$

Check for $M_2 = 1.0$: $T_2 = \frac{T_{02}}{1 + 0.2(1)^2} = \frac{815°R}{1.2} = 679°R$

$$V_2 = c_2 = \sqrt{kRT_2} = 1{,}280\ ft/s$$



Thus $p_2 = p_1 + \frac{\dot{m}}{A}(V_1 - V_2) = p_1 + \rho_1 V_1 (V_1 - V_2)$

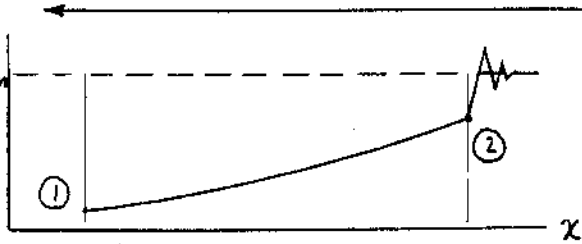
$$V_1 = M_1 c_1 = M_1\sqrt{kRT_1} = 3.0 \times 719 \frac{ft}{s} = 2{,}160\ ft/s\ ; \quad \rho_1 = \frac{p_1}{RT_1} = 0.0217\ lbm/ft^3$$

$$p_2 = 1.73 \frac{lbf}{in.^2} + 0.0217 \frac{lbm}{ft^3} \times 2160 \frac{ft}{s}(2160 - 1280)\frac{ft}{s} \times \frac{slug}{32.2\ lbm} \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \frac{ft^2}{144\ in.^2} = 10.6\ psia$$

Thus if $M_2 = 1.0$, the $p_2 < p_{atm}$, which is not possible for sonic flow. Therefore

$p_2 < p_{atm}$ and $M_2 > 1.0$ for this flow.

The pressure vs. distance plot is:



{This problem could be solved quantitatively, but only by either iteration or by use of compressible flow functions; that solution is presented on the next page.}

For $M_1 = 3.0$, from App. E.3: $\frac{T_0}{T_0^*} = 0.6540$, and $\frac{p}{p^*} = 0.1765$

Thus $T_0^* = \frac{T_{01}}{(T_0/T_0^*)_1} = \frac{613^\circ R}{0.6540} = 937^\circ R$; $p^* = \frac{p_1}{(p/p^*)_1} = \frac{1.73\ psia}{0.1765} = 9.80\ psia$

At section 2, $T_{02} = 815\ R$ and $\frac{T_{02}}{T_0^*} = \frac{815^\circ R}{937^\circ R} = 0.870$

From App. E.3 this corresponds to $M_2 = 1.74$. At this Mach number, $\frac{p}{p^*} = 0.4581$.

Thus $p_2 = p^*\left(\frac{p}{p^*}\right)_2 = 9.80\ psia \times 0.4581 = 4.49\ psia$ p_2

{ These calculations confirm that $M_2 > 1$ and $p_2 < p_{atm}$. }

Given: Aircraft cabin pressurization system, $\dot{m} = 0.75$ lbm/s

Cruise is $M = 0.85$ at $z = 40{,}000$ ft (State 1)

Air slowed isentropically to 100 ft/s w.r.to plane (State 2)

Air compressed adiabatically to pressure equivalent to 8,000 ft altitude; $\Delta T = 170°F$ (State 3); $V_3 \simeq V_2$

Air is cooled at constant pressure, with negligible friction to 70°F (State 4); $V_4 \simeq V_3$

Find: (a) Sketch a system diagram, labeling all components.
(b) Determine static and stagnation temperatures at each cross-section.
(c) Evaluate compressor work added and heat rejected in cooling process.

Solution: From the standard atmosphere, at $z_1 = 40{,}000$ ft, $T_1 = -70°F$ and $p_1 = 2.73$ psia

$$c_1 = \sqrt{kRT_1} = \left[1.4 \times 53.3 \frac{\text{ft·lbf}}{\text{lbm·°R}} \times (460-70)°R \times 32.2 \frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug·ft}}{\text{lbf·s}^2}\right]^{\frac{1}{2}} = 968 \frac{\text{ft}}{\text{s}}$$

$$V_1 = M_1 c_1 = 0.85 \times 968 \frac{\text{ft}}{\text{s}} = 823 \text{ ft/s}$$

For isentropic deceleration $T_{01} = T_1 + \frac{V_1^2}{2c_p} = T_{02} = T_2 + \frac{V_2^2}{2c_p}$

$$T_2 = T_1 + \frac{1}{2c_p}(V_1^2 - V_2^2)$$

$$T_2 = 390°R + \frac{1}{2} \times \frac{\text{lbm·°R}}{0.240 \text{ Btu}} \left[(823)^2 - (100)^2\right] \frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{Btu}}{778 \text{ ft·lbf}} \times \frac{\text{lbf·s}^2}{\text{slug·ft}} \times \frac{\text{slug}}{32.2 \text{ lbm}} = 445°R$$

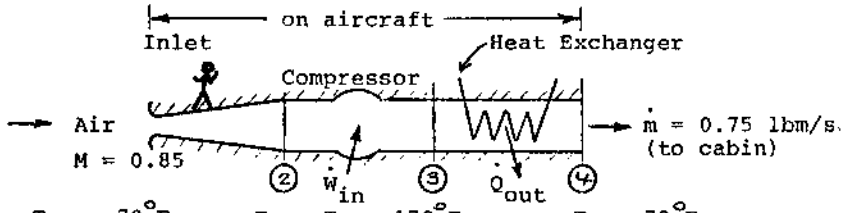
For isentropic deceleration,

$$p_2 = p_1 \left(\frac{T_2}{T_1}\right)^{k/k-1} = 2.73 \text{ psia} \left(\frac{445}{390}\right)^{3.5} = 4.33 \text{ psia}$$

The compressor raises the air the equivalent of $z_3 = z_4 = 8{,}000$ ft; from the Standard Atmosphere

$$p_3 = p_4 = 10.92 \text{ psia}; \quad p_{03} = p_3\left(1 + \frac{k-1}{2} M_3^2\right)^{\frac{k}{k-1}} = 10.92\left(1 + 0.2(0.0823)^2\right)^{3.5} = 10.97 \text{ psia}$$

Assume $V_2 \simeq V_3 \simeq V_4$ since no other data are known. The system is



Evaluating properties: $T_{01} = T_1\left(1+\frac{k-1}{2}M_1^2\right) = 390°R\,(1+0.2(0.85)^2) = 446°R = T_{02}$

$p_{01} = p_1\left(1+\frac{k-1}{2}M_1^2\right)^{\frac{k}{k-1}} = 2.73\ psia\,(1+0.2(0.85)^2)^{3.5} = 4.38\ psia = p_{02}$

$T_3 = T_2 + 170°R = 445 + 170°R = 615°R$; $T_{03} = T_{02} + 170°R = 616°R$ (since $V_2 \simeq$ const)

From the energy equation

$$\dot{W}_{in} = \dot{m}(h_3 - h_2) = \dot{m}c_p(T_3 - T_2)$$

$$\dot{W}_{in} = 0.75\frac{lbm}{s}\cdot 0.240\frac{Btu}{lbm\cdot°R}(170°R \times 778\frac{ft\cdot lbf}{Btu} \times \frac{hp\cdot s}{550\ ft\cdot lbf} = 43.3\ hp \quad \dot{W}_{in}$$

$$\dot{Q}_{in} = \dot{m}(h_4 - h_3) = \dot{m}c_p(T_4 - T_3)$$

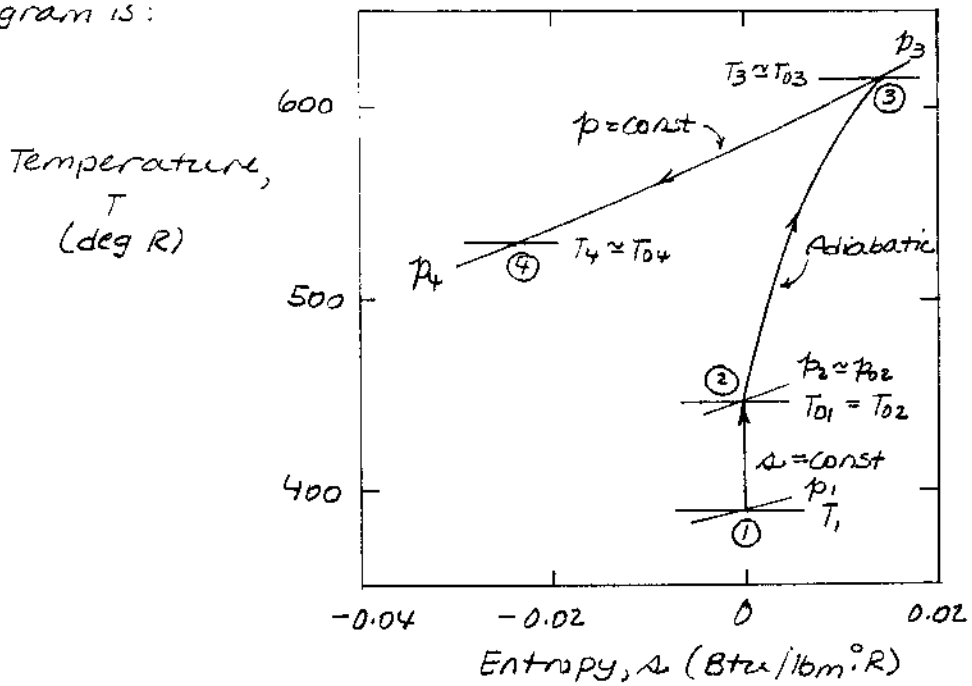
$$\dot{Q}_{in} = 0.75\frac{lbm}{s}\cdot 0.240\frac{Btu}{lbm\cdot°R}(530 - 615)°R = -15.3\ Btu/s \quad (out) \quad \dot{Q}_{in}$$

The entropy changes are computed using $Tds = dh - vdp$, as

$$\Delta s_{23} = c_p \ln\frac{T_{03}}{T_{02}} - R\ln\frac{p_{03}}{p_{02}} = 0.240\frac{Btu}{lbm\cdot°R}\ln\left(\frac{616}{446}\right) - 53.3\frac{ft\cdot lbf}{lbm\cdot°R}\ln\left(\frac{10.97}{4.38}\right)\frac{Btu}{778\ ft\cdot lbf} = 0.0146\frac{Btu}{lbm\cdot°R}$$

$$\Delta s_{34} = c_p \ln\frac{T_{04}}{T_{03}} - R\ln\frac{p_{04}}{p_{03}} = 0.240\frac{Btu}{lbm\cdot°R}\ln\left(\frac{530}{616}\right) - \frac{53.3}{778}\frac{Btu}{lbm\cdot°R}\ln\left(\frac{10.97}{10.97}\right) = -0.0361\ Btu/lbm\cdot°R$$

The Ts diagram is:



Given: Pitot-static tube inserted in supersonic flow. Normal shock occurs ahead of tube.



Find: (a) V_1 ahead of shock.
(b) Ts diagram.

Solution: Apply momentum equation to steady, 1-D compressible flow.

Computing equation: $p_1 - p_2 = \frac{\dot{m}}{A}(V_2 - V_1)$ or $p_1(1+kM_1^2) = p_2(1+kM_2^2)$

Assumptions: (1) No friction
(2) Ideal gas

$$\frac{p_0}{p} = \left(1 + \frac{k-1}{2}M^2\right)^{\frac{k}{k-1}}$$

Then:

$$M_1 = \left\{\frac{1}{k}\left[\frac{p_2}{p_1}(1+kM_2^2) - 1\right]\right\}^{1/2}$$

$$M_2 = \left\{\frac{2}{k-1}\left[\left(\frac{p_{02}}{p_2}\right)^{\frac{k-1}{k}} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{68.1}{54.8}\right)^{\frac{0.4}{1.4}} - 1\right]\right\}^{1/2} = 0.566$$

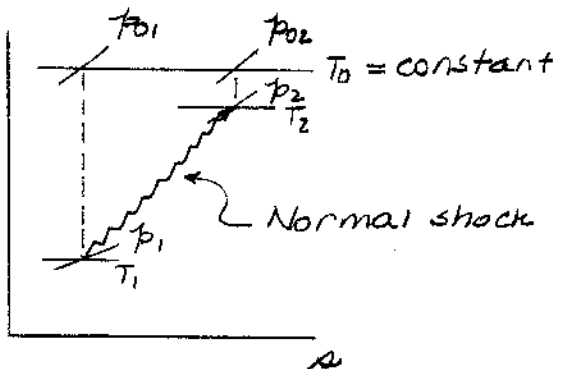
$$M_1 = \left\{\frac{1}{1.4}\left[\frac{54.8}{11.0}(1 + 1.4(0.566)^2 - 1\right]\right\}^{1/2} = 2.11$$

Sonic speed is

$$C_1 = \sqrt{kRT_1} = \left[1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 160\ K \times \frac{kg\cdot m}{N\cdot s^2}\right]^{1/2} = 254\ m/s$$

$$V_1 = M_1 C_1 = 2.11 \times 254 \frac{m}{s} = 536\ m/s \qquad V_1$$

The Ts diagram is:

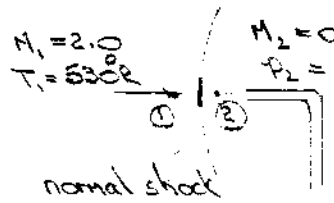


* Using Normal-Shock Flow Functions (Appendix E.4), $p_2/p_1 = \frac{54.8}{11} = 4.982$

From Eq. 12.36 by iteration, $M_1 = 2.10$

From Eq. 12.34b, $M_2 = 0.561$ This is reasonable agreement!

Given: A total-pressure probe is placed in a supersonic flow, $M_1 = 2.0$. Behind the shock $M_2 = 0.5774$, $p_2 = 5.76$ psia



Find: a) T_{o_2}, p_{o_2} b) $p_{o_1}, T_{o_1}, p_1, \rho_1, V_1$

Solution:

Computing equations: $p(1+kM^2) = \text{const.}$ (across shock)

$$\frac{T_o}{T} = 1 + \frac{k-1}{2}M^2 \qquad \frac{p_o}{p} = \left(\frac{T_o}{T}\right)^{k/k-1}$$

Assumptions: (1) steady flow (3) uniform flow at a section (2) ideal gas

Across the shock $p(1+kM^2) = \text{const}$

$$\therefore p_1 = p_2\frac{(1+kM_2^2)}{(1+kM_1^2)} = 5.76\text{ psia}\,\frac{[1+1.4(0.5774)^2]}{[1+1.4(2)^2]} = 1.28\text{ psia} \longleftarrow p_1$$

$$\rho_1 = \frac{p_1}{RT_1} = 1.28\frac{\text{lbf}}{\text{in}^2}\cdot 53.3\frac{\text{ft·lbf}}{\text{lbm·°R}}\times\frac{1}{530°\text{R}}\times 144\frac{\text{in}^2}{\text{ft}^2} = 0.006653\ \text{lbm/ft}^3 \longleftarrow \rho_1$$

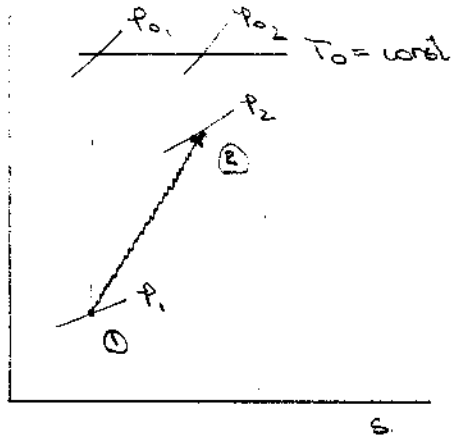
$$V_1 = M_1c_1 = M_1(kRT_1)^{1/2} = 2.0\left[1.4\cdot 53.3\frac{\text{ft·lbf}}{\text{lbm·°R}}\cdot 530°\text{R}\cdot 32.2\frac{\text{lbm}}{\text{slug}}\cdot\frac{\text{slug·ft}}{\text{lbf·s}^2}\right]^{1/2} = 2260\text{ ft/s} \longleftarrow V_1$$

$$T_{o_1} = T_1\left(1+\frac{k-1}{2}M_1^2\right) = 530°\text{R}[1+0.2(2)^2] = 954°\text{R} \longleftarrow T_{o_1}$$

$$p_{o_1} = p_1\left(1+\frac{k-1}{2}M_1^2\right)^{k/k-1} = 1.28\text{ psia}\,(1.8)^{3.5} = 10.1\text{ psia} \longleftarrow p_{o_1}$$

$$p_{o_2} = p_2\left(1+\frac{k-1}{2}M_2^2\right)^{k/k-1} = 5.76\text{ psia}\,[1+0.2(0.5774)^2]^{3.5} = 7.22\text{ psia} \longleftarrow p_{o_2}$$

$$T_{o_2} = T_{o_1} = 954\text{ R} \longleftarrow$$



Compressible-Flow Functions (Appendix E)

For $M_1 = 2.0$, from App. E.4

$M_2 = 0.577$ (12.34b)

$p_2/p_1 = 4.50$ (12.33b) $\therefore p_1 = 1.28$ psia

$p_{o_2}/p_{o_1} = 0.721$ (12.37)

For $M_1 = 2.0$, from App. E.1

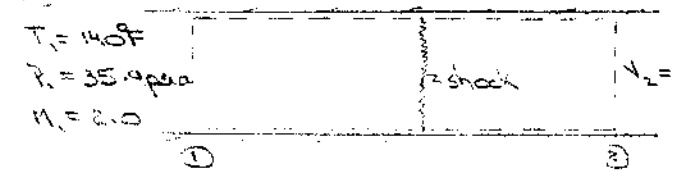
$p_1/p_{o_1} = 0.1278$ (11.17a) $\therefore p_{o_1} = 10.0$ psia

and $p_{o_2} = 0.721\,p_{o_1} = 7.21$ psia

$T/T_o = 0.556$ (11.17b) $\therefore T_{o_1} = 954°\text{R}$

Note: In using the tables it is not necessary to know the downstream Mach number.

Given: Steady flow of air through a constant area duct



Find: P_2, M_2, sketch process in T-s and p-v coordinates

Solution:

Basic equation: $h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$

Assumptions: (1) steady flow (4) $Q = 0$
(2) uniform flow at ① & ② (5) ideal gas
(3) $\dot{Q} = \dot{W}_s = \dot{m}_{leak} = 0$ (6) $A_1 = A_2 = A$

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 2.0 \left(1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot °R} \times 600°R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right)^{1/2} = 2400 \text{ ft/s}$$

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} \qquad T_2 = T_1 + \frac{1}{2c_p}(V_1^2 - V_2^2)$$

$$T_2 = 600°R + \frac{1}{2}\left[(2.4)^2 - (1.08)^2\right] \times 10^6 \frac{ft^2}{s^2} \times \frac{lbm °R}{0.24 \text{ Btu}} \times \frac{\text{Btu}}{778 \text{ ft lbf}} \times \frac{slug}{32.2 lbm} \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

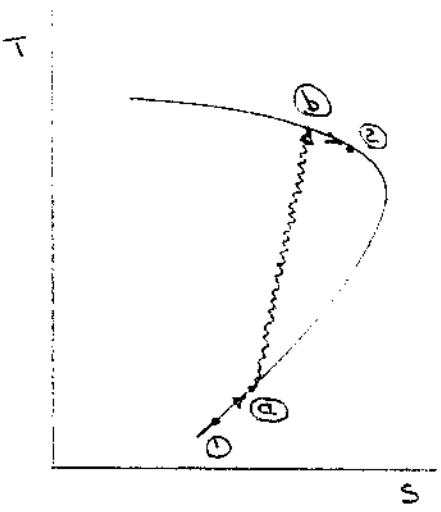
$$T_2 = 982°R$$

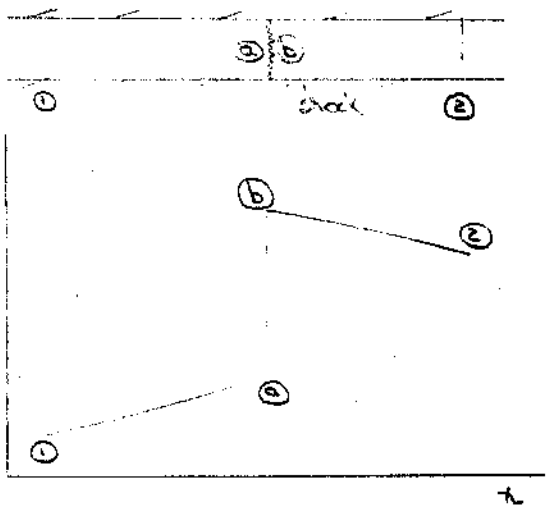
$$M_2 = \frac{V_2}{c_2} \qquad c_2 = (kRT_2)^{1/2} = \left(1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot °R} \times 982°R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right)^{1/2} = 1540 \text{ ft/s}$$

$$M_2 = \frac{1080}{1540} = 0.701 \quad \longleftarrow M_2$$

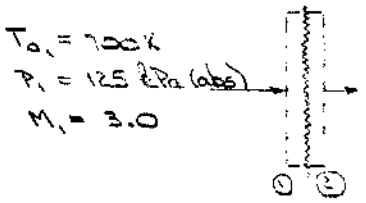
From continuity $\rho_1 V_1 = \rho_2 V_2$ $\qquad \rho_2 = \frac{V_1}{V_2}\rho_1 = \frac{V_1}{V_2}\frac{P_1}{RT_1}$

$$\rho_2 = \frac{2400}{1080} \times 35.9 \frac{lbf}{in^2} \times \frac{lbm °R}{53.3 ft \cdot lbf} \times \frac{1}{600°F} \times \frac{144 in^2}{1 ft^2} = 0.359 \text{ lbm/ft}^3 \quad \longleftarrow \rho_2$$





Given: Air flow through a normal shock as shown



Find: V_2, T_2, $s_2 - s_1$

Solution:

Compressible flow functions (Appendix E) to be used in solution

Basic equation: $T ds = dh - v\,dP$

Assumptions: (1) steady flow (5) $F_{B_x} = 0$
(2) uniform flow at a section (6) no friction forces
(3) $\dot{Q} = \dot{W}_s = \dot{W}_{shear} = 0$ (7) ideal gas
(4) $\Delta z = 0$ (8) $A_1 = A_2 = A$

$M_1 = 3.0$ From App. E.1 $T_1/T_{o_1} = 0.3571$ (11.17b) $\therefore T_1 = 250K$

From App E.4, $M_2 = 0.4752$ (12.34b)

$T_2/T_1 = 2.679$ $\therefore T_2 = 670K$ ← T_2

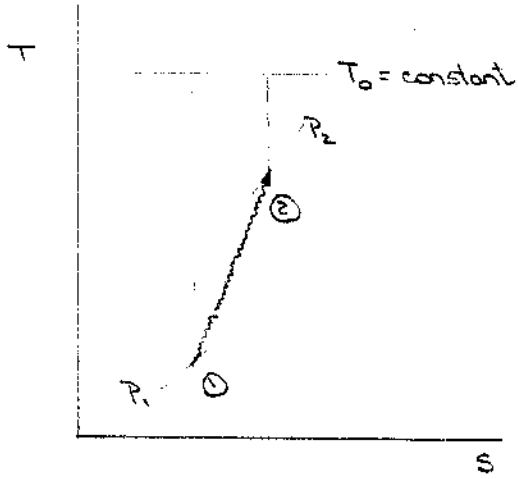
$P_2/P_1 = 10.33$

$$V_2 = M_2 c_2 = M_2 (kRT_2)^{1/2} = 0.4752 \left[1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 670K \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 247 \text{ m/s}$$ ← V_2

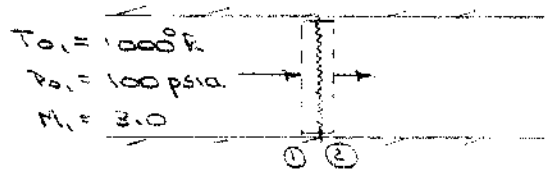
$$T\,ds = dh - v\,dP = c_p\,dT - \frac{1}{\rho}\,dP \qquad ds = c_p \frac{dT}{T} - R \frac{dP}{P}$$

$$s_2 - s_1 = \int_{s_1}^{s_2} ds = \int_{T_1}^{T_2} c_p \frac{dT}{T} - \int_{P_1}^{P_2} R \frac{dP}{P} = c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1}$$

$$s_2 - s_1 = 1.00 \frac{kJ}{kg \cdot K} \ln 2.679 - 0.287 \frac{kJ}{kg \cdot K} \ln 10.33 = 0.315 \frac{kJ}{kg \cdot K}$$ ← $s_2 - s_1$



Given: Air flow in a constant area duct undergoes a normal shock



Find: p_2, p_{2_s} for isentropic

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (5) $F_{B_x} = 0$
(2) uniform flow at a section (6) no friction forces
(3) $Q = W_s = W_{shear} = 0$ (7) ideal gas
(4) $\Delta z = 0$ (8) $A_1 = A_2 = A$

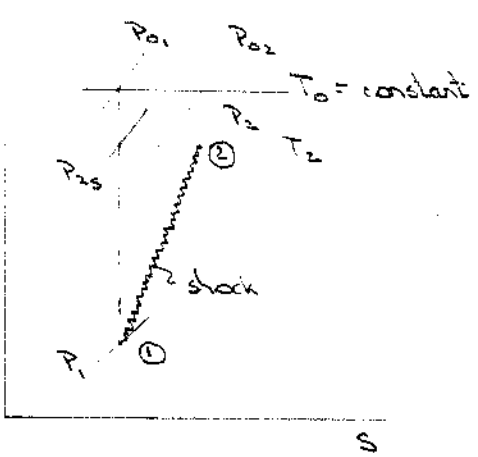
$M_1 = 3.0$ From App. E.1, $p_1/p_{o_1} = 0.02722$

From App. E.4, $p_2/p_1 = 10.33$ $M_2 = 0.475$

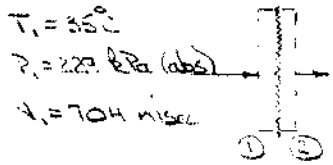
$p_2 = \frac{p_2}{p_1} \cdot \frac{p_1}{p_{o_1}} \cdot p_{o_1} = 10.33 \times 0.02722 \times 100$ psia $= 28.1$ psia ← p_2

For $M_2 = 0.475$, from App. E.1, $p_2/p_{o_2} = 0.8568$

Then $p_{2_s} = \frac{p_2}{p_{o_2}} \times p_{o_2} \times \frac{p_{o_1}}{p_{o_2}} = 0.8568 \times 100$ psia $= 85.7$ psia ← p_{2_s}



Given: Air flow through a normal shock as shown.



Find: T_2, p_{o_2}.

Solution:

Compressible flow functions (Appendix E) to be used in solution.

Assumptions: (1) steady flow; (2) uniform flow at sections; (3) $Q = \dot{W}_s = W_{shear} = 0$; (4) $\Delta z = 0$; (5) $F_{s_x} = 0$; (6) no friction forces; (7) ideal gas; (8) $A_1 = A_2 = A$

$$M_1 = \frac{V_1}{c_1} \qquad c_1 = (kRT_1)^{1/2} = \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 308\ K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 352\ m/s$$

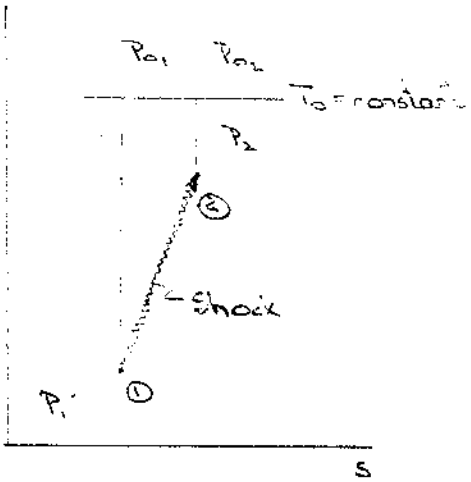
$$M_1 = \frac{704}{352} = 2.0$$

From App. E.1 $\quad p_1/p_{o_1} = 0.1278$

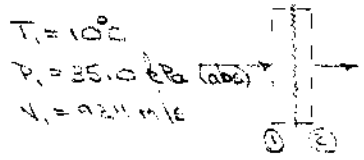
From App. E.4 $\quad p_{o_2}/p_{o_1} = 0.7209 \qquad T_2/T_1 = 1.687$

$$T_2 = \frac{T_2}{T_1} \cdot T_1 = 1.687 \times 308\ K = 520\ K \qquad \underline{T_2}$$

$$p_{o_2} = \frac{p_{o_2}}{p_{o_1}} \cdot \frac{p_{o_1}}{p_1} \cdot p_1 = 0.7209 \times \frac{1}{0.1278} \times 229\ kPa = 1.29\ MPa\ (abs) \qquad \underline{p_{o_2}}$$



Given: Air flow through a normal shock as shown:



Find: V_2, M_2, $p_{o_1} - p_{o_2}$

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) Steady flow (5) $F_{B_x} = 0$
(2) uniform flow at sections (6) no friction forces
(3) $\dot{Q} = \dot{W}_s = \dot{W}_{shear} = 0$ (7) ideal gas
(4) $\Delta g = 0$ (8) $A_1 = A_2 = A$

$$c_1 = (kRT_1)^{1/2} = \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 283\,K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 337\ m/s$$

$$M_1 = \frac{V_1}{c_1} = \frac{924}{337} = 2.74$$

From App. E.4, $M_2 = 0.493$ ← M_2

$p_2/p_1 = 8.001$ $\quad p_{o_2}/p_{o_1} = 0.4097$

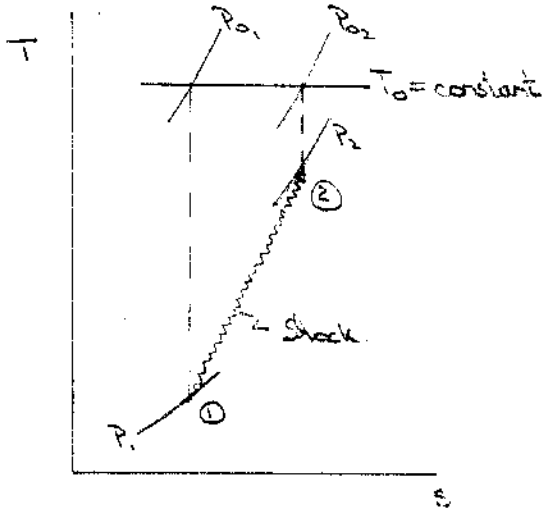
Then $V_2 = \frac{p_1}{p_2} V_1 = \frac{1}{8.001} \times 924 \frac{m}{s} = 257\ m/s$ ← V_2

From Table E.1, $p_1/p_{o_1} = 0.04027$

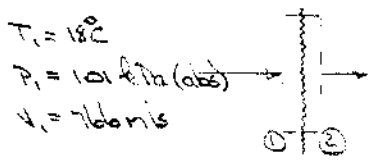
Then $p_{o_2} = \frac{p_{o_2}}{p_{o_1}} \times \frac{p_{o_1}}{p_1} \times p_1 = 0.4097 \times \frac{1}{0.04027} \times 35\,kPa = 355\ kPa$

and $p_{o_1} = \frac{p_{o_1}}{p_1} \times p_1 = \frac{1}{0.04027} \times 35\,kPa = 867\ kPa$

$p_{o_1} - p_{o_2} = (867 - 355)\ kPa = 512\ kPa$ ← $p_{o_1} - p_{o_2}$



Given: Air flow through a normal shock as shown



Find: V_2, $p_2 - p_1$, $(p_2 - p_1)_{is}$

Solution:

Compressible flow tables to be used in the solution

Assumptions: (1) steady flow (5) $F_{B_x} = 0$
(2) uniform flow at a section (6) no friction forces
(3) $\dot{Q} = \dot{W}_s = \dot{W}_{shear} = 0$ (7) ideal gas
(4) $\frac{\partial}{\partial t} = 0$ (8) $A_1 = A_2 = A$

$M_1 = \frac{V_1}{c_1}$ $\quad c_1 = (kRT_1)^{1/2} = \left(1.4 \times 287\ \frac{N\cdot m}{kg\cdot K} \times 291K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 342\ m/s$

$M_1 = \frac{766}{342} = 2.24$

From App. E.4, $p_2/p_1 = 5.687$ $\quad \rho_2/\rho_1 = 3.005$ $\quad M_2 = 0.542$

$\rho_1 V_1 = \rho_2 V_2$ $\quad V_2 = \frac{\rho_1}{\rho_2} V_1 = \frac{1}{3.005} \times 766\ \frac{m}{s} = 255\ m/s$ ← V_2

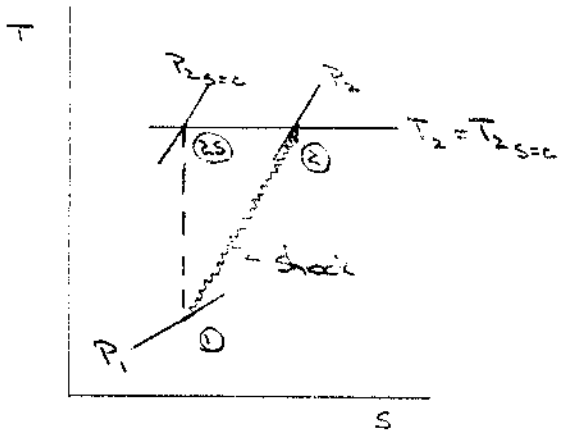
$p_2 = 5.687\, p_1$ $\quad p_2 - p_1 = 4.687\, p_1 = 4.687 \times 101\ kPa = 473\ kPa$ ← $p_2 - p_1$

From App. E.1, for $M_1 = 2.24$ $\quad p_1/p_{0_1} = 0.08784$

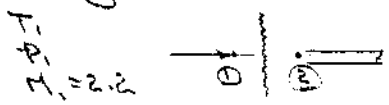
for $M_2 = 0.54$ $\quad p_2/p_{0_2} = 0.8201$

then $\frac{p_2}{p_{0_2}} \times \frac{p_{0_1}}{p_1} \times p_1 = 0.8201 \times \frac{1}{0.08784}\, p_1 = 9.336\, p_1$

$(p_2 - p_1)_{is} = 8.336\, p_1 = 8.336 \times 101\ kPa = 842\ kPa$ ← $(p_2 - p_1)_{is}$



Given: Supersonic aircraft cruises at $M = 2.2$ at 12 km altitude. Normal shock stands in front of a pitot tube which senses a stagnation pressure, p_{o_2}.



Find: (a) T_{o_1}, p_{o_1} (b) p_{o_2}

Solution: Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (2) uniform flow at a section
(3) thin shock
(4) ideal gas

Use table H.2 to determine properties at state ① At 12 km altitude,

$T_1 = 216.7$ K $\quad p_1 = 19.4$ kPa

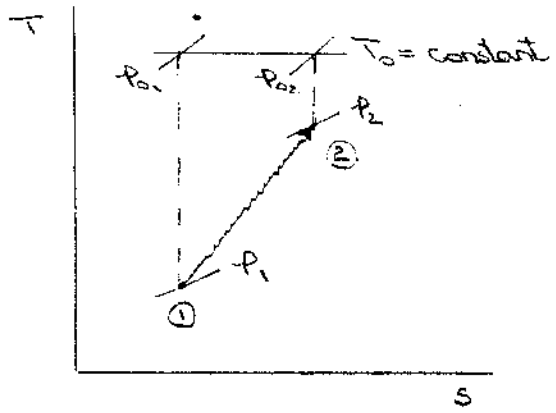
From App. E.3, for $M_1 = 2.2$,

$T_1/T_{o_1} = 0.5081 \quad \therefore T_{o_1} = 426$ K ____ T_{o_1}

$p_1/p_{o_1} = 0.09352 \quad \therefore p_{o_1} = 207$ kPa (abs) ____ p_{o_1}

From App. E.4, for $M_1 = 2.2$, $M_2 = 0.5471$

$p_{o_2}/p_{o_1} = 0.6281 \quad \therefore p_{o_2} = 130$ kPa (abs) ____ p_{o_2}



Given: Supersonic aircraft flies at 35000 ft. Normal shock stands in front of stagnation temperature probe. Probe reads $T_{o_2} = 420°F$



Find: (a) M_1, V_1 (b) p_2, p_{o_2}

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (2) uniform flow at a section
(3) thin shock, $k_s = 0$
(4) ideal gas

Use table A.3 to determine properties at state ①.

Altitude = $35000 \text{ ft} \times 0.3048 \frac{\text{m}}{\text{ft}} = 10{,}670 \text{ m}$

From Table A.3 $T_1 = 219 \text{ K} = -54°C = -65°F$

$$p_1 = 23.9 \times 10^3 \frac{\text{N}}{\text{m}^2} \times \frac{1 \text{ lbf}}{4.448 \text{ N}} \times \left(0.3048 \frac{\text{m}}{\text{ft}}\right)^2 = 499 \text{ psfa}$$

At state ① $T_1 / T_{o_1} = \frac{395}{880} = 0.4489$

From App. E.1, for $T_1/T_{o_1} = 0.4489$, $M_1 = 2.48$ ← M_1

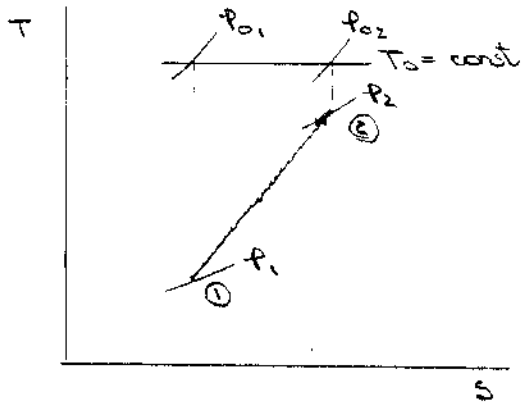
$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 2.48 \left[1.4 \times 53.3 \frac{\text{ft·lbf}}{\text{lbm°R}} \times 395°R \times 32.2 \frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug·ft/s}^2}{\text{lbf}}\right]^{1/2} = 2420 \text{ ft/s}$$ ← V_1

From App. E.1, for $M_1 = 2.48$, $p_1/p_{o_1} = 0.06038$ ∴ $p_{o_1} = 57.4$ psia

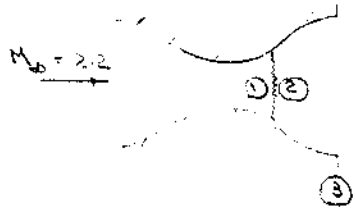
From App. E.4, for $M_1 = 2.48$, $M_2 = 0.5149$

$p_{o_2}/p_{o_1} = 0.5071$ ∴ $p_{o_2} = 29.1$ psia ← p_{o_2}

$p_2/p_1 = 7.009$ ∴ $p_2 = 24.3$ psia ← p_2



Given: Concorde flies at $M = 2.2$ at an altitude of 20 km. In the engine inlet system the air is decelerated isentropically to a Mach number of 1.3, the air undergoes a normal shock and then decelerates isentropically to $M = 0.4$.



Find: T_3, P_3, P_{o3}

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: 1 steady flow
2 ideal gas
3 flow is isentropic except across the shock.

At 20 km altitude $T_\infty = 217$ K, $P_\infty = 5.53$ kPa. (Table A.3)

$M_\infty = 2.2$ From App. E.1, $T_\infty/T_o = 0.5081$ $\therefore T_o = 427$ K; $P_\infty/P_o = 0.09352$ $P_o = 59.1$ kPa

$M_1 = 1.3$ From App. E.1, $T_1/T_o = 0.7774$ $\therefore T_1 = 332$ K; $P_1/P_o = 0.3667$ $P_1 = 21.7$ kPa.

From App. E.4, $M_2 = 0.786$ $P_{o2}/P_{o1} = 0.9794$ $\frac{T_2}{T_1} = 1.191$ $\frac{P_2}{P_1} = 1.805$

$\therefore P_{o2} = 57.9$ kPa, $T_2 = 380$ K, $P_2 = 38.4$ kPa.

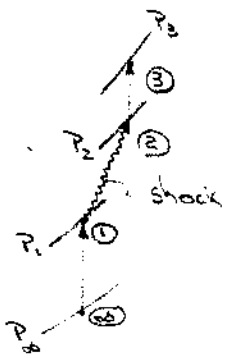
$P_{o3} = P_{o2} = 57.9$ kPa (abs) P_{o3}

$M_3 = 0.4$ From App. E.1, $T_3/T_o = 0.9690$ $P_3/P_{o3} = 0.8956$

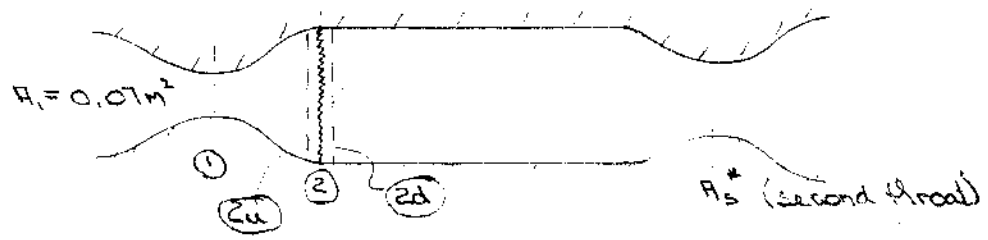
$\therefore T_3 = 414$ K T_3

$P_3 = 0.8956\, P_{o3} = 0.8956\, P_{o2} = 51.9$ kPa (abs) P_3

T



Given: Supersonic wind tunnel, supplied with air at $T_o = 500K$ and $p_o = 1.0 MPa (abs)$, is to operate at a test section mach number $M = 2.2$. Normal shock stands in exit plane of inlet nozzle.



Find: a) M_{2d} b) p_{2d} c) p_{o2d} d) A_s

Solution:

Compressible flow functions (Appendix E) to be used in solution.

Assumptions: (1) steady flow (2) uniform flow at a section (3) isentropic flow in nozzles, adiabatic flow across shock (4) ideal gas.

At $M_{2u} = 2.2$, from App E.1, $p_{2u}/p_o = 0.09352$ $\therefore p_{2u} = 93.5 kPa$

At $M_{2u} = 2.2$, from App E.4, $M_{2d} = 0.547$ (12.34b). ← M_{2d}

$p_2/p_1 = 5.480$ $\therefore p_{2d} = 512 kPa$ ← p_{2d}

$p_{o2}/p_{o1} = 0.6281$ $\therefore p_{o2d} = 628 kPa$ ← p_{o2d}

At $M_{2u} = 2.2$, from App E.1, $A_2/A^* = A_2/A_1 = 2.005$ $\therefore A_2 = 0.1404 m^2$

At $M_{2d} = 0.547$, from App E.1, $A_2/A^* = A_2/A_s^* = 1.259$ $\therefore A_s^* = 0.111 m^2$ ← A_s^*

Given: Supersonic wind tunnel starting as shown.

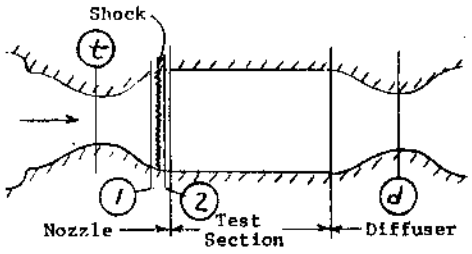
$A_t = 1.25\ ft^2$

$A_1 = A_2 = 3.05\ ft^2$

$M_{design} = 2.50$

$T_0 = 1080\ R$

$p_0 = 115\ psia$



Find: (a) Minimum possible A_d at this condition.
(b) Entropy increase across the shock.

Solution: Use functions for steady, one-dimensional compressible flow.

Computing equations: A/A^* vs. M from isentropic flow functions (App. E.1)

p_{02}/p_{01} vs. M from shock flow functions (App. E.4)

Assumptions: (1) Steady flow
(2) Uniform flow at each section
(3) Ideal gas
(4) Isentropic except across shock
(5) Adiabatic flow
(6) $F_{ex} = 0$
(7) $\Delta z = 0$

Then from App. E.1, $M_1 = 2.416$ at $\frac{A_1}{A^*} = \frac{A_1}{A_t} = \frac{3.05\ ft^2}{1.25\ ft^2} = 2.44$.

From App. E.4, at $M_1 = 2.416$, $\frac{p_{02}}{p_{01}} = 0.5395$. Thus $p_{0d} = 0.5395\ p_{01} = 62.0\ psia$.

For adiabatic flow, $T_0 = \text{constant}$ and $T^* = \frac{T_0}{1.2} = \frac{1080\ R}{1.2} = 900\ R = \text{constant}$.

From continuity, $\dot m = \rho_t A_t V_t = \rho_d V_d A_d$. Substituting $\rho = \frac{p}{RT}$ and $V = M\sqrt{kRT}$,

$$\frac{p_t}{RT_t}\sqrt{kRT_t}\,A_t = \frac{p_d}{RT_d}\sqrt{kRT_d}\,A_d \;;\; A_d = A_t\frac{p_t}{p_d} = A_t\frac{p_{0t}}{p_{0d}} = \frac{A_t}{0.5395}$$

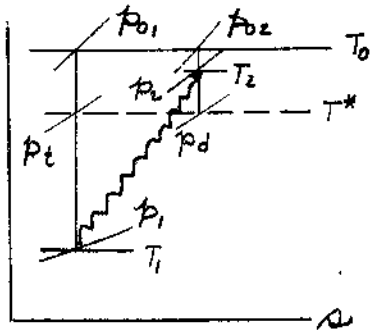
$$A_d = \frac{1.25\ ft^2}{0.5395} = 2.32\ ft^2$$ A_d

From the Gibbs equation,

$$Tds = dh - vdp \;;\; ds = c_p\frac{dT}{T} - R\frac{dp}{p}$$

$$\Delta s = c_p \ln\frac{T_{02}}{T_{01}} - R\ln\frac{p_{02}}{p_{01}}$$

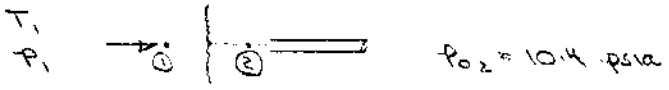
Since $T_0 = \text{constant}$, $\ln(T_{02}/T_{01}) = 0$, and



$$\Delta s = -53.3\frac{ft\cdot lbf}{lbm\cdot R}\times\ln(0.5395)\times\frac{Btu}{778\ ft\cdot lbf} = 0.0423\ Btu/lbm\cdot R$$ Δs

{ Note a slightly stronger shock could occur at $M = 2.50$; a larger diffuser throat ($A_d \approx 2.51\ ft^2$) would be needed to start this tunnel. } 1405

Given: Supersonic aircraft cruises at $M = 2.7$ at 60,000 ft altitude. Normal shock stands in front of a pitot tube which senses a stagnation pressure of 10.4 psia



Find: (a) T_2, p_2 (b) $p_{o_2} - p_{o_1}$ (c) $s_2 - s_1$

Solution.

Compressible flow functions (Appendix E) to be used in solution

Basic equation: $Tds = dh - vdp$

Assumptions: (1) steady flow (2) uniform flow at a section
(3) thin shock, $h_x = 0$
(4) ideal gas

Use table A.3 to determine properties at state ①

$$\text{Altitude} = 60{,}000 \text{ ft} \times 0.3048 \frac{\text{m}}{\text{ft}} = 18{,}290 \text{ m}$$

From table A.3 $T_1 = 216.7 \text{ K} = -56.3^\circ\text{C} = -69^\circ\text{F}$

$$p_1 = 7.25 \times 10^3 \frac{\text{N}}{\text{m}^2} \times \frac{1 \text{ lb}}{4.448 \text{ N}} \times \left(\frac{0.3048 \text{ m}}{\text{ft}}\right)^2 = 151 \text{ lb/ft}^2$$

From App. E.1 for $M_1 = 2.7$ $\quad p_1/p_{o_1} = 0.04295 \quad \therefore p_{o_1} = 24.4 \text{ lb/in}^2$

From App. E.4 for $M_1 = 2.7$ $\quad M_2 = 0.4935$

$p_{o_2}/p_{o_1} = 0.4236 \quad \therefore p_{o_2} = 10.3$ psia

$T_2/T_1 = 2.343 \quad \therefore T_2 = 911 \text{ K}$ ← T_2

$p_2/p_1 = 8.338 \quad \therefore p_2 = 1.74$ psia ← p_2

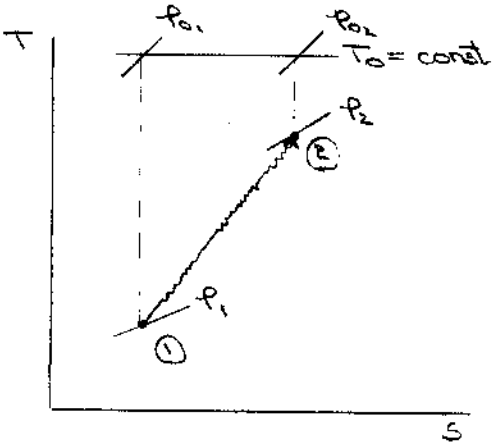
$p_{o_2} - p_{o_1} = 10.3 \text{ psi} - 24.4 \text{ psi} = -14.1 \text{ psi}$ ← $p_{o_2} - p_{o_1}$

From the Tds equation, $Tds = dh - vdp = c_p dT - RT\frac{dp}{p}$

$$\therefore ds = c_p \frac{dT}{T} - R\frac{dp}{p}$$

Then $s_2 - s_1 = s_{o_2} - s_{o_1} = c_p \ln \frac{T_{o_2}}{T_{o_1}} - R \ln \frac{p_{o_2}}{p_{o_1}}$

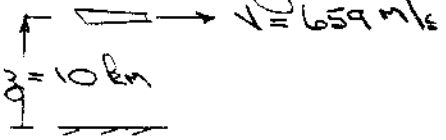
$$s_2 - s_1 = -53.3 \frac{\text{ft·lb}}{\text{lbm·°R}} \ln \frac{10.3}{24.4} \times \frac{\text{Btu}}{778 \text{ ft·lb}} = 0.0591 \frac{\text{Btu}}{\text{lbm·°R}}$$ ← $s_2 - s_1$



Given: Aircraft in supersonic flight on standard day

Total-head tube senses stagnation pressure.

Mach number computed assuming $s = c$, i.e. ignoring shock in front of tube



Find: (a) flight Mach number
(b) pressure sensed by total-head tube
(c) air speed computed assuming s = constant

Solution: (using compressible flow functions - Appendix E)

From Table A.3 at $z = 10$ km, $T = 223.3$ K, $p/p_{SL} = 0.2615$

Thus $T = 223$ K, $p = 0.2615\, p_{SL} = 0.2615 \times 101.3$ kPa $= 26.5$ kPa

$c_1 = (kRT_1)^{1/2} = \left[1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 223\,K \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 299$ m/s

M_2 ← shock ← M_1

$$M_1 = \frac{V_1}{c_1} = \frac{659}{299} = 2.20 \quad \leftarrow M_1$$

From App. E.1 at $M_1 = 2.20$, $p/p_0 = 0.09352$ $\therefore p_{0_1} = 283$ kPa

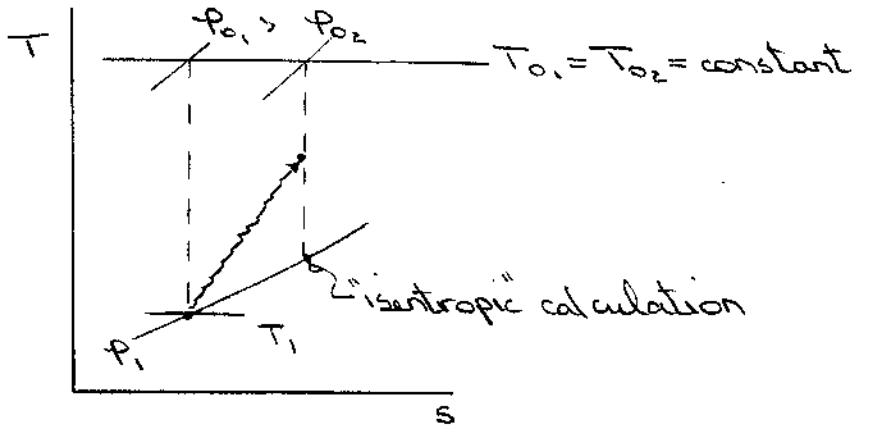
From App E.4 at $M_1 = 2.20$, $p_{0_2}/p_{0_1} = 0.6281$, $p_2/p_1 = 5.480$

$\therefore p_{0_2} = 0.6281\, p_{0_1} = 0.6281 \times 283$ kPa $= 178$ kPa (abs) $\leftarrow p_{0_2}$

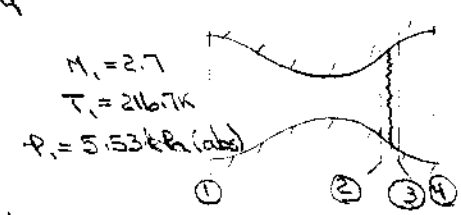
If Mach number is calculated neglecting the shock then M is calculated with $p = 26.5$ kPa and $p_0 = 178$ kPa.

For $p/p_0 = 26.5/178 = 0.149$, from App. E.1, $M_s = 1.90$

$$V_{1_s} = M_{1_s} c_1 = 1.90 \times 299 \text{ m/s} = 568 \text{ m/s} \quad \leftarrow V_{1_s}$$



Given: Supersonic aircraft flies at $M = 2.7$ at 20km altitude. Air is slowed isentropically in inlet system to $M = 1.3$. A normal shock occurs at that location. Following the shock, the flow is decelerated adiabatically, but not isentropically, to $p = 104$ kPa and $M = 0.4$



Find: a) T_0 b) $p_3 - p_2$ c) $s_4 - s_1$ d) p_{04}

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (2) uniform flow at a section
(3) thin shock, $A_2 = A_3$
(4) ideal gas

For $M_1 = 2.7$, from App. E.1, $T_{01}/T_1 = 2.458$ $\therefore T_{01} = 533$ K
$p_{01}/p_1 = 23.283$ $\therefore p_{01} = 128.8$ kPa

Note: T_0 = constant, $p_{02} = p_{01}$

T_0

For $M_2 = 1.3$, from App E.1, $T_{02}/T_2 = 1.338$ $\therefore T_2 = 398$ K
$p_{02}/p_2 = 2.771$ $p_2 = 46.48$ kPa

For $M_2 = 1.3$, from App E.4, $M_3 = 0.786$
$p_3/p_2 = 1.805$ $\therefore p_3 = 83.9$ kPa
$p_{03}/p_{02} = 0.9794$ $p_{03} = 126$ kPa
$T_3/T_2 = 1.191$ $T_3 = 474$ K

$p_3 - p_2 = (83.9 - 46.5)$ kPa $= 37.4$ kPa

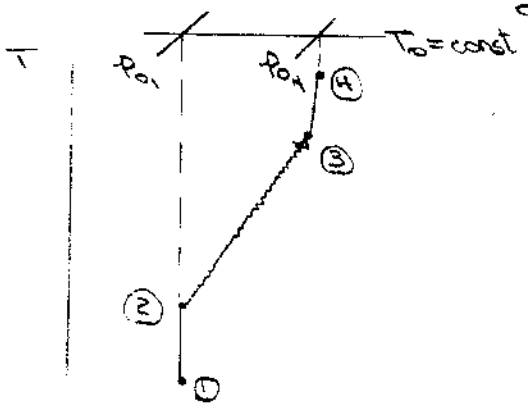
$p_3 - p_2$

For $M_4 = 0.4$, from App. E.1, $p_{04}/p_4 = 1.117$ $\therefore p_{04} = 116$ kPa

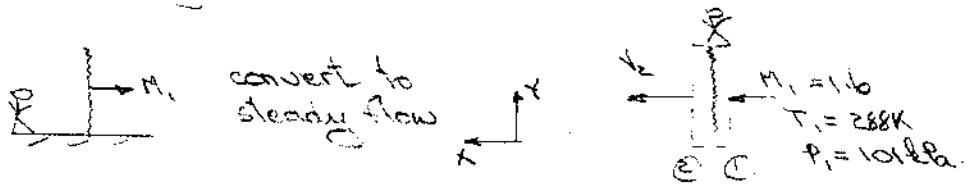
$$s_4 - s_1 = s_{04} - s_{01} = c_p \ln \frac{T_{04}}{T_{01}} - R \ln \frac{p_{04}}{p_{01}} = -287 \frac{J}{kg \cdot K} \ln \frac{116}{129}$$

$s_4 - s_1 = -30.5$ J/kg·K

$s_4 - s_1$



Given: Blast wave propagates outward from an explosion. Model the wave as strong normal shock. Wave front travels at $M=1.6$ through undisturbed air



Find: (a) V_2 relative to wave (b) V_2 relative to ground.

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow as seen by an observer on the wave
(2) uniform flow at a section
(3) thin shock, $R_x = 0$
(4) ideal gas

At state (1)

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 1.6 \left[1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 288K \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 544 \text{ m/s}$$

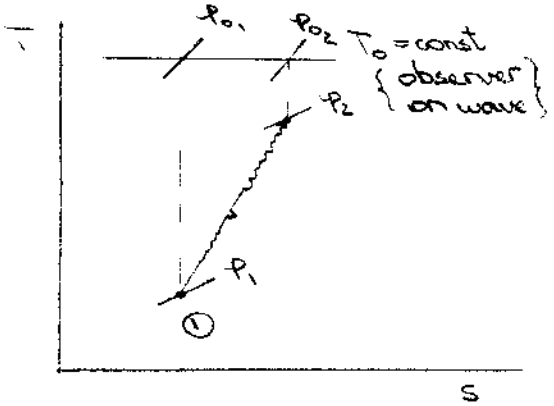
For $M_1 = 1.6$, from Appendix E.4, $V_1 / V_2 = p_2/p_1 = 2.032$

$$\therefore \vec{V}_2 = 268\hat{i} \text{ m/s} \quad \leftarrow \vec{V}_{2 \text{ rel. wave}}$$

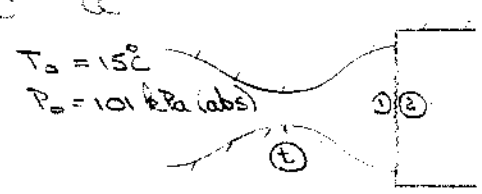
The wave moves to the right at $V_1 = 544$ m/s. Air moves to the left with respect to the wave at $V_{2rel} = 268$ m/s.

$$\vec{V}_{2\,abs} = \vec{V}_{2\,wave} + \vec{V}_{2\,rel} = -544\hat{i} + 268\hat{i} = -276\hat{i} \text{ m/s.} \quad \leftarrow \vec{V}_{2\,abs}$$

{relative to ground, air behind wave moves to right}.



Given Steady, adiabatic flow of air through a converging-diverging nozzle with a normal shock in the exit plane



Find: P_2, V_2

Solution:

Compressible flow tables to be used in the solution

Assumptions: (1) Steady flow (2) uniform flow at a section (3) ideal gas (4) isentropic flow except across the shock

$A_1/A^* = 3.5$

From Fig. E.1 and Eq. 12.6, $M_1 = 2.80$; then $T_1/T_o = 0.3894$ $P_1/P_{o_1} = 0.03665$

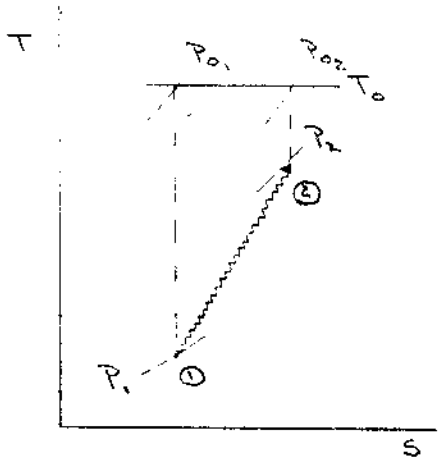
$\therefore T_1 = 112$ K $P_1 = 3.72$ kPa (abs)

For $M_1 = 2.80$, from App. E.4, $M_2 = 0.468$ $T_2/T_1 = 2.451$ $P_2/P_1 = 8.980$

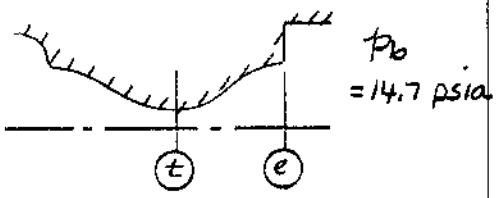
$\therefore P_2 = 8.98 P_1 = 8.98 \times 3.72$ kPa $= 33.4$ kPa (abs) P_2

$T_2 = 2.451 T_1 = 275$ K

$V_2 = M_2 c_2 = M_2 (kRT_2)^{1/2} = 0.468 \cdot 1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 275 K \times \frac{kg \cdot m}{N \cdot s^2})^{1/2} = 162$ m/s V_2



Given: C-D nozzle expanding air as shown.



$T_0 = 250°F$

$p_0 = 50.5$ psia

$A_t = 0.801$ in.^{2} $A_e = 0.917$ in.^{2}

Find: (a) Exit Mach number.
(b) Mass flow rate.

Solution: Use functions for steady, one-dimensional compressible flow.

Computing equations: A/A^*, p/p_0, and T/T_0 from isentropic (Appendix E.1)

Assumptions: (1) Steady flow
(2) Uniform flow at each section
(3) Isentropic if no shock
(4) Ideal gas
(5) $F_{Bx} = 0$
(6) $\Delta z = 0$

Check the exit condition: $\frac{A_e}{A^*} = \frac{A_e}{A_t} = \frac{0.917}{0.801} = 1.145 \rightarrow M_e = 1.452$ (App. E.1, Eq. 12.6)

Also from App. E.1, at $M = 1.452$, $\frac{p}{p_0} = 0.2919$; $p_e = 0.2919\, p_0 = 14.74$ psia.

Thus p_e is just slightly above $p_{atm} = 14.7$ psia; flow at exit is supersonic, so $M_t = 1.0$. From continuity

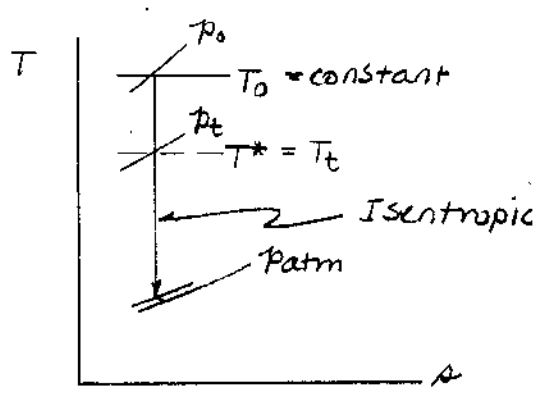
$$\dot{m} = \rho_t V_t A_t = \frac{p_t}{RT_t} M_t \sqrt{kRT_t}\, A_t = p_t \sqrt{\frac{k}{RT_t}}\, A_t$$

From App. E.1, at $M = 1$, $T/T_0 = 0.8333$ and $p/p_0 = 0.5283$, so

$$T_t = 0.8333 \times (460 + 250)°R = 592°R \quad \text{and} \quad p_t = 0.5283 \times 50.5 \text{ psia} = 26.7 \text{ psia}$$

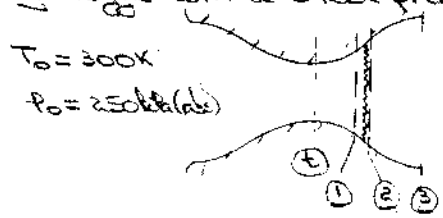
$$\dot{m} = 26.7 \frac{\text{lbf}}{\text{in.}^2} \left[1.4 \times \frac{16\text{m} \cdot °R}{53.3 \text{ ft} \cdot \text{lbf}} \times \frac{1}{592°R} \times \frac{\text{slug}}{32.2 \text{ lbm}} \times \frac{16\text{f} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} \right]^{1/2} 0.801 \text{ in.}^2 \times \frac{\text{slug} \cdot \text{ft}}{16\text{f} \cdot \text{s}^2} \times 32.2 \frac{\text{lbm}}{\text{slug}}$$

$$\dot{m} = 0.808 \text{ lbm/s}$$



{ Flow in the nozzle is just slightly underexpanded, since $p_{exit} > p_{back}$. }

Given: Steady, adiabatic air flow from a reservoir through a converging-diverging nozzle with a shock present.



Find: (a) M_2 (b) p_2 (c) $s_2 - s_1$ (d) Sketch T-s diagram

Solution: Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (2) uniform flow at each section
(3) ideal gas (4) isentropic flow, except across shock

At section 1: $p_0 = 250 kPa(abs)$, $p_1 = 68.1 kPa(abs)$ $\therefore p_1/p_0 = 0.2724$
From App. E.1 $M_1 = 1.50$

From App. E.4 with $M_1 = 1.50$, $M_2 = 0.701$ ← M_2

$p_2/p_1 = 2.458$ $\therefore p_2 = 167 kPa(abs)$ ← p_2

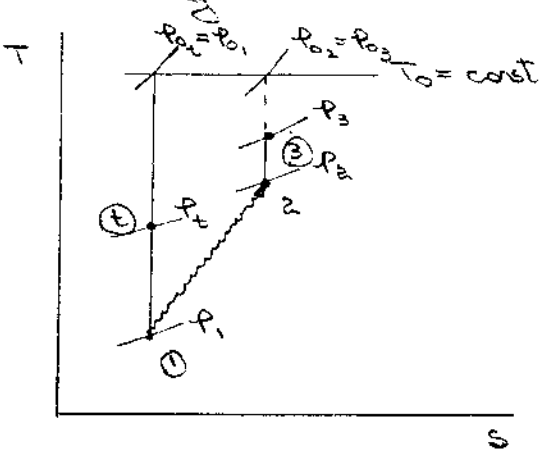
$p_{02}/p_{01} = 0.9298$

From the T-ds equation, $Tds = dh - vdp = c_p dT - RT\frac{dp}{p}$

$\therefore ds = c_p dT - R\frac{dp}{p}$

$s_2 - s_1 = s_{02} - s_{01} = c_p \ln \frac{T_{02}}{T_{01}} - R \ln \frac{p_{02}}{p_{01}}$

$s_2 - s_1 = -287 \frac{J}{kg \cdot K} \ln 0.9298 = 20.9 \frac{J}{kg \cdot K}$ ← $s_2 - s_1$



Given: Steady, adiabatic air flow from a reservoir through a converging-diverging nozzle.

$A_t = 1.0\ in^2$

$A_e = 1.58\ in^2$

$T_0 = 600°R$

$P_0 = 100\ psia$

p_0, T_0, $V_0 \simeq 0$

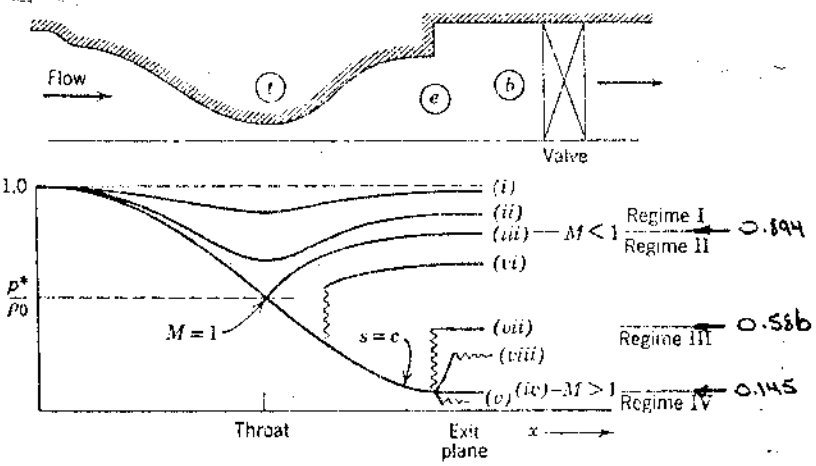


Fig. 12.20 *Pressure distributions for flow in a converging-diverging nozzle as a function of back pressure.*

Find: M_e at design conditions; P_b corresponding to regime boundaries of Fig 12.20, sketch $P(x)$

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (2) ideal gas
(3) uniform flow at a section
(4) isentropic flow except across a shock

For $A_e/A^* = 1.58$,

from Fig. E.1 and Eq 12.6, $M_e = 0.402$ $\quad P_e/P_0 = 0.8940$

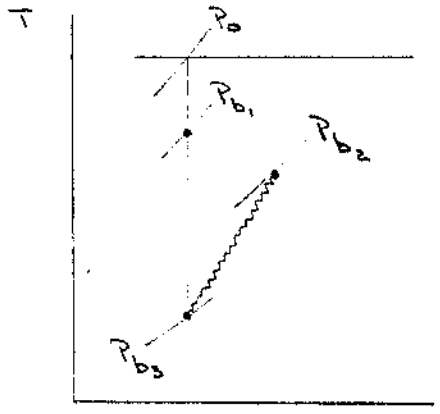
or $M_e = 1.92$ $\quad P_e/P_0 = 0.1447$ ← M_e

$P_{b_1} = \frac{P_{e_1}}{P_0} \times P_0 = 0.8940 \times 100\ psia = 89.4\ psia$ ← P_{b_1}

$P_{b_2} = \frac{P_{e_3}}{P_0} \times P_0 = 0.1447 \times 100\ psia = 14.5\ psia$ ← P_{b_3}

For $M_e = 1.92$, from App. E.4, $\frac{P_2}{P_1} = 4.045$

$P_{b_2} = \frac{P_2}{P_1} \times P_{e_3} = 4.045 \times 14.5\ psia = 58.6\ psia$ ← P_{b_2}



Given: Steady, adiabatic air flow from a reservoir through a converging-diverging nozzle; nozzle is designed to discharge to atmosphere.

$A_e/A^* = 4.0$

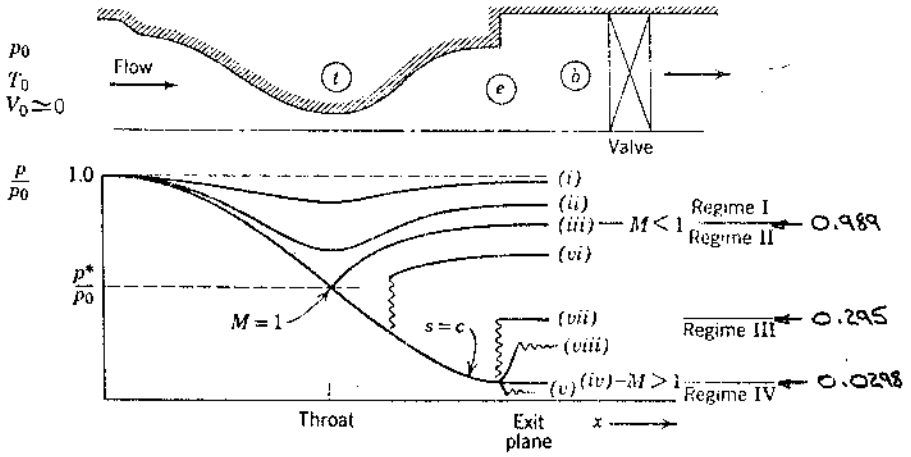


Fig. 12.20 *Pressure distributions for flow in a converging-diverging nozzle as a function of back pressure.*

Find: M_e at design conditions, P_0; P_b corresponding to regime boundaries of Fig 12.20, sketch $P(x)$

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (2) ideal gas
(3) uniform flow at a section
(4) isentropic flow except across a shock

For $A_e/A^* = 4.00$,

from Fig. E.1 and Eq. 12.6, $M_e = 2.94$ $P/P_0 = 0.0298$

or $M_e = 0.147$ $P/P_0 = 0.9887$

Then

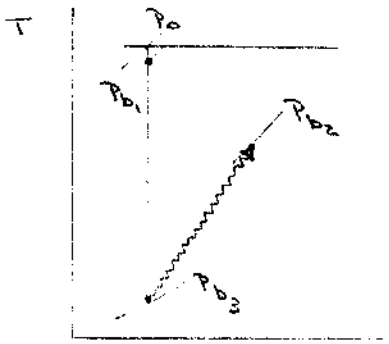
$M_{ed} = 2.94$ $P_{0d} = \frac{P_e}{0.0298} = \frac{101 \text{ kPa}}{0.0298} = 3.39 \text{ MPa (abs)}$ M_e, P_0

$P_{b_1} = \frac{P_{e_1}}{P_0} \times P_0 = 0.9887 \times 3.39 \text{ MPa} = 3.35 \text{ MPa abs}$ P_{b_1}

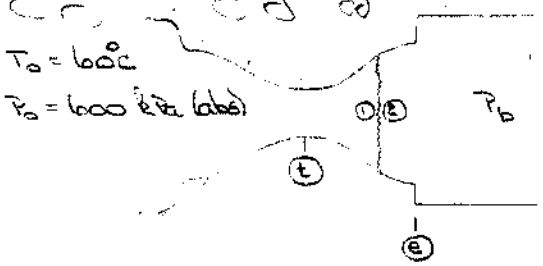
$P_{b_3} = 101 \text{ kPa (abs)}$ P_{b_3}

For $M_e = 2.94$ from App. E.4, $P_2/P_1 = 9.918$

$P_{b_2} = \frac{P_2}{P_1} \times P_{e_3} = 9.918 \times 101 \text{ kPa} = 1.00 \text{ MPa (abs)}$ P_{b_2}



Given: Steady, adiabatic air flow from a reservoir through a converging-diverging nozzle with a shock present



Find: P_b, sketch the pressure distribution.

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow
(2) uniform flow at a section
(3) ideal gas
(4) isentropic flow except across the shock

$M_1 = 2.42$ From App. E.1, $P/P_o = 0.01630$ $A_1/A_1^* = A_1/A_t = 2.448$

$\therefore P_1 = 39.8$ kPa (abs)

From App. E.4, $M_2 = 0.521$ $P_2/P_1 = 6.666$ $P_{o2}/P_{o1} = 0.5318$

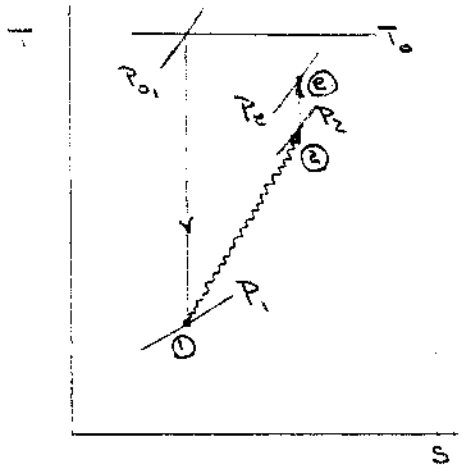
$\therefore P_2 = 265$ kPa (abs) $P_{o2} = 319$ kPa (abs)

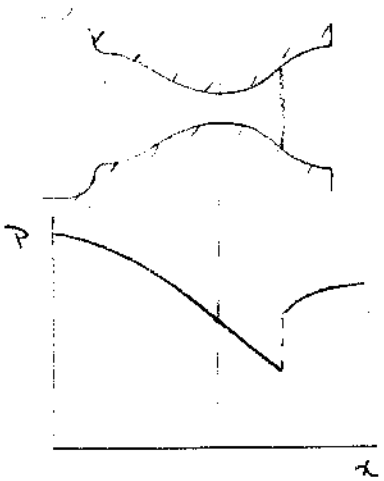
$M_2 = 0.521$ From App. E.1, $A_2/A_2^* = 1.301 = A_1/A_2^*$

Then $\dfrac{A_e}{A_2^*} = \dfrac{A_e}{A_1^*} \times \dfrac{A_1^*}{A_1} \times \dfrac{A_1}{A_2^*} = 4.0 \times \dfrac{1}{2.448} \times 1.301 = 2.126$

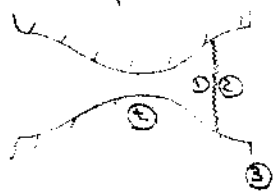
For $A_e/A^* = 2.126$, from Fig. E.1 and Eq. 12.6, $M_e = 0.284$, $P_e/P_o = 0.9447$

Then $P_b = P_e = 0.9447\, P_o = 0.9447 \times 319$ kPa $= 301$ kPa ← P_e





Given: Steady, adiabatic flow of air through a converging-diverging nozzle with a shock present



$A_1 = A_2 = 4.0\ in^2$
$M_1 = 2.50$
$T_{o_1} = 1000°R$
$P_{o_1} = 100\ psia$
$A_3 = 6.0\ in^2$

Find: P_3, $\dot{m}$, A_t

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (2) uniform flow at a section (3) ideal gas (4) isentropic flow except across the shock

$M_1 = 2.50$ From App. E.1, $T_1/T_{o_1} = 0.4444$ $P_1/P_{o_1} = 0.05853$ $A_1/A_1^* = 2.637$

$\therefore T_1 = 444°R$ $P_1 = 5.85\ psia$

$A_1^* = A_t = \frac{A_1}{2.637} = \frac{4.0\ in^2}{2.637} = 1.52\ in^2$ ← A_t

$\rho_1 = \frac{P_1}{RT_1} = 5.85 \frac{lbf}{in^2} \cdot \frac{lbm°R}{53.3\ ft\cdot lbf} \cdot \frac{1}{444°R} \cdot \frac{144\ in^2}{ft^2} = 0.03356\ lbm/ft^3$

$V_1 = M_1 c_1 = M_1 (\gamma RT_1)^{1/2} = 2.5 \left(1.4 \cdot 53.3 \frac{ft\cdot lbf}{lbm°R} \cdot 444°R \cdot 32.2 \frac{lbm}{slug} \cdot \frac{slug\cdot ft/s^2}{lbf}\right)^{1/2} = 2580\ ft/s$

$\dot{m} = \rho_1 V_1 A_1 = 0.03356 \frac{lbm}{ft^3} \cdot 2580 \frac{ft}{s} \cdot 4.0\ in^2 \cdot \frac{ft^2}{144\ in^2} = 2.55\ lbm/s$ ← $\dot{m}$

$M_1 = 2.50$ From App. E.4, $A_1^*/A_2^* = 0.513$ $P_{o_2}/P_{o_1} = 0.4990$

$\therefore P_{o_2} = 49.9\ psia$

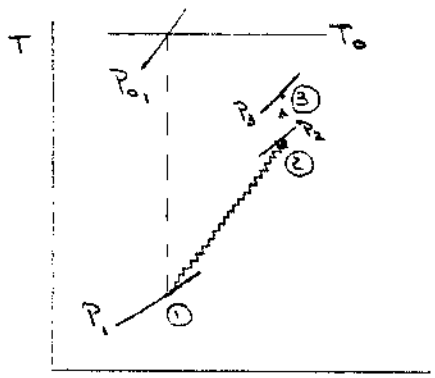
$M_2 = 0.513$ From App. E.1, $A_2/A_2^* = 1.316$

Since flow from ① → ② is isentropic, $A_3^* = A_2^*$ and $P_{o_3} = P_{o_2}$

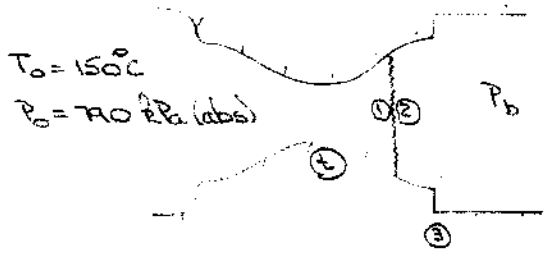
$$\frac{A_3}{A_3^*} = \frac{A_3}{A_2^*} = A_3 \cdot \frac{1.316}{A_2} = 6.0\ in^2 \cdot \frac{1.316}{4.0\ in^2} = 1.974$$

For $\frac{A_3}{A_3^*} = 1.974$, from Fig. E.1 and Eq. 12.6, $M_3 = 0.311$; then $P_3/P_{o_3} = 0.9355$

$\therefore P_3 = 0.9355\ P_{o_3} = 0.9355\ P_{o_2} = 46.7\ psia$ ← P_3



Given: Steady, adiabatic air flow from a reservoir through a converging-diverging nozzle; nozzle is designed to discharge to atmospheric pressure



Find: P_b, A_3, A_t

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (2) uniform flow at a section (3) ideal gas (4) isentropic flow except across the shock

At design conditions,

$P_3/P_o = 101/790 = 0.1278$. From App. E.1 $M_{3\,design} = 2.00$ $\frac{A_3}{A_t} = 1.688$

At section ①, $\frac{P_1}{P_o} = \frac{160}{790} = 0.2025$. From App. E.1, $M_1 = 1.70$ $A_1/A_1^* = 1.338$

$$A_t = A_1^* = \frac{A_1}{1.338} = \frac{600\ mm^2}{1.338} = 448\ mm^2$$ A_t

Then $A_3 = 1.688\ A_t = 1.688\,(448\ mm^2) = 756\ mm^2$ A_3

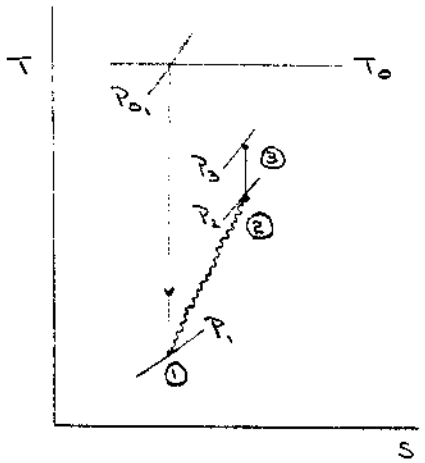
$M_1 = 1.70$ From App. E.1, $M_2 = 0.641$ $P_{o2}/P_{o1} = 0.8567$ $P_2/P_1 = 3.205$

$\therefore P_{o2} = 676$ kPa (abs) $P_2 = 513$ kPa (abs)

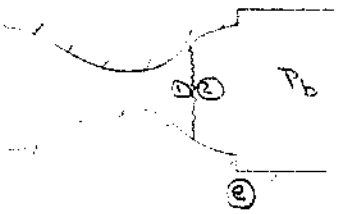
$M_2 = 0.641$ From App. E.1, $A_2/A_2^* = 1.145$ $\therefore A_2^* = \frac{A_2}{1.145} = 524\ mm^2$

$A_3/A_2^* = \frac{756}{524} = 1.443$. From Fig. E.1 and Eq 12.6, $M_3 = 0.453$; $\frac{P_3}{P_{o3}} = 0.8688$

$\therefore P_b = P_3 = 0.8688\,P_{o3} = 0.8688\,P_{o2} = 587$ kPa (abs) P_b



Given: Steady, adiabatic flow of air through a converging-diverging nozzle with a shock present; $P_b/P_{o_1} = 0.8200$. At design conditions $P_e/P_{o_1} = 0.1278$



$P_b/P_{o_1} = P_e/P_{o_1} = 0.8200$

Find: M_1

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: 1) steady flow 2) uniform flow at a section 3) ideal gas 4) isentropic flow except across the shock

At design conditions, $P_e/P_{o_1} = 0.1278$

From App. E.1, $M_d = 2.0$ and $A_e/A_1^* = 1.688$

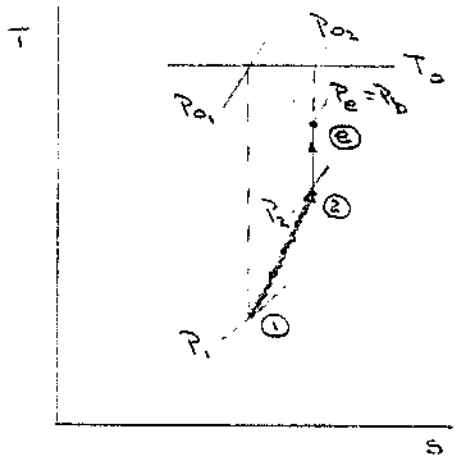
Now $\frac{P_e}{P_{o_1}} = \frac{P_e}{P_{o_2}} \cdot \frac{P_{o_2}}{P_{o_1}}$ and $\frac{A_e}{A_1^*} = \frac{A_e}{A_2^*} \cdot \frac{A_2^*}{A_1^*}$; $\frac{A_2^*}{A_1^*} = \frac{A_2^*}{A_x^*} \cdot \frac{A_x^*}{A_1^*}$

($\frac{P_e}{P_{o_2}}$: $f(M_e)$; $\frac{P_{o_2}}{P_{o_1}}$: $f(M_1)$; $\frac{A_e}{A_2^*}$: $f(M_e)$; $\frac{A_2^*}{A_1^*}$: $f(M_1)$)

We thus have a trial and error solution to determine M_1.

M_1	$\frac{A_1}{A_1^*}$ (a)	M_2 (b)	$\frac{P_{o_2}}{P_{o_1}}$ (c)	$\frac{A_2}{A_2^*}$ (d)	$\frac{A_2^*}{A_1^*}$	$\frac{A_e}{A_2^*}$	M_e (e)	$\frac{P_e}{P_{o_2}}$ (f)	$\frac{P_e}{P_{o_1}} = 0.8200$?
1.60	1.25	0.67	0.8952	1.119	1.117	1.511	0.42	0.8857	0.7927
1.40	1.150	0.74	0.9582	1.068	1.077	1.567	0.41	0.8707	0.8344
1.50	1.176	0.70	0.9298	1.094	1.075	1.570	0.403	0.8923	0.820 ✓

$\therefore M_1 = 1.50$ ← M_1

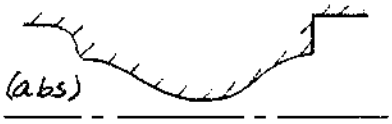


a. App. E.1 (12.6)
b. App E.4 (12.34b).
c. App E.4 (12.37)
d. App. E.1 (12.6).
e. App. E.1 Fig. E.1 ≈ Eq 12.6
f. App. E.1 (11.17a)

Given: Air flow through a converging-diverging nozzle. $A_e/A_t = 3.5$.

$T_0 = 15°C$

$p_0 = 101$ kPa (abs)



$A_t = 500\ mm^2$

Find: Range of back pressure for which a normal shock will occur in the nozzle, and the corresponding mass flowrate.

Solution: Use compressible flow functions in solution.

Computing equation: $\dot{m} = \rho V A$

Assumptions: (1) Steady flow (3) Uniform flow at a section
(2) Ideal gas (4) Isentropic, except across shock

A normal shock will occur within the nozzle for back pressure conditions in Regime II of Fig. 12.20. For isentropic flow, with $A_e/A_t = 3.5$, from Fig. E.1 and Eq. 12.6,

M	p/p_0	p_b
0.169	0.9858	99.6 kPa
2.80	0.03685	

From Appendix E.4,

M_1	M_2	p_2/p_1
2.80	0.4882	8.980

Thus $p_b = p_2 = p_0 \frac{p_1}{p_0} \frac{p_2}{p_1} = 101\ kPa\,(0.03685)(8.980) = 33.4\ kPa$

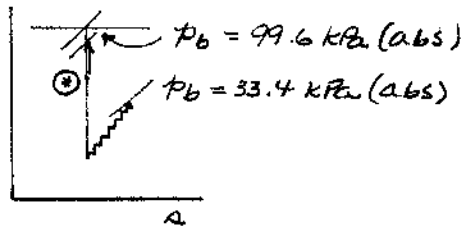
$33.4\ kPa < p_b < 99.6\ kPa\ (abs)$ (for normal shock in nozzle) p_b

Flow is choked throughout this regime. Thus

$$\dot{m} = \rho_t V_t A_t \qquad \rho_t = \frac{p_t}{RT_t} = (0.5283)\,1.01\times10^5 \frac{N}{m^2} \times \frac{kg\cdot K}{287\ N\cdot m} \times \frac{1}{(0.8333)288\ K} = 0.775 \frac{kg}{m^3}$$

$$V_t = c_t = \sqrt{kRT_t} = \left[1.4 \times 287 \frac{N\cdot m}{kg\cdot K}(0.8333)288\ K \times \frac{kg\cdot m}{N\cdot s^2}\right]^{\frac{1}{2}} = 311\ m/s$$

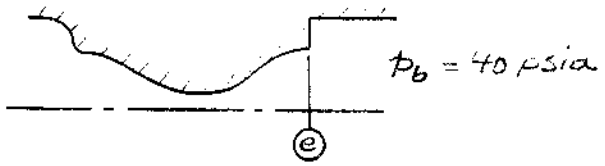
$$\dot{m} = 0.775 \frac{kg}{m^3} \times 311 \frac{m}{s} \times 500\ mm^2 \times \frac{m^2}{10^6\ mm^2} = 0.121 \frac{kg}{s}$$ $\dot{m}$



Given: Air flow through a converging-diverging nozzle. $A_e/A_t = 1.87$.

$T_0 = 240°F$

$p_0 = 100$ psia



Find: Mach number and flow velocity in exit plane.

Solution: Use compressible flow functions in solution. Assume ideal gas.

For isentropic flow through the nozzle to $A_e/A_t = 1.87$, from Fig. E.1 and Eq. 12.6,

M	T/T_0	p/p_0	p_d (psia)
2.12	0.5266	0.1060	10.6
~0.33	~0.98	~0.92	~92.0

Neither of these conditions matches the back pressure. Check the case of a shock (at $M = 2.12$) in the exit plane. From Appendix E.4,

M_1	M_2	p_2/p_1
2.12	0.5583	5.077

Then $p_e = 5.077\, p_d = 5.077\,(10.6 \text{ psia}) = 53.8$ psia.

The back pressure of 40 psia is therefore between the design pressure and the pressure that would exist downstream from a normal shock in the exit plane. The flow is in regime III of Fig. 12.20: supersonic in the exit plane with external compression. Thus

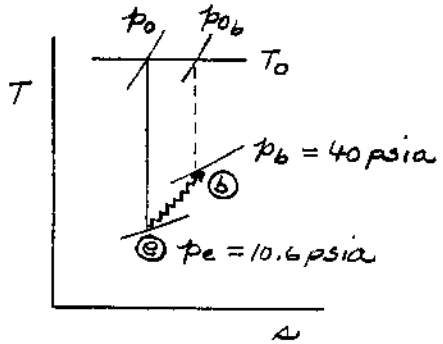
$$M_e = M_d = 2.12 \qquad \text{Mc}$$

$$V_e = M_e c_e = M_e \sqrt{kRT_e}$$

$$T_e = \frac{T}{T_0} T_0 = 0.5266\,(460 + 240)°R = 369°R$$

$$V_e = 2.12 \left[1.4 \times 53.3 \frac{\text{ft}\cdot\text{lbf}}{\text{lbm}\cdot°R} \times 369°R \times 32.2 \frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2} \right]^{\frac{1}{2}} = 2000 \frac{\text{ft}}{\text{s}} \qquad V_e$$

The Ts diagram is



Given: Converging-diverging nozzle with $A_e/A_t = 1.633$, designed to operate at atmospheric pressure at the exit plane.

Find: Ranges of stagnation pressure for which nozzle will be free from shocks.

Solution: Use compressible flow tables in solution.

Assume flow in nozzle is isentropic when shock-free. From Appendix E.1 and Eq. 12.6,

M	p/p_0	A/A^*
1.96	0.1360	1.633
0.38	0.9052	1.659
0.40	0.8956	1.590

At design conditions, $M_e = 1.96$, and

$$p_0 > \frac{p_e}{(p/p_0)_e} = \frac{101\ kPa}{0.1360} = 743\ kPa\ (abs)$$

By iteration, then the given area ratio corresponds to isentropic choked flow with $M_e = 0.388$ and $(p/p_0)_e = 0.9014$. The corresponding stagnation pressure is

$$p_0 = \frac{p_e}{(p/p_0)_e} = \frac{101\ kPa}{0.9014} = 112\ kPa\ (abs)$$

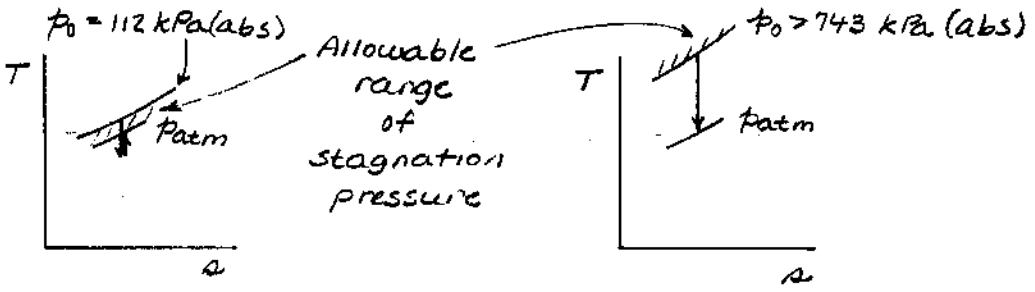
Flow will be isentropic and shock-free for

(a) $p_{atm} < p_0 < 112\ kPa\ (abs)$ $(0 < M_e < 0.388)$

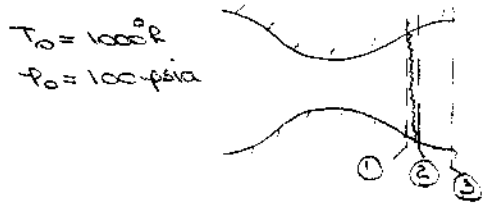
(b) $p_0 > 743\ kPa\ (abs)$ $(M_e = 1.96)$

p_0

The corresponding Ts diagrams are:



Given: Steady, adiabatic flow of air through a converging-diverging nozzle with shock in diverging section under conditions shown



Find: p_3

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (3) uniform flow at each section
(2) ideal gas (4) isentropic flow, except across shock.

For $M_1 = 2.0$, from App. E.1, $p_1/p_{o_1} = 0.1278$ $\therefore p_1 = 12.78$ psia.

For $M_1 = 2.10$, from App. E.4, $M_2 = 0.5774$
$p_2/p_1 = 4.50$ $\therefore p_2 = 57.5$ psia
$p_{o_2}/p_{o_1} = 0.7209$ $\therefore p_{o_2} = 72.1$ psia

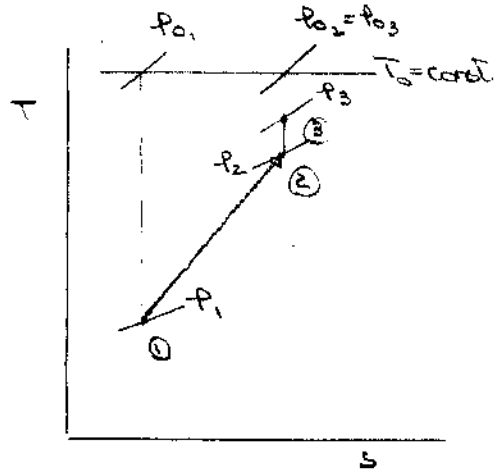
For $M_2 = 0.5774$, from App. E.1, $A_2/A_2^* = 1.216$ $\therefore A_2^* = 3.288$ in²

Then $A_3/A_2^* = 6.0/3.288 = 1.825$

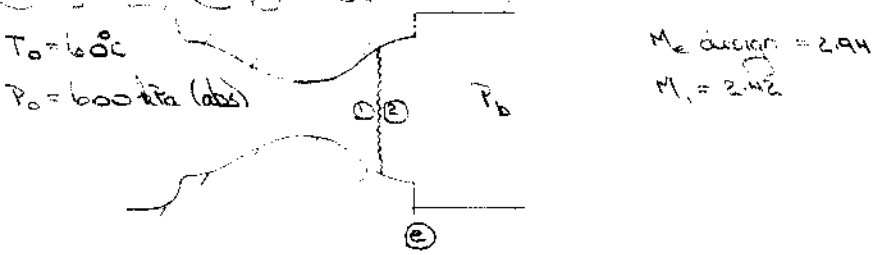
With $A_3/A^* = 1.825$, from App. E.1 (Fig. E.1 and Eq. 12.16) $M_3 = 0.340$

With $M_3 = 0.340$, from App. E.1, $p_{o_3}/p_3 = 1.083$

$$p_3 = \frac{p_{o_3}}{1.083} = \frac{p_{o_2}}{1.083} = \frac{72.1 \text{ psia}}{1.083} = 66.6 \text{ psia} \quad \underline{\quad p_3 \quad}$$



Given: Steady, adiabatic air flow from a reservoir through a converging-diverging nozzle with a shock present



Find: p_b, sketch the pressure distribution

Solution:

Compressible flow tables (Appendix E) to be used in solution

Assumptions: (1) steady flow (3) uniform flow at each section
(2) ideal gas (4) isentropic flow, except across shock

At design, $M_e = 2.94$, from Appendix E.1, $A_e/A_1^* = 3.999$

For $M_1 = 2.142$, from Appendix E.1, $p_1/p_{o_1} = 0.06630$
$A_1/A_1^* = 2.448$

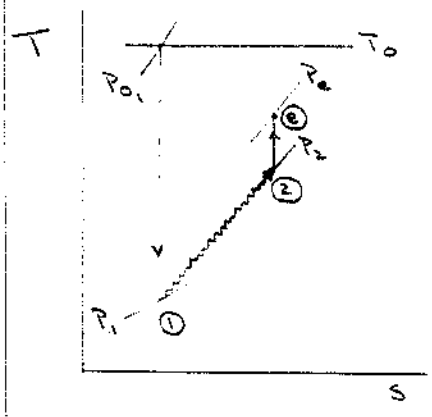
For $M_1 = 2.142$, from Appendix E.4, $M_2 = 0.521$
$p_{o_2}/p_{o_1} = 0.5318$
$p_2/p_1 = 6.666$

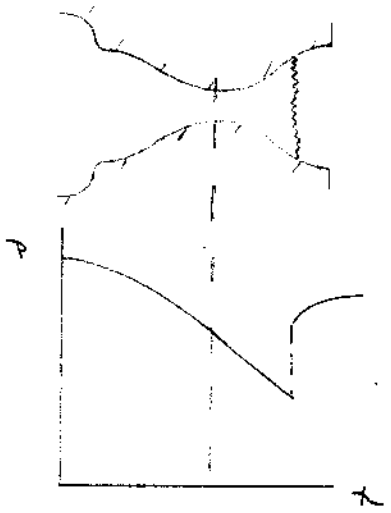
For $M_2 = 0.521$, from Appendix E.1, $A_2/A_2^* = A_1/A_2^* = 1.302$

Then, $\frac{A_e}{A_2^*} = \frac{A_e}{A_1^*} \times \frac{A_1^*}{A_1} \times \frac{A_1}{A_2^*} = 3.999 \times \frac{1}{2.448} \times 1.302 = 2.127$

For $A_e/A_2^* = 2.127$, from App. E.1 (Fig E.1 or Eq 12.6), $M_e = 0.2886$, $\frac{p_e}{p_{o_2}} = 0.9447$

$\therefore p_b = p_e = \frac{p_e}{p_{o_2}} \times \frac{p_{o_2}}{p_{o_1}} \times p_{o_1} = 0.9447 \times 0.5318 \times 600$ kPa $= 301$ kPa

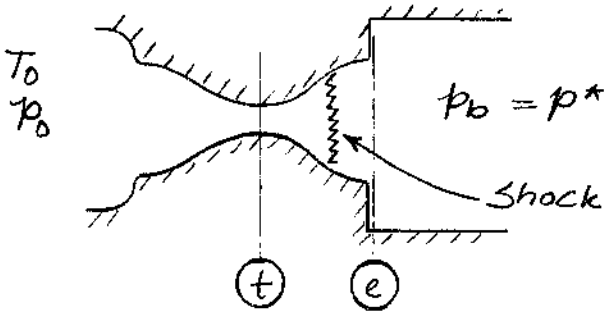


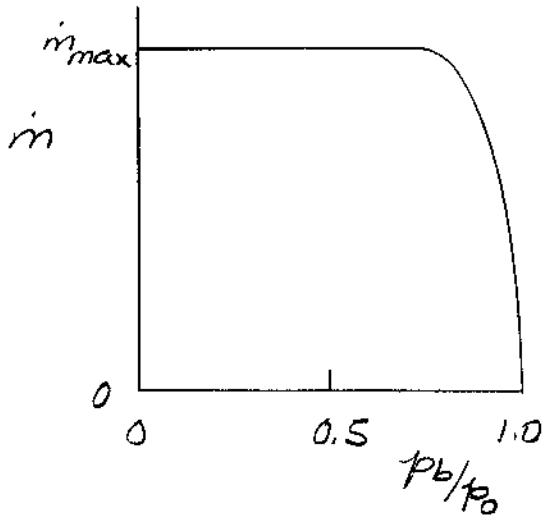


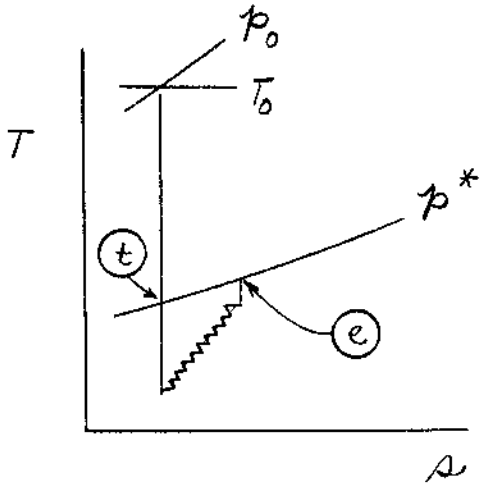
Given: Flow through a converging-diverging nozzle, $p_b - p^*$

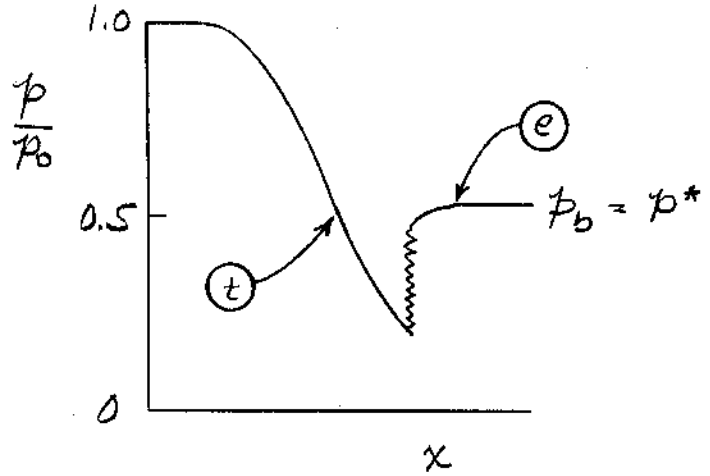
Find: Sketch (a) mass flow rate vs. pressure ratio
(b) pressure vs. distance along nozzle
(c) Ts diagram

Solution: When $p_b = p^*$, flow in the nozzle will be choked and a shock will stand in the diverging section.

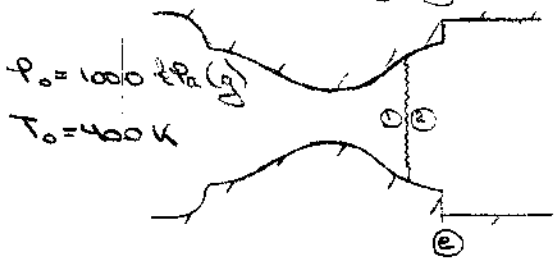








Given: Steady, adiabatic flow of air through a converging-diverging nozzle with shock in diverging section under conditions shown.



$M_1 = 3.0$
$A_1 = A_t = 500\ mm^2$
$A_e = 600\ mm^2$

Find: M_2, p_{o2}, p_2, A_t, $s_2 - s_1$, M_e

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (3) uniform flow at each section
(2) ideal gas (4) isentropic flow, except across shock

For $M_1 = 3.0$, from Appendix E.1, $p_1/p_o = 0.02722$ $\therefore p_1 = 29.97$ kPa
$A_1/A^* = 4.235$ $\therefore A^* = A_t = 118\ mm^2$ — A_t

For $M_1 = 3.0$, from Appendix E.4, $M_2 = 0.475$ — M_2
$p_{o2}/p_{o1} = 0.3283$ $\therefore p_{o2} = 361$ kPa (abs) — p_{o2}
$p_2/p_1 = 10.33$ $\therefore p_2 = 310$ kPa (abs) — p_2

Since $T_{o2} = T_{o1}$, then

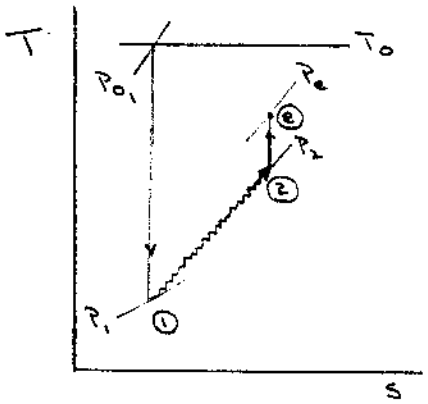
$$s_2 - s_1 = s_{o2} - s_{o1} = c_p \ln \frac{T_{o2}}{T_{o1}} - R \ln \frac{p_{o2}}{p_{o1}} = -287 \frac{N \cdot m}{kg \cdot K} \ln 0.3283$$

$$s_2 - s_1 = 0.320\ kJ/kg \cdot K$$ — $s_2 - s_1$

At $M_2 = 0.475$, from App. E.1 $A_2/A_2^* = 1.391$ $\therefore A_2^* = 359.5\ mm^2$

At exit $A_e/A_2^* = \frac{600}{359.5} = 1.669$. From App E.1 (Fig E.1) or Eq 12.6

$M_e = 0.377$ — M_e



The actual exit Mach number would be higher than the estimate based on isentropic flow downstream from the shock.

Flow downstream from the shock is subsonic. Flow slows in the diverging passage, which acts as a subsonic diffuser, causing pressure to increase in the direction of flow.

The result will be rapid growth of boundary layers on the channel walls. The boundary layers reduce the effective flow area of the passage. Because the boundary layers thicken rapidly, the area ratio for slowing the flow will be less than for isentropic flow. Therefore the actual flow will not slow as much as the isentropic model predicts.

The actual exit Mach number will be higher than the estimate based on isentropic flow.

Open-Ended Problem Statement: A supersonic wind tunnel must have two throats, with the second throat larger than the first. Explain why this must be so.

Discussion: The first throat is located in the supersonic nozzle from which flow enters the test section. The second throat is located in the supersonic diffuser that slows flow leaving the test section to subsonic speed for re-compression and re-circulation.

The second throat must be larger than the first for two reasons. First, it is impossible to slow a real flow to a Mach number of exactly one in a supersonic diffuser. The minimum Mach number that can be achieved with stable flow is about $M = 1.3$. Therefore even if the flow were isentropic everywhere the second throat would have to be larger than the first by the area ratio A/A^* corresponding to $M = 1.3$ at the throat.

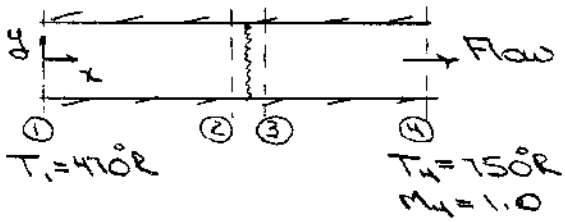
The second reason is that flow is not isentropic through the tunnel. Some friction exists, which must reduce the stagnation pressure of the flow stream. This also reduces the stagnation density. Therefore a larger area is needed to carry the mass flow at any given flow speed.

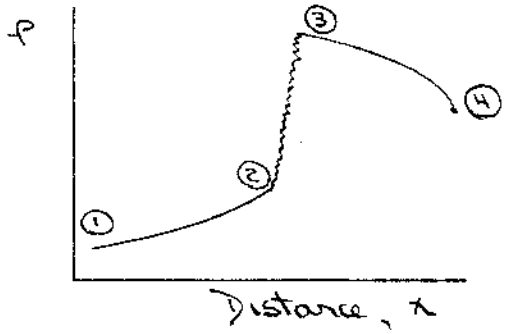
For these reasons the second throat area must be larger than the first throat area.

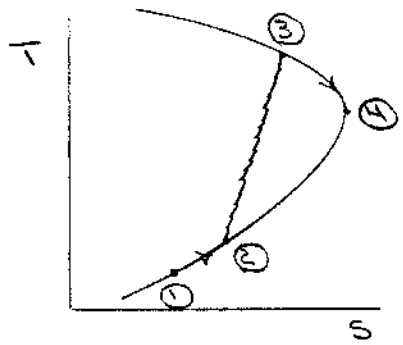
Given: A normal shock stands in a section of insulated constant-area duct; conditions immediately upstream and downstream of the shock are denoted by subscripts 2 and 3, respectively. Flow in the duct is frictional. Conditions at ① (some distance upstream) are $T_1 = 470°R$ and at ④ (some distance downstream) are $T_4 = 750°R$ and $M_4 = 1.0$

Find: (a) Sketch pressure distribution all the duct
(b) Sketch a Ts diagram
(c) Determine M_1

Solution:







For adiabatic flow, T_0 = constant (from energy eq.)

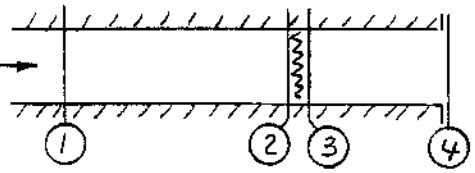
Thus, $T_{0_1} = T_{0_4} = T_4\left[1 + \frac{k-1}{2}M_4^2\right] = 750°R\,[1.2] = 900°R$

Since $T_{0_1} = T_1\left[1 + \frac{k-1}{2}M_1^2\right]$, then

$$M_1^2 = \frac{2}{k-1}\left[\frac{T_{0_1}}{T_1} - 1\right] = \frac{2}{0.4}\left[\frac{900R}{470R} - 1\right] = 4.57$$

$$M_1 = \sqrt{4.57} = 2.14$$ ← M_1

Given: Flow with shock in insulated constant-area duct, as shown.



Find: (a) Speed before shock, V_2.
(b) Entropy change, $s_4 - s_1$.

Solution: Use functions for steady, one-dimensional compressible flow.

Computing equations: $\frac{T}{T_0}$ from isentropic functions (Appendix E.1)

$\frac{p}{p^*}$ from Fanno-line functions (Appendix E.2)

Assumptions: (1) Steady flow
(2) Uniform flow at each cross-section
(3) Ideal gas
(4) Fanno line flow
(5) $F_{Bx} = 0$
(6) $\Delta z = 0$

For adiabatic flow on Fanno line and across shock, T_0 = constant. At $M_1 = 2.05$, $T/T_0 = 0.5433$ (Eq. 11.17b). Thus

$$T_0 = T_{01} = \frac{T_1}{(T/T_0)_1} = \frac{668°R}{0.5433} = 1230°R$$

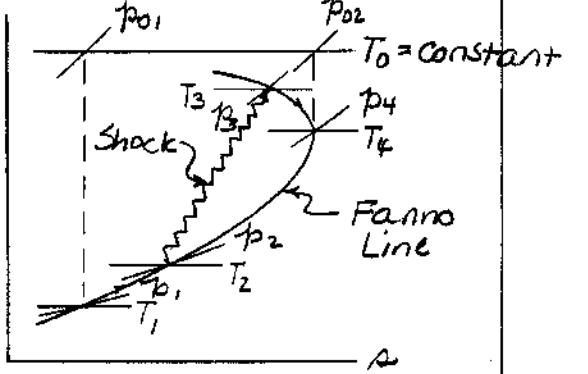
Using $T_2 = 388$ F $(848°R)$, $\frac{T}{T_0} = \frac{848°R}{1230°R} = 0.6894$ and $M_2 = 1.50$ (Table E.1).

$$V_2 = M_2 c_2 = M_2\sqrt{kRT_2} = 1.50\left[1.4 \times 53.3\frac{ft\cdot lbf}{lbm\,°R} \times 848°R \times 32.2\frac{lbm}{slug} \times \frac{slug\cdot ft}{lbf\cdot s^2}\right]^{\frac{1}{2}}$$

$$V_2 = 2140 \text{ ft/s} \qquad V_2$$

Flow must stay on the same Fanno line. Thus $(p_0/p_0^*)_1 = 1.1760$ at $M_1 = 2.05$ (Eq. 12.18e). Thus

$$p_0^* = \frac{p_{01}}{(p_0/p_0^*)_1} = \frac{78.2 \text{ psia}}{1.760} = 44.4 \text{ psia}$$



From the Gibbs equation

$$Tds = dh - vdp \; ; \; ds = c_p\frac{dT}{T} - R\frac{dp}{p}$$

$$\Delta s = c_p \ln\frac{T_{04}}{T_{01}}^{0} - R\ln\frac{p_{04}}{p_{01}}$$

$$\Delta s = -53.3\frac{ft\cdot lbf}{lbm\,°R} \times \ln\left(\frac{44.4}{78.2}\right) \times \frac{Btu}{778\ ft\cdot lbf} = 0.0388 \text{ Btu/lbm °R} \qquad \Delta s$$

{This problem could be solved without using functions, but the solution would require more calculations.}