Ejercicios de Álgebra II A

Práctica 1

6) $$\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1q} \\ \vdots & & \\ a_{p1} & \cdots & a_{pq} \end{pmatrix}}_{A\in\mathbb{R}^{p\times q}} \cdot \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix}}_{X\in\mathbb{R}^{q}} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_p \end{pmatrix}}_{b\in\mathbb{R}^{p}}$$

$\Rightarrow$ veo si $0_{\mathbb{R}^q} \in S \Rightarrow A.0 = b \Rightarrow b = 0_{\mathbb{R}^p}$ (esta es una condición)

$$X, Y \in S \Rightarrow A(X+Y) = b \Rightarrow \underbrace{\underbrace{A.X}_{b} + \underbrace{A.Y}_{b}}_{\text{pues } X \text{ e } Y \in S} = b \Rightarrow 2b = b \Rightarrow b = 0_{\mathbb{R}^p}$$

$X \in S\ ;\ \lambda \in \mathbb{K} \Rightarrow A.(\lambda.X) = b \Rightarrow \lambda.\underbrace{(A.X)}_{b} = b \Rightarrow \lambda.b = b \Rightarrow b = 0_{\mathbb{R}^p}$ (pues se debe cumplir $\forall \lambda$)

Entonces se debe cumplir $\boxed{b = 0_{\mathbb{R}^p}}$

7) - a) $S = \{X \in \mathbb{R}^3 \,/\, 2x_1 + x_2 + x_3 = 0 \ \wedge\ x_2 - x_1 = 0\}$

Es equivalente a $\underbrace{\begin{pmatrix} 2 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{0_{\mathbb{R}^2} \to b} \Rightarrow AX = 0 \rightarrow$ se debe cumplir

$\Rightarrow$ • $A.0_{\mathbb{R}^3} = 0_{\mathbb{R}^2}$ ✓ $\Rightarrow 0_{\mathbb{R}^3} \in S$

- $X, Y \in S \Rightarrow A.(X+Y) = 0 \Rightarrow \underbrace{A.X}_{0} + \underbrace{A.Y}_{0} = 0$ ✓ $\Rightarrow X+Y \in S$
- $X \in S\ ,\ \lambda \in \mathbb{K} \Rightarrow A.(\lambda.X) = 0 \Rightarrow \lambda.\underbrace{(A.X)}_{0} = 0$ ✓ $\lambda.X \in S$

S es subespacio de $\mathbb{R}^3$. (concuerda con la respuesta al ej. anterior)

b) $S = \{X \in \mathbb{R}^n \,/\, x_1 = x_2 = \cdots = x_r = 0\}$ ($x_1; \ldots; x_r$ son las coordenadas de X)

Es equivalente a decir que S es el vector nulo de $\mathbb{R}^n$.

todas las coord. de $X+Y$ son cero $\Downarrow$ $X+Y \in S$

- $0_{\mathbb{R}^n} \in S$ ✓ pues es el mismo vector.
- $X, Y \in S \Rightarrow X+Y = (x_1; \ldots; x_r) + (y_1; \ldots; y_r) = (\underbrace{x_1+y_1}_{0}; \underbrace{\ldots}_{0}; \underbrace{x_r+y_r}_{0})$ ✓

pues X e $Y \in$ a S y sus coordenadas son cero, entonces la suma de coordenadas también dará cero.

• $X \in S \,;\, \lambda \in \mathbb{K} \Rightarrow \lambda . X = \lambda . (X_1 ; \ldots ; X_r) = (\underset{\underset{0}{\underbrace{0}}}{\underset{\downarrow}{\lambda . X_1}} ; \underset{0}{\ldots} ; \underset{\underset{0}{\underbrace{0}}}{\underset{\downarrow}{\lambda . X_r}})$ ✓ $\lambda . X \in S$

S es subespacio de $\mathbb{R}^n$. $(A.\overset{\overset{S}{\uparrow}}{0} = b \Rightarrow b = 0$ se cumple también en este caso)

c) $S = \{X \in \mathbb{R}^4 / X_1 + X_2 = 0 \wedge X_1 - 2X_3 + X_4 = 1\}$

$\Rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -2 & 1 \end{pmatrix} . \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ esto es de la forma $A.X = b$ y ya se vio en el ej. anterior que un conjunto así sólo puede ser subespacio si $\underset{\substack{\downarrow \\ \text{vector} \\ \text{nulo}}}{b = 0}$, y en este caso $b \neq 0$.

S no es subespacio.

9) - a) $S = \{v \in \mathbb{R}^3 / v = \overset{\overset{\text{matriz}}{\nearrow}}{(}1 + r \quad r \quad 4r)^t \,;\, r \in \mathbb{R}\}$

Entonces $v = \begin{pmatrix} 1 + r \\ r \\ 4r \end{pmatrix}$

veo si $0_{\mathbb{R}^3} \in S \Rightarrow v = \begin{pmatrix} 1 + r \\ r \\ 4r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 1 + r = 0 \\ r = 0 \\ 4r = 0 \end{matrix}$ → este sistema no tiene solución, por lo tanto $0_{\mathbb{R}^3} \notin S$

S no es subespacio.

b) $S = \{v \in \mathbb{R}^2 / v = (r \;\, 2r)^t \,;\, r \geq 0\}$

$v = \begin{pmatrix} r \\ 2r \end{pmatrix} \Rightarrow \bullet \begin{pmatrix} r \\ 2r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow r = 0 \Rightarrow$ tiene solución, por lo tanto $0_{\mathbb{R}^2} \in S$ ✓

• $v, u \in S \Rightarrow v + u = \begin{pmatrix} r_1 \\ 2r_1 \end{pmatrix} + \begin{pmatrix} r_2 \\ 2r_2 \end{pmatrix} = \begin{pmatrix} r_1 + r_2 \\ 2.(r_1 + r_2) \end{pmatrix} = \begin{pmatrix} r' \\ 2.r' \end{pmatrix}$

y como $r \geq 0$, $r_1 + r_2$ debe ser ≥ 0 y $v + u \in S$ ✓

• $v \in S$ y $\lambda \in \mathbb{K} \Rightarrow \lambda . v = \lambda . \begin{pmatrix} r \\ 2r \end{pmatrix} = \begin{pmatrix} \lambda . r \\ 2\lambda . r \end{pmatrix} = \begin{pmatrix} r^* \\ 2.r^* \end{pmatrix}$ pero r^* sólo será $\geq 0 \Leftrightarrow \lambda \geq 0$

como $r^* \ngeq 0 \; \forall \lambda \Rightarrow \lambda . v \notin S$

S no es subespacio.

c) $S = \{A \in \mathbb{R}^{2\times 2} \,/\, A$ es singular$\}$

A es singular $\Leftrightarrow \det(A) = 0$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow \det(A) = a_{11}.a_{22} - a_{12}.a_{21} = 0$$

$$\Rightarrow \bullet\ 0.0 - 0.0 = 0 \Rightarrow \underset{\substack{\downarrow \\ \text{matriz} \\ \text{nula}}}{0_{\mathbb{R}^{2\times2}}} \in S \ ✓$$

$$\bullet\ A, B \in S \Rightarrow A+B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} \Rightarrow \det(A+B) = (a_{11}+b_{11}).(a_{22}+b_{22}) - (a_{12}+b_{12}).(a_{21}+b_{21}) =$$

$$= a_{11}.a_{22} + a_{11}.b_{22} + b_{11}.a_{22} + b_{11}.b_{22} - a_{12}.a_{21} - a_{12}.b_{21} - b_{12}.a_{21} - b_{12}.b_{21} =$$

$$= \underbrace{(a_{11}.a_{22} - a_{12}.a_{21})}_{0} + \underbrace{(b_{11}.b_{22} - b_{12}.b_{21})}_{0} + a_{11}.b_{22} + b_{11}.a_{22} - a_{12}.b_{21} - b_{12}.a_{21}$$

$$\Rightarrow a_{11}.b_{22} + b_{11}.a_{22} - a_{12}.b_{21} - b_{12}.a_{21} \neq 0 \Rightarrow A+B \notin S$$

↳ solo puede ser $= 0$ para ciertos A y B

S no es subespacio.

d) $S = \{P \in \mathcal{P}_3 \,/\, P(1) = P(2)\}$

- $0_{\mathcal{P}_3}(1) = 0_{\mathcal{P}_3}(2) = 0 \nearrow 0_{\mathcal{P}_3} \in S$ ✓ ; • $P, Q \in S \Rightarrow (P+Q)(1) = P(1) + Q(1) = P(2) + Q(2) =$

  ↳ polinomio nulo $= (P+Q)(2) \Rightarrow P+Q \in S$ ✓

- $P \in S$; $\lambda \in \mathbb{K} \Rightarrow (\lambda.P)(1) = \lambda.P(1) = \lambda.P(2) = (\lambda.P)(2) \Rightarrow \lambda.P \in S$ ✓

S es subespacio.

e) $S = \{f \in C\underbrace{[0,1]}_{\text{dominio}} \,/\, f(0) = f(1) = 0\}$

- $0_{C[0,1]}(0) = 0_{C[0,1]}(1) = 0 \Rightarrow 0_{C[0,1]} \in S$ ✓
- $f, g \in S \Rightarrow (f+g)(0) = f(0) + g(0) = f(1) + g(1) = 0 + 0 = 0 = (f+g)(1) \Rightarrow f+g \in S$ ✓
- $f \in S$; $\lambda \in \mathbb{K} \Rightarrow (\lambda.f)(0) = \lambda.f(0) = \lambda.f(1) = (\lambda f)(1) = \underset{\forall \lambda \in \mathbb{K}}{\lambda.0 = 0}$ ✓ $\Rightarrow \lambda.f \in S$ ✓

S es subespacio.

f) $S = \{ f \in C(\mathbb{R}) \,/\, [f(x)]^2 = f(x) \}$

- $[0_{C(\mathbb{R})}(x)]^2 = 0_{C(\mathbb{R})}(x) = 0 \Rightarrow 0_{C(\mathbb{R})} \in S$ ✓

- $f, g \in S \Rightarrow [(f+g)(x)]^2 = (f+g)(x) \Rightarrow [f(x) + g(x)]^2 = f(x) + g(x) \Rightarrow$

$\Rightarrow [f(x)]^2 + 2 \cdot f(x)\, g(x) + [g(x)]^2 = f(x) + g(x) \Rightarrow$ esto no se cumple $\forall f, g$, por lo tanto $f+g \notin S$.
(sólo se cumple para $f(x) = g(x) = 0$)

S no es subespacio.

11) $v_1 = (1\ -1\ 2)^t$, $v_2 = (-1\ 0\ 3)^t$; $v_3 = (0\ -1\ 5)^t$, $v_4 = (3\ -2\ 2)^t$

$$\Rightarrow \lambda_1 \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + \lambda_3 \cdot \begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix} + \lambda_4 \cdot \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbb{R}^3}$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 0 & 3 & x_1 \\ -1 & 0 & -1 & -2 & x_2 \\ 2 & 3 & 5 & 2 & x_3 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 0 & 3 & x_1 \\ 0 & -1 & -1 & 1 & x_1 + x_2 \\ 0 & -5 & -5 & 4 & 2x_1 - x_3 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 0 & 3 & x_1 \\ 0 & -1 & -1 & 1 & x_1 + x_2 \\ 0 & 0 & 0 & 1 & 3x_1 + 5x_2 + x_3 \end{array}\right) \Rightarrow \text{es un SCI}$$

∴ genera $\mathbb{R}^3$

otra forma de hacerlo es probando que hay sólo 3 vectores l.i.

$$\begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & 3 \\ 0 & -1 & 5 \\ 3 & -2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 5 \\ \cancel{0} & \cancel{-1} & \cancel{5} \\ 0 & -5 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & 21 \end{pmatrix} \rightarrow \text{son 3 l.i.}$$ ✓ (no usé la definición formal de dependencia lineal)

12) $(1\ k\ 0)^t$; $(1\ k-1\ k)^t$; $(2\ \ 2k-1\ \ k^2+k+1)^t$

$$\lambda_1 \cdot \begin{pmatrix} 1 \\ k \\ 0 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} 1 \\ k-1 \\ k \end{pmatrix} + \lambda_3 \cdot \begin{pmatrix} 2 \\ 2k-1 \\ k^2+k+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ k & k-1 & 2k-1 & 0 \\ 0 & k & k^2+k+1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & k & k^2+k+1 & 0 \end{array}\right)$$

⊛ para que sean l.d., esto no debe tener única solución ($\lambda_1 = \lambda_2 = \lambda_3 = 0$)

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -k^2-1 & 0 \end{array}\right) \circledast \Rightarrow -k^2 - 1 = 0 \Rightarrow k^2 = -1$$

Ningún $k \in \mathbb{R}$ cumple esa condición, ∴ $\nexists\, k \in \mathbb{R}$ tal que sean l.d.

13) - a) $\{(1 \quad i)^t, (i \quad -1)^t\}$

Como $\mathbb{R}$ espacio vectorial: $\lambda_1 . \begin{pmatrix} 1 \\ i \end{pmatrix} + \lambda_2 . \begin{pmatrix} i \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \lambda_1, \lambda_2 \in \mathbb{R}$

No puedo intentar triangular porque debería multiplicar una de las filas por un escalar complejo.

$$\Rightarrow \begin{matrix} \lambda_1 + i . \lambda_2 = 0 \\ i . \lambda_1 - \lambda_2 = 0 \end{matrix} \Rightarrow \lambda_2 = i . \lambda_1$$ pero λ_1 y $\lambda_2 \in \mathbb{R}$, entonces solo puede pasar que $\lambda_1 = \lambda_2 = 0$

($i . \lambda_1$ solo $\in \mathbb{R}$ si $\lambda_1 \in \mathbb{C}$ ó $\lambda_1 = 0$)

El conjunto es l.i.

Como $\mathbb{C}$ espacio vectorial:

$$\Rightarrow \left(\begin{array}{cc|c} 1 & i & 0 \\ i & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & i & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \Rightarrow SCI;$$ ∞ soluciones. El conjunto es l.d.

b) $\{2, 3 + t, 2 - t^2\}$ en P

$\lambda_1 . 2 + \lambda_2 . (3 + t) + \lambda_3 . (2 - t^2) = 0_P = 0$

$\Rightarrow \lambda_1 2 + \lambda_2 3 + \lambda_2 . t + \lambda_3 . 2 - \lambda_3 . t^2 = 0 \quad \forall t \in \mathbb{R}$

$$\Rightarrow \begin{cases} -\lambda_3 = 0 \\ \lambda_2 = 0 \\ 2\lambda_1 + 3\lambda_2 + 2\lambda_3 = 0 \end{cases} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$$ Es l.i.

c) $\{1; 2 + 2t; 1 - t + t^2; 2 - t^2\}$ en P.

$\lambda_1 . 1 + \lambda_2 . (2 + 2t) + \lambda_3 . (1 - t + t^2) + \lambda_4 . (2 - t^2) = 0 \Rightarrow$

$\Rightarrow \lambda_1 + 2\lambda_2 + 2t\lambda_2 + \lambda_3 - t . \lambda_3 + t^2 . \lambda_3 + 2 . \lambda_4 - t^2 . \lambda_4 = 0 \quad \forall t \in \mathbb{R} \Rightarrow$

$\Rightarrow (\lambda_3 - \lambda_4) . t^2 + (2\lambda_2 - \lambda_3) . t + \lambda_1 + 2\lambda_2 + \lambda_3 + 2\lambda_4 = 0 \quad \forall t \in \mathbb{R}$

$$\Rightarrow \begin{cases} \lambda_3 - \lambda_4 = 0 \\ 2\lambda_2 - \lambda_3 = 0 \\ \lambda_1 + 2\lambda_2 + \lambda_3 + 2\lambda_4 = 0 \end{cases}$$

Hay 3 ecuaciones y 4 incógnitas, por lo tanto hay ∞ soluciones $\Rightarrow$ es l.d.

d) $\{sen(x); cos(x)\}$ en $C(\mathbb{R})$

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$$\text{derivo} \begin{cases} \lambda_1 . sen(x) + \lambda_2 . cos(x) = 0 \\ \lambda_1 . cos(x) - \lambda_2 . sen(x) = 0 \end{cases} \Rightarrow W(sen(x), cos(x)) = \det \begin{pmatrix} sen(x) & cos(x) \\ cos(x) & -sen(x) \end{pmatrix} =$$

$= -sen^2(x) - cos^2(x) = -1 \neq 0 \Rightarrow$ El conjunto es l.i.

e) $\{ \text{sen}^2(x); \cos^2(x); 1 \}$ en $C(\mathbb{R})$

$\lambda_1 . \text{sen}^2(x) + \lambda_2 . \cos^2(x) + \lambda_3 = 0$

Si tomo $\lambda_1 = -1 = \lambda_2$; $\lambda_3 = 1 \Rightarrow -1.(\overbrace{\text{sen}^2(x) + \cos^2(x)}^{1}) + 1 = 0$

$\therefore \lambda_1 = \lambda_2 = \lambda_3 = 0$ no es la única solución y el conjunto es l.d.

14)

1°- dim = n
2°- dim = 2n ¿?

3°- dim = p.q

4°- dimensión no finita
5°- " "

16) - a) $S = \{ X \in \mathbb{R}^4 / x_1 + 2x_2 + x_4 = 0 \land 2x_1 - x_2 + x_4 = 0 \}$

$\Rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 & 0 \end{array}\right) \Rightarrow \begin{cases} x_1 + 2x_2 + x_4 = 0 \\ 5x_2 + x_4 = 0 \end{cases} \Rightarrow \begin{matrix} x_1 = 3x_2 \\ x_4 = -5x_2 \end{matrix}$

$\Rightarrow X = \begin{pmatrix} 3x_2 \\ x_2 \\ x_3 \\ -5x_2 \end{pmatrix} = x_2 . \begin{pmatrix} 3 \\ 1 \\ 0 \\ -5 \end{pmatrix} + x_3 . \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

Base $= \boxed{\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ -5 \end{pmatrix} ; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}}$ $\Rightarrow$ Pongo los vectores en fila y triangulo $\Rightarrow \begin{pmatrix} 3 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ $\left.\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\}$ al agregar estos 2 vectores no se va ninguna fila, lo cual prueba que no son c.l. de los otros 2 y los 4 son l.i. entre sí.

Base de $\mathbb{R}^4 = \boxed{\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ -5 \end{pmatrix} ; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}}$

b) $S = \{ P \in P_3 / P(0) = P(1) \}$

$P(x) = a_3 . x^3 + a_2 . x^2 + a_1 . x + a_0$

$\Rightarrow a_3 . 1^3 + a_2 . 1^2 + a_1 . 1 + \cancel{a_0} = a_3 . 0^3 + a_2 . 0^2 + a_1 . 0 + \cancel{a_0} \Rightarrow a_3 + a_2 + a_1 = 0 \Rightarrow$

$\Rightarrow a_1 = -a_3 - a_2 \Rightarrow a_3 . x^3 + a_2 . x^2 + (-a_3 - a_2) . x + a_0 =$

$= a_3 . (x^3 - x) + a_2 (x^2 - x) + a_0 . 1 \Rightarrow$ base $= \underbrace{\boxed{\{ x^3 - x ; x^2 - x ; 1 \}}}_{\dim 3}$

Nota: el E.V. P_3 tiene dim $= 3 + 1 = 4$

Un elemento que no es c.l. de los demás es $x \Rightarrow$ base de $P_3 = \boxed{\{ 1 ; x ; x^2 - x ; x^3 - x \}}$

c) $S=\{P\in P_4 / \int_0^1 P(x)\,dx=0\}$

$P(x)=a_4.x^4+a_3.x^3+a_2.x^2+a_1.x+a_0$

$$\int_0^1 P(x)\,dx=\left(a_4.\frac{x^5}{5}+a_3.\frac{x^4}{4}+a_2.\frac{x^3}{3}+a_1.\frac{x^2}{2}+a_0.x\right)\Bigg|_0^1=$$

$$=\frac{a_4}{5}+\frac{a_3}{4}+\frac{a_2}{3}+\frac{a_1}{2}+a_0=0 \Rightarrow a_0=-\frac{a_4}{5}-\frac{a_3}{4}-\frac{a_2}{3}-\frac{a_1}{2}$$

$$\Rightarrow P(x)=a_4.x^4+a_3.x^3+a_2.x^2+a_1.x-\frac{a_4}{5}-\frac{a_3}{4}-\frac{a_2}{3}-\frac{a_1}{2}=$$

$$=a_4.\left(x^4-\frac{1}{5}\right)+a_3.\left(x^3-\frac{1}{4}\right)+a_2.\left(x^2-\frac{1}{3}\right)+a_1.\left(x-\frac{1}{2}\right)$$

$$Base=\boxed{\left\{x-\frac{1}{2};\ x^2-\frac{1}{3};\ x^3-\frac{1}{4};\ x^4-\frac{1}{5}\right\}}$$

Es seguro que "1" no es c.l. de los otros elementos.

$$Base\ de\ P_4=\boxed{\left\{1;\ x-\frac{1}{2};\ x^2-\frac{1}{3};\ x^3-\frac{1}{4};\ x^4-\frac{1}{5}\right\}}$$

d) $S=\{A\in\mathbb{R}^{3\times3} / A=A^t\}$

$$\Rightarrow\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}=\begin{pmatrix}a_{11}&a_{21}&a_{31}\\a_{12}&a_{22}&a_{32}\\a_{13}&a_{23}&a_{33}\end{pmatrix}\Rightarrow\begin{matrix}a_{12}=a_{21}\\a_{13}=a_{31}\\a_{23}=a_{32}\end{matrix}$$

$$\Rightarrow A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{12}&a_{22}&a_{23}\\a_{13}&a_{23}&a_{33}\end{pmatrix}=a_{11}.\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}+a_{12}.\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}+a_{13}.\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}+$$

$$+a_{22}.\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}+a_{23}.\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}+a_{33}.\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$$

$$Base=\boxed{\left\{\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix};\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix};\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix};\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix};\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix};\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right\}}$$

17) $\begin{pmatrix}1&-1\\-1&2\end{pmatrix};\begin{pmatrix}-1&2\\3&1\end{pmatrix};\begin{pmatrix}2&-3\\-3&2\end{pmatrix};\begin{pmatrix}1&1\\1&6\end{pmatrix}$

Pongo las matrices en filas para armar otra matriz y triangular:

$$\begin{pmatrix}1&-1&-1&2\\-1&2&3&1\\2&-3&-3&2\\1&1&1&6\end{pmatrix}\rightarrow\begin{pmatrix}1&-1&-1&2\\0&1&2&3\\0&1&1&2\\0&-2&-2&-4\end{pmatrix}\longrightarrow\begin{pmatrix}1&-1&-1&2\\0&1&2&3\\0&0&1&1\\\cancel{0}&\cancel{0}&\cancel{2}&\cancel{2}\end{pmatrix}\begin{matrix}\left.\begin{matrix}\\\\\\\end{matrix}\right\}l.i.\\\rightarrow\text{es una c.l. de los demás.}\end{matrix}$$

Por lo tanto, una base posible es: $\boxed{B=\left\{\begin{pmatrix}1&-1\\-1&2\end{pmatrix};\begin{pmatrix}-1&2\\3&1\end{pmatrix};\begin{pmatrix}2&-3\\-3&2\end{pmatrix}\right\}}$

18) - I) $A = \begin{pmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{pmatrix}$

$Col(A) = \left\{ \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}; \begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix}; \begin{pmatrix} 9 \\ -4 \\ 10 \end{pmatrix}, \begin{pmatrix} -7 \\ 1 \\ 7 \end{pmatrix} \right\}$ pero no es base.

$$\begin{pmatrix} 1 & -1 & 5 \\ -4 & 2 & -6 \\ 9 & -4 & 10 \\ -7 & 1 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 5 \\ 0 & -2 & 14 \\ 0 & -5 & 35 \\ 0 & -6 & 42 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 5 \\ 0 & -2 & 14 \\ \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix} \Rightarrow \boxed{Col(A) = \left\{ \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}; \begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix} \right\}}$$

$$Fil(A) = \left\{ \begin{pmatrix} 1 \\ -4 \\ 9 \\ -7 \end{pmatrix}; \begin{pmatrix} -1 \\ 2 \\ -4 \\ 1 \end{pmatrix}; \begin{pmatrix} 5 \\ -6 \\ 10 \\ 7 \end{pmatrix} \right\} \Rightarrow \begin{pmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & -14 & 35 & -42 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix} \circledast$$

$$\boxed{Fil(A) = \left\{ \begin{pmatrix} 1 \\ -4 \\ 9 \\ -7 \end{pmatrix}; \begin{pmatrix} -1 \\ 2 \\ -4 \\ 1 \end{pmatrix} \right\}}$$

$$Nul(A) = \{ X \in \mathbb{R}^4 / A.X = 0_{\mathbb{R}^3} \} \Rightarrow \left(\begin{array}{cccc|c} 1 & -4 & 9 & -7 & 0 \\ -1 & 2 & -4 & 1 & 0 \\ 5 & -6 & 10 & 7 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & -4 & 9 & -7 & 0 \\ 0 & -2 & 5 & -6 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \Rightarrow$$

$$\Rightarrow \begin{cases} x_1 - 4x_2 + 9x_3 - 7x_4 = 0 \\ \quad -2x_2 + 5x_3 - 6x_4 = 0 \end{cases} \Rightarrow \begin{array}{l} x_1 = 10x_3 - 12x_4 - 9x_3 + 7x_4 = x_3 - 5x_4 \\ x_2 = \frac{5}{2}x_3 - 3x_4 \end{array}$$

$$X = \begin{pmatrix} x_3 - 5x_4 \\ \frac{5}{2}x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \cdot \begin{pmatrix} 1 \\ \frac{5}{2} \\ 1 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} -5 \\ -3 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \boxed{Nul(A) = \left\{ \begin{pmatrix} 1 \\ \frac{5}{2} \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -5 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}}$$

$$A^t = \begin{pmatrix} 1 & -1 & 5 \\ -4 & 2 & -6 \\ 9 & -4 & 10 \\ -7 & 1 & 7 \end{pmatrix} \Rightarrow Nul(A^t) = \{ X \in \mathbb{R}^3 / A^t . X = 0_{\mathbb{R}^4} \}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 5 & 0 \\ -4 & 2 & -6 & 0 \\ 9 & -4 & 10 & 0 \\ -7 & 1 & 7 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 5 & 0 \\ 0 & -2 & 14 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \Rightarrow \begin{cases} x_1 - x_2 + 5x_3 = 0 \\ \quad -2x_2 + 14x_3 = 0 \end{cases} \Rightarrow \begin{array}{l} x_1 = 7x_3 - 5x_3 = 2x_3 \\ x_2 = 7x_3 \end{array}$$

$\boxed{Nul(A^t) = \left\{ \begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix} \right\}}$; $rg(A) = \dim col(A) = \boxed{2}$ nº de columnas l.i.

$rg(A^t) = \boxed{2}$ $\circledast$

II)

$$A=\begin{pmatrix}1&-1&1\\-2&1&0\\3&1&-4\\1&-2&0\end{pmatrix}$$

$$\boxed{Col(A)=\left\{\begin{pmatrix}1\\-2\\3\\1\end{pmatrix};\begin{pmatrix}-1\\1\\1\\-2\end{pmatrix};\begin{pmatrix}1\\0\\-4\\0\end{pmatrix}\right\}} \Rightarrow \begin{pmatrix}1&-2&3&1\\-1&1&1&-2\\1&0&-4&0\end{pmatrix}\rightarrow\begin{pmatrix}1&-2&3&1\\0&-1&4&-1\\0&-2&7&1\end{pmatrix}\rightarrow\begin{pmatrix}1&-2&3&1\\0&-1&4&-1\\0&0&1&-3\end{pmatrix}\rightarrow \text{son l.i.}$$

$$Fil(A)=\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix};\begin{pmatrix}-2\\1\\0\end{pmatrix};\begin{pmatrix}3\\1\\-4\end{pmatrix};\begin{pmatrix}1\\-2\\0\end{pmatrix}\right\} \Rightarrow \begin{pmatrix}1&-1&1\\-2&1&0\\3&1&-4\\1&-2&0\end{pmatrix}\rightarrow\begin{pmatrix}1&-1&1\\0&-1&2\\0&-4&7\\0&1&1\end{pmatrix}\rightarrow\begin{pmatrix}1&-1&1\\0&-1&2\\0&0&1\\0&0&3\end{pmatrix}\rightarrow\left.\begin{pmatrix}1&-1&1\\0&-1&2\\0&0&1\\\cancel{0}&\cancel{0}&\cancel{0}\end{pmatrix}\right\}\text{l.i.}$$

$$\boxed{Fil(A)=\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix},\begin{pmatrix}-2\\1\\0\end{pmatrix},\begin{pmatrix}3\\1\\-4\end{pmatrix}\right\}}$$

$$Nul(A)=\{X\in\mathbb{R}^3 \,/\, A.X=0_{\mathbb{R}^4}\}$$

$$\Rightarrow \begin{pmatrix}1&-1&1\\-2&1&0\\3&1&-4\\1&-2&0\end{pmatrix}\cdot\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\Rightarrow\left(\begin{array}{ccc|c}1&-1&1&0\\-2&1&0&0\\3&1&-4&0\\1&-2&0&0\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}1&-1&1&0\\0&-1&2&0\\0&0&1&0\\\cancel{0}&\cancel{0}&\cancel{0}&\cancel{0}\end{array}\right)$$

es un SCD,
por lo tanto la única solución es $(0;0;0)$

$$\boxed{Nul(A)=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}}\ ;\ A^t=\begin{pmatrix}1&-2&3&1\\-1&1&1&-2\\1&0&-4&0\end{pmatrix}$$

$$Nul(A^t)=\{X\in\mathbb{R}^4 \,/\, A^t.X=0_{\mathbb{R}^3}\} \Rightarrow \left(\begin{array}{cccc|c}1&-2&3&1&0\\-1&1&1&-2&0\\1&0&-4&0&0\end{array}\right)\rightarrow\left(\begin{array}{cccc|c}1&-2&3&1&0\\0&-1&4&-1&0\\0&0&1&-3&0\end{array}\right)$$

$$\Rightarrow\begin{cases}X_1-2X_2+3X_3+X_4=0\\ -X_2+4X_3-X_4=0\\ X_3-3X_4=0\end{cases}\Rightarrow\begin{array}{l}X_1=22X_4-9X_4-X_4=12X_4\\X_2=12X_4-X_4=11X_4\\X_3=3X_4\end{array}$$

$$\boxed{Nul(A^t)=\left\{\begin{pmatrix}12\\11\\3\\1\end{pmatrix}\right\}}\ ;\ \begin{array}{l}rg(A)=\dim Col(A)=\boxed{3}\\rg(A^t)=\boxed{3}\end{array}$$

III)

$$A=\begin{pmatrix}-1&1&0&2\\1&1&-4&1\end{pmatrix}\Rightarrow Col(A)=\left\{\begin{pmatrix}-1\\1\end{pmatrix};\begin{pmatrix}1\\1\end{pmatrix};\begin{pmatrix}0\\-4\end{pmatrix};\begin{pmatrix}2\\1\end{pmatrix}\right\}$$

$$\Rightarrow\begin{pmatrix}-1&1\\1&1\\0&-4\\2&1\end{pmatrix}\rightarrow\begin{pmatrix}-1&1\\0&2\\0&-4\\0&3\end{pmatrix}\rightarrow\left.\begin{pmatrix}-1&1\\0&2\\\cancel{0}&\cancel{0}\\\cancel{0}&\cancel{0}\end{pmatrix}\right\}\text{l. i.}\Rightarrow\boxed{Col(A)=\left\{\begin{pmatrix}-1\\1\end{pmatrix};\begin{pmatrix}1\\1\end{pmatrix}\right\}}$$

$$\boxed{fil(A)=\left\{\begin{pmatrix}-1\\1\\0\\2\end{pmatrix};\begin{pmatrix}1\\1\\-4\\1\end{pmatrix}\right\}}$$ (son l.i. porque no son múltiplos)

$Nul(A)=\{X\in\mathbb{R}^4 / AX=0_{\mathbb{R}^2}\}$

$$\Rightarrow\left(\begin{array}{cccc|c}-1&1&0&2&0\\1&1&-4&1&0\end{array}\right)\rightarrow\left(\begin{array}{cccc|c}-1&1&0&2&0\\0&2&-4&3&0\end{array}\right)\Rightarrow\begin{cases}-X_1+X_2+2X_4=0\\2X_2-4X_3+3X_4=0\end{cases}$$

$$\Rightarrow\begin{matrix}X_1=X_2+2X_4\\X_3=\frac{1}{2}X_2+\frac{3}{4}X_4\end{matrix}\Rightarrow X=\begin{pmatrix}X_2+2X_4\\X_2\\\frac{1}{2}X_2+\frac{3}{4}X_4\\X_4\end{pmatrix}\Rightarrow\boxed{Nul(A)=\left\{\begin{pmatrix}1\\1\\\frac{1}{2}\\0\end{pmatrix};\begin{pmatrix}2\\0\\\frac{3}{4}\\1\end{pmatrix}\right\}}$$

$$A^t=\begin{pmatrix}-1&1\\1&1\\0&-4\\2&1\end{pmatrix}\Rightarrow Nul(A^t)=\{X\in\mathbb{R}^2 / AX=0_{\mathbb{R}^4}\}$$

$$\Rightarrow\left(\begin{array}{cc|c}-1&1&0\\1&1&0\\0&-4&0\\2&1&0\end{array}\right)\rightarrow\left(\begin{array}{cc|c}-1&1&0\\0&2&0\\\cancel{0}&\cancel{0}&\cancel{0}\\\cancel{0}&\cancel{0}&\cancel{0}\end{array}\right)\Rightarrow\begin{cases}-X_1+X_2=0\\2X_2=0\end{cases}$$

$X_1=X_2=0$

$\boxed{Nul(A^t)=\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}}$; $rg(A)=\boxed{2}$; $rg(A^t)=\boxed{2}$

1) - a) Si $A.X=b \Rightarrow b=X_1.u_1+X_2.u_2+\cdots+X_m.u_m$ es decir que es una combinación lineal de los elementos de $col(A)=\{u_1;u_2;\ldots;u_m\}$ $\therefore b\in col(A)$.

b) b es una c.l. de las columnas de A; si estas son l.i. entre sí, los coeficientes que forman la c.l. son únicos y $Ax=b$ tiene una sola solución.

c) mismo razonamiento que b), $rango(A)=m$ quiere decir que tiene m columnas l.i., lo que es equivalente a decir que todas las columnas de A son l.i.

d) $AX=0$ tiene como única solución el vector nulo si las columnas de A son l.i., pues $A.X=X_1.u_1+X_2.u_2+\cdots+X_m u_m=0$. Si las columnas son l.d. quedará un SCI que tendrá ∞ soluciones además de la trivial.

2) $S_1=gen\{(1\ 1\ 2\ 0)^t;(2\ 0\ 3\ -1)^t\}$; $S_2=\{X\in\mathbb{R}^4 / x_1+x_2+x_3+x_4=0\}$

Para $S_1\cap S_2$ puedo tomar, por ej., los vectores de S_1 y ver que cumplan la ec. de S_2:

Los vectores de S_1 son de la forma $v=\alpha.\begin{pmatrix}1\\1\\2\\0\end{pmatrix}+\beta.\begin{pmatrix}2\\0\\3\\-1\end{pmatrix}=\begin{pmatrix}\alpha+2\beta\\\alpha\\2\alpha+3\beta\\-\beta\end{pmatrix}$

Si además $v\in S_2\Rightarrow\alpha+2\beta+\alpha+2\alpha+3\beta-\beta=0\Rightarrow4\alpha+4\beta=0\Rightarrow\boxed{\beta=-\alpha}$

$$\Rightarrow v\in S_1\cap S_2 \text{ cumple } v=\begin{pmatrix}-\alpha\\\alpha\\-\alpha\\\alpha\end{pmatrix}=\alpha.\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}\Rightarrow\boxed{B_{S_1\cap S_2}=\left\{\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}\right\}}$$

Paso S_2 a generadores: $x_1 = -x_2 - x_3 - x_4$

$$\Rightarrow v \in S,\ v = \begin{pmatrix} -x_2-x_3-x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow S = \text{gen}\{(-1\ 1\ 0\ 0)^t; (-1\ 0\ 1\ 0)^t; (-1\ 0\ 0\ 1)^t\}$$

$$\Rightarrow S_1 + S_2 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 3 \\ -1 \end{pmatrix}; \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow \begin{pmatrix} 1 & 1 & 2 & 0 \\ 2 & 0 & 3 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 3 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 4 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{4} \end{pmatrix}$$ → son 4 l.i., por lo tanto $\dim(S_1+S_2) = 4$ y como están en $\mathbb{R}^4$, necesariamente $S_1 + S_2 = \mathbb{R}^4$

$$\Rightarrow B_{S_1+S_2} = \boxed{\{(1\ 0\ 0\ 0)^t; (0\ 1\ 0\ 0)^t; (0\ 0\ 1\ 0)^t; (0\ 0\ 0\ 1)^t\}} = \mathbb{R}^4$$

↓ es más simple usar la base canónica

22) $\boxed{\dim(S_1+S_2) = \dim S_1 + \dim S_2 - \dim(S_1 \cap S_2)}$

23) $S_1 = \{A \in \mathbb{R}^{3\times3} / A = A^t\}; S_2 = \{A \in \mathbb{R}^{3\times3} / A = -A^t\}$

Una forma de hacer esto es probar que $S_1 \cap S_2 = 0_W$ y que $\dim S_1 + \dim S_2 = 9 = \dim \mathbb{R}^{3\times3}$.

$\Rightarrow$ Si $A \in (S_1 \cap S_2) \Rightarrow A = A^t \wedge A = -A^t \Rightarrow A^t = -A^t \Rightarrow 2A^t = 0 \Rightarrow$

$\Rightarrow A^t = 0 \Rightarrow A = 0$ (matriz nula) $\Rightarrow S_1 \cap S_2 = \{0_{\mathbb{R}^{3\times3}}\}$ ✓

En S_1: $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \Rightarrow \begin{matrix} a_{12} = a_{21} \\ a_{13} = a_{31} \\ a_{23} = a_{32} \end{matrix} \Rightarrow S_1 \subset \mathbb{R}^{3\times3}$ y tiene 3 restricciones, por lo tanto $\dim S_1 = 6$

En S_2: $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} -a_{11} & -a_{21} & -a_{31} \\ -a_{12} & -a_{22} & -a_{32} \\ -a_{13} & -a_{23} & -a_{33} \end{pmatrix} \Rightarrow \begin{matrix} a_{11} = -a_{11} \\ a_{12} = -a_{21} \\ a_{13} = -a_{31} \\ a_{23} = -a_{32} \\ a_{22} = -a_{22} \\ a_{33} = -a_{33} \end{matrix}$ $S_2 \subset \mathbb{R}^{3\times3}$ y tiene 6 restricciones, → $\dim S_2 = 3$

$\Rightarrow \dim S_1 + \dim S_2 = 9$ ✓ $\Rightarrow$ partiendo del teorema de la dimensión, esto quiere decir que $\dim(S_1+S_2) = 9$ y dado el E.V., solo puede pasar que $S_1 \oplus S_2 = \mathbb{R}^{3\times3}$.
(y es directa porque $S_1 \cap S_2 = \{0_{\mathbb{R}^{3\times3}}\}$)

Todo el razonamiento se puede usar para probar lo mismo en $\mathbb{R}^{n\times n}$.

24) - I) $S_1 = \{(1\ 1\ 2\ 0\ 1)^t ; (2\ 0\ 3\ 0\ 1)^t\}$

$S_2 \overset{gen}{=} \{(-1\ 1\ -2\ 1\ 1)^t\}$; $S_3 = gen\{(0\ 1\ 0\ 1\ 1)^t\}$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 & 1 \\ -1 & 1 & -2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow \text{son todos l.i.}$$

ya que todos los subespacios están dados por bases de los mismos, todos los elementos de cada subesp. son l.i. entre sí. Pero además se probó que ninguno de los elementos de cada subesp. es c.l. de los elementos de los otros, por lo tanto $S_1 \cap S_2 \cap S_3 = \{0_{R^5}\}$

$\Rightarrow$ La suma es directa. $\boxed{B = \{(1\ 1\ 2\ 0\ 1)^t ; (2\ 0\ 3\ 0\ 1)^t ; (-1\ 1\ -2\ 1\ 1)^t ; (0\ 1\ 0\ 1\ 1)^t\}}$

II) $S_1 = gen\{(1\ 1\ 2\ 0\ 1)^t ; (2\ 0\ 3\ 0\ -1)^t\}$; $S_2 = gen\{(-1\ 0\ -2\ 1\ 1)^t\}$

$S_3 = gen\{(1\ 1\ 1\ 2\ 2)^t\}$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 & -1 \\ -1 & 0 & -2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & -1 \end{pmatrix} \rightarrow \left.\begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 & 3 \\ 0 & 0 & -1 & 2 & 1 \\ \cancel{0} & \cancel{0} & \cancel{1} & \cancel{-2} & \cancel{-1} \end{pmatrix}\right\} \text{son l.d.}$$

Como todos los subesp. son bases, no cabe la posibilidad de que algún vector sea l.d. con los vectores del mismo subesp., debe ser l.d. con los de los otros subesp. Entonces $S_1 \cap S_2 \cap S_3 \neq \{0_{R^5}\}$

$\Rightarrow$ La suma no es directa. $\boxed{B = \{(1\ 1\ 2\ 0\ 1)^t ; (2\ 0\ 3\ 0\ -1)^t ; (-1\ 0\ -2\ 1\ 1)^t\}}$

26) - a) $v = (1\ 2\ 3)^T$; $B = \{(1\ 1\ 0)^t ; (1\ 0\ 1)^t ; (0\ 1\ 1)^t\}$

$$\Rightarrow \alpha \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \gamma \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow \begin{cases} \alpha + \beta = 1 \\ \alpha + \gamma = 2 \\ \beta + \gamma = 3 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 1 & 3 \end{array}\right) \rightarrow$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -2 & -4 \end{array}\right) \Rightarrow \begin{cases} \alpha + \beta = 1 \\ \beta - \gamma = -1 \\ -2\gamma = -4 \end{cases} \Rightarrow \begin{matrix} \alpha = 0 \\ \beta = 1 \\ \gamma = 2 \end{matrix} \Rightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_B = \boxed{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}$$

b) $v = a + b.x + c.x^2$; $B = \{1 + x + x^2 ; 1 + x ; 1\}$

$\Rightarrow \alpha.(1 + x + x^2) + \beta.(1 + x) + \gamma = a + b.x + c.x^2 \Rightarrow$

$\Rightarrow \alpha.x^2 + (\alpha + \beta).x + (\alpha + \beta + \gamma) = c.x^2 + b.x + a$

$$\Rightarrow \begin{cases} \boxed{\alpha = c} \\ \alpha + \beta = b \\ \alpha + \beta + \gamma = a \end{cases} \Rightarrow \begin{matrix} \boxed{\beta = b - c} \\ \boxed{\gamma = a - b} + \cancel{\alpha} - \cancel{\alpha} \end{matrix}$$

$\Rightarrow (a + b.x + c.x^2)_B = \boxed{(c\quad b-c\quad a-b)^t}$

27) - a) - I - $C_B(v) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} \Rightarrow v = \alpha_1 . v_1 + \cdots + \alpha_m . v_m$

$$C_B(v') = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} \Rightarrow v' = \beta_1 . v_1 + \cdots + \beta_m . v_m$$

$$\Rightarrow C_B(v) + C_B(v') = \begin{pmatrix} \alpha_1 + \beta_1 \\ \vdots \\ \alpha_m + \beta_m \end{pmatrix} \Rightarrow v + v' = (\alpha_1 + \beta_1) . v_1 + \cdots + (\alpha_m + \beta_m) . v_m$$

$$\Rightarrow C_B(v + v') = \begin{pmatrix} \alpha_1 + \beta_1 \\ \vdots \\ \alpha_m + \beta_m \end{pmatrix} \quad ✓$$

$$C_B(\alpha . v) = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \Rightarrow \alpha . v = a_1 . v_1 + \cdots + a_m . v_m \Rightarrow v = \frac{a_1}{\alpha} v_1 + \cdots + \frac{a_m}{\alpha} . v_m$$

$$\Rightarrow C_B(v) = \begin{pmatrix} \frac{a_1}{\alpha} \\ \vdots \\ \frac{a_m}{\alpha} \end{pmatrix} = \frac{1}{\alpha} . \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = \frac{1}{\alpha} . C_B(\alpha . v) \Rightarrow \alpha . C_B(v) = C_B(\alpha . v) \quad ✓$$

29) $E = \{e_1; \ldots; e_m\}$; $B = \{v_1; \ldots; v_m\}$

La matriz de cambio de base de B a E tiene como columnas los vectores de B escritos en base E.

$$\Rightarrow \boxed{C_{BE} = \begin{pmatrix} \vdots & \vdots & & \vdots \\ C_E(v_1) & C_E(v_2) & \cdots\cdots & C_E(v_m) \\ \vdots & \vdots & & \vdots \end{pmatrix}}$$

30)

$$\boxed{C_{BB''} = C_{B'B''} \cdot C_{BB'}}$$

31) - a) $B = \{(1\ 2\ 3)^t ; (1\ 0\ 1)^t ; (3\ 4\ 6)^t\}$; $B' = \{(1\ -1\ 0)^t ; (1\ -2\ 3)^t ; (1\ 1\ 0)^t\}$

$$\Rightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \lambda_1 . \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda_2 . \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda_3 . \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -1 & -2 & 1 & 2 \\ 0 & 3 & 0 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & 3 & 0 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 6 & 12 \end{array}\right)$$

$$\Rightarrow \begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ -\lambda_2 + 2\lambda_3 = 3 \\ 6\lambda_3 = 12 \end{cases} \Rightarrow \begin{matrix} \lambda_1 = -2 \\ \lambda_2 = 1 \\ \lambda_3 = 2 \end{matrix} \Rightarrow C_{B'}(1\ 2\ 3)^t = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \lambda_1 . \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda_2 . \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda_3 . \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -1 & -2 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 6 & 4 \end{array}\right) \Rightarrow \begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ -\lambda_2 + 2\lambda_3 = 1 \\ 6\lambda_3 = 4 \end{cases} \Rightarrow \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = \frac{1}{3} \\ \lambda_3 = \frac{2}{3} \end{matrix}$$

$C_{B'}(1 \quad 0 \quad 1)^t = \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$

$$\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} = \lambda_1 . \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda_2 . \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda_3 . \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ -1 & -2 & 1 & 4 \\ 0 & 3 & 0 & 6 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 7 \\ 0 & 0 & 6 & 27 \end{array}\right) \Rightarrow \begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 3 \\ -\lambda_2 + 2\lambda_3 = 7 \\ 6\lambda_3 = 27 \end{cases}$$

$$\Rightarrow \begin{matrix} \lambda_1 = -\frac{7}{2} \\ \lambda_2 = 2 \\ \lambda_3 = \frac{9}{2} \end{matrix} \Rightarrow C_{B'}(3 \quad 4 \quad 6)^t = \begin{pmatrix} -\frac{7}{2} \\ 2 \\ \frac{9}{2} \end{pmatrix} \Rightarrow \boxed{C_{BB'} = \begin{pmatrix} -2 & 0 & -\frac{7}{2} \\ 1 & \frac{1}{3} & 2 \\ 2 & \frac{2}{3} & \frac{9}{2} \end{pmatrix}}$$

(también se puede triangular la matriz con los 3 resultados a la vez para simplificar:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 3 \\ -1 & -2 & 1 & 2 & 0 & 4 \\ 0 & 3 & 0 & 3 & 1 & 6 \end{array}\right)$$

b) $B = \{1 ; x-1 ; (x-1)^2\}$; $B' = \{1 ; x-2 ; (x-2)^2\}$

$C_{B'}(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $C_{B'}(x-1) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$\Rightarrow (x-1)^2 = \lambda_1 . 1 + \lambda_2 .(x-2) + \lambda_3 .(x-2)^2 \Rightarrow x^2 - 2x + 1 = \lambda_1 + \lambda_2 . x - 2\lambda_2 + \lambda_3 . x^2 - \lambda_3 . 4x + \lambda_3 . 4$$

$$\Rightarrow x^2 - 2x + 1 = \lambda_3 . x^2 + (\lambda_2 - 4\lambda_3) . x + (\lambda_1 - 2\lambda_2 + 4\lambda_3)$$

$$\begin{cases} \lambda_3 = 1 \\ \lambda_2 - 4\lambda_3 = -2 \\ \lambda_1 - 2\lambda_2 + 4\lambda_3 = 1 \end{cases} \Rightarrow \begin{matrix} \lambda_2 = 2 \\ \lambda_1 = 1 \end{matrix} \Rightarrow C_{B'}((x-1)^2) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \boxed{C_{BB'} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}}$$

Adicionales "Wronskiano"

1) - a) $1 ; x ; \dots ; x^n$

$$W(1; x; \dots ; x^n) = \det \begin{pmatrix} 1 & x & \cdots & \cdots & x^n \\ 0 & 1 & 2x & \cdots & n x^{(n-1)} \\ 0 & 0 & 2 & \cdots & n.(n-1).x^{(n-2)} \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & \cdots & \underbrace{x^{(n-n)}}_{1} \end{pmatrix} \neq 0 \therefore \text{es l.i.}$$

(el det de una matriz triangulada es el producto de los elementos de la diagonal, y en este caso son todos $\neq 0$)

($n+1$ funciones $\Rightarrow$ $n+1$ filas $\Rightarrow$ n derivadas (partiendo de la 1° dva.))

b) $x ; \frac{1}{x}$

$$W(x; \tfrac{1}{x}) = \det \begin{pmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{pmatrix} = \left(\cancel{x} . \left(-\frac{1}{x^{\cancel{2}}}\right) - \frac{1}{x} . 1\right) = -\frac{1}{x} - \frac{1}{x} = -\frac{2}{x} \neq 0 \ \forall x \therefore \text{es l.i.}$$

c) $e^{\alpha . x} ; x . e^{\alpha . x}$, $(\alpha \in \mathbb{R})$

$$W(e^{\alpha x} , x . e^{\alpha . x}) = \det \begin{pmatrix} e^{\alpha . x} & x . e^{\alpha . x} \\ e^{\alpha x} . \alpha & e^{\alpha . x} + x . \alpha . e^{\alpha x} \end{pmatrix} = e^{2\alpha . x} + \cancel{x . \alpha . e^{2\alpha x}} - \cancel{x . \alpha . e^{2\alpha x}} \neq 0 \ \forall x$$

$\therefore$ el conjunto es l.i.

d) e^x; $x.e^x$; e^{-x}

$$W(e^x; x.e^x; e^{-x}) = \det\begin{pmatrix} e^x & x e^x & e^{-x} \\ e^x & e^x + x.e^x & -e^{-x} \\ e^x & e^x + e^x + x.e^x & e^{-x} \end{pmatrix}$$

triangulo la matriz: $\begin{pmatrix} e^x & x.e^x & e^{-x} \\ e^x & e^x + x.e^x & -e^{-x} \\ e^x & 2e^x + x.e^x & e^{-x} \end{pmatrix} \rightarrow \begin{pmatrix} e^x & x.e^x & e^{-x} \\ 0 & -e^x & 2e^{-x} \\ 0 & -2e^x & 0 \end{pmatrix} \rightarrow$

$\rightarrow \begin{pmatrix} e^x & x.e^x & e^{-x} \\ 0 & -e^x & 2e^{-x} \\ 0 & 0 & 4e^{-x} \end{pmatrix}$ y el determinante de esto es $e^x.(-e^x).4.e^{-x} \neq 0 \ \forall x$

$\therefore$ el conjunto es l.i.

e) $sen(x)$; $sen(x + \frac{\pi}{4})$

$$W(sen(x); sen(x+\tfrac{\pi}{4})) \underset{\swarrow \text{det}}{=} \begin{pmatrix} sen(x) & sen(x+\frac{\pi}{4}) \\ cos(x) & cos(x+\frac{\pi}{4}) \end{pmatrix} = sen(x).cos(x+\tfrac{\pi}{4}) - cos(x).sen(x+\tfrac{\pi}{4}) =$$

$$= sen(x).\left(-sen(x).sen(\tfrac{\pi}{4}) + cos(x).cos(\tfrac{\pi}{4})\right) - cos(x).\left(sen(x).\underbrace{cos(\tfrac{\pi}{4})}_{\frac{1}{\sqrt{2}}} + cos(x).\underbrace{sen(\tfrac{\pi}{4})}_{\frac{1}{\sqrt{2}}}\right) =$$

$$= -sen^2(x).\frac{1}{\sqrt{2}} + \cancel{sen(x).cos(x)}.\frac{1}{\sqrt{2}} - \cancel{cos(x).sen(x)}.\frac{1}{\sqrt{2}} - cos^2(x).\frac{1}{\sqrt{2}} =$$

$$= -\frac{1}{\sqrt{2}}.(sen^2(x) + cos^2(x)) = -\frac{1}{\sqrt{2}} \neq 0 \quad \therefore \quad \text{es l.i.}$$

f) $arc\,cos(x)$; $arc\,sen(x)$

$$W(f_1; f_2) = \det\begin{pmatrix} arc\,cos(x) & arc\,sen(x) \\ -\frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-x^2}} \end{pmatrix} = \frac{arc\,cos(x)}{\sqrt{1-x^2}} + \frac{arc\,sen(x)}{\sqrt{1-x^2}} \neq 0 \therefore \text{es l.i.}$$

g) 1; $sen^2(x)$; $cos(2x)$

$$W(f_1; f_2; f_3) \underset{\swarrow \text{det}}{=} \begin{pmatrix} 1 & sen^2(x) & cos(2x) \\ 0 & 2.sen(x).cos(x) & -sen(2x).2 \\ 0 & 2.cos^2(x) - 2.sen^2(x) & -cos(2x).4 \end{pmatrix} \underset{\substack{\downarrow \\ \text{triangulo} \\ \text{la matriz}}}{\longrightarrow} \begin{pmatrix} 1 & sen^2(x) & cos(2x) \\ 0 & 2.sen(x).cos(x) & -sen(2x).2 \\ 0 & 0 & * \end{pmatrix}$$

$$* = 2.(cos^2(x) - sen^2(x)).(-sen(2x)).2 - 2.cos(x).sen(x).(-cos(2x).4) = -4.(cos^2(x) - sen^2(x)).sen(2x) +$$

$$+ 8.cos(x).sen(x).cos(2x) \Rightarrow \begin{array}{l} cos(2x) = cos^2(x) - sen^2(x) \\ sen(2x) = 2.sen(x).cos(x) \end{array}$$

$-8.(\cos^2(x)-\operatorname{sen}^2(x)].\operatorname{sen}(x).\cos(x)+8.(\cos^2(x)-\operatorname{sen}^2(x)).\cos(x).\operatorname{sen}(x) = 0\forall x$ (se anula una fila)

$\downarrow$

el det = 0

∴ no puedo saber si son l.i. con este método.

h) $\operatorname{sen}(x)$; $\cos(x)$; $\cos(x+\frac{\pi}{3})$

$$W(\operatorname{sen}(x);\cos(x);\cos(x+\tfrac{\pi}{3}))=\det\begin{pmatrix}\operatorname{sen}(x) & \cos(x) & \cos(x+\frac{\pi}{3})\\ \cos(x) & -\operatorname{sen}(x) & -\operatorname{sen}(x+\frac{\pi}{3})\\ -\operatorname{sen}(x) & -\cos(x) & -\cos(x+\frac{\pi}{3})\end{pmatrix}=0$$

pues la 3° fila es múltiplo de la 1°; el determinante se anula

∴ no puedo deducir nada a partir del wronskiano.

i) $e^{-3x}.\operatorname{sen}(2x)$; $e^{-3x}.\cos(2x)$

$$W(f_1;f_2)=\det\begin{pmatrix}e^{-3x}.\operatorname{sen}(2x) & e^{-3x}.\cos(2x)\\ e^{-3x}.(-3).\operatorname{sen}(2x)+e^{-3x}.\cos(2x).2 & e^{-3x}.(-3).\cos(2x)-e^{-3x}.\operatorname{sen}(2x).2\end{pmatrix}=$$

$$=(-3)e^{-6x}.\operatorname{sen}(2x).\cos(2x)-e^{-6x}.\operatorname{sen}^2(2x).2-((-3).e^{-6x}.\cos(2x).\operatorname{sen}(2x)+e^{-6x}.\cos^2(2x).2)=$$

$$=\cancel{(-3)e^{-6x}.\operatorname{sen}(2x).\cos(2x)}+\cancel{3.e^{-6x}.\cos(2x).\operatorname{sen}(2x)}-2.e^{-6x}.(\operatorname{sen}^2(2x)+\cos^2(2x))=-2.e^{-6x}\neq 0\,\forall x$$

∴ es l.i.

j) $\operatorname{senh}(\alpha.x)$; $\cosh(\alpha.x)$; $(\alpha\in\mathbb{R})$

$$W(f_1;f_2)=\det\begin{pmatrix}\operatorname{senh}(\alpha.x) & \cosh(\alpha.x)\\ \cosh(\alpha.x)\alpha & \operatorname{senh}(\alpha x).\alpha\end{pmatrix}=\alpha.\operatorname{senh}^2(\alpha.x)-\alpha.(\cosh)^2(\alpha.x)=$$

$$=\alpha.\left[\left(\frac{e^{\alpha.x}-e^{-\alpha.x}}{2}\right)^2-\left(\frac{e^{\alpha.x}+e^{-\alpha.x}}{2}\right)^2\right]=\alpha.\left(\frac{e^{2\alpha.x}-2+e^{-2\alpha x}}{4}-\frac{e^{2\alpha.x}+2+e^{-2\alpha.x}}{4}\right)=$$

$$=\frac{\alpha}{4}.\left(\cancel{e^{2\alpha x}}-2+\cancel{e^{-2\alpha x}}-\cancel{e^{2\alpha x}}-2-\cancel{e^{-2\alpha x}}\right)=-\alpha \Rightarrow \text{si } \alpha\neq 0 \Rightarrow$$ l.i. el conjunto

Práctica 2

1)

$u = (1 \;\; 2 \;\; -1)^t \;;\; v = (2 \;\; 1 \;\; -3)^t$

$(u, v) = u^t . v$

$$\cos \alpha = \frac{(u,v)}{\|u\| . \|v\|} \Rightarrow (u,v) = (1 \;\; 2 \;\; -1) . \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 7 \;;\; \|u\| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$$

$\downarrow$ ángulo entre vectores

$$\|v\| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{14}$$

$$\cos(\alpha) = \frac{7}{\sqrt{6} . \sqrt{14}} = \frac{7}{\sqrt{84}} \Rightarrow \boxed{\alpha = 40^\circ \;\; 12' \;\; 11''}$$

$(u, v) = 3 . u_1 . v_1 + 2 . u_2 v_2 + u_3 . v_3$

- $(v, u) = 3 . v_1 . u_1 + 2 . v_2 \, u_2 + v_3 . u_3 = (u, v)$ ✓

  $\hookrightarrow$ por conmutatividad de los productos

- $(\alpha . u + \beta . v, z) = 3 . (\alpha . u_1 + \beta . v_1) . z_1 + 2 . (\alpha . u_2 + \beta . v_2) . z_2 + (\alpha . u_3 + \beta . v_3) . z_3 =$

  $= \alpha . (3 . u_1 . z_1 + 2 \, u_2 . z_2 + u_3 . z_3) + \beta . (3 . v_1 . z_1 + 2 \, v_2 . z_2 + v_3 . z_3) =$

  $= \alpha . (u, z) + \beta . (v, z)$ ✓

- $(u, u) = 3 . u_1^2 + 2 . u_2^2 + u_3^2 \geq 0$ ✓

Si $(u, u) = 3 . u_1^2 + 2 . u_2^2 + u_3^2 = 0$, como todos los elementos son ≥ 0, no queda otra que $u_1 = u_2 = u_3 = 0 \Rightarrow u = 0_{\mathbb{R}^3}$ ✓

Si $u = 0_{\mathbb{R}^3} \Rightarrow (0, 0) = 3 . 0 + 2 . 0 + 0 = 0 \Rightarrow (u, u) = 0 \Leftrightarrow u = 0_{\mathbb{R}^3}$ ✓

Es un P.i.

$(u, v) = 3 . 1 . 2 + 2 . 2 . 1 + (-1) . (-3) = 13 \;,\; \|u\| = \sqrt{(u, u)} = \sqrt{3 . 1^2 + 2 . 2^2 + (-1)^2} = \sqrt{12}$

$\|v\| = \sqrt{(v, v)} = \sqrt{3 . 2^2 + 2 . 1^2 + (-3)^2} = \sqrt{23}$ (tener en cuenta que la norma de un vector se calcula diferente si el P.i. no es canónico, como en este caso)

$$\Rightarrow \cos(\alpha) = \frac{13}{\sqrt{12} \, \sqrt{23}} = \frac{13}{\sqrt{276}} \Rightarrow \boxed{\alpha = 38^\circ \;\; 30' \;\; 33''}$$

2) $u = (1 \;\; 2 \;\; -1)^t \Rightarrow$ Busco $v \,/\, (v, u) = 0$

$$(v, u) = v^t . u = (v_1 \;\; v_2 \;\; v_3) . \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = v_1 + 2v_2 - v_3 = 0 \Rightarrow v_3 = v_1 + 2v_2$$

$$\Rightarrow v = (v_1 \;\; v_2 \;\; v_3)^t = (v_1 \;\; v_2 \;\; v_1 + 2v_2)^t = v_1 . \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + v_2 . \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} ; v_1, v_2 \in \mathbb{R}$$

$$\Rightarrow \boxed{v = \{ (x_1 \;\; x_2 \;\; x_3)^t \in \mathbb{R}^3 \,/\, (x_1 \;\; x_2 \;\; x_3)^t = \alpha . (1 \;\; 0 \;\; 1)^t + \beta . (0 \;\; 1 \;\; 2)^t ; \alpha, \beta \in \mathbb{R} \}}$$

4) - a) $(u,v) = a_{11}.u_1.v_1 + a_{12}.u_1.v_2 + a_{21}.u_2.v_1 + a_{22}.u_2.v_2$

• $(v,u) = (u,v) \Rightarrow (v,u) = \cancel{a_{11}.v_1.u_1} + a_{12}.v_1.u_2 + a_{21}.v_2.u_1 + \cancel{a_{22}.v_2.u_2} =$

$= \cancel{a_{11}.u_1.v_1} + a_{12}.u_1.v_2 + a_{21}.u_2.v_1 + \cancel{a_{22}.u_2.v_2}$

$\Rightarrow \boxed{a_{12} = a_{21}}$

• $(\alpha.u + \beta.v, z) = \alpha.(u,z) + \beta.(v,z)$

$\Rightarrow (\alpha.u + \beta.v, z) = a_{11}.(\alpha.u_1 + \beta.v_1).z_1 + a_{12}.(\alpha.u_1 + \beta.v_1).z_2 + a_{21}.(\alpha.u_2 + \beta.v_2).z_1 +$

$+ a_{22}.(\alpha.u_2 + \beta.v_2).z_2$

$\alpha.(u,z) + \beta.(v,z) = \alpha.(a_{11}.u_1 z_1 + a_{12}.u_1.z_2 + a_{21}.u_2.z_1 + a_{22}.u_2.z_2) +$

$+ \beta.(a_{11}.v_1.z_1 + a_{12}.v_1.z_2 + a_{21}.v_2.z_1 + a_{22}.v_2.z_2)$

Se cumple independientemente de los coeficientes.

• $(u,u) \neq 0 \Rightarrow (u,u) = a_{11}.u_1^2 + a_{12}.u_1.u_2 + a_{21}.u_2.u_1 + a_{22}.u_2^2 \neq 0$

$\Rightarrow a_{11}.u_1^2 + (a_{12} + a_{21}).u_1.u_2 + a_{22}.u_2^2 \neq 0$, la única solución de esto debe ser $u = 0_{R^2}$

$\Rightarrow \boxed{a_{11} \neq 0 , a_{22} \neq 0}$ (pues si $(a_{12} + a_{21}) \neq 0$, todavía u puede ser $\neq 0_{R^2}$ y anularse el p.i.)

7)

$(a_0 + a_1.x + a_2.x^2 , b_0 + b_1.x + b_2.x^2) = a_0.b_0 + a_1.b_1 + a_2.b_2$

$B = \{1; x; x^2\}$

$P = a_0 + a_1.x + a_2.x^2$
$Q = b_0 + b_1.x + b_2.x^2$

$\Rightarrow (P,Q) = a_0.b_0 + a_1.b_1 + a_2.b_2$

• $(Q,P) = b_0.a_0 + b_1.a_1 + b_2.a_2 = (P,Q)$ ✓ por conmutatividad (como estoy en P_2, $(Q,P) = \overline{(P,Q)} = (P,Q)$)

• $(\alpha.P + \beta.Q, H) = (\alpha.a_0 + \beta.b_0).h_0 + (\alpha.a_1 + \beta.b_1).h_1 + (\alpha.a_2 + \beta.b_2).h_2 =$

$= \alpha.(a_0.h_0 + a_1.h_1 + a_2.h_2) + \beta.(b_0.h_0 + b_1.h_1 + b_2.h_2) = \alpha.(P,H) + \beta(Q,H)$ ✓

• $(P,P) = a_0^2 + a_1^2 + a_2^2 \geq 0$ ✓

• Si $(P,P) = 0 \Rightarrow a_0^2 + a_1^2 + a_2^2 = 0 \Rightarrow a_0 = a_1 = a_2 = 0$ (no hay otra opción)

Si $P = 0_{P_2} \Rightarrow (P,P) = 0^2 + 0^2 + 0^2 = 0$ ✓ $\Rightarrow$ es un p.i.

Ahora, considerando el dato que se da,

$$(a_0 + a_1.x + a_2.x^2, b_0 + b_0.x + b_0.x^2) = \Big(\underbrace{(a_0\ \ a_1\ \ a_2)^t}_{\text{coord. en base } B}, (b_0\ \ b_1\ \ b_2)^t \Big) =$$

$$= (C_B(p), C_B(q)) = (a_0\ \ a_1\ \ a_2) \cdot \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} \quad ¿?$$

8)-a) $(f,g) = \int_0^1 f(x).g(x)\,dx$

- $(g,f) = \int_0^1 g.f\,dx = \int_0^1 f.g\,dx = (f,g)$ ✓
- $(\alpha.f + \beta.g, h) = \int_0^1 (\alpha.f(x) + \beta.g(x)).h(x)\,dx = \int_0^1 (\alpha.f(x).h(x) + \beta.g(x).h(x))\,dx =$
  $= \alpha \int_0^1 f(x).h(x)\,dx + \beta.\int_0^1 g(x).h(x)\,dx = \alpha.(f,h) + \beta.(g,h)$ ✓
- $(f,f) \geq 0 \Rightarrow (f,f) = \int_0^1 (f(x))^2\,dx \geq 0$ (la integral de algo ≥ 0)
- Si $(f,f) = 0 \Rightarrow \int_0^1 (f(x))^2\,dx = 0 \Rightarrow f(x) = 0$ pues la integral solo puede ser ≥ 0.

El recíproco también se cumple.

Es P.i.

b) Como los polinomios son un determinado tipo de función, se cumplen todas las condiciones anteriores y conforman un P.i.

c) $\cos\alpha = \dfrac{(f,g)}{\|f\|.\|g\|} \Rightarrow (f,g) = (x, x^2-x+1) = \int_0^1 x.(x^2-x+1)\,dx = \int_0^1 x^3-x^2+x\,dx =$

$$= \left(\frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_0^1 = \frac{5}{12}$$

$\|f\| = \sqrt{(f,f)} = \sqrt{\int_0^1 x^2\,dx} = \sqrt{\frac{1}{3}}$; $\|g\| = \sqrt{\int_0^1 (x^2-x+1).(x^2-x+1)\,dx} =$

$$= \sqrt{\int_0^1 x^4 - 2x^3 + 3x^2 - 2x + 1\,dx} = \sqrt{\left(\frac{x^5}{5} - \frac{x^4}{2} + x^3 - x^2 + x\right)\Big|_0^1} = \sqrt{\frac{7}{10}}$$

$$\Rightarrow \cos\alpha = \frac{5/12}{\sqrt{1/3}.\sqrt{7/10}} \Rightarrow \boxed{\alpha = 30° \ 23' \ 32''}$$

$f(x) = x^2 + a.x \; ; \; g(x) = x - 1$

$(f,g) = 0 \Rightarrow (f,g) = \int_0^1 (x^2 + a.x).(x-1)\,dx = \int_0^1 x^3 - x^2 + a.x^2 - a.x\,dx =$

$= \left(\frac{x^4}{4} - \frac{x^3}{3} + \frac{a}{3}.x^3 - \frac{a}{2}x^2\right)\Big|_0^1 = -\frac{1}{12} - \frac{a}{6} = 0 \Rightarrow \boxed{a = -\frac{1}{2}}$

no importa que no sean ⊥ geométricamente.

9) - d) $V = P_2$, $B = \{1; x, x^2\}$, $(f,g) = \int_0^1 f(x).g(x)\,dx$

$\Rightarrow G_B = \begin{pmatrix} (v_1,v_1) & (v_1,v_2) & (v_1,v_3) \\ (v_2,v_1) & (v_2,v_2) & (v_2,v_3) \\ (v_3,v_1) & (v_3,v_2) & (v_3,v_3) \end{pmatrix}$ con v_i representando a los vectores de B en la posición i.

$\Rightarrow (1,1) = \int_0^1 1.1\,dx = 1 \; ; \; (1,x) = \int_0^1 x\,dx = \frac{1}{2} \; ; \; (1,x^2) = \int_0^1 x^2\,dx = \frac{1}{2}$

$(x,1) = (1,x) = \frac{1}{2} \; ; \; (x,x) = \int_0^1 x^2\,dx = \frac{1}{3} \; ; \; (x,x^2) = \int_0^1 x^3\,dx = \frac{1}{4}$

$(x^2,1) = (1,x^2) = \frac{1}{3} \; ; \; (x^2,x) = (x,x^2) = \frac{1}{4} \; ; \; (x^2,x^2) = \int_0^1 x^4\,dx = \frac{1}{5}$

$\Rightarrow \boxed{G_B = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}}$

$C_B(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \; ; \; C_B(x^2 - x + 1) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$(x, x^2 - x + 1) = (0 \;\; 1 \;\; 0).\,G_B.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \left(\frac{1}{2} \;\; \frac{1}{3} \;\; \frac{1}{4}\right).\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \boxed{\frac{5}{12}}$

f) $B = \{(1 \;\; i \;\; 0)^t \; ; \; (1 \;\; 1 \;\; 1)^t \; ; \; (i \;\; 0 \;\; 0)^t\}$, $V = \mathbb{C}^3$

$(v_1,v_1) = v_1^H.v_1 = \overline{v_1}^t.v_1 = (1 \;\; -i \;\; 0).\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = 2$

$(v_1,v_2) = (1 \;\; -i \;\; 0).\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 - i \; ; \; (v_1,v_3) = (1 \;\; -i \;\; 0).\begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix} = i$

$(v_2,v_1) = \overline{(v_1,v_2)} = 1 + i \; ; \; (v_2,v_2) = (1 \;\; 1 \;\; 1).\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3$

estoy en $\mathbb{C}$-e.v.

$(v_2,v_3) = (1 \;\; 1 \;\; 1).\begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix} = i \; ; \; (v_3,v_1) = \overline{(v_1,v_3)} = -i \; ; \; (v_3,v_2) = \overline{(v_2,v_3)} = -i$

$(v_3,v_3) = (-i \;\; 0 \;\; 0).\begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix} = 1$

$\Rightarrow \boxed{G_B = \begin{pmatrix} 2 & 1-i & i \\ 1+i & 3 & i \\ -i & -i & 1 \end{pmatrix}}$

10) $\mathbb{R}^3$; $B=\{\overbrace{(1\ 1\ -1)^t}^{u_1} ; \overbrace{(0\ 1\ 1)^t}^{u_2} ; \overbrace{(-1\ 1\ 0)^t}^{u_3}\}$; $v_1=(1\ -1\ 1)^t$; $v_2=(-1\ 2\ 2)^t$

a) calculo G en base B.

$(u_1,u_1)=\|u_1\|^2 \underset{\Downarrow}{=} 1^2=1$; $(u_2,u_2)=(u_3,u_3)=1$

⇒ de la base ortonormal (independientemente del P.i., vale 1)

Todos los productos cruzados son 0 por ser B ortonormal.

$(v_1,v_2)=C_B(v_1)^t \cdot G_B \cdot C_B(v_2)$;

$$\begin{pmatrix}-1\\2\\2\end{pmatrix}\Bigg|\begin{pmatrix}1\\-1\\1\end{pmatrix}=\alpha\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}+\beta\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}+\gamma\cdot\begin{pmatrix}-1\\1\\0\end{pmatrix} \Rightarrow \left(\begin{array}{ccc|cc}1&0&-1&1&-1\\1&1&1&-1&2\\-1&1&0&1&2\end{array}\right)\rightarrow\left(\begin{array}{ccc|cc}1&0&-1&1&-1\\0&-1&-2&2&-3\\0&1&-1&2&1\end{array}\right)\rightarrow\left(\begin{array}{ccc|cc}1&0&-1&1&-1\\0&-1&-2&2&-3\\0&0&-3&4&-2\end{array}\right)$$

$$\Rightarrow\begin{cases}\alpha-\gamma=1\\-\beta-2\gamma=2\\-3\gamma=4\Rightarrow\gamma=-\frac{4}{3}\end{cases}\quad\beta=\frac{3}{3}\quad\Rightarrow\alpha=-\frac{1}{3}\qquad C_B(v_1)=\begin{pmatrix}-1/3\\2/3\\-4/3\end{pmatrix}$$

$$\begin{cases}\alpha-\gamma=-1\\-\beta-2\gamma=-3\\-3\gamma=-2\end{cases}\Rightarrow\begin{matrix}\alpha=-\frac{1}{3}\\\beta=\frac{5}{3}\\\gamma=\frac{2}{3}\end{matrix}\qquad C_B(v_2)=\begin{pmatrix}-1/3\\5/3\\2/3\end{pmatrix}$$

$$\Rightarrow(v_1,v_2)=\left(-\frac{1}{3}\ \ \frac{2}{3}\ \ -\frac{4}{3}\right)\cdot\underbrace{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}}_{I}\cdot\begin{pmatrix}-1/3\\5/3\\2/3\end{pmatrix}=\boxed{\frac{1}{3}}$$

b) Como no conozco el P.i., no puedo armar G_E directamente

$$(u,v)=C_B(u)^t\cdot G_B\cdot C_B(v)=(C_{EB}\cdot u)^t\cdot G_B\cdot(C_{EB}\cdot v)=u^t\cdot\underbrace{C_{EB}^t\cdot G_B\cdot C_{EB}}_{G_E}\cdot v$$

(porque lo que está entre los vec. u^t y v define G, y en la misma base que u y v.

$C_{EB}=(C_{BE})^{-1}$

$$C_{BE}=\begin{pmatrix}1&0&-1\\1&1&1\\-1&1&0\end{pmatrix}$$

$$\Rightarrow\left(\begin{array}{ccc|ccc}1&0&-1&1&0&0\\1&1&1&0&1&0\\-1&1&0&0&0&1\end{array}\right)\rightarrow\left(\begin{array}{ccc|ccc}1&0&-1&1&0&0\\0&-1&-2&1&-1&0\\0&1&-1&1&0&1\end{array}\right)\rightarrow\left(\begin{array}{ccc|ccc}1&0&-1&1&0&0\\0&-1&-2&1&-1&0\\0&0&-3&2&-1&1\end{array}\right)\rightarrow\left(\begin{array}{ccc|ccc}3&0&0&1&1&-1\\0&-3&0&-1&-1&-2\\0&0&-3&2&-1&1\end{array}\right)\rightarrow$$

$$\rightarrow\left(\begin{array}{ccc|ccc}1&0&0&1/3&1/3&-1/3\\0&1&0&1/3&1/3&2/3\\0&0&1&-2/3&1/3&-1/3\end{array}\right)\Rightarrow C_{EB}=\frac{1}{3}\cdot\begin{pmatrix}1&1&-1\\1&1&2\\-2&1&-1\end{pmatrix}$$

$$G_E=\frac{1}{3}\cdot\begin{pmatrix}1&1&-2\\1&1&1\\-1&2&-1\end{pmatrix}\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\cdot\frac{1}{3}\cdot\begin{pmatrix}1&1&-1\\1&1&2\\-2&1&-1\end{pmatrix}=\boxed{\frac{1}{9}\cdot\begin{pmatrix}6&0&3\\0&3&0\\3&0&6\end{pmatrix}}$$

13) $B=\{\overbrace{1}^{v_1};\ \overbrace{x-\frac{1}{2}}^{v_2};\ \overbrace{x^2-x+\frac{1}{6}}^{v_3}\};\ (f,g)=\int_0^1 f(x).g(x)\,dx$

$(v_1,v_2)=\int_0^1 x-\frac{1}{2}\,dx=\left(\frac{x^2}{2}-\frac{x}{2}\right)\Big|_0^1=0=(v_2,v_1)$ (es un P.i.)

$(v_1,v_3)=\int_0^1 x^2-x+\frac{1}{6}\,dx=\left(\frac{x^3}{3}-\frac{x^2}{2}+\frac{x}{6}\right)\Big|_0^1=0=(v_3,v_1)$

$(v_2,v_3)=\int_0^1(x-\frac{1}{2}).(x^2-x+\frac{1}{6})\,dx=\int_0^1 x^3-x^2+\frac{x}{6}-\frac{1}{2}x^2+\frac{x}{2}-\frac{1}{12}\,dx=\int_0^1 x^3-\frac{3}{2}x^2+\frac{2}{3}x-\frac{1}{12}\,dx=$

$=\left(\frac{x^4}{4}-\frac{x^3}{2}+\frac{x^2}{3}-\frac{x}{12}\right)\Big|_0^1=0=(v_3,v_2)$

Como todos los productos entre los 3 vectores son nulos, se demuestra que es una base ortogonal.

$\Rightarrow C_B(x^2-1)=\boxed{\left(-\frac{2}{3}\quad 1\quad 1\right)^t}$

$\|P\|=\sqrt{(P,P)}=\sqrt{C_B(P)^t.G_B.C_B(P)}\Rightarrow (v_1,v_1)=1$

$(v_2,v_2)=\int_0^1 x^2-\frac{1}{2}x-\frac{1}{2}x+\frac{1}{4}\,dx=\left(\frac{x^3}{3}-\frac{x^2}{2}+\frac{x}{4}\right)\Big|_0^1=\frac{1}{12}$

$(v_3,v_3)=\int_0^1(x^2-x+\frac{1}{6}).(x^2-x+\frac{1}{6})\,dx=\int_0^1 x^4-x^3+\frac{x^2}{6}-x^3+x^2-\frac{x}{6}+\frac{x^2}{6}-\frac{x}{6}+\frac{1}{36}\,dx=\left(\frac{x^5}{5}-\frac{x^4}{2}+\frac{4.x^3}{9}-\right.$

$\left.-\frac{x^2}{6}+\frac{x}{36}\right)\Big|_0^1=\frac{1}{180}$

$\|P\|=\sqrt{\left(-\frac{2}{3}\quad 1\quad 1\right).\begin{pmatrix}1&0&0\\0&1/12&0\\0&0&1/180\end{pmatrix}.\begin{pmatrix}-\frac{2}{3}\\1\\1\end{pmatrix}}=\sqrt{\left(-\frac{2}{3}\quad\frac{1}{12}\quad\frac{1}{180}\right).\begin{pmatrix}-\frac{2}{3}\\1\\1\end{pmatrix}}=\boxed{\sqrt{\frac{8}{15}}}$

otro método:

$\|x^2-1\|^2=\|(-\frac{2}{3}).1+1.(x-\frac{1}{2})+1.(x^2-x+\frac{1}{6})\|^2=|\frac{2}{3}|^2.\overbrace{\|1\|^2}^{(v_1,v_1)}+|1|^2.\overbrace{\|x-\frac{1}{2}\|^2}^{(v_2,v_2)}+|1|^2.\overbrace{\|x^2-x+\frac{1}{6}\|^2}^{(v_3,v_3)}=$

$=\frac{4}{9}.1+1.\frac{1}{12}+1.\frac{1}{180}=\frac{8}{15}\Rightarrow\|x^2-1\|=\sqrt{\frac{8}{15}}$

14) $u=(1\ 1\ 1)^t,\ v=(1\ 2\ -1)^t$

El conjunto de los vec $\parallel v$ es $A=\{u\in\mathbb{R}^3 / u=k.v \text{ con } k\in\mathbb{R}\}=\text{gen}\{(1\ 2\ -1)^t\}$

El otro conjunto se obtiene: $(v,x)=0=(1\ 2\ -1).\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=x_1+2x_2-x_3$

$\Rightarrow x_1=-2x_2+x_3\Rightarrow x=\begin{pmatrix}-2x_2+x_3\\x_2\\x_3\end{pmatrix}=x_2.\begin{pmatrix}-2\\1\\0\end{pmatrix}+x_3.\begin{pmatrix}1\\0\\1\end{pmatrix};\ x_2,x_3\in\mathbb{R}$

$B=\text{gen}\{(-2\ 1\ 0)^t;(1\ 0\ 1)^t\}$

$\Rightarrow\begin{pmatrix}1\\1\\1\end{pmatrix}=\underbrace{\alpha.\begin{pmatrix}1\\2\\-1\end{pmatrix}}_{\parallel v}+\overbrace{\beta.\begin{pmatrix}-2\\1\\0\end{pmatrix}+\gamma.\begin{pmatrix}1\\0\\1\end{pmatrix}}^{\perp v}\Rightarrow\left(\begin{array}{ccc|c}1&-2&1&1\\2&1&0&1\\-1&0&1&1\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}1&-2&1&1\\0&-5&2&1\\0&-2&2&2\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}1&-2&1&1\\0&-5&2&1\\0&0&-6&-8\end{array}\right)$

$\Rightarrow\begin{cases}\alpha-2\beta+\gamma=1\\-5\beta+2\gamma=1\\-6\gamma=-8\end{cases}\Rightarrow\begin{matrix}\alpha=\frac{1}{3}\\\beta=\frac{1}{3}\\\gamma=\frac{4}{3}\end{matrix}\Rightarrow\boxed{u=\underbrace{\begin{pmatrix}1/3\\2/3\\-1/3\end{pmatrix}}_{\parallel v}+\underbrace{\begin{pmatrix}2/3\\1/3\\4/3\end{pmatrix}}_{\perp v}}$

15) - a) $S=\text{gen}\{\overbrace{(1\;\;1\;\;1)^*}^{u_1};\overbrace{(1\;\;-1\;\;0)^*}^{u_2}\}$; $v=(1\;\;0\;\;0)^*$

El pto de S más cercano a v es la proyección ortogonal de v sobre S.

$\text{Proy.}_S^v=\frac{(u_1,v)}{(u_1,u_1)}\cdot u_1+\frac{(u_2,v)}{(u_2,u_2)}\cdot u_2$

$(u_1,v)=(1\;\;1\;\;1)\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=1$; $(u_1,u_1)=(1\;\;1\;\;1)\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=3$

$(u_2,v)=(1\;\;-1\;\;0)\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=1$, $(u_2,u_2)=(1\;\;-1\;\;0)\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}=2$

$\text{Proy}_S^v=\frac{1}{3}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}+\frac{1}{2}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}=\boxed{\begin{pmatrix}5/6\\-1/6\\1/3\end{pmatrix}}=v'$

$d(v,S)=\|v-v'\|=\|(1/6\;\;1/6\;\;-1/3)\|=\sqrt{(1/6\;\;1/6\;\;-1/3)\cdot\begin{pmatrix}1/6\\1/6\\-1/3\end{pmatrix}}=\boxed{\frac{1}{\sqrt{6}}}$

b) $S=\text{gen}\{\overbrace{(i\;\;-1\;\;1+i)^*}^{u_1}\}$; $v=(a\;\;b\;\;c)^*$

$\text{Proy.}_S^v=\frac{(u_1,v)}{(u_1,u_1)}\cdot u_1 \Rightarrow (u_1,v)=u_1^{\#}\cdot v=\overline{u_1}^t\cdot v=(-i\;\;-1\;\;1-i)\cdot\begin{pmatrix}a\\b\\c\end{pmatrix}=$

$=-a\cdot i-b+c\cdot(1-i)=-(a+c)\cdot i+c-b$

$(u_1,u_1)=(-i\;\;-1\;\;1-i)\cdot\begin{pmatrix}i\\-1\\1+i\end{pmatrix}=1+1+2=4$

$\text{Proy}_S^v=\frac{-(a+c)\cdot i+c-b}{4}\cdot\begin{pmatrix}i\\-1\\1+i\end{pmatrix}=\boxed{\frac{1}{4}\cdot\begin{pmatrix}a+c+(c-b)\cdot i\\b-c+(a+c)\cdot i\\a+2c-b+(-a-b)\cdot i\end{pmatrix}}$

$d(v,S)=\|-(a\;\;b\;\;c)^*+\frac{1}{4}(a+c+(c-b)i\;\;\;b-c+(a+c)i\;\;\;a+2c-b+(-a-b)\cdot i)^*\|=$

$=\|\frac{1}{4}\cdot(c+(c-b)\cdot i\;\;\;-c+(a+c)\cdot i\;\;\;a+c-b+(-a-b)\cdot i)^*\|=|\frac{1}{4}|\cdot\|*\|$

$\Rightarrow(c+(c-b)\cdot i\;\;\;-c+(a+c)\cdot i\;\;\;a+c-b-(a+b)\cdot i)\cdot\begin{pmatrix}c+(c-b)\cdot i\\-c+(a+c)\cdot i\\a+c-b-(a+b)\cdot i\end{pmatrix}=$

$=(c+(c-b)\cdot i)\cdot(c+(c-b)\cdot i]+(-c+(a+c)\cdot i)\cdot(-c+(a+c)\cdot i]+(a+c-b-(a+b)\cdot i)\cdot(a+c-b-(a+b)\cdot i)=$

$=c^2+c(c-b)\cdot i+c(c-b)\cdot i-(c-b)^2+c^2-2c\cdot(a+c)\cdot i-(a+c)^2+a^2+a\cdot c-a\cdot b$

c) $S = \text{gen}\{\overset{w_1}{\widetilde{1}}, \overset{w_2}{\widetilde{x - \frac{1}{2}}}\}$, $v = x^2 + x + 1$

$(w_1, v) = \int_0^1 x^2 + x + 1 \, dx = \left(\frac{x^3}{3} + \frac{x^2}{2} + x\right)\Big|_0^1 = \frac{11}{6}$; $(w_1, w_1) = 1$

$(w_2, v) = \int_0^1 (x - \frac{1}{2}) \cdot (x^2 + x + 1) \, dx = \int_0^1 \underbrace{x^3 + x^2 + x - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{2}}_{x^3 + \frac{x^2}{2} + \frac{x}{2} - \frac{1}{2}} dx = \left(\frac{x^4}{4} + \frac{x^3}{6} + \frac{x^2}{4} - \frac{x}{2}\right)\Big|_0^1 = \frac{1}{6}$

$(w_2, w_2) = \int_0^1 x^2 - x + \frac{1}{4} \, dx = \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right)\Big|_0^1 = \frac{1}{12}$

$\text{Proy}_S^v = \frac{11}{6} \cdot 1 + \frac{\frac{1}{6}}{\frac{1}{12}} \cdot (x - \frac{1}{2}) = \boxed{2x + \frac{5}{6}}$

$d(v, S) = \| x^2 + x + 1 - 2x - \frac{5}{6} \| = \| x^2 - x + \frac{1}{6} \| = \sqrt{\int_0^1 (x^2 - x + \frac{1}{6})^2 \, dx} =$

$= \sqrt{\int_0^1 \underbrace{x^4 - x^3 + \frac{x^2}{6} - x^3 + x^2 - \frac{x}{6} + \frac{x^2}{6} - \frac{x}{6} + \frac{1}{36}}_{x^4 - 2x^3 + \frac{4}{3} \cdot x^2 - \frac{x}{3} + \frac{1}{36}} dx} = \sqrt{\left(\frac{x^5}{5} - \frac{x^4}{2} + \frac{4 \cdot x^3}{9} - \frac{x^2}{6} + \frac{x}{36}\right)\Big|_0^1} = \boxed{\frac{1}{\sqrt{180}}}$

19) $u = (1 \;\; 1 \;\; 0)^t$; $v_1 = (1 \; -1 \;\; 0)^t$; $v_2 (1 \;\; 0 \;\; 1)^t$

$\text{Proy}_S^u = w$ y w debe cumplir:
- $(u - w) \perp S$
- $w \in S$.

$w = \alpha \cdot (1 \; -1 \;\; 0)^t + \beta \cdot (1 \;\; 0 \;\; 1)^t = (\alpha + \beta \;\; -\alpha \;\; \beta)^t$

$u - w = (1 - \alpha - \beta \;\;\; 1 + \alpha \;\;\; -\beta)^t$

$\Rightarrow (u - w, v_1) = 0 \underset{\downarrow}{=} (1 - \alpha - \beta \;\;\; 1 + \alpha \;\;\; -\beta) \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1 - \alpha - \beta - 1 - \alpha = -2\alpha - \beta \Rightarrow \beta = -2\alpha$

usando el
P.i. canónico

$(u - w, v_2) = (1 - \alpha - \beta \;\;\; 1 + \alpha \;\;\; -\beta) \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1 - \alpha - \beta - \beta = 1 - \alpha - 2\beta = 0$

$\Rightarrow 1 - \alpha + 4\alpha = 0 \Rightarrow \alpha = -\frac{1}{3} \Rightarrow \beta = \frac{2}{3} \Rightarrow \boxed{\text{Proy}_S^u = w = (\frac{1}{3} \;\; \frac{1}{3} \;\; \frac{2}{3})^t}$

21) - a)

$S = \text{gen}\{(1 \; -1 \;\; 1 \;\; 0)^t ; (1 \; -1 \;\; 0 \;\; 0)^t\}$

$\Rightarrow \exists ; v_1, v_2 \in S^{\perp} \Rightarrow \begin{matrix} ((1 \; -1 \;\; 1 \;\; 0)^t, v_1) = 0 \\ ((1 \; -1 \;\; 0 \;\; 0)^t, v_2) = 0 \end{matrix}$; $v = (x_1 \;\; x_2 \;\; x_3 \;\; x_4)^t$

$\Rightarrow (1 \; -1 \;\; 1 \;\; 0) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \Rightarrow x_1 - x_2 + x_3 = 0$

$(1 \; -1 \;\; 0 \;\; 0) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \Rightarrow x_1 - x_2 = 0$

$\therefore \boxed{S^{\perp} = \left\{ v \in \mathbb{R}^4 / \underbrace{\begin{matrix} x_1 - x_2 + x_2 = 0 \\ x_1 - x_2 = 0 \end{matrix}}_{} \right\}}$

pues los vec que $\in S^{\perp}$
deben cumplir estas
condiciones

b) $(u,v) = 2.u_1.v_1 + u_2.v_2 + 3.u_3.v_3 + u_4.v_4$

mismo procedimiento:

$$((1\ \ -1\ \ 1\ 0)^*, v_1) = 2\ 1.x_1 - 1.x_2 + 3.1.x_3 + 0.x_4 = 0$$

$\Rightarrow 2x_1 - x_2 + 3x_3 = 0$

$((1\ \ -1\ \ 0\ \ 0)^*, v_2) = 2.1.x_1 - 1.x_2 + 3.0.x_3 + 0.x_4 = 0 \Rightarrow 2x_1 - x_2 = 0$

$\Rightarrow x_3 = 0$

$$\therefore \boxed{S^\perp = \{v \in \mathbb{R}^4 / 2x_1 - x_2 = x_3 = 0\}}$$

c)

$S = gen\{(1\ \ 0\ \ 1+i)^*; (2\ \ 1\ \ i)^*\}$

$v = (x_1\ \ x_2\ \ x_3)$; $((1\ \ 0\ \ 1+i)^*, v_1) = \overline{(1\ \ 0\ \ 1+i)} . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 + (1-i).x_3 = 0$

$((2\ \ 1\ \ i)^*, v_2) = \overline{(2\ \ 1\ \ i)} . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2x_1 + x_2 - i.x_3 = 0$

Si quiero representar $S^\perp$ con sus generadores:

$x_1 = -(1-i).x_3 \Rightarrow -2.(1-i).x_3 + x_2 - i.x_3 = -2x_3 + 2i.x_3 + x_2 - i.x_3 = 0$

$\Rightarrow x_2 = (2-i)x_3 \Rightarrow v = ((-1+i).x_3\ \ (2-i)x_3\ \ x_3)^* = x_3.(-1+i\ \ 2-i\ \ 1), x_3 \in \mathbb{C}$

$\Rightarrow \boxed{S^\perp = gen\{(-1+i\ \ 2-i\ \ 1)^*\}}$

22) - b) - I:

$S = \{x \in \mathbb{R}^3 / 2x_1 + x_2 - x_3 = 0\}$

$\Rightarrow 2x_1 + x_2 - x_3 = (2\ \ 1\ \ -1). \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = 0 \Rightarrow (2\ \ 1\ \ -1)$ es el vector que cumple ser ortogonal a los elementos x de S. $\therefore \boxed{S^\perp = gen\{(2\ \ 1\ \ -1)^*\}}$

II: $S = \{x \in \mathbb{R}^n / x_1 = \cdots = x_r = 0\}$ (tener en cuenta que $n \neq r$)

$\Rightarrow S$ equivale a
$(1\ 0\ 0\ 0\ \ \ 0).x = 0$
$(0\ 1\ 0 \cdots 0)\ x = 0$ hasta el vector canónico r.
$(0\ 0\ 1\ 0 \cdots 0).x = 0$

$$\therefore \boxed{S^\perp \underset{\underset{gen}{\downarrow}}{=} \{e_1; \ldots; e_r\}}$$

III: $S=\{X\in \mathbb{C}^4 / x_1 - i.x_2 + (1-i).x_3=0 \ \wedge\ (2+i).x_2 + x_4 = 0\}$

$$\Rightarrow (1 \ \ -i \ \ (1-i), \ 0) . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 = (u_1, X) = u_1^H . X \Rightarrow u_1^H = (1 \ \ -i \ \ 1-i \ \ 0) \Rightarrow$$

$$\Rightarrow u_1 = \begin{pmatrix} 1 \\ i \\ 1+i \\ 0 \end{pmatrix} ; (0 \ \ 2+i \ \ 0 \ \ 1) . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 = (u_2, X) = u_2^H . X \Rightarrow u_2^H = (0 \ \ 2+i \ \ 0 \ \ 1)$$

$$\Rightarrow u_2 = \begin{pmatrix} 0 \\ 2-i \\ 0 \\ 1 \end{pmatrix} \therefore \boxed{S^{\perp} = gen\{(1 \ \ i \ \ 1+i)^t ; (0 \ \ 2-i \ \ 0 \ \ 1)^t\}}$$

c) $(1 \ \ 1 \ \ 1) . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 + x_2 + x_3 = 0 \ ; \ (1 \ -1 \ \ 0) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 - x_2 = 0$

$S^{\perp} = \{x \in \mathbb{R}^3 / \ x_1 + x_2 + x_3 = 0 = x_1 - x_2\}$

$$\Rightarrow \begin{array}{l} x_1 = x_2 \\ 2x_2 + x_3 = 0 \Rightarrow x_3 = -2x_2 \end{array} \Rightarrow x = (x_2 \ \ x_2 \ \ -2x_2)^t \Rightarrow S^{\perp} = gen\{(1 \ \ 1 \ \ -2)\}$$

$$(1 \ \ 1 \ \ -2) . \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{X} = 0 \ , \text{con } X \in S \Rightarrow \boxed{S = \{X \in \mathbb{R}^3 / x_1 + x_2 - 2x_3 = 0\}}$$

todos los ⊥ a $(1 \ 1 \ -2)^t$ forman a S

otra forma:

Pienso a $X\in S$ como c.l. de los generadores:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha . \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta . \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha + \beta \\ \alpha - \beta \\ \alpha \end{pmatrix} \Rightarrow \begin{array}{l} x_1 = \alpha + \beta \\ x_2 = \alpha - \beta \\ x_3 = \alpha \end{array}$$

y busco una relación entre x_1, x_2 y x_3 que cumpla esas 3 ecuaciones.

$$\Rightarrow \begin{array}{l} \beta = x_1 - x_3 \\ \beta = x_3 - x_2 \end{array} \Rightarrow x_1 - x_3 = x_3 - x_2 \Rightarrow \boxed{x_1 + x_2 - 2x_3 = 0}$$

El número mínimo de ec. en este caso es 1, porque $dim(S)=2$ y $V=\mathbb{R}^3$ y con cada restricción (ec.) disminuye en 1 la dim.

17) $\overbrace{(1\ 2\ 2\ 1)^t}^{v_1}, \overbrace{(1\ 1\ -1\ -1)^t}^{v_2}, \overbrace{(2\ -1\ -1\ 2)^t}^{v_3}\ ;\ u=(1\ 1\ 0\ 0)^t$

1° veo si es ortogonal:

$(v_1, v_2) = (1\ 2\ 2\ 1).\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} = 0 \quad ; (v_1, v_3) = (1\ 2\ 2\ 1).\begin{pmatrix}2\\-1\\-1\\2\end{pmatrix} = 0$

$(v_2, v_3) = (1\ 1\ -1\ -1).\begin{pmatrix}2\\-1\\-1\\2\end{pmatrix} = 0$

$\Rightarrow \boxed{B_S = \{(1\ 2\ 2\ 1)^t ; (1\ 1\ -1\ -1)^t ; (2\ -1\ -1\ 2)^t\}}$ es base ortogonal

$\Rightarrow P_S(u) = \frac{(v_1, u)}{(v_1, v_1)} . v_1 + \frac{(v_2, u)}{(v_2, v_2)} . v_2 + \frac{(v_3, u)}{(v_3, v_3)} . v_3$

$(v_1, u) = 3 \ ; (v_2, u) = 2 \ ; (v_3, u) = 1$

$(v_1, v_1) = 10 \ ; (v_2, v_2) = 4 \ ; (v_3, v_3) = 10$

$P_S(u) = \frac{3}{10}.(1\ 2\ 2\ 1)^t + \frac{1}{2}.(1\ 1\ -1\ -1)^t + \frac{1}{10}.(2\ -1\ -1\ 2)^t = \boxed{(1\ 1\ 0\ 0)^t = u}$

Esto sucede porque $u \in S$. (es c.l. de los elem. de la base).

18) $(f, g) = \int_0^1 f.g\, dx \ ,\ B = \{\overset{v_1}{1} ; \overset{v_2}{x} ; \overset{v_3}{x^2}\}$

$\Rightarrow (1, x) = \int_0^1 x\, dx = \frac{1}{2} \neq 0 \Rightarrow$ no es base ortogonal.

La base que busco esta compuesta por u_1, u_2 y u_3. y llamo $S_1 = \text{gen}\{u_1\}$
$S_2 = \text{gen}\{u_1, u_2\}$

Tomo $u_1 = 1 \ ; u_2 = v_2 - P_{S_1}(v_2) = v_2 - \frac{(u_1, v_2)}{(u_1, u_1)} u_1 =$

$(u_1, v_2) = \int_0^1 1 . x\, dx = \frac{1}{2} \ ; (u_1, u_1) = \int_0^1 1\, dx = 1$

$\Rightarrow u_2 = x - \frac{\frac{1}{2}}{1} . 1 = x - \frac{1}{2}$

$u_3 = v_3 - P_{S_2}(v_3) = v_3 - \frac{(u_1, v_3)}{(u_1, u_1)} . u_1 - \frac{(u_2, v_3)}{(u_2, u_2)} u_2$

$(u_1, v_3) = \int_0^1 x^2\, dx = \frac{1}{3} \ ; (u_2, v_3) = \int_0^1 (x - \frac{1}{2}) . x^2\, dx = \left(\frac{x^4}{4} - \frac{x^3}{6}\right)\Big|_0^1 = \frac{1}{12}$

$(u_2, u_2) = \int_0^1 x^2 - x + \frac{1}{4}\, dx = \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right)\Big|_0^1 = \frac{1}{12}$

$\Rightarrow u_3 = x^2 - \frac{1/3}{1} . 1 - \frac{1/12}{1/12} . (x - \frac{1}{2}) = x^2 - x + 1/6$

Pero no es una base ortonormal, debo dividir los elementos por su norma.

$\|u_1\|^2 = (u_1, u_1) = 1$ ✓

$\|u_2\|^2 = (u_2, u_2) = \frac{1}{12} \Rightarrow \|u_2\| = \frac{1}{\sqrt{12}}$

$\|u_3\|^2 = (u_3, u_3) = \int_0^1 (x^2 - x + 1/6).(x^2 - x + 1/6)\, dx = \int_0^1 x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36}\, dx =$

$= \left(\frac{x^5}{5} - \frac{1}{2}.x^4 + \frac{4x^3}{9} - \frac{x^2}{6} + \frac{x}{36}\right)\Big|_0^1 = \frac{1}{180} \Rightarrow \|u_3\| = \frac{1}{\sqrt{180}}$

$\Rightarrow \boxed{B' = \{1 \,;\, \sqrt{12}.(x - \frac{1}{2}) \,;\, \sqrt{180}.(x^2 - x + 1/6)\}}$

25) $(f, g) = \int_0^1 f(x).g(x)\, dx$

$S = \{P \in V \,/\, \int_0^1 P(x).(1 + x)\, dx = 0 \;\wedge\; \int_0^1 P(x).(1 - x)\, dx = 0\}$

a) otra forma de escribir esas ecuaciones es $(P, 1+x) = 0$, $(P, 1-x) = 0$

Lo cual indica que una base de $S^\perp$ es $\{\overset{v_1}{1+x} \,;\, \overset{v_2}{1-x}\}$

$(1 + x, 1 - x) = \int_0^1 1 - x^2\, dx = \left(-\frac{x^3}{3} + x\right)\Big|_0^1 = \frac{2}{3} \neq 0$ (no es base ortogonal)

Gram Schmidt: $u_1 = v_1 \Rightarrow u_2 = v_2 - P_{S_1}(v_2) = v_2 - \frac{(v_1, v_2)}{(v_1, v_1)}.v_1 = 1 - x - \frac{2/3}{7/3}.(1 + x) = -\frac{9}{7}x + \frac{5}{7}$

$S_1 = \overset{\to gen}{\{v_1\}}$

$B_{S^\perp} = \{1 + x \,;\, -\frac{9}{7}x + \frac{5}{7}\}$

b) $g(x) = x^2$, $P_{S^\perp}(x^2) = \frac{(u_1, x^2)}{(u_1, u_1)}.u_1 + \frac{(u_2, x^2)}{(u_2, u_2)}.u_2$

$(u_1, x^2) = \int_0^1 x^2 + x^3\, dx = \left(\frac{x^4}{4} + \frac{x^3}{3}\right)\Big|_0^1 = \frac{7}{12}$, $(u_1, u_1) = 7/3$

$(u_2, x^2) = \int_0^1 -\frac{9}{7}.x^3 + \frac{5}{7}.x^2\, dx = \left(-\frac{9}{28}.x^4 + \frac{5}{21}.x^3\right)\Big|_0^1 = -\frac{1}{12}$

$(u_2, u_2) = \int_0^1 \frac{81}{49}.x^2 - \frac{90}{49}.x + \frac{25}{49}\, dx = \left(\frac{81}{147}.x^3 - \frac{90}{98}.x^2 + \frac{25}{49}.x\right)\Big|_0^1 = \frac{1}{7}$

$P_{S^\perp}(x^2) = \frac{7/12}{7/3}.(1 + x) + \frac{-1/12}{1/7}.\left(-\frac{9}{7}.x + \frac{5}{7}\right) = \boxed{x - \frac{1}{6}}$

c) El elemento de S que mejor aproxima a g es $P_S(x^2)$

y $P_S(x^2) = x^2 - P_{S^\perp}(x^2) = \boxed{x^2 - x + \frac{1}{6}}$

$d(g, S) = \|x^2 - P_S(x^2)\| = \|P_{S^\perp}(x^2)\| = \sqrt{(x - 1/6, x - 1/6)} = \sqrt{\int_0^1 x^2 - \frac{x}{3} + \frac{1}{36}\, dx} = \boxed{\sqrt{\frac{7}{36}}}$

26) $S = gen\{(1\ 1\ 0)^t, (1\ 1\ -1)^t\}$

$P_S(v) = (2\ 2\ -1)^t$, $\|v\| = 5$

$$(1\ 1\ 0) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow x_1 + x_2 = 0 \qquad x_1 = -x_2$$

$$\Rightarrow$$

$$(1\ 1\ -1) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow x_1 + x_2 - x_3 = 0 \qquad x_3 = 0$$

$$\Rightarrow x = \begin{pmatrix} -x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, x_2 \in \mathbb{R} \Rightarrow S^\perp = gen\{(-1\ 1\ 0)^t\}$$

$$P_{S^\perp}(v) = \underbrace{\frac{(-1\ 1\ 0) \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}}{(-1\ 1\ 0) \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}}_{\alpha} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \alpha \cdot (-1\ 1\ 0)^t$$

$$v = P_S(v) + P_{S^\perp}(v) = (2\ 2\ -1)^t + \alpha \cdot (-1\ 1\ 0)^t$$

$$\|v\|^2 = \|(2\ 2\ -1)^t\|^2 + \|\alpha \cdot (-1\ 1\ 0)^t\|^2 = 25 = 9 + |\alpha|^2 \cdot \|(-1\ 1\ 0)^t\|^2 =$$

$\Downarrow$
pitágoras
$(v = (2\ 2\ -1)^t + \alpha \cdot (-1\ 1\ 0)^t$ y ambos vectores son $\perp$ entre sí$)$

$$= 9 + |\alpha|^2 \cdot 2 \Rightarrow |\alpha|^2 = 8 \Rightarrow |\alpha| = \sqrt{8} \Rightarrow \alpha_1 = -\sqrt{8}$$

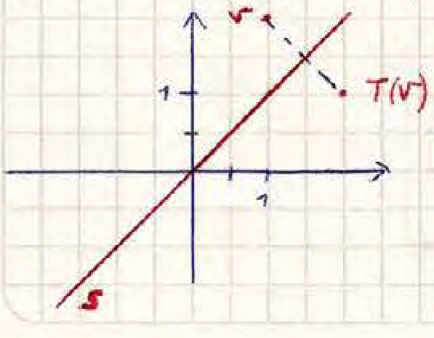
$$\alpha_2 = \sqrt{8}$$

$$\boxed{\begin{aligned} v_1 &= (2\ 2\ -1)^t + \sqrt{8} \cdot (-1\ 1\ 0)^t \\ v_2 &= (2\ 2\ -1)^t - \sqrt{8} \cdot (-1\ 1\ 0)^t \end{aligned}}$$

28)-a) $u = (1\ -1)^t$; $v = (1\ 2)^t$, $S = \{x \in \mathbb{R}^2 / (1\ -1) \cdot x = 0\}$

$\Rightarrow S = gen\{(1\ 1)^t\}$; $S^\perp = gen\{(1\ -1)^t\}$

$$T(v) = v - 2 \cdot \frac{u^t \cdot v}{u^t \cdot u} u = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 \cdot \frac{(1\ -1)\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{(1\ -1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$



(es el reflejo de V respecto de S)

b) $T(v) = v - 2\cdot\frac{u^t\cdot v}{u^t\cdot u}u = P_S(v) + P_{S^\perp}(v) - 2\cdot\frac{u^t\cdot v}{u^t\cdot u}\cdot u = P_S(v) + P_{S^\perp}(v) - 2\cdot P_{S^\perp}(v) =$

$= P_S(v) - P_{S^\perp}(v)$ ✓

c) En este caso, T refleja a v respecto del conjunto S (da la posición de su pto. reflejado).

29) - a) $S = \{x \in \mathbb{R}^3 / 2x_1 + x_2 - x_3 = 0\}$, $v = (a\ b\ c)^t$

$S^\perp = \text{gen}\{(2\ 1\ -1)^t\}$

$$T(v) = H.v = \left(I - 2\cdot\frac{u.u^t}{u^t u}\right).v = \left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{1}{3}\cdot\begin{pmatrix}4&2&-2\\2&1&-1\\-2&-1&1\end{pmatrix}\right)\cdot\begin{pmatrix}a\\b\\c\end{pmatrix} = \boxed{\frac{1}{3}\begin{pmatrix}-a-2b+2c\\-2a+2b+c\\2a+b+2c\end{pmatrix}}$$

$(u \in S^\perp) \Rightarrow u = (2\ 1\ -1)^t$

b) $S = \text{gen}\{(-2\ 1\ 0\ 0)^t; (-1\ 0\ 1\ 0)^t; (0\ 0\ 0\ -1)^t\}$, $v = (1\ 1\ 0\ 1)^t$

$$\begin{cases}-2x_1 + x_2 = 0\\ -x_1 + x_3 = 0\\ \quad -x_4 = 0\end{cases} \Rightarrow \boxed{\begin{matrix}x_2 = 2x_1\\ x_1 = x_3\end{matrix}} \quad \Rightarrow \boxed{x_4 = 0} \Rightarrow x = \begin{pmatrix}x_1\\2x_1\\x_1\\0\end{pmatrix} = x_1\cdot\begin{pmatrix}1\\2\\1\\0\end{pmatrix},\ x_1 \in \mathbb{R}$$

↓
busco $S^\perp$

$S^\perp = \text{gen}\{(1\ 2\ 1\ 0)^t\}$, $u \in S^\perp \Rightarrow$ tomo $u = (1\ 2\ 1\ 0)^t$

$$T(v) = H.v = \left(I - 2\,\underbrace{\frac{u.u^t}{u^t.u}}\right).v = \left(\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} - \frac{1}{3}\cdot\begin{pmatrix}1&2&1&0\\2&4&2&0\\1&2&1&0\\0&0&0&0\end{pmatrix}\right)\cdot\begin{pmatrix}1\\1\\0\\1\end{pmatrix} =$$

$$= \begin{pmatrix}2/3&-2/3&-1/3&0\\-2/3&-1/3&-2/3&0\\-1/3&-2/3&2/3&0\\0&0&0&1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\\1\end{pmatrix} = \boxed{\begin{pmatrix}0\\-1\\-1\\1\end{pmatrix}}$$

30) El segmento que une a $(1\ 1\ -1)^t$ y $(\sqrt{3}\ 0\ 0)^t$ es la recta ortogonal al plano respecto del cual se está reflejando.

Con las denominaciones que se usaron antes, el plano sería S y la recta $S^\perp$, además $u \in S^\perp$

$\Rightarrow (1\ 1\ -1)^t - (\sqrt{3}\ 0\ 0)^t = (1-\sqrt{3}\ 1\ -1)^t$ es el vec. dirección de $S^\perp$.

$\therefore S^\perp = \text{gen}\{(1-\sqrt{3}\ 1\ -1)^t\}$ y $u = (1-\sqrt{3}\ 1\ -1)^t$

$$H = I - 2\cdot\frac{u.u^t}{u^t.u} = \boxed{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{2}{6-\sqrt{12}}\cdot\begin{pmatrix}4-\sqrt{12}&1-\sqrt{3}&-1+\sqrt{3}\\1-\sqrt{3}&1&-1\\-1+\sqrt{3}&-1&1\end{pmatrix}}$$

Práctica 3

3) $P = u . u^T$; en un $\mathbb{R}$-e.v. se debe cumplir:
- $P = P^T$
- $P = P^2$

$P^T = (u . u^T)^T = (u^T)^T . u^T = u . u^T = P$ ✓

$P^2 = (u . u^T)^2 = u . \underbrace{u^T . u}_{1} . u^T = u . u^T = P$ ✓

1 → dado que $u^T . u = (u, u)$ es Pi canónico y $(u, u) = \|u\|^2 = 1$

P es matriz de proyección.

Por definición, u debe ser la matriz cuyas columnas son los elementos de una base ortonormal del subespacio sobre el cual se proyecta.

En este caso u es una sola columna, por lo tanto P proyecta sobre $S = \text{gen}\{u\}$; además P siempre proyecta sobre col(P), ∴ $\text{col}(P) = \text{gen}\{u\}$ y con esto sé que el rango de P es 1.

5) - a) F

b) F

c) V (por definición)

d) V (hay una única base ortonormal para un subespacio de dimensión finita, por lo tanto hay una única Q y lo mismo con P)

e) V (como P proyecta sobre col(P), col(P) no puede ser todo $\mathbb{R}^n$ porque en ese caso no se podría hablar de una proyección)

f) V $(P^2 = P \Rightarrow P^2 . P^{-1} = P . P^{-1} \Rightarrow P = I)$
↓
sólo sucede si existe P^{-1}

→ en realidad sólo se efectúa multiplicando por I al [illegible] para el cual se busca la proyección

6) - a) $S = \text{gen}\{(1\ 2\ 2)^t ; (-2\ \ 2\ \ -1)^t\}$

uso el P.i. canónico:

$$(1\ 2\ 2) . \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} = 0$$ (es una base ortogonal)

⇒ la normalizo: $\|(1\ 2\ 2)^t\| = \sqrt{(1\ 2\ 2)(1\ 2\ 2)^t} = \sqrt{1+4+4} = 3$

$\|(-2\ \ 2\ \ -1)^t\| = \sqrt{4+4+1} = 3$

$B_S = \{(\frac{1}{3}\ \ \frac{2}{3}\ \ \frac{2}{3})^t, (-\frac{2}{3}\ \ \frac{2}{3}\ \ -\frac{1}{3})^t\} \Rightarrow P = \frac{1}{9} . \begin{pmatrix} 1 & -2 \\ 2 & 2 \\ 2 & -1 \end{pmatrix} . \begin{pmatrix} 1 & 2 & 2 \\ -2 & 2 & -1 \end{pmatrix} = \boxed{\frac{1}{9} \begin{pmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{pmatrix}}$

b) $S^{\perp} = \text{gen}\{(1\ 1\ 3)^t ; (-1\ 1\ -1)^t\}$

$\Rightarrow S = \left\{X \in \mathbb{R}^3 / \begin{array}{l}(1\ 1\ 3).X = 0 \\ (-1\ 1\ -1).X = 0\end{array}\right\} \Rightarrow \left\{\begin{array}{l}X_1 + X_2 + 3X_3 = 0 \\ -X_1 + X_2 - X_3 = 0\end{array}\right. \Rightarrow \left(\begin{array}{ccc|c}1 & 1 & 3 & 0 \\ -1 & 1 & -1 & 0\end{array}\right) \rightarrow \left(\begin{array}{ccc|c}1 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0\end{array}\right)$

$\Rightarrow \left\{\begin{array}{l}X_1 + X_2 + 3X_3 = 0 \Rightarrow X_1 - X_3 + 3X_3 = 0 \Rightarrow X_1 = -2X_3 \\ 2X_2 + 2X_3 = 0 \Rightarrow X_2 = -X_3\end{array}\right.$

$\Rightarrow S = \text{gen}\{(-2\ -1\ 1)^t\}$ (n ortogonal) $\Rightarrow \|(-2\ -1\ 1)^t\| = \sqrt{4+1+1} = \sqrt{6}$

$\Rightarrow P = \frac{1}{6} \cdot \begin{pmatrix}-2 \\ -1 \\ 1\end{pmatrix} \cdot (-2\ -1\ 1) = \boxed{\frac{1}{6} \cdot \begin{pmatrix}4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1\end{pmatrix}}$

c) $S = \{X \in \mathbb{R}^4 / X_1 + 2X_2 - X_3 + X_4 = 0\}$

$\Rightarrow S^{\perp} = \text{gen}\{(1\ 2\ -1\ 1)^t\}$

$\Rightarrow \|(1\ 2\ -1\ 1)^t\| = \sqrt{1+4+1+1} = \sqrt{7}$

$\Rightarrow P_{S^{\perp}} = \frac{1}{7} \cdot \begin{pmatrix}1 \\ 2 \\ -1 \\ 1\end{pmatrix} \cdot (1\ 2\ -1\ 1) = \frac{1}{7}\begin{pmatrix}1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ -1 & -2 & 1 & -1 \\ 1 & 2 & -1 & 1\end{pmatrix} \Rightarrow P_{S^{\perp}} + P_S = I_{\mathbb{R}^4} \Rightarrow P_S = I_{\mathbb{R}^{4\times4}} - P_{S^{\perp}} =$

$= \begin{pmatrix}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{pmatrix} - \frac{1}{7} \cdot \begin{pmatrix}1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ -1 & -2 & 1 & -1 \\ 1 & 2 & -1 & 1\end{pmatrix} = \boxed{\frac{1}{7} \cdot \begin{pmatrix}6 & -2 & 1 & -1 \\ -2 & 3 & 2 & -2 \\ 1 & 2 & 6 & 1 \\ -1 & -2 & 1 & 6\end{pmatrix}}$

7) Como se cumple que $\text{Col}(AB) \subset \text{col}(A)$ ~~y~~ que $\text{rg}(A) = 2 = \text{rg}(AB)$

$\Rightarrow \text{col}(A) = \text{col}(AB)$

$\Rightarrow \text{Col}(A) = \{\underbrace{(1\ -1\ 0)^t}_{V_1}, \underbrace{(1\ 2\ 3)^t}_{V_2}\} \Rightarrow (1\ -1\ 0). \begin{pmatrix}1 \\ 2 \\ 3\end{pmatrix} = -1 \neq 0$

uso teorema de Gram-Schmidt para hallar una base ortogonal.

$u_1 = (1\ -1\ 0)^t \ ; S_1 = \text{gen}\{u_1\}$

$\Rightarrow u_2 = V_2 - P_{S_1}(V_2) = (1\ 2\ 3)^t - \underbrace{\frac{((1\ -1\ 0)^t ; (1\ 2\ 3)^t)}{((1\ -1\ 0)^t ; (1\ -1\ 0)^t)}}_{-\frac{1}{2}} \cdot (1\ -1\ 0)^t = \begin{pmatrix}1 \\ 2 \\ 3\end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix}1 \\ -1 \\ 0\end{pmatrix} = \begin{pmatrix}3/2 \\ 3/2 \\ 3\end{pmatrix}$

$\Rightarrow \|(1\ -1\ 0)^t\| = \sqrt{2} \ , \|(3/2\ \ 3/2\ \ 3)\| = \sqrt{27/2} \Rightarrow B_{\text{Col}(A)} = \{\frac{1}{\sqrt{2}} \cdot (1\ -1\ 0)^t ; \sqrt{2/27} \cdot (3/2\ \ 3/2\ \ 3)^t\}$

$$\Rightarrow P = \begin{pmatrix} 1/\sqrt{2} & 3/\sqrt{54} \\ -1/\sqrt{2} & 3/\sqrt{54} \\ 0 & \sqrt{2/27}\cdot 3 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 3/\sqrt{54} & 3/\sqrt{54} & 3\cdot\sqrt{2/27} \end{pmatrix} = \boxed{\begin{pmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \\ 1/3 & 1/3 & 2/3 \end{pmatrix}}$$

8) X es $\perp$ a cada columna de A.

$\Rightarrow X \in (col(A))^{\perp}$; además, si $rg(A)=3$, $dim(col(A))=3$

Con lo cual $dim(col(A)^{\perp})=1 \Rightarrow col(A)^{\perp} = gen\{(1\ 1\ 1\ 1)^t\}$

$\|(1\ 1\ 1\ 1)^t\| = 2 \Rightarrow$ una BON de $col(A)^{\perp}$ es $B=\{\frac{1}{2}\cdot(1\ 1\ 1\ 1)^t\}$

$$\Rightarrow P_{col(A)^{\perp}} = \frac{1}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\cdot(1\ 1\ 1\ 1) = \frac{1}{4}\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}$$

$$\Rightarrow P_{col(A)^{\perp}} + P_{col(A)} = I_{\mathbb{R}^{4\times4}} \Rightarrow P_{col(A)} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} - \frac{1}{4}\cdot\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix} = \boxed{\frac{1}{4}\cdot\begin{pmatrix}3&-1&-1&-1\\-1&3&-1&-1\\-1&-1&3&-1\\-1&-1&-1&3\end{pmatrix}}$$

10) - 1)

$$P = \begin{pmatrix}1/3 & 1/3 & 1/3\\ 1/3 & 5/6 & -1/6\\ 1/3 & -1/6 & 5/6\end{pmatrix} \Rightarrow P\cdot P = P^2 = \frac{1}{9}\cdot\begin{pmatrix}1&1&1\\1&5/2&-1/2\\1&-1/2&5/2\end{pmatrix}\cdot\begin{pmatrix}1&1&1\\1&5/2&-1/2\\1&-1/2&5/2\end{pmatrix} = \frac{1}{9}\cdot\begin{pmatrix}3&3&3\\3&15/2&-3/2\\3&-3/2&15/2\end{pmatrix} = P \ ✓$$

$\wedge\ P = P^t$ porque P es simétrica.

P es matriz de proyección.

11)

$$\Rightarrow \begin{pmatrix}1/3&1/3&1/3\\1/3&5/6&-1/6\\1/3&-1/6&5/6\end{pmatrix} \to \begin{pmatrix}1/3&1/3&1/3\\0&-1/2&1/2\\ \cancel{0}&\cancel{1/2}&\cancel{-1/2}\end{pmatrix} \to rg(P)=2$$

Como P es la matriz de proyección sobre $col(P)$, tengo que hallar una BON de $col(P)$.

el $(1\ 1\ 1)^t \in col(P)$
$(0\ -1\ 1)^t \in col(P)$
$\Rightarrow$ con el P.I. canónico: $(1\ 1\ 1)\cdot\begin{pmatrix}0\\-1\\1\end{pmatrix} = 0$ (son ortogonales)

(son múltiplos de las columnas l.i.)

$\|(1\ 1\ 1)^t\| = \sqrt{3}$
$\|(0\ -1\ 1)^t\| = \sqrt{2}$
$\Rightarrow B_{col(P)} = \{\frac{1}{\sqrt{3}}\cdot(1\ 1\ 1)^t ; \frac{1}{\sqrt{2}}\cdot(0\ -1\ 1)^t\}$

Con lo cual Q tiene que ser $\boxed{Q = \begin{pmatrix}1/\sqrt{3} & 0\\ 1/\sqrt{3} & -1/\sqrt{2}\\ 1/\sqrt{3} & 1/\sqrt{2}\end{pmatrix}}$

16) - a)

$$A=\begin{pmatrix}-1 & 2\\ 2 & -3\\ -1 & 3\end{pmatrix};\ b=\begin{pmatrix}4\\1\\2\end{pmatrix} \Rightarrow \left(\begin{array}{cc|c}-1 & 2 & 4\\ 2 & -3 & 1\\ -1 & 3 & 2\end{array}\right) \to \left(\begin{array}{cc|c}-1 & 2 & 4\\ 0 & 1 & 9\\ 0 & 5 & 6\end{array}\right) \to \left(\begin{array}{cc|c}-1 & 2 & 4\\ 0 & 1 & 9\\ 0 & 0 & 39\end{array}\right) \to S.I.$$

Resuelvo por cuadrados mínimos.

$$\Rightarrow AX=b \Rightarrow A^t.A.\hat{X}=A^t.b \quad , A^tA=\begin{pmatrix}-1 & 2 & -1\\ 2 & -3 & 3\end{pmatrix}\cdot\begin{pmatrix}-1 & 2\\ 2 & -3\\ -1 & 3\end{pmatrix}=\begin{pmatrix}6 & -11\\ -11 & 22\end{pmatrix}$$

$$A^t.b=\begin{pmatrix}-1 & 2 & -1\\ 2 & -3 & 3\end{pmatrix}\cdot\begin{pmatrix}4\\1\\2\end{pmatrix}=\begin{pmatrix}-4\\11\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}6 & -11\\ -11 & 22\end{pmatrix}.\hat{X}=\begin{pmatrix}-4\\11\end{pmatrix} \Rightarrow \left(\begin{array}{cc|c}6 & -11 & -4\\ -11 & 22 & 11\end{array}\right) \to \left(\begin{array}{cc|c}6 & -11 & -4\\ 0 & 11 & 22\end{array}\right) \to \left(\begin{array}{cc|c}6 & -11 & -4\\ 0 & 1 & 2\end{array}\right)$$

$$\Rightarrow \begin{cases}6x_1-11x_2=4\\ \boxed{x_2=2}\end{cases} \Rightarrow \boxed{x_1=3} \Rightarrow \boxed{\hat{X}=\begin{pmatrix}3\\2\end{pmatrix}}$$

b)

$$A=\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\\ 1 & 0 & 1\end{pmatrix};\ b=\begin{pmatrix}1\\3\\8\\2\end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c}1 & 1 & 0 & 1\\ 1 & 1 & 0 & 3\\ 1 & 0 & 1 & 8\\ 1 & 0 & 1 & 2\end{array}\right)$$

→ las filas que son iguales deberían tener el mismo resultado, pero no es así. Es seguro que es S.I.

$$\Rightarrow A^tA=\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\\ 1 & 0 & 1\end{pmatrix}=\begin{pmatrix}4 & 2 & 2\\ 2 & 2 & 0\\ 2 & 0 & 2\end{pmatrix}$$

$$A^t.b=\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{pmatrix}\cdot\begin{pmatrix}1\\3\\8\\2\end{pmatrix}=\begin{pmatrix}14\\4\\10\end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c}4 & 2 & 2 & 14\\ 2 & 2 & 0 & 4\\ 2 & 0 & 2 & 10\end{array}\right) \to \left(\begin{array}{ccc|c}4 & 2 & 2 & 14\\ 0 & 2 & -2 & -6\\ \cancel{0} & \cancel{-2} & \cancel{2} & \cancel{6}\end{array}\right) \to \left(\begin{array}{ccc|c}2 & 1 & 1 & 7\\ 0 & 1 & -1 & -3\end{array}\right)$$

$$\begin{cases}2x_1+x_2+x_3=7\\ x_2-x_3=-3\end{cases} \Rightarrow \begin{array}{l}2x_1-3+x_3+x_3=7 \Rightarrow x_1=5-x_3\\ x_2=-3+x_3\end{array}$$

$$\Rightarrow \hat{X}=\begin{pmatrix}5-x_3\\ -3+x_3\\ x_3\end{pmatrix}=\boxed{x_3.\begin{pmatrix}-1\\1\\1\end{pmatrix}+\begin{pmatrix}5\\-3\\0\end{pmatrix}\ ;\ x_3\in R}$$

17)

$$A=\begin{pmatrix}3 & 4\\ -2 & 1\\ 3 & 4\end{pmatrix};\ b=\begin{pmatrix}11\\-9\\5\end{pmatrix};\ u=\begin{pmatrix}5\\-1\end{pmatrix},\ v=\begin{pmatrix}5\\-2\end{pmatrix}$$

$$\Rightarrow A.u=\begin{pmatrix}3 & 4\\ -2 & 1\\ 3 & 4\end{pmatrix}\cdot\begin{pmatrix}5\\-1\end{pmatrix}=\begin{pmatrix}11\\-11\\11\end{pmatrix}\quad , A.v=\begin{pmatrix}3 & 4\\ -2 & 1\\ 3 & 4\end{pmatrix}\cdot\begin{pmatrix}5\\-2\end{pmatrix}=\begin{pmatrix}7\\-12\\7\end{pmatrix}$$

$$\|b-u\|=\|(0\quad 2\quad -6)^t\|=\sqrt{40}=d(b,u)\quad , d(b,v)=\|b-v\|=\|(4\quad 3\quad -2)^t\|=\sqrt{29}$$

↓ con P.I. canónico

$$\Rightarrow d(b,v)<d(b,u)$$

∴ u no puede ser solución por cuad. mín. porque no es el pto más cercano a b.

19) - a) V ; b) V ; c) F $\|b-A.\hat{x}\| \leq \|b-A.x\|$ pues $A.\hat{x}$ es la proyección de b, el pto. a menor distancia de este.

d) V ; e) F ; f) V ; g) V

↳ siempre existe una proyección

20) - I) $A=\begin{pmatrix}1&1&2\\-1&2&1\\-1&1&0\end{pmatrix}$ => busca $nul(A)=nul(A^t A)$

$$\Rightarrow \left(\begin{array}{ccc|c}1&1&2&0\\-1&2&1&0\\-1&1&0&0\end{array}\right) \rightarrow \left(\begin{array}{ccc|c}1&1&2&0\\0&3&3&0\\ \cancel{0}&\cancel{2}&\cancel{2}&\cancel{0}\end{array}\right) \rightarrow \left(\begin{array}{ccc|c}1&1&2&0\\0&1&1&0\end{array}\right) \rightarrow \begin{cases}x_1+x_2+2x_3=0\\ x_2+x_3=0\end{cases} \Rightarrow x_2=-x_3$$

$$\Rightarrow x_1-x_3+2x_3=0 \Rightarrow x_1=-x_3 \Rightarrow Nul(A)=gen\{(-1\ \ -1\ \ 1)^t\}$$

y las soluciones del problema de cuad. mínimos pueden escribirse como $\hat{x}=v+u$ con $v \in Nul(A)=Nul(A^t A)$ y u solución particular de cuadrados mín

$$\Rightarrow \boxed{\hat{x}=\alpha.\begin{pmatrix}-1\\-1\\1\end{pmatrix}+\begin{pmatrix}1\\2\\-1\end{pmatrix}}$$

II) como $\underset{\downarrow}{A.\hat{x}} = P_{col(A)}(b) \Rightarrow b = A.\overset{(1\ 2\ 1)^t}{\overset{\nearrow}{\hat{x}}} + \underset{\downarrow}{c}$, con $c \in (col(A))^\perp$

esto se cumple para cualquiera de los $\hat{x}$, así que puedo usar $(1\ 2\ -1)^t$ por ej

pues $b \notin col(A)$, entonces necesito otra coordenada que no esté en col(A)

$$\Rightarrow Col(A)=gen\{(1\ \ -1\ \ -1)^t;(1\ \ 2\ \ 1)^t\}$$

$\Rightarrow (Col(A))^\perp$ debe cumplir (con P.i canónico) : $(1\ \ -1\ \ -1)\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0 \quad x_1-x_2-x_3=0$

$\Rightarrow$

$(1\ \ 2\ \ 1)\cdot\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0 \quad x_1+2x_2+x_3=0$

$$\Rightarrow \left(\begin{array}{ccc|c}1&-1&-1&0\\1&2&1&0\end{array}\right) \rightarrow \left(\begin{array}{ccc|c}1&-1&-1&0\\0&-3&-2&0\end{array}\right)$$

$$\Rightarrow \begin{cases}x_1-x_2-x_3=0\\ -3x_2-2x_3=0\end{cases} \Rightarrow \begin{array}{l}x_1+\frac{2}{3}x_3-x_3=0 \Rightarrow x_1=\frac{1}{3}x_3\\ x_2=-\frac{2}{3}x_3\end{array} \Rightarrow (col(A))^\perp = gen\{(1\ \ -2\ \ 3)^t\}$$

$$\Rightarrow b = K.\begin{pmatrix}1\\-2\\3\end{pmatrix}+\begin{pmatrix}1\\2\\1\end{pmatrix} \ ; \ \underset{\downarrow}{\|b\|=3} \Rightarrow (K+1)^2+(-2K+2)^2+(3K+1)^2=9$$

P.i. canónico

$\Rightarrow k^2 + 2k + 1 + 4k^2 \cancel{-8k} + 4 + 9k^2 \cancel{+6k} + 1 = 9 = 14k^2 + 6$

$\Rightarrow 14k^2 - 3 = 0 \Rightarrow k = \pm \frac{\sqrt{168}}{28} = \pm \frac{\sqrt{42}}{14}$

$$\boxed{b_1 = \frac{\sqrt{42}}{14} \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \; ; \; b_2 = -\frac{\sqrt{42}}{14} \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}$$

22) $A^{\#} = (A^* A)^{-1} A^*$

a)

$$A^* A = \begin{pmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{pmatrix} = \underset{\substack{\downarrow \\ \text{busco} \\ \text{inverso}}}{\begin{pmatrix} 6 & -11 \\ -11 & 22 \end{pmatrix}} \Rightarrow \left(\begin{array}{cc|cc} 6 & -11 & 1 & 0 \\ -11 & 22 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 6 & -11 & 1 & 0 \\ 0 & 11 & 11 & 6 \end{array}\right) \rightarrow$$

$$\rightarrow \left(\begin{array}{cc|cc} 6 & 0 & 12 & 6 \\ 0 & 11 & 11 & 6 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 6/11 \end{array}\right) \rightarrow (A^* A)^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 6/11 \end{pmatrix}$$

$$\Rightarrow A^{\#} = (A^* A)^{-1} A^* = \begin{pmatrix} 2 & 1 \\ 1 & 6/11 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 1 & 1 \\ 1/11 & 4/11 & 7/11 \end{pmatrix}}$$

b)

$$A^* A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

→ esa matriz no es inversible porque tiene columnas l.d.

$\Rightarrow \nexists A^{\#}$

24) Resuelvo por cuadrados mínimos.

La ecuación de la recta es $y = m.x + b$

a)

$$\Rightarrow \begin{array}{l} 1 = 0 + b \\ 1 = m + b \\ 2 = 2m + b \\ 2 = 3m + b \end{array} \equiv \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}}_{A} \cdot \begin{pmatrix} m \\ b \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}}_{b}$$

→ 1° reviso que efectivamente sea un S.I.

$$\Rightarrow \left(\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{array}\right) \rightarrow \left.\left(\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{array}\right)\right\} \neq \text{resultados, es un S.I.}$$

Resuelvo las ec normales:

$$A^* \cdot A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix}$$

$$A^* \cdot b = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{cc|c} 14 & 6 & 11 \\ 6 & 4 & 6 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 14 & 6 & 11 \\ 0 & -20 & -18 \end{array}\right) \rightarrow \begin{cases} 14x_1 + 6x_2 = 11 \\ \quad -20x_2 = -18 \end{cases} \Rightarrow \begin{array}{l} x_1 = 2/5 \\ x_2 = \frac{9}{10} \end{array}$$

Donde $x_1 = m$ y $x_2 = b$ ⇒ La recta que mejor ajusta a los ptos es:

$$\boxed{y = \frac{2}{5} x + 9/10}$$

b)

$$\begin{aligned} 1 &= m + b \\ 2 &= m + b \\ -1 &= m + b \end{aligned} \equiv \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}}_{A} \cdot \begin{pmatrix} m \\ b \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}}_{b} \rightarrow \text{seguro es S.I.}$$

$$A^{t}.A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \quad ; \quad A^{t}.b = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \cdot \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 3 & 3 & 2 \\ \cancel{3} & \cancel{3} & \cancel{2} \end{array}\right) \longrightarrow 3m + 3b = 2 \Rightarrow b = -m + \frac{2}{3}$$

$$Y = m.x - m + \frac{2}{3} \Rightarrow \boxed{Y = m.(x-1) + \frac{2}{3}} \quad (\text{son } \infty \text{ rectas})$$

26) a)

$$\text{Con } v = \text{cte} \Rightarrow P = P_0 + v.t$$

$$\Rightarrow \begin{aligned} 5{,}07 &= P_0 + v.1 \\ 10{,}43 &= P_0 + v.2 \\ 15{,}94 &= P_0 + v.3 \\ 21{,}63 &= P_0 + v.4 \\ 27{,}49 &= P_0 + v.5 \end{aligned} \equiv \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} P_0 \\ v \end{pmatrix}}_{\hat{x}} = \underbrace{\begin{pmatrix} 5{,}07 \\ 10{,}43 \\ 15{,}94 \\ 21{,}63 \\ 27{,}49 \end{pmatrix}}_{b} \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 5{,}07 \\ 1 & 2 & 10{,}43 \\ 1 & 3 & 15{,}94 \\ 1 & 4 & 21{,}63 \\ 1 & 5 & 27{,}49 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 5{,}07 \\ 0 & -1 & -5{,}36 \\ 0 & -2 & -10{,}87 \\ 0 & -3 & \vdots \\ 0 & -4 & \vdots \end{array}\right)$$

} una de estos valores debería ser el doble del otro, pero no es así, => es un S.I.

$$A^{t}.A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 15 \\ 15 & 55 \end{pmatrix} \quad ; \quad A^{t}.b = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 5{,}07 \\ 10{,}43 \\ 15{,}94 \\ 21{,}63 \\ 27{,}49 \end{pmatrix} = \begin{pmatrix} 80{,}56 \\ 297{,}72 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{cc|c} 5 & 15 & 80{,}56 \\ 15 & 55 & 297{,}72 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 5 & 15 & 80{,}56 \\ 0 & -10 & -56{,}04 \end{array}\right) \rightarrow \begin{cases} 5.P_0 + 15.v = 80{,}56 \\ +10.v = +56{,}04 \end{cases} \Rightarrow \begin{aligned} P_0 &= -0{,}7 \\ v &= 5{,}604 \end{aligned}$$

La función de posición que mejor ajusta es: $\boxed{P(t) = 5{,}604.t - 0{,}7}$

(v es 5,604 m/s)

b)

Ahora, con $a = cte \Rightarrow P(t) = P_0 + V_0\cdot(t - \underset{0}{\underset{\downarrow}{t_0}}) + \frac{1}{2}\cdot a\cdot(t - \underset{0}{\underset{\downarrow}{t_0}})^2$

$$\begin{aligned} 5{,}07 &= P_0 + V_0\cdot 1 + a\cdot\tfrac{1}{2} \\ 10{,}43 &= P_0 + 2V_0 + 2a \\ 15{,}94 &= P_0 + 3V_0 + \tfrac{9}{2}a \\ 21{,}63 &= P_0 + 4V_0 + 8a \\ 27{,}49 &= P_0 + 5V_0 + \tfrac{25}{2}a \end{aligned} \equiv \underbrace{\begin{pmatrix} 1 & 1 & 1/2 \\ 1 & 2 & 2 \\ 1 & 3 & 9/2 \\ 1 & 4 & 8 \\ 1 & 5 & 25/2 \end{pmatrix}}_{A} \cdot \underset{\tilde{x}}{\begin{pmatrix} P_0 \\ V_0 \\ a \end{pmatrix}} = \underbrace{\begin{pmatrix} 5{,}07 \\ 10{,}43 \\ 15{,}94 \\ 21{,}63 \\ 27{,}49 \end{pmatrix}}_{b}$$

$$\Rightarrow A^t\cdot A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1/2 & 2 & 9/2 & 8 & 25/2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1/2 \\ 1 & 2 & 2 \\ 1 & 3 & 9/2 \\ 1 & 4 & 8 \\ 1 & 5 & 25/2 \end{pmatrix} = \begin{pmatrix} 5 & 15 & 55/2 \\ 15 & 55 & \frac{225}{2} \\ 55/2 & \frac{225}{2} & 979/4 \end{pmatrix} ;\ A^t\cdot b = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1/2 & 2 & 9/2 & 8 & 25/2 \end{pmatrix} \cdot \begin{pmatrix} 5{,}07 \\ 19{,}43 \\ 15{,}94 \\ 21{,}63 \\ 27{,}49 \end{pmatrix} = \begin{pmatrix} 80{,}56 \\ 297{,}72 \\ 611{,}79 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 5 & 15 & 55/2 & 80{,}56 \\ 15 & 55 & 225/2 & 297{,}72 \\ 55/2 & \frac{225}{2} & 979/4 & 611{,}79 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 5 & 15 & 55/2 & 80{,}56 \\ 0 & -50 & -150 & -280{,}2 \\ 0 & -150 & -\frac{335}{2} & -843{,}55 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 5 & 15 & 55/2 & 80{,}56 \\ 0 & -50 & -150 & -280{,}2 \\ 0 & 0 & 35/2 & 2{,}95 \end{array}\right)$$

$$\Rightarrow \begin{cases} 5\cdot P_0 + 15\cdot V_0 + 55/2\cdot a = 80{,}56 \\ \quad -50V_0 - 150\cdot a = -280{,}2 \\ \quad\quad 35/2\ a = 2{,}95 \end{cases} \Rightarrow \begin{aligned} P_0 &= -0{,}1102 \\ V_0 &= 5{,}0983 \\ a &= 0{,}1686 \end{aligned}$$

$\Rightarrow$ La función que mejor ajusta es: $\boxed{\begin{aligned} &P(t) = -0{,}1102 + 5{,}0983\cdot t + 0{,}1686\cdot\tfrac{1}{2}\cdot t^2 \\ &a = 0{,}1686\ m/s^2 \end{aligned}}$

c)

Por lo de b), ya que con $a \neq 0$ se logra una mejor aproximación a la situación real.

También se puede saber viendo qué proyección está más cerca al valor de b.

En a), $\underset{P_{col(A)}(b)}{\underset{\|}{A\cdot\hat{x}}} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} -0{,}7 \\ 5{,}604 \end{pmatrix} = \begin{pmatrix} 4{,}904 \\ 10{,}508 \\ 16{,}112 \\ 21{,}716 \\ 27{,}32 \end{pmatrix} \Rightarrow \|b - P_{col(A)}(b)\| \underset{\text{P.i. canónico}}{\underset{\downarrow}{=}} 0{,}3155$

A

En b), $\underset{P_{col(A)}(b)'}{\underset{\|}{A\cdot\hat{x}}} = \begin{pmatrix} 1 & 1 & 1/2 \\ 1 & 2 & 2 \\ 1 & 3 & 9/2 \\ 1 & 4 & 8 \\ 1 & 5 & 25/2 \end{pmatrix} \cdot \begin{pmatrix} -0{,}1102 \\ 5{,}0983 \\ 0{,}1686 \end{pmatrix} = \begin{pmatrix} 5{,}0724 \\ 10{,}4236 \\ 15{,}9434 \\ 21{,}6318 \\ 27{,}4888 \end{pmatrix} \Rightarrow \|b - P_{col(A)}(b)'\| = 0{,}007935 < \|b - P_{col(A)}(b)\|$

Práctica 4

1) - a) $T:\mathbb{R}^2\to\mathbb{R}$, $T((x_1\ \ x_2)^t)=x_1+2x_2$

=> • $T\left(\begin{pmatrix}u_1\\u_2\end{pmatrix}+\begin{pmatrix}v_1\\v_2\end{pmatrix}\right)=T\begin{pmatrix}u_1+v_1\\u_2+v_2\end{pmatrix}=(u_1+v_1)+2.(u_2+v_2)=u_1+2.u_2+v_1+2.v_2=$

$=T\begin{pmatrix}u_1\\u_2\end{pmatrix}+T\begin{pmatrix}v_1\\v_2\end{pmatrix}$ ✓

• $T\left(\lambda.\begin{pmatrix}x_1\\x_2\end{pmatrix}\right)=T\begin{pmatrix}\lambda.x_1\\\lambda.x_2\end{pmatrix}=\lambda.x_1+2.(\lambda.x_2)=\lambda.(x_1+2x_2)=\lambda.T\begin{pmatrix}x_1\\x_2\end{pmatrix}$ ✓

∴ es t.l.

b) $T:\mathbb{R}^2\to\mathbb{R}^2$, $T((x_1\ \ x_2)^t)=(1+x_2\ \ \ x_1)^t$

• $T\begin{pmatrix}u_1+v_1\\u_2+v_2\end{pmatrix}=\begin{pmatrix}1+u_2+v_2\\u_1+v_1\end{pmatrix}=\begin{pmatrix}1+u_2\\u_1\end{pmatrix}+\begin{pmatrix}v_2\\v_1\end{pmatrix}\neq T\begin{pmatrix}u_1\\u_2\end{pmatrix}+T\begin{pmatrix}v_1\\v_2\end{pmatrix}$ ∴ no es t.l.

c) $T:\mathbb{C}\to\mathbb{C}$, $T(z)=\bar{z}$

$\mathbb{C}$ como $\mathbb{C}$ - e.v.:

• $T(z_1+z_2)=\overline{z_1+z_2}=\bar{z}_1+\bar{z}_2=T(z_1)+T(z_2)$ ✓

• $T(k.z)=\overline{k.z}=\bar{k}.\bar{z}=\bar{k}.T(z)\neq k.T(z)$ ⊗ ∴ no es t.l.
$k\in\mathbb{C}$

$\mathbb{C}$ como $\mathbb{R}$ - e.v.

• $T(z_1+z_2)=T(z_1)+T(z_2)$ ✓ (se hace igual)

• $T(k.z)=\overline{k.z}=\bar{k}.\bar{z}=k.\bar{z}=k.T(z)$ ✓ ∴ es t.l.
$k\in\mathbb{R}$
↓
si k es real
$k=\bar{k}$

d) $T:\mathbb{R}^{m\times m}\to\mathbb{R}$; $T(A)=tr(A)$ $\quad(tr(A)=a_{11}+a_{22}+\cdots+a_{mm})$

• $T(A+B)=tr(A+B)=(a_{11}+b_{11})+(a_{22}+b_{22})+\cdots+(a_{mm}+b_{mm})=$

$=(a_{11}+\cdots+a_{mm})+(b_{11}+\cdots+b_{mm})=tr(A)+tr(B)=T(A)+T(B)$ ✓

• $T(\lambda.A)=tr(\lambda.A)=\lambda.a_{11}+\cdots+\lambda.a_{mm}=\lambda.(a_{11}+\cdots+a_{mm})=\lambda.tr(A)=$
$\lambda\in\mathbb{K}$

$=\lambda.T(A)$ ✓ ∴ es t.l.

2) Si $T(\alpha_1.v_1 + \alpha_2.v_2) = \alpha_1.T(v_1) + \alpha_2.T(v_2)$ se están verificando las 2 condiciones necesarias para que T sea t.l.:

- $T(v_1 + v_2) = T(v_1) + T(v_2)$
- $T(\lambda.v) = \lambda.T(v)$

$\Rightarrow T(\alpha_1.v_1 + \alpha_2.v_2) = T(\alpha_1.v_1) + T(\alpha_2.v_2) = \alpha_1.T(v_1) + \alpha_2.T(v_2)$

3) - a) $T: \mathbb{C}^m \to \mathbb{C}^n$; $T(v) = A.v$; $A \in \mathbb{C}^{n\times m}$

- $v; w \in \mathbb{C}^m$; $T(v+w) = A.(v+w) = A.v + A.w = T(v) + T(w)$ ✓
- $v \in \mathbb{C}^m$; $\lambda \in \mathbb{K}$; $T(\lambda.v) = A.(\lambda.v) = \lambda.(A.v) = \lambda.T(v)$ ✓ ∴ es t.l.

b) $T: P \to P$; $T(P) = (x^2+1).P$

- $P, Q \in P$; $T(P+Q) = (x^2+1).(P+Q) = (x^2+1).P + (x^2+1).Q = T(P) + T(Q)$ ✓
- $P \in P$, $\lambda \in \mathbb{K}$; $T(\lambda.P) = (x^2+1).(\lambda.P) = \lambda.(x^2+1).P = \lambda.T(P)$ ✓ ∴ es t.l.

c) $T_f: C[a;b] \to C[a;b]$, $T_f(g) = f.g$ con $f \in C[a;b]$

- $g, h \in C[a;b]$; $T_f(g+h) = f.(g+h) = f.g + f.h = T_f(g) + T_f(h)$
- $g \in C[a;b]$, $\lambda \in \mathbb{K}$; $T_f(\lambda.g) = f.(\lambda.g) = \lambda.(f.g) = \lambda.T_f(g)$ ∴ es t.l.

d) $T: C[0;1] \to \mathbb{R}$; $T(f) = \int_0^1 f(x)\,dx$

- $f, g \in C[0;1]$; $T(f+g) = \int_0^1 (f+g)(x)\,dx = \int_0^1 (f(x) + g(x))\,dx = \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx =$
$= T(f) + T(g)$
- $f \in C[0;1]$, $\lambda \in \mathbb{K}$; $T(\lambda.f) = \int_0^1 (\lambda.f)(x)\,dx = \int_0^1 \lambda.f(x)\,dx = \lambda.\int_0^1 f(x)\,dx = \lambda.T(f)$

∴ es t.l.

e) $T: C^2(\mathbb{R}) \to C(\mathbb{R})$; $T(f) = f'' - f$

- $f, g \in C^2(\mathbb{R})$; $T(f+g) = (f+g)'' - (f+g) = f'' + g'' - f - g = f'' - f + g'' - g = T(f) + T(g)$
- $f \in C^2(\mathbb{R})$, $\lambda \in \mathbb{K}$; $T(\lambda.f) = (\lambda f)'' - (\lambda.f) = \lambda.f'' - \lambda.f = \lambda.(f'' - f) = \lambda.T(f)$

∴ es t.l.

f) $T: \mathbb{C}^{n\times m} \to \mathbb{C}^{k\times 1}$, $T(X) = A.X.B$; $A \in \mathbb{C}^{k\times n}$; $B \in \mathbb{C}^{m\times 1}$

- $X, Y \in \mathbb{C}^{n\times m}$; $T(X+Y) = A.(X+Y).B = A.X.B + A.Y.B = T(X) + T(Y)$
- $X \in \mathbb{C}^{n\times m}$; $\lambda \in \mathbb{K}$; $T(\lambda.X) = A.(\lambda.X).B = \lambda.(A.X.B) = \lambda.T(X)$ ∴ es t.l.

(en cada caso se aplican propiedades de producto y suma de polinomios, matrices y funciones)

4) - a)

- $v, w \in \mathbb{V}$; $I(v+w) = (v+w) = I(v) + I(w)$
- $v \in \mathbb{V}$; $\lambda \in \mathbb{K}$; $I(\lambda . v) = \lambda . v = \lambda . I(v)$ $\therefore$ es t.l.

b) $v, w \in \mathbb{V}$; $O(v+w) = \underset{\mathbb{W}}{0} = \overset{0}{\widehat{O(v)}} + \overset{0}{\widehat{O(w)}}$

$v \in \mathbb{V}$, $\lambda \in \mathbb{K}$; $O(\lambda . v) = 0 = \lambda . \underbrace{O(v)}_{0}$ $\therefore$ es t.l.

c) $T_\lambda : \mathbb{V} \to \mathbb{V}$; $T_\lambda(v) = \lambda . v$

- $v, w \in \mathbb{V}$; $T_\lambda(v+w) = \lambda . (v+w) = \lambda . v + \lambda . w = T_\lambda(v) + T_\lambda(w)$
- $v \in \mathbb{V}$; $\alpha \in \mathbb{K}$; $T_\lambda(\alpha v) = \lambda \, \alpha . v = \alpha . \lambda . v = \alpha . T_\lambda(v)$ $\therefore$ es t.l.

5) $T(X) = A . X$

I) $A = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ $(k \in \mathbb{R})$

$T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} . \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} k \\ k \end{pmatrix}$

$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} . \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} k \\ 0 \end{pmatrix}$; $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} . \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ k \end{pmatrix}$

La imagen del cuadrado tiene vértices: $\boxed{(0 \ 0)^t ; (k \ k)^t ; (k \ 0)^t ; (0 \ k)^t}$

II) $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = A . \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$; $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Los vértices de la imagen son: $\boxed{(0 \ 0)^t ; (1 \ 0)^t ; (0 \ -1)^t ; (1 \ -1)^t}$

III) $A = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ $(k \in \mathbb{R}) \Rightarrow T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} k \\ 0 \end{pmatrix}$; $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} k \\ 1 \end{pmatrix}$

vértices: $\boxed{(0 \ 0)^t ; (k \ 0)^t ; (0 \ 1)^t ; (k \ 1)^t}$

IV) $A = \begin{pmatrix} \cos(\alpha) & -\text{sen}(\alpha) \\ \text{sen}(\alpha) & \cos(\alpha) \end{pmatrix} \Rightarrow T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\alpha) \\ \text{sen}(\alpha) \end{pmatrix}$; $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\text{sen}(\alpha) \\ \cos(\alpha) \end{pmatrix}$; $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\alpha) - \text{sen}(\alpha) \\ \text{sen}(\alpha) + \cos(\alpha) \end{pmatrix}$

$(\alpha \in \mathbb{R})$

vértices: $\boxed{\begin{pmatrix} \cos(\alpha) \\ \text{sen}(\alpha) \end{pmatrix} ; \begin{pmatrix} 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} -\text{sen}(\alpha) \\ \cos(\alpha) \end{pmatrix} ; \begin{pmatrix} \cos(\alpha) - \text{sen}(\alpha) \\ \text{sen}(\alpha) + \cos(\alpha) \end{pmatrix}}$

V)

$A=\begin{pmatrix}0&-1\\-1&0\end{pmatrix} \Rightarrow T\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix};\ T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\-1\end{pmatrix};\ T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix};\ T\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}-1\\-1\end{pmatrix}$

vértices: $\boxed{(0\ \ 0)^t;\ (0\ \ -1)^t;\ (-1\ \ 0)^t;\ (-1\ \ -1)^t}$

VI)

$A=\begin{pmatrix}1&0\\0&0\end{pmatrix} \Rightarrow T\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix};\ T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix};\ T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix};\ T\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$

vértices: $\boxed{(0\ \ 0)^t;\ (0\ \ 0)^t;\ (1\ \ 0)^t;\ (1\ \ 0)^t}$ (una recta)

7) - a) $T: \mathbb{R}^2 \to \mathbb{R}^3;\ T(v)=A.v\ ;\ A=\begin{pmatrix}1&2\\0&-1\\2&1\end{pmatrix}$

$\Rightarrow \bullet\, Nu(T)=\{v\in\mathbb{R}^2 \,/\, T(v)=0_{\mathbb{R}^3}\} \Rightarrow \begin{pmatrix}1&2\\0&-1\\2&1\end{pmatrix}.\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \left(\begin{array}{cc|c}1&2&0\\0&-1&0\\2&1&0\end{array}\right) \to \left(\begin{array}{cc|c}1&2&0\\0&-1&0\\\cancel{0}&\cancel{3}&\cancel{0}\end{array}\right)$

$\Rightarrow \begin{cases}x_1+2x_2=0\\-x_2=0\end{cases} \Rightarrow x_1=x_2=0$

$\Rightarrow \boxed{Nu(T)=\{0_{\mathbb{R}^2}\}}$

$\bullet\ Im(T)=\{u\in\mathbb{R}^3 \,/\, T(v)=u\ ;\ v\in\mathbb{R}^2\} \Rightarrow \begin{pmatrix}1&2\\0&-1\\2&1\end{pmatrix}.\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} \Rightarrow$

$\Rightarrow \left(\begin{array}{cc|c}1&2&u_1\\0&-1&u_2\\2&1&u_3\end{array}\right) \longrightarrow \underset{2F_1-F_3}{\left(\begin{array}{cc|c}1&2&u_1\\0&-1&u_2\\0&3&2u_1-u_3\end{array}\right)} \longrightarrow \underset{3F_2+F_3}{\left(\begin{array}{cc|c}1&2&u_1\\0&-1&u_2\\0&0&2u_1+3u_2-u_3\end{array}\right)}$

$\Rightarrow -2.u_1+3.u_2-u_3=0 \Rightarrow u_3=2u_1+3u_2$

$u\in\begin{pmatrix}u_1\\u_2\\2u_1+3u_2\end{pmatrix}=u_1.\begin{pmatrix}1\\0\\2\end{pmatrix}+u_2.\begin{pmatrix}0\\1\\3\end{pmatrix} \Rightarrow \boxed{Im(T)=\left\{\begin{pmatrix}1\\0\\2\end{pmatrix};\begin{pmatrix}0\\1\\3\end{pmatrix}\right\}}$ (son l.i.)

otra forma más fácil es:

$A.X=u \Rightarrow (A_1\ \ A_2).\begin{pmatrix}x_1\\x_2\end{pmatrix}=A_1.x_1+A_2.x_2=\begin{pmatrix}1\\0\\2\end{pmatrix}.x_1+\begin{pmatrix}2\\-1\\1\end{pmatrix}.x_2=u$, lo cual indica que el u que busco es combinación lineal de esos 2 vectores.

$\Rightarrow Im(T)=\left\{\begin{pmatrix}1\\0\\2\end{pmatrix};\begin{pmatrix}2\\-1\\1\end{pmatrix}\right\}$ y $(2\ \ -1\ \ 1)^t$ es c.l. de los vec. de la base hallada antes, por lo tanto es el mismo subespacio.

b) $T: \mathbb{R}^{2\times2} \to \mathbb{R}$, $T(A) = X.A.Y$; $X = (1 \;\; 2)$; $Y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$A \in \mathbb{R}^{2\times2} / T(A) = 0_{\mathbb{R}} \Rightarrow (1 \;\; 2) . \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \Rightarrow (a_{11} + 2a_{21} \;\;\; a_{12} + 2a_{22}) . \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \Rightarrow$

$\Rightarrow a_{11} + 2a_{21} - a_{12} - 2.a_{22} = 0 \Rightarrow a_{11} = a_{12} + 2a_{22} - 2a_{21}$

$\Rightarrow A = \begin{pmatrix} a_{12} + 2a_{22} - 2a_{21} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{12} . \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + a_{22} . \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + a_{21} . \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix}$

Por la forma de las matrices, son l.i.

$\boxed{Nu(T) = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} ; \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} ; \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix} \right\}}$

$\Rightarrow u = a_{11} + 2a_{21} - a_{12} - 2.a_{22}$ es una c.l. de $\{1; 2; -1; -2\}$, pero en $\mathbb{R}$ todos son múltiplos, entonces queda un espacio generado por un solo escalar.

$\boxed{Im(T) = \{1\}} = \mathbb{R}$

c) $T: P \to \mathbb{R}^2$, $T(P) = (P(0) \;\; P'(0))^*$

$\begin{pmatrix} P(0) \\ P'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} P(0) = 0 \\ P'(0) = 0 \end{matrix}$; Los $P \in P$ tiene la forma $P(x) = a_0 + a_1.x + \cdots + a_m.x^m$
$\Rightarrow P'(x) = a_1 + a_2.2x + \cdots + a_m.m.x^{(m-1)}$

$\Rightarrow P(0) = a_0$; $P'(0) = a_1 \Rightarrow a_0 = a_1 = 0$

$\therefore \boxed{Nu(T) = \{ P \in P / P(x) = a_2.x^2 + \cdots + a_m.x^m \}}$ (no tiene dim finita)

si $u = (x_1 \;\; x_2)^* \Rightarrow \begin{matrix} P(0) = a_0 = x_1 \\ P'(0) = a_1 = x_2 \end{matrix}$

$\Rightarrow u = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = a_0 . \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_1 . \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $a_1, a_0 \in \mathbb{R} \Rightarrow \boxed{Im(T) = \mathbb{R}^2}$

d) $T: C[0;1] \to C[0;1]$, $T(f)(x) = (1 + x^2) . f(x)$ (el dominio es $f(x)$, no x)

$\Rightarrow \underbrace{(1 + x^2)}_{\neq 0} . f(x) = 0_{C[0;1]} \Rightarrow f(x) = O(x)$ (función nula)

$Nu(T) = \{ f(x) \in C[0;1] / T(f)(x) = 0_{C[0;1]} = \underset{\text{función nula}}{O(x)} \} \Rightarrow \boxed{Nu(T) = \{O(x)\}}$

$T(f)(x) = g(x)$ con $g \in C[0;1]$ (codominio) $\Rightarrow (1 + x^2) . f(x) = g(x)$ pero estas son todas las funciones continuas

$\Rightarrow \boxed{Im(T) = C[0;1]}$

e) $T: \mathcal{C}^2(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$; $T(P) = P''$

$P'' = 0 \underset{\substack{\downarrow \\ \text{integrando}}}{\Rightarrow} P' = C \quad (C\ \text{cte}) \Rightarrow P = C.X + a$

$Nu(T) = \{P \in \mathcal{C}^2(\mathbb{R}) \,/\, P = C.X + a \,;\, C, a \in \mathbb{R}\} = \boxed{\{1; x\}}$

$T(P) = P'' = g$ con $g \in \mathcal{C}(\mathbb{R})$, esto quiere decir que g son todas las funciones continuas, porque $P \in \mathcal{C}^2(\mathbb{R})$. (funciones continuas hasta su der. 2°)

$$\boxed{Im(T) = \mathcal{C}(\mathbb{R})}$$

9) $T: \mathbb{C}^m \to \mathbb{C}^n$, con $T(v) = A.v$; $A \in \mathbb{C}^{n \times m}$

$Im(T) \Rightarrow A.v = u$ con $u \in \mathbb{C}^n \Rightarrow$

$$A.v = \begin{pmatrix} A_1 & \cdots & A_m \end{pmatrix} . \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = A_1.v_1 + \cdots + A_m.v_m = u$$

$\Rightarrow v_1; \ldots; v_m \in \mathbb{R} \Rightarrow \boxed{Im(T) = \{A_1; \cdots; A_m\} = Col(A)}$

$Nu(T) \Rightarrow \boxed{\underset{\substack{\downarrow \\ \text{por definición}}}{Nul(A)} = \{v \in \mathbb{C}^m \,/\, A.v = 0\} = Nu(T)}$

El dominio tiene dim finita "m"; se debe cumplir $dim(Nu(T)) + dim(Im(T)) =$

$= \underbrace{dim\, \mathbb{C}^m}_{m}$

Si rango$(A) = K \Rightarrow dim(Im(T)) = K$ (habrá K columnas l.i.)

$\Rightarrow dim(Nu(T)) + K = m \Rightarrow \boxed{dim(Nu(T)) = m - K}$

10) - a) $T: \mathbb{R}^5 \to \mathbb{R}^4$; $T(v) = A.v$;

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 & -3 \\ 3 & 6 & 4 & -1 & 2 \\ 4 & 8 & 5 & 1 & -1 \\ -2 & -4 & -3 & 3 & -5 \end{pmatrix}$$

$Nu(T) = \{v \in \mathbb{R}^5 \,/\, A.v = 0_{\mathbb{R}^4}\}$

$$\Rightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & -3 & 0 \\ 3 & 6 & 4 & -1 & 2 & 0 \\ 4 & 8 & 5 & 1 & -1 & 0 \\ -2 & -4 & -3 & 3 & -5 & 0 \end{array}\right) \longrightarrow \begin{array}{r} \\ 3F_1 - F_2 \\ 4F_1 - F_3 \\ 2F_1 + F_4 \end{array} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & -3 & 0 \\ 0 & 0 & -1 & 7 & -11 & 0 \\ \cancel{0} & \cancel{0} & \cancel{-1} & \cancel{7} & \cancel{-11} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{-1} & \cancel{7} & \cancel{-11} & \cancel{0} \end{array}\right)$$

$$\Rightarrow \begin{cases} X_1 + 2X_2 + X_3 + 2X_4 - 3X_5 = 0 \\ \quad -X_3 + 7X_4 - 11X_5 = 0 \end{cases} \Rightarrow \begin{array}{l} X_1 + 2X_2 + 7X_4 - 11X_5 + 2X_4 - 3X_5 = 0 \Rightarrow X_1 = -2X_2 - 9X_4 + 14X_5 \\ X_3 = 7X_4 - 11X_5 \end{array}$$

$$\Rightarrow v = X_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + X_4 . \begin{pmatrix} -9 \\ 0 \\ 7 \\ 1 \\ 0 \end{pmatrix} + X_5 . \begin{pmatrix} 14 \\ 0 \\ -11 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \boxed{Nu(T) = \{(-2\ 1\ 0\ 0\ 0)^t; (-9\ 0\ 7\ 1\ 0)^t; (14\ 0\ -11\ 0\ 1)^t\}}$$

$Im(T) = \{ u \in \mathbb{R}^4 / T(v) = u \ ; \ v \in \mathbb{R}^5 \}$

$\Rightarrow A.v = u = A_1.x_1 + \cdots + A_5.x_5 \quad (x_1, \ldots ; x_5 \in \mathbb{R}) \Rightarrow Im(T) = gen \left\{ \begin{pmatrix} 1 \\ 3 \\ 4 \\ -2 \end{pmatrix} ; \begin{pmatrix} 2 \\ 6 \\ 8 \\ -4 \end{pmatrix} ; \begin{pmatrix} 1 \\ 4 \\ 5 \\ -3 \end{pmatrix} ; \begin{pmatrix} 2 \\ -1 \\ 1 \\ 3 \end{pmatrix} ; \begin{pmatrix} -3 \\ 2 \\ -1 \\ -5 \end{pmatrix} \right\}$

$$\Rightarrow \begin{pmatrix} 1 & 3 & 4 & -2 \\ 2 & 6 & 8 & -4 \\ 1 & 4 & 5 & -3 \\ 2 & -1 & 1 & 3 \\ -3 & 2 & -1 & -5 \end{pmatrix} \longrightarrow \begin{array}{r} \\ 2F_1 - F_2 \\ F_1 - F_3 \\ 2F_1 - F_4 \\ 3F_1 + F_5 \end{array} \begin{pmatrix} 1 & 3 & 4 & -2 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & -1 & -1 & 1 \\ \cancel{0} & \cancel{7} & \cancel{7} & \cancel{-7} \\ \cancel{0} & \cancel{11} & \cancel{11} & \cancel{-11} \end{pmatrix} \Rightarrow \boxed{Im(T) = \left\{ \begin{pmatrix} 1 \\ 3 \\ 4 \\ -2 \end{pmatrix} ; \begin{pmatrix} 1 \\ 4 \\ 5 \\ -3 \end{pmatrix} \right\}}$$

b) $T: \mathbb{C}^3 \to \mathbb{C}^2 \ ; \ T((z_1 \ z_2 \ z_3)^*) = (i.z_1 + z_2 + z_3 \quad -2.z_1 + 2.i.z_2 + 2.i.z_3)^*$

Como $\mathbb{C}$ - e.v.:

$Nu(T)$:

$$\Rightarrow \begin{pmatrix} i.z_1 + z_2 + z_3 \\ -2.z_1 + 2.i.z_2 + 2.i.z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c} i & 1 & 1 & 0 \\ -2 & 2i & 2i & 0 \end{array} \right) \to \begin{array}{r} \\ 2.F_1 + i.F_2 \end{array} \left(\begin{array}{ccc|c} i & 1 & 1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array} \right)$$

$\Rightarrow i.z_1 + z_2 + z_3 = 0 \Rightarrow z_3 = -i.z_1 - z_2$

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ -i.z_1 - z_2 \end{pmatrix} = z_1 . \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} + z_2 . \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \boxed{Nu(T) = \left\{ \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} ; \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}} \text{ (son l.i.)}$$

$Im(T)$:

$$u \in \mathbb{C}^2 \ ; \ u = \begin{pmatrix} i.z_1 + z_2 + z_3 \\ -2.z_1 + 2i.z_2 + 2i.z_3 \end{pmatrix} = z_1 . \begin{pmatrix} i \\ -2 \end{pmatrix} + z_2 . \begin{pmatrix} 1 \\ 2i \end{pmatrix} + z_3 . \begin{pmatrix} 1 \\ 2i \end{pmatrix}$$ y son l.d. los 3

Por lo tanto $\boxed{Im(T) = \left\{ \begin{pmatrix} i \\ -2 \end{pmatrix} \right\}}$

Como $\mathbb{R}$ - e.v.:

$Nu(T)$: Planteo a los escalares complejos como suma de una parte real y otra imaginaria.

Del núcleo anterior: $z_1 . \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} + z_2 . \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = (a + b.i) . \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} + (c + d.i) . \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} =$

$$= a . \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} + b . \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} + c . \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + d . \begin{pmatrix} 0 \\ i \\ -i \end{pmatrix} \quad a, b, c, d \in \mathbb{R} \Rightarrow \boxed{Nu(T) = \left\{ \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} ; \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} ; \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} ; \begin{pmatrix} 0 \\ i \\ -i \end{pmatrix} \right\}}$$

$Im(T)$:

$$z . \begin{pmatrix} i \\ -2 \end{pmatrix} = (a + b.i) . \begin{pmatrix} i \\ -2 \end{pmatrix} = a . \begin{pmatrix} i \\ -2 \end{pmatrix} + b . \begin{pmatrix} -1 \\ -2i \end{pmatrix} \Rightarrow \boxed{Im(T) = \left\{ \begin{pmatrix} i \\ -2 \end{pmatrix} ; \begin{pmatrix} -1 \\ -2i \end{pmatrix} \right\}}$$

12) - a) $T: P_3 \to \mathbb{R}^2$; $T(P) = (P(0) \quad P(1))^t$

$Nu(T): \begin{pmatrix} P(0) \\ P(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; los $P \in P_3$ son de la forma $P(x) = a_0 + a_1.x + a_2.x^2 + a_3.x^3$

$$\Rightarrow P(0) = a_0 = 0$$

$$P(1) = \underset{\substack{\downarrow \\ 0}}{a_0} + a_1 + a_2 + a_3 = 0 \Rightarrow a_1 = -a_2 - a_3$$

$$P(x) = (-a_2 - a_3).x + a_2.x^2 + a_3.x^3 = a_2.(x^2 - x) + a_3.(x^3 - x)$$

$$\boxed{Nu(T) = \{x^2 - x \,;\, x^3 - x\}}$$

$Im(T)$:

$$u \in \mathbb{R}^2 \,;\, u = \begin{pmatrix} P(0) \\ P(1) \end{pmatrix} = \begin{pmatrix} a_0 \\ a_0 + a_1 + a_2 + a_3 \end{pmatrix} = a_0.\begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_1.\begin{pmatrix} 0 \\ 1 \end{pmatrix} + a_2.\begin{pmatrix} 0 \\ 1 \end{pmatrix} + a_3.\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad a_0; a_1; a_2; a_3 \in \mathbb{R}$$

$$\Rightarrow Im(T) = \boxed{\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}} = \mathbb{R}^2$$

b) $T: P_2 \to \mathbb{R}^3$; $T(P) = (P(t_0) \quad P'(t_0) \quad P''(t_0))^t$; $t_0 \in \mathbb{R}$ (lo llamo x)

$$Nu(T): \begin{pmatrix} P(x) \\ P'(x) \\ P''(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \; ; \begin{array}{l} P(x) = a_0 + a_1.x + a_2.x^2 \\ P'(x) = a_1 + 2a_2.x \\ P''(x) = 2.a_2 \end{array} \Rightarrow \begin{cases} a_0 + a_1.x + a_2.x^2 = 0 \Rightarrow \boxed{a_0 = 0} \\ a_1 + 2.a_2 x = 0 \Rightarrow \boxed{a_1 = 0} \\ 2.a_2 = 0 \Rightarrow \boxed{a_2 = 0} \end{cases}$$

$$\boxed{Nu(T) = \{0_{P_2}\}}$$

$Im(T)$:

$$u \in \mathbb{R}^3 \,;\, u = \begin{pmatrix} P(x) \\ P'(x) \\ P''(x) \end{pmatrix} = \begin{pmatrix} a_0 + a_1 x + a_2 x^2 \\ a_1 + 2a_2 x \\ 2a_2 \end{pmatrix} = a_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a_1 \begin{pmatrix} x \\ 1 \\ 0 \end{pmatrix} + a_2.\begin{pmatrix} x^2 \\ 2x \\ 2 \end{pmatrix}$$

$$\boxed{Im(T) = \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} x \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} x^2 \\ 2x \\ 2 \end{pmatrix}\right\}} = \mathbb{R}^3 \quad \text{(son l.i.)}$$

c) $T: \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$; $T(X) = A.X - X.A$; $A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$

$Nu(T)$:

$$A.X - X.A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} . \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} - \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} . \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} X_{11} + X_{21} & X_{12} + X_{22} \\ 2X_{11} - X_{21} & 2X_{12} - X_{22} \end{pmatrix} -$$

$$- \begin{pmatrix} X_{11} + 2.X_{12} & X_{11} - X_{12} \\ X_{21} + 2.X_{22} & X_{21} - X_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} X_{21} - 2X_{12} & 2X_{12} + X_{22} - X_{11} \\ 2X_{11} - 2X_{21} - 2X_{22} & 2X_{12} - X_{21} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} X_{22} - 2X_{12} = 0 \\ 2X_{12} + X_{22} - X_{11} = 0 \\ \cancel{2X_{11} - 2X_{21} - 2X_{22} = 0} \\ \cancel{2X_{12} - X_{21} = 0} \end{cases} \Rightarrow \begin{array}{l} X_{21} = 2X_{12} \\ X_{11} = X_{22} + 2X_{12} \end{array} \Rightarrow \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \begin{pmatrix} X_{22} + 2X_{12} & X_{12} \\ 2.X_{12} & X_{22} \end{pmatrix} =$$

se repiten

$$= X_{12}.\begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} + X_{22}.\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \boxed{Nu(T) = \left\{\begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}}$$

$Im(T)$:

$$B \in \mathbb{R}^{2\times 2} ; \begin{pmatrix} x_{21}-2x_{12} & 2x_{12}+x_{22}-x_{11} \\ 2x_{11}-2x_{21}-2x_{22} & 2x_{12}-x_{21} \end{pmatrix} = B =$$

$$= x_{12}.\begin{pmatrix} -2 & 2 \\ 0 & 2 \end{pmatrix} + x_{11}.\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} + x_{21}.\begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} + x_{22}.\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$$

l.d.

l.d.

$$\boxed{Im(T)=\left\{\begin{pmatrix} -2 & 2 \\ 0 & 2 \end{pmatrix} ; \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}\right\}}$$

13) - a) verdadera. Si $\{v_1; \dots ; v_r\}$ es l.d. $\Rightarrow \alpha_1.v_1 + \dots + \alpha_r.v_r = 0$, $\alpha_i \in \mathbb{R}$ / algún $\alpha \neq 0$.

$$\Rightarrow T(\underbrace{\alpha_1.v_1 + \dots + \alpha_r.v_r}_{0}) \underset{\substack{\downarrow \\ T \text{ es t.l.}}}{=} \alpha_1.T(v_1) + \dots + \alpha_r.T(v_r) = 0 \quad (T(0)=0 \text{ siempre})$$

ya que algún α es $\neq 0 \Rightarrow \{T(v_1); \dots ; T(v_r)\}$ es l.d.

b) ver~~da~~dera. Si $\{T(v_1); \dots ; T(v_r)\}$ es l.d. $\Rightarrow \alpha_1.T(v_1) + \dots + \alpha_r.T(v_r) = 0$, $\alpha_i \in \mathbb{R}$ / algún $\alpha \neq 0$

$$\Rightarrow \alpha_1.T(v_1) + \dots + \alpha_r.T(v_r) \underset{\substack{\downarrow \\ T \text{ es t.l.}}}{=} T(\alpha_1.v_1 + \dots + \alpha_r.v_r) = 0 \Rightarrow$$

$\Rightarrow \alpha_1.v_1 + \dots + \alpha_r.v_r \in Nu(T) \Rightarrow Nu(T) = \{v_1; \dots ; v_r\}$ pero como se sabe que $\exists$ algún $\alpha \neq 0$ resulta que el conjunto $\{v_1; \dots ; v_r\}$ es l.d.

c)-d) ver~~da~~deros. (mismo razonamiento que antes)

e) Si $Nu(T) = \{0\} \Rightarrow$ la solución de $T(\underbrace{\alpha_1.v_1 + \dots + \alpha_m.v_m}_{\text{c.l. genérica de los elementos de } V}) = 0$ es

$\alpha_1.v_1 + \dots + \alpha_m.v_m = 0$, y si $\alpha_1 = \dots = \alpha_m = 0 \Rightarrow \{v_1; \dots ; v_m\}$ es l.i.

$\Rightarrow T(\alpha_1.v_1 + \dots + \alpha_m.v_m) = \alpha_1.T(v_1) + \dots + \alpha_m.T(v_m) = 0$ y por lo anterior, $\{T(v_1); \dots ; T(v_m)\}$ es l.i.

verdadera.

f) $\{v_1; \dots ; v_n\}$ es base (es l.i.) $\Rightarrow \alpha_1.v_1 + \dots + \alpha_n.v_m = 0 \Leftrightarrow \alpha_1 = \dots = \alpha_m = 0$

$\Rightarrow T(\alpha_1.v_1 + \dots + \alpha_m.v_m) = \alpha_1.T(v_1) + \dots + \alpha_n.T(v_m) \underset{\circledast}{\Rightarrow} \{T(v_1), \dots , T(v_m)\}$ es l.i.

Por $\circledast$, $T(\alpha_1.v_1 + \dots + \alpha_n.v_m) = 0 \Leftrightarrow \alpha_1 = \dots = \alpha_n = 0$ lo cual indica que sólo $T(0) = 0$

$\Rightarrow Nu(T) = \{0\}$ (verdadera)

14) - c) V. Si v y v' son 2 soluciones de $T(v) = u \Rightarrow T(v) = T(v') = u$

$\Rightarrow T(v) - T(v') = u - u = 0_W = T(v - v') \Rightarrow v - v' \in Nu(T) = \{0_V\} \Rightarrow v - v' = 0_V \Rightarrow v = v'$

d) V. Si $Im(T) = W$, T es epimorfismo. Si $Nu(T) = \{0\}$, T es monomorfismo (ya que sólo el 0_V va al 0_W y ningún otro elemento de V lo hace)

$\therefore$ T es un isomorfismo y admite inversa.

f) V. $\dim(Nu(T)) + \dim(Im(T)) = \dim V \Rightarrow$ si $Nu(T) = \{0\} \Rightarrow \dim(Im(T)) = \dim V = \dim W$

y no queda otra más que $Im(T) = W$ pues $Im(T) \in W$

Si $Im(T) = W \Rightarrow \dim(Im(T)) = \dim W = \dim V = n$

$\Rightarrow \dim(Nu(T)) + \cancel{n} = \cancel{n} \Rightarrow \dim(Nu(T)) = 0 \Rightarrow Nu(T) = \{0\}$

(esto solo sirve para espacios vectoriales de dimensión n finito.)

g) V. (es el mismo ejercicio anterior)

h) $\dim Nu(T) + \dim Im(T) = \dim V \Rightarrow \dim Nu(T) = \dim V - \dim Im(T) > 0$ pues $Im(T) \in W$ y a lo sumo $\dim Im(T) = \dim W < \dim V$.

Si $\dim Nu(T) > 0$, $Nu(T) \neq \{0\}$. (V)

i) $\dim Nu(T) + \dim Im(T) = \dim V \Rightarrow \dim Im(T) = \dim V - \dim Nu(T) \leq \dim V < \dim W$

ya que $\dim Nu(T) \leq \dim V$ $(Nu(T) \in V)$

Como $\dim Im(T) < \dim W \Rightarrow Im(T) \neq W$ (V)

j) Si T es biyectiva, $\underset{(Nu(T) = \{0\})}{\dim Nu(T) = 0}$ y $\underset{(Im(T) = W)}{\dim Im(T) = \dim W}$

$\Rightarrow 0 + \dim W = \dim V$ (V) (el teorema solo se puede aplicar si dim de los e.v. es n finito)

15) - a)

$B.A = \begin{vmatrix} B.A_1 & B.A_2 & \cdots & B.A_m \end{vmatrix}$ siendo $\{A_1; \ldots; A_m\} = Col(A)$

$\Rightarrow \{B.A_1; \ldots; B.A_m\} = Col(BA)$; si $Y \in Col(BA) \Rightarrow (B.A).X = Y = B.(\underset{K}{\underbrace{AX}}) =$

$= (B_1 \cdots B_m).K = \underset{(k_1; \ldots; k_m \in R)}{B_1 k_1 + \cdots + B_m . k_m} \Rightarrow Y \in Col(B) \Rightarrow Col(B.A) \subseteq Col(B)$ ✓

17) $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(x_1 \; x_2 \; x_3)^t = (x_1 - x_2 + x_3 \quad 2x_1 \quad x_1 + x_2)^t$

1° veo si admite inversa:

$Im(T)$: $u \in \mathbb{R}^3$, $u = \begin{pmatrix} x_1 - x_2 + x_3 \\ 2x_1 \\ x_1 + x_2 \end{pmatrix} = x_1 . \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + x_2 . \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + x_3 . \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $x_1, x_2, x_3 \in \mathbb{R}$

y claramente son l.i. => $Im(T) = \{(1 \; 2 \; 1)^t ; (-1 \; 0 \; 1)^t ; (1 \; 0 \; 0)^t\}$

Como el dominio tiene dim. finita y $\dim Im(T) = 3$ => $\dim Nu(T) = 0$, es decir,

$Im(T) = \mathbb{R}^3$ (porque $Im(T) \subset \mathbb{R}^3$ y tiene dim3) y $Nu(T) = \{0_{\mathbb{R}^3}\}$

=> T es iso. y admite inversa.

Transformo los vectores de la base canónica de $\mathbb{R}^3$.

$T(1 \; 0 \; 0)^t = (1 \quad 2 \quad 1)^t$
$T(0 \; 1 \; 0)^t = (-1 \quad 0 \quad 1)^t$
$T(0 \; 0 \; 1)^t = (1 \quad 0 \quad 0)^t$

(que son justamente todos los vectores posibles de la imagen)

Como T es iso., la transformación de una base me da otra base.

Entonces defino T^{-1} sobre esa base:

$$\boxed{\begin{array}{l} T^{-1}(1 \; 2 \; 1)^t = (1 \quad 0 \quad 0)^t \\ T^{-1}(-1 \; 0 \; 1)^t = (0 \quad 1 \quad 0)^t \\ T^{-1}(1 \; 0 \; 0)^t = (0 \quad 0 \quad 1)^t \end{array}}$$

Si quiero la fórmula de T^{-1}:

Planteo $\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = \alpha . \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \beta . \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \gamma . \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \alpha - \beta + \gamma = x_1 \\ 2\alpha = x_2 \\ \alpha + \beta = x_3 \end{cases}$

$\gamma = x_1 - \frac{x_2}{2} + x_3 - \frac{x_2}{2} = x_1 - x_2 + x_3$

$\Rightarrow \alpha = \frac{x_2}{2}$

$\beta = x_3 - \frac{x_2}{2}$

$\Rightarrow T^{-1}(x) = T^{-1}(\alpha . (1 \; 2 \; 1)^t + \beta . (-1 \; 0 \; 1)^t + \gamma . (1 \; 0 \; 0)^t) =$

$= \alpha . T^{-1}(1 \; 2 \; 1)^t + \beta \; T^{-1}(-1 \; 0 \; 1)^t + \gamma . T^{-1}(1 \; 0 \; 0)^t =$

$= \frac{x_2}{2} . (1 \; 0 \; 0)^t + (x_3 - \frac{x_2}{2}) . (0 \; 1 \; 0)^t + (x_1 - x_2 + x_3) . (0 \; 0 \; 1)^t$

$\Rightarrow \boxed{T^{-1}(x) = \left(\frac{x_2}{2} \quad x_3 - \frac{x_2}{2} \quad x_1 - x_2 + x_3 \right)^t}$

18) $T: C[0;1] \to C[0,1]$, $T(f)(t) = (1+t^2)\cdot f(t)$

$Nu(T)$: $\underbrace{(1+t^2)}_{\neq 0}\cdot f(t) = \underset{\substack{\downarrow \\ \text{función nula}}}{0_{C[0,1]}} \Rightarrow f(t) = 0_{C[0,1]}$; $Nu(T) = \{0_{C[0,1]}\}$

$Im(T)$: $g \in C[0,1] \Rightarrow (1+t^2)\cdot f(t) = g(t)$ como g es composición de 2 funciones continuas, g es continua, más precisamente, g son todas las continuas.

$Im(T) = C[0,1]$

T es mono. y epi. ∴ es iso. y admite inversa (T^{-1})

Planteo: $h(t) = (1+t^2)\cdot f(t) \Rightarrow f(t) = \dfrac{h(t)}{\underbrace{1+t^2}_{\neq 0}}$

$\Rightarrow$ Defino $T^{-1}: C[0,1] \to C[0,1]$ / $\boxed{T^{-1}(h)(t) = \dfrac{h(t)}{1+t^2}}$

19) $T: P_2 \to \mathbb{R}^3$, $T(p) = (p(t_0) \quad p'(t_0) \quad p''(t_0))^t$, $t_0 \in \mathbb{R}$

Sé que $Nu(T) = \{0_{P_2}\}$ e $Im(T) = \mathbb{R}^3$ ∴ es iso. y admite inversa.

Transformo los elementos de la base canónica de P_2:

$T(1) = (1 \quad 0 \quad 0)^t$

$T(t) = (t_0 \quad 1 \quad 0)^t$; ya que T es iso., sé que el conjunto obtenido es base.

$T(t^2) = (t_0^2 \quad 2t_0 \quad 2)^t$

Defino T^{-1}:

$$\boxed{\begin{aligned} T^{-1}(1 \quad 0 \quad 0)^t &= 1 \\ T^{-1}(t_0 \quad 1 \quad 0)^t &= t \\ T^{-1}(t_0^2 \quad 2t_0 \quad 2)^t &= t^2 \end{aligned}}$$

20) $Nu(S \circ T) = \{v \in V \,/\, (S\circ T)(v) = 0_Z\}$

$\Rightarrow (S\circ T)(v) = S(T(v)) = 0_Z \Rightarrow T(v) \in Nu(S)$ pero como S es inversible, S es mono. y $Nu(S) = \{0_W\} \Rightarrow T(v) = 0_W \Rightarrow v \in Nu(T)$, como T es inversible, $Nu(T) = \{0_V\} \Rightarrow v = 0_V$ ✓ $Nu(S\circ T) = \{0_V\}$, $S\circ T$ es mono.

$Im(S\circ T) = \{z \in Z \,/\, (S\circ T)(v) = z \text{ , con } v \in V\}$

Usando teorema de la dimensión para t.l. con dim V finita.

$\underbrace{\dim Nu(S\circ T)}_{0} + \dim Im(S\circ T) = \dim V = \dim Z$ (para ser inversible todos los e.v. deben tener la misma dim)

como $Im(S\circ T) \subset Z \Rightarrow Im(S\circ T) = Z \Rightarrow S\circ T$ es epi. ∴ es iso. ✓

22) $T: P_2 \to P_3$; $P \in P_2$; $T(P) = x.P + \frac{(P - P(0))}{x}$; $B = \{1; 1+x; 1+x+x^2\}$

$C = \{1; 1-x; 1+2x+x^2; 1-3x+3x^2-x^3\}$

Busco los transformados por T de la base B.

$T(1) = x + 0$; $T(1+x) = x + x^2 + \frac{1+x-1}{x} = x^2+x+1$

$T(1+x+x^2) = x^3+x^2+x + \frac{1+x+x^2-1}{x} = x^3+x^2+2x+1$

Busco sus coordenadas en base C: $C_c(x) = (1 \ \ -1 \ \ 0 \ \ 0)^t$

$C_c(x^2+x+1) = (-1 \ \ 1 \ \ 1 \ \ 0)^t$

$x^3+x^2+2x+1 = \alpha.(1) + \beta.(1-x) + \gamma.(1+2x+x^2) + \delta.(1-3x+3x^2-x^3) =$

$= (-\delta).x^3 + (3\delta+\gamma).x^2 + (-3\delta+2\gamma-\beta).x + \alpha+\beta+\gamma+\delta$

$$\Rightarrow \begin{cases} -\delta = 1 \\ 3\delta+\gamma = 1 \\ -3\delta+2\gamma-\beta = 2 \\ \alpha+\beta+\gamma+\delta = 1 \end{cases} \Rightarrow \begin{matrix} \boxed{\delta=-1} \\ \boxed{\gamma=4} \\ \boxed{\beta=9} \\ \boxed{\alpha=-11} \end{matrix} \Rightarrow C_c(x^3+x^2+2x+1) = (-11 \ \ 9 \ \ 4 \ \ -1)^t$$

$$\Rightarrow \boxed{[T]_{BC} = \begin{pmatrix} 1 & -1 & -11 \\ -1 & 1 & 9 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}}$$

1° forma: $T(-2+3x-x^2) = x.(-2+3x-x^2) + \frac{\cancel{-2}+3x-x^2+\cancel{2}}{x} =$

$= -x^3+3x^2-2x-x+3 = \boxed{-x^3+3x^2-3x+3}$

2° forma:

$C_B(-2+3x-x^2) = (-5 \ \ 4 \ \ -1)^t \Rightarrow [T]_{BC}.C_B(P) = C_c(T(P))$

$$\Rightarrow \begin{pmatrix} 1 & -1 & -11 \\ -1 & 1 & 9 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix} . \begin{pmatrix} -5 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow C_c(T(-2+3x-x^2)) = (2 \ \ 0 \ \ 0 \ \ 1)^t$$

$\Rightarrow T(-2+3x-x^2) = 2.1 + 1.(1-3x+3x^2-x^3) = \boxed{-x^3+3x^2-3x+3}$

23) $T : P_n \longrightarrow P_n \ , \ T(p) = p'$

$E_{P_n} = \{ 1 ; x ; x^2 ; \ldots\ldots ; x^n \}$

$T(1) = 0 \ ; \ T(x) = 1 \ ; \ T(x^2) = 2x \ ; \ \ldots\ldots ; T(x^n) = n . x^{(n-1)}$

$C_{E_{P_n}}(T(1)) = (0 \cdots\cdots 0)^t \ ; \ C_{E_{P_n}}(T(x)) = (1 \ 0 \cdots\cdots 0)^t \ , \ C_{E_{P_n}}(T(x^2)) = (0 \ 2 \ 0 \cdots\cdots 0)^t$

$C_{E_{P_n}}(T(x^n)) = (0 \cdots\cdots 0 \ n \ 0)^t$

$$\boxed{[T]_{E_{P_n}} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & 2 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & 0 & 3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & 0 \\ \vdots & \vdots & \vdots & \vdots & & & n \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}}$$

24) $T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (1 \ 2) . \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (1 \ 0) . \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1$

$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (1 \ 2) . \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (0 \ 1) . \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1$

$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = (1 \ 2) . \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (2 \ 0) . \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2$

$T\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{E_{R^{2x2}}} = (1 \ 2) . \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (0 \ 2) . \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2$

$E_{\mathbb{R}} = \{1\}$

$\Rightarrow C_{E_{\mathbb{R}}}\left(T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = 1$, etc.

$\Rightarrow \boxed{[T]_{E_{\mathbb{R}^{2x2}} E_{\mathbb{R}}} = (1 \ \ -1 \ \ 2 \ \ -2)}$

27) - a)

$v \in Nu(T) \Rightarrow T(v) = 0_W \ ; \ C_B(v) = (k_1 , \ldots\ldots ; k_n)$

$\Rightarrow A . C_B(v) = C_C(T(v)) = C_C(0_W) = (0 \cdots\cdots 0)^t$

$\Rightarrow A . C_B(v) = 0 \ \therefore \ C_B(v) \in Nul(A)$

La demostración en el sentido opuesto es casi igual.

b)

$w \in Im(T) \Rightarrow T(v) = w \ ; \ C_C(w) = (k_1 ; \ldots ; k_n) \ , \ C_B(v) = (\lambda_1 ; \ldots ; \lambda_k)$

$A . C_B(v) = C_C(T(v)) = C_C(w) = (A_1 \cdots\cdots A_k) . \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \lambda_1 . A_1 + \cdots\cdots + \lambda_k A_k$

$\therefore C_C(w) \in Col(A)$ (es c.l. de la columnas)

28) $B=\{v_1; v_2; v_3\}$; $T: V \to V$

$$[T]_B = \begin{pmatrix} r & -1 & r \\ 1 & r & 1 \\ r & -1 & 2r \end{pmatrix}$$

a) Como las columnas de $[T]_B$ son las coordenadas en base B de los elementos de B transformados por T, solo tengo que asegurar que son l.d., pues los vectores son l.d. $\Leftrightarrow$ sus coord. son l.d.
Con probar eso se verifica que $Im(T) \neq \mathbb{R}^3$ (no es sobreyectiva)

$$\Rightarrow \begin{pmatrix} r & -1 & r \\ 0 & r^2+1 & 0 \\ 0 & 0 & r \end{pmatrix} \Rightarrow \text{propongo } \det \begin{vmatrix} r & -1 & r \\ 0 & r^2+1 & 0 \\ 0 & 0 & r \end{vmatrix} = 0$$

$\Rightarrow r.(r^2+1).r=0 \Rightarrow r=0 \Rightarrow$ T no será sobreyectiva para $\boxed{r=0.}$

b) $r^2 \cdot (r^2+1)=0$, $r_1 = \boxed{0}$

$\Rightarrow r^2+1=0 \Rightarrow r^2=-1 \Rightarrow r_2 = \boxed{i}$; $r_3 = \boxed{-i}$

$$\boxed{r_1=0;\ r_2=i;\ r_3=-i}$$

c) $v=-v_1+i.v_2+(i-1).v_3 \Rightarrow C_B(V)=(-1 \quad i \quad i-1)^t$

$\Rightarrow [T]_B \cdot C_B(x) = C_B(v)$ → debo ver si el sist. es compatible

$v \to T(x) \to$ imagen

$r=0$:

$$\left(\begin{array}{ccc|c} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & i \\ 0 & -1 & 0 & i-1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & i \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & i-1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & i \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & i \end{array}\right) \to S.I.,\ \text{con } r=0\ \ v \notin Im(T)$$

$r=i$:

$$\left(\begin{array}{ccc|c} i & -1 & i & -1 \\ 1 & i & 1 & i \\ i & -1 & 2i & i-1 \end{array}\right) \to \left(\begin{array}{ccc|c} i & -1 & i & -1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & -i & -i \end{array}\right) \to SCI,\ \text{con } r=i\ \ v \in Im(T)$$

$r=-i$:

$$\left(\begin{array}{ccc|c} -i & -1 & -i & -1 \\ 1 & -i & 1 & i \\ -i & -1 & -2i & i-1 \end{array}\right) \to \left(\begin{array}{ccc|c} -i & -1 & -i & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & i & -i \end{array}\right) \to SI;\ \text{con } r=-i\ \ v \notin Im(T)$$

$$\Rightarrow [T]_B \cdot C_B(u) = C_B(v) \Rightarrow \left(\begin{array}{ccc|c} i & -1 & i & -1 \\ 0 & 0 & -i & -i \end{array}\right) \Rightarrow \begin{cases} i.x_1 - x_2 + i.x_3 = -1 \\ -i x_3 = -i \end{cases} \Rightarrow \begin{array}{l} x_2 = i + i.x_1 + 1 \\ x_3 = 1 \end{array}$$

$\Rightarrow C_B(u) = x_1.(1 \quad i \quad 0)^t + (0 \quad i+1 \quad 1)^t$; $x_1 \in \mathbb{C}$

$\Rightarrow \boxed{u = \alpha.(v_1 + i.v_2) + ((i+1).v_2 + v_3)}$; $\alpha \in \mathbb{C}$

2) $T: \mathbb{R}^3 \rightarrow P_2$

$T(1\ 1\ 1)^t = 2\beta + \alpha.x$; $T(0\ -1\ 1)^t = \alpha.x + \beta.x^2$; $T(0\ 0\ 1)^t = \beta + (\alpha-1).x$

a) $Nu(T) \neq \{0_{\mathbb{R}^3}\}$

Llamo B a la base $\{(1\ 1\ 1)^t ; (0\ -1\ 1)^t ; (0\ 0\ 1)^t\}$; $E_{P_2} = \{1 ; x ; x^2\}$

$[T]_{BE_{P_2}} = \begin{pmatrix} 2\beta & 0 & \beta \\ \alpha & \alpha & \alpha-1 \\ 0 & \beta & 0 \end{pmatrix}$; voy a buscar α y β tales las columnas sean l.d., pues son las coordenadas de los vec. trans por T, y si son l.d. querrá decir que dim $Im(T) < 3$, con lo cual dim $Nu(T) > 0$ por teo. de la dim. Entonces $Nu(T) \neq \{0\}$.

uso el det.:

$$\det\begin{pmatrix} 2\beta & 0 & \beta \\ \alpha & \alpha & \alpha-1 \\ 0 & \beta & 0 \end{pmatrix} \underset{\substack{\downarrow \\ 3^{\circ}\text{ fila}}}{=} (-\beta).\det\begin{pmatrix} 2\beta & \beta \\ \alpha & \alpha-1 \end{pmatrix} = (-\beta).((2\beta).(\alpha-1) - \beta.\alpha] = 0 = (-\beta^2).(2.(\alpha-1)-\alpha)$$

$\Rightarrow \boxed{\beta=0}$; $2\alpha - 2 - \alpha = 0 \Rightarrow \boxed{\alpha = 2} \Rightarrow \boxed{\alpha = 2 ; \beta = 0}$ (pero no es necesario que ambos tengan esos valores al mismo tiempo)

b) $Nu(T) = \{v \in \mathbb{R}^3 / T(v) = 0_{P_2}\}$

$$\Rightarrow [T]_{BE_{P_2}} . C_B(v) = C_{E_{P_2}}(\underbrace{T(v)}_{0_{P_2}}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 2\beta & 0 & \beta & 0 \\ 2 & 2 & 1 & 0 \\ 0 & \beta & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 2\beta & 0 & \beta & 0 \\ 0 & 2\beta & 0 & 0 \\ 0 & \beta & 0 & 0 \end{array}\right)$$

$\alpha = 2 ; \beta \neq 0$:

$$\Rightarrow \begin{cases} 2\beta.x_1 + \beta.x_3 = 0 & \Rightarrow x_3 = -2x_1 \\ 2\beta.x_2 = 0 & \Rightarrow x_2 = 0\ (\beta \neq 0) \Rightarrow C_B(v) = \begin{pmatrix} x_1 \\ 0 \\ -2x_1 \end{pmatrix} = x_1 . \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} ; x_1 \in R \\ 0 = 0 \end{cases}$$

$\boxed{Nu(T) = \{(1\ 1\ -1)^t\}}$

$Im(T) = \{P \in P_2 / T(v) = P ; v \in \mathbb{R}^3\} \Rightarrow$
$T(1\ 1\ 1)^t = 2\beta + 2x$
$T(0\ -1\ 1)^t = 2x + \beta.x^2$ múltiplos
$T(0\ 0\ 1)^t = \beta + x$

$\Rightarrow \boxed{Im(T) = \{\beta + x ; 2.x + \beta.x^2\}}$

$\alpha \neq 2 \; ; \; \beta = 0$:

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ \alpha & \alpha & \alpha-1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left\{ \alpha . x_1 + \alpha . x_2 + (\alpha-1) . x_3 = 0 \Rightarrow x_1 = \frac{-\alpha . x_2 - (\alpha-1) . x_3}{\alpha} = \right.$$

$$= -x_2 - x_3 + \frac{1}{\alpha} x_3 = -x_2 + \left(\frac{1}{\alpha} - 1\right) . x_3$$

$$\Rightarrow C_B(v) = \begin{pmatrix} -x_2 + \left(\frac{1}{\alpha}-1\right) . x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 . \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 . \begin{pmatrix} \frac{1}{\alpha} - 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \boxed{\mathcal{N}u(T) = \{ (-1 \;\; -2 \;\; 0)^t \; ; \; (-\alpha + 1 \;\; -\alpha+1 \;\; 1)^t \}}$$

$$\left.\begin{array}{l} T(1 \;\; 1 \;\; 1)^t = \alpha . t \\ T(0 \;\; -1 \;\; 1)^t = \alpha . t \\ T(0 \;\; 0 \;\; 1)^t = (\alpha - 1)\, t \end{array}\right\} \text{todos son múltiplos} \Rightarrow \boxed{Im(T) = \{t\}}$$

$\alpha = 2 \; ; \; \beta = 0$:

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \left\{ 2x_1 + 2x_2 + x_3 = 0 \Rightarrow x_3 = -2x_1 - 2x_2 \right.$$

$$C_B(v) = \begin{pmatrix} x_1 \\ x_2 \\ -2x_1 - 2x_2 \end{pmatrix} = x_1 . \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + x_2 . \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \; ; \; x_1, x_2 \in \mathbb{R}$$

$$\Rightarrow \begin{array}{l} v_1 = 1.(1 \;\; 1 \;\; 1)^t - 2 (0 \;\; 0 \;\; 1)^t = (1 \;\; 1 \;\; -1)^t \\ v_2 = 1.(0 \;\; -1 \;\; 1)^t - 2.(0 \;\; 0 \;\; 1)^t = (0 \;\; -1 \;\; -1)^t \end{array} \Rightarrow \boxed{\mathcal{N}u(T) = \{ (1 \;\; 1 \;\; -1)^t ; (0 \;\; -1 \;\; -1)^t \}}$$

$$\begin{array}{l} T(1 \;\; 1 \;\; 1)^t = 2t \\ T(0 \;\; -1 \;\; 1)^t = 2t \\ T(0 \;\; 0 \;\; 1)^t = t \end{array} \Rightarrow \boxed{Im(T) = \{t\}}$$

c) $U = \{ x \in \mathbb{R}^3 / \; x_1 + x_3 = x_2 + x_3 = 0 \} \Rightarrow \; u = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 . \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$C_B(u) = \begin{pmatrix} \lambda \\ 0 \\ -2\lambda \end{pmatrix} \; ; \; \lambda \in \mathbb{R}$ → para que sea claro que busco la imagen de un conjunto de vectores

$$\begin{pmatrix} 2\beta & 0 & \beta \\ \alpha & \alpha & \alpha-1 \\ 0 & \beta & 0 \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ 0 \\ -2\lambda \end{pmatrix} = C_{E_{P_2}} (T(U)) \Rightarrow C_{E_{P_2}} (T(U)) = \begin{pmatrix} 2\beta . \lambda - 2\beta . \lambda \\ \alpha . \lambda + (-2\alpha + 2) . \lambda \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ (-\alpha + 2) . \lambda \\ 0 \end{pmatrix}$$

$$\Rightarrow T(U) = \underset{\in \mathbb{R}}{\underbrace{(-\alpha + 2) . \lambda}} . \; t = \boxed{gen\{ (2 - \alpha) . t \}}$$

30) $T:\mathbb{C}^2 \to \mathbb{C}^2$; $T(1+i \quad 1-i)^t = (1 \quad i)^t$; $T(1 \quad i)^t = (-1 \quad 0)^t$

Como T está definida sobre una base de 2 elementos, la base canónica también debe tener 2 elementos, por lo tanto se trabaja en $\mathbb{C}$ como $\mathbb{C}$-e.v.

$B = \{(1+i \quad 1-i)^t ; (1 \quad i)^t\}$ $\Rightarrow [T]_{BE_{\mathbb{C}^2}} = \begin{pmatrix} 1 & -1 \\ i & 0 \end{pmatrix}$

$E_{\mathbb{C}^2} = \{(1 \quad 0)^t ; (0 \quad 1)^t\}$

$$\Rightarrow [T]_{E_{\mathbb{C}^2}} = [T]_{BE_{\mathbb{C}^2}} \cdot \underbrace{C_{E_{\mathbb{C}^2}B}}_{\text{matriz de cambio de base}} \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1+i \\ 1-i \end{pmatrix} + \beta \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{cc|c} 1+i & 1 & 1 \\ 1-i & i & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1+i & 1 & 1 \\ 0 & 2-2i & 1-i \end{array}\right)$$

$$\Rightarrow \begin{cases} (1+i).\alpha + \beta = 1 \\ (2-2i)\beta = 1-i \Rightarrow \beta = \dfrac{\cancel{1-i}}{2.(\cancel{1-i})} = \dfrac{1}{2} \end{cases} \Rightarrow \alpha = \frac{1}{2.(1+i)}$$

$$\Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha \cdot \begin{pmatrix} 1+i \\ 1-i \end{pmatrix} + \beta \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 1+i & 1 & 0 \\ 0 & 2-2i & -1-i \end{array}\right) \Rightarrow \begin{cases} (1+i).\alpha + \beta = 0 \\ (2-2i).\beta = -1-i \Rightarrow \beta = \dfrac{-1-i}{2-2i} \end{cases}$$

$$\Rightarrow \alpha = \frac{\cancel{1+i}}{(2-2i).(\cancel{1+i})} = \frac{1}{2-2i} \Rightarrow C_{E_{\mathbb{C}^2}B} = \begin{pmatrix} \dfrac{1}{2+2i} & \dfrac{1}{2-2i} \\ \dfrac{1}{2} & -\dfrac{(1+i)}{2-2i} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ i & 0 \end{pmatrix} \cdot \begin{pmatrix} \dfrac{1}{2+2i} & \dfrac{1}{2-2i} \\ \dfrac{1}{2} & -\dfrac{(1+i)}{2-2i} \end{pmatrix} = \boxed{\begin{pmatrix} -\dfrac{i}{2+2i} & \dfrac{2+i}{2-2i} \\ \dfrac{i}{2+2i} & \dfrac{i}{2-2i} \end{pmatrix} = [T]_{E_{\mathbb{C}^2}}}$$

32) $B = \{v_1 ; v_2 ; v_3\}$; $C = \{w_1 ; w_2 ; w_3 ; w_4\}$;

$T(v_1) = w_1 + w_2 + w_3 - w_4$; $T(v_2) = w_1 - w_2 + 2w_3 + 3w_4$

$T(v_3) = 2.w_1 + 3.w_3 + 2.w_4$

a) $[T]_{BC} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & 3 \\ -1 & 3 & 2 \end{pmatrix} \Rightarrow Nu(T):$ $\quad [T]_{BC} \cdot C_B(v) = C_C(T(v)) = C_C(0_W) = (0 \quad 0 \quad 0 \quad 0)^t$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ -1 & 3 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ \cancel{0} & \cancel{2} & \cancel{2} & \cancel{0} \\ 0 & -1 & -1 & 0 \\ \cancel{0} & \cancel{4} & \cancel{4} & \cancel{0} \end{array}\right) \Rightarrow \begin{cases} x_1 + x_2 + 2x_3 = 0 \\ -x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{array}{l} x_1 = x_2 \\ x_3 = -x_2 \end{array} \Rightarrow C_B(v) = x_2 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \boxed{Nu(T) = \{v_1 + v_2 - v_3\}}$$

$Im(T) \doteq gen\{T(v_1); T(v_2); T(v_3)\}$ → veo si son l.i. mirando sus coordenadas en C.

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 2 & 3 \\ 2 & 0 & 3 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & -1 & -4 \\ \cancel{0} & \cancel{2} & \cancel{-1} & \cancel{-4} \end{pmatrix} \Rightarrow \boxed{Im(T) = \{w_1 + w_2 + w_3 - w_4 ; w_1 - w_2 + 2w_3 + 3w_4\}}$$

b) 1° veo si se $T(v) \in Im(T)$, y lo será si es l.d. con los de la imagen, mirando sus coord. en C.

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & -1 & -4 \\ 0 & 2 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 0 & -3 \end{pmatrix} \rightarrow \text{son l.i.} \;\therefore\; T(v) \notin Im(T)$$

$\therefore \nexists$ v que cumpla eso.

c) $B' = \{v_1 ; 2v_2 + v_3 ; v_2 + v_3\}$; $C' = \{w_1 ; w_2 ; w_3 + w_4 ; w_3 - w_4\}$

Son bases porque cada uno de sus elementos son c.l. de los elem. de las B y C respectivamente.

$$\Rightarrow [T]_{B'C'} = C_{CC'} \cdot [T]_{BC} \cdot C_{B'B} \quad ; \quad C_{B'B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$C_{CC'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \Rightarrow [T]_{BC} \cdot C_{B'B} = \begin{pmatrix} 1 & 4 & 3 \\ 1 & -2 & -1 \\ 1 & 7 & 5 \\ -1 & 8 & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 & 3 \\ 1 & -2 & -1 \\ 1 & 7 & 5 \\ -1 & 8 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ 1 & -2 & -1 \\ 0 & \frac{15}{2} & 5 \\ 1 & -\frac{1}{2} & 0 \end{pmatrix}$$

$$\Rightarrow \boxed{[T]_{B'C'} = \begin{pmatrix} 1 & 4 & 3 \\ 1 & -2 & -1 \\ 0 & \frac{15}{2} & 5 \\ 1 & -\frac{1}{2} & 0 \end{pmatrix}}$$

3.9) $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$; $T(X) = A.X$; $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 3 & 6 & -3 \\ -1 & -2 & 0 \end{pmatrix}$

$B = \{v_1 ; v_2 ; v_3\}$; $B' = \{w_1 ; w_2 ; w_3 ; w_4)$

$$[T]_{BB'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow T(v_1) = w_1 \; ; \; T(v_2) = w_2 \; ; \; T(v_3) = (0 \; 0 \; 0 \; 0)^t$$

$$\Rightarrow v_3 \in Nu(T) \Rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 3 & 6 & -3 \\ -1 & -2 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 6 & -3 & 0 \\ -1 & -2 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{-1} & \cancel{0} \end{array}\right)$$

$\Rightarrow \begin{cases} x_1+2x_2-x_3=0 \\ \boxed{x_3=0} \end{cases} \Rightarrow \boxed{x_1=-2x_2} \Rightarrow v = \begin{pmatrix} -2x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2 \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$Nu(T) = \{(-2 \ \ 1 \ \ 0)^t\}$; puedo tomar $v_3 = (-2 \ \ 1 \ \ 0)^t$

Tomo 2 vec. l.i. con ese, $v_1 = (1 \ \ 0 \ \ 0)^t$; $v_2 = (0 \ \ 0 \ \ 1)^t$

$$\boxed{B = \{ (1 \ \ 0 \ \ 0)^t \, ; (0 \ \ 0 \ \ 1)^t \, ; (-2 \ \ 1 \ \ 0)^t \}}$$

$\Rightarrow T(1 \ \ 0 \ \ 0)^t = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 3 & 6 & -3 \\ -1 & -2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \\ -1 \end{pmatrix}$; $T(0 \ \ 0 \ \ 1)^t = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 3 & 6 & -3 \\ -1 & -2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -3 \\ 0 \end{pmatrix}$

$u_1 = (1 \ \ 0 \ \ 3 \ \ -1)^t$; $u_2 = (-1 \ \ 1 \ \ -3 \ \ 0)^t$

y elijo u_3 y u_4 l.i. con u_1 y $u_2 \Rightarrow u_3 = (0 \ \ 0 \ \ 1 \ \ 0)^t$
$u_4 = (0 \ \ 0 \ \ 0 \ \ 1)^t$

⊛

$\begin{pmatrix} 1 & 0 & 3 & -1 \\ -1 & 1 & 3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$ ⊛ → de acuerdo a la forma triangulada deduzco u_3 y u_4.

$$\boxed{B' = \{ (1 \ \ 0 \ \ 3 \ \ -1)^t ; (-1 \ \ 1 \ \ -3 \ \ 0)^t ; (0 \ \ 0 \ \ 1 \ \ 0)^t ; (0 \ \ 0 \ \ 0 \ \ 1)^t \}}$$

con $[T]_{BB'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ no existen B y B', pues esta matriz indicaría que $\dim Im(T) = 3$ porque las coordenadas (las columnas) son l.i.

$[T]_{BB'}$ debería representar la misma t.l., pero la t.l. anterior tiene $\dim Im(T) = 2$, eso demuestra que no pueden existir B y B'.

Práctica 4
(2° parte)

1) - a) $f(t) = (2+i).t + (1-i).sen\,t$

$\Rightarrow f(x) = 2x + i.x + sen(x) - i\,sen(x) = 2x + sen(x) + i.(x - sen(x))$

$$\boxed{Re(f) = 2x + sen(x) \;;\; Im(f) = x - sen(x)}$$

$$\boxed{f'(x) = 2 + cos(x) + \underset{\underset{cte}{\uparrow}}{i}.(1 - cos(x))}$$

$\int_a^b f(x)dx = \int_a^b 2x + sen(x) + i.(x - sen(x))\,dx = \left(x^2 - cos(x) + i.\frac{x^2}{2} + i.cos(x)\right)\Big|_a^b =$

$$= \boxed{\left(x^2 - cos(x)\right)\Big|_a^b + i.\left(\frac{x^2}{2} + cos(x)\right)\Big|_a^b}$$

b) $f(x) = (1 + 3.i.x)^3$

$\Rightarrow f(x) = (1+3i\,x).(1+3i.x).(1+3i.x) = (1+3i.x+3i.x-9.x^2).(1+3i.x) =$

$= 1 + 6i.x - 9x^2 + 3i.x - 18x^2 - 27.i.x^3 = -27.x^2 + 1 + i.(-27x^3 + 9x)$

$$\boxed{Re(f) = -27.x^2 + 1 \;;\; Im(f) = -27.x^3 + 9.x}$$

$$\boxed{f'(x) = -54.x + i.(-81.x^2 + 9)}$$

$\int_a^b f(x)dx = \int_a^b -27x^2 + 1 - i.27.x^3 + i.9x\,dx = \boxed{\left(-9x^3 + x\right)\Big|_a^b + i.\left(-\frac{27}{4}.x^4 + \frac{9}{2}.x^2\right)\Big|_a^b}$

c) $f(x) = \frac{1}{1+i.x} \Rightarrow f(x) = \frac{1}{1+i.x}\cdot\frac{1-i.x}{1-i.x} = \frac{1-i.x}{1+x^2} = \frac{1}{1+x^2} + i.\left(\frac{-x}{1+x^2}\right)$

$$\boxed{Re(f) = \frac{1}{1+x^2} \;;\; Im(f) = -\frac{x}{1+x^2}}$$

$f'(x) = -\frac{2x}{(x^2+1)^2} - i.\left(\frac{1+x^2-2x^2}{(1+x^2)^2}\right) = \boxed{-\frac{2x}{(x^2+1)^2} - i.\left(\frac{1-x^2}{(1+x^2)^2}\right)}$

d) $\int_0^{\pi} e^{(2+i).x}\,dx = \frac{1}{2+i}\cdot\left(e^{(2+i).x}\right)\Big|_0^{\pi} = \frac{e^{(2+i)\pi} - 1}{2+i}$

Se cumple que: $e^{(2+i).x} = e^{2x}.e^{i.x} = e^{2x}.(cos(x) + i.sen(x)) = e^{2x}.cos(x) + i.e^{2x}.sen(x)$

$e^{(2+i).\pi} = e^{2\pi}.e^{i.\pi} = e^{2\pi}.\left(\underbrace{cos(\pi)}_{-1} + i.\underbrace{sen(\pi)}_{0}\right) = -e^{2\pi}$

$$\int_0^{\pi} e^{(2+i).t}\,dt = \frac{-e^{2\pi}-1}{2+i} = \frac{-e^{2\pi}-1}{2+i}\cdot\frac{2-i}{2-i} = \frac{-2.(e^{2\pi}+1)+i.(e^{2\pi}+1)}{5} = -\frac{2}{5}.(e^{2\pi}+1)+i.\left(\frac{e^{2\pi}+1}{5}\right)$$

$$\Rightarrow \int_0^{\pi} e^{(2+i).t}\,dt = \int_0^{\pi} e^{2t}.\cos(t)+i.e^{2t}.\operatorname{sen}(t)\,dt = \int_0^{\pi} e^{2t}.\cos(t)\,dt + i.\int_0^{\pi} e^{2t}.\operatorname{sen}(t)\,dt =$$

$$= -\frac{2}{5}.(e^{2\pi}+1) + i.\left(\frac{e^{2\pi}+1}{5}\right) \Rightarrow \boxed{\int_0^{\pi} e^{2t}.\cos(t)\,dt = -\frac{2}{5}.(e^{2\pi}+1)}$$

$$\boxed{\int_0^{\pi} e^{2t}.\operatorname{sen}(t)\,dt = \frac{e^{2\pi}+1}{5}}$$

7) I) $y'-2y=0$

variables separables

$$\Rightarrow y'=2y \Rightarrow \frac{dy}{dx}=2y \Rightarrow \frac{1}{\underset{\to y\neq 0}{2y}}dy = dx \Rightarrow \int\frac{1}{2y}dy=\int dx \Rightarrow$$

$$\Rightarrow \frac{1}{2}.\ln|y| = x + \underset{\to cte}{c} \Rightarrow |y| = e^{2x+2c} = e^{2x}.e^{2c} \Rightarrow y = \underbrace{\pm e^{2c}}_{k}.e^{2x}$$

$\boxed{y=k.e^{2x}}$ $(k\in\mathbb{R})$

II) $y'=y+(1+i)x \Rightarrow y'-y=(1+i).x$

Planteo $y=u.v$; $y'=u'.v+u.v'$

$$\Rightarrow u'.v+u.v'-u.v=(1+i).x \Rightarrow u'.v+u.\underbrace{(v'-v)}=(1+i).x$$

debe ser solución de la ED homogénea

$$\Rightarrow v'-v=0 \Rightarrow \frac{dv}{dx}=v \Rightarrow \frac{dv}{\underset{\to v\neq 0}{v}}=dx \Rightarrow \int\frac{1}{v}dv=\int dx \Rightarrow \ln|v| = x+c$$

$$\Rightarrow |v|=e^{x}.e^{c} \Rightarrow v=\underbrace{\pm e^{c}}_{k}.e^{x}$$, tomo $k=1 \Rightarrow \boxed{v=e^{x}}$ (solo necesito una solución)

$$\Rightarrow u'.v+0=(1+i).x \Rightarrow u'.e^{x}=(1+i)x \Rightarrow u'=(1+i).\frac{x}{e^{x}} \Rightarrow \frac{du}{dx}=(1+i).\frac{x}{e^{x}}$$

$$\Rightarrow \int du=\int(1+i)\frac{x}{e^{x}}dx \Rightarrow u=(1+i).\int\frac{x}{e^{x}}dx = (1+i).(-e^{-x}).(x+1)+A \quad (A=cte)$$

$$\Rightarrow y(x)=u(x).v(x)=\left((1+i).(-e^{-x}).(x+1)+A\right).e^{x} = \boxed{-(1+i).(x+1)+A.e^{x}}$$

III) $y'+2x.y=y \Rightarrow y'+y.(2x-1)=0 \Rightarrow \frac{dy}{dx}=-y.(2x-1) \Rightarrow \frac{dy}{\underset{y\neq 0}{y}}=-(2x-1)dx$

(si $y=0$ la ED se cumple siempre)

$$\Rightarrow \int\frac{1}{y}dy=-\int 2x-1\,dx \Rightarrow \ln|y| = -x^2+x+c \; ; c=cte$$

$$\Rightarrow y=\pm e^{-x^2+x}.e^{c} = \boxed{k.e^{-x^2+x}}$$

IV) $x.y' + y = 3x^3 - 1 \quad (x>0)$

Planteo $y = u.v$, $y' = u'.v + u.v'$

1° divido la ED por x.

$\Rightarrow y' + \frac{y}{x} = 3x^2 - \frac{1}{x} \Rightarrow u'.v + u.v' + \frac{u.v}{x} = 3x^2 - \frac{1}{x}$

$\Rightarrow u'.v + u.\underbrace{\left(v' + \frac{v}{x}\right)}_{0} = 3x^2 - \frac{1}{x} \Rightarrow v' + \frac{v}{x} = 0 \Rightarrow \frac{dv}{dx} = -\frac{v}{x} \Rightarrow -\frac{dv}{v} = \frac{dx}{x}$

$\Rightarrow \int \frac{dv}{v} = -\int \frac{dx}{x} \Rightarrow \ln|v| = -\underset{\underset{x>0}{\downarrow}}{\ln|x|} + c$, $c = cte \Rightarrow |v| = e^{\overbrace{-\ln(x)}^{\ln(x^{-1})}} . e^c = x^{-1} e^c \Rightarrow$

$\Rightarrow v = \underbrace{\pm e^c}_{K} . x^{-1}$, tomo $K = 1 \Rightarrow \boxed{v = x^{-1}}$

$\Rightarrow u'.x^{-1} = 3x^2 - \frac{1}{x} \Rightarrow \frac{du}{dx} = 3x^3 - 1 \Rightarrow \int du = \int 3x^3 - 1\, dx \Rightarrow u = \frac{3}{4}x^4 - x + A$, $A = cte$

$\Rightarrow y(x) = u(x).v(x) = \left(\frac{3}{4}x^4 - x + A\right).(x^{-1}) = \boxed{\frac{3}{4}.x^3 - 1 + \frac{A}{x}}$

8) - I) $x.y' = (1+x).y$; $y(1) = 3$

si $x = 0 \Rightarrow y = 0$

si $x \neq 0$

$\Rightarrow \frac{y'}{y} = \frac{1+x}{x} \Rightarrow \frac{dy}{y} = \frac{1+x}{x} dx \Rightarrow \int \frac{dy}{y} = \int \frac{1+x}{x} dx \Rightarrow$

$\Rightarrow \ln|y| = \ln|x| + x + c$, $c = cte \Rightarrow |y| = e^{\ln|x| + x + c} = |x|.e^x.e^c \Rightarrow$

$\Rightarrow y = \underbrace{\pm e^c}_{K} . x.e^x$; $y(1) = K.e = 3 \Rightarrow K = \frac{3}{e} \Rightarrow \boxed{y = \frac{3x}{e}.e^x}$

II) $y' + y = \operatorname{sen} x$, $y(0) = 0$

$\Rightarrow \underset{y(x)}{\underset{\downarrow}{y}} = \underset{u(x)}{\underset{\downarrow}{u}}.\underset{v(x)}{v}$; $y' = u'.v + u.v'$

$\Rightarrow u'.v + u.v' + u.v = \operatorname{sen} x \Rightarrow u'.v + u.\underbrace{(v' + v)}_{0} = \operatorname{sen} x$

$\Rightarrow v' + v = 0 \Rightarrow \frac{dv}{v} = -dx \Rightarrow \int \frac{dv}{v} = -\int dx \Rightarrow \ln|v| = -x + c$, $c = cte$

$\Rightarrow |v| = e^{-x}.e^c \Rightarrow v = \underbrace{\pm e^c}_{K} . e^{-x}$, elijo $K = 1 \Rightarrow v = e^{-x}$

$\Rightarrow u'.e^{-x} = \operatorname{sen} x \Rightarrow \frac{du}{dx} = \operatorname{sen}(x)\, e^x \Rightarrow \int du = \int \operatorname{sen}(x).e^x dx \Rightarrow$

$$\Rightarrow u = \frac{e^{x}(\operatorname{sen}(x)-\cos(x))}{2} + A,\ A = \text{cte}$$

$$\Rightarrow y = u.v = \left(\frac{e^{x}(\operatorname{sen}(x)-\cos(x))}{2} + A\right)\cdot e^{-x} = \frac{\operatorname{sen}(x)-\cos(x)}{2} + \frac{A}{e^{x}}$$

$$y(0) = \frac{\operatorname{sen}(0)-\cos(0)}{2} + \frac{A}{e^{0}} = 0 \Rightarrow A = \frac{1}{2} \Rightarrow \boxed{y = \frac{\operatorname{sen}(x)-\cos(x)}{2} + \frac{1}{2.e^{x}}}$$

III) $(x+1).y' + x^2.y = e^{-x/2}\ ;\ y(0)=1$

$$\Rightarrow y(x) = u(x).v(x) \Rightarrow y' = u'.v + u.v'$$

$$\Rightarrow (x+1).u'.v + (x+1).u.v' + x^2.u.v = e^{-\frac{x}{2}} \Rightarrow (x+1).u'.v + u.\underbrace{((x+1)v' + x^2.v)}_{0} = e^{-\frac{x}{2}}$$

$$(x+1).v' + x^2.v = 0 \Rightarrow \frac{dv}{dx} = -\frac{x^2}{\underset{\substack{\downarrow\\ x\neq -1}}{x+1}}.v \Rightarrow \int\frac{dv}{v} = -\int\frac{x^2}{x+1}\,dx \Rightarrow$$

si $x=-1 \Rightarrow v=0$

$$\Rightarrow \ln|v| = -\frac{(x+1)^2}{2} + 2(x+1) - \ln(x+1) = -\frac{x^2}{2} + x + \frac{3}{2} - \ln(x+1) \pm c,\ c = \text{cte} \Rightarrow$$

$$\Rightarrow |v| = e^{-\frac{x^2}{2}}.e^{x}.e^{\frac{3}{2}}.\underbrace{(x+1)^{-1}}_{e^{\ln((x+1)^{-1})} = e^{-\ln(x+1)}}.e^{c} \Rightarrow v = \underbrace{\pm e^{c}.e^{\frac{3}{2}}}_{k}.e^{-\frac{x^2}{2}+x}.(x+1)^{-1}$$

$\Rightarrow$ tomando $k=1 \Rightarrow v = e^{-\frac{x^2}{2}+x}.(x+1)^{-1}$

$$\Rightarrow \cancel{(x+1)}.u'.e^{-\frac{x^2}{2}+x}\,\cancel{(x+1)}^{-1} = e^{-\frac{x}{2}} \Rightarrow u' = \frac{e^{-\frac{x}{2}}}{e^{-\frac{x^2}{2}+x}} \Rightarrow \int du = \int\frac{e^{-\frac{x}{2}}}{e^{-\frac{x^2}{2}+x}}\,dx \Rightarrow$$

10) - a) $y'' + 3y' + 2y = 0 \Rightarrow$ el polinomio asociado es $P(\lambda) = \lambda^2 + 3\lambda + 2$

Las raíces son: $\lambda_1 = -1$, $\lambda_2 = -2$ $\Rightarrow$ una base del espacio de solucions de la ED es:

$$\boxed{S.F. = \text{gen}\{e^{-x}\ ;\ e^{-2x}\}}$$

↓ sistema fundamental de solucions

b) $2y'' - 18y = 0 \Rightarrow 2\lambda^2 - 18 = 0 \Rightarrow \lambda^2 = 9 \Rightarrow \lambda_1 = 3$, $\lambda_2 = -3$

$$\Rightarrow \boxed{S.F. = \text{gen}\{e^{3x}\ ;\ e^{-3x}\}}$$

c) $y'' - 8y' + 16y = 0 \Rightarrow \lambda^2 - 8\lambda + 16 = 0 \Rightarrow \lambda_1 = \lambda_2 = 4$

$$\Rightarrow \boxed{S.F. = \text{gen}\{e^{4x}\ ;\ x.e^{4x}\}}$$

d) $y'' + 9y = 0 \Rightarrow \lambda^2 + 9 = 0 \Rightarrow \lambda^2 = -9 \Rightarrow \lambda_1 = 3i$
$\lambda_2 = -3i$

$\Rightarrow S.F. = \text{gen}\{e^{3ix}; e^{-3ix}\} = \boxed{\text{gen}\{\cos(3x); \text{sen}(3x)\}}$
↳ no se lo puede dejar expresado en complejos.

e) $2y'' + 2.y' + 2y = 0 \Rightarrow 2\lambda^2 + 2\lambda + 2 = 0 \Rightarrow$ las raíces son complejas

$\Rightarrow$ Planteo: $W^2 = b^2 - 4.a.c = -12 \Rightarrow W_1 = \sqrt{12}.i \; ; W_2 = -\sqrt{12}.i \Rightarrow -\dfrac{b+W}{2.a} = \lambda$

$\Rightarrow \lambda_1 = \dfrac{-2+\sqrt{12}.i}{4} = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}.i \; , \lambda_2 = -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}.i \Rightarrow \boxed{S.F. = \text{gen}\{e^{-\frac{1}{2}x}.\cos(\tfrac{\sqrt{3}}{2}x); e^{-\frac{1}{2}x}.\text{sen}(\tfrac{\sqrt{3}}{2}x)\}}$

f) $y'' - k^2.y = 0 \; , k \in \mathbb{R} \Rightarrow \lambda^2 - k^2 = 0 \Rightarrow \lambda^2 = k^2 \Rightarrow \lambda_1 = k$
$\lambda_2 = -k$

$\Rightarrow \boxed{S.F. = \text{gen}\{e^{kx}; e^{-kx}\}}$; si $k = 0$, $\lambda_1 = \lambda_2 = 0 \Rightarrow \boxed{S.F. = \text{gen}\{1; x\}}$
$\downarrow k \neq 0$ $\qquad \downarrow e^{0x} \quad \downarrow x.e^{0.x}$

g) $2y'' + 10y' + 25y = 0 \Rightarrow 2\lambda^2 + 10\lambda + 25 = 0 \Rightarrow$ el determinante es $b^2 - 4.a.c < 0$

Las raíces son complejas.

Planteo: $W^2 = b^2 - 4.a.c = -100 \Rightarrow W_1 = 10.i \; , W_2 = -10.i$

$\dfrac{-b \pm W}{2.a} = \lambda \Rightarrow \lambda_1 = \dfrac{-10 + 10i}{4} = -\dfrac{5+5i}{2}$

$\lambda_2 = \dfrac{-5-5i}{2}$

$\Rightarrow e^{(-\frac{5}{2}+\frac{5}{2}i)x} = e^{-\frac{5}{2}x}.e^{\frac{5}{2}ix} = e^{-\frac{5}{2}x}.(\cos(\tfrac{5}{2}x) + i.\text{sen}(\tfrac{5}{2}x))$

$e^{(-\frac{5}{2}-\frac{5}{2}i)x} = e^{-\frac{5}{2}x}.e^{-\frac{5}{2}ix} = e^{-\frac{5}{2}x}.(\cos(-\tfrac{5}{2}x) + i.\text{sen}(-\tfrac{5}{2}x)) = e^{-\frac{5}{2}x}.(\cos(\tfrac{5}{2}x) - i.\text{sen}(\tfrac{5}{2}x))$

se ve que ambas soluciones son C.L. de $\cos(\frac{5}{2}x).e^{-\frac{5}{2}x}$ y $\text{sen}(\frac{5}{2}x).e^{-\frac{5}{2}x}$

entonces $\boxed{S.F. = \text{gen}\{\cos(\tfrac{5}{2}x).e^{-\frac{5}{2}x}; \text{sen}(\tfrac{5}{2}x).e^{-\frac{5}{2}x}\}}$ (se nota así para no dejar expresado en complejos, ya que son funciones)

h) $y'' + 2y' + (\omega^2 + 1).y = 0 \; ; (\omega \geq 0)$

$\Rightarrow \lambda^2 + 2\lambda + \omega^2 + 1 = 0 \Rightarrow 2^2 - 4.1.(\omega^2+1) = 4 - 4\omega^2 - 4 = -4\omega^2$

si $\omega = 0 \Rightarrow$ 2 raíces reales iguales
si $\omega > 0 \Rightarrow$ 2 raíces complejas

$\Rightarrow \omega = 0$: $\lambda_1 = \lambda_2 = -1 \Rightarrow \boxed{S.F. = \text{gen}\{e^{-x}; x.e^{-x}\}}$

$\omega > 0$: $W^2 = -4.\omega^2 \Rightarrow W_1 = 2.\omega.i \; ; W_2 = -2.\omega.i$

$\Rightarrow \lambda = \dfrac{-b+W}{2.a} \Rightarrow \lambda_1 = \dfrac{-2+2\omega i}{2} = -1 + \omega.i \; ; \lambda_2 = -1 - \omega.i$

$\Rightarrow e^{(-1+m.i).x} = e^{-x}.e^{m.i.x} = e^{-x}(\cos(m.x) + i\,\mathrm{sen}(m.x))$

$e^{(-1-m.i)x} = e^{-x}.e^{-m.i.x} = e^{x}(\cos(-m.x) + i.\mathrm{sen}(-m.x)) = e^{x}.(\cos(m.x) - i.\mathrm{sen}(m.x))$

$\Rightarrow \boxed{S.F. = \mathrm{gen}\{e^{-x}.\cos(m.x);\ e^{-x}.\mathrm{sen}(m.x)\}}$

11) - a) $y'' + y' = 3x^2$

1° busco una solución de la ec homogénea usando el pol. asociado

$\lambda^2 + \lambda = 0 \Rightarrow \lambda.(\lambda+1) = 0 \Rightarrow \lambda_1 = 0\ ;\ \lambda_2 = -1$

$\Rightarrow S.F. = \mathrm{gen}\{1;\ e^{-x}\}$

↗ por la forma del resultado de la ED

Ahora busco una solución particular que sea de la forma $Y_p = Q(x)$

$\Rightarrow Y_p' = Q'(x)\ ;\ Y_p'' = Q''(x)$

↗ porque en la ED Y está mult. por 0

Reemplazo en la ED: $Q''(x) + Q'(x) = 3x^2$ (Q debería ser de grado 3)

$(Q = a.x^3 + b.x^2 + c.x + d)$ ↓ $gr(Q) = K \Rightarrow gr(Q') = K - 1 = n$ ← $gr(3x^2)$

$\Rightarrow K = n + 1$ (n: el gr más alto que aparece de Q)

$\Rightarrow 6a.x + 2b + 3ax^2 + 2bx + c = 3x^2$

$\Rightarrow 3ax^2 + (6a + 2b).x + (2b + c) = 3x^2 \Rightarrow \begin{cases} 3a = 3 \\ 6a + 2b = 0 \\ 2b + c = 0 \end{cases} \Rightarrow \begin{matrix} a = 1 \\ b = -3 \\ c = 6 \end{matrix}$

Con d no hay condiciones, pero como necesito una Y_p cualquiera, tomo d=0

$\Rightarrow Y_p = x^3 - 3x^2 + 6x \Rightarrow Y_G(x) = Y_p(x) + Y_H(x) = \boxed{C_1.e^{-X} + C_2 + x^3 - 3x^2 + 6X}$

b) $Y'' + 6.Y' + 9Y = 18.\cos(3X)$

$\Rightarrow \lambda^2 + 6\lambda + 9 = 0 \Rightarrow \lambda_1 = \lambda_2 = -3 \Rightarrow S.F. = \mathrm{gen}\{e^{-3X};\ X.e^{-3X}\}$

$\Rightarrow$ Propongo $Y_p = Q(x).(\alpha.\cos(3x) + \beta.\mathrm{sen}(3x))$, como Y está mult. por un N° $\neq 0$, $gr(Q) = gr(18) = 0$

$\therefore Q(x) = cte = K$

$\Rightarrow Y_p' = K.(-3\alpha.\mathrm{sen}(3X) + 3\beta.\cos(3X))$

$Y_p'' = K.(-9\alpha.\cos(3X) - 9\beta.\mathrm{sen}(3X))$

$\Rightarrow \cancel{-9\alpha.K.\cos(3X)} \cancel{-9K.\beta.\mathrm{sen}(3X)} - 18\alpha.K.\mathrm{sen}(3x) + 18.K\beta.\cos(3x) \cancel{+9.K\alpha.\cos(3x)} \cancel{+9.K\beta.\mathrm{sen}(3X)} = 18.\cos(3X)$

$\Rightarrow$ Como necesito alguna Y_p, tomo $\alpha = 0$, $\beta = 1$ y $K = 1$ que cumplen la ec. $\forall x$

$\Rightarrow Y_p = \mathrm{sen}(3X)$

$\Rightarrow \boxed{Y_H(X) = C_1.e^{-3X} + C_2.X.e^{-3X} + \mathrm{sen}(3X)}$

c) $y'' - 4y = e^{2x}$

$\Rightarrow x^2 - 4 = 0 \Rightarrow x_1 = 2;\ x_2 = -2 \quad \Rightarrow S.F. = \text{gen}\{e^{2x}; e^{-2x}\}$

$Y_p = e^{2x}.Q(x)\ ,\ gr(Q) = \underset{\substack{\downarrow\\0}}{n} + 1 = 1$ (pues e^{2x} ∈ al conjunto de soluciones homogéneas, entonces debo subir un grado a $Q(x)$ respecto de $P(x)=1$)

$\Rightarrow Q(x) = a.x + b \Rightarrow Y_p' = 2.e^{2x}.(a.x+b) + e^{2x}.a$

$Y_p'' = 4e^{2x}.(ax+b) + 2e^{2x}.a + 2a.e^{2x}$

$\Rightarrow \cancel{e^{2x}}.(4ax + 4b + 2a + 2a) - \cancel{e^{2x}}.(4ax + 4b) = \cancel{e^{2x}} \Rightarrow$

$\Rightarrow 4\cancel{a}x + \cancel{4b} + 4a - 4\cancel{a}x - \cancel{4b} = 1 \Rightarrow a = \frac{1}{4}$

Para b no hay condición así que $b = 0$

$Y_p(x) = \frac{x}{4}.e^{2x}$

$\Rightarrow \boxed{Y_G(x) = \frac{x}{4}.e^{2x} + c_1.e^{2x} + c_2.e^{-2x}}$

d) $Y'' - Y' - 2Y = e^x + x$

$\Rightarrow x^2 - x - 2 = 0 \Rightarrow x_1 = 2;\ x_2 = -1 \quad \Rightarrow S.F. = \text{gen}\{e^{2x}; e^{-x}\}$

Propongo:

$Y_p = a.e^x + b.x + c$

$Y_p' = a.e^x + b \qquad \Rightarrow \cancel{a.e^x} - \cancel{a.e^x} - b - 2.a.e^x - 2bx - 2c = e^x + x$

$Y_p'' = a.e^x$

$$\Rightarrow \begin{cases} -2a = 1 \\ -2b = 1 \\ -b - 2c = 0 \end{cases} \Rightarrow \begin{matrix} a = -\frac{1}{2} \\ b = -\frac{1}{2} \\ c = -\frac{1}{4} \end{matrix}$$

$Y_p = -\frac{1}{2}.e^x - \frac{1}{2}x - \frac{1}{4} \Rightarrow \boxed{Y_G = c_1.e^{2x} + c_2.e^{-x} - \frac{1}{2}.e^x - \frac{1}{2}x - \frac{1}{4}}$

e) $y'' + y = \cos(w.x)\ ,\ w > 0$

$\Rightarrow x^2 + 1 = 0 \Rightarrow x_1 = i;\ x_2 = -i \quad \Rightarrow e^{ix} = \cos(x) + i.\text{sen}(x)$

$e^{-ix} = \cos(-x) + i.\text{sen}(-x) = \cos(x) - i.\text{sen}(x)$

$\Rightarrow S.F. = \text{gen}\{\cos(x); \text{sen}(x)\}$ $\quad K.(\alpha.\cos(wx) + \beta.\text{sen}(wx))$

Propongo $Y_p =$ (pues $P(x) = 1$ y $gr(Q) = gr(P) = 0 \Rightarrow Q(x) = K = cte$)

Pero esto es sólo el caso que $w \neq 1$, pues $\cos(wx)$ no sería sol. del homog.

Si $w = 1$, hay que subir un grado a $Q \Rightarrow Q(x) = a.x + b$

Para $w \neq 1$:

$$y_p = K.(\alpha.\cos(w.x) + \beta.\text{sen}(w.x))$$

$$y_p' = K.(-\alpha.w.\text{sen}(wx) + \beta.w.\cos(wx))$$

$$y_p'' = K.(-\alpha.w^2.\cos(wx) - \beta.w^2.\text{sen}(wx))$$

$$\Rightarrow -K.\alpha.w^2.\cos(wx) - K.\beta.w^2\,\text{sen}(wx) + K.\alpha.\cos(w.x) + K.\beta.\text{sen}(wx) = \cos(wx)$$

$$\Rightarrow \cos(wx).(-K.\alpha.w^2 + K.\alpha) + \text{sen}(wx).(-K.\beta.w^2 + K.\beta) = \cos(wx)$$

$$\Rightarrow \begin{cases} -K.\alpha.w^2 + K.\alpha = 1 \Rightarrow K.\alpha.(-w^2+1) = 1 \Rightarrow K.\alpha = \dfrac{1}{\underset{\hookrightarrow\, w \neq 1}{-w^2+1}} \\ -K.\beta.w^2 + K.\beta = 0 \Rightarrow \beta = 0 \text{ cumple esta ec.} \end{cases}$$

$$\Rightarrow y_p = \frac{1}{-w^2+1}.\cos(wx)$$

$$\Rightarrow \boxed{y_G(x) = c_1.\cos(x) + c_2.\text{sen}(x) + \frac{\cos(wx)}{-w^2+1}}$$

Para $w = 1$:

$$y_p = (a.x + b).(\alpha.\cos(wx) + \beta.\text{sen}(wx)) = a x\,\alpha.\cos(wx) + b.\alpha.\cos(wx) + a x.\beta.\text{sen}(wx) + $$
$$+ b.\beta.\text{sen}(wx)$$

$$y_p' = a.\alpha.\cos(wx) + a.\alpha x.w.\text{sen}(wx) - b.\alpha.w.\text{sen}(wx) + a.\beta.\text{sen}(wx) +$$
$$+ a.\beta.x.\cos(wx).m + b.\beta.w.\cos(wx)$$

$$y_p'' = -a.\alpha.w.\text{sen}(wx) - a.\alpha.w.\text{sen}(wx) - a\alpha.w^2.x.\cos(wx) - b.\alpha.w^2.\cos(wx) +$$
$$+ a.\beta.w.\cos(wx) + a.\beta.\cos(wx).m - a.\beta.w^2.x.\text{sen}(wx) - b.\beta.w^2.\text{sen}(wx)$$

$$\Rightarrow \cos(wx).(-a.\alpha.w^2.x - b.\alpha.w^2 + a.\beta.w + a.\beta.w) + \text{sen}(wx).(-a.\alpha.w - a.\alpha.w -$$
$$- a.\beta.w^2 x - b.\beta.w^2) + \cos(wx).(a x.\alpha + b.\alpha) + \text{sen}(wx).(a x.\beta + b.\beta) = \cos(wx)$$

reemplazo w por $1 \Rightarrow$

$$\cos(x).(-a.\alpha.x - b.\alpha + a.\beta + a.\beta + a.x.\alpha + b.\alpha) + \text{sen}(x).(-a.\alpha - a.\alpha - a\beta.x - b.\beta +$$
$$+ a x.\beta + b.\beta) = \cos(x)$$

$$\Rightarrow \begin{cases} -\cancel{a\alpha x} - \cancel{b.\alpha} + 2a.\beta + \cancel{a x \alpha} + \cancel{b\alpha} = 1 \\ -2a\alpha - \cancel{a.\beta x} - \cancel{b\beta} + \cancel{a x.\beta} + \cancel{b\beta} = 0 \end{cases} \Rightarrow \begin{cases} 2a.\beta = 1 \\ -2a\alpha = 0 \end{cases}$$

Algunos valores que cumplen eso son: $\alpha = 0$, $\beta = 1$, $a = \frac{1}{2}$, $b = 0$

Así que una y_p es $y_p = \frac{1}{2}x.\text{sen}(x)$

$$\Rightarrow \boxed{y_G(x) = c_1.\cos(x) + c_2.\text{sen}(x) + \frac{1}{2}x.\text{sen}(x)}$$

12) - a) $y'' - y' = 3 . e^{2x}$; $y(0) = 0$; $y'(0) = -2$

$\Rightarrow \lambda^2 - \lambda = 0 \Rightarrow \lambda.(\lambda - 1) = 0 \Rightarrow \lambda_1 = 0 , \lambda_2 = 1$

$\Rightarrow S.F. = gen\{1 ; e^{x}\}$

$y_p = e^{2x} . Q(x)$, $P(x) = 3$, como y está mult. por 0 , $gr(Q) = m + 1 = 1$
$(gr(P) = 0 = m)$

$Q(x) = a . x + b$

$y_p' = 2 . e^{2x} . (ax + b) + e^{2x} . a$; $y_p'' = 4e^{2x} . (ax + b) + 2e^{2x} . a + 2a . e^{2x}$

$\Rightarrow 4 . \cancel{e^{2x}} . (ax + b) + 4a . \cancel{e^{2x}} - 2\cancel{e^{2x}} (ax + b) - a . \cancel{e^{2x}} = 3 . \cancel{e^{2x}} \Rightarrow$

$\Rightarrow 4ax + 4b + 4a - 2ax - 2b - a = 3 \Rightarrow (2a) . x + 3a + 2b = 3$

$\Rightarrow \begin{cases} 2a = 0 \\ 3a + 2b = 3 \end{cases} \Rightarrow \begin{matrix} a = 0 \\ b = \frac{3}{2} \end{matrix}$

$\Rightarrow y_p = \frac{3}{2} . e^{2x}$

$\Rightarrow y_G = c_1 + c_2 . e^{x} + \frac{3}{2} . e^{2x}$; $y_G(0) = c_1 + c_2 + \frac{3}{2} = 0$

$y_G' = c_2 . e^{x} + 3 . e^{2x}$; $y_G'(0) = c_2 + 3 = -2 \Rightarrow c_2 = -5$; $c_1 = \frac{7}{2}$

$\Rightarrow \boxed{y(x) = \frac{7}{2} - 5 . e^{x} + \frac{3}{2} . e^{2x}}$

b) $y'' - y' - 2y = 10 . sen(x)$; $y(\frac{\pi}{2}) = -3$; $y'(\frac{\pi}{2}) = -1$

$\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_1 = 2 ; \lambda_2 = -1 \Rightarrow S.F. = gen\{e^{2x} ; e^{-x}\}$

$y_p = K . (\alpha . cos(x) + \beta . sen(x))$; K, α, β ctes
$= K . \alpha . cos(x) + K . \beta \; sen(x)$

$y_p' = -K . \alpha . sen(x) + K . \beta . cos(x)$
$y_p'' = -K . \alpha . cos(x) - K . \beta \; sen(x)$

$\Rightarrow -K . \alpha . cos(x) - K . \beta \; sen(x) + K . \alpha . sen(x) - K . \beta . cos(x) - 2K . \alpha . cos(x) - 2K . \beta . sen(x) =$
$= 10 . sen(x)$

$\Rightarrow cos(x) . (-K . \alpha - K . \beta - 2K . \alpha) + sen(x) . (-K . \beta + K\alpha - 2K . \beta) = 10 . sen(x)$

$\Rightarrow \begin{cases} -3K\alpha - K . \beta = 0 \\ -3K . \beta + K . \alpha = 10 \end{cases} \Rightarrow \begin{cases} K . (-3\alpha - \beta) = 0 \\ K . (-3\beta + \alpha) = 10 \end{cases} \Rightarrow \begin{matrix} -3\alpha - \beta = 0 \Rightarrow \beta = -3\alpha \\ K . (9\alpha + \alpha) = 10 \Rightarrow K\alpha = 1 \end{matrix}$

↳ $K \neq 0$ para que se cumplan ambos

entonces tomo $K = 1 , \alpha = 1 , \beta = -3$

$y_p = cos(x) - 3 . sen(x)$

$\Rightarrow y_G(x) = C_1 . e^{2x} + c_2\, e^{-x} + \cos(x) - 3.\operatorname{sen}(x)$; $y_G(\frac{\pi}{2}) = C_1 . e^{\pi} + c_2\, e^{-\frac{\pi}{2}} - 3 = -3$

$y_G'(x) = 2.C_1\, e^{2x} - C_2.e^{-x} - \operatorname{sen}(x) - 3.\cos(x)$; $y_G'(\frac{\pi}{2}) = 2.C_1.e^{\pi} - C_2.e^{-\frac{\pi}{2}} - 1 = -1$

$\Rightarrow C_1 = C_2 = 0$

$\Rightarrow \boxed{y(x) = \cos(x) - 3.\operatorname{sen}(x)}$

d) $y'' + 4y' + 4y = 9.\cosh(x)$; $y(0) = 0$; $y'(0) = 0$

$x^2 + 4x + 4 = 0 \Rightarrow x_1 = x_2 = -2 \Rightarrow$ S.F. $= \operatorname{gen}\{e^{-2x} ; x\,e^{-2x}\}$

$\cosh(x) = \dfrac{e^{x} + e^{-x}}{2}$ así que no es c.l. de los elementos de S.F.

Propongo:

$y_p = K.(\alpha.\cosh(x) + \beta.\operatorname{senh}(x))$; K, α, β = ctes

$y_p' = K.\alpha.\operatorname{senh}(x) + K.\beta.\cosh(x)$

$y_p'' = K.\alpha.\cosh(x) + K\beta.\operatorname{senh}(x)$

$\Rightarrow K.\alpha.\cosh(x) + K.\beta\operatorname{senh}(x) + 4K.\alpha.\operatorname{senh}(x) + 4K.\beta.\cosh(x) + 4K\alpha.\cosh(x) + 4K.\beta\operatorname{senh}(x) =$

$= 9.\cosh(x)$

$\Rightarrow \cosh(x).(5K.\alpha + 4K.\beta) + \operatorname{senh}(x).(5K.\beta + 4K.\alpha) = 9.\cosh(x)$

$\begin{cases} 5K.\alpha + 4K.\beta = 9 \\ 5K.\beta + 4K.\alpha = 0 \end{cases} \Rightarrow \begin{array}{l} K.(-\frac{25}{4}\beta + 4\beta) = 9 \Rightarrow K.\beta = -4 \\ \underbrace{K}_{\neq 0}.(5\beta + 4\alpha) = 0 \Rightarrow \alpha = -\frac{5}{4}\beta \end{array}$ $\Rightarrow$ elijo $K = 1$; $\beta = -4$; $\alpha = 5$

$\Rightarrow y_p = 5.\cosh(x) - 4.\operatorname{senh}(x)$

$\Rightarrow y_G(x) = C_1 . e^{-2x} + C_2 . x\, e^{-2x} + 5.\cosh(x) - 4.\operatorname{senh}(x)$

$y_G(0) = C_1 + 5 = 0 \Rightarrow C_1 = -5$

$y_G'(x) = -2\, C_1 . e^{-2x} + C_2 . e^{-2x} - 2.C_2 x\, e^{-2x} + 5.\operatorname{senh}(x) - 4.\cosh(x)$

$y_G'(0) = -2.C_1 + C_2 - 4 = 0 \Rightarrow C_2 = -6$

$\Rightarrow \boxed{y(x) = -5.e^{-2x} - 6.x.e^{-2x} + 5.\cosh(x) - 4.\operatorname{senh}(x)}$

HOJA *

FECHA

Práctica 5

1) - I)

$$A=\begin{pmatrix}12 & -4\\ -8 & 8\end{pmatrix} \Rightarrow \text{Busco autovalores} \Rightarrow \det(\lambda.I-A)=\det\begin{pmatrix}\lambda-12 & 4\\ 8 & \lambda-8\end{pmatrix}=$$

$$=(\lambda-12).(\lambda-8)-32=\lambda^2-8\lambda-12\lambda+96-32=\lambda^2-20\lambda+64=0$$

$$\Rightarrow \boxed{\lambda_1=16;\ \lambda_2=4}$$

$$\Rightarrow (\lambda.I-A).v=0 \Rightarrow \begin{pmatrix}16-12 & 4\\ 8 & 16-8\end{pmatrix}.\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \left(\begin{array}{cc|c}4 & 4 & 0\\ \cancel{8} & \cancel{8} & \cancel{0}\end{array}\right) \rightarrow x_1+x_2=0 \Rightarrow x_1=-x_2$$

$v=x_2.\begin{pmatrix}-1\\ 1\end{pmatrix}$, $x_2\in\mathbb{R}$ $\Rightarrow$ el auto espacio asociado a $\lambda=16$ es $\boxed{S_{\lambda=16}=gen\{(-1\ \ 1)^t\}}$

Para $\lambda=4 \Rightarrow \left(\begin{array}{cc|c}4-12 & 4 & 0\\ 8 & 4-8 & 0\end{array}\right) \rightarrow \left(\begin{array}{cc|c}-8 & 4 & 0\\ \cancel{8} & \cancel{4} & \cancel{0}\end{array}\right) \rightarrow -2x_1+x_2=0 \Rightarrow x_2=2x_1$

$v=x_1.\begin{pmatrix}1\\ 2\end{pmatrix}$, $x_1\in\mathbb{R}$ $\Rightarrow \boxed{S_{\lambda=4}=gen\{(1\ \ 2)^t\}}$

Para $\mathbb{C}$ queda igual.

II)

$\underbrace{\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}}_{A}$ Como un autovalor cumple que $A.v=\lambda.v$
si $\lambda=0 \Rightarrow A.v=0$, el autovector $\in Nul(A)$

y en este caso $\dim(Nul(A))>0$ pues $rg(A)<2$, y hay un autovec. asociado a $\lambda=0$
Entonces puedo asegurar que existe un autovalor $\boxed{\lambda=0}$

Además, $\underset{\substack{\downarrow\\ \text{propiedad}}}{tr(A)=1-1}=\underset{\substack{\downarrow\\ 0}}{\lambda_1}+\lambda_2 \Rightarrow \lambda_2=0 \Rightarrow$ entonces $\lambda=0$ es un autovalor de mult. algebraica 2.

Busco $Nul(A) \Rightarrow \left(\begin{array}{cc|c}1 & -1 & 0\\ \cancel{1} & \cancel{-1} & \cancel{0}\end{array}\right) \rightarrow x_1-x_2=0 \Rightarrow x_1=x_2$
$\Rightarrow \boxed{S_{\lambda=0}=gen\{(1\ \ 1)^t\}}$

III)

$\underset{\substack{\uparrow\\ A}}{\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}} \Rightarrow p(\lambda)=\det(\lambda.I-A)=\det\begin{pmatrix}\lambda & -1\\ 1 & \lambda\end{pmatrix}=\lambda^2+1=0$

$\Rightarrow$ en $\mathbb{R}$ no tiene autovalores
considerando $\mathbb{C}$-e.v.: $\boxed{\lambda_1=i;\ \lambda_2=-i}$

NOTA

$\Rightarrow (\lambda . I - A) . v = 0$

$\lambda = i \Rightarrow \left(\begin{array}{cc|c} i & -1 & 0 \\ \cancel{1} & \cancel{i} & \cancel{0} \end{array}\right) \rightarrow i . X_1 - X_2 = 0 \Rightarrow X_2 = i . X_1$

$\Rightarrow \boxed{S_{\lambda = i} = \text{gen}\{(1 \quad i)^t\}}$

$\lambda = -i \Rightarrow \left(\begin{array}{cc|c} -i & -1 & 0 \\ \cancel{1} & \cancel{-i} & \cancel{0} \end{array}\right) \rightarrow -i . X_1 - X_2 = 0 \Rightarrow X_2 = -i X_1 \quad \Rightarrow \boxed{S_{\lambda = -i} = \text{gen}\{(1 \quad -i)^t\}}$

IV)

$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix} \Rightarrow P(\lambda) = \det(\lambda . I - A) = \det \begin{pmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 1 & \lambda + 1 \end{pmatrix} = (\lambda - 2) . \det \begin{pmatrix} \lambda & -1 \\ 1 & \lambda + 1 \end{pmatrix} =$

$= (\lambda - 2) . \left(\lambda . (\lambda + 1) + 1\right) = (\lambda^2 + \lambda + 1) . (\lambda - 2) = 0 \quad \boxed{\lambda_1 = 2} \quad (\text{en } \mathbb{R} - \text{e.v})$

$W^2 = b^2 - 4 . a . c = -3 \Rightarrow W_1 = \sqrt{3} . i \; ; W_2 = -\sqrt{3} . i$

$\Rightarrow \lambda = \frac{-b + W}{2 . a} \Rightarrow \boxed{\lambda_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \; ; \; \lambda_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i} \quad (\text{en } \mathbb{C} - \text{e.v})$

Para $\mathbb{R}$:

$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right) \rightarrow \begin{cases} 2X_2 - X_3 = 0 \\ 7X_3 = 0 \end{cases} \Rightarrow X_3 = 0 = X_2$

$\boxed{S_{\lambda = 2} = \text{gen}\{(1 \quad 0 \quad 0)^t\}}$

Para $\mathbb{C}$:

$\lambda = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \Rightarrow \left(\begin{array}{ccc|c} -\frac{5}{2} + \frac{\sqrt{3}}{2} i & 0 & 0 & 0 \\ 0 & -\frac{1}{2} + \frac{\sqrt{3}}{2} i & -1 & 0 \\ \cancel{0} & \cancel{1} & \cancel{\frac{1}{2} + \frac{\sqrt{3}}{2} i} & \cancel{0} \end{array}\right) \Rightarrow \begin{cases} \left(-\frac{5}{2} + \frac{\sqrt{3}}{2} i\right) . X_1 = 0 \Rightarrow X_1 = 0 \\ \text{son múltiplos} \Rightarrow X_3 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) X_2 \end{cases}$

$\Rightarrow \boxed{S_{\lambda = -\frac{1}{2} + \frac{\sqrt{3}}{2} i} = \text{gen}\left\{\left(0 \quad 1 \quad -\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^t\right\}}$

$\lambda = -\frac{1}{2} - \frac{\sqrt{3}}{2} i \Rightarrow \left(\begin{array}{ccc|c} -\frac{5}{2} - \frac{\sqrt{3}}{2} i & 0 & 0 & 0 \\ 0 & -\frac{1}{2} - \frac{\sqrt{3}}{2} i & -1 & 0 \\ 0 & 1 & \frac{1}{2} - \frac{\sqrt{3}}{2} i & 0 \end{array}\right) \rightarrow \begin{cases} X_1 = 0 \\ X_2 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} . i\right) . X_3 \end{cases}$

$\Rightarrow \boxed{S_{\lambda = -\frac{1}{2} - \frac{\sqrt{3}}{2} i} = \text{gen}\left\{\left(0 \quad 1 \quad -\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^t\right\}}$

v)

$$A=\begin{pmatrix}2&2&0&0\\0&1&0&0\\0&0&12&-4\\0&0&-8&8\end{pmatrix} \Rightarrow P_A(\lambda)=\det(\lambda I-A)\overset{\det}{=}\begin{pmatrix}\lambda-2&-2&0&0\\0&\lambda-1&0&0\\0&0&\lambda-12&4\\0&0&8&\lambda-8\end{pmatrix}=$$

$$=\det\begin{pmatrix}\lambda-2&-2\\0&\lambda-1\end{pmatrix}\cdot\det\begin{pmatrix}\lambda-12&4\\8&\lambda-8\end{pmatrix}=(\lambda-2).(\lambda-1)\left[(\lambda-12).(\lambda-8)-32\right]=0$$

$\Rightarrow \boxed{\lambda_1=2\,;\,\lambda_2=1}$; $(\lambda-12).(\lambda-8)-32=0 \Rightarrow \lambda^2-8\lambda-12\lambda+96-32=\lambda^2-20\lambda+64=0$

$\Rightarrow \boxed{\lambda_3=16\,;\,\lambda_4=4}$

$$\Rightarrow \lambda_1=2:\ \left(\begin{array}{cccc|c}\cancel{0}&\cancel{2}&\cancel{0}&\cancel{0}&\cancel{0}\\0&1&0&0&0\\0&0&-10&4&0\\0&0&8&-6&0\end{array}\right)\rightarrow\left(\begin{array}{cccc|c}0&0&0&0&0\\0&1&0&0&0\\0&0&-10&4&0\\0&0&0&-28&0\end{array}\right)\rightarrow\begin{cases}x_2=0\\x_3=0\\x_4=0\end{cases}$$

$$\Rightarrow \boxed{S_{\lambda=2}=\text{gen}\{(1\ \ 0\ \ 0\ \ 0)^t\}}$$

$$\lambda=1:\ \left(\begin{array}{cccc|c}-1&-2&0&0&0\\0&0&0&0&0\\0&0&-11&4&0\\0&0&8&-7&0\end{array}\right)\rightarrow\left(\begin{array}{cccc|c}-1&-2&0&0&0\\0&0&0&0&0\\0&0&-11&4&0\\0&0&0&-45&0\end{array}\right)\rightarrow\begin{cases}x_1=-2x_2\\x_3=0\\x_4=0\end{cases}$$

$$\Rightarrow \boxed{S_{\lambda=1}=\text{gen}\{(-2\ \ 1\ \ 0\ \ 0)^t\}}$$

$$\lambda=16:\ \left(\begin{array}{cccc|c}14&-2&0&0&0\\0&15&0&0&0\\0&0&4&4&0\\\cancel{0}&\cancel{0}&\cancel{8}&\cancel{8}&\cancel{0}\end{array}\right)\Rightarrow\begin{cases}x_2=0\\x_1=0\\x_3+x_4=0\Rightarrow x_3=-x_4\end{cases}$$

$$\Rightarrow \boxed{S_{\lambda=16}=\text{gen}\{(0\ \ 0\ \ -1\ \ 1)^t\}}$$

$$\lambda=4:\ \left(\begin{array}{cccc|c}2&-2&0&0&0\\0&3&0&0&0\\0&0&-8&4&0\\\cancel{0}&\cancel{0}&\cancel{8}&\cancel{-4}&\cancel{0}\end{array}\right)\rightarrow\begin{cases}x_1=x_2=0\\x_2=0\\-8x_3+4x_4=0\Rightarrow x_4=2x_3\end{cases}$$

$$\Rightarrow \boxed{S_{\lambda=4}=\text{gen}\{(0\ \ 0\ \ 1\ \ 2)^t\}}$$

3) $A=\begin{pmatrix}2&2&-6\\2&-1&-3\\-2&-1&1\end{pmatrix}$; $A.v=\lambda.v \Rightarrow A.v=\begin{pmatrix}2&2&-6\\2&-1&-3\\-2&-1&1\end{pmatrix}.\begin{pmatrix}1\\-2\\0\end{pmatrix}=\begin{pmatrix}-2\\4\\0\end{pmatrix}=\lambda.v$

$\Rightarrow \lambda.\begin{pmatrix}1\\-2\\0\end{pmatrix}=\begin{pmatrix}-2\\4\\0\end{pmatrix} \Rightarrow \boxed{\lambda_1=-2}$;

Pruebo si λ_1 tiene mult. alg. >1

$\Rightarrow (\lambda.I-A).v=0 \Rightarrow \left(\begin{array}{ccc|c}-2-2&-2&6&0\\-2&-2+1&3&0\\2&1&-2-1&0\end{array}\right) \rightarrow \left(\begin{array}{ccc|c}\cancel{-4}&\cancel{-2}&\cancel{6}&\cancel{0}\\-2&-1&3&0\\\cancel{2}&\cancel{1}&\cancel{-3}&\cancel{0}\end{array}\right) \Rightarrow -2x_1-x_2+3x_3=0$

$\Rightarrow x_2=-2x_1+3x_3$

$\Rightarrow \boxed{S_{\lambda=-2}=gen\left\{\begin{pmatrix}1\\-2\\0\end{pmatrix};\begin{pmatrix}0\\3\\1\end{pmatrix}\right\}}$, la mult. alg. de λ_1 es al menos 2.

$\Rightarrow 2-1+1=\underset{-2}{\underset{\downarrow}{\lambda_1}}+\underset{-2}{\underset{\downarrow}{\lambda_2}}+\lambda_3 \Rightarrow \boxed{\lambda_3=6}$

Para $\lambda_3=6$: $\left(\begin{array}{ccc|c}6-2&-2&6&0\\-2&6+1&3&0\\2&1&6-1&0\end{array}\right) \rightarrow \left(\begin{array}{ccc|c}4&-2&6&0\\-2&7&3&0\\2&1&5&0\end{array}\right) \rightarrow \left(\begin{array}{ccc|c}4&-2&6&0\\\cancel{0}&\cancel{12}&\cancel{12}&\cancel{0}\\0&4&4&0\end{array}\right) \rightarrow \left(\begin{array}{ccc|c}2&-1&3&0\\0&1&1&0\end{array}\right)$

$\Rightarrow \begin{cases}2x_1-x_2+3x_3=0\\ \quad x_2+x_3=0\end{cases} \Rightarrow \begin{array}{l}2x_1+x_3+3x_3=0 \Rightarrow x_1=-2x_3\\ x_2=-x_3\end{array}$

$\Rightarrow \boxed{S_{\lambda=6}=gen\left\{\begin{pmatrix}-2\\-1\\1\end{pmatrix}\right\}}$ mult. alg. y geom. de 6 es 1
" " " " " -2 es 2

4) $A=\begin{pmatrix}7&1&2\\-1&7&0\\1&-1&6\end{pmatrix} \Rightarrow \det(\lambda.I-A)=\det\begin{vmatrix}\lambda-7&-1&-2\\1&\lambda-7&0\\-1&1&\lambda-6\end{vmatrix}=(1).(-1).\begin{vmatrix}-1&-2\\1&\lambda-6\end{vmatrix}+(\lambda-7).\begin{vmatrix}\lambda-7&-2\\-1&\lambda-6\end{vmatrix}=$

$\downarrow$ 2° fila

$=(-1).(-\lambda+6+2)+(\lambda-7).((\lambda-7).(\lambda-6)-2)=\lambda^3-20\lambda^2+132\lambda-288=0$

$\Rightarrow \boxed{\lambda_1=6} \Rightarrow$

$$\begin{array}{c|cccc} & 1 & -20 & 132 & -288\\ 6 & & 6 & -84 & 288\\\hline & 1 & -14 & 48 & \boxed{0}\end{array}$$

$\Rightarrow P_A(\lambda)=(\lambda-6).(\lambda^2-14\lambda+48)$

$\boxed{\lambda_2=8}$ $\boxed{\lambda_3=6}$

$\lambda=6$: $A.v=6.v \Rightarrow (6.I-A).v=0$

$$\Rightarrow \left(\begin{array}{ccc|c} -1 & -1 & -2 & 0 \\ 1 & -1 & 0 & 0 \\ \cancel{-1} & \cancel{1} & \cancel{0} & \cancel{0} \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -1 & -1 & -2 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right) \longrightarrow \begin{cases} -x_1-x_2-2x_3=0 \\ -x_2-x_3=0 \end{cases} \rightarrow \begin{array}{l} x_1=-x_3 \\ x_2=-x_3 \end{array}$$

$$\Rightarrow \boxed{S_{\lambda=6}=\text{gen}\{(-1\ \ -1\ \ 1)^t\}}$$ mult. alg. es 2
" geom. es 1

$\lambda=8$: $(8.I-A).v=0$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 1 & 1 & 0 & 0 \\ \cancel{-1} & \cancel{1} & \cancel{2} & \cancel{0} \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right) \rightarrow \begin{cases} x_1-x_2-2x_3=0 \\ x_2+x_3=0 \end{cases} \rightarrow \begin{array}{l} x_1=x_3 \\ x_2=-x_3 \end{array}$$

$$\Rightarrow \boxed{S_{\lambda=8}=\text{gen}\{(1\ \ -1\ \ 1)^t\}}$$ Mult. alg. y geom. es 1

7)

$(k\times k)$ $\nearrow A_{11}$ $\nearrow A_{12}$

$$A=\begin{pmatrix} 2 & 2 & -6 & 0 & 0 \\ 2 & -1 & -3 & 0 & 0 \\ -2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$, por propiedad $\det(A)=\det(A_{11}).\det(A_{22})-\underbrace{\det(A_{12}).\det(A_{21})}_{0}$

$\swarrow$ A_{21} $\downarrow$ $A_{22}\ (n-k)\times(n-k)$

$$\Rightarrow \det(\lambda.I-A)=\det(\underset{\substack{\downarrow \\ k\times k}}{\lambda.I-A_{11}}).\det(\underset{\substack{\downarrow \\ (n-k)\times(n-k)}}{\lambda.I-A_{22}})=0 \Leftrightarrow \det(\lambda.I-A_{11})=0\ \vee\ \det(\lambda.I-A_{22})=0$$

$$\Rightarrow \det(\lambda.I-A_{11})=\det\begin{pmatrix} \lambda-2 & -2 & 6 \\ -2 & \lambda+1 & 3 \\ -2 & 1 & \lambda-1 \end{pmatrix} \underset{\substack{\downarrow \\ 1^{a}\text{ fila}}}{=} (\lambda-2).\begin{vmatrix} \lambda+1 & 3 \\ 1 & \lambda-1 \end{vmatrix} + (-2).(-1).\begin{vmatrix} -2 & 3 \\ 2 & \lambda-1 \end{vmatrix} +$$

$$+\ 6.\begin{vmatrix} -2 & \lambda+1 \\ 2 & 1 \end{vmatrix} = (\lambda-2).\left((\lambda+1).(\lambda-1)-3\right)+2.\left(-2\lambda+2-6\right)+6.\left(-2-2\lambda-2\right)=$$

$$=(\lambda-2).(\lambda^2-\cancel{\lambda}+\cancel{\lambda}-1-3)\underbrace{-4\lambda+4-12-12-12\lambda-12}_{-16\lambda-32}=\lambda^3-4\lambda-2\lambda^2+8-16\lambda-32=$$

$$=\lambda^3-2\lambda^2-20\lambda-24=0 \Rightarrow \boxed{\lambda_1=6}$$

$$\Rightarrow \begin{array}{r|rrrr} & 1 & -2 & -20 & -24 \\ 6 & & 6 & 24 & 24 \\ \hline & 1 & 4 & 4 & \boxed{0} \end{array} \Rightarrow p_{A_{11}}(\lambda)=(\lambda-6).\underset{\substack{\downarrow \\ \boxed{\lambda_2=\lambda_3=-2}}}{(\lambda^2+4\lambda+4)}$$

Los autovectores asociados a este autoval. cumplen que las coordenadas x_4 y x_5 son nulos, y puedo usar las 3 primeras filas para calcularlos.

$\lambda=6$; $\left(\begin{array}{ccccc|c} 4 & -2 & 6 & 0 & 0 & 0 \\ -2 & 7 & 3 & 0 & 0 & 0 \\ 2 & 1 & 5 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccccc|c} 4 & -2 & 6 & 0 & 0 & 0 \\ \cancel{0} & \cancel{12} & \cancel{12} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 4 & 4 & 0 & 0 & 0 \end{array}\right) \rightarrow \begin{cases} 2X_1 - X_2 + 3X_3 = 0 \\ X_2 + X_3 = 0 \end{cases} \rightarrow \begin{array}{l} X_1 = -2X_3 \\ X_2 = -X_3 \end{array}$

$\Rightarrow \boxed{S_{\lambda=6} = \text{gen}\{(-2 \quad -1 \quad 1 \quad 0 \quad 0)^t\}}$

$\lambda=-2$: (uso lo desarrollado en el ej. 3)

$\boxed{S_{\lambda=-2} = \text{gen}\{(1 \quad -2 \quad 0 \quad 0 \quad 0)^t ; (0 \quad 3 \quad 1 \quad 0 \quad 0)^t\}}$

$\det(\lambda I - A_{22}) = \det\begin{pmatrix} \lambda-1 & 1 \\ -1 & \lambda+1 \end{pmatrix} = (\lambda-1)\cdot(\lambda+1)+1 = \lambda^2 + \cancel{\lambda} - \cancel{\lambda} - \cancel{1} + \cancel{1} = 0$

$\Rightarrow \boxed{\lambda_4 = \lambda_5 = 0}$

Los autovec. para estos autoval. cumplen que las 3 primeras coordenadas son nulas, y puedo calcularlos sólo con la 4ª y 5ª fila.

$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & -1 & 1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{-1} & \cancel{1} & \cancel{0} \end{array}\right) \rightarrow \{X_4 = X_5 \Rightarrow \boxed{S_{\lambda=0} = \text{gen}\{(0 \quad 0 \quad 0 \quad 1 \quad 1)^t\}}$

$\lambda=6$: mult. alg. y geom. 1
$\lambda=-2$: mult. " " " 2
$\lambda=0$: mult. alg. 2 y mult. geom. 1

10)

$A = \begin{pmatrix} 2 & k & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow p_A(\lambda) = 0 \Rightarrow \boxed{\lambda_1 = \lambda_2 = 2} ; \boxed{\lambda_3 = 1}$ $\forall k$ (matriz triangular)

mult. alg. 2 es 2
mult. alg. 1 es 1

$\lambda=2$: $\left(\begin{array}{ccc|c} 2-2 & -k & -1 & 0 \\ 0 & 2-2 & -1 & 0 \\ 0 & 0 & 2-1 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 0 & -k & -1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{-1} & \cancel{0} \\ 0 & 0 & 1 & 0 \end{array}\right) \rightarrow$ si $K=0$, mult. geom. 2 es 2
si $K \neq 0$, " " " es 1

$K=0 \rightarrow X_3 = 0 \Rightarrow \boxed{S_{\lambda=2} = \text{gen}\{(1 \quad 0 \quad 0)^t ; (0 \quad 1 \quad 0)^t\}}$

$k \neq 0 \rightarrow \begin{cases} X_3 = 0 \\ kX_2 + X_3 = 0 \Rightarrow X_2 = 0 \quad (k \neq 0) \end{cases} \Rightarrow \boxed{S_{\lambda=2} = \text{gen}\{(1 \quad 0 \quad 0)^t\}}$

$\lambda=1$: $\left(\begin{array}{ccc|c} -1 & -k & -1 & 0 \\ 0 & -1 & -1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \Rightarrow \begin{cases} -X_1 + kX_2 + X_3 = 0 \\ X_2 + X_3 = 0 \end{cases} \rightarrow \begin{array}{l} X_1 = (-1+k)\cdot X_3 \\ X_2 = -X_3 \end{array}$

$\Rightarrow \boxed{S_{\lambda=1} = \text{gen}\{(-1+k \quad -1 \quad 1)^t\}}$ (mult. alg. y geom. de 1 es 1 $\forall k$)

NOTA

HOJA

FECHA

11) si rango(A) = k < n ; dim Nul(A) = n − k y habrán n−k autovalores nulos

∴ hay k autovalores no nulos.

12) $A=\begin{pmatrix}1&-1&2\\1&-1&2\\1&-1&2\end{pmatrix}$, seguro que un autovalor es $\boxed{\lambda=0}$ y tendrá mult. alg 2 por lo menos, ya que tiene 2 autovec asociados.

Además, $\underset{\underset{0}{\downarrow}}{\lambda_1}+\underset{\underset{0}{\downarrow}}{\lambda_2}+\lambda_3 = 1-1+2 \Rightarrow \boxed{\lambda=2}$ es el otro autovalor.

Busco Nul(A) $\Rightarrow \left(\begin{array}{ccc|c}1&-1&2&0\\ \cancel{1}&\cancel{-1}&\cancel{2}&\cancel{0}\\ \cancel{1}&\cancel{-1}&\cancel{2}&\cancel{0}\end{array}\right) \rightarrow x_1-x_2+2x_3=0 \Rightarrow x_1 = x_2-2x_3$

$\Rightarrow \boxed{S_{\lambda=0}=\text{gen}\{(1\ \ 1\ \ 0)^t;(-2\ \ 0\ \ 1)^t\}}$

Para $\lambda=2$: $(2I-A)v=0 \Rightarrow \left(\begin{array}{ccc|c}1&1&-2&0\\-1&3&-2&0\\-1&1&0&0\end{array}\right) \rightarrow \left(\begin{array}{ccc|c}1&1&-2&0\\ \cancel{0}&\cancel{4}&\cancel{-4}&\cancel{0}\\0&2&-2&0\end{array}\right) \rightarrow \begin{cases}x_1+x_2-2x_3=0\\ x_2-x_3=0\end{cases}$

$\Rightarrow \begin{matrix}x_2=x_3\\ x_1=x_3\end{matrix} \Rightarrow \boxed{S_{\lambda=2}=\text{gen}\{(1\ \ 1\ \ 1)^t\}}$

14) Caso r=0 ⇒ rango(A) = 1 , dim Nul(A) = n−1 , λ=0 es un autovalor de mult. alg y geom. n−1

$tr(A) = n\cdot 1 = n = \underset{\underset{\text{multiplicidad}}{\downarrow}}{(n-1)}\cdot \underset{\hookrightarrow\text{autovalor}}{0} + \lambda_x$ → autoval ≠ 0

$\Rightarrow \lambda_x = n$ (mult alg. 1)

Autovectores: Busco Nul(A) $\Rightarrow \left(\begin{array}{ccc|c}1&\cdots&1&0\\ \cancel{\vdots}&&\cancel{\vdots}&\cancel{\vdots}\\ \cancel{1}&\cancel{\cdots}&\cancel{1}&\cancel{0}\end{array}\right) \rightarrow x_1 = -x_2 - \cdots - x_n$

$\Rightarrow \boxed{S_{\lambda=0}=\text{gen}\{(-1\ \ 1\ \ 0\ \cdots\ 0)^t,(-1\ \ 0\ \ 1\ \ 0\ \cdots\ 0)^t;\ldots\ldots,(-1\ \ 0\ \cdots\ 0\ \ 1)^t\}}$
(dim = n−1)

Para λ=n, hay un solo autovec y debe ser l.i. con los de $S_{\lambda=0}$.
Propongo $\boxed{S_{\lambda=n}=\text{gen}\{(1\ \ 1\ \ \cdots\cdots\ \ 1)^t\}}$ porque $S_{\lambda=n}=(S_{\lambda=0})^{\perp}$
$\hookrightarrow x_1+\cdots+x_n=0$

NOTA

si $A = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \Rightarrow \underset{\substack{\downarrow \\ \text{matriz dada en el enunciado}}}{B} = A + r.I$ y los autovd. de A son λ_1 y λ_2

Entonces los autovalores de B son $\lambda_1 + r$ y $\lambda_2 + r$

$\therefore$ Los autovd. pedidos son: $\boxed{\lambda'_1 = 0 + r = r}$

$\boxed{\lambda'_2 = n + r}$

Los autovectores son los mismos que para A.

19) Los autovd. de A eran: $\lambda = 6 ; \lambda = 8$

Los autovd. de la matriz dada son $\mu_1 = 6^3 + 2.6^2 - 3.6 + 1 = \boxed{271}$

$\mu_2 = 8^3 + 2.8^2 - 3.8 + 1 = \boxed{617}$

(se usa que $P(\lambda)$ son los autovd. de $P(A)$)

20) Lo que se pide es equivalente a decir que los autovd. de $\underbrace{A^3 + r.A^2 - I}_{B}$ sean $\neq$ de 0, es decir $\dim \mathrm{Nul}(B) = 0 \rightarrow$ rango de B máximo

Los autovd. de B son: $\mu_1 = (-2)^3 + r(-2)^2 - 1 = 4r - 9$

$\mu_2 = 6^3 + r(6)^2 - 1 = 36r + 275$

$4r - 9 = 0 \Rightarrow r = \frac{9}{4}$

$36r + 275 \Rightarrow r = -\frac{275}{36}$

$\Rightarrow$ La matriz resulta inversible para $\boxed{r \in \mathbb{R} - \{\frac{9}{4} ; -\frac{275}{36}\}}$

21) $A.v = \lambda.v \quad (v \neq 0) \Rightarrow A.v = A^2.v = A.A.v = A\,\lambda.v = \lambda.A v = \lambda.\lambda.v = \lambda^2.v$

$\Rightarrow \lambda.v = \lambda^2 v \Rightarrow \lambda = \lambda^2 \Rightarrow \boxed{\lambda_1 = 1 \ ; \ \lambda_2 = 0}$

22) - I) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \rightarrow p_A(\lambda) = \det(\lambda.I - A) = \det\begin{pmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{pmatrix} = (\lambda-2)^2 - 1 = 0$

$\Rightarrow \lambda_1 = 3 ; \lambda_2 = 1 \Rightarrow (\lambda.I - A).v = 0$

$\lambda = 3 : \left(\begin{array}{cc|c} 1 & -1 & 0 \\ \cancel{-1} & \cancel{1} & \cancel{0} \end{array}\right) \rightarrow x_1 = x_2 \Rightarrow S_{\lambda=3} = \mathrm{gen}\{(1 \quad 1)^t\}$

$\lambda = 1 : \left(\begin{array}{cc|c} -1 & -1 & 0 \\ \cancel{-1} & \cancel{-1} & \cancel{0} \end{array}\right) \rightarrow x_1 = -x_2 \Rightarrow S_{\lambda=1} = \mathrm{gen}\{(-1 \quad 1)^t\}$

$Q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow$ Busco Q^{-1}: $\left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & -2 & 1 & -1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 2 & 0 & 1 & 1 \\ 0 & -2 & 1 & -1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right)$

$Q^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \Rightarrow \boxed{A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} . \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}}$

$\underbrace{}_{Q} \quad \underbrace{}_{\Lambda} \quad \underbrace{}_{Q^{-1}}$

NOTA

II) $A=\begin{pmatrix}1&-1\\2&4\end{pmatrix} \Rightarrow P_A(\lambda)=\det(\lambda I-A)=\det\begin{pmatrix}\lambda-1&1\\-2&\lambda-4\end{pmatrix}=(\lambda-1)\cdot(\lambda-4)+2=0$

$\Rightarrow \lambda^2-5\lambda+6=0 \quad , \lambda_1=3 \, , \lambda_2=2$

$\lambda=3: \left(\begin{array}{cc|c}2&1&0\\\cancel{-2}&\cancel{-1}&\cancel{0}\end{array}\right) \rightarrow X_2=-2X_1 \Rightarrow S_{\lambda=3}=\text{gen}\{(1 \;\; -2)^t\}$

$\lambda=2: \left(\begin{array}{cc|c}1&1&0\\\cancel{-2}&\cancel{-1}&\cancel{0}\end{array}\right) \rightarrow X_1=-X_2 \Rightarrow S_{\lambda=2}=\text{gen}\{(-1 \;\; 1)^t\}$

$Q=\begin{pmatrix}1&-1\\-2&1\end{pmatrix} \Rightarrow \left(\begin{array}{cc|cc}1&-1&1&0\\-2&1&0&1\end{array}\right) \rightarrow \left(\begin{array}{cc|cc}1&-1&1&0\\0&-1&2&1\end{array}\right) \rightarrow \left(\begin{array}{cc|cc}-1&0&1&1\\0&-1&2&1\end{array}\right) \rightarrow \left(\begin{array}{cc|cc}1&0&-1&-1\\0&1&-2&-1\end{array}\right)$

$\Rightarrow \boxed{A=\begin{pmatrix}1&-1\\-2&1\end{pmatrix}\cdot\begin{pmatrix}3&0\\0&2\end{pmatrix}\cdot\begin{pmatrix}-1&-1\\-2&-1\end{pmatrix}}$

III) $A=\begin{pmatrix}1&i\\-3i&-1\end{pmatrix} \Rightarrow P_A(\lambda)=\det\begin{pmatrix}\lambda-1&-i\\3i&\lambda+1\end{pmatrix}=(\lambda-1)\cdot(\lambda+1)-3=0$

$\Rightarrow \lambda^2-4=0 \Rightarrow \lambda_1=2 \, , \lambda_2=-2$

$\lambda=2: \left(\begin{array}{cc|c}1&-i&0\\\cancel{3i}&\cancel{3}&\cancel{0}\end{array}\right) \rightarrow X_1=iX_2 \Rightarrow S_{\lambda=2}=\text{gen}\{(i \;\; 1)^t\}$

$\lambda=-2: \left(\begin{array}{cc|c}-3&-i&0\\\cancel{3i}&\cancel{-1}&\cancel{0}\end{array}\right) \rightarrow -3X_1-iX_2=0 \Rightarrow X_2=-\frac{3}{i}X_1$

$\Rightarrow S_{\lambda=-2}=\text{gen}\{(1 \;\; -3/i)^t\}$

$Q=\begin{pmatrix}i&1\\1&-3/i\end{pmatrix} \Rightarrow \left(\begin{array}{cc|cc}i&1&1&0\\1&-3/i&0&1\end{array}\right) \rightarrow \left(\begin{array}{cc|cc}i&1&1&0\\0&-4&-1&i\end{array}\right) \rightarrow \left(\begin{array}{cc|cc}4i&0&3&i\\0&-4&-1&i\end{array}\right) \rightarrow \left(\begin{array}{cc|cc}1&0&\frac{3}{4}i&1/4\\0&1&1/4&-\frac{i}{4}\end{array}\right)$

$\Rightarrow \boxed{A=\begin{pmatrix}i&1\\1&-3/i\end{pmatrix}\cdot\begin{pmatrix}2&0\\0&-2\end{pmatrix}\cdot\begin{pmatrix}3/4\,i&1/4\\1/4&-i/4\end{pmatrix}}$

23) $A=\begin{pmatrix}0&1\\-1&0\end{pmatrix} \Rightarrow P_A(\lambda)=\det\begin{pmatrix}\lambda&-1\\1&\lambda\end{pmatrix}=\lambda^2+1=0 \Rightarrow \lambda_1=i \, ; \lambda_2=-i$

$\lambda=i: \left(\begin{array}{cc|c}\cancel{i}&\cancel{-1}&\cancel{0}\\1&i&0\end{array}\right) \rightarrow X_1=-iX_2 \Rightarrow S_{\lambda=i}=\text{gen}\{(-i \;\; 1)^t\}$

$\lambda=-i: \left(\begin{array}{cc|c}\cancel{-i}&\cancel{-1}&\cancel{0}\\1&-i&0\end{array}\right) \rightarrow X_1=iX_2 \Rightarrow S_{\lambda=-i}=\text{gen}\{(i \;\; 1)^t\}$

$Q=\begin{pmatrix}-i&i\\1&1\end{pmatrix} \Rightarrow \left(\begin{array}{cc|cc}-i&i&1&0\\1&1&0&1\end{array}\right) \rightarrow \left(\begin{array}{cc|cc}-i&i&1&0\\0&2i&1&i\end{array}\right) \rightarrow \left(\begin{array}{cc|cc}-2i&0&2i-1&-i\\0&2i&1&i\end{array}\right) \rightarrow \left(\begin{array}{cc|cc}1&0&\frac{2}{3}i-1&1/2\\0&1&\frac{2}{3}i&1/2\end{array}\right)$

$$\Rightarrow \boxed{A = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3}i-1 & \frac{1}{2} \\ -\frac{2}{3}i & \frac{1}{2} \end{pmatrix}}$$

24)

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \Rightarrow P_A(\lambda) = \det\begin{pmatrix} \lambda-2 & -1 \\ 0 & \lambda-2 \end{pmatrix} = (\lambda-2)^2 = 0$$

$\Rightarrow \lambda = 2$ (mult. alg es 2)

$\left(\begin{array}{cc|c} 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \Rightarrow x_2 = 0 \Rightarrow S_{\lambda=2} = \text{gen}\{(1\ 0)^t\} \Rightarrow$ mult. geom. es $1 \neq$ mult. alg.

∴ A no es diagonalizable, no existe Q cuadrada.

25) -a) Los autovalores son: $\lambda_1 = -2$ (mult alg. 2) y $\lambda_2 = 6$

y los autoespacios son: $S_{\lambda=-2} = \text{gen}\{(1\ -2\ 0)^t; (0\ 3\ 1)^t\}$

$S_{\lambda=6} = \text{gen}\{(2\ 1\ -1)^t\}$

La matriz es diagonalizable.

$$Q = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 3 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{8} & \frac{5}{8} \\ \frac{1}{4} & \frac{1}{8} & -\frac{3}{8} \end{pmatrix} \Rightarrow \boxed{A = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 3 & 1 \\ 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{8} & \frac{5}{8} \\ \frac{1}{4} & \frac{1}{8} & -\frac{3}{8} \end{pmatrix}}$$

b) $\lambda_1 = 2$; $\lambda_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}$; $\lambda_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2} \cdot i$ (es diagonalizable porque la matriz es de 3x3 y tiene 3 autoval. ≠)

$S_{\lambda=2} = \text{gen}\{(1\ 0\ 0)^t\}$; $S_{\lambda=-\frac{1}{2}+\frac{\sqrt{3}}{2}i} = \text{gen}\left\{\left(0\ \ 1\ \ -\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^t\right\}$

$S_{\lambda=-\frac{1}{2}-\frac{\sqrt{3}}{2}i} = \text{gen}\left\{\left(0\ \ 1\ \ -\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^t\right\}$

$$\Rightarrow \boxed{A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -\frac{1}{2}+\frac{\sqrt{3}}{2}i & -\frac{1}{2}-\frac{\sqrt{3}}{2}i \end{pmatrix}}_{Q} \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2}+\frac{\sqrt{3}}{2}i & 0 \\ 0 & 0 & -\frac{1}{2}-\frac{\sqrt{3}}{2}i \end{pmatrix} \cdot Q^{-1}}$$

c) Es diagonalizable, matriz de nxn con n autoval. ≠.

$$Q = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & -1 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \Rightarrow \boxed{A = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}}$$

d) $\lambda_1=6$, $\lambda_2=8$, $\lambda=6$ tiene mult. alg. 2 y geom 1, así que esta matriz no es diagonalizable.

26)

$$A=\begin{pmatrix} 2\alpha+4 & 1-\alpha & -2\alpha-\alpha^2 \\ 0 & 4-\alpha & 0 \\ 0 & 0 & 4-\alpha^2 \end{pmatrix}$$

a) Como A está triangulada, los autovalores son $\lambda_1=2\alpha+4$
$\lambda_2=4-\alpha$
$\lambda_3=4-\alpha^2$

Busco los α para los cuales los 3 autoval son ≠:

$\lambda_1=\lambda_2 \Rightarrow 2\alpha+4=4-\alpha \Rightarrow \alpha=0$
$\lambda_1=\lambda_3 \Rightarrow 2\alpha+4=4-\alpha^2 \Rightarrow \alpha^2+2\alpha=0 \Rightarrow \alpha.(\alpha+2)=0 \Rightarrow \alpha=-2$
$\lambda_2=\lambda_3 \Rightarrow 4-\alpha=4-\alpha^2 \Rightarrow \alpha^2-\alpha=0 \Rightarrow \alpha(\alpha-1)=0 \Rightarrow \alpha=1$

veo los casos particulares:

si $\alpha=0$:

$$A=\begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \Rightarrow A.v=\lambda.v \Rightarrow (\lambda.I-A).v=0$$ y $\lambda=4$ es autoval. de mult alg. 3

$\lambda=4$ $$\Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \rightarrow x_2=0$$

$\Rightarrow S_{\lambda=4}=\text{gen}\{(1\ 0\ 0)^t, (0\ 0\ 1)^t\}$ → mult geom es 2

A no es diagonalizable

si $\alpha=-2$; $\lambda=0$ es autoval. de mult. alg. 2.

$$A=\begin{pmatrix} 0 & 3 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$; el autoespacio asociado a $\lambda=0$ es Nul(A) y se puede apreciar que rg(A) = 1 ⇒ dim Nul(A) = 2 ⇒ mult. geom. de $\lambda=0$ es 2

∴ A es diagonalizable

si $\alpha=1$, $\lambda=3$ es autoval. de mult. alg. 2.

$$A=\begin{pmatrix} 6 & 0 & -3 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \Rightarrow \text{Para } \lambda=3 \Rightarrow \left(\begin{array}{ccc|c} -3 & 0 & 6 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \Rightarrow x_1=2x_3$$

$\Rightarrow S_{\lambda=3}=\text{gen}\{(0\ 1\ 0)^t; (2\ 0\ 1)^t\}$

$\lambda=3$ tiene mult geom. 2 ∴ A es diagonalizable.

A es diagonalizable para $\boxed{\alpha \in \mathbb{R}-\{0\}}$

b) Para $\alpha=1$: $S_{\lambda=3}=\text{gen}\{(0\ 1\ 0)^t ; (2\ 0\ 1)^t\}$

$\lambda=6$: $\Rightarrow \left(\begin{array}{ccc|c} 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{3} & \cancel{0} \end{array}\right) \Rightarrow \begin{array}{l} x_3=0 \\ x_2=0 \end{array} \Rightarrow S_{\lambda=6}=\text{gen}\{(1\ 0\ 0)^t\}$

$$\Rightarrow \boxed{Q=\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ; D=\begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} ; Q^{-1}=\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \Rightarrow \boxed{A=Q.D.Q^{-1}}$$

Para $\alpha=2$:

$A=\begin{pmatrix} 8 & -1 & -8 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} \lambda_1=8 \\ \lambda_2=2 \\ \lambda_3=0 \end{array} \Rightarrow$ $\lambda=8$: $\Rightarrow (8.I-A).V=0 \Rightarrow \left(\begin{array}{ccc|c} \cancel{0} & \cancel{1} & \cancel{8} & \cancel{0} \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 8 & 0 \end{array}\right) \Rightarrow x_2=x_3=0$

$\Rightarrow S_{\lambda=8}=\text{gen}\{(1\ 0\ 0)^t\}$

$\lambda=2$:

$\left(\begin{array}{ccc|c} -6 & 1 & 8 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & 2 & 0 \end{array}\right) \Rightarrow \begin{cases} -6x_1+x_2+8x_3=0 \Rightarrow x_2=6x_1 \\ x_3=0 \end{cases} \Rightarrow S_{\lambda=2}=\text{gen}\{(1\ 6\ 0)^t\}$

$\lambda=0$:

$Nul(A) \Rightarrow \left(\begin{array}{ccc|c} 8 & -1 & -8 & 0 \\ 0 & 2 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \Rightarrow \begin{cases} x_2=0 \\ 8x_1-8x_3=0 \Rightarrow x_1=x_3 \end{cases} \Rightarrow S_{\lambda=0}=\text{gen}\{(1\ 0\ 1)^t\}$

$$\Rightarrow \boxed{A=\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 6 \\ 0 & 1 & 0 \end{pmatrix}}_{Q} \cdot \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot Q^{-1}}$$

27) - a) V

b) F (Contraejemplo: $A=\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}}_{P} \cdot \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}}_{\Lambda} . P^{-1}$ ó $A=\underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}}_{S} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} . S^{-1}$)

c) V

d) -

e) V

29) - a) - F , pues para hallar el autovector se procede haciendo:

$$T(v)=\lambda v \Rightarrow \underbrace{[T]_B \cdot C_B(v)}_{C_B(T(v))} = \underbrace{\lambda \cdot C_B(v)}_{C_B(\lambda \cdot v)} \Rightarrow (\lambda . I - [T]_B) . C_B(v) = 0$$

El autovector va a cambiar dependiendo de la base a partir de la cual se toman las coordenadas. Mientras que λ no varía; esto justifica el sgte. ítem

b) - V

c) - V (definición de condición de diagonalización de una T.L.)

d) - V (análogo a una matriz A diagonalizable)

30) - a) $T(X) = (x_1+x_2-x_3 \quad x_1+x_2+x_3 \quad -x_1+x_2+x_3)^t$

Considero base $E_{\mathbb{R}^3}=\{e_1, e_2, e_3\}$

$[T]_E = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$; Busco autoval. de $[T]_E \Rightarrow \det(\lambda I - [T]_E) = \det \begin{pmatrix} \lambda-1 & -1 & 1 \\ -1 & \lambda-1 & -1 \\ 1 & -1 & \lambda-1 \end{pmatrix} =$

$$= (\lambda-1) . \begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 1 & \lambda-1 \end{vmatrix} + \begin{vmatrix} -1 & \lambda-1 \\ 1 & -1 \end{vmatrix} = (\lambda-1) . \left((\lambda-1)^2-1\right) - \lambda + 1 + 1 + 1 - \lambda + 1 =$$

$$= (\lambda-1) . \left((\lambda-1)^2-1\right) - 2\lambda + 4 = (\lambda-1) . (\lambda^2 - 2\lambda + \cancel{1} - \cancel{1}) - 2\lambda + 4 = \lambda^3 - 2\lambda^2 - \lambda^2 + \cancel{2\lambda} - \cancel{2\lambda} + 4 =$$

$= \lambda^3 - 3\lambda^2 + 4 = 0 \Rightarrow \lambda = 2$ es raíz

$$\begin{array}{c|cccc} & 1 & -3 & 0 & 4 \\ 2 & & 2 & -2 & -4 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$$\Rightarrow P_{[T]_E}(\lambda) = (\lambda-2) . \underbrace{(\lambda^2-\lambda-2)}_{\lambda=2 \quad \lambda=-1} \quad \Rightarrow \boxed{\begin{array}{l} \lambda_1 = 2 \text{ (doble)} \\ \lambda_2 = -1 \end{array}}$$

$\Rightarrow \lambda = 2$:

$$(2 . I - [T]_E) . \underbrace{C_E(v)}_{v} = 0 \Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ \cancel{-1} & \cancel{1} & \cancel{-1} & \cancel{0} \\ \cancel{1} & \cancel{-1} & \cancel{1} & \cancel{0} \end{array}\right) \Rightarrow x_1 = x_2 - x_3$$

$$\Rightarrow \boxed{S_{\lambda=2} = \text{gen}\{(1 \;\; 1 \;\; 0)^t ; (-1 \;\; 0 \;\; 1)^t\}}$$

$\lambda=-1$:

$$(-I-[T]_E).\underbrace{C_E(v)}_{v}=0 \Rightarrow \left(\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ -1 & -2 & -1 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ \cancel{0} & \cancel{-3} & \cancel{-3} & \cancel{0} \end{array}\right) \rightarrow \begin{cases} -2X_1-X_2+X_3=0 \\ -3X_2-3X_3=0 \end{cases}$$

$\Rightarrow X_2=-X_3$

$-2X_1+2X_3=0 \Rightarrow X_1=X_3$

$\Rightarrow \boxed{S_{\lambda=-1}=\text{gen}\{(1\ \ -1\ \ 1)^t\}}$

mult. alg. y geom. de $\lambda=2$ y $\lambda=-1$ son 2 y 1 respectivamente

$\boxed{B=\{(1\ \ -1\ \ 1)^t;(1\ \ 1\ \ 0)^t;(-1\ \ 0\ \ 1)^t\}}$

b) $T(a+b.x+c.x^2)=2.a+c.x+(-b-c).x^2$

$E_{P_2}=\{1,x;x^2\} \Rightarrow [T]_E=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$ $(T(1)=2+0+0$ pues $a=1, b=0, c=0$; etc$)$

$$\Rightarrow P_{[T]_E}(\lambda)=\det(\lambda.I-[T]_E)=\det\begin{pmatrix} \lambda-2 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 1 & \lambda+1 \end{pmatrix}=(\lambda-2).(\lambda.(\lambda+1)+1)=0$$

$\Rightarrow \boxed{\lambda=2}$ es autoval.; $\lambda^2+\lambda+1=0 \Rightarrow W^2=b^2-4.a.c=-3$

$\Rightarrow W=\sqrt{3}.i$; $\lambda=\frac{-b\pm W}{2.a} \Rightarrow \boxed{\begin{array}{l} \lambda_1=-\frac{1}{2}+\frac{\sqrt{3}}{2}.i \\ \lambda_2=-\frac{1}{2}-\frac{\sqrt{3}}{2}.i \end{array}}$

$\lambda=2$:

$$(2I-[T]_E).\underbrace{C_{E_{P_2}}(v)}_{v}=0 \Rightarrow \left(\begin{array}{ccc|c} \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 2 & -1 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 0 & 2 & -1 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right) \rightarrow \begin{array}{l} X_2=0 \\ X_3=0 \end{array}$$

$\Rightarrow C_{E_{P_2}}(v)=X_1.(1\ \ 0\ \ 0)^t \Rightarrow \boxed{S_{\lambda=2}=\text{gen}\{1\}}$

$\Rightarrow$ En $\mathbb{R}$ T no es diagonalizable, en $\mathbb{C}$ sí. (3 autoval. $\neq$); la mult. alg. y geom. de cada autoval. es 1.

$\rightarrow$ no existe la base pedida

$\lambda=-\frac{1}{2}+\frac{\sqrt{3}}{2}.i$:

$$\Rightarrow \left(\begin{array}{ccc|c} -\frac{5}{2}+\frac{\sqrt{3}}{2}.i & 0 & 0 & 0 \\ 0 & -\frac{1}{2}+\frac{\sqrt{3}}{2}.i & -1 & 0 \\ \cancel{0} & \cancel{1} & \cancel{\frac{1}{2}+\frac{\sqrt{3}}{2}.i} & \cancel{0} \end{array}\right) \rightarrow \begin{array}{l} X_1=0 \\ X_3=\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}.i\right).X_2 \end{array}$$

$\Rightarrow C_{E_{P_2}}(v)=X_2.\left(0\ \ 1\ \ -\frac{1}{2}+\frac{\sqrt{3}}{2}.i\right)^t$

$\Rightarrow \boxed{S_{\lambda=-\frac{1}{2}+\frac{\sqrt{3}}{2}.i}=\text{gen}\left\{x+\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}.i\right).x^2\right\}}$

. . . . .

c) $[T]_B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow P_{[T]_B}(\lambda) = \det(\lambda . I - [T]_B) = \det\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} = \lambda^2 + 1 = 0$

$\Rightarrow \boxed{\lambda_1 = i \ ; \ \lambda_2 = -i}$; en $\mathbb{R}$ - e.v. no hay autoval. y no existe la base pedida. (pues no hay autovec. para armarla). en $\mathbb{C}$ sí pues mult. alg. y geom. de ambos autoval. es 1.

En $\mathbb{C}$ - e.v.:

$\lambda = i : \left(\begin{array}{cc|c} \cancel{i} & \cancel{-1} & \cancel{0} \\ 1 & i & 0 \end{array}\right) \rightarrow X_1 = -i . X_2 \Rightarrow C_B(v) = X_2 . (-i \quad 1)^t$

$\Rightarrow \boxed{S_{\lambda = i} = \text{gen}\{ -i . v_1 + v_2 \}}$ $\qquad X_2 \in \mathbb{C}$

$\lambda = -i : \left(\begin{array}{cc|c} \cancel{-i} & \cancel{-1} & \cancel{0} \\ 1 & -i & 0 \end{array}\right) \rightarrow X_1 = i . X_2 \Rightarrow C_B(v) = X_2 . (i \quad 1)^t$

$\Rightarrow \boxed{S_{\lambda = -i} = \text{gen}\{ i . v_1 + v_2 \}}$

la base pedida es $\boxed{A = \{ -i . v_1 + v_2 \ ; \ i . v_1 + v_2 \}}$

d) $T(P) = P'' - P \quad , \quad T : P_3 \rightarrow P_3$

$E_{P_3} = \{1 ; x ; x^2 ; x^3\} \Rightarrow \begin{array}{l} T(1) = 0 - 1 = -1 \\ T(x) = 0 - x = -x \\ T(x^2) = 2 - x^2 \\ T(x^3) = 6x - x^3 \end{array} \Rightarrow [T]_E = \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 6 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$\Rightarrow$ El autovalor es $\boxed{\lambda = -1}$ (cuádruple) (matriz triangulada).

$\Rightarrow (-I - [T]_E) . C_{E_{P_3}}(v) = 0 \Rightarrow \left(\begin{array}{cccc|c} 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \Rightarrow X_3 = X_4 = 0$

$C_{E_{P_3}}(v) = X_1 . (1 \ 0 \ 0 \ 0)^t + X_2 . (0 \ 1 \ 0 \ 0)^t \Rightarrow \boxed{S_{\lambda = -1} = \text{gen}\{1 , x\}}$

$X_1 , X_2 \in \mathbb{R}$

mult. alg de $\lambda = -1$ es 4 y mult. geom. es 2. $\therefore \nexists$ la base pedida.

37)

$$[T]_B = \begin{pmatrix} 1+\alpha & -\alpha & \alpha \\ 2+\alpha & -\alpha & \alpha-1 \\ 2 & -1 & 0 \end{pmatrix}, \ \alpha \in \mathbb{R}$$

a) $$P_{[T]_B}(\lambda) = \det \begin{pmatrix} \lambda-1-\alpha & \alpha & -\alpha \\ -2-\alpha & \lambda+\alpha & -\alpha+1 \\ -2 & 1 & \lambda \end{pmatrix} = (-2).\begin{vmatrix} \alpha & -\alpha \\ \lambda+\alpha & -\alpha+1 \end{vmatrix} - \begin{vmatrix} \lambda-1-\alpha & -\alpha \\ -2-\alpha & -\alpha+1 \end{vmatrix} +$$

$$+\lambda.\begin{vmatrix} \lambda-1-\alpha & \alpha \\ -2-\alpha & \lambda+\alpha \end{vmatrix} = (-2).\left(\alpha.(-\alpha+1)+\alpha.(\lambda+\alpha)\right) - (\lambda-1-\alpha).(-\alpha+1)+\alpha.(2+\alpha)+$$

$$+\lambda.\left((\lambda-1-\alpha).(\lambda+\alpha)+\alpha.(2+\alpha)\right) = (-2).(-\cancel{\alpha^2}+\alpha+\alpha.\lambda+\cancel{\alpha^2}) - (-\lambda\alpha+\lambda+\alpha-1+\alpha^2-\alpha] +$$

$$+2\alpha+\alpha^2+\lambda.(\lambda^2+\cancel{\lambda\alpha}-\lambda-\alpha-\cancel{\alpha}\lambda-\cancel{\alpha^2}+2\alpha+\cancel{\alpha^2}) = -\cancel{2\alpha}-2\cancel{\alpha}.\lambda+\cancel{\lambda\alpha}-\lambda-\cancel{\alpha}+1-$$

$$-\cancel{\alpha^2}+\cancel{\alpha}+\cancel{2\alpha}+\cancel{\alpha^2}+\lambda^3-\lambda^2-\cancel{\alpha}\lambda+\lambda\cancel{2\alpha} = \lambda^3-\lambda^2-\lambda+1=0$$

$\Rightarrow \boxed{\lambda=1}$ es autoval.

$$\Rightarrow \begin{array}{c|cccc} & 1 & -1 & -1 & 1 \\ 1 & & 1 & 0 & -1 \\ \hline & 1 & 0 & -1 & \boxed{0} \end{array} \Rightarrow P(\lambda)=(\lambda-1).(\lambda^2-1)=0$$

$\lambda=1$ $\quad \boxed{\lambda=-1}$

b) $$\lambda=1: \left(\begin{array}{ccc|c} -\alpha & \alpha & -\alpha & 0 \\ -2-\alpha & 1+\alpha & -\alpha+1 & 0 \\ -2 & 1 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -\alpha & \alpha & -\alpha & 0 \\ -2-\alpha & 1+\alpha & -\alpha+1 & 0 \\ 0 & -\alpha & 3\alpha & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -\alpha & \alpha & -\alpha & 0 \\ 0 & -\alpha & 3\alpha & 0 \\ \cancel{0} & \cancel{-\alpha} & \cancel{3\alpha} & \cancel{0} \end{array}\right)$$

$$\Rightarrow \begin{cases} -\alpha.x_1+\alpha.x_2-\alpha.x_3=0 \\ -\alpha.x_2+3\alpha.x_3=0 \end{cases} \Rightarrow \begin{array}{l} -x_1+3x_3-x_3=0 \Rightarrow x_1=2x_3 \quad (\alpha\neq 0) \\ x_2=3x_3 \end{array}$$

$\Rightarrow C_B(v)=x_3.(2 \ \ 3 \ \ 1)^t; \ x_3\in\mathbb{R} \Rightarrow \boxed{S_{\lambda=1}=\text{gen}\{2v_1+3v_2+v_3\}}$

T no es diagonalizable. ($\lambda=1$ tiene mult. alg. 2 y geom. 1)

$$\lambda=-1: \left(\begin{array}{ccc|c} -2-\alpha & \alpha & -\alpha & 0 \\ -2-\alpha & \alpha-1 & -\alpha+1 & 0 \\ -2 & 1 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -2-\alpha & \alpha & -\alpha & 0 \\ 0 & 1 & -1 & 0 \\ \cancel{0} & \cancel{-\alpha+2} & \cancel{\alpha-2} & \cancel{0} \end{array}\right)$$

$$\Rightarrow \begin{cases} (-2-\alpha).x_1+\alpha.x_2-\alpha.x_3=0 \\ x_2-x_3=0 \end{cases} \Rightarrow \begin{array}{l} (-2-\alpha).x_1=0 \Rightarrow x_1=0 \quad (\alpha\neq -2) \\ x_2=x_3 \end{array}$$

$\Rightarrow C_B(v)=x_2.(0 \ \ 1 \ \ 1)^t, \ x_2\in\mathbb{R}$

$\Rightarrow \boxed{S_{\lambda=-1}=\text{gen}\{v_2+v_3\}}$

c) $\dim(S_{\lambda=1})=2$ para que sea diagonalizable

si $\alpha=0$: $\left(\begin{array}{ccc|c} \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ -2 & 1 & 1 & 0 \\ \cancel{-2} & \cancel{1} & \cancel{1} & \cancel{0} \end{array}\right)$ y es el único valor de α que funciona

En este caso $C_B(v)=x_1.(1 \ \ 0 \ \ 2)^t+x_2.(0 \ \ 1 \ \ -1)^t$

$\Rightarrow S_{\lambda=1}=\text{gen}\{v_1+2v_3; v_2-v_3\} \Rightarrow \boxed{B'=\{v_1+2v_3; v_2-v_3; v_2+v_3\}}$

NOTA

33)

$T(v) = v - 3 \cdot \frac{u u^t}{u^t . u} . v \quad ; u = (1 \ -1 \ 2)^t \quad ; u^t u = 6$

$u . u^t = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix}$

$T(v) = v - \frac{1}{2} \cdot \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix} . v = \begin{pmatrix} 1/2 & 1/2 & -1 \\ 1/2 & 1/2 & 1 \\ -1 & 1 & -1 \end{pmatrix} . v$

$E = \{e_1; e_2; e_3\}$

$[T]_E = \begin{pmatrix} 1/2 & 1/2 & -1 \\ 1/2 & 1/2 & 1 \\ -1 & 1 & -1 \end{pmatrix} \Rightarrow P_T(\lambda) = \det(\lambda I - [T]_E) = \det \begin{pmatrix} \lambda - 1/2 & -1/2 & 1 \\ -1/2 & \lambda - 1/2 & -1 \\ 1 & -1 & \lambda + 1 \end{pmatrix} =$

$= \begin{vmatrix} -1/2 & \lambda - 1/2 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} \lambda - 1/2 & -1/2 \\ 1 & -1 \end{vmatrix} + (\lambda+1) . \begin{vmatrix} \lambda - 1/2 & -1/2 \\ -1/2 & \lambda - 1/2 \end{vmatrix} = \frac{1}{2} - \lambda + 1/2 - \lambda + \frac{1}{2} + \frac{1}{2} +$

$+ (\lambda+1) . \left((\lambda - 1/2)^2 - \frac{1}{4} \right) = 2 - 2\lambda + (\lambda+1) . (\lambda^2 - \lambda + \cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{4}}) =$

$= 2 - 2\lambda + \lambda^3 - \cancel{\lambda^2} + \cancel{\lambda^2} - \lambda = \lambda^3 - 3\lambda + 2 = 0 \Rightarrow \boxed{\lambda = 1}$ es autovalor

$\Rightarrow \begin{array}{c|cccc} & 1 & 0 & -3 & 2 \\ 1 & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array} \Rightarrow P_T(\lambda) = (\lambda - 1) . (\lambda^2 + \lambda - 2) = 0$

$\lambda = 1 \qquad \boxed{\lambda = -2}$

Para $\lambda = 1$:

$\left(\begin{array}{ccc|c} \cancel{1/2} & \cancel{-1/2} & \cancel{1} & \cancel{0} \\ \cancel{-1/2} & \cancel{1/2} & \cancel{-1} & \cancel{0} \\ 1 & -1 & 2 & 0 \end{array}\right)$ → queda una sola condición $\Rightarrow \dim(S_{\lambda=1}) = 2$

∴ T es diagonalizable (y para cualquier base pues lo único que cambia es la forma de los autovec., pero no la cantidad)

35) Para que S sea invariante por A $\Rightarrow A . v = \lambda . v \ \forall v \in S$, pues $A . v$ debe $\in S$ si $v \in S$.

S es un autoespacio asociado a ese λ, y λ tiene mult. alg. ≥ 1.

$S^\perp = \{X \in R^3 / X_1 - X_2 = 0\} = \text{gen}\{(1 \ 1 \ 0)^t ; (0 \ 0 \ 1)^t\}$

si $\lambda = 2$ es el asociado a $S^\perp$, la multiplicidad algebraica de ese autoval. es 2, ya que hay otro autoval. $\neq$

A es diagonalizable por lo anterior

$$A = Q.D.Q^{-1} \Rightarrow Q = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \rightarrow$$

$$\rightarrow Q^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$, elijo $D = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$\Rightarrow A = \begin{pmatrix} \lambda & 2 & 0 \\ -\lambda & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\lambda+1 & -\frac{1}{2}\lambda+1 & 0 \\ -\frac{1}{2}\lambda+1 & \frac{1}{2}\lambda+1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\det(A) = 12 \Rightarrow \det(A) = 2.(-1)^6 . \begin{vmatrix} \frac{1}{2}\lambda+1 & -\frac{1}{2}\lambda+1 \\ -\frac{1}{2}\lambda+1 & \frac{1}{2}\lambda+1 \end{vmatrix} = 2.\left(\left(\frac{1}{2}\lambda+1\right)^2 - \left(-\frac{1}{2}\lambda+1\right)^2\right) =$$

$$= 2.\left(\cancel{\frac{1}{4}\lambda^2} + \lambda \cancel{+1} - \cancel{\frac{1}{4}\lambda^2} + \lambda \cancel{-1}\right) = 4\lambda = 12 \Rightarrow \lambda = 3$$

$$\boxed{A = \begin{pmatrix} 5/2 & -1/2 & 0 \\ -1/2 & 5/2 & 0 \\ 0 & 0 & 2 \end{pmatrix}}$$

otra forma es acordarse de que el determinante de una matriz es igual al producto de sus autoval.

3b)

De I) saco que $\lambda = 1$ es autovalor de T, con $S_{\lambda=1} = gen\{6v_1 + 2v_2 + 5v_3\}$

En II), $C_B(v) = (a \ \ b \ \ c)^t \Rightarrow (2 \ \ 11 \ \ -7) . \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow 2a + 11b - 7c = 0$

$a, b, c \in \mathbb{R}$

$$\Rightarrow c = \frac{2}{7}a + \frac{11}{7}b \Rightarrow C_B(v) = \begin{pmatrix} a \\ b \\ \frac{2}{7}a + \frac{11}{7}b \end{pmatrix} \Rightarrow v = a.v_1 + b.v_2 + \left(\frac{2}{7}a + \frac{11}{7}b\right).v_3 =$$

$$= a.\left(v_1 + \frac{2}{7}v_3\right) + b.\left(v_2 + \frac{11}{7}v_3\right) \Rightarrow S = gen\left\{v_1 + \frac{2}{7}v_3 ; v_2 + \frac{11}{7}v_3\right\}$$

Por lo tanto el autoval asociado a S tiene mult. alg 2

De III) saco que la suma de los autoval. de T es igual a la traza. Además, 2 autoval. son iguales.

$$\lambda_1 + \lambda_2 + \lambda_3 = \underset{[T]_B}{\underset{\downarrow}{tr(A)}} \Rightarrow 1 + \lambda + \lambda = 5 \Rightarrow 2\lambda = 4 \Rightarrow \lambda = 2$$

a) $\boxed{\lambda_1 = 1 \ ; \ \lambda_2 = 2}$

b) Como las mult. alg. y geom coinciden para ambos autoval, T es diagonalizable.

$B' = \{6v_1 + 2v_2 + 5v_3 ; v_1 + \frac{2}{7}v_3 ; v_2 + \frac{11}{7}v_3\}$ es la base de autovectores.

$$[T]_{B'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow [T]_B = C_{B'B} . [T]_{B'} . C_{BB'} \ ; \ C_{B'B} = \begin{pmatrix} 6 & 1 & 0 \\ 2 & 0 & 1 \\ 5 & 2/7 & 11/7 \end{pmatrix}$$

NOTA

$$C_{BB'} = (C_{B'B})^{-1} = \begin{pmatrix} -2 & -11 & 7 \\ 13 & 66 & -42 \\ 4 & 23 & -14 \end{pmatrix}$$

$$[T]_B = \begin{pmatrix} 6 & 2 & 0 \\ 2 & 0 & 2 \\ 5 & 4/4 & 23/4 \end{pmatrix} \cdot \begin{pmatrix} -2 & -11 & 7 \\ 13 & 66 & -42 \\ 4 & 23 & -14 \end{pmatrix} = \boxed{\begin{pmatrix} 14 & 66 & -42 \\ 4 & 24 & -14 \\ 10 & 55 & -33 \end{pmatrix}}$$

Práctica 5 (2° parte)

1) - I) $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$\Rightarrow$ Busco autoval. de A: $P_A(\lambda) = \det\begin{pmatrix} \lambda-3 & -1 \\ -1 & \lambda-3 \end{pmatrix} = (\lambda-3)^2 - 1 = \lambda^2 - 6\lambda + 8 = 0$

$\lambda_1 = 4$ $\Rightarrow$ como $\lambda_1 \neq \lambda_2$ A s diagonalizable
$\lambda_2 = 2$

$A = P.D.P^{-1}$

Autovec.: $\lambda = 4: \left(\begin{array}{cc|c} 1 & -1 & 0 \\ \cancel{-1} & \cancel{1} & \cancel{0} \end{array}\right) \rightarrow a = b \Rightarrow S_{\lambda=4} = gen\{(1 \ \ 1)^T\}$

$\lambda = 2: \left(\begin{array}{cc|c} -1 & -1 & 0 \\ \cancel{-1} & \cancel{-1} & \cancel{0} \end{array}\right) \rightarrow a = -b \Rightarrow S_{\lambda=2} = gen\{(-1 \ \ 1)^T\}$

$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$; $D = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$

$\Rightarrow X' = A.X = P.D\,P^{-1}.X \Rightarrow (P.Y)' = P.D.\underbrace{P^{-1}.P}_{I}.Y \Rightarrow P.Y' = P.D.Y$

$\downarrow$
$X = P.Y$ (cambio de variable)

$\Rightarrow Y' = D.Y$

$\Rightarrow \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} . \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \begin{cases} y_1' = 4\, y_1 \\ y_2' = 2.y_2 \end{cases}$

La solución de este tipo de ED se obtiene:

$y' = \alpha.y \Rightarrow \frac{dy}{dx} = \alpha.y \Rightarrow \frac{dy}{y} = \alpha.dx \Rightarrow \ln|y| = \alpha.x + c \Rightarrow y = \underbrace{\pm e^c}_{c_1} . e^{\alpha.x} = c_1 e^{\alpha.x}$

$\downarrow$
suponiendo que depende de x

$\therefore y_1 = c_1 . e^{4.x}$, $y_2 = c_2 . e^{2x}$

$\Rightarrow X = P.Y \Rightarrow \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} . \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 - y_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} c_1 . e^{4x} - c_2 . e^{2x} \\ c_1 . e^{4x} + c_2 . e^{2x} \end{pmatrix}$

$$\Rightarrow \boxed{\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = c_1 . e^{4x} . \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 . e^{2x} . \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

II) $A=\begin{pmatrix}1 & -2\\ 1 & -2\end{pmatrix}$ => Busco autoval.: $P_A(\lambda)=\det\begin{pmatrix}\lambda-1 & 2\\ -1 & \lambda+2\end{pmatrix}=(\lambda-1)\cdot(\lambda+2)+2=$

$=\lambda^2+\lambda=0 \Rightarrow \begin{matrix}\lambda_1=0\\ \lambda_2=-1\end{matrix}$ => es diagonalizable.

$\lambda=0:\left(\begin{array}{cc|c}-1 & 2 & 0\\ \cancel{-1} & \cancel{2} & \cancel{0}\end{array}\right) \rightarrow a=2b \rightarrow S_{\lambda=0}=\text{gen}\{(2 \quad 1)^T\}$

$\lambda=-1:\left(\begin{array}{cc|c}-2 & 2 & 0\\ \cancel{-1} & \cancel{1} & \cancel{0}\end{array}\right) \rightarrow a=b \rightarrow S_{\lambda=-1}=\text{gen}\{(1 \quad 1)^T\}$

$P=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$; $D=\begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}$; $A=P.D.P^{-1}$ (no hace falta que sea una diagonalización ortogonal)

$X'=A.X \Rightarrow P\,Y'=P.D.P^{-1}.P.Y \Rightarrow Y'=D.Y$

$\Rightarrow \begin{cases}y_1'=0\\ y_2'=-y_2\end{cases} \rightarrow \begin{matrix}y_1=c_1\\ y_2=c_2.e^{-t}\end{matrix} \rightarrow X=P.Y=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\cdot\begin{pmatrix}y_1\\ y_2\end{pmatrix}=\begin{pmatrix}2.c_1+c_2e^{-t}\\ c_1+c_2.e^{-t}\end{pmatrix}$

$$\Rightarrow \boxed{\begin{pmatrix}x_1\\ x_2\end{pmatrix}=c_1.\begin{pmatrix}2\\ 1\end{pmatrix}+c_2.e^{-t}.\begin{pmatrix}1\\ 1\end{pmatrix}}$$

III) $A=\begin{pmatrix}2 & 2 & -6\\ 2 & -1 & -3\\ -2 & -1 & 1\end{pmatrix} \longrightarrow P_A(\lambda)=\det\begin{pmatrix}\lambda-2 & -2 & 6\\ -2 & \lambda+1 & 3\\ 2 & 1 & \lambda-1\end{pmatrix}=$

$=(\lambda-2).\det\begin{pmatrix}\lambda+1 & 3\\ 1 & \lambda-1\end{pmatrix}+2.\det\begin{pmatrix}-2 & 6\\ 1 & \lambda-1\end{pmatrix}+2.\det\begin{pmatrix}-2 & 6\\ \lambda+1 & 3\end{pmatrix}=$

$=(\lambda-2)\cdot\left((\lambda+1).(\lambda-1)-3\right)-4\lambda+4-12-12-12\lambda-12=\lambda^3-2.\lambda^2-20\lambda-24=0$

$\lambda_1=6 \Rightarrow$ $\begin{array}{c|cccc} & 1 & -2 & -20 & -24\\ 6 & & 6 & 24 & 24\\ \hline & 1 & 4 & 4 & \boxed{0}\end{array}$ $\Rightarrow P_A(\lambda)=(\lambda-6).(\lambda^2+4\lambda+4)=0 \Rightarrow \begin{matrix}\lambda_2=-2\\ \lambda_3=-2\end{matrix}$

$\lambda=6:\left(\begin{array}{ccc|c}4 & -2 & 6 & 0\\ -2 & 7 & 3 & 0\\ 2 & 1 & 5 & 0\end{array}\right) \rightarrow \left(\begin{array}{ccc|c}4 & -2 & 6 & 0\\ \cancel{0} & \cancel{12} & \cancel{12} & \cancel{0}\\ 0 & -4 & -4 & 0\end{array}\right) \rightarrow \begin{cases}2.a-b+3.c=0 \Rightarrow a=-2c\\ b=-c\end{cases}$

$S_{\lambda=6}=\text{gen}\{(-2 \quad -1 \quad 1)^T\}$

$\lambda=-2:\left(\begin{array}{ccc|c}\cancel{-4} & \cancel{-2} & \cancel{6} & 0\\ \cancel{-2} & \cancel{-1} & \cancel{3} & 0\\ 2 & 1 & -3 & 0\end{array}\right) \Rightarrow b=-2a+3c \Rightarrow S_{\lambda=-2}=\text{gen}\{(1 \quad -2 \quad 0)^T;(0 \quad 3 \quad 1)^T\}$ (A es diagonalizable)

$$P=\begin{pmatrix}2&1&0\\1&-2&3\\-1&0&1\end{pmatrix};\ D=\begin{pmatrix}6&0&0\\0&-2&0\\0&0&-2\end{pmatrix}$$

$$A=P.D.P^{-1} \Rightarrow \underset{\substack{\downarrow\\ X=P.Y}}{X'=A.X} \Rightarrow P.Y'=P.D.\underbrace{P^{-1}P}_{I}.Y \Rightarrow Y'=D.Y$$

$$\begin{cases}Y_1'=6.Y_1\\ Y_2'=-2.Y_2\\ Y_3'=-2.Y_3\end{cases} \Rightarrow \begin{matrix}Y_1=C_1.e^{6.t}\\ Y_2=C_2.e^{-2.t}\\ Y_3=C_3.e^{-2.t}\end{matrix}$$

$$\Rightarrow X=\begin{pmatrix}2&1&0\\1&-2&3\\-1&0&1\end{pmatrix}.\begin{pmatrix}Y_1\\Y_2\\Y_3\end{pmatrix}=\begin{pmatrix}2.C_1.e^{6.t}+C_2.e^{-2.t}\\ C_1e^{6t}-2.C_2.e^{-2t}+3.C_3.e^{-2t}\\ -C_1e^{6t}+C_3.e^{-2t}\end{pmatrix}$$

$$\Rightarrow \boxed{\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix}=C_1.e^{6t}.\begin{pmatrix}2\\1\\-1\end{pmatrix}+C_2.e^{-2t}.\begin{pmatrix}1\\-2\\0\end{pmatrix}+C_3.e^{-2t}.\begin{pmatrix}0\\3\\1\end{pmatrix}}$$

2) - a) $X(0)=(1\ \ -2)^T$, $A=\begin{pmatrix}2&-1\\-1&2\end{pmatrix}$

$$P_A(\lambda)=\det\begin{vmatrix}\lambda-2&1\\1&\lambda-2\end{vmatrix}=(\lambda-2)^2-1=\lambda^2-4\lambda+3=0 \Rightarrow \begin{matrix}\lambda_1=3\\ \lambda_2=1\end{matrix}$$

$$\lambda=3:\begin{pmatrix}1&1&|&0\\ \cancel{1}&\cancel{1}&|&\cancel{0}\end{pmatrix}\rightarrow a=-b\rightarrow S_{\lambda=3}=\text{gen}\{(-1\ \ 1)^T\}$$

$$\lambda=1:\begin{pmatrix}-1&1&|&0\\ \cancel{1}&\cancel{-1}&|&\cancel{0}\end{pmatrix}\rightarrow a=b\rightarrow S_{\lambda=1}=\text{gen}\{(1\ \ 1)^T\}$$

$$P=\begin{pmatrix}1&-1\\1&1\end{pmatrix},\ D=\begin{pmatrix}1&0\\0&3\end{pmatrix} \Rightarrow \underset{\substack{\downarrow\\ X=P.y\\ A=P.D.P^{-1}}}{X'=A.X} \Rightarrow Y'=D.Y \Rightarrow \begin{pmatrix}Y_1'\\Y_2'\end{pmatrix}=\begin{pmatrix}Y_1\\3Y_2\end{pmatrix}$$

$$\Rightarrow y_1=C_1.e^{t};\ y_2=C_2.e^{3t} \Rightarrow X=P.y=\begin{pmatrix}1&-1\\1&1\end{pmatrix}.\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}C_1.e^{t}-C_2.e^{3t}\\ C_1e^{t}+C_2.e^{3t}\end{pmatrix}$$

$$\overset{y}{X}=C_1.e^{t}.\begin{pmatrix}1\\1\end{pmatrix}+C_2.e^{3t}.\begin{pmatrix}-1\\1\end{pmatrix};\ X(0)=C_1.\begin{pmatrix}1\\1\end{pmatrix}+C_2.\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}1\\-2\end{pmatrix}\Rightarrow \begin{matrix}C_1-C_2=1\\ C_1+C_2=-2\end{matrix}$$

$$\Rightarrow \begin{matrix}C_1=-\frac{1}{2}\\ C_2=-\frac{3}{2}\end{matrix} \Rightarrow \boxed{X=\left(-\frac{1}{2}\right).e^{t}.\begin{pmatrix}1\\1\end{pmatrix}+\left(-\frac{3}{2}\right).e^{3t}.\begin{pmatrix}-1\\1\end{pmatrix}}$$

b) $X(0) = (1 \quad -2 \quad 1)^T$, A del ej. 1)-III)

$$X = c_1 . e^{6t} . \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + c_2 . e^{-2t} . \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + c_3 . e^{-2t} . \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \Rightarrow X(0) = \begin{pmatrix} 2c_1 + c_2 \\ c_1 - 2c_2 + 3c_3 \\ -c_1 + c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 1 & -2 & 3 & -2 \\ -1 & 0 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 0 & -5 & 6 & -5 \\ 0 & 1 & 2 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 0 & -5 & 6 & -5 \\ 0 & 0 & 16 & 10 \end{array}\right) \rightarrow \begin{cases} 2.c_1 + c_2 = 1 \Rightarrow c_1 = -3/8 \\ -5.c_2 + 6.c_3 = -5 \Rightarrow c_2 = 7/4 \\ c_3 = 5/8 \end{cases}$$

$$\boxed{X = (-3/8) . e^{6t} . \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + (7/4) . e^{-2t} . \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + (5/8) . e^{-2t} . \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}}$$

3) - I)

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow P_A(\lambda) = \det\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} = \lambda^2 + 1 = 0 \Rightarrow \begin{matrix} \lambda_1 = i \\ \lambda_2 = -i \end{matrix}$$

$$\lambda = i : \left(\begin{array}{cc|c} i & -1 & 0 \\ \cancel{1} & \cancel{i} & \cancel{0} \end{array}\right) \rightarrow b = i.a \rightarrow S_{\lambda=i} = gen\{(1 \quad i)^T\}$$

$$\lambda = -i : \left(\begin{array}{cc|c} \cancel{-i} & \cancel{-1} & \cancel{0} \\ 1 & -i & 0 \end{array}\right) \rightarrow a = i.b \Rightarrow S_{\lambda=-i} = gen\{(i \quad 1)^T\}$$

$$P = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} ; D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \Rightarrow \underset{\substack{A = P.D.P^{-1} \\ X = P.Y}}{X' = A.X \Rightarrow Y' = D.Y} \Rightarrow \begin{cases} Y_1' = i.Y_1 \\ Y_2' = -i.Y_2 \end{cases}$$

$$\begin{matrix} Y_1 = c_1 . e^{it} \\ Y_2 = c_2 . e^{-i.t} \end{matrix} \Rightarrow X = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} . \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} c_1 . e^{it} + i.c_2 . e^{-it} \\ c_2 . e^{-it} + i.c_1 . e^{i.t} \end{pmatrix} =$$

$$= \begin{pmatrix} c_1 . (\cos(t) + i.\operatorname{sen}(t)) + i.c_2 . (\overbrace{\cos(-t)}^{\cos(t)} + i.\overbrace{\operatorname{sen}(-t)}^{-\operatorname{sen}(t)}) \\ i.c_1 . (\cos(t) + i.\operatorname{sen}(t)) + c_2 . (\underbrace{\cos(-t)}_{\cos(t)} + i.\underbrace{\operatorname{sen}(-t)}_{-\operatorname{sen}(t)}) \end{pmatrix} =$$

$$= c_1 . \begin{pmatrix} \cos(t) + i.\operatorname{sen}(t) \\ -\operatorname{sen}(t) + i.\cos(t) \end{pmatrix} + c_2 . \begin{pmatrix} \operatorname{sen}(t) + i.\cos(t) \\ \cos(t) - i.\operatorname{sen}(t) \end{pmatrix}$$

→ expresan 2 combinaciones lineales entre parte real e imaginaria; son equivalentes así que expreso X como una de ellas.

$$\boxed{X = \alpha . \begin{pmatrix} \cos(t) \\ -\operatorname{sen}(t) \end{pmatrix} + \beta . \begin{pmatrix} \operatorname{sen}(t) \\ \cos(t) \end{pmatrix}}$$

II)

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -2 & 2 \end{pmatrix} \Rightarrow p_A(\lambda) = \det \begin{pmatrix} \lambda-2 & -1 & 1 \\ 0 & \lambda & -1 \\ 0 & 2 & \lambda-2 \end{pmatrix} = (\lambda-2) . \det \begin{pmatrix} \lambda & -1 \\ 2 & \lambda-2 \end{pmatrix} =$$

$$= (\lambda-2) . (\lambda . (\lambda-2) + 2) = (\lambda-2) . (\underset{\substack{\downarrow \\ \lambda \in \mathbb{C}}}{\lambda^2 - 2\lambda + 2}) = 0 \Rightarrow \lambda_1 = 2$$

$$W^2 = b^2 - 4 . a . c = -4$$

$$W = \pm 2i \qquad \Rightarrow \lambda_2 = \frac{-b + W_1}{2 . a} = 1 + i$$

$$\lambda_3 = 1 - i$$

$$\lambda = 2 : \left(\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right) \rightarrow \begin{matrix} b = 0 \\ c = 0 \\ a \in \mathbb{R} \end{matrix} \Rightarrow S_{\lambda=2} = gen\{(1 \quad 0 \quad 0)^T\}$$

$$\lambda = 1 + i : \left(\begin{array}{ccc|c} i-1 & -1 & 1 & 0 \\ 0 & i+1 & -1 & 0 \\ 0 & 2 & i-1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} i-1 & -1 & 1 & 0 \\ 0 & i+1 & -1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \rightarrow \begin{cases} (i-1) . a - b + (i+1) . b = 0 \\ c = (i+1) . b \end{cases}$$

$$\Rightarrow a = \frac{-1 + i}{2} . b$$

$$S_{\lambda=1+i} = gen\{(\tfrac{-1+i}{2} \quad 1 \quad i+1)^T\}$$

$$\lambda = 1 - i : \left(\begin{array}{ccc|c} -i-1 & -1 & 1 & 0 \\ 0 & 1-i & -1 & 0 \\ \cancel{0} & \cancel{2} & \cancel{i-1} & 0 \end{array}\right) \rightarrow \begin{cases} -(i+1) . a - b + c = 0 \\ c = (1-i) . b \end{cases} \Rightarrow a = -\frac{(1+i)}{2} . b$$

$$S_{\lambda=1-i} = gen\{(\tfrac{-1-i}{2} \quad 1 \quad 1-i)^T\}$$

$$P = \begin{pmatrix} 1 & \frac{-1+i}{2} & \frac{-1-i}{2} \\ 0 & 1 & 1 \\ 0 & i+1 & 1-i \end{pmatrix} ; D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{pmatrix}$$

$$X' = A . X \Rightarrow Y' = DY = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{pmatrix} . \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \Rightarrow \begin{cases} Y_1' = 2 . Y_1 \\ Y_2' = (1+i) . Y_2 \\ Y_3' = (1-i) . Y_3 \end{cases}$$

$$\begin{matrix} \downarrow \\ A = P . D . P^{-1} \\ X = P Y \end{matrix}$$

$$\Rightarrow \begin{matrix} Y_1 = C_1 . e^{2t} \\ Y_2 = C_2 . e^{(1+i) . t} = C_2 . e^t . (\cos(t) + i . sen(t)) \\ Y_3 = C_3 . e^{(1-i) . t} = C_3 . e^t . (\cos(t) - i . sen(t)) \end{matrix}$$

$$X = \begin{pmatrix} 1 & \frac{-1+i}{2} & \frac{-1-i}{2} \\ 0 & 1 & 1 \\ 0 & i+1 & 1-i \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} C_1 . e^{2t} + \left(\frac{-1+i}{2}\right) . C_2 . e^{t} . (\cos(t) + i . \operatorname{sen}(t)) & \boxed{} \\ C_2 . e^{t} . (\cos(t) + i . \operatorname{sen}(t)) + C_3 . e^{t} . (\cos(t) - i . \operatorname{sen}(t)) \\ (i+1) . C_2 . e^{t} . (\cos(t) + i . \operatorname{sen}(t)) + (1-i) . C_3 . e^{t} . (\cos(t) - i . \operatorname{sen}(t)) \end{pmatrix} =$$

$$= C_1 . \begin{pmatrix} e^{2t} \\ 0 \\ 0 \end{pmatrix} + C_2 . \begin{pmatrix} -\frac{1}{2} . e^{t} . (\cos(t) + \operatorname{sen}(t)) + i . e^{t} . \frac{1}{2} . (\cos(t) - \operatorname{sen}(t)) \\ e^{t} . \cos(t) + i . e^{t} . \operatorname{sen}(t) \\ e^{t} . (\cos(t) - \operatorname{sen}(t)) + i . e^{t} . (\operatorname{sen}(t) + \cos(t)) \end{pmatrix} +$$

$$+ C_3 . \begin{pmatrix} 0 \\ e^{t} . \cos(t) - i . e^{t} . \operatorname{sen}(t) \\ e^{t} . (\cos(t) - \operatorname{sen}(t)) - i . e^{t} . (\cos(t) + \operatorname{sen}(t)) \end{pmatrix}$$

$$\Rightarrow \boxed{X = \alpha . \begin{pmatrix} e^{2t} \\ 0 \\ 0 \end{pmatrix} + \beta . \begin{pmatrix} -\frac{1}{2} . e^{t} . (\cos(t) + \operatorname{sen}(t)) \\ e^{t} . \cos(t) \\ e^{t} . (\cos(t) - \operatorname{sen}(t)) \end{pmatrix} + \gamma . \begin{pmatrix} \frac{1}{2} . e^{t} . (\cos(t) - \operatorname{sen}(t)) \\ e^{t} . \operatorname{sen}(t) \\ e^{t} . (\operatorname{sen}(t) + \cos(t)) \end{pmatrix}}$$

4) $X(0) = (1 \quad 0)^T$; $A = \begin{pmatrix} 0 & 5 \\ -1 & 4 \end{pmatrix} \rightarrow P_A(\lambda) = \begin{vmatrix} \lambda & -5 \\ 1 & \lambda - 4 \end{vmatrix} = \lambda^2 - 4\lambda + 5 = 0$

$\lambda_1 = 2 + i$

$\lambda_2 = 2 - i$

$\Rightarrow \lambda = 2 + i : \left(\begin{array}{cc|c} 2+i & -5 & 0 \\ 1 & -2+i & 0 \end{array}\right) \rightarrow a = (2-i) . b$

$w = \pm 2i$

$S_{\lambda = 2+i} = \text{gen}\{(2-i \quad 1)^T\}$

$\Rightarrow S_{\lambda = 2-i} = \overline{(S_{\lambda = 2+i})} = \text{gen}\{(2+i \quad 1)^T\}$

$A = P . D . P^{-1}$ con $P = \begin{pmatrix} 2-i & 2+i \\ 1 & 1 \end{pmatrix}$; $D = \begin{pmatrix} 2+i & 0 \\ 0 & 2-i \end{pmatrix}$

$$\underset{\substack{\downarrow \\ X = P.Y}}{X' = A . X} \Rightarrow P . Y' = P . D . \underbrace{P^{-1} . P}_{I} . Y \Rightarrow Y' = D . Y \Rightarrow \begin{cases} Y_1' = (2+i) . Y_1 \\ Y_2' = (2-i) . Y_2 \end{cases}$$

$Y_1 = C_1 . e^{(2+i).t} = C_1 . e^{2t} . (\cos(t) + i . \operatorname{sen}(t))$

$Y_2 = C_2 . e^{(2-i).t} = C_2 . e^{2t} . (\cos(t) - i . \operatorname{sen}(t))$

$$X = \begin{pmatrix} 2-i & 2+i \\ 1 & 1 \end{pmatrix} . \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} (2-i) . C_1 . e^{2t} . (\cos(t) + i . \operatorname{sen}(t)) + (2+i) . C_2 . e^{2t} . (\cos(t) - i . \operatorname{sen}(t)) \\ C_1 . e^{2t} . (\cos(t) + i . \operatorname{sen}(t)) + C_2 . e^{2t} . (\cos(t) - i . \operatorname{sen}(t)) \end{pmatrix} =$$

$$= C_1 . \begin{pmatrix} e^{2t} . (2 . \cos(t) + \operatorname{sen}(t)) + i . e^{2t} . (2 . \operatorname{sen}(t) - \cos(t)) \\ e^{2t} . \cos(t) + i . e^{2t} . \operatorname{sen}(t) \end{pmatrix} +$$

$$C_2 . \begin{pmatrix} e^{2t} . (2 . \cos(t) + \operatorname{sen}(t)) + i . e^{2t} . (\cos(t) - 2 . \operatorname{sen}(t)) \\ e^{2t} . \cos(t) - i . e^{2t} . \operatorname{sen}(t) \end{pmatrix}$$

$$X = \alpha \cdot \begin{pmatrix} e^{2t} \cdot (2 \cdot \cos(t) + \operatorname{sen}(t)) \\ e^{2t} \cdot \cos(t) \end{pmatrix} + \beta \cdot \begin{pmatrix} e^{2t} \cdot (2 \cdot \operatorname{sen}(t) - \cos(t)) \\ e^{2t} \operatorname{sen}(t) \end{pmatrix}$$

$$X(0) = \alpha \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \beta \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} 2\alpha - \beta = 1 \Rightarrow \beta = -1 \\ \alpha = 0 \end{matrix}$$

$$\Rightarrow \boxed{X = (-e^{2t}) \cdot \begin{pmatrix} 2 \cdot \operatorname{sen}(t) - \cos(t) \\ \operatorname{sen}(t) \end{pmatrix}}$$

5) - a)

$$\begin{cases} y_1' = 2 \cdot y_1 - y_2 + e^{t} \\ y_2' = -y_1 + 2y_2 - e^{2t} \end{cases} = Y' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + F(t)$$

Diagonalizo la matriz $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = A \Rightarrow P_A(\lambda) = \det \begin{pmatrix} \lambda-2 & 1 \\ 1 & \lambda-2 \end{pmatrix} =$

$$= (\lambda-2)^2 - 1 = \lambda^2 - 4\lambda + 3 = 0 \Rightarrow \begin{matrix} \lambda_1 = 3 \\ \lambda_2 = 1 \end{matrix}$$

$$\lambda = 3: \left(\begin{array}{cc|c} 1 & 1 & 0 \\ \cancel{1} & \cancel{1} & \cancel{0} \end{array}\right) \rightarrow a = -b \rightarrow S_{\lambda=3} = \operatorname{gen}\{(-1 \quad 1)^T\}$$

$$\lambda = 1: \left(\begin{array}{cc|c} -1 & 1 & 0 \\ \cancel{1} & \cancel{-1} & \cancel{0} \end{array}\right) \rightarrow b = a \rightarrow S_{\lambda=1} = \operatorname{gen}\{(1 \quad 1)^T\}$$

$A = P.D\,P^{-1}$ con $P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$; $D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ (normalizo P para que sea más fácil invertirla luego, pues P queda ortogonal)

$$\Rightarrow \underset{\substack{\downarrow \\ \text{cambio de} \\ \text{variable: } Y = P.X}}{Y' = P.D\,P^{-1}.Y + F(t) \Rightarrow} \quad P.X' = P.D.\underbrace{P^{-1}.P}_{I}.X + F(t) \Rightarrow$$

$$\Rightarrow \underbrace{P^{-1}.P}_{I}.X' = \underbrace{P^{-1}.P}_{I}.D.X + \underbrace{P^{-1}.F(t)}_{\tilde{F}(t)} \Rightarrow X' = D.X + \tilde{F}(t)$$

↓
multiplico por P^{-1} a ambos lados

$$\Rightarrow \tilde{F}(t) = P^{-1}.F(t) = P^{T}.F(t) = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} e^{t} \\ -e^{2t} \end{pmatrix} =$$

$$= \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} e^{t} - e^{2t} \\ -e^{t} - e^{2t} \end{pmatrix} \Rightarrow \begin{cases} X_1' = X_1 + 1/\sqrt{2} \cdot (e^{t} - e^{2t}) \\ X_2' = 3X_2 - 1/\sqrt{2} \cdot (e^{t} + e^{2t}) \end{cases}$$

$x'_1=x_1+1/\sqrt{2}.(e^t-e^{2t}) \rightarrow$ la solución del homogéneo es $x_{1H}=c_1.e^t$

Propongo solución particular $x_{1p}=c(t).e^t$ $\searrow x_1'=x_1$

Reemplazo en la ED:

$$c'(t).e^t+\cancel{c(t)}.e^t=\cancel{c(t)}.e^t+1/\sqrt{2}.(e^t-e^{2t})$$

$$\Rightarrow c'(t)=1/\sqrt{2}.(1-e^t) \Rightarrow c(t)=1/\sqrt{2}\int(1-e^t)\,dt=1/\sqrt{2}.\left(\int 1\,dt-\int e^t\,dt\right)=$$

$$=1/\sqrt{2}.(t-e^t)+\underset{\hookrightarrow \text{cte}}{k} \Rightarrow x_1=c_1.e^t+1/\sqrt{2}.(t-e^t).e^t+k.e^t$$

$$=(\tilde{c}_1+1/\sqrt{2}.(t-e^t)).e^t$$ → a c_1+k la dejo expresada como otra cte $\tilde{c}_1$.

$$x_2'=3x_2-1/\sqrt{2}.(e^t+e^{2t}) \Rightarrow x_{2H}=c_2.e^{3t}, \ x_{2p}=c(t).e^{3t}$$

$$\Rightarrow c'(t).e^{3t}+\cancel{c(t).3.e^{3t}}=\cancel{3.c(t).e^{3t}}-1/\sqrt{2}.(e^t+e^{2t})$$

$$\Rightarrow c(t)=-1/\sqrt{2}.\int(e^{-2t}+e^{-t})\,dt=-1/\sqrt{2}.\left(\int e^{-2t}\,dt+\int e^{-t}\,dt\right)=$$

$$=-1/\sqrt{2}.\left(\frac{e^{-2t}}{-2}+\frac{e^{-t}}{-1}\right)+\underset{\hookrightarrow \text{cte}}{H}=1/\sqrt{2}.\left(\frac{e^{-2t}}{2}+e^{-t}\right)+H$$

$$x_2=c_2.e^{3t}+1/\sqrt{2}.\left(\frac{e^{-2t}}{2}+e^{-t}\right).e^{3t}+H.e^{3t}=\left(\tilde{c}_2+1/\sqrt{2}.\left(\frac{e^{-2t}}{2}+e^{-t}\right)\right).e^{3t}$$

(c_2+H queda $\tilde{c}_2$)

$$y=P.X=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}.\begin{pmatrix}x_1\\x_2\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}(\tilde{c}_1+1/\sqrt{2}.(t-e^t)).e^t-\left(\tilde{c}_2+1/\sqrt{2}.\left(\frac{e^{-2t}}{2}+e^{-t}\right)\right).e^{3t}\\(\tilde{c}_1+1/\sqrt{2}.(t-e^t)).e^t+\left(\tilde{c}_2+1/\sqrt{2}.\left(\frac{e^{-2t}}{2}+e^{-t}\right)\right).e^{3t}\end{pmatrix}$$

$$\boxed{\begin{pmatrix}y_1\\y_2\end{pmatrix}=\frac{1}{\sqrt{2}}.\begin{pmatrix}(\tilde{c}_1+1/\sqrt{2}.(t-e^t)).e^t-\left(\tilde{c}_2+1/\sqrt{2}.\left(\frac{1}{2}.e^{-2t}+e^{-t}\right)\right).e^{3t}\\(\tilde{c}_1+1/\sqrt{2}.(t-e^t)).e^t+\left(\tilde{c}_2+1/\sqrt{2}.\left(\frac{1}{2}.e^{-2t}+e^{-t}\right)\right).e^{3t}\end{pmatrix}}$$

b) $y'+A.y=F(x)$; $A=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$; $F(t)=(t \ \ -t)^T$

$\Rightarrow y'=-A.y+F(t) \Rightarrow$ diagonalizo A: $P_A(\lambda)=\det\begin{pmatrix}\lambda&1\\1&\lambda\end{pmatrix}=\lambda^2-1=0$

$\lambda_1=1$

$\lambda_2=-1 \Rightarrow \lambda=1: \left(\begin{array}{cc|c}1&1&0\\\cancel{1}&\cancel{1}&\cancel{0}\end{array}\right) \rightarrow a=-b \Rightarrow S_{\lambda=1}=\text{gen}\{(-1 \ \ 1)^T\}$

$\lambda=-1: \left(\begin{array}{cc|c}-1&1&0\\1&-1&0\end{array}\right) \rightarrow a=b \Rightarrow S_{\lambda=-1}=\text{gen}\{(1 \ \ 1)^T\}$

$P=\frac{1}{\sqrt{2}}.\begin{pmatrix}1&-1\\1&1\end{pmatrix}$; $D=\begin{pmatrix}-1&0\\0&1\end{pmatrix} \Rightarrow A=P.D.P^{-1}$; $y=P.X \Rightarrow P.X'=-P.D.P^{-1}.P.X+F(t) \Rightarrow$

$$\Rightarrow \underbrace{P^{-1}.P}_{I}.X' = -\underbrace{P^{-1}.P}_{I}.D\,\underbrace{P^{-1}.P}_{I}.X + \underbrace{P^{-1}F(t)}_{\tilde{F}(t)} \Rightarrow X' = -D.X + \tilde{F}(t)$$

$P^{-1} = P^T$ (P es ortogonal)

$$\Rightarrow \tilde{F}(t) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} . \frac{1}{\sqrt{2}} . \begin{pmatrix} t \\ -t \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{2}{\sqrt{2}}.t \end{pmatrix}$$

$$\begin{cases} X_1' = X_1 \\ X_2' = -X_2 - \frac{2}{\sqrt{2}}.t \end{cases} \Rightarrow X_1 = C_1.e^{t}$$

$X_{2H} = C_2.e^{-t}$, Propongo $X_{2P} = C(t).e^{-t}$

$$\Rightarrow C'(t).e^{-t} - \cancel{C(t).e^{-t}} = -\cancel{C(t).e^{-t}} - \frac{2}{\sqrt{2}}.t \Rightarrow C'(t) = -\frac{2}{\sqrt{2}}.t.e^{t} \Rightarrow$$

$$\Rightarrow C(t) = -\frac{2}{\sqrt{2}}.\int t.e^{t}\,dt = \left(-\frac{2}{\sqrt{2}}\right).e^{t}.(t-1) + \underset{\hookrightarrow cte}{K}$$

$$\Rightarrow X_2 = C_2.e^{-t} + \left(-\frac{2}{\sqrt{2}}\right).\cancel{e^{t}}.(t-1).\cancel{e^{-t}} + K.e^{-t} = \tilde{C}_2.e^{-t} - \frac{2}{\sqrt{2}}.t + \frac{2}{\sqrt{2}}$$

$$Y = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} . \frac{1}{\sqrt{2}} . \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} C_1.e^{t} - \tilde{C}_2.e^{-t} + \frac{2}{\sqrt{2}}.t - \frac{2}{\sqrt{2}} \\ C_1.e^{t} + \tilde{C}_2.e^{-t} - 2/\sqrt{2}.t + 2/\sqrt{2} \end{pmatrix} = \boxed{\begin{pmatrix} 1/\sqrt{2}.C_1.e^{t} - 1/\sqrt{2}.\tilde{C}_2.e^{-t} + t - 1 \\ 1/\sqrt{2}.C_1.e^{t} + 1/\sqrt{2}.\tilde{C}_2.e^{-t} - t + 1 \end{pmatrix}}$$

6)

$$X = \begin{pmatrix} C_1.e^{-t} + 5.C_2.e^{3t} \\ 2.C_1.e^{-t} + C_2.e^{3t} \end{pmatrix} = C_1.e^{-t}.\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2.e^{3t}.\begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Dada la forma de la solución, los autovalores son $\lambda_1 = -1$ y $\lambda_2 = 3$ y están asociados a los autoespacios

$$S_{\lambda=-1} = gen\{(1 \ \ 2)^T\} \text{ y } S_{\lambda=3} = gen\{(5 \ \ 1)^T\}$$

$\Rightarrow$ A es diagonalizable:

$$A = P.D.P^{-1} \text{ con } P = \begin{pmatrix} 1 & 5 \\ 2 & 1 \end{pmatrix} \text{ y } D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -1/9 & 5/9 \\ 2/9 & -1/9 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 1 & 5 \\ 2 & 1 \end{pmatrix} . \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} . \begin{pmatrix} -1 & 5 \\ 2 & -1 \end{pmatrix} . \frac{1}{9} = \begin{pmatrix} -1 & 15 \\ -2 & 3 \end{pmatrix} . \begin{pmatrix} -1 & 5 \\ 2 & -1 \end{pmatrix} . \frac{1}{9} = \boxed{\frac{1}{9}.\begin{pmatrix} 31 & -20 \\ 8 & -13 \end{pmatrix}}$$

Es una de las A posibles

8) - I) $A=\begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix} \Rightarrow P_A(\lambda)=\det\begin{pmatrix} \lambda-1 & 4 \\ -1 & \lambda+3 \end{pmatrix} = (\lambda-1).(\lambda+3)+4 = \lambda^2+2\lambda+1=0$

$\lambda_1=\lambda_2=-1$

$\lambda=-1: \left(\begin{array}{cc|c} \cancel{-2} & \cancel{4} & \cancel{0} \\ -1 & 2 & 0 \end{array}\right) \rightarrow a=2b \Rightarrow S_{\lambda=-1}=gen\{(2\ \ 1)^T\}$

A no es diagonalizable, pero es posible escribirla como

$A=P.J.P^{-1}$ siendo $J=\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ una matriz de Jordan.

$A=P.J.P^{-1} \Rightarrow A.P=P.J.\underbrace{P^{-1}.P}_{I}$; $P=(v_1\ \ v_2) \Rightarrow A.(v_1\ \ v_2)=(v_1\ \ v_2).J \Rightarrow$

$\Rightarrow \begin{cases} A.v_1=\lambda.v_1 \\ A.v_2=v_1+\lambda.v_2 \end{cases} \Rightarrow v_1$ es el autovector asoc a $\lambda=-1$

$v_1=(2\ \ 1)^T$ (una posibilidad)

$\Rightarrow (A-\overset{-1}{\lambda} I).v_2=v_1 \Rightarrow \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}.v_2=\begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow v_2=\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow P=\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

$\underset{\substack{\downarrow \\ X=P.Y}}{X'=P.J.P^{-1}.X} \Rightarrow P.Y'=P.J.\underbrace{P^{-1}.P}_{I}.Y \Rightarrow \underbrace{P^{-1}.P}_{I}.Y'=\underbrace{P^{-1}.P}_{I}.J.Y \Rightarrow Y'=J.Y$

$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix}=\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \equiv \begin{cases} y_1'=-y_1+y_2 \\ y_2'=-y_2 \end{cases} \Rightarrow y_2=c_2.e^{-t}$

$y_1'=-y_1+c_2.e^{-t} \Rightarrow y_{1H}=c_1.e^{-t}$; Propongo $y_{1p}=c(t).e^{-t}$

Reemplazo:

$c'(t).\cancel{e^{-t}}-\cancel{c(t)}.e^{-t}=-\cancel{c(t)}.e^{-t}+c_2.\cancel{e^{-t}} \Rightarrow c'(t)=c_2 \Rightarrow c(t)=c_2.t+\underset{\hookrightarrow cte}{k}$

$y_{1p}=c_2\,t\,e^{-t}+k\,e^{-t}$

$y_1=c_1.e^{-t}+c_2.t.e^{-t}+k.e^{-t}=(\underset{\substack{\downarrow \\ c_1+k}}{\tilde{c}_1}+c_2\,t).e^{-t}$

$X=\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}=\begin{pmatrix} 2.(\tilde{c}_1+c_2.t).e^{-t}+c_2.e^{-t} \\ (\tilde{c}_1+c_2\,t).e^{-t} \end{pmatrix}=\tilde{c}_1.e^{-t}.\begin{pmatrix} 2 \\ 1 \end{pmatrix}+c_2.e^{-t}.\begin{pmatrix} 2t+1 \\ t \end{pmatrix}$

$=\boxed{(\tilde{c}_1+c_2.t).e^{-t}.\begin{pmatrix} 2 \\ 1 \end{pmatrix}+c_2.e^{-t}.\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

I)
$$A=\begin{pmatrix}0&-1&0\\1&-2&0\\0&0&-1\end{pmatrix} \Rightarrow P_A(\lambda)=\det\begin{vmatrix}\lambda&1&0\\-1&\lambda+2&0\\0&0&\lambda+1\end{vmatrix}=(\lambda+1)..\left(\lambda.(\lambda+2)+1\right)=$$

$$=(\lambda+1).(\lambda^2+2\lambda+1)=0 \Rightarrow \lambda=-1 \text{ (triple)}$$

$$\lambda=-1:\left(\begin{array}{ccc|c}-1&1&0&0\\ \cancel{-1}&\cancel{1}&\cancel{0}&\cancel{0}\\ \cancel{0}&\cancel{0}&\cancel{0}&\cancel{0}\end{array}\right)\longrightarrow \begin{array}{l}a=b\\ c\in R\end{array} \Rightarrow S_{\lambda=-1}=\text{gen}\{(1\ \ 1\ \ 0)^T;(0\ \ 0\ \ 1)^T\}$$

A no es diagonalizable y $\dim(S_{\lambda=-1})=2$

$$A=P.J.P^{-1}\ ,\ J=\begin{pmatrix}\lambda&1&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}=\begin{pmatrix}-1&1&0\\0&-1&0\\0&0&-1\end{pmatrix}$$

$$A.P=P.J.\underbrace{P^{-1}.P}_{I} \Rightarrow A.(v_1\ \ v_2\ \ v_3)=(v_1\ \ v_2\ \ v_3).\begin{pmatrix}-1&1&0\\0&-1&0\\0&0&-1\end{pmatrix}$$

$$\Rightarrow\begin{cases}A.v_1=-v_1\\A.v_2=v_1-v_2\\A.v_3=-v_3\end{cases} \Rightarrow v_1 \text{ y } v_3 \text{ son autove. asoc. a } \lambda=-1$$

$$v_1=(1\ \ 1\ \ 0)^T\ ;\ v_3=(0\ \ 0\ \ 1)^T$$

$$(A+I).v_2=v_1 \Rightarrow \begin{pmatrix}1&-1&0\\1&-1&0\\0&0&0\end{pmatrix}.v_2=\begin{pmatrix}1\\1\\0\end{pmatrix} \Rightarrow v_2=(1\ \ 0\ \ 0)^T$$

$$P=\begin{pmatrix}1&1&0\\1&0&0\\0&0&1\end{pmatrix} \Rightarrow \begin{array}{l}A=P.J.P^{-1}\\X=P.Y\end{array} \Rightarrow X'=A.X \Rightarrow P.Y'=P.J.P^{-1}.P.Y \Rightarrow Y'=J.Y$$

$$\begin{cases}Y_1'=-Y_1+Y_2\\Y_2'=-Y_2\\Y_3'=-Y_3\end{cases} \Rightarrow \begin{array}{l}Y_1'=-Y_1+C_2.e^{-t}\\Y_2=C_2.e^{-t}\\Y_3=C_3.e^{-t}\end{array}$$

$$Y_{1H}=C_1.e^{-t}\ ,\ Y_{1P}=C(t).e^{-t} \Rightarrow C'(t).\cancel{e^{-t}}-\cancel{C(t).e^{-t}}=-\cancel{C(t).e^{-t}}+C_2.\cancel{e^{-t}}$$

$$\Rightarrow C(t)=C_2.t \Rightarrow Y_1=C_1.e^{-t}+C_2.t.e^{-t}$$

→ la solución final queda de la misma sin plantear otra cte k.

$$X=P.Y=\begin{pmatrix}1&1&0\\1&0&0\\0&0&1\end{pmatrix}.\begin{pmatrix}Y_1\\Y_2\\Y_3\end{pmatrix}=\begin{pmatrix}C_1.e^{-t}+C_2.t.e^{-t}+C_2.e^{-t}\\C_1.e^{-t}+C_2.t.e^{-t}\\C_3.e^{-t}\end{pmatrix}=$$

$$=\boxed{C_1.e^{-t}.\begin{pmatrix}1\\1\\0\end{pmatrix}+C_2.e^{-t}.\begin{pmatrix}t+1\\t\\0\end{pmatrix}+C_3.e^{t}.\begin{pmatrix}0\\0\\1\end{pmatrix}}$$

Práctico 6

1) $P=\begin{pmatrix} \frac{\sqrt{6}}{3} & -\frac{\sqrt{3}}{3} & a \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} & b \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} & c \end{pmatrix}$, $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ debe ser ortogonal a las otras 2 columnas y de norma uno

$$(a\ b\ c).\begin{pmatrix} \frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix}.\underbrace{\sqrt{6}}_{\neq 0}=0 \Rightarrow a.\frac{1}{3}+b.\frac{1}{6}+c.\frac{1}{6}=0$$

$\Rightarrow$ sumo ecuaciones:

$$(a\ b\ c).\begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}.\underbrace{\sqrt{3}}_{\neq 0}=0 \Rightarrow -a.\frac{1}{3}+b.\frac{1}{3}+c.\frac{1}{3}=0 \qquad b.\frac{1}{2}+c.\frac{1}{2}=0 \Rightarrow b=-c$$

$\Rightarrow a.\frac{1}{3}=0 \Rightarrow a=0$

Además, $\sqrt{a^2+b^2+c^2}=1 \Rightarrow a^2+b^2+c^2=1 \Rightarrow 2.c^2=1 \Rightarrow c=\pm\frac{1}{\sqrt{2}}$

$\boxed{a=0,\ b=-\frac{1}{\sqrt{2}},\ c=\frac{1}{\sqrt{2}}}$ ó $\boxed{a=0,\ b=\frac{1}{\sqrt{2}},\ c=-\frac{1}{\sqrt{2}}}$ (También se puede hacer pensando en que $P^{-1}=P^t$)

$$U=\begin{pmatrix} \frac{1+i}{2} & a \\ \frac{1-i}{2} & b \end{pmatrix} \Rightarrow \frac{1}{2}.(1+i\ \ 1-i).\begin{pmatrix} a \\ b \end{pmatrix}=0 \Rightarrow a.(1+i)+b.(1-i)=0 \Rightarrow \begin{matrix} a=1+i \\ b=1-i \end{matrix}$$

Esos valores cumplen la ec. pero el vector debe tener norma 1.

$$\|\underbrace{(1+i\ \ 1-i)^t}_{v}\|=\sqrt{\underset{\rightarrow \text{canónico } \mathbb{C}^2}{(v,v)}}=\sqrt{v^H.v}=\sqrt{(1-i\ \ 1+i).\begin{pmatrix} 1+i \\ 1-i \end{pmatrix}}=\sqrt{4}=2 \Rightarrow \boxed{\begin{matrix} a=\frac{1+i}{2} \\ b=\frac{1-i}{2} \end{matrix}}$$

2)
- I) es ortogonal si $k=\pm1$
- II) " "
- III) sí, si $k=\pm1$
- IV) sí ($\forall\alpha$)
- V) sí
- VI) no (las columnas no forman una BON)

Para el 6:
- I) no
- II) sí
- III) sí
- IV) sí $\}\ \forall\alpha$
- V) sí $\}\ \forall\alpha$

3) $$H=I-\frac{2}{u^t.u}\cdot u.u^t \Rightarrow H^t=I^t-\left(\frac{2}{u^t.u}.u.u^t\right)^t=I-\frac{2.}{\underset{\overset{\|}{(u^t.u)^t}}{\underbrace{u^t.u}}}\overset{\overset{(u.u^t)^t}{\|}}{\overbrace{u.u^t}}=H$$

$$H.H=I-2.\frac{u.u^t}{u^t.w}-2.\frac{u.u^t}{u^t.u}+4\frac{(u.u^t)^2}{(u^t.u)^2}=$$

$$=I-4.\frac{u.u^t}{u^t.u}+4.\frac{u\,\cancel{u^t.u}\,u^t}{(u^t.u)^2}=I \Rightarrow H=H^{-1}=H^t \therefore H \text{ es ortogonal.}$$

4) - a) $\underset{\substack{\downarrow \\ U \text{ es unitaria}}}{U^{-1}=U^{H}} \Rightarrow (U^{H})^{-1} = (U^{-1})^{-1} = U = (U^{H})^{H}$ ✓ ∴ U^{H} es unitaria

$(U^{T})^{-1} = (U^{-1})^{T} = (U^{H})^{T} = (U^{T})^{H}$ ✓ ∴ U^{T} es unitaria

b) $U.V.(U.V)^{H} = U.\underbrace{V.V^{H}}_{\substack{I \\ (V \text{ es unit.})}}.U^{H} = \underset{\substack{\downarrow \\ (U \text{ es unit.})}}{U.U^{H} = I}$ ✓ ∴ U.V es unitaria

9) $S = \{X \in \mathbb{R}^3 / X_1 - X_2 + X_3 = 0\}$; $\|T(x)\| = \|X\|$

Dada la condición de la norma, $T(X) = U.X$ con U unitaria. Entonces T debe ser unitaria y transformar BON en BON.

$S = gen\{(1 \;\; 1 \;\; 0)^{t} ; (0 \;\; 1 \;\; 1)^{t}\} \rightarrow$ S es invariante por T $\Rightarrow T(S) = a.\begin{pmatrix}1\\1\\0\end{pmatrix} + b.\begin{pmatrix}0\\1\\1\end{pmatrix}$

$a, b \in \mathbb{R}$

$S^{\perp} = gen\{(1 \;\; -1 \;\; 1)^{t}\}$

uso Gram - Schmidt para obtener una base ortogonal de S

$S_1 = gen\{(1 \;\; 1 \;\; 0)^{t}\}$, $u_1 = (1 \;\; 1 \;\; 0)^{t} = v_1$; $v_2 = (0 \;\; 1 \;\; 1)^{t}$

$u_2 = v_2 - P_{S_1}(v_2) = (0 \;\; 1 \;\; 1)^{t} - \dfrac{(v_1, v_2)}{(v_1, v_1)}.v_1 = (0 \;\; 1 \;\; 1)^{t} - \dfrac{1}{2}.(1 \;\; 1 \;\; 0)^{t} =$

$= \begin{pmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 1\end{pmatrix} \Rightarrow$ la BON de salida es $B_1 = \{(\frac{1}{\sqrt{2}} \;\; \frac{1}{\sqrt{2}} \;\; 0)^{t}; (-\frac{\sqrt{2}}{2.\sqrt{3}} \;\; \frac{\sqrt{2}}{2.\sqrt{3}} \;\; \sqrt{\frac{2}{3}})^{t}; (\frac{1}{\sqrt{3}} \;\; -\frac{1}{\sqrt{3}} \;\; \frac{1}{\sqrt{3}})^{t}\}$

$$\Rightarrow \boxed{T\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix} = \begin{pmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix} ; \; T\begin{pmatrix}-\frac{1}{2}.\sqrt{\frac{2}{3}}\\ \frac{1}{2}.\sqrt{\frac{2}{3}}\\ \sqrt{\frac{2}{3}}\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix} ; \; T\begin{pmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix} = \begin{pmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}}$$

11)
$$P=\begin{pmatrix}\frac{\sqrt{6}}{3} & -\frac{\sqrt{3}}{3} & 0\\ \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} & -1/\sqrt{2}\\ \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} & 1/\sqrt{2}\end{pmatrix} \Rightarrow P_p(\lambda)=\det(\lambda.I-P)=\det\begin{vmatrix}\lambda-\sqrt{6}/3 & \sqrt{3}/3 & 0\\ -\sqrt{6}/6 & \lambda-\sqrt{3}/3 & 1/\sqrt{2}\\ -\sqrt{6}/6 & -\sqrt{3}/3 & \lambda-1/\sqrt{2}\end{vmatrix}=$$

$$=\left(\frac{1}{\sqrt{2}}\right).(-1)^5.\det\begin{vmatrix}\lambda-\sqrt{6}/3 & \sqrt{3}/3\\ -\sqrt{6}/6 & -\sqrt{3}/3\end{vmatrix}+(\lambda-1/\sqrt{2}).(-1)^6\det\begin{vmatrix}\lambda-\sqrt{6}/3 & \sqrt{3}/3\\ -\sqrt{6}/6 & \lambda-\sqrt{3}/3\end{vmatrix}=$$

$$=-\frac{1}{\sqrt{2}}\cdot\left((\lambda-\sqrt{6}/3).(-\sqrt{3}/3)+\frac{\sqrt{3}}{3}\cdot\frac{\sqrt{6}}{6}\right)+(\lambda-1/\sqrt{2}).\left((\lambda-\sqrt{6}/3).(\lambda-\sqrt{3}/3)+\frac{\sqrt{3}}{3}\cdot\frac{\sqrt{6}}{6}\right)=$$

$$=\left(-\frac{1}{\sqrt{2}}\right).\left(-\frac{\sqrt{3}}{3}\cdot\lambda+\underbrace{\frac{\sqrt{18}}{9}+\frac{\sqrt{18}}{18}}_{\frac{1}{\sqrt{2}}}\right)+(\lambda-1/\sqrt{2}).\left(\lambda^2-\frac{\sqrt{3}}{3}\lambda-\frac{\sqrt{6}}{3}\cdot\lambda+\underbrace{\frac{\sqrt{18}}{9}+\frac{\sqrt{18}}{18}}_{\frac{1}{\sqrt{2}}}\right)=$$

$$=\frac{1}{3}\cdot\sqrt{\frac{3}{2}}\cdot\lambda-\frac{1}{2}+\lambda^3+\frac{1}{\sqrt{2}}\cdot\left(\frac{\sqrt{3}+\sqrt{6}}{3}\right).\lambda-\frac{1}{\sqrt{2}}\cdot\lambda^2-\left(\frac{\sqrt{3}+\sqrt{6}}{3}\right).\lambda^2+\frac{1}{\sqrt{2}}\cdot\lambda-\frac{1}{2}=$$

$$=\lambda^3+\left(-\frac{\sqrt{3}}{3}-\frac{\sqrt{6}}{3}-\frac{1}{\sqrt{2}}\right).\lambda^2+\left(\frac{2\sqrt{3}+\sqrt{6}}{\sqrt{2}.3}+\frac{1}{\sqrt{2}}\right).\lambda-1=0$$

$$\boxed{\begin{array}{l}\lambda_1=1\ ;\ \lambda_2=\left(\frac{\sqrt{3}+\sqrt{6}}{6}+\frac{\sqrt{2}}{4}-\frac{1}{2}\right)-i.\left(\frac{\sqrt{2}}{12}+\frac{\sqrt{6}}{12}+\frac{3}{8}\right)\\ \qquad\lambda_3=\overline{\lambda_2}\qquad \text{(el módulo de cada uno es 1)}\end{array}}$$

El P.i. debe dar 0 para cualquier par de autovectores.

13) - 1)
$$A=\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix} \Rightarrow P_A(\lambda)=\det(\lambda\ I-A)=\det\begin{pmatrix}\lambda-3 & -1\\ -1 & \lambda-3\end{pmatrix}=$$

$$=(\lambda-3)^2-1=0=\lambda^2-6\lambda+8\Rightarrow \begin{array}{l}\lambda_1=4\\ \lambda_2=2\end{array}$$

$$\lambda=4:\ (4.I-A).v=0\Rightarrow\left(\begin{array}{cc|c}1 & -1 & 0\\ \cancel{-1} & \cancel{1} & \cancel{0}\end{array}\right)\rightarrow x_1=x_2\Rightarrow S_{\lambda=4}=\text{gen}\{(1\ \ 1)^t\}$$

$$\lambda=2:\ (2.I-A).v=0\Rightarrow\left(\begin{array}{cc|c}-1 & -1 & 0\\ \cancel{-1} & \cancel{-1} & \cancel{0}\end{array}\right)\rightarrow x_1=-x_2\Rightarrow S_{\lambda=2}=\text{gen}\{(-1\ \ 1)^t\}$$

Los autoespacios son ortogonales entre sí, sólo falta normalizar

$$P=\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix},\ \text{como P es ortogonal}\ P^{-1}=P^t\Rightarrow A=P.\Lambda.P^t=\boxed{\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}.\begin{pmatrix}4 & 0\\ 0 & 2\end{pmatrix}.\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}}$$

II) $A=\begin{pmatrix}1&1&3\\1&3&1\\3&1&1\end{pmatrix} \Rightarrow P_A(\lambda)=\det(\lambda I-A)=\det\begin{pmatrix}\lambda-1&-1&-3\\-1&\lambda-3&-1\\-3&-1&\lambda-1\end{pmatrix}=$

$=(\lambda-1)\cdot\begin{vmatrix}\lambda-3&-1\\-1&\lambda-1\end{vmatrix}+\begin{vmatrix}-1&-1\\-3&\lambda-1\end{vmatrix}+(-3)\cdot\begin{vmatrix}-1&\lambda-3\\-3&-1\end{vmatrix}=(\lambda-1)\cdot\left((\lambda-3)\cdot(\lambda-1)-1\right)-\lambda+1-3+(-3)\left(1+3\cdot(\lambda-3)\right)=$

$=(\lambda-1)\cdot\left((\lambda-3)\cdot(\lambda-1)-1\right)-\lambda-2-9\lambda+24=\lambda^3-5\lambda^2-4\lambda+20=0$

$\lambda=2$ es autoval. $\Leftrightarrow$ $\begin{array}{r|rrrr} & 1 & -5 & -4 & 20\\ 2 & & 2 & -6 & -20\\ \hline & 1 & -3 & -10 & \boxed{0}\end{array}$ $\Rightarrow P_A(\lambda)=(\lambda-2)\cdot(\lambda^2-3\lambda-10)=0$

$\swarrow$ $\lambda=5$ $\searrow$ $\lambda=-2$

$\lambda=2$:

$\left(\begin{array}{ccc|c}1&-1&-3&0\\-1&-1&-1&0\\-3&-1&1&0\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}1&-1&-3&0\\0&-2&-4&0\\\cancel{0}&\cancel{-4}&\cancel{-8}&\cancel{0}\end{array}\right)\rightarrow\begin{cases}X_1-X_2-3X_3=0\\-2X_2-4X_3=0\end{cases}\rightarrow\begin{array}{l}X_1+2X_3-3X_3=0\Rightarrow X_1=X_3\\X_2=-2X_3\end{array}$

$S_{\lambda=2}=\text{gen}\{(1\quad -2\quad 1)^t\}$

$\lambda=5$:

$\left(\begin{array}{ccc|c}4&-1&-3&0\\-1&2&-1&0\\-3&-1&4&0\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}4&-1&-3&0\\0&7&-7&0\\\cancel{0}&\cancel{-7}&\cancel{7}&\cancel{0}\end{array}\right)\rightarrow\begin{cases}4X_1-X_2-3X_3=0\\X_2-X_3=0\end{cases}\rightarrow\begin{array}{l}4X_1-X_3-3X_3=0\\X_2=X_3\end{array}$

$\Rightarrow X_1=X_3$

$S_{\lambda=5}=\text{gen}\{(1\quad 1\quad 1)^t\}$

$\lambda=-2$:

$\left(\begin{array}{ccc|c}-3&-1&-3&0\\-1&-5&-1&0\\\cancel{-3}&\cancel{-1}&\cancel{-3}&\cancel{0}\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}3&1&3&0\\1&5&1&0\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}3&1&3&0\\0&14&0&0\end{array}\right)\rightarrow\begin{array}{l}3X_1+3X_3=0\Rightarrow X_1=-X_3\\X_2=0\end{array}$

$S_{\lambda=-2}=\text{gen}\{(-1\quad 0\quad 1)^t\}$

$P=\begin{pmatrix}1/\sqrt6&1/\sqrt3&-1/\sqrt2\\-2/\sqrt6&1/\sqrt3&0\\1/\sqrt6&1/\sqrt3&1/\sqrt2\end{pmatrix} \Rightarrow \boxed{A=\begin{pmatrix}1/\sqrt6&1/\sqrt3&-1/\sqrt2\\-2/\sqrt6&1/\sqrt3&0\\1/\sqrt6&1/\sqrt3&1/\sqrt2\end{pmatrix}\cdot\begin{pmatrix}2&0&0\\0&5&0\\0&0&-2\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt6&-2/\sqrt6&1/\sqrt6\\1/\sqrt3&1/\sqrt3&1/\sqrt3\\-1/\sqrt2&0&1/\sqrt2\end{pmatrix}}$

III) $A=\begin{pmatrix}-2 & -36 & 0\\ -36 & -23 & 0\\ 0 & 0 & 3\end{pmatrix} \Rightarrow P_A(\lambda)=\det(\lambda . I-A)=\det\begin{pmatrix}\lambda+2 & 36 & 0\\ 36 & \lambda+23 & 0\\ 0 & 0 & \lambda-3\end{pmatrix}=$

$=(\lambda-3) . \det\begin{pmatrix}\lambda+2 & 36\\ 36 & \lambda+23\end{pmatrix}=(\lambda-3) . \left((\lambda+2) . (\lambda+23)-36^2\right)=0$

$\lambda_1=3$; $\lambda^2+23\lambda+2\lambda+46-36^2=\lambda^2+25\lambda-1250=0 \Rightarrow \lambda_2=25$
$\lambda_3=-50$

$\lambda=3$:

$\left(\begin{array}{ccc|c}5 & 36 & 0 & 0\\ 36 & 26 & 0 & 0\\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0}\end{array}\right) \rightarrow \begin{cases}5x_1+36x_2=0\\ 36x_1+26x_2=0\end{cases} \rightarrow$ sistema homogéneo compatible determinado, solución trivial: $x_1=x_2=0$; $x_3\in\mathbb{R}$

$S_{\lambda=3}=\text{gen}\{(0 \quad 0 \quad 1)^{t}\}$

$\lambda=25$:

$\left(\begin{array}{ccc|c}27 & 36 & 0 & 0\\ \cancel{36} & \cancel{48} & \cancel{0} & \cancel{0}\\ 0 & 0 & 22 & 0\end{array}\right) \rightarrow \begin{cases}27x_1+36 . x_2=0 \Rightarrow x_1=-\frac{4}{3}x_2\\ x_3=0\end{cases}$

$S_{\lambda=25}=\text{gen}\{(-\frac{4}{3} \quad 1 \quad 0)^{t}\}$

$\lambda=-50$:

$\left(\begin{array}{ccc|c}\cancel{-48} & \cancel{36} & \cancel{0} & \cancel{0}\\ 36 & -27 & 0 & 0\\ 0 & 0 & -53 & 0\end{array}\right) \rightarrow \begin{cases}36x_1-27x_2=0\\ x_3=0\end{cases} \rightarrow x_1=\frac{3}{4}x_2$

$S_{\lambda=-50}=\text{gen}\{(\frac{3}{4} \quad 1 \quad 0)^{t}\}$

$$\Rightarrow \boxed{A=\begin{pmatrix}0 & -4/5 & 3/5\\ 0 & 3/5 & 4/5\\ 1 & 0 & 0\end{pmatrix} . \begin{pmatrix}3 & 0 & 0\\ 0 & 25 & 0\\ 0 & 0 & -50\end{pmatrix} . \begin{pmatrix}0 & 0 & 1\\ -4/5 & 3/5 & 0\\ 3/5 & 4/5 & 0\end{pmatrix}}$$

IV) $A=\begin{pmatrix}2 & 0 & 1 & 0\\ 0 & 2 & 0 & 1\\ 1 & 0 & 2 & 0\\ 0 & 1 & 0 & 2\end{pmatrix} \Rightarrow P_A(\lambda)=\det\begin{pmatrix}\lambda-2 & 0 & -1 & 0\\ 0 & \lambda-2 & 0 & -1\\ -1 & 0 & \lambda-2 & 0\\ 0 & -1 & 0 & \lambda-2\end{pmatrix}=$

$=(-1) . \det\begin{pmatrix}0 & -1 & 0\\ \lambda-2 & 0 & -1\\ -1 & 0 & \lambda-2\end{pmatrix}+(\lambda-2) . \det\begin{pmatrix}\lambda-2 & 0 & -1\\ 0 & \lambda-2 & 0\\ -1 & 0 & \lambda-2\end{pmatrix}=(-1) . \det\begin{pmatrix}\lambda-2 & -1\\ -1 & \lambda-2\end{pmatrix}+$

$+(\lambda-2) . \left[(-1) . \det\begin{pmatrix}0 & -1\\ \lambda-2 & 0\end{pmatrix}+(\lambda-2) . \det\begin{pmatrix}\lambda-2 & 0\\ 0 & \lambda-2\end{pmatrix}\right]=$

$= (-1).\left((\lambda-2)^2-1\right) + (\lambda-2).\left[-\lambda+2 \;\; + (\lambda-2)^3\right] = (-1).(\lambda-2)^2 + 1 + (\lambda-2)^2.(-1) + (\lambda-2)^4 =$

$= (\lambda-2)^2.\left(-2 + (\lambda-2)^2\right) + 1 = 0 \Rightarrow \lambda_1 = 1$ (doble)
$\lambda_2 = 3$ (doble)

$\lambda = 1: \left(\begin{array}{cccc|c} -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ \cancel{-1} & \cancel{0} & \cancel{-1} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{-1} & \cancel{0} & \cancel{-1} & \cancel{0} \end{array}\right) \rightarrow \begin{cases} X_1 = -X_3 \\ X_2 = -X_4 \end{cases} \rightarrow S_{\lambda=1} = \text{gen}\{(-1 \;\; 0 \;\; 1 \;\; 0)^t; (0 \;\; -1 \;\; 0 \;\; 1)^t\}$

$\lambda = 3: \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ \cancel{-1} & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{-1} & \cancel{0} & \cancel{1} & \cancel{0} \end{array}\right) \rightarrow \begin{cases} X_1 = X_3 \\ X_2 = X_4 \end{cases} \rightarrow S_{\lambda=3} = \text{gen}\{(1 \;\; 0 \;\; 1 \;\; 0)^t; (0 \;\; 1 \;\; 0 \;\; 1)^t\}$

$$\boxed{A = \begin{pmatrix} -1/\sqrt{2} & 0 & 2/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} . \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}}$$

14) - I)

$A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \Rightarrow P_A(\lambda) = \det(\lambda.I - A) = \det\begin{pmatrix} \lambda & -i \\ i & \lambda \end{pmatrix} = \lambda^2 + i^2 = \lambda^2 - 1 = 0$

$\lambda_1 = 1 \;,\; \lambda_2 = -1$

$\lambda = 1:$

$\left(\begin{array}{cc|c} \cancel{1} & \cancel{-i} & \cancel{0} \\ i & 1 & 0 \end{array}\right) \rightarrow i.X_1 + X_2 = 0 \Rightarrow X_2 = -i.X_1 \Rightarrow S_{\lambda=1} = \text{gen}\{(1 \;\; -i)^t\}$

$\lambda = -1:$

$\left(\begin{array}{cc|c} \cancel{-1} & \cancel{-i} & \cancel{0} \\ i & -1 & 0 \end{array}\right) \rightarrow X_2 = i.X_1 \Rightarrow S_{\lambda=-1} = \text{gen}\{(1 \;\; i)^t\}$

$\left((1 \;\; -i)^t, (1 \;\; i)^t\right) = (1 \;\; i).\begin{pmatrix} 1 \\ i \end{pmatrix} = 0$ ✓ (son ortogonals)

$\|(1 \;\; i)^t\| = \sqrt{(1 \;\; -i).\begin{pmatrix} 1 \\ i \end{pmatrix}} = \sqrt{2} \;;\; \|(1 \;\; -i)^t\| = \sqrt{(1 \;\; i).\begin{pmatrix} 1 \\ -i \end{pmatrix}} = \sqrt{2}$

$\Rightarrow U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & i/\sqrt{2} \end{pmatrix} \Rightarrow \boxed{A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & i/\sqrt{2} \end{pmatrix} . \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}}$

II)

$A = \begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow P_A(\lambda) = \det\begin{vmatrix} \lambda-1 & -i & 0 \\ i & \lambda-1 & 0 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1).\det\begin{pmatrix} \lambda-1 & -i \\ i & \lambda-1 \end{pmatrix} =$

$= (\lambda-1).\left((\lambda-1)^2-1\right) = 0 \Rightarrow \lambda_1 = 1 \;;\; \lambda^2 - 2\lambda = 0 \Rightarrow \lambda_2 = 0 \;,\; \lambda_3 = 2$

NOTA

$\lambda = 1$;

$$\left(\begin{array}{ccc|c} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \rightarrow \begin{array}{l} X_2 = 0 \\ X_1 = 0 \\ X_3 \in \mathbb{C} \end{array} \Rightarrow S_{\lambda=1} = gen\{(0 \quad 0 \quad 1)^t\}$$

$\lambda = 0$:

$$\left(\begin{array}{ccc|c} \cancel{1} & \cancel{i} & \cancel{0} & \cancel{0} \\ i & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) \rightarrow \begin{array}{l} X_3 = 0 \\ X_2 = i X_1 \end{array} \Rightarrow S_{\lambda=0} = gen\{(1 \quad i \quad 0)^t\}$$

$\lambda = 2$:

$$\left(\begin{array}{ccc|c} 1 & -i & 0 & 0 \\ \cancel{i} & \cancel{1} & \cancel{0} & \cancel{0} \\ 0 & 0 & 1 & 0 \end{array}\right) \rightarrow \begin{array}{l} X_1 = i . X_2 \\ X_3 = 0 \end{array} \rightarrow S_{\lambda=2} = gen\{(i \quad 1 \quad 0)^t\}$$

$$\|(1 \quad i \quad 0)^t\| = \|(i \quad 1 \quad 0)^t\| = \sqrt{2}$$

$$\boxed{A = \begin{pmatrix} 0 & 1/\sqrt{2} & i/\sqrt{2} \\ 0 & i/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} . \begin{pmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ -i/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}}$$

15) – I)

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow P_A(\lambda) = \det \begin{pmatrix} \lambda-1 & 1 \\ -1 & \lambda-1 \end{pmatrix} = (\lambda-1)^2 + 1 = 0 \Rightarrow$$

$$\Rightarrow \lambda_1 = i+1 \; ; \; \lambda_2 = -i+1 \Rightarrow \lambda = i+1 : \left(\begin{array}{cc|c} \cancel{i} & \cancel{1} & \cancel{0} \\ -1 & i & 0 \end{array}\right) \rightarrow X_1 = i . X_2$$

$$S_{\lambda=i+1} = gen\{(i \quad 1)^t\}$$

$$\lambda = -i+1 : \left(\begin{array}{cc|c} -i & 1 & 0 \\ \cancel{-1} & \cancel{-i} & \cancel{0} \end{array}\right) \rightarrow X_2 = i . X_1 \rightarrow S_{\lambda=-i+1} = gen\{(1 \quad i)^T\}$$

$$\|(i \quad 1)^T\| = \|(1 \quad i)^T\| = \sqrt{2} \Rightarrow \boxed{A = \begin{pmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix} . \begin{pmatrix} i+1 & 0 \\ 0 & i-1 \end{pmatrix} . \begin{pmatrix} -i/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}}$$

II)

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow P_A(\lambda) = \det \begin{vmatrix} \lambda & 1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1) . (\lambda^2 + 1) = 0$$

$\lambda_1 = 1$; $\lambda_2 = i$; $\lambda_3 = -i$

$$\lambda = 1 : \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \rightarrow \begin{array}{l} X_1 = X_2 = 0 \\ X_3 \in \mathbb{C} \end{array} \Rightarrow S_{\lambda=1} = gen\{(0 \quad 0 \quad 1)^T\}$$

$\lambda = i:$ $\left(\begin{array}{ccc|c} \cancel{i} & \cancel{1} & \cancel{0} & \cancel{0} \\ -1 & i & 0 & 0 \\ 0 & 0 & i-1 & 0 \end{array}\right) \rightarrow \begin{array}{l} i \cdot x_2 = x_1 \\ x_3 = 0 \end{array} \Rightarrow S_{\lambda=i} = gen\{(i \quad 1 \quad 0)^T\}$

$\lambda = -i:$ $\left(\begin{array}{ccc|c} -i & 1 & 0 & 0 \\ \cancel{-1} & \cancel{-i} & \cancel{0} & \cancel{0} \\ 0 & 0 & -i-1 & 0 \end{array}\right) \rightarrow \begin{array}{l} x_2 = i \cdot x_1 \\ x_3 = 0 \end{array} \rightarrow S_{\lambda=-i} = gen\{(1 \quad i \quad 0)^T\}$

$$\Rightarrow \boxed{A = \begin{pmatrix} 0 & i/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & i/\sqrt{2} \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ -i/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -i/\sqrt{2} & 0 \end{pmatrix}}$$

16) - a) - V $A^T = (P \cdot D \cdot P^T)^T = P \cdot D^T \cdot P^T = P \cdot D \cdot P^T = A$

b) - F I_n solo tiene a 1 como autovalor

c) - V (una matriz simétrica siempre es diagonalizable)

d) - F

e) - V con autoval. reales se cumple ⊛ $A^H = (U \cdot D \cdot U^H)^H = U \cdot \underset{\sim}{D^H} \cdot U^H = A$

⊛ D

f) - V $A = U \cdot D \cdot U^H \Rightarrow A^k = U \cdot D^k \cdot U^H$

g) - V $A = U \cdot D \cdot U^H \Rightarrow A^{-1} = U \cdot D^{-1} \cdot U^H$

17) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

18) $(B^T \cdot A \cdot B)^T = B^T \cdot A^T \cdot B \underset{\substack{\downarrow \\ A \text{ es simétrica}}}{=} B^T \cdot A \cdot B$ ✓

$(B^T \cdot B)^T = B^T (B^T)^T = B^T \cdot B$ ✓ (lo mismo con $B \cdot B^T$)

19) - a) $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow P_C(\lambda) = \det\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda_1 = -i \, ; \, \lambda_2 = i$ ✓

$\lambda = i:$

$\left(\begin{array}{cc|c} i & -1 & 0 \\ \cancel{1} & \cancel{i} & \cancel{0} \end{array}\right) \rightarrow x_2 = i \cdot x_1 \Rightarrow S_{\lambda=i} = gen\{(1 \quad i)^T\}$

$\lambda = -i:$ $\left(\begin{array}{cc|c} \cancel{-i} & \cancel{-1} & \cancel{0} \\ 1 & -i & 0 \end{array}\right) \rightarrow x_1 = i \cdot x_2 \Rightarrow S_{\lambda=-i} = gen\{(i \quad 1)^T\}$

Es posible diagonalizarla unitariamente.

b) $(i.C)^H = C^H.(-i) \underset{\substack{\downarrow \\ C\in\mathbb{R}^{n\times n}}}{=} C^T.(-i) = (-C).(-i) = i.C$ Ⓥ

20) Si A es simétrica, debería poder escribirse como $P.D.P^T$ con P ortogonal.

Busco $(S_{\lambda_1})^\perp \Rightarrow (S_{\lambda_1})^\perp = \{X\in\mathbb{R}^3 / x_1+x_2+x_3=0\}$

$x_1=-x_2-x_3 \Rightarrow (S_{\lambda_1})^\perp = gen\{(-1\ \ 1\ \ 0)^T ; (-1\ \ 0\ \ 1)^T\}$

Propongo que $\lambda=2$ sea autovalor doble y $\underbrace{S_{\lambda_2}=(S_{\lambda_1})^\perp}$ → si A es simétrica esto debe cumplirse

Para armar P todavía falta que los autovec. de S_{λ_2} sean $\perp$ entre sí.

Gram-Schmidt: $v_1=(-1\ \ 1\ \ 0)^T = u_1$, $v_2=(-1\ \ 0\ \ 1)^T$

$S_1 = gen\{(-1\ \ 1\ \ 0)^T\} \Rightarrow u_2 = v_2 - P_{S_1}(v_2) = v_2 - \frac{(v_1,v_2)}{(v_1,v_1)}.v_1 =$

$= v_2 - \frac{1}{2}.v_1 = (-\frac{1}{2}\ \ -\frac{1}{2}\ \ 1) \rightarrow \|\ \| = \sqrt{\frac{3}{2}}$ → P.i. canónico en $\mathbb{R}^3$

Normalizo los autovectores para armar A:

$$A = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -\frac{1}{2}.\sqrt{\frac{2}{3}} \\ 1/\sqrt{3} & 1/\sqrt{2} & -\frac{1}{2}.\sqrt{\frac{2}{3}} \\ 1/\sqrt{3} & 0 & \sqrt{\frac{2}{3}} \end{pmatrix} . \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} . \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -\frac{1}{2}.\sqrt{\frac{2}{3}} & -\frac{1}{2}.\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{3} & -2/\sqrt{2} & -\sqrt{\frac{2}{3}} \\ -1/\sqrt{3} & 2/\sqrt{2} & -\sqrt{\frac{2}{3}} \\ -1/\sqrt{3} & 0 & 2.\sqrt{\frac{2}{3}} \end{pmatrix} . \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -\frac{1}{2}\sqrt{\frac{2}{3}} & -\frac{1}{2}\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} \end{pmatrix} =$$

$$= \boxed{\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}}$$

21) Si S es invariante por A, se puede pensar a S como un autoespacio de A. Debe estar asociado a $\lambda=2$, pues de esa forma la mult. algebraica y geométrica coincidirían.

$S = S_{\lambda=2} = gen\{(i\ \ 1\ \ 0)^T, (-1\ \ 0\ \ 1)^T\}$

Busco $(S_{\lambda=2})^\perp$: $(S_{\lambda=2})^\perp = gen\{(1\ \ -i\ \ 1)^T\}$ → sale de los coef. de la ec. de S

B es singular $\Leftrightarrow$ no es inversible $\Leftrightarrow \det(B)=0$

si μ es autoval. de B $\Rightarrow \mu = \lambda^3-\lambda^2+\lambda-1$ con λ autoval. de A.
El autovalor doble de B es $\mu = 2^3-2^2+2-1 = 5$

Además, el determinante de una matriz es igual al producto de sus autovalores.

$\det(B) = 5 \cdot 5 \cdot \mu_2 = 0 \Rightarrow \mu_2 = 0 \Rightarrow \lambda^3 - \lambda^2 + \lambda - 1 = 0 \Rightarrow \lambda_2 = 1$

$$\begin{array}{c|cccc} & 1 & -1 & 1 & -1 \\ 1 & & 1 & 0 & 1 \\ \hline & 1 & 0 & 1 & 0 \end{array} \Rightarrow (\lambda - 1)(\lambda^2 + 1) = 0$$

$\Rightarrow$ los otros soluciones son complejos, pero si A es hermítica no puede tener autovalores complejos.

$S_{\lambda_2} = \text{gen}\{(1 \quad -i \quad 1)^T\}$

Busco BOG de S_{λ_1} ($\lambda_1 = 2$):

$v_1 = (i \quad 1 \quad 0)^T = c_1 \quad , v_2 = (-1 \quad 0 \quad 1)^T$

$S_1 = \text{gen}\{v_1\}$

$$\Rightarrow c_2 = v_2 - P_{S_1}(v_2) = v_2 - \frac{(v_1, v_2)}{(v_1, v_1)} \cdot v_1 = v_2 - \frac{i}{2} \cdot v_1 =$$

$\to$ P. i. canónico en $\mathbb{C}^3$

$$= \begin{pmatrix} -\frac{1}{2} \\ -\frac{i}{2} \\ 1 \end{pmatrix} \to \| \; \| = \sqrt{\frac{3}{2}}$$

$$A = \begin{pmatrix} i/\sqrt{2} & -\frac{1}{2} \cdot \sqrt{\frac{2}{3}} & 1/\sqrt{3} \\ 1/\sqrt{2} & -\frac{i}{2} \sqrt{\frac{2}{3}} & -i/\sqrt{3} \\ 0 & \sqrt{\frac{2}{3}} & 1/\sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{bmatrix} -i/\sqrt{2} & 1/\sqrt{2} & 0 \\ -\frac{1}{2} \cdot \sqrt{\frac{2}{3}} & \frac{i}{2} \cdot \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} \\ 1/\sqrt{3} & i/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = \begin{pmatrix} 2i/\sqrt{2} & -\sqrt{\frac{2}{3}} & 1/\sqrt{3} \\ \frac{2}{\sqrt{2}} & -i \cdot \sqrt{\frac{2}{3}} & -i/\sqrt{3} \\ 0 & 2 \cdot \sqrt{\frac{2}{3}} & 1/\sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} -i/\sqrt{2} & 1/\sqrt{2} & 0 \\ -\frac{1}{2} \cdot \sqrt{\frac{2}{3}} & \frac{i}{2} \cdot \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} \\ 1/\sqrt{3} & i/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} =$$

$$= \boxed{\frac{1}{3} \cdot \begin{pmatrix} 5 & 3i & -1 \\ -3i & 5 & -3i \\ -1 & 3i & 5 \end{pmatrix}}$$

usando diferentes combinaciones lineales para los autoespacios (manteniendo la ortogonalidad) se pueden obtener otros A.

22) - a) $10x_1^2 - 6 \cdot x_1 \cdot x_2 - 3x_2^2 \Rightarrow \boxed{A = \begin{pmatrix} 10 & -3 \\ -3 & -3 \end{pmatrix}}$

b) $5x_1^2 + 3x_1 \cdot x_2 \Rightarrow \boxed{A = \begin{pmatrix} 5 & 3/2 \\ 3/2 & 0 \end{pmatrix}}$

c) $x_1 \cdot x_2 \Rightarrow \boxed{A = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}}$

23) - a) $4x_1 \cdot x_2 + 6 \cdot x_1 \cdot x_3 - 8 \cdot x_2 \cdot x_3 \Rightarrow \boxed{A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & -4 \\ 3 & -4 & 0 \end{pmatrix}}$

b) $8 \cdot x_1^2 + 7 \cdot x_2^2 - 3x_3^2 - 6 \cdot x_1 \cdot x_2 + 4 \cdot x_1 \cdot x_3 - 2 \cdot x_2 \cdot x_3 \Rightarrow \boxed{A = \begin{pmatrix} 8 & -3 & 2 \\ -3 & 7 & -1 \\ 2 & -1 & -3 \end{pmatrix}}$

c) $x_1^2 - x_1 \cdot x_3 + x_3^2 \Rightarrow \boxed{A = \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1 \end{pmatrix}}$

NOTA

24) - a) $X_1^2 + 10 . X_1 . X_2 + X_2^2$

$\Rightarrow F(X) = X^T . A . X$ con $A = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$, $a_{11} = 1 > 0$, $\det(A) = -24 < 0$

$\therefore$ F(X) es indefinido

A puede escribirse como $P.D.P^T$ por ser simétrica (P ortogonal)

Busco autoval. de A:

$$P_A(\lambda) = \det(\lambda . I - A) = \det\begin{pmatrix} \lambda-1 & -5 \\ -5 & \lambda-1 \end{pmatrix} = (\lambda-1)^2 - 25 = 0$$

$\Rightarrow \lambda_1 = 6$, $\lambda_2 = -4$ $\Rightarrow$ $\lambda = 6$: $\left(\begin{array}{cc|c} 5 & -5 & 0 \\ \cancel{-5} & \cancel{5} & \cancel{0} \end{array}\right) \to X_1 = X_2 \Rightarrow S_{\lambda=6} = gen\{(1 \quad 1)^T\}$

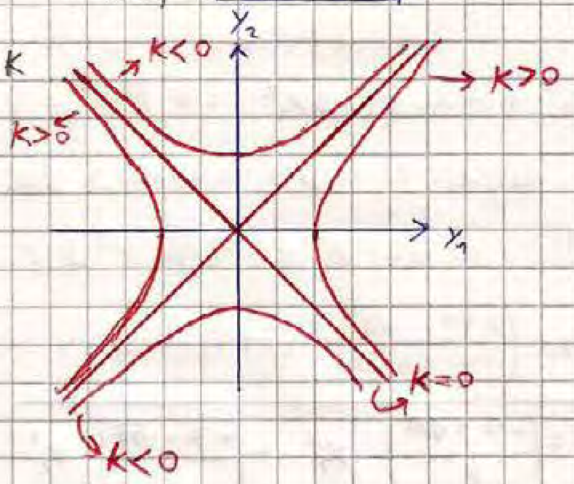
$\lambda = -4$: $\left(\begin{array}{cc|c} -5 & -5 & 0 \\ \cancel{-5} & \cancel{-5} & \cancel{0} \end{array}\right) \to X_1 = -X_2 \Rightarrow S_{\lambda=-4} = gen\{(-1 \quad 1)^T\}$

cambio de variable: $X = P.Y \Rightarrow X^T . P.D.P^T . X = (P.Y)^T . P.D. \underbrace{P^T . P}_{I} . Y =$

$= Y^T . \underbrace{P^T . P}_{I} . D . Y = Y^T . D . Y$

$F(Y) = (Y_1 \quad Y_2) \cdot \begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix} \cdot \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = (6 Y_1 \quad -4 Y_2) \cdot \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \boxed{6 Y_1^2 - 4 . Y_2^2}$

conjuntos de nivel: $6 . Y_1^2 - 4 . Y_2^2 = K$
$(K \in \mathbb{R})$



b) $3 . X_1^2 - 4 . X_1 . X_2 + 6 . X_2^2$, $F(X) = X^T . A . X$, $A = \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}$

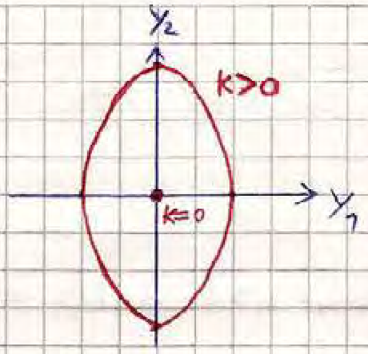
$a_{11} > 0$, $\det(A) = 18 - 4 > 0$ $\Rightarrow$ la función es definida positiva

$A = P.D.P^T$, con $X = P.Y \Rightarrow F(Y) = Y^T . D . Y$

Autovalores: $P_A(\lambda) = \det\begin{pmatrix} \lambda-3 & 2 \\ 2 & \lambda-6 \end{pmatrix} = (\lambda-3).(\lambda-6) - 4 = 0 = \lambda^2 - 9\lambda + 14$

$\lambda_1 = 7$
$\lambda_2 = 2$ $\Rightarrow$ $\boxed{F(Y) = 7 . Y_1^2 + 2 . Y_2^2}$

$1.Y_1^2 + 2.Y_2^2 = K$



c) $X_1^2 + 2.X_1.X_2 + X_2^2 \Rightarrow F(X) = X^T.A.X \quad , A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

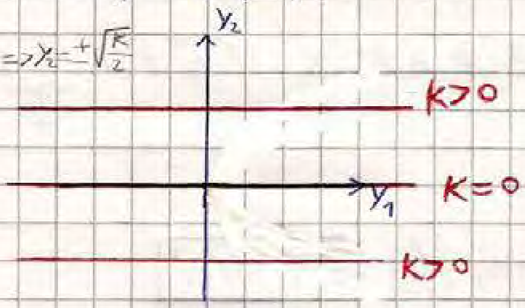
$a_{11} > 0$, $\det(A) = 0 \Rightarrow F(X)$ es semidefinido positivo.

$A = P.D.P^T \quad , \text{ si } X = P.Y \Rightarrow F(Y) = Y^T.D.Y$

$P_A(\lambda) = \det\begin{pmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{pmatrix} = (\lambda-1)^2 - 1 = 0 \Rightarrow \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 2 \end{matrix}$

$F(Y) = (Y_1 \ Y_2) . \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} . \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \boxed{2.Y_2^2}$

$2.Y_2^2 = K \Rightarrow Y_2 = \pm\sqrt{\frac{K}{2}}$
$(Y_1 \in \mathbb{R})$



d) $-4.X_1^2 + 4X_1.X_2 - 3X_2^2 \Rightarrow F(X) = X^T.A.X \quad , A = \begin{pmatrix} -4 & 2 \\ 2 & -3 \end{pmatrix}$

$a_{11} < 0$; $\det(A) = 12 - 4 > 0 \Rightarrow F(X)$ es def neg.

$A = P.D.P^T \quad , \text{ si } X = PY \Rightarrow F(Y) = Y^T D.Y$

$P_A(\lambda) = \det\begin{pmatrix} \lambda+4 & -2 \\ -2 & \lambda+3 \end{pmatrix} = (\lambda+4).(\lambda+3) - 4 = 0 = \lambda^2 + 7\lambda + 8 \Rightarrow \lambda_1 = \frac{-7+\sqrt{17}}{2}$

$\lambda_2 = \frac{-7-\sqrt{17}}{2}$

$\Rightarrow \boxed{F(Y) = \left(\frac{-7+\sqrt{17}}{2}\right).Y_1^2 + \left(\frac{-7-\sqrt{17}}{2}\right).Y_2^2}$

Con K negativo las curvas de nivel son hipérbolas.

2.5) - a) $5X_1^2 + 6.X_2^2 + 7.X_3^2 + 4.X_1.X_2 - 4.X_2.X_3$

$A = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 7 \end{pmatrix} \Rightarrow P_A(\lambda) = \det\begin{pmatrix} \lambda-5 & -2 & 0 \\ -2 & \lambda-6 & 2 \\ 0 & 2 & \lambda-7 \end{pmatrix} = (-2).\det\begin{pmatrix} \lambda-5 & 0 \\ -2 & 2 \end{pmatrix} + (\lambda-7).\det\begin{pmatrix} \lambda-5 & -2 \\ -2 & \lambda-6 \end{pmatrix} =$

NOTA

FECHA

$= (-2).2.(\lambda-5) + (\lambda-7).\left((\lambda-5).(\lambda-6) - 4\right) = \lambda^3 - 18.\lambda^2 + 99\lambda - 162 = 0$

$\lambda = 3$ es autoval. $\Rightarrow$

$$\begin{array}{r|rrr|r} & 1 & -18 & 99 & -162 \\ 3 & & 3 & -45 & 162 \\ \hline & 1 & -15 & 54 & \boxed{0} \end{array} \Rightarrow P_A(\lambda) = (\lambda-3).(\lambda^2 - 15\lambda + 54)$$

$\lambda = 9$ $\quad$ $\lambda = 6$

$\boxed{F(Y) = 3.Y_1^2 + 6.Y_2^2 + 9.Y_3^2}$; los autovalores son $> 0 \Rightarrow$ la función es def. pos.

b) $3X_1^2 + 2.X_2^2 + 2.X_3^2 + 2.X_1.X_2 + 2.X_1.X_3 + 4.X_2.X_3$

$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix} \rightarrow P_A(\lambda) = \det\begin{pmatrix} \lambda-3 & -1 & -1 \\ -1 & \lambda-2 & -2 \\ -1 & -2 & \lambda-2 \end{pmatrix} =$

$= (\lambda-3)\det\begin{pmatrix} \lambda-2 & -2 \\ -2 & \lambda-2 \end{pmatrix} + \det\begin{pmatrix} -1 & -1 \\ -2 & \lambda-2 \end{pmatrix} - \det\begin{pmatrix} -1 & -1 \\ \lambda-2 & -2 \end{pmatrix} =$

$= (\lambda-3).\left((\lambda-2)^2 - 4\right) - \lambda \cancel{+2} \cancel{-2} \cancel{-2} - \lambda \cancel{+2} = \lambda^3 - 7\lambda^2 + 10\lambda = \lambda.(\lambda^2 - 7\lambda + 10) = 0$

$\underbrace{\lambda_1 = 0\ ,\ \lambda_2 = 5\ ,\ \lambda_3 = 2}_{F \text{ es semidef. positiva}} \Rightarrow \boxed{F(Y) = 5.Y_2^2 + 2.Y_3^2}$

c) $X_1.X_2 + X_2 X_3 + X_1.X_3$; $A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$

$P_A(\lambda) = \det\begin{pmatrix} \lambda & -1/2 & -1/2 \\ -1/2 & \lambda & -1/2 \\ -1/2 & -1/2 & \lambda \end{pmatrix} = \lambda.\det\begin{pmatrix} \lambda & -1/2 \\ -1/2 & \lambda \end{pmatrix} + (\frac{1}{2}).\det\begin{pmatrix} -1/2 & -1/2 \\ -1/2 & \lambda \end{pmatrix} + (-\frac{1}{2}).\det\begin{pmatrix} -1/2 & -1/2 \\ \lambda & -1/2 \end{pmatrix} =$

$= \lambda.\left(\lambda^2 - \frac{1}{4}\right) - \frac{1}{4}\lambda - \frac{1}{8} - \frac{1}{8} - \frac{1}{4}\lambda = \lambda.\left(\lambda^2 - \frac{1}{4}\right) - \frac{1}{2}\lambda - \frac{1}{4} = \lambda^3 - \frac{3}{4}\lambda - \frac{1}{4} = 0$

$\lambda_1 = 1 \Rightarrow$

$$\begin{array}{r|rrr|r} & 1 & 0 & -\frac{3}{4} & -\frac{1}{4} \\ 1 & & 1 & 1 & \frac{1}{4} \\ \hline & 1 & 1 & \frac{1}{4} & \boxed{0} \end{array} \Rightarrow P_A(\lambda) = (\lambda-1).(\lambda^2 + \lambda + \frac{1}{4}) = 0$$

$\hookrightarrow \lambda = -\frac{1}{2}$ (doble)

F es indefinida.

$\boxed{F(Y) = Y_1^2 - \frac{1}{2}.Y_2^2 - \frac{1}{2}.Y_3^2}$

29) - a) $x_1.y_1 + x_1.y_2 + x_2.y_4 + 3x_2.y_2 = X^T. \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}}_{A}. Y$

Si A es def. pos. se trata de un producto interno.

$a_{11} > 0$, $\det(A) = 2 > 0 \Rightarrow$ A es def. pos. ∴ es un P.i.

b) = a)

c) $7.x_1.y_1 - 4.x_1.y_2 + 4.x_1.y_3 - 4.x_2.y_1 + 5x_2.y_2 + 4x_3.y_1 + 9.x_3.y_3$

$$A = \begin{pmatrix} 7 & -4 & 4 \\ -4 & 5 & 0 \\ 4 & 0 & 9 \end{pmatrix} \Rightarrow a_{11} > 0 \; , \det\begin{pmatrix} 7 & -4 \\ -4 & 5 \end{pmatrix} = 19 > 0 \; , \det\begin{vmatrix} 7 & -4 & 4 \\ -4 & 5 & 0 \\ 4 & 0 & 9 \end{vmatrix} =$$

$$= 4.\det\begin{pmatrix} -4 & 4 \\ 5 & 0 \end{pmatrix} + 9.\det\begin{pmatrix} 7 & -4 \\ -4 & 5 \end{pmatrix} = 4.(-20) + 9.(35-16) = 91 > 0$$

$\Rightarrow$ A es def. pos. ∴ es un P.i.

30) $(x, y) = 2.\bar{x}_1.y_1 + i.\bar{x}_1.y_2 - i.\bar{x}_2.y_1 + \bar{x}_2.y_2 + \bar{x}_3.y_3$

$$(x,y) = x^H.A.y = (\bar{x}_1 \; \bar{x}_2 \; \bar{x}_3).\begin{pmatrix} 2 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$P_A(\lambda) = \det\begin{pmatrix} \lambda-2 & -i & 0 \\ i & \lambda-1 & 0 \\ 0 & 0 & \lambda-1 \end{pmatrix} = (\lambda-1).\det\begin{pmatrix} \lambda-2 & -i \\ i & \lambda-1 \end{pmatrix} = (\lambda-1).\left((\lambda-2).(\lambda-1) - 1\right) =$$

$$= (\lambda-1).(\lambda^2 - 3\lambda + 1) = 0 \Rightarrow \lambda_1 = 1 , \lambda_2 = \frac{3+\sqrt{5}}{2} , \lambda_3 = \frac{3-\sqrt{5}}{2}$$

Los tres son positivos $\Rightarrow$ A es def. pos. ∴ es un P.i. en $\mathbb{C}^3$

32) $x^T x = \|Y\|^2 = 1 \Rightarrow \|X\| = 1$

Ej.24 - a) $F(Y) = 6.Y_1^2 - 4.Y_2^2$ el máximo de esta función es $\boxed{\lambda = 6}$ y el mínimo es $\boxed{\lambda = -4}$

Esto se comprueba porque $6.Y_1^2 - 4.Y_2^2 \leq 6.Y_1^2 + 6Y_2^2 = 6.\underbrace{(Y_1^2 + Y_2^2)}_{1}$ → la norma se mantiene con el cambio de variable

y $F(\underset{\hookrightarrow (1\;0)^T}{e_1}) = 6$; $6Y_1^2 - 4.Y_2 \geq (-4).\underbrace{(Y_1^2 + Y_2^2)}_{1}$ y $F(e_2) = -4$

Obviamente el máximo se encuentra en $Y = (\pm 1 \;\; 0)^T$ y el mínimo en $Y = (0 \; \pm 1)^T$.

NOTA

Pero para poder expresarlo en la variable X:

$X = P\,Y$ (P ortogonal)

Así que debo armar P primero:

Dato: para armar P se debe tener en cuenta el D que se usó para armar F(Y), si $F(Y) = 6.Y_1^2 - 4.Y_2^2$, D es $\begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix}$ y se debe usar la matriz P correspondiente a ese D.

$$P = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \Rightarrow X_{máx} = P.Y_{máx} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} = \boxed{\pm\frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

(donde se alcanza el máximo)

$$X_{mín} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} = \boxed{\pm\frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

b) máximo: $\boxed{\lambda = 7}$

mínimo: $\boxed{\lambda = 2}$

(usando los mismos argumentos que antes)

Autovectores:

$\lambda = 7$: $\lambda I - A = \begin{pmatrix} \lambda-3 & 2 \\ 2 & \lambda-6 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} \cancel{4} & \cancel{2} & \cancel{0} \\ 2 & 1 & 0 \end{array}\right) \rightarrow b = -2.a \quad (a \in \mathbb{R})$

$S_{\lambda=7} = \text{gen}\{(1 \quad -2)^T\}$

$\lambda = 2$:

$\left(\begin{array}{cc|c} -1 & 2 & 0 \\ \cancel{2} & \cancel{-4} & \cancel{0} \end{array}\right) \rightarrow a = 2b \Rightarrow S_{\lambda=2} = \text{gen}\{(2 \quad 1)^T\}$

$\Rightarrow P = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$; $Y_{máx} = (\pm 1 \quad 0)^T$, $Y_{mín} = (0 \quad \pm 1)^T$ $\Rightarrow X_{máx} = \frac{1}{\sqrt{5}} \cdot \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} = \boxed{\pm\frac{1}{\sqrt{5}} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$

$$X_{mín} = \frac{1}{\sqrt{5}} \cdot \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} = \boxed{\pm\frac{1}{\sqrt{5}} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

c) máximo: $\boxed{\lambda = 2}$

mínimo: $\boxed{\lambda = 0}$

$\lambda.I - A = \begin{pmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{pmatrix} \Rightarrow \lambda = 2 : \left(\begin{array}{cc|c} 1 & -1 & 0 \\ \cancel{-1} & \cancel{1} & \cancel{0} \end{array}\right) \rightarrow a = b \Rightarrow S_{\lambda=2} = \text{gen}\{(1 \quad 1)^T\}$

$\lambda = 0 : \left(\begin{array}{cc|c} -1 & -1 & 0 \\ \cancel{-1} & \cancel{-1} & \cancel{0} \end{array}\right) \quad a = -b \Rightarrow S_{\lambda=0} = \text{gen}\{(-1 \quad 1)^T\}$

$P = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$; $Y_{máx} = (0 \quad \pm 1)^T$; para hallar $Y_{mín}$ planteo las condiciones que debe cumplir:

$$\begin{cases} F(Y) = 2.y_2^2 = 0 \\ y_1^2 + y_2^2 = 1 \end{cases} \Rightarrow Y_{mín} = (\pm 1 \quad 0)^T$$

También $Y_{máx}$ debe cumplir $\begin{cases} Y_1^2 + Y_2^2 = 1 \\ 2 Y_2^2 = 2 \end{cases}$ pero en ciertos casos, como este, se puede deducir directamente sin hacer cuentas.

$$X_{máx} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} = \boxed{\pm \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \; ; \; X_{mín} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} = \boxed{\pm \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

Ej. 25 - a) máximo: $\boxed{\lambda = 9}$

mínimo: $\boxed{\lambda = 3}$

$$\lambda . I - A = \begin{pmatrix} \lambda - 5 & -2 & 0 \\ -2 & \lambda - 6 & 2 \\ 0 & 2 & \lambda - 7 \end{pmatrix} \Rightarrow \lambda = 9 : \left(\begin{array}{ccc|c} 4 & -2 & 0 & 0 \\ -2 & 3 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 4 & -2 & 0 & 0 \\ \cancel{0} & \cancel{4} & \cancel{4} & \cancel{0} \\ 0 & 2 & 2 & 0 \end{array}\right)$$

$$\Rightarrow \begin{array}{l} 2 . a - b = 0 \Rightarrow a = \frac{b}{2} \\ b + c = 0 \Rightarrow c = -b \end{array} \Rightarrow S_{\lambda=9} = gen\{(\tfrac{1}{2} \quad 1 \quad -1)^T\}$$

$$\lambda = 6 : \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ \cancel{0} & \cancel{-4} & \cancel{2} & \cancel{0} \\ 0 & 2 & -1 & 0 \end{array}\right) \rightarrow \begin{cases} a - 2b = 0 \\ 2b - c = 0 \end{cases} \Rightarrow \begin{array}{l} a = 2b \\ c = 2b \end{array} \Rightarrow S_{\lambda=6} = gen\{(2 \quad 1 \quad 2)^T\}$$

$$\lambda = 3 : \left(\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ -2 & -3 & 2 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ \cancel{0} & \cancel{2} & \cancel{-4} & \cancel{0} \end{array}\right) \rightarrow \begin{cases} a = -b \\ c = \frac{b}{2} \end{cases} \Rightarrow S_{\lambda=3} = gen\{(-1 \quad 1 \quad \tfrac{1}{2})^T\}$$

$P = \frac{1}{3} \cdot \begin{pmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{pmatrix}$ considerando que el D usado para $F(Y)$ es $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}$

$$\begin{array}{l} Y_{máx} = (0 \quad 0 \quad \pm 1)^T \\ Y_{mín} = (\pm 1 \quad 0 \quad 0)^T \end{array} \Rightarrow X_{máx} = \frac{1}{3} \cdot \begin{pmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix} = \boxed{\pm \frac{1}{3} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}$$

$$X_{mín} = \frac{1}{3} \cdot \begin{pmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{pmatrix} \cdot \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix} = \boxed{\pm \frac{1}{3} \cdot \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}}$$

b) máximo: $\boxed{\lambda = 5}$

mínimo: $\boxed{\lambda = 0}$; $\lambda . I - A = \begin{pmatrix} \lambda - 3 & -1 & -1 \\ -1 & \lambda - 2 & -2 \\ -1 & -2 & \lambda - 2 \end{pmatrix}$

$$\lambda = 5 : \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 3 & -2 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 5 & -5 & 0 \\ \cancel{0} & \cancel{-5} & \cancel{5} & \cancel{0} \end{array}\right) \rightarrow \begin{cases} 2a - b - c = 0 \Rightarrow a = c \\ b = c \end{cases}$$

$$S_{\lambda=5} = gen\{(1 \quad 1 \quad 1)^T\}$$

NOTA

HOJA *

FECHA

$\lambda=2$: $\left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & 0 & -2 & 0 \\ -1 & -2 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ \cancel{0} & \cancel{1} & \cancel{-1} & \cancel{0} \end{array}\right) \rightarrow \begin{cases} a+b+c=0 \rightarrow a=-2c \\ b=c \end{cases}$

$S_{\lambda=2}=\text{gen}\{(-2 \quad 1 \quad 1)^T\}$

$\lambda=0$: $\left(\begin{array}{ccc|c} -3 & -1 & -1 & 0 \\ -1 & -2 & -2 & 0 \\ \cancel{-1} & \cancel{-2} & \cancel{-2} & \cancel{0} \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 3 & 1 & 1 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 3 & 1 & 1 & 0 \\ 0 & 5 & 5 & 0 \end{array}\right) \rightarrow \begin{cases} 3a+b+c=0 \Rightarrow a=0 \\ b=-c \end{cases}$

$S_{\lambda=0}=\text{gen}\{(0 \quad -1 \quad 1)^T\}$

$P=\begin{pmatrix} 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}$; $Y_{máx}=(0 \quad \pm 1 \quad 0)^T$

$Y_{mín}=(\pm 1 \quad 0 \quad 0)^T$

$X_{máx}=\begin{pmatrix} 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix} = \boxed{\pm \frac{1}{\sqrt{3}} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$; $X_{mín}=\begin{pmatrix} 0 & 1/\sqrt{5} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix} \cdot \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix} = \boxed{\pm \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}$

c) máximo: $\boxed{\lambda=1}$

mínimo : $\boxed{\lambda=-\frac{1}{2}}$

$\lambda=1$: $\left(\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ -1/2 & 1 & -1/2 & 0 \\ -1/2 & -1/2 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & 3/2 & -3/2 & 0 \\ \cancel{0} & \cancel{-3/2} & \cancel{3/2} & \cancel{0} \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) \rightarrow \begin{cases} 2a-b-c=0 \\ b=c \end{cases}$

$\Rightarrow a=c$

$S_{\lambda=1}=\text{gen}\{(1 \quad 1 \quad 1)^T\}$

$\lambda=-1/2$: $\left(\begin{array}{ccc|c} 1/2 & -1/2 & -1/2 & 0 \\ \cancel{-1/2} & \cancel{-1/2} & \cancel{-1/2} & \cancel{0} \\ \cancel{1/2} & \cancel{-1/2} & \cancel{-1/2} & \cancel{0} \end{array}\right) \rightarrow a+b+c=0 \Rightarrow a=-b-c$

$S_{\lambda=-\frac{1}{2}}=\text{gen}\{\underbrace{(-1 \quad 1 \quad 0)^T;(-1 \quad 0 \quad 1)^T}_{\not\perp}\}$

Para P debo tener una BOG de autovec.:

Gram-Schmidt:

$v_1=(-1 \quad 1 \quad 0)^T=u_1$, $v_2=(-1 \quad 0 \quad 1)^T$; $S_1=\text{gen}\{v_1\}$

$u_2=v_2-P_{S_1}(v_2)=v_2-\frac{(v_1,v_2)}{(v_1,v_1)}\cdot v_1=v_2-\frac{1}{2}\cdot v_1=(-1/2 \quad -1/2 \quad 1)$; $\| \ \|=\sqrt{\frac{3}{2}}$

$\rightarrow$ P.i. canónica $\mathbb{R}^3$

$P=\begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -\frac{1}{2}\cdot\sqrt{\frac{2}{3}} \\ 1/\sqrt{3} & 1/\sqrt{2} & -\frac{1}{2}\cdot\sqrt{\frac{2}{3}} \\ 1/\sqrt{3} & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}$; $Y_{máx}=(\pm 1 \quad 0 \quad 0)^T$

NOTA

$$Y_{mín}\begin{cases} Y_1^2-\frac{1}{2}\cdot(Y_2^2+Y_3^2)=-\frac{1}{2} \\ Y_1^2+Y_2^2+Y_3^2=1 \end{cases} \Rightarrow \frac{3}{\cancel{2}}\cdot(Y_2^2+Y_3^2)=\frac{\cancel{3}}{\cancel{2}} \Rightarrow Y_2^2+Y_3^2=1$$

$$Y_1^2+Y_2^2+Y_3^2=1 \Rightarrow Y_1^2+\cancel{1}=\cancel{1} \Rightarrow Y_1=0$$

$$X_{máx}=\begin{pmatrix} 1/\sqrt{3} & & \\ 1/\sqrt{3} & & \\ 1/\sqrt{3} & & \end{pmatrix}\cdot\begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}=\boxed{\pm\frac{1}{\sqrt{3}}\cdot\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

33) $Q(X)=-2.X_1^2+2.X_1.X_2-2.X_2^2-10.X_3^2 \quad ; \quad X_1^2+X_2^2+9.X_3^2=9$

$\Rightarrow \frac{1}{9}.X_1^2+\frac{1}{9}.X_2^2+X_3^2=1 \Rightarrow$ Cambio de variable: $Z_1=\frac{1}{3}.X_1$

$Z_2=\frac{1}{3}.X_2$

$\Rightarrow$ La restricción queda $Z_1^2+Z_2^2+Z_3^2=1$

$Z_3=X_3$

$Q(Z)=-18.Z_1^2+18.Z_1.Z_2-18.Z_2^2-10.Z_3^2$

Ahora necesito otro cambio de variable:

$Z=P.Y$ con P ortogonal.

$Q(Z)=Z^T.A.Z$ con $A=\begin{pmatrix} -18 & 9 & 0 \\ 9 & -18 & 0 \\ 0 & 0 & -10 \end{pmatrix}$, como A es simétrica puede escribirse como $A=P.D.P^T$

$$\Rightarrow Q(Y)=(P.Y)^T.P.D.\underbrace{P^T.P}_{I}.Y=Y^T.\underbrace{P^T.P}_{I}.D.Y=Y^T.D.Y$$

Busco autoval. de A: $P_A(\lambda)=\det\begin{pmatrix} \lambda+18 & -9 & 0 \\ -9 & \lambda+18 & 0 \\ 0 & 0 & \lambda+10 \end{pmatrix}=(\lambda+10).\det\begin{pmatrix} \lambda+18 & -9 \\ -9 & \lambda+18 \end{pmatrix}=$

$=(\lambda+10).\left((\lambda+18)^2-81\right)=0 \Rightarrow (\lambda+10).(\lambda^2+36\lambda+243)=0 \Rightarrow \lambda_1=-10$

$\lambda_2=-9$

$\lambda_3=-27$

El máximo es: $\boxed{\lambda=-9}$

El mínimo es: $\boxed{\lambda=-27}$

Aclaración: el 1° cambio de variable se hace para lograr la condición $\|Z\|^2=1 \Rightarrow \|Z\|=1$. Ya que en el 2° cambio de variable la norma se conserva (por ser P ortogonal $\Rightarrow$ $\Rightarrow \|Z\|=\|P.Y\|=\|Y\|$) es posible pensar:

$$Q(Y)=-9.Y_1^2-10.Y_2^2-27.Y_3^2 \leq -9.Y_1^2-9.Y_2^2-9.Y_3^2=(-9).\underbrace{\|Y\|^2}_{1}=-9$$

De donde se deduce que -9 es el máximo. Mismo procedimiento para el mínimo.

NOTA

Para saber dónde se alcanzan los extremos debo hallar los autovectores.

Pero en lugar de armar P, hallar $Y_{máx}$ e $Y_{mín}$ y luego hallar $Z_{máx}$ y $Z_{mín}$, puedo usar que:

$Z_{máx} = S_{\lambda_{máx}} \cap \{Z / \|Z\| = 1\}$, $Z_{mín} = S_{\lambda_{mín}} \cap \{Z / \|Z\| = 1\}$

$\lambda=-9: \left(\begin{array}{ccc|c} 9 & -9 & 0 & 0 \\ \cancel{-9} & \cancel{9} & \cancel{0} & \cancel{0} \\ 0 & 0 & 1 & 0 \end{array}\right) \rightarrow \begin{matrix} a=b \\ c=0 \end{matrix} \Rightarrow S_{\lambda=-9} = gen\{(1 \ \ 1 \ \ 0)^T\}$

$\lambda=-27: \left(\begin{array}{ccc|c} -9 & -9 & 0 & 0 \\ \cancel{-9} & \cancel{-9} & \cancel{0} & \cancel{0} \\ 0 & 0 & -17 & 0 \end{array}\right) \rightarrow \begin{matrix} a=-b \\ c=0 \end{matrix} \rightarrow S_{\lambda=-27} = gen\{(-1 \ \ 1 \ \ 0)^T\}$

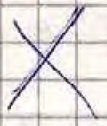
$\Rightarrow Z_{máx} = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$; $Z_{mín} = \pm \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Sin embargo, es recomendable usar $Z = P.Y$

$X_1 = 3\ Z_1$; $X_2 = 3 . Z_2$; $X_3 = Z_3$

$$\boxed{X_{máx} = \pm \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \ ; \ X_{mín} = \pm \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix}}$$

34) $Q(X) = X_1^2 + X_2^2$; $2 . X_1^2 - 2 . X_1 . X_2 + 2 . X_2^2 = 4$



36) - a) $A = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow A^T . A = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix}$

Autovalores: $\lambda_1 = 9$, $\lambda_2 = 0$, los valores singulares son

$\sigma_1 = \sqrt{\lambda_1} = 3$, $\sigma_2 = \sqrt{\lambda_2} = 0$

Busco autovec.: $\lambda=9: \left(\begin{array}{cc|c} \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 9 & 0 \end{array}\right) \rightarrow X_2 = 0 \Rightarrow S_{\lambda=9} = gen\{(1 \ \ 0)^T\}$

$\lambda=0: \left(\begin{array}{cc|c} -9 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \rightarrow X_1 = 0 \Rightarrow S_{\lambda=0} = gen\{(0 \ \ 1)^T\}$

La matriz ortogonal V es $V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$

Recordar que los σ deben estar de mayor a menor en la diagonal, y armar U y V a partir de Σ.

Para armar U ortogonal formada por las columnas u:

$u_1 = \frac{1}{\sigma_1} \underset{\substack{\downarrow \\ \text{columna 1 de V}}}{A . v_1} = \frac{1}{3} \cdot \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$; para u_2 sola busco alguno que complete una BON de $\mathbb{R}^2$ $(\sigma_2 = 0)$

$u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \boxed{A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$

$A = U . \Sigma . V^T$

b) $A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \Rightarrow A^H A = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix}$

$P_{A^H.A}(\lambda) = \det \begin{pmatrix} \lambda-2 & -2i \\ 2i & \lambda-2 \end{pmatrix} = (\lambda-2)^2 - 4 = 0 = \lambda^2 - 4\lambda = \lambda . (\lambda - 4)$

$\lambda_1 = 0 \qquad \sigma_2 = 0$
$\lambda_2 = 4 \rightarrow \sigma_1 = 2$

$\lambda = 0 : \left(\begin{array}{cc|c} -2 & -2i & 0 \\ 2i & -2 & 0 \end{array}\right) \rightarrow 2iX_1 - 2X_2 = 0 \Rightarrow X_2 = iX_1$
$\Rightarrow S_{\lambda=0} = \text{gen}\{(1 \quad i)^T\}$

$\lambda = 4 : \left(\begin{array}{cc|c} 2 & -2i & 0 \\ \cancel{2i} & \cancel{2} & \cancel{0} \end{array}\right) \rightarrow 2.X_1 - 2i.X_2 = 0 \Rightarrow X_1 = i\,X_2$
$\Rightarrow S_{\lambda=4} = \text{gen}\{(i \quad 1)^T\}$

$V = \begin{pmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix}$; $\Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$; $U = (u_1 | u_2)$

$u_1 = \frac{1}{\sigma_1} \cdot A . V_1 = \frac{1}{2} \cdot \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \cdot \begin{pmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} 2i/\sqrt{2} \\ -2/\sqrt{2} \end{pmatrix} = \begin{pmatrix} i/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

Completo BON de $\mathbb{C}^2$: $u_2 = \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix} \Rightarrow U = \begin{pmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}$

$A = U . \Sigma . V^H = \boxed{\frac{1}{2} \cdot \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}}$

c) $A = \begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix} \Rightarrow A^T . A = \begin{pmatrix} 2 & 2 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix}$

$P_{A^T.A}(\lambda) = \det \begin{pmatrix} \lambda-8 & -2 \\ -2 & \lambda-5 \end{pmatrix} = (\lambda-8)(\lambda-5) - 4 = \lambda^2 - 13\lambda + 36 = 0 \Rightarrow \begin{matrix} \lambda_1 = 9 \\ \lambda_2 = 4 \end{matrix} \Rightarrow \begin{matrix} \sigma_1 = 3 \\ \sigma_2 = 2 \end{matrix}$

$\lambda = 9 : \left(\begin{array}{cc|c} 1 & -2 & 0 \\ \cancel{-2} & \cancel{4} & \cancel{0} \end{array}\right) \rightarrow X_1 = 2X_2 \Rightarrow S_{\lambda=9} = \text{gen}\{(2 \quad 1)^T\}$

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$\lambda=4: \left(\begin{array}{cc|c} \cancel{-4} & \cancel{-2} & \cancel{0} \\ -2 & -1 & 0 \end{array}\right) \rightarrow x_2=-2x_1 \Rightarrow S_{\lambda=4}=gen\{(1 \ \ -2)^T\}$

$\Rightarrow V=\begin{pmatrix} 2/\sqrt5 & 1/\sqrt5 \\ 1/\sqrt5 & -2/\sqrt5 \end{pmatrix} \ ; \ \Sigma=\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \ ; \ u_1=\frac{1}{\sigma_1}\cdot A\cdot v_1=\frac{1}{3}\cdot\begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix}\cdot\begin{pmatrix} 2/\sqrt5 \\ 1/\sqrt5 \end{pmatrix}=\begin{pmatrix} 1/\sqrt5 \\ 2/\sqrt5 \end{pmatrix}$

$u_2=\frac{1}{\sigma_2}\cdot A\cdot v_2=\frac{1}{2}\cdot\begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix}\cdot\begin{pmatrix} 1/\sqrt5 \\ -2/\sqrt5 \end{pmatrix}=\begin{pmatrix} 2/\sqrt5 \\ -1/\sqrt5 \end{pmatrix}$

$$\boxed{A=\frac{1}{5}\cdot\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\cdot\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\cdot\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}}$$

d) $A=\begin{pmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{pmatrix} \Rightarrow A^T\cdot A=\begin{pmatrix} 7 & 0 & 5 \\ 1 & 0 & 5 \end{pmatrix}\cdot\begin{pmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{pmatrix}=\begin{pmatrix} 74 & 32 \\ 32 & 26 \end{pmatrix}$

$P_{A^T\cdot A}(\lambda)=\det\begin{pmatrix} \lambda-74 & -32 \\ -32 & \lambda-26 \end{pmatrix}=(\lambda-74)\cdot(\lambda-26)-1024=\lambda^2-100\lambda+900=0$

$\lambda_1=90 \quad \Rightarrow \quad \sigma_1=\sqrt{90}$
$\lambda_2=10 \quad \quad \sigma_2=\sqrt{10}$

$\lambda=90: \left(\begin{array}{cc|c} 16 & -32 & 0 \\ \cancel{-32} & \cancel{64} & \cancel{0} \end{array}\right) \rightarrow x_1=2x_2 \Rightarrow S_{\lambda=90}=gen\{(2 \ \ 1)^T\}$

$\lambda=10: \left(\begin{array}{cc|c} \cancel{-64} & \cancel{-32} & \cancel{0} \\ -32 & -16 & 0 \end{array}\right) \rightarrow x_2=-2x_1 \Rightarrow S_{\lambda=10}=gen\{(1 \ \ -2)^T\}$

$V=\begin{pmatrix} 2/\sqrt5 & 1/\sqrt5 \\ 1/\sqrt5 & -2/\sqrt5 \end{pmatrix} \ ; \ \Sigma=\begin{pmatrix} \sqrt{90} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{pmatrix} \rightarrow$ se completa la matriz para que la dimensión sea la correcta

$u_1=\frac{1}{\sigma_1}\ A\cdot v_1=\frac{1}{\sqrt{90}}\cdot\begin{pmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{pmatrix}\cdot\begin{pmatrix} 2/\sqrt5 \\ 1/\sqrt5 \end{pmatrix}=\begin{pmatrix} 1/\sqrt2 \\ 0 \\ 1/\sqrt2 \end{pmatrix} \ ; \ u_2=\frac{1}{\sqrt{10}}\cdot\begin{pmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{pmatrix}\cdot\begin{pmatrix} 1/\sqrt5 \\ -2/\sqrt5 \end{pmatrix}=\begin{pmatrix} 5/\sqrt{50} \\ 0 \\ -5/\sqrt{50} \end{pmatrix}$

Completo BON de $\mathbb{R}^3$: $u_3=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow U=\begin{pmatrix} 1/\sqrt2 & 5/\sqrt{50} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt2 & -5/\sqrt{50} & 0 \end{pmatrix}$

$$\Rightarrow \boxed{A=\begin{pmatrix} 1/\sqrt2 & 5/\sqrt{50} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt2 & -5/\sqrt{50} & 0 \end{pmatrix}\cdot\begin{pmatrix} \sqrt{90} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} 2/\sqrt5 & 1/\sqrt5 \\ 1/\sqrt5 & -2/\sqrt5 \end{pmatrix}}$$

e) $A=\begin{pmatrix}3&2&2\\2&3&-2\end{pmatrix} \Rightarrow A^T\cdot A=\begin{pmatrix}3&2\\2&3\\2&-2\end{pmatrix}\cdot\begin{pmatrix}3&2&2\\2&3&-2\end{pmatrix}=\begin{pmatrix}13&12&2\\12&13&-2\\2&-2&8\end{pmatrix}$

$$\rho_{A^T.A}(\lambda)=\det\begin{pmatrix}\lambda-13&-12&-2\\-12&\lambda-13&2\\-2&2&\lambda-8\end{pmatrix}=(\lambda-13)\cdot\det\begin{pmatrix}\lambda-13&2\\2&\lambda-8\end{pmatrix}+12\cdot\det\begin{pmatrix}-12&-2\\2&\lambda-8\end{pmatrix}+$$

$$+(-2)\cdot\det\begin{pmatrix}-12&-2\\\lambda-13&2\end{pmatrix}=(\lambda-13)\cdot\left((\lambda-13)\cdot(\lambda-8)-4\right)+12\cdot\left(-12\cdot\lambda+96+4\right)+$$

$$+(-2)\cdot\left(-24+2\lambda-26\right)=(\lambda-13)\cdot\left(\lambda^2-21\lambda+100\right)-144\lambda+1200-4\lambda+100=$$

$$=\lambda^3-13\lambda^2-21\lambda^2+273\cdot\lambda+100\lambda-\cancel{1360}-148\lambda+\cancel{1360}=\lambda^3-34\lambda^2+225\lambda=$$

$$=\lambda\cdot(\lambda^2-34\lambda+225)=0 \Rightarrow \lambda_1=0;\ \lambda_2=25;\ \lambda_3=9 \Rightarrow \sigma_1=5,\ \sigma_2=3,\ \sigma_3=0$$

$$\lambda=25:\ \left(\begin{array}{ccc|c}12&-12&-2&0\\\cancel{-12}&\cancel{12}&\cancel{2}&\cancel{0}\\-2&2&17&0\end{array}\right)\rightarrow\begin{cases}6X_1-6X_2-X_3=0\\-2X_1+2X_2+17X_3=0\end{cases}\Rightarrow \begin{array}{l}X_3=6\cdot X_1-6X_2\Rightarrow X_3=0\\100X_1-100X_2=0\Rightarrow X_1=X_2\end{array}$$

$$S_{\lambda=25}=\operatorname{gen}\{(1\ \ 1\ \ 0)^T\}$$

$$\lambda=9:\ \left(\begin{array}{ccc|c}-4&-12&-2&0\\-12&-4&2&0\\-2&2&1&0\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}2&6&1&0\\-6&-2&1&0\\-2&2&1&0\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}2&6&1&0\\\cancel{0}&\cancel{16}&\cancel{4}&\cancel{0}\\0&8&2&0\end{array}\right)$$

$$\rightarrow\begin{cases}2X_1+6X_2+X_3=0\\4X_2+X_3=0\end{cases}\rightarrow\begin{array}{l}X_1=-X_2\\X_3=-4X_2\end{array}\Rightarrow S_{\lambda=9}=\operatorname{gen}\{(-1\ \ 1\ \ -4)^T\}$$

$$\lambda=0:\ \left(\begin{array}{ccc|c}-13&-12&-2&0\\-12&-13&2&0\\-2&2&-8&0\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}13&12&2&0\\12&13&-2&0\\1&-1&4&0\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}13&12&2&0\\0&25&-50&0\\\cancel{0}&\cancel{-25}&\cancel{50}&\cancel{0}\end{array}\right)$$

$$\rightarrow\begin{cases}13X_1+12X_2+2X_3=0\\X_2=2\cdot X_3\end{cases}\Rightarrow X_1=-2X_3\Rightarrow S_{\lambda=0}=\operatorname{gen}\{(-2\ \ 2\ \ 1)^T\}$$

$$V=\begin{pmatrix}1/\sqrt{2}&-1/\sqrt{18}&-2/3\\1/\sqrt{2}&1/\sqrt{18}&2/3\\0&-4/\sqrt{18}&1/3\end{pmatrix};\ \Sigma=\begin{pmatrix}5&0&0\\0&3&0\end{pmatrix};\ \mu_1=\frac{1}{5}\cdot\begin{pmatrix}3&2&2\\2&3&-2\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}=\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$$

$$\mu_2=\frac{1}{3}\cdot\begin{pmatrix}3&2&2\\2&3&-2\end{pmatrix}\cdot\begin{pmatrix}-1/\sqrt{18}\\1/\sqrt{18}\\-4/\sqrt{18}\end{pmatrix}=\begin{pmatrix}-3/\sqrt{18}\\3/\sqrt{18}\end{pmatrix}=\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$$

$$A=U\cdot\Sigma\cdot V^T=\boxed{\begin{pmatrix}1/\sqrt{2}&-1/\sqrt{2}\\1/\sqrt{2}&1/\sqrt{2}\end{pmatrix}\cdot\begin{pmatrix}5&0&0\\0&3&0\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt{2}&1/\sqrt{2}&0\\-1/\sqrt{18}&1/\sqrt{18}&-4/\sqrt{18}\\-2/3&2/3&1/3\end{pmatrix}}$$

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37) $A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ -3 & 1 \end{pmatrix} \Rightarrow T(X) = A\,X = \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ -3 & 1 \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_1 + X_2 \\ X_1 + 3X_2 \\ -3X_1 + X_2 \end{pmatrix}$

$\|T(X)\| = \sqrt{(X_1+X_2)^2 + (X_1+3X_2)^2 + (-3X_1+X_2)^2}$ → los máximos y mínimos serán los mismos que para $\|T(X)\|^2$

$\|T(X)\|^2 = X_1^2 + 2X_1X_2 + X_2^2 + X_1^2 + \cancel{6X_1X_2} + 9X_2^2 + 9X_1^2 - \cancel{6X_1X_2} + X_2^2 =$

$= 11X_1^2 + 2X_1.X_2 + 11.X_2^2 = Q(X)$

La matriz que representa a esta función es $B = \begin{pmatrix} 11 & 1 \\ 1 & 11 \end{pmatrix}$

Como B es simétrica, puede escribirse $B = P.D.P^T$ con P ortogonal.

Hago un cambio de variable $X = P.Y$

$Q(X) = X^T.B.X \Rightarrow Q(Y) = (P.Y)^T.P.D\underbrace{P^T.P}_{I}.Y = Y^T.\underbrace{P^T P}_{I}.D.Y = Y^T.D.Y$

Busco autoval de B:

$P_B(\lambda) = \det\begin{pmatrix} \lambda-11 & -1 \\ -1 & \lambda-11 \end{pmatrix} = (\lambda-11)^2 - 1 = \lambda^2 - 22\lambda + 120 = 0$

$\lambda_1 = 12$

$\lambda_2 = 10 \Rightarrow Q(Y) = (Y_1 \;\; Y_2).\begin{pmatrix} 12 & 0 \\ 0 & 10 \end{pmatrix}.\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = 12.Y_1^2 + 10.Y_2^2$

$12.Y_1^2 + 10.Y_2^2 \leq 12.Y_1^2 + 12Y_2^2 = 12.\underbrace{\|Y\|^2}_{1} = 12$ → 12 es el máximo

$12.Y_1^2 + 10.Y_2^2 \geq 10.Y_1^2 + 10.Y_2^2 = 10.\|Y\|^2 = 10$ → 10 es el mínimo

Se puede ver que los vectores para los cuales se alcanza el máximo son:

$Y_{máx} = (\pm 1 \;\; 0)^T$ y el mínimo: $Y_{mín} = (0 \;\; \pm 1)^T$

Busco autovec para armar P:

$\lambda = 12: \left(\begin{array}{cc|c} 1 & -1 & 0 \\ \cancel{-1} & \cancel{1} & \cancel{0} \end{array}\right) \to a = b \Rightarrow S_{\lambda=12} = gen\{(1 \;\; 1)^T\}$

$\lambda = 10: \left(\begin{array}{cc|c} -1 & -1 & 0 \\ \cancel{-1} & \cancel{-1} & \cancel{0} \end{array}\right) \to a = -b \Rightarrow S_{\lambda=10} = gen\{(-1 \;\; 1)^T\}$

$\Rightarrow X_{máx} = P.Y_{max} = \underbrace{\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{\text{ubicar columnas de acuerdo a la matriz D propuesta}} . \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} = \boxed{\pm \frac{1}{\sqrt{2}} . \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

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$X_{mín} = P \cdot Y_{mín} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} = \boxed{\pm \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

$A^T \cdot A = \begin{pmatrix} 1 & 1 & -3 \\ 1 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 1 \\ 1 & 11 \end{pmatrix}$ esta matriz coincide con B.

Por lo tanto sus valores singulares son $\sigma_1 = \sqrt{\lambda_1} = \sqrt{12}$, $\sigma_2 = \sqrt{\lambda_2} = \sqrt{10}$

Se aprecia que al máximo autovalor le corresponde el máximo valor singular; lo mismo para el mínimo.

39)

c) $A = \underbrace{\begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix}}_{U} \cdot \underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}}_{\Sigma} \cdot \underbrace{\begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}}_{V^T}$

$\Rightarrow A^{-1} = (U \cdot \Sigma \cdot V^T)^{-1} = (V^T)^{-1} \cdot \Sigma^{-1} \cdot U^{-1} = V \cdot \Sigma^{-1} \cdot U^T$ como U y V^T son matrices ortogonales, $U^{-1} = U^T$ y $(V^T)^{-1} = (V^T)^T = V$

$\boxed{A^{-1} = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix} \cdot \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix}}$

e) $A = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{18} & 1/\sqrt{18} & -4/\sqrt{18} \\ -2/3 & 2/3 & 1/3 \end{pmatrix}$

$A = U \cdot \Sigma \cdot V^T \Rightarrow A^T = (U \cdot \Sigma \cdot V^T)^T = V \cdot \Sigma^T \cdot U^T$

$\boxed{A^T = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \\ 0 & -4/\sqrt{18} & 1/3 \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}$

40)

$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} \Rightarrow \overset{\nearrow \text{pseudoinversa}}{A^+} = V \, \Sigma^+ \cdot U^T$ siendo $\Sigma^+ = \begin{array}{r} \\ m-k\{ \end{array} \left(\begin{array}{c|c} \begin{matrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_k \end{matrix} & \overbrace{\quad 0 \quad}^{m-k} \\ \hline 0 & 0 \end{array}\right)$, $A \in \mathbb{R}^{m \times n}$

con $\sigma_1, \ldots, \sigma_k$ valores singulares $\neq 0$.

$\Rightarrow A = \overbrace{\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}^{U} \cdot \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \cdot \overbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{18} & 1/\sqrt{18} & -4/\sqrt{18} \\ -2/3 & 2/3 & 1/3 \end{pmatrix}}^{V^T}$

(es casi como trasponer A, pero Σ se invierte además de trasponerse)

$A^+ = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \\ 0 & -4/\sqrt{18} & 1/3 \end{pmatrix} \cdot \begin{pmatrix} 1/5 & 0 \\ 0 & 1/3 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \boxed{\begin{pmatrix} 7 & 2 \\ 2 & 7 \\ 10 & -10 \end{pmatrix} \cdot \frac{1}{45}}$

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la sol. por cuad. min. es $\hat{x} = A^{+} \cdot b = \frac{1}{45} \cdot \begin{pmatrix} 7 & 2 \\ 2 & 7 \\ 10 & -10 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \boxed{\frac{1}{9} \cdot \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}}$