Ej En $\mathbb{R}^2[x]$ con el p.i.

$$\langle a_0+a_1x+a_2x^2, b_0+b_1x+b_2x^2\rangle = a_0b_0+a_0b_1+a_1b_0+4a_1b_1+a_2b_2$$

Dado $S=\{p\in\mathbb{R}^2[x] / p(0)=-p''(0)\}$ hallar todos los $q\in\mathbb{R}^2[x]$ tales que $\|q\|=8$ y $\operatorname{Proy}_S(q)=4-x-2x^2$.

$\boxed{S}$

$$p(x)=a_0+a_1x+a_2x^2$$

$$p(0)=a_0 \longrightarrow -2a_2=a_0$$

$$p'(x)=a_1+2a_2x$$

$$p''(x)=2a_2 \Rightarrow p''(0)=2a_2$$

$$p(x)=-2a_2+a_1x+a_2x^2=a_2(x^2-2)+a_1x$$

$\Rightarrow S$ es subespacio de $\mathbb{R}^2[x]$ y

$$S=\text{gen}\{x, x^2-2\}$$

$$\operatorname{Proy}_S(q)=4-x-2x^2 \ ¿\in S?$$

$$=-2(x^2-2)-1.x$$

$$q=\underbrace{\operatorname{Proy}_S(q)}_{4-x-2x^2}+\underbrace{\operatorname{Proy}_{S^\perp}(q)}_{?}$$

$$\Rightarrow \|q\|^2=\|\operatorname{Proy}_S(q)\|^2+\|\operatorname{Proy}_{S^\perp}(q)\|^2$$

$$64 = \underbrace{\| Proy_S(q) \|^2}_{\| 4-x-2x^2 \|^2} + \underbrace{\| Proy_{S^\perp}(q) \|^2}_{?}$$

$$\| 4-x-2x^2 \|^2 = \langle 4-x-2x^2, 4-x-2x^2 \rangle =$$

$$4.4 + 4.(-1) + (-1).4 + 4(-1).(-1) + (-2)(-2) = 16.$$

$$\Rightarrow \| Proy_{S^\perp}(q) \|^2 = 64 - 16 = 48$$

$S^\perp = gen\{\ ?\ \}$ como la dim de $\mathbb{R}^2[x]$ es 3 ($\dim \mathbb{R}^2[x] = 3$) y dim de S es 2 ($\dim S = 2$ porque $S = gen\{x, x^2-2\}$ y $\{x, x^2-2\}$ es l.i.) $\Rightarrow \dim S^\perp = 3 - 2 = 1$

$S^\perp = gen\{?\}$ $\quad r(x) = a_0 + a_1 x + a_2 x^2$

- $\underbrace{\langle r, x^2-2 \rangle = 0}_{a_0(-2) + a_1 (-2) + a_2 1 = 0}$

$-2a_0 - 2a_1 + a_2 = 0$

$a_2 = 2a_0 + 2a_1 = -8a_1 + 2a_1 = -6a_1$

- $\underbrace{\langle r, x \rangle = 0}_{a_0 . 1 + 4a_1 . 1 = 0}$

$a_0 = -4a_1$

$$r(x) = -4a_1 + a_1 x - 6a_1 x^2 = a_1(-4 + x - 6x^2)$$

$$S^{\perp} = \text{gen}\{\underbrace{6x^2 - x}_{} + 4\}$$

$$\text{Proy}_{S^{\perp}}(f) = \alpha(6x^2 - x + 4) \quad y$$

$$\underbrace{\|\text{Proy}_{S^{\perp}}(f)\|^2}_{} = 48$$

$$\langle \alpha(6x^2 - x + 4), \alpha(6x^2 - x + 4) \rangle = 48$$

$$\alpha^2 \underbrace{\langle 6x^2 - x + 4, 6x^2 - x + 4 \rangle}_{} = 48$$

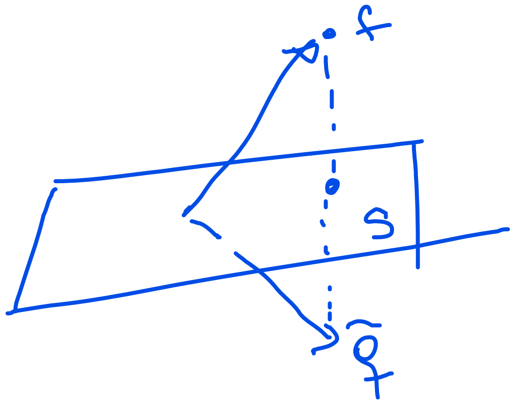
$$\alpha^2 (\underbrace{16 + 4(-1) + (-1)4 + 4(-1)(-1) + 6.6}_{48}) = 48$$

$$\Rightarrow \alpha^2 = 1 \Rightarrow \alpha = 1 \text{ ó } \alpha = -1$$

$$f = \underbrace{\text{Proy}_S(f)}_{4 - x - 2x^2} + \underbrace{\text{Proy}_{S^{\perp}}(f)}_{\alpha(4 - x + 6x^2)} =$$

si $\alpha = 1$ $\quad f(x) = 8 - 2x + 4x^2$

si $\alpha = -1$ $\quad f(x) = -8x^2$



<u>Ejercicio</u> Sea $\mathbb{V}$ un $\mathbb{C}$-espacio vectorial con producto interno $\langle f,g\rangle = \int_{-\pi}^{\pi} f(t)\bar{g}(t)\,dt$.

Sea $B = \{1, e^{it}, e^{-it}\}$ una base del subespacio $S \subseteq V$.

(a) Demuestre que $B' = \{1, \text{sen}(t), \cos(t)\}$ es una base ortogonal de S.

(b) Dado $f(t) = 2-t \in \mathbb{V}$ halle la proyección ortogonal de f sobre S. Calcule $\|f - \text{Proy}_S f\|^2$

(a) para demostrar que B' es base ortogonal de $S = \text{gen}\{1, e^{it}, e^{-it}\}$

tenemos que ver que $\dim S = 3$

$1 \in S$, $\cos(t) \in S$, $\text{sen}(t) \in S$, que

dim de $\text{gen}\{1, \cos(t), \text{sen}(t)\}$ también

es 3 y que (finalmente) $\langle 1, \cos(t)\rangle = 0$

$\langle 1, \text{sen}(t)\rangle = 0$ y $\langle \cos(t), \text{sen}(t)\rangle = 0$

• veamos que B' es base de S:

$\dim S = 3$ (nos lo dice el enunciado al avisarnos que

$B = \{1, e^{it}, e^{-it}\}$ es base de S)

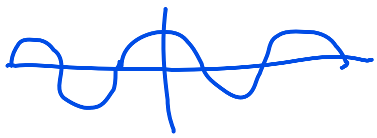
- $1 \in S \Rightarrow$ ✓
- $\cos(t) \in S \Rightarrow$ $\cos(t) = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it}$
- $\operatorname{sen}(t) \in S$

$\Downarrow$

$$\frac{1}{2i}e^{it} - \frac{1}{2i}e^{-it}$$

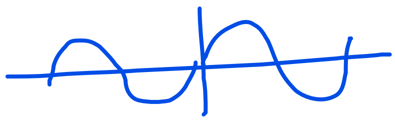
$$e^{it} = \cos(t) + i\operatorname{sen}(t)$$
$$e^{-it} = \cos(-t) + i\operatorname{sen}(-t)$$

sé que $\cos(-t) = \cos(t)$



par

$\operatorname{sen}(-t) = -\operatorname{sen}(t)$



impar

$$\Rightarrow e^{-it} = \cos(t) - i\operatorname{sen}(t)$$

- En ppio tendríamos que ver que $\{1, \cos(t), \operatorname{sen}(t)\}$ es un conjunto li para terminar de afirmar que $B = \{1, \cos(t), \operatorname{sen}(t)\}$ es base de S

Veamos que $\{1, \cos(t), \operatorname{sen}(t)\}$ es un conjunto ortogonal:

$$\langle 1, \cos(t)\rangle = \int_{-\pi}^{\pi} 1 \cdot \overline{\cos(t)}\, dt = \int_{-\pi}^{\pi} \cos(t)\, dt = 0$$

$$= \operatorname{sen}(t)\Big|_{-\pi}^{\pi} = \underbrace{\operatorname{sen}(\pi)}_{0} - \underbrace{\operatorname{sen}(-\pi)}_{0} = 0$$

$$\langle 1, \operatorname{sen}(t)\rangle = \int_{-\pi}^{\pi} 1 \cdot \overline{\operatorname{sen}(t)}\, dt = \int_{-\pi}^{\pi} \operatorname{sen}(t)\, dt = -\cos(t)\Big|_{-\pi}^{\pi} = 0$$

$$\langle \operatorname{sen}(t), \cos(t) \rangle = \int_{-\pi}^{\pi} \underbrace{\operatorname{sen}(t)\cos(t)}_{\text{impar}} dt = 0$$

$\Rightarrow$ $\{1, \operatorname{sen}(t), \cos(t)\}$ es un conjunto ortogonal y $0 \notin$ al conjunto

$\Rightarrow$ podemos afirmar que $\{1, \operatorname{sen}(t), \cos(t)\}$ es un conjunto l.i.

$\Rightarrow$ $B' = \{1, \operatorname{sen}(t), \cos(t)\}$ es base ortogonal de S.

$$\operatorname{Proy}_S (2-t) = \frac{\langle 2-t, 1 \rangle}{\|1\|^2} \cdot 1 + \frac{\langle 2-t, \operatorname{sen}(t) \rangle}{\|\operatorname{sen}(t)\|^2} \operatorname{sen}(t)$$

$$+ \frac{\langle 2-t, \cos(t) \rangle}{\|\cos(t)\|^2} \cos(t)$$

$$\langle 2-t, 1 \rangle = \int_{-\pi}^{\pi} (2-t) \cdot \bar{1}\, dt = \int_{-\pi}^{\pi} (2-t) dt = 4\pi$$

$$\|1\|^2 = \langle 1, 1 \rangle = \int_{-\pi}^{\pi} 1 \cdot \bar{1}\, dt = \int_{-\pi}^{\pi} 1\, dt = 2\pi$$

$$\langle 2-t, \operatorname{sen}(t) \rangle = \int_{-\pi}^{\pi} (2-t) \overline{\operatorname{sen}(t)}\, dt = \int_{-\pi}^{\pi} (2-t) \operatorname{sen}(t) dt = -2\pi$$

$$\int_{-\pi}^{\pi}(2-t)\,\mathrm{sen}(t)\,dt = (2-t)(-\cos(t))\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(-\cos(t))(-1)\,dt$$

$$= \underbrace{-(2-t)\cos(t)\Big|_{-\pi}^{\pi}}_{\underbrace{-(2-\pi)(-1)+(2+\pi)(-1)}_{-2\pi}} - \underbrace{\int_{-\pi}^{\pi}\cos(t)\,dt}_{0} = -2\pi$$

$$\|\mathrm{sen}(t)\|^2 = \pi = \langle \mathrm{sen}(t), \mathrm{sen}(t)\rangle$$

$$\|\cos(t)\|^2 = \pi = \langle \cos(t), \cos(t)\rangle$$

$$\int_{-\pi}^{\pi}(2-t)\cos(t)\,dt = 0$$

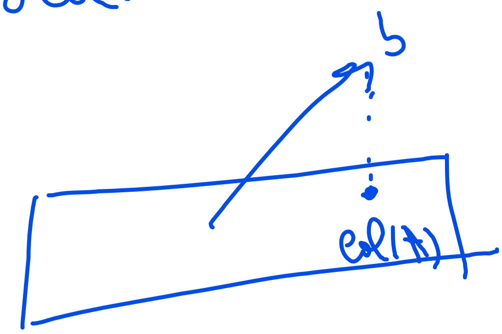
$$\mathrm{Proy}_S(2-t) = \frac{4\pi}{2\pi}\cdot 1 + \frac{-2\pi}{\pi}\,\mathrm{sen}(t)$$

$$= \boxed{2 - 2\,\mathrm{sen}(t)}$$

Ej Sean $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$ y $x_0 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$.

Determinar todos los b de norma 6 para los cuales x_0 es una solución por cuadrados mínimos de $Ax=b$. Para cada uno de los b hallados calcular todas las soluciones por cuadrados mínimos de $Ax=b$.

$Ax_0 = \text{Proy}_{col(A)}(b)$ $\qquad$ $Ax = b$



$$Ax_0 = \begin{pmatrix}1&1&1\\-1&1&0\\0&2&1\end{pmatrix}\begin{pmatrix}0\\2\\0\end{pmatrix} = \begin{pmatrix}2\\2\\4\end{pmatrix} = \text{Proy}_{col(A)}(b)$$

$$b = \underbrace{\begin{pmatrix}2\\2\\4\end{pmatrix}}_{\text{Proy}_{col(A)}(b)} + \underbrace{\;?\;}_{\text{Proy}_{col(A)^{\perp}}(b)} \qquad \longleftarrow \alpha\begin{pmatrix}1\\1\\-1\end{pmatrix}$$

$col(A)^{\perp}$

$$\begin{pmatrix}1\\-1\\0\end{pmatrix} \rightarrow \begin{pmatrix}1 & -1 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$

$$\begin{pmatrix}1\\1\\2\end{pmatrix} \rightarrow \begin{pmatrix}1 & 1 & 2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$

$$\begin{pmatrix}1\\0\\1\end{pmatrix} \rightarrow \begin{pmatrix}1 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$

$$\left(\begin{array}{ccc|c}1&-1&0&0\\1&1&2&0\\1&0&1&0\end{array}\right) \rightarrow \left(\begin{array}{ccc|c}1&-1&0&0\\0&2&2&0\\0&1&1&0\end{array}\right) \rightarrow \left(\begin{array}{ccc|c}1&-1&0&0\\0&1&1&0\\0&0&0&0\end{array}\right)$$

$x - y = 0$ $\qquad$ $x = y$ $\qquad$ $\alpha\begin{pmatrix}1\\1\\-1\end{pmatrix}$

$y + z = 0$ $\qquad$ $z = -y$

$$\|b\|^2 = 36 = \underbrace{\left\| \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} \right\|^2}_{24} + \left\| \alpha \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\|^2$$

$$+ \alpha^2 \cdot 3$$

$$12 = \alpha^2 \cdot 3 \implies \alpha = 2 \quad o \quad \alpha = -2$$

$$b = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$$

$$b = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$$

$$Ax = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix} \qquad Ax = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$$

$A \in \mathbb{R}^{3\times 3}$ $\quad rg(A) = 2$

$\implies$ infinitas soluciones

$$\hat{x} = x_0 + y \quad \text{con} \quad y \in Nul(A)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x+y+z=0 \longrightarrow x=-y-z=y$

$2y+z=0 \rightarrow z=-2y$

$\text{Nul}(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}$ ✓

$$\boxed{\hat{x} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}$$