Ej En $\mathbb{R}^2[x]$ con el p.i.

$\langle a_0+a_1x+a_2x^2, b_0+b_1x+b_2x^2\rangle = a_0b_0+a_0b_1+a_1b_0+4a_1b_1+a_2b_2$

Dado $S=\{p\in\mathbb{R}^2[x] / p(0)=-p''(0)\}$ hallar todos los $q\in\mathbb{R}^2[x]$ tales que $\|q\|=8$ y $\text{Proy}_S(q)=4-x-2x^2$.

$p(x)=a_0+a_1x+a_2x^2$

$p(0)=a_0$ ⟶ $-2a_2=a_0$

$p'(x)=a_1+2a_2x$

$p''(x)=2a_2 \Rightarrow p''(0)=2a_2$

$p(x)=-2a_2+a_1x+a_2x^2=a_2(x^2-2)+a_1x$

$\Rightarrow S$ es subespacio de $\mathbb{R}^2[x]$ y

$S=\text{gen}\{x, x^2-2\}$

$\text{Proy}_S(q)=4-x-2x^2 \overset{?}{\in} S$

$=-2(x^2-2)-1\cdot x$

$q=\underbrace{\text{Proy}_S(q)}_{4-x-2x^2}+\underbrace{\text{Proy}_{S^\perp}(q)}_{?}$

$\Rightarrow \|q\|^2=\|\text{Proy}_S(q)\|^2+\|\text{Proy}_{S^\perp}(q)\|^2$

$$64 = \underbrace{\|\text{Proy}_S(q)\|^2}_{\|4-x-2x^2\|^2} + \underbrace{\|\text{Proy}_{S^\perp}(q)\|^2}_{?}$$

$$\|4-x-2x^2\|^2 = \langle 4-x-2x^2, 4-x-2x^2\rangle =$$

$$4\cdot 4 + 4\cdot(-1) + (-1)\cdot 4 + 4(-1)\cdot(-1) + (-2)(-2) = 16.$$

$$\Rightarrow \|\text{Proy}_{S^\perp}(q)\|^2 = 64 - 16 = 48$$

$S^\perp = \text{gen}\{\ ?\ \}$ como la dim de $\mathbb{R}^2[x]$ es 3 ($\dim \mathbb{R}^2[x] = 3$) y dim de S es 2 ($\dim S = 2$ porque $S = \text{gen}\{x, x^2-2\}$ y $\{x, x^2-2\}$ es l.i.) $\Rightarrow \dim S^\perp = 3-2 = 1$

$S^\perp = \text{gen}\{r\}$, $r(x) = a_0 + a_1 x + a_2 x^2$

- $\langle r, x^2-2\rangle = 0$

$a_0(-2) + a_1\cdot 0 + a_2 1 = 0$

$-2a_0 - 2a_1 + a_2 = 0$

$a_2 = 2a_0 + 2a_1 = -8a_1 + 2a_1 = -6a_1$

- $\langle r, x\rangle = 0$

$a_0\cdot 1 + 4a_1\cdot 1 = 0$

$a_0 = -4a_1$

$r(x) = -4a_1 + a_1 x - 6a_1 x^2 = a_1(-4 + x - 6x^2)$

$S^\perp = \text{gen}\{6x^2 - x + 4\}$

$\text{Proy}_{S^\perp}(p) = \alpha(6x^2 - x + 4)$ y

$\|\text{Proy}_{S^\perp}(p)\|^2 = 48$

$\langle \alpha(6x^2 - x + 4), \alpha(6x^2 - x + 4) \rangle = 48$

$\alpha^2 \langle 6x^2 - x + 4, 6x^2 - x + 4 \rangle = 48$

$\alpha^2 (16 + 4(-1) + (-1)4 + 4(-1)(-1) + 6 \cdot 6) = 48$

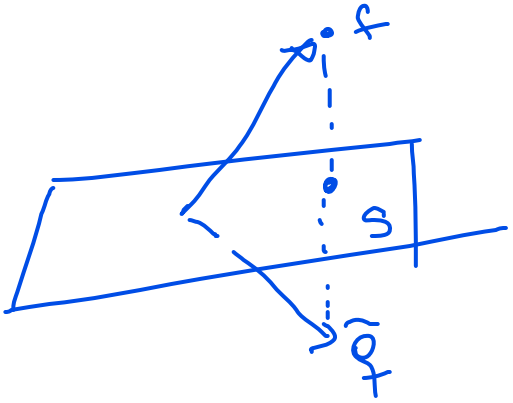
48

$\Rightarrow \alpha^2 = 1 \Rightarrow \alpha = 1$ ó $\alpha = -1$.

$q = \underbrace{\text{Proy}_S(p)}_{4 - x - 2x^2} + \underbrace{\text{Proy}_{S^\perp}(p)}_{\alpha(4 - x + 6x^2)} =$

si $\alpha = 1$ $q(x) = 8 - 2x + 4x^2$

si $\alpha = -1$ $q(x) = -8x^2$



Ejercicio Sea V un $\mathbb{C}$-espacio vectorial con producto interno $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)\overline{g}(t)\,dt$.

Sea $B = \{1, e^{it}, e^{-it}\}$ una base del subespacio $S \subseteq V$.

(a) Demuestre que $B' = \{1, \operatorname{sen}(t), \cos(t)\}$ es una base ortogonal de S.

(b) Dada $f(t) = 2 - t \in V$ halle la proyección ortogonal de f sobre S. Calcule $\|f - \operatorname{Proy}_S f\|^2$

(a) para demostrar que B' es base ortogonal de $S = \operatorname{gen}\{1, e^{it}, e^{-it}\}$

tenemos que ver que $\dim S = 3$

$1 \in S$, $\cos(t) \in S$, $\operatorname{sen}(t) \in S$, que $\dim$ de $\operatorname{gen}\{1, \cos(t), \operatorname{sen}(t)\}$ también es 3 y que (fácilmente) $\langle 1, \cos(t)\rangle = 0$ $\langle 1, \operatorname{sen}(t)\rangle = 0$ y $\langle \cos(t), \operatorname{sen}(t)\rangle = 0$

• vemos que B' es base de S:

$\dim S = 3$ (nos lo dice el enunciado al principio) que

$B = \{1, e^{it}, e^{-it}\}$ es base de $\mathcal{S}$)

- $1 \in \mathcal{S} \Rightarrow$ ✓
- $\cos(t) \in \mathcal{S} \Rightarrow \cos(t) = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it}$
- $\operatorname{sen}(t) \in \mathcal{S}$

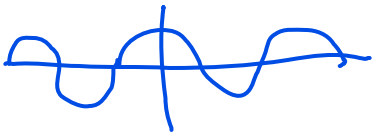
$$\Downarrow$$

$$\frac{1}{2i}e^{it} - \frac{1}{2i}e^{-it}$$

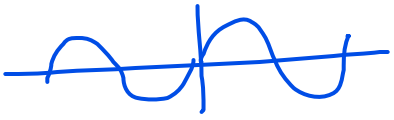
$e^{it} = \cos(t) + i\operatorname{sen}(t)$

$e^{-it} = \cos(-t) + i\operatorname{sen}(-t)$

sé que $\cos(-t) = \cos(t)$ par



$\operatorname{sen}(-t) = -\operatorname{sen}(t)$ impar



$\Rightarrow e^{-it} = \cos(t) - i\operatorname{sen}(t)$

- En ppio tendríamos que ver que $\{1, \cos(t), \operatorname{sen}(t)\}$ es un conjunto li para terminar de afirmar que $B = \{1, \cos(t), \operatorname{sen}(t)\}$ es base de $\mathcal{S}$

Veamos que $\{1, \cos(t), \operatorname{sen}(t)\}$ es un conjunto ortogonal:

$$\langle 1, \cos(t)\rangle = \int_{-\pi}^{\pi} 1 \cdot \overline{\cos(t)}\, dt = \int_{-\pi}^{\pi} \cos(t)\, dt = 0$$

$$= \operatorname{sen}(t)\Big|_{-\pi}^{\pi} = \underbrace{\operatorname{sen}(\pi)}_{0} - \underbrace{\operatorname{sen}(-\pi)}_{0} = 0$$

$$\langle 1, \operatorname{sen}(t)\rangle = \int_{-\pi}^{\pi} 1 \cdot \overline{\operatorname{sen}(t)}\, dt = \int_{-\pi}^{\pi} \operatorname{sen}(t)\, dt = -\cos(t)\Big|_{-\pi}^{\pi} = 0$$

$$\langle \operatorname{sen}(t), \cos(t) \rangle = \int_{-\pi}^{\pi} \underbrace{\operatorname{sen}(t)\cos(t)}_{\text{impar}} dt = 0$$

$\Rightarrow \{1, \operatorname{sen}(t), \cos(t)\}$ es un conjunto ortogonal y $0 \notin$ al conjunto

$\Rightarrow$ podemos afirmar que $\{1, \operatorname{sen}(t), \cos(t)\}$ es un conjunto l.i.

$\Rightarrow B' = \{1, \operatorname{sen}(t), \cos(t)\}$ es base ortogonal de S.

$$\operatorname{Proy}_S(2-t) = \frac{\langle 2-t, 1 \rangle}{\|1\|^2} \cdot 1 + \frac{\langle 2-t, \operatorname{sen}(t) \rangle}{\|\operatorname{sen}(t)\|^2} \operatorname{sen}(t)$$

$$+ \frac{\langle 2-t, \cos(t) \rangle}{\|\cos(t)\|^2} \cos(t)$$

$$\langle 2-t, 1 \rangle = \int_{-\pi}^{\pi} (2-t) \cdot \overline{1}\, dt = \int_{-\pi}^{\pi} (2-t)\, dt = 4\pi$$

$$\|1\|^2 = \langle 1, 1 \rangle = \int_{-\pi}^{\pi} 1 \cdot \overline{1}\, dt = \int_{-\pi}^{\pi} 1\, dt = 2\pi$$

$$\langle 2-t, \operatorname{sen}(t) \rangle = \int_{-\pi}^{\pi} (2-t)\overline{\operatorname{sen}(t)}\, dt = \int_{-\pi}^{\pi} (2-t)\operatorname{sen}(t)\, dt = -2\pi$$

$$\int_{-\pi}^{\pi}(2-t)\operatorname{sen}(t)\,dt = (2-t)\cdot(-\cos(t))\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(-\cos(t))(-1)\,dt$$

$$= -(2-t)\cos(t)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\cos(t)\,dt = -2\pi$$

$$-(2-\pi)(-1) + (2+\pi)(-1) = -2\pi$$

$$\|\operatorname{sen}(t)\|^2 = \pi = \langle \operatorname{sen}(t), \operatorname{sen}(t)\rangle$$

$$\|\cos(t)\|^2 = \pi = \langle \cos(t), \cos(t)\rangle$$

$$\int_{-\pi}^{\pi}(2-t)\cos(t)\,dt = 0$$

$$\operatorname{Proy}_S(2-t) = \frac{4\pi}{2\pi}\cdot 1 + \frac{-2\pi}{\pi}\operatorname{sen}(t)$$

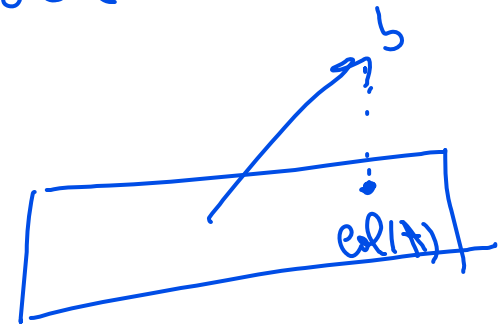
$$= \boxed{2 - 2\operatorname{sen}(t)}$$

Ej Sean $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$ y $x_0 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$.

Determinar todos los b de norma 6 para los cuales x_0 es una solución por cuadrados mínimos de $Ax = b$.

Para cada uno de los b hallados calcular todas las soluciones por cuadrados mínimos de $Ax = b$.

$Ax_0 = Proy_{col(A)}(b)$ $\quad Ax = b$



$$Ax_0 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = Proy_{col(A)}(b)$$

$$b = \underbrace{\begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}}_{Proy_{col(A)}(b)} + \underbrace{?}_{Proy_{col(A)^\perp}(b)} \leftarrow \alpha\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$col(A)^\perp$

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x - y = 0 \quad x = y$

$y + z = 0 \quad z = -y$

$$\alpha\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\|b\|^2 = 36 = \underbrace{\left\|\begin{pmatrix}2\\2\\4\end{pmatrix}\right\|^2}_{24} + \left\|\alpha\begin{pmatrix}1\\1\\-1\end{pmatrix}\right\|^2 + \alpha^2 \cdot 3$$

$$12 = \alpha^2 \cdot 3 \Rightarrow \alpha = 2 \text{ o } \alpha = -2$$

$$b = \begin{pmatrix}2\\2\\4\end{pmatrix} + 2\begin{pmatrix}1\\1\\-1\end{pmatrix} = \begin{pmatrix}4\\4\\2\end{pmatrix}$$

$$b = \begin{pmatrix}2\\2\\4\end{pmatrix} - 2\begin{pmatrix}1\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\6\end{pmatrix}$$

$$Ax = \begin{pmatrix}4\\4\\2\end{pmatrix} \qquad \left\{ \quad AX = \begin{pmatrix}0\\0\\6\end{pmatrix} \right.$$

$A \in \mathbb{R}^{3\times 3}$ $\quad rg(A) = 2$

$\Rightarrow$ infinitas soluciones

$$\hat{X} = X_0 + y \quad \text{con} \quad y \in Nul(A)$$

$$\left(\begin{array}{ccc|c}1&1&1&0\\-1&1&0&0\\0&2&1&0\end{array}\right) \to \left(\begin{array}{ccc|c}1&1&1&0\\0&2&1&0\\0&2&1&0\end{array}\right) \to \left(\begin{array}{ccc|c}1&1&1&0\\0&2&1&0\\0&0&0&0\end{array}\right)$$

$x+y+z=0 \longrightarrow x=-y-z=y$

$2y+z=0 \rightarrow z=-2y$

$\text{Nul}(A)=\text{gen}\left\{\begin{pmatrix}1\\1\\-2\end{pmatrix}\right\}$ ✓

$$\hat{x}=\begin{pmatrix}0\\2\\0\end{pmatrix}+\beta\begin{pmatrix}1\\1\\-2\end{pmatrix}$$